Lecture 37.

Introduction to classical oscillation and resonance II.

(Ch. 1-2 of Unit 3   4.30.12)

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

Beat, lifetimes, and quality factor effects

Common Lorentzian (a.k.a. Witch of Agnesi)

Review of Lecture 36

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

Methods for treating 2D harmonic oscillator equations and eigen-solutions

- *Geometric method*
- *Matrix-algebraic method*
  - *Idempotent projectors and spectral decomposition*

Lecture 37 ends here

Methods for analyzing 2D-HO beats and mixed mode dynamics

- *Geometric method*
- *Algebraic method*

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0}L(\Delta - i\Gamma)$$

Define *complex detuning-decay* $\delta = \Delta - i\Gamma$ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\,L \quad + i\,\mathrm{Im}\,L \quad = \quad \frac{\Delta}{\Delta^2 + \Gamma^2} \quad + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = \quad |L|^2\,\Delta \quad + i\,|L|^2\,\Gamma$$

$$= |L|e^{i\rho} = |L|\cos\rho + i\,|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

$$|L| = \frac{1}{\Gamma}\sin\rho$$

$$|L| = \frac{1}{\Delta}\cos\rho$$

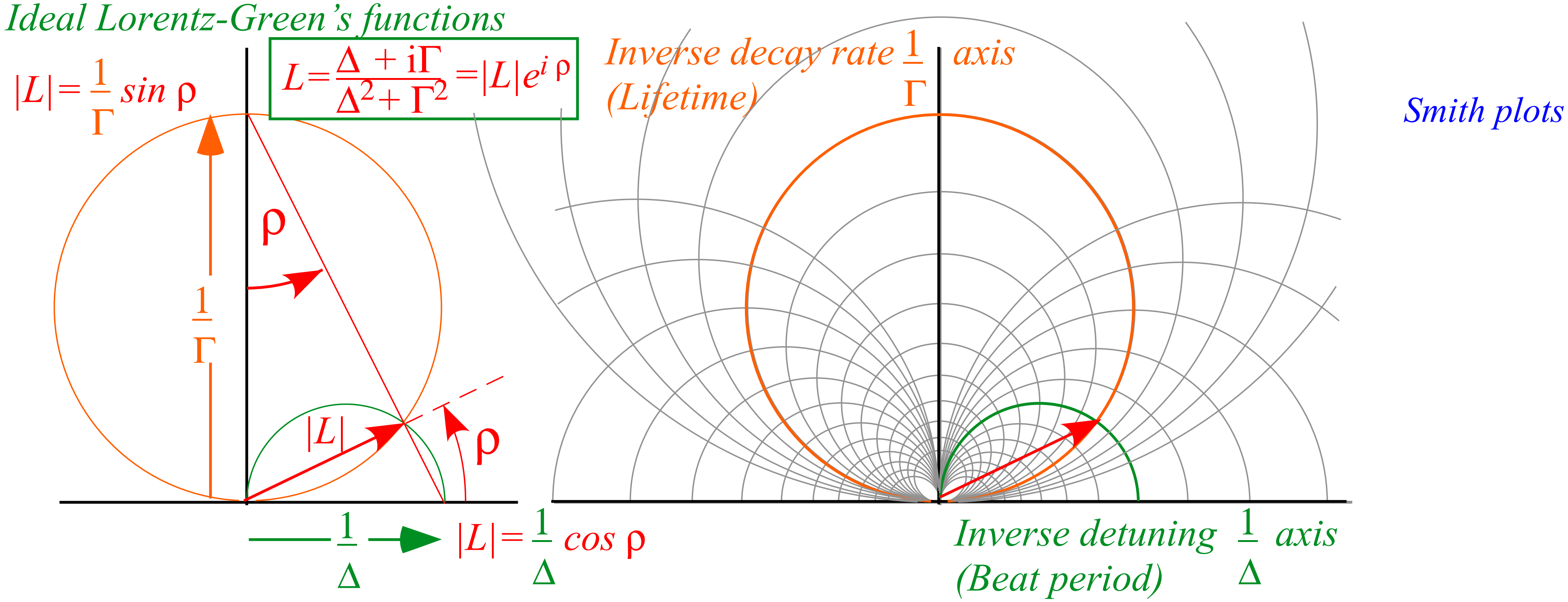
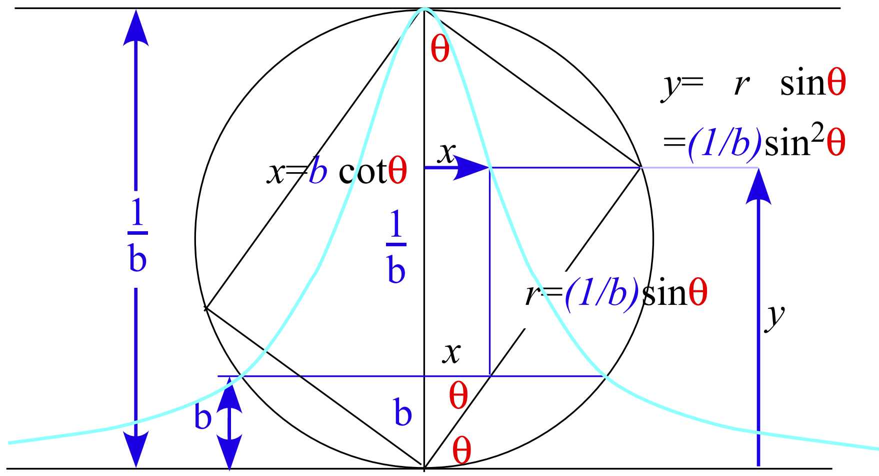
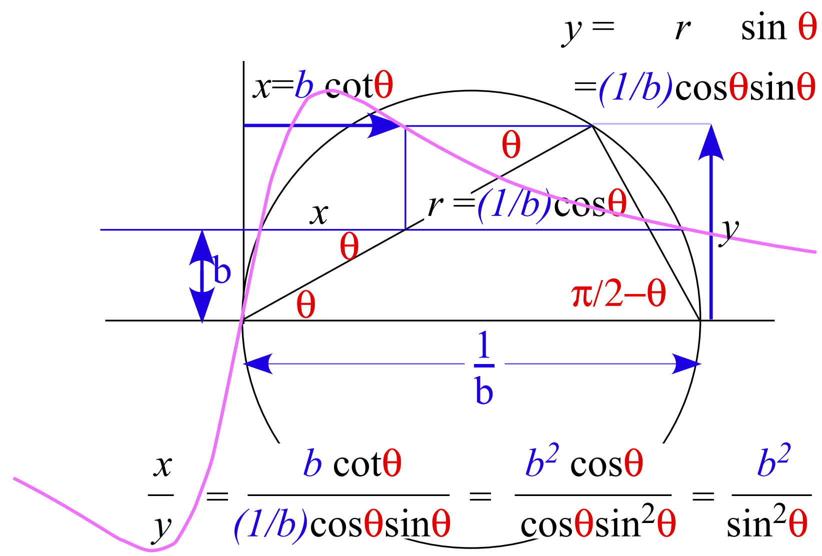


Fig. 3.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time $1/\Gamma$ *vs. beat-period* $1/\Delta$ *coordinates)*

Constant Δ and Γ curves in Fig. 3.2.13 are orthogonal circles of *1/z- dipolar coordinates*. Recall Fig. 1.10.11.

The Common Lorentzian (a.k.a. *The Witch of Agnesi*)





$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

Maria Gaetana Agnesi

Born	May 16, 1718
Died	January 9, 1799 (aged 80)
Residence	Italy
Nationality	Italy
Fields	Mathematics

$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}-b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

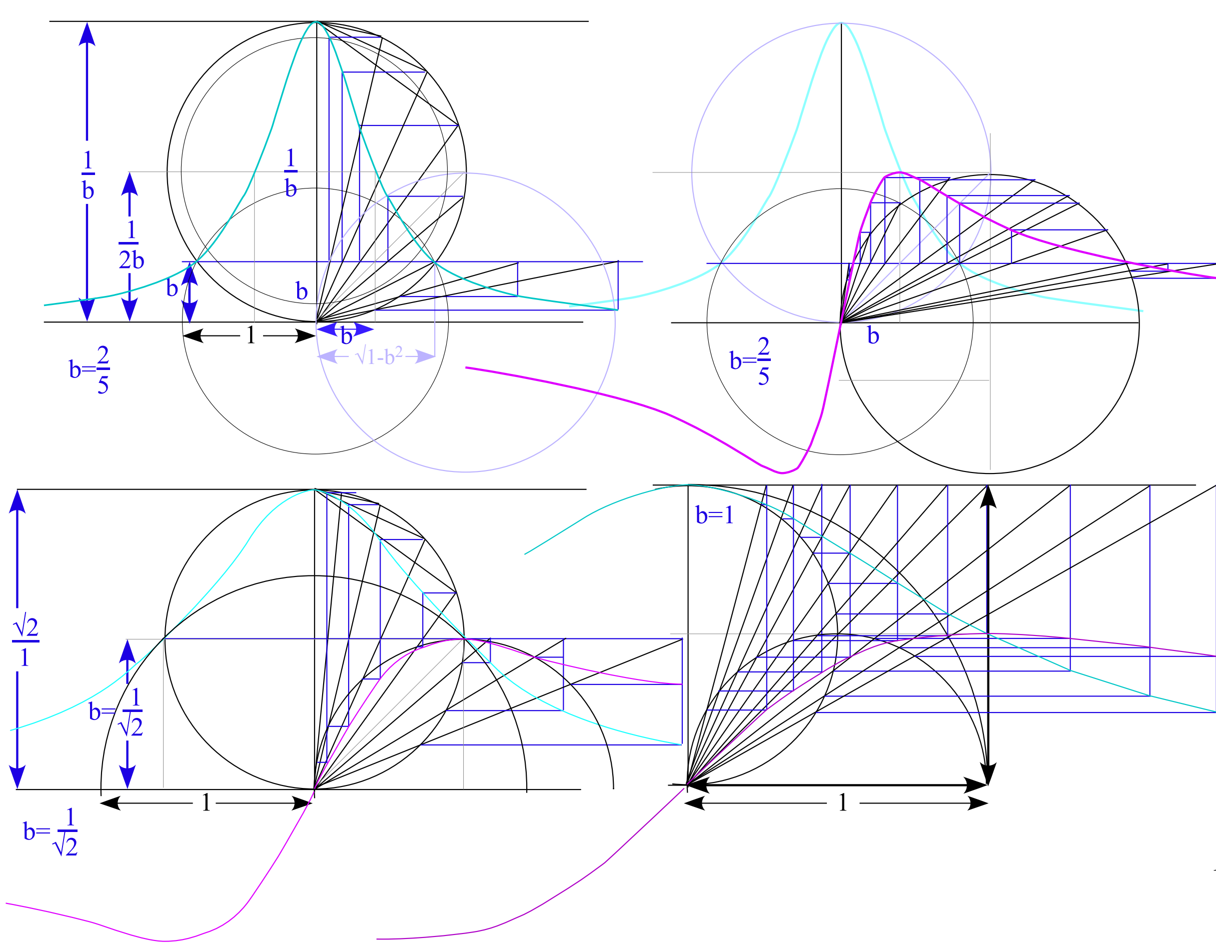
$$y=\frac{b}{x^2+b^2}$$

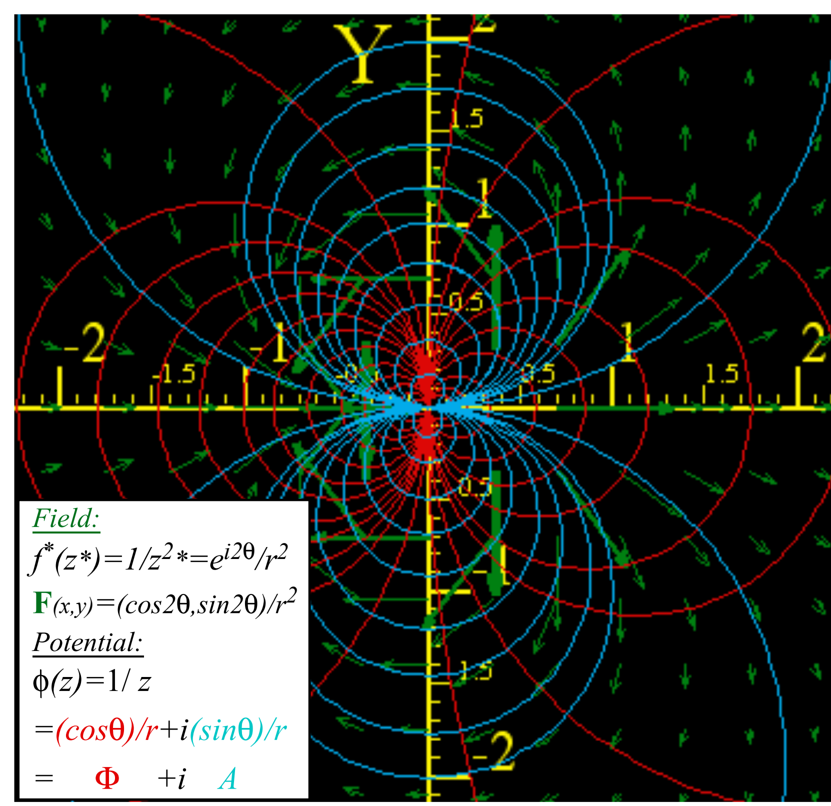
Common Lorentzian function I.
(imaginary "absorbtive" part)

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

$$y=\frac{x}{x^2+b^2}$$

Common Lorentzian function II.
(real "refractory" part)





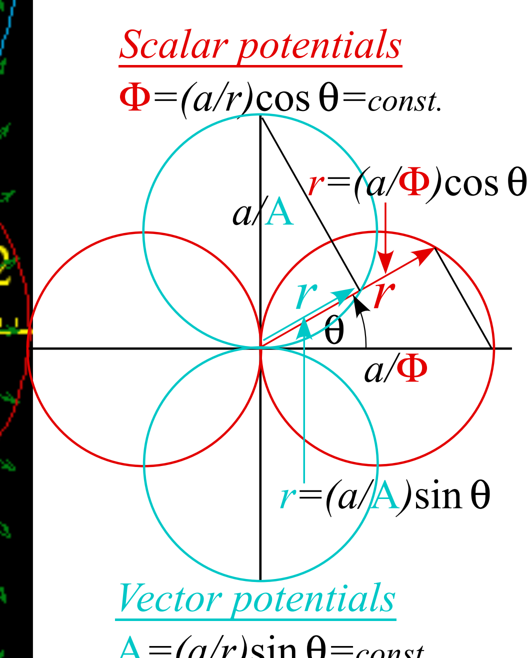


*Fig. 10.11 Dipole **F**-field $f(z)=1/z^2$ and scalar potential (Φ=const.)-circles orthogonal to (A=const.)-circles.*

2D harmonic oscillators

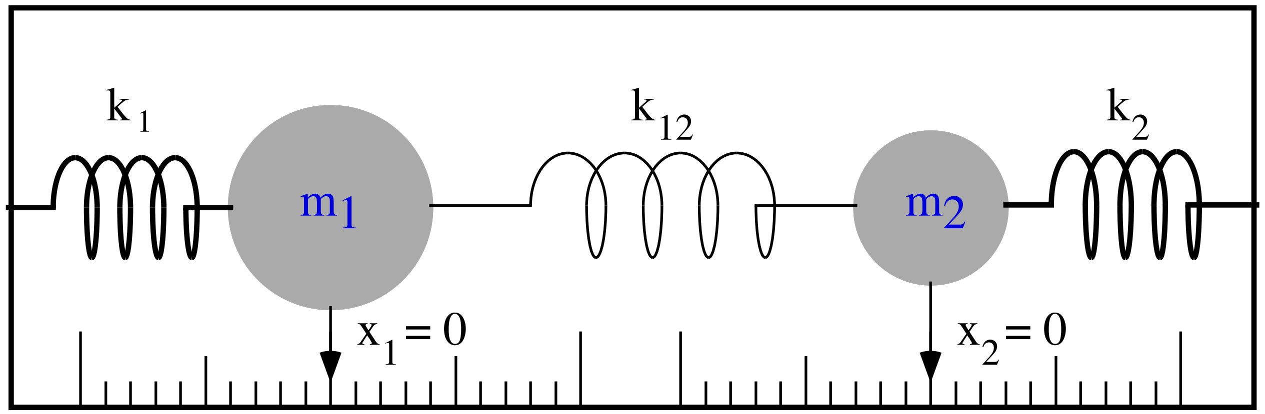
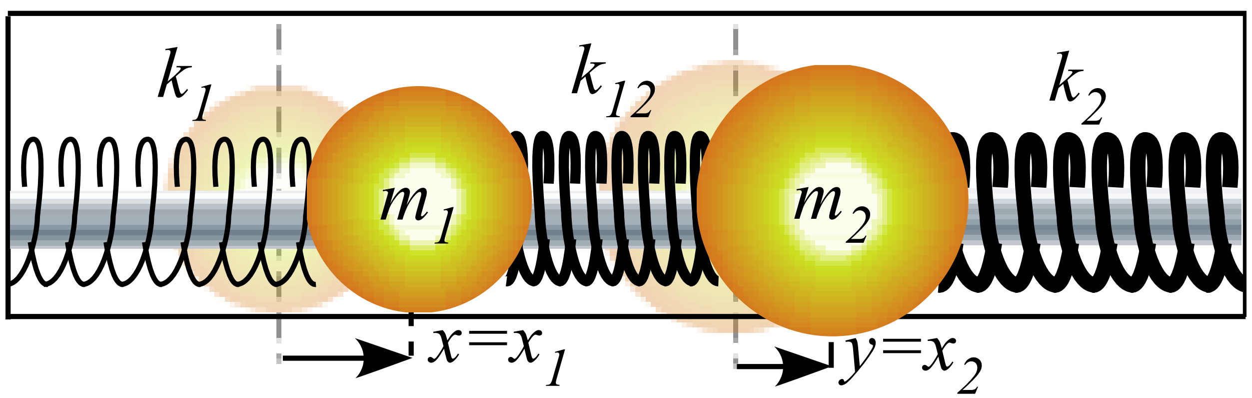


Fig. 3.3.1 Two 1-dimensional coupled oscillators



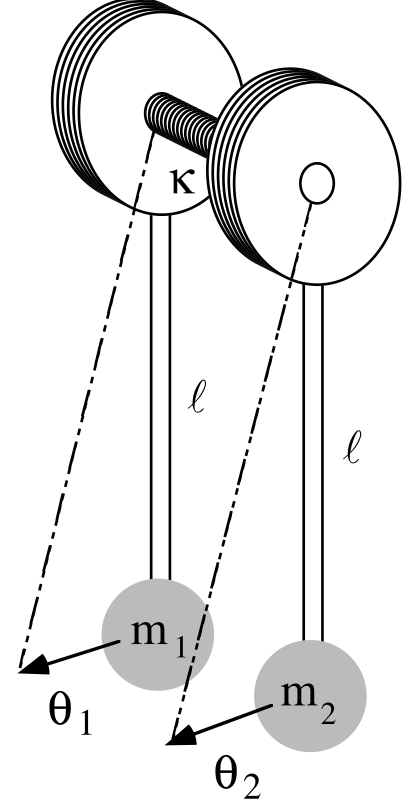


Fig. 3.3.2 Coupled pendulums

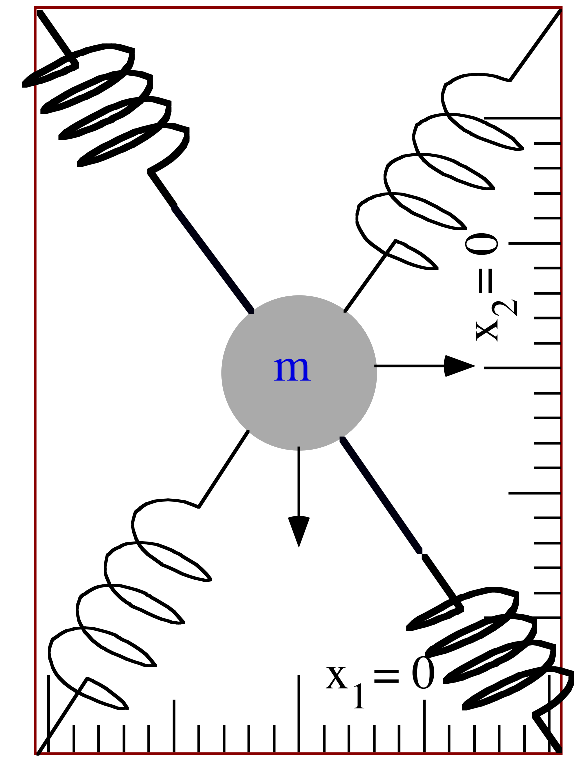


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D harmonic oscillator energy

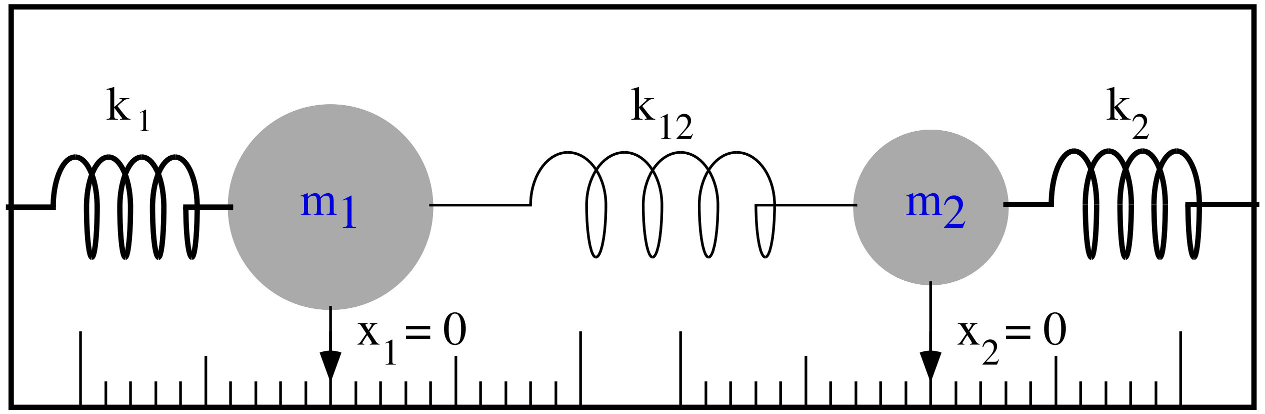
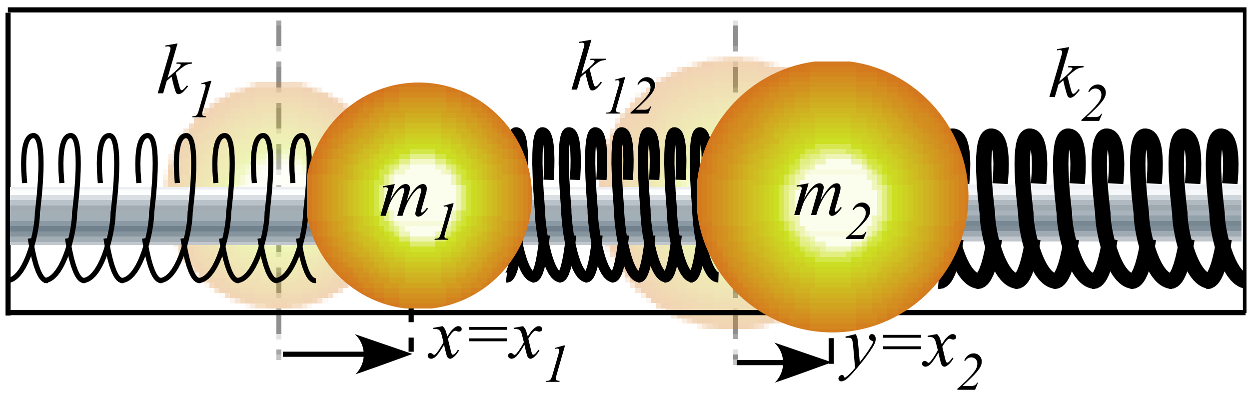
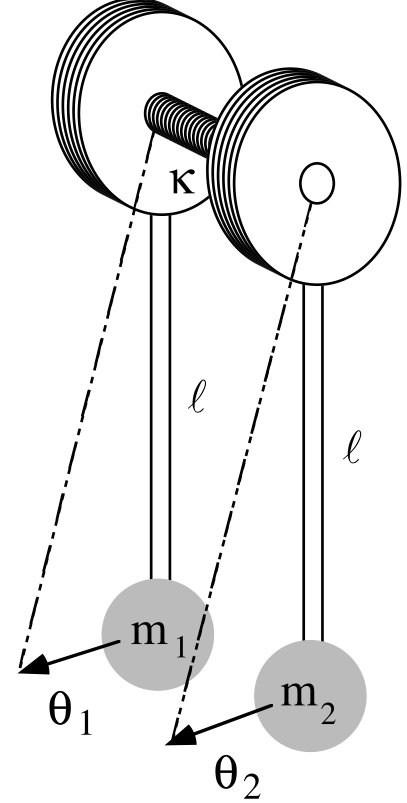


Fig. 3.3.1 Two 1-dimensional coupled oscillators





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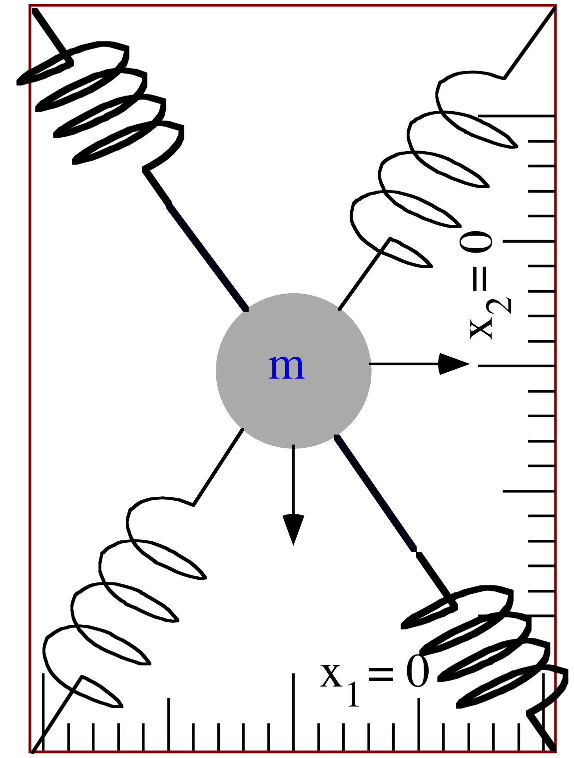


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

2D harmonic oscillator energy

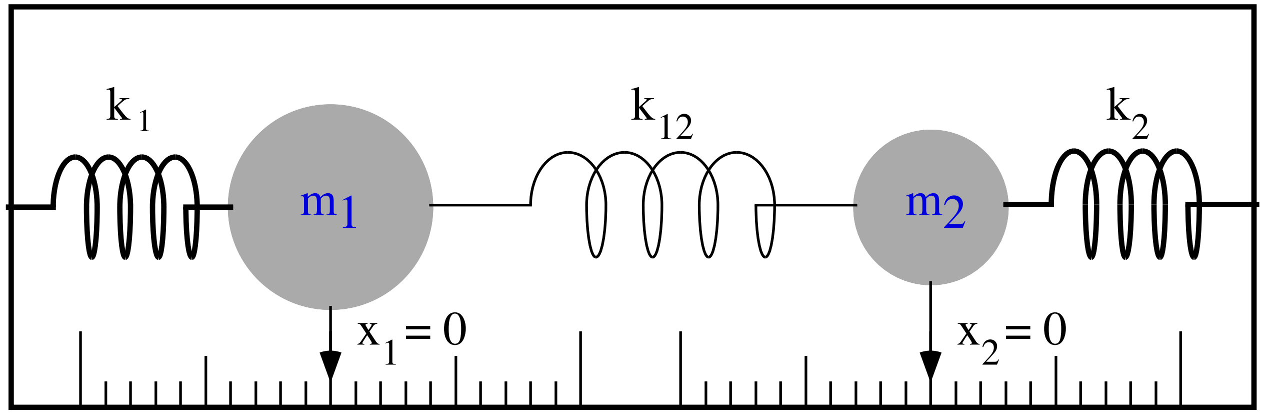
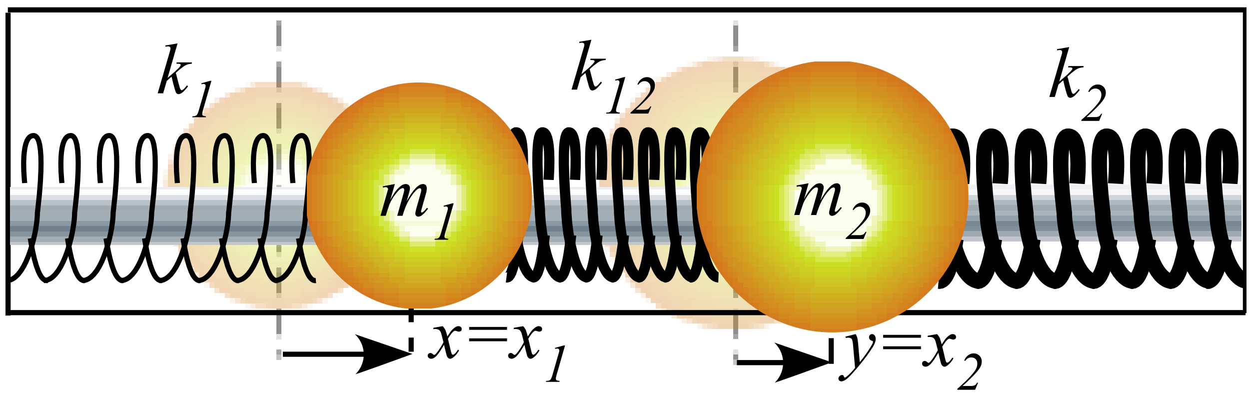


Fig. 3.3.1 Two 1-dimensional coupled oscillators



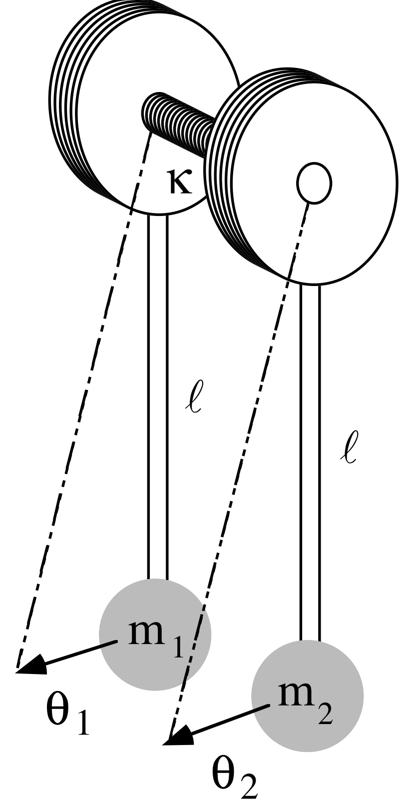
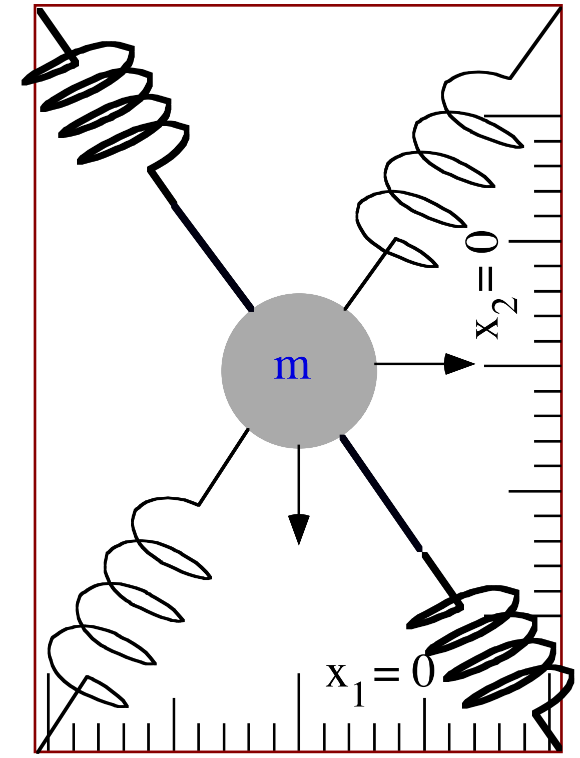


Fig. 3.3.2 Coupled pendulums



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2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}k_1 x_1^2 + \frac{1}{2}k_2 x_2^2 + \frac{1}{2}k_{12}\left(x_1 - x_2\right)^2$$

$$= \frac{1}{2}\left(k_1 + k_{12}\right)x_1^2 - k_{12}x_1 x_2 + \frac{1}{2}\left(k_2 + k_{12}\right)x_2^2$$

Lagrangian $L=T-V$

2D harmonic oscillator equations

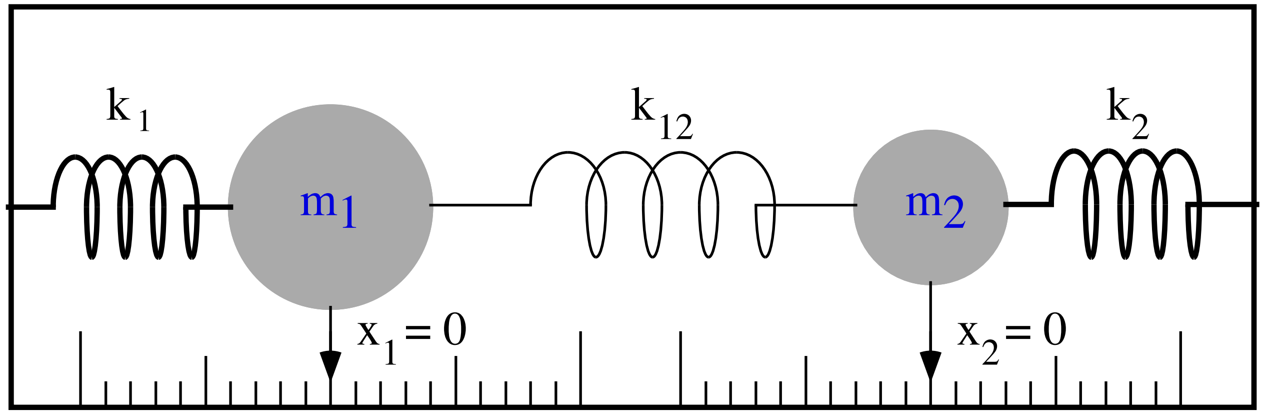
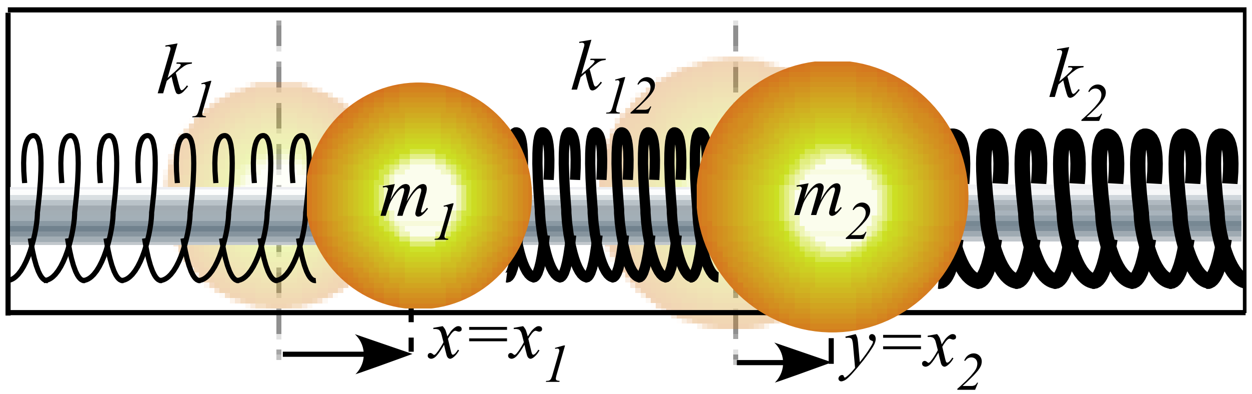


Fig. 3.3.1 Two 1-dimensional coupled oscillators



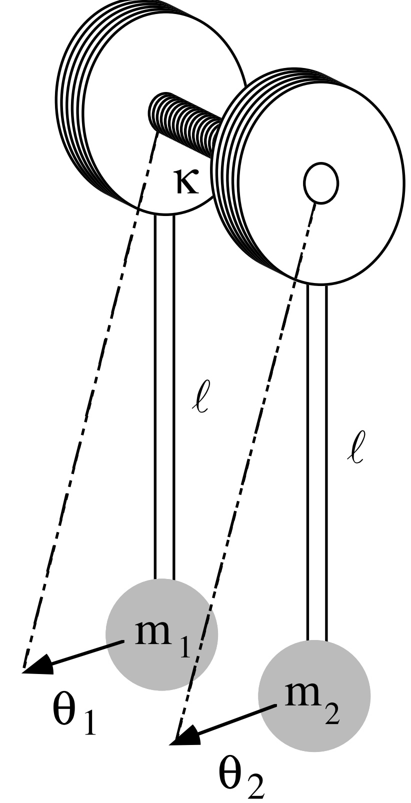
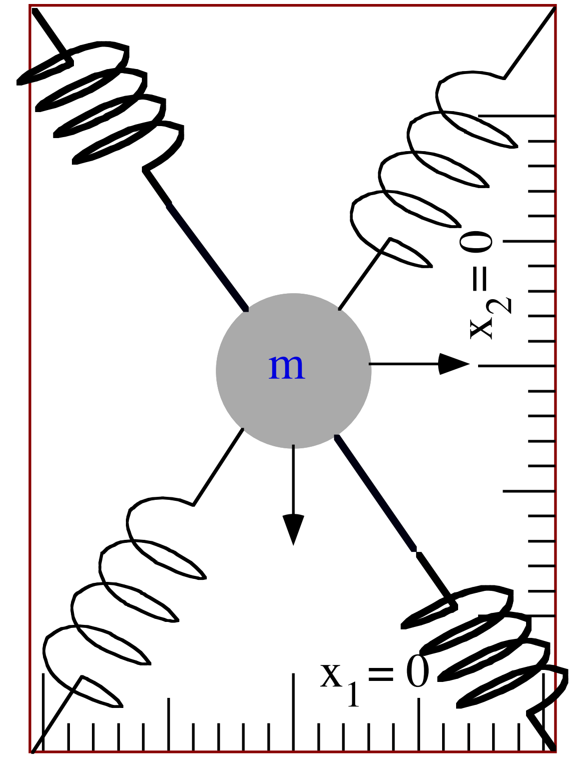


Fig. 3.3.2 Coupled pendulums



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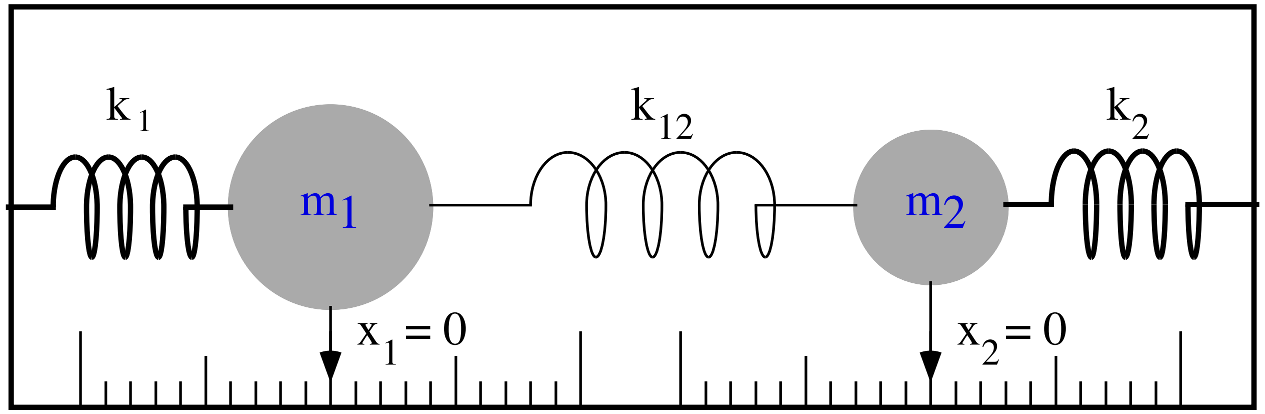
Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -\left(k_1 + k_{12}\right)x_1 + k_{12}x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12}x_1 - \left(k_2 + k_{12}\right)x_2$$

2D harmonic oscillator equations



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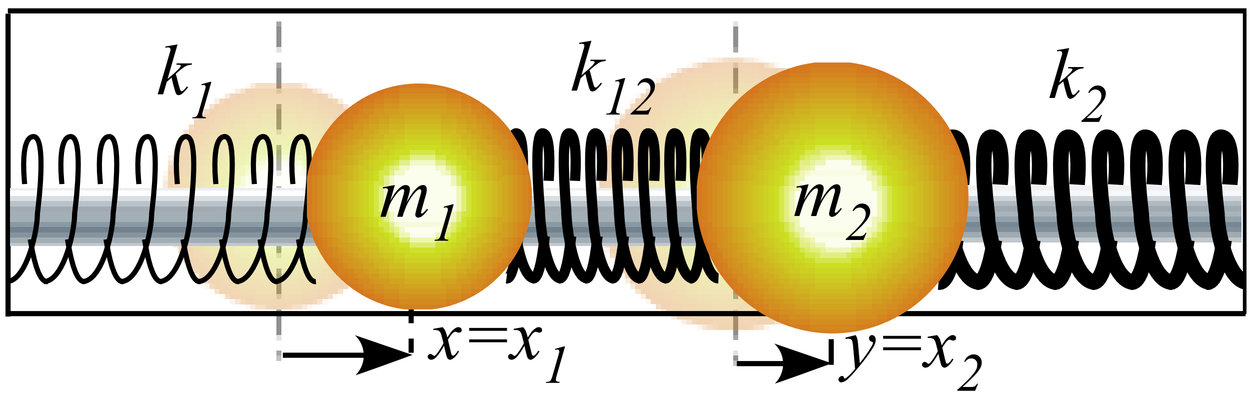

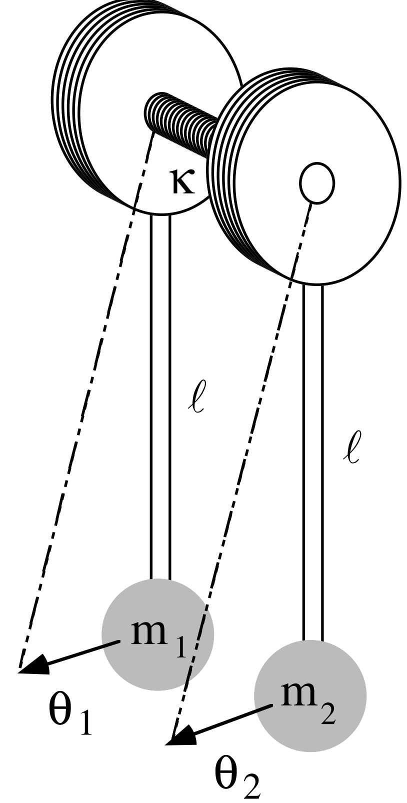

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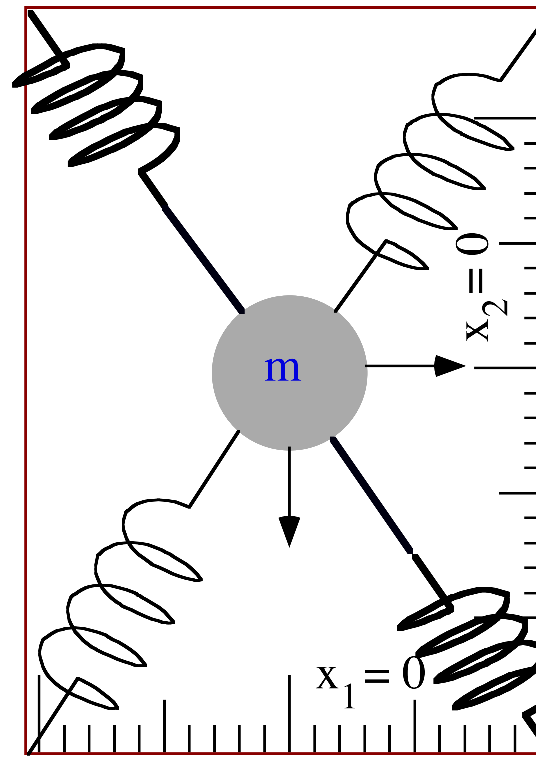

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2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2D harmonic oscillator equations

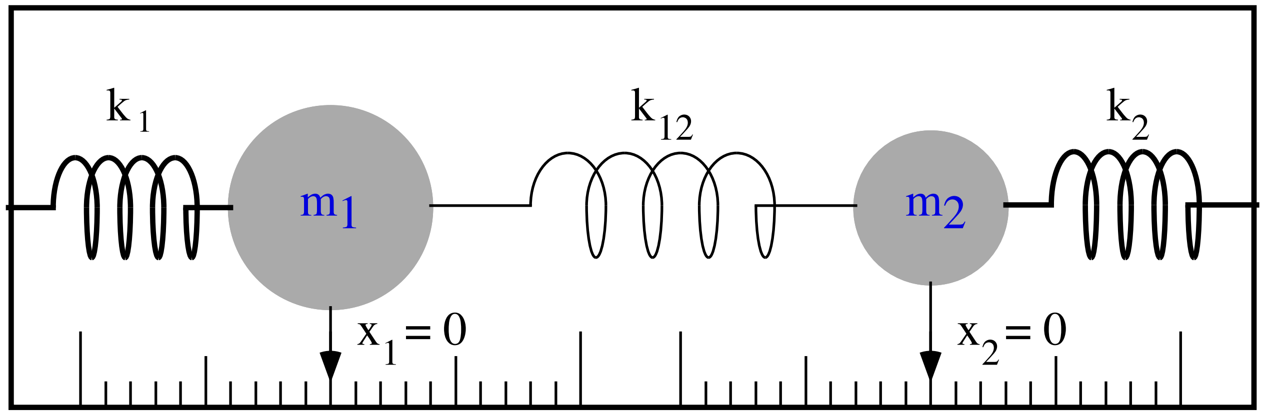
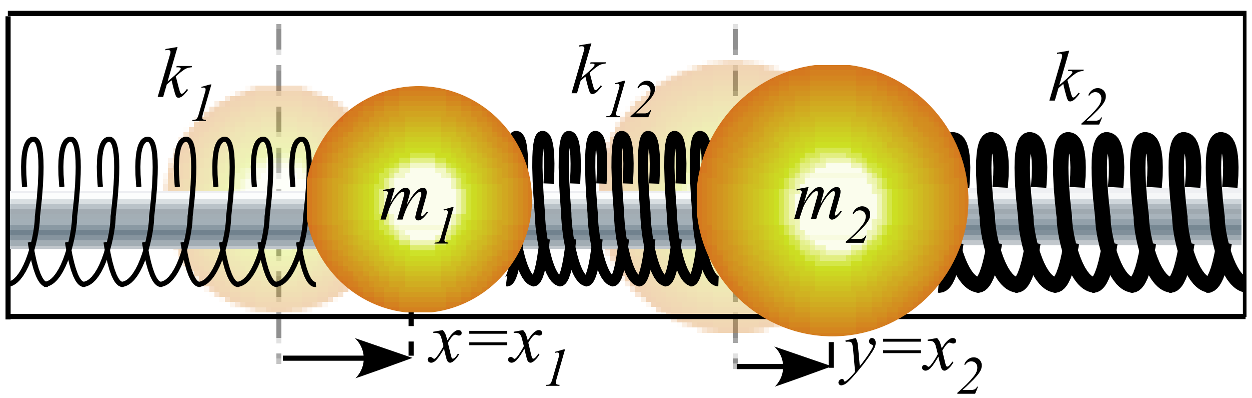


Fig. 3.3.1 Two 1-dimensional coupled oscillators



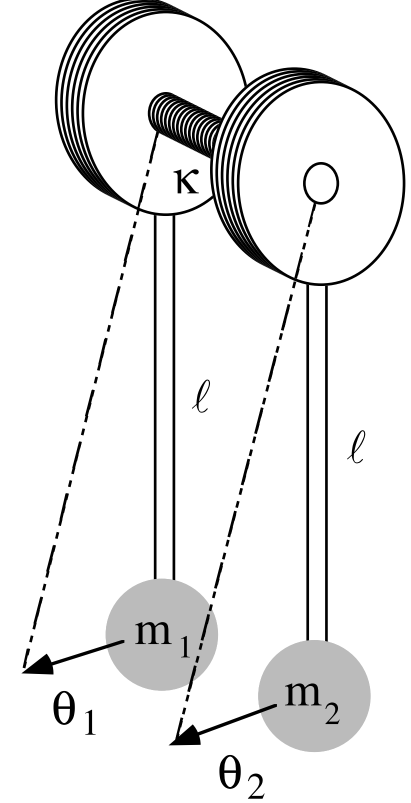


Fig. 3.3.2 Coupled pendulums

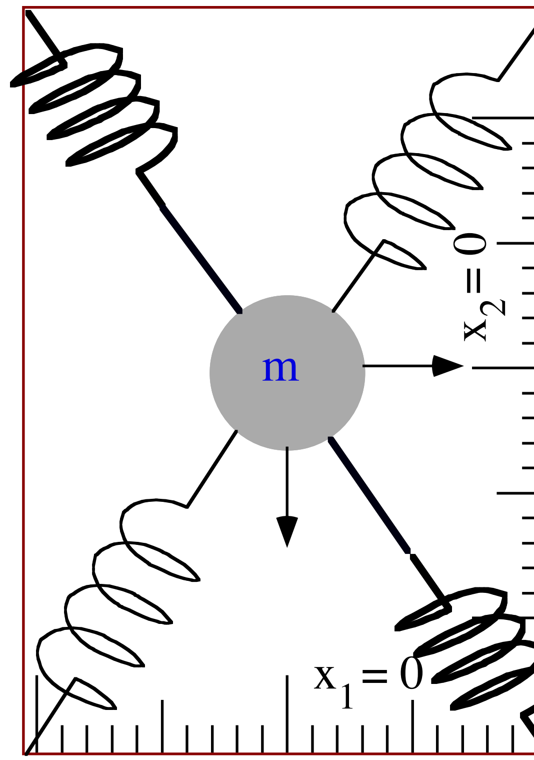


Fig. 3.3.3 One 2-dimensional coupled oscillator

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Lagrangian L=T-V

2D HO Lagrange equations

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2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M}\bullet\left|\ddot{\mathbf{x}}\right\rangle = -\mathbf{K}\bullet\left|\mathbf{x}\right\rangle$$

2D harmonic oscillator equations

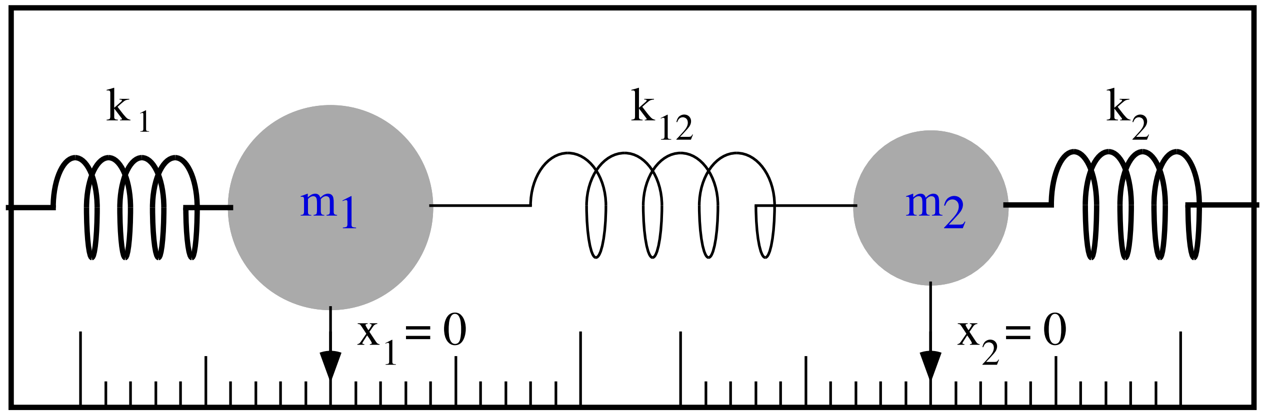
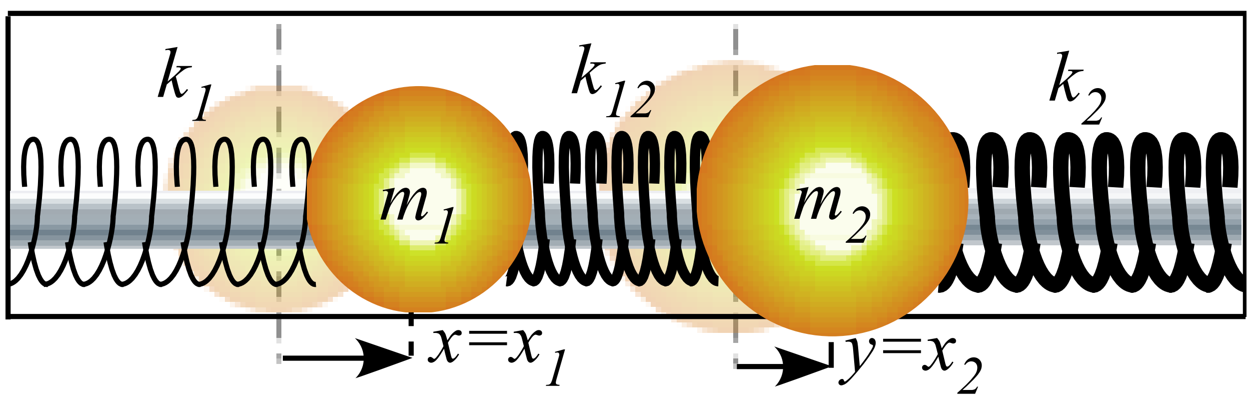


Fig. 3.3.1 Two 1-dimensional coupled oscillators



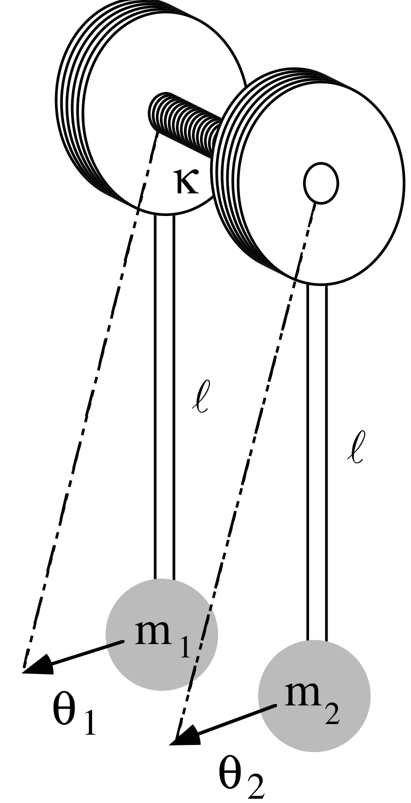


Fig. 3.3.2 Coupled pendulums

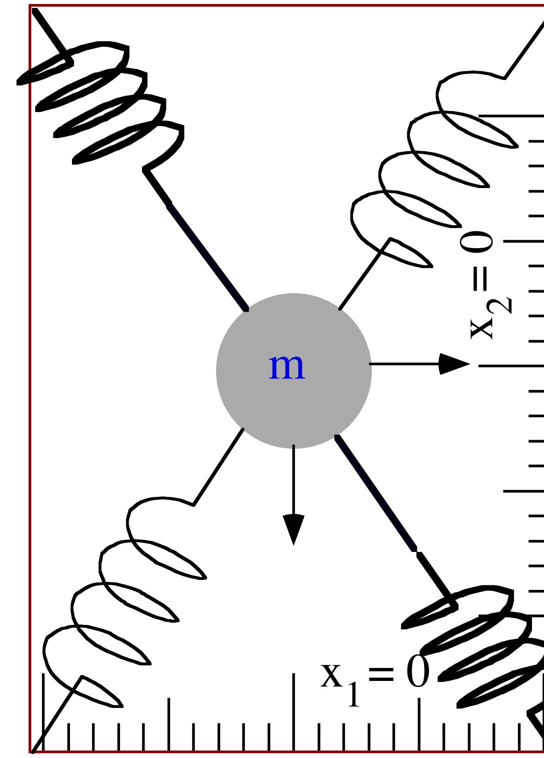


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2}(k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrangian L=T-V

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \bullet |\ddot{\mathbf{x}}\rangle \quad = \quad - \quad \mathbf{K} \bullet |\mathbf{x}\rangle$$

2D harmonic oscillator equation solutions

1. Need to rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1 + k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

To introduce eigensolutions we take a simple case of unit masses *($m_1=1=m_2$)*

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

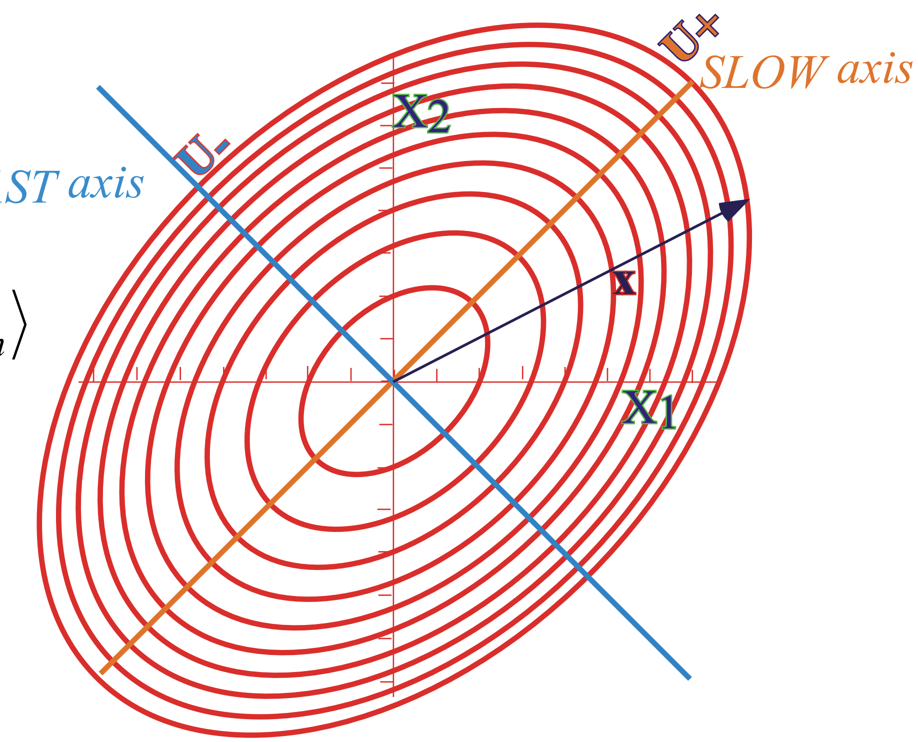
Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V=\frac{1}{2}\left(k_1+k_{12}\right)x_1^2-k_{12}x_1x_2+\frac{1}{2}\left(k_2+k_{12}\right)x_2^2=\frac{1}{2}\langle\mathbf{x}|\mathbf{K}|\mathbf{x}\rangle=\frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x}=\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours

What direction $|\mathbf{x}\rangle=|\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??



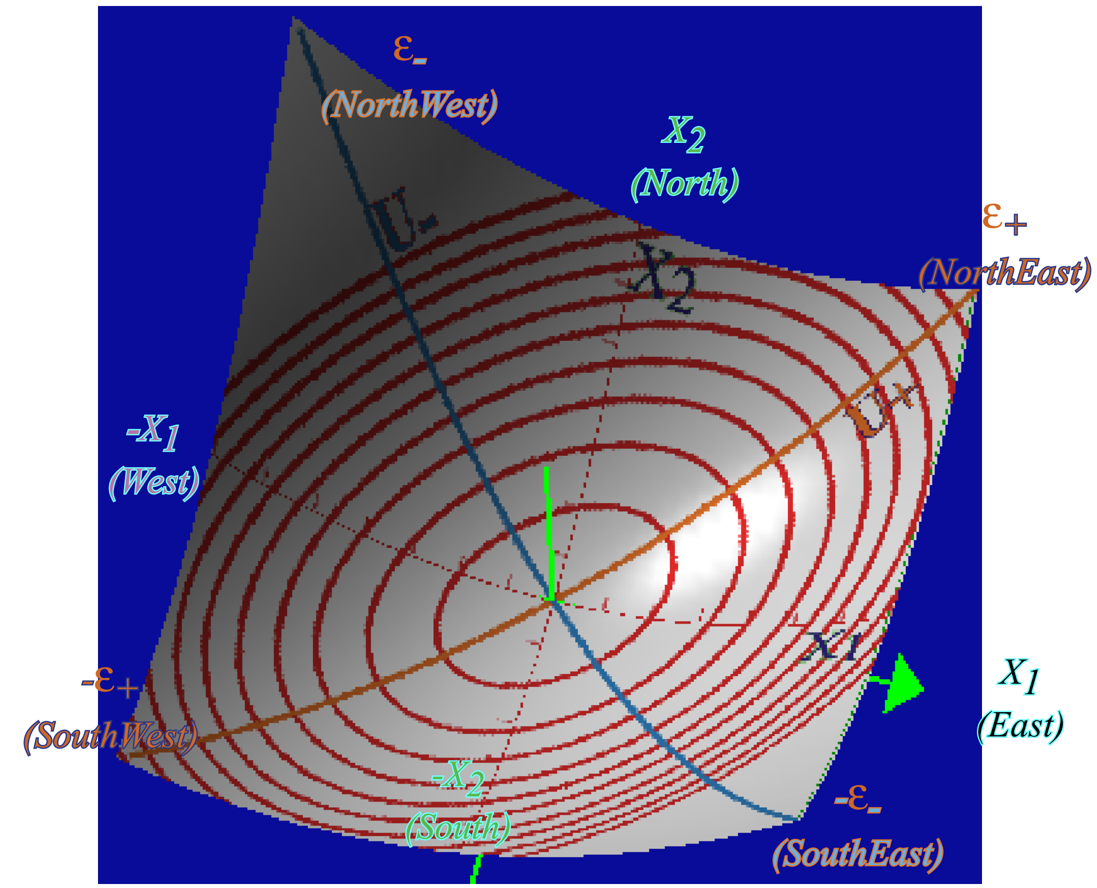
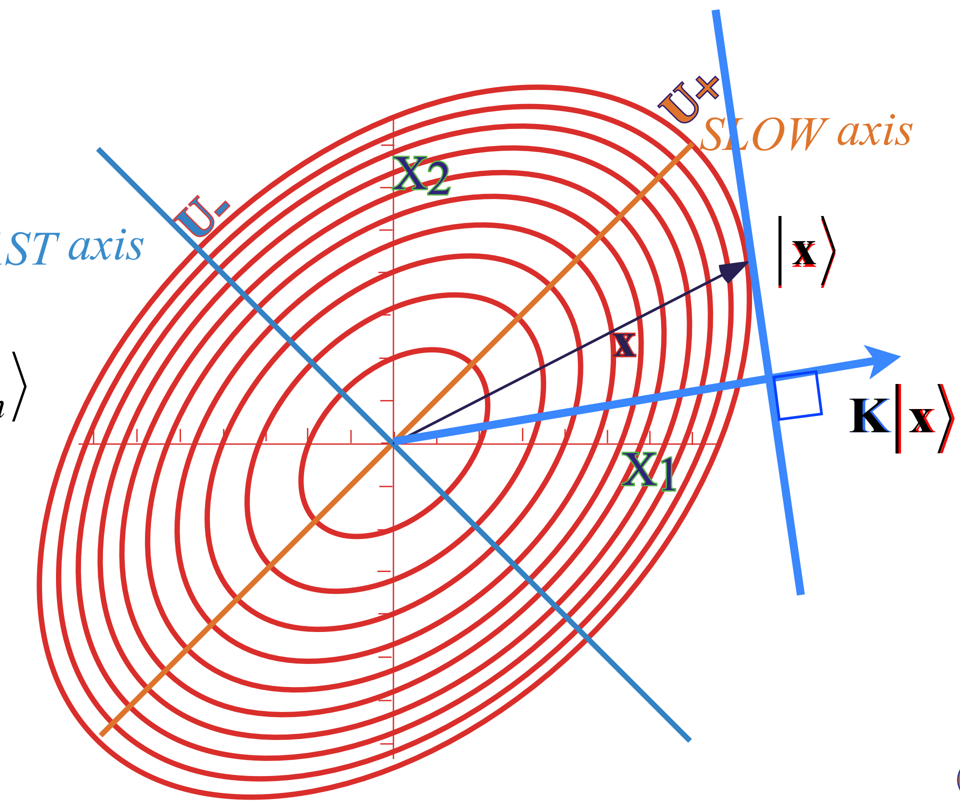


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

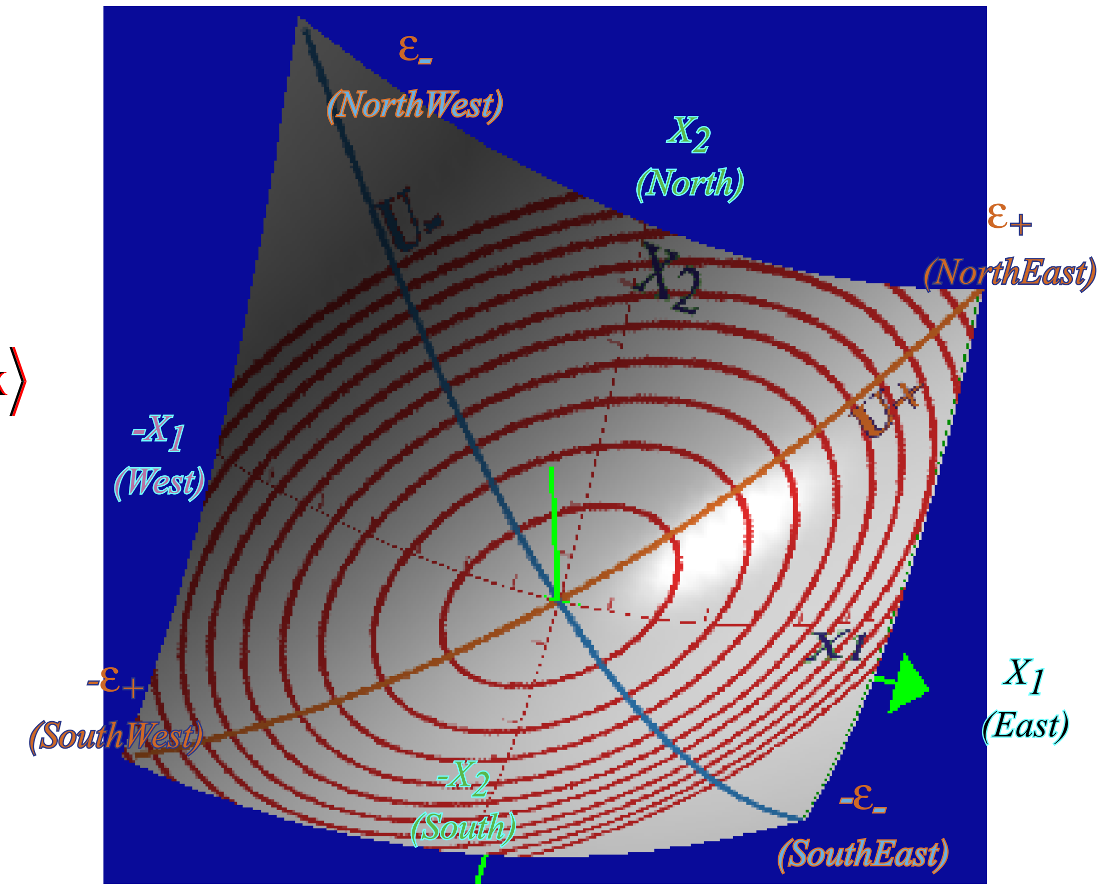
2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$
is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not most directions!



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2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours

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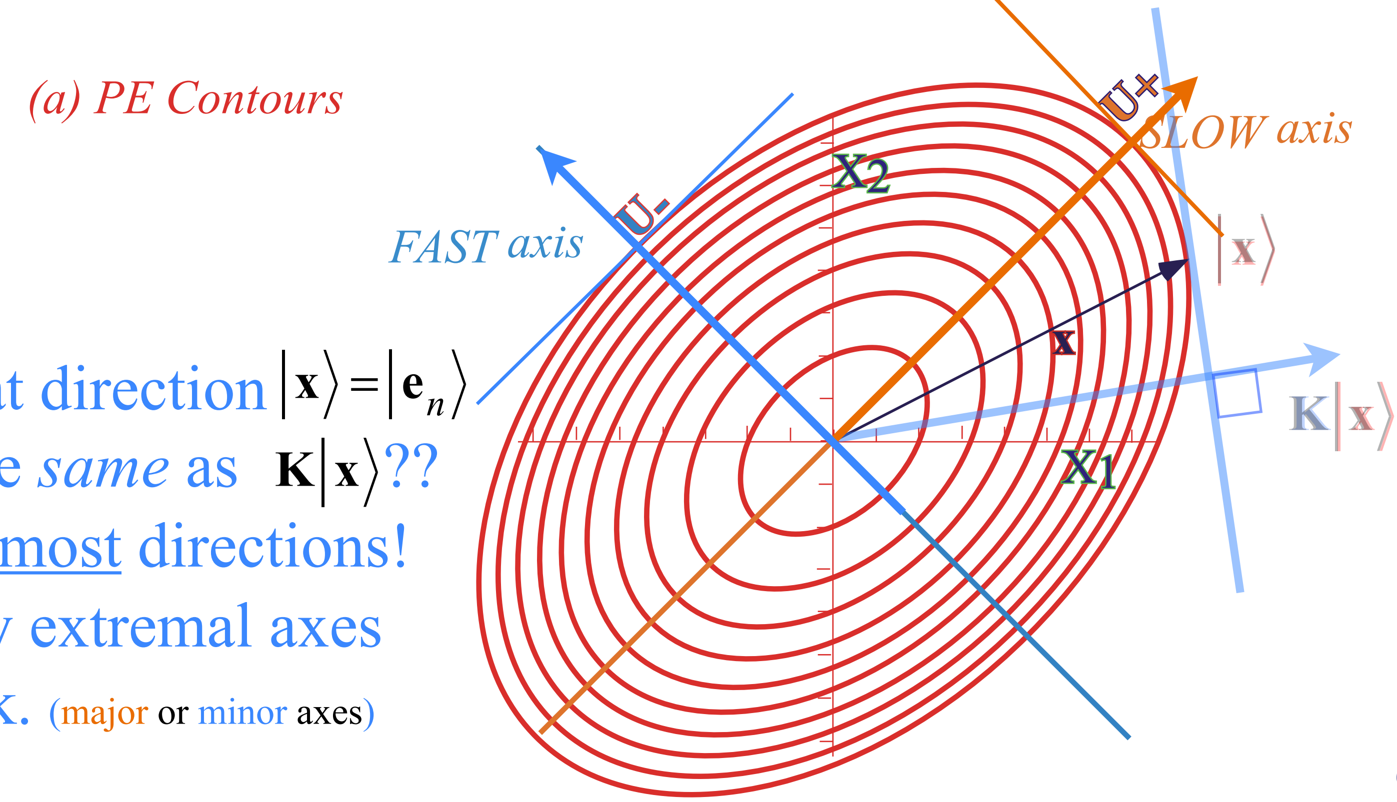


What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$??
Not <u>most</u> directions!
Only extremal axes work. (major or minor axes)

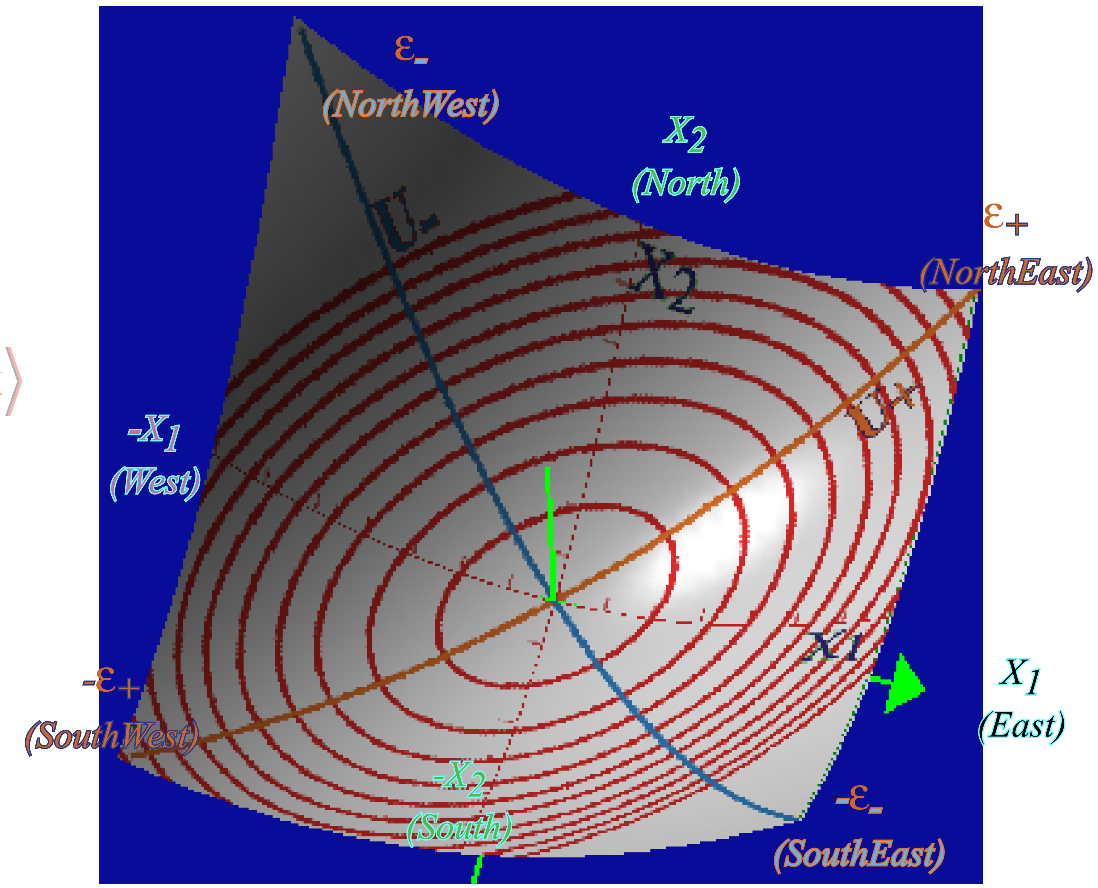
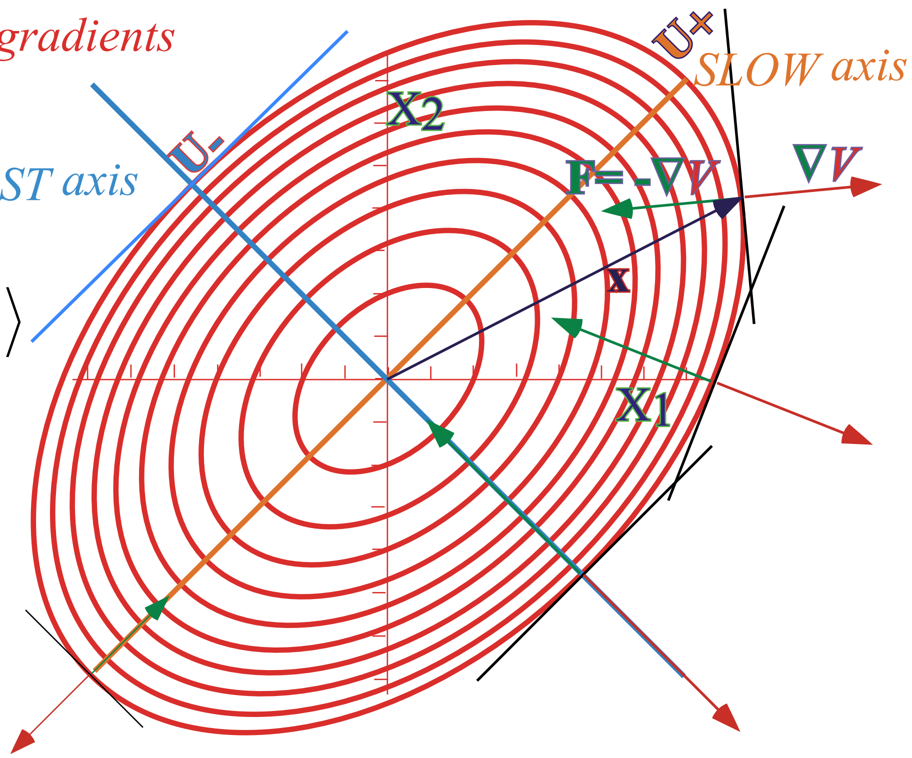


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



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Not most directions!

Only extremal axes work. (major or minor axes)

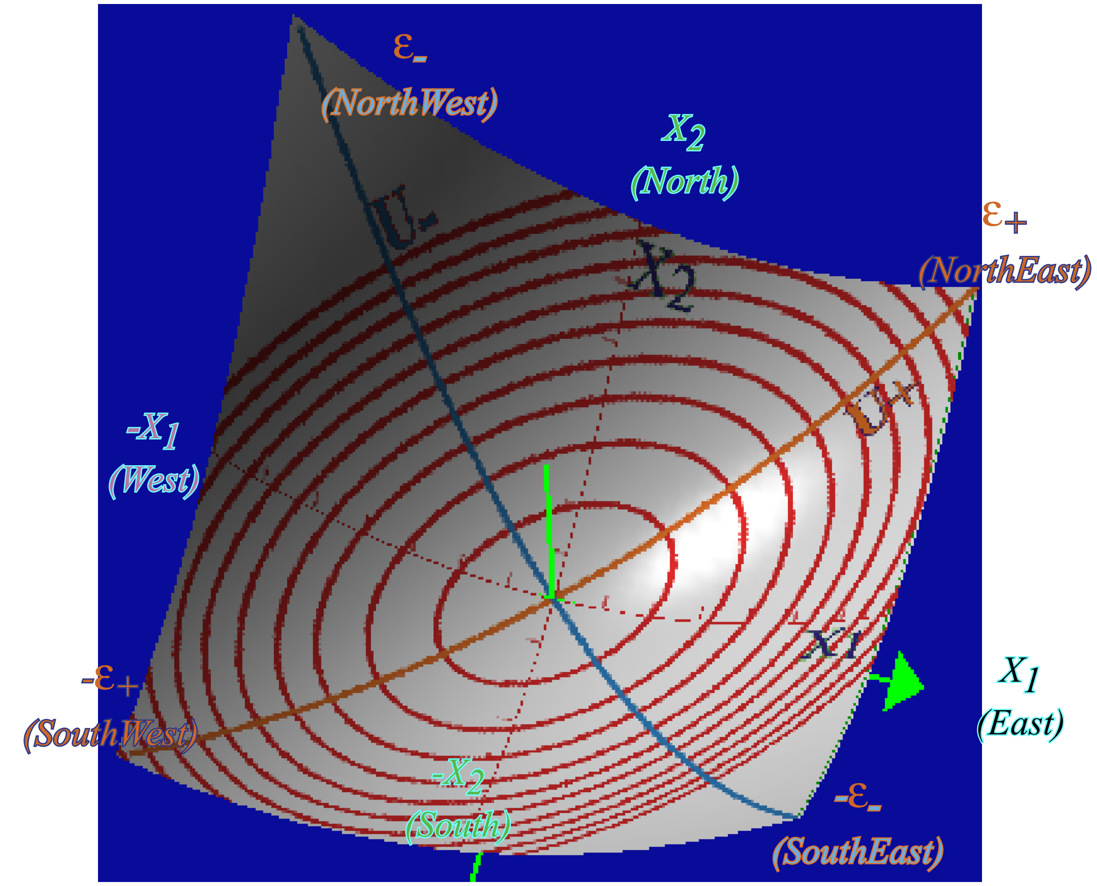
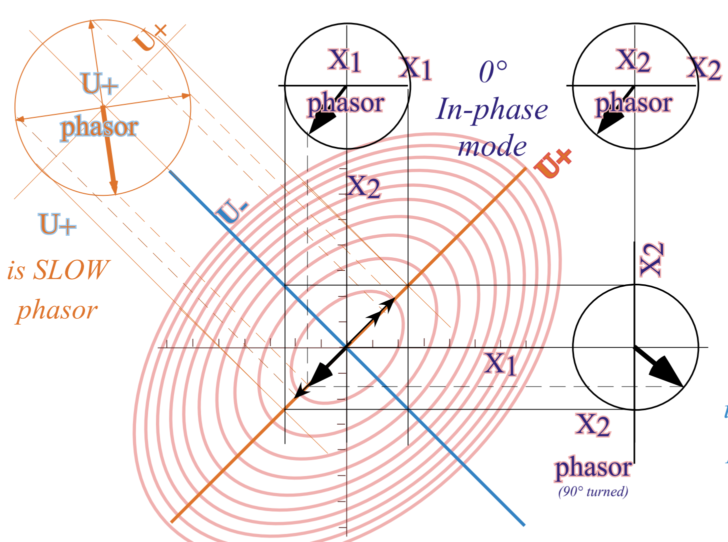
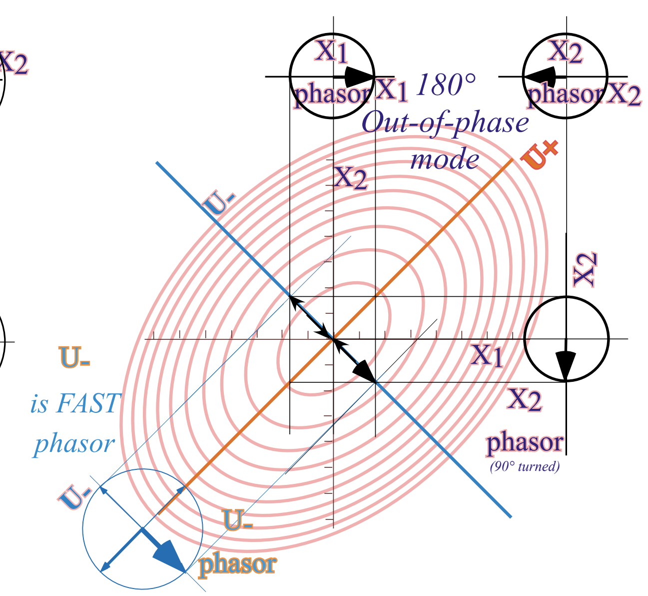


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode



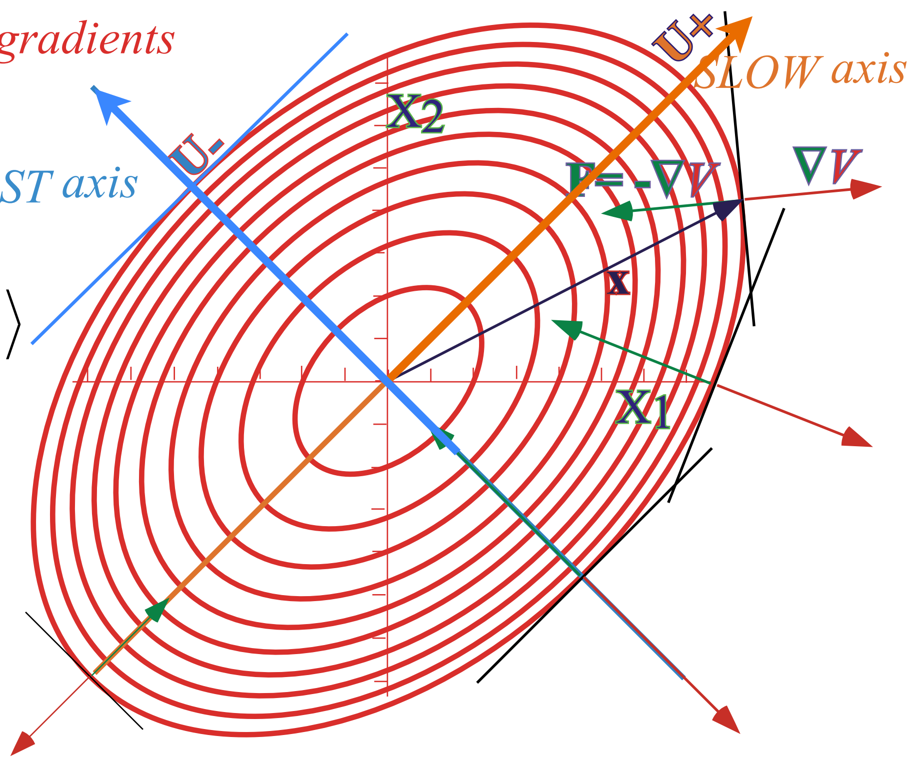
(c) Anti-symmetric U- Coordinate FAST Mode



2D HO potential energy $V(x_1, x_2)$ quadratic form defines layers of elliptical V-contours (Here: $k_1 = k = k_2$)

$$V = \frac{1}{2}(k + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k + k_{12})x_2^2 = \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle = \frac{1}{2}\mathbf{x}\bullet\mathbf{K}\bullet\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(a) PE Contours and gradients



What direction $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ is the *same* as $\mathbf{K}|\mathbf{x}\rangle$?? Not most directions! Only extremal axes work. (major or minor axes)

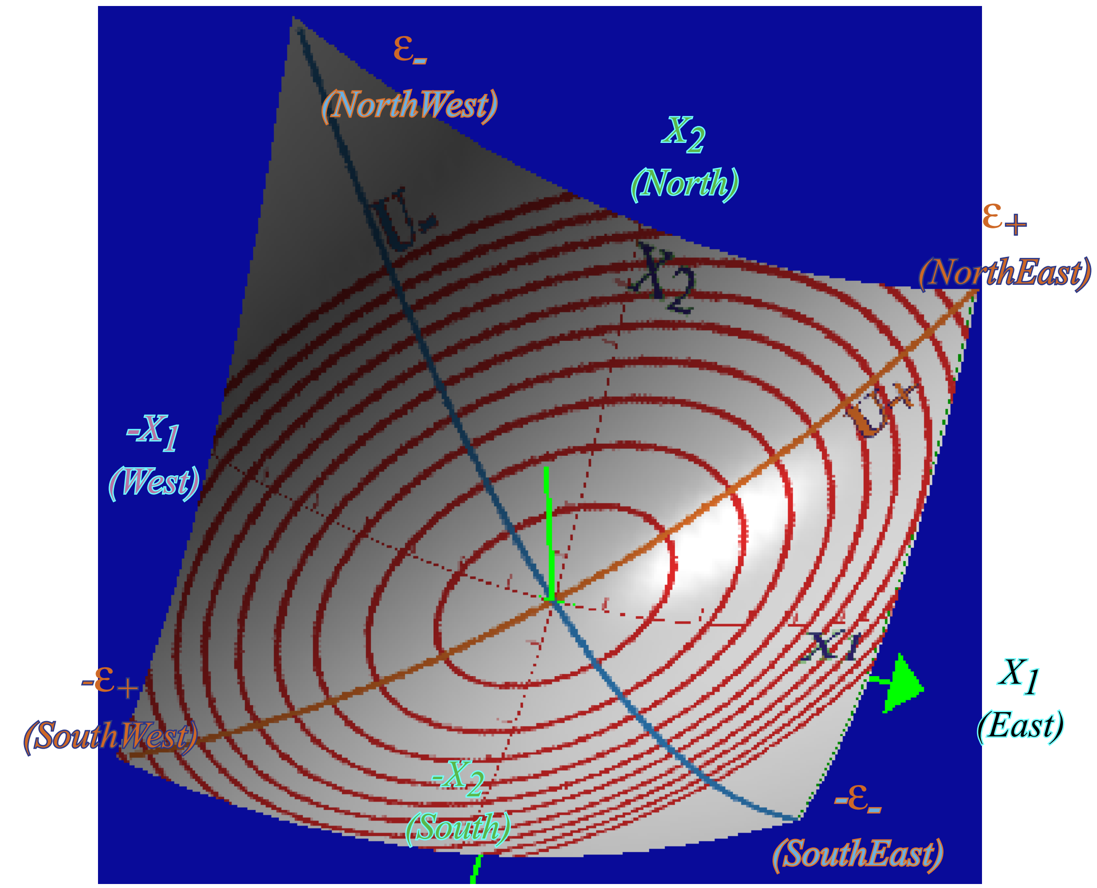
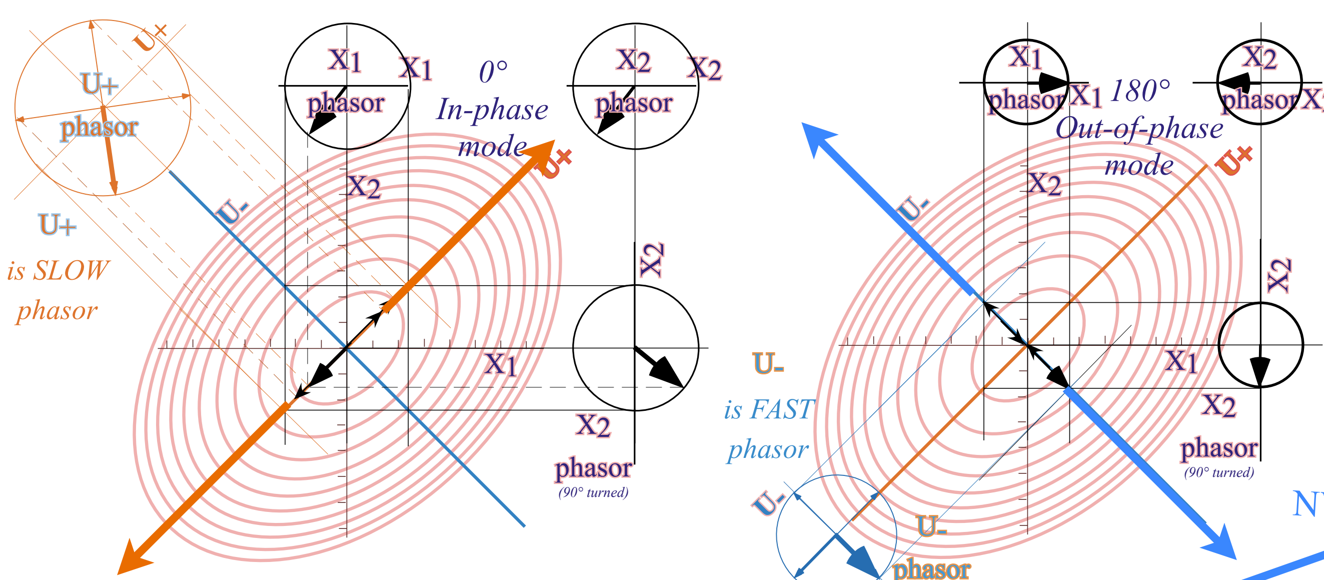
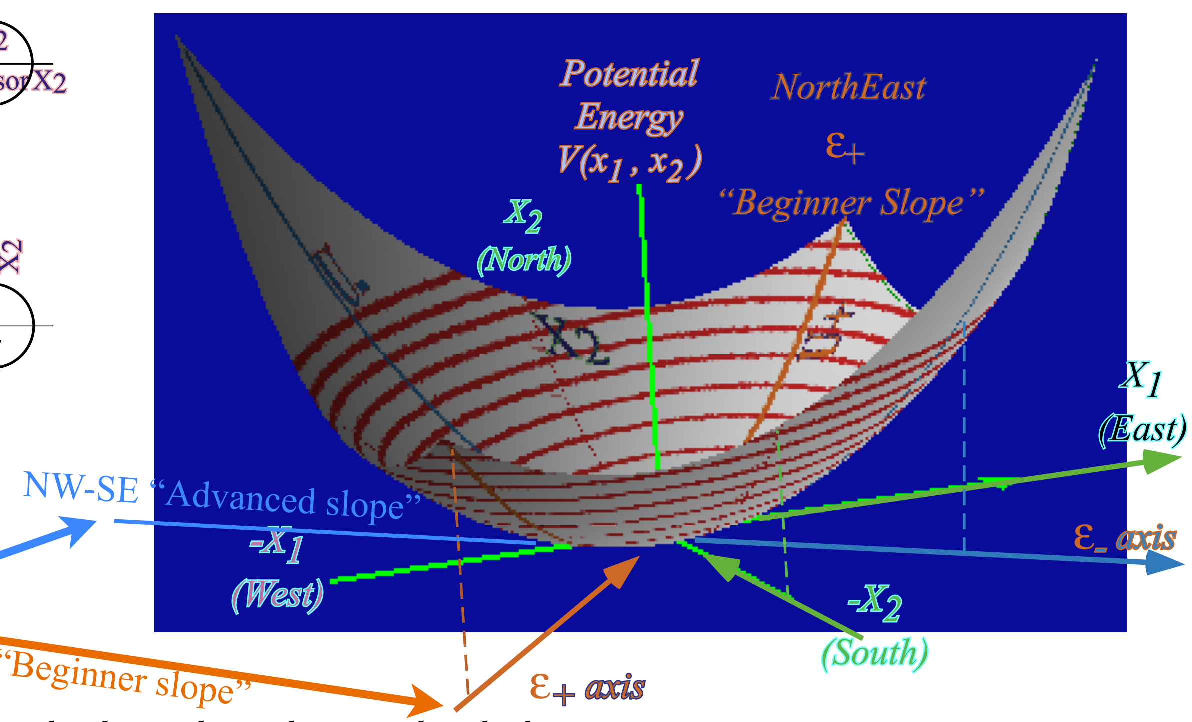


Fig. 3.3.4 Plot of potential function $V(x_1,x_2)$ showing elliptical $V(x_1,x_2)$=const. level curves.

(b) Symmetric U+ Coordinate SLOW Mode

(c) Anti-symmetric U- Coordinate FAST Mode





With Bilateral symmetry ($k_1 = k = k_2$) the extremal axes lie at ±45°

Fig. 3.3.5 Topography lines of potential function $V(x_1,x_2)$ and orthogonal ε_+ and ε_- normal mode slopes

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle, \text{ or: } (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\varepsilon\begin{pmatrix} x \\ y \end{pmatrix} \text{ or: } \begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix} \langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle \\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle \end{pmatrix}=\begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

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$$\begin{pmatrix} \langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle \\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle \end{pmatrix}=\begin{pmatrix} \varepsilon_1 & 0 & \cdots & 0 \\ 0 & \varepsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n \end{pmatrix}$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\varepsilon\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or: } \begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix} 0 & 1 \\ 0 & 2-\varepsilon \end{pmatrix}\right|}{\det\left|\begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\right|} \quad \text{and} \quad y=\frac{\det\left|\begin{pmatrix} 4-\varepsilon & 0 \\ 3 & 0 \end{pmatrix}\right|}{\det\left|\begin{pmatrix} 4-\varepsilon & 1 \\ 3 & 2-\varepsilon \end{pmatrix}\right|}$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

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ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1 & 0 & \cdots & 0\\ 0 & \varepsilon_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ \ a_n=(-1)^n\det|\mathbf{M}|$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\ \text{ or: }\ \begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

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$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

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A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1 & 0 & \cdots & 0\\ 0 & \varepsilon_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon_n\end{pmatrix}$$

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where:

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Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\quad\text{or:}\quad\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

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$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\text{ or: }(\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\\langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\\vdots&\vdots&\ddots&\vdots\\\langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle&\langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle&\cdots&\langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1&0&\cdots&0\\0&\varepsilon_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\varepsilon_n\end{pmatrix}$$

First step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ a_n=(-1)^n\det|\mathbf{M}|$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

Each ε replaced by **M** and each ε_k by $\varepsilon_k\mathbf{1}$ gives *Hamilton-Cayley* matrix equation.

$$\mathbf{0}=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$

Obviously true if **M** has diagonal form. (But, that's circular logic. Faith needed!)

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\quad\text{or:}\quad\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

$$0=\mathbf{M}^2-6\mathbf{M}+5\mathbf{1}=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})$$

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{or:}\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1&0&\cdots&0\\0&\varepsilon_2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&\varepsilon_n\end{pmatrix}$$

1st step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

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Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

Each ε replaced by **M** and each ε_k by $\varepsilon_k\mathbf{1}$ gives *Hamilton-Cayley* matrix equation.

$$\mathbf{0}=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$

Obviously true if **M** has diagonal form. (But, that's circular logic. Faith needed!)

Replace j^{th} HC-factor by (**1**) to make *projection operators* $\mathbf{p}_k=\prod_{j\neq k}(\mathbf{M}-\varepsilon_j\mathbf{1})$.

$$\mathbf{p}_1=(\quad\mathbf{1}\quad)(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\mathbf{p}_2=(\mathbf{M}-\varepsilon_1\mathbf{1})(\quad\mathbf{1}\quad)\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\vdots$$
$$\mathbf{p}_n=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\quad\mathbf{1}\quad)$$

(Assume distinct e-values here: *Non-degeneracy clause*) $\varepsilon_j\neq\varepsilon_k\neq\ldots$

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\quad\text{or:}\quad\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\left|\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}\quad\text{and}\quad y=\frac{\det\left|\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\right|}{\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$
$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{ so let: }\ \varepsilon_1=1\ \text{ and: }\ \varepsilon_2=5$$

$$0=\mathbf{M}^2-6\mathbf{M}+5\mathbf{1}=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})$$

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}4-5&1\\3&2-5\end{pmatrix}=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{1})=\begin{pmatrix}4-1&1\\3&2-1\end{pmatrix}=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

An *eigenvector* $|\varepsilon_k\rangle$ of **M** is in a direction that is left unchanged by **M**.

$$\mathbf{M}|\varepsilon_k\rangle=\varepsilon_k|\varepsilon_k\rangle,\ \text{or:}\ (\mathbf{M}-\varepsilon_k\mathbf{1})|\varepsilon_k\rangle=\mathbf{0}$$

ε_k is *eigenvalue* associated with eigenvector $|\varepsilon_k\rangle$ direction.

A change of basis to $\{|\varepsilon_1\rangle,|\varepsilon_2\rangle,\cdots|\varepsilon_n\rangle\}$ called *diagonalization* gives

$$\begin{pmatrix}\langle\varepsilon_1|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_1|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_1|\mathbf{M}|\varepsilon_n\rangle\\ \langle\varepsilon_2|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_2|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_2|\mathbf{M}|\varepsilon_n\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\varepsilon_n|\mathbf{M}|\varepsilon_1\rangle & \langle\varepsilon_n|\mathbf{M}|\varepsilon_2\rangle & \cdots & \langle\varepsilon_n|\mathbf{M}|\varepsilon_n\rangle\end{pmatrix}=\begin{pmatrix}\varepsilon_1 & 0 & \cdots & 0\\ 0 & \varepsilon_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varepsilon_n\end{pmatrix}$$

1st step in finding eigenvalues: Solve *secular equation*

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n\left(\varepsilon^n+a_1\varepsilon^{n-1}+a_2\varepsilon^{n-2}+\ldots+a_{n-1}\varepsilon\ +a_n\right)$$

where:

$$a_1=-Trace\mathbf{M},\cdots,\ a_k=(-1)^k\sum\text{diagonal k-by-k minors of }\mathbf{M},\cdots,\ a_n=(-1)^n\det|\mathbf{M}|$$

Secular equation has n-factors, one for each eigenvalue.

$$\det|\mathbf{M}-\varepsilon\mathbf{1}|=0=(-1)^n(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)\cdots(\varepsilon-\varepsilon_n)$$

Each ε replaced by **M** and each ε_k by $\varepsilon_k\mathbf{1}$ gives *Hamilton-Cayley* matrix equation.

$$\mathbf{0}=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$

Obviously true if **M** has diagonal form. (But, that's circular logic. Faith needed!)

Replace j^{th} HC-factor by (**1**) to make *projection operators* $\mathbf{p}_k=\prod_{j\neq k}(\mathbf{M}-\varepsilon_j\mathbf{1})$.

$$\mathbf{p}_1=(\quad\mathbf{1}\quad)(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\mathbf{p}_2=(\mathbf{M}-\varepsilon_1\mathbf{1})(\quad\mathbf{1}\quad)\cdots(\mathbf{M}-\varepsilon_n\mathbf{1})$$
$$\vdots$$
$$\mathbf{p}_n=(\mathbf{M}-\varepsilon_1\mathbf{1})(\mathbf{M}-\varepsilon_2\mathbf{1})\cdots(\quad\mathbf{1}\quad)$$

(Assume distinct e-values here: *Non-degeneracy clause*)
$\varepsilon_j\neq\varepsilon_k\neq\ldots$

Each $\mathbf{p}_k$ contains *eigen-bra-kets* since: $(\mathbf{M}-\varepsilon_k\mathbf{1})\mathbf{p}_k=0$ or: $\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k=\mathbf{p}_k\mathbf{M}$.

$$\mathbf{M}|\varepsilon\rangle=\begin{pmatrix}4&1\\3&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\varepsilon\begin{pmatrix}x\\y\end{pmatrix}\quad\text{or:}\quad\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Trying to solve by Kramer's inversion:

$$x=\frac{\det\begin{vmatrix}\begin{pmatrix}0&1\\0&2-\varepsilon\end{pmatrix}\end{vmatrix}}{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\end{vmatrix}}\quad\text{and}\quad y=\frac{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&0\\3&0\end{pmatrix}\end{vmatrix}}{\det\begin{vmatrix}\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\end{vmatrix}}$$

Only possible non-zero $\{x,y\}$ if denominator is zero, too!

$$0=\det|\mathbf{M}-\varepsilon\cdot\mathbf{1}|=\det\left|\begin{pmatrix}4&1\\3&2\end{pmatrix}-\varepsilon\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=\det\left|\begin{pmatrix}4-\varepsilon&1\\3&2-\varepsilon\end{pmatrix}\right|$$

$$0=(4-\varepsilon)(2-\varepsilon)-1\cdot3=8-6\varepsilon+\varepsilon^2-1\cdot3=\varepsilon^2-6\varepsilon+5$$

Non-degeneracy clause

$$0=\varepsilon^2-Trace(\mathbf{M})\varepsilon+\det(\mathbf{M})=\varepsilon^2-6\varepsilon+5$$

$$0=(\varepsilon-1)(\varepsilon-5)\ \text{so let:}\ \varepsilon_1=1\ \text{and:}\ \varepsilon_2=5$$

$$0=\mathbf{M}^2-6\mathbf{M}+5\mathbf{M}=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})$$

$$\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}4&1\\3&2\end{pmatrix}^2-6\begin{pmatrix}4&1\\3&2\end{pmatrix}+5\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{p}_1=(\mathbf{1})(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}4-5&1\\3&2-5\end{pmatrix}=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})(\mathbf{1})=\begin{pmatrix}4-1&1\\3&2-1\end{pmatrix}=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

$$\mathbf{M}\mathbf{p}_1=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\begin{pmatrix}-1&1\\3&-3\end{pmatrix}=1\cdot\mathbf{p}_1$$

$$\mathbf{M}\mathbf{p}_2=\begin{pmatrix}4&1\\3&2\end{pmatrix}\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\begin{pmatrix}3&1\\3&1\end{pmatrix}=5\cdot\mathbf{p}_2$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

$$\mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{p}_1\mathbf{p}_2=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k=\prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j)=\mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m)=\begin{cases}\mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix*

$$\mathbf{M}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})\qquad \mathbf{M}\mathbf{p}_k=\varepsilon_k\mathbf{p}_k$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}\qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k=\prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j)=\mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m)=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step is to make *Idempotent Projectors:* $\mathbf{P}_k=\frac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}=\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\mathbf{P}_j\mathbf{P}_k=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}\qquad \mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$$

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases}\mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step is to make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases}\mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k\end{cases} \qquad \mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$$

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix}-1&1\\3&-3\end{pmatrix}$$

$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix}3&1\\3&1\end{pmatrix}$$

$$\mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

Factoring bra-kets into “Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix} = |1\rangle\langle 1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix} = |2\rangle\langle 2|$$

Matrix-algebraic method for finding eigenvector and eigenvalues

With example matrix $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}\qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k=\prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j)=\mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m)=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Factoring bra-kets into "Ket-Bras:

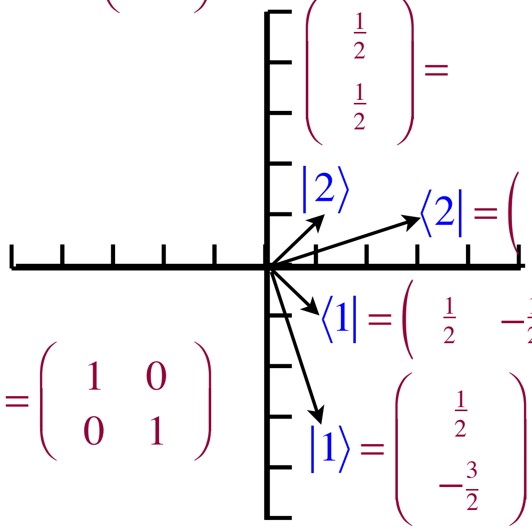
Last step is to make *Idempotent Projectors:* $\mathbf{P}_k=\frac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}=\frac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}=|1\rangle\langle 1|$$

$$\mathbf{P}_j\mathbf{P}_k=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}\qquad \mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}=|2\rangle\langle 2|$$

They're *Ortho-Normal*



$$|2\rangle=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\qquad \langle 2|=\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}\qquad \langle 1|=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}\qquad |1\rangle=\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}$$

$$\begin{pmatrix}\langle 1|1\rangle & \langle 1|2\rangle\\ \langle 2|1\rangle & \langle 2|2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

Multiplication properties of $\mathbf{p}_j$:

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Last step is to make *Idempotent Projectors:* $\mathbf{P}_k=\dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}=\dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}_j\mathbf{P}_k=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}\qquad \mathbf{M}\mathbf{P}_k=\varepsilon_k\mathbf{P}_k$$

They're *Ortho-Normal* and satisfy *Completeness Relation* $\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}\qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

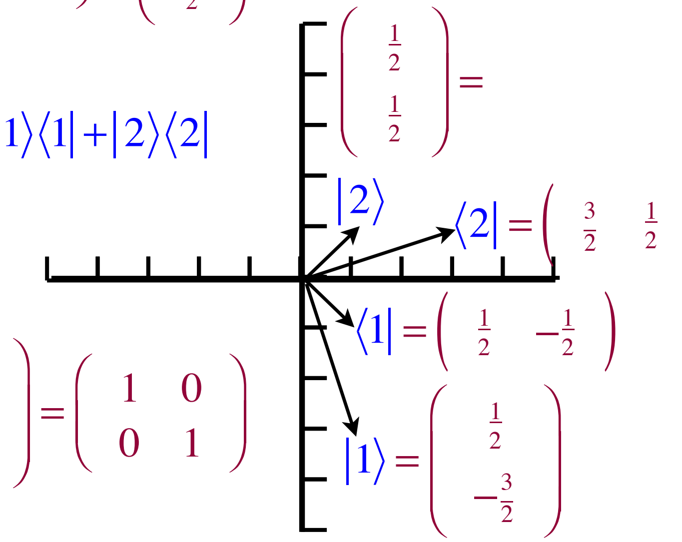
$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}=|1\rangle\langle 1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}=|2\rangle\langle 2|$$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|1\rangle\langle 1|+|2\rangle\langle 2|$$



$$\begin{pmatrix}\langle 1|1\rangle&\langle 1|2\rangle\\\langle 2|1\rangle&\langle 2|2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues *With example matrix* $\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}$

$$\mathbf{p}_j\mathbf{p}_k=\mathbf{p}_j\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})=\prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

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$$\mathbf{p}_j\mathbf{p}_k=\prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j)=\mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m)=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k\end{cases}$$

Last step is to make *Idempotent Projectors:* $\mathbf{P}_k=\dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}=\dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}_j\mathbf{P}_k=\begin{cases}\mathbf{0} & if: j\neq k\\ \mathbf{P}_k & if: j=k\end{cases}\qquad \mathbf{MP}_k=\varepsilon_k\mathbf{P}_k$$

They're *Ortho-Normal* and satisfy *Completeness Relation* $\mathbf{1}=\mathbf{P}_1+\mathbf{P}_2+...+\mathbf{P}_n$

Eigen-operators $\mathbf{MP}_k=\varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator $\mathbf{M}$

$$\mathbf{M}=\mathbf{MP}_1+\mathbf{MP}_2+...+\mathbf{MP}_n=\varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+...+\varepsilon_n\mathbf{P}_n$$

$$\mathbf{p}_1=(\mathbf{M}-5\cdot\mathbf{1})=\begin{pmatrix}-1&1\\3&-3\end{pmatrix}\qquad \mathbf{p}_1\mathbf{p}_2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

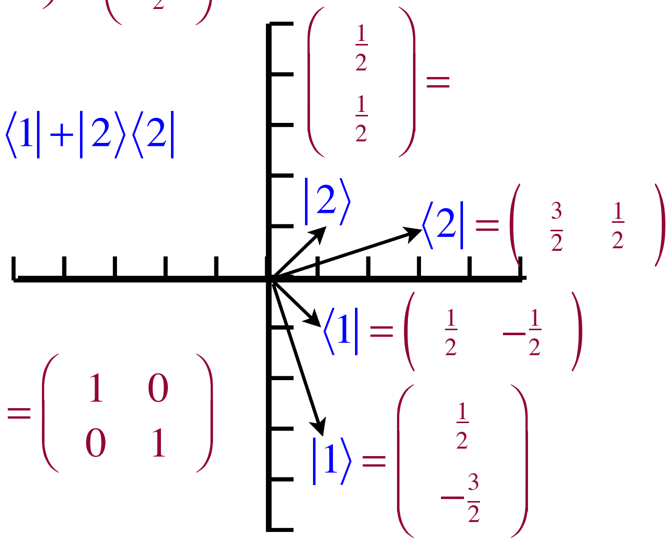
$$\mathbf{p}_2=(\mathbf{M}-1\cdot\mathbf{1})=\begin{pmatrix}3&1\\3&1\end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1=\frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)}=\frac{1}{4}\begin{pmatrix}1&-1\\-3&3\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\-\frac{3}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\end{pmatrix}=|1\rangle\langle 1|$$

$$\mathbf{P}_2=\frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)}=\frac{1}{4}\begin{pmatrix}3&1\\3&1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix}\otimes\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\end{pmatrix}=|2\rangle\langle 2|$$

$$\mathbf{P}_1+\mathbf{P}_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=|1\rangle\langle 1|+|2\rangle\langle 2|$$



$$\begin{pmatrix}\langle 1|1\rangle & \langle 1|2\rangle\\ \langle 2|1\rangle & \langle 2|2\rangle\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\mathbf{M}=\begin{pmatrix}4&1\\3&2\end{pmatrix}=1\mathbf{P}_1+5\mathbf{P}_2=1|1\rangle\langle 1|+5|2\rangle\langle 2|$$

$$=1\begin{pmatrix}\frac{1}{4}&-\frac{1}{4}\\-\frac{3}{4}&\frac{3}{4}\end{pmatrix}+5\begin{pmatrix}\frac{3}{4}&\frac{1}{4}\\\frac{3}{4}&\frac{1}{4}\end{pmatrix}$$

Matrix-algebraic method for finding eigenvector and eigenvalues

$$\mathbf{p}_j\mathbf{p}_k = \mathbf{p}_j \prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1}) = \prod_{m\neq k}(\mathbf{p}_j\mathbf{M}-\varepsilon_m\mathbf{p}_j\mathbf{1})$$

Multiplication properties of $\mathbf{p}_j$:

$$\mathbf{p}_j\mathbf{p}_k = \prod_{m\neq k}(\varepsilon_j\mathbf{p}_j-\varepsilon_m\mathbf{p}_j) = \mathbf{p}_j\prod_{m\neq k}(\varepsilon_j-\varepsilon_m) = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{p}_k\prod_{m\neq k}(\varepsilon_k-\varepsilon_m) & if: j=k \end{cases}$$

Last step is to make *Idempotent Projectors:* $\mathbf{P}_k = \dfrac{\mathbf{p}_k}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)} = \dfrac{\prod_{m\neq k}(\mathbf{M}-\varepsilon_m\mathbf{1})}{\prod_{m\neq k}(\varepsilon_k-\varepsilon_m)}$

$$\mathbf{P}_j\mathbf{P}_k = \begin{cases} \mathbf{0} & if: j\neq k \\ \mathbf{P}_k & if: j=k \end{cases} \qquad \mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$$

They're *Ortho-Normal* and satisfy *Completeness Relation* $\mathbf{1} = \mathbf{P}_1+\mathbf{P}_2+\ldots+\mathbf{P}_n$

Eigen-operators $\mathbf{M}\mathbf{P}_k = \varepsilon_k\mathbf{P}_k$ then give *Spectral Decomposition* of operator $\mathbf{M}$

$$\mathbf{M} = \mathbf{M}\mathbf{P}_1+\mathbf{M}\mathbf{P}_2+\ldots+\mathbf{M}\mathbf{P}_n = \varepsilon_1\mathbf{P}_1+\varepsilon_2\mathbf{P}_2+\ldots+\varepsilon_n\mathbf{P}_n$$

...and *Functional Spectral Decomposition* of any function $f(\mathbf{M})$ of $\mathbf{M}$

$$f(\mathbf{M}) == f(\varepsilon_1)\mathbf{P}_1+f(\varepsilon_2)\mathbf{P}_2+\ldots+f(\varepsilon_n)\mathbf{P}_n$$

Lecture 37 ends here

With example matrix $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

$$\mathbf{p}_1 = (\mathbf{M}-5\cdot\mathbf{1}) = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix} \qquad \mathbf{p}_1\mathbf{p}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

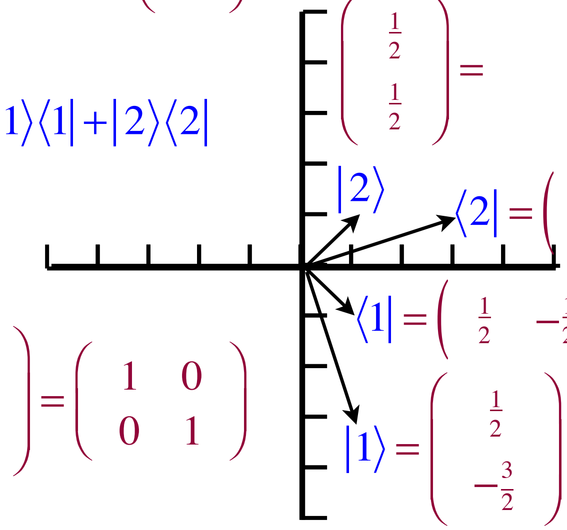
$$\mathbf{p}_2 = (\mathbf{M}-1\cdot\mathbf{1}) = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$

Factoring bra-kets into "Ket-Bras:

$$\mathbf{P}_1 = \frac{(\mathbf{M}-5\cdot\mathbf{1})}{(1-5)} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = |1\rangle\langle 1|$$

$$\mathbf{P}_2 = \frac{(\mathbf{M}-1\cdot\mathbf{1})}{(5-1)} = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} = |2\rangle\langle 2|$$

$$\mathbf{P}_1+\mathbf{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| + |2\rangle\langle 2|$$



$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = |2\rangle \qquad \langle 2| = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \end{pmatrix} \qquad \langle 1| = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix}$$

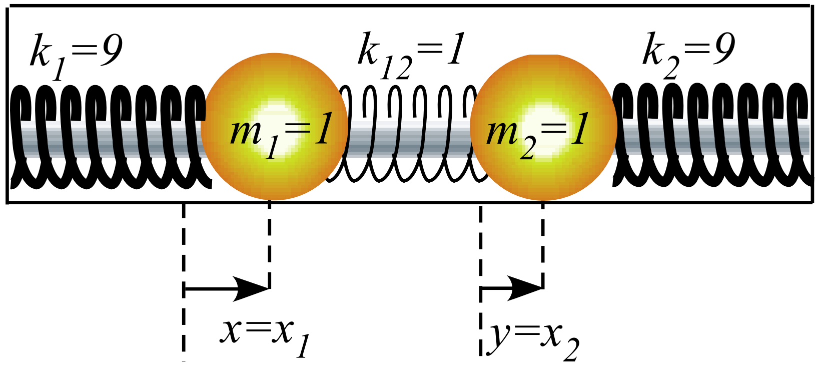
$$\begin{pmatrix} \langle 1|1\rangle & \langle 1|2\rangle \\ \langle 2|1\rangle & \langle 2|2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1\mathbf{P}_1 + 5\mathbf{P}_2 = 1|1\rangle\langle 1| + 5|2\rangle\langle 2|$$

$$= 1\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Examples with $\mathbf{M}^{50} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = 1^{50}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} + 5^{50}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$

$$\sqrt{\mathbf{M}} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix} = \pm\sqrt{1}\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{pmatrix} \pm\sqrt{5}\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$



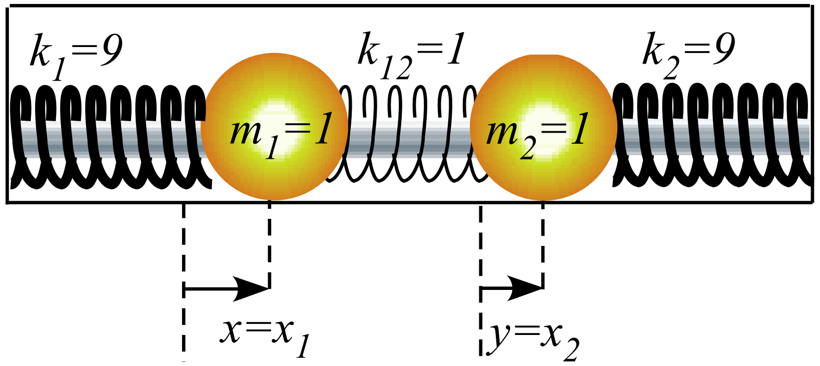
Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11) = (K-9)(K-11)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$

Analyzing 2D-HO beats and mixed mode eigen-solutions



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

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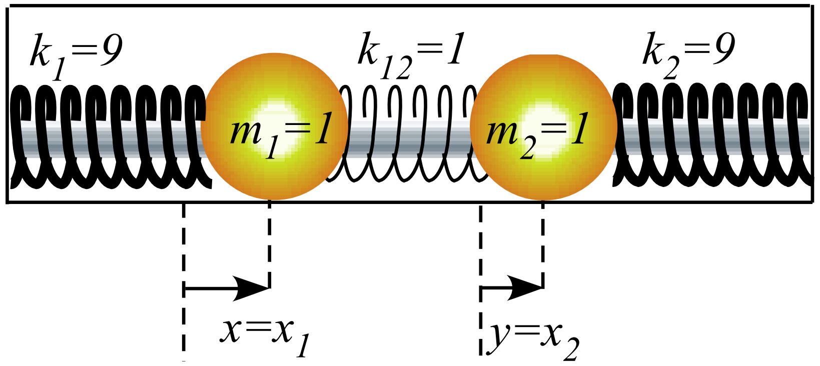
Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11} - K_2 & K_{12} \\ K_{12} & K_{22} - K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11} - K_1 & K_{12} \\ K_{12} & K_{22} - K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11)$

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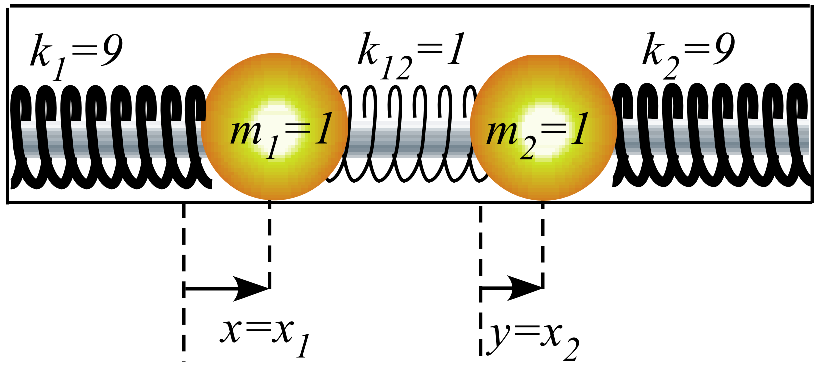
$$= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$
$$= \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+99=0=(K-9)(K-11)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=9,\quad K_2=\omega_0^2(\varepsilon_2)=11,$

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$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11 \end{pmatrix}}{9-11}=\frac{\begin{pmatrix} 1 & +1 \\ +1 & 1 \end{pmatrix}}{2}$$
$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

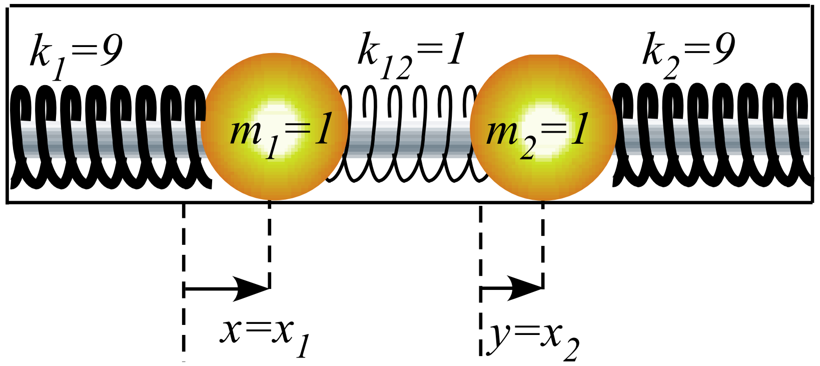
$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9 \end{pmatrix}}{11-9}=\frac{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}{2}$$
$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+|\varepsilon_2\rangle\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t}+\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$



Analyzing 2D-HO beats and mixed mode eigen-solutions

$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 10 & -1 \\ -1 & 10\end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 99 = 0 = (K-9)(K-11)$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 9, \quad K_2 = \omega_0^2(\varepsilon_2) = 11,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2} = \frac{\begin{pmatrix} 10-11 & -1 \\ -1 & 10-11\end{pmatrix}}{9-11} = \frac{\begin{pmatrix} 1 & +1 \\ +1 & 1\end{pmatrix}}{2}$$

$$=\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1} = \frac{\begin{pmatrix} 10-9 & -1 \\ -1 & 10-9\end{pmatrix}}{11-9} = \frac{\begin{pmatrix} 1 & -1 \\ -1 & 1\end{pmatrix}}{2}$$

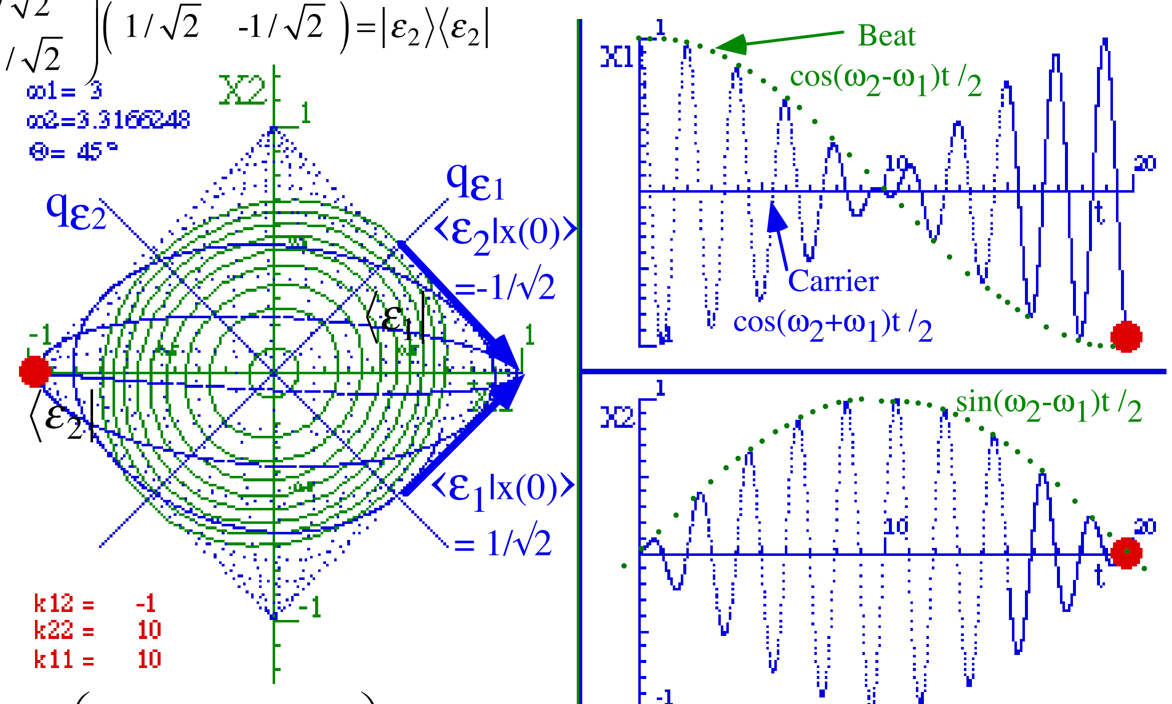
$$=\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} 1/\sqrt{2} & +1/\sqrt{2}\end{pmatrix}, \quad \langle\varepsilon_2| = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$

Mixed mode dynamics

$$|x(t)\rangle = |\varepsilon_1\rangle \langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + |\varepsilon_2\rangle \langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_1|x(0)\rangle e^{-i\omega_1 t} + \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix}\langle\varepsilon_2|x(0)\rangle e^{-i\omega_2 t}$$

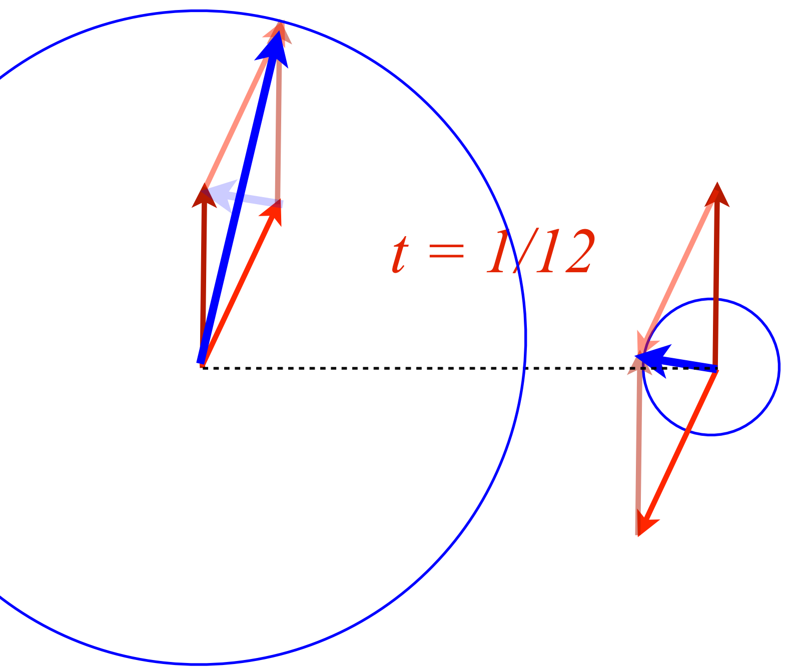


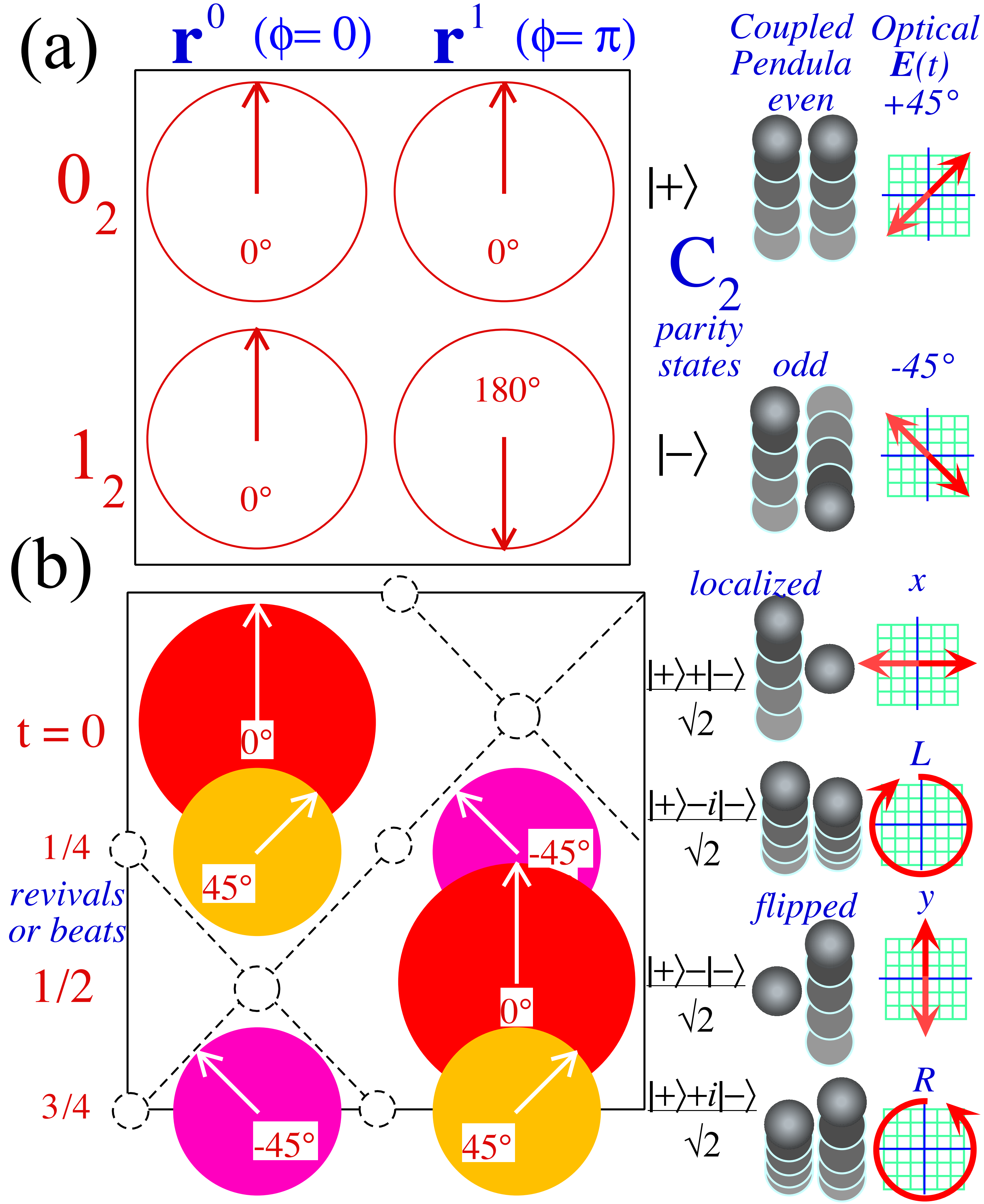
Full modulation (SWR=0)

$$\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = \begin{pmatrix} \frac{e^{-i\omega_1 t}+e^{-i\omega_2 t}}{2} \\ \frac{e^{-i\omega_1 t}-e^{-i\omega_2 t}}{2}\end{pmatrix} = \frac{e^{-i\frac{(\omega_1+\omega_2)}{2}t}}{2}\begin{pmatrix} e^{-i\frac{(\omega_1-\omega_2)}{2}t} + e^{i\frac{(\omega_1-\omega_2)}{2}t} \\ e^{-i\frac{(\omega_1-\omega_2)}{2}t} - e^{i\frac{(\omega_1-\omega_2)}{2}t}\end{pmatrix} = e^{-i\frac{(\omega_1+\omega_2)}{2}t}\begin{pmatrix} \cos\frac{(\omega_2-\omega_1)t}{2} \\ i\sin\frac{(\omega_2-\omega_1)t}{2}\end{pmatrix}$$

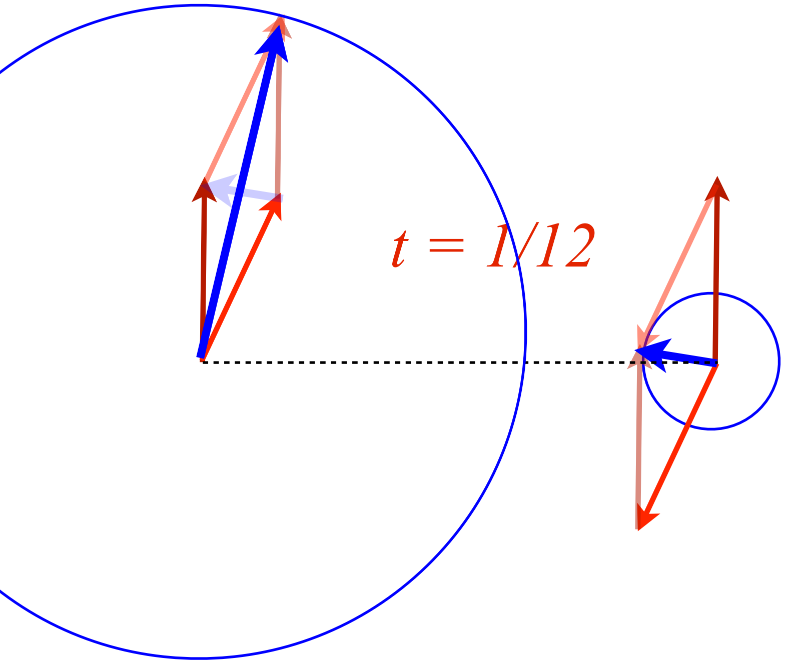
Fig. 3.3.9 Beats in weakly coupled symmetric oscillators with equal mode magnitudes.

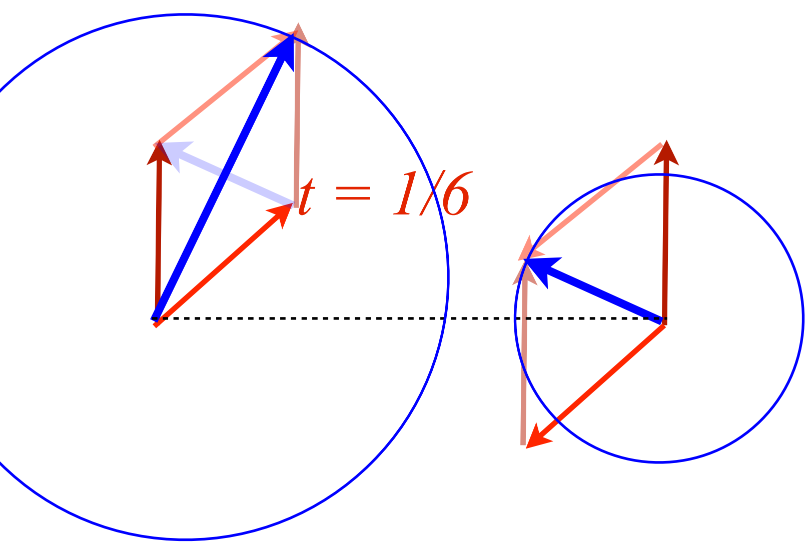
2D-HO beats and mixed mode geometry

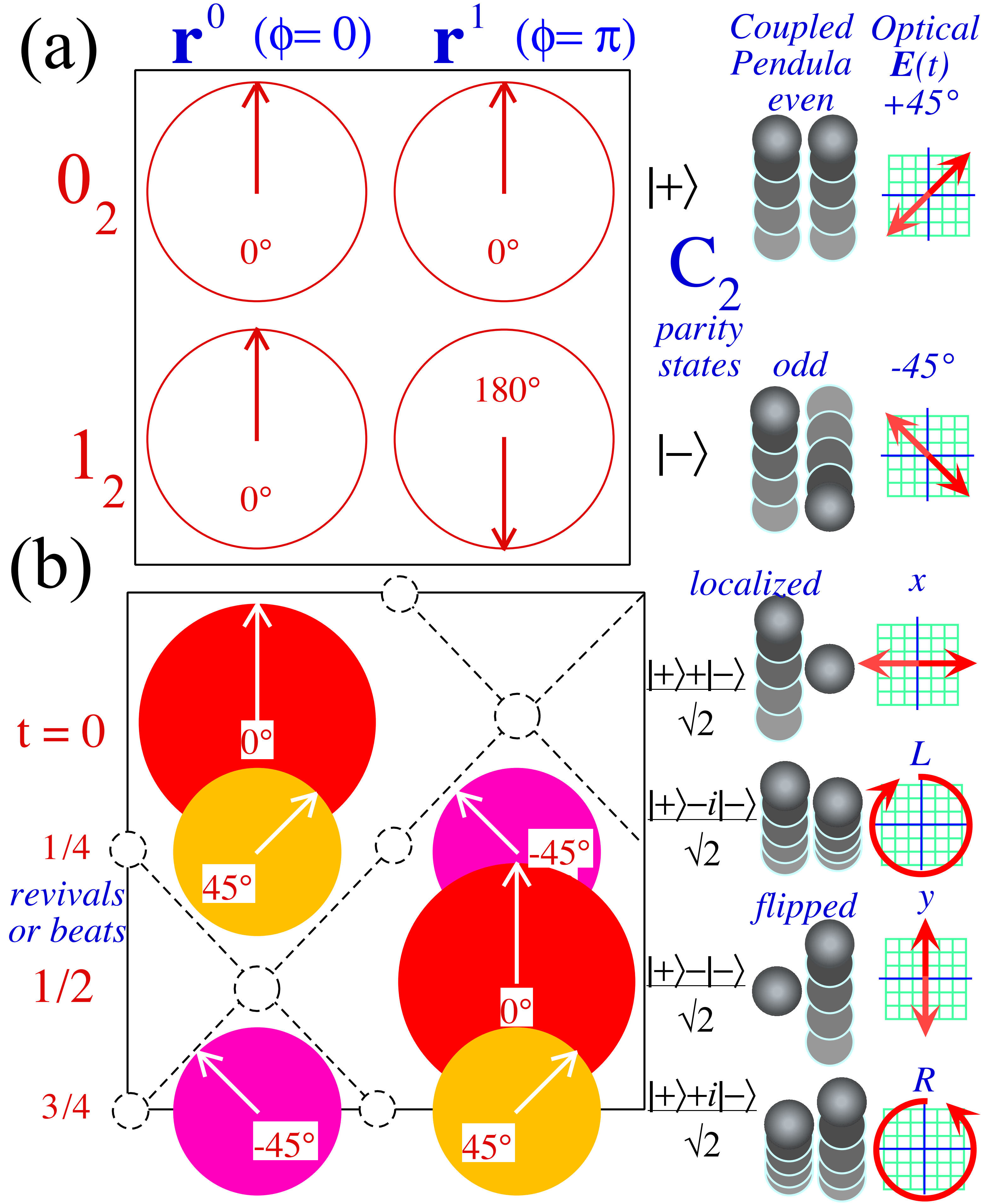




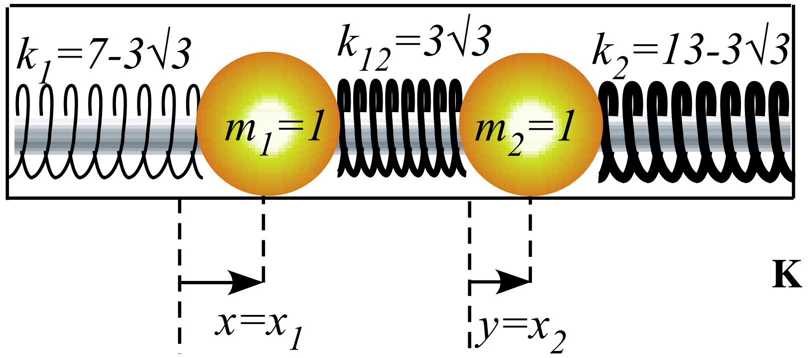
2D-HO beats and mixed mode geometry







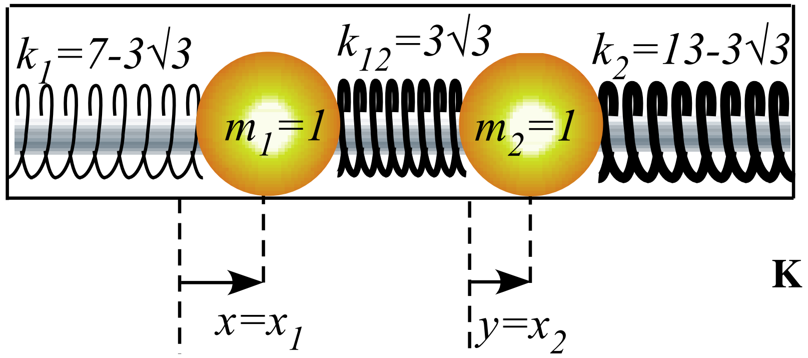
Analyzing 2D-HO mixed modes for lower symmetry



$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} = \begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$ $\qquad Det(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\qquad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$

$Det(\mathbf{K}) = 7 \cdot 13 - 27 = 91 - 27 = 64$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1 = \omega_0^2(\varepsilon_1) = 4, \quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

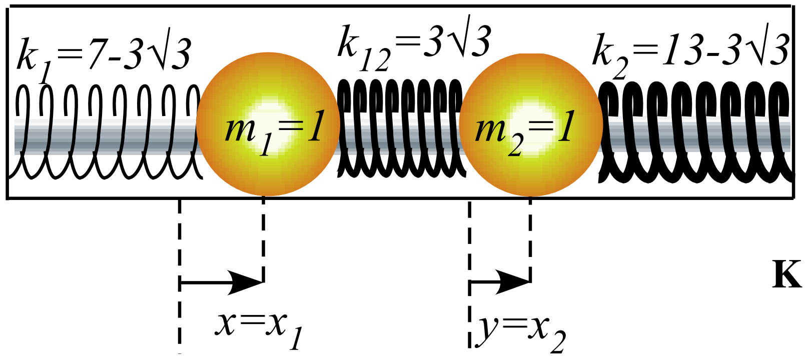
Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1 = \frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1 - K_2} = \frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16} = \frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix} = |\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2 = \frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2 - K_1} = \frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4} = \frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

$$= \frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} = |\varepsilon_2\rangle\langle\varepsilon_2|$$



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22}\end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12}\end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13\end{pmatrix}$$

The **K** secular equation $K^2 - Trace(\mathbf{K})K + Det(\mathbf{K}) = K^2 - 20K + 64 = 0 = (K-4)(K-16)$ $Det(\mathbf{K}) = 7\cdot 13 - 27 = 91 - 27 = 64$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $K_1 = \omega_0^2(\varepsilon_1) = 4,\quad K_2 = \omega_0^2(\varepsilon_2) = 16,$

Eigen-projectors $\mathbf{P}_k$

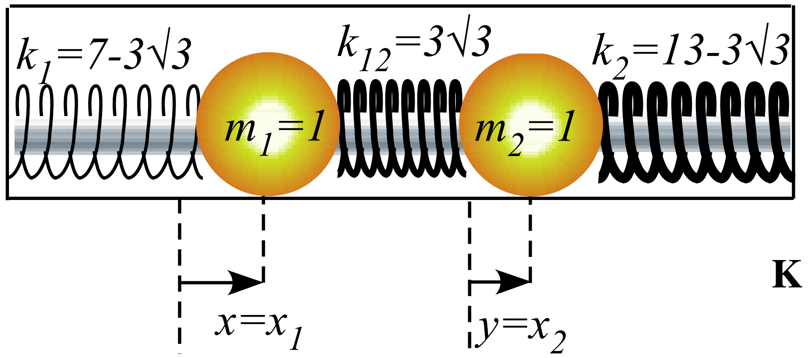
$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2\end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16\end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1\end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2\end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1\end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4\end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9\end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3\end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2\end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1| = \begin{pmatrix} \sqrt{3}/2 & 1/2\end{pmatrix},\quad \langle\varepsilon_2| = \begin{pmatrix} -1/2 & \sqrt{3}/2\end{pmatrix}$



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$

$Det(\mathbf{K})=7\cdot 13-27=91-27=64$

Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=4,\quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$

$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$

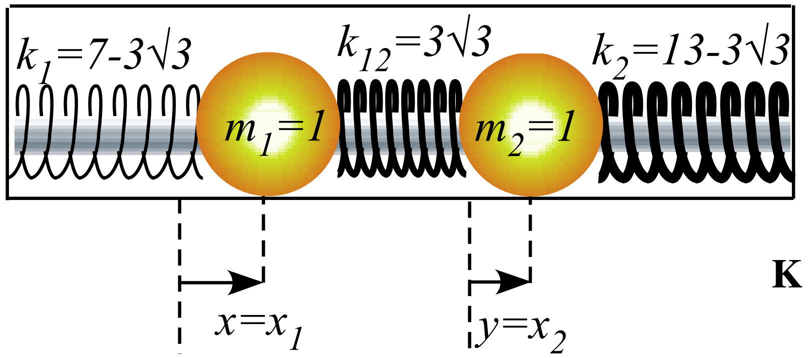
$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix},\quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t,\quad q_2(t)=-\frac{1}{2}\cos 4t \right)$$

Using $\cos 4t=2\cos^2 2t-1$ derives a parabolic trajectory!

$$q_2=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}q_1^2+\frac{1}{2}$$



$$\mathbf{K}=\begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix}=\begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}=\begin{pmatrix} 7 & -3\sqrt{3} \\ -3\sqrt{3} & 13 \end{pmatrix}$$

The **K** secular equation $K^2-Trace(\mathbf{K})K+Det(\mathbf{K})=K^2-20K+64=0=(K-4)(K-16)$ $\quad Det(\mathbf{K})=7\cdot 13-27=91-27=64$

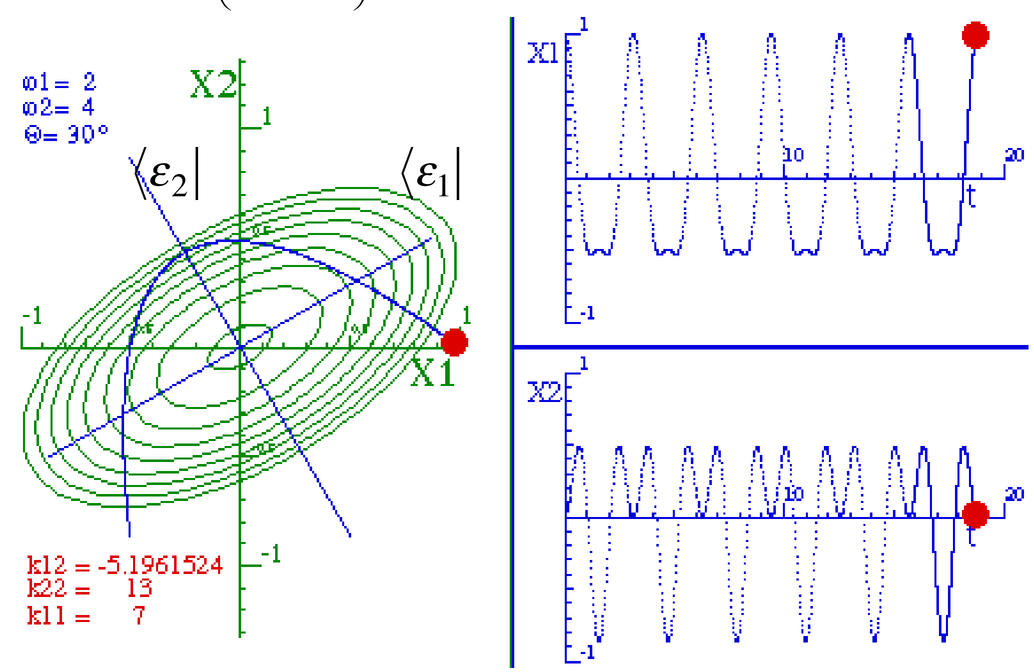
Eigenvalues K_k and squared eigenfrequencies $\omega_0(\varepsilon_k)^2$ $\quad K_1=\omega_0^2(\varepsilon_1)=4, \quad K_2=\omega_0^2(\varepsilon_2)=16,$

Eigen-projectors $\mathbf{P}_k$

$$\mathbf{P}_1=\frac{\begin{pmatrix} K_{11}-K_2 & K_{12} \\ K_{12} & K_{22}-K_2 \end{pmatrix}}{K_1-K_2}=\frac{\begin{pmatrix} 7-16 & -3\sqrt{3} \\ -3\sqrt{3} & 13-16 \end{pmatrix}}{4-16}=\frac{\begin{pmatrix} 9 & +3\sqrt{3} \\ +3\sqrt{3} & 3 \end{pmatrix}}{12}$$
$$=\frac{\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}}{4}=\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}=|\varepsilon_1\rangle\langle\varepsilon_1|$$

$$\mathbf{P}_2=\frac{\begin{pmatrix} K_{11}-K_1 & K_{12} \\ K_{12} & K_{22}-K_1 \end{pmatrix}}{K_2-K_1}=\frac{\begin{pmatrix} 7-4 & -3\sqrt{3} \\ -3\sqrt{3} & 13-4 \end{pmatrix}}{16-4}=\frac{\begin{pmatrix} 3 & -3\sqrt{3} \\ -3\sqrt{3} & 9 \end{pmatrix}}{12}$$
$$=\frac{\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}}{4}=\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}=|\varepsilon_2\rangle\langle\varepsilon_2|$$

Eigenbra vectors: $\langle\varepsilon_1|=\begin{pmatrix} \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \langle\varepsilon_2|=\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix}$

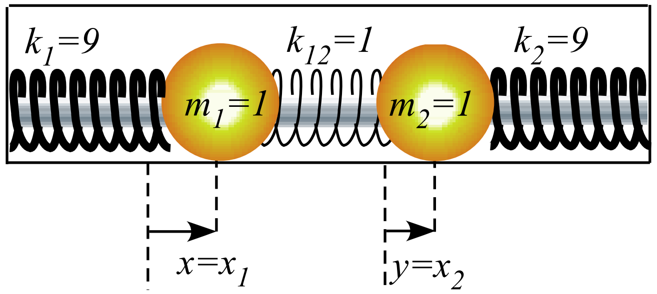


$$\left(q_1(t)=\frac{\sqrt{3}}{2}\cos 2t, \quad q_2(t)=-\frac{1}{2}\cos 4t \right)$$

Using $\cos 4t=2\cos^2 2t-1$ derives a parabolic trajectory!

$$q_2=-\frac{1}{2}2\cos^2 2t+\frac{1}{2}=-\frac{4}{3}q_1^2+\frac{1}{2}$$

Fig. 3.3.6 Normal coordinate axes, coupled oscillator trajectories and equipotential (V=const.) ovals for an integral 1:2 eigenfrequency ratio ($\omega_0(\varepsilon_1)$=2.0, $\omega_0(\varepsilon_2)$= 4.0) and zero initial velocity.



k_1=9
k_{12}=1
k_2=9
m_1=1
m_2=1
x=x_1
y=x_2