Lecture 20
Mon. 3.12.2012

Complex Variables, Series, and Field Coordinates II.

(Ch. 10 of Unit 1)

From Part I:

1. Review of source-free (analytic) fields

Easy 2D source-free field theory

Easy 2D vector field-potential theory

2. Review of basic Riemann-Cauchy conditions

1. Complex numbers provide "automatic trigonometry"
2. Complex numbers add like vectors.
3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.
4. Complex products provide 2D rotation operations.
5. Complex products provide 2D "dot"(•) and "cross"(x) products.
6. Complex derivative contains "divergence"($\nabla \cdot \mathbf{F}$) and "curl"($\nabla \times \mathbf{F}$) of 2D vector field
7. Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F}=0$ and $\nabla \times \mathbf{F}=0$]
8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla \times \mathbf{A}$) potentials

End of Part I. Lecture 19 Thur. 3.08.2012

3. 2D source-field-potential-coordinate analysis

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

9. Complex integrals $\int f(z)dz$ count 2D "circulation"($\int \mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int \mathbf{F} \times d\mathbf{r}$)
10. Complex integrals define 2D monopole fields and potentials
11. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field
12. Complex derivatives give 2D dipole fields
13. More derivatives give 2D 2^N-pole fields…
14. ...and 2^N-pole multipole expansions of fields and potentials...
15. ...and Laurent Series...
16. ...and non-analytic source analysis.

13-16 Not covered in class on 3.12.12

From Part I:

1. Review of source-free (analytic) fields

→ *Easy 2D source-free field theory*

Easy 2D vector field-potential theory

2. Review of basic Riemann-Cauchy conditions

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of *(z,z*)* to *(x=*Re*z,y=*Im*z)* defines a *z*-derivative $\frac{df}{dz}$ and "star" *z**-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}$$

Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\bullet\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz}(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{1}{2}\nabla\bullet\mathbf{f} + \frac{i}{2}|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function *f(z)*, conjugate it (change all *i*'s to *–i*) to give $f^*(z^*)$ for which $\frac{df^*}{dz} = 0$.

For example: if *f(z)=a·z* then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ has *zero divergence*: $\nabla\bullet\mathbf{F}=0$ and has *zero curl*: $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0$$

A *DFL* field **F** (*Divergence-Free-Laminar*)

From Part I:

1. Review of source-free (analytic) fields

Easy 2D source-free field theory

→ *Easy 2D vector field-potential theory*

2. Review of basic Riemann-Cauchy conditions

*8. Complex potential ϕ contains "scalar"(**F=**$\nabla\Phi$) and "vector"(**F=**∇x**A**) potentials*

Any *DFL* field **F** is a gradient of a *scalar potential field* **Φ** or a curl of a *vector potential field* **A**.

$$\mathbf{F} = \nabla\Phi \qquad\qquad \mathbf{F} = \nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\Phi(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\Phi(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*) = d\phi^*/dz^*$ giving *DFL* field **F**.

Now if you have a field f(z) you *<u>integrate</u>* *to get the potential $\phi(z)$ field:* $\phi(z) = \int f(z)\, dz$

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

Any *DFL* field **F** is a gradient of a *scalar potential field* Φ or a curl of a *vector potential field* **A**.

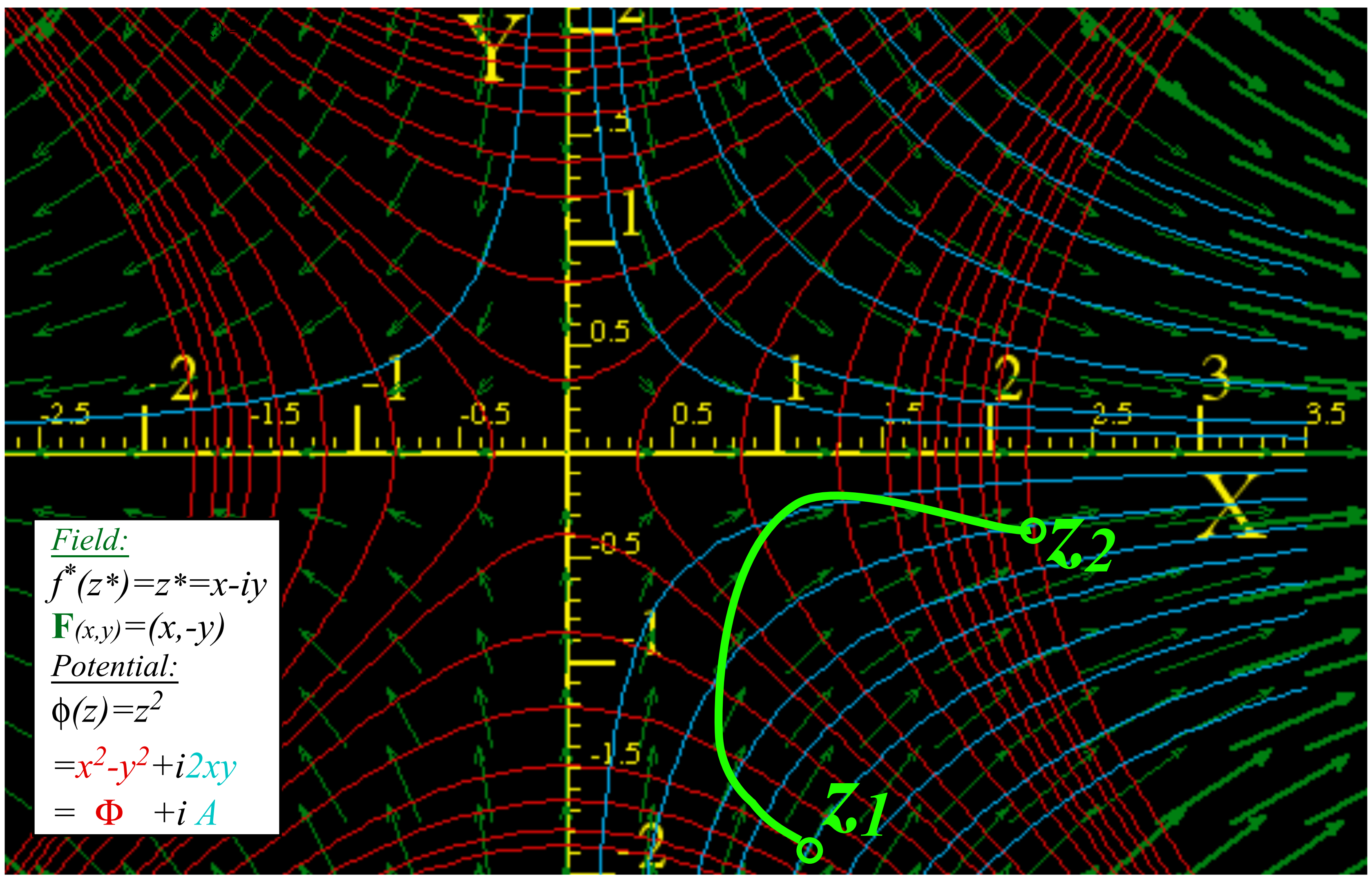
$$\mathbf{F}=\nabla\Phi \qquad\qquad \mathbf{F}=\nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\Phi(x,y)+i\mathbf{A}(x,y)$ exists whose *z*-derivative is $f(z)=d\phi/dz$.
Its complex conjugate $\phi^*(z^*)=\Phi(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field **F**.

To find $\phi=\Phi+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\mathrm{Re}\,\phi=\Phi$) and imaginary ($\mathrm{Im}\,\phi=\mathbf{A}$) parts.

$$\phi=\underbrace{\Phi}_{=\frac{1}{2}a(x^2-y^2)}+i\underbrace{\mathbf{A}}_{axy}=\int f\cdot dz=\int az\cdot dz=\tfrac{1}{2}az^2=\tfrac{1}{2}a(x+iy)^2$$

$$=\tfrac{1}{2}a(x^2-y^2)+i\,axy$$



Unit 1
Fig. 10.7

But, if you have a potential $\phi(z)$ *you <u>differentiate</u> to get the field* $f(z) = \dfrac{d}{dz}\phi(z)$

What Good Are Complex Exponentials? (contd.)

8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial y}\end{pmatrix}$ of vector $\mathbf{A}$ (*and they're equal!*)

The half-n'-half result

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\Phi$ replaces usual physicist's definition $\mathbf{F}=-\nabla\Phi$

What Good Are Complex Exponentials? (contd.)

*8. (contd.) Complex potential ϕ contains "scalar"(**F=$\nabla\Phi$**) and "vector"(**F=$\nabla\times$A**) potentials*

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial A}{\partial y}\\ -\frac{\partial A}{\partial x}\end{pmatrix}$ of vector **A** (*and they're equal!*)

The half-n'-half result

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field **F**=+$\nabla\Phi$ replaces usual physicist's definition **F**=-$\nabla\Phi$

Given ϕ:

$$\phi=\Phi+i\,\mathbf{A}=\tfrac{1}{2}a(x^2-y^2)+i\,axy$$

The half-n'-half result

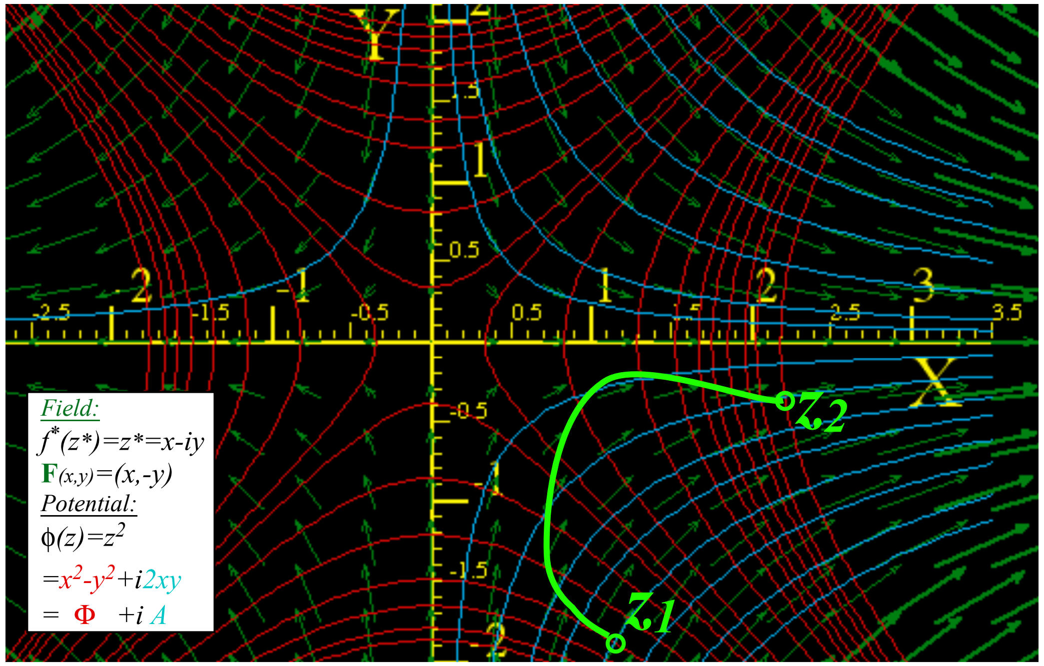
find:

$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

$$\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial\mathbf{A}}{\partial y}\\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial y}axy\\ -\frac{\partial}{\partial x}axy\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* **A**=*const.* define *DFL field-net*.



The half-n'-half results

are called

Riemann-Cauchy

Derivative Relations

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\mathbf{A}}{\partial y}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial x}=\frac{\partial\,\mathrm{Im}f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y}=-\frac{\partial\mathbf{A}}{\partial x}\quad\text{is:}\quad\frac{\partial\,\mathrm{Re}f(z)}{\partial y}=-\frac{\partial\,\mathrm{Im}f(z)}{\partial x}$$

From Part I:

1. Review of source-free (analytic) fields

Easy 2D source-free field theory

Easy 2D vector field-potential theory

→ *2. Review of basic Riemann-Cauchy conditions*

What's analytic? (...and what's not?)

Review (z,z) to (x,y) transformation relations*

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$$

Criteria for a field function $f = f_x(x,y) + i\, f_y(x,y)$ to be an ***analytic function f(z)*** *of $z=x+iy$:*

First, f(z) must <u>not</u> be a function of $z^*=x-iy$, that is: $\frac{df}{dz^*} = 0$

This implies f(z) satisfies differential equations known as the ***Riemann-Cauchy conditions***

$$\frac{df}{dz^*} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) \text{ implies : } \frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \quad \text{and :} \quad \frac{\partial f_y}{\partial x} = -\frac{\partial f_x}{\partial y}$$

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

Review (z,z) to (x,y) transformation relations*

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

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$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$

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$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = \frac{\partial f_y}{\partial y} - i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = \frac{\partial}{\partial iy}(f_x + i f_y)$$

Criteria for a field function $f = f_x(x,y) + i f_y(x,y)$ to be an ***analytic function f(z*)*** *of $z^*=x-iy$:*

First, f(z) must not be a function of $z=x+iy$, that is:* $\frac{df}{dz} = 0$

This implies f(z) satisfies differential equations we call* ***Anti-Riemann-Cauchy conditions***

$$\frac{df}{dz} = 0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right) = \text{implies: } \frac{\partial f_x}{\partial x} = -\frac{\partial f_y}{\partial y} \text{ and: } \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}$$

$$\frac{df}{dz^*} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial f_y}{\partial x} + \frac{\partial f_x}{\partial y}\right) = \frac{\partial f_x}{\partial x} + i\frac{\partial f_y}{\partial x} = -\frac{\partial f_y}{\partial y} + i\frac{\partial f_x}{\partial y} = \frac{\partial}{\partial x}(f_x + i f_y) = -\frac{\partial}{\partial iy}(f_x + i f_y)$$

What's analytic? (...and what's <u>not</u>?)

Example: Is $f(x,y) = 2x + iy$ an analytic function of $z=z+iy$?

What's analytic? (...and what's not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$
or: $x = (z+z^*)/2$ and: $y =-i(z-z^*)/2$

What's analytic? (...and what's <u>not</u>?)

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$$f(x,y) = 2x + i4y = 2\,(z+z^*)/2 + i4(-i(z-z^*)/2)$$

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$$f(x,y) = 2x + i4y = 2\,(z+z^*)/2 + i4(-i(z-z^*)/2)$$
$$= z+z^* + (2z-2z^*)$$

What's analytic? (...and what's <u>not</u>?)

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$$f(x,y) = 2x + i4y = 2\,(z+z^*)/2 + i4(-i(z-z^*)/2)$$
$$= \quad z+z^* \quad + \quad (2z-2z^*)$$
$$= \quad 3z-z^*$$

What's analytic? (...and what's <u>not</u>?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

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$$= z+z^* + (2z-2z^*)$$
$$= 3z-z^*$$

A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

What's analytic? (...and what's <u>not</u>?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$
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$$f(x,y) = 2x + i4y = 2\,(z+z^*)/2 + i4(-i(z-z^*)/2)$$
$$= \quad z+z^* \quad + \quad (2z-2z^*)$$
$$= \quad 3z-z^*$$

A: *NO!* *It's a function of z <u>and</u> z* so not analytic for <u>either</u>.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=z+iy$?

A: *NO!* *r(xy)=z*z is a function of z <u>and</u> z* so not analytic for <u>either</u>.*

What's analytic? (...and what's not?)

Example: Q: Is $f(x,y) = 2x + i4y$ an analytic function of $z=z+iy$?

Well, test it using definitions: $z = x + iy$ and: $z^* = x - iy$

or: $x = (z+z^*)/2$ and: $y = -i(z-z^*)/2$

$$f(x,y) = 2x + i4y = 2\,(z+z^*)/2 + i4(-i(z-z^*)/2)$$
$$= z+z^* + (2z-2z^*)$$
$$= 3z-z^*$$

A: *NO!* *It's a function of z and z* so not analytic for either.*

Example 2: Q: Is $r(x,y) = x^2 + y^2$ an analytic function of $z=z+iy$?

A: *NO!* $r(xy)=z^*z$ *is a function of z and z* so not analytic for either.*

Example 3: Q: Is $s(x,y) = x^2-y^2 + 2ixy$ an analytic function of $z=z+iy$?

A: *YES!* $s(xy)=(x+iy)^2 = z^2$ *is analytic function of z.* (Yay!)

3. 2D source-field-potential-coordinate analysis

→ *Easy 2D circulation and flux integrals*

Easy 2D curvilinear coordinate discovery

Easy 2D monopole, dipole, and 2^n-pole analysis

*9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫**F**•d**r**) and "flux"(∫**F**xd**r**)*

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi = \phi(z_2) - \phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\boldsymbol{\Phi}(x_2, y_2) - \boldsymbol{\Phi}(x_1, y_1)}_{\Delta\boldsymbol{\Phi}} + i\,\underbrace{[\mathbf{A}(x_2, y_2) - \mathbf{A}(x_1, y_1)]}_{\Delta\mathbf{A}}$$

$$\Delta\phi = \Delta\boldsymbol{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL* field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫F•dr) and "flux"(∫Fxdr)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi=\phi(z_2)-\phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2,y_2) - \mathbf{\Phi}(x_1,y_1)}_{\Delta\mathbf{\Phi}} + i\,[\underbrace{\mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \Delta\mathbf{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL* field **F**, Δϕ is independent of the integration path $z(t)$ connecting z_1 and z_2.

$$\int f(z)dz = \int\left(f^*(z^*)\right)^* dz = \int\left(f^*(z^*)\right)^*\left(dx + i\,dy\right) = \int\left(f_x^* + i\,f_y^*\right)^*\left(dx + i\,dy\right) = \int\left(f_x^* - i\,f_y^*\right)\left(dx + i\,dy\right)$$

$$= \int(f_x^* dx + f_y^* dy) + i\int(f_x^* dy - f_y^* dx)$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\times d\mathbf{r}\bullet\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\bullet d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} + i\int \mathbf{F}\bullet d\mathbf{S} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫F•dr) and "flux"(∫Fxdr)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi=\phi(z_2)-\phi(z_1)$

$$\Delta\phi = \phi(z_2)-\phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\boldsymbol{\Phi}(x_2,y_2)-\boldsymbol{\Phi}(x_1,y_1)}_{\Delta\boldsymbol{\Phi}} + i\,[\underbrace{\mathbf{A}(x_2,y_2)-\mathbf{A}(x_1,y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \Delta\boldsymbol{\Phi} + i \; \Delta\mathbf{A}$$

In *DFL* field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.

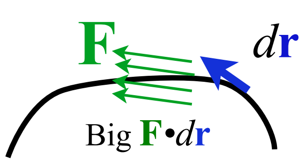
$$\int f(z)dz = \int\left(f^*(z^*)\right)^* dz = \int\left(f^*(z^*)\right)^*\left(dx+i\,dy\right) = \int\left(f_x^* + i\,f_y^*\right)^*\left(dx+i\,dy\right) = \int\left(f_x^* - i\,f_y^*\right)\left(dx+i\,dy\right)$$

$$= \int(f_x^* dx + f_y^* dy) + i\int(f_x^* dy - f_y^* dx)$$

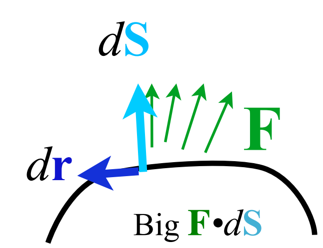
$$= \int \mathbf{F}\bullet d\mathbf{r} \quad + i\int \mathbf{F}\times d\mathbf{r}\bullet\hat{\mathbf{e}}_Z$$

$$= \int \mathbf{F}\bullet d\mathbf{r} \quad + i\int \mathbf{F}\bullet d\mathbf{r}\times\hat{\mathbf{e}}_Z$$

$$= \boxed{\int \mathbf{F}\bullet d\mathbf{r}} \quad + i\boxed{\int \mathbf{F}\bullet d\mathbf{S}} \qquad \text{where:} \quad d\mathbf{S} = d\mathbf{r}\times\hat{\mathbf{e}}_Z$$



Real part $\int_1^2 \mathbf{F}\bullet d\mathbf{r} = \Delta\boldsymbol{\Phi}$
sums **F** projections *along* path $d\mathbf{r}$ that is, *circulation* on path to get $\Delta\boldsymbol{\Phi}$.



Imaginary part $\int_1^2 \mathbf{F}\bullet d\mathbf{S} = \Delta\mathbf{A}$
sums **F** projection *across* path $d\mathbf{r}$ that is, *flux* thru surface elements $d\mathbf{S}=d\mathbf{r}\times\mathbf{e}_Z$ normal to $d\mathbf{r}$ to get $\Delta\mathbf{A}$.

3. 2D source-field-potential-coordinate analysis

Easy 2D circulation and flux integrals

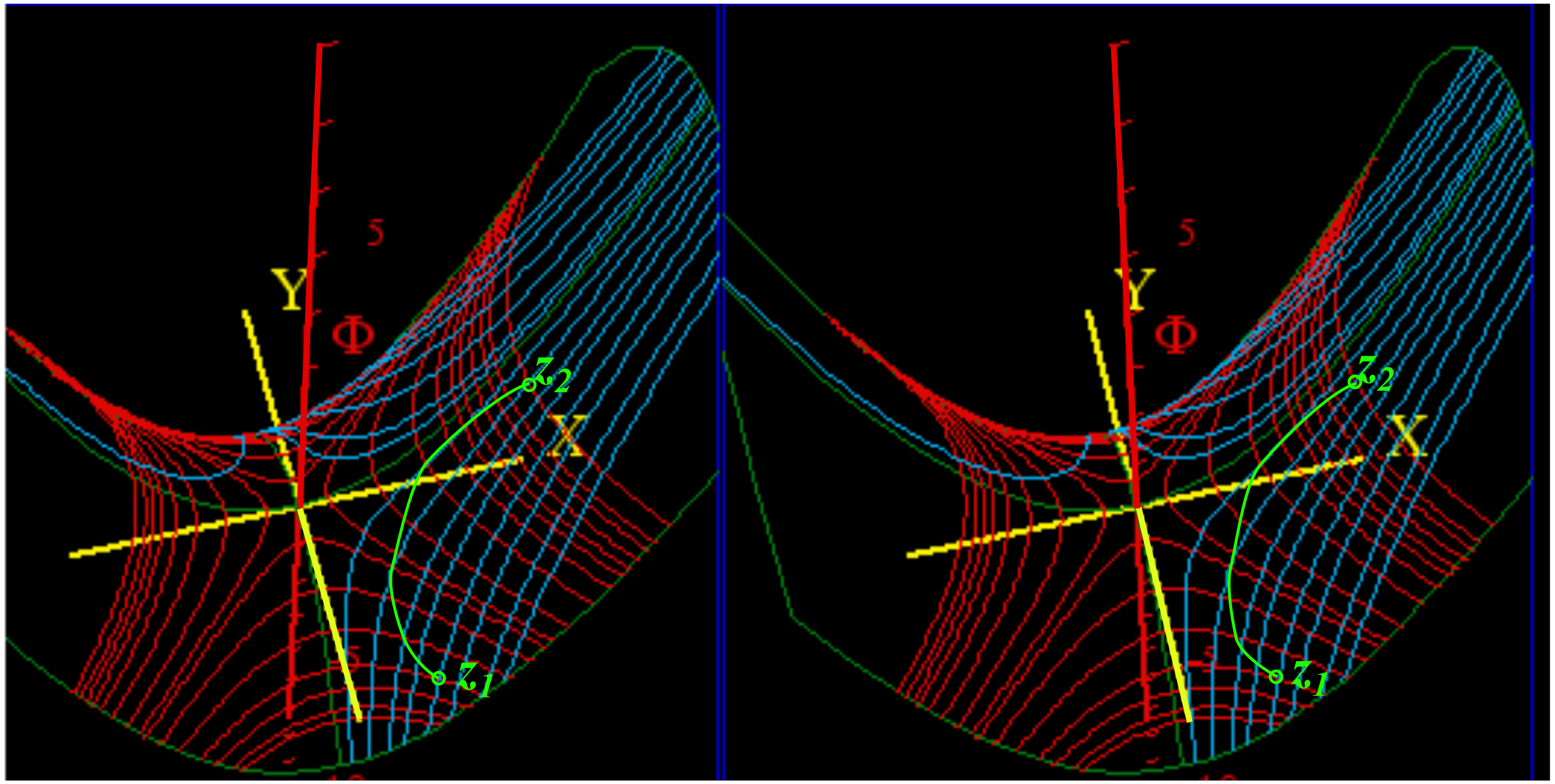
→ *Easy 2D curvilinear coordinate discovery*

Easy 2D monopole, dipole, and 2^n-pole analysis

Here the scalar potential $\Phi=(x^2-y^2)/2$ is stereo-plotted vs. (x,y)

The $\Phi=(x^2-y^2)/2=const.$ curves are topography lines

The $A=(xy)=const.$ curves are streamlines normal to topography lines



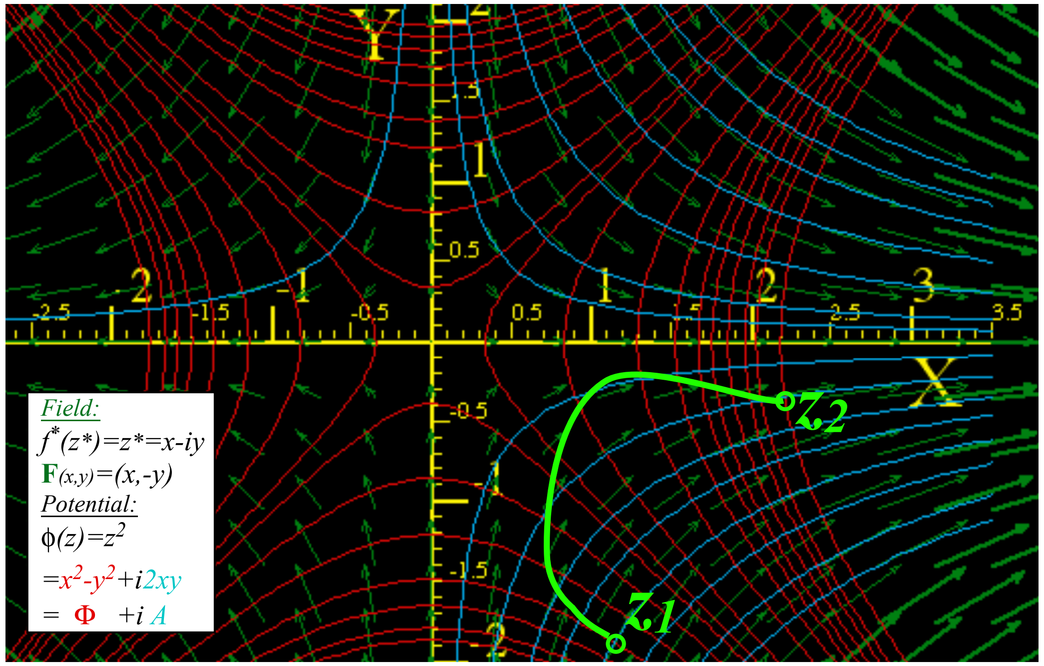
11. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field

The (Φ,A) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2 - y^2)/2 = const.$

$q^2 = A = (xy) = const.$

*Actually it's OCC.



$$Kajobian = \begin{pmatrix} \frac{\partial q^1}{\partial x} & \frac{\partial q^1}{\partial y} \\ \frac{\partial q^2}{\partial x} & \frac{\partial q^2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} & \frac{\partial \Phi}{\partial y} \\ \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\Phi} \\ \leftarrow \mathbf{E}^{A} \end{matrix}$$

$$Jacobian = \begin{pmatrix} \frac{\partial x}{\partial q^1} & \frac{\partial x}{\partial q^2} \\ \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \Phi} & \frac{\partial x}{\partial A} \\ \frac{\partial y}{\partial \Phi} & \frac{\partial y}{\partial A} \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix} \qquad \begin{matrix} \uparrow & \uparrow \\ \mathbf{E}_{\Phi} & \mathbf{E}_{A} \end{matrix}$$

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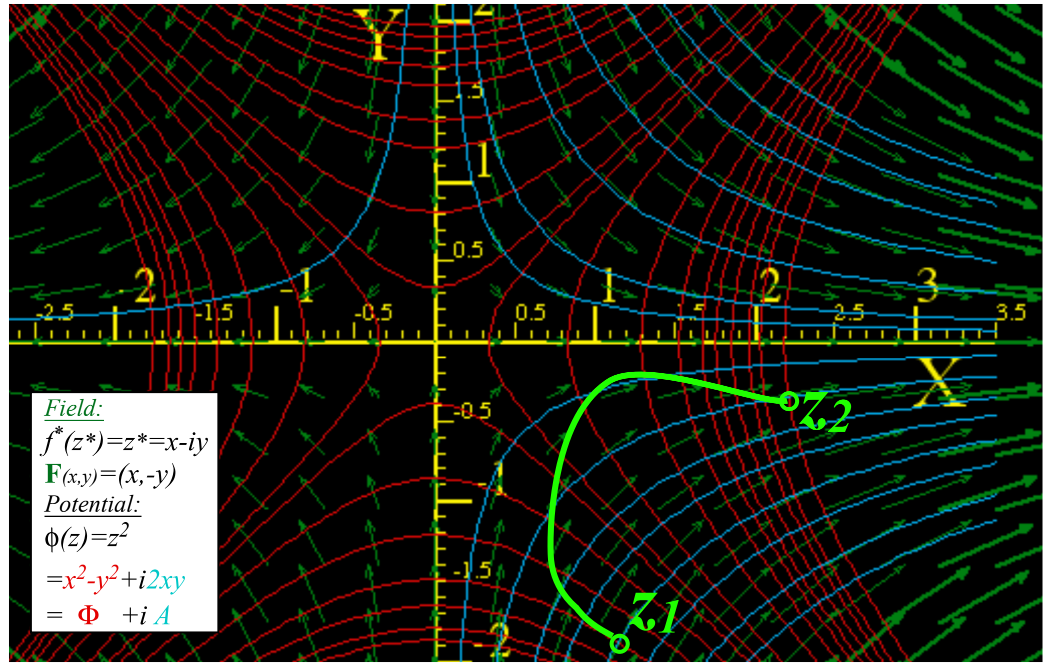

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Riemann-Cauchy Derivative Relations make coordinates orthogonal

$$\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2 - y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2 - y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

The half-n'-half results assure

$$\mathbf{E}_{\Phi}\bullet\mathbf{E}_{A} = \frac{\partial\Phi}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial A}{\partial y} = -\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y} + \frac{\partial\Phi}{\partial y}\frac{\partial\Phi}{\partial x} = 0$$

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial A}{\partial y} \\ -\frac{\partial A}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

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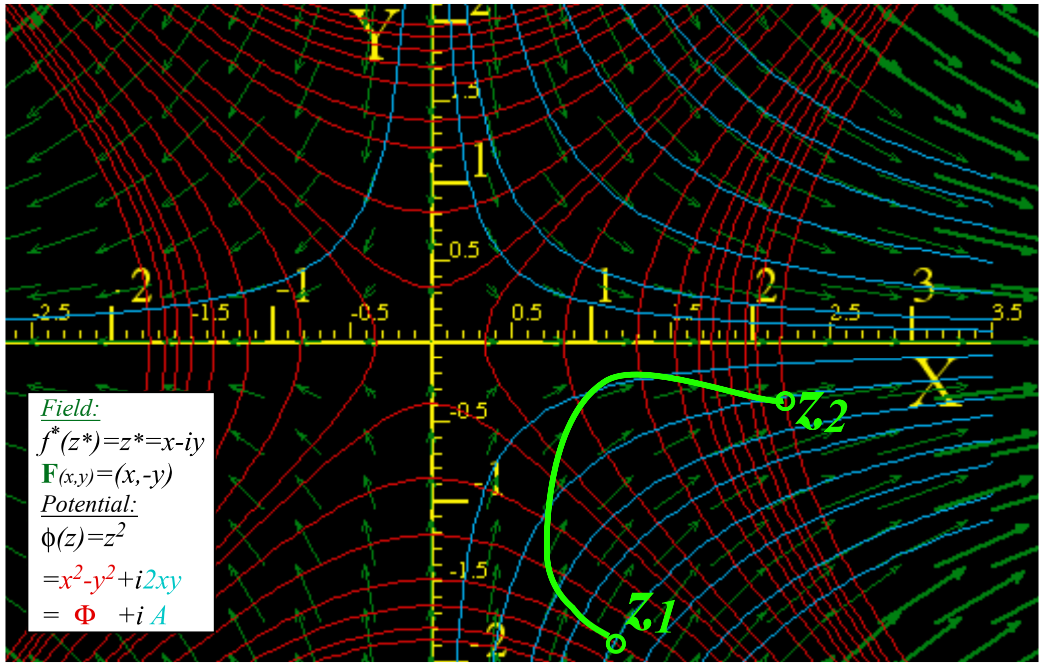

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or Riemann-Cauchy

Zero divergence requirement: $0 = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial y} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ *potential Φ obeys Laplace equation*

and so does **A**

3. 2D source-field-potential-coordinate analysis

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

→ *Easy 2D monopole, dipole, and 2^n-pole analysis*

10. Complex integrals define 2D monopole fields and potentials

Of all power-law fields $f(z)=az^n$ one lacks a power-law potential $\phi(z)=\frac{a}{n+1}z^{n+1}$. It is the $n = -1$ case.

Unit monopole field: $f(z)=\frac{1}{z}=z^{-1}$

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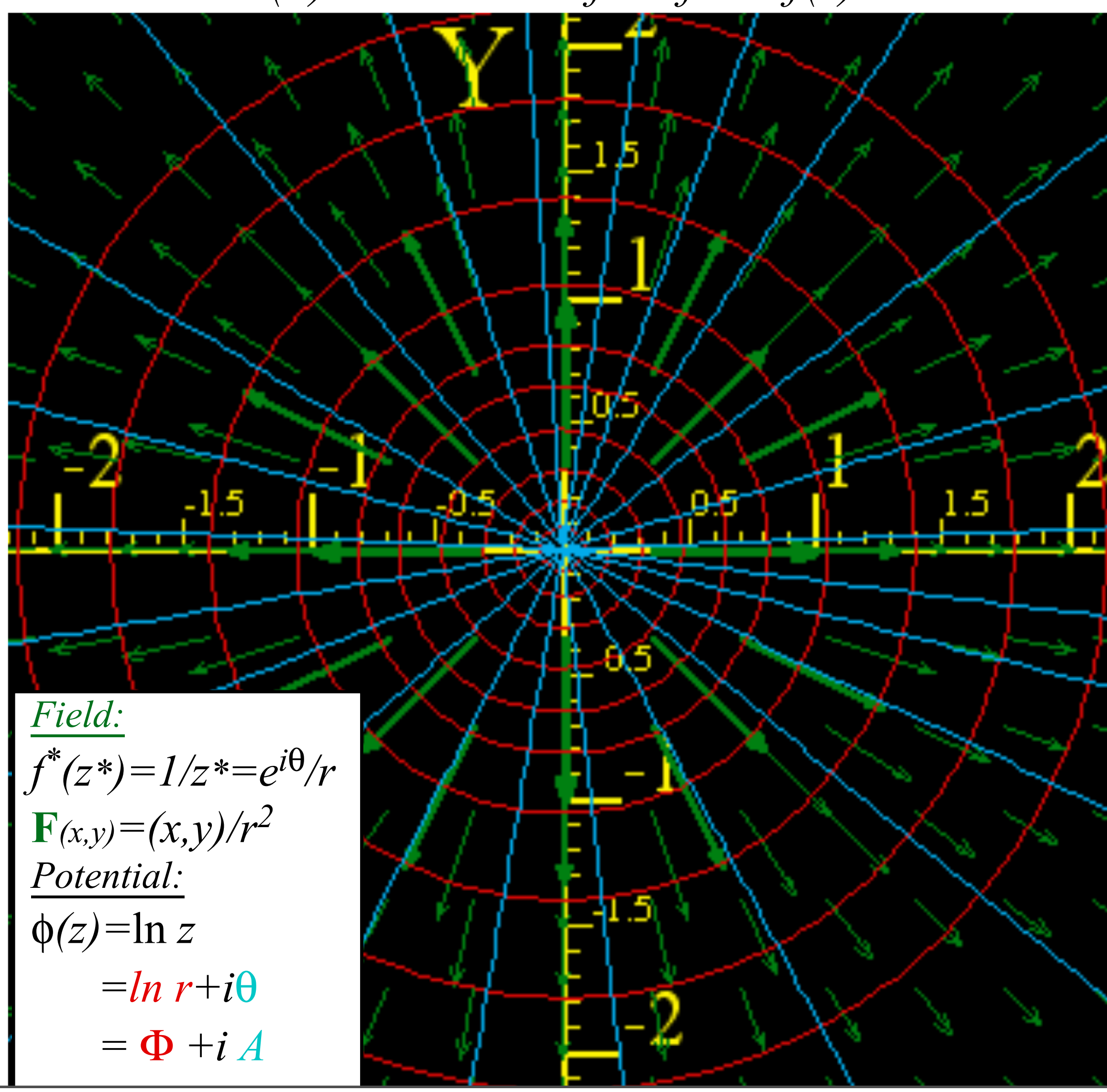
(a) Unit Z-line-flux field $f(z)=1/z$



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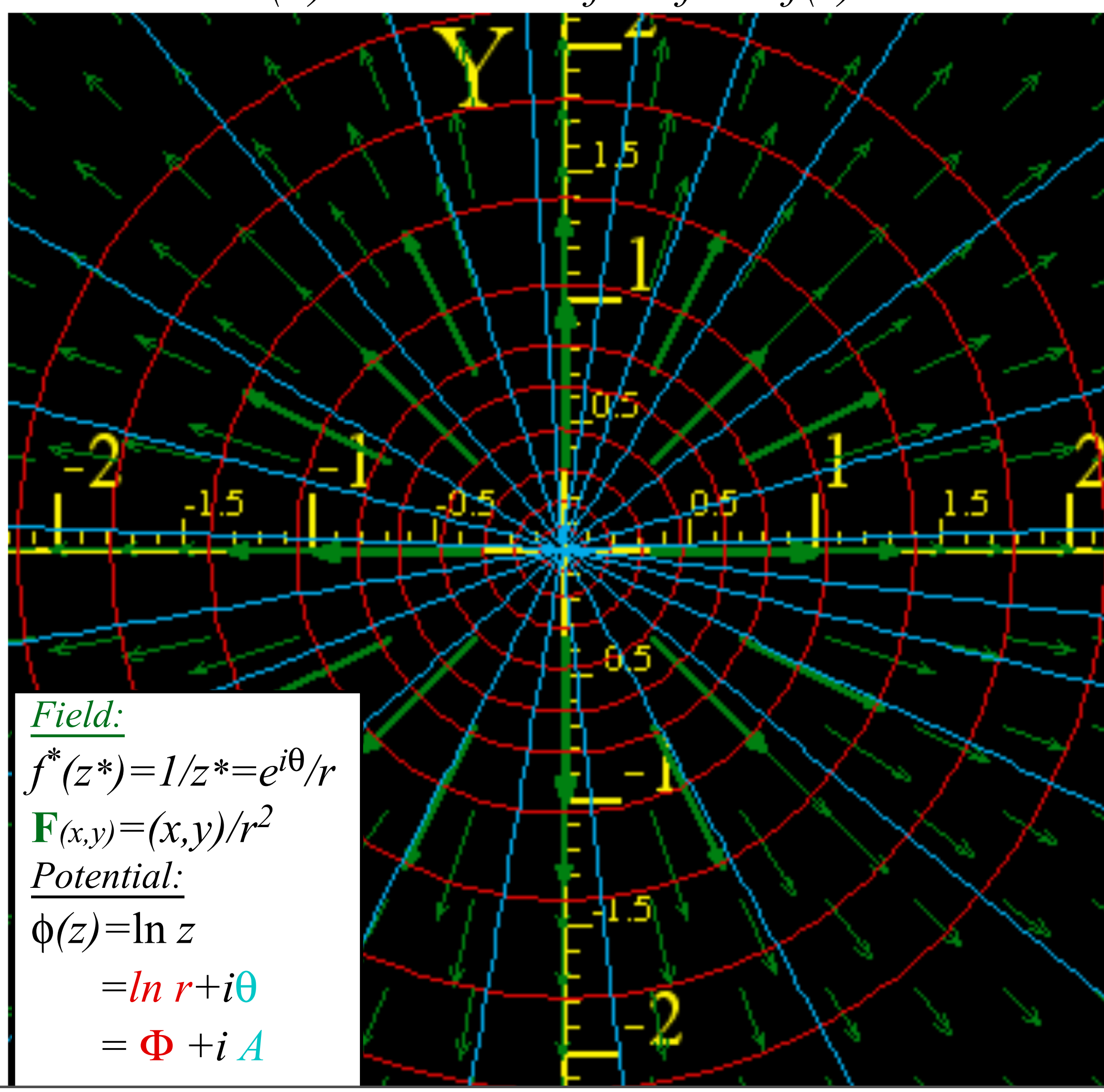
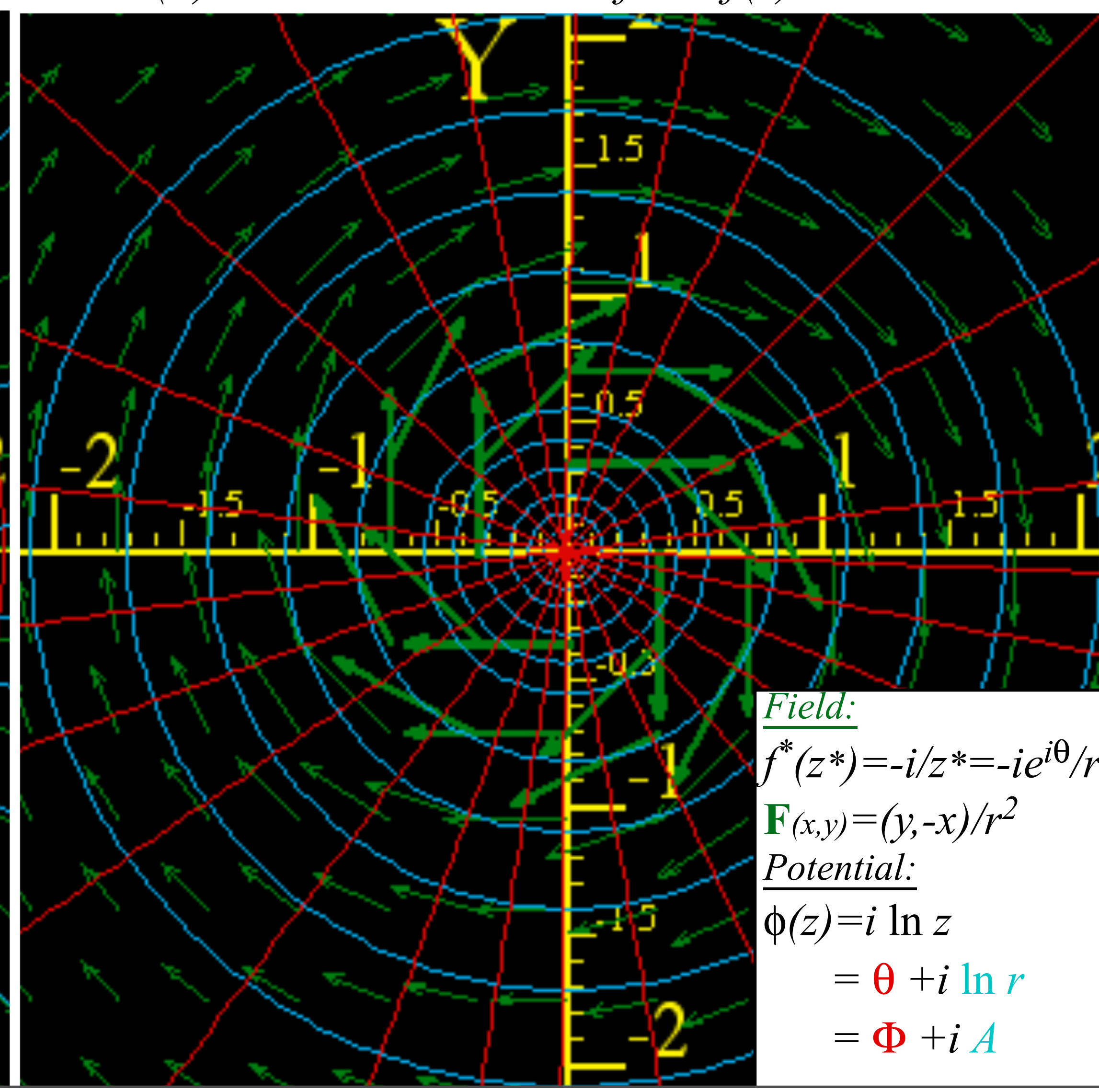


(b) Unit Z-line-vortex field f(z)=i/z



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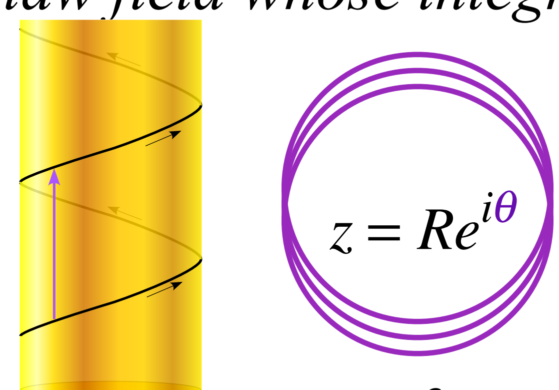
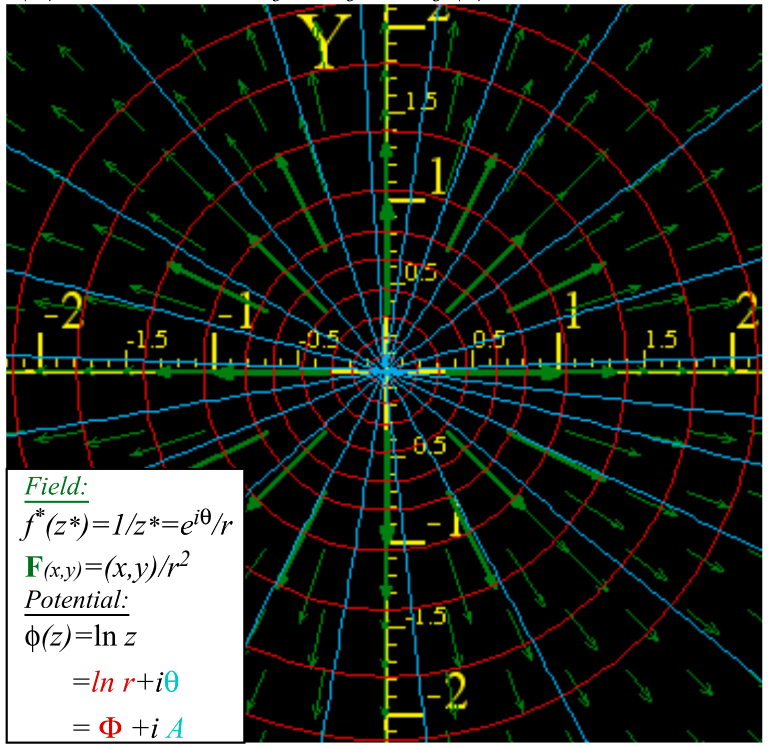
A monopole field is the only power-law field whose integral (potential) depends on path of integration.



path that goes N times around origin (r=0) at constant r = R.

$$\Delta\phi=\oint f(z)dz=a\oint\frac{dz}{z}=a\int_{\theta=0}^{\theta=2\pi N}\frac{d(Re^{i\theta})}{Re^{i\theta}}=a\int_{\theta=0}^{\theta=2\pi N}id\theta=ai\,\theta\Big|_0^{2\pi N}=2a\pi iN$$

(a) Unit Z-line-flux field f(z)=1/z



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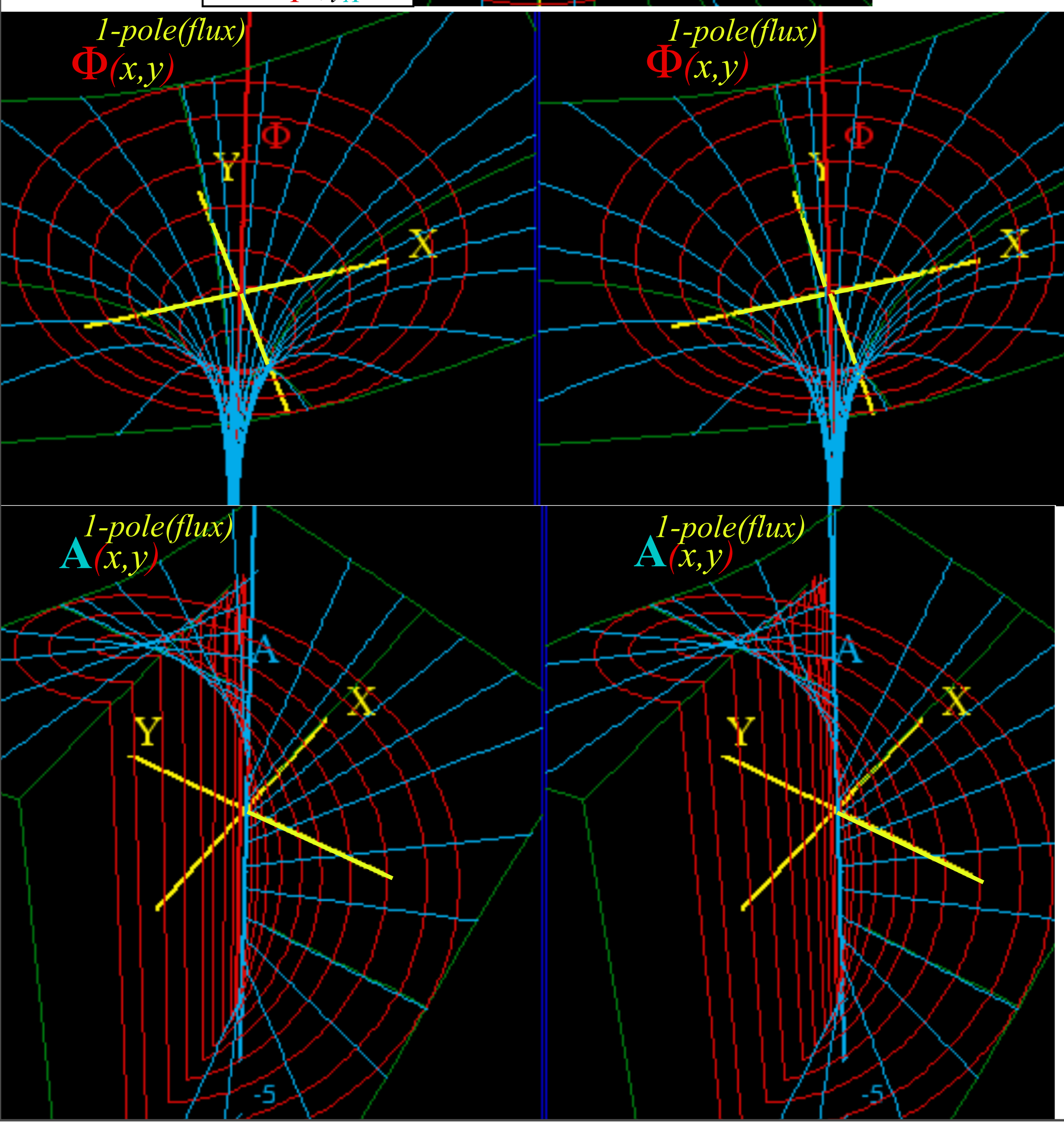
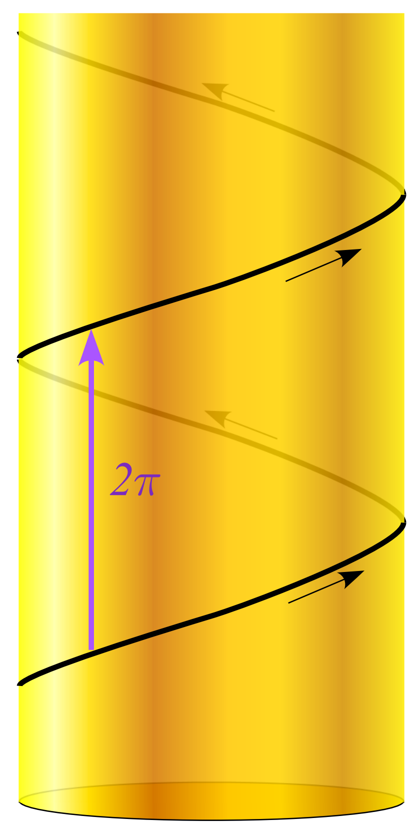


Each turn around origin
*adds 2πi to vector potential i***A**



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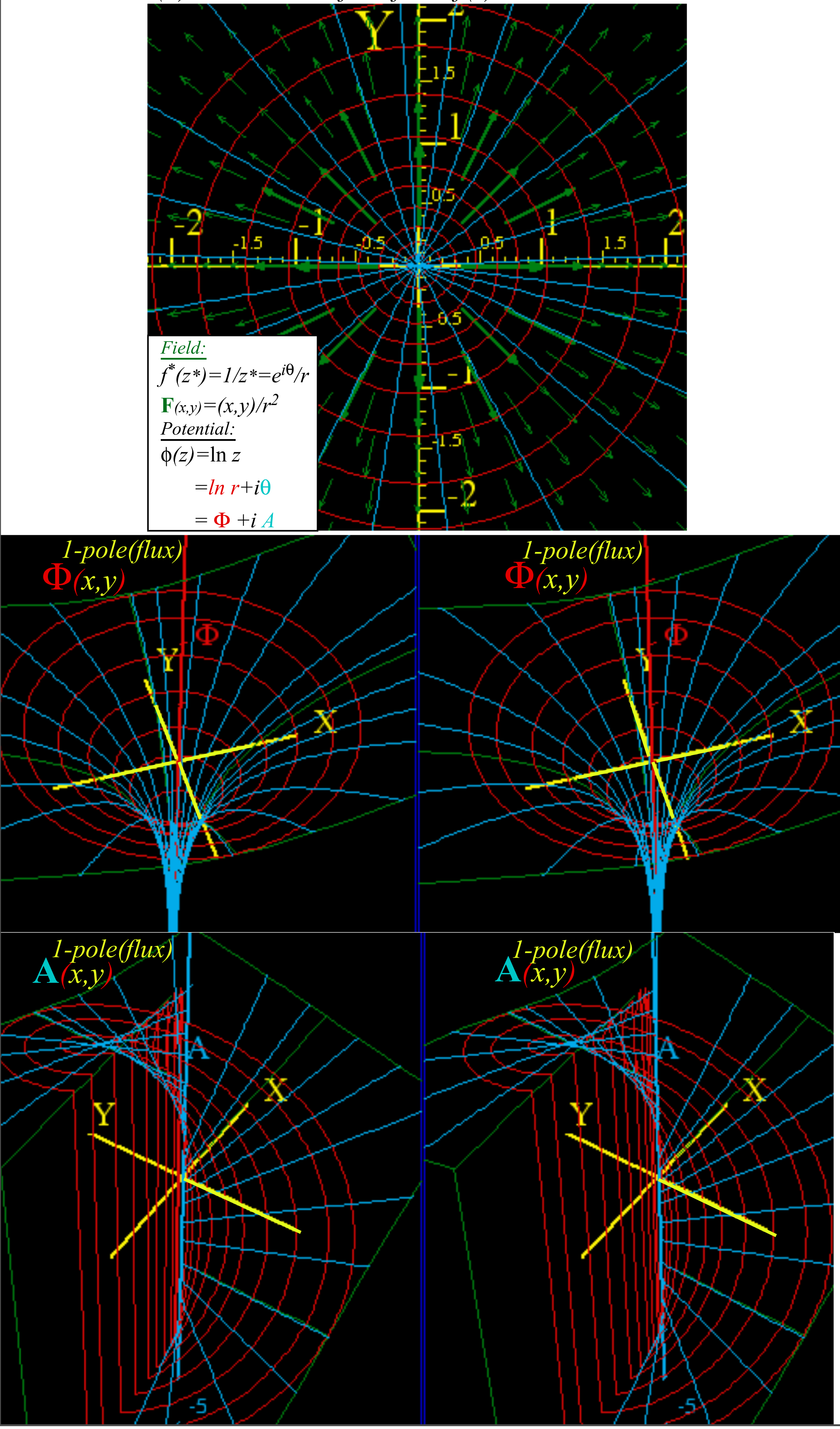

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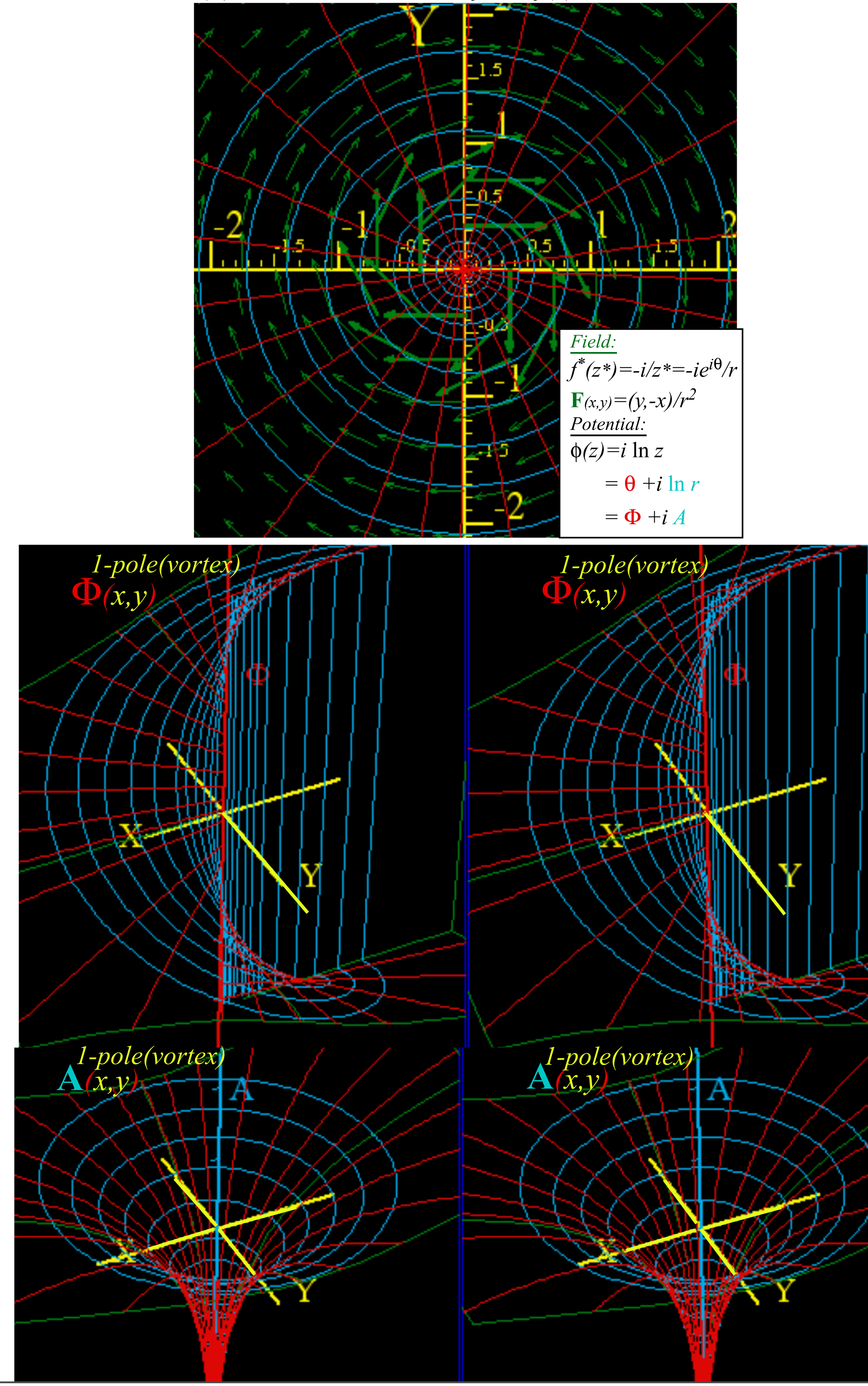
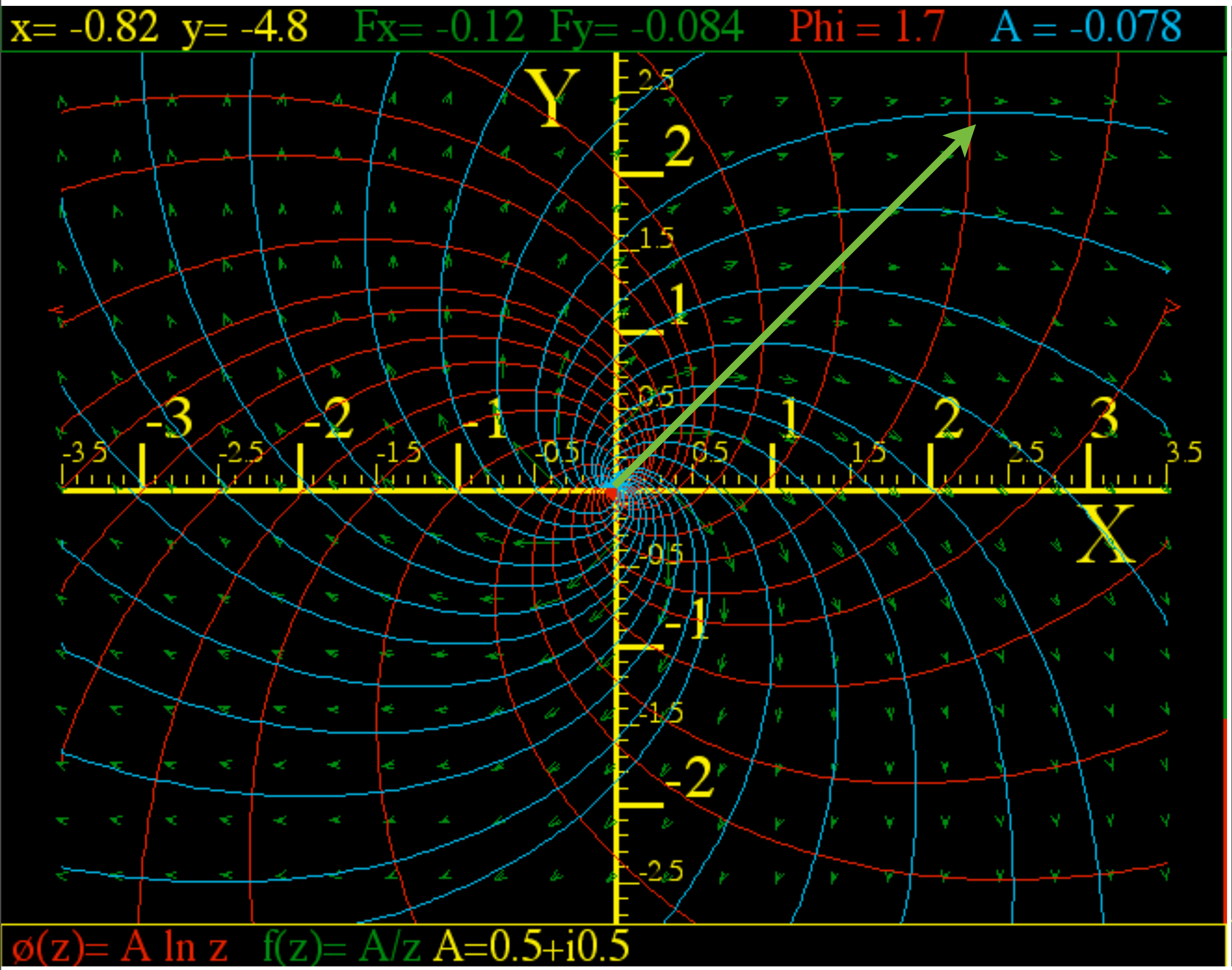


What Good Are Complex Exponentials? (contd.)

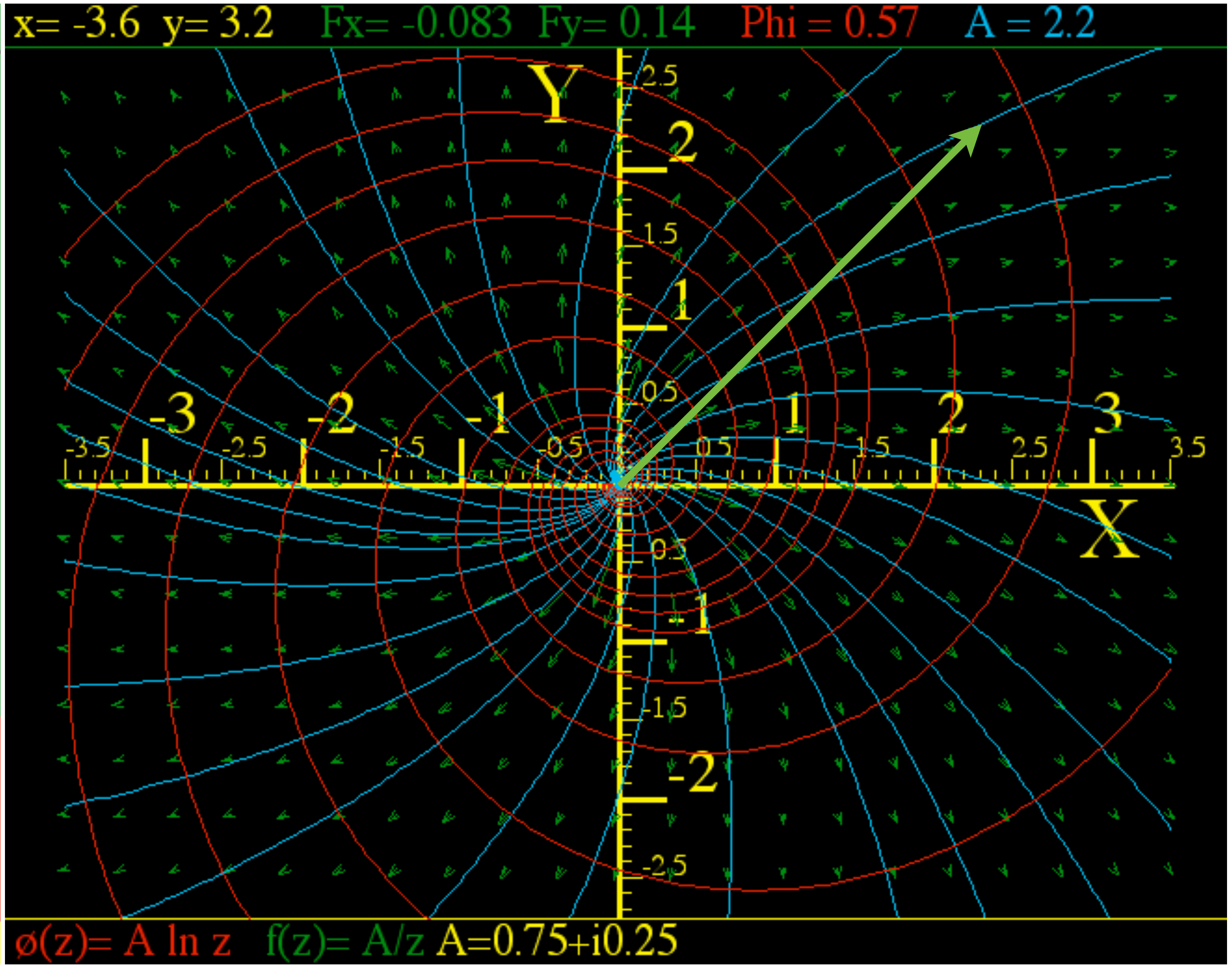
$f(z) = (0.5 + i0.5)/z = e^{i\pi/4}/z\sqrt{2}$

$f(z) = (0.75 + i0.25)/z = e^{i18^\circ}/z\sqrt{n}$

"Vortex"



"Hurricane"



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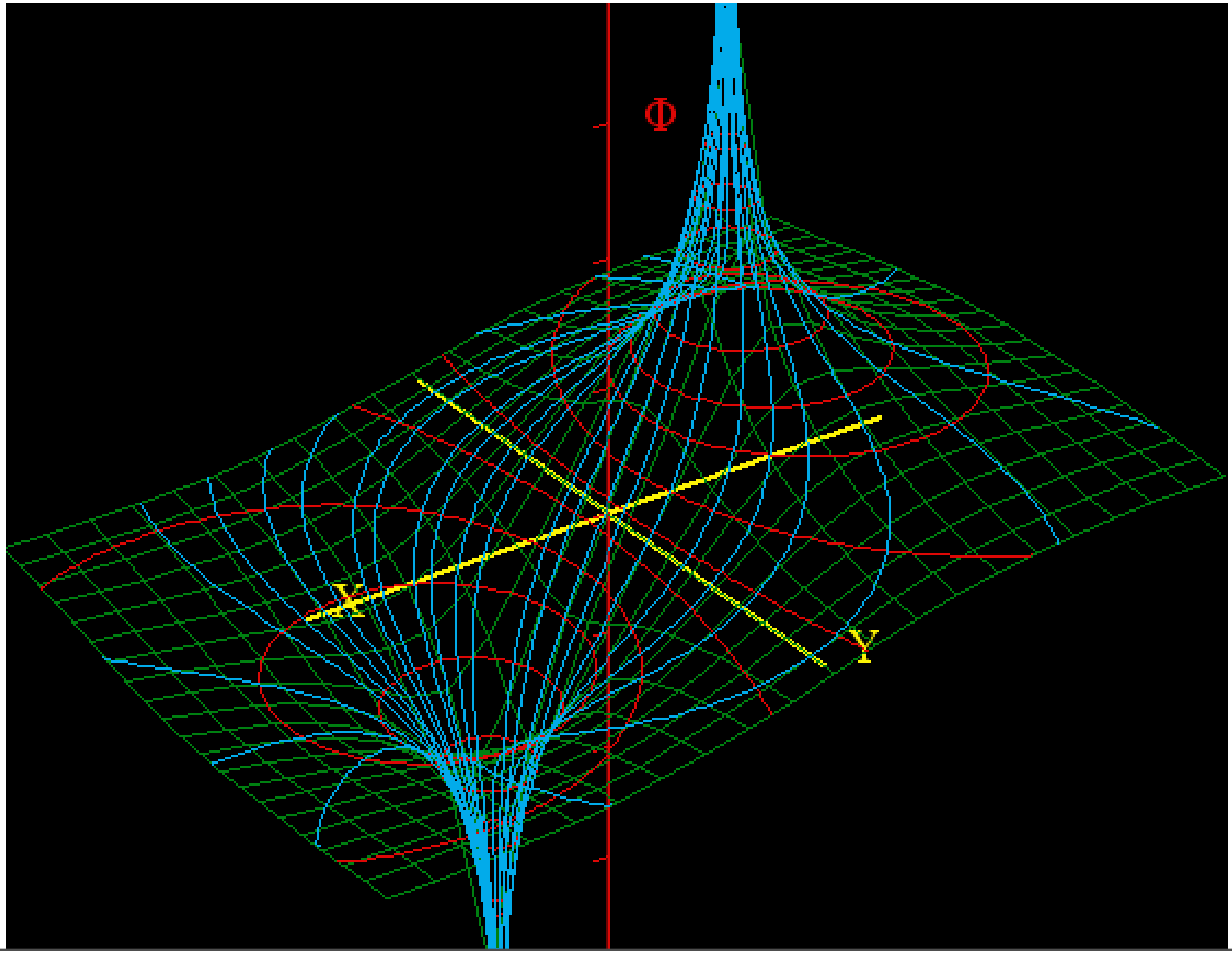
Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

Now let these two line-sources of equal but opposite source constants $+a$ and $-a$ be located at $z=\pm\Delta/2$ separated by a small interval Δ. This sum (actually difference) of $f^{1\text{-}pole}$-fields is called a *dipole field*.

$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$



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If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$

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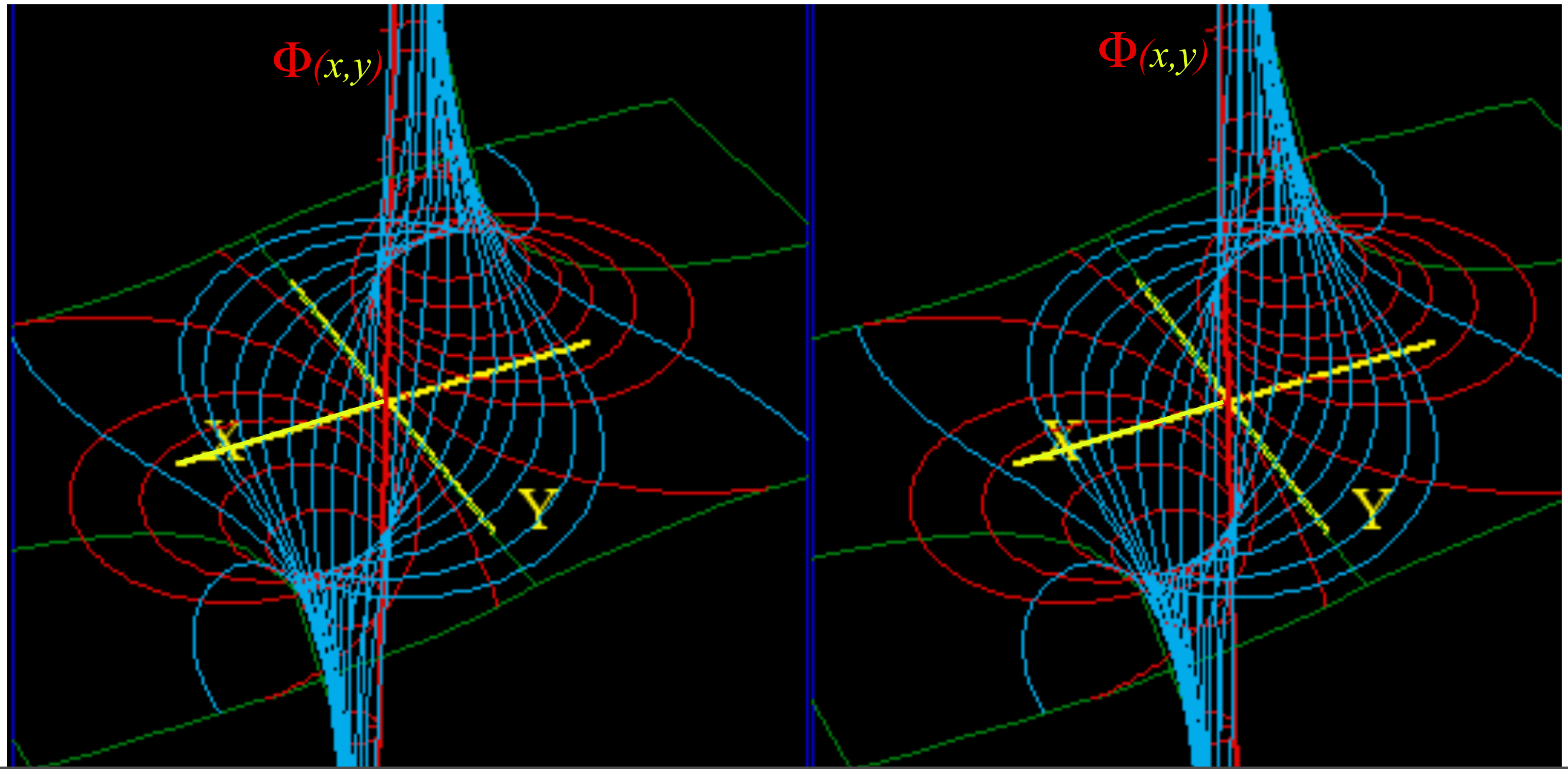
$$f^{dipole}(z)=\frac{a}{z+\frac{\Delta}{2}}-\frac{a}{z-\frac{\Delta}{2}}=\frac{-a\cdot\Delta}{z^2-\frac{\Delta^2}{4}} \qquad \phi^{dipole}(z)=a\ln(z-\tfrac{\Delta}{2})-a\ln(z+\tfrac{\Delta}{2})=a\ln\frac{z-\frac{\Delta}{2}}{z+\frac{\Delta}{2}}$$

If interval Δ is *tiny* and is divided out we get a *point-dipole field* $f^{2\text{-}pole}$ that is the z-derivative of $f^{1\text{-}pole}$.

$$f^{2\text{-}pole}=\frac{-a}{z^2}=\frac{df^{1\text{-}pole}}{dz}=\frac{d\phi^{2\text{-}pole}}{dz} \qquad \phi^{2\text{-}pole}=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz}$$



12. Complex derivatives give 2D dipole fields

Start with $f(z)=az^{-1}$: 2D line *monopole field* and is its *monopole potential* $\phi(z)=a\ln z$ of source strength a.

$$f^{1\text{-}pole}(z)=\frac{a}{z}=\frac{d\phi^{1\text{-}pole}}{dz} \qquad \phi^{1\text{-}pole}(z)=a\ln z$$

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A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose z-derivative is $f^{2\text{-}pole}$) is a z-derivative of $\phi^{1\text{-}pole}$.

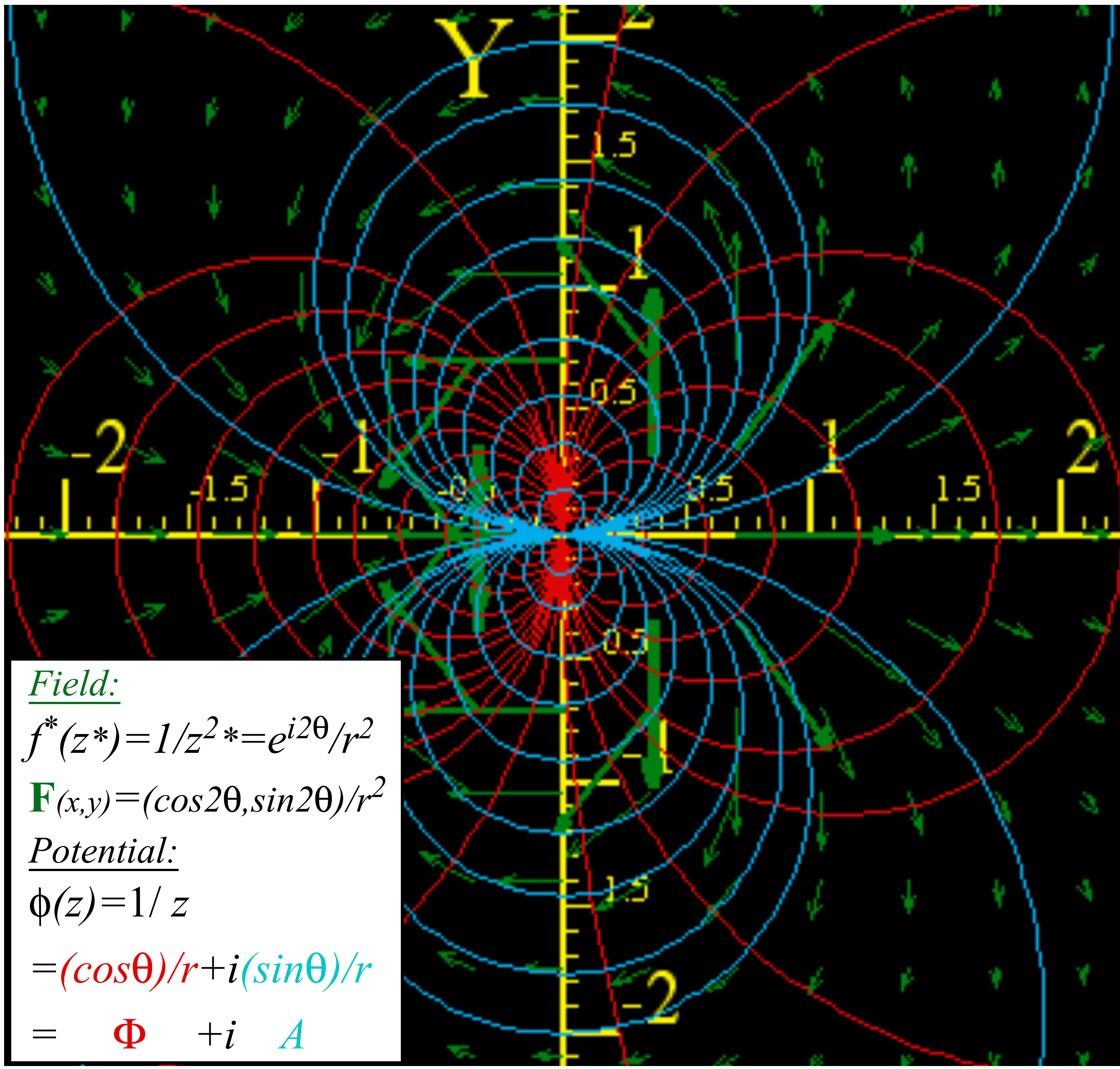
$$\phi^{2\text{-}pole}=\frac{a}{z}=\frac{a}{x+iy}=\frac{a}{x+iy}\frac{x-iy}{x-iy}=\frac{ax}{x^2+y^2}+i\frac{-ay}{x^2+y^2}=\frac{a}{r}\cos\theta-i\frac{a}{r}\sin\theta$$

$$=\Phi^{2\text{-}pole}+i\,\mathbf{A}^{2\text{-}pole}$$

A *point-dipole potential* $\phi^{2\text{-}pole}$ (whose z-derivative is $f^{2\text{-}pole}$) is a z-derivative of $\phi^{1\text{-}pole}$.

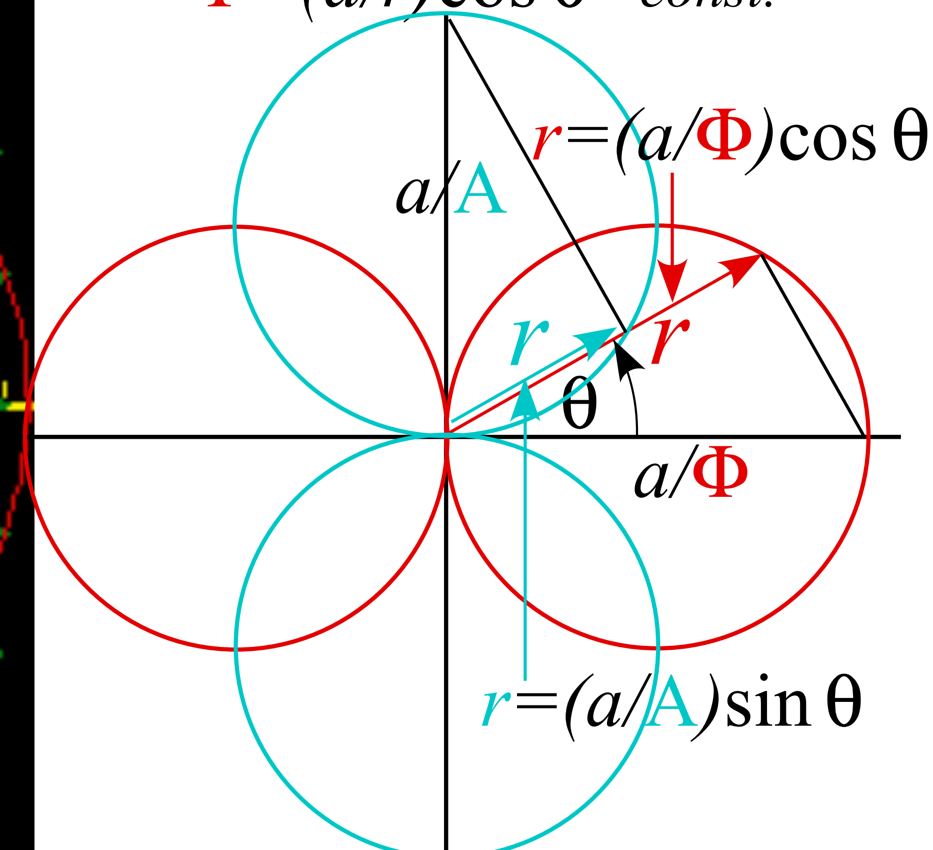
$$\phi^{2\text{-}pole} = \frac{a}{z} = \frac{a}{x+iy} = \frac{a}{x+iy}\frac{x-iy}{x-iy} = \frac{ax}{x^2+y^2} + i\frac{-ay}{x^2+y^2} = \frac{a}{r}\cos\theta - i\frac{a}{r}\sin\theta$$

$$= \Phi^{2\text{-}pole} + i\,A^{2\text{-}pole}$$



Scalar potentials

$\Phi=(a/r)\cos\theta=const.$



Vector potentials

$A=(a/r)\sin\theta=const.$

3. 2D source-field-potential-coordinate analysis

Easy 2D circulation and flux integrals

Easy 2D curvilinear coordinate discovery

→ *Easy 2D monopole, dipole, and 2^n-pole analysis*

2^n-pole analysis *(quadrupole: 2^2=4-pole, octapole: 2^3=8-pole, ..., pole dancer,*

What if we put a *(-)*copy of a *2-pole* near its original?

Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a z-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{\text{4-pole}} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{\text{2-pole}}}{dz} = \frac{d\phi^{\text{4-pole}}}{dz}$$

$$\phi^{\text{4-pole}} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{\text{2-pole}}}{dz}$$

2^n-pole analysis *(quadrupole: 2^2=4-pole, octapole: 2^3=8-pole, ..., pole dancer,*

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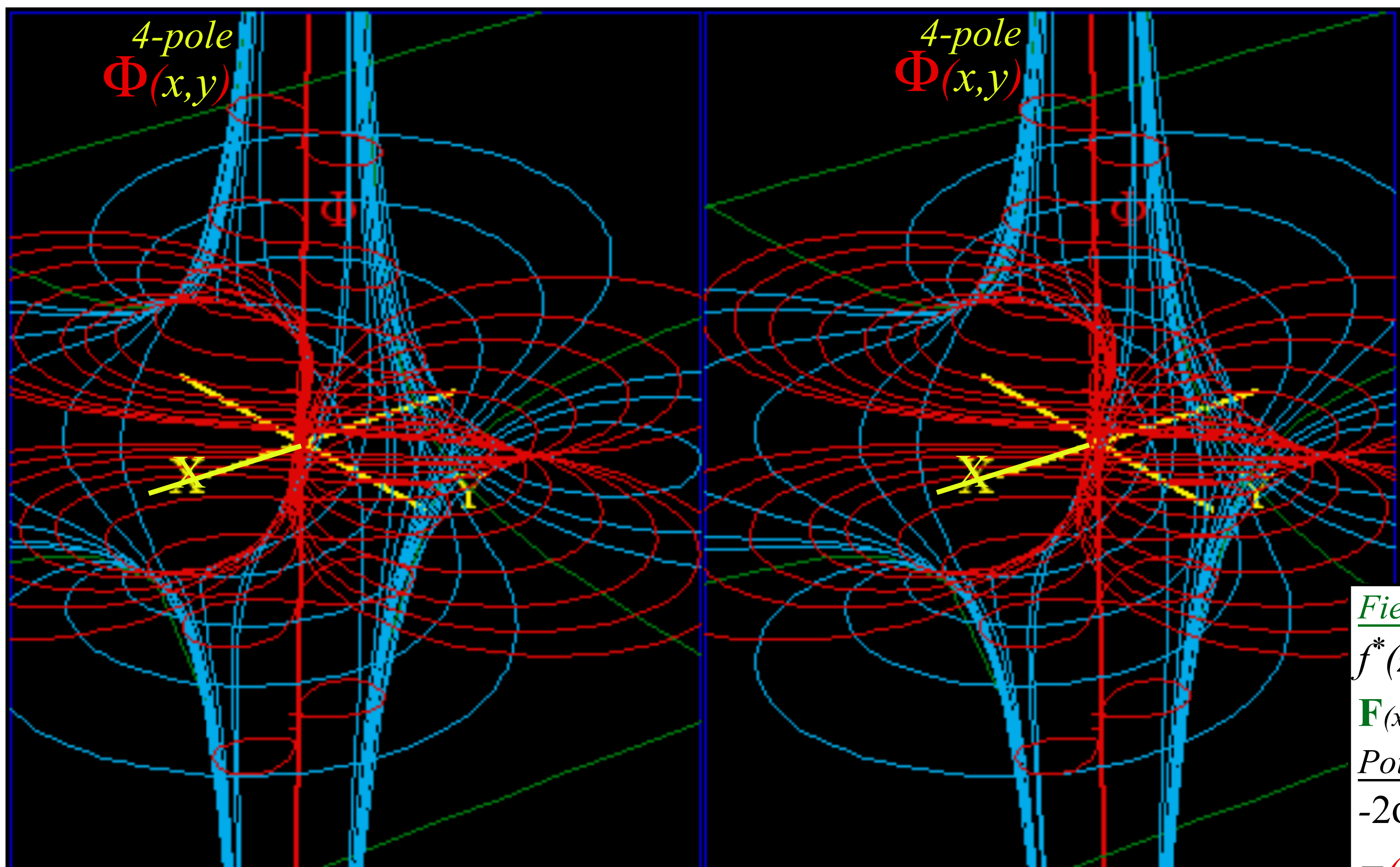
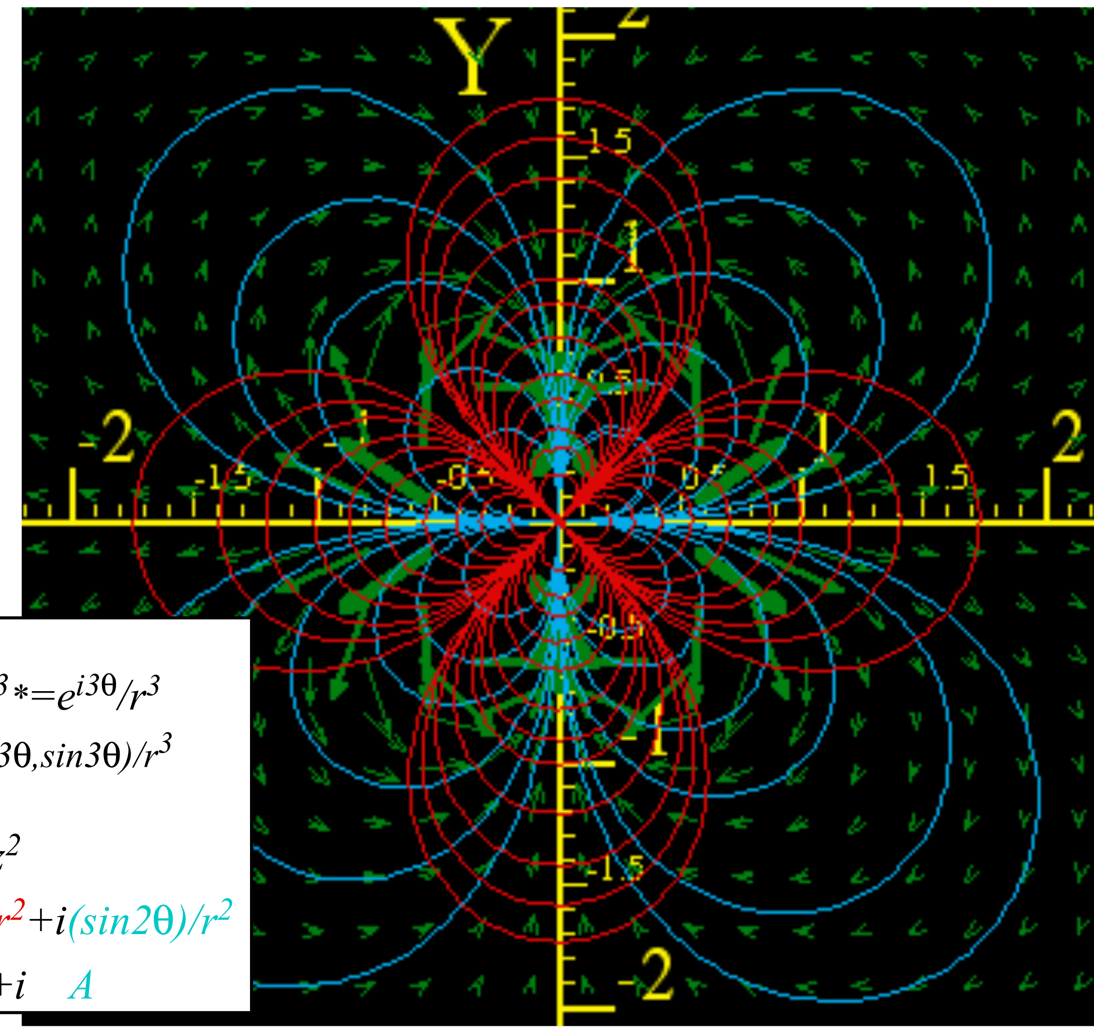
Well, the result is *4-pole* or *quadrupole* field $f^{\text{4-pole}}$ and potential $\phi^{\text{4-pole}}$.

Each a *z*-derivative of $f^{\text{2-pole}}$ and $\phi^{\text{2-pole}}$.

$$f^{4\text{-}pole} = \frac{a}{z^3} = \frac{1}{2}\frac{df^{2\text{-}pole}}{dz} = \frac{d\phi^{4\text{-}pole}}{dz} \qquad \phi^{4\text{-}pole} = -\frac{a}{2z^2} = \frac{1}{2}\frac{d\phi^{2\text{-}pole}}{dz}$$





Field:

$f^*(z^*) = 1/z^{3*} = e^{i3\theta}/r^3$

$\mathbf{F}(x,y) = (\cos 3\theta, \sin 3\theta)/r^3$

Potential:

$-2\phi(z) = 1/z^2$

$= (\cos 2\theta)/r^2 + i(\sin 2\theta)/r^2$

$= \quad \Phi \quad + i \quad A$

2^n-pole analysis: Laurent series *(Generalization of Maclaurin-Taylor series)*

Laurent series or *multipole expansion* of a given complex field function *f(z)* around *z=0*.

$$f(z) = \ldots a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + \ldots$$

$\cdots 2^2$-pole	2^1-pole	2^0-pole	2^1-pole	2^2-pole	2^3-pole	2^4-pole	2^5-pole	2^6-pole $\cdots$
at $z=0$	at $z=0$	at $z=0$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$	at $z=\infty$

$$\phi(z) = \ldots \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \frac{a_3}{4}z^4 + \frac{a_4}{5}z^5 + \frac{a_5}{6}z^6 + \ldots$$

All field terms $a_{m-1}z^{m-1}$ except *1-pole* $\frac{a_{-1}}{z}$ have potential term $a_{m-1}z^m/m$ of a *2^m-pole.*

These are located at $z=0$ for $m<0$ and at $z=\infty$ for $m>0$.

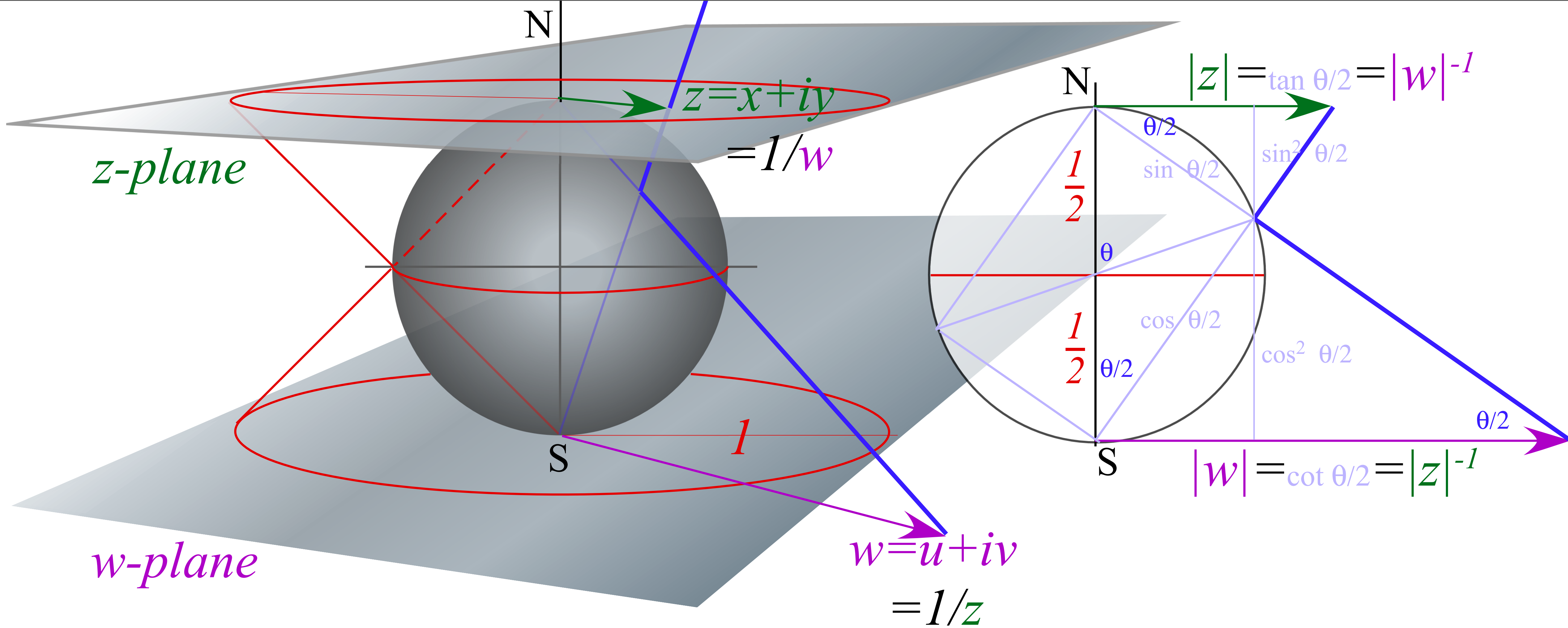
$$\phi(z) = \ldots \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-3}}{-2}z^{-2} + \frac{a_{-2}}{-1}z^{-1} + a_{-1}\ln z + a_0 z + \frac{a_1}{2}z^2 + \frac{a_2}{3}z^3 + \ldots$$

$$\phi(w) = \ldots \frac{a_{-3}}{-2}w^{-2} + \frac{a_{-3}}{-2}w^{-2} + \frac{a_{-2}}{-1}w^{-1} + a_{-1}\ln w + a_0 w + \frac{a_1}{2}w^2 + \frac{a_2}{3}w^3 + \ldots$$

(with $z=w^{-1}$)

$$= \ldots \frac{a_2}{3}z^{-2} + \frac{a_1}{2}z^{-2} + a_0 z^{-1} - a_{-1}\ln z + \frac{a_{-2}}{-1}z + \frac{a_{-3}}{-2}z^2 + \frac{a_{-3}}{-2}z^3 + \ldots$$

(with $w=z^{-1}$)



$$\phi(z) = \ldots \frac{a_{-3}}{-2} z^{-2} + \frac{a_{-3}}{-2} z^{-2} + \frac{a_{-2}}{-1} z^{-1} + a_{-1} \ln z + a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \ldots$$

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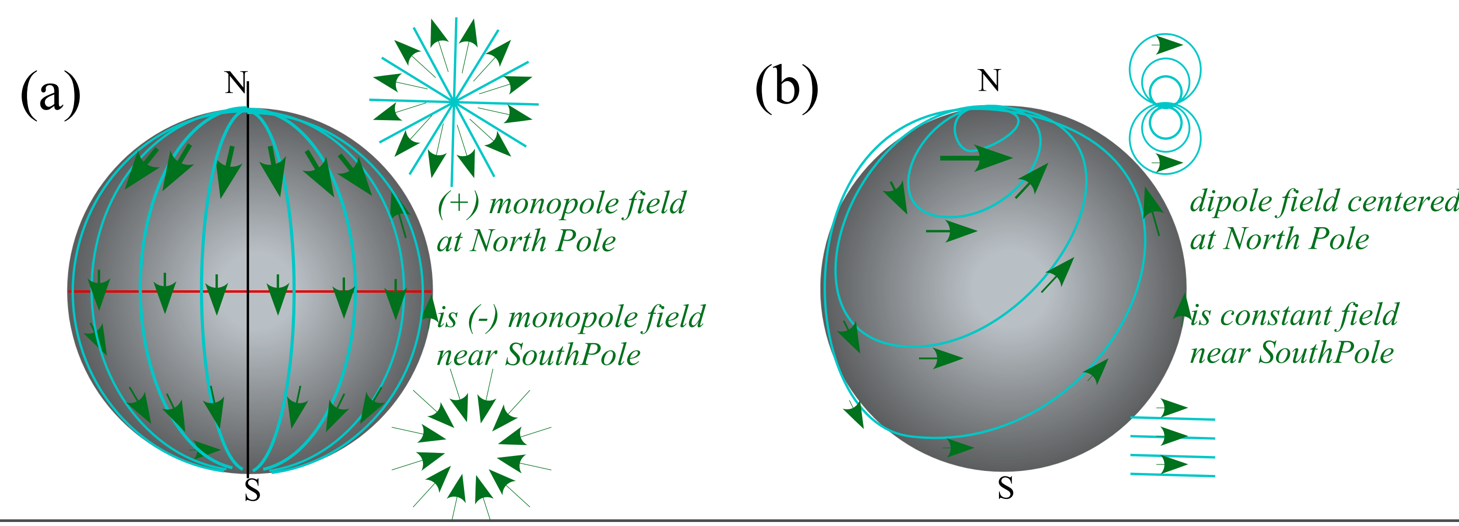

Of all 2^m-*pole* field terms $a_{m-1}z^{m-1}$, only the $m=0$ monopole $a_{-1}z^{-1}$ has a non-zero loop integral (10.39).

$$\oint f(z)dz = \oint a_{-1}z^{-1}dz = 2\pi i a_{-1} \qquad a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz$$

This $m=1$-*pole* constant-a_{-1} formula is just the first in a series of Laurent coefficient expressions.

$$\cdots a_{-3} = \tfrac{1}{2\pi i}\oint z^2 f(z)dz \ , \ a_{-2} = \tfrac{1}{2\pi i}\oint z^1 f(z)dz \ , \ a_{-1} = \tfrac{1}{2\pi i}\oint f(z)dz \ , \ a_0 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z}dz \ , \ a_1 = \tfrac{1}{2\pi i}\oint \frac{f(z)}{z^2}dz \ ,\cdots$$

Source analysis starts with 1-pole loop integrals $\oint z^{-1}dz = 2\pi i$ or, with origin shifted $\oint (z-a)^{-1}dz = 2\pi i$.

They hold for any loop about point-a. Function $f(z)$ is just $f(a)$ on a tiny circle around point-a.

$$\oint \frac{f(z)}{z-a}dz = \oint \frac{f(a)}{z-a}dz = f(a)\oint \frac{1}{z-a}dz = 2\pi i f(a) \qquad f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}dz$$

The $f(a)$ result is called a *Cauchy integral*. Then repeated a-derivatives gives a sequence of them.

$$\frac{df(a)}{da} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^2}dz \ , \ \frac{d^2 f(a)}{da^2} = \frac{2}{2\pi i}\oint \frac{f(z)}{(z-a)^3}dz \ , \ \frac{d^3 f(a)}{da^3} = \frac{3!}{2\pi i}\oint \frac{f(z)}{(z-a)^4}dz, \ \cdots, \frac{d^n f(a)}{da^n} = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz$$

This leads to a general *Taylor-Laurent* power series expansion of function $f(z)$ around point-a.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \qquad where: \ a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}dz \left(= \frac{1}{n!}\frac{d^n f(a)}{da^n} \quad for: \ n \geq 0 \right)$$

End of this Lecture