Lecture 19
Thur. 3.08.2012

Complex Variables, Series, and Field Coordinates I.

(Ch. 10 of Unit 1)

1. The Story of e (A Tale of Great $Interest$)
How good are those power series?

2. What good are complex exponentials?
Easy trig
Easy 2D vector analysis
Easy oscillator phase analysis
Easy 2D vector derivatives
Easy 2D source-free field theory
Easy 2D vector field-potential theory
The half-n'-half results: (Riemann-Cauchy Derivative Relations)

End of Part I. Lecture 19 Thur. 3.08.2012

Easy 2D circulation and flux integrals
Easy 2D curvilinear coordinate discovery
Easy 2D monopole, dipole, and 2^n-pole analysis

1. Complex numbers provide "automatic trigonometry"
2. Complex numbers add like vectors.
3. Complex exponentials $Ae^{-i\omega t}$ track position and velocity using Phasor Clock.
4. Complex products provide 2D rotation operations.
5. Complex products provide 2D "dot"(•) and "cross"(x) products.
6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field
7. Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ and $\nabla\times\mathbf{F}=0$]
8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials
9. Complex integrals $\int f(z)dz$ count 2D "circulation"($\int\mathbf{F}\cdot d\mathbf{r}$) and "flux"($\int\mathbf{F}\times d\mathbf{r}$)
10. Complex integrals define 2D monopole fields and potentials
11. Complex potentials define 2D Orthogonal Curvilinear Coordinates (OCC) of field
12. Complex derivatives give 2D dipole fields

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$$p^{\frac{1}{2}}(t)=(1+r\cdot\tfrac{t}{2})p(\tfrac{t}{2})=(1+r\cdot\tfrac{t}{2})\cdot(1+r\cdot\tfrac{t}{2})p(0)=\tfrac{3}{2}\cdot\tfrac{3}{2}\cdot 1=\tfrac{9}{4}=2.25$$

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Trimester compounded interest gives $p(\frac{t}{3})=(1+r\cdot\frac{t}{3})p(0)$ at the *1/3*rd-period $\frac{t}{3}$ or 1st trimester and then use that to figure the 2nd trimester and so on. Now \$1.00 at rate $r=1$ earns \$2.37.

$$p^{\frac{1}{3}}(t)=(1+r\cdot\tfrac{t}{3})p(2\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})p(\tfrac{t}{3})=(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})\cdot(1+r\cdot\tfrac{t}{3})p(0)=\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{3}\cdot 1=\tfrac{64}{27}=2.37$$

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So if you compound interest more and more frequently, do you approach **INFININTEREST***?*

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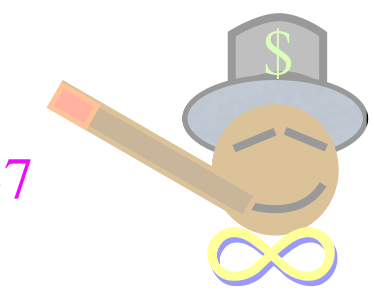

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+25¢

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$$p^{\frac{1}{4}}(t)=(1+r\cdot\tfrac{t}{4})^4 p(0)=\left(\tfrac{5}{4}\right)^4\cdot 1=\tfrac{625}{256}=2.44$$

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Monthly: $p^{\frac{1}{12}}(t)=(1+r\cdot\frac{t}{12})^{12}p(0)=\left(\frac{13}{12}\right)^{12}\cdot 1=2.613$

Weekly: $p^{\frac{1}{52}}(t)=(1+r\cdot\frac{t}{52})^{52}p(0)=\left(\frac{53}{52}\right)^{52}\cdot 1=2.693$

Daily: $p^{\frac{1}{365}}(t)=(1+r\cdot\frac{t}{365})^{365}p(0)=\left(\frac{366}{365}\right)^{365}\cdot 1=\mathbf{2.71}45$

Hrly: $p^{\frac{1}{8760}}(t)=(1+r\cdot\frac{t}{8760})^{8760}p(0)=\left(\frac{8761}{8760}\right)^{8760}\cdot 1=\mathbf{2.718}1$

Interest product formula is <u>really</u> inefficient: 10^6 products for 6-figures! .. .10^9 products for 9 ...

$$p^{1/m}(1) = (1+\tfrac{1}{m})^m \xrightarrow[m\to\infty]{} \mathbf{2.718281828459..} = e$$

Let: $m \cdot r \cdot t = n$

or: $1/m = r \cdot t/n$

$$(1+\tfrac{1}{m})^{m \cdot r \cdot t} \xrightarrow[m\to\infty]{} e^{r \cdot t}$$

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$p^{1/m}(1) = \mathbf{2.71}69239322$	*for m = 1,000*
$p^{1/m}(1) = \mathbf{2.7181}459268$	*for m = 10,000*
$p^{1/m}(1) = \mathbf{2.7182}682372$	*for m = 100,000*
$p^{1/m}(1) = \mathbf{2.71828}04693$	*for m = 1,000,000*
$p^{1/m}(1) = \mathbf{2.7182816}925$	*for m = 10,000,000*
$p^{1/m}(1) = \mathbf{2.71828181}49$	*for m = 100,000,000*
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$$(x+y)^n = x^n + n\cdot x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots + n\cdot xy^{n-1} + y^n$$

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Define: *Factorials*(!):

$0! = 1 = 1!,\quad 2! = 1\cdot 2,\quad 3! = 1\cdot 2\cdot 3, \ldots$

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As $n\to\infty$ *let* :

$$n(n-1)\to n^2,$$

$$n(n-1)(n-2)\to n^3, \textit{etc}.$$

$$\boxed{e^{r\cdot t} = 1 + r\cdot t + \frac{1}{2!}(r\cdot t)^2 + \frac{1}{3!}(r\cdot t)^3 + \ldots = \sum_{p=0}^{o}\frac{(r\cdot t)^p}{p!}}$$

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Precision order:

(o=1)-e-series = **2.**00000 =1+1
(o=2)-e-series = **2.**50000 =1+1+1/2
(o=3)-e-series = **2.**66667 =1+1+1/2+1/6
(o=4)-e-series = **2.7**0833 =1+1+1/2+1/6+1/24
(o=5)-e-series = **2.71**667 =1+1+1/2+1/6+1/24+1/120
(o=6)-e-series = **2.718**05 =1+1+1/2+1/6+1/24+1/120+1/720
(o=7)-e-series = **2.7182**5
(o=8)-e-series = **2.71828**

About 12 summed quotients
for 6-figure precision (A lot better!)

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.* Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

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Rate of change of position $x(t)$ is *velocity $v(t)$*.

Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + ... + nc_n t^{n-1} +$$

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Change of velocity $v(t)$ is *acceleration $a(t)$*.

Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2\cdot 3c_3 t + 3\cdot 4c_4 t^2 + 4\cdot 5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$

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$$a(t) = \frac{d}{dt} v(t) = 0 + 2c_2 + 2{\cdot}3c_3 t + 3{\cdot}4c_4 t^2 + 4{\cdot}5c_5 t^3 + \ldots + n(n-1)c_n t^{n-2} +$$

Change of acceleration $a(t)$ is *jerk* $j(t)$. (*Jerk* is NASA term.)

Set $t=0$ to get $c_3 = \frac{1}{3!} j(0)$.

$$j(t) = \frac{d}{dt} a(t) = 0 + 2{\cdot}3c_3 + 2{\cdot}3{\cdot}4c_4 t + 3{\cdot}4{\cdot}5c_5 t^2 + \ldots + n(n-1)(n-2)c_n t^{n-3} +$$

Change of jerk $j(t)$ is *inauguration* $i(t)$. (Be silly like NASA!)

Set $t=0$ to get $c_4 = \frac{1}{4!} i(0)$.

$$i(t) = \frac{d}{dt} j(t) = 0 + 2{\cdot}3{\cdot}4c_4 + 2{\cdot}3{\cdot}4{\cdot}5c_5 t + \ldots + n(n-1)(n-2)(n-3)c_n t^{n-4} +$$

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.* Set $t=0$ to get $c_0 = x(0)$.

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + \ldots + c_n t^n +$$

Rate of change of position $x(t)$ is *velocity* $v(t)$. Set $t=0$ to get $c_1 = v(0)$.

$$v(t) = \frac{d}{dt} x(t) = 0 + c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + 5c_5 t^4 + \ldots + nc_n t^{n-1} +$$

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Gives Maclaurin (or Taylor) power series

$$\boxed{x(t) = x(0) + v(0)t + \frac{1}{2!} a(0)t^2 + \frac{1}{3!} j(0)t^3 + \frac{1}{4!} i(0)t^4 + \frac{1}{5!} r(0)t^5 + \ldots + \frac{1}{n!} x^{(n)} t^n +}$$

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Change of velocity $v(t)$ is *acceleration $a(t)$*.

Set $t=0$ to get $c_2 = \frac{1}{2} a(0)$.

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Good old UP I formula!

Power Series Good! *Need general power series development*

Start with a general power series with constant coefficients c_0, c_1, *etc.* Set $t=0$ to get $c_0 = x(0)$.

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$$\boxed{\boxed{x(t) = x(0) + v(0)t + \tfrac{1}{2!} a(0)t^2} + \tfrac{1}{3!} j(0)t^3 + \tfrac{1}{4!} i(0)t^4 + \tfrac{1}{5!} r(0)t^5 + \ldots + \tfrac{1}{n!} x^{(n)} t^n +}$$

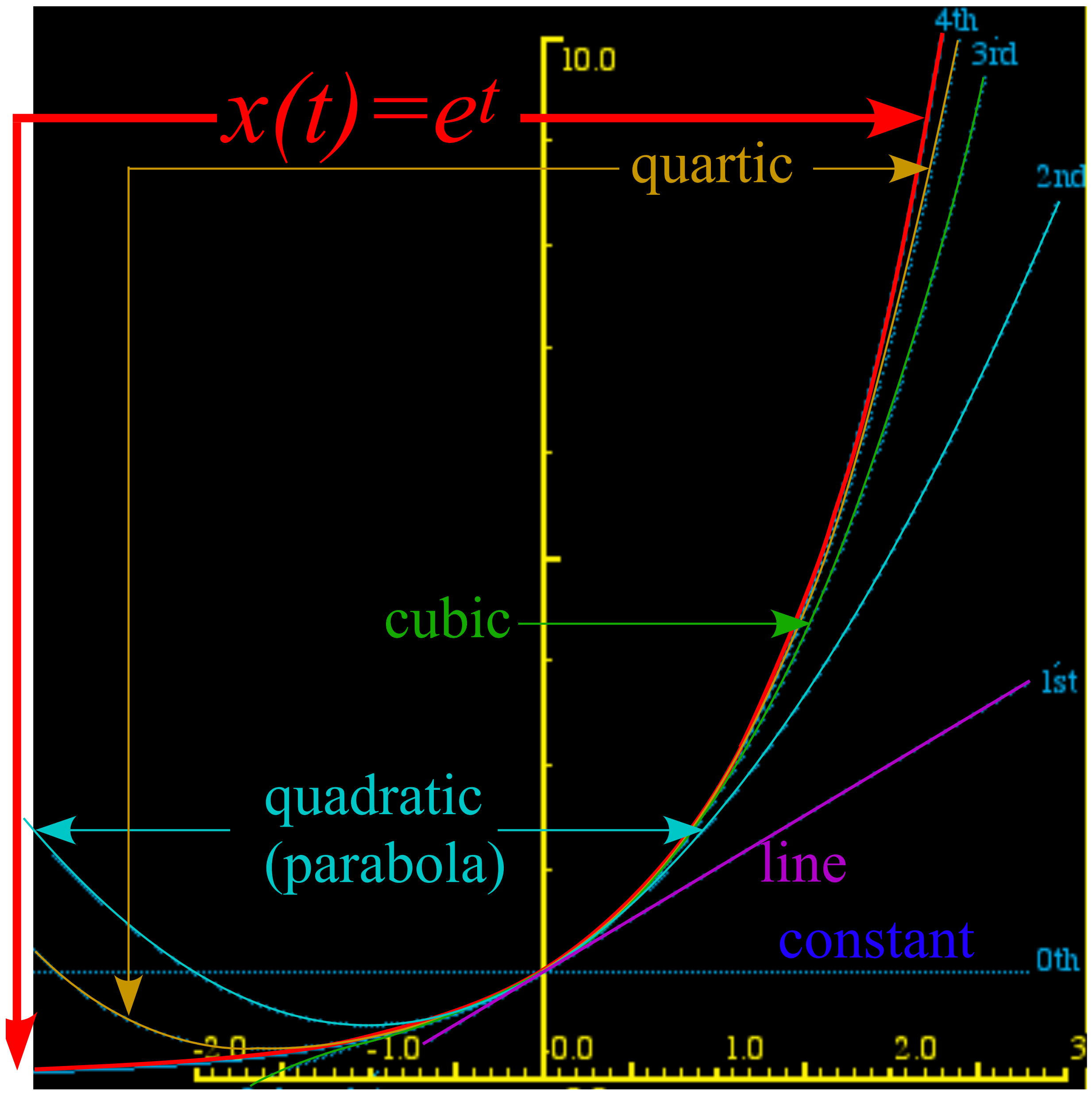
Setting all initial values to $1 = x(0) = v(0) = a(0) = j(0) = i(0) = \ldots$

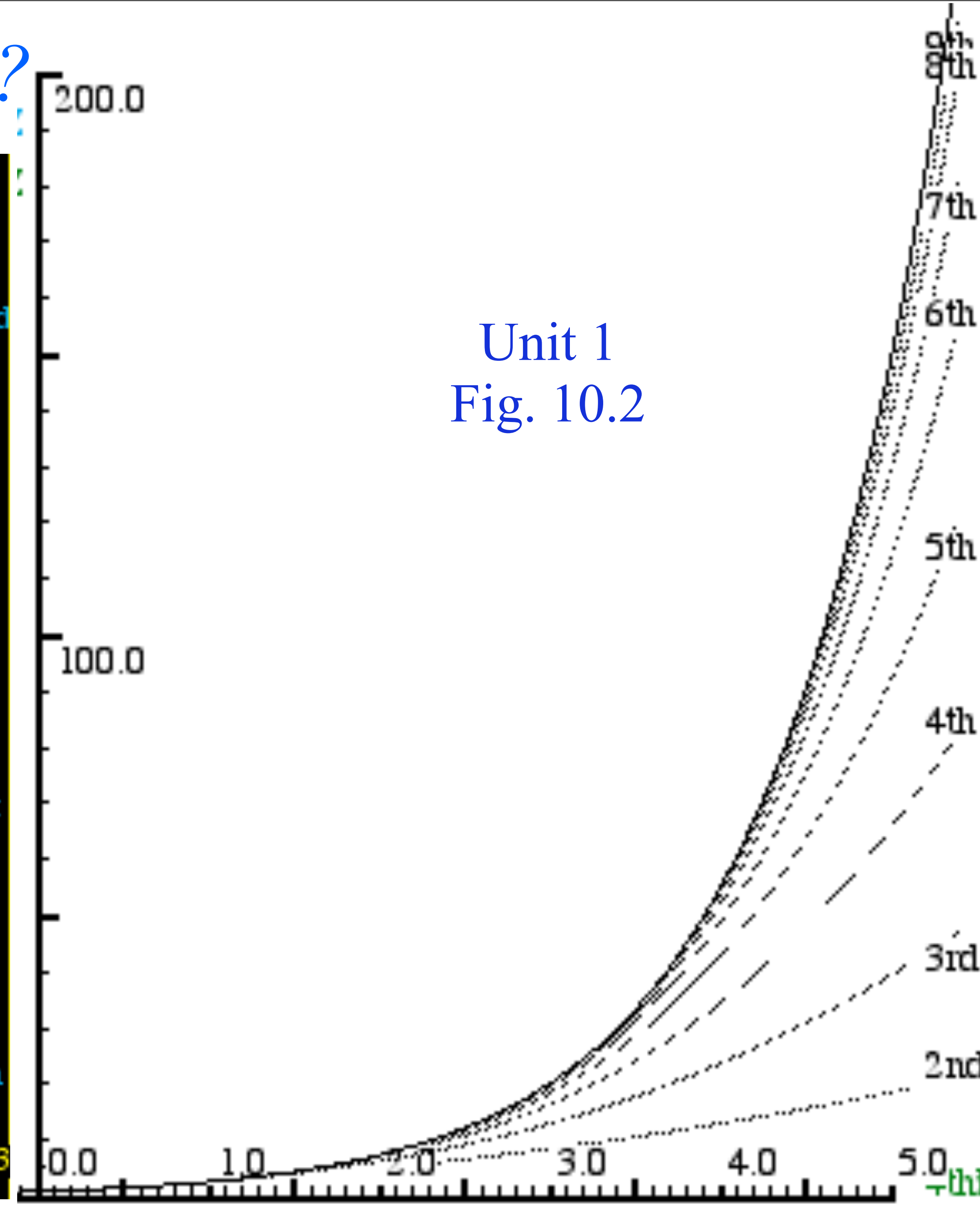
Good old UP I formula!

gives *exponential:*

$$e^t = 1 + t + \tfrac{1}{2!} t^2 + \tfrac{1}{3!} t^3 + \tfrac{1}{4!} t^4 + \tfrac{1}{5!} t^5 + \ldots + \tfrac{1}{n!} t^n +$$

But, how good are power series?





Unit 1
Fig. 10.2

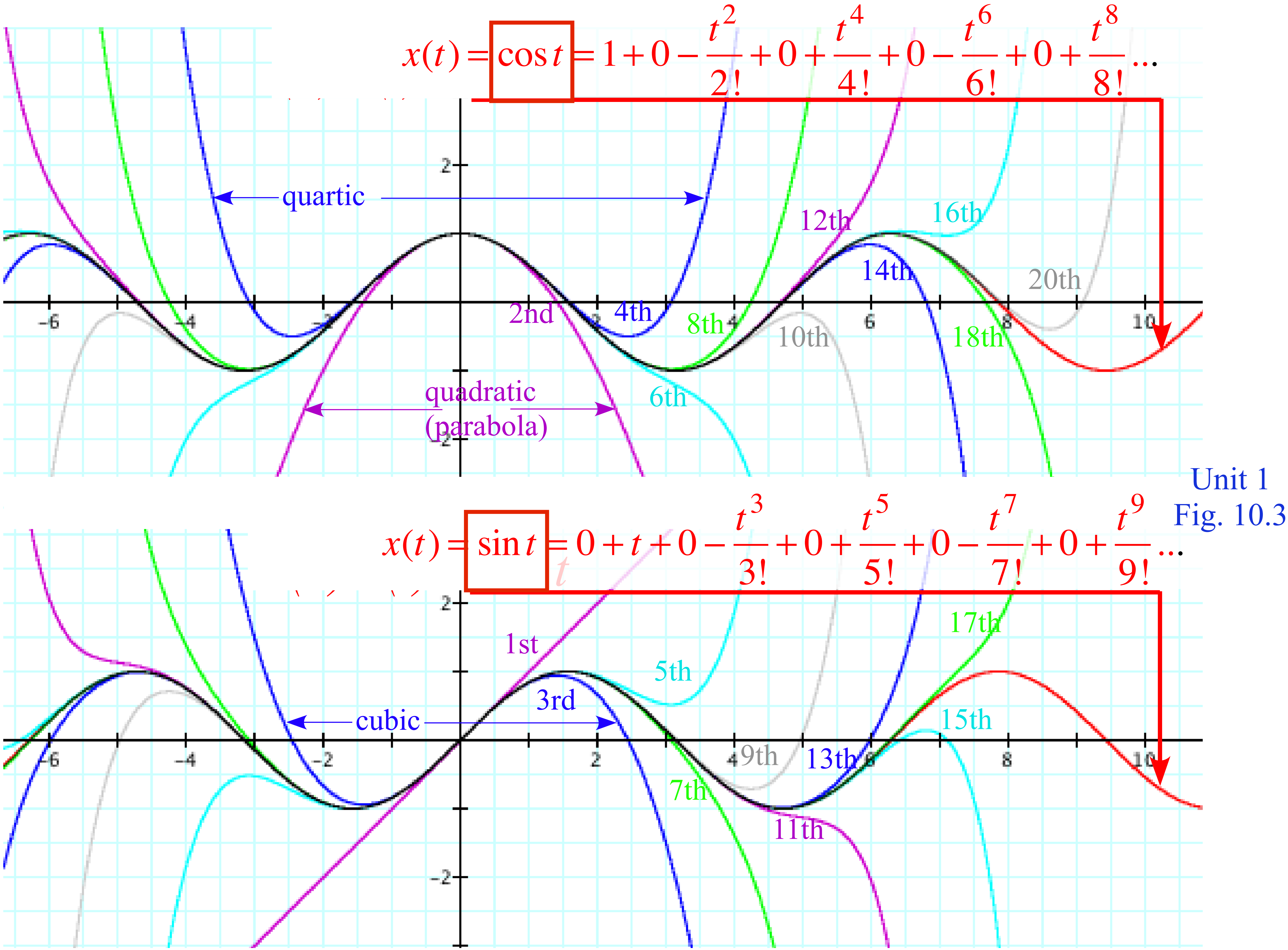
Gives Maclaurin (or Taylor) power series

$$x(t) = x(0) + v(0)t + \tfrac{1}{2!}\, a(0)t^2 + \tfrac{1}{3!}\, j(0)t^3 + \tfrac{1}{4!}\, i(0)t^4 + \tfrac{1}{5!}\, r(0)t^5 + \ldots + \tfrac{1}{n!}\, x^{(n)}t^n +$$

Setting all initial values to *1= x(0) = v(0) = a(0) = j(0) = i(0)* = ...

gives *exponential:* $$e^t = 1 + t + \tfrac{1}{2!}\, t^2 + \tfrac{1}{3!}\, t^3 + \tfrac{1}{4!}\, t^4 + \tfrac{1}{5!}\, t^5 + \ldots + \tfrac{1}{n!}\, t^n +$$

How good are power series? Depends...



Suppose the fancy bankers really went bonkers and made interest rate *r* an *imaginary number* $r = i\theta$.

Imaginary number $i = \sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1, etc...$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + ... \qquad \text{(From exponential series)}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - ... \qquad (i = \sqrt{-1} \text{ imples: } i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = +1,\ i^5 = i, ...)$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - ...\right)$$

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1$*,etc...*

$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots$$ (From exponential series)

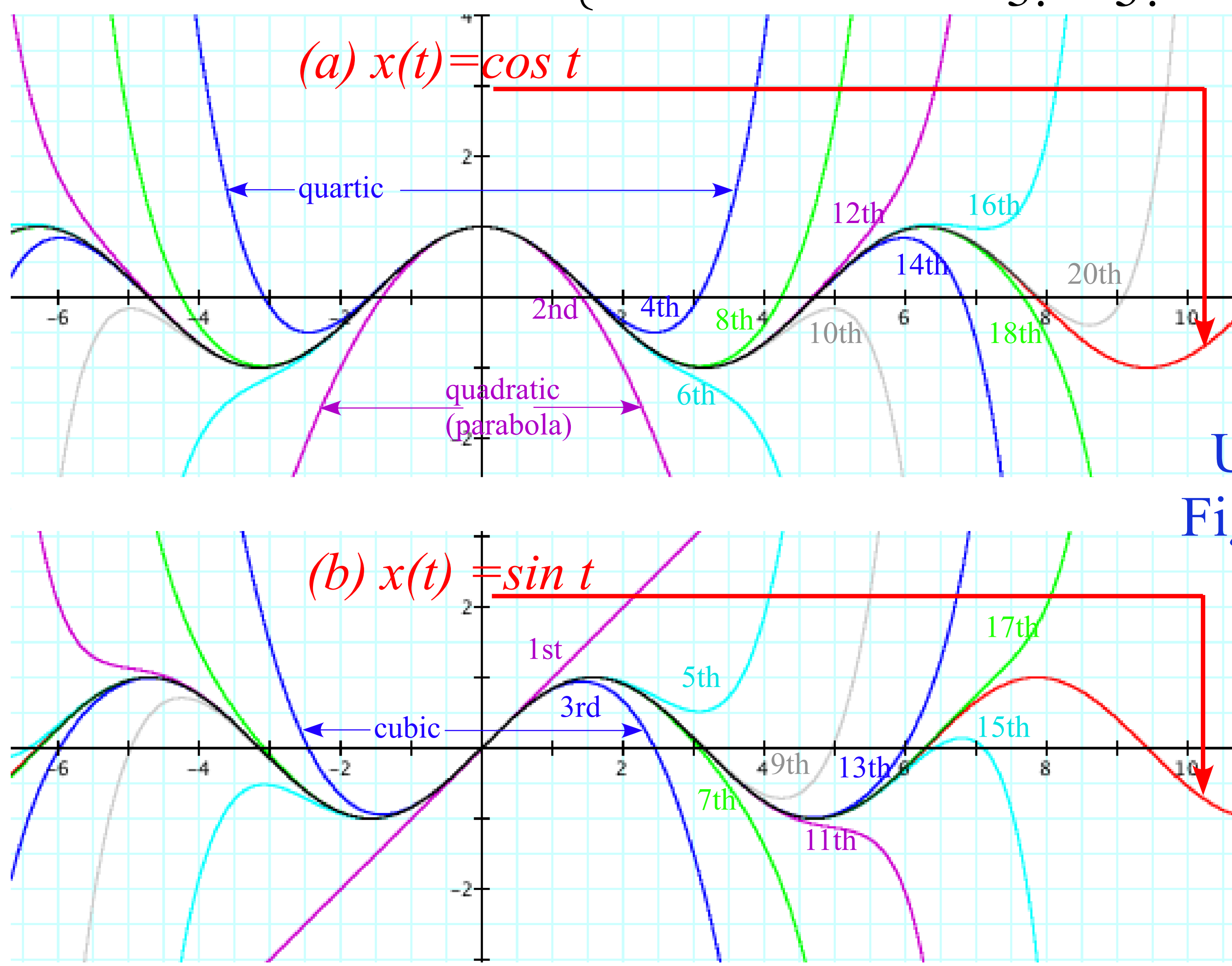
$$=1+i\theta-\frac{\theta^2}{2!}\quad -i\frac{\theta^3}{3!}\quad +\frac{\theta^4}{4!}\quad +i\frac{\theta^5}{5!}\quad -\ldots$$ ($i=\sqrt{-1}$ imples: $i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots$)

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right)$$ To match series for

$$\begin{cases} \textit{cosine}: \cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots \\ \textit{sine}: \sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots \end{cases}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Euler-DeMoivre Theorem



Unit 1
Fig. 10.3

Suppose the fancy bankers really went bonkers and made interest rate r an *imaginary number* $r=i\theta$.

Imaginary number $i=\sqrt{-1}$ powers have *repeat-after-4-pattern:* $i^0=1,\ i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=1, etc...$

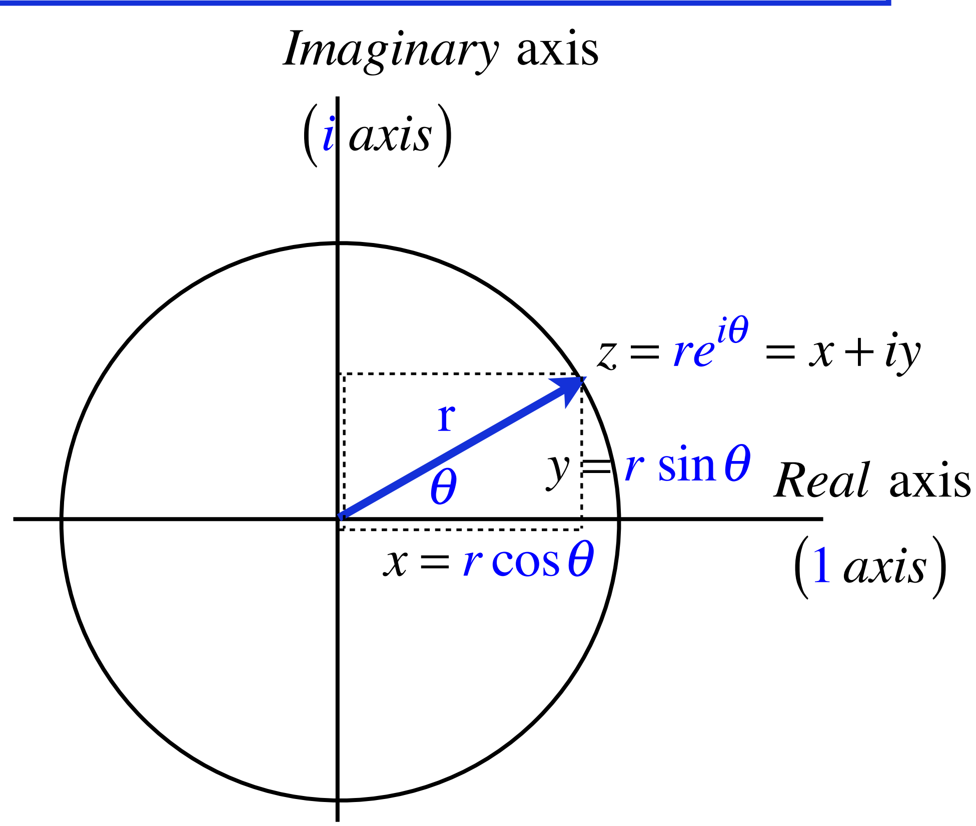
$$e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\frac{(i\theta)^4}{4!}+\frac{(i\theta)^5}{5!}+\ldots \qquad \text{(From exponential series)}$$

$$=1+i\theta-\frac{\theta^2}{2!}\quad -i\frac{\theta^3}{3!}\quad +\frac{\theta^4}{4!}\quad +i\frac{\theta^5}{5!}\quad -\ldots \qquad (i=\sqrt{-1}\text{ imples: } i^1=i,\ i^2=-1,\ i^3=-i,\ i^4=+1,\ i^5=i,\ldots)$$

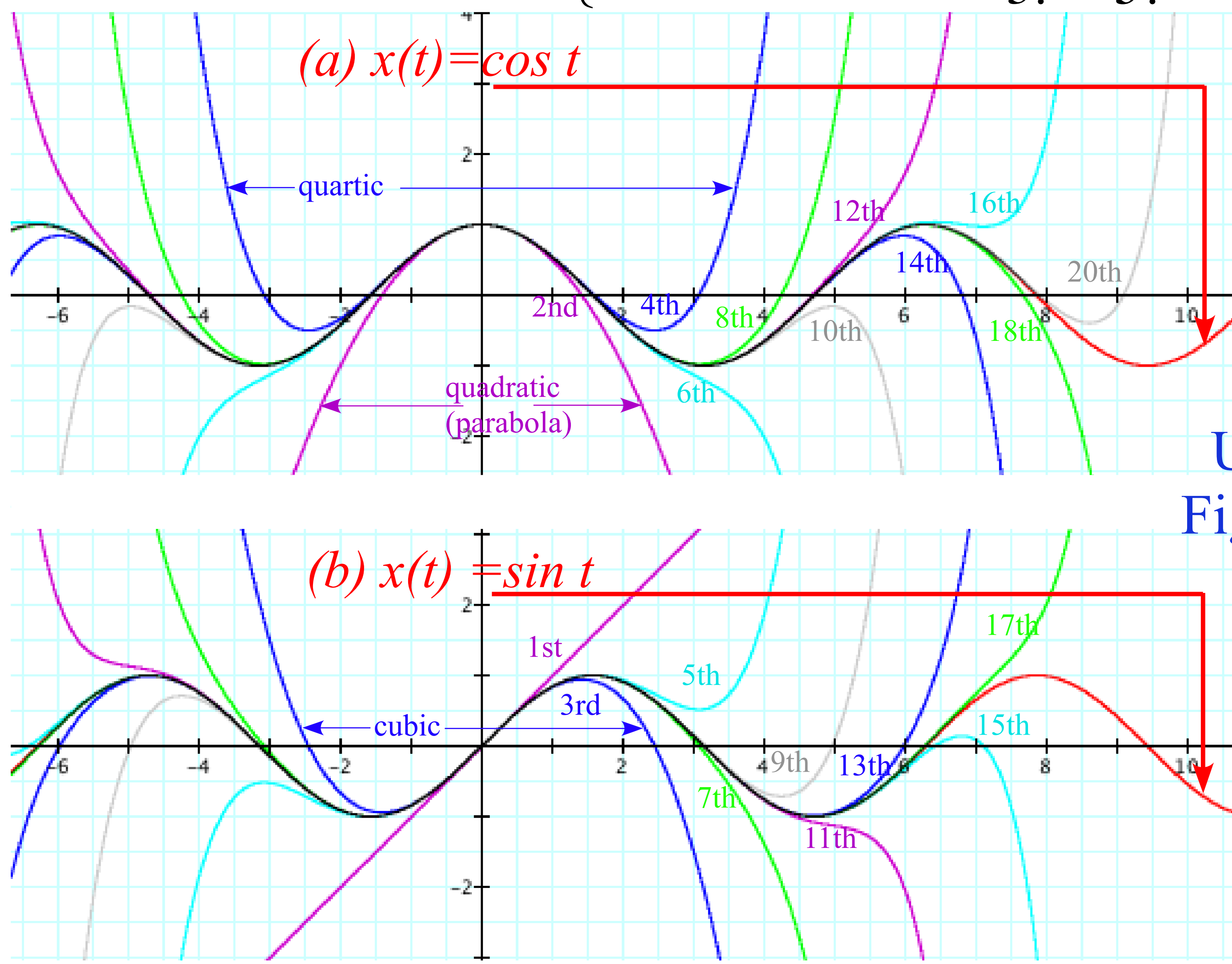
$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+\left(i\theta-i\frac{\theta^3}{3!}+i\frac{\theta^5}{5!}-\ldots\right) \qquad \text{To match series for } \begin{cases} cosine: \cos x = 1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots \\ sine: \sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\cdots \end{cases}$$

$$\boxed{e^{i\theta} \quad = \quad \cos\theta \quad + \quad i\sin\theta}$$

Euler-DeMoivre Theorem



$$re^{i\theta}=r\cos\theta+i\sin\theta$$



Unit 1
Fig. 10.3

2. What Good Are Complex Exponentials?

What Good Are Complex Exponentials?

1. Complex numbers provide "automatic trigonometry"

Can't remember is cos$(a+b)$ *or* sin$(a+b)$? Just factor $e^{i(a+b)} = e^{ia}e^{ib}$...

$$e^{i(a+b)} = e^{ia} \quad e^{ib}$$

$$\cos(a+b) + i\sin(a+b) = (\cos a + i\sin a)(\cos b + i\sin b)$$

$$\boxed{\cos(a+b)} + i\,\boxed{\sin(a+b)} = \boxed{[\cos a\cos b - \sin a\sin b]} + i\,\boxed{[\sin a\cos b + \cos a\sin b]}$$

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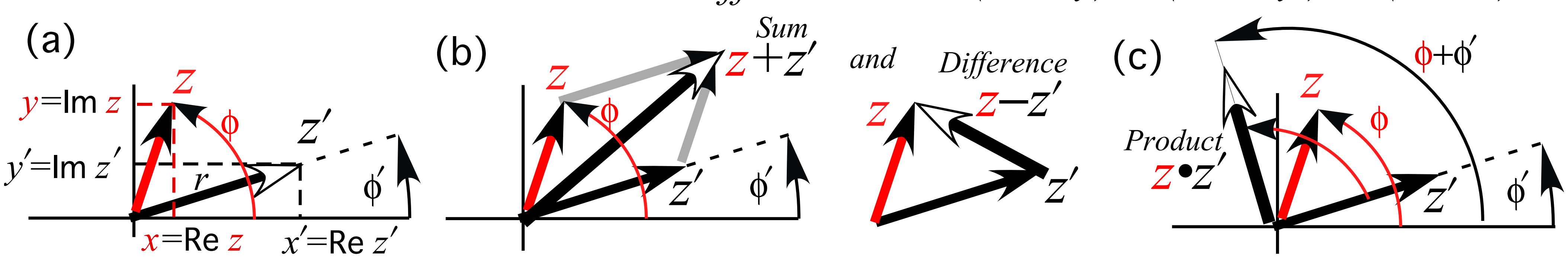
2. Complex numbers add like vectors.

$$zsum = z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y')$$

$$zdiff = z - z' = (x + iy) - (x' + iy') = (x - x') + i(y - y')$$



Unit 1
Fig. 10.6

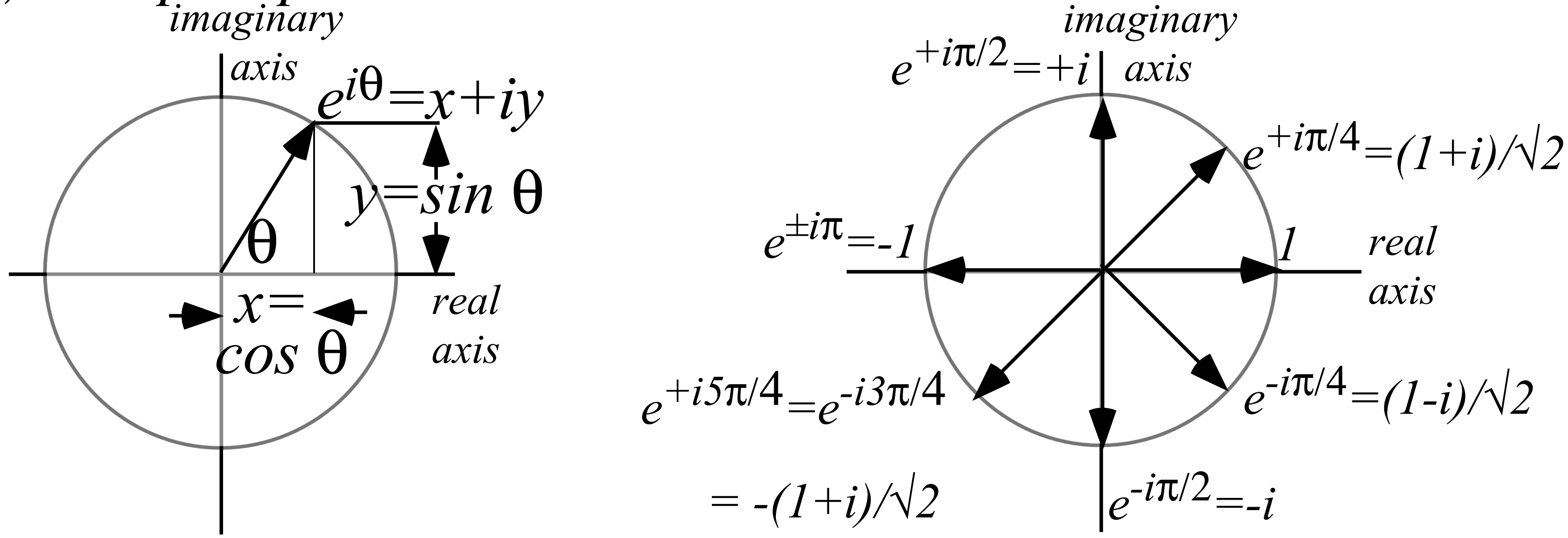
$$|z_{SUM}| = \sqrt{(z+z')^*(z+z')} = \sqrt{\left(re^{i\phi} + r'e^{i\phi'}\right)^*\left(re^{i\phi} + r'e^{i\phi'}\right)} = \sqrt{\left(re^{-i\phi} + r'e^{-i\phi'}\right)\left(re^{i\phi} + r'e^{i\phi'}\right)}$$

$$= \sqrt{r^2 + r'^2 + rr'\left(e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}\right)} = \sqrt{r^2 + r'^2 + 2rr'\cos(\phi-\phi')}$$

(quick derivation of Cosine Law)

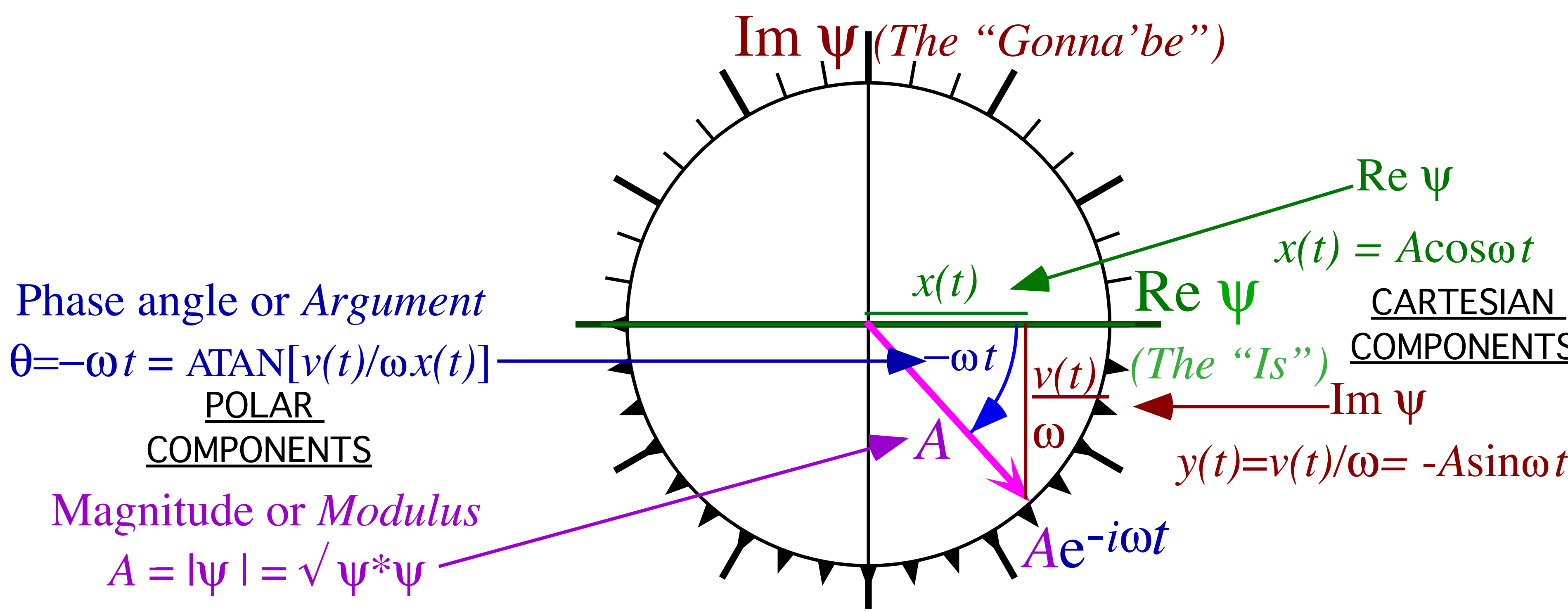
3. Complex exponentials $Ae^{-i\omega t}$ track position <u>and</u> velocity using Phasor Clock.

(a) Complex plane and unit vectors



(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - i\,A\sin\omega t = x+iy$

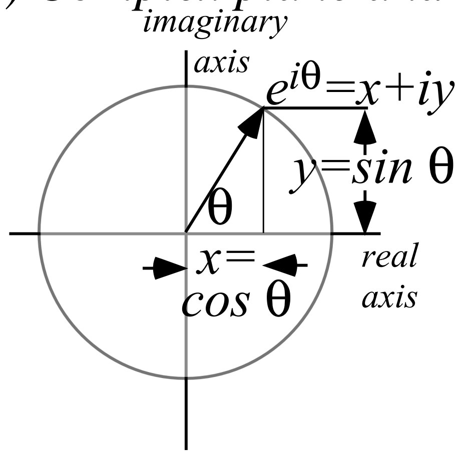
Unit 1
Fig. 10.5

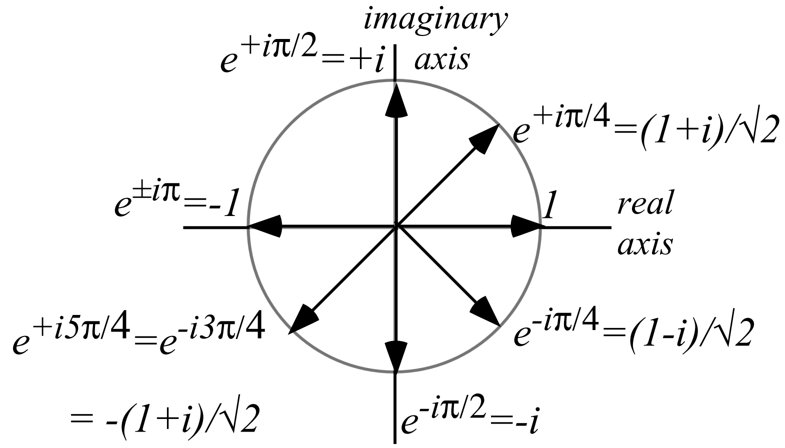


What Good Are Complex Exponentials? (contd.)

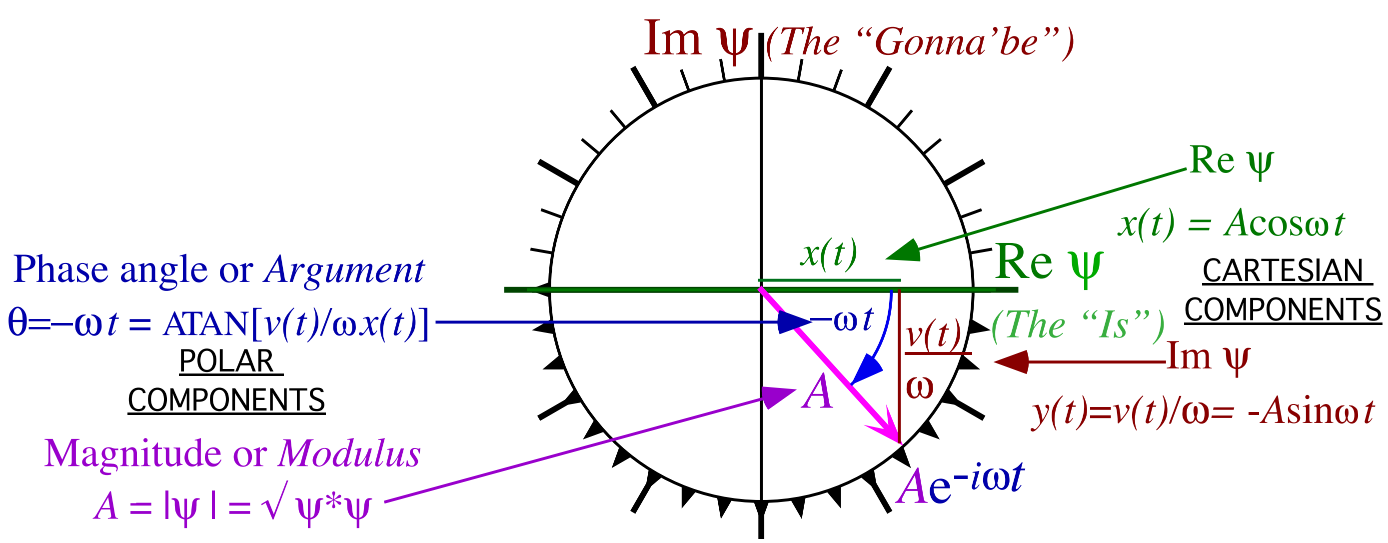
3. Complex exponentials $Ae^{-i\omega t}$ track position <u>and</u> velocity using Phasor Clock.

(a) Complex plane and unit vectors





(b) Quantum Phasor Clock $\psi = Ae^{-i\omega t} = A\cos\omega t - i\,A\sin\omega t = x+iy$



Unit 1
Fig. 10.5

Some Rect-vs-Polar *relations worth remembering*

Cartesian (x,y) *form*

$$\psi_x = \mathrm{Re}\,\psi(t) = x(t) = A\cos\omega t = \frac{\psi+\psi^*}{2}$$

$$\psi_y = \mathrm{Im}\,\psi(t) = \frac{v(t)}{\omega} = -A\sin\omega t = \frac{\psi-\psi^*}{2i}$$

$$\psi = re^{+i\theta} = re^{-i\omega t} = r(\cos\omega t - i\sin\omega t)$$

$$\psi^* = re^{-i\theta} = re^{+i\omega t} = r(\cos\omega t + i\sin\omega t)$$

Polar (r,θ) *form*

$$r = A = |\psi| = \sqrt{\psi_x^{\,2}+\psi_y^{\,2}} = \sqrt{\psi^*\psi}$$

$$\theta = -\omega t = \arctan(\psi_y/\psi_x)$$

$$\cos\theta = \tfrac{1}{2}(e^{+i\theta}+e^{-i\theta})$$

$$\sin\theta = \tfrac{1}{2i}(e^{+i\theta}-e^{-i\theta})$$

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x + iy) = x\cos\phi - y\sin\phi \quad + i \quad (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} = (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$

4. Complex products provide 2D rotation operations.

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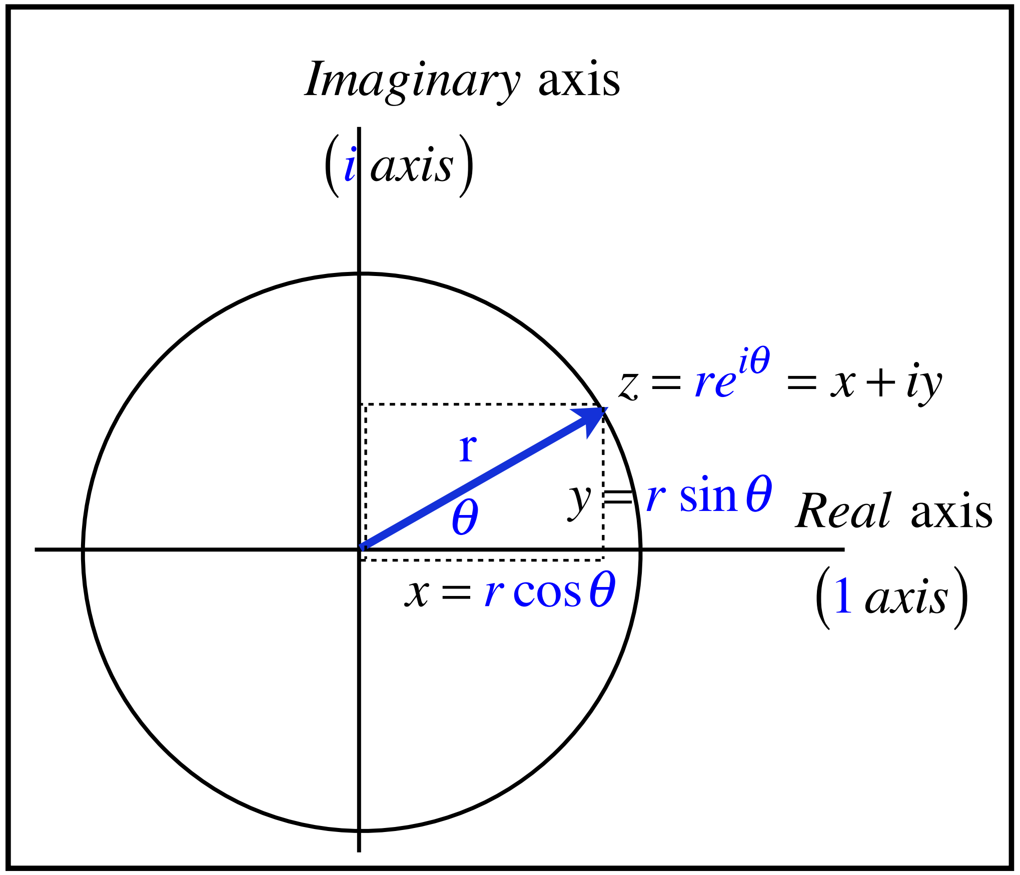
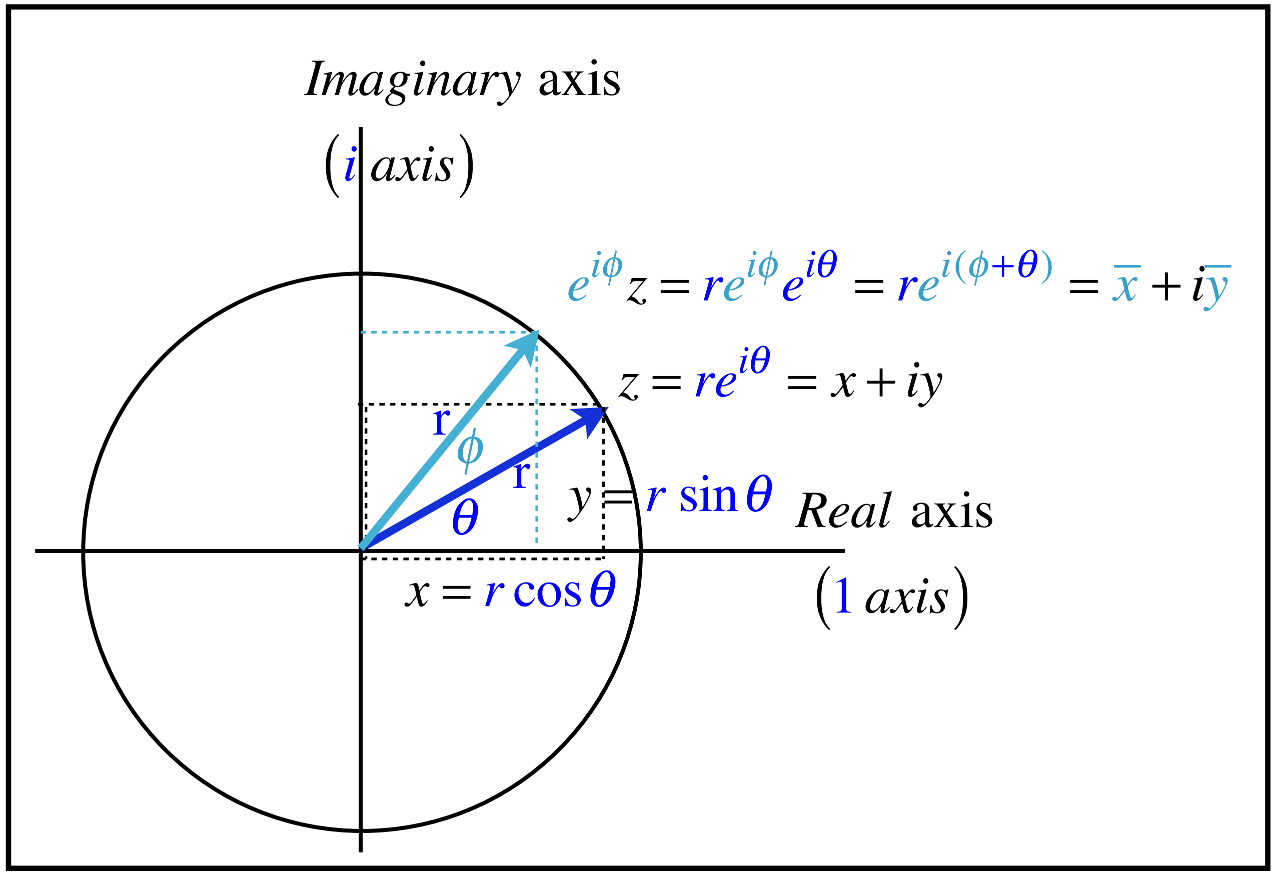
$e^{i\phi}$ acts on this: $z = re^{i\theta}$

to give this: $e^{i\phi}$ $e^{i\phi}z = re^{i\phi}e^{i\theta}$





4. Complex products provide 2D rotation operations.

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5. Complex products provide 2D “dot”(•) and “cross”(x) products.

Two complex numbers $A=A_x+iA_y$ and $B=B_x+iB_y$ and their “star” (*)-product $A{*}B$.

$$A*B = (A_x + iA_y)^*(B_x + iB_y) = (A_x - iA_y)(B_x + iB_y)$$
$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x) = \mathbf{A}\bullet\mathbf{B} + i\,|\,\mathbf{A}\times\mathbf{B}\,|_{Z\perp(x,y)}$$

Real part is scalar or “dot”(•) product **A•B**.

Imaginary part is vector or “cross”(×) product, but just the *Z*-component *normal* to *xy*-plane.

Rewrite $A{*}B$ in polar form.

$$A*B = (|A|e^{i\theta_A})^*(|B|e^{i\theta_B}) = |A|e^{-i\theta_A}|B|e^{i\theta_B} = |A||B|e^{i(\theta_B-\theta_A)}$$
$$= |A||B|\cos(\theta_B - \theta_A) + i|A||B|\sin(\theta_B - \theta_A) = \mathbf{A}\bullet\mathbf{B} + i\,|\,\mathbf{A}\times\mathbf{B}\,|_{Z\perp(x,y)}$$

4. Complex products provide 2D rotation operations.

$$e^{i\phi}\cdot z = (\cos\phi + i\sin\phi)\cdot(x+iy) = x\cos\phi - y\sin\phi \quad + i \quad (x\sin\phi + y\cos\phi)$$

$$\mathbf{R}_{+\phi}\bullet\mathbf{r} = (x\cos\phi - y\sin\phi)\hat{\mathbf{e}}_x + (x\sin\phi + y\cos\phi)\hat{\mathbf{e}}_y$$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix}$$

5. Complex products provide 2D "dot"(•) and "cross"(x) products.

Two complex numbers $A=A_x+iA_y$ and $B=B_x+iB_y$ and their "star" (*)-product A^*B.

$$A^*B = (A_x+iA_y)^*(B_x+iB_y) = (A_x-iA_y)(B_x+iB_y)$$
$$= (A_xB_x + A_yB_y) + i(A_xB_y - A_yB_x) = \mathbf{A}\bullet\mathbf{B} + i\,|\,\mathbf{A}\times\mathbf{B}\,|_{Z\perp(x,y)}$$

Real part is scalar or "dot"(•) product **A•B**.

Imaginary part is vector or "cross"(×) product, but just the *Z*-component *normal* to *xy*-plane.

Rewrite A^*B in polar form.

$$A^*B = (|A|e^{i\theta_A})^*(|B|e^{i\theta_B}) = |A|e^{-i\theta_A}|B|e^{i\theta_B} = |A||B|e^{i(\theta_B-\theta_A)}$$
$$= |A||B|\cos(\theta_B-\theta_A) + i|A||B|\sin(\theta_B-\theta_A) = \mathbf{A}\bullet\mathbf{B} + i\,|\,\mathbf{A}\times\mathbf{B}\,|_{Z\perp(x,y)}$$

$$\mathbf{A}\bullet\mathbf{B} = |A||B|\cos(\theta_B-\theta_A)$$
$$= |A|\cos\theta_A|B|\cos\theta_B + |A|\sin\theta_A|B|\sin\theta_B$$
$$= A_xB_x + A_yB_y$$

$$|\,\mathbf{A}\times\mathbf{B}\,| = |A||B|\sin(\theta_B-\theta_A)$$
$$= |A|\cos\theta_A|B|\sin\theta_B - |A|\sin\theta_A|B|\cos\theta_B$$
$$= A_xB_y - A_yB_x$$

6. Complex derivative contains "divergence"($\nabla\cdot\mathbf{F}$) and "curl"($\nabla\times\mathbf{F}$) of 2D vector field

Relation of (z,z^*) to $(x=\mathrm{Re}\,z, y=\mathrm{Im}\,z)$ defines a z-derivative $\frac{df}{dz}$ and "star" z^*-derivative. $\frac{df}{dz^*}$

$$z = x + iy \qquad x = \tfrac{1}{2}(z + z^*)$$

$$z^* = x - iy \qquad y = \tfrac{1}{2i}(z - z^*)$$

Applying chain-rule

$$\frac{df}{dz} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y}$$

$$\frac{df}{dz^*} = \frac{\partial x}{\partial z^*}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z^*}\frac{\partial f}{\partial y} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}$$

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Derivative chain-rule shows real part of $\frac{df}{dz}$ has 2D divergence $\nabla\cdot\mathbf{f}$ and imaginary part has curl $\nabla\times\mathbf{f}$.

$$\frac{df}{dz} = \frac{d}{dz}(f_x + i f_y) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(f_x + i f_y) = \frac{1}{2}\overbrace{\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)} + \frac{i}{2}\overbrace{\left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)} = \frac{1}{2}\nabla\bullet\mathbf{f} + \frac{i}{2}|\nabla\times\mathbf{f}|_{Z\perp(x,y)}$$

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7. Invent source-free 2D vector fields [$\nabla \cdot \mathbf{F}=0$ and $\nabla \times \mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function $f(z)$, conjugate it (change all i's to $-i$) to give $f^*(z^*)$ for which $\boxed{\frac{df^*}{dz} = 0}$.

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We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function *f(z)*, conjugate it (change all *i*'s to –*i*) to give $f^*(z^*)$ for which $\boxed{\frac{df^*}{dz} = 0}$.

For example: if *f(z)=a·z* then $f^*(z^*)=a\cdot z^*=a(x-iy)$ is not function of *z* so it has *zero z-derivative*.

$\mathbf{F}=(F_x,F_y)=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ *has zero divergence:* $\nabla\bullet\mathbf{F}=0$ *and has zero curl:* $|\nabla\times\mathbf{F}|=0$.

$$\nabla\bullet\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial(ax)}{\partial x} + \frac{\partial F(-ay)}{\partial y} = 0 \qquad |\nabla\times\mathbf{F}|_{Z\perp(x,y)} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial(-ay)}{\partial x} - \frac{\partial F(ax)}{\partial y} = 0$$

7. (contd.) Invent source-free 2D vector fields [$\nabla\cdot\mathbf{F}=0$ *and* $\nabla\times\mathbf{F}=0$]

We can invent *source-free 2D vector fields* that are both *zero-divergence* and *zero-curl*.
Take any function $f(z)$, conjugate it (change all i's to $-i$) to give $f^*(z^*)$ for which $\boxed{\frac{df^*}{dz}=0}$.

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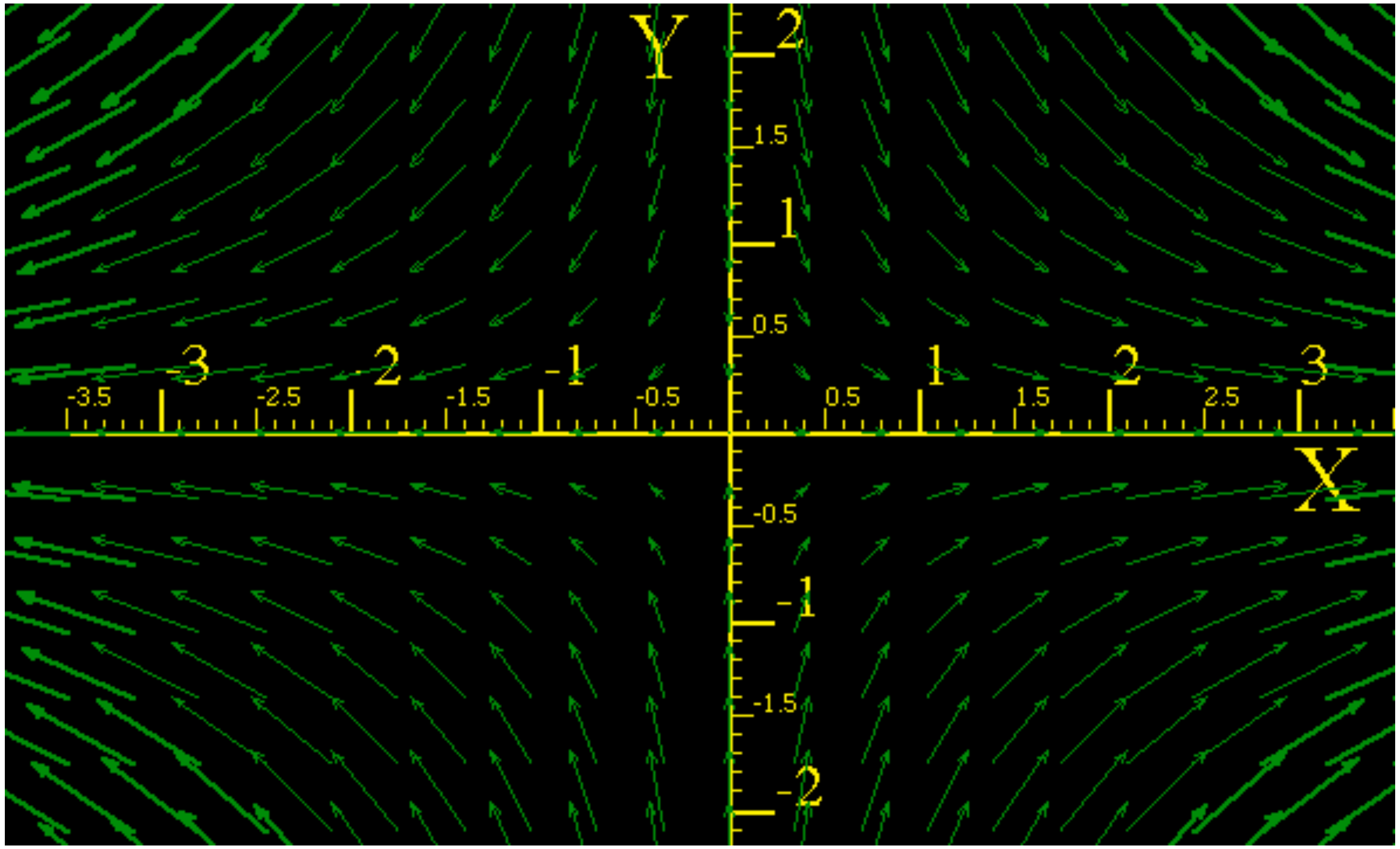

precursor to Unit 1 Fig. 10.7

$\mathbf{F}=(f^*_x,f^*_y)=(a\cdot x,-a\cdot y)$ is a *divergence-free laminar (DFL)* field.

8. Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla x\mathbf{A}$) potentials

Any *DFL* field **F** is a gradient of a *scalar potential field* $\mathbf{\Phi}$ or a curl of a *vector potential field* **A**.

$$\mathbf{F}=\nabla\mathbf{\Phi} \qquad\qquad \mathbf{F}=\nabla\times\mathbf{A}$$

A *complex potential* $\phi(z)=\mathbf{\Phi}(x,y)+i\mathbf{A}(x,y)$ exists whose z-derivative is $f(z)=d\phi/dz$.

Its complex conjugate $\phi^*(z^*)=\mathbf{\Phi}(x,y)-i\mathbf{A}(x,y)$ has z^*-derivative $f^*(z^*)=d\phi^*/dz^*$ giving *DFL* field **F**.

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To find $\phi=\mathbf{\Phi}+i\mathbf{A}$ integrate $f(z)=a\cdot z$ to get ϕ and isolate real ($\mathrm{Re}\phi=\mathbf{\Phi}$) and imaginary ($\mathrm{Im}\phi=\mathbf{A}$) parts.

$$\phi=\overbrace{\tfrac{1}{2}a(x^2-y^2)}^{\mathbf{\Phi}}+i\overbrace{axy}^{\mathbf{A}}=\int f\cdot dz=\int az\cdot dz=\tfrac{1}{2}az^2=\tfrac{1}{2}a(x+iy)^2$$

*8. Complex potential ϕ contains "scalar"(**F=**$\nabla\Phi$) and "vector"(**F=**∇x**A**) potentials*

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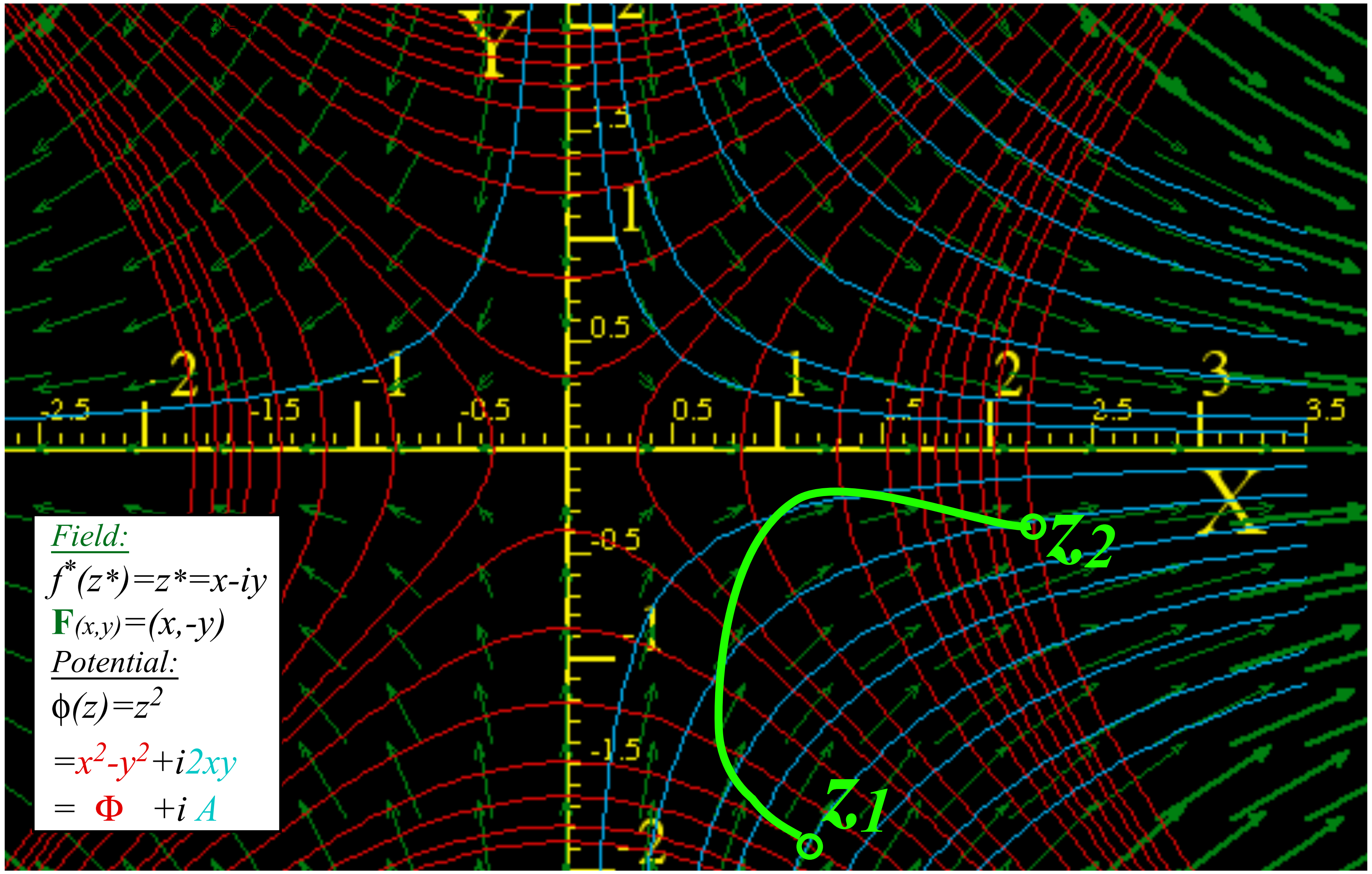
Unit 1
Fig. 10.7



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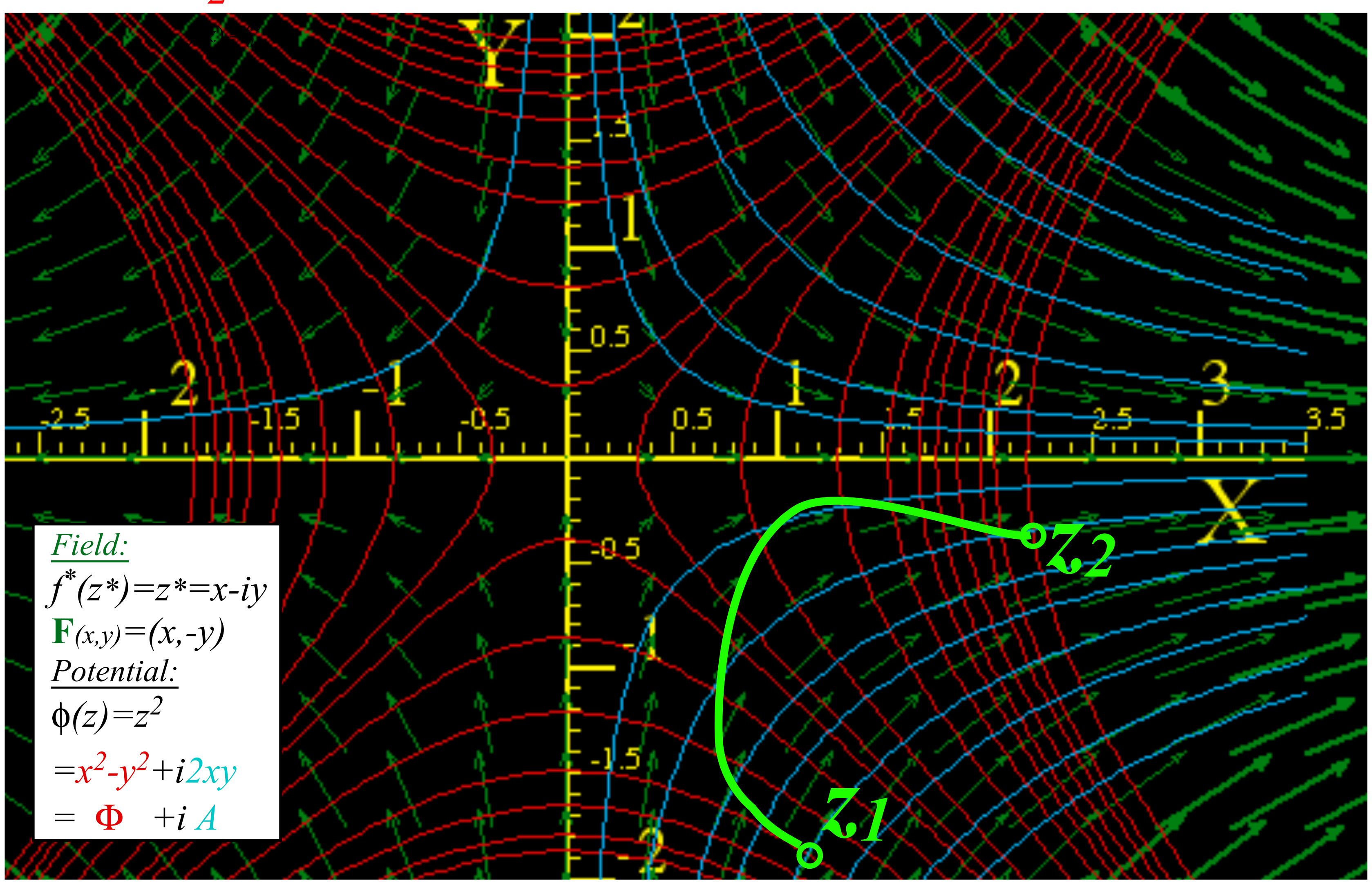
BONUS!
Get a free
coordinate
system!



Unit 1
Fig. 10.7

The (Φ,$\mathbf{A}$) grid is a GCC coordinate system*:

$q^1 = \Phi = (x^2-y^2)/2 = const.$

$q^2 = \mathbf{A} = (xy) = const.$

*Actually it's OCC.

What Good Are Complex Exponentials? (contd.)

*8. (contd.) Complex potential ϕ contains "scalar"(**F=**$\nabla\Phi$) and "vector"(**F=**∇x**A**) potentials*

...and either one (or half-n'-half!) works just as well.

Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial\mathbf{A}}{\partial y} \\ -\frac{\partial\mathbf{A}}{\partial y} \end{pmatrix}$ of vector **A** (*and they're <u>equal</u>!*)

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y} - i\frac{\partial\mathbf{A}}{\partial x}\right) = \frac{1}{2}\nabla\Phi + \frac{1}{2}\nabla\times\mathbf{A}$$

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$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\underbrace{\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)}+\frac{1}{2}\underbrace{\left(\frac{\partial\mathbf{A}}{\partial y}-i\frac{\partial\mathbf{A}}{\partial x}\right)}=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field $\mathbf{F}=+\nabla\Phi$ replaces usual physicist's definition $\mathbf{F}=-\nabla\Phi$

What Good Are Complex Exponentials? (contd.)

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$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y} - i\frac{\partial\mathbf{A}}{\partial x}\right) = \frac{1}{2}\nabla\Phi + \frac{1}{2}\nabla\times\mathbf{A}$$

Note, *mathematician definition* of force field **F**=+$\nabla\Phi$ replaces usual physicist's definition **F**=-$\nabla\Phi$

Given ϕ:

$$\phi = \Phi + i\mathbf{A} = \frac{1}{2}a(x^2-y^2) + i\,axy$$

The half-n'-half result

find:

$$\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y}\end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial\mathbf{A}}{\partial y} \\ -\frac{\partial\mathbf{A}}{\partial x}\end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy\end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

8. (contd.) Complex potential ϕ contains "scalar"($\mathbf{F}=\nabla\Phi$) and "vector"($\mathbf{F}=\nabla\times\mathbf{A}$) potentials

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Derivative $\frac{d\phi^*}{dz^*}$ has 2D gradient $\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}$ of scalar Φ and curl $\nabla\times\mathbf{A}=\begin{pmatrix}\frac{\partial A}{\partial y}\\ -\frac{\partial A}{\partial x}\end{pmatrix}$ of vector $\mathbf{A}$ (*and they're equal!*)

$$\frac{d}{dz^*}\phi^*=\frac{d}{dz^*}(\Phi-i\mathbf{A})=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\Phi-i\mathbf{A})=\frac{1}{2}\underbrace{\left(\frac{\partial\Phi}{\partial x}+i\frac{\partial\Phi}{\partial y}\right)}_{\nabla\Phi}+\frac{1}{2}\underbrace{\left(\frac{\partial A}{\partial y}-i\frac{\partial A}{\partial x}\right)}_{\nabla\times\mathbf{A}}=\frac{1}{2}\nabla\Phi+\frac{1}{2}\nabla\times\mathbf{A}$$

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Given ϕ: $\boxed{\begin{aligned}\phi=&\quad\Phi &&+i\ \ \mathbf{A}\\ =&\tfrac{1}{2}a(x^2-y^2) &&+i\ \ axy\end{aligned}}$

The half-n'-half result

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$$\nabla\Phi=\begin{pmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2)\\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2)\end{pmatrix}=\begin{pmatrix}ax\\ -ay\end{pmatrix}=\mathbf{F}$$

or find:

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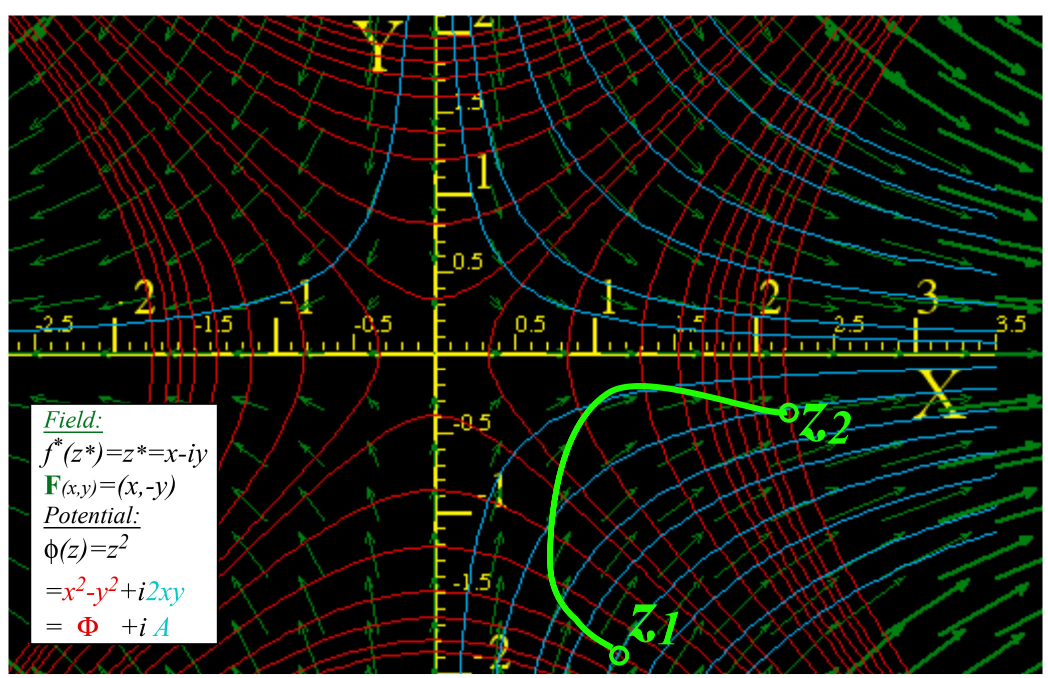
Scalar *static potential lines* Φ=*const.* and vector *flux potential lines* $\mathbf{A}$=*const.* define *DFL field-net*.



What Good Are Complex Exponentials? (contd.)

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The half-n'-half result

$$\frac{d}{dz^*}\phi^* = \frac{d}{dz^*}(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(\Phi - i\mathbf{A}) = \frac{1}{2}\left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Phi}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial\mathbf{A}}{\partial y} - i\frac{\partial\mathbf{A}}{\partial x}\right) = \frac{1}{2}\nabla\Phi + \frac{1}{2}\nabla\times\mathbf{A}$$

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Given ϕ: $\boxed{\begin{aligned}\phi &= \Phi + i\,\mathbf{A} \\ &= \tfrac{1}{2}a(x^2-y^2) + i\,axy\end{aligned}}$

The half-n'-half result

find:

$$\nabla\Phi = \begin{pmatrix} \frac{\partial\Phi}{\partial x} \\ \frac{\partial\Phi}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{a}{2}(x^2-y^2) \\ \frac{\partial}{\partial y}\frac{a}{2}(x^2-y^2) \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

or find:

$$\nabla\times\mathbf{A} = \begin{pmatrix} \frac{\partial\mathbf{A}}{\partial y} \\ -\frac{\partial\mathbf{A}}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y}axy \\ -\frac{\partial}{\partial x}axy \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix} = \mathbf{F}$$

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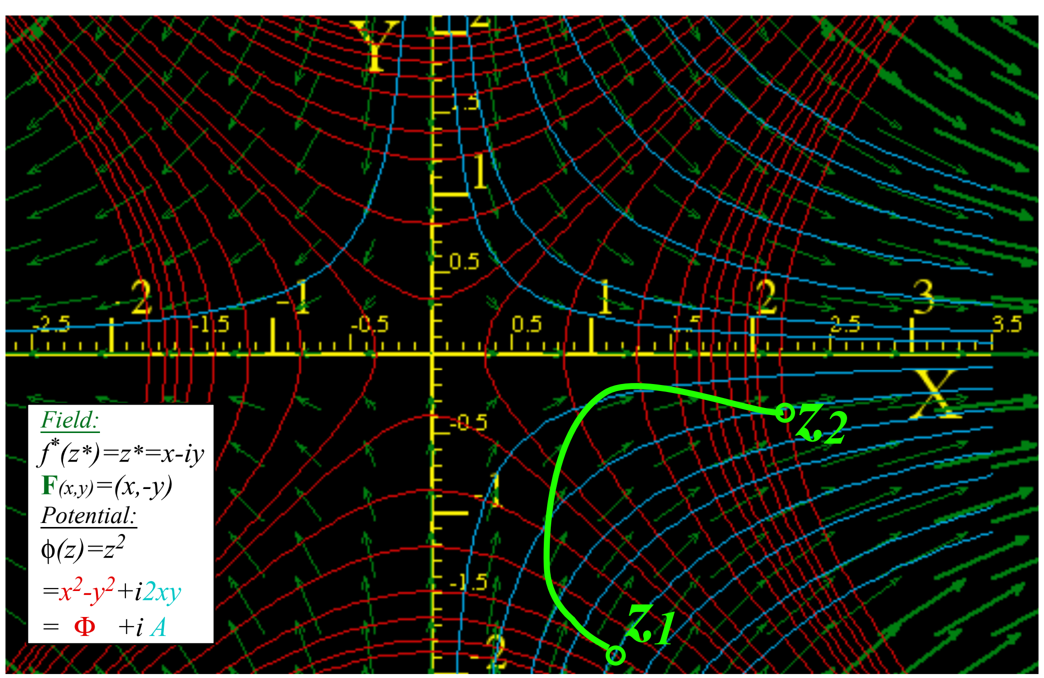

The half-n'-half results

are called

Riemann-Cauchy

Derivative Relations

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\mathbf{A}}{\partial y} \quad \text{is:} \quad \frac{\partial \mathrm{Re} f(z)}{\partial x} = \frac{\partial \mathrm{Im} f(z)}{\partial y}$$

$$\frac{\partial\Phi}{\partial y} = -\frac{\partial\mathbf{A}}{\partial x} \quad \text{is:} \quad \frac{\partial \mathrm{Re} f(z)}{\partial y} = -\frac{\partial \mathrm{Im} f(z)}{\partial x}$$

9. Complex integrals ∫ f(z)dz count 2D "circulation"(∫F•dr) and "flux"(∫Fxdr)

Integral of $f(z)$ between point z_1 and point z_2 is potential difference $\Delta\phi=\phi(z_2)-\phi(z_1)$

$$\Delta\phi = \phi(z_2) - \phi(z_1) = \int_{z_1}^{z_2} f(z)dz = \underbrace{\mathbf{\Phi}(x_2,y_2) - \mathbf{\Phi}(x_1,y_1)}_{\Delta\mathbf{\Phi}} + i\,[\underbrace{\mathbf{A}(x_2,y_2) - \mathbf{A}(x_1,y_1)}_{\Delta\mathbf{A}}]$$

$$\Delta\phi = \Delta\mathbf{\Phi} + i\,\Delta\mathbf{A}$$

In *DFL* field **F**, $\Delta\phi$ is independent of the integration path $z(t)$ connecting z_1 and z_2.