Lecture 8

Revised 12.21.12 from 9.13.2012

Kepler Geometry of IHO (Isotropic Harmonic Oscillator) Elliptical Orbits

(Ch. 9 and Ch. 11 of Unit 1)

Constructing 2D IHO orbits by phasor plots

Review of phasor "clock" geometry (From Lecture 7)
Integrating IHO equations by phasor geometry

Constructing 2D IHO orbits using Kepler anomaly plots

Mean-anomaly and eccentric-anomaly geometry
Calculus and vector geometry of IHO orbits
A confusing introduction to Coriolis-centrifugal force geometry

Some Kepler's "laws" for central (isotropic) force F(r)

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ *with* $U(r)=k\cdot r^2/2$ *(Derived rigorously)*
Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ *with* $U(r)=-GMm\cdot/r$ *(Derived later)*
Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ *(Derived rigorously)*
Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ *(Derived later)*

Brief introduction to matrix quadratic form geometry

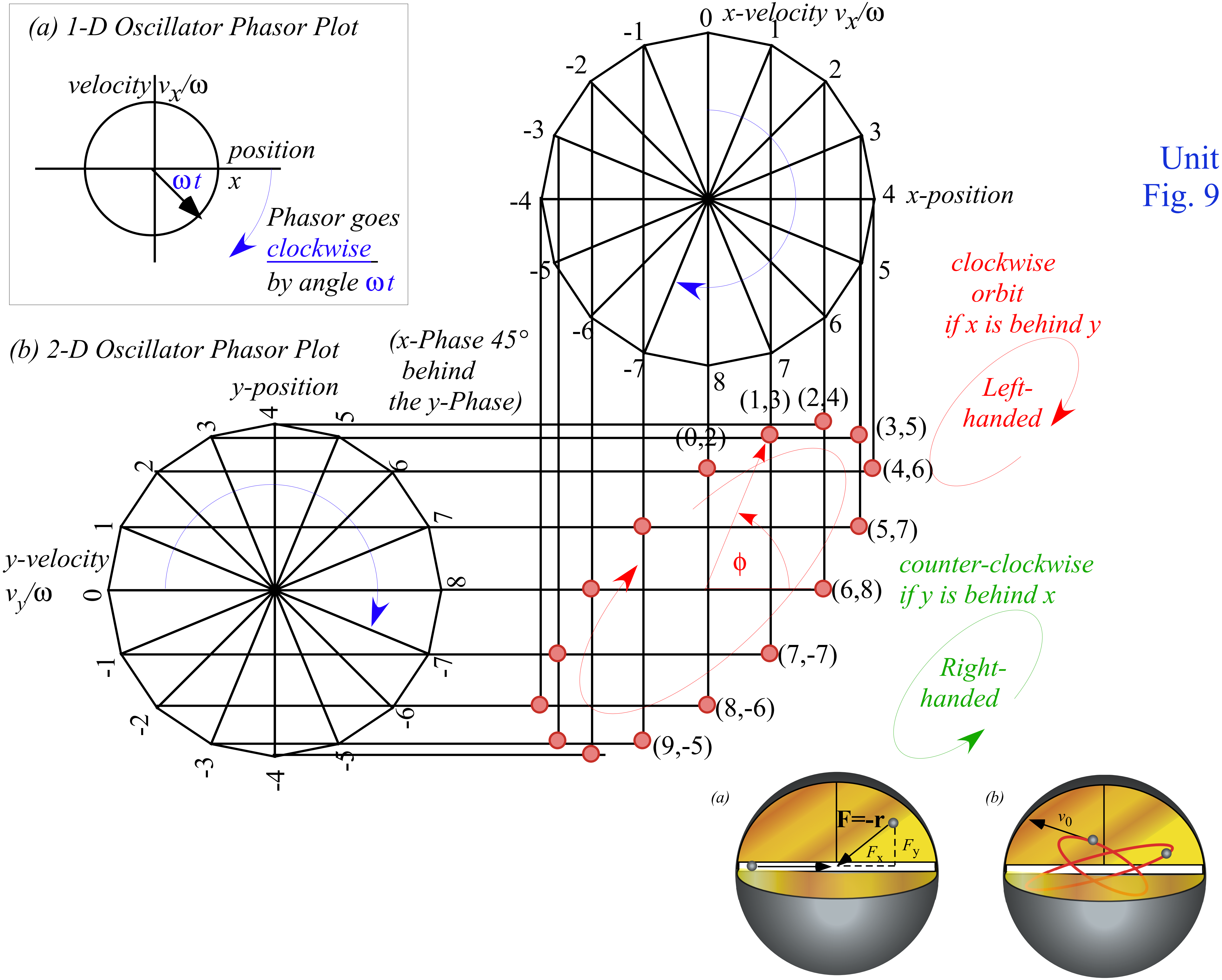
Constructing 2D IHO orbits by phasor plots

→ *Review of phasor "clock" geometry (From Lecture 7)*

Integrating IHO equations by phasor geometry

Isotropic Harmonic Oscillator phase dynamics in uniform-body

Unit 1
Fig. 9.10



(a) 1-D Oscillator Phasor Plot

Phasor goes clockwise by angle ωt

(b) 2-D Oscillator Phasor Plot

(x-Phase 45° behind the y-Phase)

clockwise orbit if x is behind y

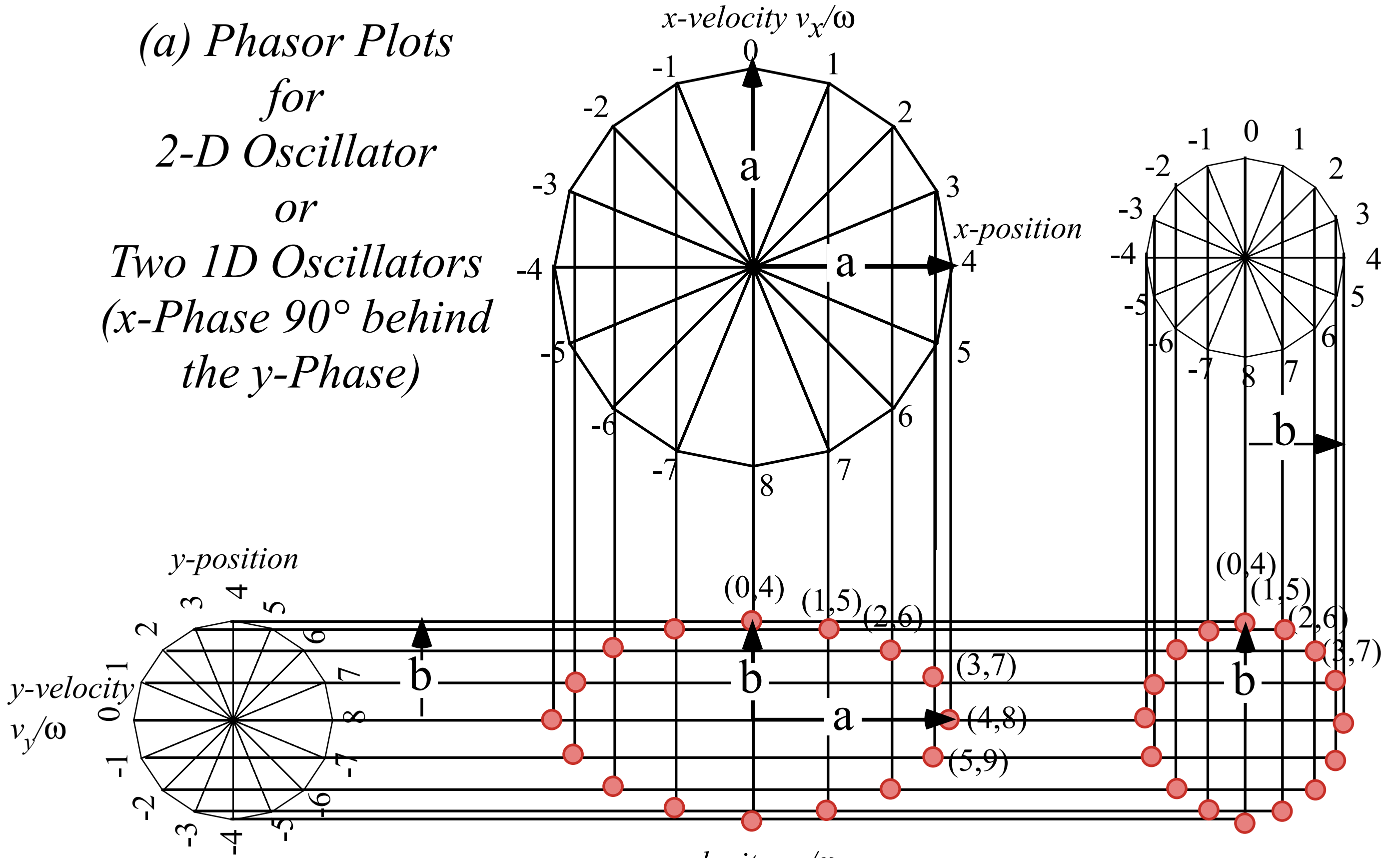
Left-handed

counter-clockwise if y is behind x

Right-handed

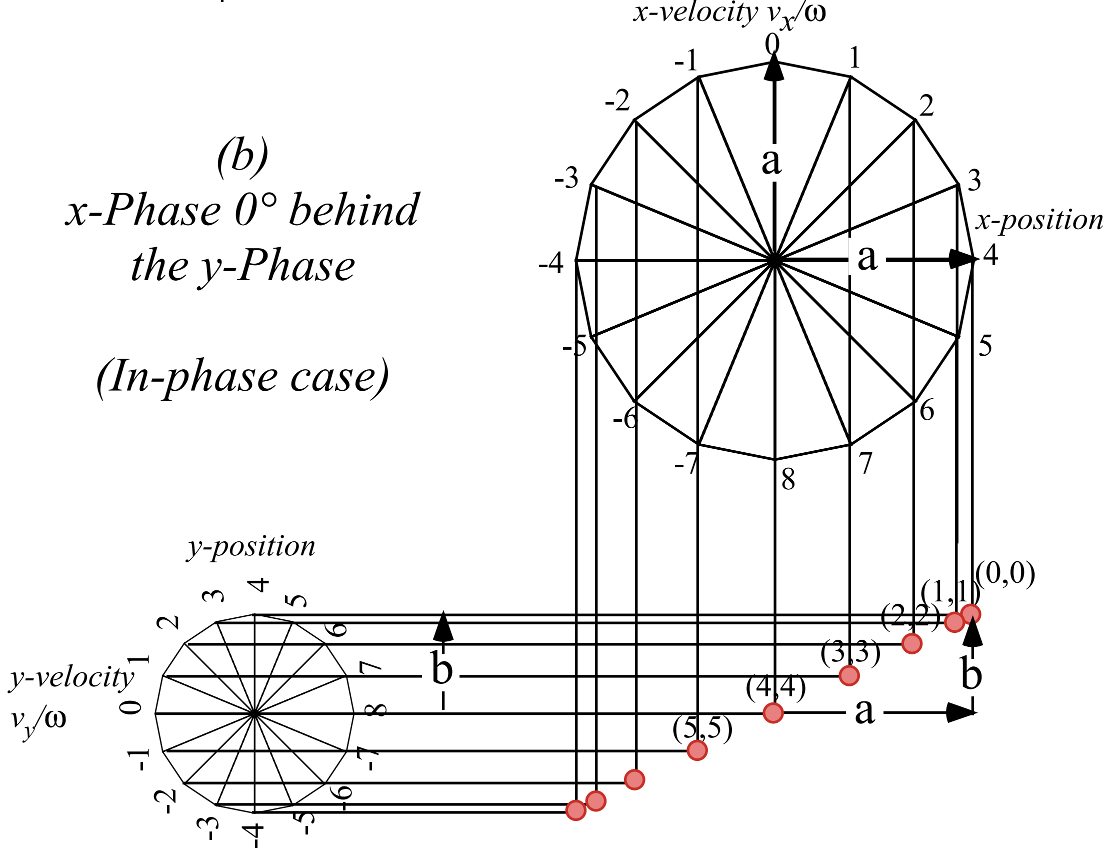
Unit 1
Fig. 9.12

(a) Phasor Plots
for
2-D Oscillator
or
Two 1D Oscillators
(x-Phase 90° behind
the y-Phase)

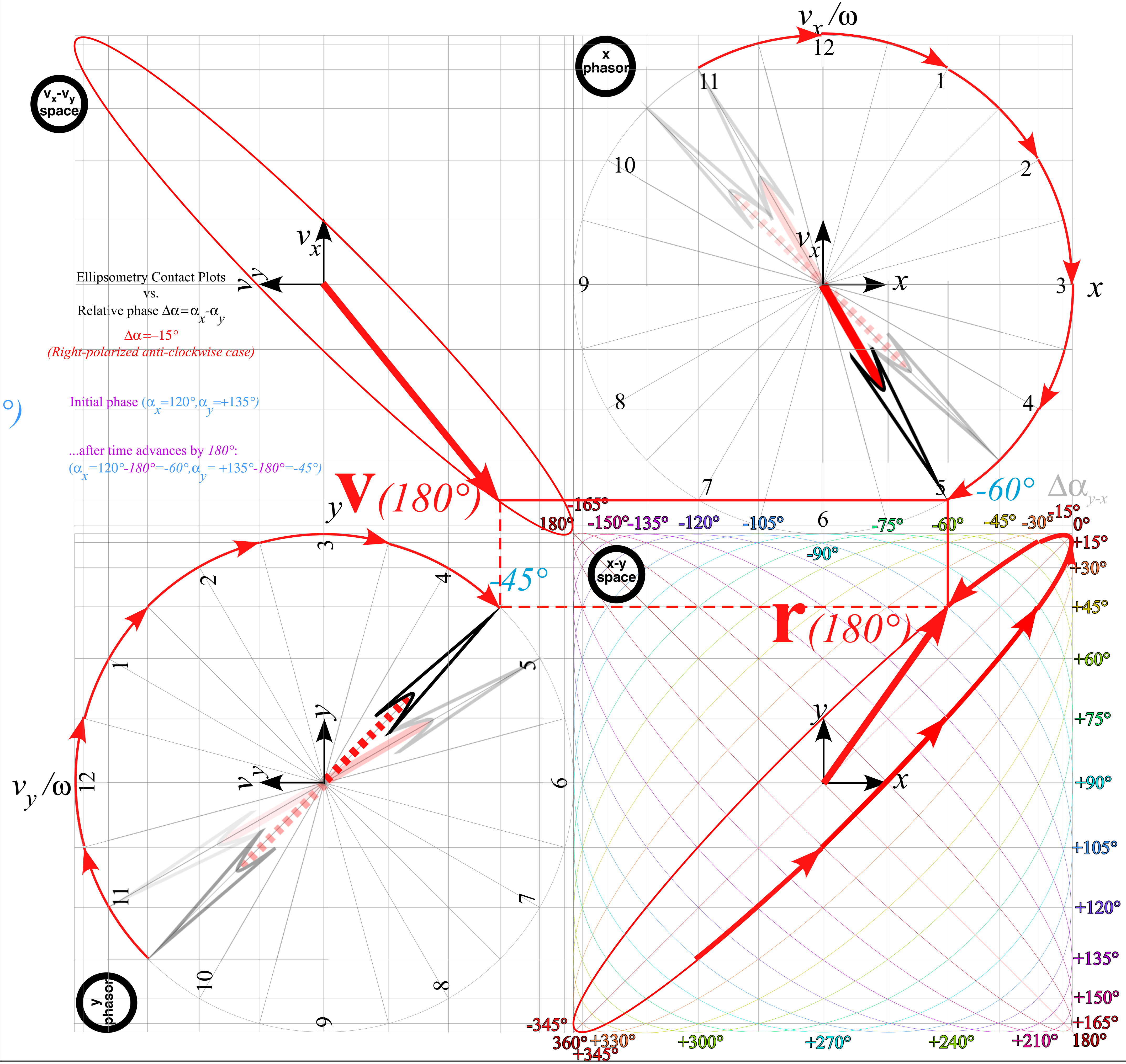


(b)
x-Phase 0° behind
the y-Phase

(In-phase case)



These are more generic examples with radius of x-phasor differing from that of the y-phasor.



Ellipsometry Contact Plots
vs.
Relative phase $\Delta\alpha=\alpha_x-\alpha_y$

$\Delta\alpha=-15°$
(Right-polarized anti-clockwise case)

Initial phase $(\alpha_x=120°,\alpha_y=+135°)$

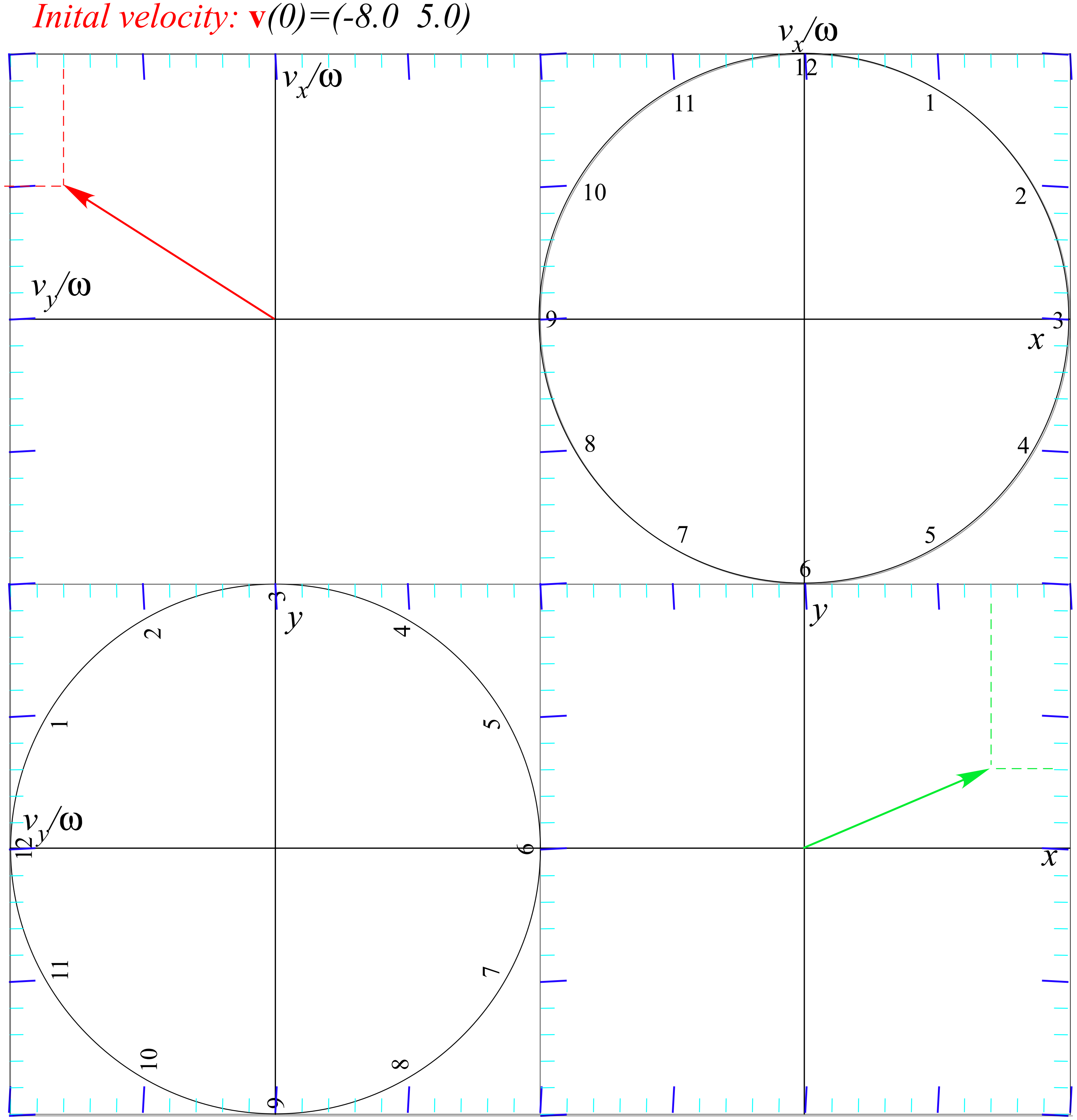
...after time advances by *180°*:
$(\alpha_x=120°-180°=-60°,\alpha_y=+135°-180°=-45°)$

Constructing 2D IHO orbits by phasor plots

Review of phasor "clock" geometry (From Lecture 7)

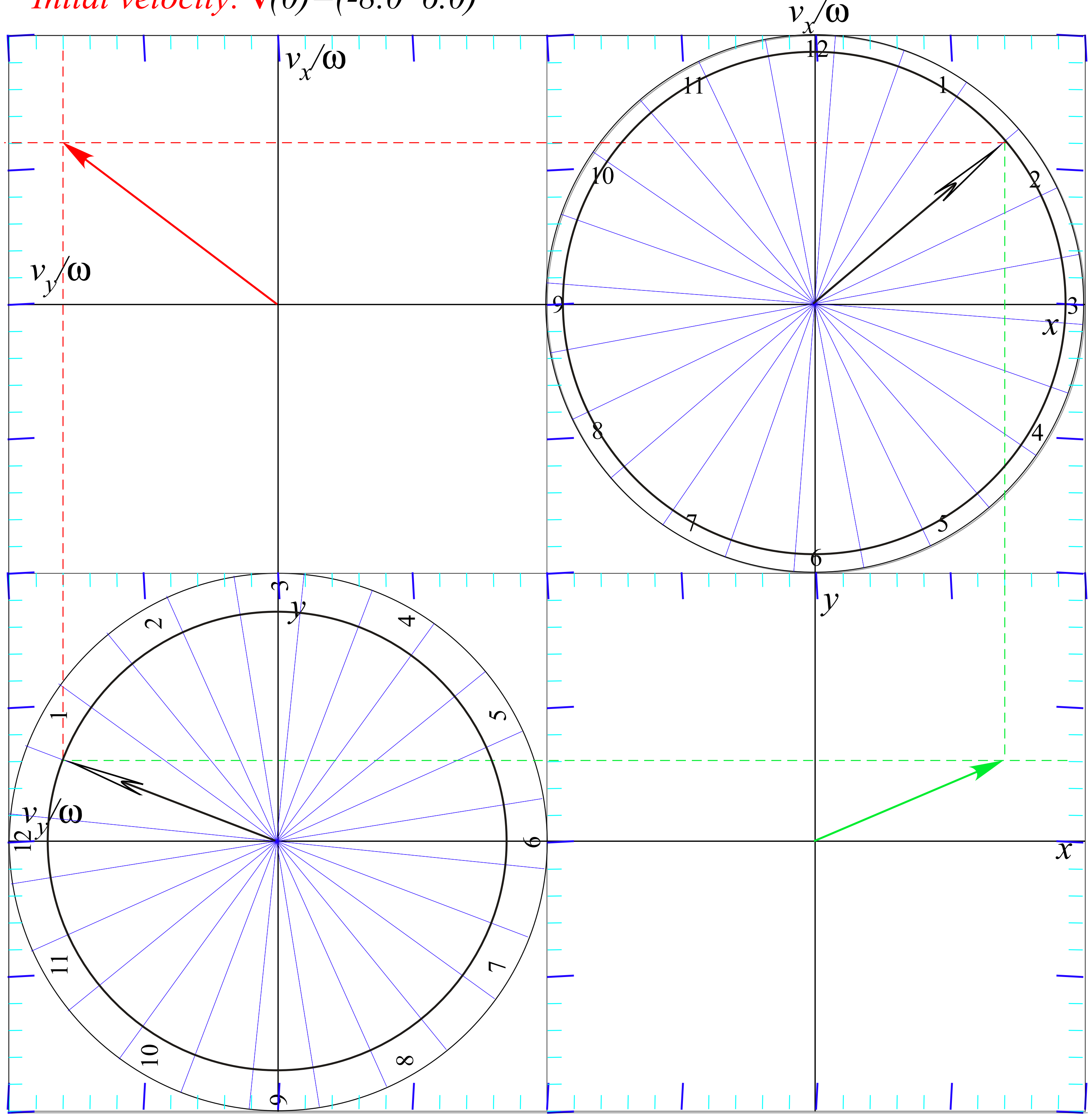
→ *Integrating IHO equations by phasor geometry*

Inital velocity: **v**$(0)=(-8.0\ \ 5.0)$



Initial position: **r**$(0)=(7.0\ \ 3.0)$

Inital velocity: **v***(0)=(-8.0 6.0)*



Initial position: **r***(0)=(7.0 3.0)*

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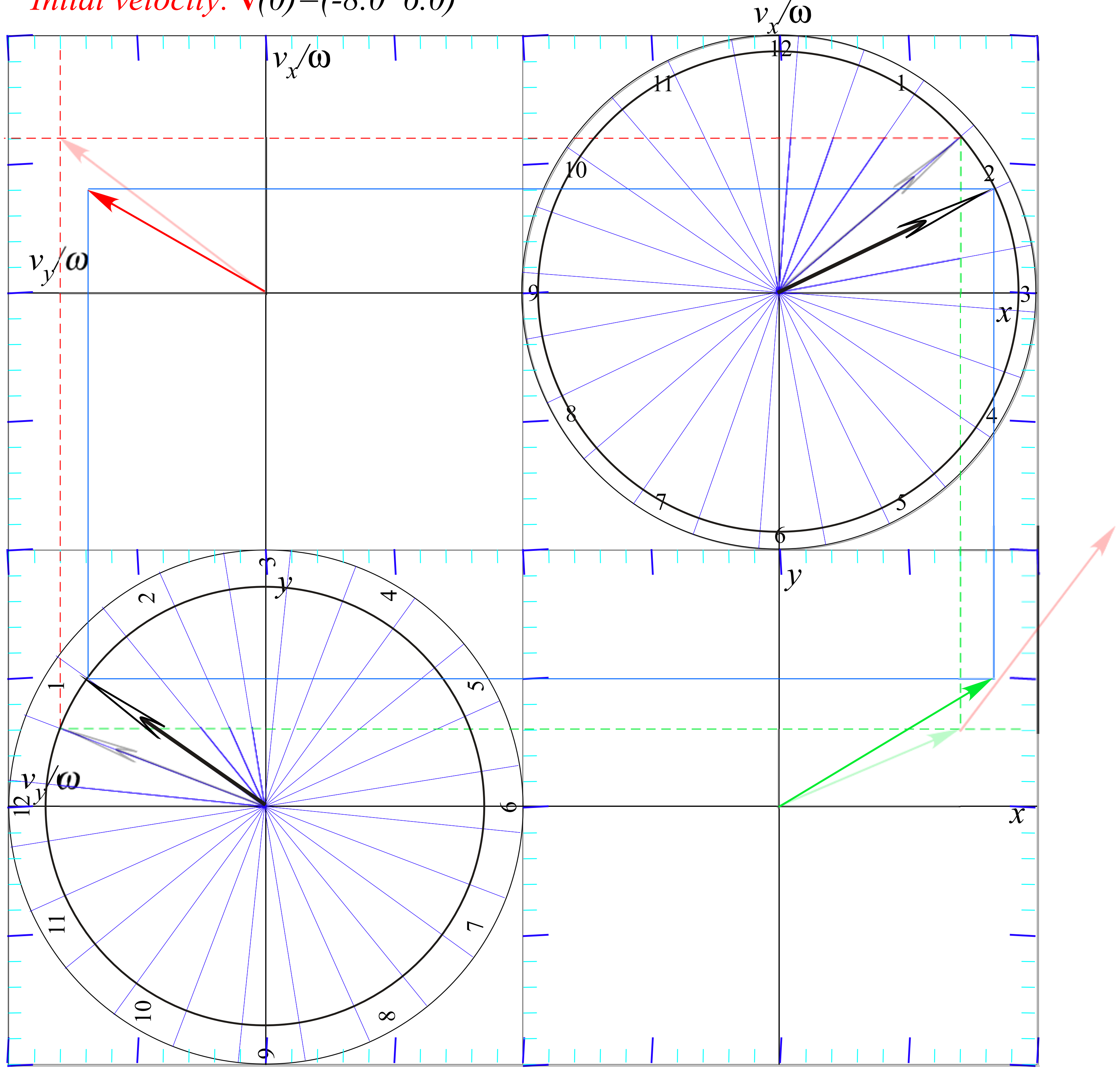

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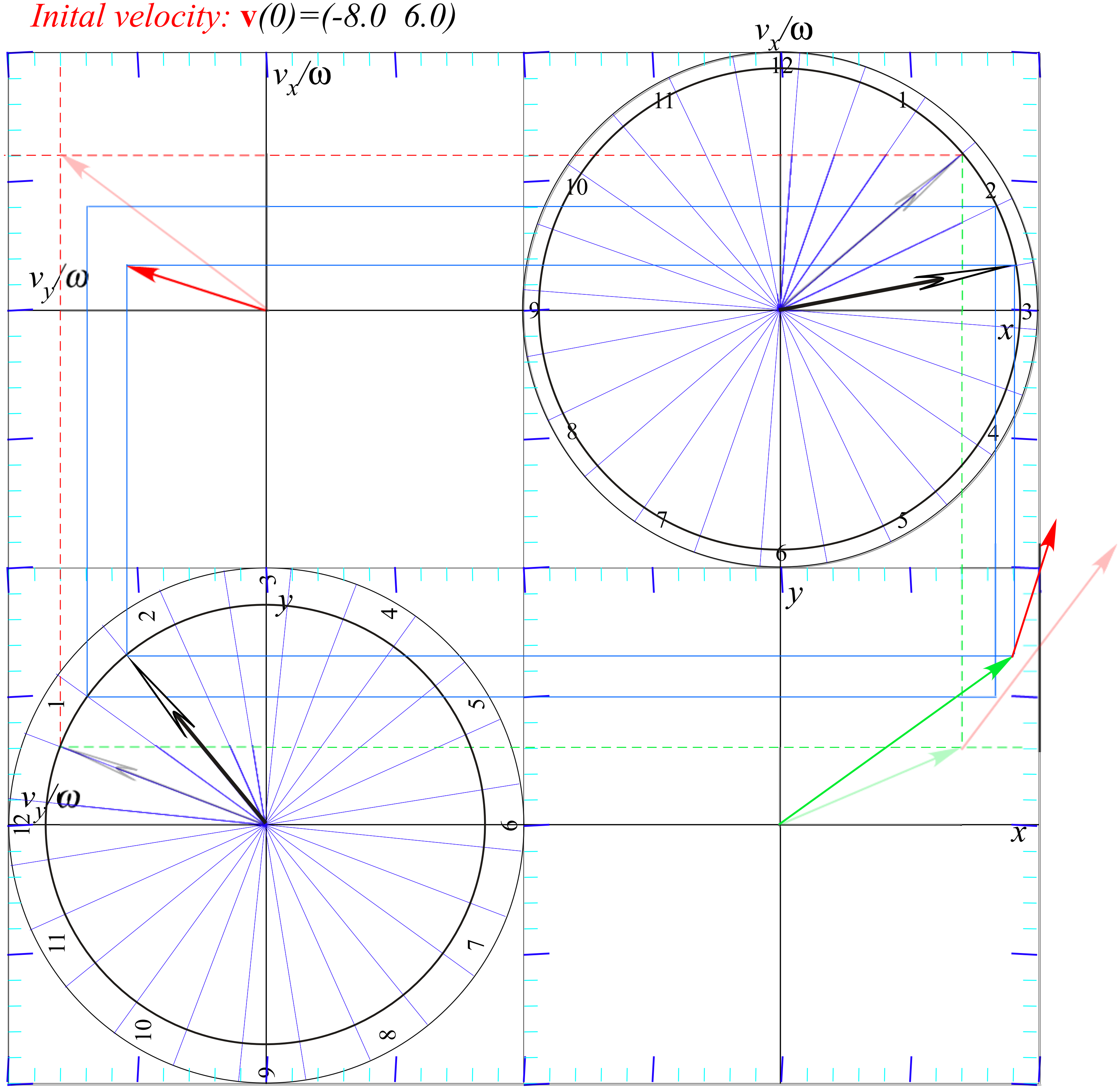

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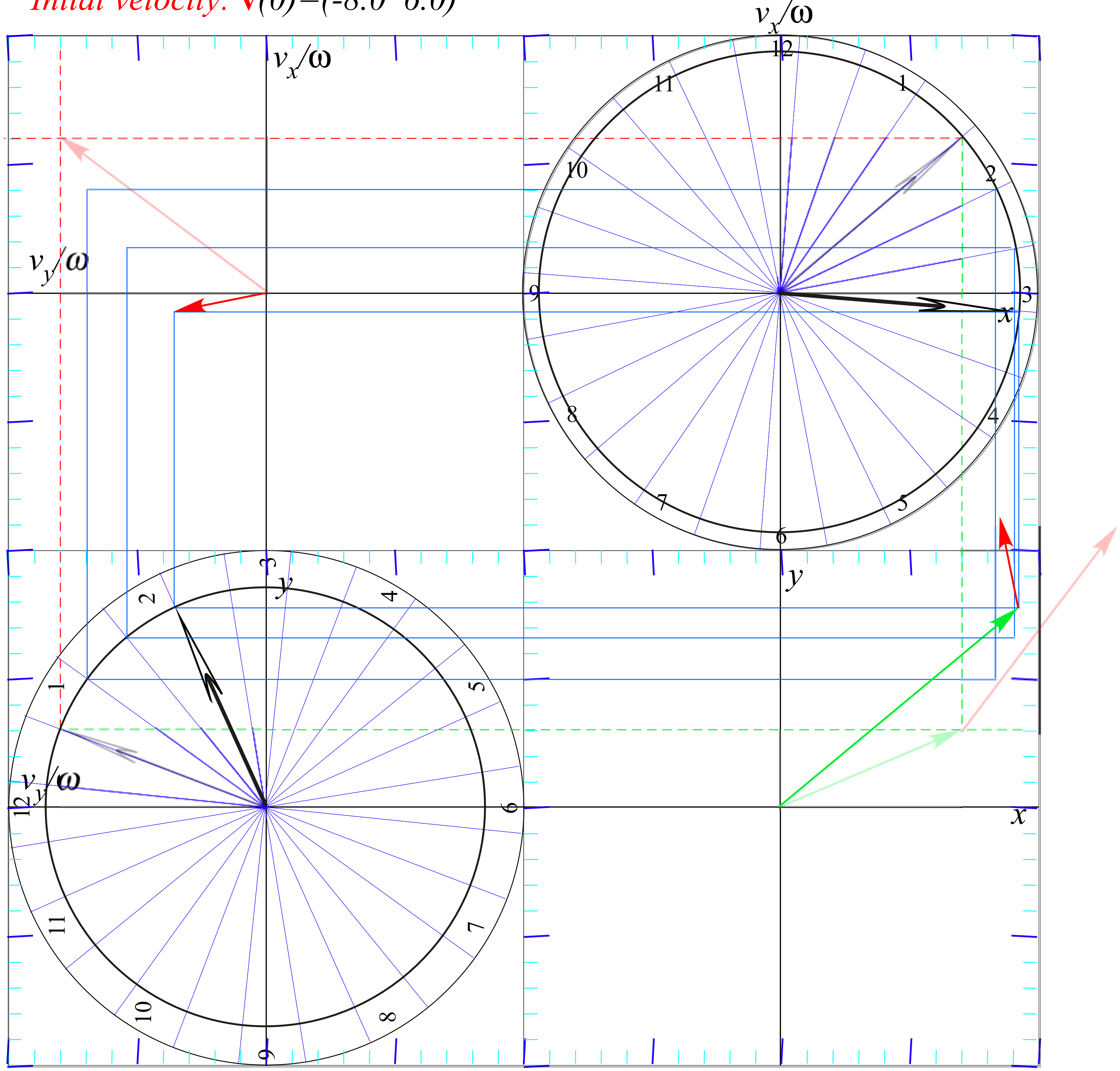

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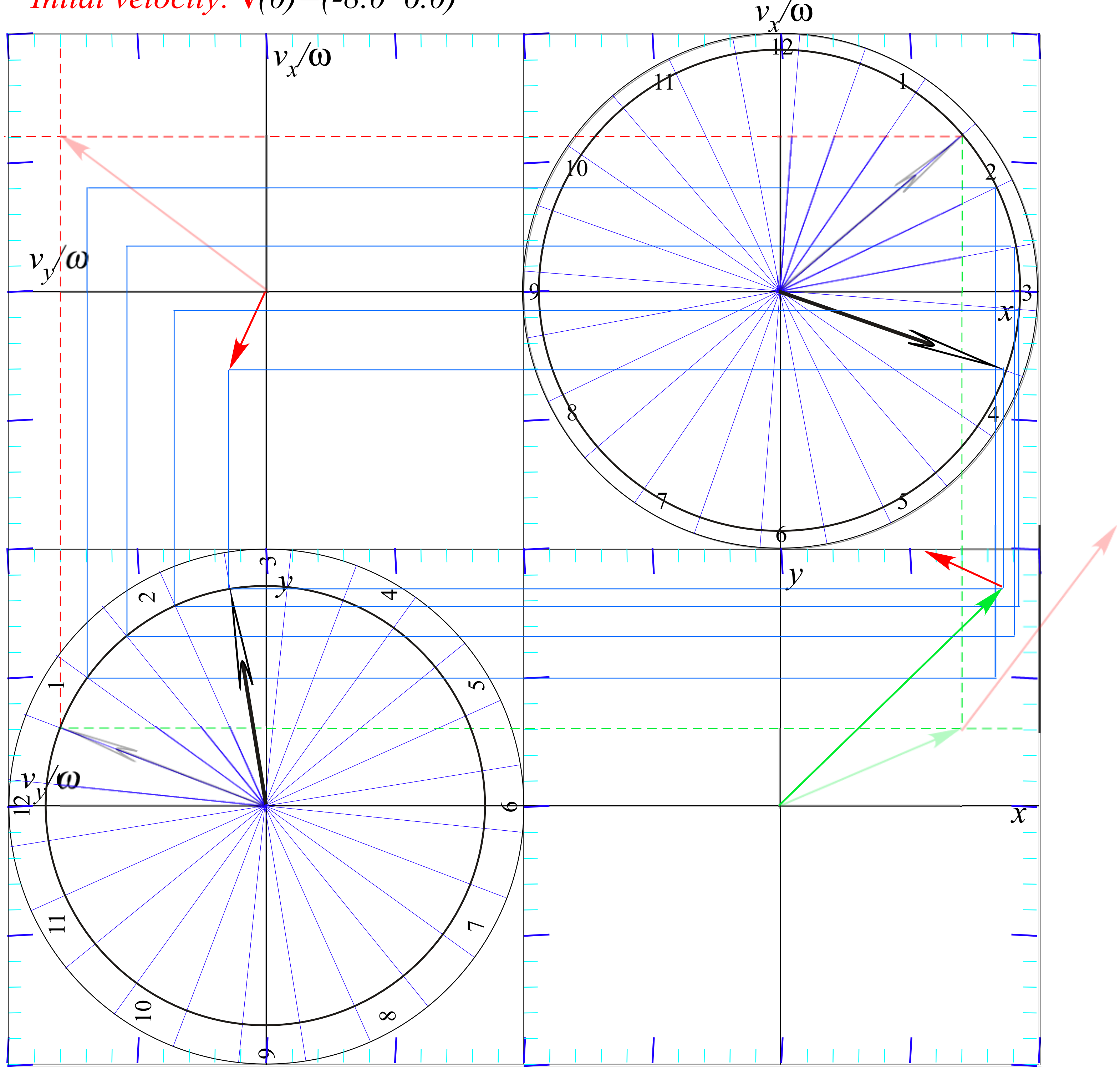

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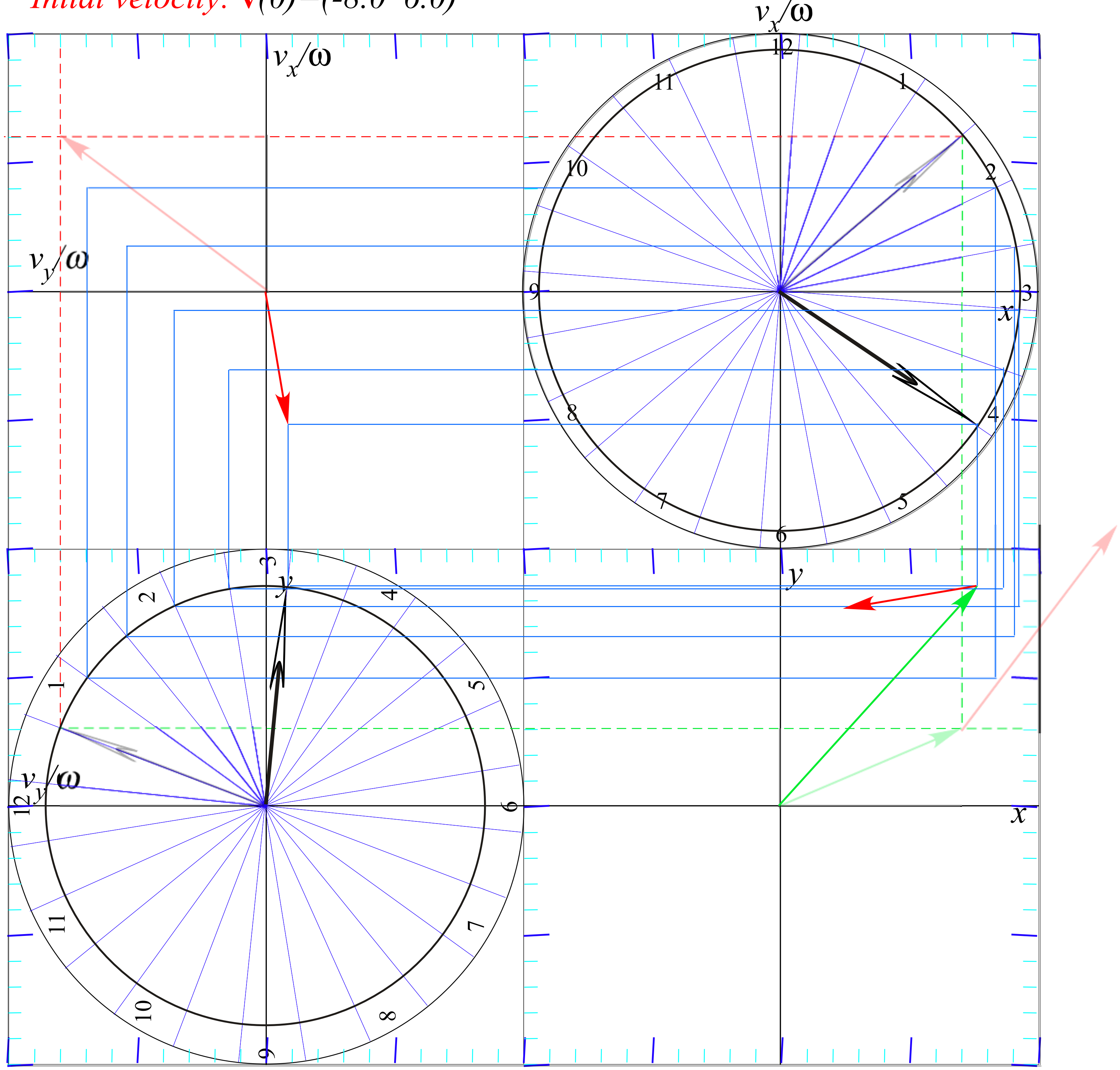

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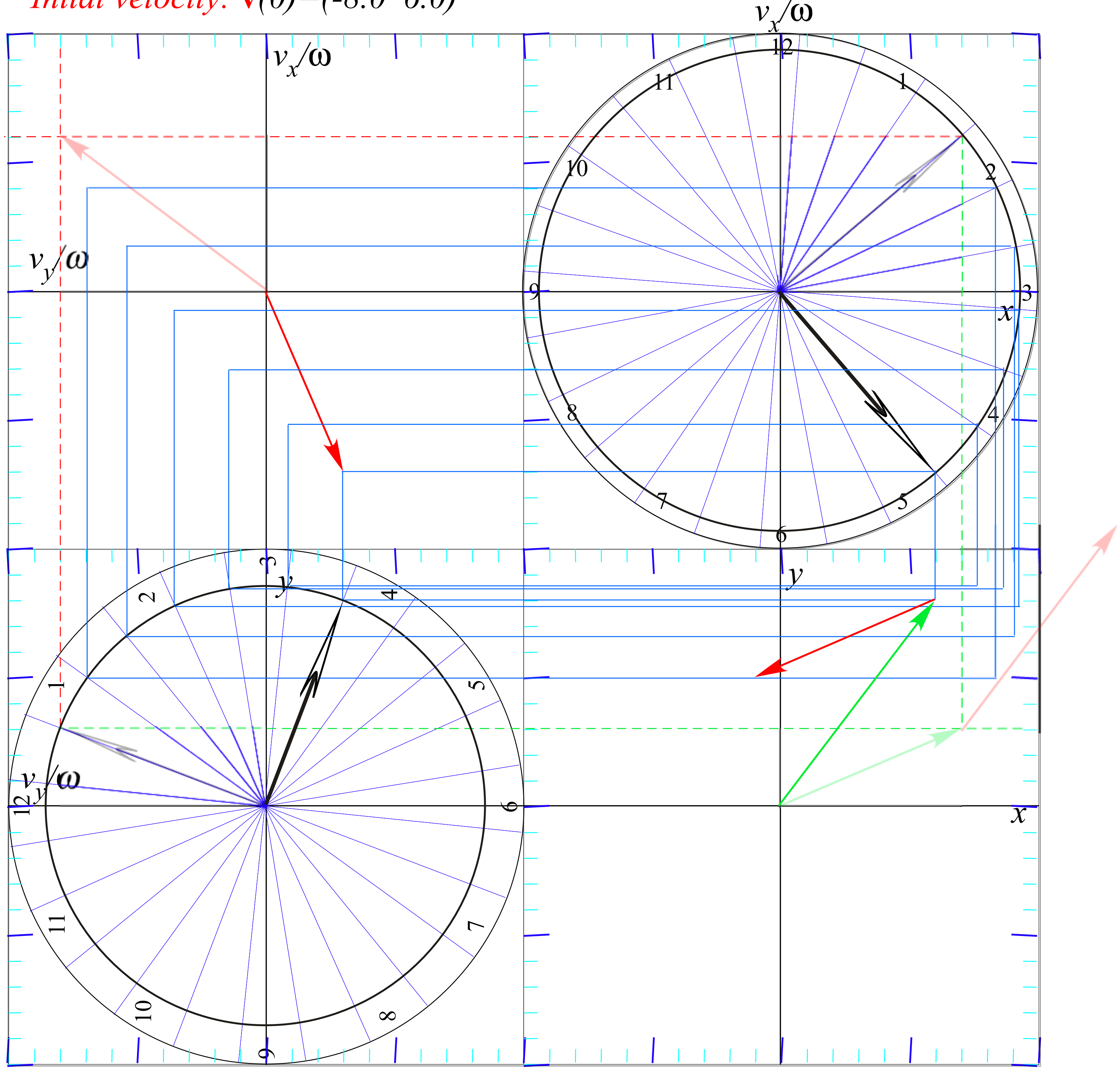

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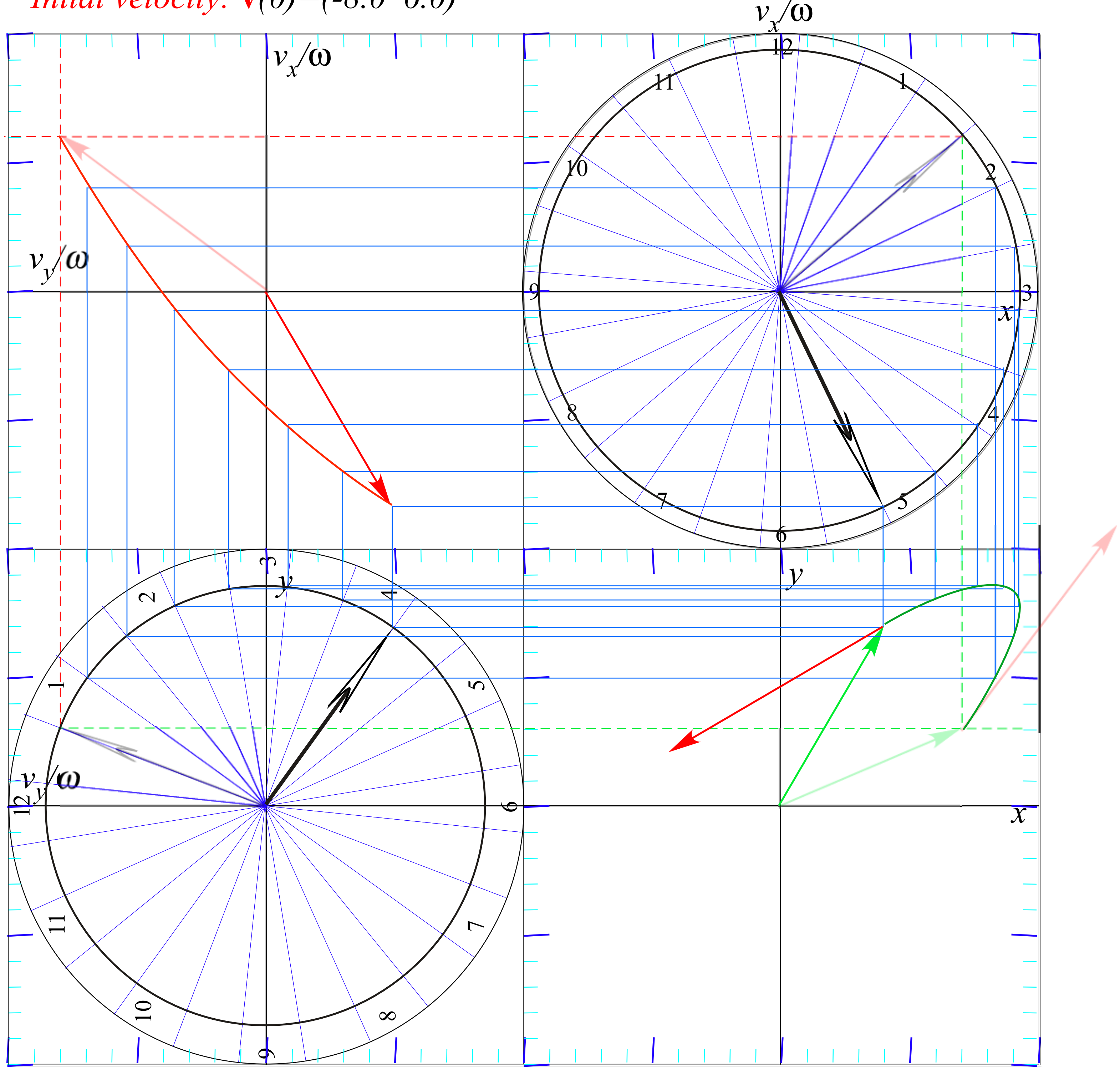


Initial position: **r***(0)=(7.0 3.0)*

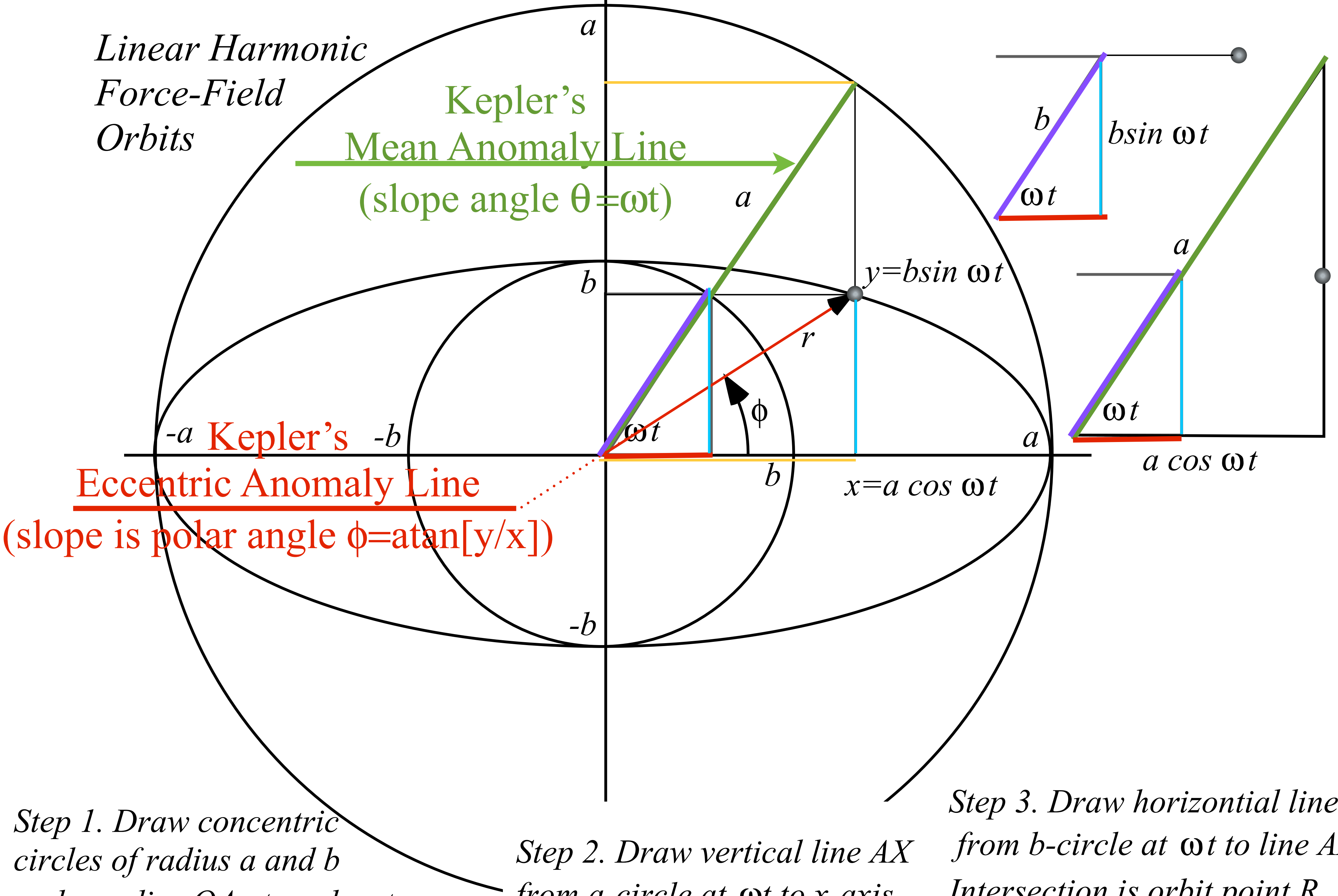
Constructing 2D IHO orbits using Kepler anomaly plots

→ *Mean-anomaly and eccentric-anomaly geometry*

Calculus and vector geometry of IHO orbits

A confusing introduction to Coriolis-centrifugal force geometry

Linear Harmonic Force-Field Orbits

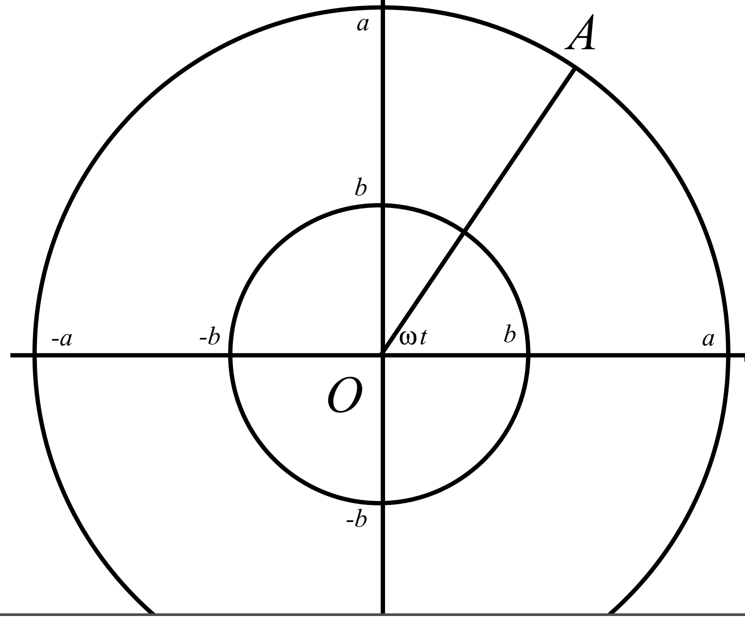


Kepler's Mean Anomaly Line (slope angle θ=ωt)

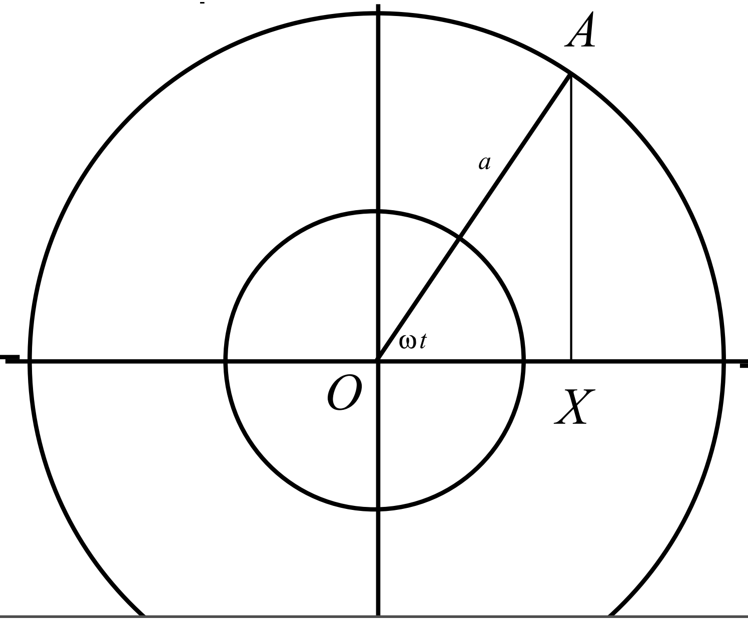
Kepler's Eccentric Anomaly Line (slope is polar angle φ=atan[y/x])

Unit 1
Fig. 11.1
(top 2/3's)

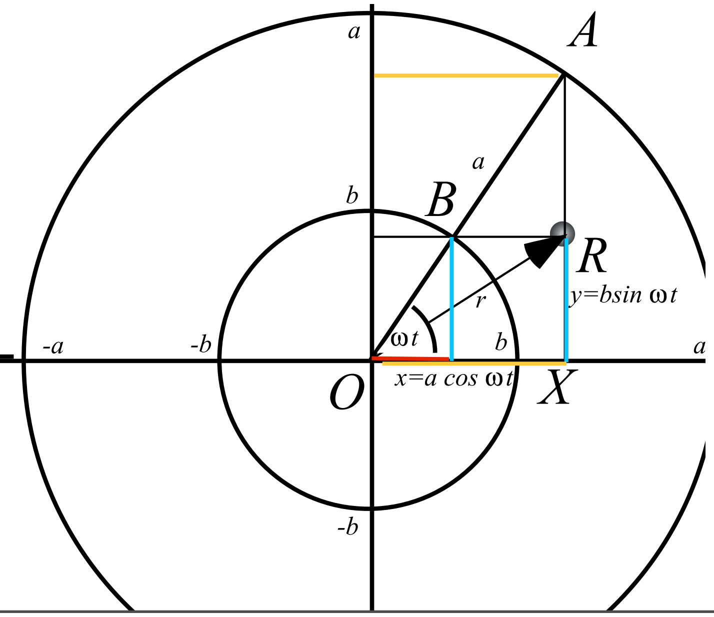
Step 1. Draw concentric circles of radius a and b and a radius OA at angle ωt

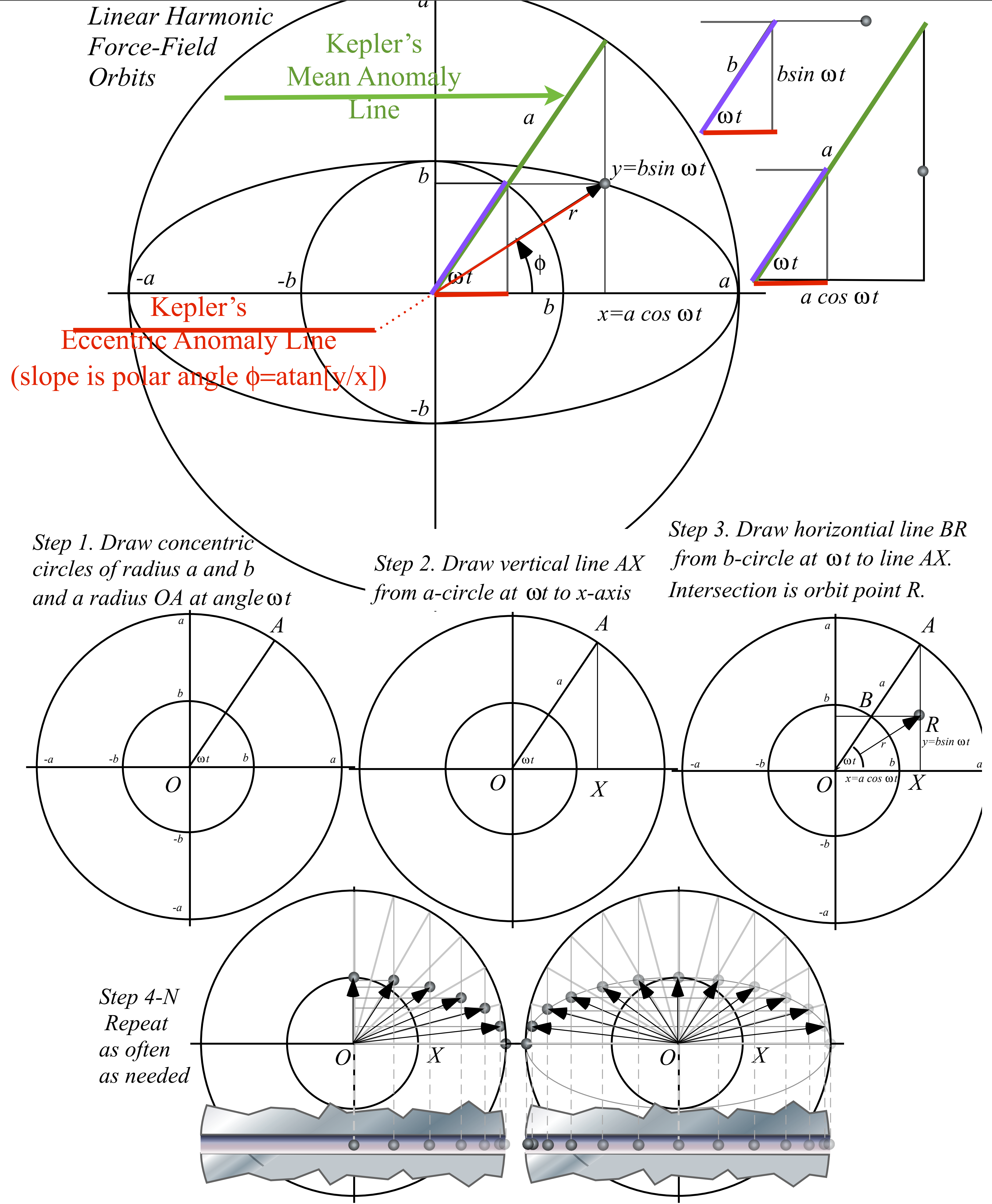


Step 2. Draw vertical line AX from a-circle at ωt to x-axis



Step 3. Draw horizontial line BR from b-circle at ωt to line AX. Intersection is orbit point R.





Step 1. Draw concentric circles of radius a and b and a radius OA at angle ωt

Step 2. Draw vertical line AX from a-circle at ωt to x-axis

Step 3. Draw horizontal line BR from b-circle at ωt to line AX. Intersection is orbit point R.

Step 4-N Repeat as often as needed

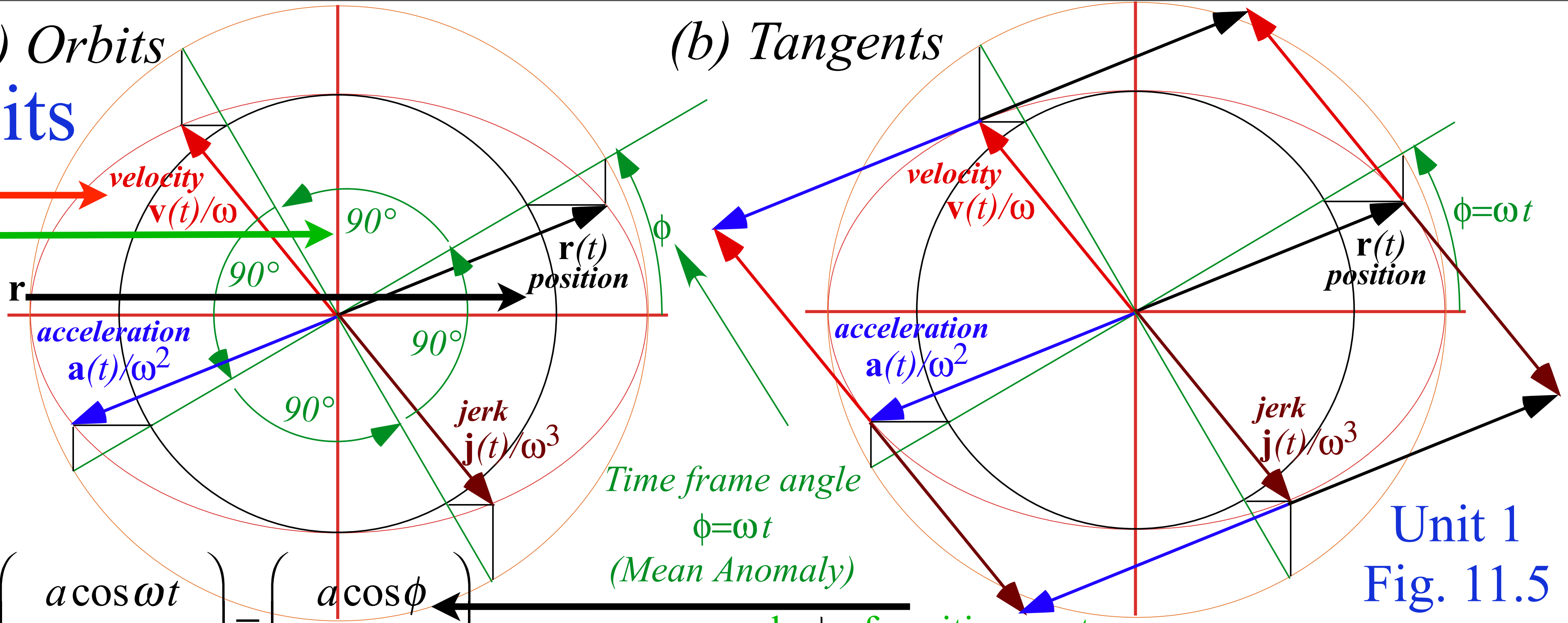
Unit 1
Fig. 11.1

Constructing 2D IHO orbits using Kepler anomaly plots

Mean-anomaly and eccentric-anomaly geometry
→ *Calculus and vector geometry of IHO orbits*
A confusing introduction to Coriolis-centrifugal force geometry

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly ϕ of position vector **r**



Unit 1
Fig. 11.5

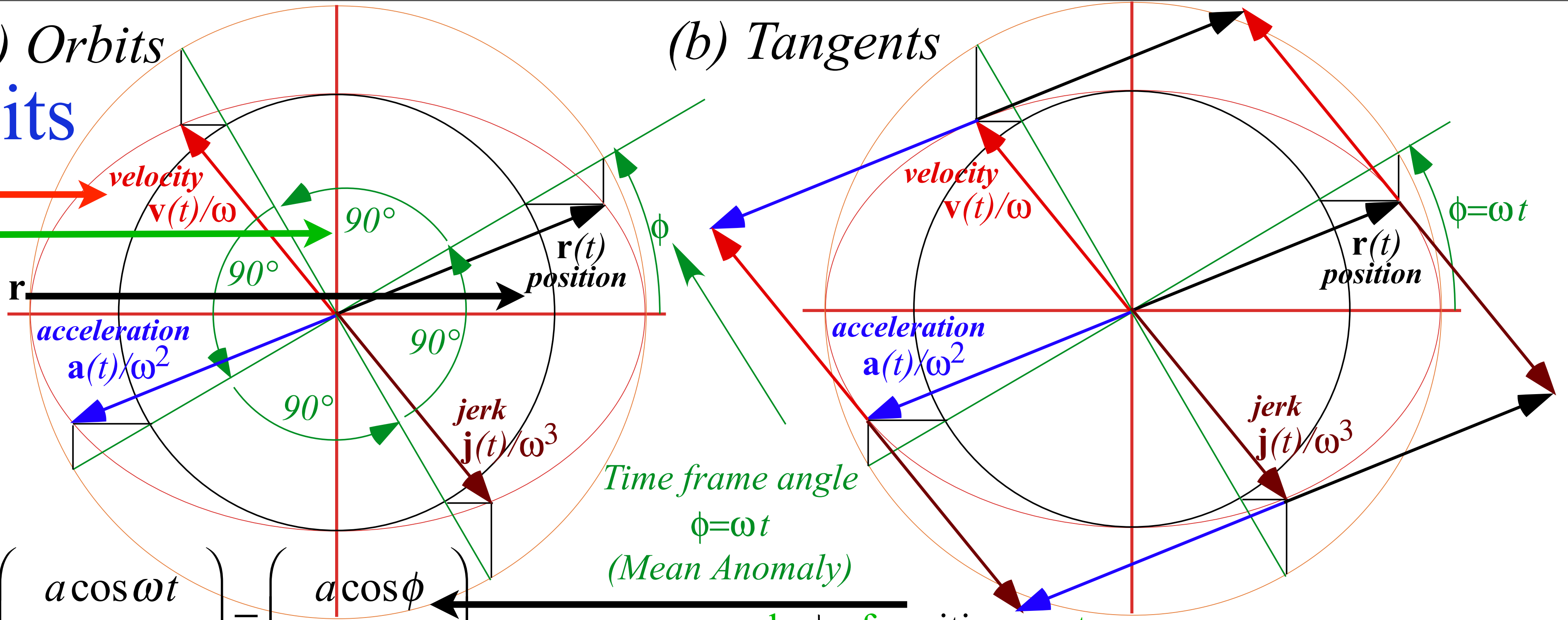
$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly ϕ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**


(a) Orbits

(b) Tangents

$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly φ of position vector **r**
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Unit 1
Fig. 11.5

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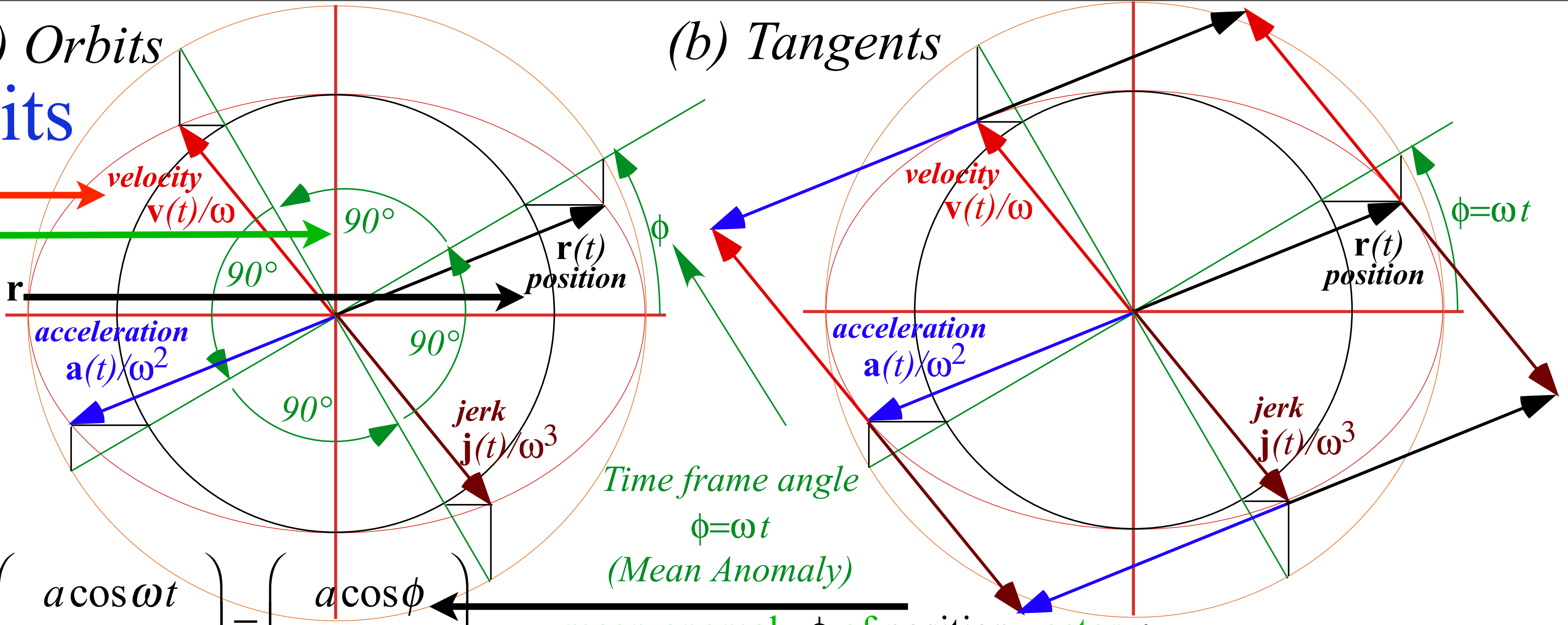
m.a. φ+π/2 of vector **v** rotated by
another π/2 is *m.a.* of vector **a**

$$\textit{acceleration or force vector}: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi + \frac{2\pi}{2}\right) \\ b\sin\left(\phi + \frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**



Unit 1
Fig. 11.5

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mean-anomaly φ of position vector **r**
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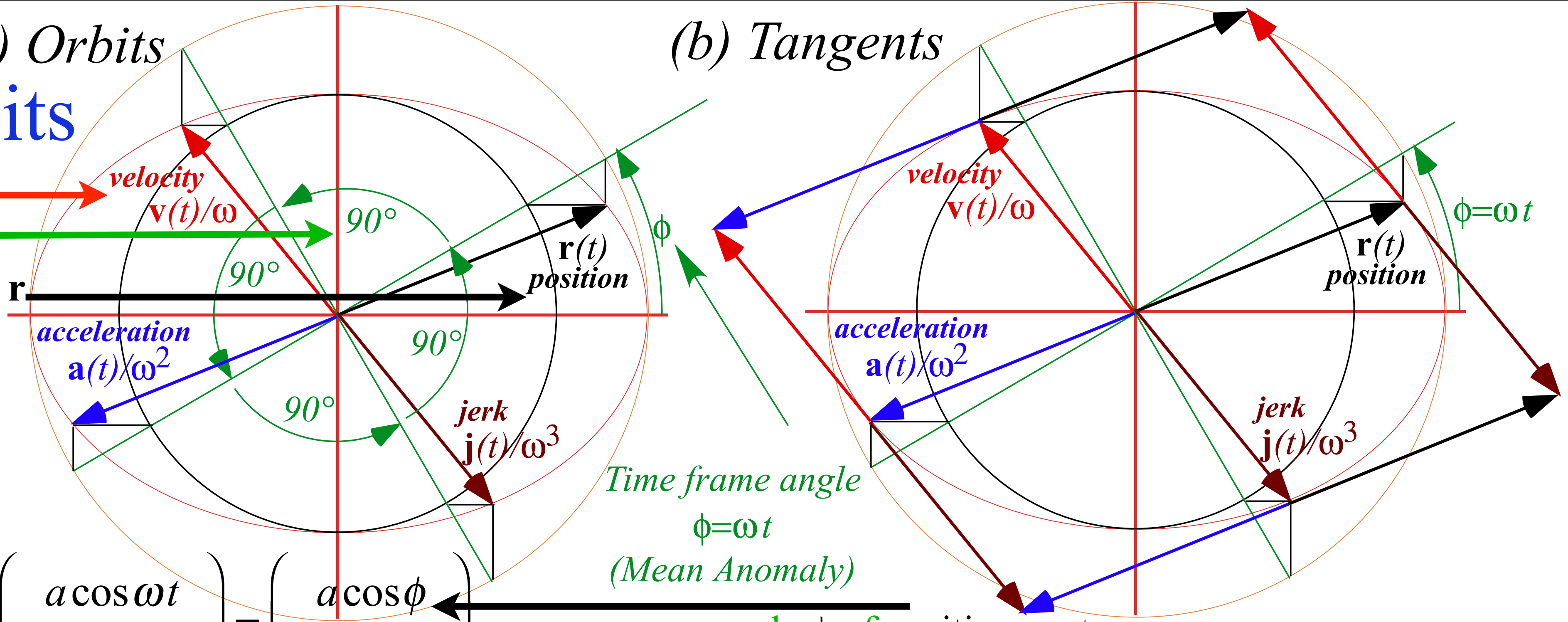
or change of velocity

$$\text{jerk or change of acceleration}: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi + \frac{3\pi}{2}\right) \\ b\sin\left(\phi + \frac{3\pi}{2}\right) \end{pmatrix}$$

...and so forth...

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**



Time frame angle
φ=ωt
(Mean Anomaly)

$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly φ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

m.a. φ+π/2 of vector **v** rotated by
another π/2 is *m.a.* of vector **a**

$$\begin{array}{l} \textit{acceleration or force vector}: \dfrac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \dfrac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \dfrac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi + \frac{2\pi}{2}\right) \\ b\sin\left(\phi + \frac{2\pi}{2}\right) \end{pmatrix} \\ \qquad \textit{or change of velocity} \end{array}$$

$$\textit{jerk or change of acceleration}: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi + \frac{3\pi}{2}\right) \\ b\sin\left(\phi + \frac{3\pi}{2}\right) \end{pmatrix}$$

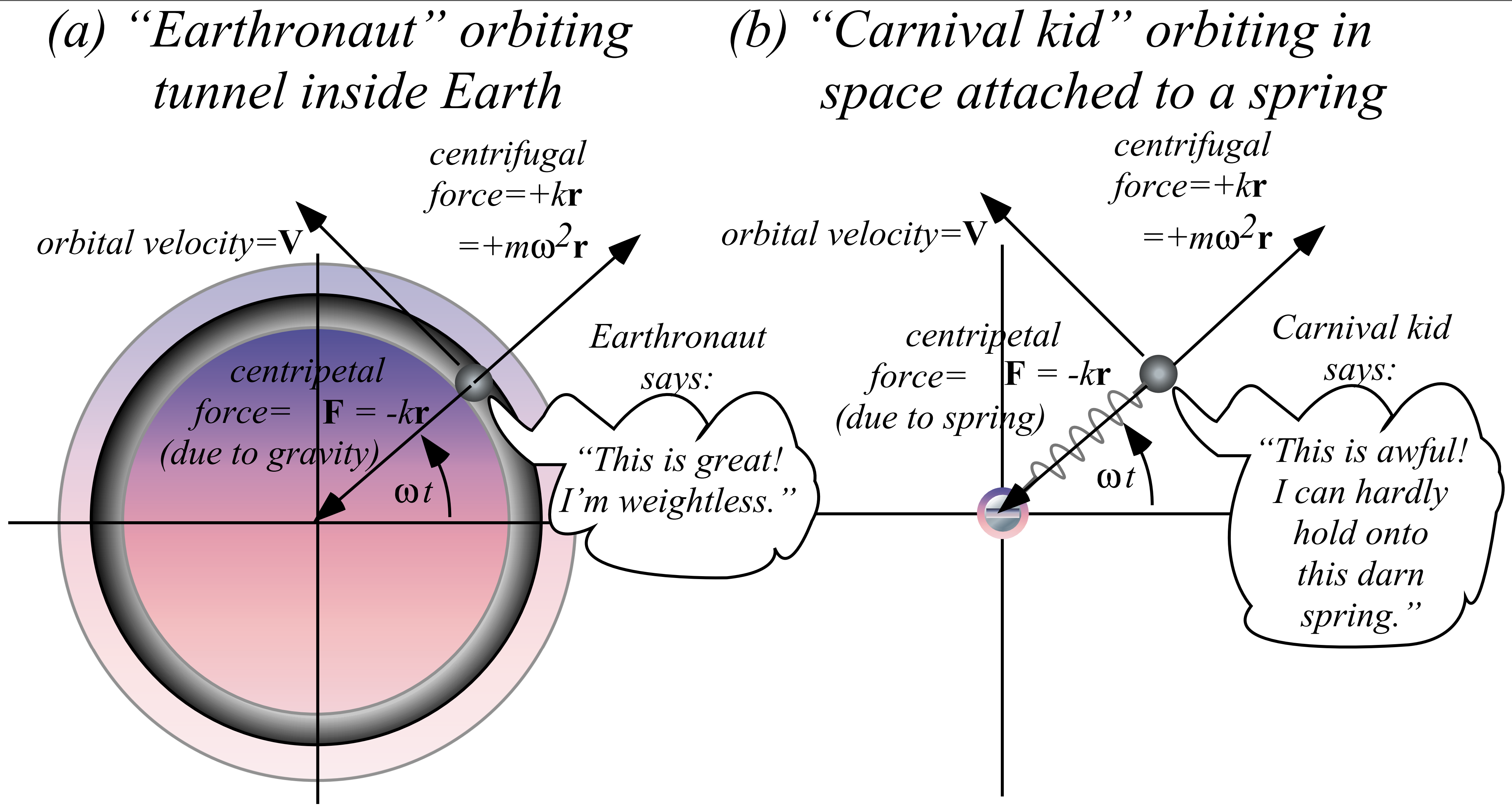
...and so forth...

...and so on...

$$\textit{inauguration or change of jerk}: \mathbf{i} = \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \begin{pmatrix} +a\omega^4\cos\omega t \\ +b\omega^4\sin\omega t \end{pmatrix} = \frac{d\mathbf{j}}{dt} = \dot{\mathbf{j}} = \ddot{\mathbf{a}} = \dddot{\mathbf{v}} = \ddddot{\mathbf{r}} = \frac{d^4\mathbf{r}}{dt^4} = \begin{pmatrix} a\cos\left(\phi + \frac{4\pi}{2}\right) \\ b\sin\left(\phi + \frac{4\pi}{2}\right) \end{pmatrix}$$

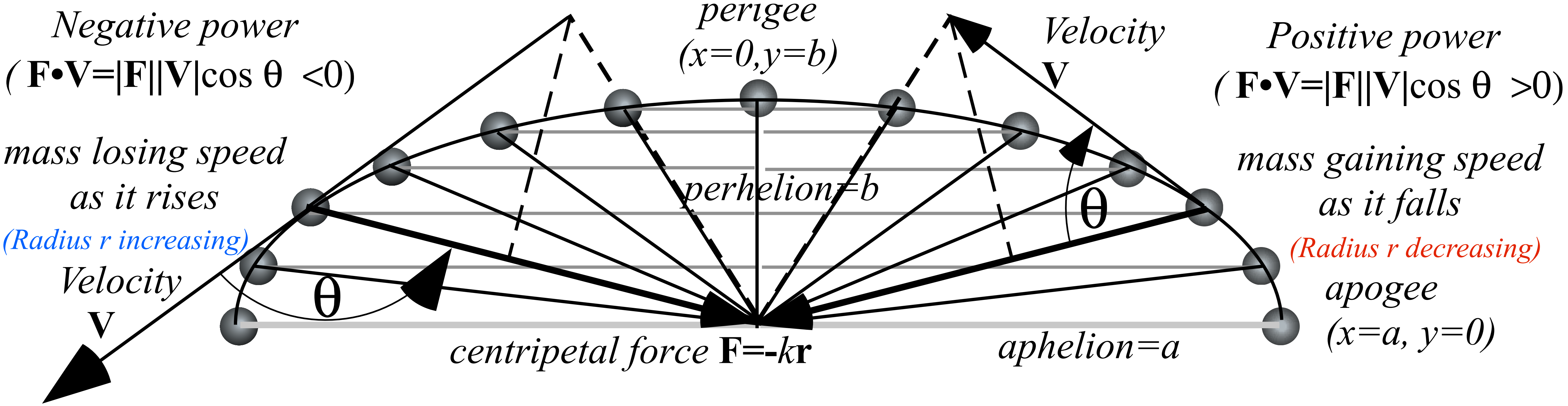
Constructing 2D IHO orbits using Kepler anomaly plots

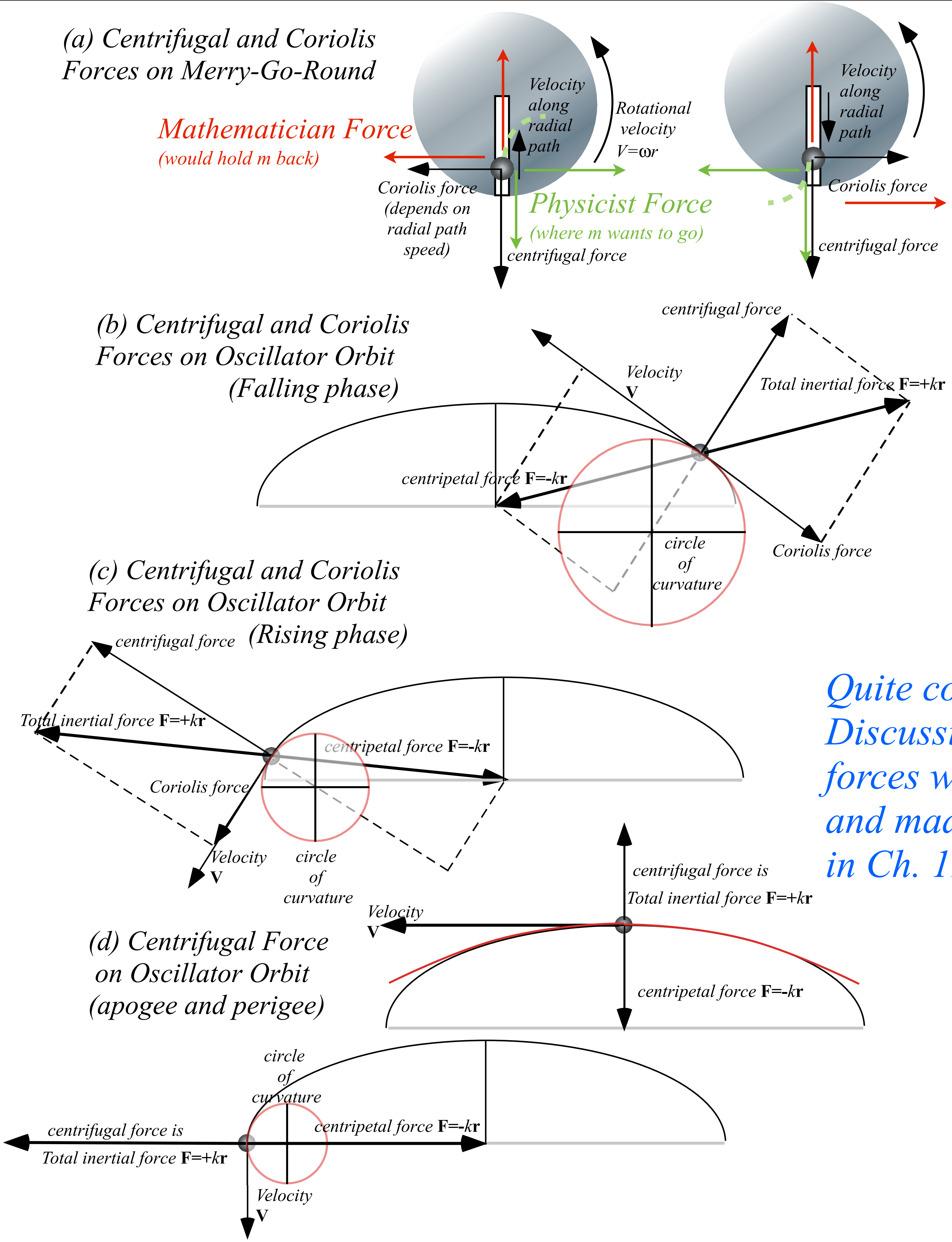
- *Mean-anomaly and eccentric-anomaly geometry*
- *Calculus and vector geometry of IHO orbits*
- → *A confusing introduction to Coriolis-centrifugal force geometry*



Unit 1
Fig. 11.2

Unit 1
Fig. 11.3





(a) Centrifugal and Coriolis Forces on Merry-Go-Round

(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)

(c) Centrifugal and Coriolis Forces on Oscillator Orbit (Rising phase)

(d) Centrifugal Force on Oscillator Orbit (apogee and perigee)

Unit 1
Fig. 11.4
a-d

Quite confusing?
Discussion of Coriolis forces will be done more elegantly and made more physically intuitive in Ch. 12 of Unit1 and in Unit 6.

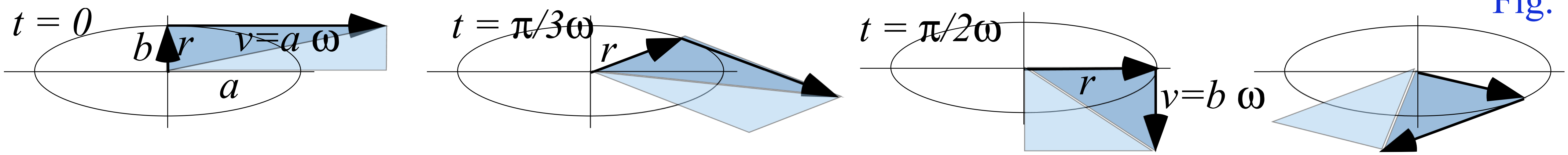
Some Kepler's "laws" for central (isotropic) force F(r)

→ Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Derived rigorously)
Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ (Derived later)
Total energy $E=KE+PE$ invariance of IHO: $F(r)=-k\cdot r$ (Derived rigorously)
Total energy $E=KE+PE$ invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived later)

Some Kepler's "laws" for central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle^{\mathbf{v}}_{\mathbf{r}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot\left(b\omega\cos\omega t\right) - b\sin\omega t\cdot\left(-a\omega\sin\omega t\right) = ab\cdot\omega$$

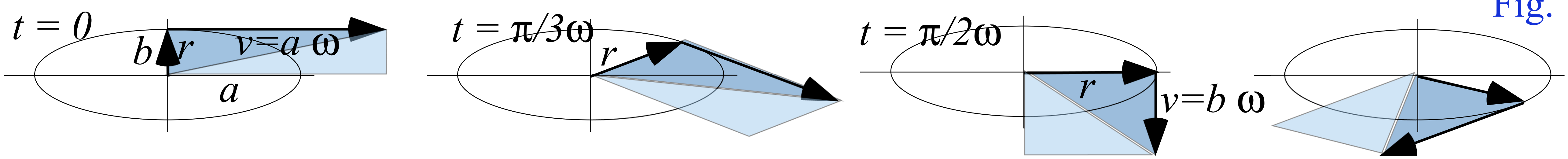
✓ *for IHO*

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} \qquad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ *with* $U(r)=k\cdot r^2/2$ (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_\oplus$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - a\sin\omega t\cdot(-b\omega\sin\omega t) = ab\cdot\omega$$

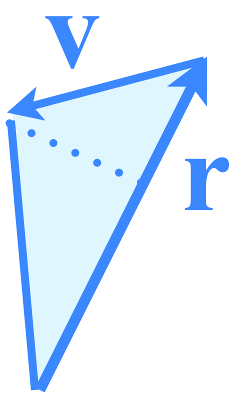
✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega$$

✓ *for IHO*

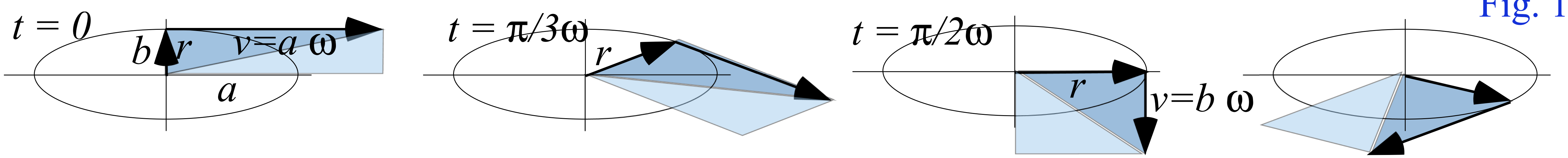
$|\mathbf{r}\times\mathbf{v}| = r\cdot v\cdot \sin\measuredangle_r^v$



Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - a\sin\omega t\cdot(-b\omega\sin\omega t) = ab\cdot\omega$$

✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega$$

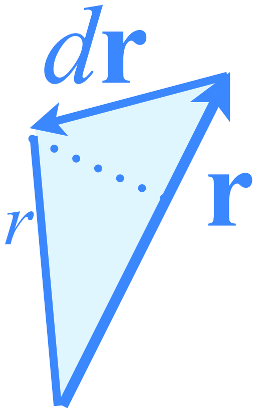
✓ *for IHO*

3. Equal area is swept by radius vector in each equal time interval T

$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{\underset{\boxed{\text{by 2.}}}{2m}}\int_0^T dt = \frac{L}{2m}T$$

✓ *for IHO*

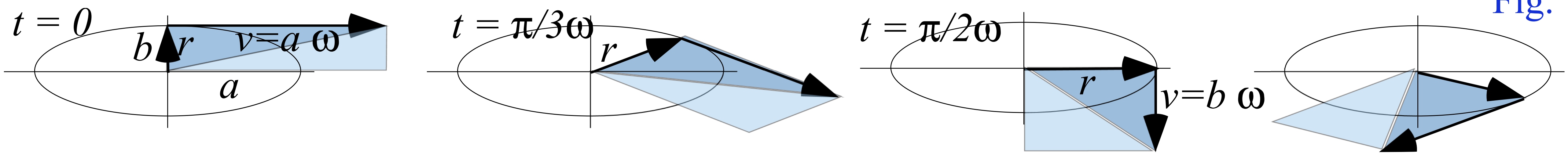
$$|\mathbf{r}\times d\mathbf{r}| = r\cdot dr\cdot\sin\measuredangle_r^{dr}$$



Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 7: $k = Gm\frac{4\pi}{3}\rho_\oplus$)

Unit 1
Fig. 11.8



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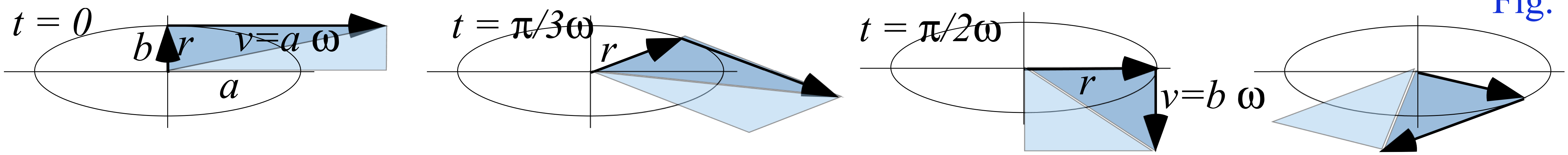
✓ for IHO

In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ $(= ab\cdot\pi$ for ellipse orbit)

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ (Recall from Lecture 7: $k=Gm\frac{4\pi}{3}\rho_\oplus$)

Unit 1
Fig. 11.8



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(Recall from Lecture 7: $\omega = \sqrt{k/m} = \sqrt{G\rho_\oplus 4\pi/3}$)

Some Kepler's "laws" for central (isotropic) force F(r)

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ *with* $U(r)=k\cdot r^2/2$ *(Derived rigorously)*

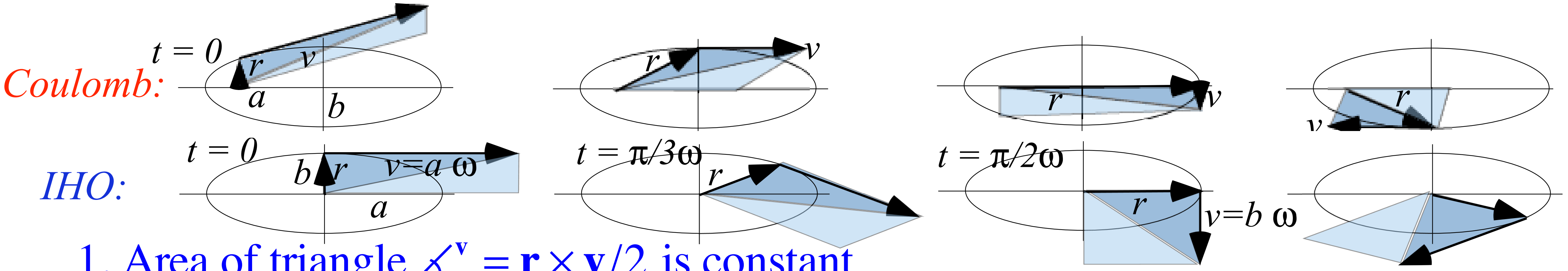
→ *Angular momentum invariance of Coulomb:* $F(r)=-GMm/r^2$ *with* $U(r)=-GMm\cdot/r$ *(Derived later)*

Total energy $E=KE+PE$ *invariance of IHO:* $F(r)=-k\cdot r$ *(Derived rigorously)*

Total energy $E=KE+PE$ *invariance of Coulomb:* $F(r)=-GMm/r^2$ *(Derived later)*

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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✓ *for IHO*

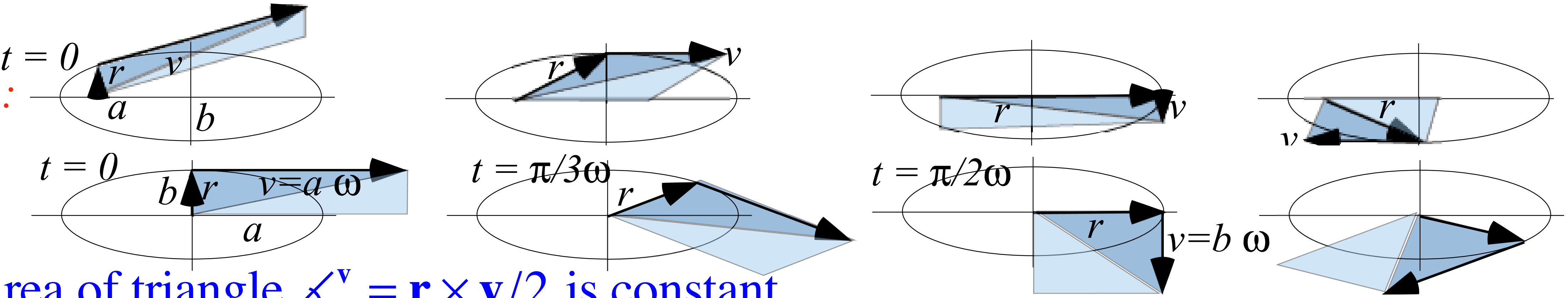
✓ *for Coul.*

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$

Coulomb:

IHO:



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✓ *for IHO*

✓ *for Coul.*

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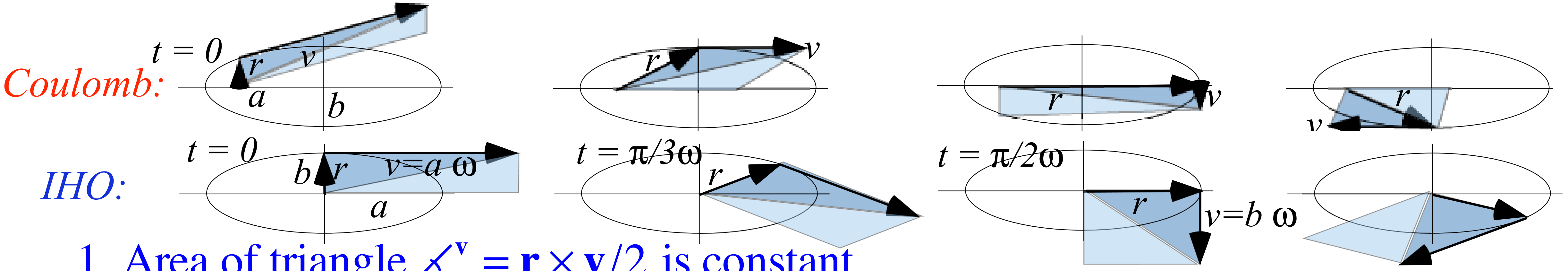
$$L = m\,\mathbf{r}\times\mathbf{v} = m\left(r_x v_y - r_y v_x\right) = \begin{cases} m\cdot ab\cdot\sqrt{G\rho_\oplus 4\pi/3} & \textit{for IHO} \\ m\cdot a^{-1/2}b\sqrt{GM_\oplus} & \textit{for Coul.} \end{cases}$$

✓ *for IHO*

✓ *for Coul.*

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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✓ for Coul.

3. Equal area is swept by radius vector in each equal time interval T

In one period:

$$\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L} = \frac{2m\cdot ab\cdot\pi}{L} = \begin{cases} \dfrac{2m\cdot ab\cdot\pi}{m\cdot ab\cdot\sqrt{G\rho_\oplus 4\pi/3}} = \dfrac{2\pi}{\sqrt{G\rho_\oplus 4\pi/3}} & \text{for IHO} \\ \dfrac{2m\cdot ab\cdot\pi}{m\cdot a^{-1/2}b\sqrt{GM_\oplus}} = \dfrac{2\pi}{a^{-3/2}\sqrt{GM_\oplus}} & \text{for Coul.} \end{cases}$$

Applies to any central F(r)

Applies to IHO and Coulomb

that is ω_{IHO}

that is ω_{Coul}

Some Kepler's "laws" for central (isotropic) force F(r)

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ *with* $U(r)=k\cdot r^2/2$ *(Derived rigorously)*
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Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix}=\begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} \qquad \begin{pmatrix} r_x \\ r_y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix}$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=$KE + PE$ is constant

$$KE + PE = \frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$= \frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}kr_x^2 + \frac{1}{2}kr_y^2$$

$$= \frac{1}{2}m(-a\omega\sin\omega t)^2 + \frac{1}{2}m(b\omega\cos\omega t)^2 + \frac{1}{2}k(a\cos\omega t)^2 + \frac{1}{2}k(b\sin\omega t)^2$$

$$= \frac{1}{2}ma^2\omega^2(\sin^2\omega t) + \frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2 + \frac{1}{2}ka^2(\cos^2\omega t) + \frac{1}{2}kb^2(\sin^2\omega t)$$

$$= \frac{1}{2}m\omega^2(a^2 + b^2)$$

$$Given : k = m\omega^2$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)\qquad Given: k=m\omega^2$$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)\quad \text{since: } \omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3}\quad \text{or: } m\omega^2=k$$

Some Kepler's "laws" for central (isotropic) force F(r)

Angular momentum invariance of IHO: $F(r)=-k \cdot r$ with $U(r)=k \cdot r^2/2$ (Derived rigorously)

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm \cdot /r$ (Derived later)

Total energy $E=KE+PE$ invariance of IHO: $F(r)=-k \cdot r$ (Derived rigorously)

→ *Total energy $E=KE+PE$ invariance of Coulomb: $F(r)=-GMm/r^2$ (Derived later)*

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

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$$=\frac{1}{2}\begin{pmatrix}v_x & v_y\end{pmatrix}\bullet\begin{pmatrix}m & 0\\ 0 & m\end{pmatrix}\bullet\begin{pmatrix}v_x\\ v_y\end{pmatrix}+\begin{pmatrix}r_x & r_y\end{pmatrix}\bullet\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}\bullet\begin{pmatrix}r_x\\ r_y\end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$Given: k=m\omega^2$

$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3}$ or: $m\omega^2=k$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

$$E=KE+PE=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2-\frac{k}{r}=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2-\frac{GM_\oplus m}{r}=-\frac{GM_\oplus m}{a}$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite

$$\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} \qquad\qquad =1$$

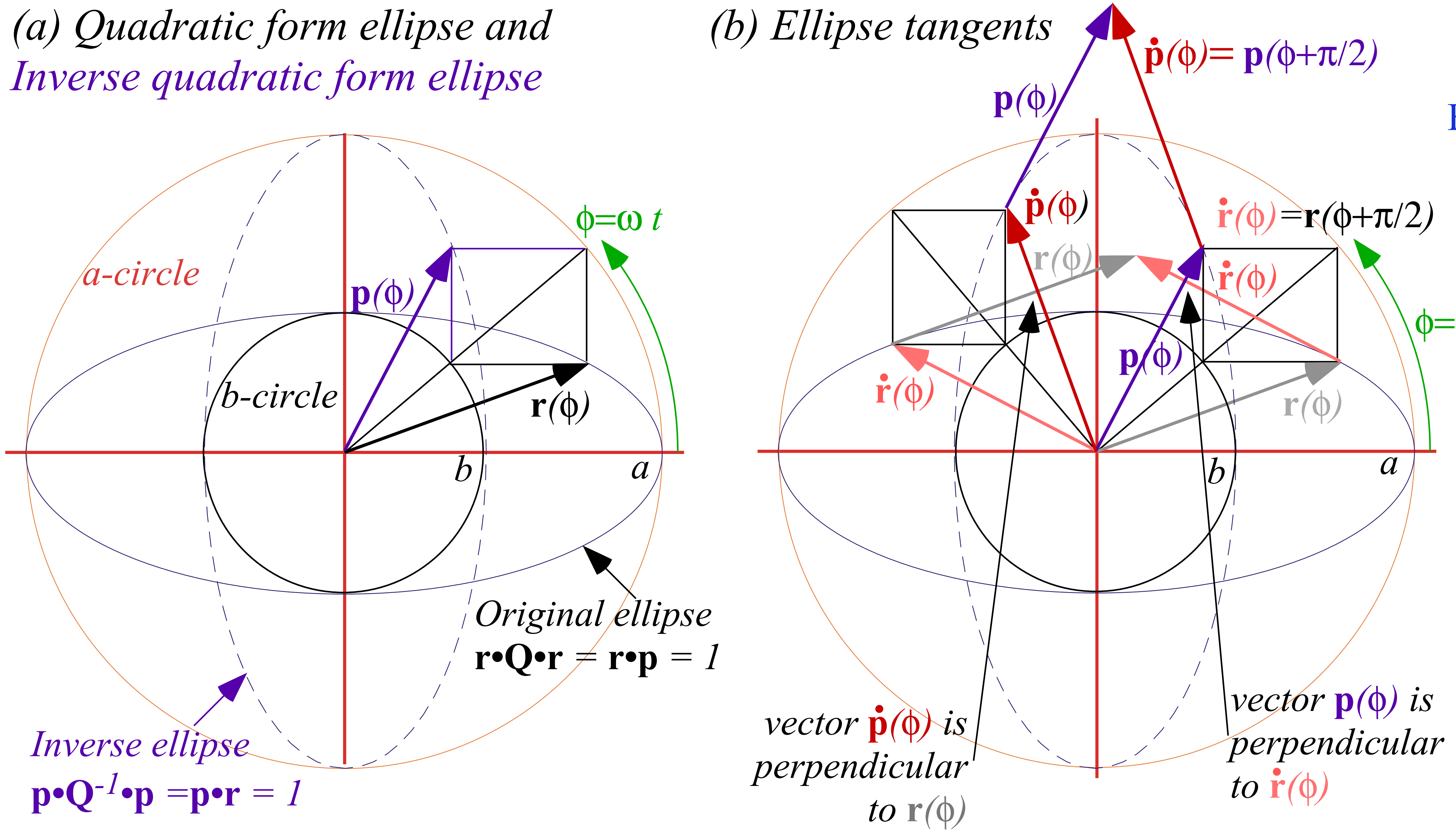
$$\begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix}=1=\begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

A inverse matrix Q^{-1} generates an ellipse by **p**•*Q^{-1}*•**p**=*1 called* inverse *or* dual *ellipse:*

$$\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} \qquad\qquad =1$$

$$\begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\bullet\begin{pmatrix} p_x \\ p_y \end{pmatrix}=1=\begin{pmatrix} p_x & p_y \end{pmatrix}\bullet\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}=a^2 p_x^2+b^2 y^2$$

Unit 1
Fig. 11.6


(a) Quadratic form ellipse and
Inverse quadratic form ellipse

(b) Ellipse tangents

Note some quadratic form mutual duality relations:

unit mutual projection

$$\mathbf{p} = \mathbf{Q}\bullet\mathbf{r} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where: } \begin{matrix} x = r_x = a\cos\phi = a\cos\omega t \\ y = r_y = b\sin\phi = b\sin\omega t \end{matrix} \quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r} = 1}$$

p *is perpendicular to velocity* $\mathbf{v}=\dot{\mathbf{r}}$ *, a mutual orthogonality*

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p} = 0} = \begin{pmatrix} \dot r_x & \dot r_y \end{pmatrix}\bullet\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} -a\sin\phi & b\cos\phi \end{pmatrix}\bullet\begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where: } \begin{matrix} \dot r_x = -a\sin\phi \\ \dot r_y = b\cos\phi \end{matrix} \quad \text{and: } \begin{matrix} p_x = (1/a)\cos\phi \\ p_y = (1/b)\sin\phi \end{matrix}$$