

Group Theory in Quantum Mechanics

Lecture 8 (2.9.17)

Spinor and vector representations of $U(2)$ and $R(3)$ Operators

(Quantum Theory for Computer Age - Ch. 10A-B of Unit 3)

(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5)

Review: How “Crazy-Thing”-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ of spinor σ_μ -operators

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ of group-operators

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Harter and Patterson

U(2) Models for vibronic dynamics

J. Chem. Phys. **85** 5560 (1986)

Harter and Dos Santos

Double-Group Theory on the Half-Shell I and II

Am. J. Phys. **46** 251 (1986)

➔ Review: How “Crazy-Thing”-Theorem makes spinor and vector representation matrices
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$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Lecture 7 Review: The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\begin{aligned}
 \mathbf{H} &= \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} && \text{Notation for } 2D \text{ Spinor space} \\
 &= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0 \sigma_0 + \vec{\omega} \cdot \vec{\sigma} = \omega_0 \mathbf{1} + \omega \sigma_\omega \\
 &= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega} \cdot \vec{\mathbf{S}} \\
 &= \frac{A+D}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} && \text{Notation for } 3D \text{ Vector space} \\
 &\text{0th component unchanged} \quad \text{components } A, B, C \text{ switch } 1/2\text{-factor from } \omega\text{-velocity to } S\text{-momentum}
 \end{aligned}$$

Symmetry archetypes: A (Asymmetric[↑]diagonal) | B (Bilateral[↑]balanced) | C (Chiral[↑]circular-complex...)

"Crank" vector (2D-Spinor)

The $\{\sigma_I, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_I = \sigma_0, \sigma_B = \sigma_X, \sigma_C = \sigma_Y, \sigma_A = \sigma_Z\}$

The $\{\mathbf{1}, \mathbf{S}_A, \mathbf{S}_B, \mathbf{S}_C\}$ are the *Jordan-Angular-Momentum operators* $\{\mathbf{1} = \sigma_0, \mathbf{S}_B = \mathbf{S}_X, \mathbf{S}_C = \mathbf{S}_Y, \mathbf{S}_A = \mathbf{S}_Z\}$

(Often labeled $\{J_X, J_Y, J_Z\}$)

$$\vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega} \cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix} \cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix} \cdot t$$

Notation for 2D Spinor space

where: $\vec{\varphi} = \vec{\omega} \cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix} \cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix} \cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\mathbf{H}t} = e^{-i \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} t} = e^{-i(\omega_0 \sigma_0 + \vec{\omega} \cdot \vec{\sigma})t} = e^{-i\omega_0 t} e^{-i \vec{\omega} \cdot \vec{\sigma} t} = e^{-i\omega_0 t} e^{-i \sigma_\omega \omega t} = e^{-i\omega_0 t} (\mathbf{1} \cos \omega t - i \sigma_\omega \sin \omega t)$$

"Crank" vector (3D-Vector)

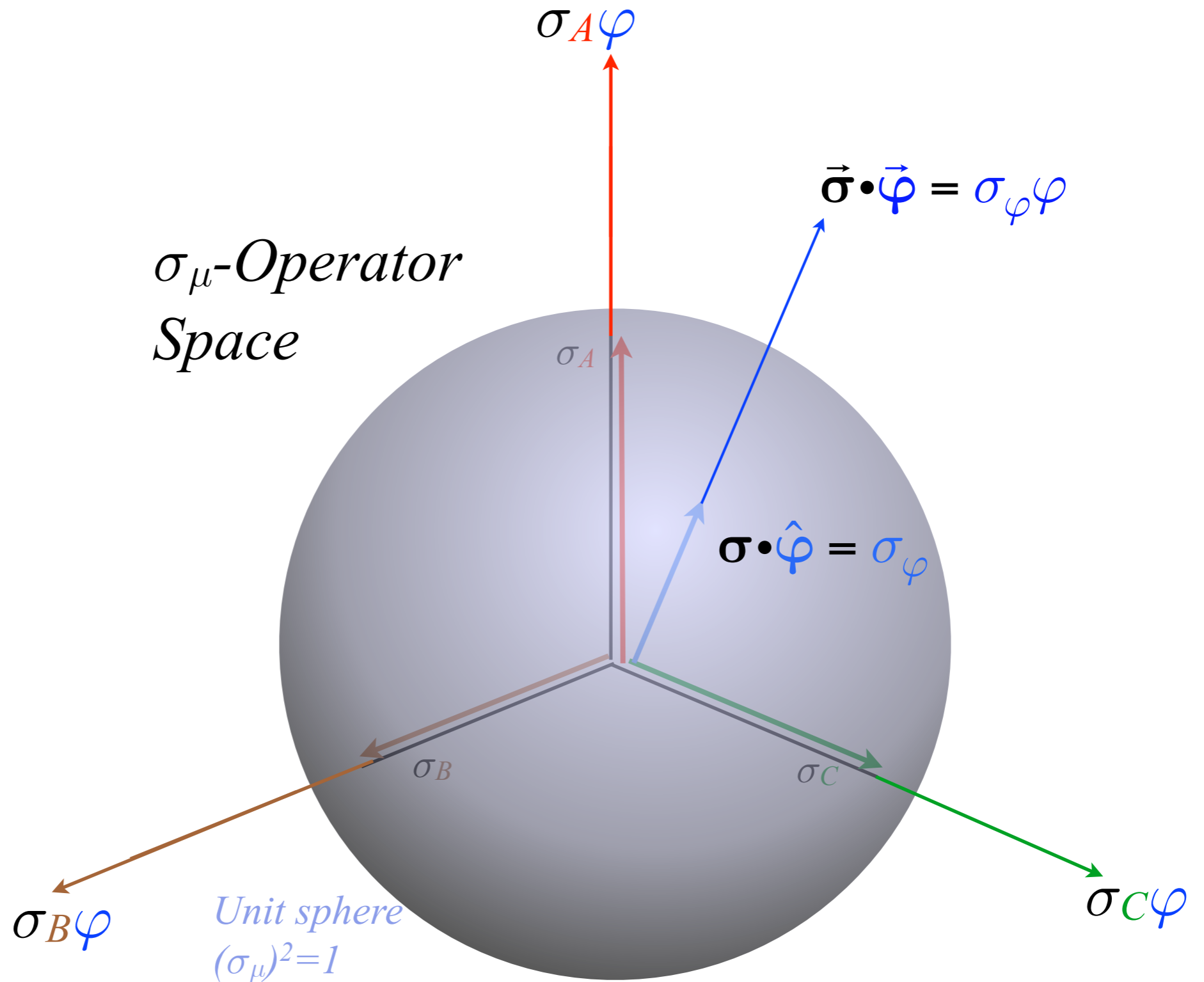
$$= e^{-i(\Omega_0 \mathbf{1} + \vec{\Omega} \cdot \vec{\mathbf{S}})t} = e^{-i\Omega_0 t} e^{-i \vec{\Omega} \cdot \vec{\mathbf{S}} t} = e^{-i\Omega_0 t} \left(\mathbf{1} \cos \frac{\Omega t}{2} - i \sigma_\omega \sin \frac{\Omega t}{2} \right)$$

$$\vec{\Theta} = \begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix} = \vec{\Omega} \cdot t = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} \cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix} \cdot t$$

Notation for 3D Vector space

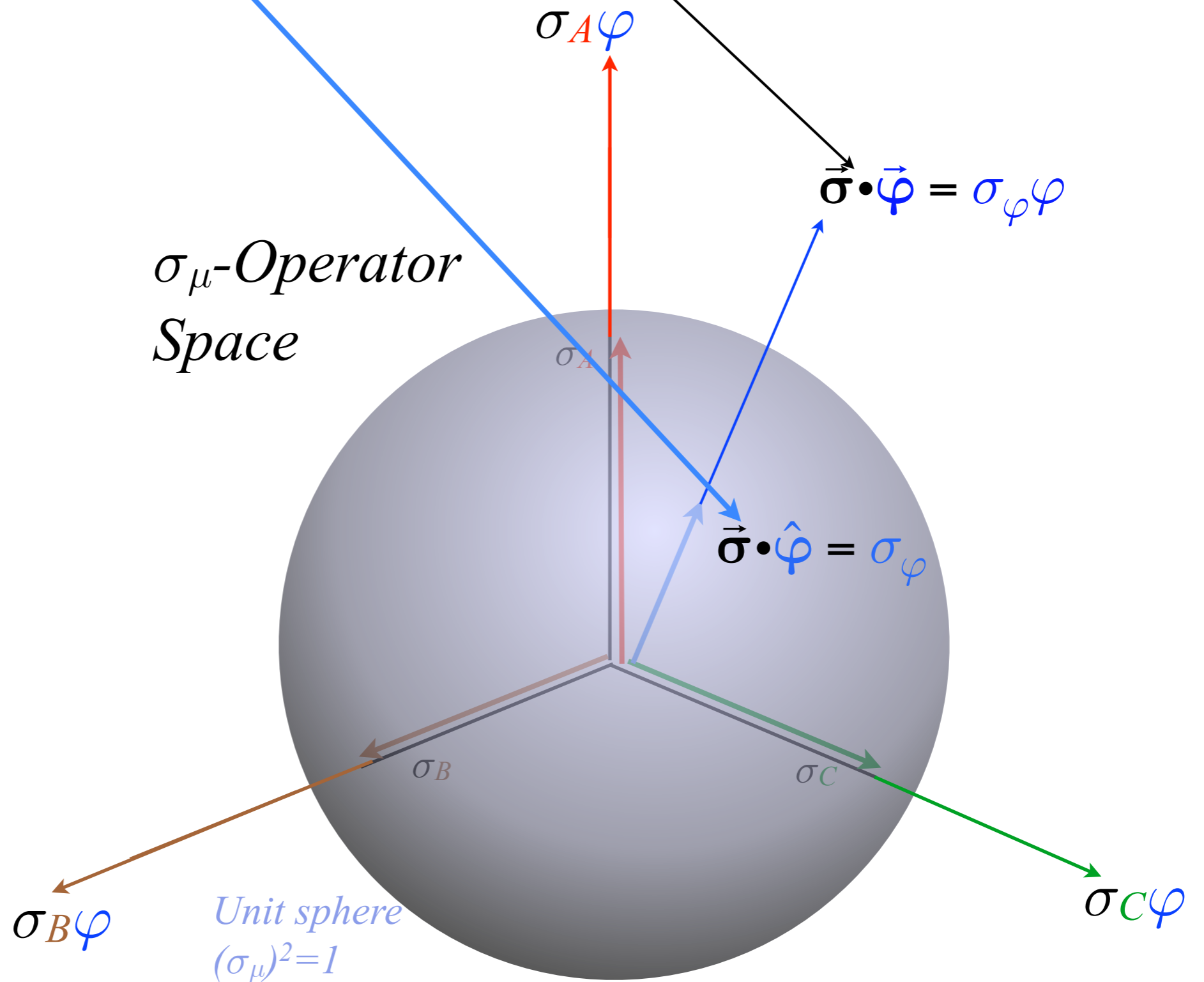
where: $\vec{\Theta} = \vec{\Omega} \cdot t = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} \cdot t = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix} \cdot t$ and: $\Omega_0 = \frac{A+D}{2}$

Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \vec{\sigma} \cdot \hat{\varphi} = \vec{\sigma} \cdot \vec{\varphi} / \varphi$ where: $(\sigma_\varphi)^2 = 1$



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \vec{\sigma} \cdot \hat{\varphi} = \vec{\sigma} \cdot \vec{\varphi} / \varphi$ where: $(\sigma_\varphi)^2 = 1$

$$\sigma_\varphi \varphi = \sigma_A \varphi_A + \sigma_B \varphi_B + \sigma_C \varphi_C = \vec{\sigma} \cdot \varphi = \vec{\sigma} \cdot \vec{\varphi} \text{ where: } (\sigma_\varphi)^2 = 1 \varphi^2$$

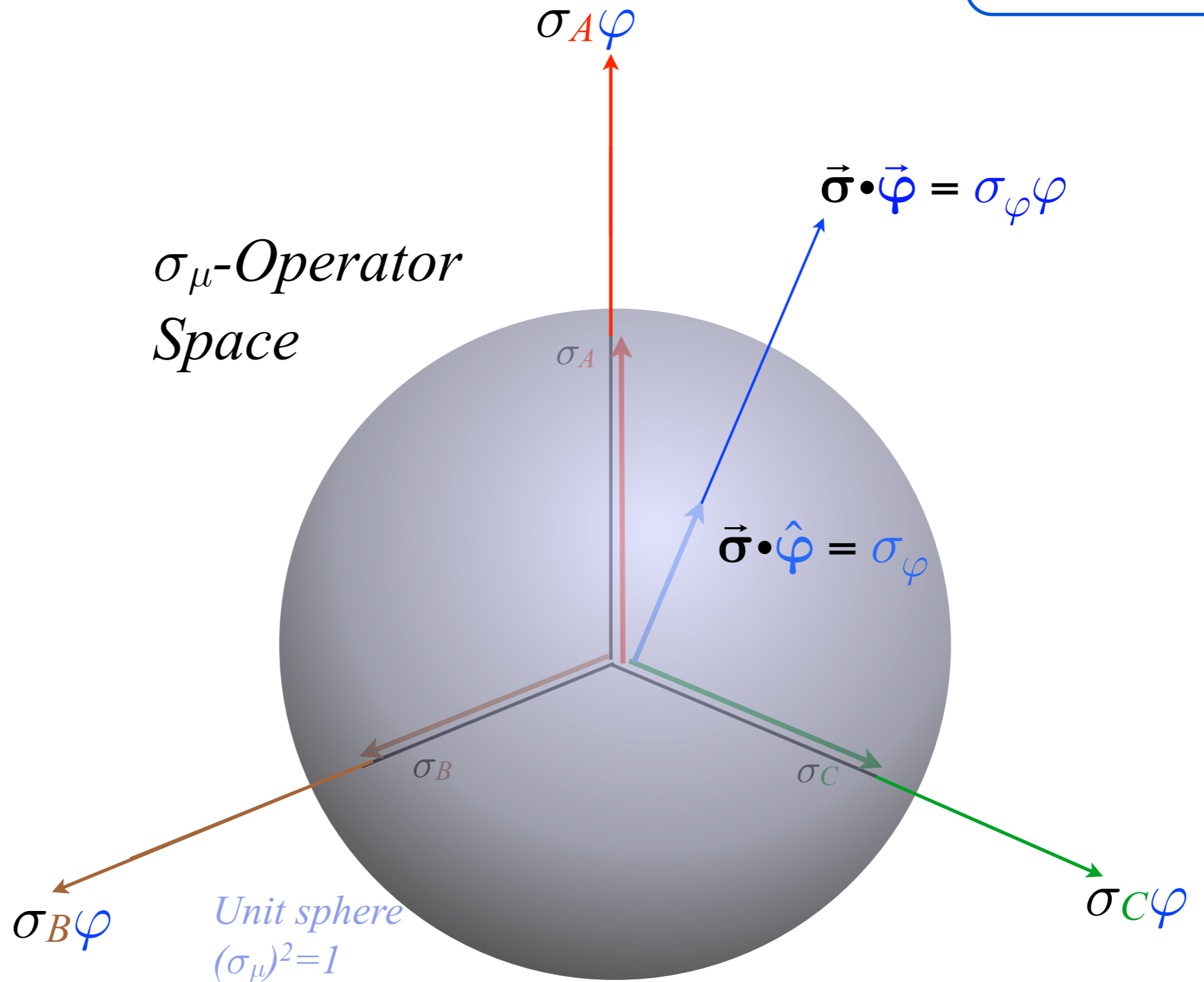


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Crazy Thing: $i\sigma_\varphi = -i\sigma_A \hat{\varphi}_A - i\sigma_B \hat{\varphi}_B - i\sigma_C \hat{\varphi}_C = -i\vec{\sigma} \cdot \hat{\varphi} = -i\vec{\sigma} \cdot \vec{\varphi} / \varphi$

satisfies crazy requirement: $(i\sigma_\varphi)^2 = (-i\sigma_\varphi)^2 = -1$

The Crazy Thing Theorem:
 If $(i\sigma_\varphi)^2 = -1$
 Then:
 $e^{(i\sigma_\varphi)\theta} = 1 \cos\theta + (i\sigma_\varphi) \sin\theta$



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \vec{\sigma} \cdot \hat{\varphi} = \vec{\sigma} \cdot \vec{\varphi} / \varphi$ where: $(\sigma_\varphi)^2 = 1$

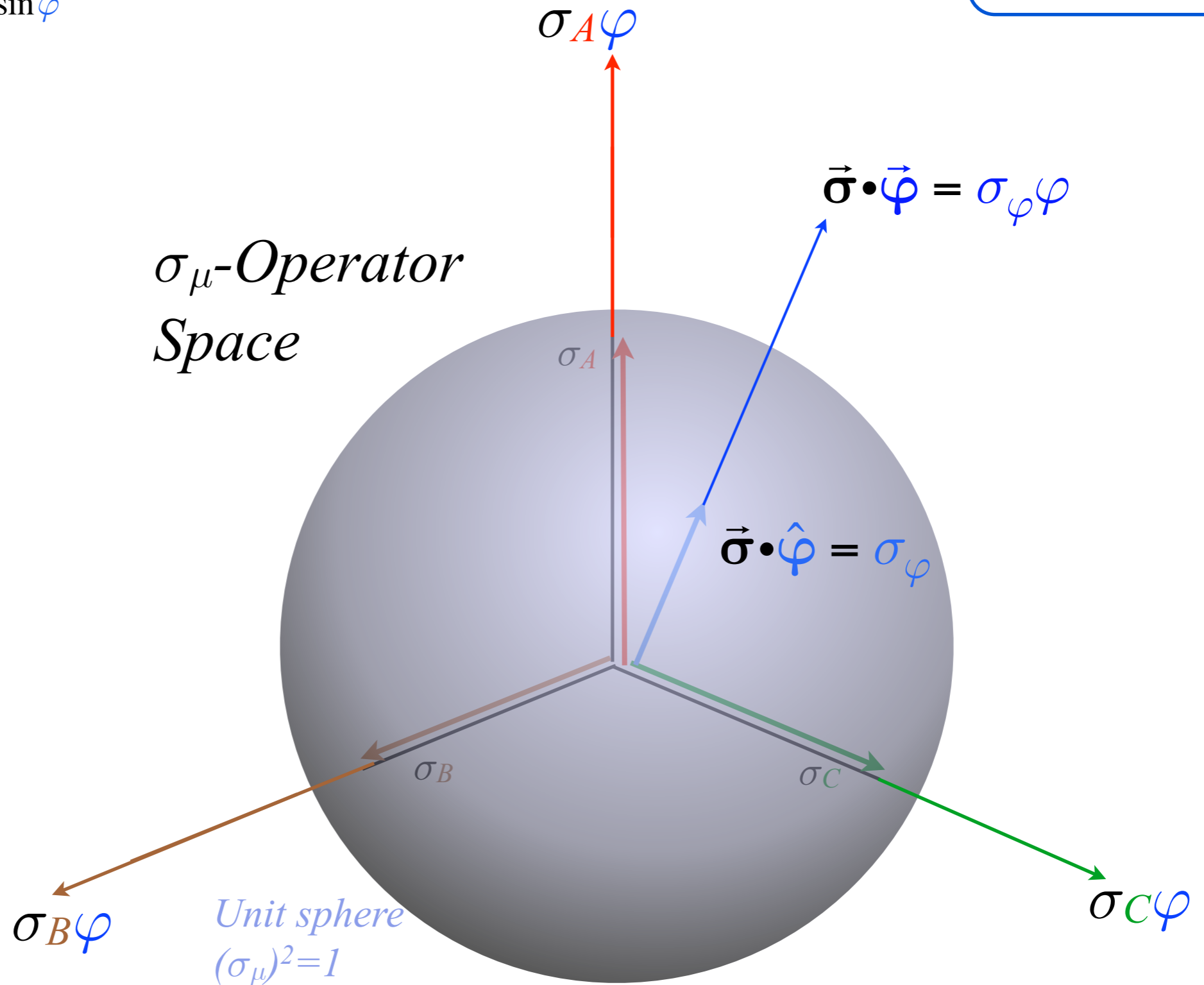
Crazy Thing: $i\sigma_\varphi = -i\sigma_A \hat{\varphi}_A - i\sigma_B \hat{\varphi}_B - i\sigma_C \hat{\varphi}_C = -i\vec{\sigma} \cdot \hat{\varphi} = -i\vec{\sigma} \cdot \vec{\varphi} / \varphi$

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So:

$$e^{-i\sigma_\varphi\varphi} = 1 \cos\varphi - i\sigma_\varphi \sin\varphi$$

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Crazy Thing: $\text{🤪} = -i\sigma_\varphi = -i\sigma_A \hat{\varphi}_A - i\sigma_B \hat{\varphi}_B - i\sigma_C \hat{\varphi}_C = -i\vec{\sigma} \cdot \hat{\varphi} = -i\vec{\sigma} \cdot \vec{\varphi} / \varphi$

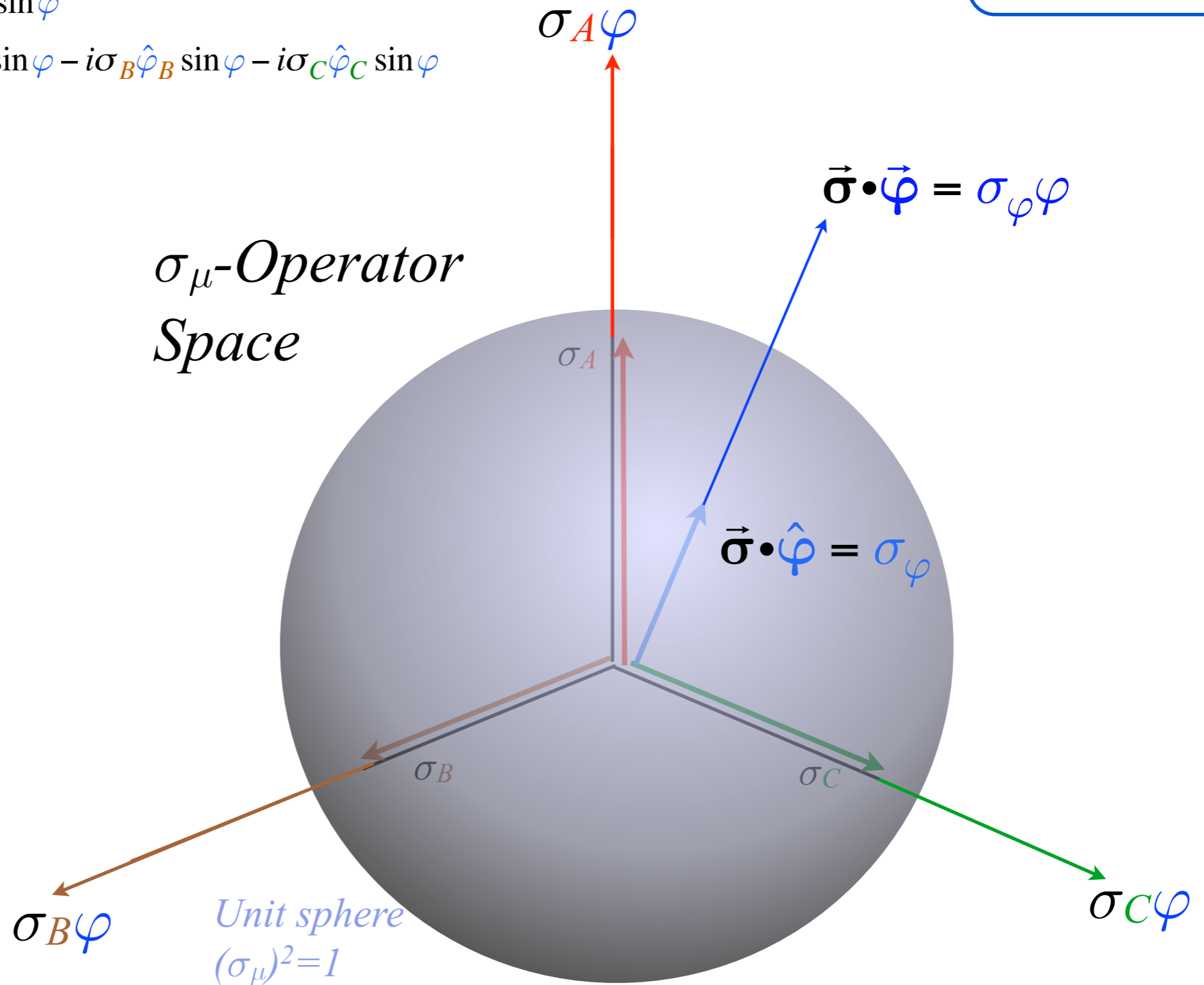
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So:

$$e^{-i\sigma_\varphi\varphi} = 1 \cos \varphi - i\sigma_\varphi \sin \varphi$$

$$= 1 \cos \varphi - i\sigma_A \hat{\varphi}_A \sin \varphi - i\sigma_B \hat{\varphi}_B \sin \varphi - i\sigma_C \hat{\varphi}_C \sin \varphi$$

The  Crazy Thing Theorem:
If $(\text{🤪})^2 = -1$
Then:
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Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \vec{\sigma} \cdot \hat{\varphi} = \vec{\sigma} \cdot \vec{\varphi} / \varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

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So:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1} \cos \varphi - i\sigma_\varphi \sin \varphi$$

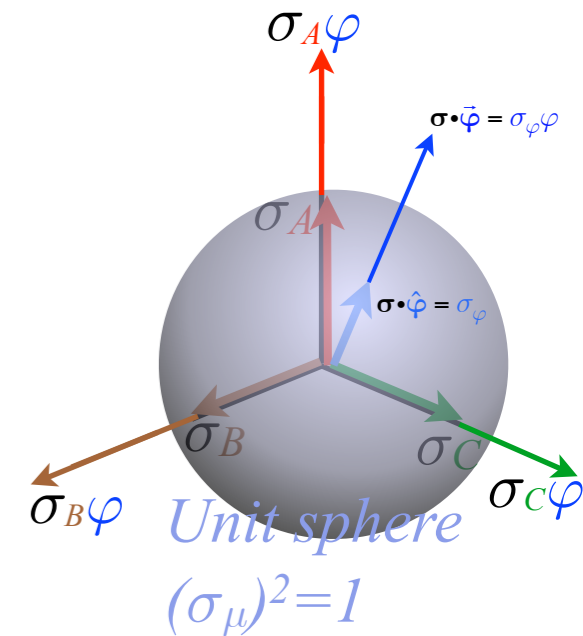
$$= \mathbf{1} \cos \varphi - i\sigma_A \hat{\varphi}_A \sin \varphi - i\sigma_B \hat{\varphi}_B \sin \varphi - i\sigma_C \hat{\varphi}_C \sin \varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \varphi - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\varphi}_A \sin \varphi - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\varphi}_B \sin \varphi - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\varphi}_C \sin \varphi$$

$$= \begin{pmatrix} \cos \varphi - i\hat{\varphi}_A \sin \varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C) \sin \varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C) \sin \varphi & \cos \varphi + i\hat{\varphi}_A \sin \varphi \end{pmatrix}$$

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 Then:
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σ_μ -Operator Space



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \vec{\sigma} \cdot \hat{\varphi} = \vec{\sigma} \cdot \vec{\varphi} / \varphi$ where: $(\sigma_\varphi)^2 = \mathbf{1}$

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$$= \begin{pmatrix} \cos \varphi - i\hat{\varphi}_A \sin \varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C) \sin \varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C) \sin \varphi & \cos \varphi + i\hat{\varphi}_A \sin \varphi \end{pmatrix}$$

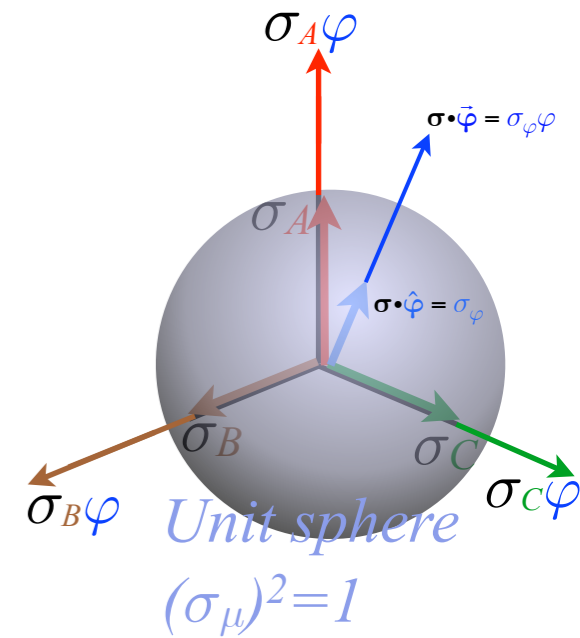
Case A

$$= \begin{pmatrix} \cos \varphi - i \sin \varphi & 0 \\ 0 & \cos \varphi + i \sin \varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_A$$

$$= \mathbf{R}_A = e^{-i\sigma_A \varphi}$$

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Then:
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σ_μ -Operator Space



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satisfies crazy requirement: $(i\sigma_\varphi)^2 = (-i\sigma_\varphi)^2 = -1$

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$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \varphi - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\varphi}_A \sin \varphi - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\varphi}_B \sin \varphi - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\varphi}_C \sin \varphi$$

$$= \begin{pmatrix} \cos \varphi - i\hat{\varphi}_A \sin \varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C) \sin \varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C) \sin \varphi & \cos \varphi + i\hat{\varphi}_A \sin \varphi \end{pmatrix}$$

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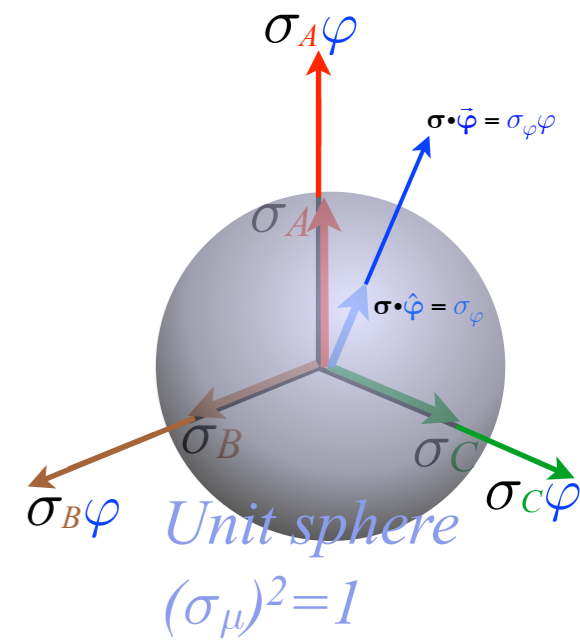
$$= \mathbf{R}_A = e^{-i\sigma_A \varphi}$$

Case B

$$= \mathbf{R}_B = e^{-i\sigma_B \varphi} = \begin{pmatrix} \cos \varphi & -i \sin \varphi \\ -i \sin \varphi & \cos \varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_B$$

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σ_μ -Operator Space



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Case C

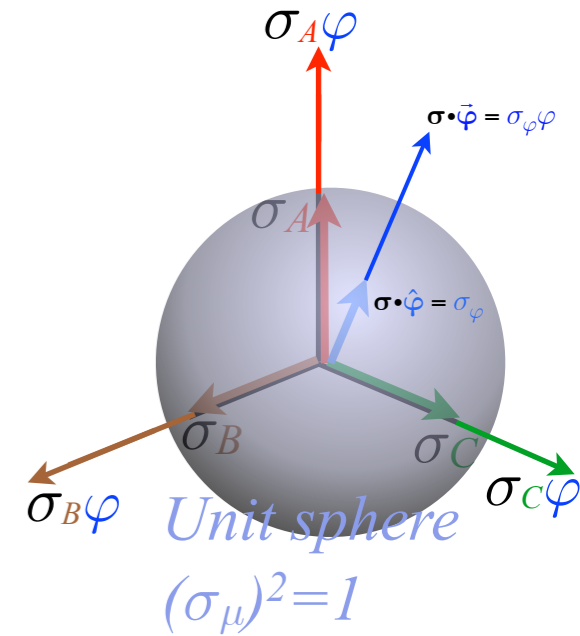
$$= \mathbf{R}_C = e^{-i\sigma_C \varphi} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_C$$

The Crazy Thing Theorem:
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Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \vec{\sigma} \cdot \hat{\varphi} = \vec{\sigma} \cdot \vec{\varphi} / \varphi$ where: $(\sigma_\varphi)^2 = 1$

Crazy Thing: $i\sigma_\varphi = -i\sigma_A \hat{\varphi}_A - i\sigma_B \hat{\varphi}_B - i\sigma_C \hat{\varphi}_C = -i\vec{\sigma} \cdot \hat{\varphi} = -i\vec{\sigma} \cdot \vec{\varphi} / \varphi$

satisfies crazy requirement: $(i\sigma_\varphi)^2 = (-i\sigma_\varphi)^2 = -1$

So:

$$e^{-i\sigma_\varphi\varphi} = 1 \cos \varphi - i\sigma_\varphi \sin \varphi$$

$$= 1 \cos \varphi - i\sigma_A \hat{\varphi}_A \sin \varphi - i\sigma_B \hat{\varphi}_B \sin \varphi - i\sigma_C \hat{\varphi}_C \sin \varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \varphi - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\varphi}_A \sin \varphi - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\varphi}_B \sin \varphi - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\varphi}_C \sin \varphi$$

$$= \begin{pmatrix} \cos \varphi - i\hat{\varphi}_A \sin \varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C) \sin \varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C) \sin \varphi & \cos \varphi + i\hat{\varphi}_A \sin \varphi \end{pmatrix}$$

Case A

$$= \begin{pmatrix} \cos \varphi - i \sin \varphi & 0 \\ 0 & \cos \varphi + i \sin \varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_A$$

$$= \mathbf{R}_A = e^{-i\sigma_A \varphi}$$

Case B

$$= \mathbf{R}_B = e^{-i\sigma_B \varphi} = \begin{pmatrix} \cos \varphi & -i \sin \varphi \\ -i \sin \varphi & \cos \varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_B$$

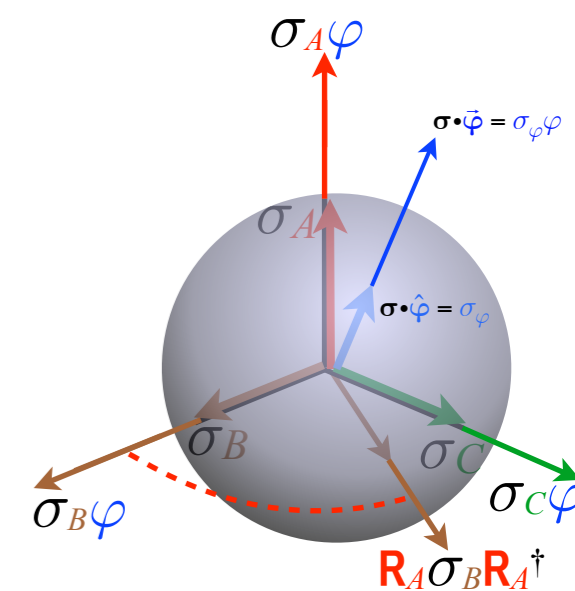
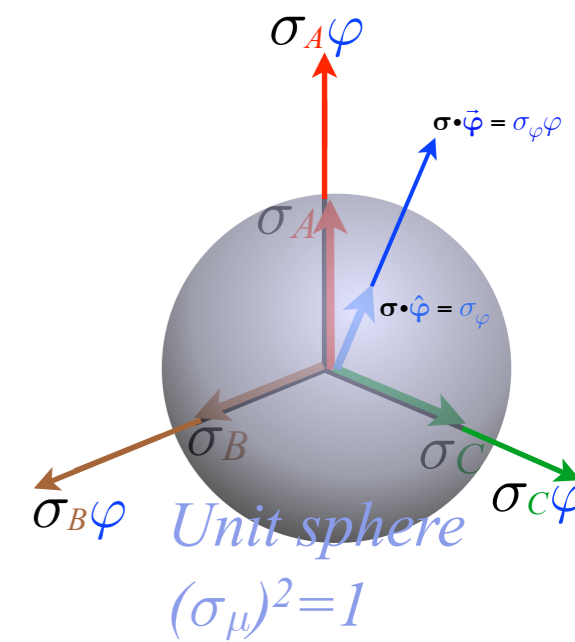
Case C

$$= \mathbf{R}_C = e^{-i\sigma_C \varphi} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_C$$

$\mathbf{R}_A \sigma_B \mathbf{R}_A^\dagger$ example: σ_B rotated by $\mathbf{R}_A = e^{-i\sigma_A \varphi}$ shows 3D-space double angle $\Theta = 2\varphi$

The Crazy Thing Theorem:
If $(i\sigma_\varphi)^2 = -1$
Then:
 $e^{(i\sigma_\varphi)\theta} = 1 \cos \theta + (i\sigma_\varphi) \sin \theta$

σ_μ -Operator Space



Not-so-Crazy Thing: $\sigma_\varphi = \sigma_A \hat{\varphi}_A + \sigma_B \hat{\varphi}_B + \sigma_C \hat{\varphi}_C = \vec{\sigma} \cdot \hat{\varphi} = \vec{\sigma} \cdot \vec{\varphi} / \varphi$ where: $(\sigma_\varphi)^2 = 1$

Crazy Thing: $\text{☺} = -i\sigma_\varphi = -i\sigma_A \hat{\varphi}_A - i\sigma_B \hat{\varphi}_B - i\sigma_C \hat{\varphi}_C = -i\vec{\sigma} \cdot \hat{\varphi} = -i\vec{\sigma} \cdot \vec{\varphi} / \varphi$

satisfies crazy requirement: $(\text{☺})^2 = (-i\sigma_\varphi)^2 = -1$

So:

$$e^{-i\sigma_\varphi\varphi} = 1 \cos \varphi - i\sigma_\varphi \sin \varphi$$

$$= 1 \cos \varphi - i\sigma_A \hat{\varphi}_A \sin \varphi - i\sigma_B \hat{\varphi}_B \sin \varphi - i\sigma_C \hat{\varphi}_C \sin \varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \varphi - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\varphi}_A \sin \varphi - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\varphi}_B \sin \varphi - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\varphi}_C \sin \varphi$$

$$= \begin{pmatrix} \cos \varphi - i\hat{\varphi}_A \sin \varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C) \sin \varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C) \sin \varphi & \cos \varphi + i\hat{\varphi}_A \sin \varphi \end{pmatrix}$$

Case A

$$= \begin{pmatrix} \cos \varphi - i \sin \varphi & 0 \\ 0 & \cos \varphi + i \sin \varphi \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_A$$

$$= \mathbf{R}_A = e^{-i\sigma_A \varphi}$$

Case B

$$= \mathbf{R}_B = e^{-i\sigma_B \varphi} = \begin{pmatrix} \cos \varphi & -i \sin \varphi \\ -i \sin \varphi & \cos \varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_B$$

Case C

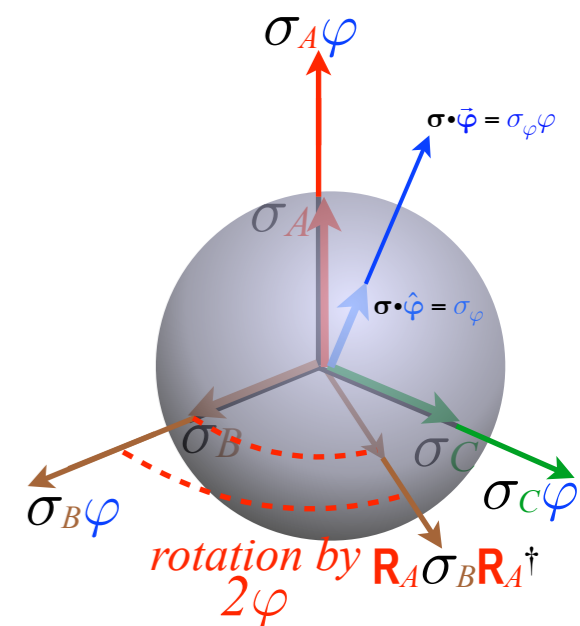
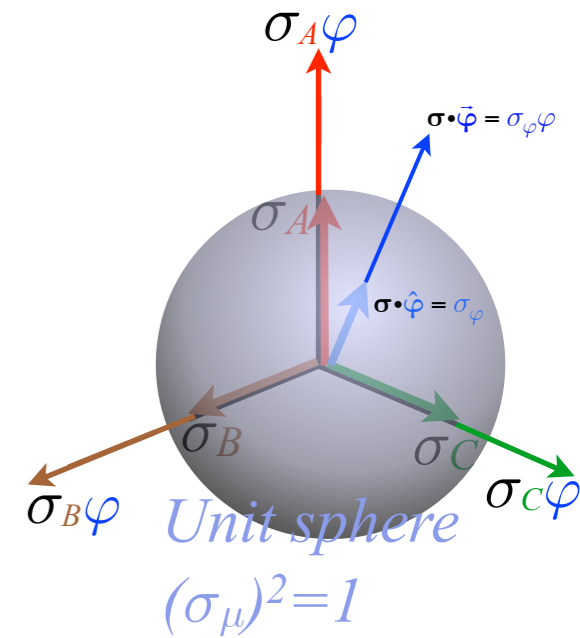
$$= \mathbf{R}_C = e^{-i\sigma_C \varphi} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \text{ for: } \hat{\varphi} = \begin{pmatrix} \hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ or: } \vec{\sigma} \cdot \hat{\varphi} = \sigma_C$$

$\mathbf{R}_A \sigma_B \mathbf{R}_A^\dagger$ example: σ_B rotated by $\mathbf{R}_A = e^{-i\sigma_A \varphi}$ shows 3D-space double angle $\Theta = 2\varphi$

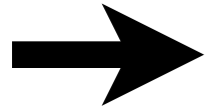
$$e^{-i\sigma_A \varphi} \sigma_B e^{+i\sigma_A \varphi} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{+i\varphi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{+i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{+i\varphi} & 0 \end{pmatrix} \begin{pmatrix} e^{+i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i2\varphi} \\ e^{+i2\varphi} & 0 \end{pmatrix} = \sigma_B \cos 2\varphi + \sigma_C \sin 2\varphi$$

The Crazy Thing Theorem:
If $(\text{☺})^2 = -1$
Then:
 $e^{(\text{☺})\theta} = 1 \cos \theta + (\text{☺}) \sin \theta$

σ_μ -Operator Space



Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices



Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $q_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ of spinor σ_μ -operators

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ of group-operators

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

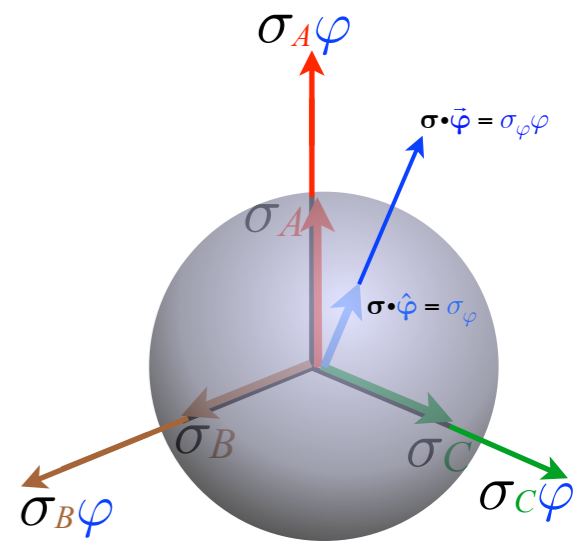
Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Half-angle $\Theta/2 = \varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\vec{\sigma} \cdot \vec{\varphi}} = e^{-i\vec{\sigma} \cdot \hat{\varphi} \varphi}$ where: $\sigma_\varphi = \vec{\sigma} \cdot \vec{\varphi} = \sigma_A \varphi_A + \sigma_B \varphi_B + \sigma_C \varphi_C = \vec{\sigma} \cdot \hat{\varphi} \varphi$

Replace spinor angle φ in: $e^{-i\vec{\sigma} \cdot \vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1} \cos \varphi - i\vec{\sigma} \cdot \hat{\varphi} \sin \varphi$

with 3D $\frac{1}{2}$ -angle $\frac{\Theta}{2} = \varphi$

σ_μ -Operator Space



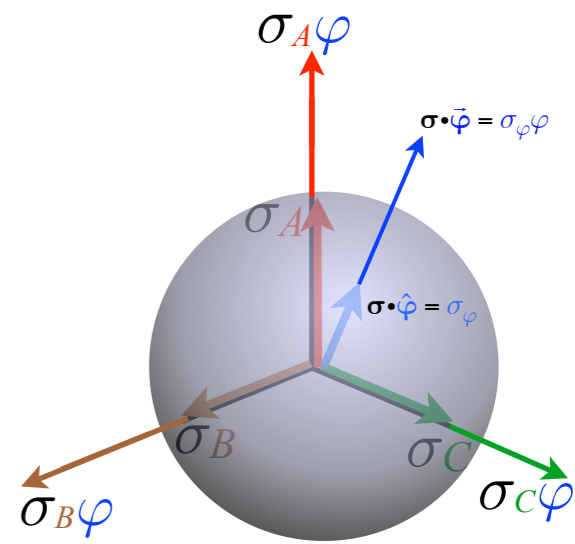
The Crazy Thing Theorem:
 If $(\text{🤪})^2 = -\mathbf{1}$
 Then:
 $e(\text{🤪})\theta = \mathbf{1} \cos \theta + (\text{🤪}) \sin \theta$

Half-angle $\Theta/2 = \varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\vec{\sigma} \cdot \vec{\varphi}} = e^{-i\vec{\sigma} \cdot \hat{\varphi} \varphi}$ where: $\sigma_\varphi = \vec{\sigma} \cdot \vec{\varphi} = \sigma_A \varphi_A + \sigma_B \varphi_B + \sigma_C \varphi_C = \vec{\sigma} \cdot \hat{\varphi} \varphi$

Replace spinor angle φ in: $e^{-i\vec{\sigma} \cdot \vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1} \cos \varphi - i\vec{\sigma} \cdot \hat{\varphi} \sin \varphi$

with 3D $\frac{1}{2}$ -angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\vec{\sigma} \cdot \frac{\vec{\Theta}}{2}} = \mathbf{R}[\frac{\vec{\Theta}}{2}] = \mathbf{1} \cos \frac{\Theta}{2} - i\vec{\sigma} \cdot \hat{\Theta} \sin \frac{\Theta}{2}$

σ_μ -Operator Space



The Crazy Thing Theorem:
 If $(\text{🤪})^2 = -\mathbf{1}$
 Then:
 $e(\text{🤪})\theta = \mathbf{1} \cos \theta + (\text{🤪}) \sin \theta$

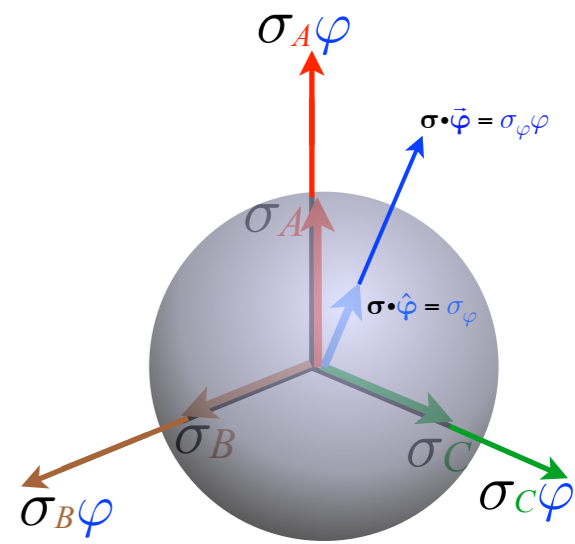
Half-angle $\Theta/2 = \varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\vec{\sigma} \cdot \vec{\varphi}} = e^{-i\vec{\sigma} \cdot \hat{\varphi} \varphi}$ where: $\sigma_\varphi = \vec{\sigma} \cdot \vec{\varphi} = \sigma_A \varphi_A + \sigma_B \varphi_B + \sigma_C \varphi_C = \vec{\sigma} \cdot \hat{\varphi} \varphi$


Replace spinor angle φ in: $e^{-i\vec{\sigma} \cdot \vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1} \cos \varphi - i\vec{\sigma} \cdot \hat{\varphi} \sin \varphi$

with 3D $\frac{1}{2}$ -angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\vec{\sigma} \cdot \frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1} \cos \frac{\Theta}{2} - i\vec{\sigma} \cdot \hat{\Theta} \sin \frac{\Theta}{2}$

Unit rotation axis vector $\hat{\Theta} = (\hat{\Theta}_X, \hat{\Theta}_Y, \hat{\Theta}_Z) = (\cos \varphi \sin \vartheta, \sin \varphi \sin \vartheta, \cos \vartheta)$ is the same as the unit $\hat{\varphi}$.

σ_μ -Operator Space



The 
Crazy Thing
Theorem:
If $(\img alt="smiley face" data-bbox="415 830 455 875})^2 = -\mathbf{1}$
Then:
 $e(\img alt="smiley face" data-bbox="350 905 390 945})\theta = \mathbf{1} \cos \theta + (\img alt="smiley face" data-bbox="480 910 520 945}) \sin \theta$

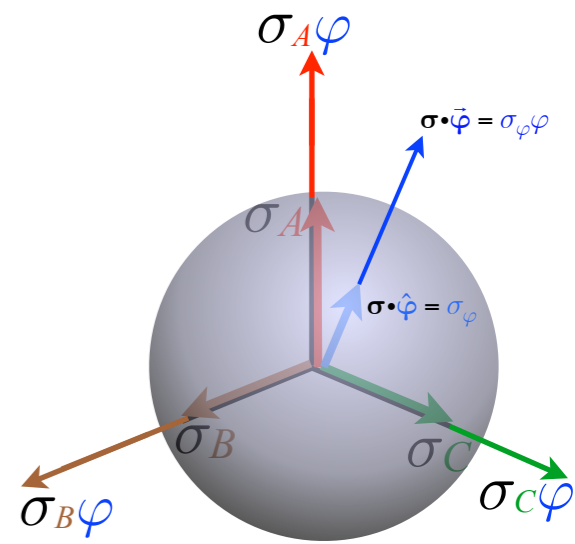
Half-angle $\Theta/2 = \varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\vec{\sigma} \cdot \vec{\varphi}} = e^{-i\vec{\sigma} \cdot \hat{\varphi} \varphi}$ where: $\sigma_\varphi = \vec{\sigma} \cdot \vec{\varphi} = \sigma_A \varphi_A + \sigma_B \varphi_B + \sigma_C \varphi_C = \vec{\sigma} \cdot \hat{\varphi} \varphi$


Replace spinor angle φ in: $e^{-i\vec{\sigma} \cdot \vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1} \cos \varphi - i\vec{\sigma} \cdot \hat{\varphi} \sin \varphi$

with 3D $\frac{1}{2}$ -angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\vec{\sigma} \cdot \frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1} \cos \frac{\Theta}{2} - i\vec{\sigma} \cdot \hat{\Theta} \sin \frac{\Theta}{2} = e^{-i\frac{\vec{\sigma}}{2} \cdot \vec{\Theta}} = e^{-i\mathbf{S} \cdot \vec{\Theta}}$

Unit rotation axis vector $\hat{\Theta} = (\hat{\Theta}_X, \hat{\Theta}_Y, \hat{\Theta}_Z) = (\cos \varphi \sin \vartheta, \sin \varphi \sin \vartheta, \cos \vartheta)$ is the same as the unit $\hat{\varphi}$.

σ_μ -Operator Space



The 
Crazy Thing
Theorem:
If $(\img alt="smiley face" data-bbox="415 835 450 880})^2 = -\mathbf{1}$
Then:
 $e(\img alt="smiley face" data-bbox="350 905 385 945})\theta = \mathbf{1} \cos \theta + (\img alt="smiley face" data-bbox="485 910 520 945}) \sin \theta$

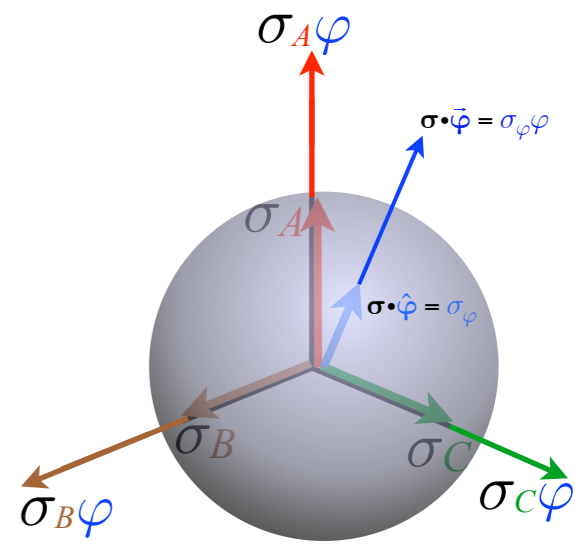
Half-angle $\Theta/2 = \varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\sigma_\varphi\varphi} = e^{-i\sigma \cdot \hat{\varphi}\varphi}$ where: $\sigma_\varphi = \vec{\sigma} \cdot \vec{\varphi} = \sigma_A\varphi_A + \sigma_B\varphi_B + \sigma_C\varphi_C = \vec{\sigma} \cdot \hat{\varphi}\varphi$


Replace spinor angle φ in: $e^{-i\vec{\sigma} \cdot \vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1} \cos \varphi - i\vec{\sigma} \cdot \hat{\varphi} \sin \varphi$

with 3D $\frac{1}{2}$ -angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\vec{\sigma} \cdot \frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1} \cos \frac{\Theta}{2} - i\vec{\sigma} \cdot \hat{\Theta} \sin \frac{\Theta}{2} = e^{-i\frac{\vec{\sigma}}{2} \cdot \vec{\Theta}} = e^{-i\mathbf{S} \cdot \vec{\Theta}} = e^{-i\sigma_\varphi\varphi} = e^{-i\vec{\sigma} \cdot \vec{\varphi}}$

Unit rotation axis vector $\hat{\Theta} = (\hat{\Theta}_X, \hat{\Theta}_Y, \hat{\Theta}_Z) = (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta)$ is the same as the unit $\hat{\varphi}$.

σ_μ -Operator Space



The 
Crazy Thing
Theorem:
If $(\img alt="smiley face" data-bbox="415 830 450 870})^2 = -\mathbf{1}$
Then:
 $e^{(\img alt="smiley face" data-bbox="350 905 385 945})\theta} = \mathbf{1} \cos\theta + (\img alt="smiley face" data-bbox="480 910 515 950}) \sin\theta$

Half-angle $\Theta/2 = \varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\sigma_\varphi\varphi} = e^{-i\vec{\sigma}\cdot\hat{\varphi}\varphi}$ where: $\sigma_\varphi = \vec{\sigma}\cdot\vec{\varphi} = \sigma_A\varphi_A + \sigma_B\varphi_B + \sigma_C\varphi_C = \vec{\sigma}\cdot\hat{\varphi}\varphi$

Replace spinor angle φ in: $e^{-i\vec{\sigma}\cdot\vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1}\cos\varphi - i\vec{\sigma}\cdot\hat{\varphi}\sin\varphi$

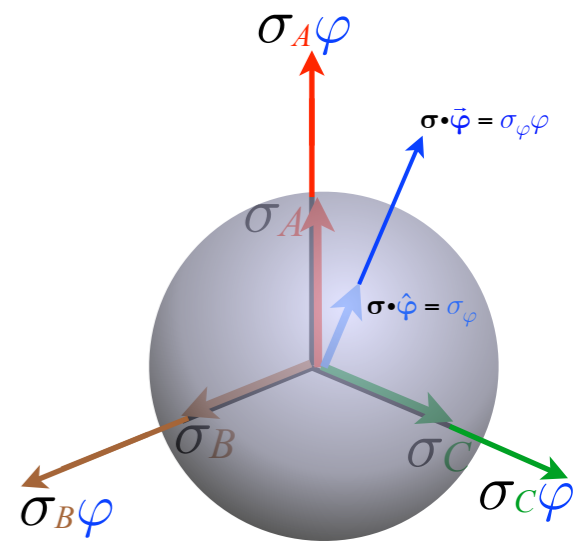
with 3D $\frac{1}{2}$ -angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\vec{\sigma}\cdot\frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1}\cos\frac{\Theta}{2} - i\vec{\sigma}\cdot\hat{\Theta}\sin\frac{\Theta}{2} = e^{-i\frac{\vec{\sigma}}{2}\cdot\vec{\Theta}} = e^{-i\mathbf{S}\cdot\vec{\Theta}} = e^{-i\sigma_\varphi\varphi} = e^{-i\vec{\sigma}\cdot\vec{\varphi}}$

Unit rotation axis vector $\hat{\Theta} = (\hat{\Theta}_X, \hat{\Theta}_Y, \hat{\Theta}_Z) = (\cos\varphi\sin\vartheta, \sin\varphi\sin\vartheta, \cos\vartheta)$ is the same as the unit $\hat{\varphi}$.

$$\mathbf{R}[\vec{\Theta}] = \cos\frac{\Theta}{2} \mathbf{1} - i\sigma_X \hat{\Theta}_X \sin\frac{\Theta}{2} - i\sigma_Y \hat{\Theta}_Y \sin\frac{\Theta}{2} - i\sigma_Z \hat{\Theta}_Z \sin\frac{\Theta}{2}$$

$$= \cos\frac{\Theta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\Theta}_X \sin\frac{\Theta}{2} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\Theta}_Y \sin\frac{\Theta}{2} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\Theta}_Z \sin\frac{\Theta}{2}$$

σ_μ -Operator Space



The Crazy Thing Theorem:
 If $(\text{smiley})^2 = -\mathbf{1}$
 Then:
 $e(\text{smiley})\theta = \mathbf{1}\cos\theta + (\text{smiley})\sin\theta$

Half-angle $\Theta/2 = \varphi$ replacement in rotation $\mathbf{R}_\varphi = e^{-i\sigma_\varphi\varphi} = e^{-i\sigma \cdot \hat{\varphi}\varphi}$ where: $\sigma_\varphi = \vec{\sigma} \cdot \vec{\varphi} = \sigma_A\varphi_A + \sigma_B\varphi_B + \sigma_C\varphi_C = \vec{\sigma} \cdot \hat{\varphi}\varphi$

Replace spinor angle φ in: $e^{-i\vec{\sigma} \cdot \vec{\varphi}} = \mathbf{R}_\varphi = \mathbf{1} \cos \varphi - i\vec{\sigma} \cdot \hat{\varphi} \sin \varphi$

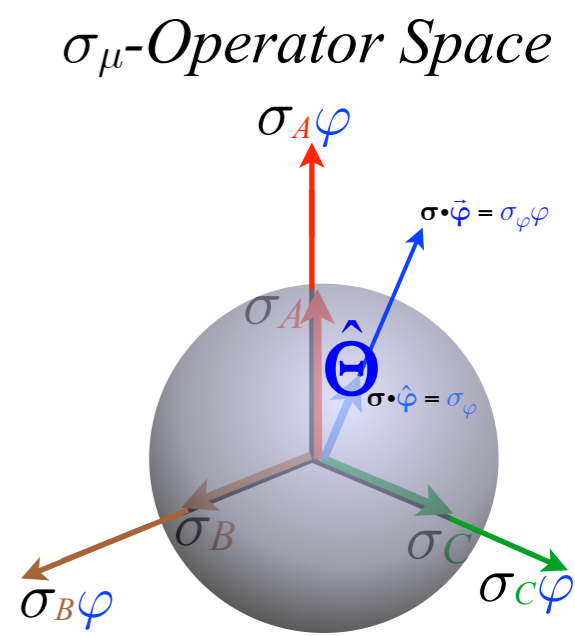
with 3D $\frac{1}{2}$ -angle $\frac{\Theta}{2} = \varphi$ in: $e^{-i\vec{\sigma} \cdot \frac{\vec{\Theta}}{2}} = \mathbf{R}[\vec{\Theta}] = \mathbf{1} \cos \frac{\Theta}{2} - i\vec{\sigma} \cdot \hat{\Theta} \sin \frac{\Theta}{2} = e^{-i\frac{\vec{\sigma}}{2} \cdot \vec{\Theta}} = e^{-i\mathbf{S} \cdot \vec{\Theta}} = e^{-i\sigma_\varphi\varphi} = e^{-i\vec{\sigma} \cdot \vec{\varphi}}$

Unit rotation axis vector $\hat{\Theta} = (\hat{\Theta}_X, \hat{\Theta}_Y, \hat{\Theta}_Z) = (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta)$ is the same as the unit $\hat{\varphi}$.

$$\mathbf{R}[\vec{\Theta}] = \cos \frac{\Theta}{2} \mathbf{1} - i \sigma_X \hat{\Theta}_X \sin \frac{\Theta}{2} - i \sigma_Y \hat{\Theta}_Y \sin \frac{\Theta}{2} - i \sigma_Z \hat{\Theta}_Z \sin \frac{\Theta}{2}$$

$$= \cos \frac{\Theta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\Theta}_X \sin \frac{\Theta}{2} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\Theta}_Y \sin \frac{\Theta}{2} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\Theta}_Z \sin \frac{\Theta}{2}$$

$$\begin{pmatrix} \langle 1 | \mathbf{R}[\vec{\Theta}] | 1 \rangle & \langle 1 | \mathbf{R}[\vec{\Theta}] | 2 \rangle \\ \langle 2 | \mathbf{R}[\vec{\Theta}] | 1 \rangle & \langle 2 | \mathbf{R}[\vec{\Theta}] | 2 \rangle \end{pmatrix} = \begin{pmatrix} \cos \frac{\Theta}{2} - i \hat{\Theta}_Z \sin \frac{\Theta}{2} & -i \sin \frac{\Theta}{2} (\hat{\Theta}_X - i \hat{\Theta}_Y) \\ -i \sin \frac{\Theta}{2} (\hat{\Theta}_X + i \hat{\Theta}_Y) & \cos \frac{\Theta}{2} + i \hat{\Theta}_Z \sin \frac{\Theta}{2} \end{pmatrix}$$



The Crazy Thing Theorem:
If $(i)^2 = -1$
Then:
 $e^{(i)\theta} = \mathbf{1} \cos \theta + (i) \sin \theta$

$$\hat{\Theta}_X = \cos \varphi \sin \vartheta$$

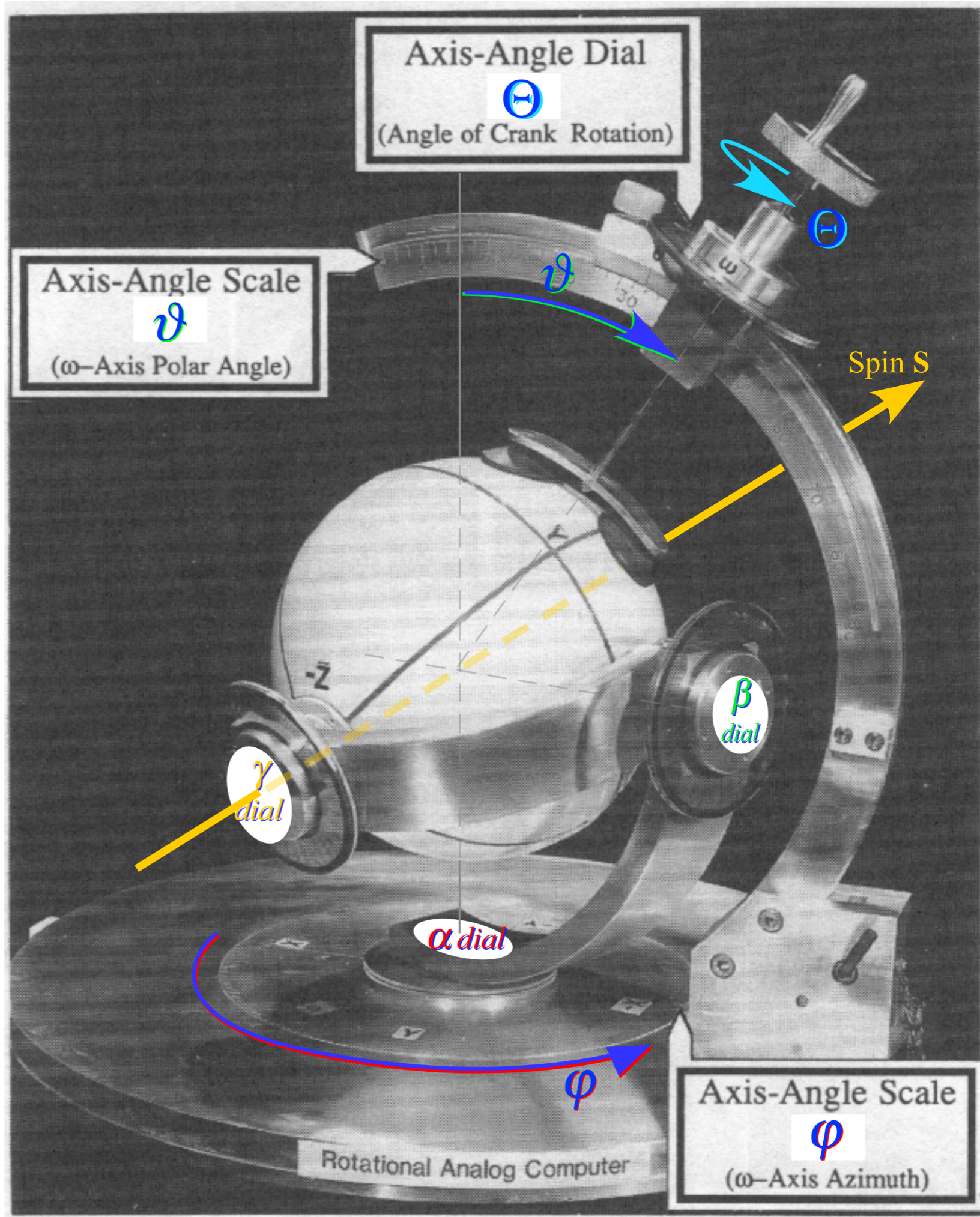
$$\hat{\Theta}_Y = \sin \varphi \sin \vartheta$$

$$\hat{\Theta}_Z = \cos \vartheta$$

Polar coordinates for unit axis vector $\hat{\Theta}$

*Polar coordinates
for unit axis vector $\hat{\Theta}$*

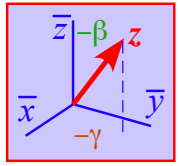
$$\begin{aligned} \hat{\Theta}_x &= \cos\varphi \sin\vartheta \\ \hat{\Theta}_y &= \sin\varphi \sin\vartheta \\ \hat{\Theta}_z &= \cos\vartheta \end{aligned}$$



Here spin-rotor S -polar coordinates are Euler angles

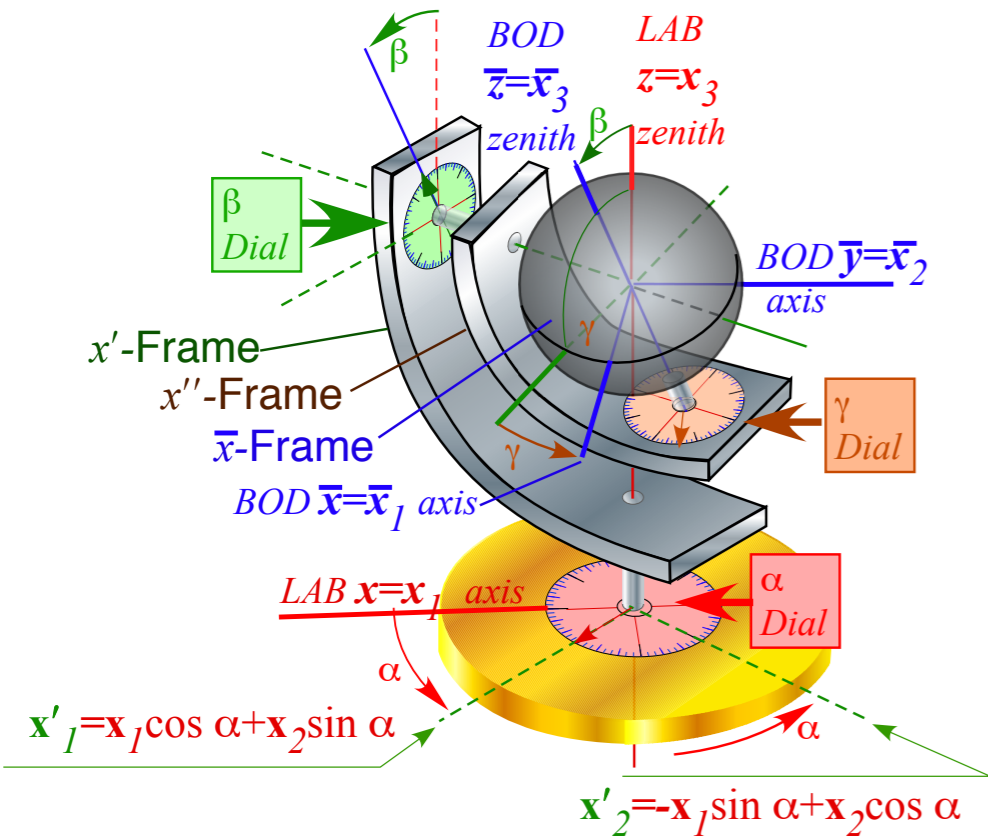
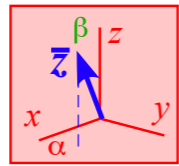
BOD frame view

Polar angles of LAB zenith $\bar{z}=\bar{x}_3$ are (azimuth angle= $-\gamma$, polar angle= $-\beta$)



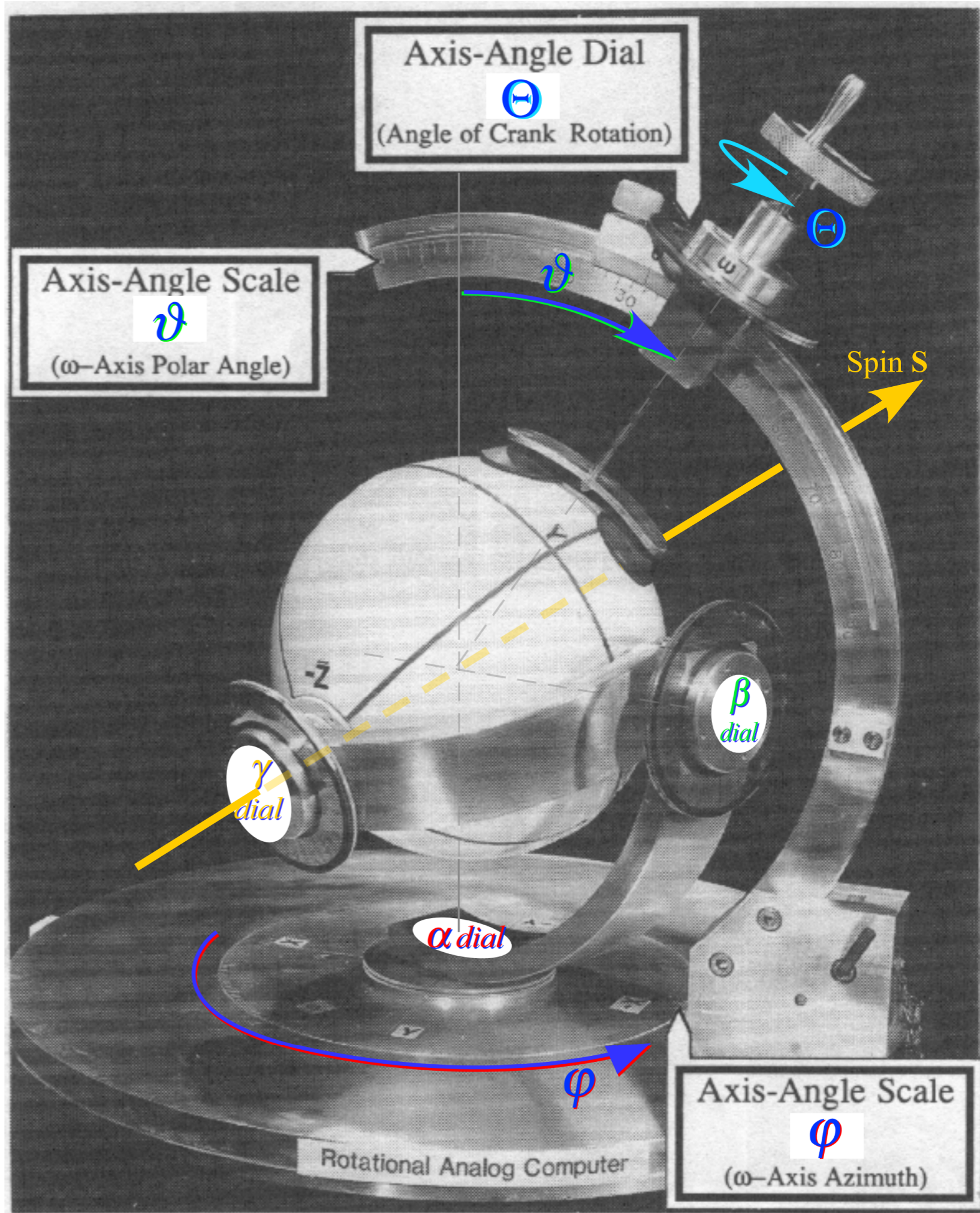
LAB frame view

Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ are (azimuth angle= α , polar angle= β)



Polar coordinates for unit axis vector $\hat{\Theta}$

$$\begin{aligned} \hat{\Theta}_x &= \cos \varphi \sin \vartheta \\ \hat{\Theta}_y &= \sin \varphi \sin \vartheta \\ \hat{\Theta}_z &= \cos \vartheta \end{aligned}$$



Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

→ Product algebra for Pauli's σ_μ and Hamilton's $q_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula

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Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's " σ_μ -quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	q_X	q_Y	q_Z		•	1	σ_X	σ_Y	σ_Z
1	1	q_X	q_Y	q_Z	,	1	1	σ_X	σ_Y	σ_Z
q_X	q_X	-1	q_Z	-q_Y		σ_X	σ_X	1	iσ_Z	-iσ_Y
q_Y	q_Y	-q_Z	-1	q_X		σ_Y	σ_Y	-iσ_Z	1	iσ_X
q_Z	q_Z	q_Y	-q_X	-1		σ_Z	σ_Z	iσ_Y	-iσ_X	1

Operator-on-Operator transformations

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•	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	•	1	σ_X	σ_Y	σ_Z
1	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	1	1	σ_X	σ_Y	σ_Z
\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$	σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X	σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
\mathbf{q}_Z	\mathbf{q}_Z	\mathbf{q}_Y	$-\mathbf{q}_X$	-1	σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu} \mathbf{1} + \varepsilon_{\mu\nu\lambda} \mathbf{q}_\lambda$$

$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu} \mathbf{1} + i \varepsilon_{\mu\nu\lambda} \sigma_\lambda$$

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1	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	1	1	σ_X	σ_Y	σ_Z
\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$	σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X	σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
\mathbf{q}_Z	\mathbf{q}_Z	\mathbf{q}_Y	$-\mathbf{q}_X$	-1	σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

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Commutation rules for Pauli ops: σ_μ
 $\sigma_\mu \sigma_\nu - \sigma_\nu \sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i \epsilon_{\mu\nu\lambda} \sigma_\lambda$

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\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X
\mathbf{q}_Z	\mathbf{q}_Z	\mathbf{q}_Y	$-\mathbf{q}_X$	-1

,

•	1	σ_X	σ_Y	σ_Z
1	1	σ_X	σ_Y	σ_Z
σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

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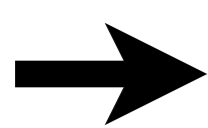
Jordan's spin-ops: $\mathbf{J}_\mu = \mathbf{S}_\mu = \sigma_\mu/2$.

$$\mathbf{S}_\mu \mathbf{S}_\nu - \mathbf{S}_\nu \mathbf{S}_\mu = [\mathbf{S}_\mu, \mathbf{S}_\nu] = i \varepsilon_{\mu\nu\lambda} \mathbf{S}_\lambda$$

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1	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	1	1	σ_X	σ_Y	σ_Z
\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$	σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X	σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
\mathbf{q}_Z	\mathbf{q}_Z	\mathbf{q}_Y	$-\mathbf{q}_X$	-1	σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

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Operator-on-Operator transformations

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1	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z		1	1	σ_X	σ_Y	σ_Z
\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$		σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X		σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
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1	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z		1	1	σ_X	σ_Y	σ_Z
\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$		σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X		σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
\mathbf{q}_Z	\mathbf{q}_Z	\mathbf{q}_Y	$-\mathbf{q}_X$	-1		σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

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Operator-on-Operator transformations

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•	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	•	1	σ_X	σ_Y	σ_Z
1	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	1	1	σ_X	σ_Y	σ_Z
\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$	σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X	σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
\mathbf{q}_Z	\mathbf{q}_Z	\mathbf{q}_Y	$-\mathbf{q}_X$	-1	σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

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$$= \cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} \mathbf{1} - i \left[\hat{\Theta}_a \sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} + \hat{\Theta}_b \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} \right] \cdot \vec{\sigma} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\vec{\sigma} \cdot \hat{\Theta}_a)(\vec{\sigma} \cdot \hat{\Theta}_b)$$

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\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$		σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X		σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
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†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma} \cdot \mathbf{a})(\vec{\sigma} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{1} + i(\mathbf{a} \times \mathbf{b}) \cdot \vec{\sigma}$ reduces $(\vec{\sigma} \cdot \hat{\Theta}_a)(\vec{\sigma} \cdot \hat{\Theta}_b)$ to: $(\hat{\Theta}_a \cdot \hat{\Theta}_b) \mathbf{1} + (\hat{\Theta}_a \times \hat{\Theta}_b) \cdot \vec{\sigma}$

Operator-on-Operator transformations

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•	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	,	•	1	σ_X	σ_Y	σ_Z
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$$- i \left\{ \hat{\Theta}_a \sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} + \hat{\Theta}_b \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\hat{\Theta}_a \times \hat{\Theta}_b) \right\} \cdot \vec{\sigma}$$

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\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X	σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
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$$\mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1} \cos \frac{\Theta_a}{2} - i(\vec{\sigma} \cdot \hat{\Theta}_a) \sin \frac{\Theta_a}{2} \right) \left(\mathbf{1} \cos \frac{\Theta_b}{2} - i(\vec{\sigma} \cdot \hat{\Theta}_b) \sin \frac{\Theta_b}{2} \right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$= \cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} \mathbf{1} - i \left[\hat{\Theta}_a \sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} + \hat{\Theta}_b \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} \right] \cdot \vec{\sigma} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\vec{\sigma} \cdot \hat{\Theta}_a)(\vec{\sigma} \cdot \hat{\Theta}_b)$$

†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma} \cdot \mathbf{a})(\vec{\sigma} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{1} + i(\mathbf{a} \times \mathbf{b}) \cdot \vec{\sigma}$ reduces $(\vec{\sigma} \cdot \hat{\Theta}_a)(\vec{\sigma} \cdot \hat{\Theta}_b)$ to: $(\hat{\Theta}_a \cdot \hat{\Theta}_b) \mathbf{1} + (\hat{\Theta}_a \times \hat{\Theta}_b) \cdot \vec{\sigma}$

$$= \left(\cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\hat{\Theta}_a \cdot \hat{\Theta}_b) \right) \mathbf{1}$$

$$- i \left\{ \hat{\Theta}_a \sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} + \hat{\Theta}_b \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\hat{\Theta}_a \times \hat{\Theta}_b) \right\} \cdot \vec{\sigma}$$

$$= \left(\cos \frac{\Theta_{ab}}{2} \right) \mathbf{1} - i \left\{ \hat{\Theta}_{ab} \sin \frac{\Theta_{ab}}{2} \right\} \cdot \vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's " σ_μ -quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z		•	1	σ_X	σ_Y	σ_Z
1	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	,	1	1	σ_X	σ_Y	σ_Z
\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$		σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X		σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
\mathbf{q}_Z	\mathbf{q}_Z	\mathbf{q}_Y	$-\mathbf{q}_X$	-1		σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu} \mathbf{1} + \epsilon_{\mu\nu\lambda} \mathbf{q}_\lambda$$

$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu} \mathbf{1} + i \epsilon_{\mu\nu\lambda} \sigma_\lambda$$

Commutation rules for Pauli ops: σ_μ
 $\sigma_\mu \sigma_\nu - \sigma_\nu \sigma_\mu = [\sigma_\mu, \sigma_\nu] = 2i \epsilon_{\mu\nu\lambda} \sigma_\lambda$

Group products: $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma} \cdot \vec{\Theta}_a)/2} e^{-i(\vec{\sigma} \cdot \vec{\Theta}_b)/2}$ This NOT just $e^{ia} e^{ib} = e^{i(a+b)}$!

$$\mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1} \cos \frac{\Theta_a}{2} - i(\vec{\sigma} \cdot \hat{\Theta}_a) \sin \frac{\Theta_a}{2} \right) \left(\mathbf{1} \cos \frac{\Theta_b}{2} - i(\vec{\sigma} \cdot \hat{\Theta}_b) \sin \frac{\Theta_b}{2} \right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$= \cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} \mathbf{1} - i \left[\hat{\Theta}_a \sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} + \hat{\Theta}_b \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} \right] \cdot \vec{\sigma} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\vec{\sigma} \cdot \hat{\Theta}_a)(\vec{\sigma} \cdot \hat{\Theta}_b)$$

†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma} \cdot \mathbf{a})(\vec{\sigma} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{1} + i(\mathbf{a} \times \mathbf{b}) \cdot \vec{\sigma}$ reduces $(\vec{\sigma} \cdot \hat{\Theta}_a)(\vec{\sigma} \cdot \hat{\Theta}_b)$ to: $(\hat{\Theta}_a \cdot \hat{\Theta}_b) \mathbf{1} + (\hat{\Theta}_a \times \hat{\Theta}_b) \cdot \vec{\sigma}$

$$= \left(\cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\hat{\Theta}_a \cdot \hat{\Theta}_b) \right) \mathbf{1} - i \left\{ \hat{\Theta}_a \sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} + \hat{\Theta}_b \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\hat{\Theta}_a \times \hat{\Theta}_b) \right\} \cdot \vec{\sigma}$$

$$= \left(\cos \frac{\Theta_{ab}}{2} \right) \mathbf{1} - i \left\{ \hat{\Theta}_{ab} \sin \frac{\Theta_{ab}}{2} \right\} \cdot \vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

1st Step: Coefficient () of unit 1
 derives angle of rotation: Θ_{ab}

Operator-on-Operator transformations

Product algebra Multiplication rules for Pauli's " σ_μ -quaternions" and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$.

•	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z		•	1	σ_X	σ_Y	σ_Z
1	1	\mathbf{q}_X	\mathbf{q}_Y	\mathbf{q}_Z	,	1	1	σ_X	σ_Y	σ_Z
\mathbf{q}_X	\mathbf{q}_X	-1	\mathbf{q}_Z	$-\mathbf{q}_Y$		σ_X	σ_X	1	$i\sigma_Z$	$-i\sigma_Y$
\mathbf{q}_Y	\mathbf{q}_Y	$-\mathbf{q}_Z$	-1	\mathbf{q}_X		σ_Y	σ_Y	$-i\sigma_Z$	1	$i\sigma_X$
\mathbf{q}_Z	\mathbf{q}_Z	\mathbf{q}_Y	$-\mathbf{q}_X$	-1		σ_Z	σ_Z	$i\sigma_Y$	$-i\sigma_X$	1

$$\mathbf{q}_\mu \mathbf{q}_\nu = -\delta_{\mu\nu} \mathbf{1} + \epsilon_{\mu\nu\lambda} \mathbf{q}_\lambda$$

$$\sigma_\mu \sigma_\nu = \delta_{\mu\nu} \mathbf{1} + i \epsilon_{\mu\nu\lambda} \sigma_\lambda$$

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$$\mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b) = \left(\mathbf{1} \cos \frac{\Theta_a}{2} - i(\vec{\sigma} \cdot \hat{\Theta}_a) \sin \frac{\Theta_a}{2} \right) \left(\mathbf{1} \cos \frac{\Theta_b}{2} - i(\vec{\sigma} \cdot \hat{\Theta}_b) \sin \frac{\Theta_b}{2} \right) = \mathbf{R}_{ab}(\vec{\Theta}_{ab})$$

$$= \cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} \mathbf{1} - i \left[\hat{\Theta}_a \sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} + \hat{\Theta}_b \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} \right] \cdot \vec{\sigma} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\vec{\sigma} \cdot \hat{\Theta}_a)(\vec{\sigma} \cdot \hat{\Theta}_b)$$

†Lecture 7 p. 38

Jordan-Pauli† identity $(\vec{\sigma} \cdot \mathbf{a})(\vec{\sigma} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{1} + i(\mathbf{a} \times \mathbf{b}) \cdot \vec{\sigma}$ reduces $(\vec{\sigma} \cdot \hat{\Theta}_a)(\vec{\sigma} \cdot \hat{\Theta}_b)$ to: $(\hat{\Theta}_a \cdot \hat{\Theta}_b) \mathbf{1} + (\hat{\Theta}_a \times \hat{\Theta}_b) \cdot \vec{\sigma}$

$$= \left(\cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\hat{\Theta}_a \cdot \hat{\Theta}_b) \right) \mathbf{1}$$

$$- i \left\{ \hat{\Theta}_a \sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} + \hat{\Theta}_b \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\hat{\Theta}_a \times \hat{\Theta}_b) \right\} \cdot \vec{\sigma}$$

$$= \left(\cos \frac{\Theta_{ab}}{2} \right) \mathbf{1} - i \left\{ \hat{\Theta}_{ab} \sin \frac{\Theta_{ab}}{2} \right\} \cdot \vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

1st Step: Coefficient () of unit $\mathbf{1}$
 derives angle of rotation: Θ_{ab}

2nd Step: Coefficient { } of $-i \cdot \vec{\sigma}$
 derives unit-vector $\hat{\Theta}_{ab}$ of rotation:

Operator-on-Operator transformations

Product algebra $\mathbf{R}_{ab}(\vec{\Theta}_{ab}) \equiv \mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b) = e^{-i(\vec{\sigma} \cdot \vec{\Theta}_a)/2} e^{-i(\vec{\sigma} \cdot \vec{\Theta}_b)/2}$

This NOT just $e^{ia}e^{ib}=e^{i(a+b)}$!

(...except when $\hat{\Theta}_a = \hat{\Theta}_b$!)

$$= \left(\cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} (\hat{\Theta}_a \cdot \hat{\Theta}_b) \right) \mathbf{1}$$

$$= \left(\cos \frac{\Theta_{ab}}{2} \right) \mathbf{1} - i \left\{ \hat{\Theta}_{ab} \sin \frac{\Theta_{ab}}{2} \right\} \cdot \vec{\sigma} = \mathbf{R}_{ab}(\vec{\Theta}_{ab}) = \mathbf{R}_a(\vec{\Theta}_a) \cdot \mathbf{R}_b(\vec{\Theta}_b)$$

Match with "Crazy-Thing" form of product $\mathbf{R}_{ab}(\vec{\Theta}_{ab})$

1st Step: Coefficient () of unit $\mathbf{1}$ derives *angle of rotation*: Θ_{ab}

2nd Step: Coefficient { } of $-i \cdot \vec{\sigma}$ derives *unit-vector* $\hat{\Theta}_{ab}$ of rotation:

Now easy to find the *product angle* Θ_{ab} and *crank unit vector* $\hat{\Theta}_{ab}$.

$$\frac{\Theta_{ab}}{2} = \cos^{-1} \left(\cos \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} - \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} \hat{\Theta}_a \cdot \hat{\Theta}_b \right)$$

U(2) and R(3) Group Product Formulae

$$\vec{\Theta}_{ab} = \left[\sin \frac{\Theta_a}{2} \cos \frac{\Theta_b}{2} \hat{\Theta}_a + \cos \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} \hat{\Theta}_b + \sin \frac{\Theta_a}{2} \sin \frac{\Theta_b}{2} \hat{\Theta}_a \times \hat{\Theta}_b \right] / \sin \frac{\Theta_{ab}}{2}$$

Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

Group product algebra

→ Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula
Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ of spinor σ_μ -operators
Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ of group-operators
Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

*Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators*

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

Group product algebra

→ *Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula
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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

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Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Transformation of spinor σ_μ -operators

$$\mathbf{R}[\vec{\Theta}] \sigma_L \mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right)^\dagger$$

Transformation of spinor σ_μ -operators

$$\begin{aligned} \mathbf{R}[\vec{\Theta}] \sigma_L \mathbf{R}[\vec{\Theta}]^\dagger &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \vec{\sigma} \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \vec{\sigma} \right)^\dagger \\ &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta}_K \sigma_K \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} + i \sin \frac{\Theta}{2} \hat{\Theta}_M \sigma_M \right) \end{aligned}$$

Sum over repeated indices is implied:

$$a_M b_M = \sum_{M=1}^3 a_M b_M$$

Transformation of spinor σ_μ -operators

$$\begin{aligned}
 \mathbf{R}[\vec{\Theta}] \sigma_L \mathbf{R}[\vec{\Theta}]^\dagger &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right)^\dagger \\
 &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta}_K \sigma_K \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} + i \sin \frac{\Theta}{2} \hat{\Theta}_M \sigma_M \right) \\
 = \sigma'_L &= \sigma_L \cos \Theta - \varepsilon_{LKN} \hat{\Theta}_K \sigma_N \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_L \left(\hat{\Theta}_M \sigma_M \right)
 \end{aligned}$$

Sum over repeated indices is implied:

$$a_M b_M = \sum_{M=1}^3 a_M b_M$$

(Left as an exercise)

Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

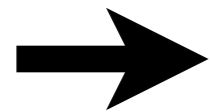
Operator-on-Operator transformations

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ of group-operators

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

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$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Transformation of spinor σ_μ -operators

$$\begin{aligned}
 \mathbf{R}[\vec{\Theta}] \sigma_L \mathbf{R}[\vec{\Theta}]^\dagger &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right)^\dagger \\
 &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta}_K \sigma_K \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} + i \sin \frac{\Theta}{2} \hat{\Theta}_M \sigma_M \right) \\
 = \sigma'_L &= \sigma_L \cos \Theta - \varepsilon_{LKN} \hat{\Theta}_K \sigma_N \sin \Theta + (1 - \cos \Theta) \hat{\Theta}_L \left(\hat{\Theta}_M \sigma_M \right)
 \end{aligned}$$

General transformation of rotational $\mathbf{R}[\vec{\Theta}']$ -operators

$$\begin{aligned}
 \mathbf{R}[\vec{\Theta}] \mathbf{R}[\vec{\Theta}'] \mathbf{R}[\vec{\Theta}]^\dagger &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right) \mathbf{R}[\vec{\Theta}'] \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right)^\dagger \\
 \mathbf{R}[\mathbf{R}[\vec{\Theta}] \cdot \vec{\Theta}'] &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta}_K \sigma_K \right) \mathbf{R}[\vec{\Theta}'] \left(\cos \frac{\Theta}{2} \mathbf{1} + i \sin \frac{\Theta}{2} \hat{\Theta}_M \sigma_M \right)
 \end{aligned}$$

Transformation of spinor σ_μ -operators

$$\begin{aligned}
 \mathbf{R}[\vec{\Theta}] \sigma_L \mathbf{R}[\vec{\Theta}]^\dagger &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta} \cdot \boldsymbol{\sigma} \right)^\dagger \\
 &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta}_K \sigma_K \right) \sigma_L \left(\cos \frac{\Theta}{2} \mathbf{1} + i \sin \frac{\Theta}{2} \hat{\Theta}_M \sigma_M \right) \\
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General transformation of rotational $\mathbf{R}[\vec{\Theta}']$ -operators

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 \mathbf{R}[\mathbf{R}[\vec{\Theta}] \cdot \vec{\Theta}'] &= \left(\cos \frac{\Theta}{2} \mathbf{1} - i \sin \frac{\Theta}{2} \hat{\Theta}_K \sigma_K \right) \mathbf{R}[\vec{\Theta}'] \left(\cos \frac{\Theta}{2} \mathbf{1} + i \sin \frac{\Theta}{2} \hat{\Theta}_M \sigma_M \right)
 \end{aligned}$$

This one is better seen geometrically. Algebra not so quick.

Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

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Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

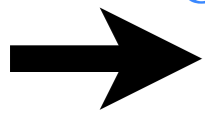
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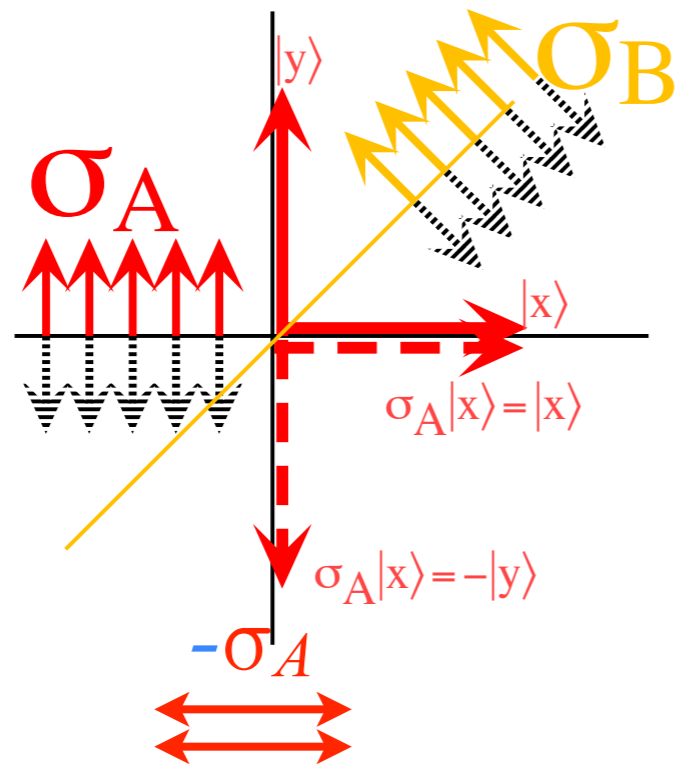
Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial



Geometry of $U(2)$ transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$+\sigma_A$ is an x -plane mirror



Note that $-\sigma_A$ is a y -plane mirror

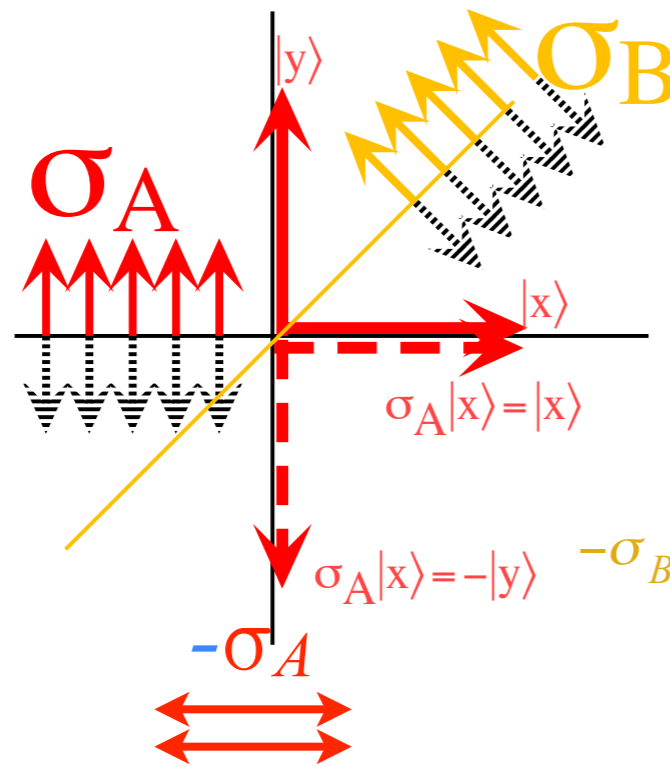
$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Geometry of $U(2)$ transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$+\sigma_B$ is an
 45° -plane
mirror

$+\sigma_A$ is an
 x -plane
mirror



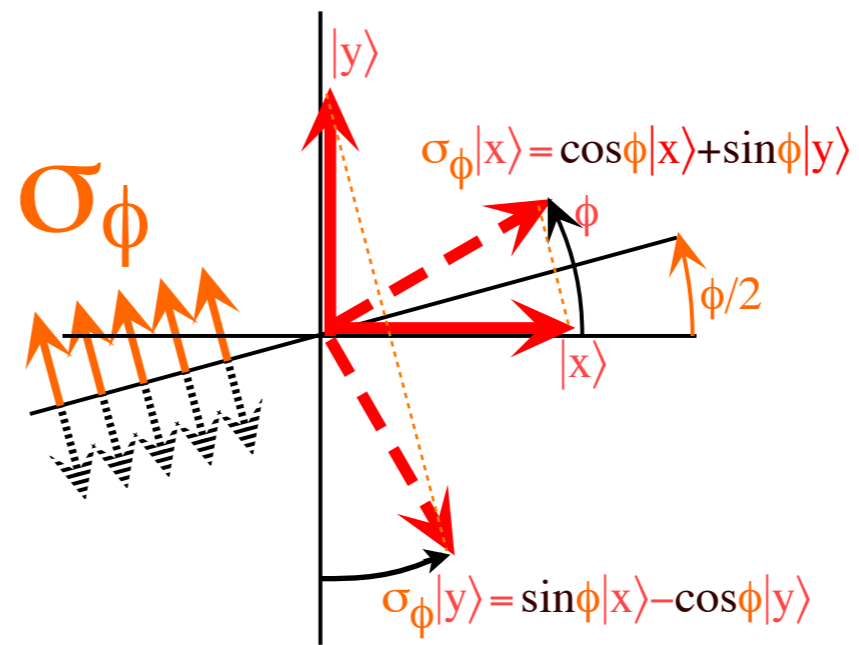
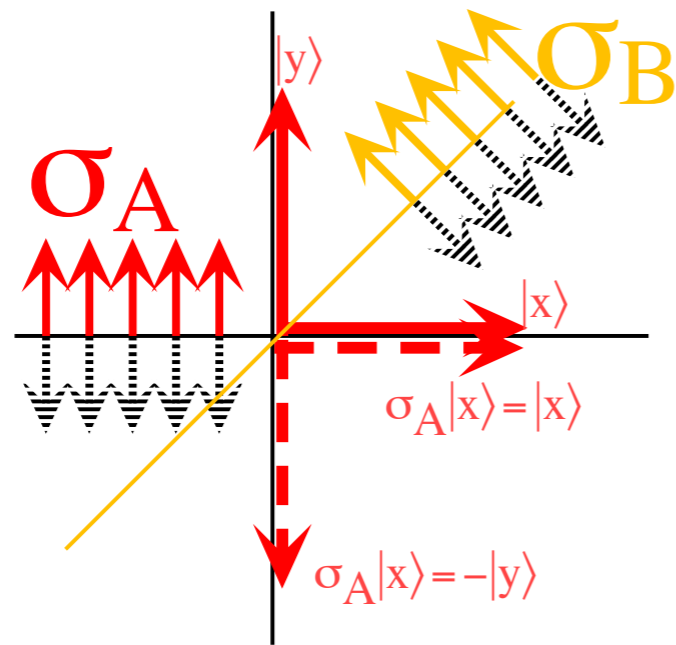
$-\sigma_B$ is an
 -45° -plane
mirror

Note that $-\sigma_A$ is a y -plane mirror

$$-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

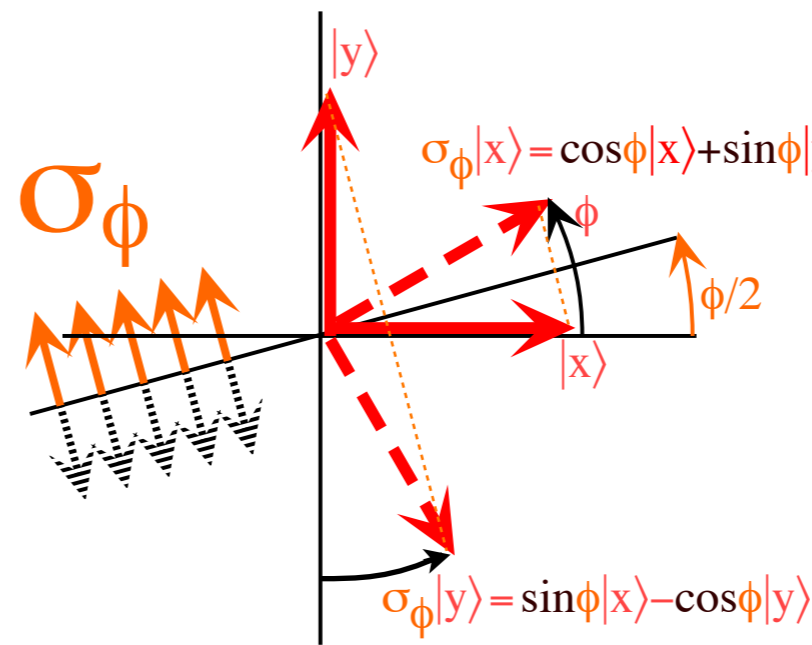
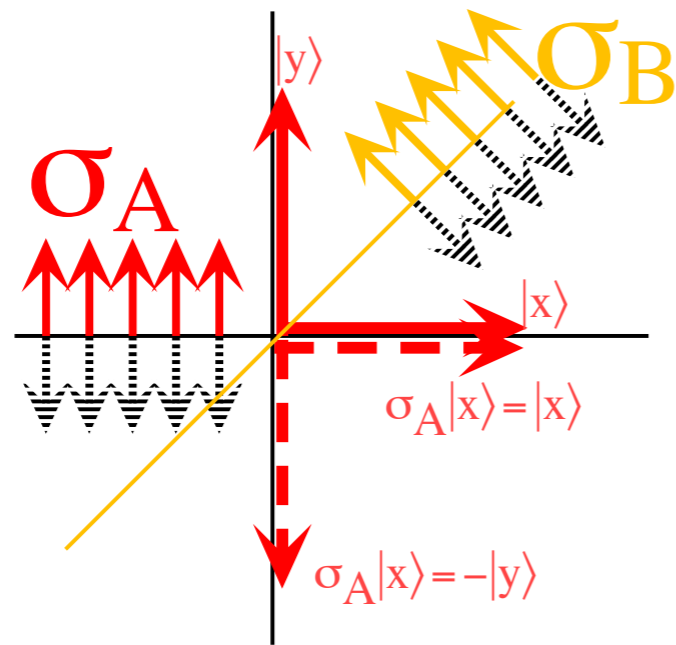
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Geometry of $U(2)$ transformations. It's all done with mirrors!

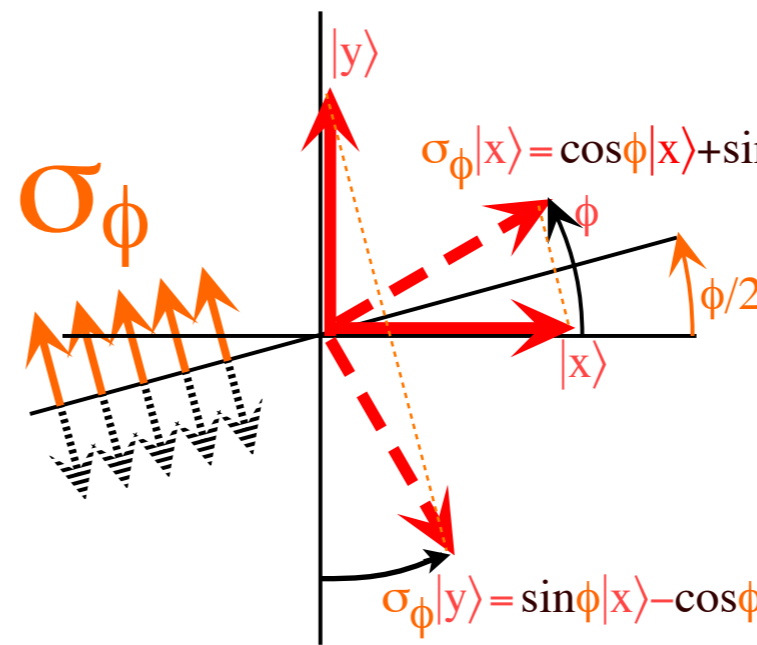
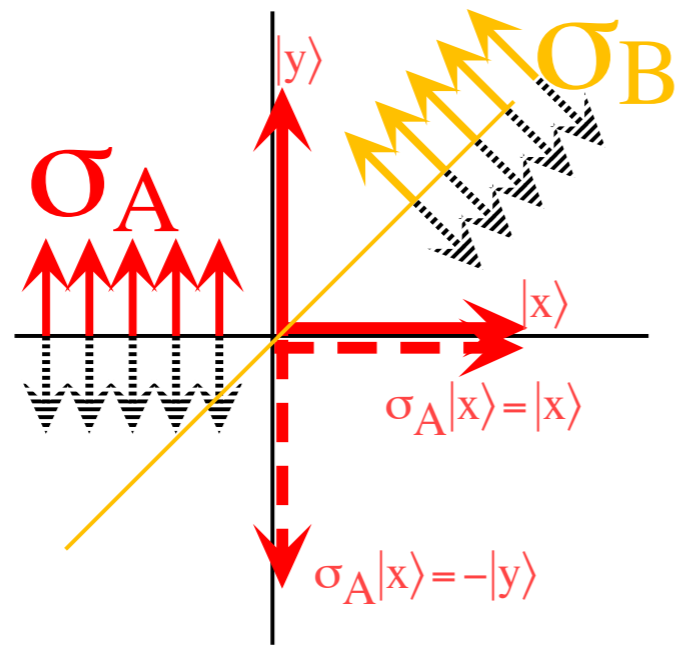
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implies: $\langle x | \sigma_\phi | x \rangle = \cos \phi$
 and: $\langle y | \sigma_\phi | x \rangle = \sin \phi$

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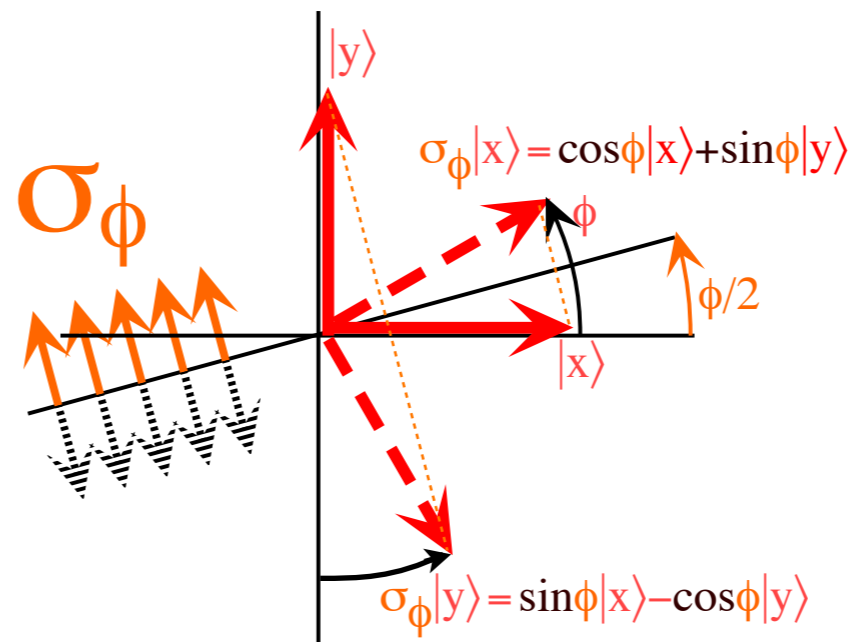
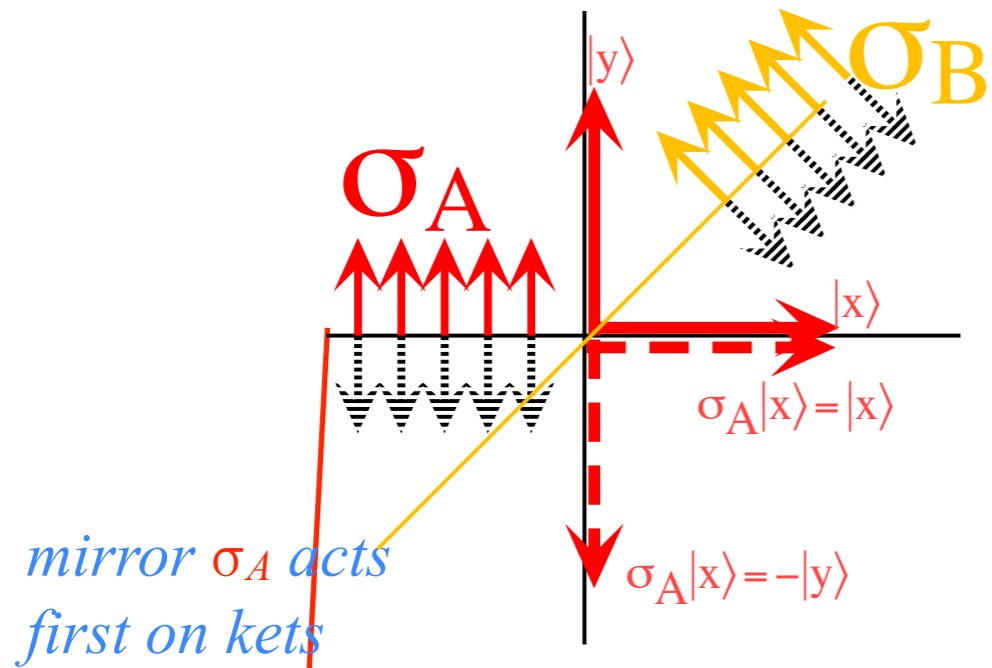


implies: $\langle x | \sigma_\phi | x \rangle = \cos \phi$
and: $\langle y | \sigma_\phi | x \rangle = \sin \phi$

implies: $\langle x | \sigma_\phi | y \rangle = \sin \phi$
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Geometry of $U(2)$ transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{pmatrix} = \sigma_A \cos \phi + \sigma_B \sin \phi$$

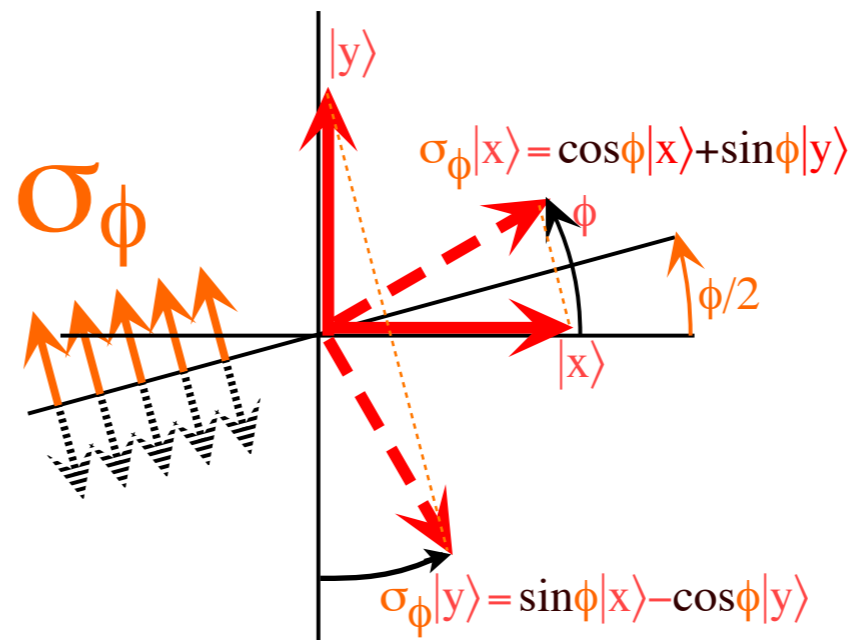
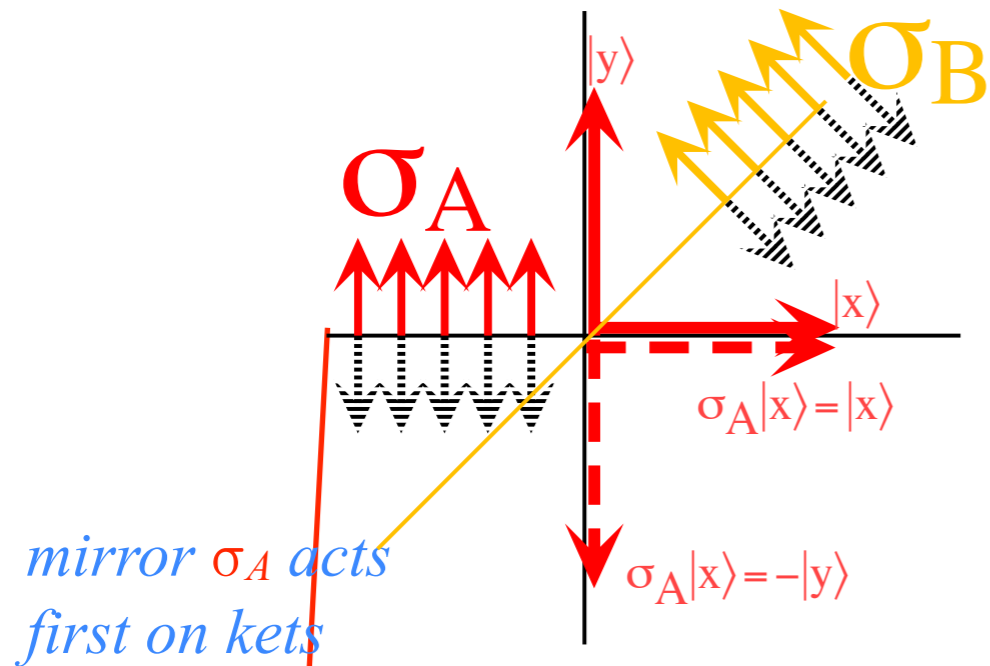


mirror σ_ϕ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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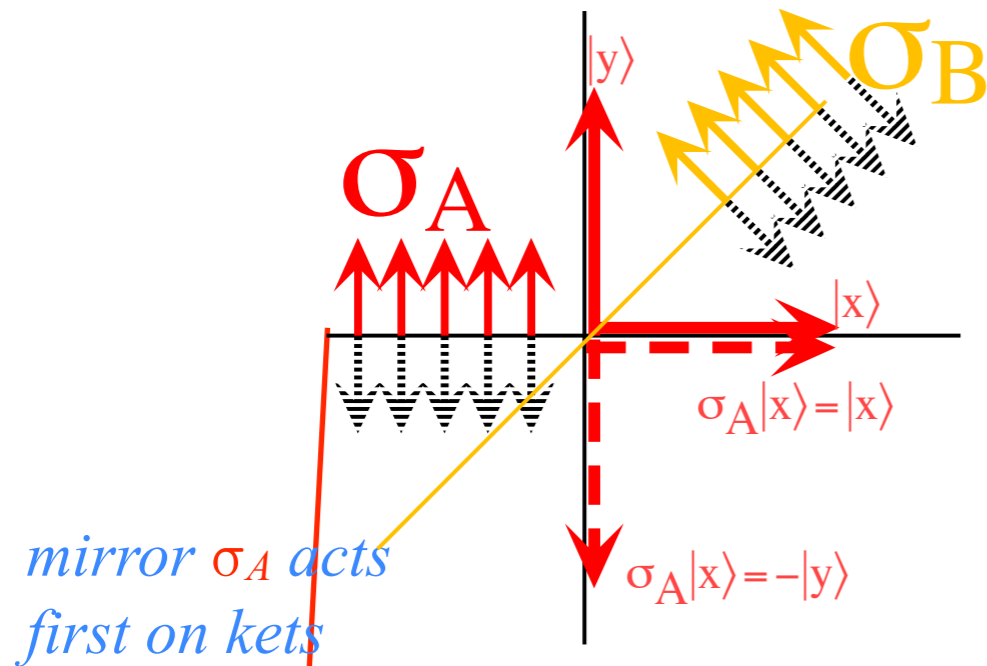


mirror σ_ϕ goes 2nd

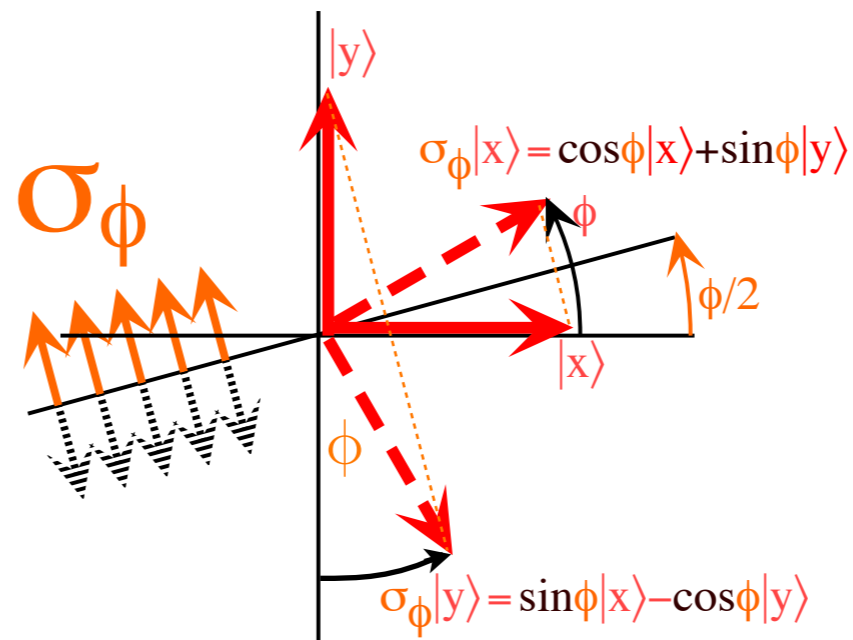
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mirror σ_A acts first on kets



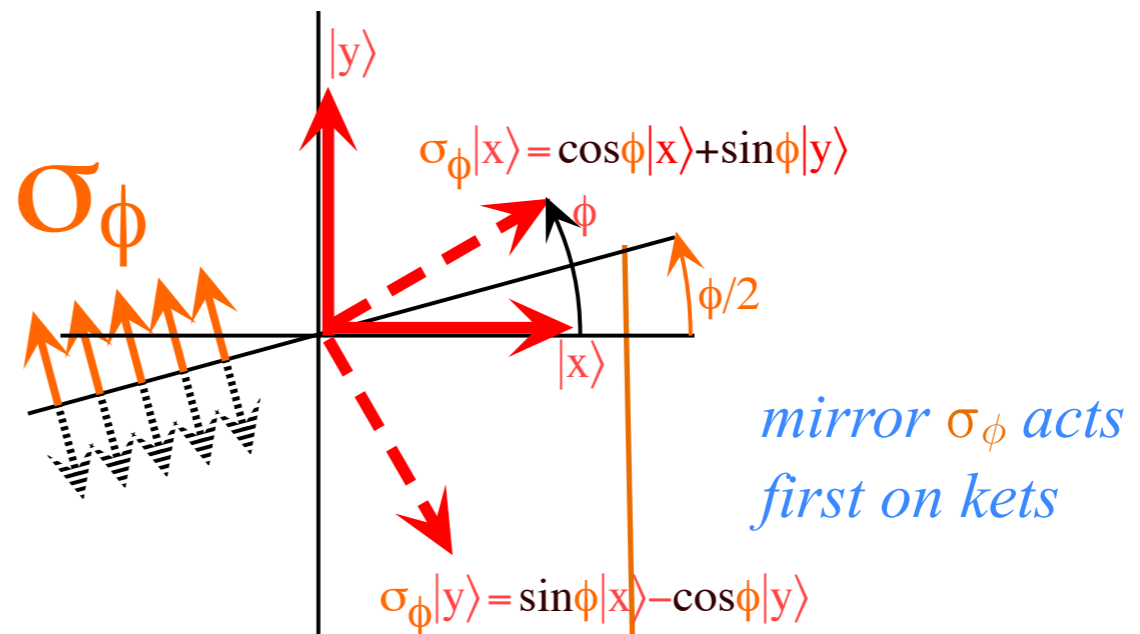
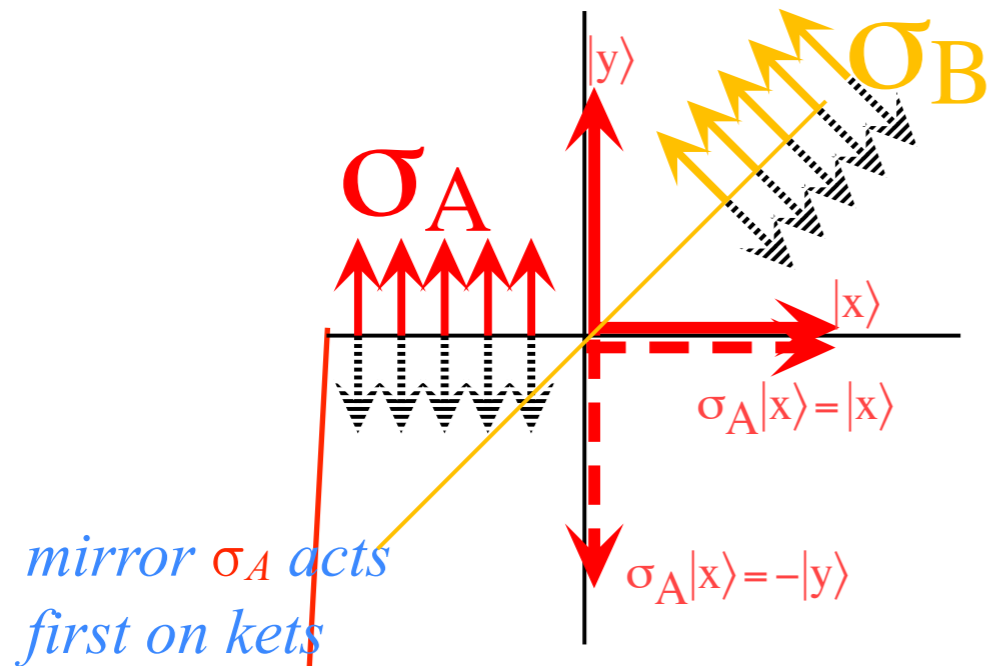
mirror σ_ϕ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi],$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

Geometry of $U(2)$ transformations. It's all done with mirrors!

$$\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \sigma_A \cos\phi + \sigma_B \sin\phi$$



mirror σ_ϕ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi],$$

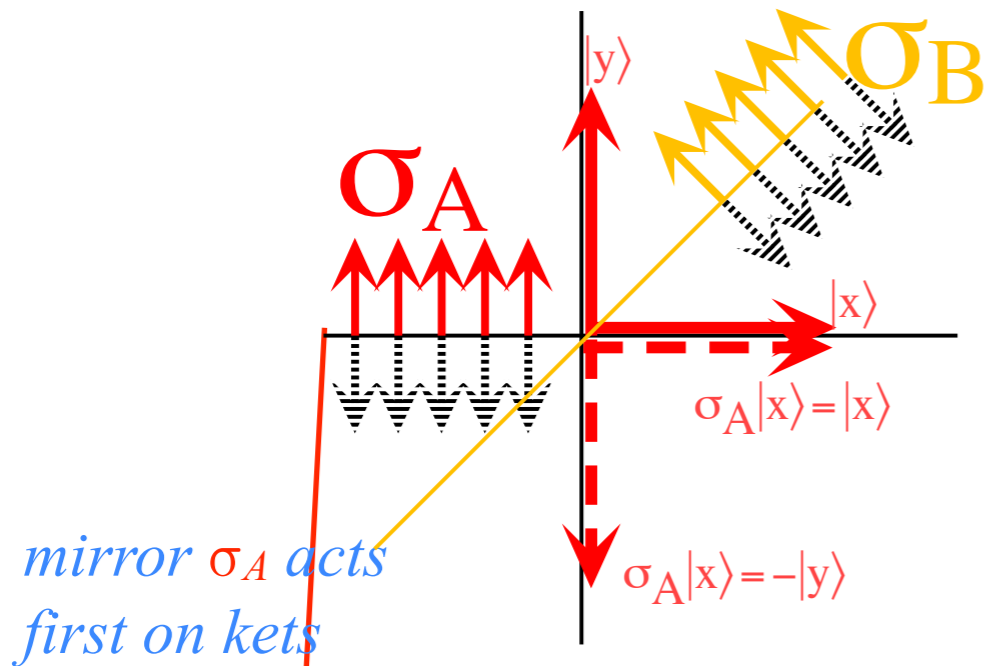
mirror σ_A goes 2nd

$$\sigma_A \sigma_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = R[-\phi]$$

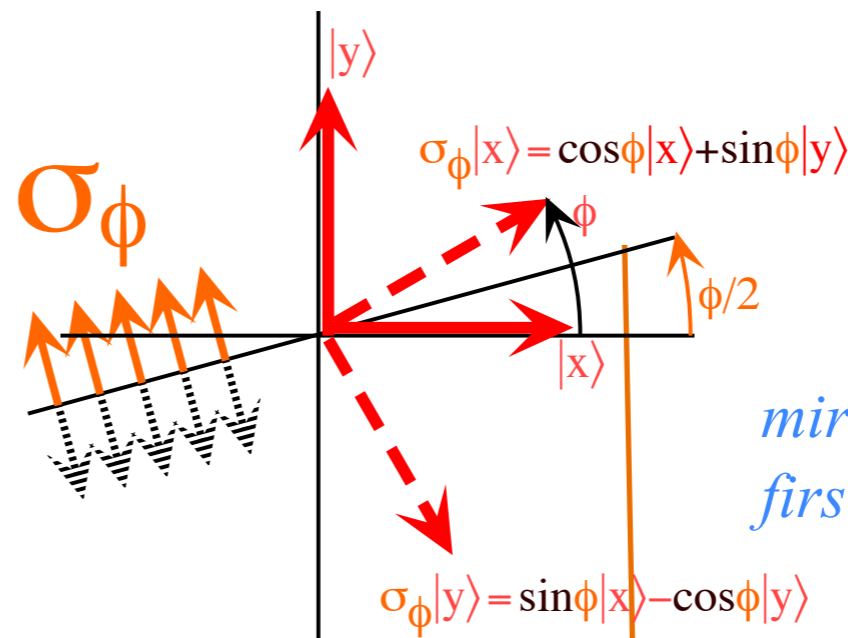
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mirror σ_A acts first on kets



mirror σ_ϕ acts first on kets

mirror σ_ϕ goes 2nd

$$\sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = R[\phi]$$

mirror σ_A goes 2nd

$$\sigma_A \sigma_\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = R[-\phi]$$

Rotation angle ϕ is TWICE the angle $\phi/2$ between mirror σ_A and mirror σ_ϕ

$$\sigma_A \sigma_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_C$$

determinant $\det = -1$ determinant $\det = -1$ determinant $\det = +1$ determinant $\det = -1$

xy -rotation by -90° imaginary reflection?

Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

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Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

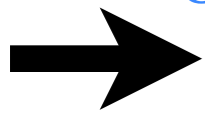
Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

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Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial



Geometry of U(2) group products: Hamilton's Turns

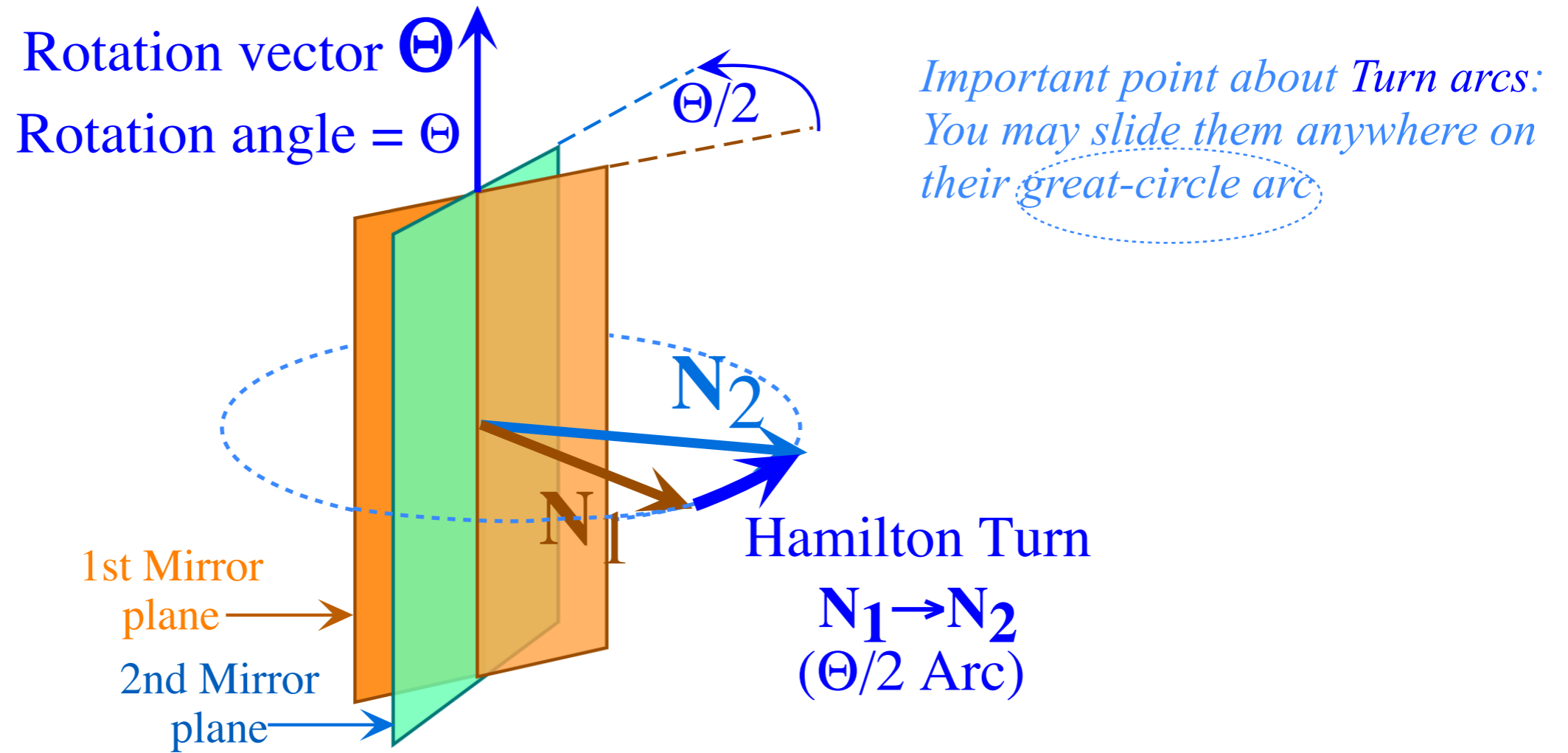
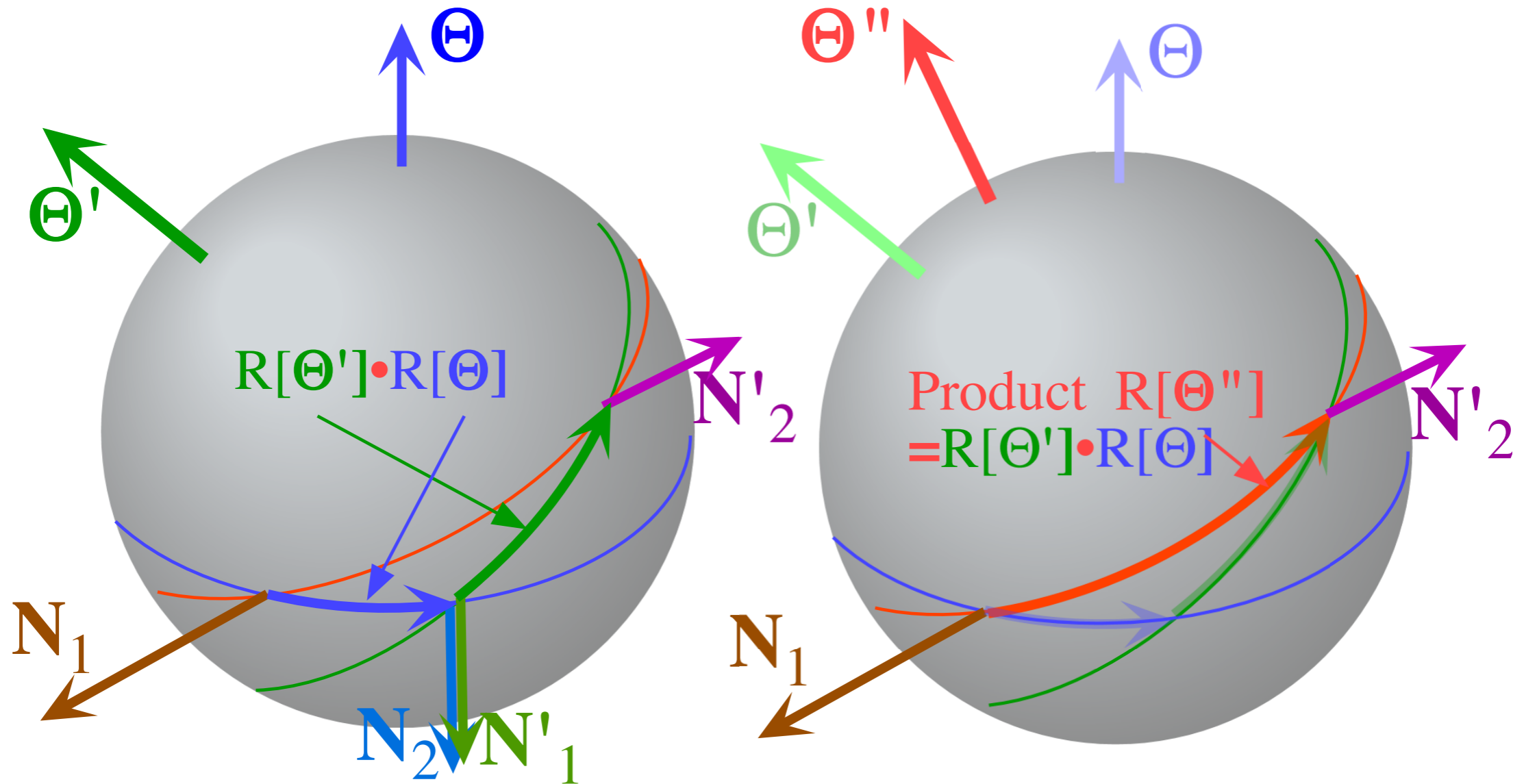


Fig. 10.A.7 Mirror reflection planes, normals, and Hamilton-turn arc vector.

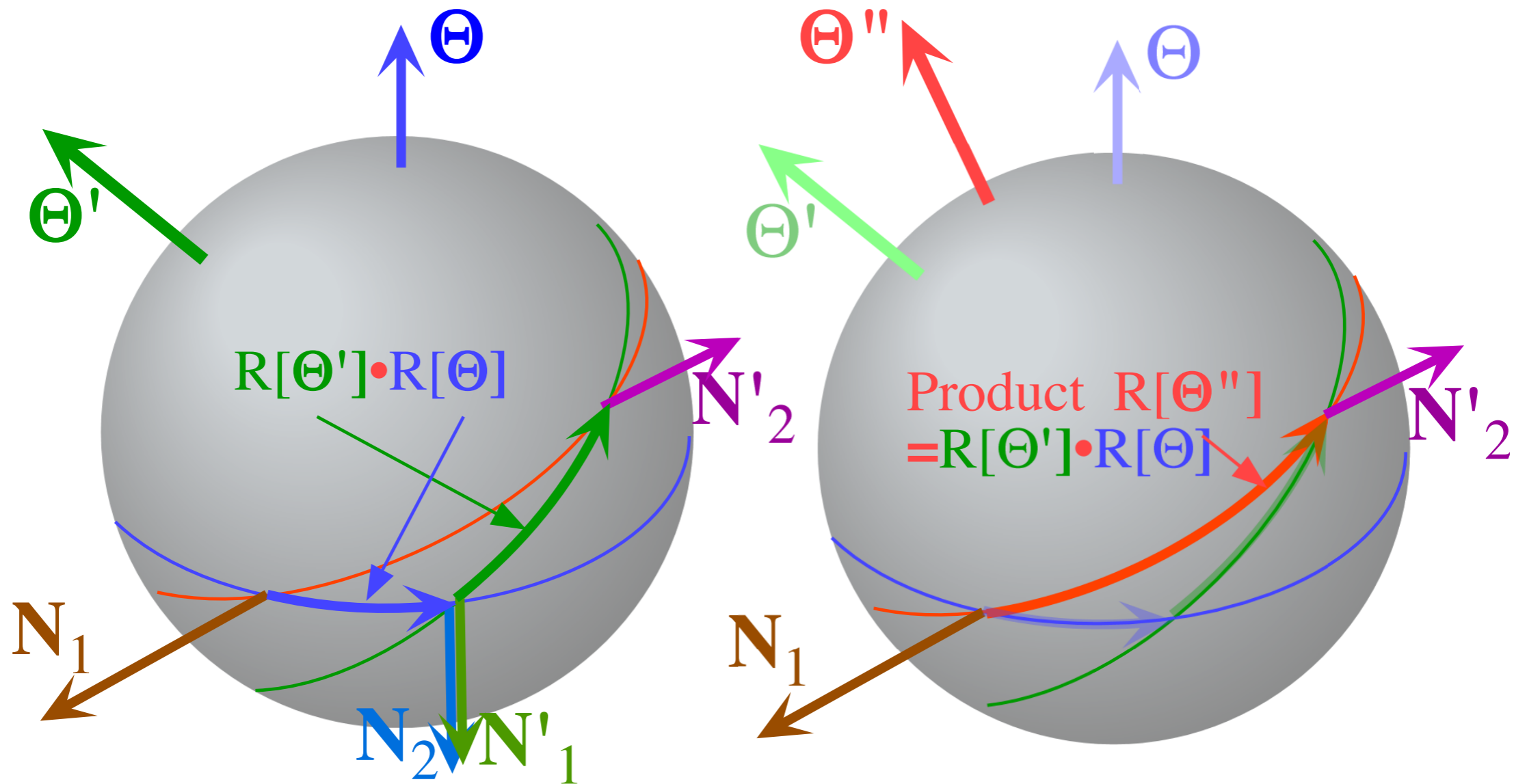
Geometry of U(2) group products: Hamilton's Turns



QTforCA Fig. 10.A.8 Adding Hamilton-turn arcs to compute a U(2) product $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$.

Each arc $\Theta/2$, $\Theta'/2$, or $\Theta''/2$ is 1/2 actual angle Θ , Θ' , or Θ'' of rotation $\mathbf{R}[\Theta]$, $\mathbf{R}[\Theta']$, or $\mathbf{R}[\Theta'']$.

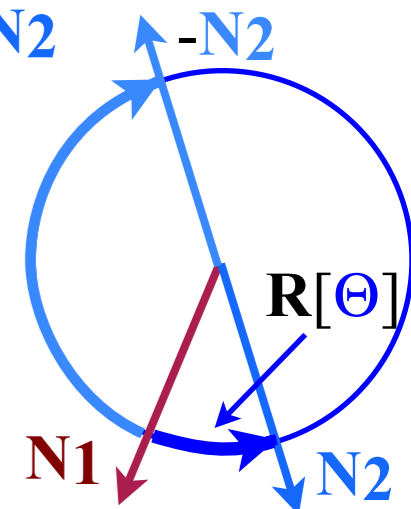
Geometry of $U(2)$ group products: Hamilton's Turns



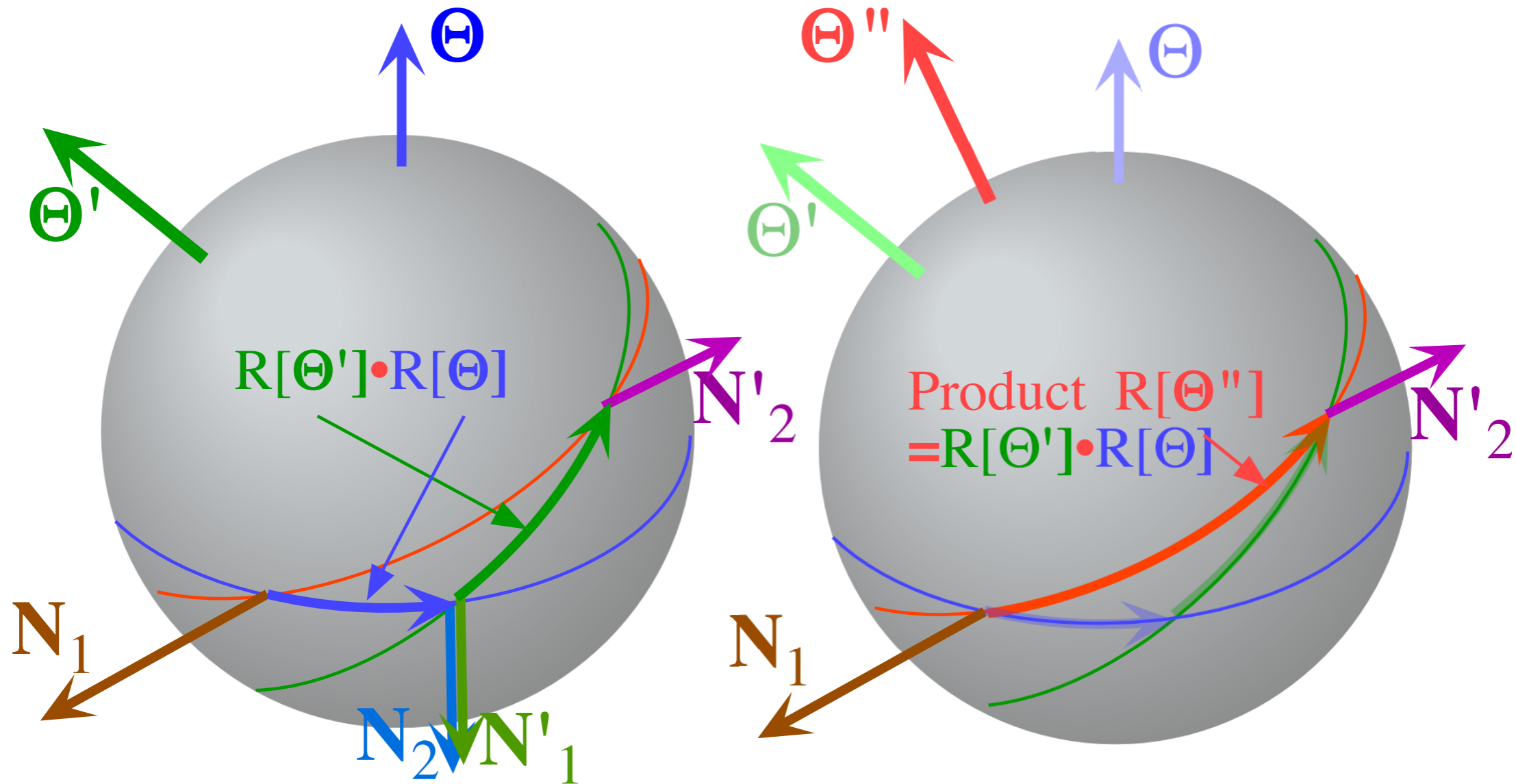
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Arc $\Theta/2$ between N_1 and N_2 and its supplement $(\Theta \pm 2\pi)/2 = \Theta/2 \pm \pi$ between N_1 and $-N_2$ represent the same classical rotation by Θ .



Geometry of $U(2)$ group products: Hamilton's Turns

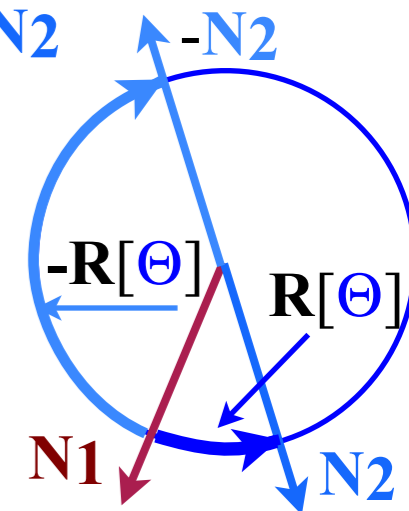


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Arc $\Theta/2$ between \mathbf{N}_1 and \mathbf{N}_2 and its supplement $(\Theta \pm 2\pi)/2 = \Theta/2 \pm \pi$ between \mathbf{N}_1 and $-\mathbf{N}_2$ represent the same classical rotation by Θ .

For quantum spin- $1/2$ object, the arc pointing from \mathbf{N}_1 to the antipodal normal $-\mathbf{N}_2$ represents a Θ -rotation with an extra π -phase factor $e^{\pm i\pi} = -1$, that is, $-\mathbf{R}[\Theta]$.



Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
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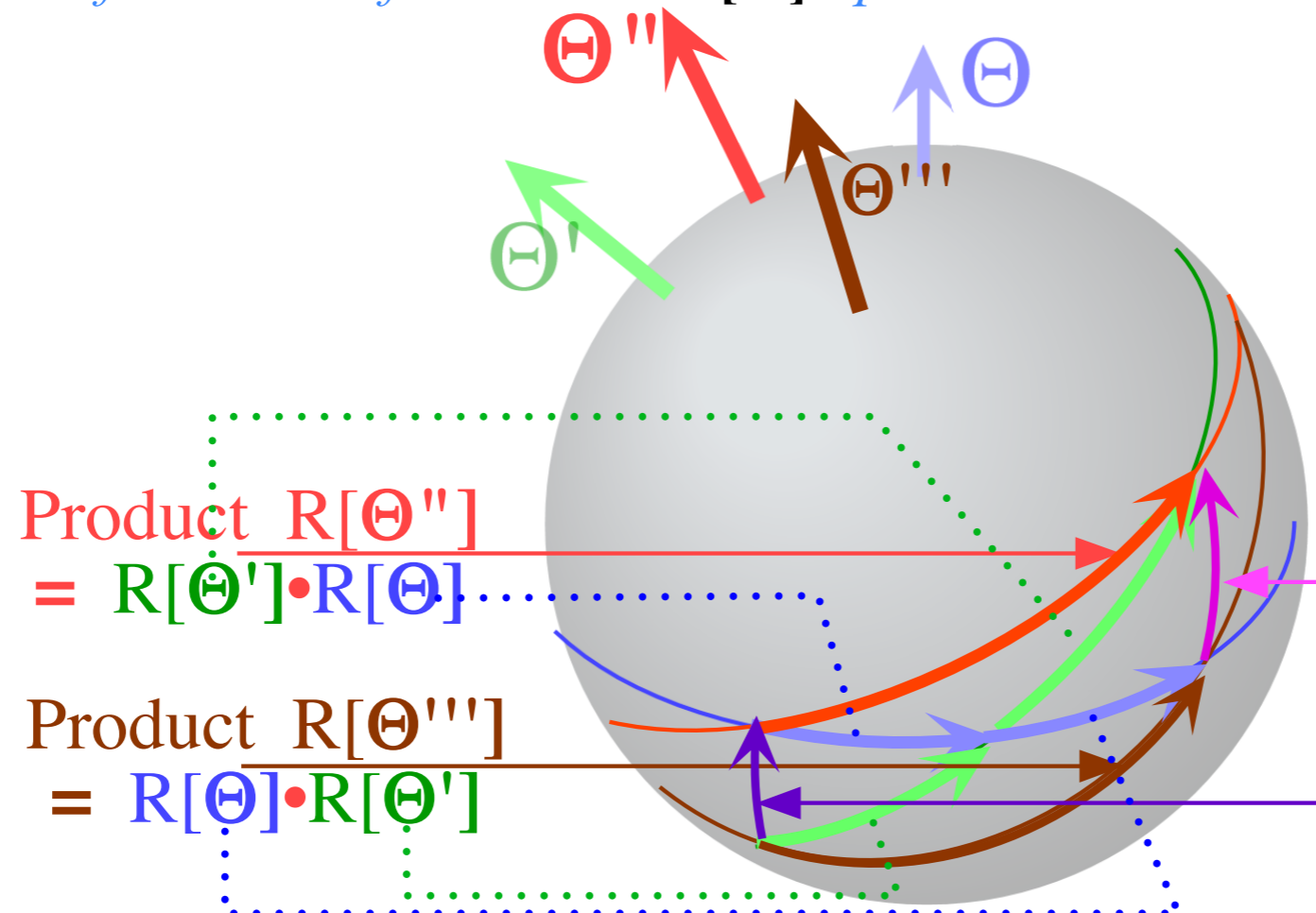
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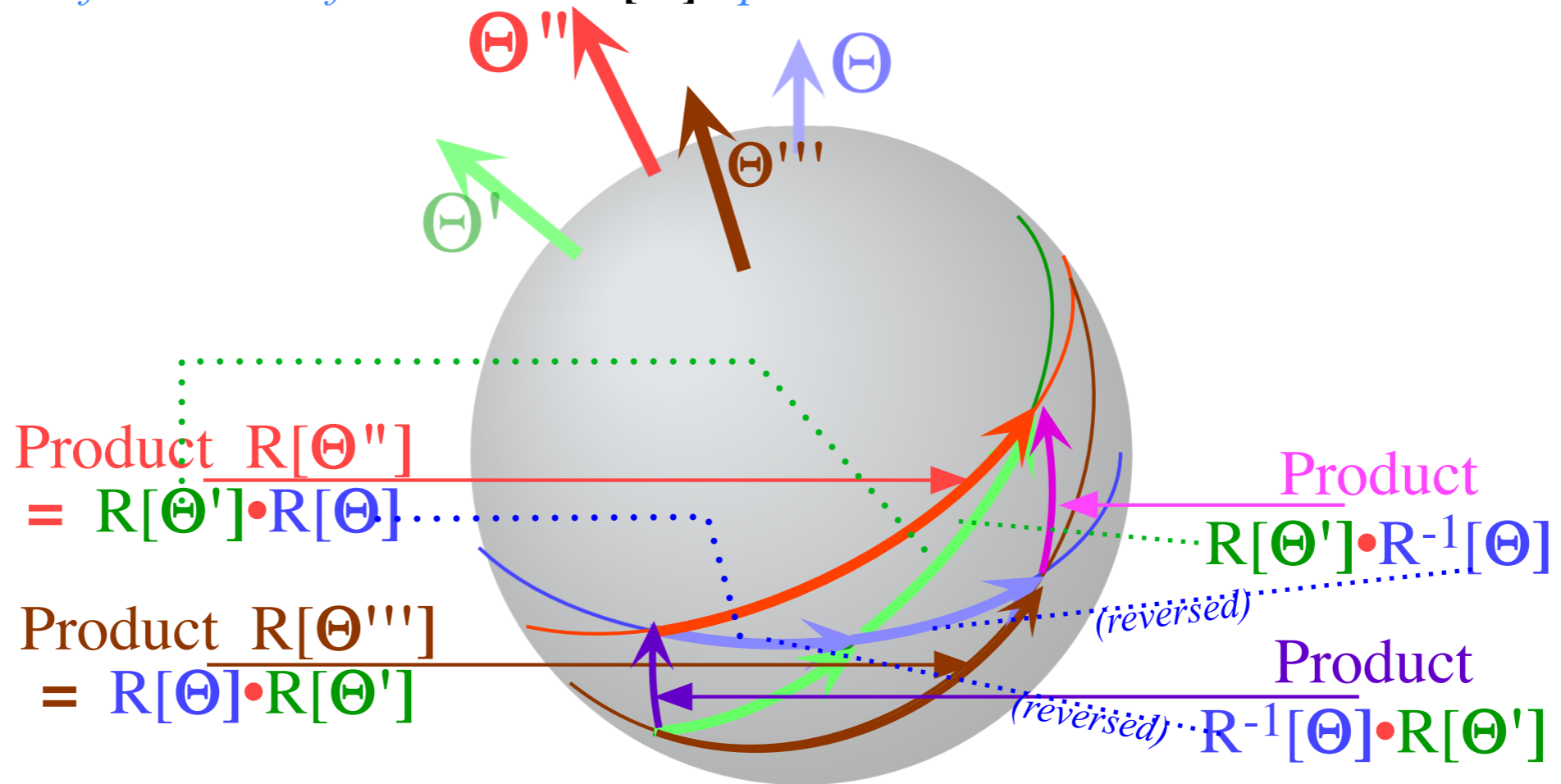
Geometry of transformation of rotational $\mathbf{R}[\Theta]$ -operators



QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$ and $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$

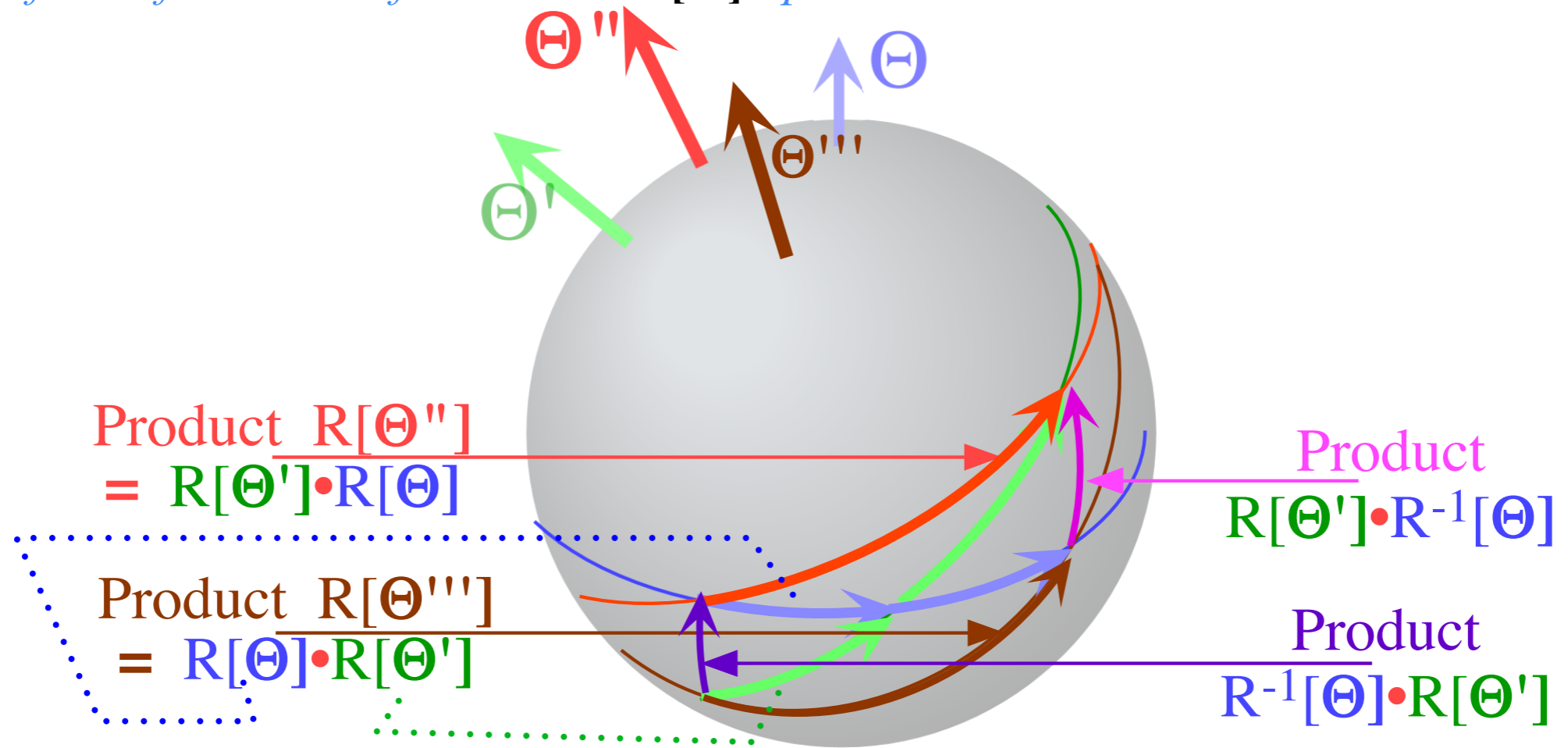
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QTforCA

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Geometry of transformation of rotational $\mathbf{R}[\Theta]$ -operators

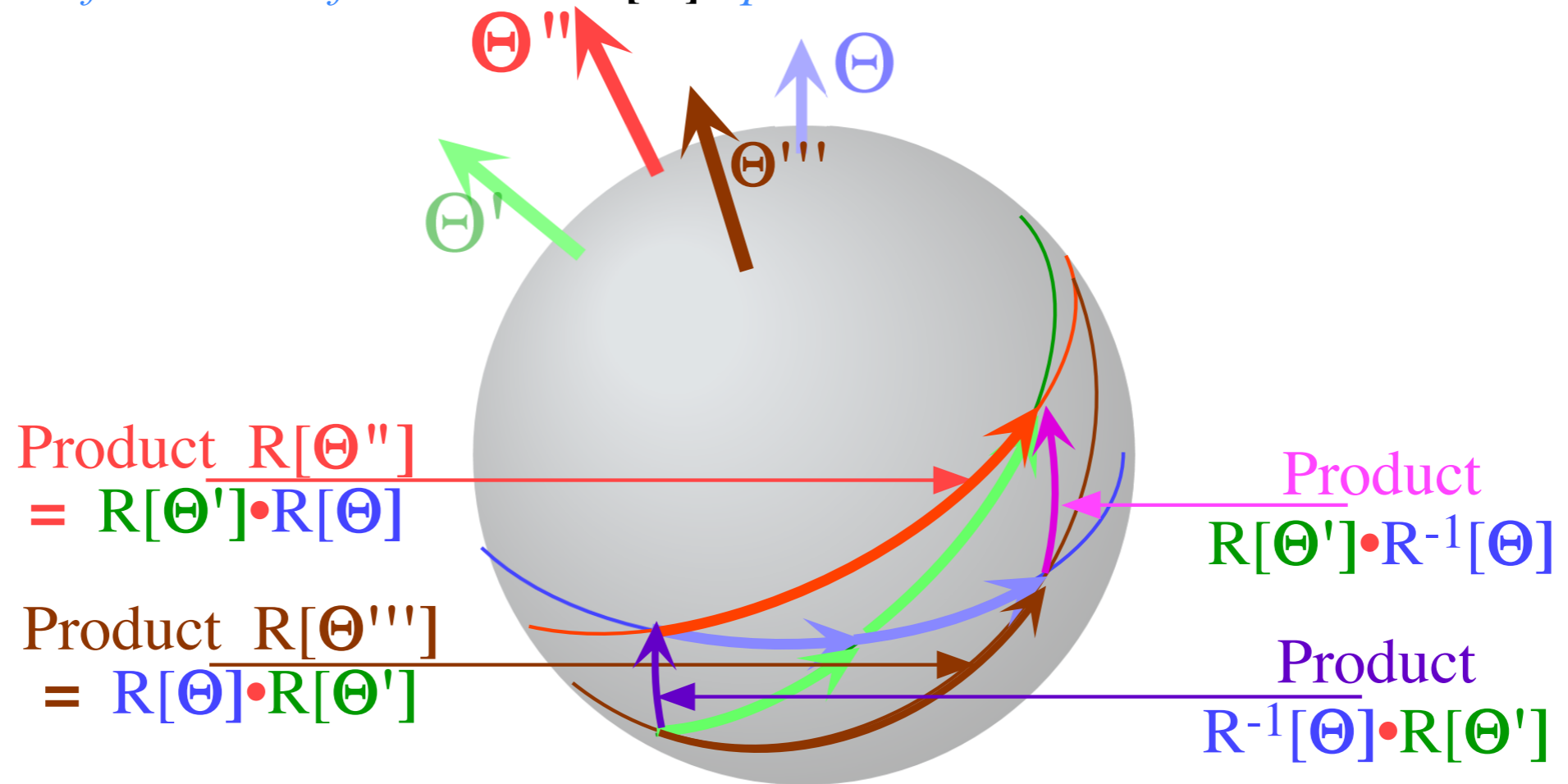


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Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$ and $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$

Vectors added in the reverse order give $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$ instead of $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$.

Geometry of transformation of rotational $\mathbf{R}[\Theta]$ -operators



QTforCA

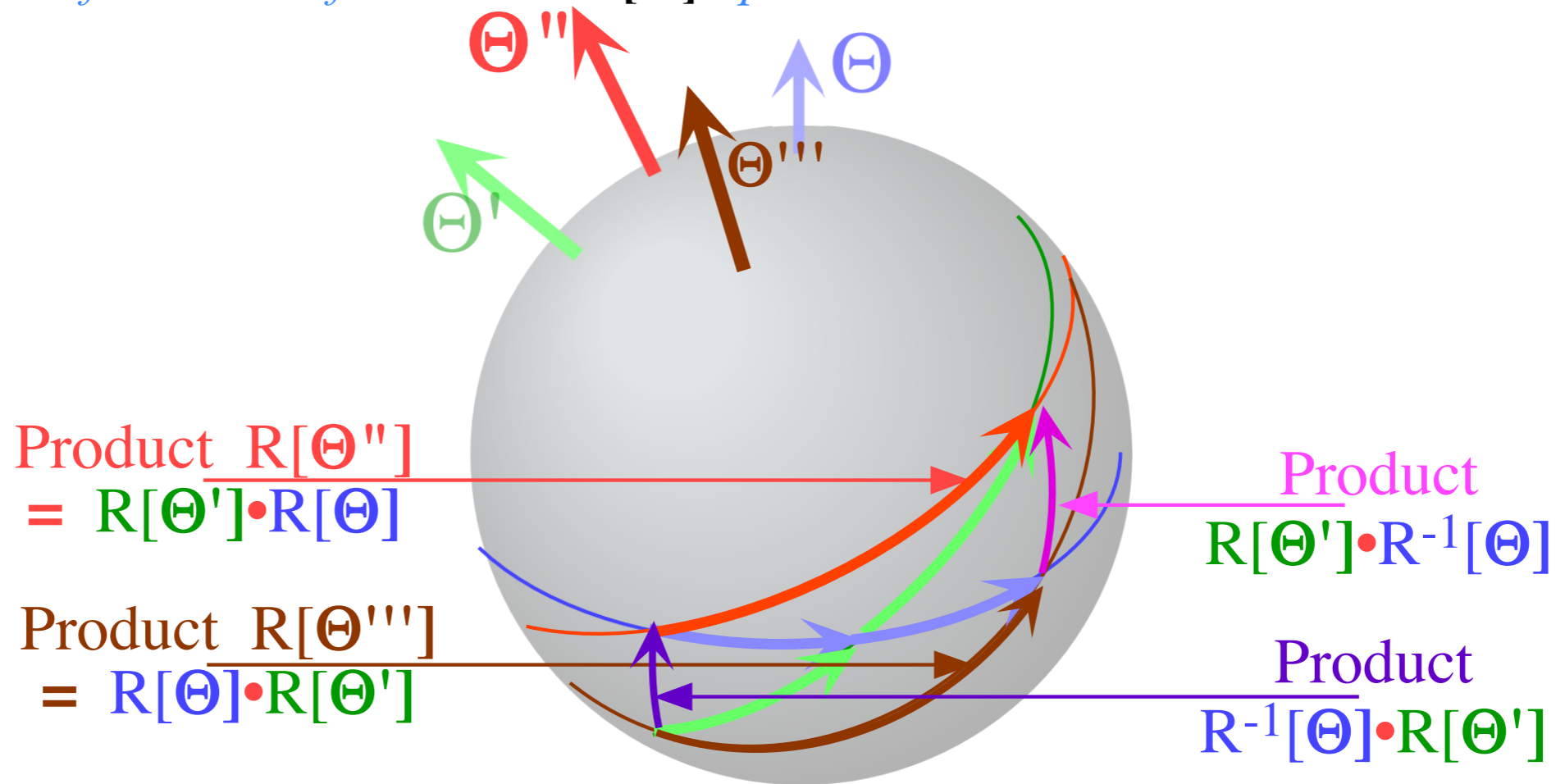
Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$ and $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$

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A similarity transformation of rotation $\mathbf{R}[\Theta'']$ by rotation $\mathbf{R}[\Theta]$ gives rotation $\mathbf{R}[\Theta''']$

$$\mathbf{R}[\Theta] \underbrace{\mathbf{R}[\Theta''] \mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta''']$$

Geometry of transformation of rotational $\mathbf{R}[\Theta]$ -operators



QTforCA

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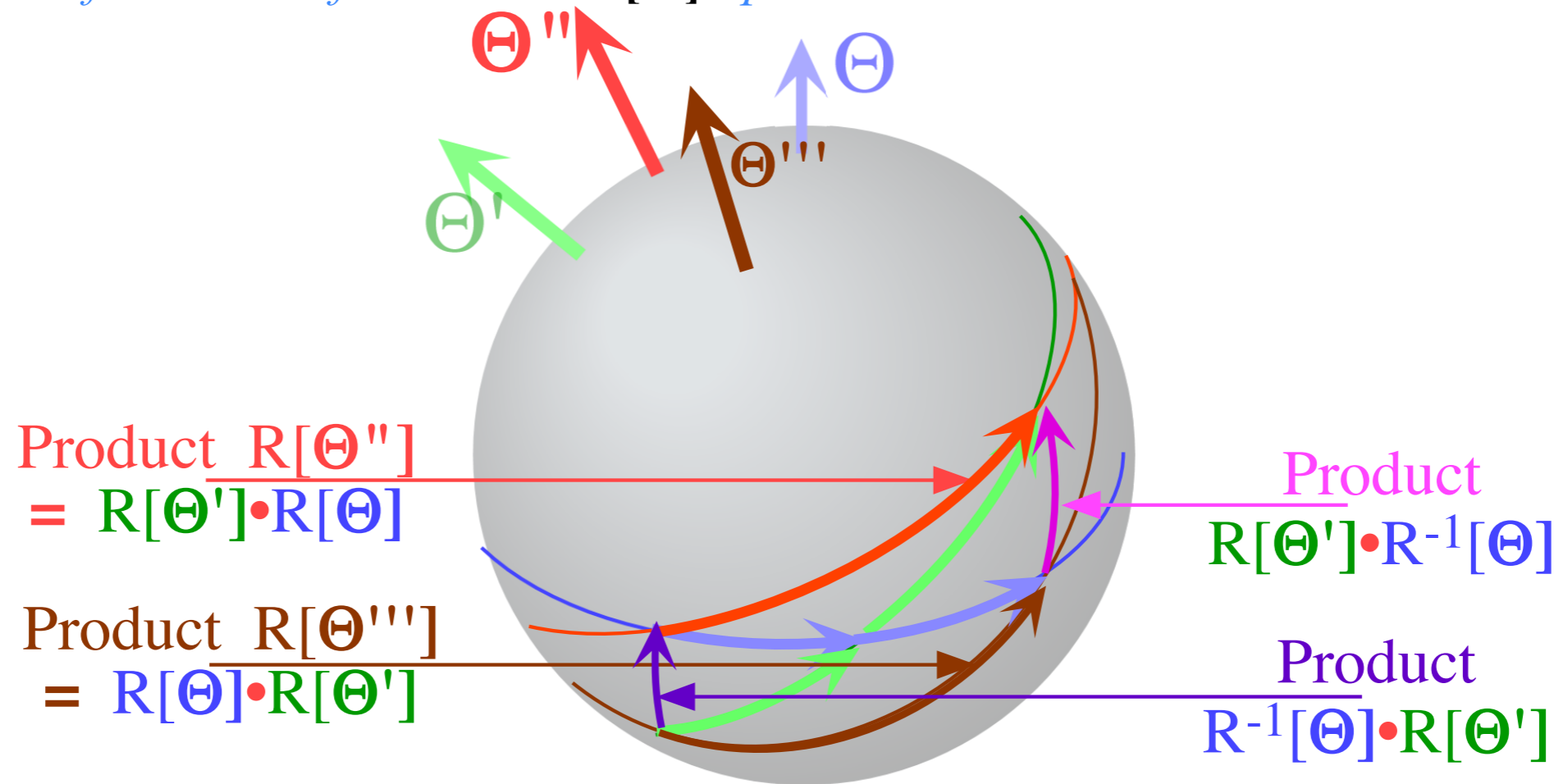
Vectors added in the reverse order give $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$ instead of $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$.

A similarity transformation of rotation $\mathbf{R}[\Theta'']$ by rotation $\mathbf{R}[\Theta]$ gives rotation $\mathbf{R}[\Theta''']$ and *vice-versa*:

$$\mathbf{R}[\Theta] \underbrace{\mathbf{R}[\Theta''] \mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta''']$$

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Geometry of transformation of rotational $\mathbf{R}[\Theta]$ -operators



QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$ and $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$

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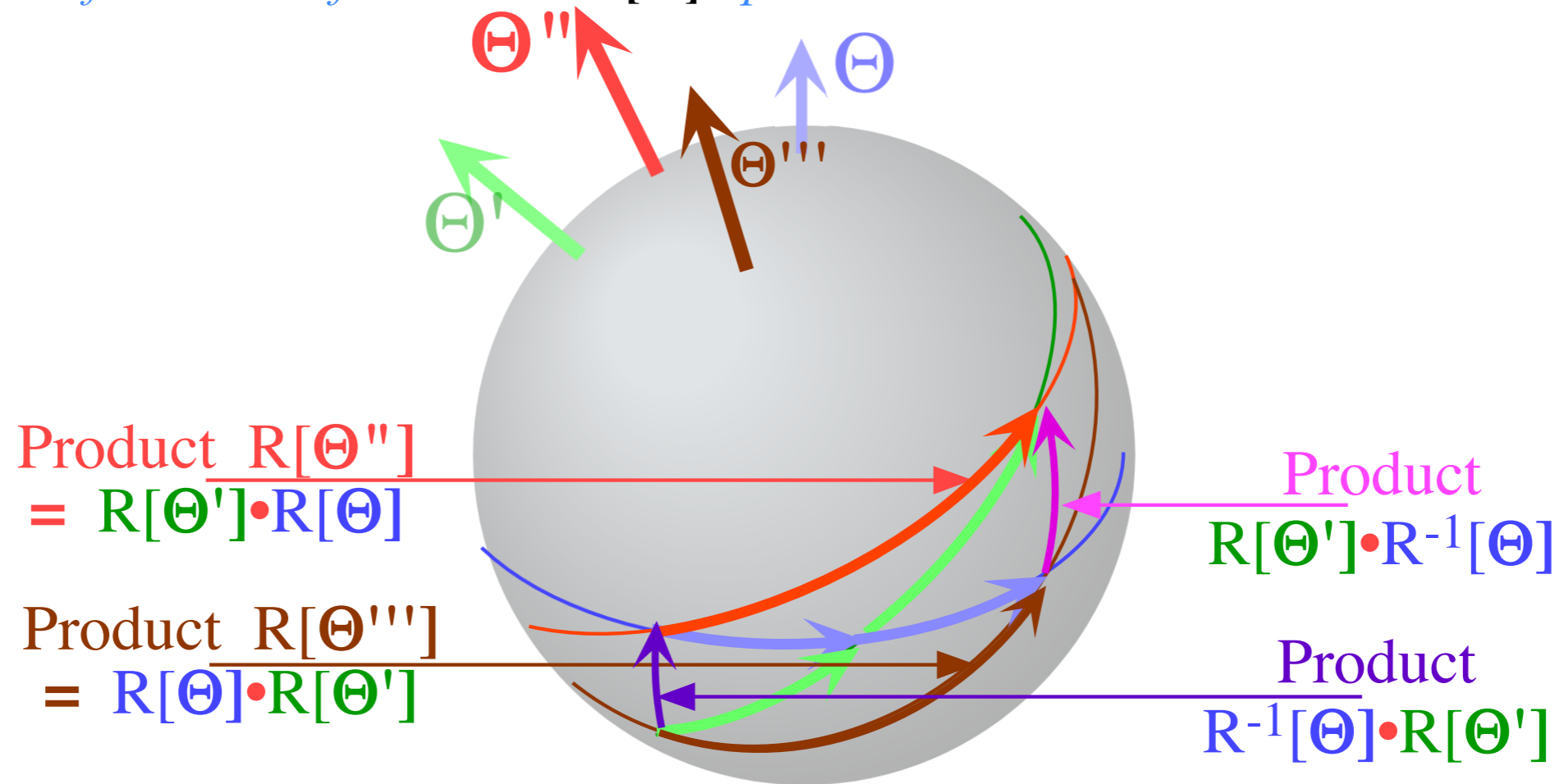
A similarity transformation of rotation $\mathbf{R}[\Theta''']$ by rotation $\mathbf{R}[\Theta]$ gives rotation $\mathbf{R}[\Theta''']$ and *vice-versa*:

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Everything associated with rotation $\mathbf{R}[\Theta''']$ is rotated by full angle Θ around axis Θ .

Geometry of transformation of rotational $\mathbf{R}[\Theta]$ -operators



QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$ and $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$

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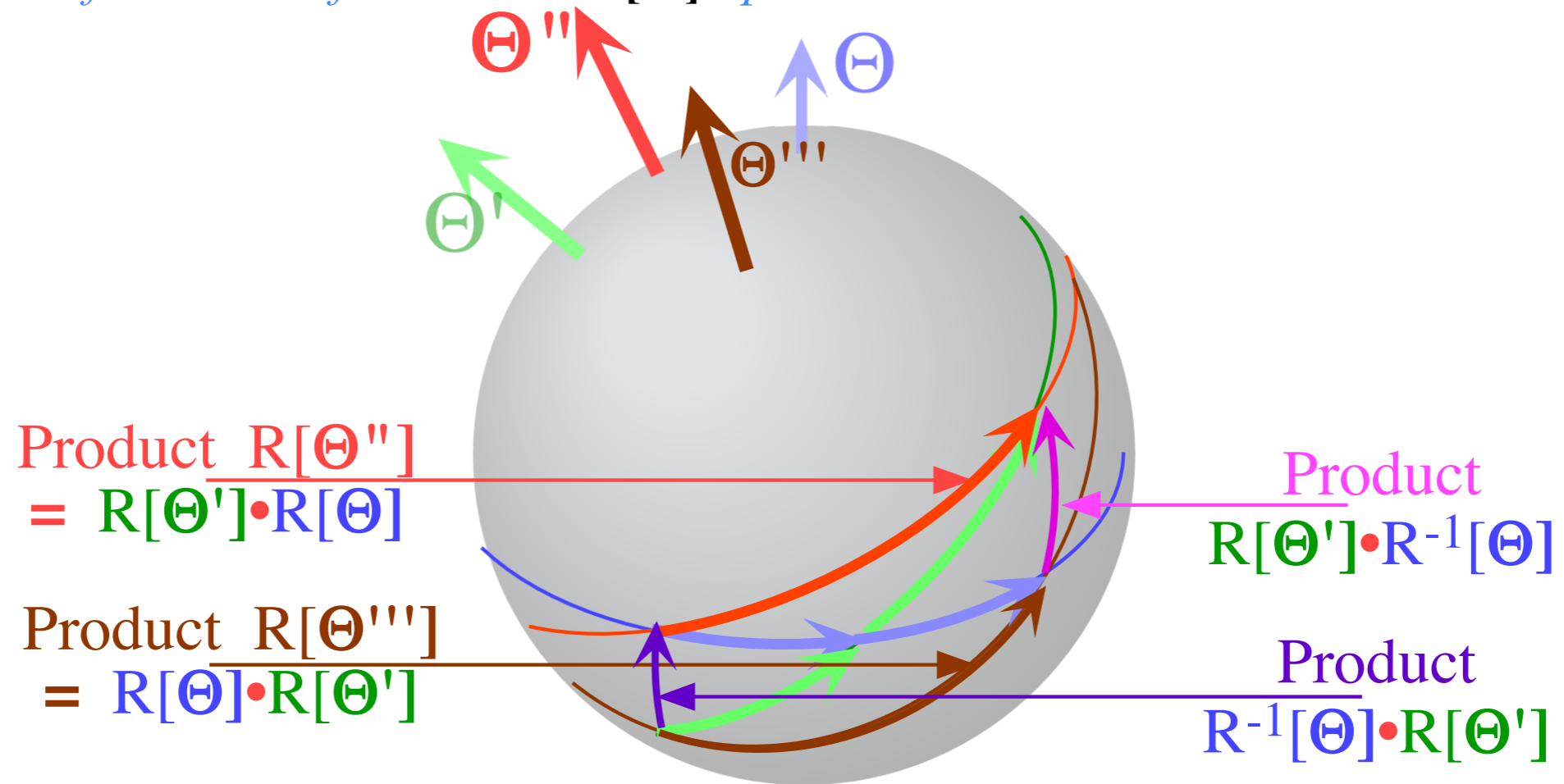
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Everything associated with rotation $\mathbf{R}[\Theta'']$ is rotated by full angle Θ around axis Θ .

Crank vector Θ and its turn arc moved by two $\mathbf{R}[\Theta]$ turn arcs into turn arc of $\mathbf{R}[\Theta''']$ below it.

Geometry of transformation of rotational $\mathbf{R}[\Theta]$ -operators



QTforCA

Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$ and $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$

Vectors added in the reverse order give $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$ instead of $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$.

A similarity transformation of rotation $\mathbf{R}[\Theta'']$ by rotation $\mathbf{R}[\Theta]$ gives rotation $\mathbf{R}[\Theta''']$ and *vice-versa*:

$$\mathbf{R}[\Theta] \underbrace{\mathbf{R}[\Theta''] \mathbf{R}[-\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'''] \quad \mathbf{R}[-\Theta] \underbrace{\mathbf{R}[\Theta'''] \mathbf{R}[\Theta]}_{\mathbf{R}[\Theta']} = \mathbf{R}[\Theta'']$$

Everything associated with rotation $\mathbf{R}[\Theta'']$ is rotated by full angle Θ around axis Θ .

Crank vector Θ and its turn arc moved by two $\mathbf{R}[\Theta]$ turn arcs into turn arc of $\mathbf{R}[\Theta''']$ below it.

Another similarity transformation of rotation $\mathbf{R}[\Theta''']$ by rotation $\mathbf{R}[\Theta']$ to $\mathbf{R}[\Theta'']$

$$\mathbf{R}[\Theta'] \underbrace{\mathbf{R}[\Theta'''] \mathbf{R}[-\Theta']}_{\mathbf{R}[\Theta]} = \mathbf{R}[\Theta'']$$

Geometry of transformation of rotational $\mathbf{R}[\Theta]$ -operators

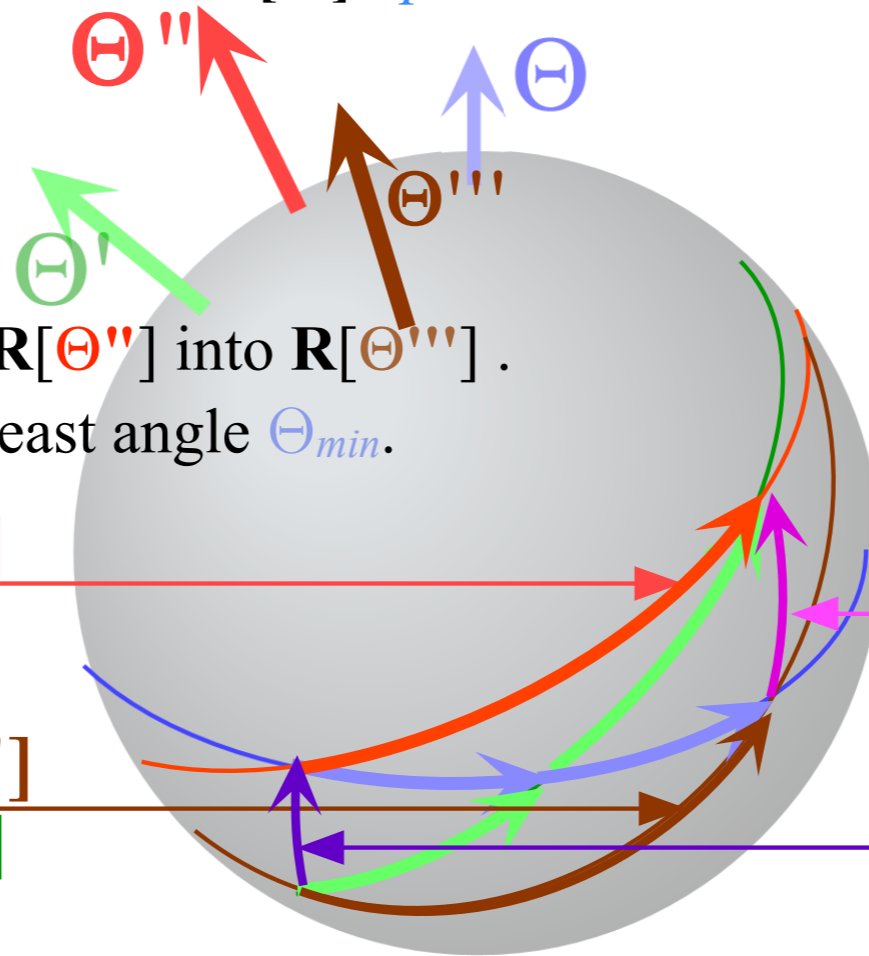
Many (∞) rotations transform $\mathbf{R}[\Theta'']$ into $\mathbf{R}[\Theta''']$.
Of these, there is one with the least angle Θ_{min} .

$$\text{Product } \mathbf{R}[\Theta''] \\ = \mathbf{R}[\Theta'] \cdot \mathbf{R}[\Theta]$$

$$\text{Product } \mathbf{R}[\Theta'''] \\ = \mathbf{R}[\Theta] \cdot \mathbf{R}[\Theta']$$

$$\text{Product} \\ \mathbf{R}[\Theta'] \cdot \mathbf{R}^{-1}[\Theta]$$

$$\text{Product} \\ \mathbf{R}^{-1}[\Theta] \cdot \mathbf{R}[\Theta']$$



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Fig. 10.A.9 Hamilton-turn arc parallelogram with $\mathbf{R}[\Theta''] = \mathbf{R}[\Theta']\mathbf{R}[\Theta]$ and $\mathbf{R}[\Theta'''] = \mathbf{R}[\Theta]\mathbf{R}[\Theta']$

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Everything associated with rotation $\mathbf{R}[\Theta'']$ is rotated by full angle Θ around axis Θ .

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*Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators*

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ of spinor σ_μ -operators

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ of group-operators

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

 *Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$*

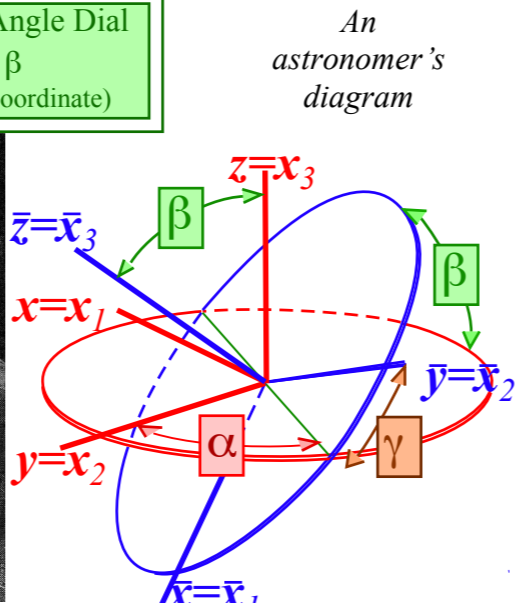
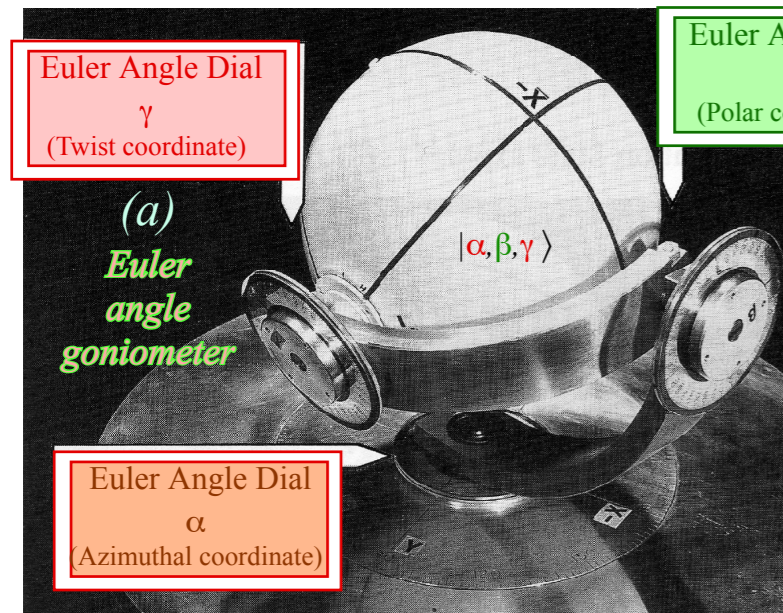
Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

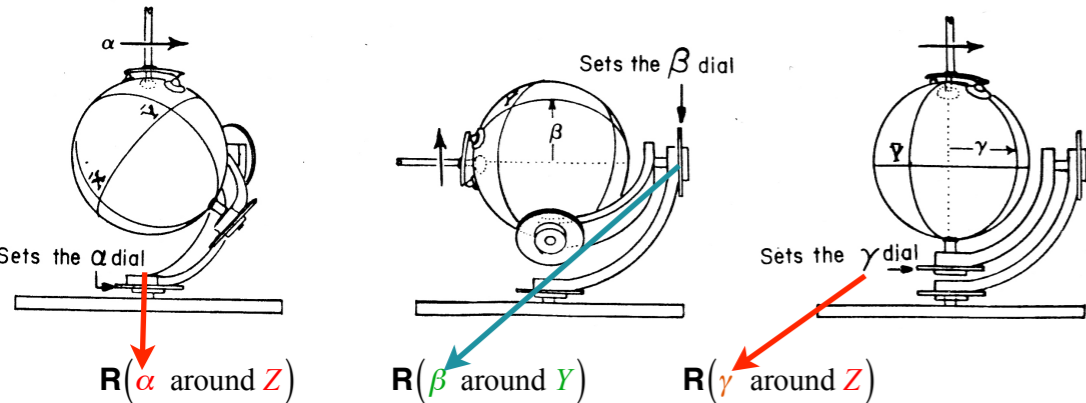
$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Euler $R(\alpha\beta\gamma)$ versus Darboux $R[\varphi\vartheta\Theta]$



Third rotation $R(\alpha 0 0)$ Second rotation $R(0 \beta 0)$ First rotation $R(0 0 \gamma)$

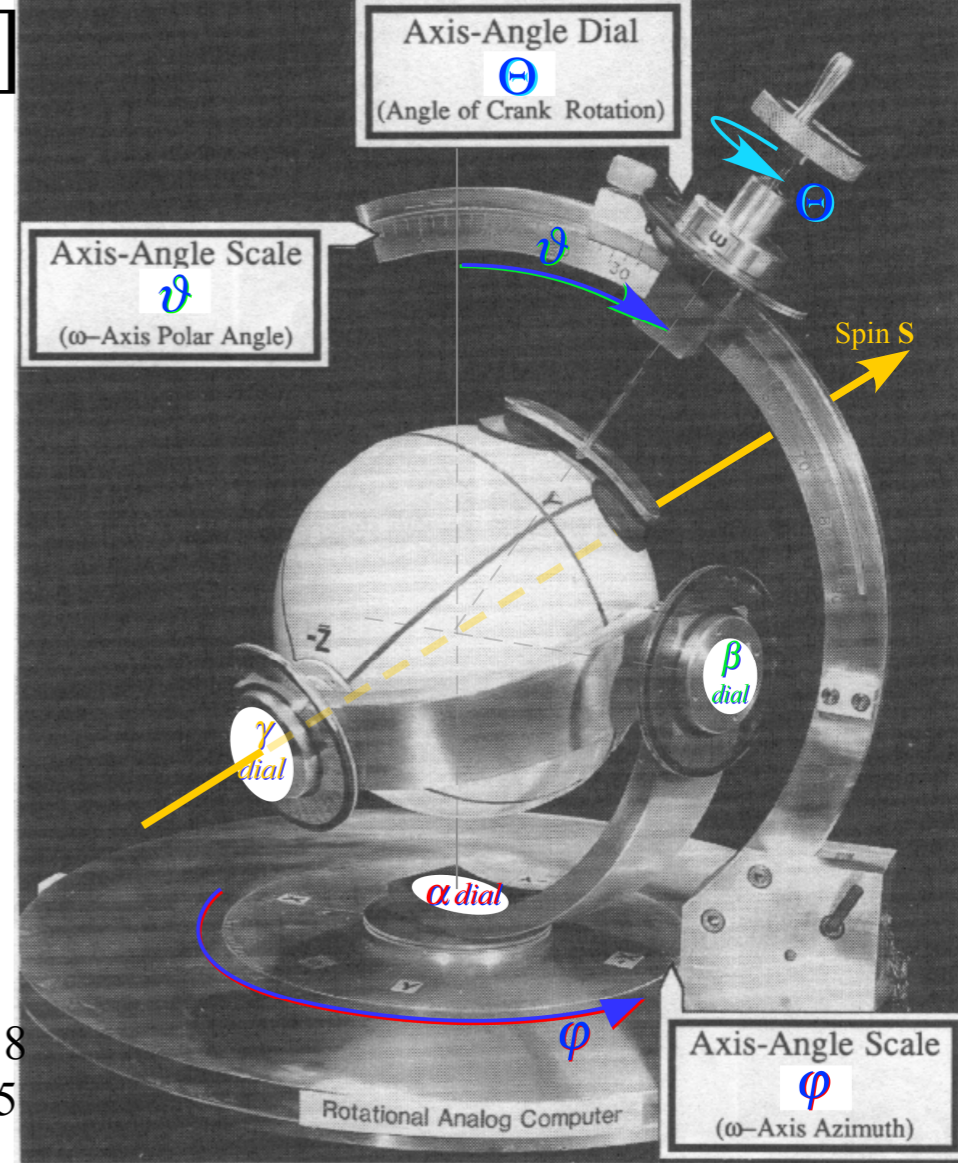


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$$R(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

$$= \cos\frac{\alpha+\gamma}{2} \cos\frac{\beta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\frac{\gamma-\alpha}{2} \sin\frac{\beta}{2} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cos\frac{\gamma-\alpha}{2} \sin\frac{\beta}{2} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin\frac{\alpha+\gamma}{2} \cos\frac{\beta}{2}$$

Euler $R(\alpha\beta\gamma)$ is simpler to form than Θ -axis Darboux $R[\varphi\vartheta\Theta]$.



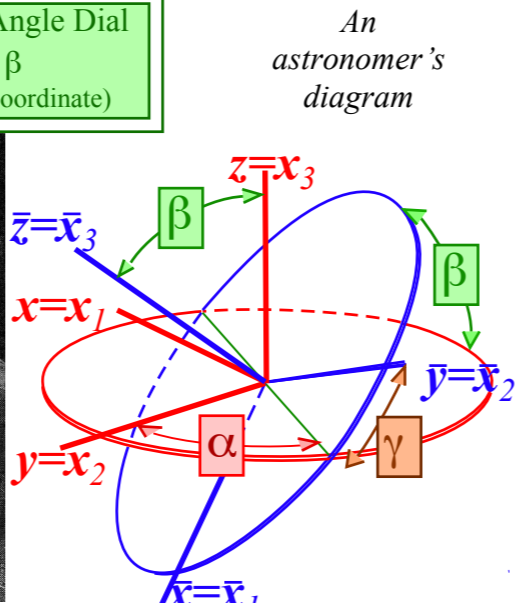
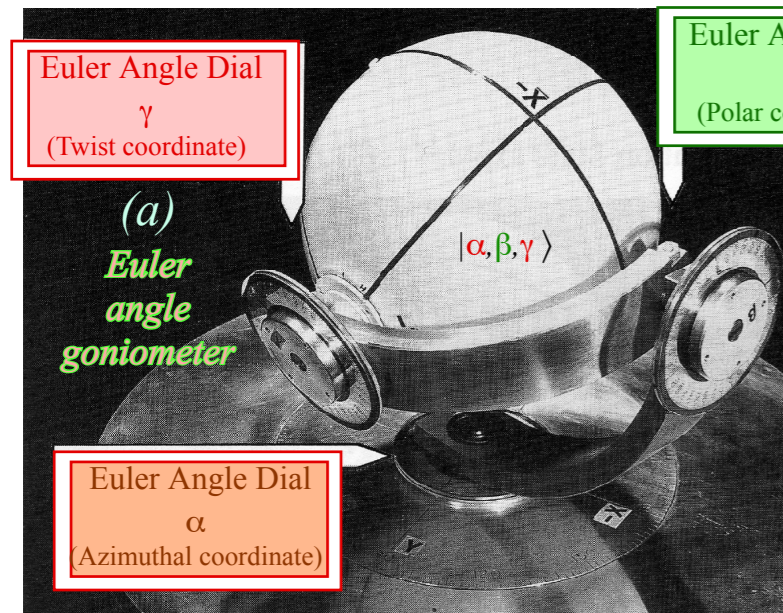
This Lecture 8 page 21 to 25

$$R[\bar{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z \sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix} = R[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\hat{\Theta}_X \sin\frac{\Theta}{2}}_{\cos\varphi \sin\vartheta} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \underbrace{\hat{\Theta}_Y \sin\frac{\Theta}{2}}_{\sin\varphi \sin\vartheta} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underbrace{\hat{\Theta}_Z \sin\frac{\Theta}{2}}_{\cos\vartheta}$$

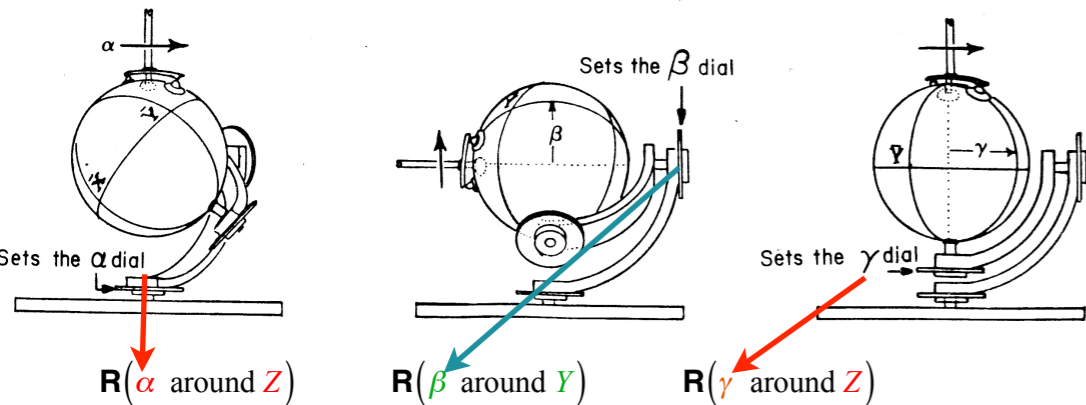
$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta \sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi \sin\vartheta + i\cos\varphi \sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi \sin\vartheta - i\cos\varphi \sin\vartheta) & \cos\frac{\Theta}{2} + i\cos\vartheta \sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $R(\alpha\beta\gamma)$ versus Darboux $R[\varphi\vartheta\Theta]$



Third rotation $R(\alpha 0 0)$ Second rotation $R(0 \beta 0)$ First rotation $R(0 0 \gamma)$

From Lecture 7 page 80 to 89



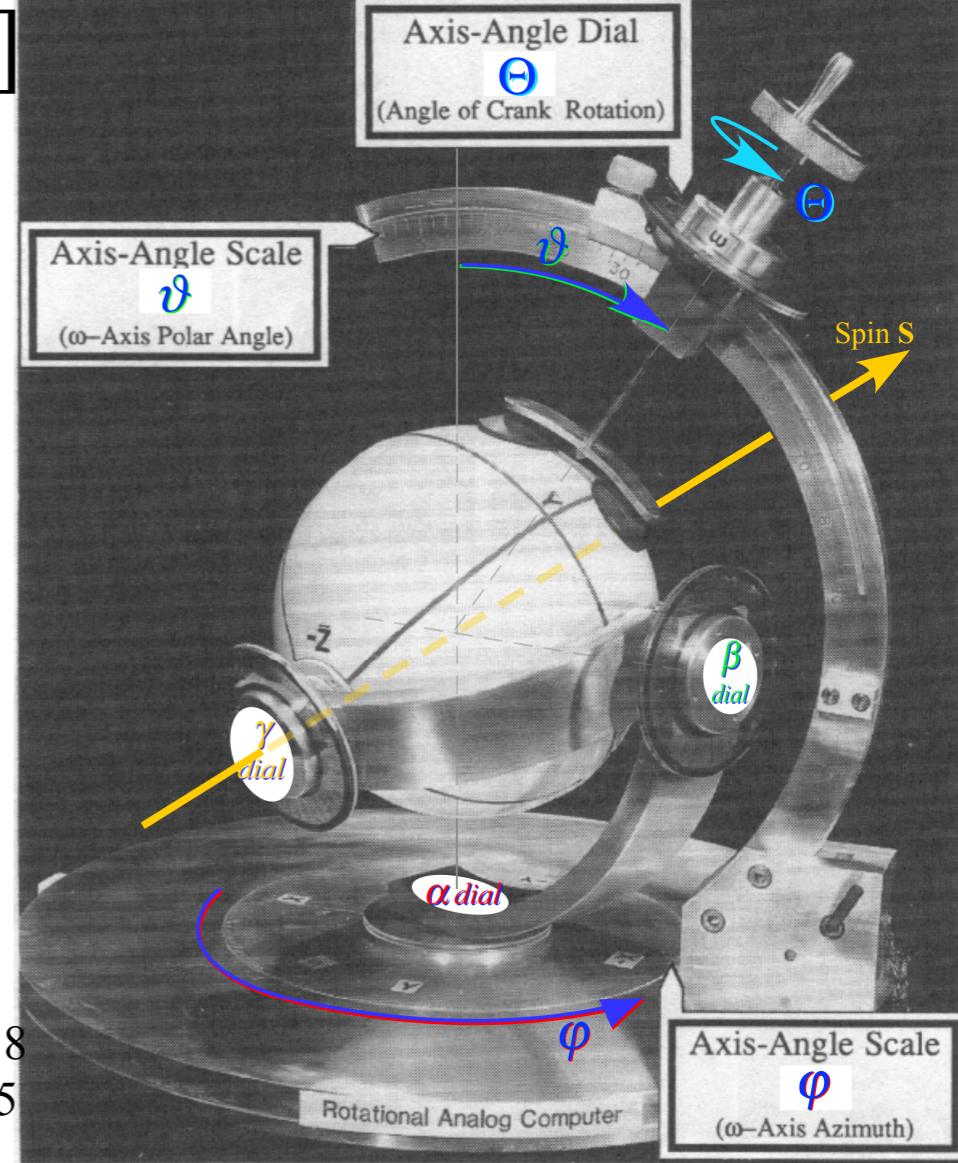
$$R(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

$$= \cos\frac{\alpha+\gamma}{2} \cos\frac{\beta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\frac{\gamma-\alpha}{2} \sin\frac{\beta}{2} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cos\frac{\gamma-\alpha}{2} \sin\frac{\beta}{2} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin\frac{\alpha+\gamma}{2} \cos\frac{\beta}{2}$$

Euler $R(\alpha\beta\gamma)$ is simpler to form than Θ -axis Darboux $R[\varphi\vartheta\Theta]$.

Euler *state definition*:

$$|\alpha\beta\gamma\rangle = R(\alpha\beta\gamma)|000\rangle \quad (\alpha\beta\gamma \text{ make better coordinates})$$



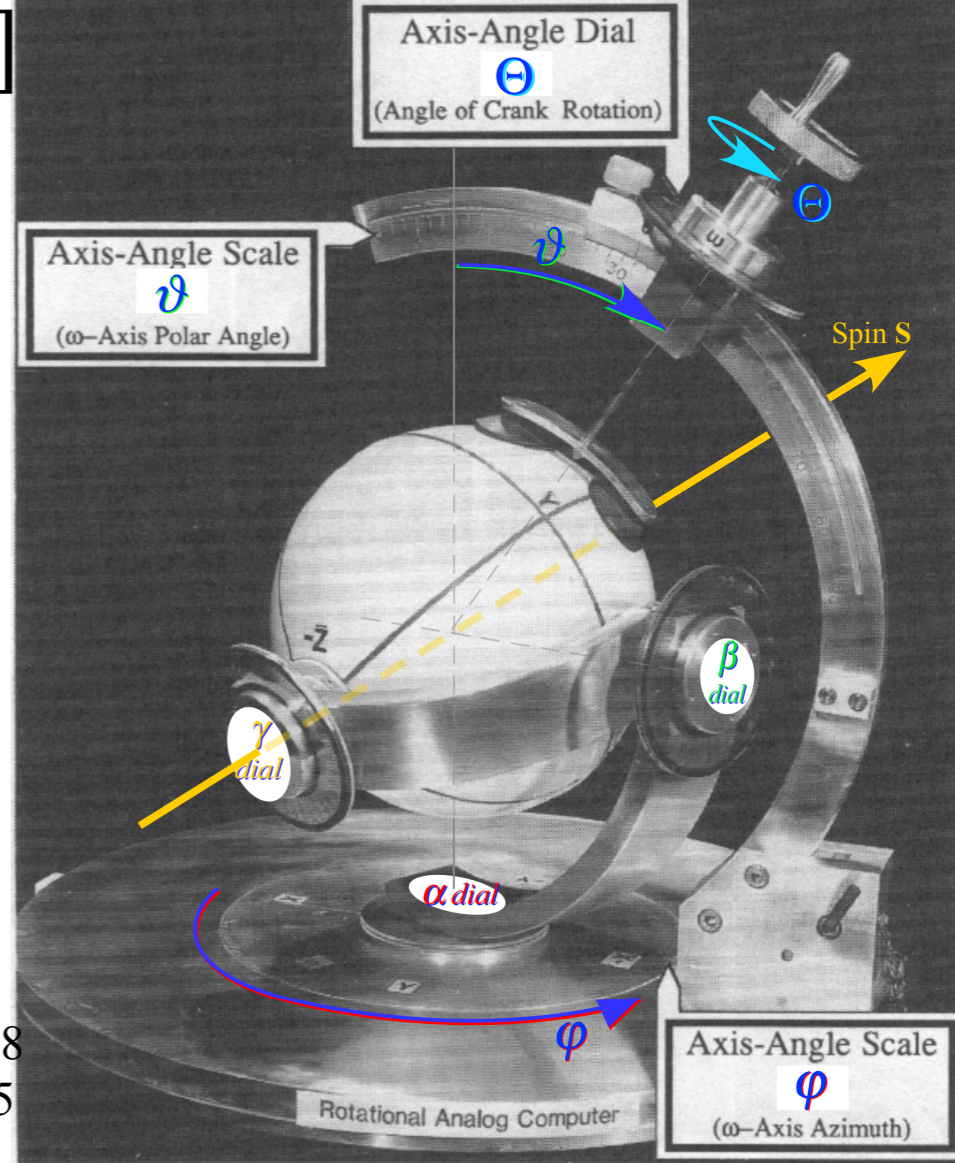
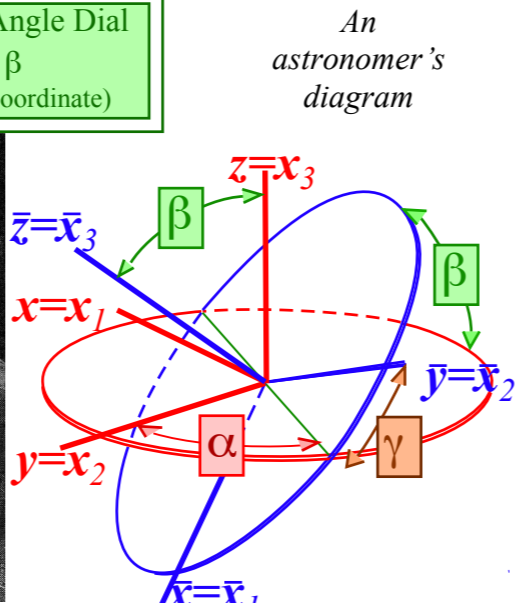
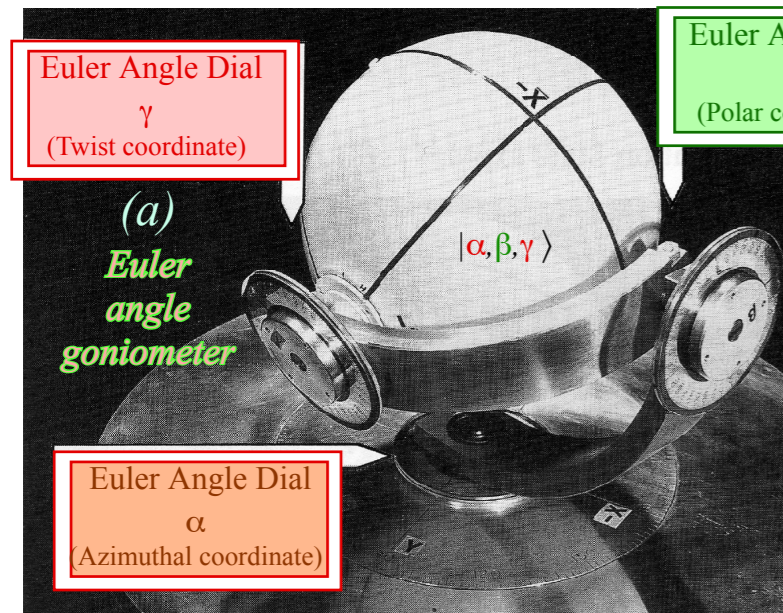
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$$R[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z \sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix} = R[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\hat{\Theta}_X \sin\frac{\Theta}{2}}_{\cos\varphi \sin\vartheta} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \underbrace{\hat{\Theta}_Y \sin\frac{\Theta}{2}}_{\sin\varphi \sin\vartheta} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underbrace{\hat{\Theta}_Z \sin\frac{\Theta}{2}}_{\cos\vartheta}$$

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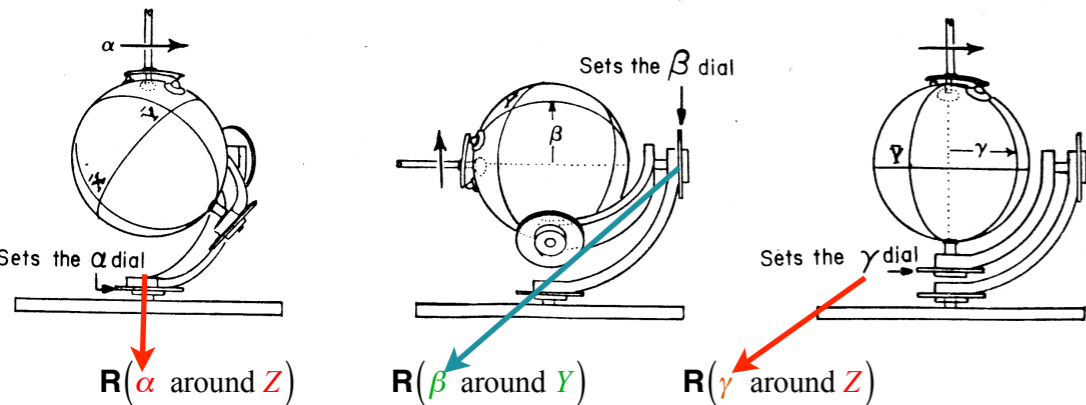
Euler $R(\alpha\beta\gamma)$ versus Darboux $R[\varphi\vartheta\Theta]$



Third rotation $R(\alpha 0 0)$ Second rotation $R(0 \beta 0)$ First rotation $R(0 0 \gamma)$

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$$R(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

$$= \cos\frac{\alpha+\gamma}{2} \cos\frac{\beta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\frac{\gamma-\alpha}{2} \sin\frac{\beta}{2} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cos\frac{\gamma-\alpha}{2} \sin\frac{\beta}{2} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin\frac{\alpha+\gamma}{2} \cos\frac{\beta}{2}$$

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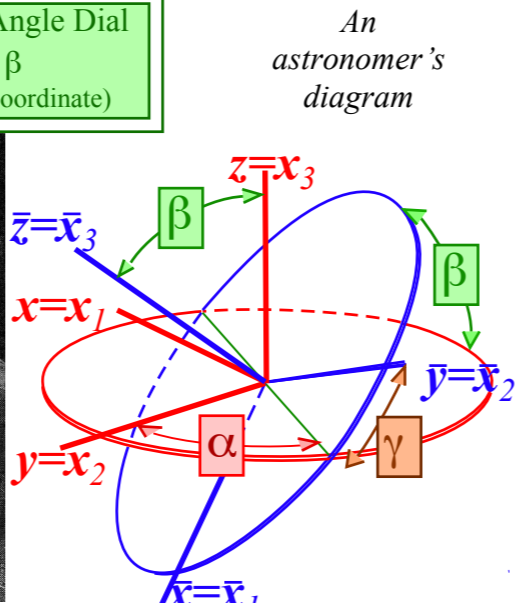
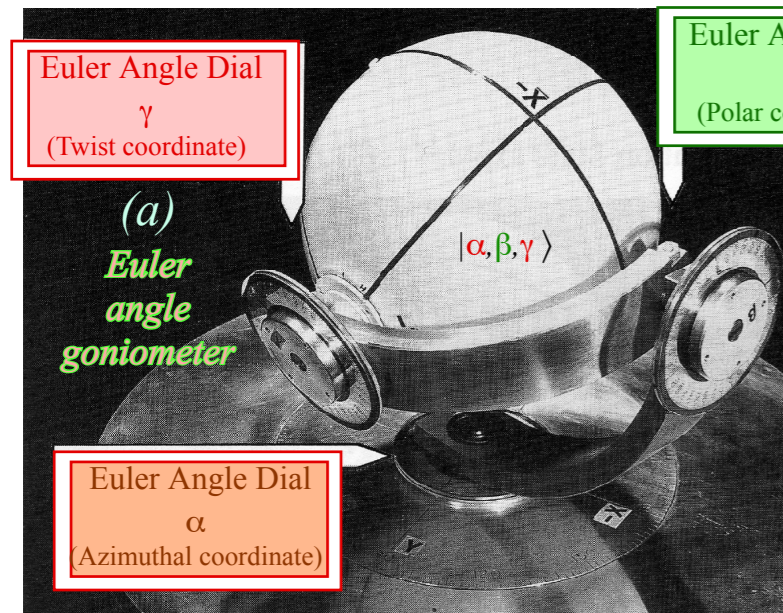
$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta \sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi \sin\vartheta + i\cos\varphi \sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi \sin\vartheta - i\cos\varphi \sin\vartheta) & \cos\frac{\Theta}{2} + i\cos\vartheta \sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $R(\alpha\beta\gamma)$ is simpler to form than Θ -axis Darboux $R[\varphi\vartheta\Theta]$.

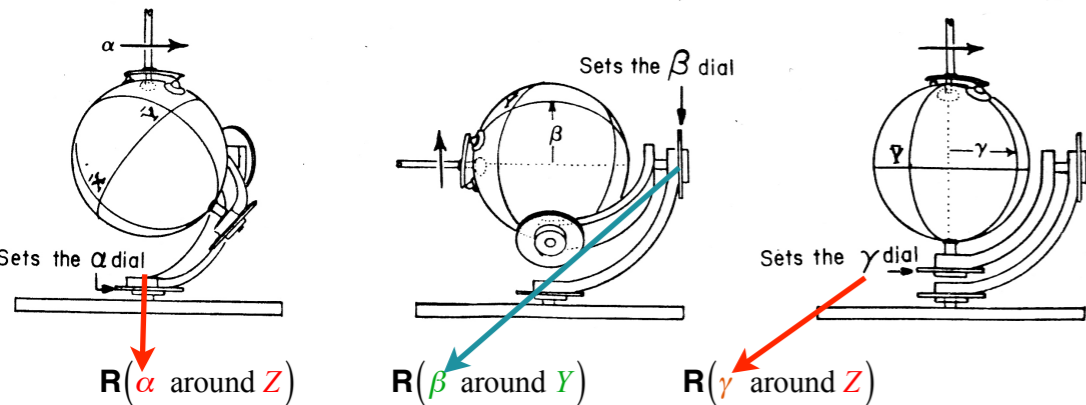
Euler *state definition* lets us relate $R(\alpha\beta\gamma)$ to $R[\varphi\vartheta\Theta]$...

$$|\alpha\beta\gamma\rangle = R(\alpha\beta\gamma)|000\rangle \quad (\alpha\beta\gamma \text{ make better coordinates})$$

Euler $R(\alpha\beta\gamma)$ versus Darboux $R[\varphi\vartheta\Theta]$



Third rotation $R(\alpha 0 0)$ Second rotation $R(0 \beta 0)$ First rotation $R(0 0 \gamma)$



From Lecture 7 page 80 to 89

$$R(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

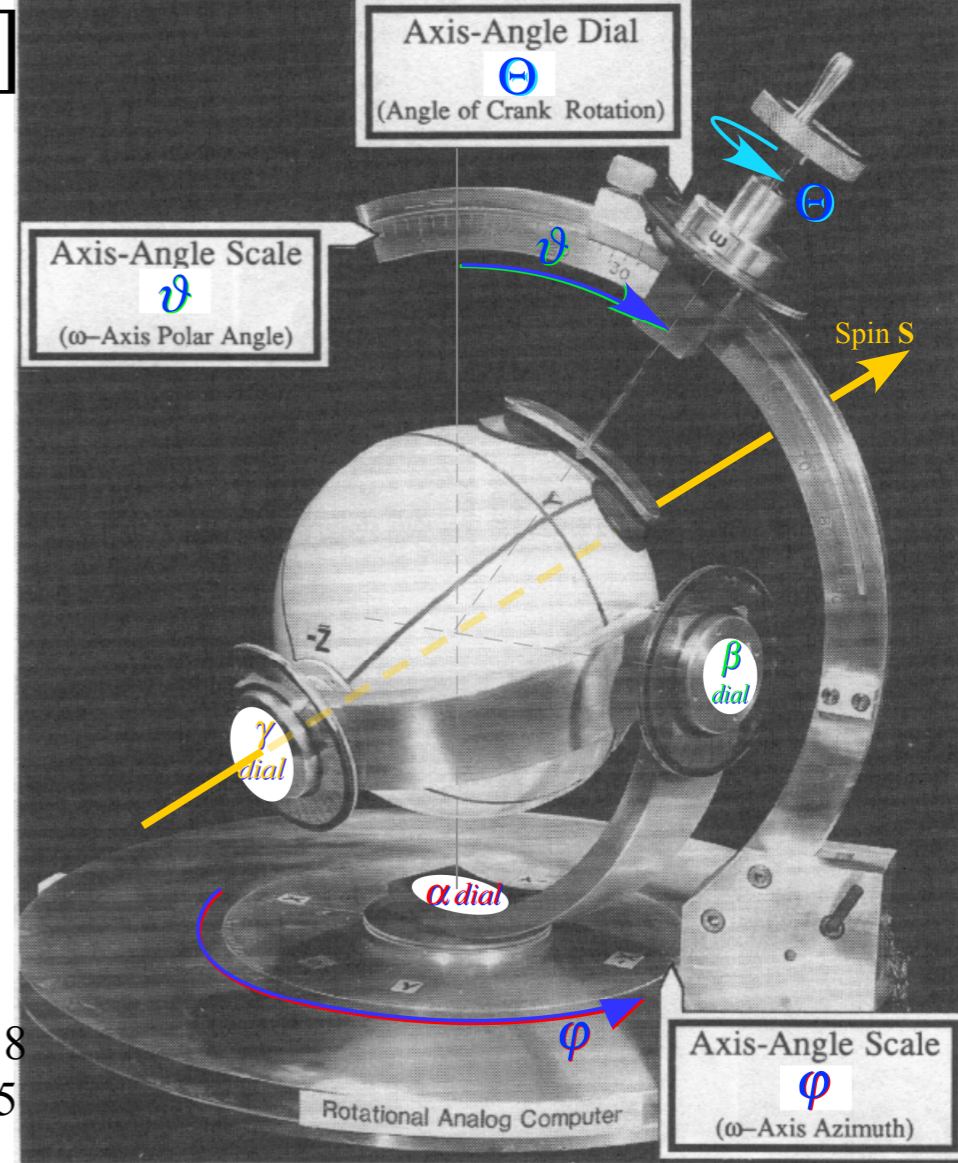
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Euler $R(\alpha\beta\gamma)$ is simpler to form than Θ -axis Darboux $R[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $R(\alpha\beta\gamma)$ to $R[\varphi\vartheta\Theta]$...

$|\alpha\beta\gamma\rangle = R(\alpha\beta\gamma)|000\rangle$ ($\alpha\beta\gamma$ make better coordinates)

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



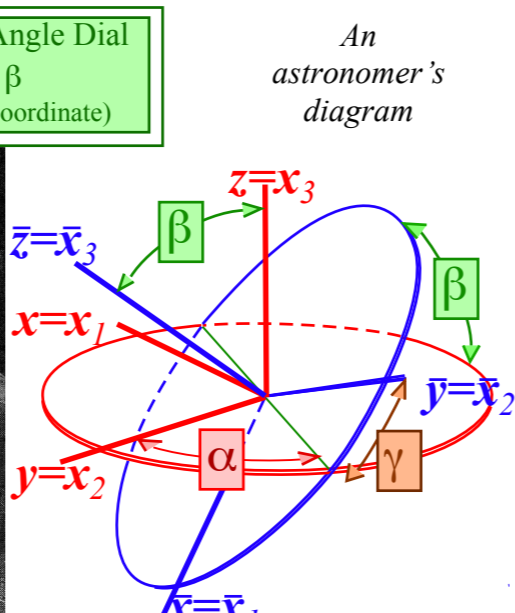
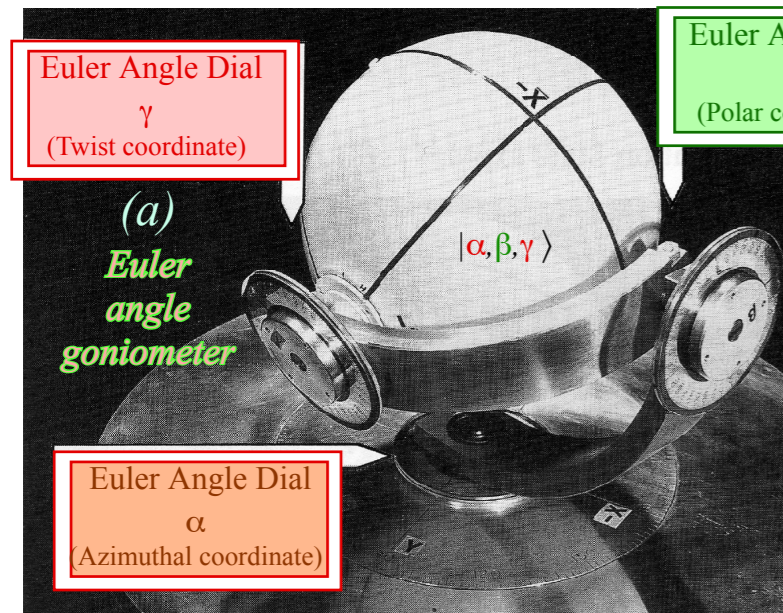
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$$R[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z \sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix} = R[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

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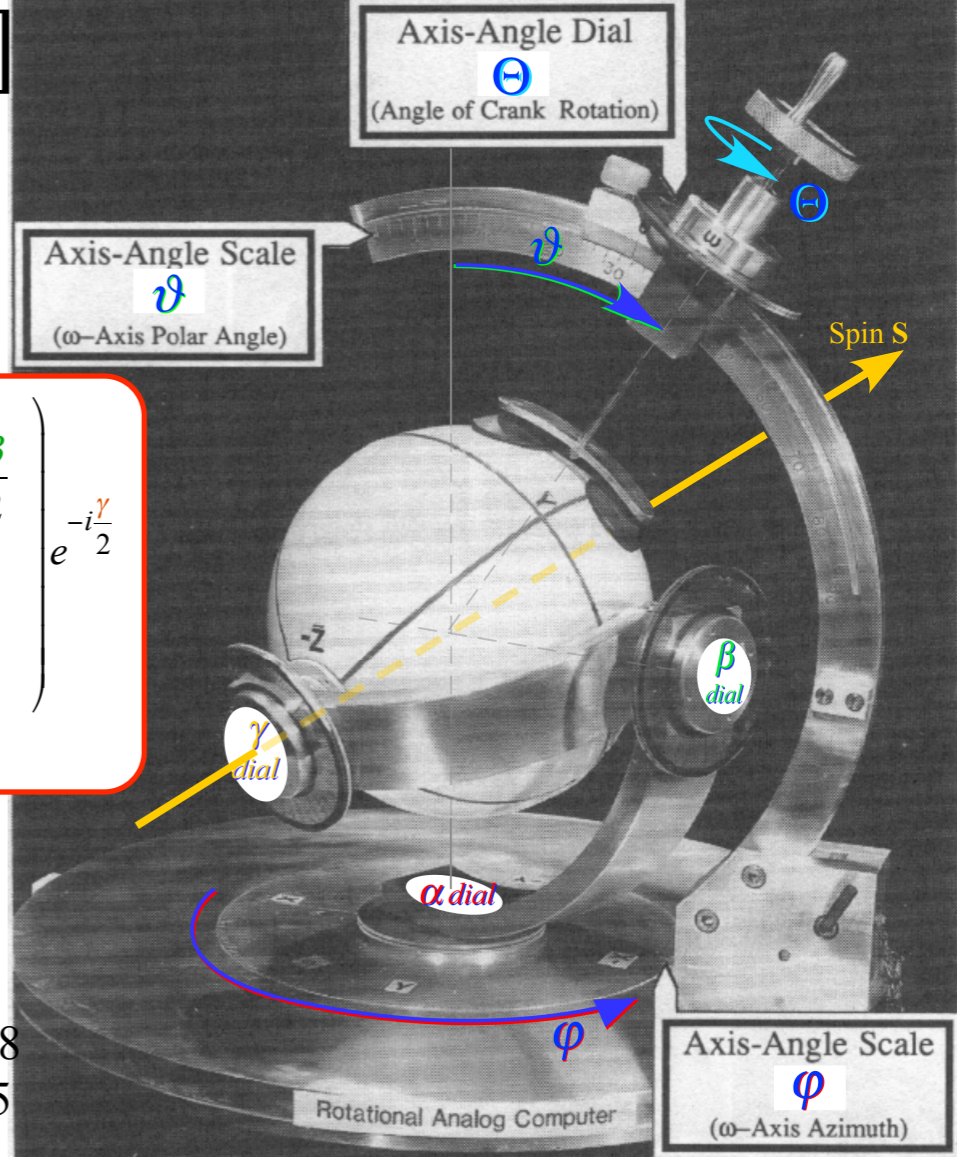
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Euler $R(\alpha\beta\gamma)$ versus Darboux $R[\varphi\vartheta\Theta]$

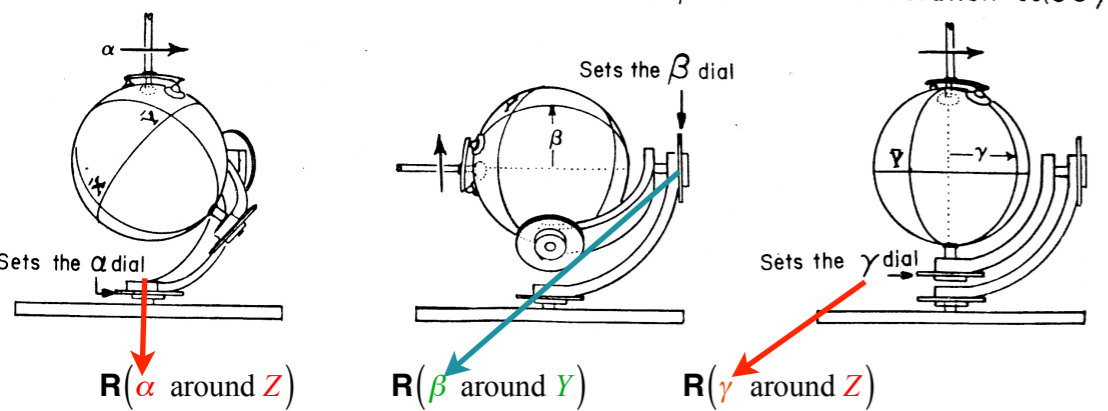


$$|\uparrow_{\alpha\beta\gamma}\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$$

$$= R(\alpha\beta\gamma)|\uparrow_{000}\rangle$$



Third rotation $R(\alpha 0 0)$ Second rotation $R(0 \beta 0)$ First rotation $R(0 0 \gamma)$



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$$R(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

$$R[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z \sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix} = R[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\Theta}_X \sin\frac{\Theta}{2} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\Theta}_Y \sin\frac{\Theta}{2} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix}$$

$\underbrace{\cos\varphi \sin\vartheta}_{\text{yellow}}$ $\underbrace{\sin\varphi \sin\vartheta}_{\text{green}}$ $\underbrace{\cos\vartheta}_{\text{red}}$

Euler $R(\alpha\beta\gamma)$ is simpler to form than Θ -axis Darboux $R[\varphi\vartheta\Theta]$. Euler *state definition* lets us relate $R(\alpha\beta\gamma)$ to $R[\varphi\vartheta\Theta]$...

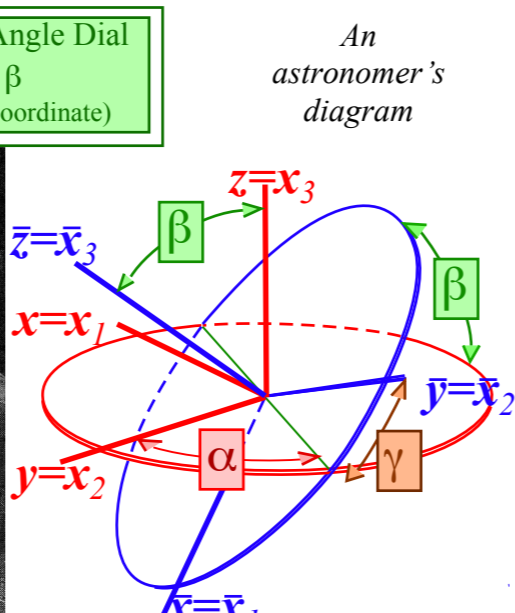
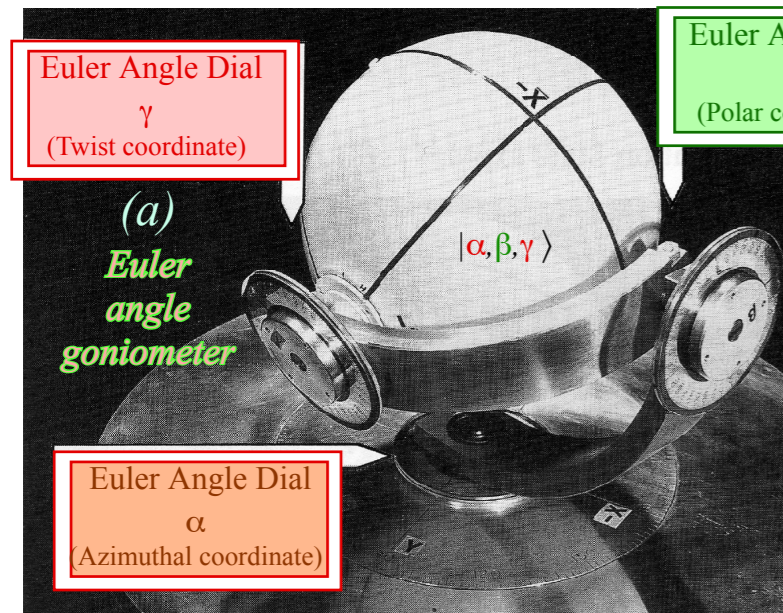
$|\alpha\beta\gamma\rangle = R(\alpha\beta\gamma)|000\rangle$ ($\alpha\beta\gamma$ make better coordinates)

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

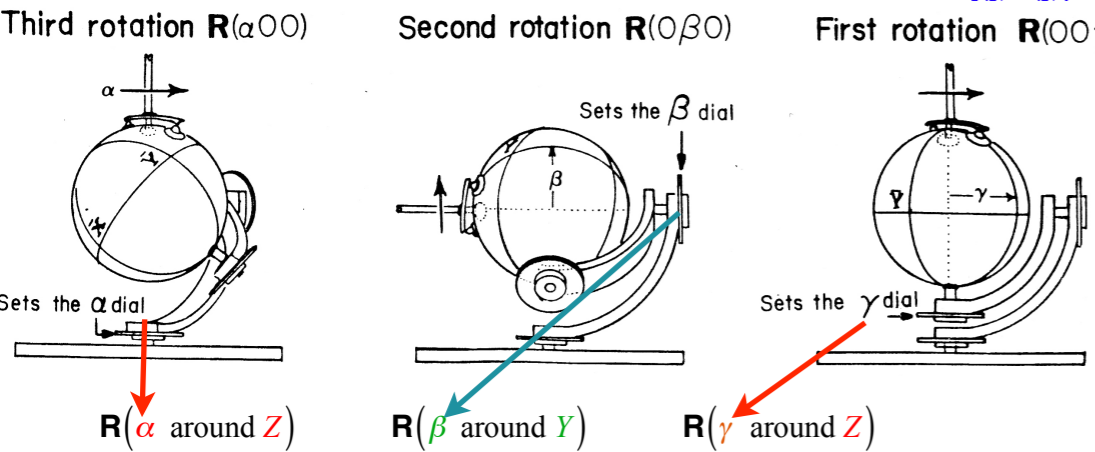
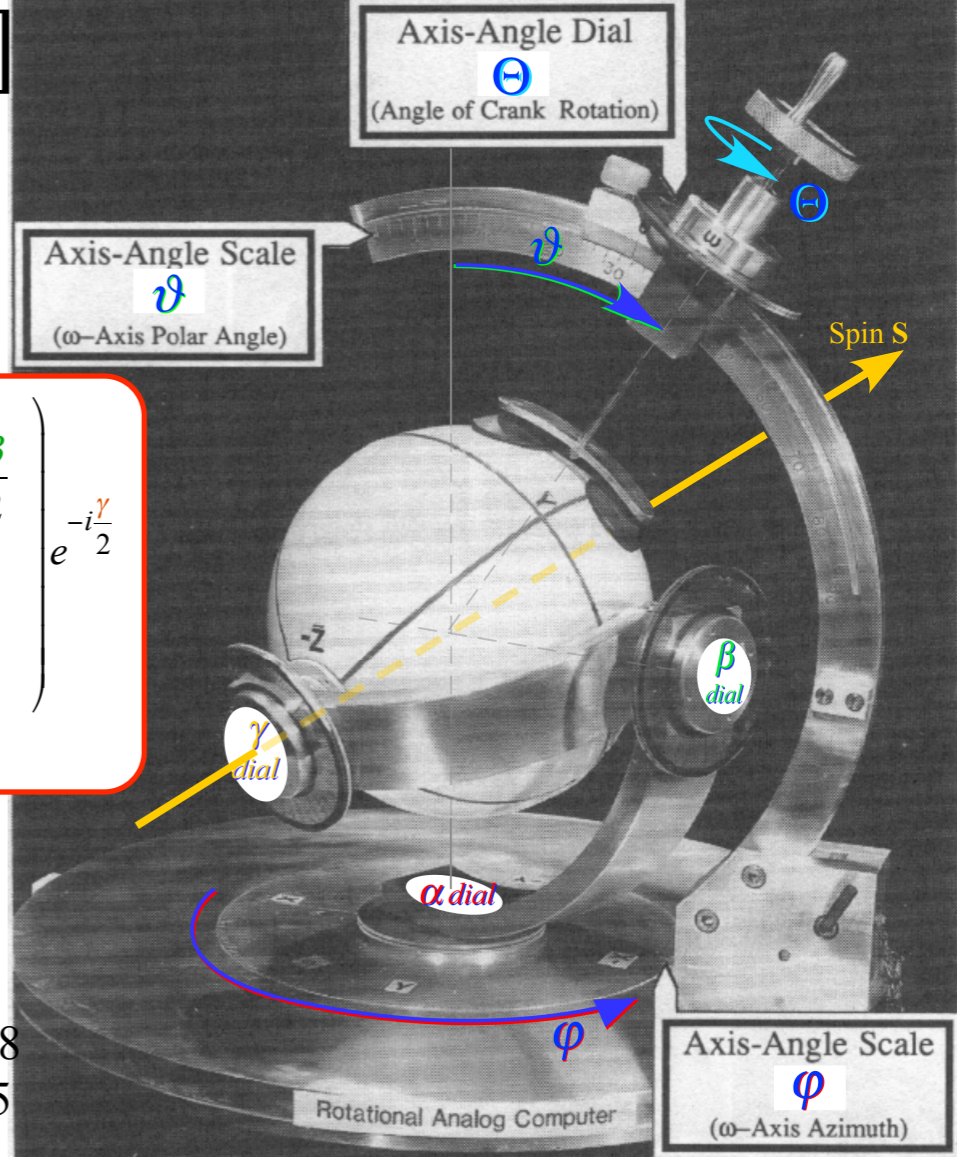
$x_1 = \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta \sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi \sin\vartheta + i\cos\varphi \sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi \sin\vartheta - i\cos\varphi \sin\vartheta) & \cos\frac{\Theta}{2} + i\cos\vartheta \sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $R(\alpha\beta\gamma)$ versus Darboux $R[\varphi\vartheta\Theta]$



$$|\uparrow_{\alpha\beta\gamma}\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \\ e^{-i\frac{\gamma}{2}} \end{pmatrix} = R(\alpha\beta\gamma)|\uparrow_{000}\rangle$$



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$$R(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

$$R[\bar{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z \sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix} = R[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\Theta}_X \sin\frac{\Theta}{2} - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \hat{\Theta}_Y \sin\frac{\Theta}{2} - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{\Theta}_Z \sin\frac{\Theta}{2} \\ \cos\varphi \sin\vartheta & \sin\varphi \sin\vartheta & \cos\vartheta \end{pmatrix}$$

Euler $R(\alpha\beta\gamma)$ is simpler to form than Θ -axis Darboux $R[\varphi\vartheta\Theta]$. Euler *state definition* lets us relate $R(\alpha\beta\gamma)$ to $R[\varphi\vartheta\Theta]$...

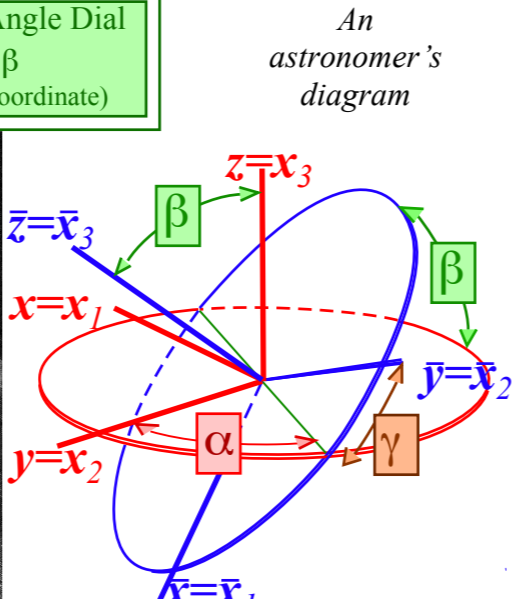
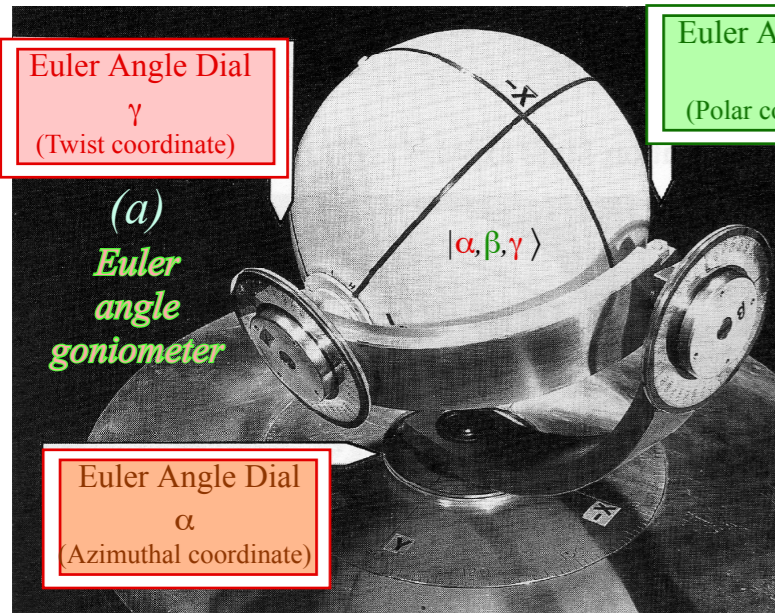
$|\alpha\beta\gamma\rangle = R(\alpha\beta\gamma)|000\rangle$ ($\alpha\beta\gamma$ make better coordinates)

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

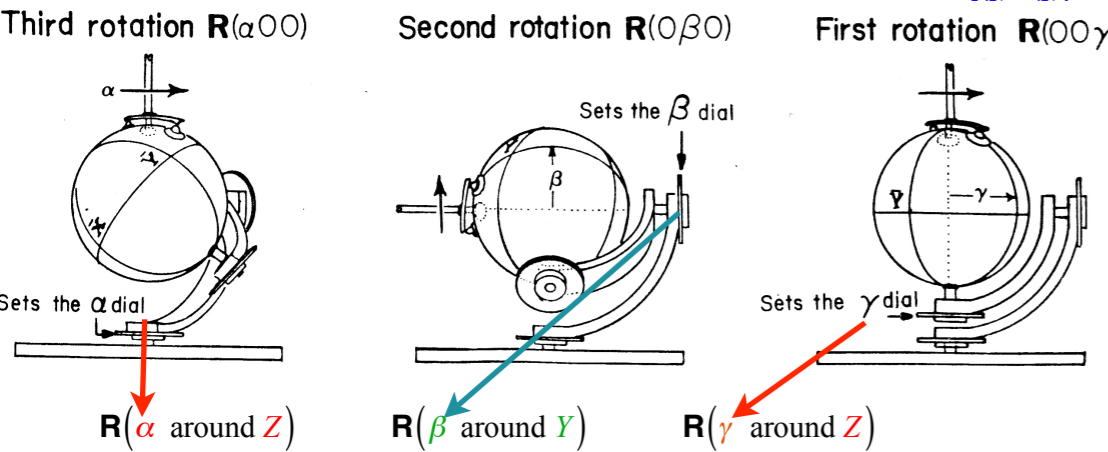
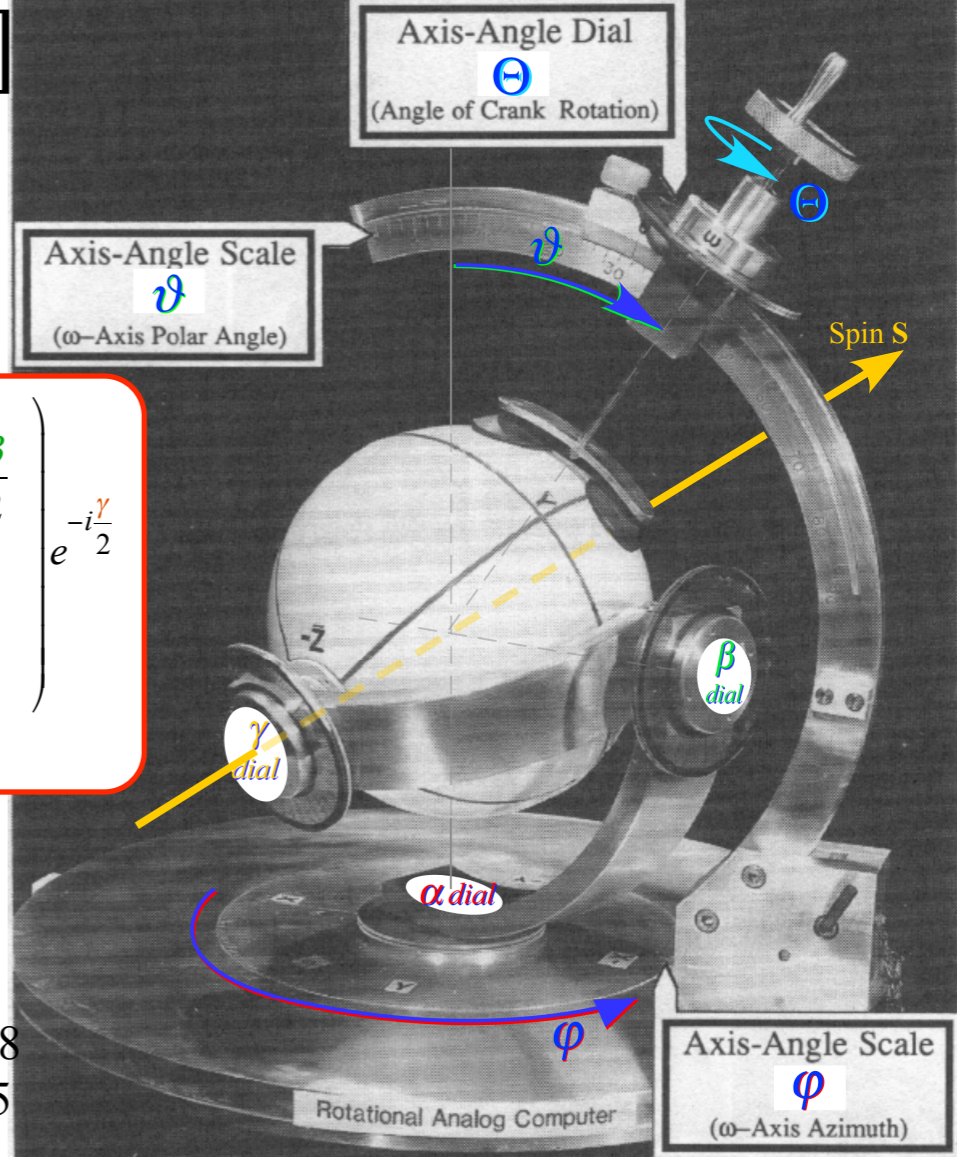
$$x_1 = \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2$$

Euler $R(\alpha\beta\gamma)$ versus Darboux $R[\varphi\vartheta\Theta]$



$$|\uparrow_{\alpha\beta\gamma}\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \\ e^{-i\frac{\gamma}{2}} \end{pmatrix} = R(\alpha\beta\gamma)|\uparrow_{000}\rangle$$



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$$R(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

$$R[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z \sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix} = R[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} & 0 & 0 \\ 0 & \cos\varphi \sin\vartheta & \sin\varphi \sin\vartheta \\ 0 & \sin\varphi \sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \hat{\Theta}_X \sin\frac{\Theta}{2} - i\hat{\Theta}_Y \sin\frac{\Theta}{2} & \hat{\Theta}_Z \sin\frac{\Theta}{2} \\ 0 & \hat{\Theta}_Y \sin\frac{\Theta}{2} + i\hat{\Theta}_X \sin\frac{\Theta}{2} & \hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix}$$

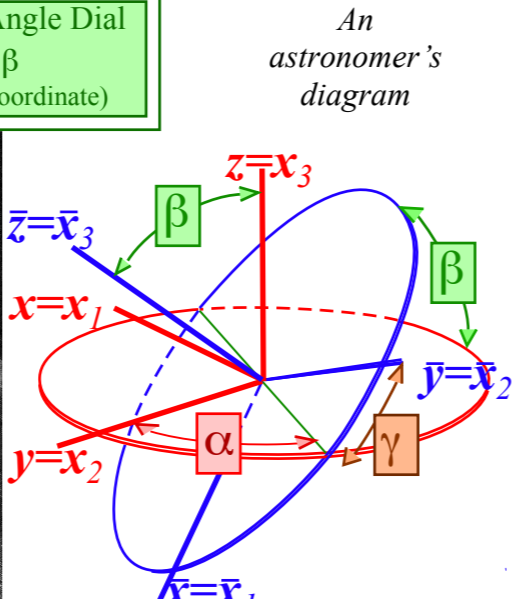
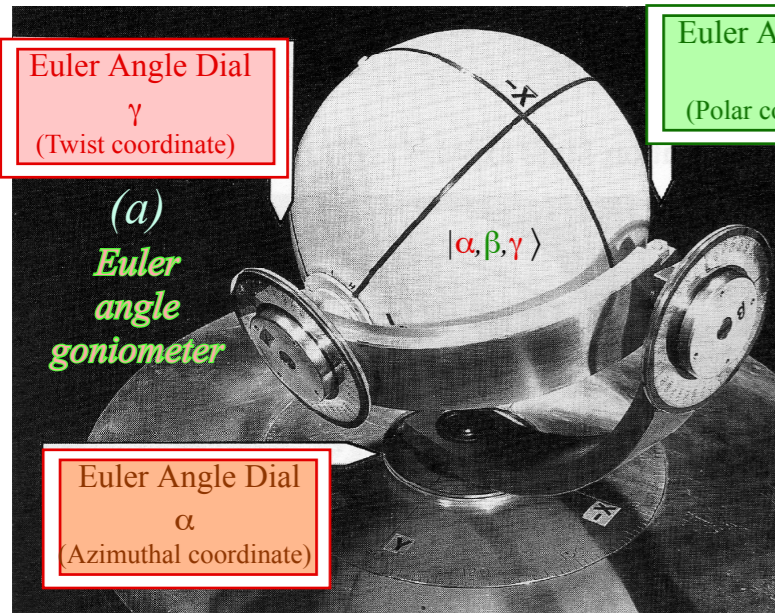
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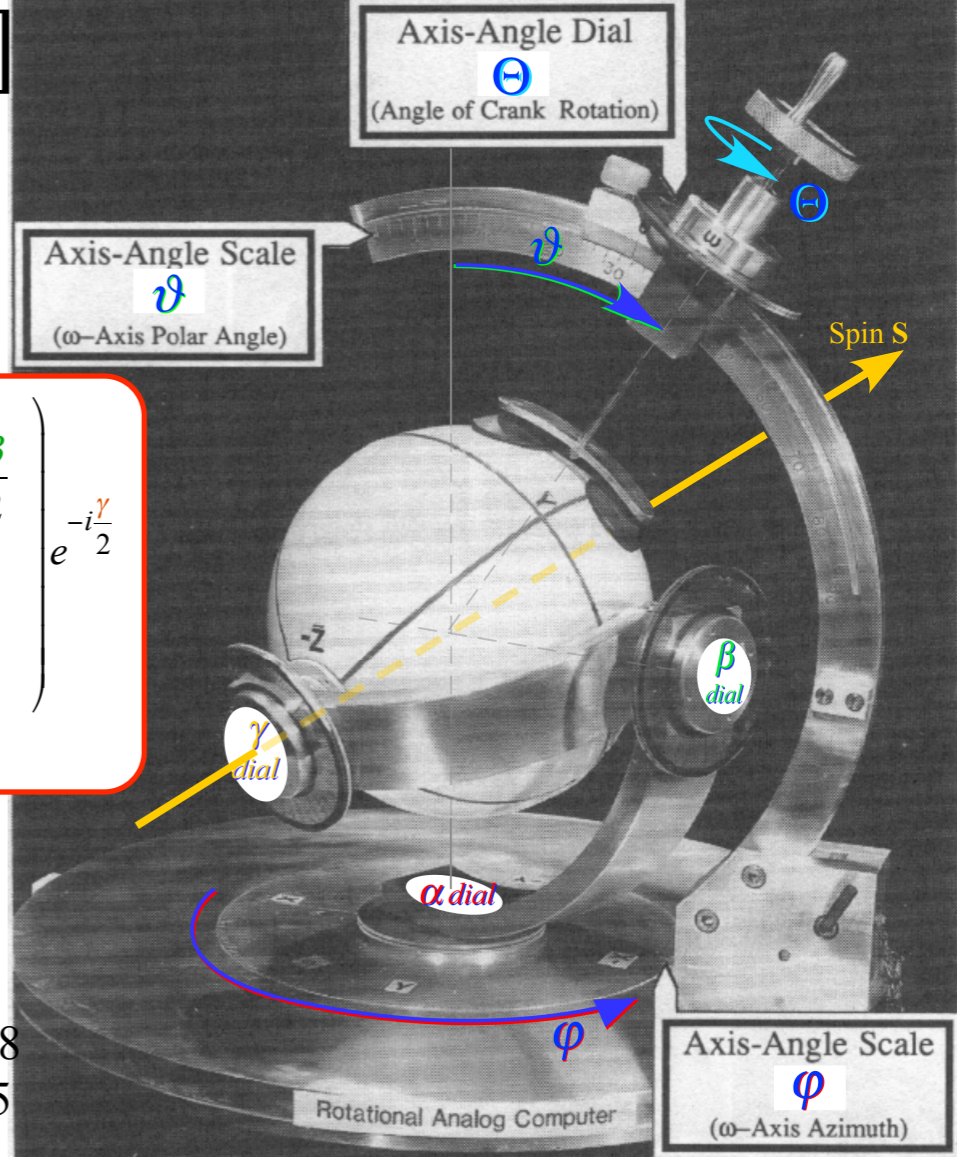
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi \sin\vartheta \sin\Theta/2 \end{aligned}$$

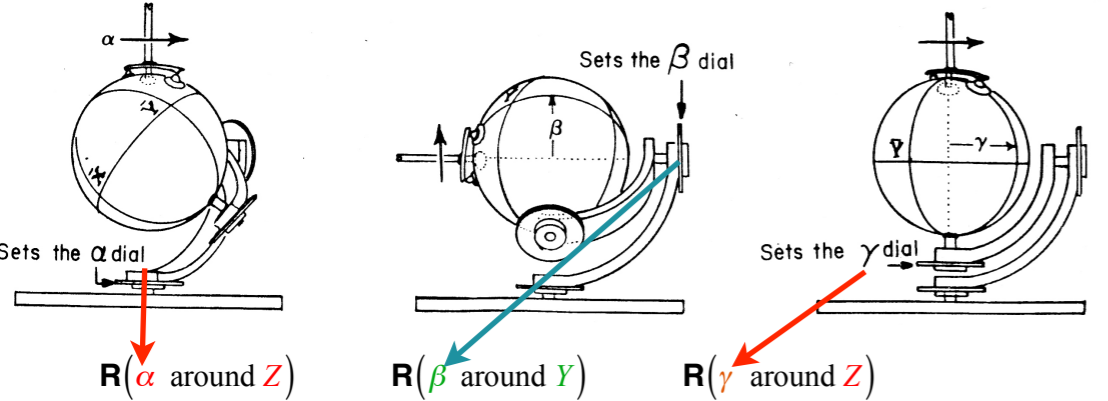
Euler $R(\alpha\beta\gamma)$ versus Darboux $R[\varphi\vartheta\Theta]$



$$|\uparrow_{\alpha\beta\gamma}\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \\ e^{-i\frac{\gamma}{2}} \end{pmatrix} = R(\alpha\beta\gamma)|\uparrow_{000}\rangle$$



Third rotation $R(\alpha 0 0)$ Second rotation $R(0 \beta 0)$ First rotation $R(0 0 \gamma)$



From Lecture 7 page 80 to 89

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$$R(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

$$R[\bar{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z \sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z \sin\frac{\Theta}{2} \end{pmatrix} = R[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} & 0 & 0 \\ 0 & \cos\varphi \sin\vartheta & \sin\varphi \sin\vartheta \\ 0 & \sin\varphi \sin\vartheta & \cos\vartheta \end{pmatrix}$$

Euler $R(\alpha\beta\gamma)$ is simpler to form than Θ -axis Darboux $R[\varphi\vartheta\Theta]$. Euler *state definition* lets us relate $R(\alpha\beta\gamma)$ to $R[\varphi\vartheta\Theta]$...

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$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi \sin\vartheta \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2] \cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta \sin\Theta/2 \end{aligned}$$

Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ of spinor σ_μ -operators

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ of group-operators

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

➔ Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$...

$$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle \quad \alpha\beta\gamma \text{ make better coordinates but: } \mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)|1\rangle = \mathbf{R}[\varphi\vartheta\Theta]|1\rangle$$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta \tan\Theta/2$$

$$x_1 = \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi \sin\vartheta \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2] \cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta \sin\Theta/2$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ make better coordinates but: $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)|1\rangle = \mathbf{R}[\varphi\vartheta\Theta]|1\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta \tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2} - \varphi]$$

$$x_1 = \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi \sin\vartheta \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2] \cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta \sin\Theta/2$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$...

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta \tan\Theta/2$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta \tan\Theta/2]$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2} - \varphi]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2} - \varphi$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi \sin\vartheta \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2] \cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta \sin\Theta/2 \end{aligned}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$\begin{aligned} \tan[(\gamma-\alpha)/2] &= \cot\varphi = \tan[\frac{\pi}{2} - \varphi] \\ (\gamma-\alpha)/2 &= \frac{\pi}{2} - \varphi \\ \sin[(\gamma-\alpha)/2] &= \sin[\frac{\pi}{2} - \varphi] = \cos\varphi \end{aligned}$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta \tan\Theta/2$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta \tan\Theta/2]$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$...

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi \sin\vartheta \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2] \cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta \sin\Theta/2 \end{aligned}$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta \tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2} - \varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta \tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2} - \varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2} - \varphi] = \cos\varphi$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta \tan\Theta/2)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta \tan\Theta/2)$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$...

$$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle \quad \alpha\beta\gamma \text{ make better coordinates but: } \mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)|1\rangle = \mathbf{R}[\varphi\vartheta\Theta]|1\rangle$$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi \sin\vartheta \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2] \cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta \sin\Theta/2 \end{aligned}$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta \tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2} - \varphi]$$

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$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2} - \varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta \tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2 \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta \tan\Theta/2)$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2] \cos\beta/2 = \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi \sin\vartheta \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2] \sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi \sin\vartheta \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2] \cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta \sin\Theta/2 \end{aligned}$$

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Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha - \gamma + \pi)/2$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2} - \varphi] = \sin\varphi$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

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$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ make better coordinates but: $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)|1\rangle = \mathbf{R}[\varphi\vartheta\Theta]|1\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$\varphi = (\alpha - \gamma + \pi)/2$$

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$$x_1 = \cos[(\gamma+\alpha)/2] \cos\beta/2 =$$

$$\cos\Theta/2$$

Example: *Euler angles* $(\alpha=50^\circ \beta=60^\circ \gamma=70^\circ)$

$$\varphi = (50^\circ - 70^\circ + 180^\circ)/2 = 80^\circ$$

$$\vartheta = \tan^{-1}[\tan 60^\circ/2 / \sin(50^\circ+70^\circ)/2] = 33.7^\circ$$

$$\Theta = 2 \cos^{-1}[\cos 60^\circ/2 \cos(50^\circ+70^\circ)/2] = 128.7^\circ$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

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$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta \tan\Theta/2)$$

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Reverse check: $(\alpha\beta\gamma)$ in terms of $[\varphi\vartheta\Theta]$

$$\alpha = 80^\circ - 90^\circ + \tan^{-1}(\tan(128.7^\circ/2) \cos 33.7^\circ) = 50.007^\circ$$

$$\beta = 2\sin^{-1}(\sin 128.7^\circ/2 \sin 33.7^\circ) = 60.022^\circ$$

$$\gamma = \pi/2 - 128.7^\circ + \tan^{-1}(\tan(128.7^\circ/2)) = 70.007^\circ$$

*Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators*

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ of spinor σ_μ -operators

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ of group-operators

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!


Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

 *Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed*

$R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

$\Theta = 0^\circ$

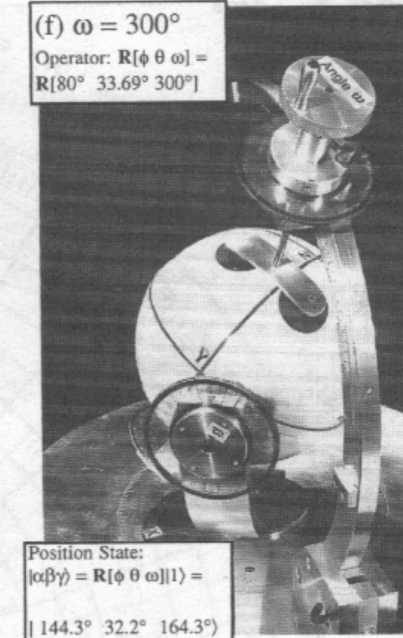
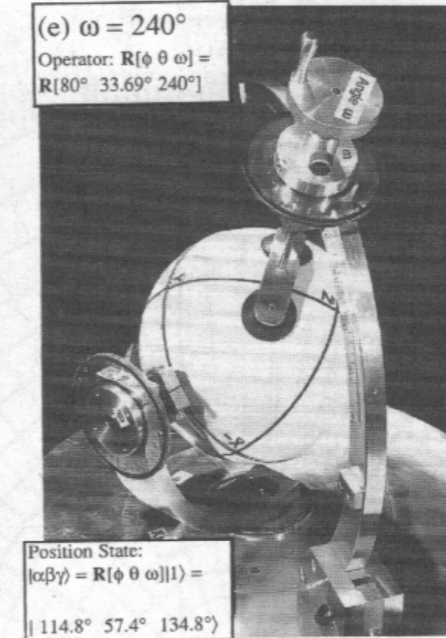
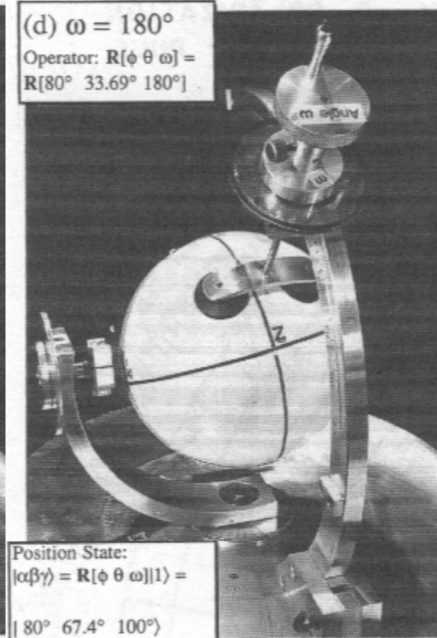
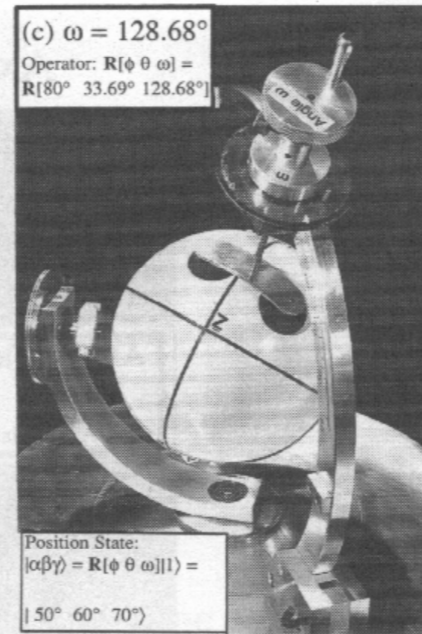
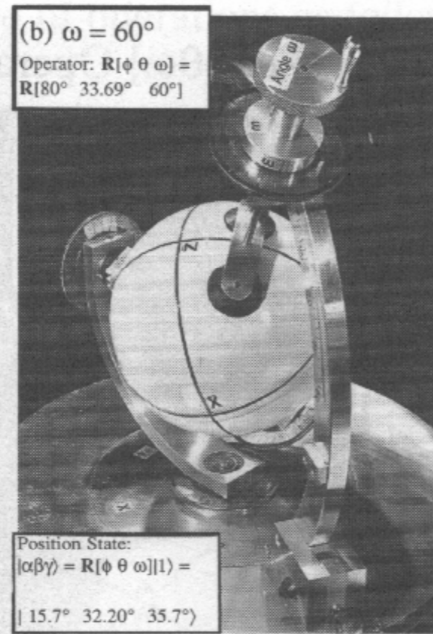
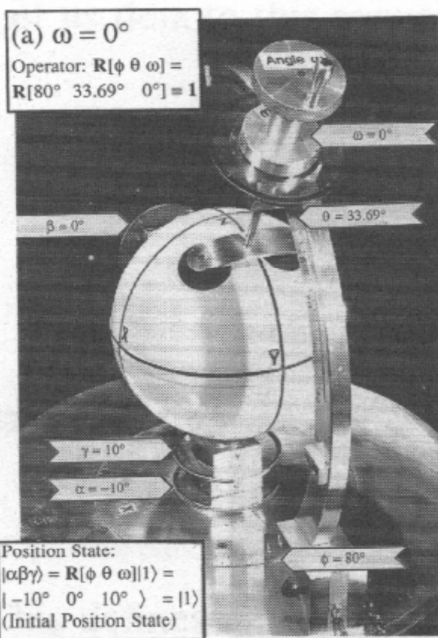
$\Theta = 60^\circ$

$\Theta = 128.7^\circ$

$\Theta = 180^\circ$

$\Theta = 240^\circ$

$\Theta = 300^\circ$



Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

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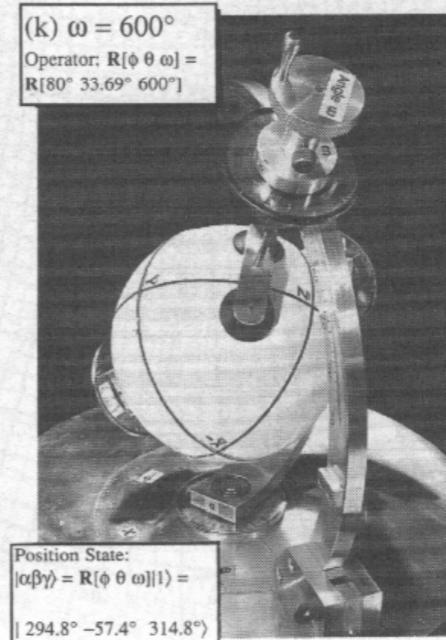
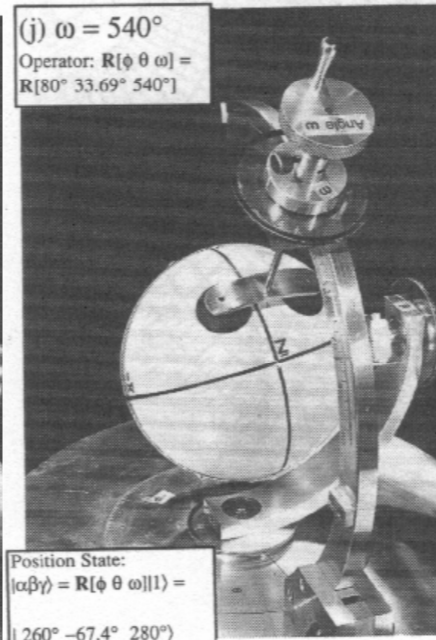
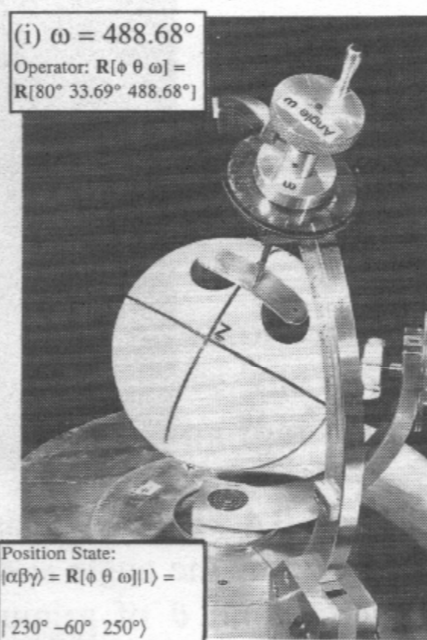
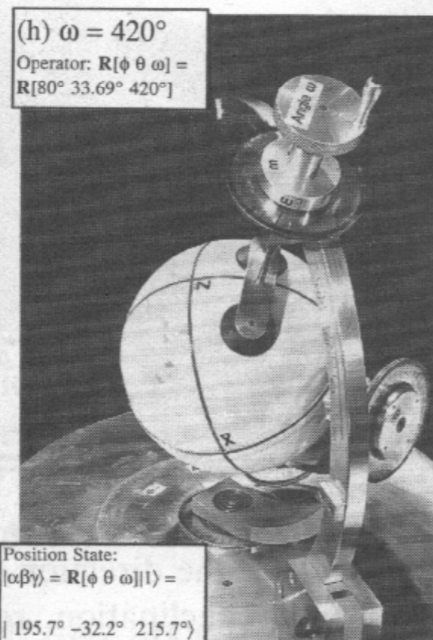
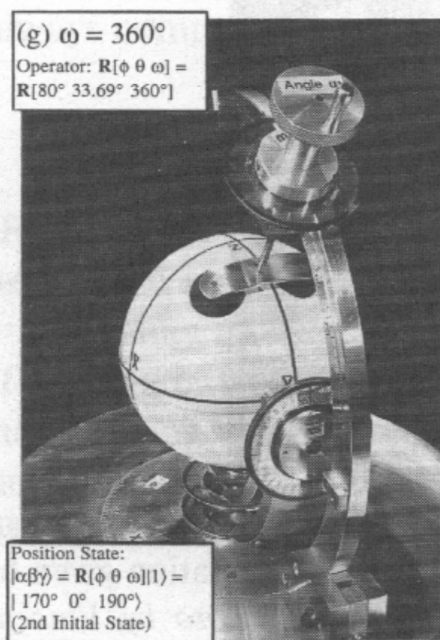
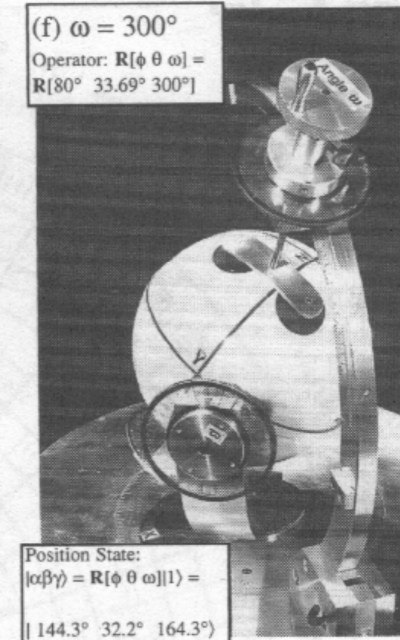
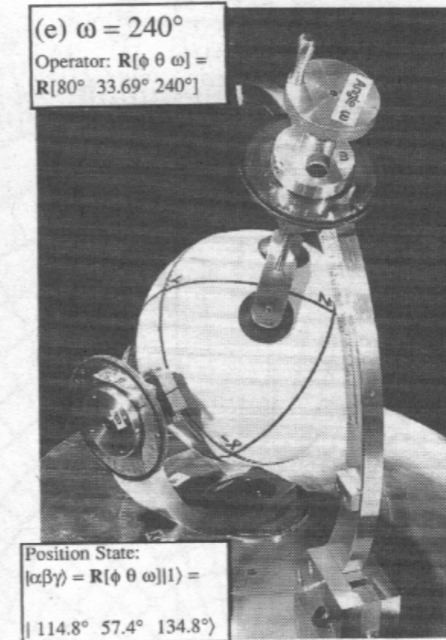
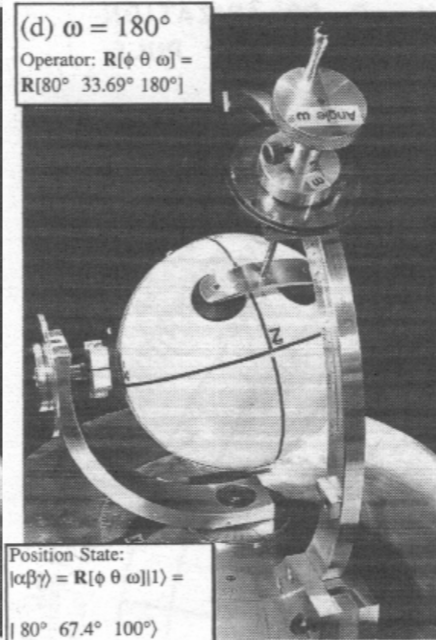
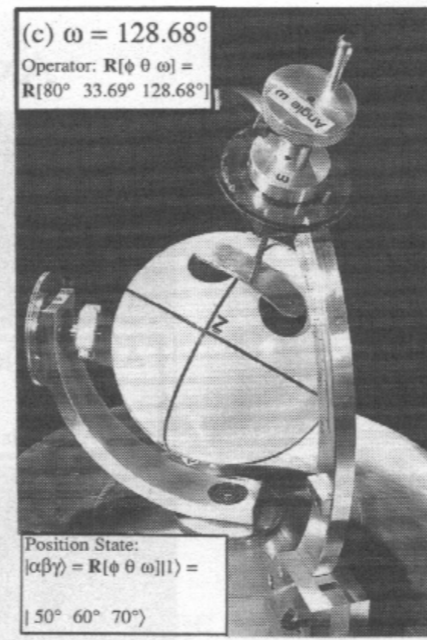
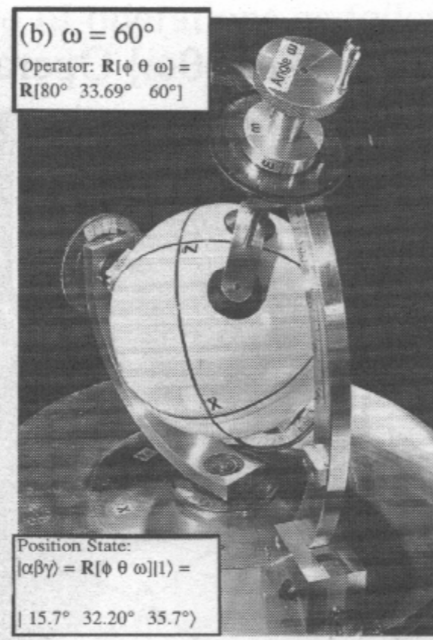
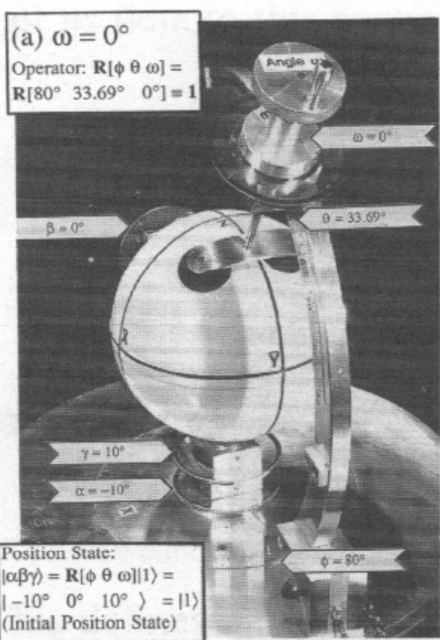
$\Theta = 60^\circ$

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$\Theta = 180^\circ$

$\Theta = 240^\circ$

$\Theta = 300^\circ$



$\Theta = 360^\circ$

$\Theta = 420^\circ$

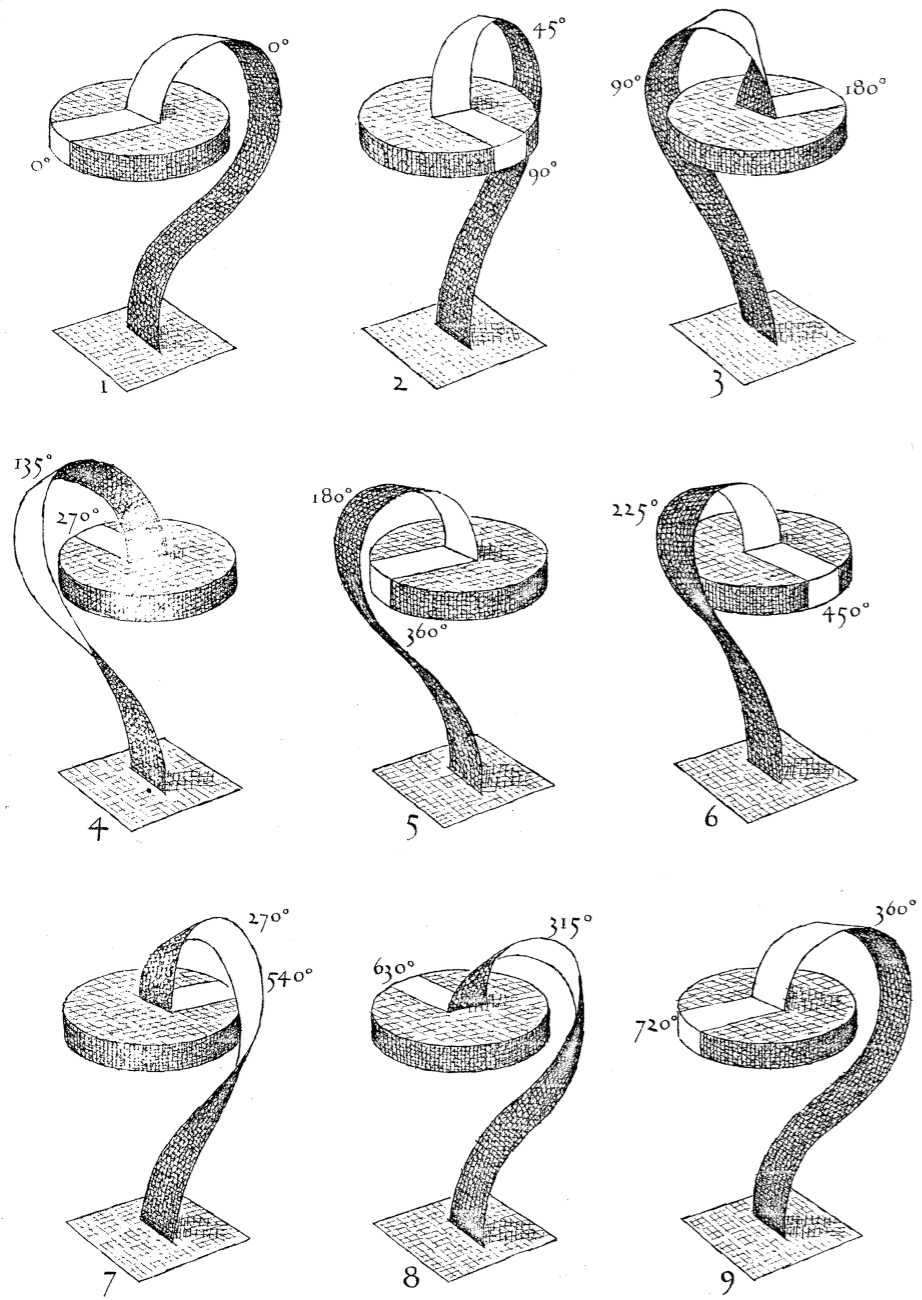
$\Theta = 488.7^\circ$

$\Theta = 540^\circ$

$\Theta = 600^\circ$

$\Theta = 660^\circ$

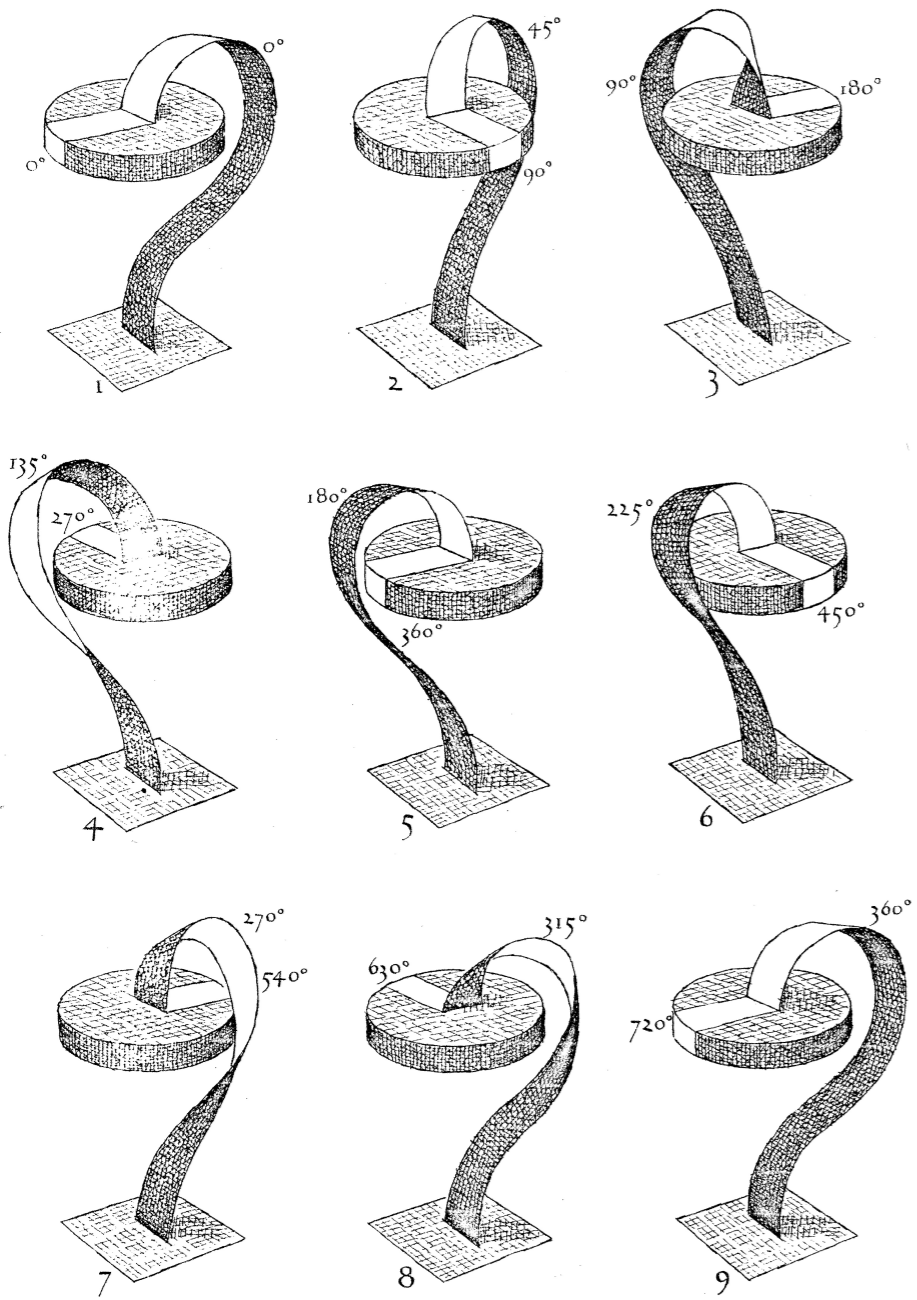
*Some "real-world" applications of
the $U(2)$ - $R(3)$ spinor-vector topology*



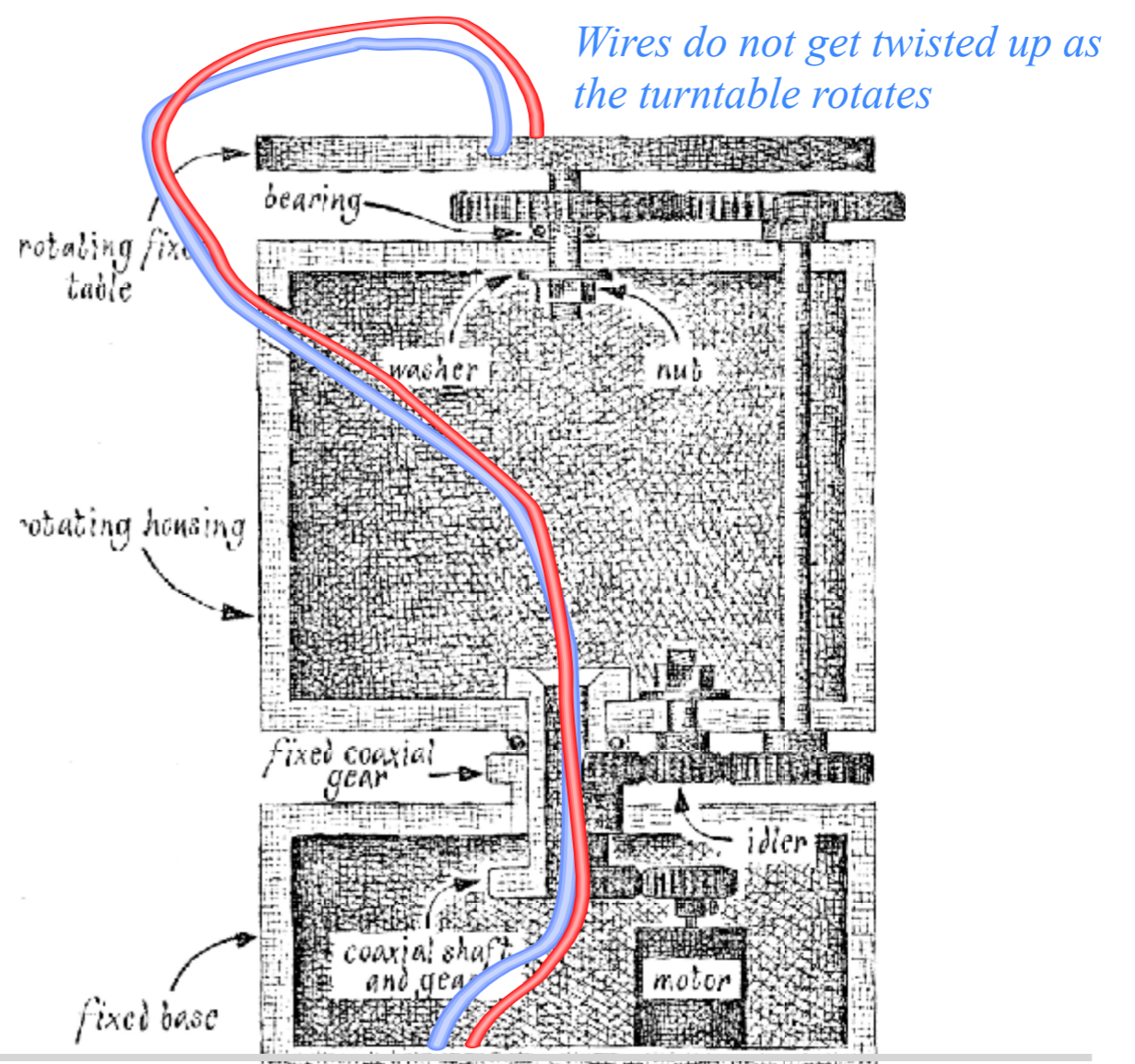
Sequential models of D. A. Adams' antitwister mechanism

*From Scientific American
December 1975-p.120-125*

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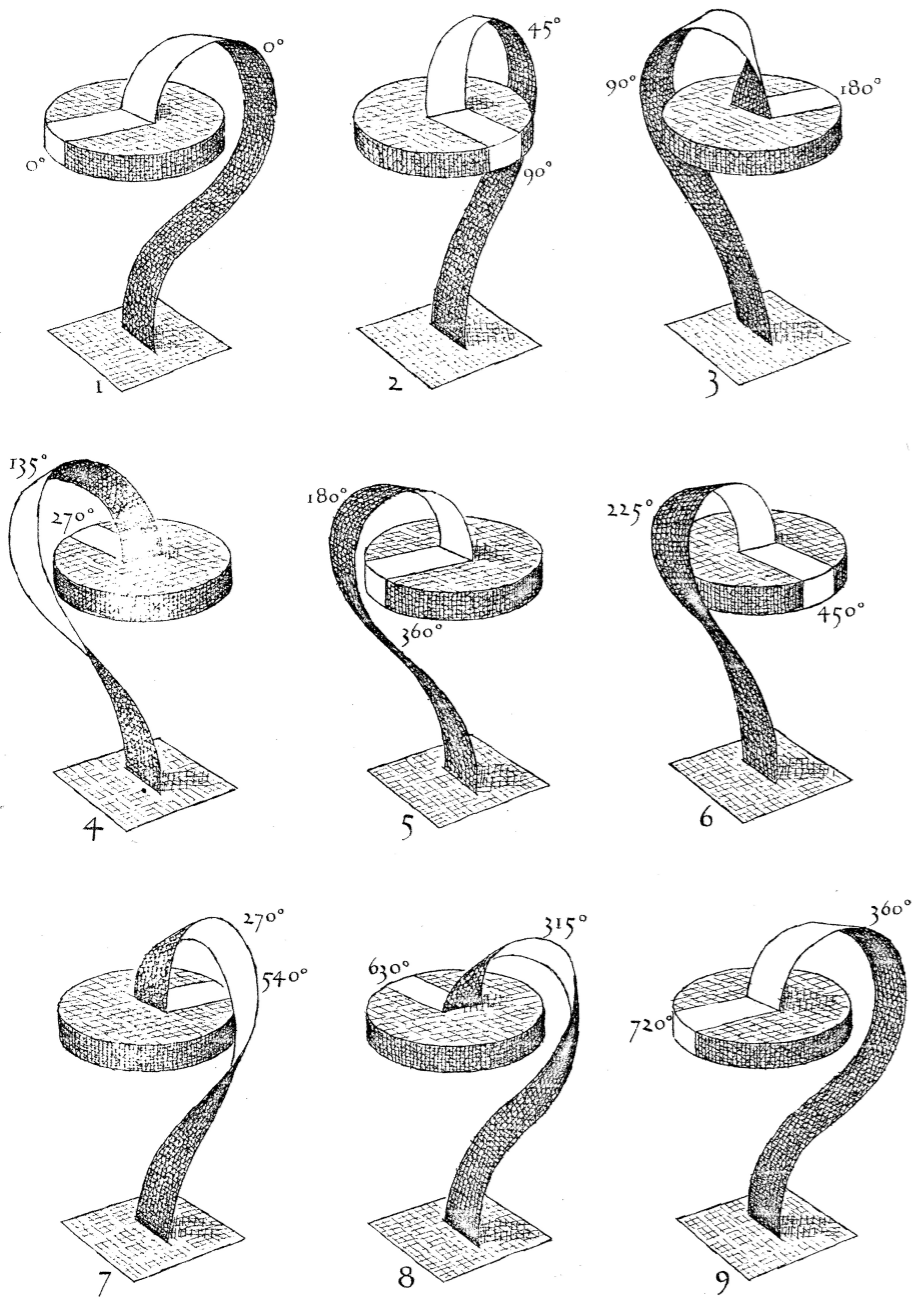


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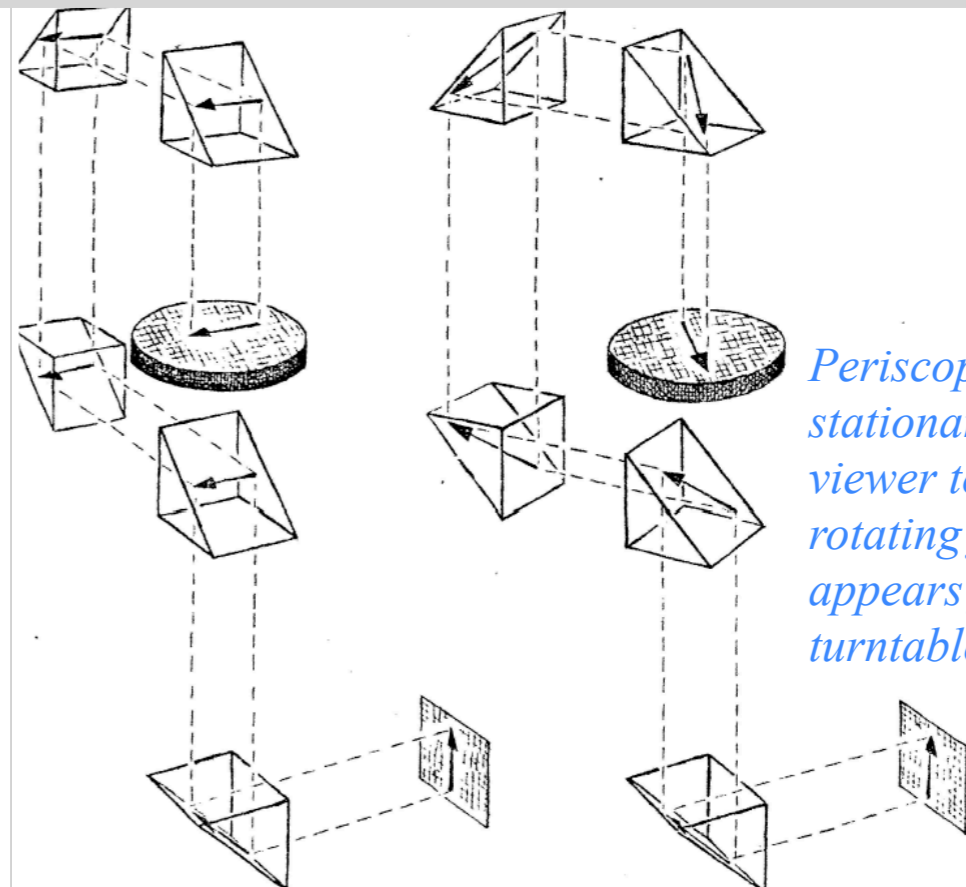
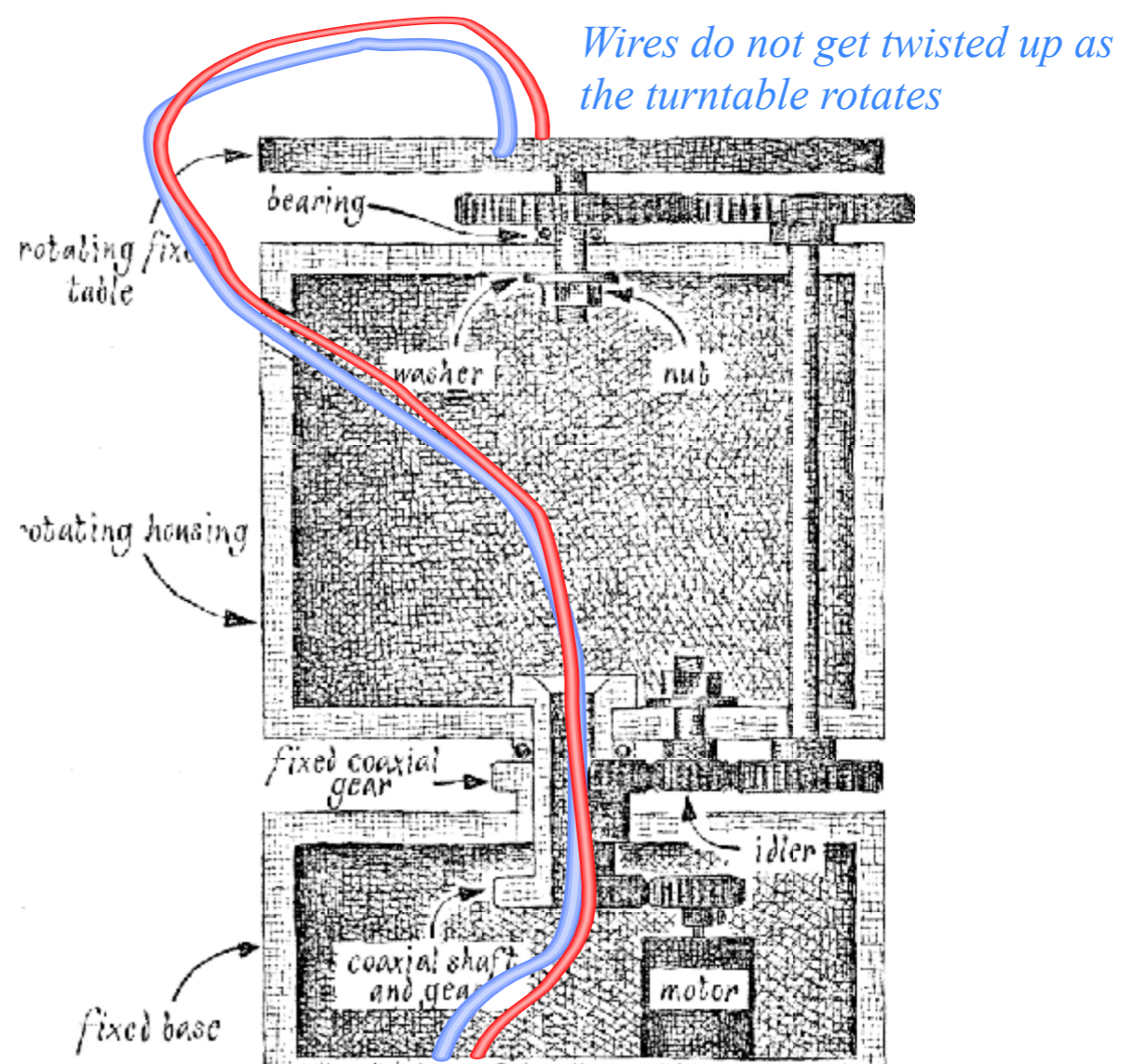
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Periscope allows stationary outside viewer to see into a rotating frame that appears fixed as the turntable rotates

Review: How "Crazy-Thing"-Theorem makes spinor and vector representation matrices
Half-angle $\Theta/2 = \varphi$ replacement and Darboux crank axis operators

Operator-on-Operator transformations

Product algebra for Pauli's σ_μ and Hamilton's $\mathbf{q}_\mu = -i\sigma_\mu$

Group product algebra

Jordan-Pauli identity and $U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - formula

Transformation $\mathbf{R}[\Theta]\sigma_\mu\mathbf{R}[\Theta]^\dagger$ of spinor σ_μ -operators

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ of group-operators

Operator-on-Operator transformations

Geometry of groups: Hamilton's turns and It's all done with mirrors!

Group product geometry

$U(2)$ product $\mathbf{R}[\Theta]\mathbf{R}[\Theta'] = \mathbf{R}[\Theta''']$ - geometry

Transformation $\mathbf{R}[\Theta]\mathbf{R}[\Theta']\mathbf{R}[\Theta]^\dagger$ geometry

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta = 0-4\pi$ -sequence $[\varphi\vartheta]$ fixed

➔ $R(3)$ - $U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

Euler $\mathbf{R}(\alpha\beta\gamma)$ Sundial

$R(3)-U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

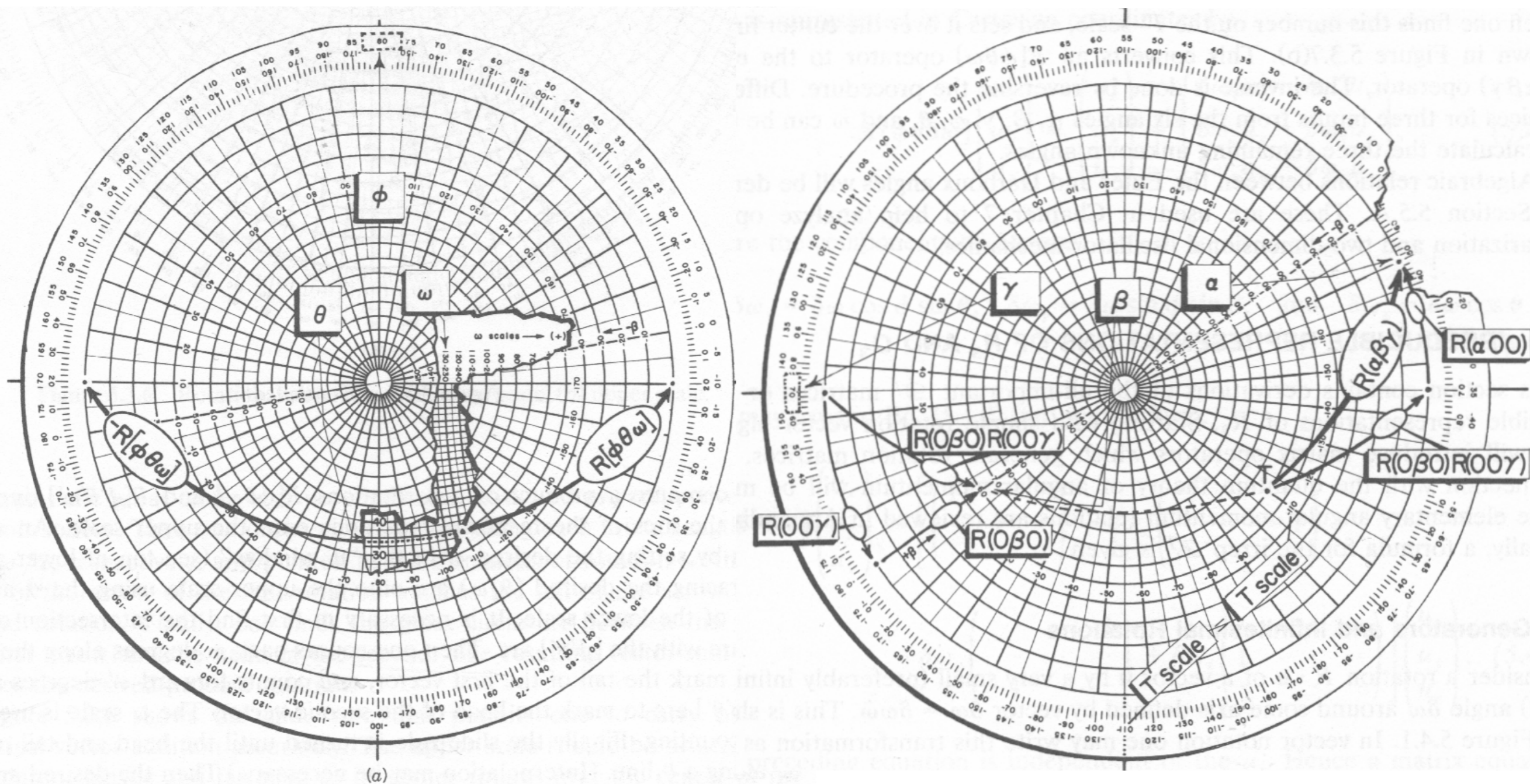


Figure 5.3.7 Setting the rotational slide rule. (a) Darboux or axis angles. (b) Euler angles.

$R(3)-U(2)$ slide rule for converting $\mathbf{R}(\alpha\beta\gamma) \leftrightarrow \mathbf{R}[\varphi\vartheta\Theta]$

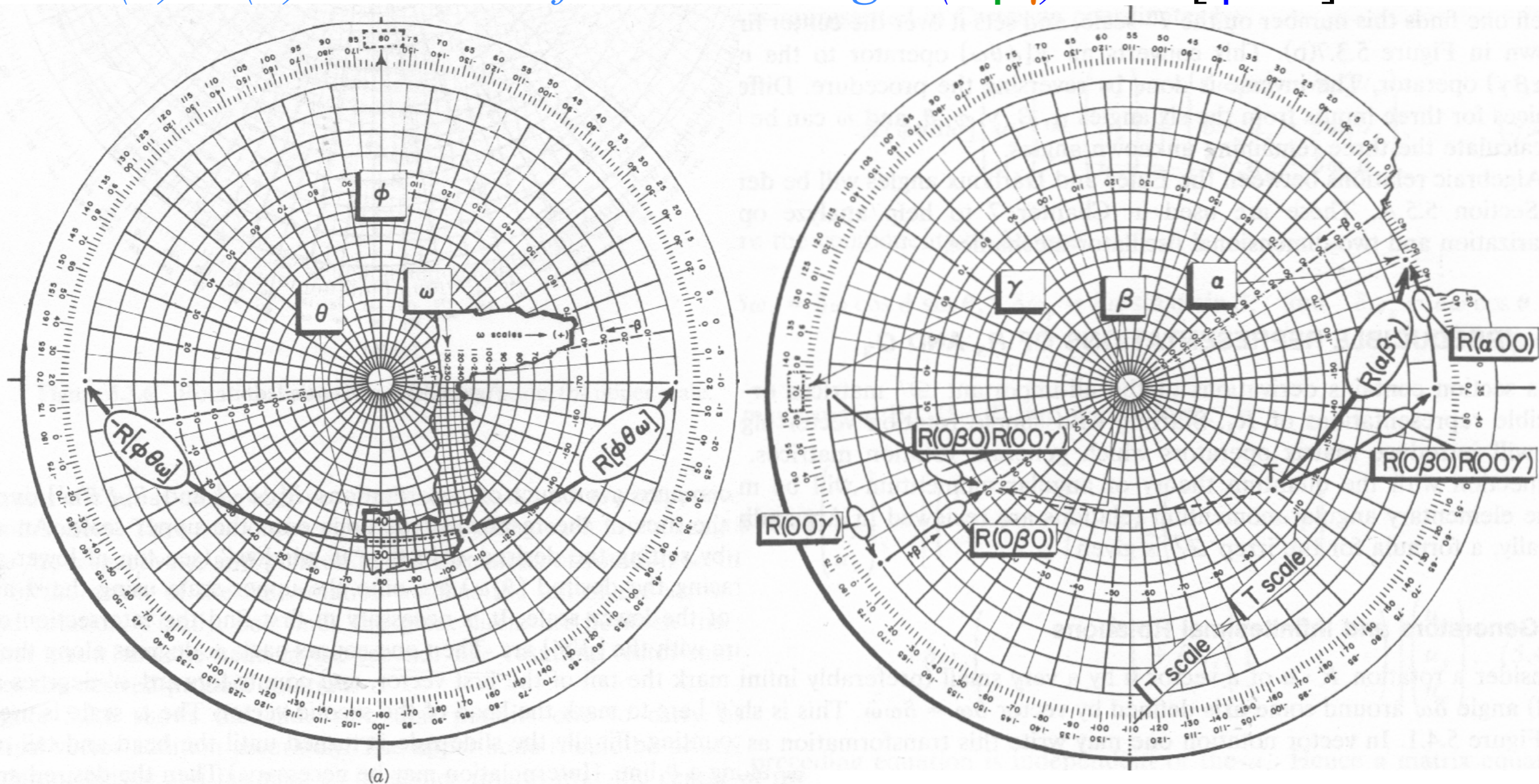
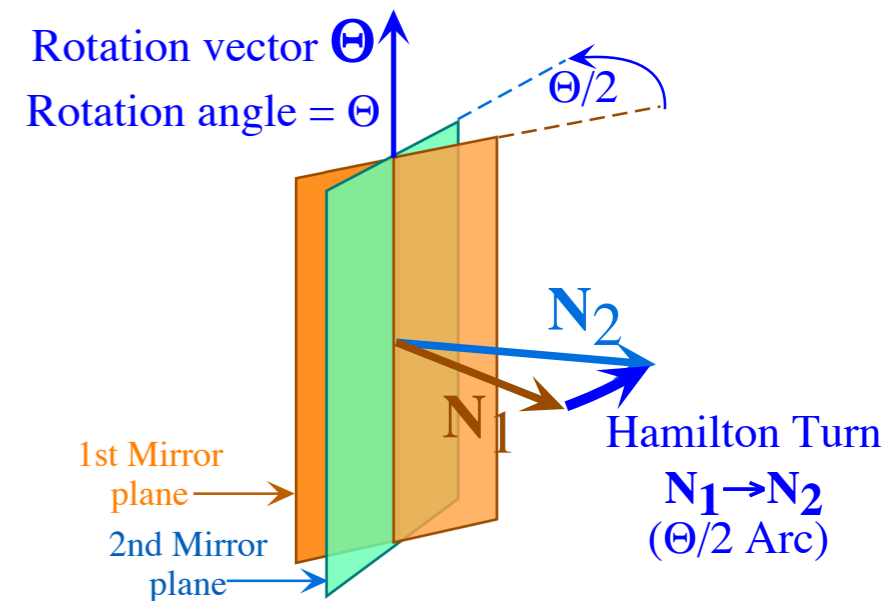
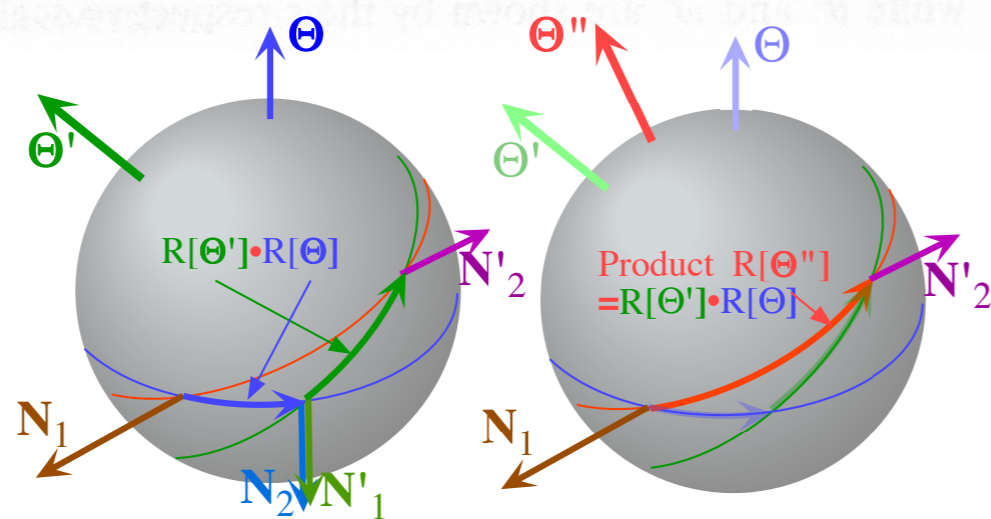


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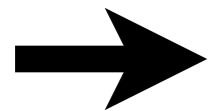
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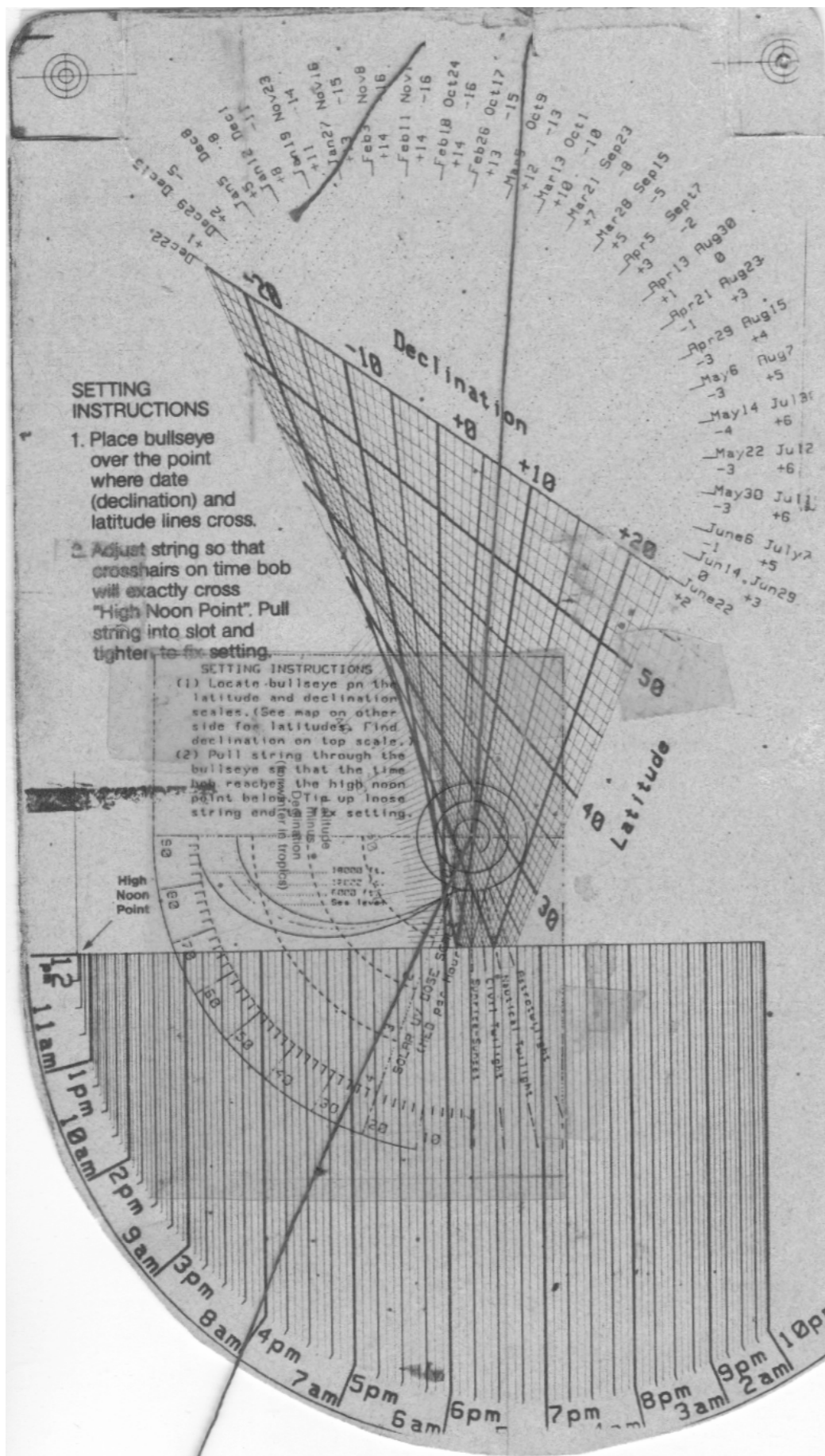
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Euler R(αβγ) Sundial

