Read Unit 2 Chapter 3 (all) and Chapter 4 thru part (b).

The following deals with spinors, matrix eigensolutions, and applications of them 2D-HO

2.4.1 Derive multiplication table for spinor matrix operators:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_C = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

	σ_0	σ_A	σ_B	σ_C
σ_0				
σ_A				
σ_B				
σ_C				

(a)Two of the operators are real mirror-plane reflections (Recall superball bounce Theory in Ch. 4-5) Describe what reflection each of these did.

Consider a normal combination $\sigma(\theta) = \sigma_A \cos\theta + \sigma_B \sin\theta$. Does it square like a reflection $\sigma(\theta)^2 = ?$

(b)What is its effect on vectors $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$? Display on graph for θ=30° and θ=45°.

(c) Use the functional spectral decomposition (Lect. 18 around p.61-63) to derive the following matrix functions of the spinor matrices: (Show that more than one answer exists for each.)

$\sqrt{\sigma_A} =$ _________, $\sqrt{\sigma_B} =$ _________, $\sqrt{\sigma_C} =$ _________, $\sqrt{\sigma(\theta)} =$ _________,

$e^{-i\phi\sigma_A} =$ _________, $e^{-i\phi\sigma_B} =$ _________, $e^{-i\phi\sigma_C} =$ _________, $e^{-i\phi\sigma(\theta)} =$ _________,

Compare last four results with what you get from the "Crazy-Thing-Theorem" (Lect. 18 p.31).

(d) 2D-HO phasor space and quantum spin ½ is described by a state vector with 4-parameters

$|\psi\rangle = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = r \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{+i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$. Express the 4 phase phase variables (x_1, p_1, x_2, p_2) in terms of the radius r, three Euler-polar angles (azimuth α, polar angle β, and phase γ) of spin (Stokes) vector **S**.

Also express (x_1, p_1, x_2, p_2) in terms of (A,B,C,D) following projector method in Lect. 18 p.74-75.

2.4.1 D2(quantum-spin ½) multiplication table: (Using standard notation: $\sigma_X=\sigma_B,\sigma_Y=\sigma_C,\sigma_Z=\sigma_A$)

$$\sigma_Z \cdot \sigma_X$$
$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=i\begin{pmatrix}0&-i\\i&0\end{pmatrix}=i\sigma_Y$$

$$\sigma_X \cdot \sigma_Z$$
$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}=-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}=-i\sigma_Y$$

	σ_X	σ_Y	σ_Z
σ_X	**1**	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	$-i\sigma_Z$	**1**	$i\sigma_X$
σ_Z	$i\sigma_Y$	$-i\sigma_X$	**1**

$\varepsilon-tensor\ form$

$$\sigma_K\sigma_L=i\varepsilon_{KLM}\sigma_M+\delta_{KL}\mathbf{1}$$

$$\sigma_X \cdot \sigma_Y$$
$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\begin{pmatrix}i&0\\0&-i\end{pmatrix}=i\sigma_Z$$

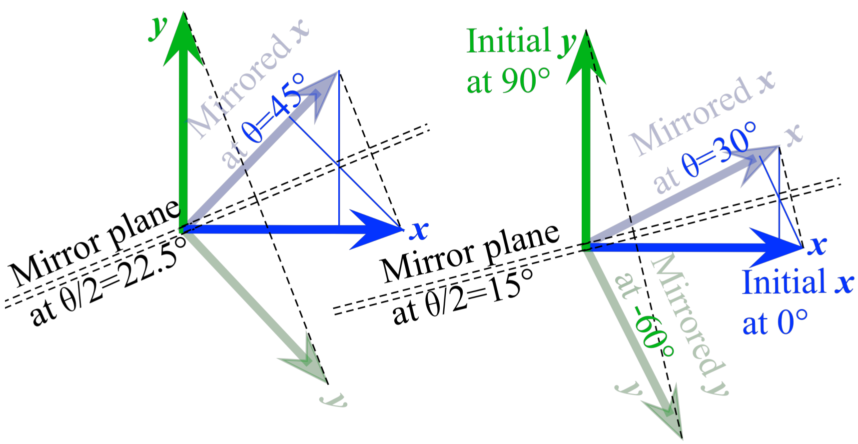
$$\sigma_Y \cdot \sigma_X$$
$$\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}-i&0\\0&i\end{pmatrix}=-i\sigma_Z$$

Part (a)₁ Both $\sigma_X=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\sigma_Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ are real 2D mirror ops. So is normal $\sigma(\theta)=\sigma_Z\cos\theta+\sigma_X\sin\theta$

$$\sigma(\theta)=\sigma_Z\cos\theta+\sigma_X\sin\theta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos\theta+\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\theta=\begin{pmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{pmatrix}$$

Note: $\sigma(\theta)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\cos\theta+y\sin\theta\\x\sin\theta-y\cos\theta\end{pmatrix}=x\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}+y\begin{pmatrix}\sin\theta\\-\cos\theta\end{pmatrix}$

Better: $\sigma(\theta)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$ and: $\sigma(\theta)\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\sin\theta\\-\cos\theta\end{pmatrix}$



Square of any reflection $\sigma(\theta)$ is unit matrix: $\sigma(\theta)^2=\begin{pmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{pmatrix}$

$$\sigma(\theta)^2=\begin{pmatrix}\cos^2\theta+\sin^2\theta&\cos\theta\sin\theta-\sin\theta\cos\theta\\\sin\theta\cos\theta-\cos\theta\sin\theta&\sin^2\theta+\cos^2\theta\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Extra-credit : Product $\sigma(\varphi)\sigma(\theta)$ of any two reflections $\sigma(\varphi)$ and $\sigma(\theta)$ is a rotation by angle $(\varphi-\theta)$:

$$\sigma(\varphi)\sigma(\theta)=\begin{pmatrix}\cos\varphi&\sin\varphi\\\sin\varphi&-\cos\varphi\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{pmatrix}=\begin{pmatrix}\cos\varphi\cos\theta+\sin\varphi\sin\theta&\cos\varphi\sin\theta-\sin\varphi\cos\theta\\\sin\varphi\cos\theta-\cos\varphi\sin\theta&\sin\varphi\sin\theta+\cos\varphi\cos\theta\end{pmatrix}$$

Sort the product sums using complex trig: $e^{i(\varphi-\theta)}=e^{i\varphi}e^{-i\theta}$

$$e^{i(\varphi-\theta)}=\cos(\varphi-\theta)+i\sin(\varphi-\theta)=e^{i\varphi}e^{-i\theta}=(\cos\varphi+i\sin\varphi)(\cos\theta-i\sin\theta)$$
$$=\cos\varphi\cos\theta+\sin\varphi\sin\theta+i(-\cos\varphi\sin\theta+\sin\varphi\cos\theta)$$

So: $\sigma(\varphi)\sigma(\theta)=\begin{pmatrix}\cos(\varphi-\theta)&-\sin(\varphi-\theta)\\\sin(\varphi-\theta)&\cos(\varphi-\theta)\end{pmatrix}$ Note *Inverse*: $\sigma(\theta)\sigma(\varphi)=\begin{pmatrix}\cos(\varphi-\theta)&+\sin(\varphi-\theta)\\-\sin(\varphi-\theta)&\cos(\varphi-\theta)\end{pmatrix}$

Part (c) 1-2 Problem: Find functions of **σ**-matrices.

$\sqrt{\sigma_A} =$ __________, $\sqrt{\sigma_B} =$ __________, $\sqrt{\sigma_C} =$ __________, $\sqrt{\sigma(\theta)} =$ __________,

$e^{-i\phi\sigma_A} =$ __________, $e^{-i\phi\sigma_B} =$ __________, $e^{-i\phi\sigma_C} =$ __________, $e^{-i\phi\sigma(\theta)} =$ __________,

Part (c)1-2 Any **σ**-operator satisfies equations $\boldsymbol{\sigma}^2=\mathbf{1}$ or $\mathbf{1}-\boldsymbol{\sigma}^2=\mathbf{0}$ or $(\mathbf{1}-\boldsymbol{\sigma})(\mathbf{1}+\boldsymbol{\sigma})=\mathbf{0}$. We combine them into projectors $\mathbf{P}^+=(\mathbf{1}+\boldsymbol{\sigma})/2$ and $\mathbf{P}^-=(\mathbf{1}-\boldsymbol{\sigma})/2$ for which $\mathbf{P}^++\mathbf{P}^-=\mathbf{1}$ and $\mathbf{P}^+\cdot\mathbf{P}^-=\mathbf{0}$ and $\mathbf{P}^+\cdot\mathbf{P}^+=\mathbf{P}^+$ and $\mathbf{P}^-\cdot\mathbf{P}^-=\mathbf{P}^-$. Eigen-operator equations $\boldsymbol{\sigma}\mathbf{P}^+=+\mathbf{P}^+$ and $\boldsymbol{\sigma}\mathbf{P}^-=-\mathbf{P}^-$ follow with spectral decomposition $\boldsymbol{\sigma}=\mathbf{P}^++\mathbf{P}^-$. Then function spectral decomposition $f(\boldsymbol{\sigma})=f(+1)\mathbf{P}^++f(-1)\mathbf{P}^-$ is used for any function $f(x)$ that exists at $x=\pm 1$.

$$\sigma_A=\begin{pmatrix}1&0\\0&-1\end{pmatrix}=(+1)\mathbf{P}^++(-1)\mathbf{P}^- \text{ has projectors } \mathbf{P}^+=\tfrac{1}{2}(\mathbf{1}+\sigma_A)=\begin{pmatrix}1&0\\0&0\end{pmatrix} \text{ and } \mathbf{P}^-=\tfrac{1}{2}(\mathbf{1}-\sigma_A)=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\sqrt{\sigma_A}=\sqrt{+1}\mathbf{P}^++\sqrt{-1}\mathbf{P}^-=\pm(\mathbf{P}^+\pm i\mathbf{P}^-)=\pm\begin{pmatrix}1&0\\0&\pm i\end{pmatrix} \qquad \text{Exponential: } e^{-i\phi\sigma_A}=e^{-i\phi\cdot 1}\mathbf{P}^++e^{+i\phi\cdot 1}\mathbf{P}^-=\begin{pmatrix}e^{-i\phi\cdot 1}&0\\0&e^{+i\phi\cdot 1}\end{pmatrix}$$

$$\sigma_B=\begin{pmatrix}0&1\\1&0\end{pmatrix} \text{ has projectors } \mathbf{P}^+=\tfrac{1}{2}(\mathbf{1}+\sigma_B)=\tfrac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} \text{ and } \mathbf{P}^-=\tfrac{1}{2}(\mathbf{1}-\sigma_B)=\tfrac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} \text{ where: } \frac{e^{i\phi}+e^{-i\phi}}{2}=\cos\phi \text{ and: } \frac{e^{i\phi}-e^{-i\phi}}{2i}=\sin\phi$$

$$\sqrt{\sigma_B}=\sqrt{+1}\mathbf{P}^++\sqrt{-1}\mathbf{P}^-=\pm(\mathbf{P}^+\pm i\mathbf{P}^-)=\pm\tfrac{1}{2}\begin{pmatrix}1\pm i&1\mp i\\1\mp i&1\pm i\end{pmatrix} \qquad \text{Exponential: } e^{-i\phi\sigma_B}=e^{-i\phi\cdot 1}\mathbf{P}^++e^{+i\phi\cdot 1}\mathbf{P}^-=\begin{pmatrix}\cos\phi&-i\sin\phi\\-i\sin\phi&\cos\phi\end{pmatrix}$$

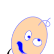
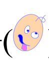
$$\sigma_C=\begin{pmatrix}0&-i\\i&0\end{pmatrix} \text{ has projectors } \mathbf{P}^+=\tfrac{1}{2}(\mathbf{1}+\sigma_C)=\tfrac{1}{2}\begin{pmatrix}1&-i\\i&1\end{pmatrix} \text{ and } \mathbf{P}^-=\tfrac{1}{2}(\mathbf{1}-\sigma_C)=\tfrac{1}{2}\begin{pmatrix}1&i\\-i&1\end{pmatrix}$$

$$\sqrt{\sigma_C}=\sqrt{+1}\mathbf{P}^++\sqrt{-1}\mathbf{P}^-=\pm(\mathbf{P}^+\pm i\mathbf{P}^-)=\pm\tfrac{1}{2}\begin{pmatrix}1\pm i&\mp 1-i\\\pm 1+i&1\pm i\end{pmatrix} \qquad \text{Exponential: } e^{-i\phi\sigma_C}=e^{-i\phi\cdot 1}\mathbf{P}^++e^{+i\phi\cdot 1}\mathbf{P}^-=\begin{pmatrix}\cos\phi&-\sin\phi\\\sin\phi&\cos\phi\end{pmatrix}$$

$\sigma(\theta)=\sigma_Z\cos\theta+\sigma_X\sin\theta$ is more complicated but treated similarly.

$$\sigma_\theta=\begin{pmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{pmatrix} \text{ has projectors } \mathbf{P}^+=\tfrac{1}{2}(\mathbf{1}+\sigma_\theta)=\tfrac{1}{2}\begin{pmatrix}1+\cos\theta&\sin\theta\\\sin\theta&1-\cos\theta\end{pmatrix} \text{ and } \mathbf{P}^-=\tfrac{1}{2}(\mathbf{1}-\sigma_\theta)=\tfrac{1}{2}\begin{pmatrix}1-\cos\theta&-\sin\theta\\-\sin\theta&1+\cos\theta\end{pmatrix}$$

$$\sqrt{\sigma_\theta}=\sqrt{+1}\mathbf{P}^++\sqrt{-1}\mathbf{P}^-=\pm(\mathbf{P}^+\pm i\mathbf{P}^-)=\pm\tfrac{1}{2}\begin{pmatrix}1\pm i+(1\mp i)\cos\theta&(1\mp i)\sin\theta\\(1\mp i)\sin\theta&1\pm i-(1\mp i)\cos\theta\end{pmatrix}$$

Part (c)3 Crazy-Thing Theorem: If ()$^2=-\mathbf{1}$ then e$^{(}$  $^{)\varphi}$ = $\mathbf{1}\cos\varphi$ +()$\sin\varphi$ gives same results. Here:

 $=-i\sigma_\varphi=-i(\boldsymbol{\sigma}\bullet\hat{\varphi})=-i(\boldsymbol{\sigma}\bullet\vec{\varphi})/\varphi$ (Crazy-Thing Theorem is easier to apply than projector forms.)

Part (c)3 $$e^{-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\varphi_A}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_A-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\varphi_A=\begin{pmatrix}\cos\varphi_A-i\sin\varphi_A&0\\0&\cos\varphi_A-i\sin\varphi_A\end{pmatrix}=\begin{pmatrix}e^{-i\varphi_A}&0\\0&e^{i\varphi_A}\end{pmatrix}$$

$$e^{-i\sigma_B\varphi_B}=e^{-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\varphi_B}=\mathbf{1}\cos\varphi_B-i(\sigma_B)\sin\varphi_B=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_B-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\varphi_B=\begin{pmatrix}\cos\varphi_B&-i\sin\varphi_B\\-i\sin\varphi_B&\cos\varphi_B\end{pmatrix}$$

$$e^{-i\sigma_C\varphi_C}=e^{-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\varphi_C}=\mathbf{1}\cos\varphi_C-i(\sigma_C)\sin\varphi_C=\begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_C-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\varphi_C=\begin{pmatrix}\cos\varphi_C&-\sin\varphi_C\\\sin\varphi_C&\cos\varphi_C\end{pmatrix}$$

Part (d)1 2D-HO phasor space and quantum spin ½ is described by a state vector with 4-parameters

$$|\psi\rangle = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = r \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{+i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

. Express the 4 phase phase variables (x_1, p_1, x_2, p_2) in terms of the radius r, three Euler-polar angles (azimuth α, polar angle β, and phase γ) of spin (Stokes) vector **S**.

For zero overall phase (γ=0) the real and imaginary parts separate easily:

$$|\psi\rangle = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = r \begin{pmatrix} \left(\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\right)\cos\frac{\beta}{2} \\ \left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)\sin\frac{\beta}{2} \end{pmatrix}$$

so:

$$x_1 = r\cos\frac{\alpha}{2}\cos\frac{\beta}{2} \qquad p_1 = -r\sin\frac{\alpha}{2}\cos\frac{\beta}{2}$$

$$x_2 = r\cos\frac{\alpha}{2}\sin\frac{\beta}{2} \qquad p_2 = r\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$

For non-zero overall phase (γ=0):

$$x_1 = r\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2} \qquad p_1 = -r\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$x_2 = r\cos\frac{\alpha-\gamma}{2}\sin\frac{\beta}{2} \qquad p_2 = r\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta}{2}$$

Part (d)2 Also express (x_1, p_1, x_2, p_2) in terms of (A,B,C,D) following projector method in Lect.18 p.74-75.

Calculate **H**-matrix scaled so its square is **1**. That means you have to divide the **H** by

$$\omega_{ABCD} = \sqrt{\omega_A{}^2 + \omega_B{}^2 + \omega_C{}^2} = \sqrt{\frac{(A-D)^2}{4} + B^2 + C^2}$$

$$\mathbf{h} + \hat{\omega}_0 \mathbf{1}$$

$$= \frac{\mathbf{H}}{\omega_{ABCD}} = \frac{A-D}{2\omega_{ABCD}} \boldsymbol{\sigma}_A + \frac{B}{\omega_{ABCD}} \boldsymbol{\sigma}_B + \frac{C}{\omega_{ABCD}} \boldsymbol{\sigma}_C + \frac{A+D}{2\omega_{ABCD}} \boldsymbol{\sigma}_0$$

$$= \hat{\omega}_A \boldsymbol{\sigma}_A + \hat{\omega}_B \boldsymbol{\sigma}_B + \hat{\omega}_C \boldsymbol{\sigma}_C + \frac{A+D}{2\omega_{ABCD}} \mathbf{1}$$

$$= \hat{\omega}_A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \hat{\omega}_B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \hat{\omega}_C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2\omega_{ABCD}} \mathbf{1}$$

$$= \begin{pmatrix} \hat{\omega}_A & \hat{\omega}_B - i\hat{\omega}_C \\ \hat{\omega}_B + i\hat{\omega}_C & -\hat{\omega}_A \end{pmatrix} + \hat{\omega}_0 \mathbf{1} = \sigma_{\hat{\omega}} + \hat{\omega}_0 \mathbf{1} = \boldsymbol{\sigma} \bullet \hat{\omega} + \hat{\omega}_0 \mathbf{1}$$

Because $\mathbf{h}^2 = \mathbf{1}$ we get eigen projectors just by writing $\mathbf{h}^{-}\mathbf{1}$ and $\mathbf{h}^{+}\mathbf{1}$ (leaving off the norm factor ½.)

$$\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}^{ABCD+} = \frac{1}{2} \begin{pmatrix} \hat{\omega}_A + 1 \\ \hat{\omega}_B + i\hat{\omega}_C \end{pmatrix}$$

high eigenfrequency $\hat{\omega}_0 + \omega_{ABCD}$

(This is just the first column of $\mathbf{h}^{-}\mathbf{1}$ and $\mathbf{h}^{+}\mathbf{1}$.)

$$\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}^{ABCD-} = \frac{1}{2} \begin{pmatrix} \hat{\omega}_A - 1 \\ \hat{\omega}_B + i\hat{\omega}_C \end{pmatrix}$$

low eigenfrequency $\hat{\omega}_0 - \omega_{ABCD}$