Lecture 39.

Fourier symmetry for discrete quantum waves

(Ch. 4-8 of Unit 3   5.3.12)

2-State Schrodinger quantum analogy with classical 2D-HO *Review of Lecture 38*

ABCD Symmetry operator analysis and U(2) spinors

ABCD Time evolution operator and U(2)~R(3) spin spaces

The "Crazy-Thing-Theorem" and generalized complex operators

Example 1: A or Z rotation

Example 2: C or Y rotation

Example 3: Any $\varphi=\omega t$-axial rotation

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Life in 2D Spinor space is "Half-Fast"

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z,)$ in field $\mathbf{B}=(B_x, B_y, B_z)$

Euler's definition of spin state of rotation

Using spin-1/2 matrix $\mathbf{R}(\alpha/2)$ and $\mathbf{R}(\beta/2)$

3D Stokes vector components S_a definition

Polarization ellipses, 2D HO orbits, and spin- states

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A+B\boldsymbol{\sigma}_B$

Matrix-operator spectral decomposition, eigenvectors, and eigenvalues

Example matrix $\mathbf{M}=\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}$

$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k=\prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j)=\mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m)=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step is to make *Idempotent Projectors:* $\mathbf{P}_k=\dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}=\dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}_j\mathbf{P}_k=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}\qquad \mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$$

They're *Ortho-Normal* and satisfy *Completeness Relation* $\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+...+\mathbf{P}_n$

Eigen-operators $\mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator $\mathbf{M}$

$$\mathbf{M}=\mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+...+\mathbf{M}\mathbf{P}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+...+\varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function $f(\mathbf{M})$ of $\mathbf{M}$

$$f(\mathbf{M})==f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+...+f(\varepsilon_n)\mathbf{P}_n$$

From Lecture 37

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1 & 1\\ 3 & -3\end{pmatrix}\qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

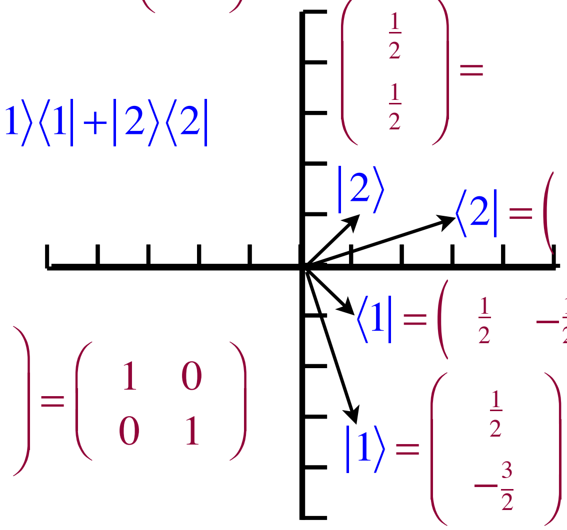
$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}=|1\rangle\langle 1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3 & 1\\ 3 & 1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}=|2\rangle\langle 2|$$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=|1\rangle\langle 1|+|2\rangle\langle 2|$$



$$|2\rangle=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}\qquad \langle 2|=\begin{pmatrix}\frac{3}{2} & \frac{1}{2}\end{pmatrix}\qquad \langle 1|=\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\end{pmatrix}\qquad |1\rangle=\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\end{pmatrix}$$

$$\begin{pmatrix}\langle 1|1\rangle & \langle 1|2\rangle\\ \langle 2|1\rangle & \langle 2|2\rangle\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

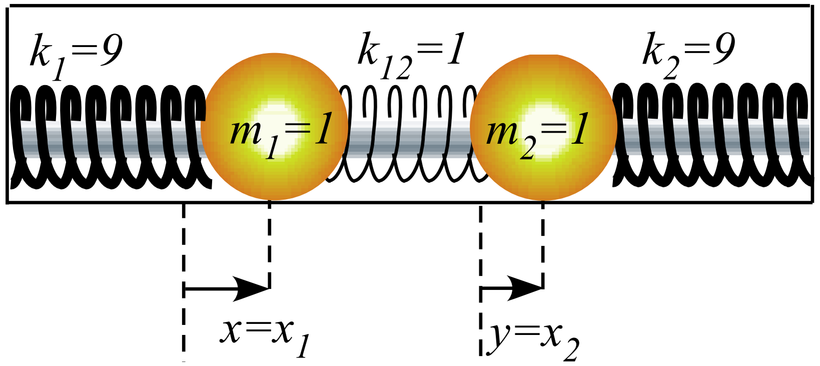
$$\mathbf{M}=\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle 1|+5|2\rangle\langle 2|$$

$$=1\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$$

Examples with $\mathbf{M}^{50}=\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}=1^{50}\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix}+5^{50}\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$

$$\sqrt{\mathbf{M}}=\begin{pmatrix}4 & 1\\ 3 & 2\end{pmatrix}=\pm\sqrt{1}\begin{pmatrix}\frac{1}{4} & -\frac{1}{4}\\ -\frac{3}{4} & \frac{3}{4}\end{pmatrix}\pm\sqrt{5}\begin{pmatrix}\frac{3}{4} & \frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$$

Spectral decomposition of 2D-HO mode dynamics



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10\end{pmatrix}$$

The **K** secular equation $\quad K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+99=0=(K-9)(K-11)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2 \qquad K_1=\omega_0^2(\varepsilon_1)=9,\quad K_2=\omega_0^2(\varepsilon_2)=11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11\end{pmatrix}}{9-11}=\frac{\begin{pmatrix} 1 & +1 \\ +1 & 1\end{pmatrix}}{2}$$

$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9\end{pmatrix}}{11-9}=\frac{\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}}{2}$$

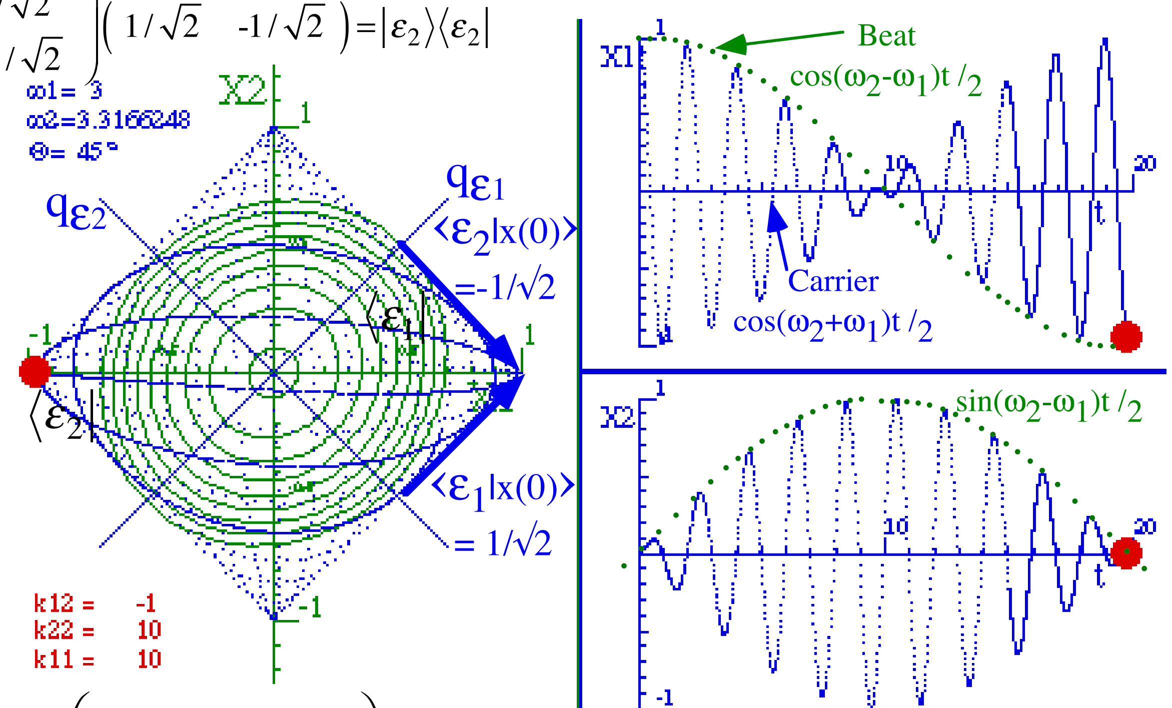
$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2}\end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$



100% modulation (SWR=0)

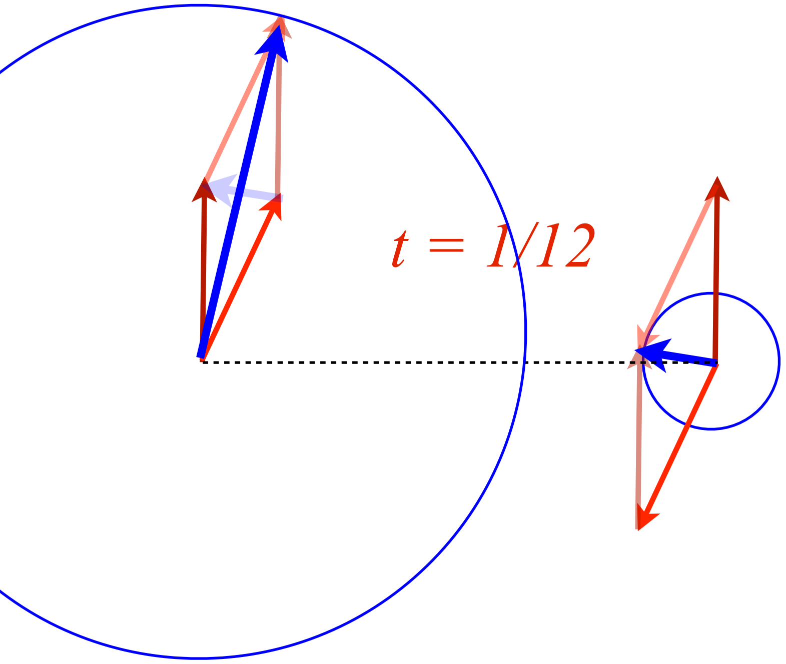
$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}=\begin{pmatrix} \dfrac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \dfrac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix}=\frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t}+e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t}-e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix}=e^{-i\frac{(\omega_1+\omega_2)}{2}t}\begin{pmatrix} \cos\dfrac{(\omega_2-\omega_1)t}{2} \\ i\sin\dfrac{(\omega_2-\omega_1)t}{2}\end{pmatrix}$$

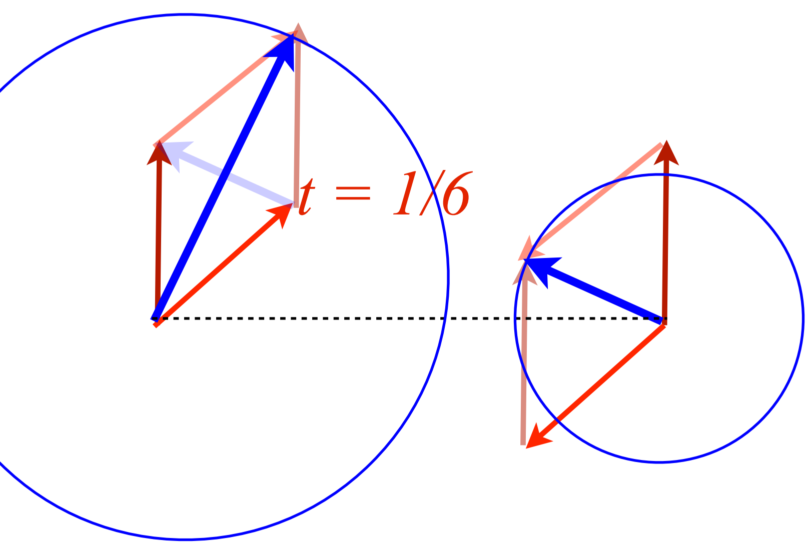
Note the i phase

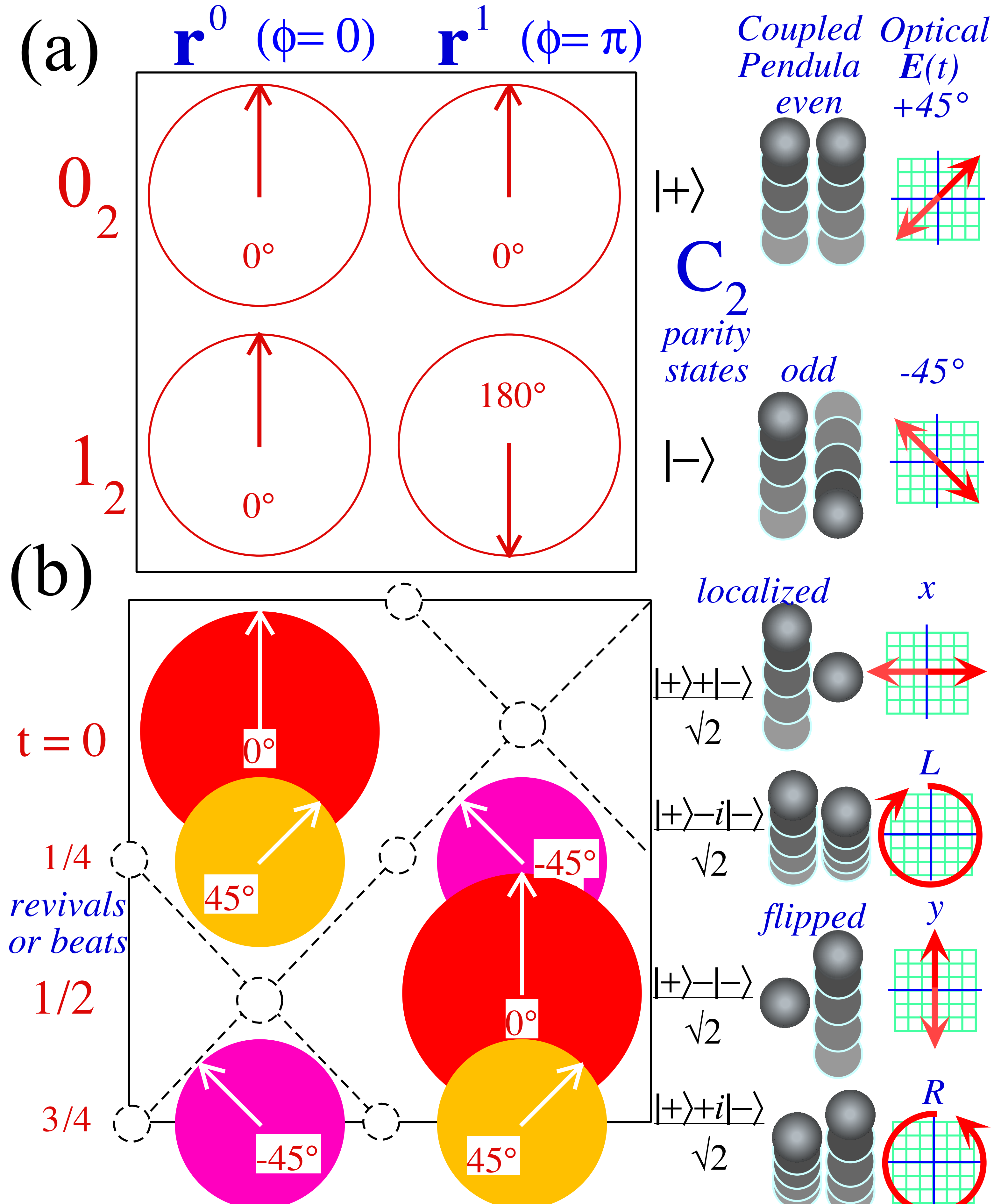
From Lecture 38

Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

2D-HO beats and mixed mode geometry







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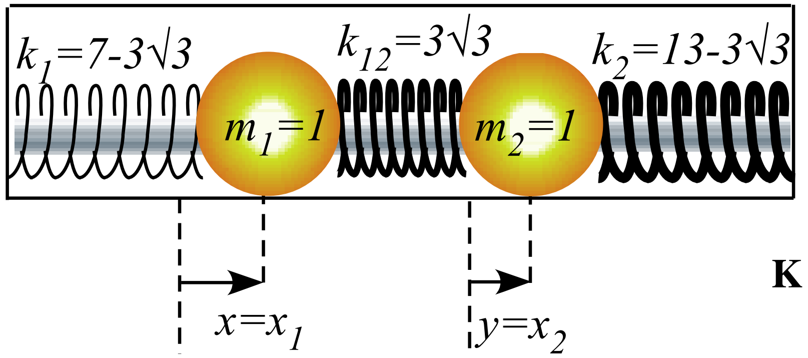
Spectral decomposition of 2D-HO mode dynamics for lower symmetry



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$ $\quad Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$=\begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4}\end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$=\begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{3}{4}\end{pmatrix}=\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix}\sqrt{3}/2 & 1/2\end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix}-1/2 & \sqrt{3}/2\end{pmatrix}$

Spectral decomposition of initial state ***x****(0)=(1,0):*

$$\mathbf{1}\cdot\mathbf{x}(0)=(\mathbf{P}_1+\mathbf{P}_2)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2}\end{pmatrix}\otimes\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix}\left(\tfrac{\sqrt{3}}{2}\right)+\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2}\end{pmatrix}\left(-\tfrac{1}{2}\right)$$

(Note projection onto eigen-axes)

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t, \quad q_2(t)=-\frac{1}{2}\cos 4t\right)$$

Using $\cos 4t = 2\cos^2 2t - 1$ derives a parabolic trajectory!

$$q_2 = -\frac{1}{2}2\cos^2 2t + \frac{1}{2} = -\frac{4}{3}q_1^2 + \frac{1}{2}$$

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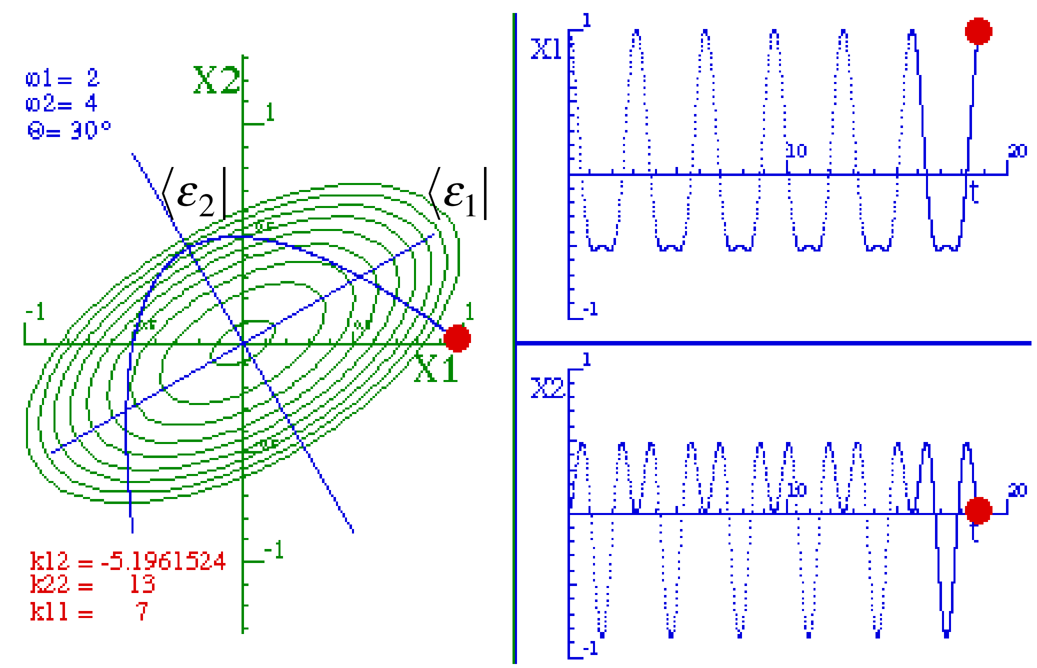


Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential (V=const.) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\varepsilon_1)$=2.0, $\omega_0(\varepsilon_2)$= 4.0) and zero initial velocity.

2-State Schrodinger equations $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ are analogous to classical 2D coupled oscillator equations.

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of real real 1st-order differential equations.

$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$
$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

Then start with classical Hamiltonian.

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1 = \frac{\partial H_c}{\partial p_1} = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -\frac{\partial H_c}{\partial x_1} = -\left(Ax_1 + Bx_2 + Cp_2\right)$$
$$\dot{x}_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

QM vs. Classical Equations are identical

Finally a 2nd time derivative (Assume *constant A, B, D,* and *let C=0*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A + D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2 &= B\dot{p}_1 + D\dot{p}_2 + C\dot{x}_1 \\ &= -B\left(Ax_1 + Bx_2 + Cp_2\right) - D\left(Bx_1 + Dx_2 - Cp_1\right) + C\left(Ap_1 + Bp_2 - Cx_2\right) \\ &= -\left(AB + BD\right)x_1 - \left(B^2 + D^2 + C^2\right)x_2 + C\left(A + D\right)p_1\end{aligned}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2+B^2 & AB+BD \\ AB+BD & B^2+D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2+B^2 & AB+BD \\ AB+BD & B^2+D^2 \end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle}$

From Lecture 38

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

$$= \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

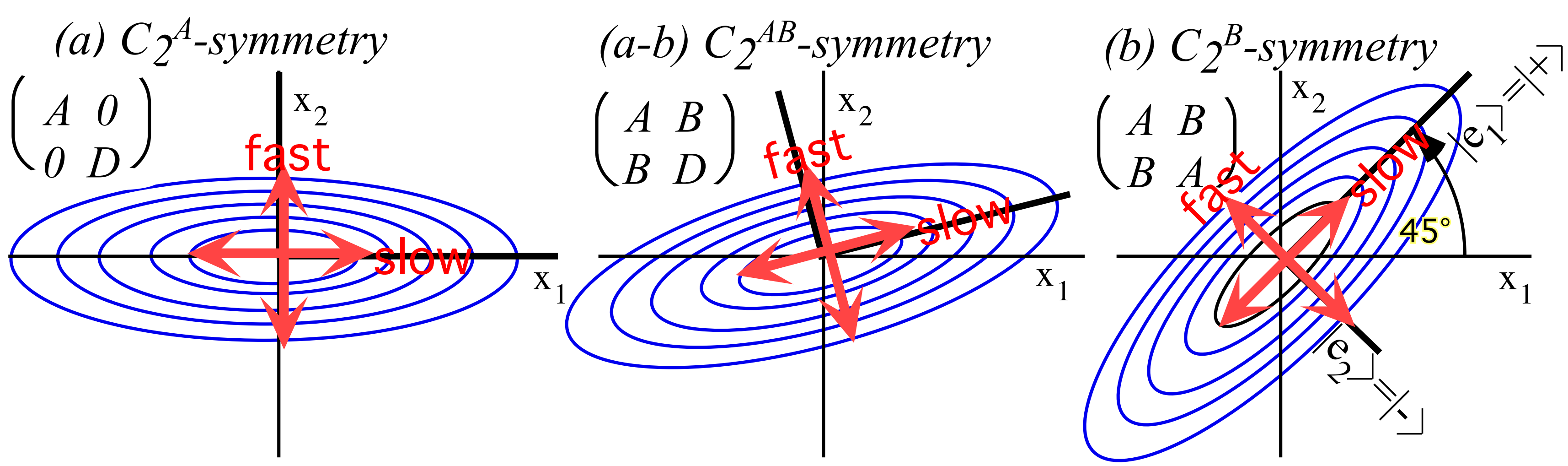
$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*
The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are best known as *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

In 1843 Hamilton discovers *quaternions* $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. They are related to σ's: $\{\sigma_1=\mathbf{1}=\sigma_0, i\sigma_B=\mathbf{i}=i\sigma_X, i\sigma_C=\mathbf{j}=i\sigma_Y, i\sigma_A=\mathbf{k}=i\sigma_Z\}$.

Each Hamilton quaternion squares to *negative*-**1** ($\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-\mathbf{1}$) like imaginary number $i^2=-1$. (They make up the Quaternion group.)

Each Pauli σ_μ squares to *positive*-**1** ($\sigma_X^2=\sigma_Y^2=\sigma_Z^2=+\mathbf{1}$) (Each makes a cyclic C_2 group $C_2^A=\{\mathbf{1},\sigma_A\}$, $C_2^B=\{\mathbf{1},\sigma_B\}$, or $C_2^C=\{\mathbf{1},\sigma_C\}$.)



From Lecture 38

Fig. 3.4.1 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral U(2)system.

ABCD Time evolution operator

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t - i\boldsymbol{\sigma}_\omega \sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad \sigma_B = \sigma_X$ $\quad \sigma_C = \sigma_Y$

where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

Symmetry relations make spinors σ_X , σ_Y , and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

σ-products do dot • *and* cross × products by symmetries: $\sigma_X\sigma_Y = i\sigma_Z = -\sigma_Y\sigma_X$, $\quad \sigma_Z\sigma_X = i\sigma_Y = -\sigma_X\sigma_Z$, $\quad \sigma_Y\sigma_Z = i\sigma_X = -\sigma_Z\sigma_Y$

$$\sigma_a\sigma_b = (\sigma\bullet\mathbf{a})(\sigma\bullet\mathbf{b}) = (a_X\sigma_X + a_Y\sigma_Y + a_Z\sigma_Z)(b_X\sigma_X + b_Y\sigma_Y + b_Z\sigma_Z)$$

$$= \begin{matrix} a_Xb_X\mathbf{1} & +a_Xb_Y\sigma_X\sigma_Y & -a_Xb_Z\sigma_Z\sigma_X \\ -a_Yb_X\sigma_X\sigma_Y & +a_Yb_Y\mathbf{1} & +a_Yb_Z\sigma_Y\sigma_Z \\ +a_Zb_X\sigma_Z\sigma_X & -a_Zb_X\sigma_Y\sigma_Z & +a_Zb_Z\mathbf{1} \end{matrix} = (a_Xb_X + a_Yb_Y + a_Zb_Z)\mathbf{1} \begin{matrix} +i(a_Yb_Z - a_Zb_Y)\sigma_X \\ +i(a_Zb_X - a_Xb_Z)\sigma_Y \\ +i(a_Xb_Y - a_Yb_X)\sigma_Z \end{matrix}$$

Write the product in Gibbs notation. (Where do you think Gibbs *got* his {**i,j,k**} notation!)

$$\sigma_a\sigma_b = (\sigma\bullet\mathbf{a})(\sigma\bullet\mathbf{b}) = \quad (\mathbf{a}\bullet\mathbf{b})\mathbf{1} \quad + \quad i(\mathbf{a}\times\mathbf{b})\bullet\sigma$$

(Recall (1.10.29). in complex variable unit.)

$$A^*B = (A_X + iA_Y)^*(B_X + iB_Y) = (A_X - iA_Y)(B_X + iB_Y)$$
$$= (A_XB_X + A_YB_Y) + i(A_XB_Y - A_YB_X) = \ (\mathbf{A}\bullet\mathbf{B}) \ + \ i(\mathbf{A}\times\mathbf{B})_Z$$

From Lecture 38

ABCD Time evolution operator

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Lecture 38 ends here

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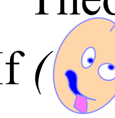
If ()^2= -**1**
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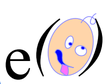
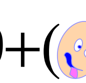
e()θ =**1**cosθ+()sinθ

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Example 1: A or Z rotation

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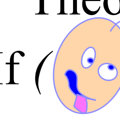
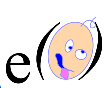
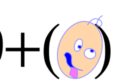
If ()^2 = -**1**

Then:

e^()θ = **1**cosθ + ()sinθ

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Example 1:
A or Z
rotation

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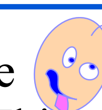
Example 2:
C or Y
rotation

The 
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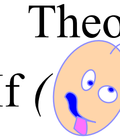
If ()^2 = -**1**
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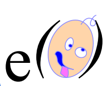
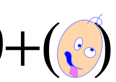
e()θ = **1**cosθ+()sinθ

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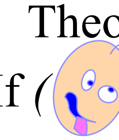
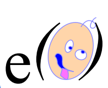
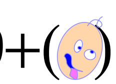
If ()^2= -**1**
Then:
e()θ = **1**cosθ+()sinθ

$$e^{-i(\vec{\omega}\bullet\vec{\varphi})} = \mathbf{1}\cos\varphi - i\boldsymbol{\sigma}_\omega \sin\varphi = \mathbf{1}\cos\varphi - i\ (\sigma\bullet\hat{\varphi})\sin\varphi = \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi \end{pmatrix}$$ where: $\vec{\varphi} = \vec{\omega}\cdot t$

Example 3:
Any $\varphi=\omega t$-axial
rotation

ABCD Time evolution operator

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t - i\sigma_\omega\sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\varphi} = 1\cos\varphi - i\sin\varphi$$

generalizes to:

$$e^{-i\sigma_a\varphi} = \mathbf{1}\cos\varphi - i\,\sigma_a\sin\varphi$$

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

We test these operators by making them rotate each other....

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

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$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2} \\ B \\ C\end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

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generalizes to:

$$e^{-i\boldsymbol{\sigma}_a\varphi} = \mathbf{1}\cos\varphi \ -\ i\ \boldsymbol{\sigma}_a \sin\varphi$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\varphi_A = \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 2: C or Y rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\sin\varphi_C = \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

3D vector $\hat{\mathbf{a}}$ defines a combination $\boldsymbol{\sigma}_a = a_A\boldsymbol{\sigma}_A + a_B\boldsymbol{\sigma}_B + a_C\boldsymbol{\sigma}_C$ of *operators* $\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B, \boldsymbol{\sigma}_C$.

These may be rotated by 2-by-2 $\boldsymbol{\sigma}_a$ matrices acting *twice*, fore and aft^{-1}

The result is rotation by *twice* the *2D* angle φ_a.

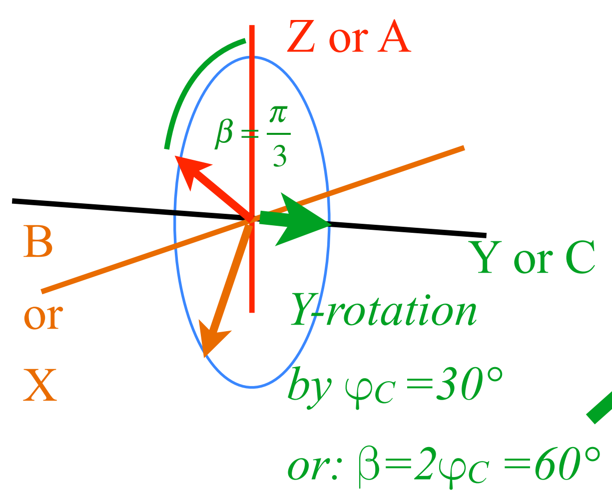
$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C \\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\sin 2\varphi_C$$

$$= \boldsymbol{\sigma}_A\ \cos 2\varphi_C + \boldsymbol{\sigma}_B\ \sin 2\varphi_C$$



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ABCD Time evolution operator

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$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\quad\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\quad\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2} \\ B \\ C\end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

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Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\varphi_A = \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 2: C or Y rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\sin\varphi_C = \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

3D vector $\hat{\mathbf{a}}$ defines a combination $\boldsymbol{\sigma}_a = a_A\boldsymbol{\sigma}_A + a_B\boldsymbol{\sigma}_B + a_C\boldsymbol{\sigma}_C$ of *operators* $\boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B, \boldsymbol{\sigma}_C$.

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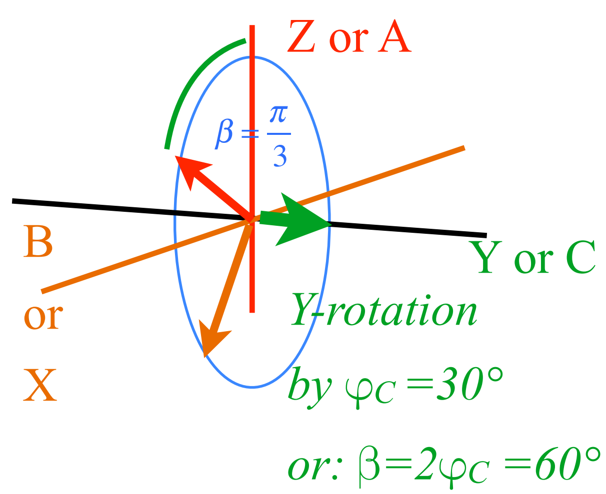
$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$
$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C \\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\sin 2\varphi_C$$
$$= \boldsymbol{\sigma}_A\cos 2\varphi_C + \boldsymbol{\sigma}_B\sin 2\varphi_C$$



$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_B\cdot\mathbf{R}^{-1}(\varphi_C)$$
$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C \\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}-2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C - \sin^2\varphi_C \\ \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}\sin 2\varphi_C + \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cos 2\varphi_C$$
$$= -\boldsymbol{\sigma}_A\sin 2\varphi_C + \boldsymbol{\sigma}_B\cos 2\varphi_C$$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Notation for 2D Spinor space

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega} \bullet \vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

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$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_I, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_I = \sigma_0,\ \sigma_B = \sigma_X,\ \sigma_C = \sigma_Y,\ \sigma_A = \sigma_Z\}$

The $\{\mathbf{1}, \mathsf{S}_A, \mathsf{S}_B, \mathsf{S}_C\}$ are the *Jordan-Angular-Momentum operators* $\{\mathbf{1} = \sigma_0,\ \mathsf{S}_B = \mathsf{S}_X,\ \mathsf{S}_C = \mathsf{S}_Y,\ \mathsf{S}_A = \mathsf{S}_Z\}$
(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Notation for 2D Spinor space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$=\omega_0\,\sigma_0+\omega_A\,\sigma_A+\omega_B\,\sigma_B+\omega_C\,\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

$$=\Omega_0\,\mathbf{1}+\Omega_A\,\mathbf{S}_A+\Omega_B\,\mathbf{S}_B+\Omega_C\,\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

Notation for 3D Vector space

$$=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+(A-D)\begin{pmatrix}\frac{1}{2}&0\\0&-\frac{1}{2}\end{pmatrix}+2B\begin{pmatrix}0&\frac{1}{2}\\\frac{1}{2}&0\end{pmatrix}+2C\begin{pmatrix}0&-\frac{i}{2}\\\frac{i}{2}&0\end{pmatrix}$$

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Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix}\omega_A\\\omega_B\\\omega_C\end{pmatrix}\cdot t=\begin{pmatrix}\frac{A-D}{2}\\B\\C\end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix}A&B-iC\\B+iC&D\end{pmatrix}\cdot t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\sigma_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\sigma_\omega\sin\omega\cdot t\right)$$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Notation for 2D Spinor space

$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

$$=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

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(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\sigma_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\boldsymbol{\sigma}_\omega\sin\omega\cdot t\right)$$

$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t}=e^{-i\Omega_0\cdot t}e^{-i\vec{\Omega}\cdot t\bullet\mathbf{S}}=e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\boldsymbol{\sigma}_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

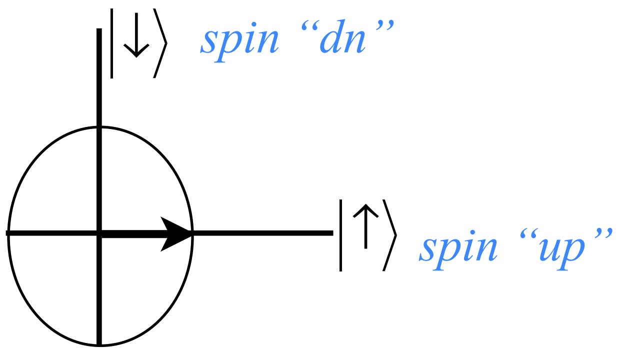
Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

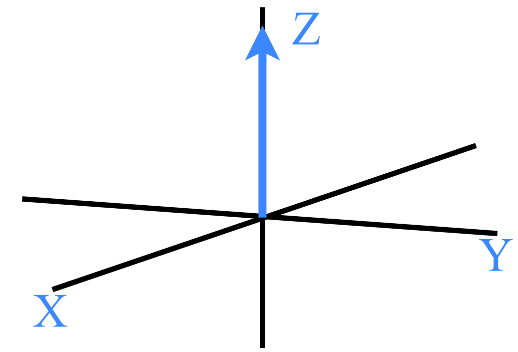
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor {|↑⟩,|↓⟩}-space (complex)

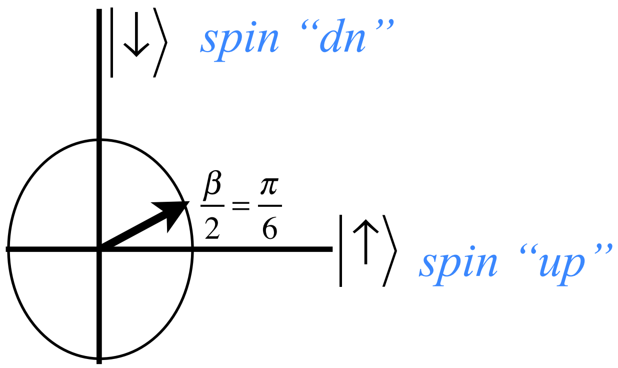
R(3):3D Spin Vector {S_X, S_Y, S_Z}-space (real)



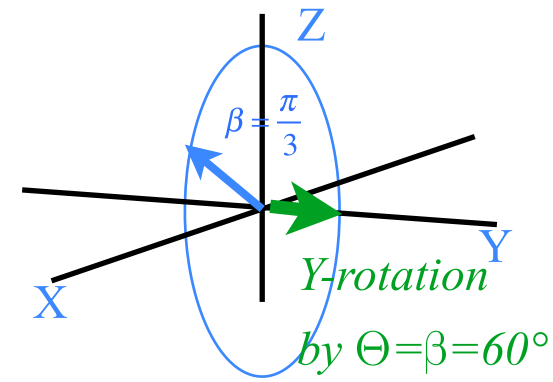
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$



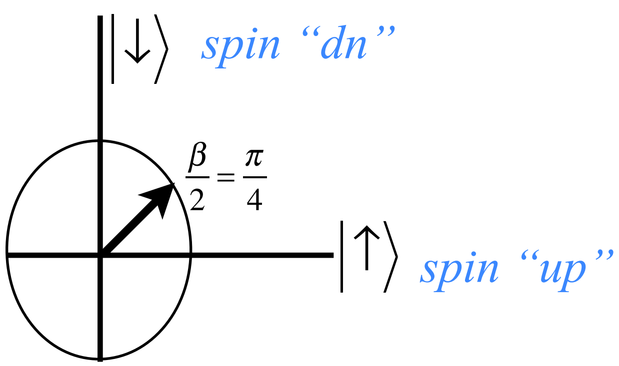
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$



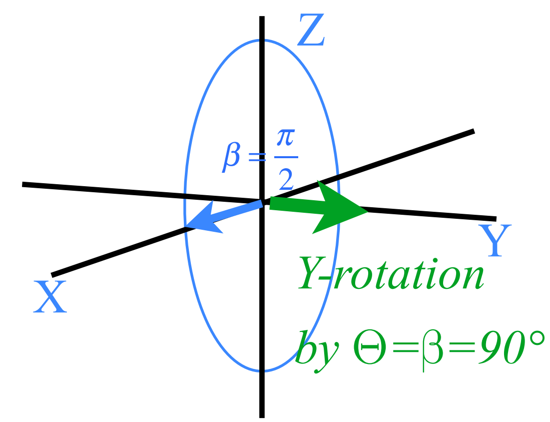
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



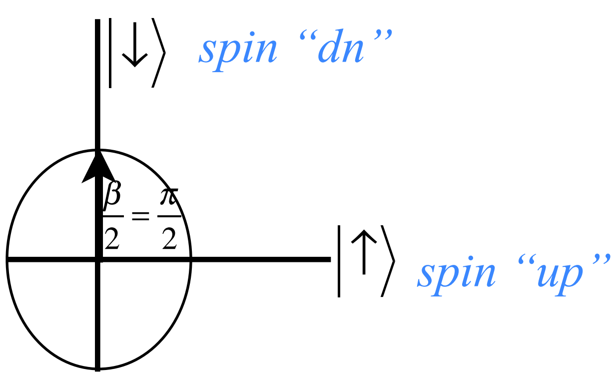
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}\sqrt{3}/2\\ 0\\ 1/2\end{pmatrix}$$



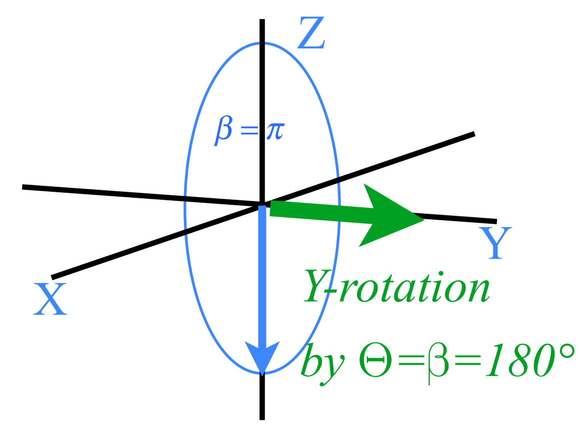
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$$

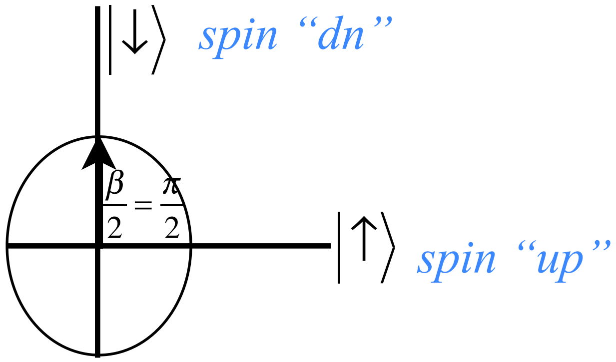


$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$

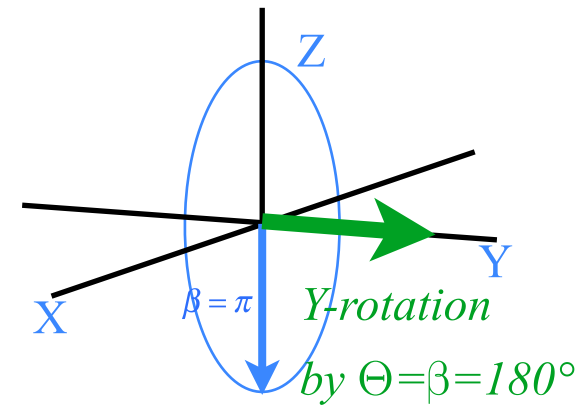
Life in 2D Spinor space is "Half-Fast"

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

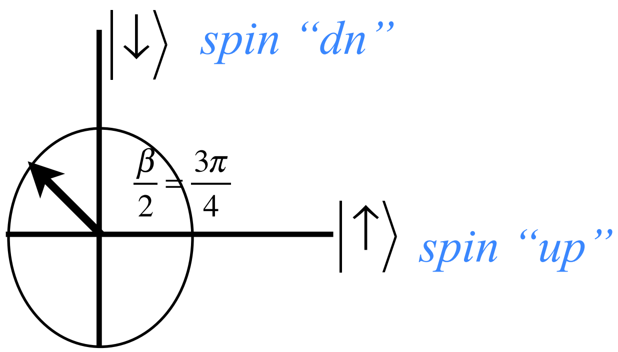
U(2):2D Spinor {|↑⟩,|↓⟩}-space (complex) — *R(3):3D Spin Vector {SX, SY, SZ}-space (real)*



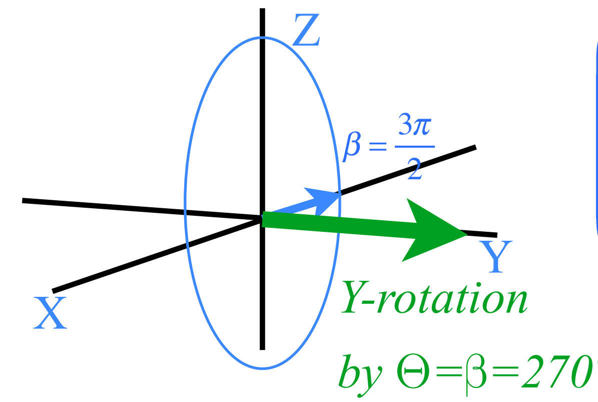
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$$



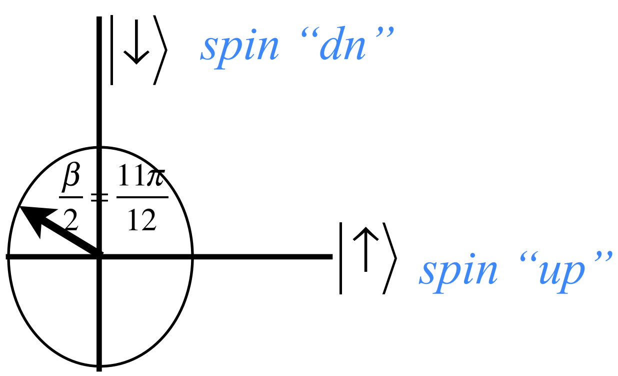
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$



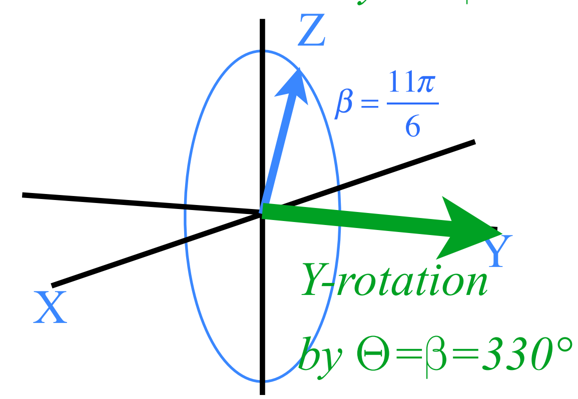
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



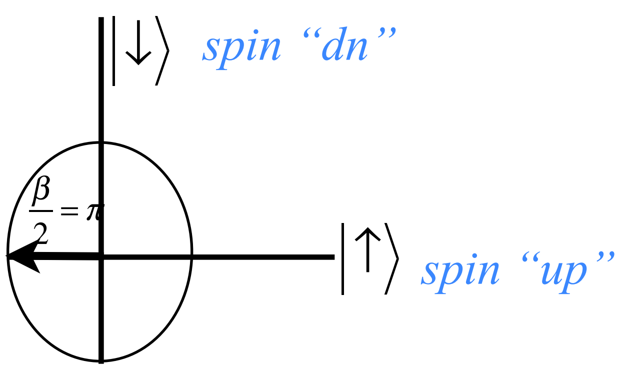
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}$$



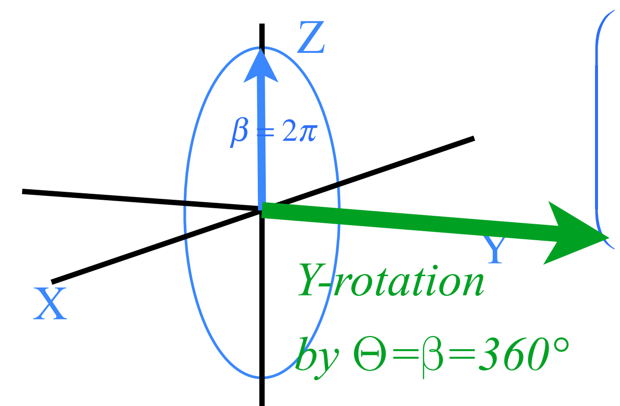
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{-\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}-1\\ 0\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast" and needs Θ=4π=720° to return to original state

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ in field $\mathbf{B}=(B_x, B_y, B_z)$

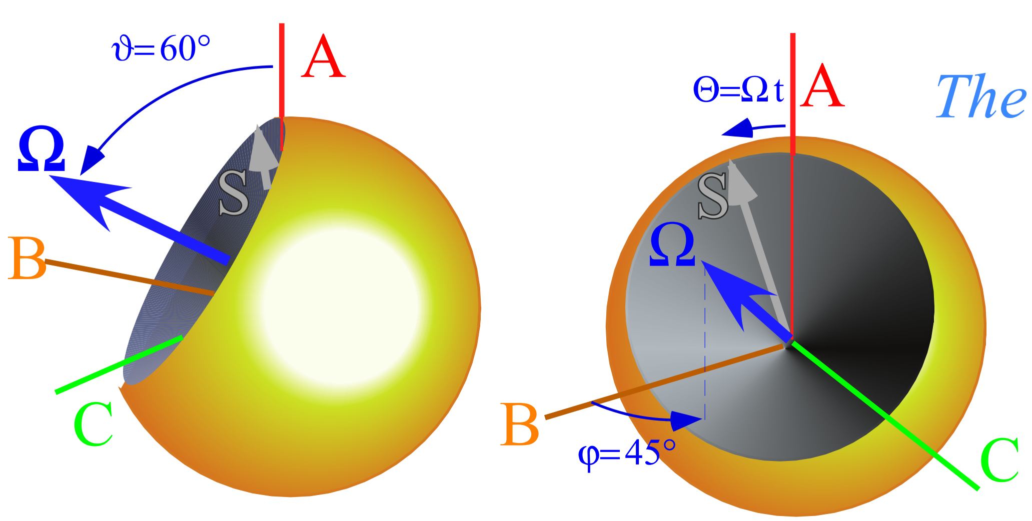
$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\mathbf{H}=\begin{pmatrix} gB_Z & gB_X-igB_Y \\ gB_X+igB_Y & -gB_Z \end{pmatrix}=gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= gB_Z\ \sigma_A + gB_X\ \sigma_X + gB_Y\ \sigma_Y = \vec{\omega}\bullet\vec{\sigma} = \omega\sigma_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$



The driving $\Theta=\Omega t$ vector is defined by the ABCD of Hamiltonian **H** .

The driven spin vector **S** *defines the state. But, how?*

*Fig. 3.4.2 Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$ whirling Stokes state vector **S** in ABC-space.*

Euler's definition of state of rotation using spin-1/2 matrix **R**$(\alpha/2)$ *and* **R**$(\beta/2)$ *)*

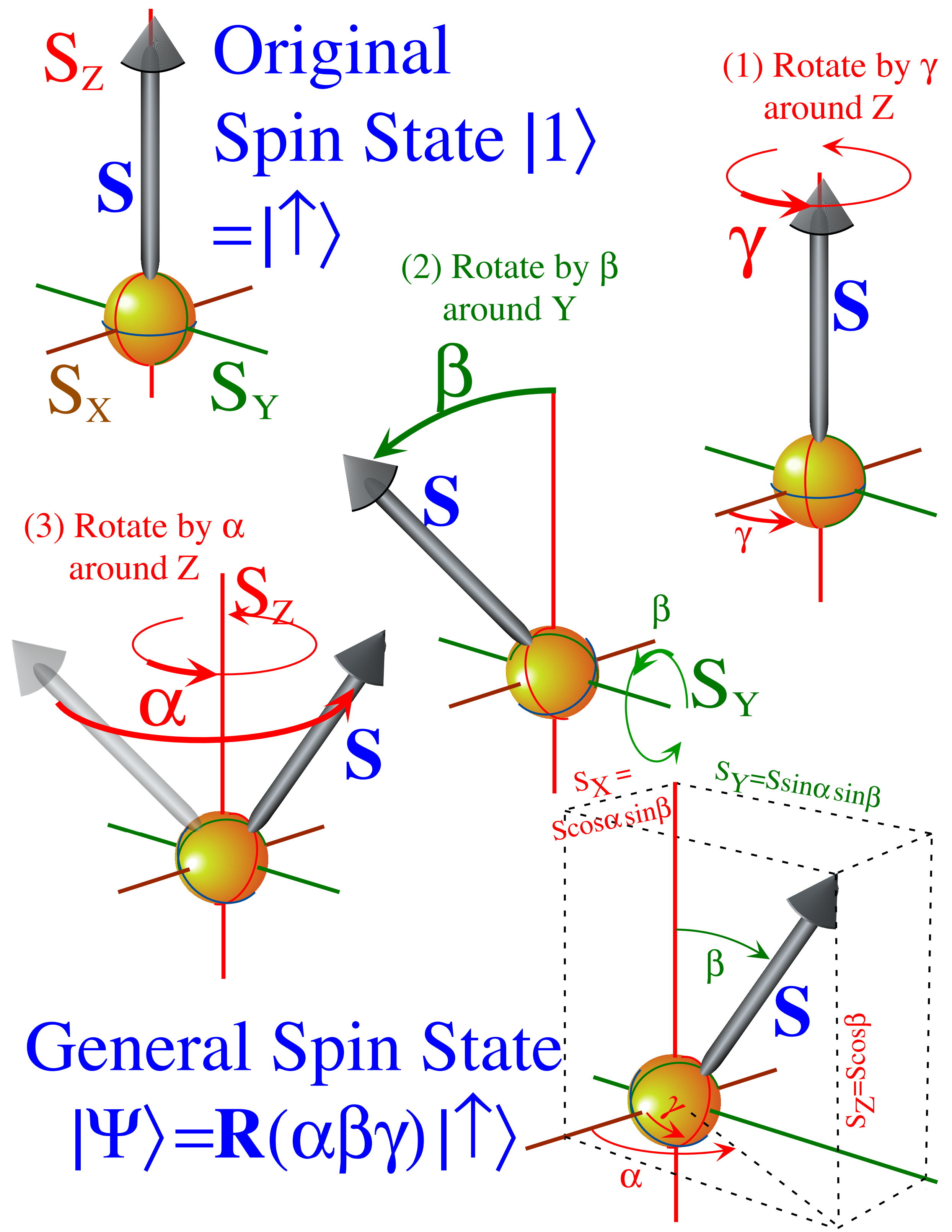
$$\left|a\right\rangle = \mathbf{R}\left(\alpha\beta\gamma\right)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha \,\text{about}\, Z]\cdot\mathbf{R}[\beta \,\text{about}\, Y]\cdot\mathbf{R}[\gamma \,\text{about}\, Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



3D *Stokes vector components* S_a define polarization ellipses, *2D* HO orbits, and spin-$\frac{1}{2}$ states.

Each point $\{E_1, E_2\}$ in complex *2D* oscillator space or in analogous to Ψ-space given by 2D array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$
maps to real 3D spin vector (S_A, S_B, S_C) in that "points" to a particular state of polarization.

Asymmetry $S_A = \frac{1}{2}(a|\sigma_A|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right]$

Balance $S_B = \frac{1}{2}(a|\sigma_B|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right]$

Chirality $S_C = \frac{1}{2}(a|\sigma_C|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right]$

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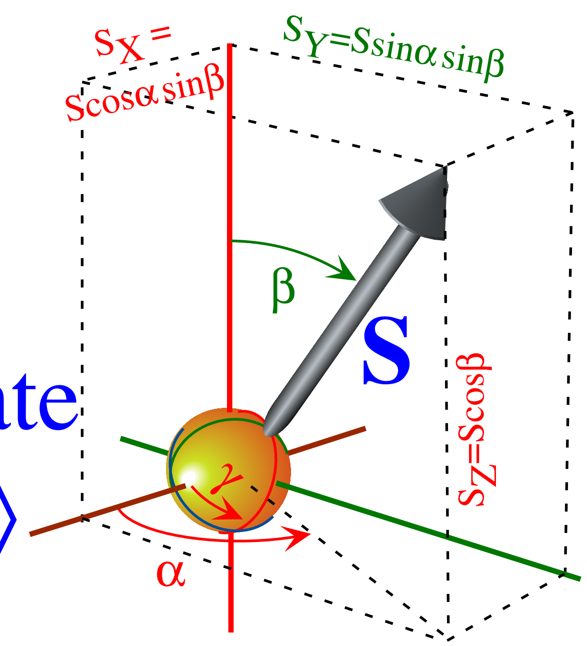
Chirality $S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$

General Spin State

$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$



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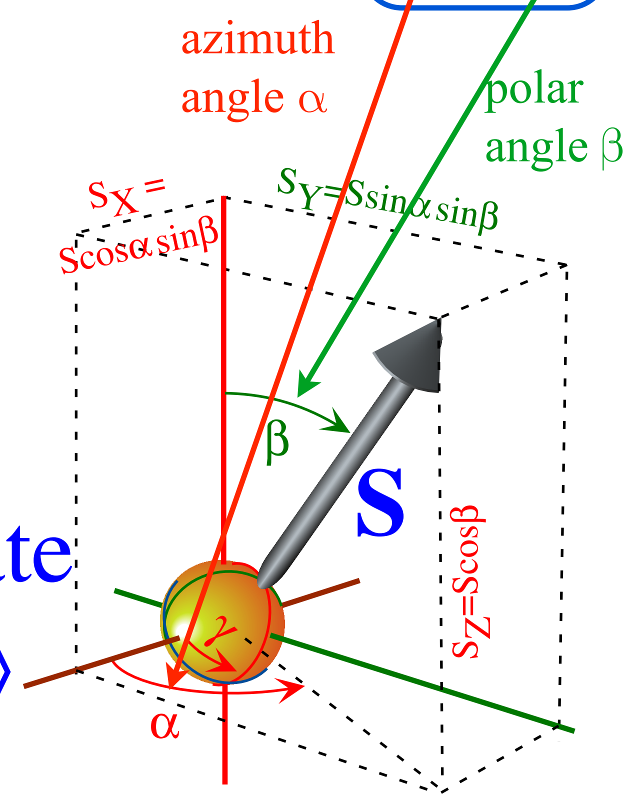

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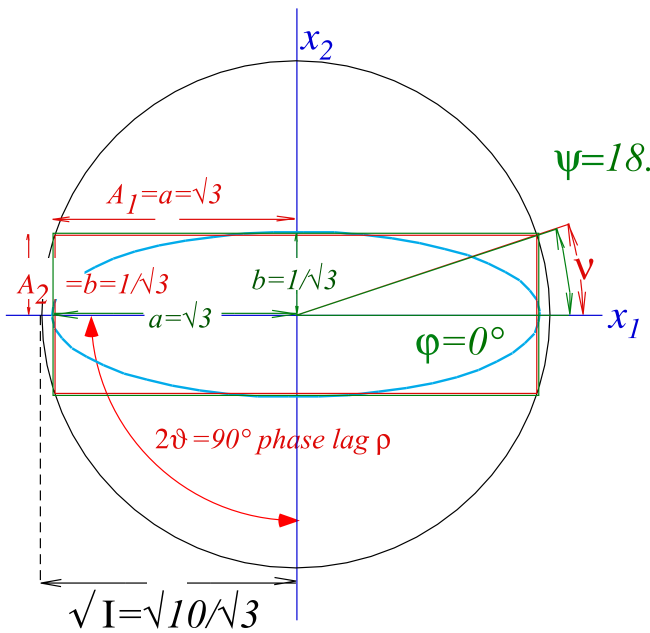
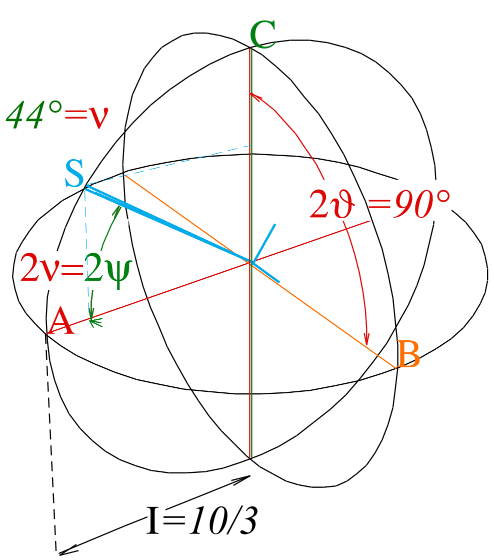
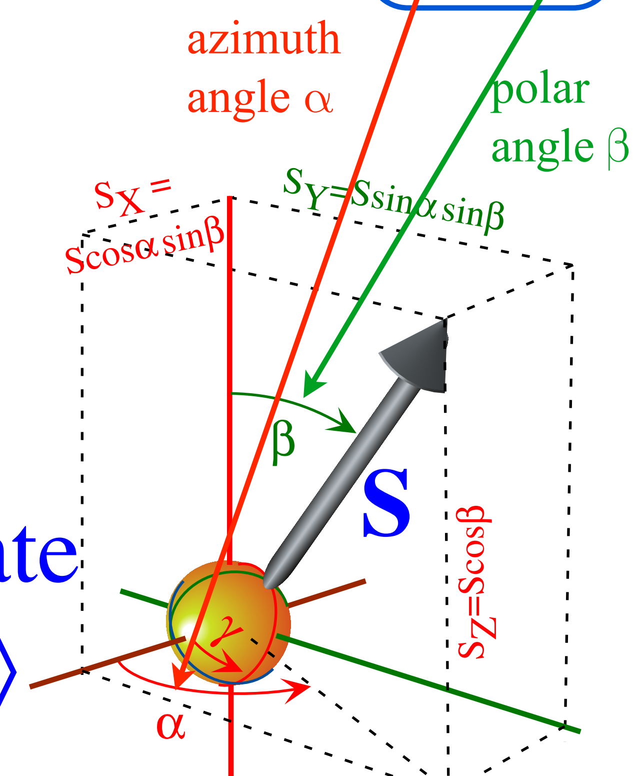




General Spin State

|Ψ⟩=R(αβγ)|↑⟩



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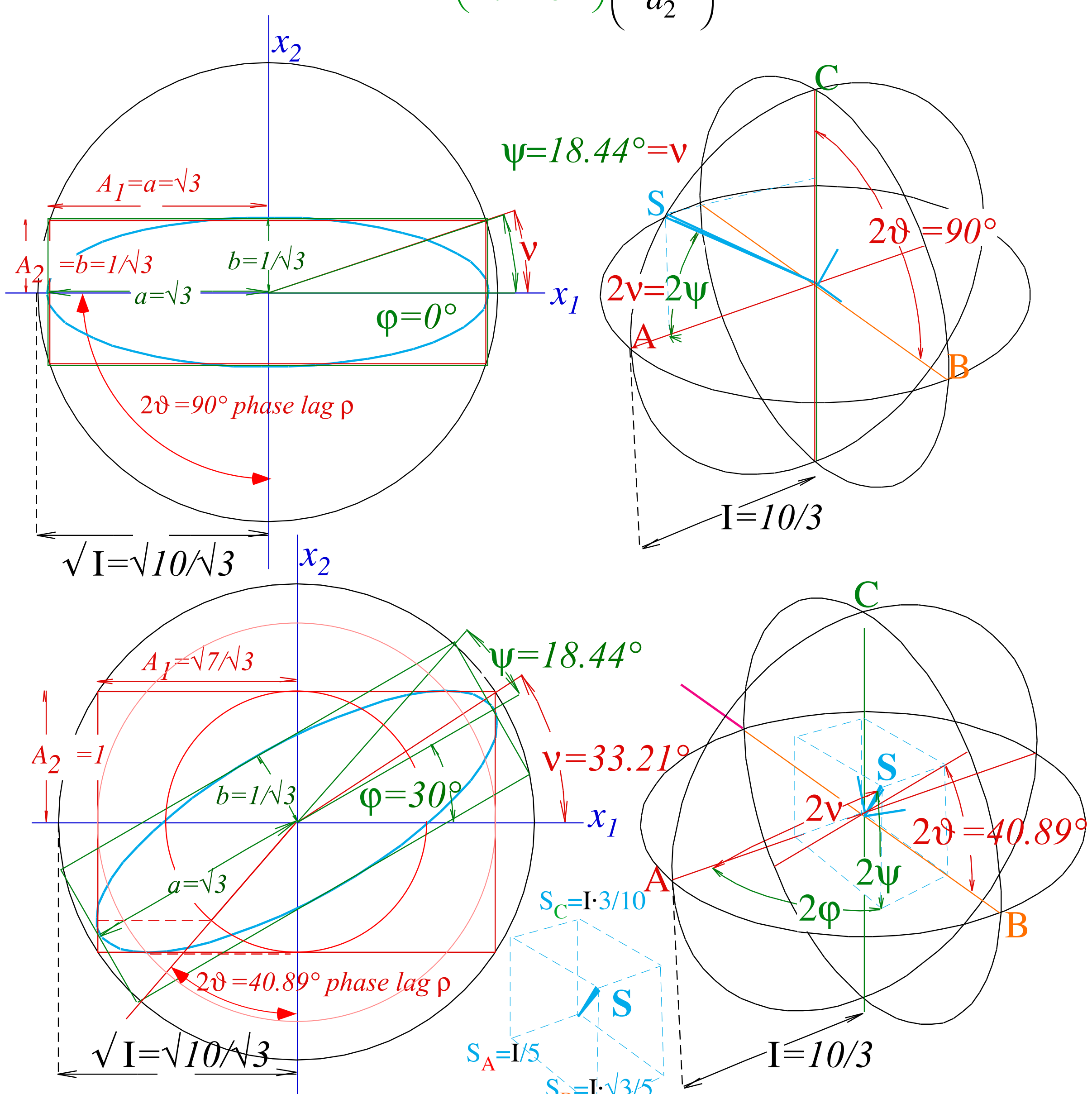

General Spin State
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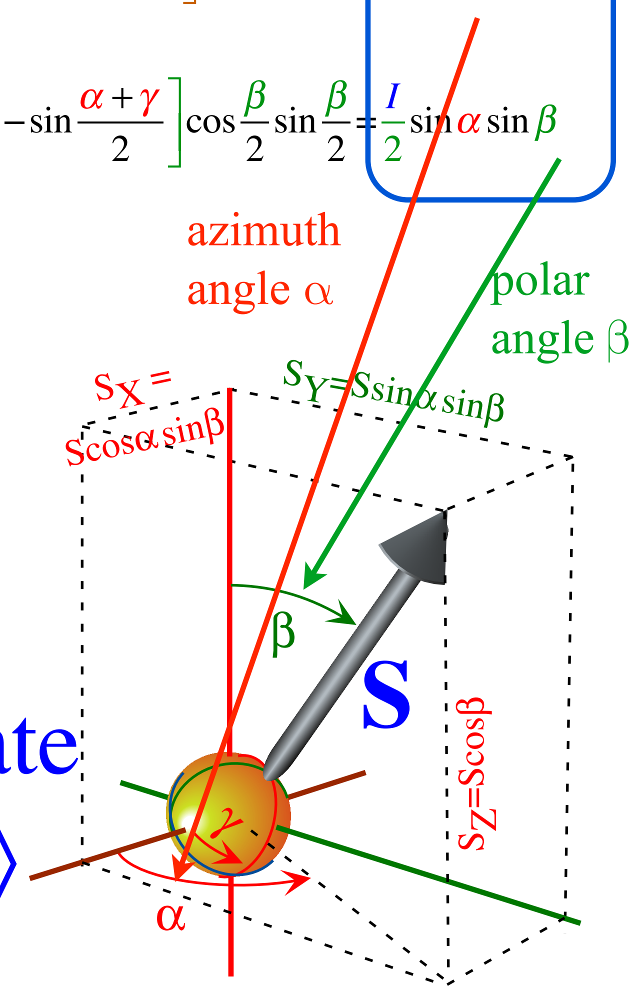


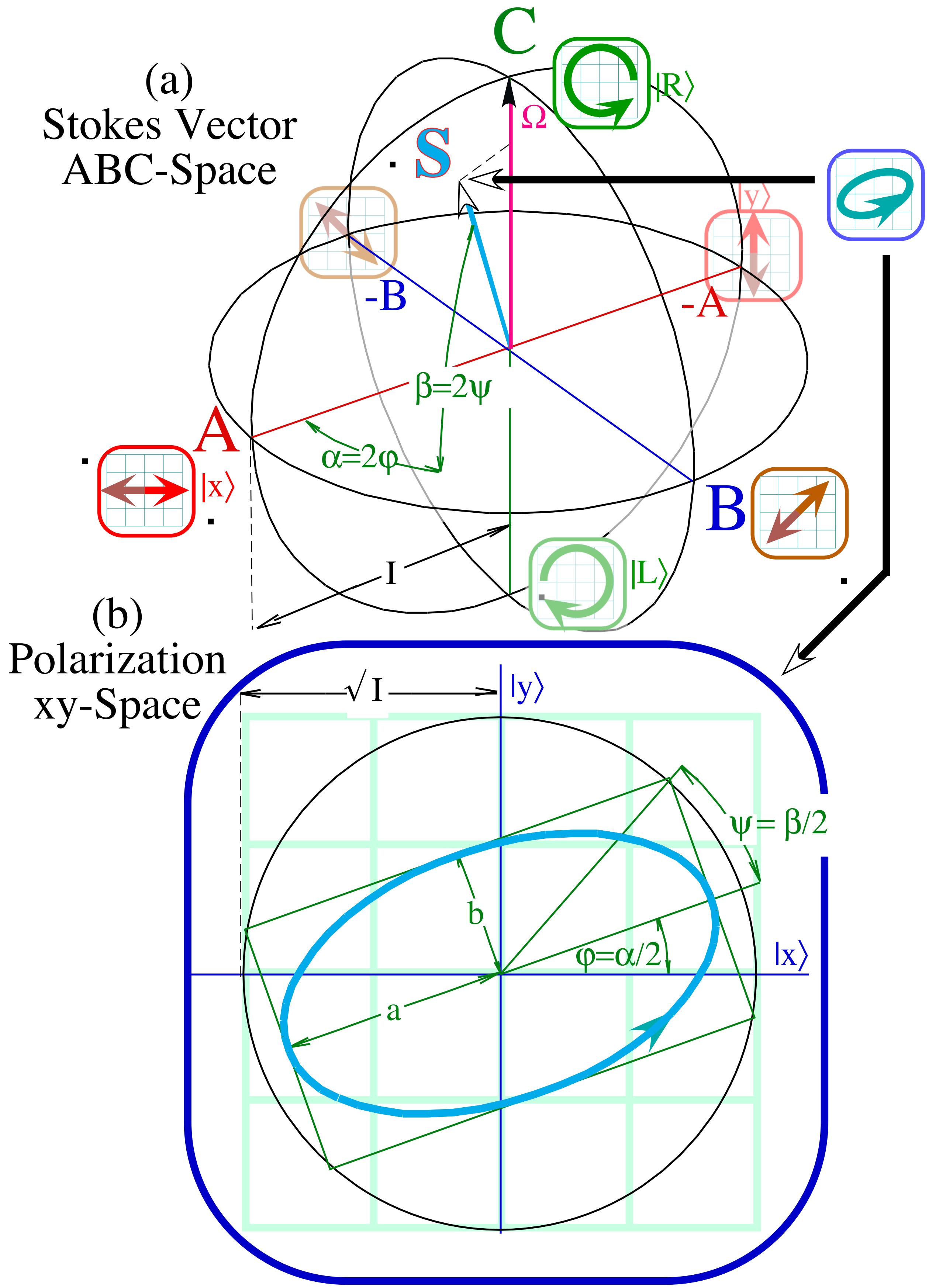
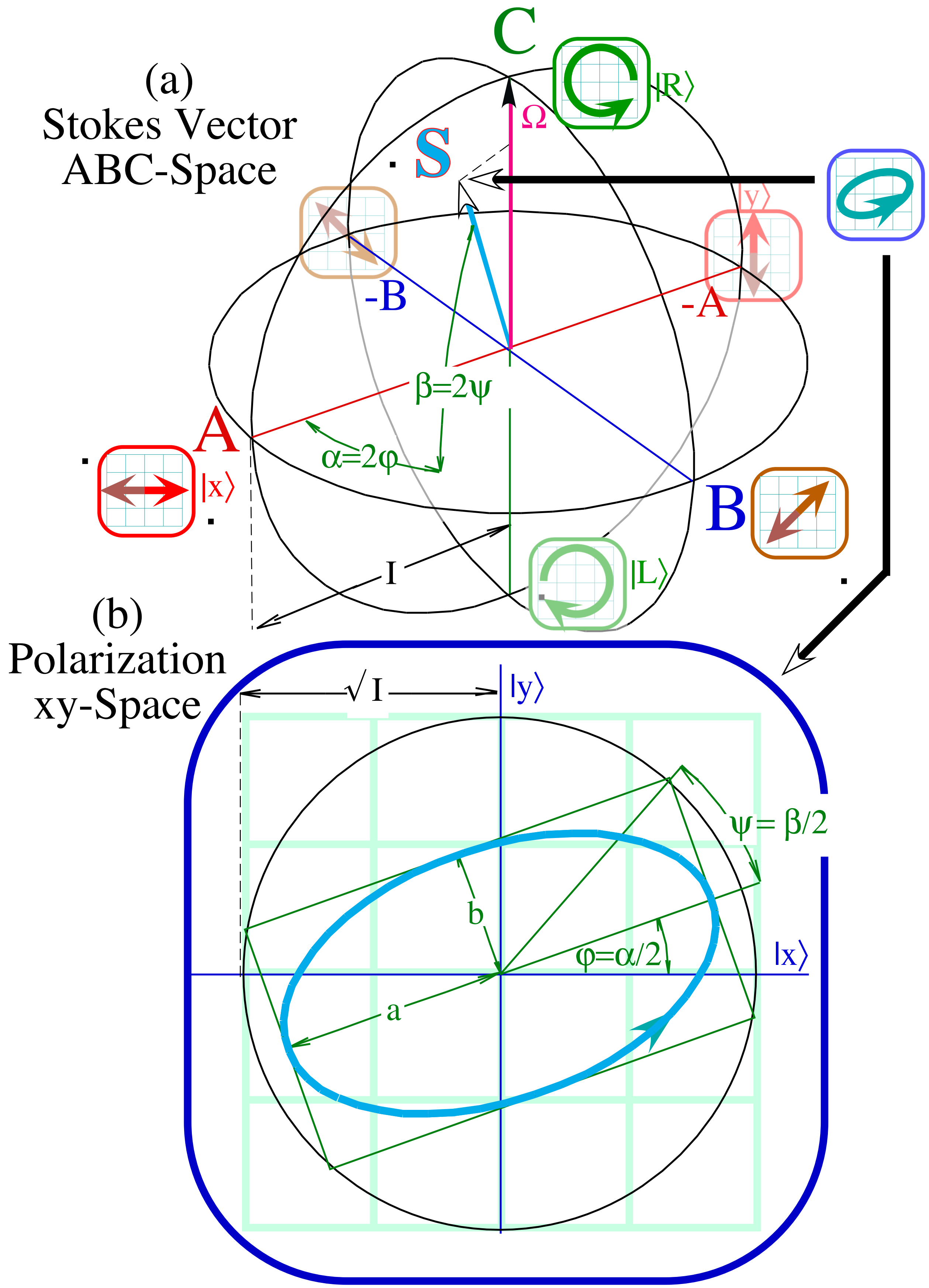


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).



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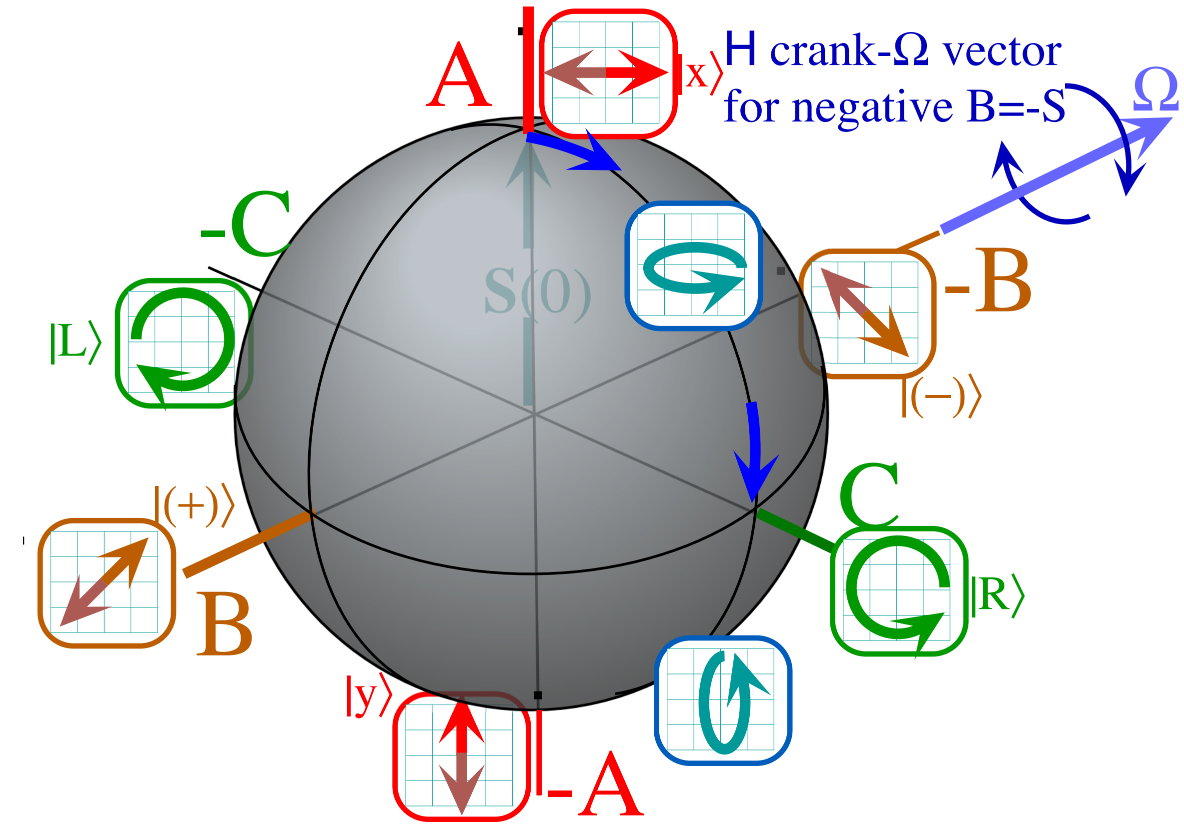


Fig. 3.4.6 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

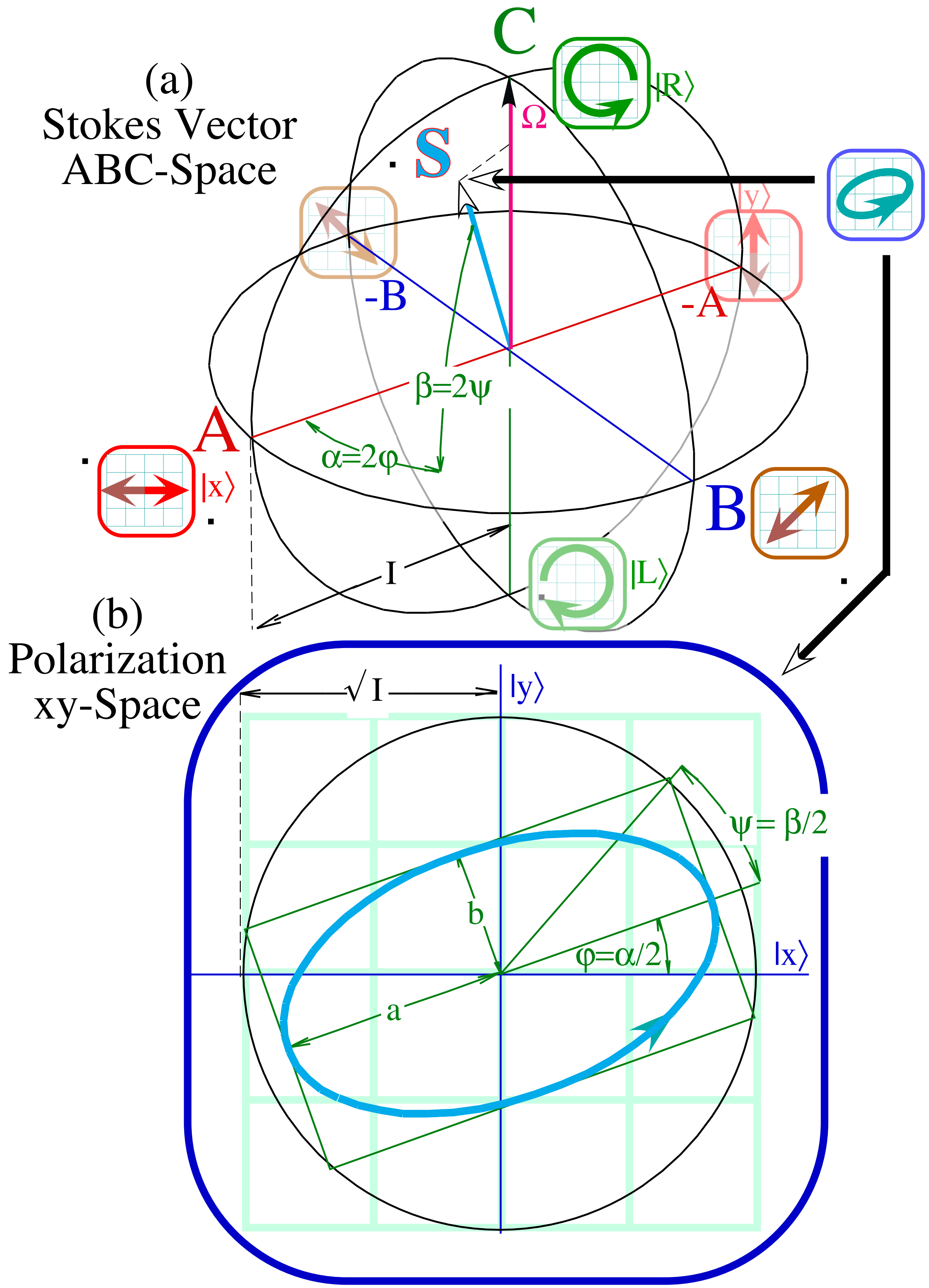


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

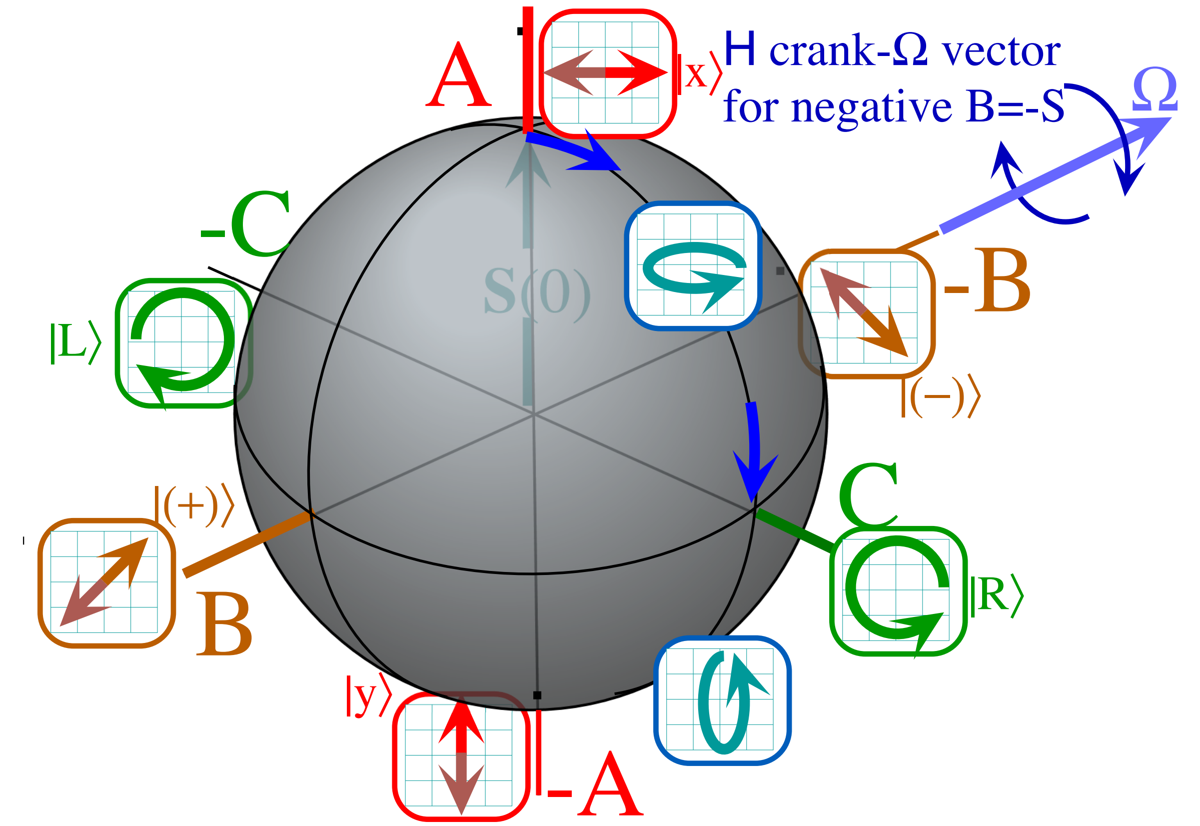


Fig. 3.4.6 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

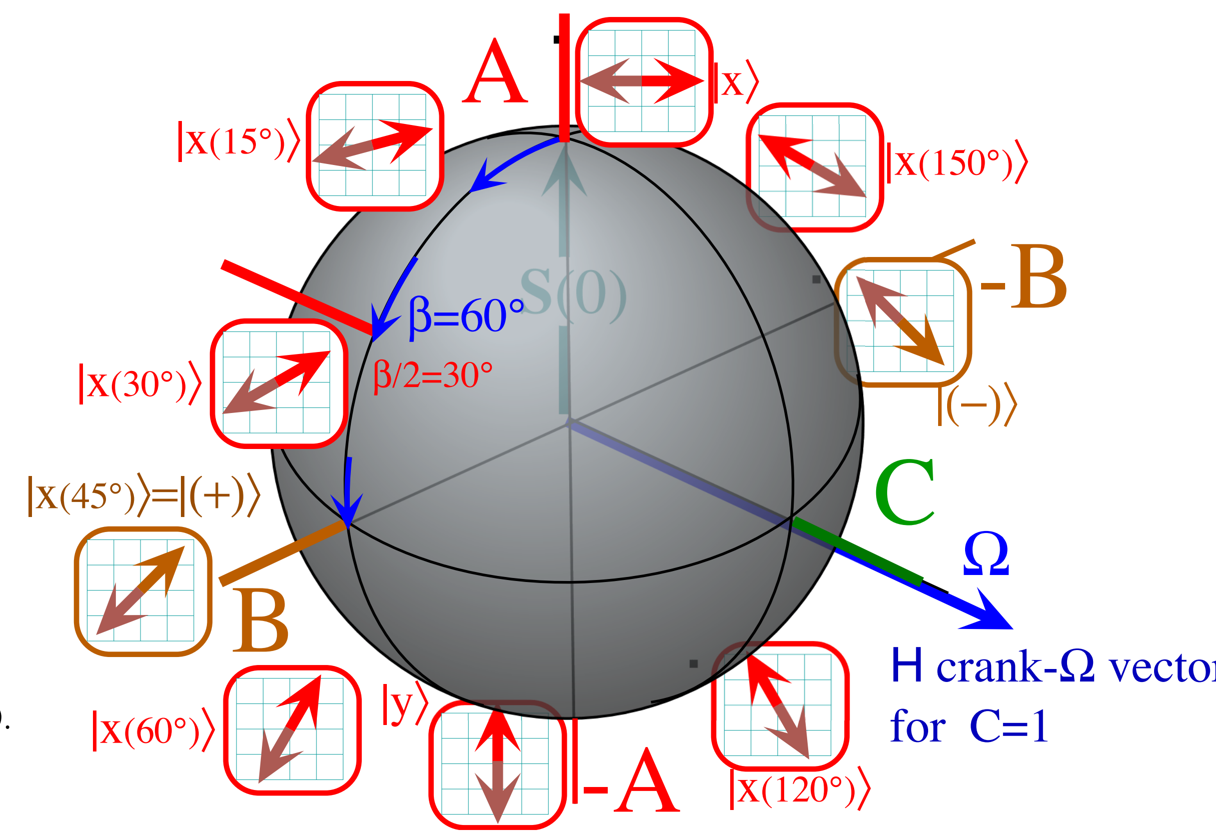
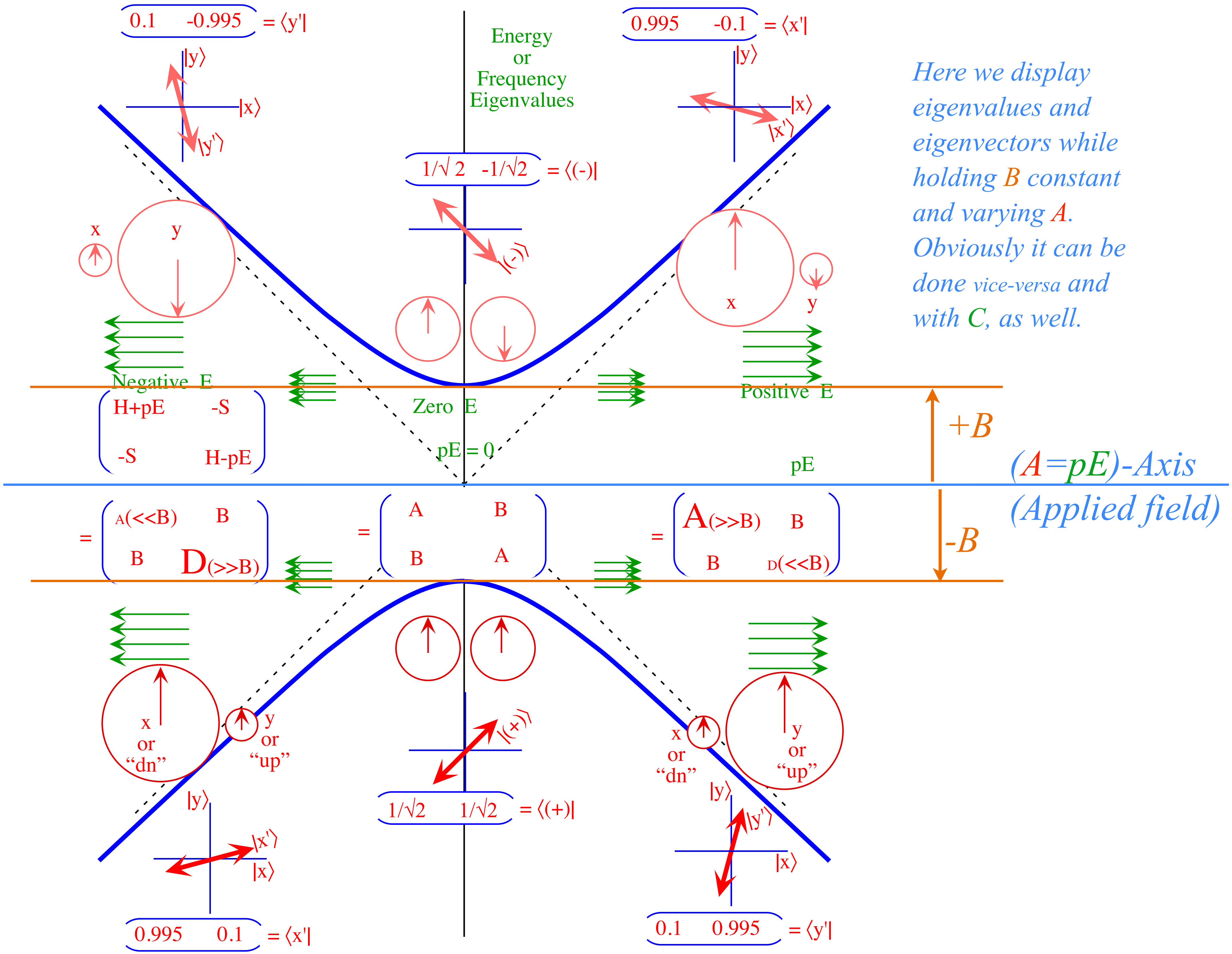


Fig. 3.4.7 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$



Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

ABCD Time evolution operator

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}\cdot t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}\cdot t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\cdot t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\cdot t - iC\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cdot t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cdot t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})\cdot t} = e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t - i\boldsymbol{\sigma}_\omega\sin\omega\cdot t\right)$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2} \\ B \\ C\end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$ and: $\vec{\Theta} = \vec{\Omega}\cdot t = \begin{pmatrix}\Omega_A \\ \Omega_B \\ \Omega_C\end{pmatrix}\cdot t = \begin{pmatrix}A-D \\ 2B \\ 2C\end{pmatrix}\cdot t$ and: $\Omega_0 = \frac{A+D}{2}$

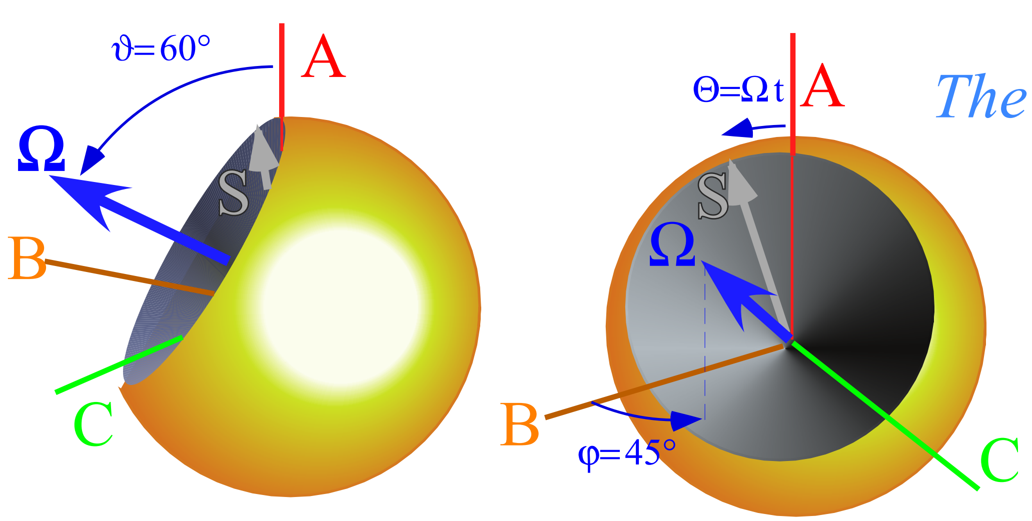
Symmetry relations make spinors σ_X, σ_Y, and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

3D crank vector $\vec{\Theta} = \vec{\Omega}\cdot t$ and *spin operator* $\mathbf{S}$ defines *3D ABC*-rotation with ratio $\frac{1}{2}$ or 2 between Θ_a and $\varphi_a = \frac{1}{2}\Theta_a$ or between $\mathbf{S}$ and $\boldsymbol{\sigma} = 2\mathbf{S}$.

$$e^{-i\boldsymbol{\sigma}\bullet\vec{\varphi}} = e^{-i\boldsymbol{\sigma}\bullet\vec{\Theta}/2} = e^{-i\mathbf{S}\bullet\vec{\Theta}} = \mathbf{1}\cos\frac{\Theta}{2} - i\,(\boldsymbol{\sigma}\bullet\hat{\Theta})\sin\frac{\Theta}{2} = \begin{pmatrix}\cos\frac{\Theta}{2} - i\hat{\Theta}_A\sin\frac{\Theta}{2} & (-i\hat{\Theta}_B - \hat{\Theta}_C)\sin\frac{\Theta}{2} \\ (-i\hat{\Theta}_B + \hat{\Theta}_C)\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} + i\hat{\Theta}_A\sin\frac{\Theta}{2}\end{pmatrix}$$

Example 3: Any $\Theta = \Omega t$-axial rotation

2D angle: $\varphi = \frac{1}{2}\Theta$ $\qquad$ *3D Crank vector*: $\vec{\Theta} = \Theta\hat{\Theta} = 2\varphi_a\hat{\mathbf{a}} = 2\vec{\varphi}$ $\qquad$ *2D spin matrix*: $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$



The driving $\mathbf{\Theta} = \mathbf{\Omega}t$ vector is defined by the ABCD of Hamiltonian **H**.

The driven spin vector **S** *defines the state. But, how?*

Fig. 3.4.2 Two views of Hamilton crank vector Ω(φ,ϑ) whirling Stokes state vector S in ABC-space.