Group Theory in Quantum Mechanics

Lecture 7 (2.3.15)

Spectral Analysis of U(2) Operators

(Quantum Theory for Computer Age - Ch. 10 of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5)

Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
- Spinor arithmetic like complex arithmetic
- Spinor vector algebra like complex vector algebra
- Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$
- The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space
  - 2D Spinor vs 3D vector rotation
  - NMR Hamiltonian: 3D Spin Moment **m** in **B** field

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$
- Spin-1 (3D-real vector) case
- Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states
- Asymmetry $S_A=S_Z$, Balance $S_B=S_X$, and Chirality $S_C=S_Y$
- Polarization ellipse and spinor state dynamics

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Polarization ellipse and spinor state dynamics

(Review of Lect. 6) 2D harmonic oscillator equation solutions

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2+k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

Note eigenvalue is square of eigenfrequency

To introduce eigensolutions we take a simple case of unit masses *(m_1=1=m_2)*

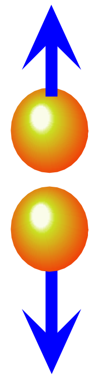
So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

 Three famous 2-state systems and two-complex-component coordinates

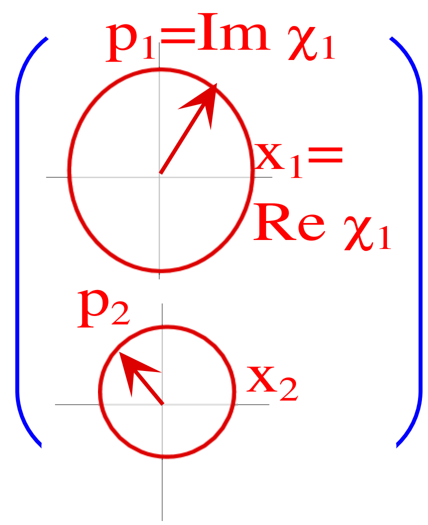
(a) Electron Spin-1/2-Polarization

Spin-up $|1\rangle = |\uparrow\rangle$



Spin-dn $|2\rangle = |\downarrow\rangle$

$$|\chi\rangle = \begin{pmatrix} \chi\uparrow \\ \chi\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\chi\rangle \\ \langle\downarrow|\chi\rangle \end{pmatrix} =$$



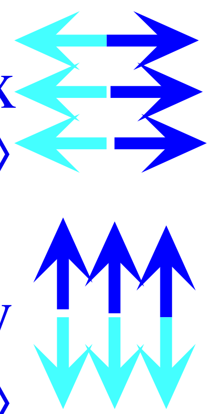
$$= |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$$

Rabi, Ramsey, and Schwinger 1954
Rev. Mod. Phys. **26** *167 (1954)*

Fig. 10.5.1
QTCA Unit 3 Chapter 10

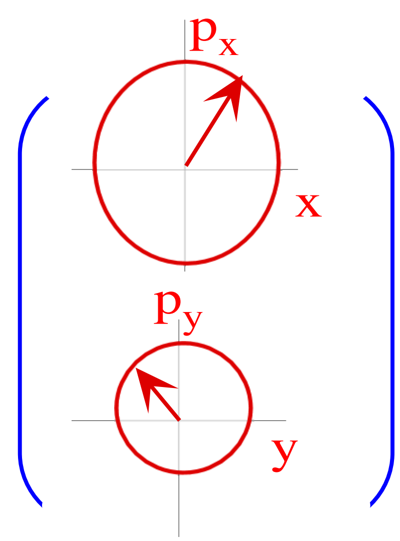
(b) Photon Spin-1-Polarization

Plane-x $|1\rangle = |x\rangle$



Plane-y $|2\rangle = |y\rangle$

$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \langle x|\psi\rangle \\ \langle y|\psi\rangle \end{pmatrix} =$$



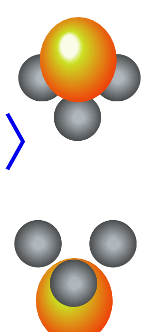
$$= |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

John Stokes 1862
Proc. Soc. London **11** *547 (1862)*

Harter and Dos Santos
Am. J. Phys. **46** *251 (1986)*
J. Chem. Phys. **85** *5560 (1986)*

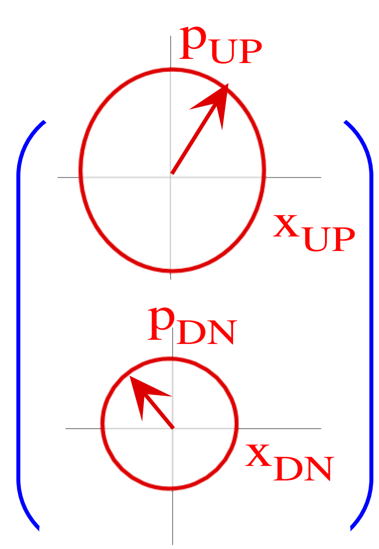
(c) Ammonia (NH_3) Inversion States

N-UP $|1\rangle = |UP\rangle$



N-DN $|2\rangle = |DN\rangle$

$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|\nu\rangle \\ \langle DN|\nu\rangle \end{pmatrix} =$$



$$= |UP\rangle\langle UP|\nu\rangle + |DN\rangle\langle DN|\nu\rangle$$

Feynman, Vernon, and Hellwarth 1957
J. Appl. Phys. **28** *49 (1957)*

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Polarization ellipse and spinor state dynamics

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

(Review of Lect. 6) $i\hbar|\dot{\Psi}(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ $|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}=\mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\begin{pmatrix}a_1\\ a_2\end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi=\mathbf{H}\Psi$ into pairs of ***real*** 1st-order differential equations.

$$\dot{x}_1=Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-Ax_1-Bx_2-Cp_2$$
$$\dot{x}_2=Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-Bx_1-Dx_2+Cp_1$$

QM vs. Classical Equations are identical

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c=\frac{A}{2}\left(p_1^2+x_1^2\right)+B\left(x_1x_2+p_1p_2\right)+C\left(x_1p_2-x_2p_1\right)+\frac{D}{2}\left(p_2^2+x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1=\frac{\partial H_c}{\partial p_1}=Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-\frac{\partial H_c}{\partial x_1}=-\left(Ax_1+Bx_2+Cp_2\right)$$
$$\dot{x}_2=\frac{\partial H_c}{\partial p_2}=Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-\frac{\partial H_c}{\partial x_2}=-\left(Bx_1+Dx_2-Cp_1\right)$$

Finally a 2nd time derivative (Assume *constant* *A, B, D,* and *let C=0*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\ddot{x}_1=A\dot{p}_1+B\dot{p}_2-C\dot{x}_2$$
$$=-A\left(Ax_1+Bx_2+Cp_2\right)-B\left(Bx_1+Dx_2-Cp_1\right)-C\left(Bp_1+Dp_2+Cx_1\right)$$
$$=-\left(A^2+B^2+C^2\right)x_1-\left(AB+BD\right)x_2-C\left(A+D\right)p_2$$

$$\ddot{x}_2=B\dot{p}_1+D\dot{p}_2+C\dot{x}_1$$
$$=-B\left(Ax_1+Bx_2+Cp_2\right)-D\left(Bx_1+Dx_2-Cp_1\right)+C\left(Ap_1+Bp_2-Cx_2\right)$$
$$=-\left(AB+BD\right)x_1-\left(B^2+D^2+C^2\right)x_2+C\left(A+D\right)p_1$$

For constant A,B,C, and D

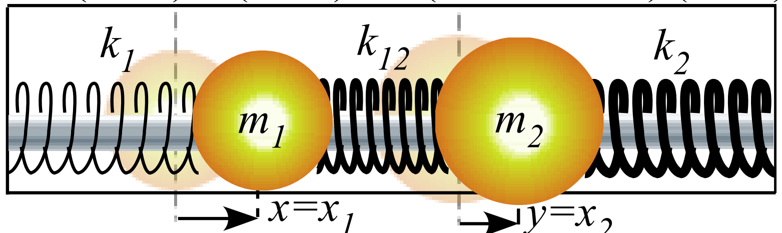
$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}A^2+B^2 & AB+BD\\ AB+BD & B^2+D^2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\equiv\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

ABD-to- K_{ij} or k_a connection formulae

$$m_1K_{11}=A^2+B^2=k_1+k_{12}, \qquad m_1K_{12}=AB+BD=-k_{12},$$
$$m_2K_{21}=AB+BD=-k_{12}, \qquad m_2K_{22}=B^2+D^2=k_2+k_{12}.$$



Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t=\mathbf{H}$ (with $C\neq 0$) and square it!

$$i\frac{\partial}{\partial t}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}\Rightarrow\left(i\frac{\partial}{\partial t}\right)^2=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}^2\Rightarrow-\frac{\partial^2}{\partial t^2}=\begin{pmatrix}A^2+B^2+C^2 & AB+BD-i(AC+CD)\\ AB+BD+i(AC+CD) & B^2+D^2+C^2\end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle}$

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$ABCD$ *Symmetry operator analysis and U(2) spinors*

Decompose the Hamiltonian operator **H** into four $ABCD$ *symmetry operators*

(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

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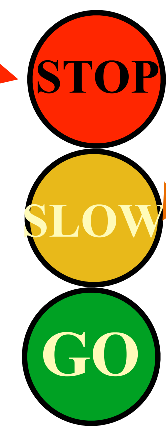
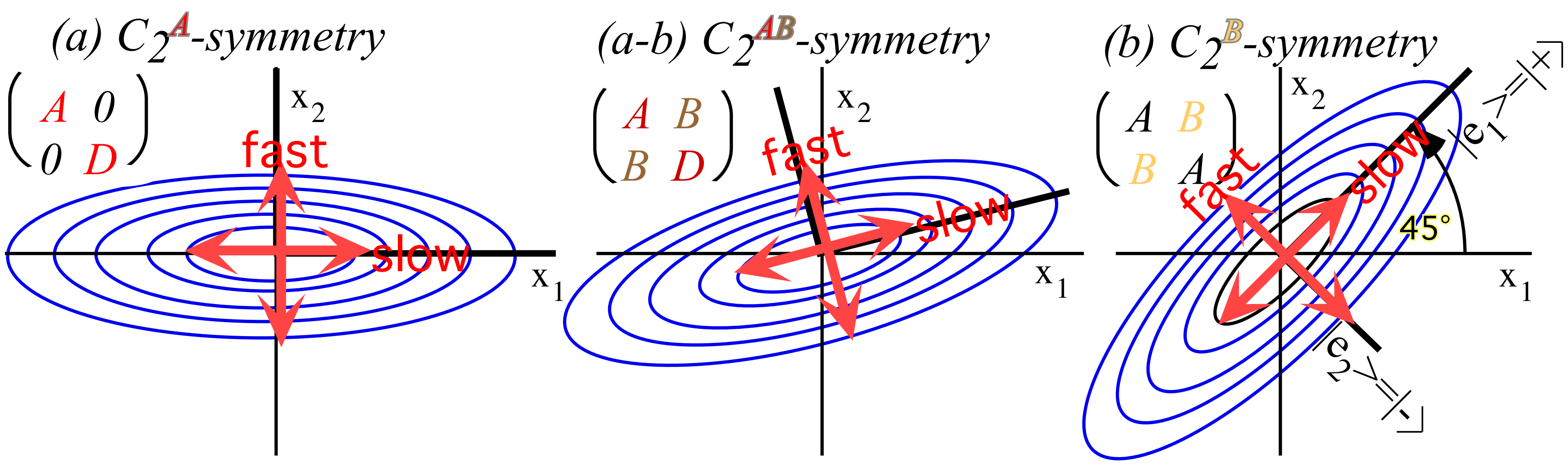
Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (complex, circular, chiral, cyclotron, Coriolis, centrifugal, curly, and circulating-current-carrying...)*

Motivation for coloring scheme:
The Traffic Signal



Standing waves

Moving waves



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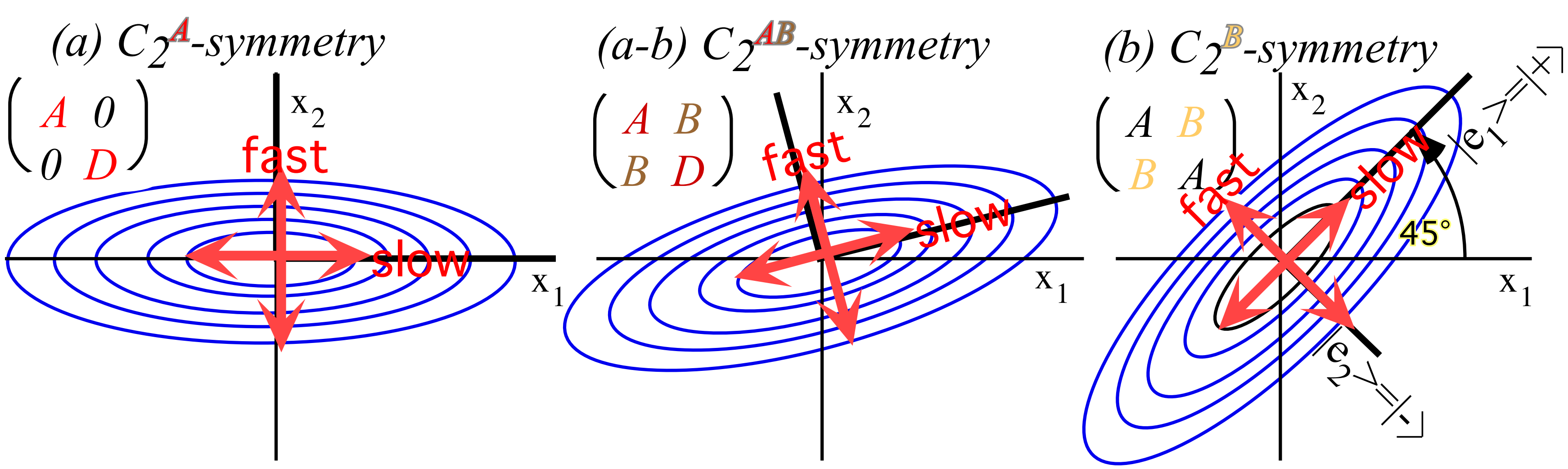

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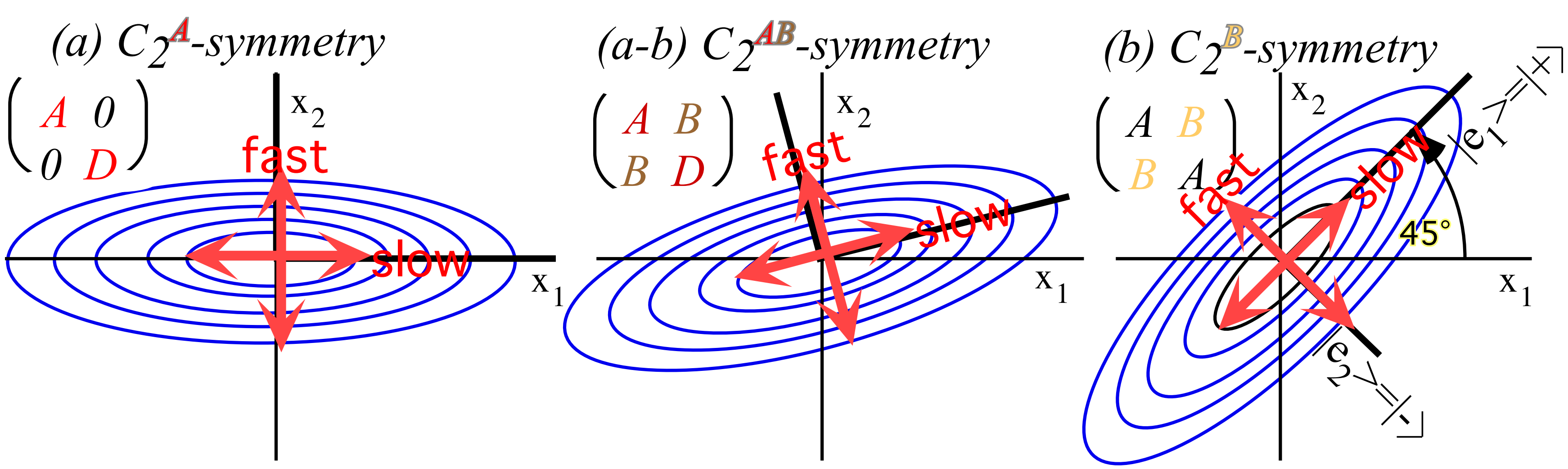
In 1843 Hamilton invents *quaternions* {**1**, **i**, **j**, **k**}. σ_μ related by *i*-factor: $\{\sigma_1=\mathbf{1}=\sigma_0, i\sigma_B=\mathbf{i}=i\sigma_X, i\sigma_C=\mathbf{j}=i\sigma_Y, i\sigma_A=\mathbf{k}=i\sigma_Z\}$.

Each Hamilton quaternion squares to *negative*-**1** ($\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-\mathbf{1}$) like imaginary number $i^2=-1$. (They make up the Quaternion group.)



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In 1843 Hamilton invents *quaternions* $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. σ_μ related by *i*-factor: $\{\sigma_1=\mathbf{1}=\sigma_0, i\sigma_B=\mathbf{i}=i\sigma_X, i\sigma_C=\mathbf{j}=i\sigma_Y, i\sigma_A=\mathbf{k}=i\sigma_Z\}$.

Each Hamilton quaternion squares to *negative*-**1** ($\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-\mathbf{1}$) like imaginary number $i^2=-1$. (They make up the Quaternion group.)

Each Pauli σ_μ squares to *positive*-**1** ($\sigma_X{}^2=\sigma_Y{}^2=\sigma_Z{}^2=+\mathbf{1}$) (Each makes a cyclic C_2 group $C_2^A=\{\mathbf{1},\sigma_A\}$, $C_2^B=\{\mathbf{1},\sigma_B\}$, or $C_2^C=\{\mathbf{1},\sigma_C\}$.)

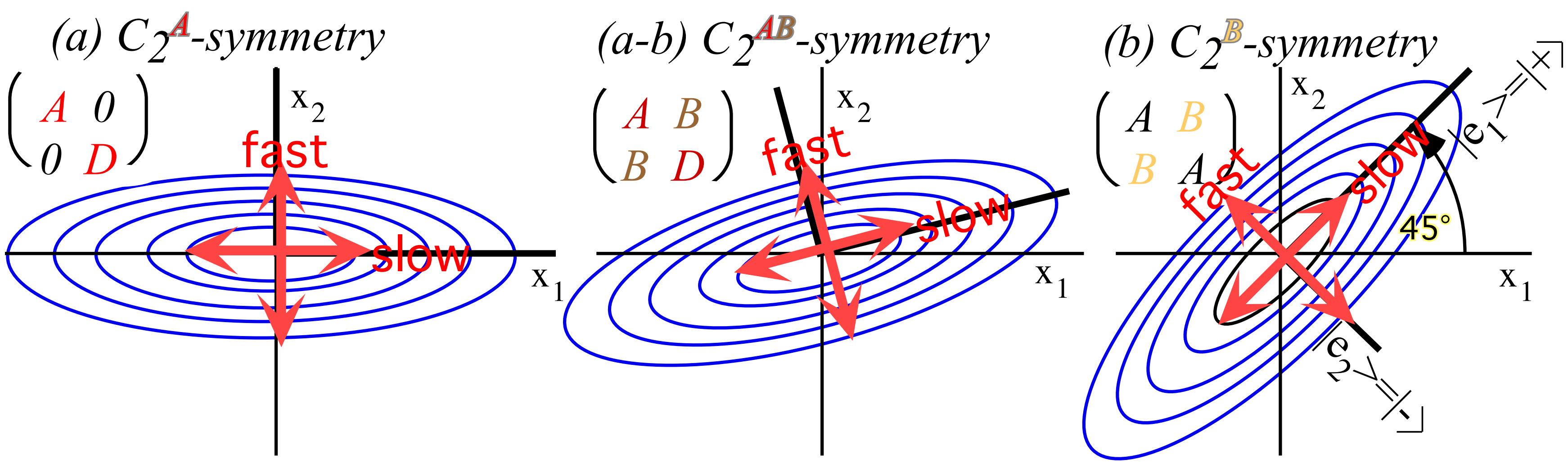

Fig. 10.1.2 Potentials for (a) $C_2{}^A$-asymmetric-diagonal, (ab) $C_2{}^{AB}$-mixed , (b) $C_2{}^B$-bilateral U(2)system.

Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
  - *2D Spinor vs 3D vector rotation*
  - *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

- *Spin-1 (3D-real vector) case*
- *Spin-1/2 (2D-complex spinor) case*

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

- *Asymmetry* $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$
- *Polarization ellipse and spinor state dynamics*

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

ABCD Time evolution operator

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$\sigma_A = \sigma_Z$ $\sigma_B = \sigma_X$ $\sigma_C = \sigma_Y$

Key pieces of mathematical bookkeeping

ABCD Time evolution operator

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$\sigma_A = \sigma_Z$ $\sigma_B = \sigma_X$ $\sigma_C = \sigma_Y$

$$= e^{-i\vec{\sigma}\bullet\vec{\omega}\cdot t}\, e^{-i\omega_0\cdot t}$$

$\vec{\sigma}\bullet\vec{\omega}\cdot t$

where: $\vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

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	σ_X	σ_Y	σ_Z
σ_X	**1**		
σ_Y		**1**	
σ_Z			**1**

U(2) generator product table

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$$\sigma_X\sigma_Y = i\sigma_Z$$

	σ_X	σ_Y	
σ_X	1	$i\sigma_Z$	
σ_Y		1	
			1

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	σ_X	σ_Y	
σ_X	1	$i\sigma_Z$	
σ_Y	$-i\sigma_Z$	1	
			1

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	σ_X	σ_Y	σ_Z
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			1

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\sigma_Y$$

$\sigma_Z \cdot \sigma_X$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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U(2) generator product table

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ABCD Time evolution operator

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions **i**$=-i\sigma_X$, **j**$=-i\sigma_Y$, and **k**$=-i\sigma_Z$ powerful.

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So-called *anti-commutation(* $\sigma_X\sigma_Y = -\sigma_Y\sigma_X$, $\sigma_X\sigma_Z = -\sigma_Z\sigma_X$ etc.) kills off-diagonal terms:

So: $\sigma_a{}^2 = \mathbf{1}$

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Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
  - *2D Spinor vs 3D vector rotation*
  - *NMR Hamiltonian: 3D Spin Moment **m** in **B** field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

- *Spin-1 (3D-real vector) case*
- *Spin-1/2 (2D-complex spinor) case*

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

- *Asymmetry $S_A=S_Z$, Balance $S_B=S_X$, and Chirality $S_C=S_Y$*
- *Polarization ellipse and spinor state dynamics*

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$$\begin{aligned}
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&= +a_Yb_X\sigma_Y\sigma_X \quad +a_Yb_Y\sigma_Y\sigma_Y \quad +a_Yb_Z\sigma_Y\sigma_Z = -a_Yb_X\sigma_X\sigma_Y \quad +a_Yb_Y\mathbf{1} \quad +a_Yb_Z\sigma_Y\sigma_Z\\
&\quad +a_Zb_X\sigma_Z\sigma_X \quad +a_Zb_Y\sigma_Z \quad +a_Zb_Z\sigma_Z\sigma_Z \qquad +a_Zb_X\sigma_Z\sigma_X \quad -a_Zb_Y\sigma_Y\sigma_Z \quad +a_Zb_Z\mathbf{1}
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$$\begin{aligned} = &\; a_Xb_X\mathbf{1} + a_Xb_Y\sigma_X\sigma_Y - a_Xb_Z\sigma_Z\sigma_X \\ &- a_Yb_X\sigma_X\sigma_Y + a_Yb_Y\mathbf{1} + a_Yb_Z\sigma_Y\sigma_Z \\ &+ a_Zb_X\sigma_Z\sigma_X - a_Zb_Y\sigma_Y\sigma_Z + a_Zb_Z\mathbf{1} \end{aligned}$$

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(Recall complex variable result.)

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Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

➡ *Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$*

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- ➡ *Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
  - *2D Spinor vs 3D vector rotation*
  - *NMR Hamiltonian: 3D Spin Moment **m** in **B** field*

Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

- *Spin-1 (3D-real vector) case*
- *Spin-1/2 (2D-complex spinor) case*

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

- *Asymmetry $S_A=S_Z$, Balance $S_B=S_X$, and Chirality $S_C=S_Y$*
- *Polarization ellipse and spinor state dynamics*

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Unit spinor vector

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Unit spinor vector

$$\sigma_\varphi = \frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi} = (\vec{\sigma}\bullet\hat{\varphi})\varphi = \frac{\varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_C\sigma_C}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}}$$

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U(2) generator product table

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$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}\cdot t}\left|\Psi(0)\right\rangle = (\mathbf{1}\cos\varphi - i\sigma_{\varphi}\sin\varphi)e^{-i\omega_0 \cdot t}$$

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Key pieces of mathematical bookkeeping

ABCD Time evolution operator

Symmetry relations make spinors $\{\sigma_X = \sigma_B,\ \sigma_Y = \sigma_C,\ \sigma_Z = \sigma_A\}$ or quaternions $\{\mathbf{i} = -i\sigma_X,\ \mathbf{j} = -i\sigma_Y,\ \mathbf{k} = -i\sigma_Z\}$ into a powerful *U(2)-algebra*.

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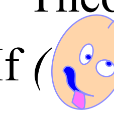
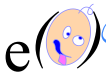
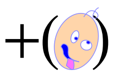
If ()^2= -**1**

Then:

e$^{(\text{face})\varphi}$  $= \mathbf{1}\cos\varphi + ($  $)\sin\varphi$

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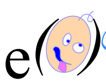
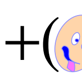
If ()^2 = -1

Then:

e$^{(}$  $^{)\varphi}$ = $\mathbf{1}\cos\varphi$ + ()$\sin\varphi$

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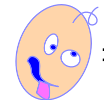
Here:  = -i

Crazy thing is just -√-1

Here:  $= -i\sigma_\varphi = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
  - *2D Spinor vs 3D vector rotation*
  - *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

- *Spin-1 (3D-real vector) case*
- *Spin-1/2 (2D-complex spinor) case*

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

- *Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$*
- *Polarization ellipse and spinor state dynamics*

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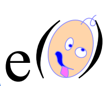
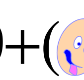
e$^{(\text{face})\theta}$  $= \mathbf{1}\cos\theta + ($  $)\sin\theta$

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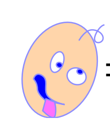
Here:  $= -i\sigma_\varphi = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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Example 1: A or Z rotation

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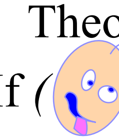
If ()^2 = -**1**

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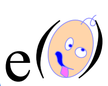
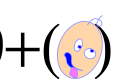
e()θ = **1**cosθ + ()sinθ

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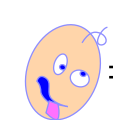
Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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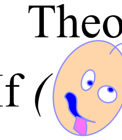
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

The 
Crazy Thing
Theorem:
If ()^2 = -**1**
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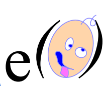
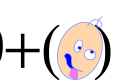
e()θ = **1**cosθ + ()sinθ

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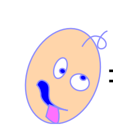
Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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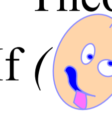
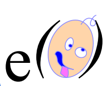
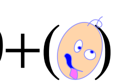
Example 2: C or Y rotation

The  Crazy Thing Theorem:

If ()^2= -**1**

Then:

e()θ =**1**cosθ+()sinθ

Let: $\vec{\varphi} = \vec{\omega}\cdot t$

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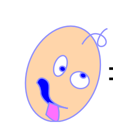
$$e^{-i(\sigma\bullet\vec{\varphi})} = e^{-i(\sigma\bullet\hat{\varphi})\varphi} = e^{-i\sigma_{\varphi}\varphi} = \mathbf{1}\cos\varphi - i\sigma_{\varphi}\sin\varphi = \mathbf{1}\cos\varphi - i\ (\boldsymbol{\sigma}\bullet\hat{\varphi})\sin\varphi$$

$$= \mathbf{1}\cos\varphi \quad - i\sigma_A\hat{\varphi}_A\sin\varphi \quad - i\sigma_B\hat{\varphi}_B\sin\varphi \quad - i\sigma_C\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A\sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B\sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C\sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A\sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A\sin\varphi \end{pmatrix}$$

Example 3: Any $\varphi = \omega t$-axial rotation

Here:  $= -i\sigma_{\varphi} = -i(\vec{\boldsymbol{\sigma}}\bullet\hat{\varphi}) = -i\dfrac{(\vec{\boldsymbol{\sigma}}\bullet\vec{\varphi})}{\varphi}$

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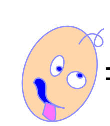
Example 2: C or Y rotation

We test these operators by making them rotate each other....

Here:  $= -i\sigma_\varphi = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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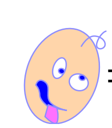
Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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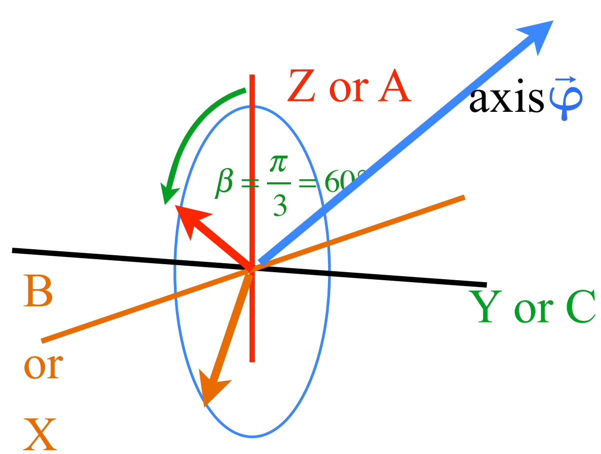
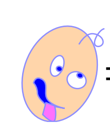


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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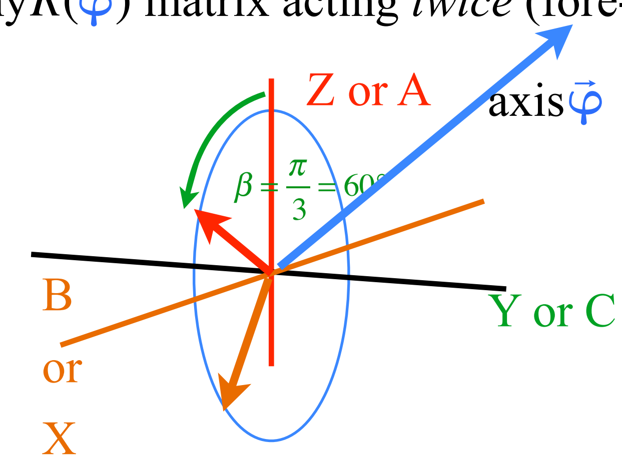
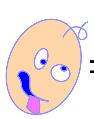


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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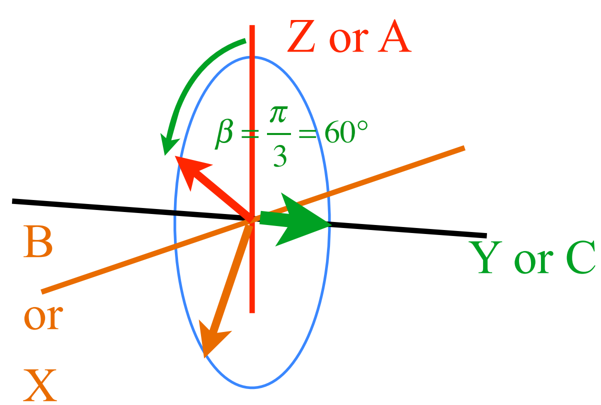
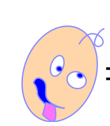


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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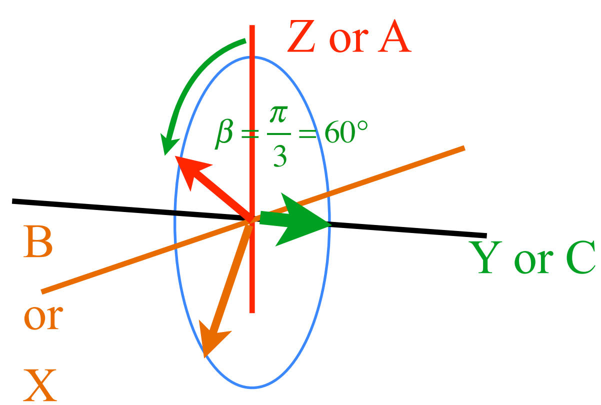
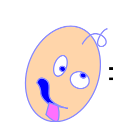


Here:  $= -i\sigma_\varphi = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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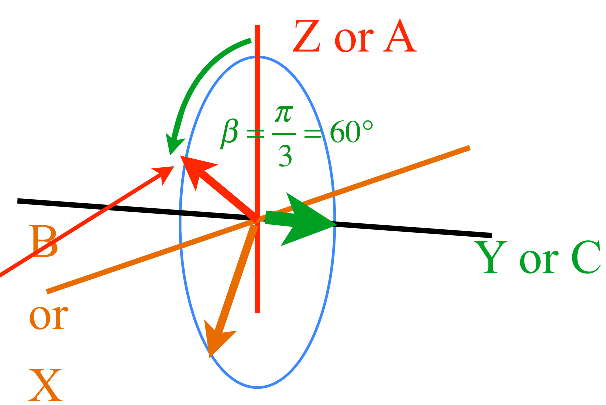
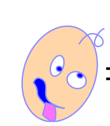


Here:  $= -i\sigma_{\varphi} = -i(\vec{\sigma}\bullet\hat{\varphi}) = -i\frac{(\vec{\sigma}\bullet\vec{\varphi})}{\varphi}$

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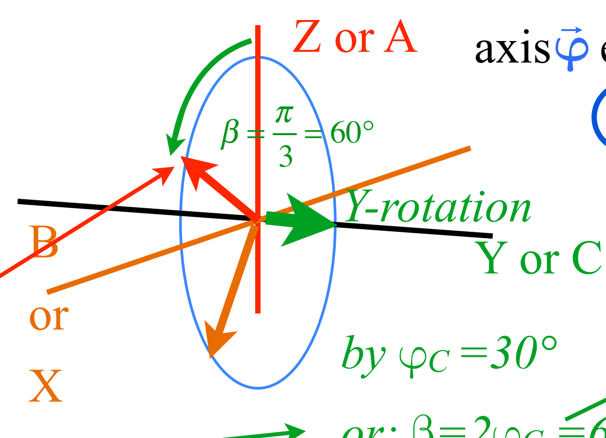

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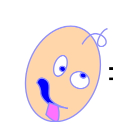
Here:  $= -i\sigma_\varphi = -i(\vec{\boldsymbol{\sigma}}\bullet\hat{\varphi}) = -i\dfrac{(\vec{\boldsymbol{\sigma}}\bullet\vec{\varphi})}{\varphi}$

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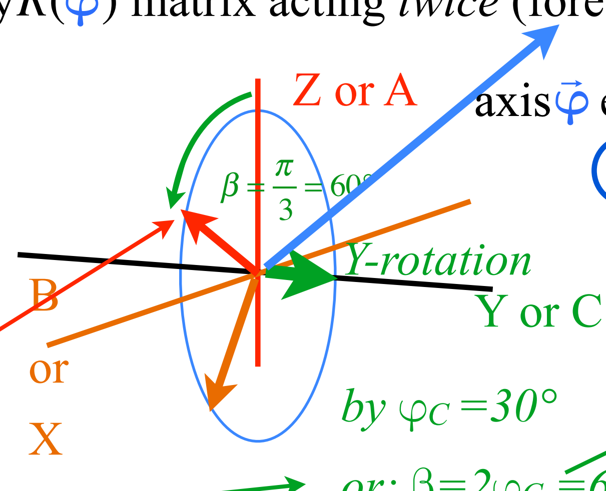

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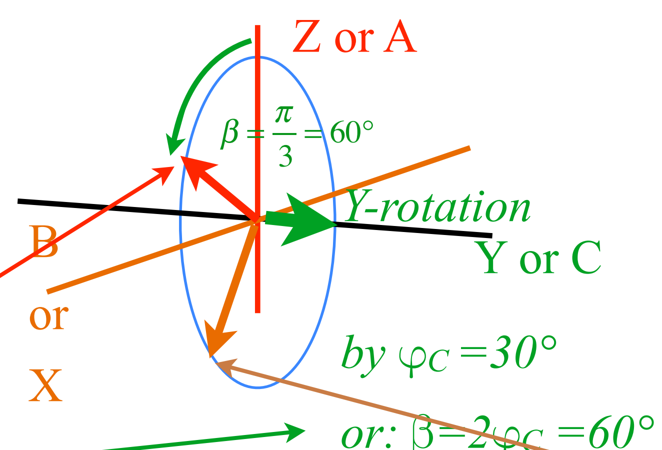

$$\mathbf{R}(\varphi_C)\cdot\sigma_B\cdot\mathbf{R}^{-1}(\varphi_C)$$
$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}-2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C - \sin^2\varphi_C\\ \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\sin 2\varphi_C + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cos 2\varphi_C$$
$$= -\sigma_A\sin 2\varphi_C + \sigma_B\cos 2\varphi_C$$

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*

→ *Geometry of U(2) evolution (or R(3) revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- → *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
  - *2D Spinor vs 3D vector rotation*
  - *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

- *Spin-1 (3D-real vector) case*
- *Spin-1/2 (2D-complex spinor) case*

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

- *Asymmetry* $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$
- *Polarization ellipse and spinor state dynamics*

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0\mathbf{1} + \omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_I, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_I=\boldsymbol{\sigma}_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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Notation for 2D Spinor space

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\boldsymbol{\Omega}} \bullet \vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

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$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0\mathbf{1} + \omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$= \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$= \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

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The $\{\mathbf{1}, \mathsf{S}_A, \mathsf{S}_B, \mathsf{S}_C\}$ are the *Jordan-Angular-Momentum operators* $\{\mathbf{1}=\sigma_0, \mathsf{S}_B=\mathsf{S}_X, \mathsf{S}_C=\mathsf{S}_Y, \mathsf{S}_A=\mathsf{S}_Z\}$

(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Notation for 2D Spinor space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\sigma_\omega$$

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Notation for 3D Vector space

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

0^{th} component unchanged *components A, B, C switch 1/2-factor from ω-velocity to S-momentum*

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Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}\cdot t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\cdot t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})\cdot t}=e^{-i\omega_0\cdot t}e^{-i\,\vec{\omega}\bullet\vec{\sigma}\cdot t}=e^{-i\omega_0\cdot t}e^{-i\sigma_\omega\omega\cdot t}=e^{-i\omega_0\cdot t}\left(\mathbf{1}\cos\omega\cdot t-i\sigma_\omega\sin\omega\cdot t\right)$$

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Notation for 2D Spinor space

$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

$$=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

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Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

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Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

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Notation for 3D Vector space

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Symmetry archetypes: *A (Asymmetric-diagonal)* | *B (Bilateral-balanced)* | *C (Chiral-circular-complex...)*

"Crank" vector (2D-Spinor)

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

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Notation for 2D Spinor space

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"Crank" vector (3D-Vector)

$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})\cdot t}=e^{-i\Omega_0\cdot t}e^{-i\,\vec{\Omega}\cdot t\bullet\mathbf{S}}=e^{-i\Omega_0\cdot t}\left(\mathbf{1}\cos\frac{\Omega\cdot t}{2}-i\sigma_\omega\sin\frac{\Omega\cdot t}{2}\right)$$

$$\vec{\Theta}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

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Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

→ *Geometry of U(2) evolution (or R(3) revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

→ *2D Spinor vs 3D vector rotation*

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

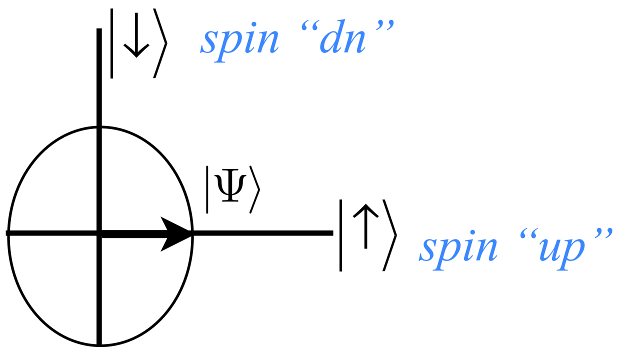
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor {|↑⟩,|↓⟩}-space (complex)

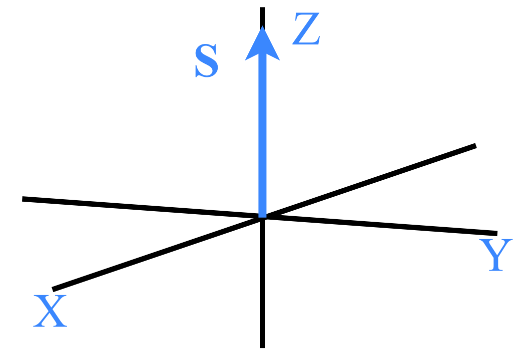
R(3):3D Spin Vector {S_X, S_Y, S_Z}-space (real)

State vector $|\Psi\rangle = |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$

Spin vector $\mathbf{S} = |X\rangle\langle X|\mathbf{S}\rangle + |Y\rangle\langle Y|\mathbf{S}\rangle + |Z\rangle\langle Z|\mathbf{S}\rangle$



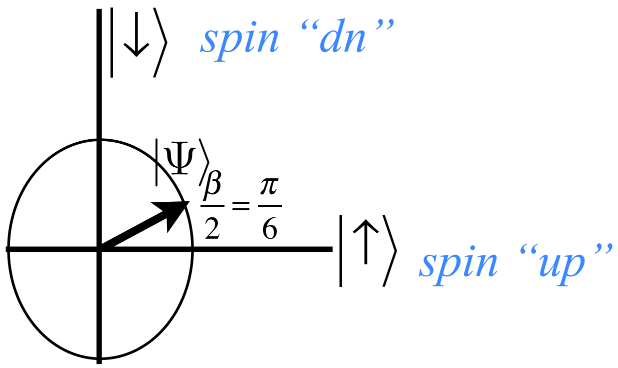
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$



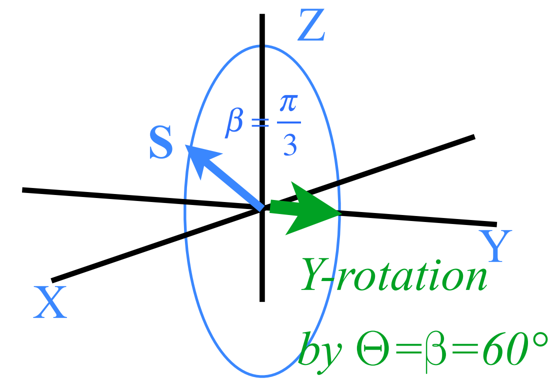
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

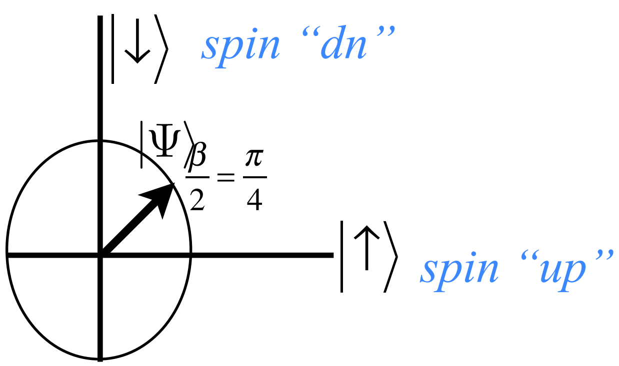
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$



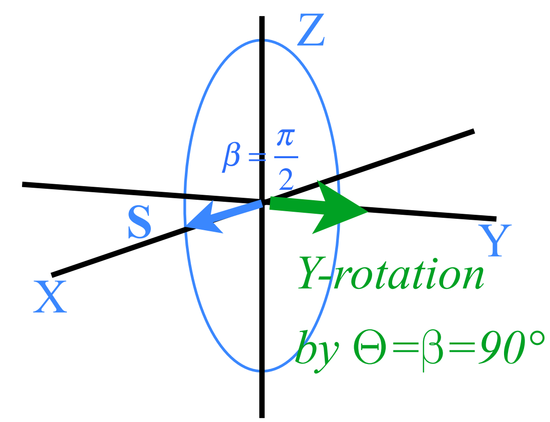
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



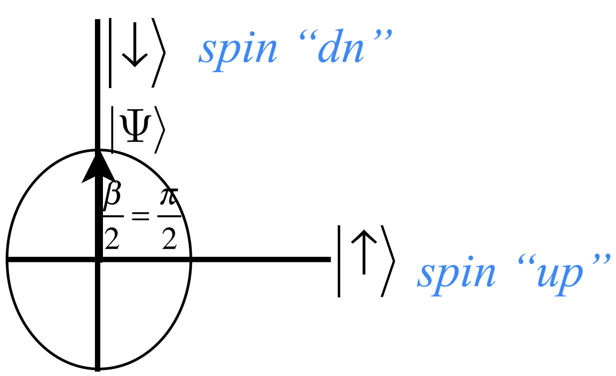
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2\\ 0\\ 1/2\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$



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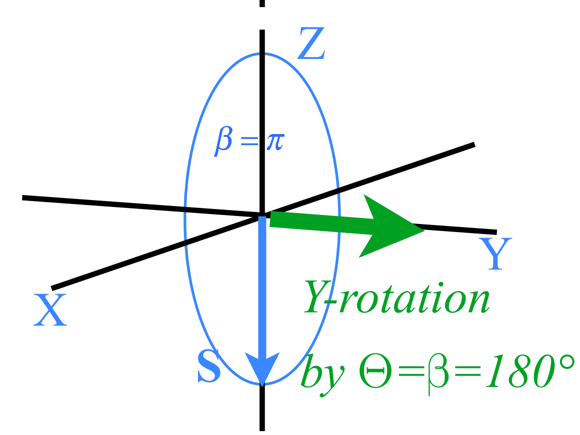


$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast"

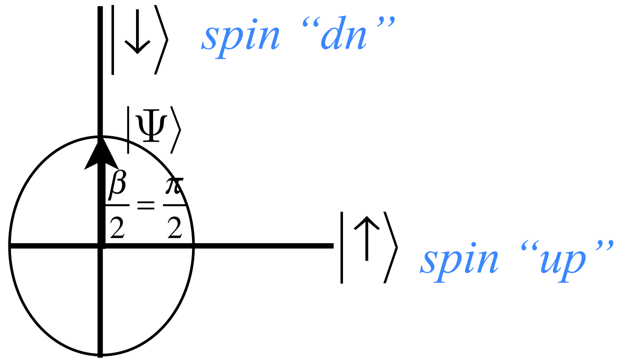
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor $\{|\uparrow\rangle,|\downarrow\rangle\}$*-space (complex)*

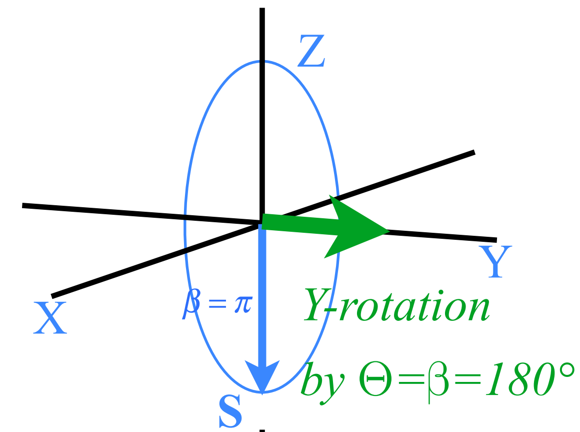
R(3):3D Spin Vector $\{S_X, S_Y, S_Z\}$*-space (real)*

State vector $|\Psi\rangle=|\uparrow\rangle\langle\uparrow|\Psi\rangle+|\downarrow\rangle\langle\downarrow|\Psi\rangle$

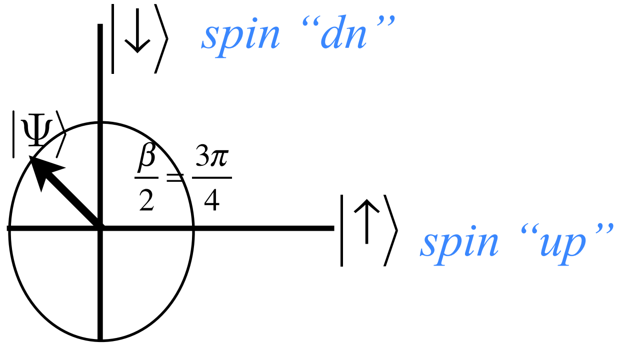
Spin vector $\mathbf{S}=|X\rangle\langle X|\mathbf{S}\rangle+|Y\rangle\langle Y|\mathbf{S}\rangle+|Z\rangle\langle Z|\mathbf{S}\rangle$



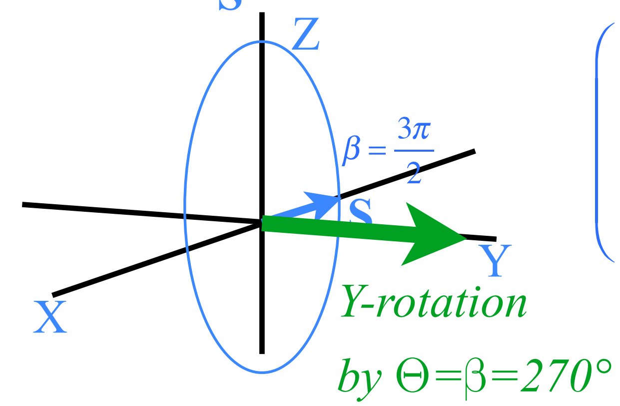
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$$



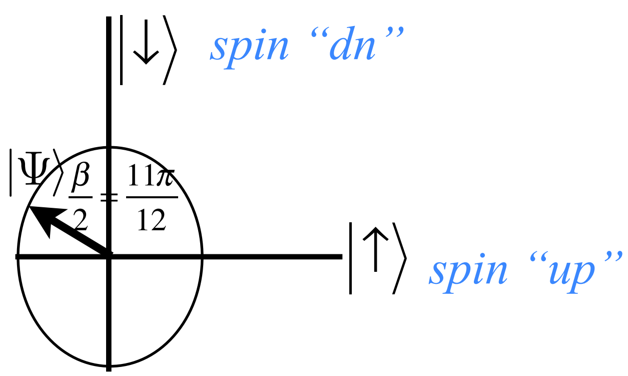
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$



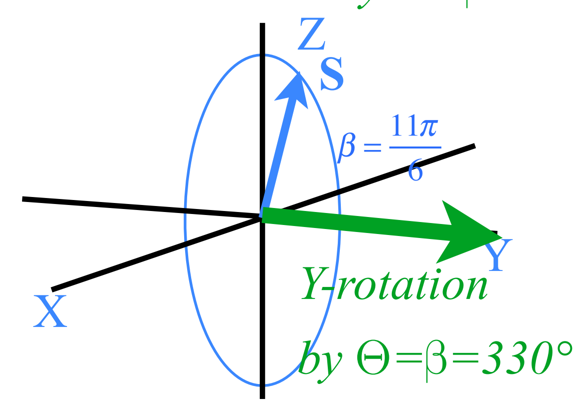
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



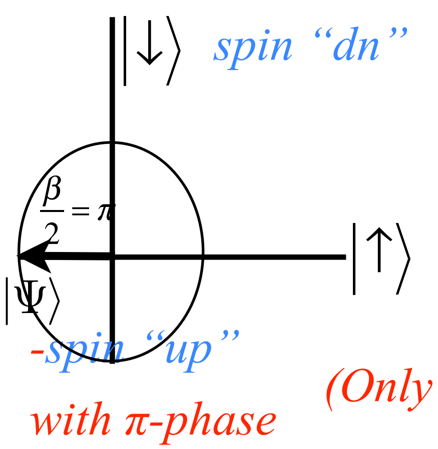
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=\begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}$$



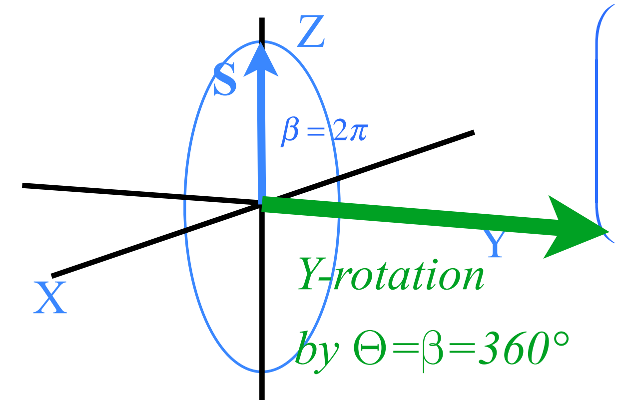
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}\frac{-\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix}=$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix}=\begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix}=\begin{pmatrix}-1\\ 0\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

(Only "half-way" home after 2π =360° rotation)

Life in 2D Spinor space is "Half-Fast" and needs Θ=4π=720° to return to original state

Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

→ *Geometry of U(2) evolution (or R(3) revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

→ *NMR Hamiltonian: 3D Spin Moment **m** in **B** field*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$, *Balance* $S_B=S_X$, *and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X - igB_Y \\ gB_X + igB_Y & -gB_Z \end{pmatrix} = gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= gB_Z\ \sigma_A + gB_X\ \sigma_X + gB_Y\ \sigma_Y = \vec{\omega}\bullet\vec{\sigma} = \omega\sigma_\omega$$

Notation for
2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

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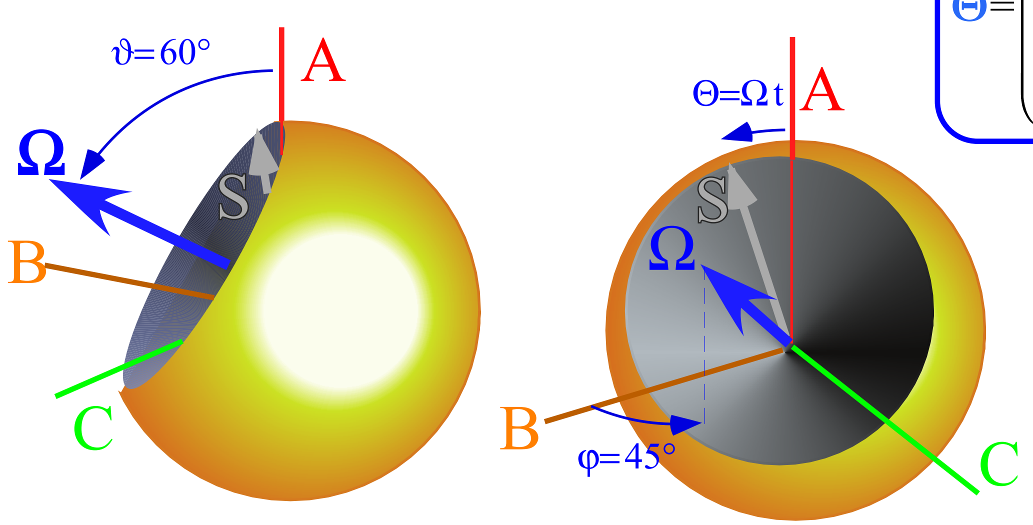
The driving $\mathbf{\Theta}=\mathbf{\Omega}t$ *crank vector defined by ABCD of Hamiltonian* $\mathbf{H}$.

Notation for 3D Vector space

$$\vec{\Theta}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$



Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$ *whirling Stokes state vector* $\mathbf{S}$ *in ABC-space.*

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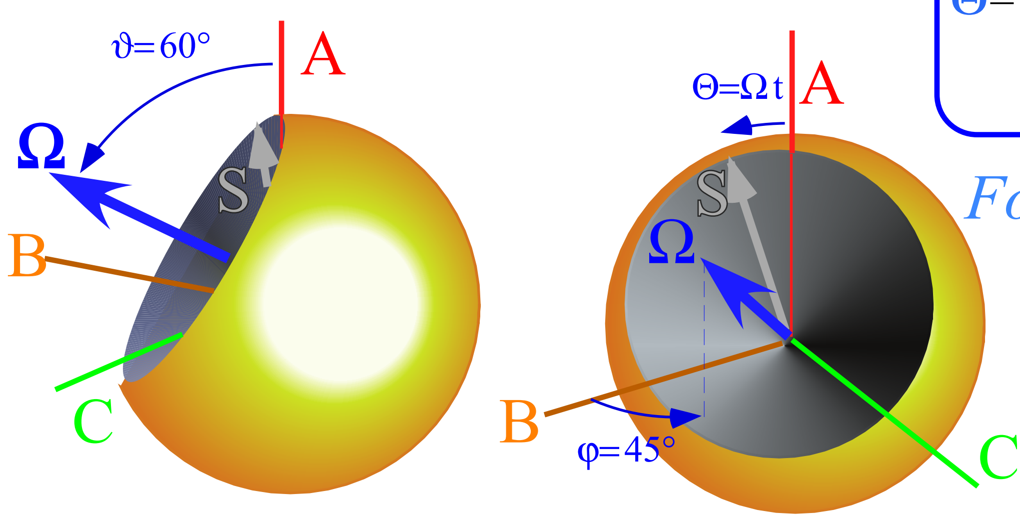
For fermion spin that $\boldsymbol{\Omega}$ *is the* g**B**-*field!*



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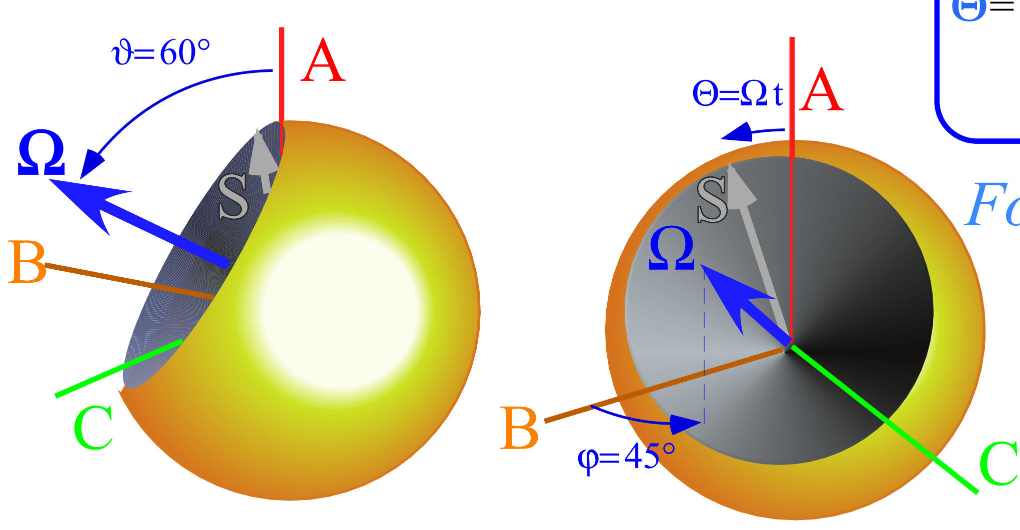
For fermion spin that $\mathbf{\Omega}$ *is the g***B***-field!*

Q: But, how is a spin state-$|\psi\rangle$ *or spin vector-***S** *defined?*



Two views of Hamilton crank vector Ω(φ,ϑ) whirling Stokes state vector S in ABC-space.

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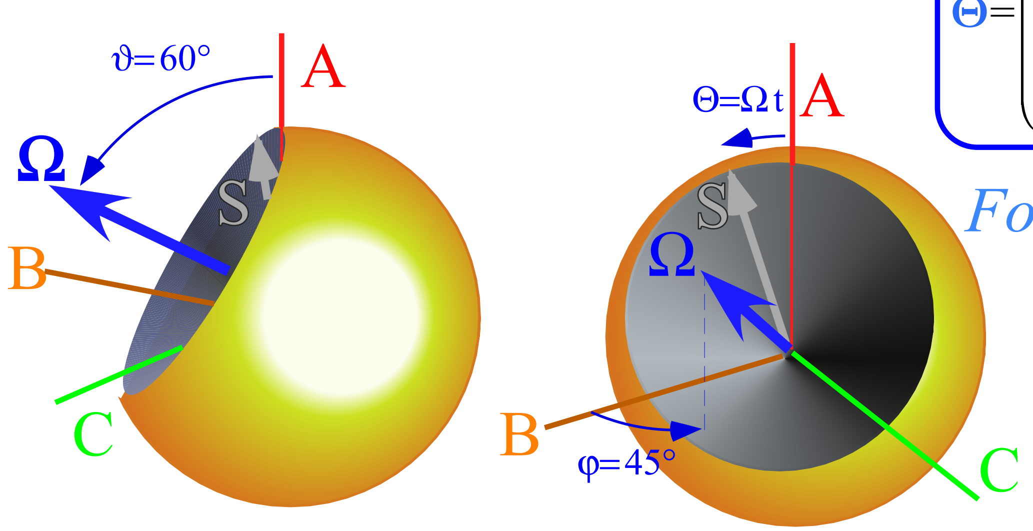

For fermion spin that $\mathbf{\Omega}$ *is the g*$\mathbf{B}$*-field!*

Q: But, how is a spin state-$|\psi\rangle$ *or spin vector-*$\mathbf{S}$ *defined?*

A: By U(2) group operator $|\psi(t)\rangle=\mathbf{R}[\Theta]|\psi(0)\rangle$.

Two views of Hamilton crank vector Ω(φ,ϑ) whirling Stokes state vector S in ABC-space.

Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

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Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- *Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

- *The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
  - *2D Spinor vs 3D vector rotation*
  - *NMR Hamiltonian: 3D Spin Moment **m** in **B** field*

→ *Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$*

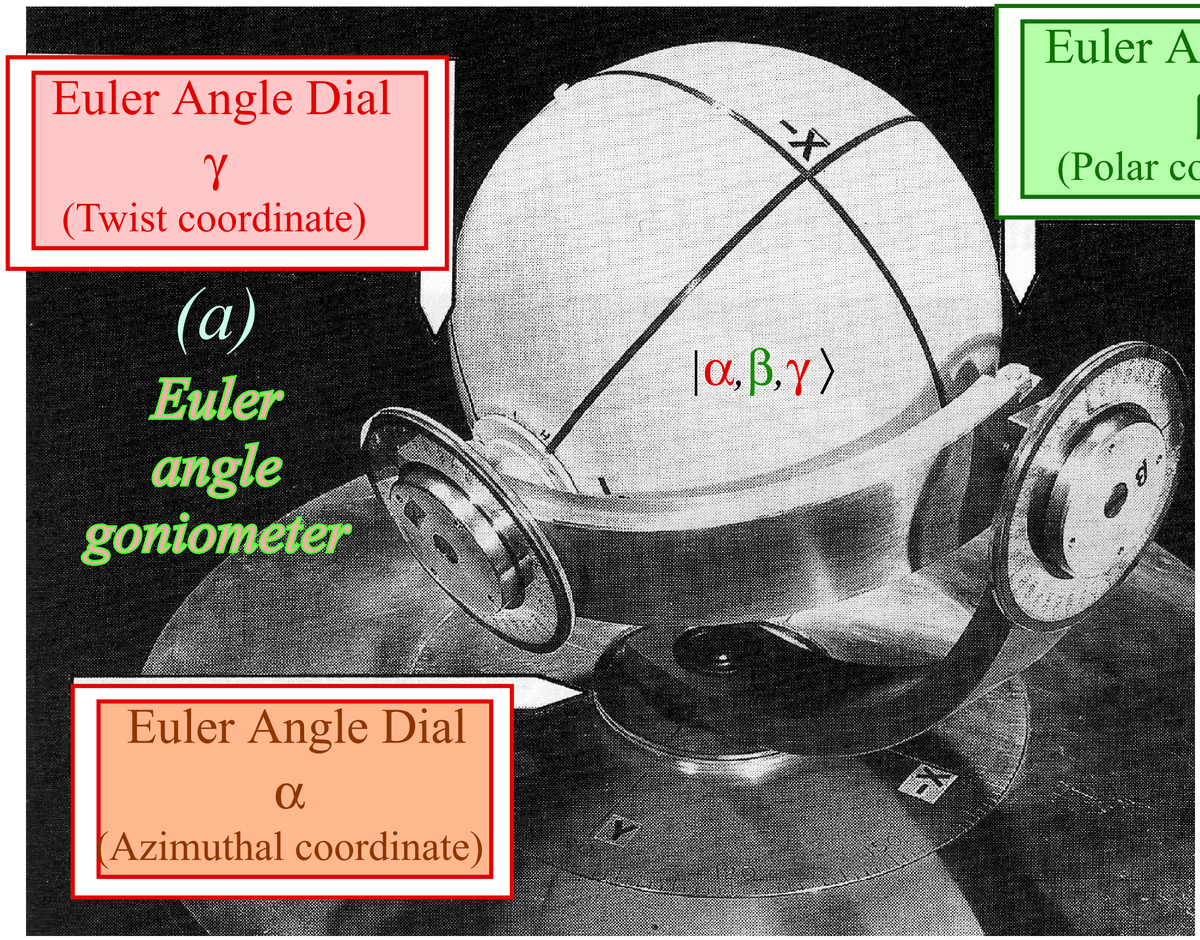
- → *Spin-1 (3D-real vector) case*
- *Spin-1/2 (2D-complex spinor) case*

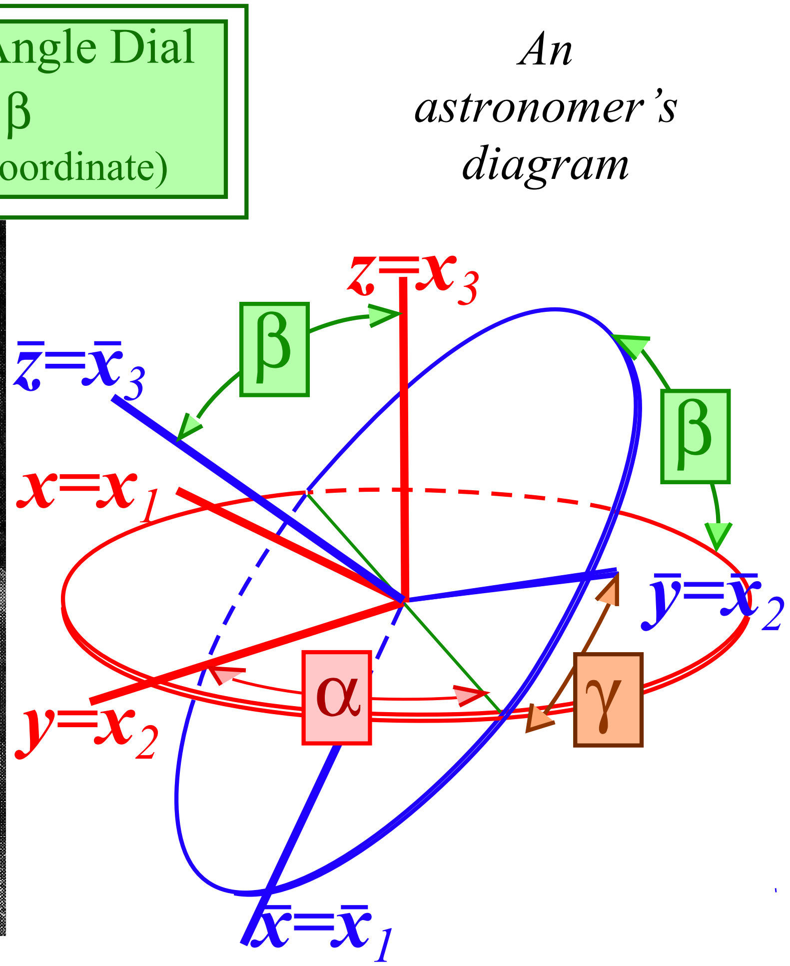
3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

- *Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$*
- *Polarization ellipse and spinor state dynamics*

Euler's state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

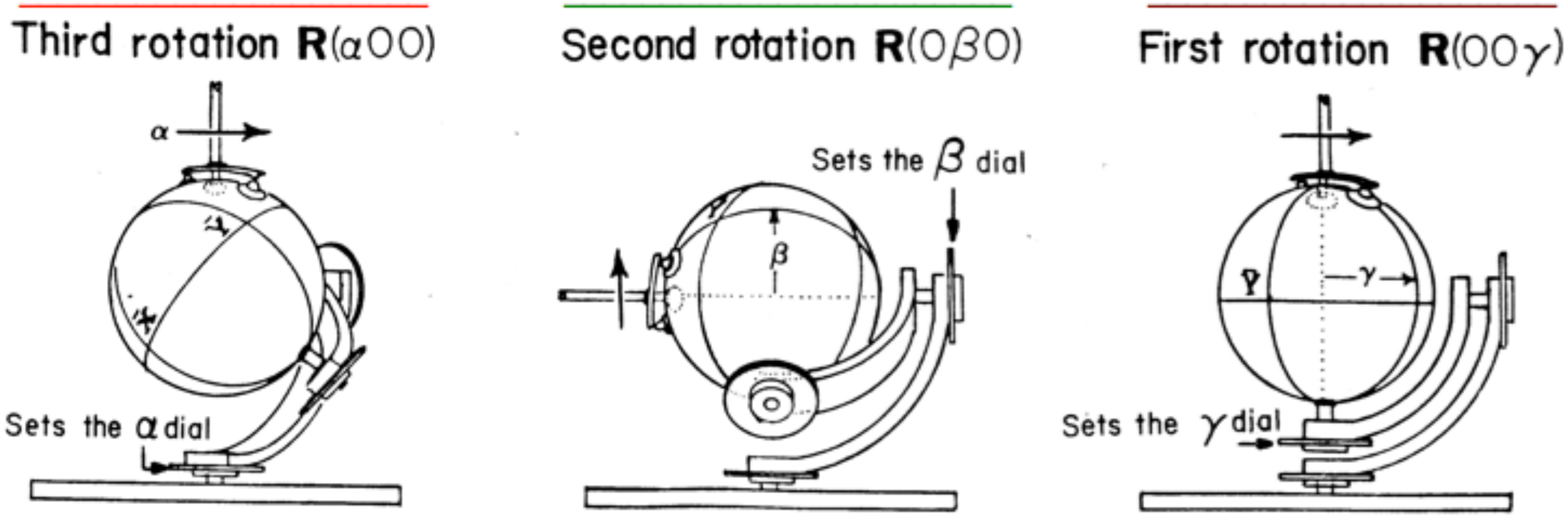
Spin-1 (3D-real vector) case





Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, and **R**$(0,0,\gamma)$

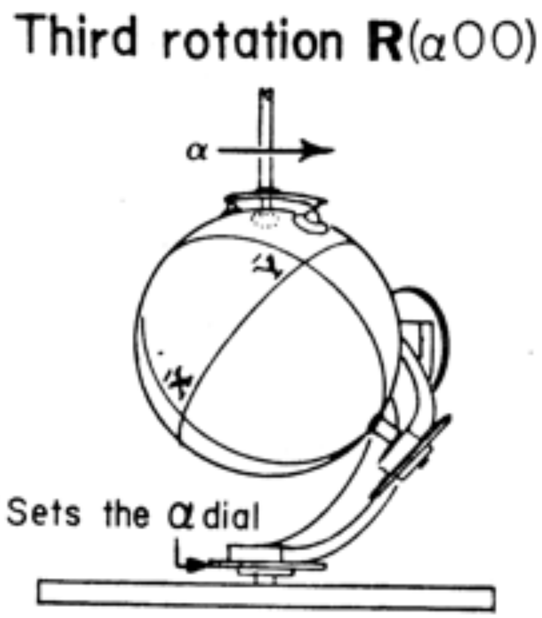
Spin-1 (3D-real vector) case

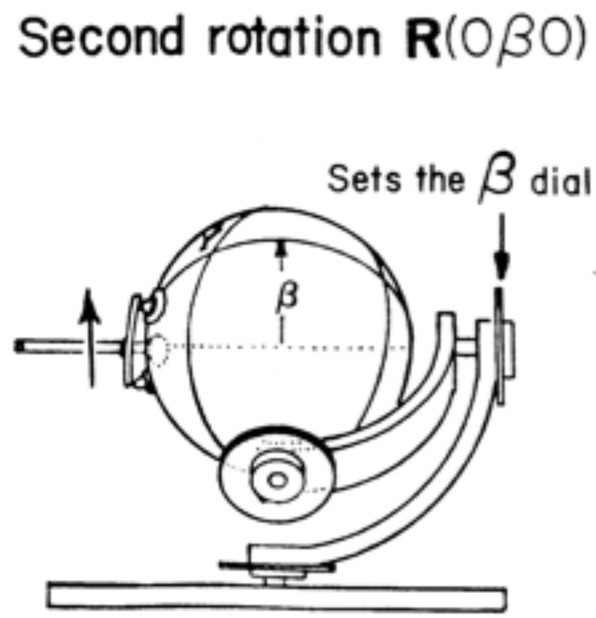


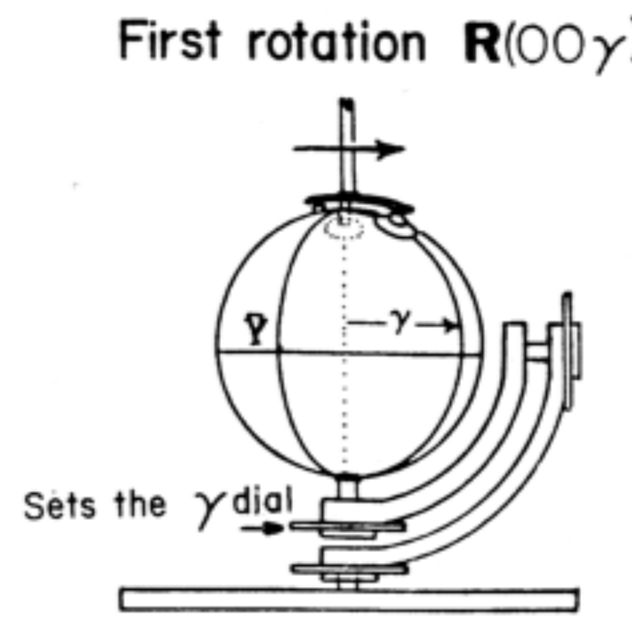
$$\left\langle R(\alpha\beta\gamma)\right\rangle = \left\langle R(\alpha 00)\right\rangle \qquad \left\langle R(0\beta 0)\right\rangle \qquad \left\langle R(00\gamma)\right\rangle$$

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$*,* $\mathbf{R}(0,\beta,0)$*, and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case





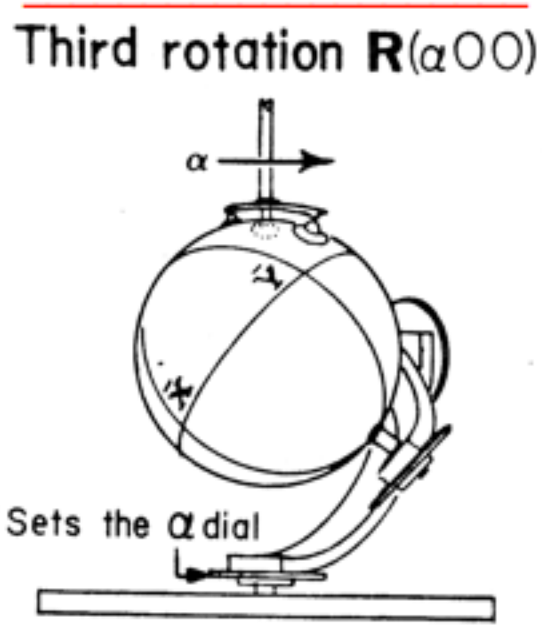


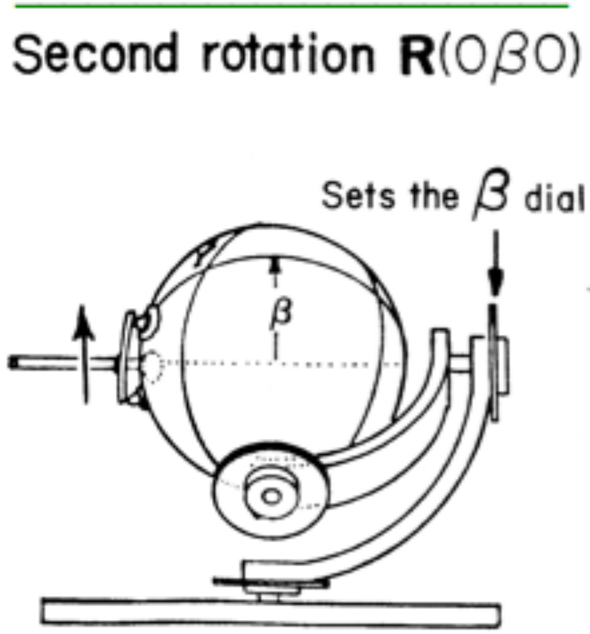
$$\left\langle R(\alpha\beta\gamma)\right\rangle = \left\langle R(\alpha 00)\right\rangle \left\langle R(0\beta 0)\right\rangle \left\langle R(00\gamma)\right\rangle$$

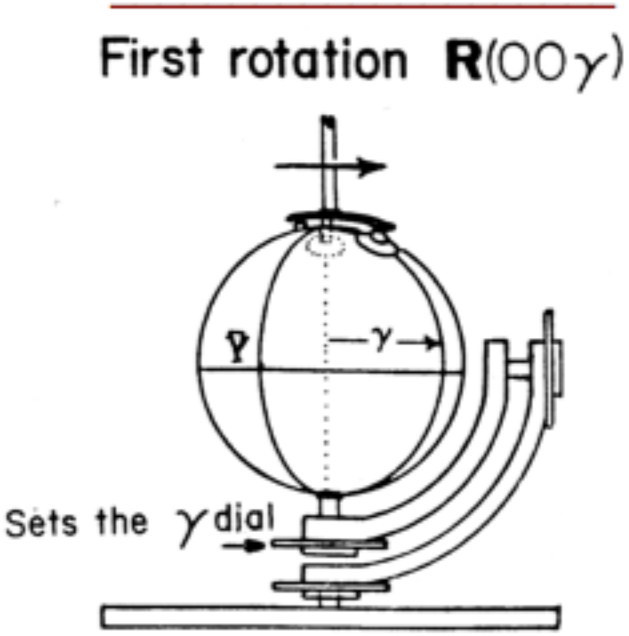
$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case







$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \qquad \langle R(0\beta 0)\rangle \qquad \langle R(00\gamma)\rangle$$

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$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

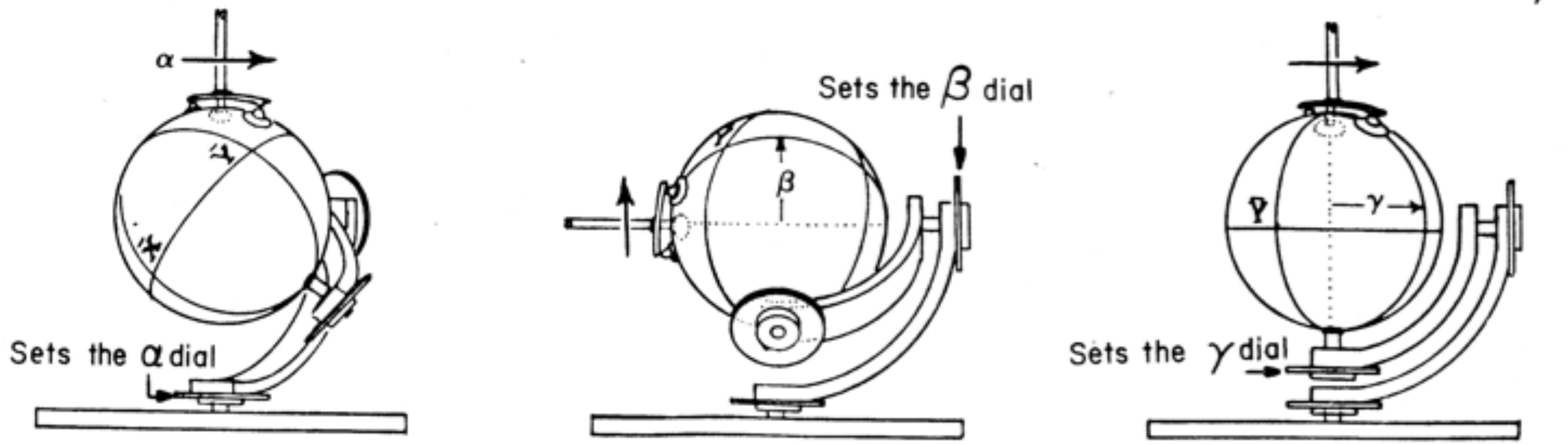
$$\left(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B \rangle\right) = \begin{matrix} \langle \mathbf{e}_X | \\ \langle \mathbf{e}_Y | \\ \langle \mathbf{e}_Z | \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \boxed{\cos\alpha\sin\beta} \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

Note lab frame polar coordinates

Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1 (3D-real vector) case

Third rotation **R**$(\alpha 00)$ — Second rotation **R**$(0\beta 0)$ — First rotation **R**(00γ)



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$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

$$\left(\langle\mathbf{e}_A|R(\alpha\beta\gamma)|\mathbf{e}_B\rangle\right) = \begin{matrix} \langle\mathbf{e}_X| \\ \langle\mathbf{e}_Y| \\ \langle\mathbf{e}_Z| \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

Note lab-frame polar coordinates of Z(body)

...and body-frame polar coordinates

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

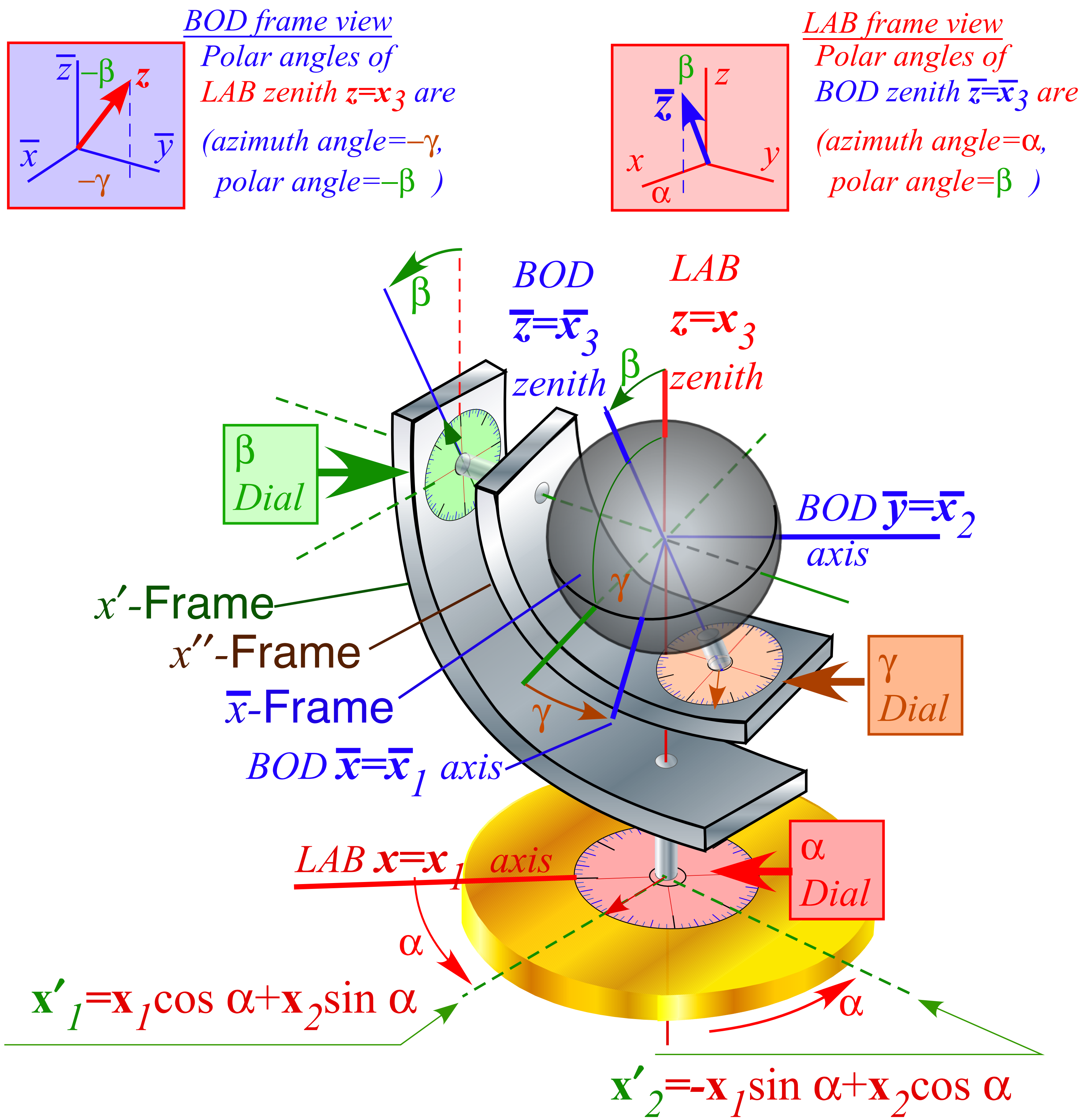
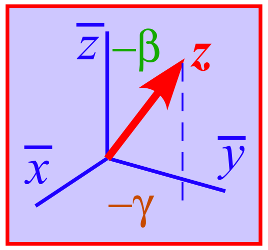


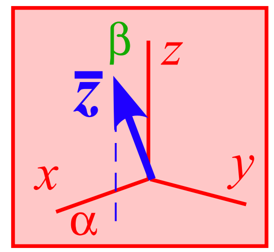
Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

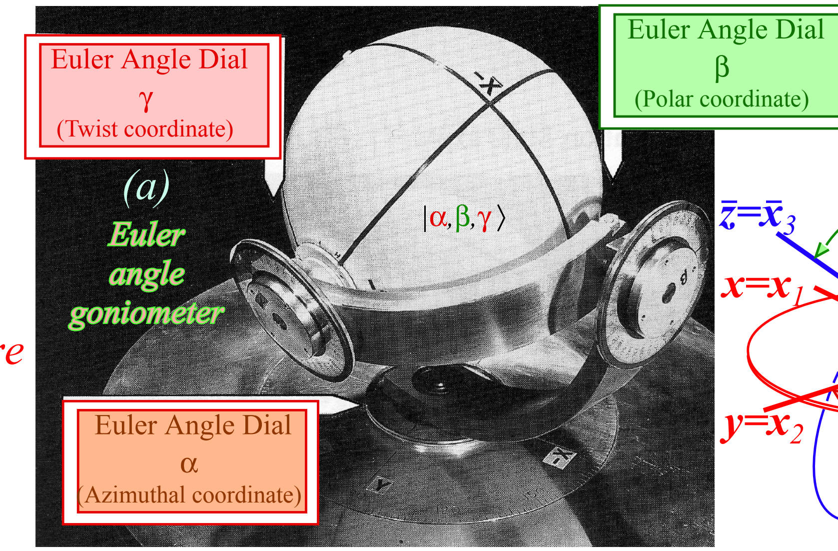
Spin-1 (3D-real vector) case

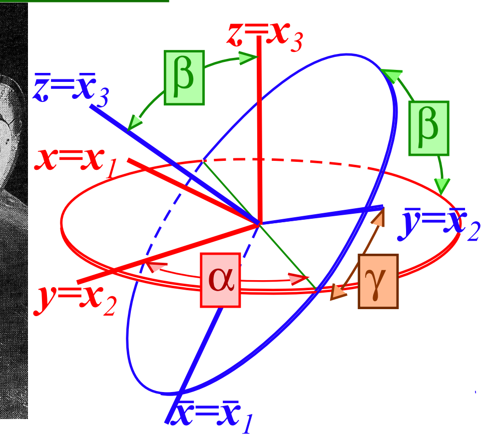


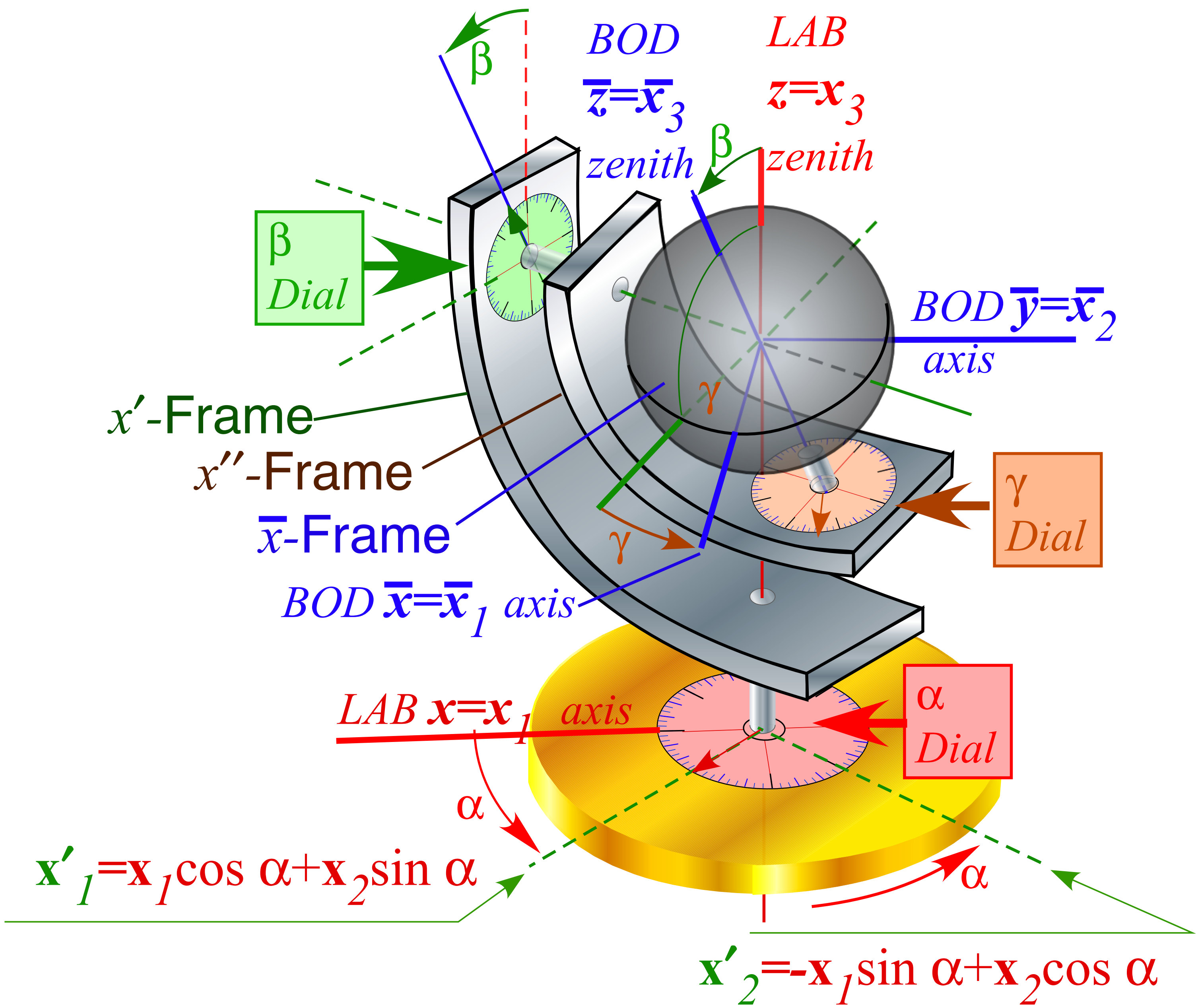
BOD frame view
Polar angles of LAB zenith $z=x_3$ *are (azimuth angle*$=-\gamma$*, polar angle*$=-\beta$ *)*



LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are (azimuth angle*$=\alpha$*, polar angle*$=\beta$ *)*







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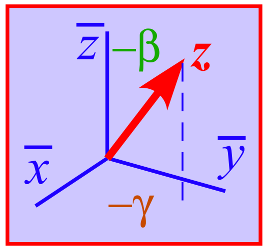

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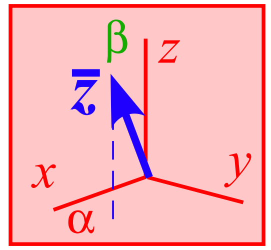
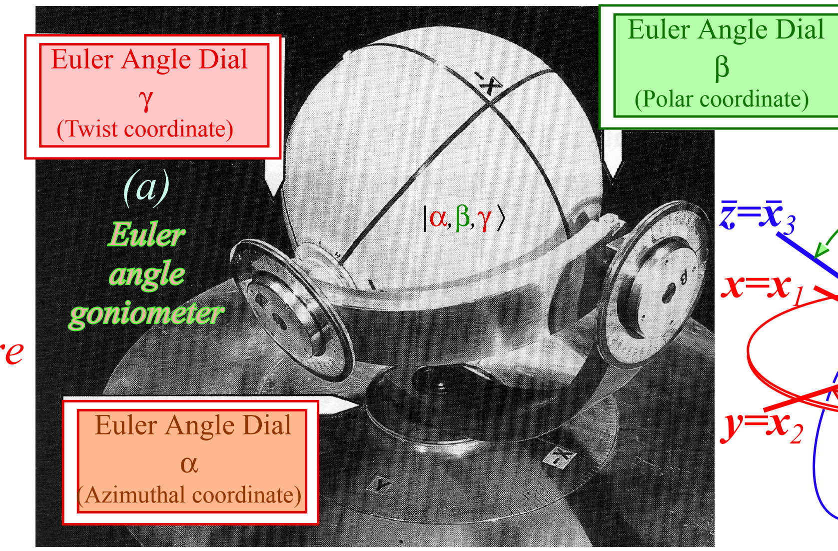
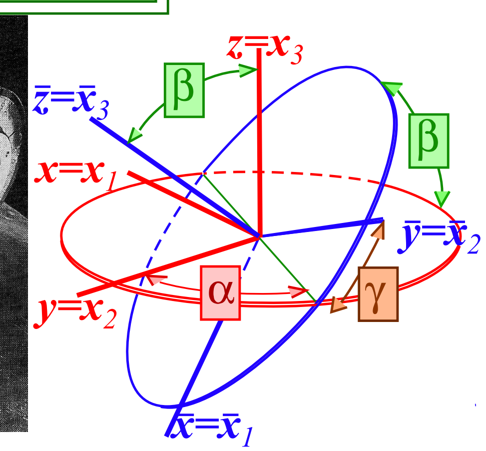
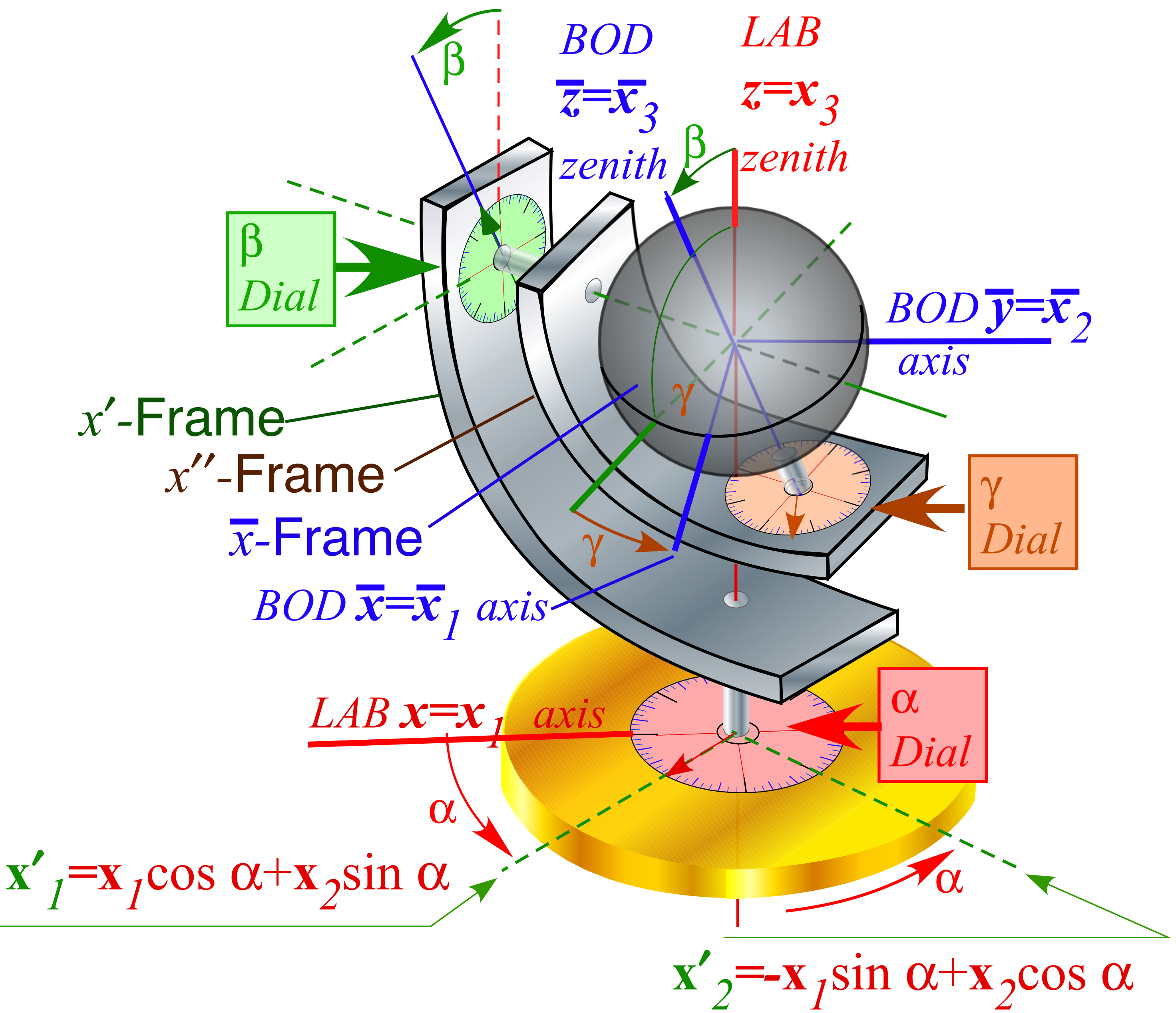


LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ *are (azimuth angle*$=\alpha$*, polar angle*$=\beta$ *)*







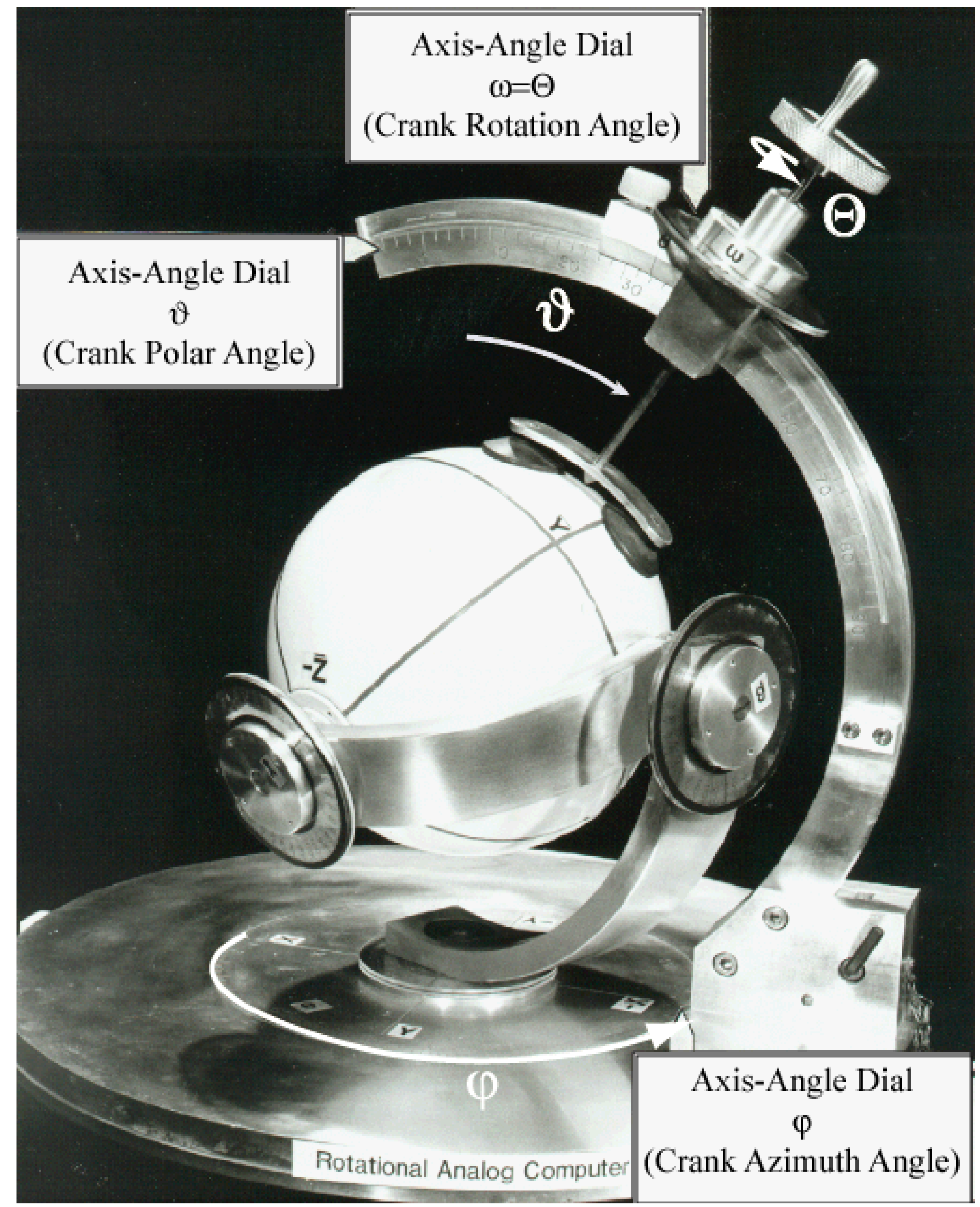


Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Review of Lecture 6: Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Deriving $\boldsymbol{\sigma}$-exponential time evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *Spinor arithmetic like complex arithmetic*
- *Spinor vector algebra like complex vector algebra*
- *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*

Geometry of U(2) evolution (or R(3) revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

- *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*
  - *2D Spinor vs 3D vector rotation*
  - *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*

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3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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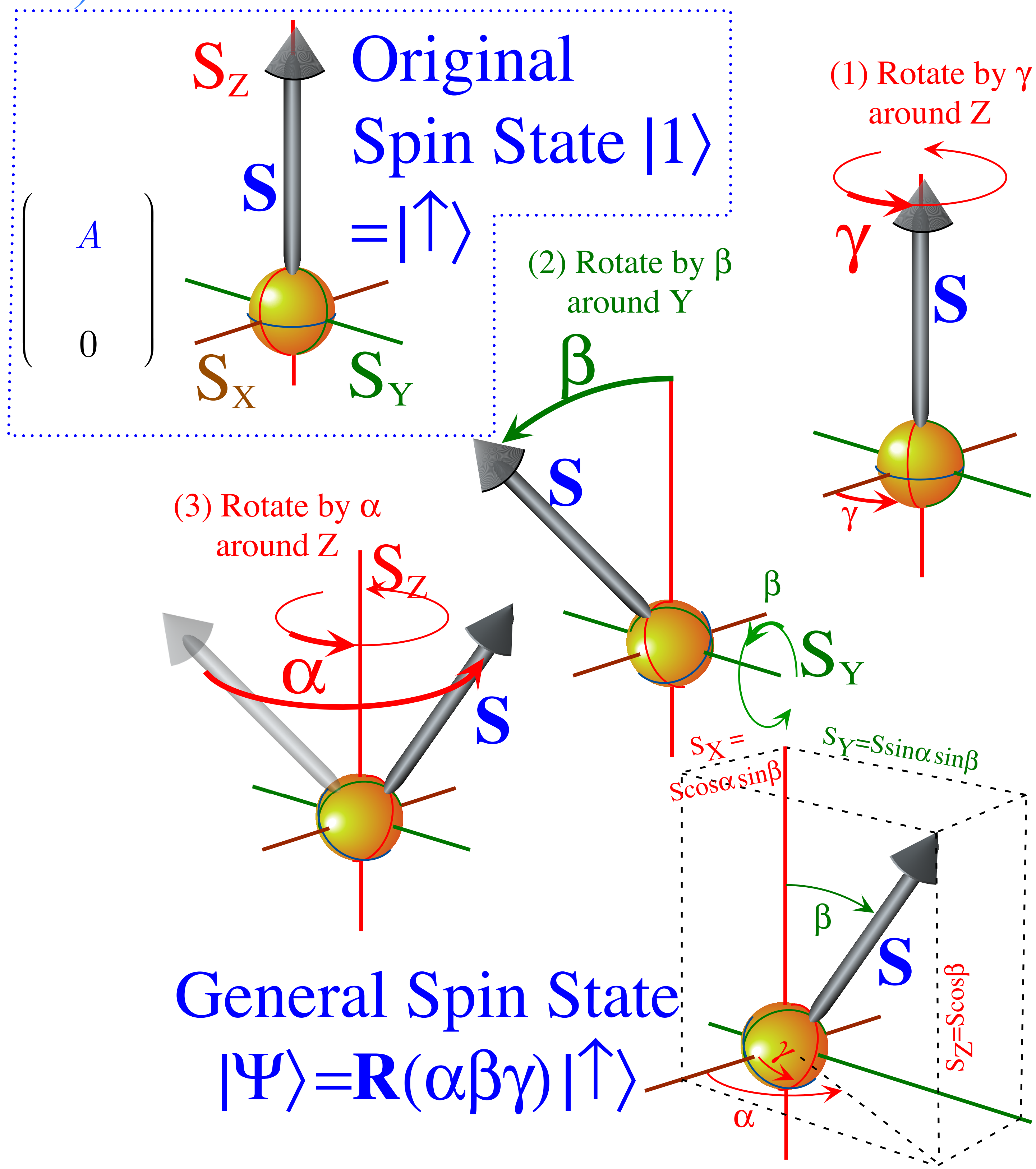
Spin-1/2 (2D-complex spinor) case

$$\left|a\right\rangle = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$



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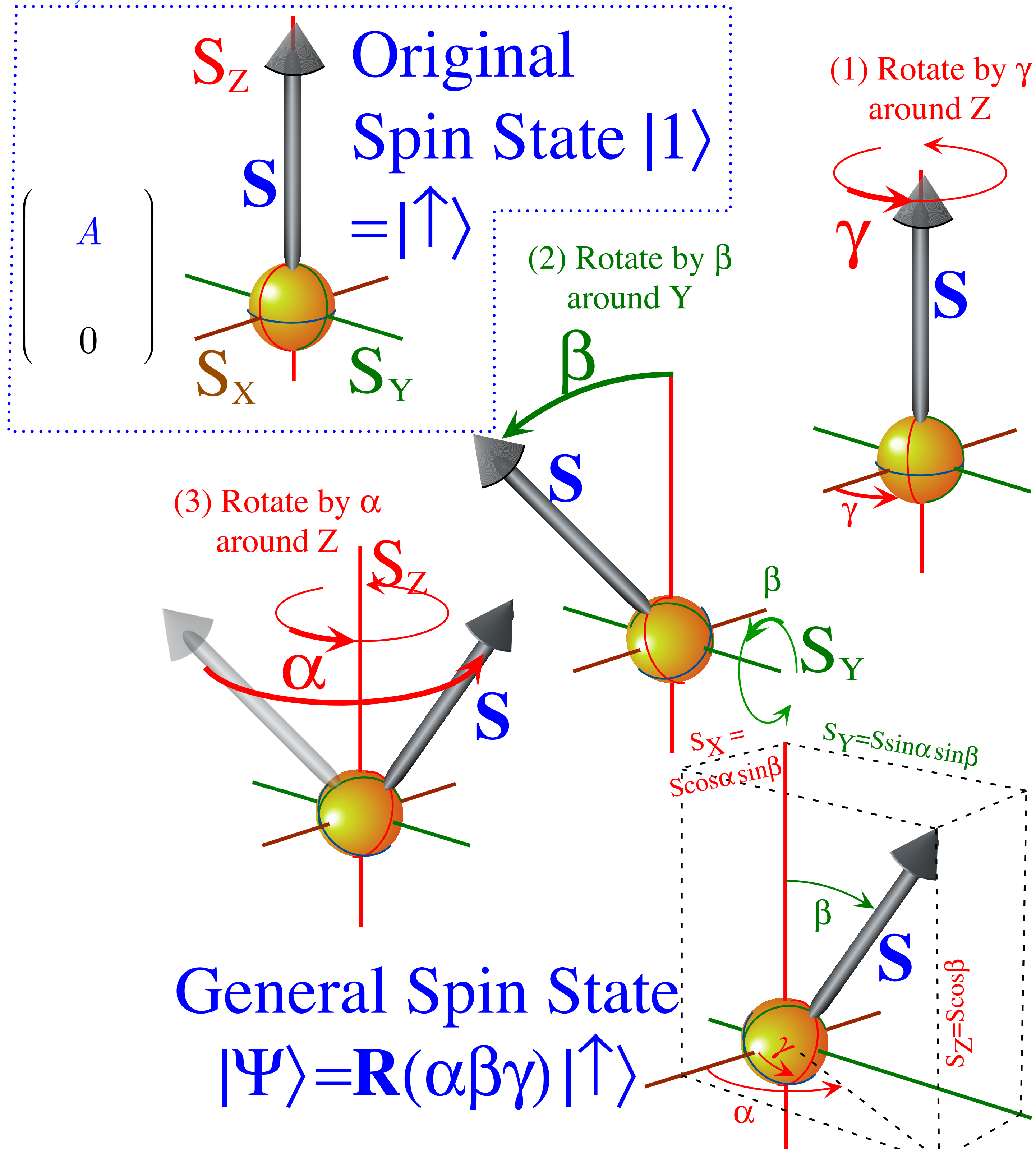
$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

$$= A \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



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→ *3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states*

→ *Asymmetry $S_A=S_Z$, Balance $S_B=S_X$, and Chirality $S_C=S_Y$*

Polarization ellipse and spinor state dynamics

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\text{Asymmetry } S_A = \frac{1}{2}\left(a\middle|\sigma_A\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right]$$

$$\text{Balance } S_B = \frac{1}{2}\left(a\middle|\sigma_B\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right]$$

$$\text{Chirality } S_C = \frac{1}{2}\left(a\middle|\sigma_C\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right]$$

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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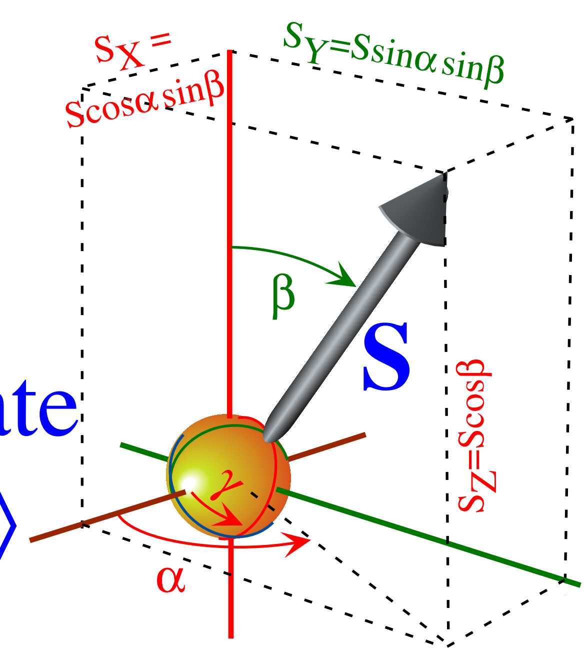
Chirality $S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2} \cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$

General Spin State
$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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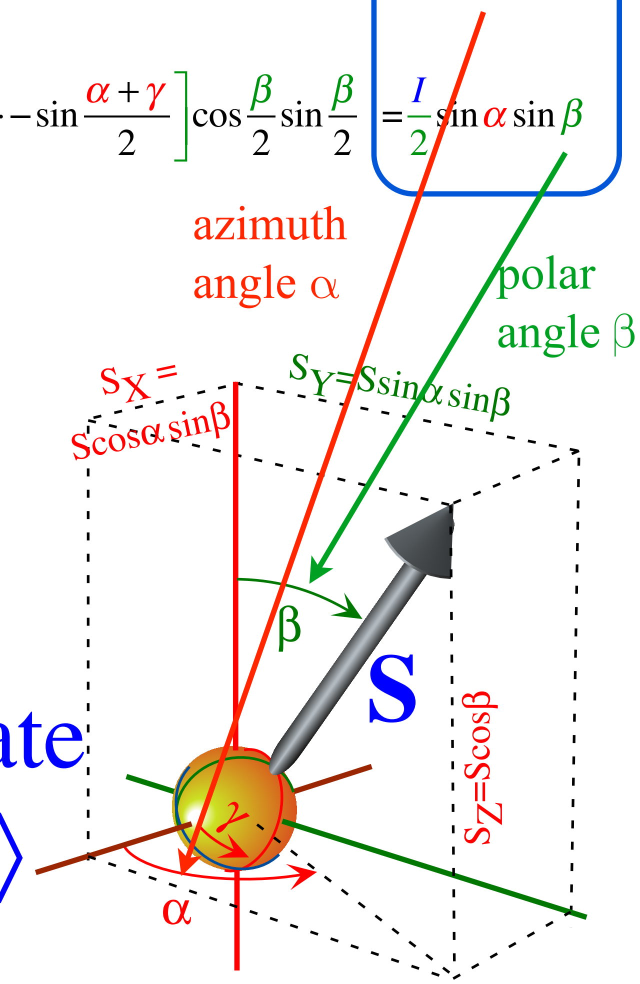

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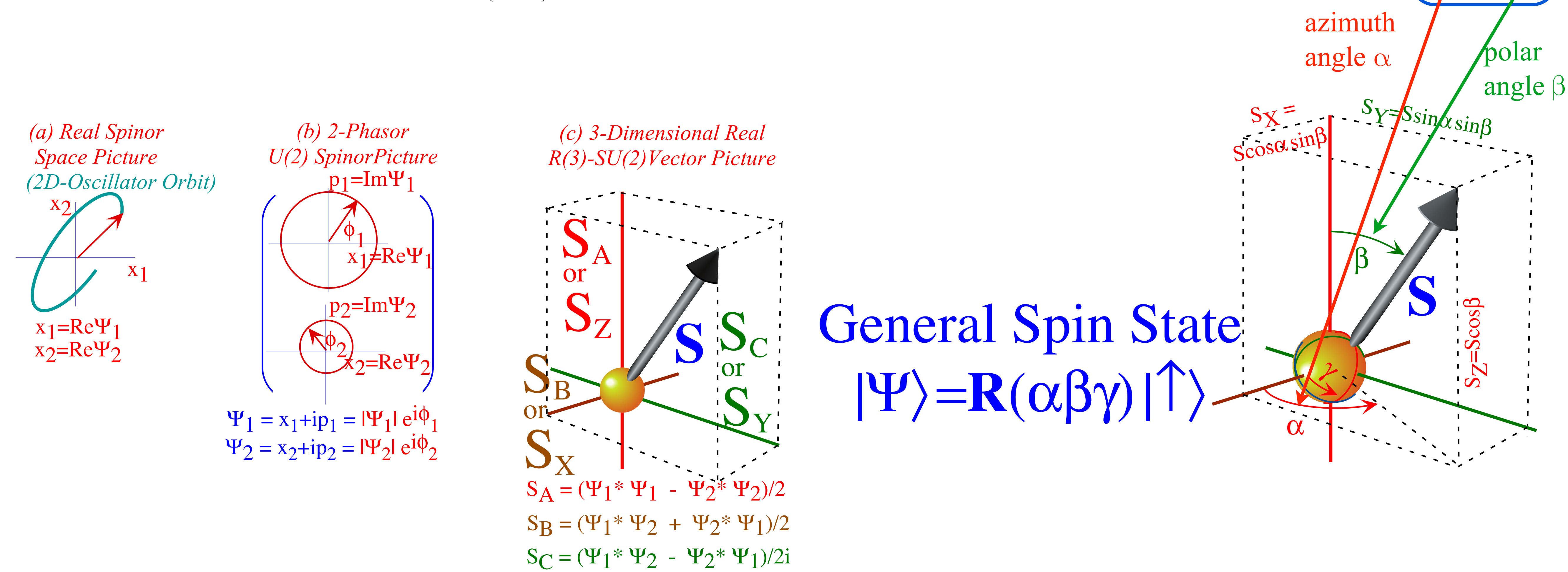

Fig. 10.5.2 Spinor, phasor, and vector descriptions of 2-state systems .

Review of Lecture 6: C_2 symmetry is 2D oscillators and three famous 2-state systems

Review of Lecture 6: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ *vs. Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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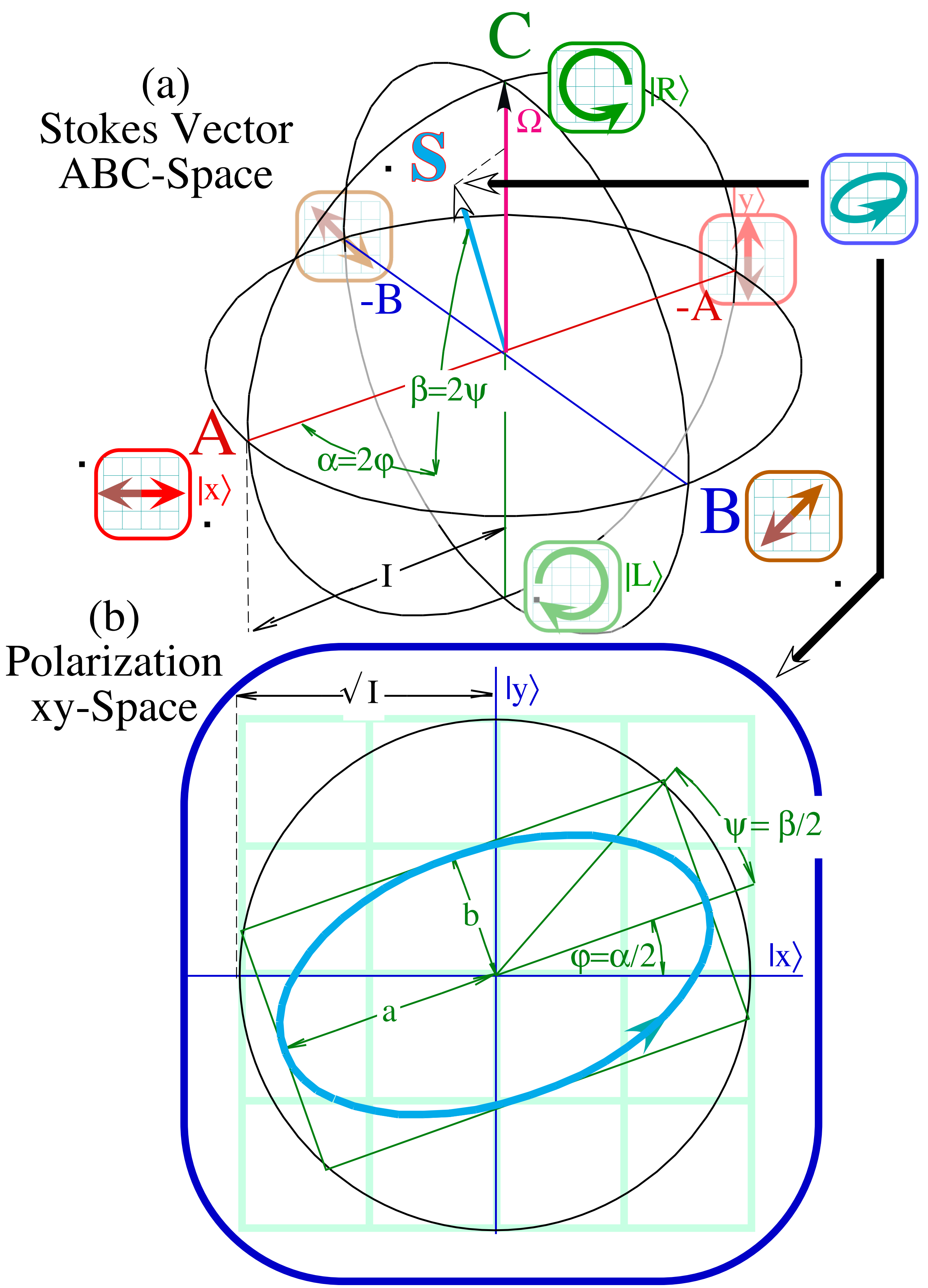
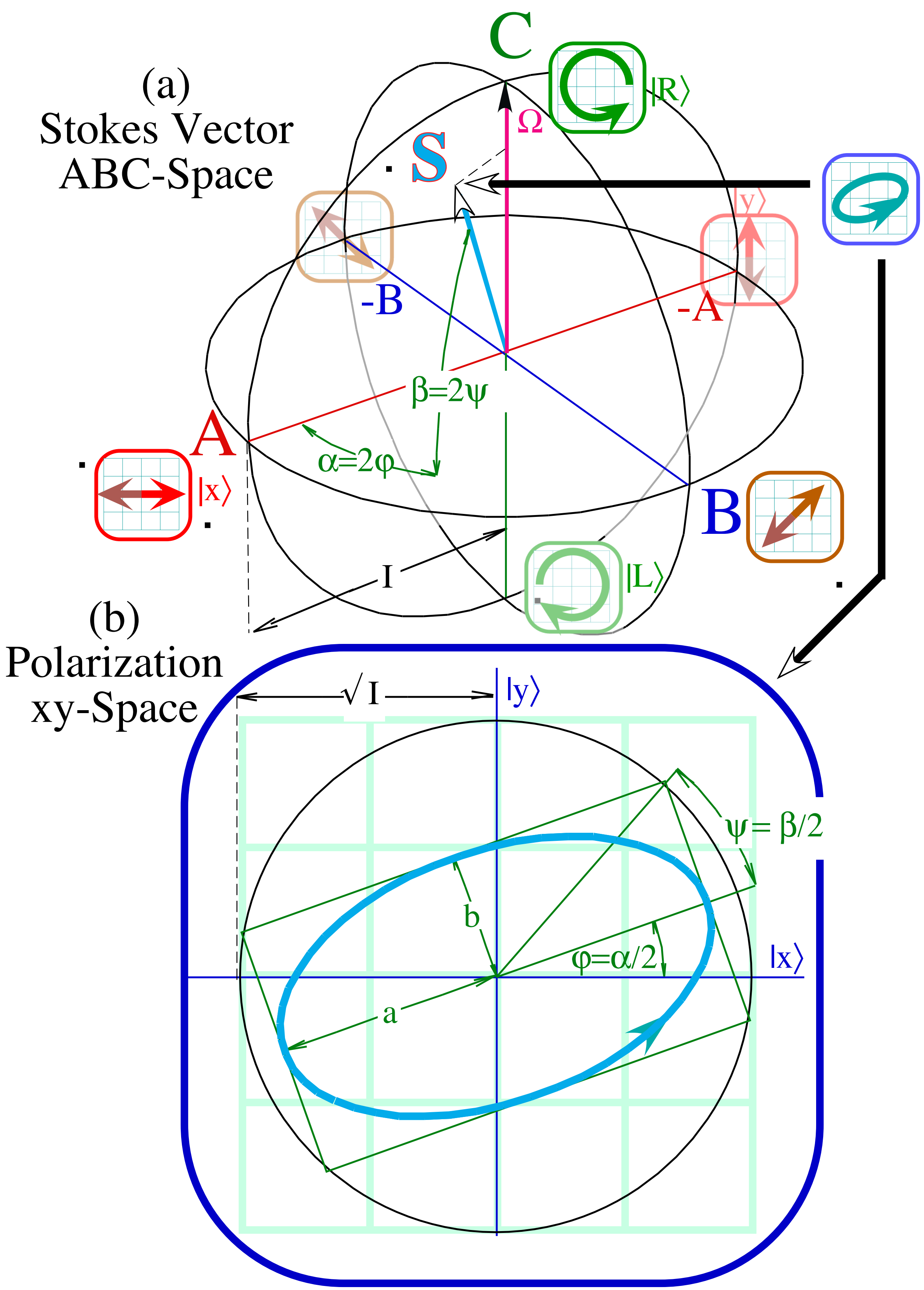


Fig. 10.B.3 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

Polarization ellipse and spinor state dynamics



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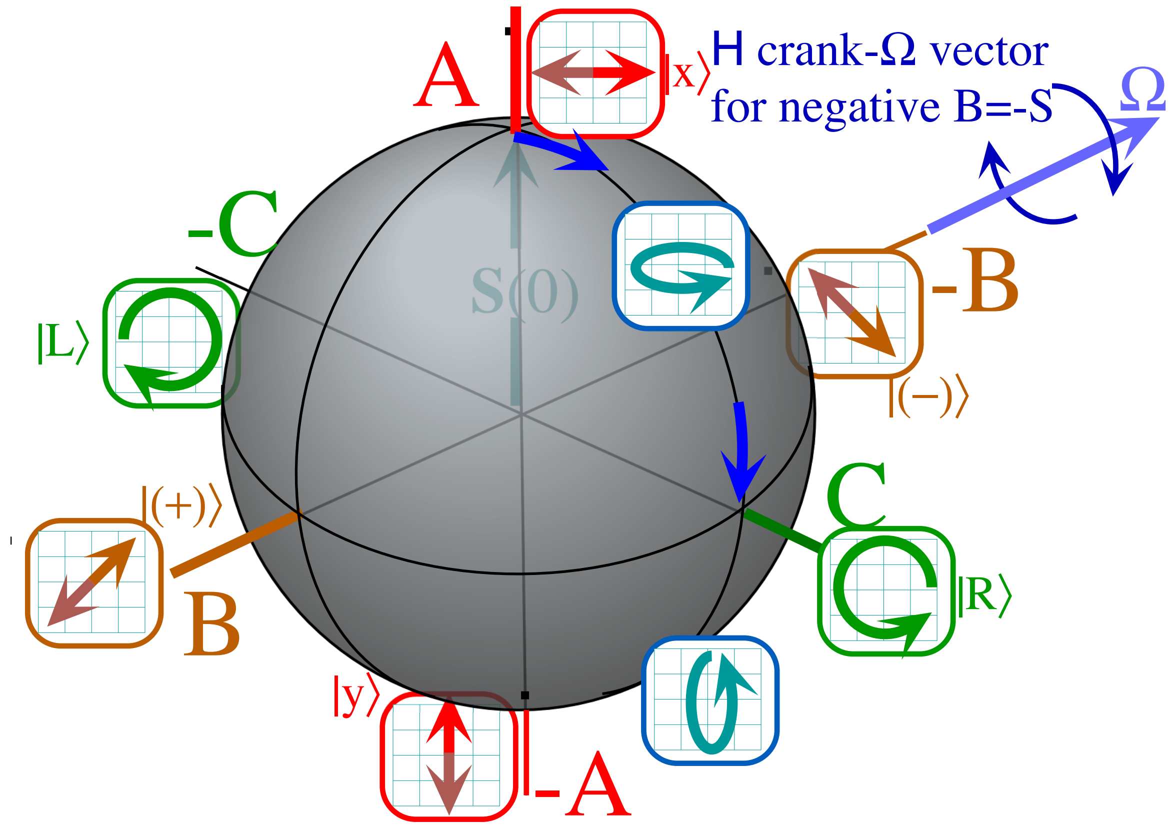
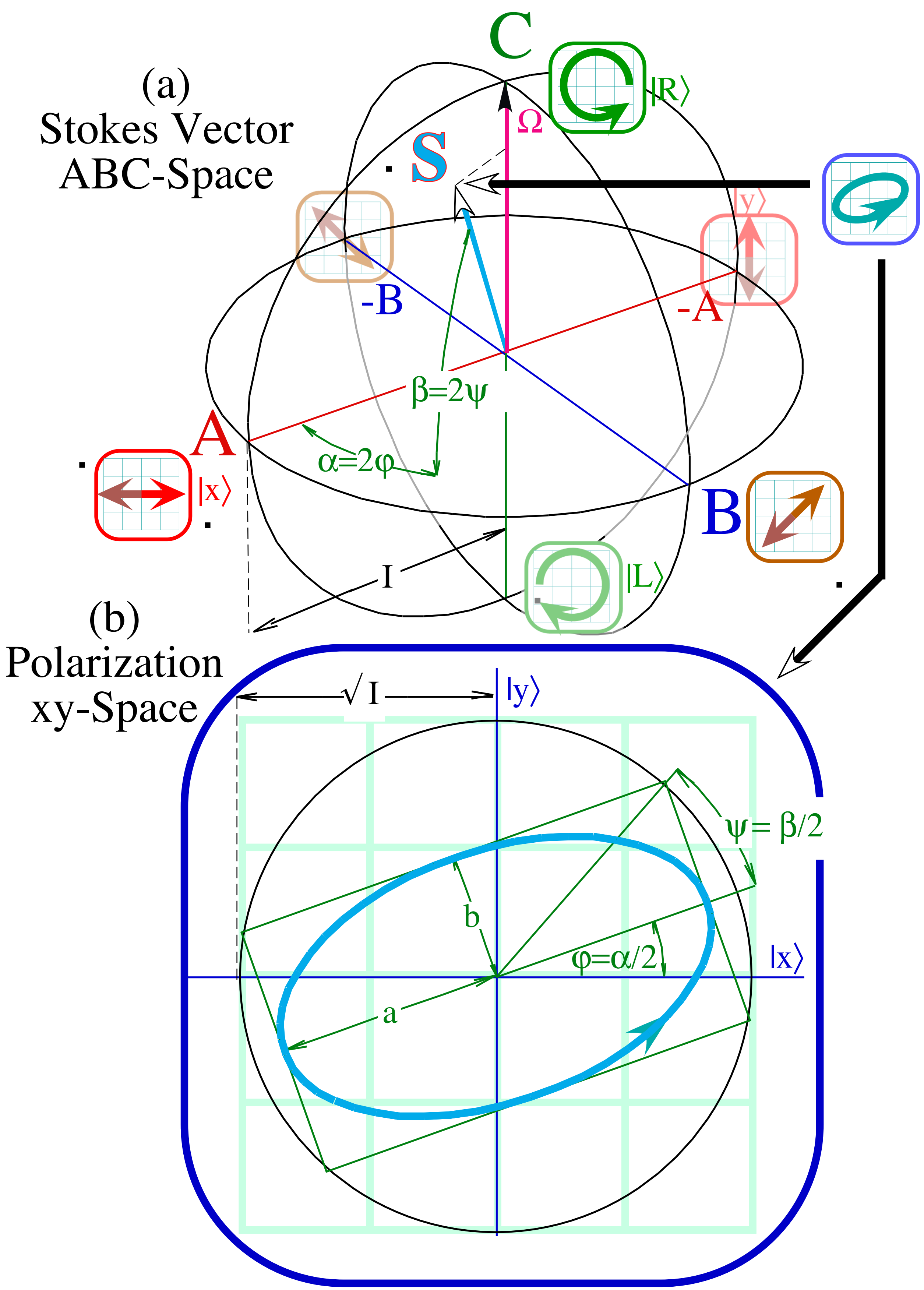


Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Polarization ellipse and spinor state dynamics



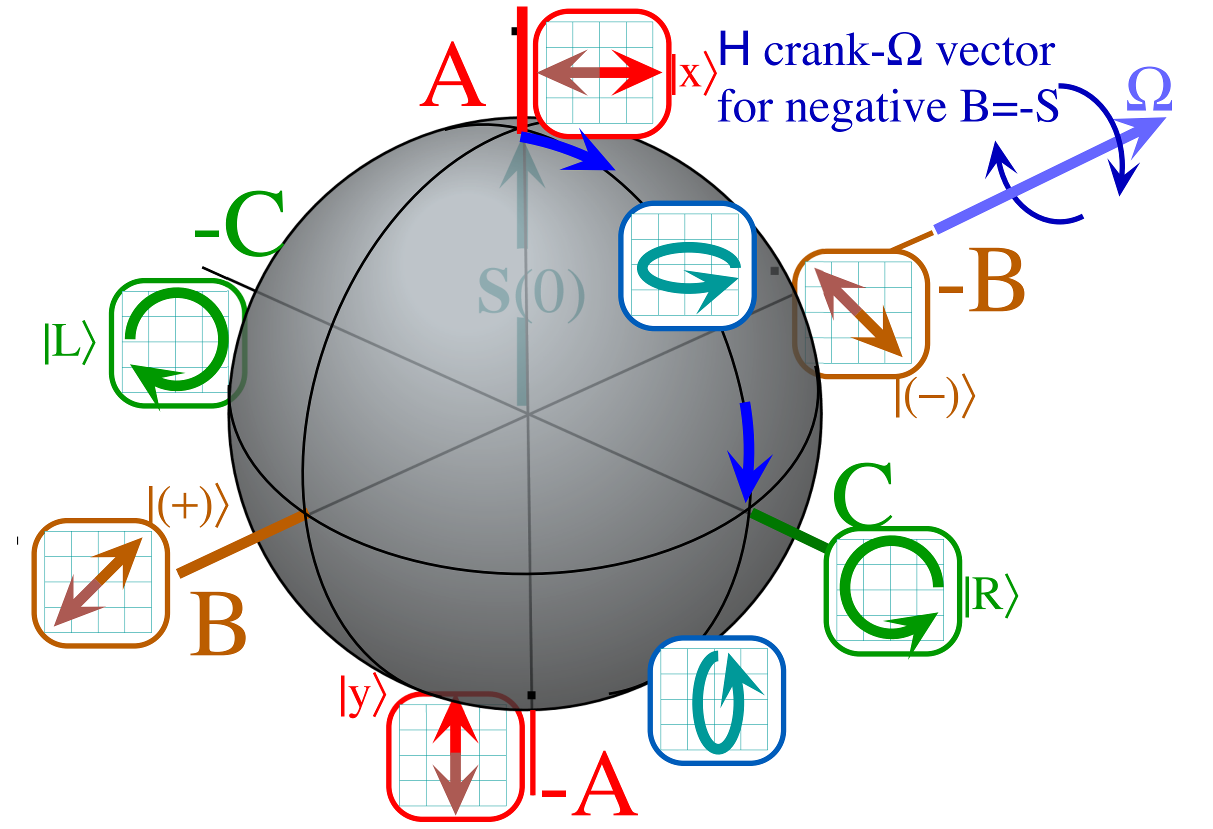


Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Fig. 10.5.6 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*

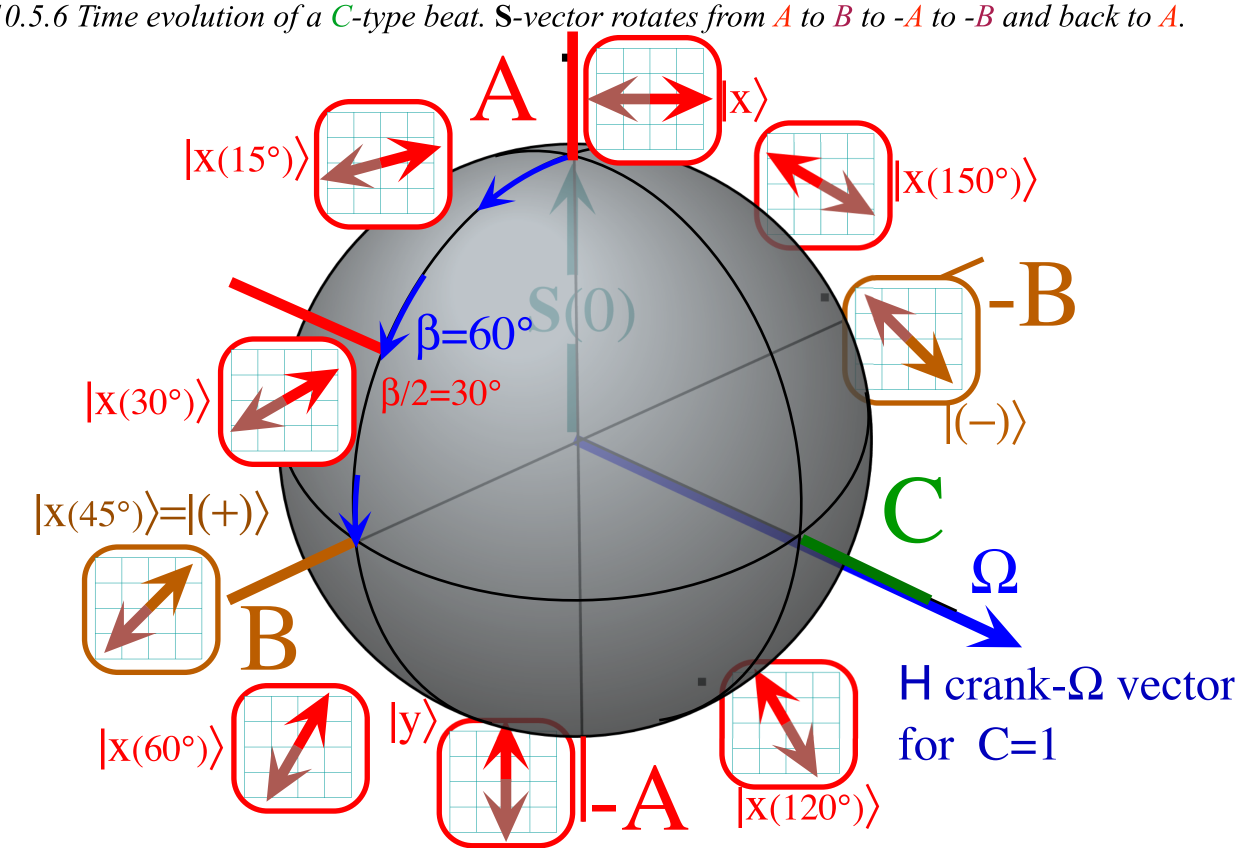
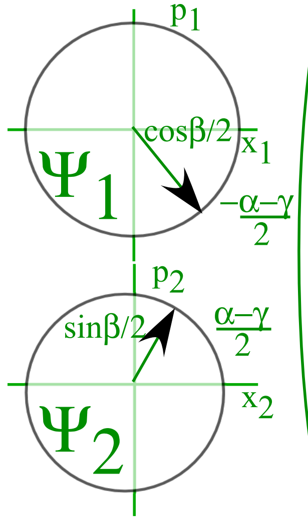


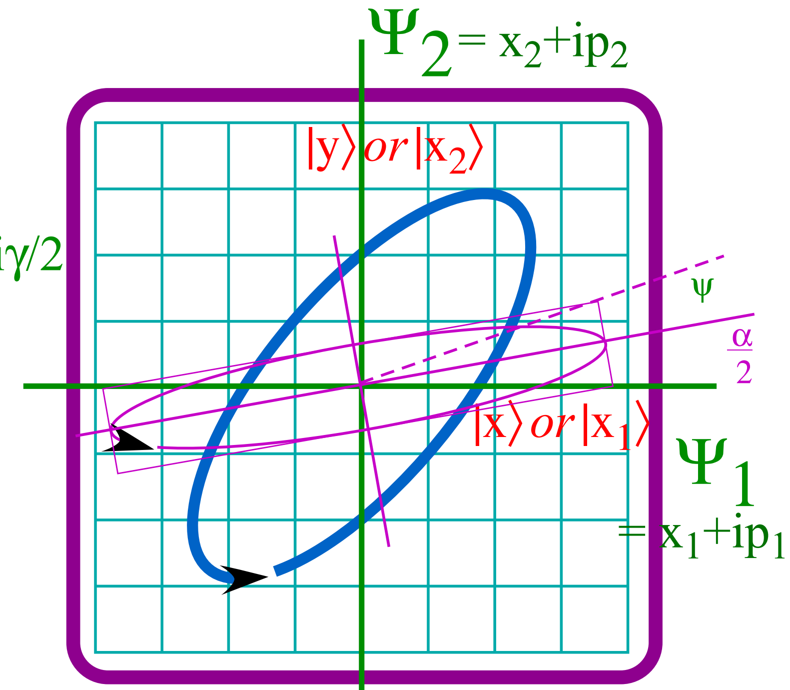
Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

U(2) World : Complex 2D Spinors

2-State ket $|\Psi\rangle =$



$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2\\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

|Ψ⟩ State

Spin Vector
S

$$\begin{pmatrix}s_B\\ s_C\\ s_A\end{pmatrix} = \begin{pmatrix}N\sin\beta\cos\alpha\\ N\sin\beta\sin\alpha\\ N\cos\beta\end{pmatrix}\frac{1}{2}$$

(for α=15° β=45°)

H-Operator $\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}$

Angular velocity

$$\Omega = \begin{pmatrix}\Omega_B\\ \Omega_C\\ \Omega_A\end{pmatrix} = \begin{pmatrix}2B\\ 2C\\ A-D\end{pmatrix} = \begin{pmatrix}\Omega\sin\vartheta\cos\varphi\\ \Omega\sin\vartheta\sin\varphi\\ \Omega\cos\vartheta\end{pmatrix}$$