

Group Theory in Quantum Mechanics

Lecture 6 (2.02.17)

Spectral Decomposition of Bi-Cyclic ($C_2 \subset U(2)$) Operators

(Quantum Theory for Computer Age - Ch. 7-9 of Unit 3)

(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 2)

Review: How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

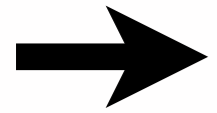
C_2 Symmetric 2DHO uncoupling and mixed mode projector algebra

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

*ANALOGY: $U(2)$ vs $R(3)$: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ vs. **Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$***

*Hamilton-Pauli spinor symmetry (σ -expansion in **ABCD**-Types) $\mathbf{H} = \omega_\mu \sigma_\mu$*



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Review: How symmetry groups become eigen-solvers

Suppose you need to diagonalize a complicated operator \mathbf{K} and knew that \mathbf{K} commutes with some other operators \mathbf{G} and \mathbf{H} for which irreducible projectors are more easily found.

$$\begin{aligned} \mathbf{KG} = \mathbf{GK} \text{ or } \mathbf{G}^\dagger \mathbf{KG} = \mathbf{K} \text{ or } \mathbf{GKG}^\dagger = \mathbf{K} & \quad (\text{Here assuming } \textit{unitary} \\ \mathbf{KH} = \mathbf{HK} \text{ or } \mathbf{H}^\dagger \mathbf{KH} = \mathbf{K} \text{ or } \mathbf{HKH}^\dagger = \mathbf{K} & \quad \mathbf{G}^\dagger = \mathbf{G}^{-1} \text{ and } \mathbf{H}^\dagger = \mathbf{H}^{-1}.) \end{aligned}$$

This means \mathbf{K} is *invariant* to the transformation by \mathbf{G} and \mathbf{H} and all their products \mathbf{GH} , \mathbf{GH}^2 , $\mathbf{G}^2\mathbf{H}$,... *etc.* and all their inverses \mathbf{G}^\dagger , \mathbf{H}^\dagger ,... *etc.*

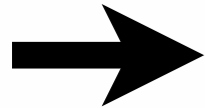
The group $\mathcal{G}_{\mathbf{K}} = \{\mathbf{1}, \mathbf{G}, \mathbf{H}, \dots\}$ so formed by such operators is called a *symmetry group* for \mathbf{K} .

In certain ideal cases a \mathbf{K} -matrix $\langle \mathbf{K} \rangle$ is a linear combination of matrices $\langle \mathbf{1} \rangle, \langle \mathbf{G} \rangle, \langle \mathbf{H} \rangle, \dots$ from $\mathcal{G}_{\mathbf{K}}$. Then spectral resolution of $\{\langle \mathbf{1} \rangle, \langle \mathbf{G} \rangle, \langle \mathbf{H} \rangle, \dots\}$ also resolves $\langle \mathbf{K} \rangle$.

We will study ideal cases first. More general cases are built from this idea.

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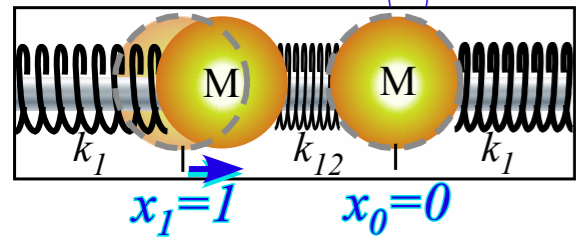
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2D HO “binary” bases and coord. $\{x_0, x_1\}$

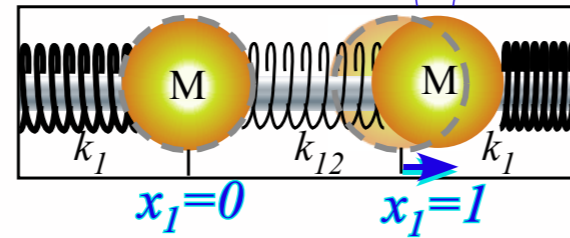
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$$|0\rangle = |x\rangle = |2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

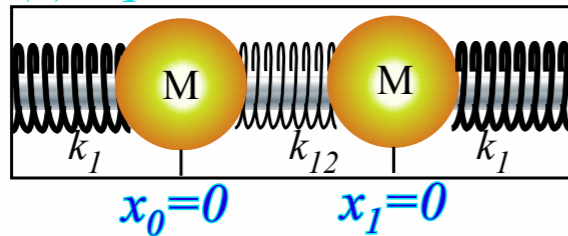


(b) unit base state

$$|1\rangle = |y\rangle = |-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



2D HO Matrix operator equations

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} \frac{k_1 + k_{12}}{M} & \frac{-k_{12}}{M} \\ \frac{-k_{12}}{M} & \frac{k_1 + k_{12}}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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To review examples of motion:

[BoxIt Simulation: 2 Uncoupled Oscillators](#)

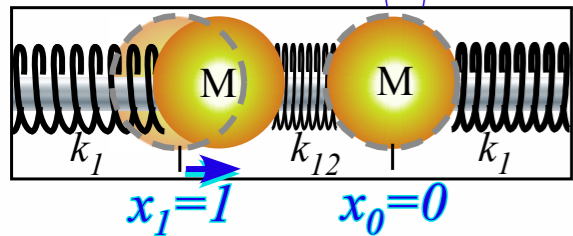
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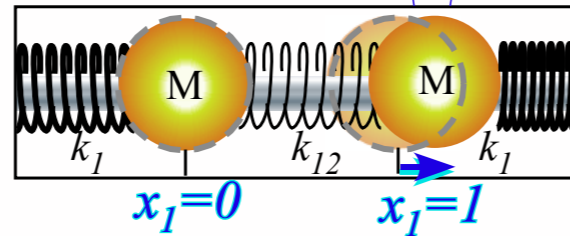
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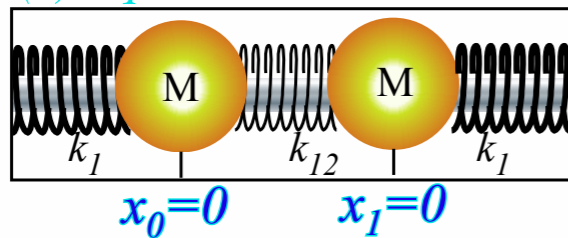


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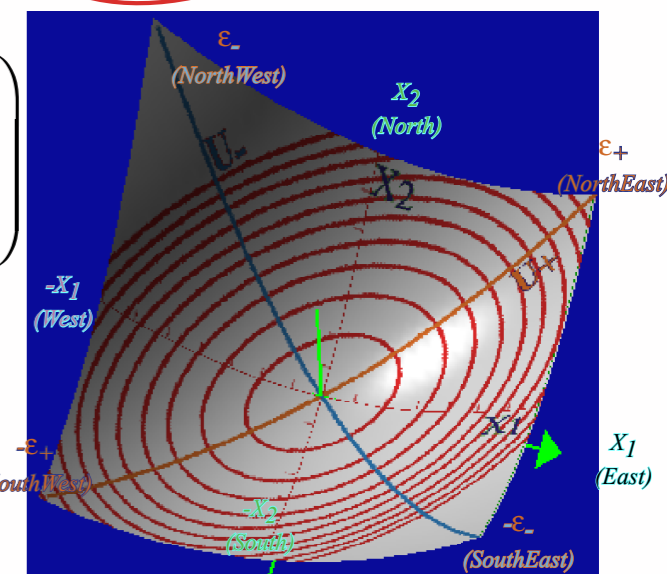
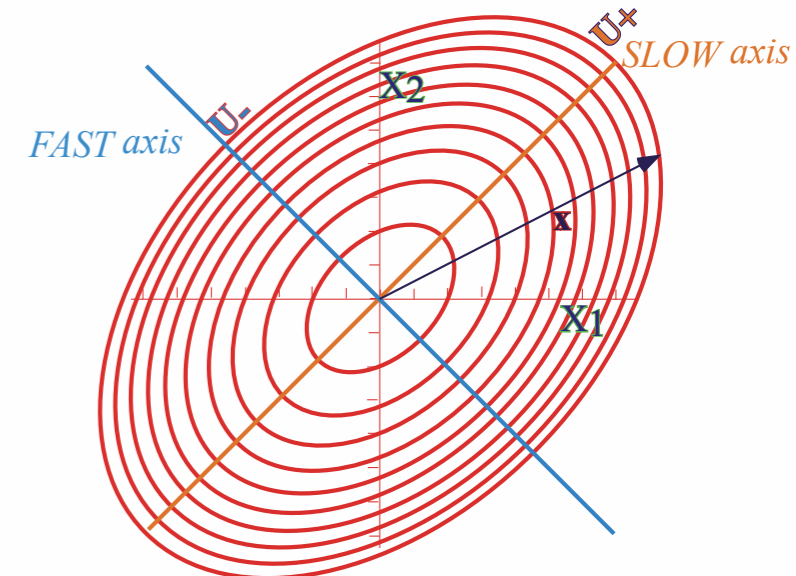
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(a) PE Contours



2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = - \frac{\partial V}{\partial x_1} = - (k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = - \frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Lagrange equations

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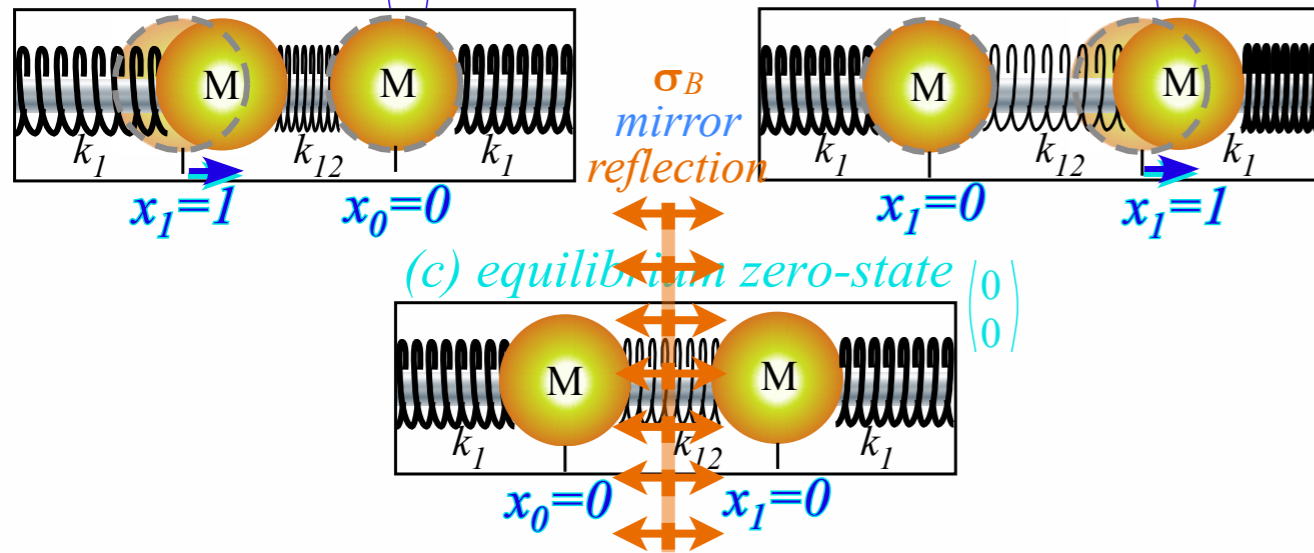
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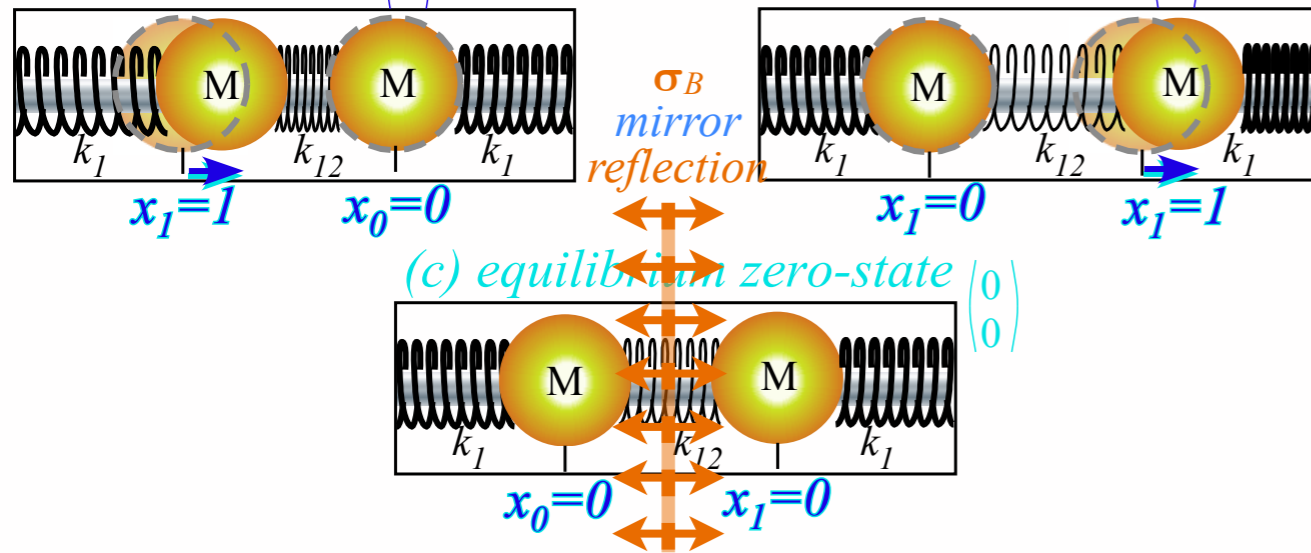
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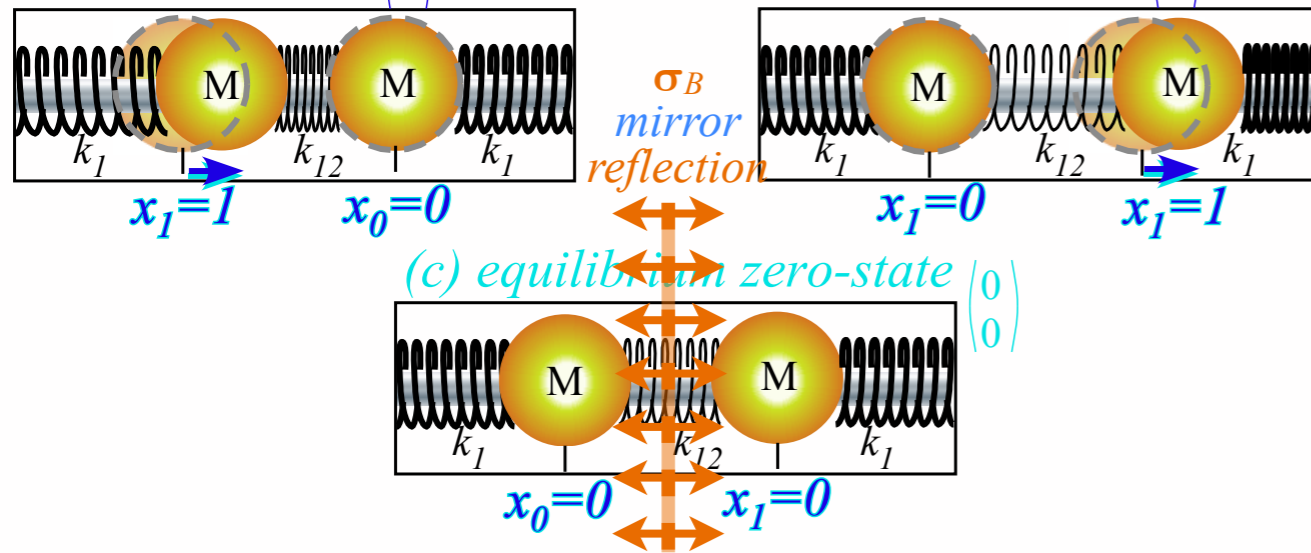
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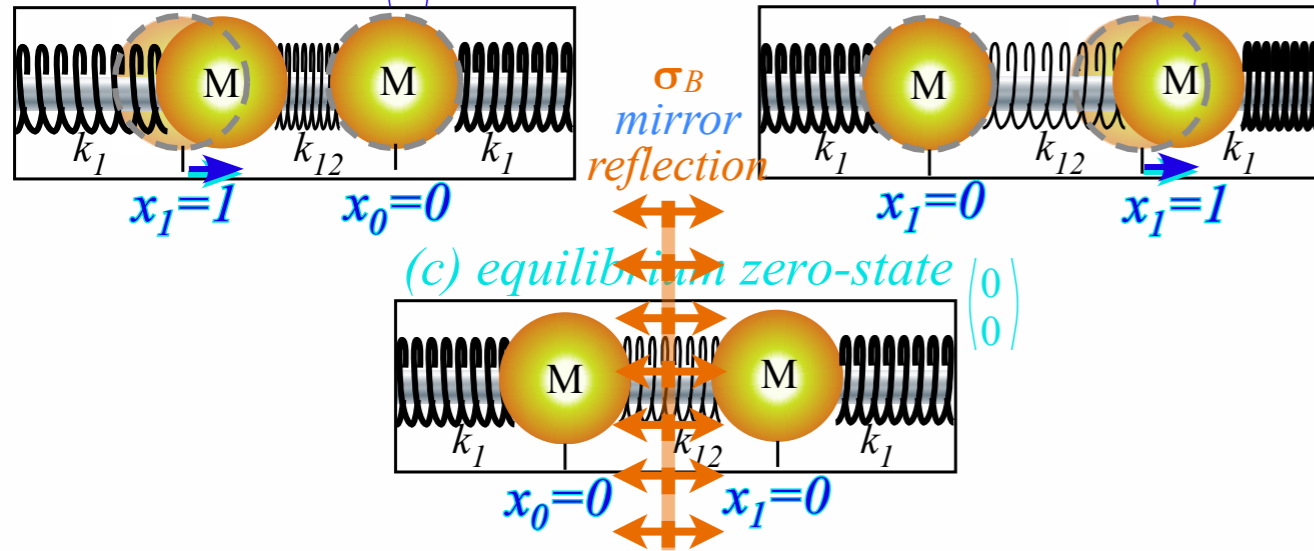
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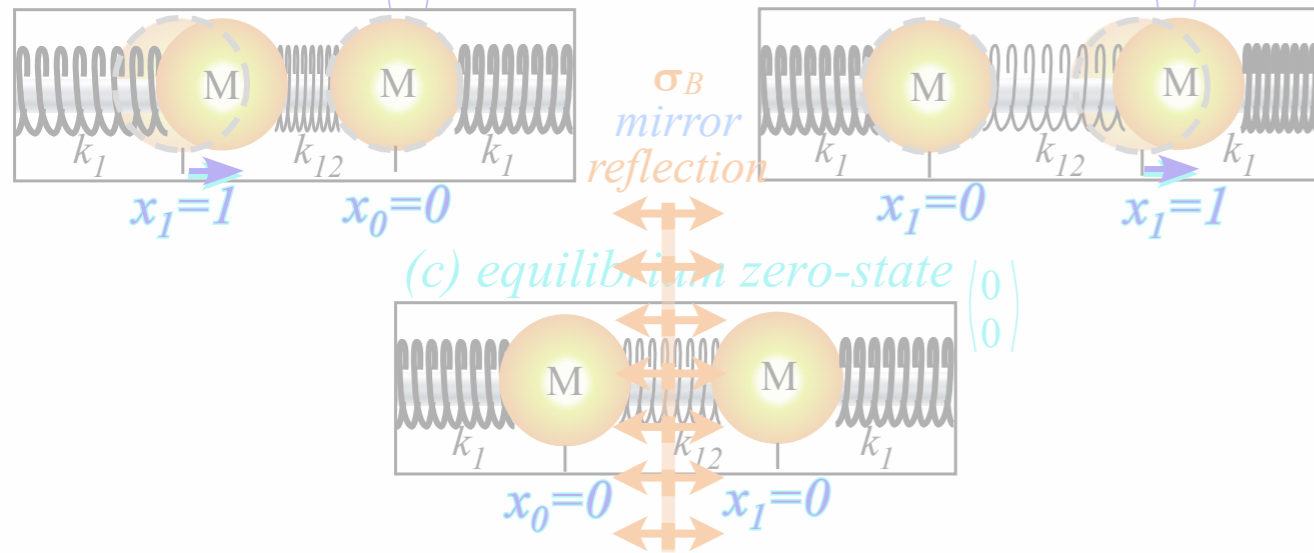
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Minimal equation of σ_B is: $\sigma_B^2 = \mathbf{1}$

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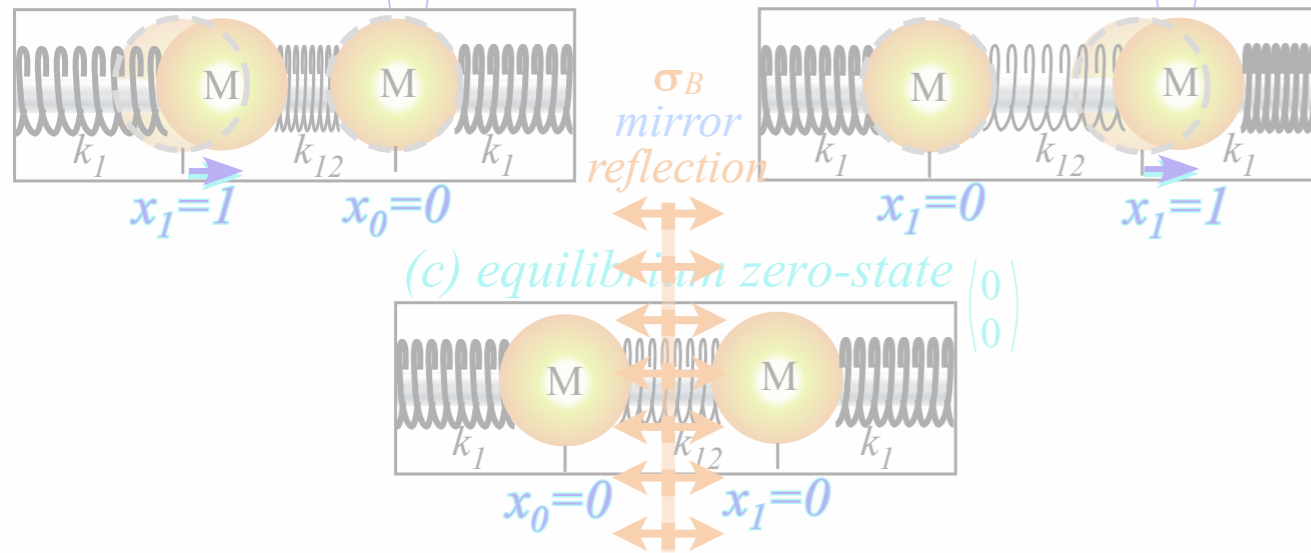
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C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

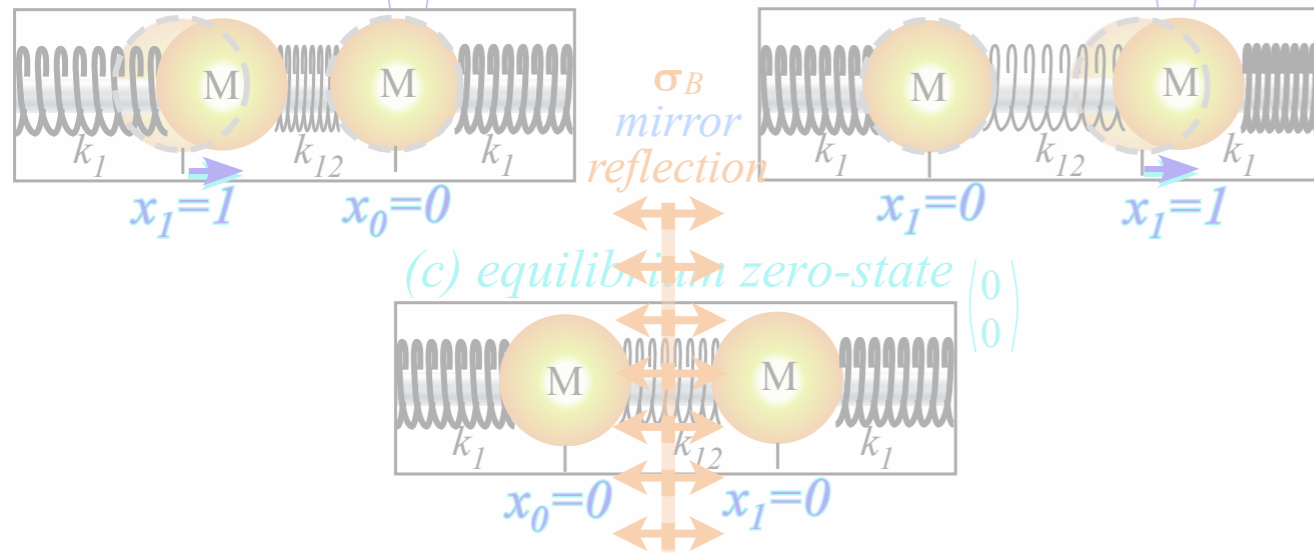
2D HO "binary" bases and coord. $\{x_0, x_1\}$

(a) unit base state

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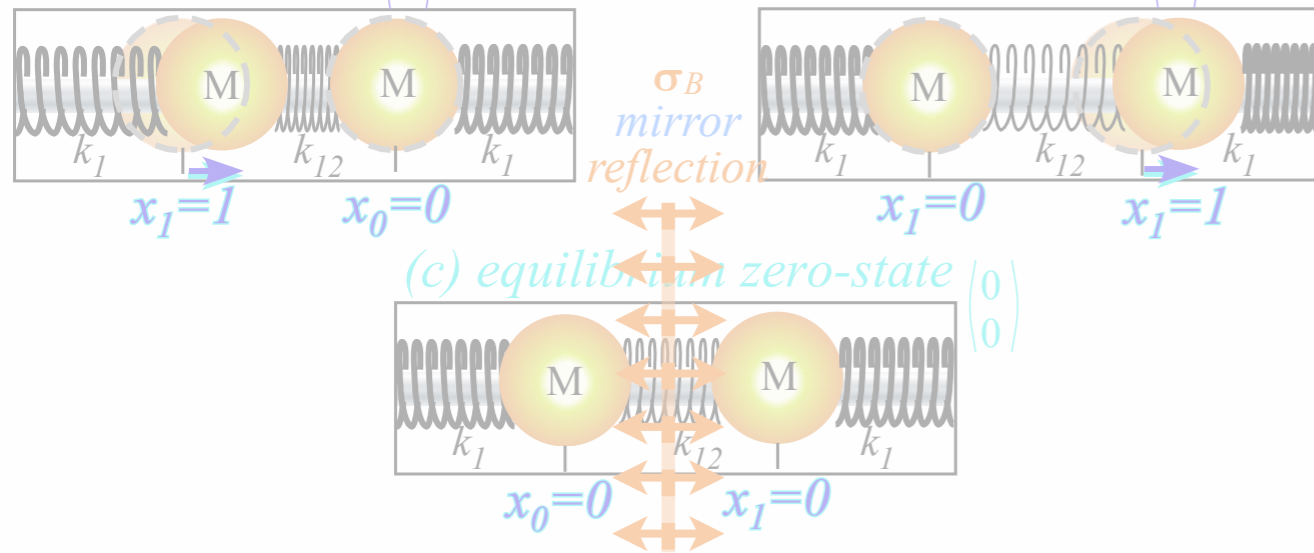
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Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+, \mathbf{P}^-\}$

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C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

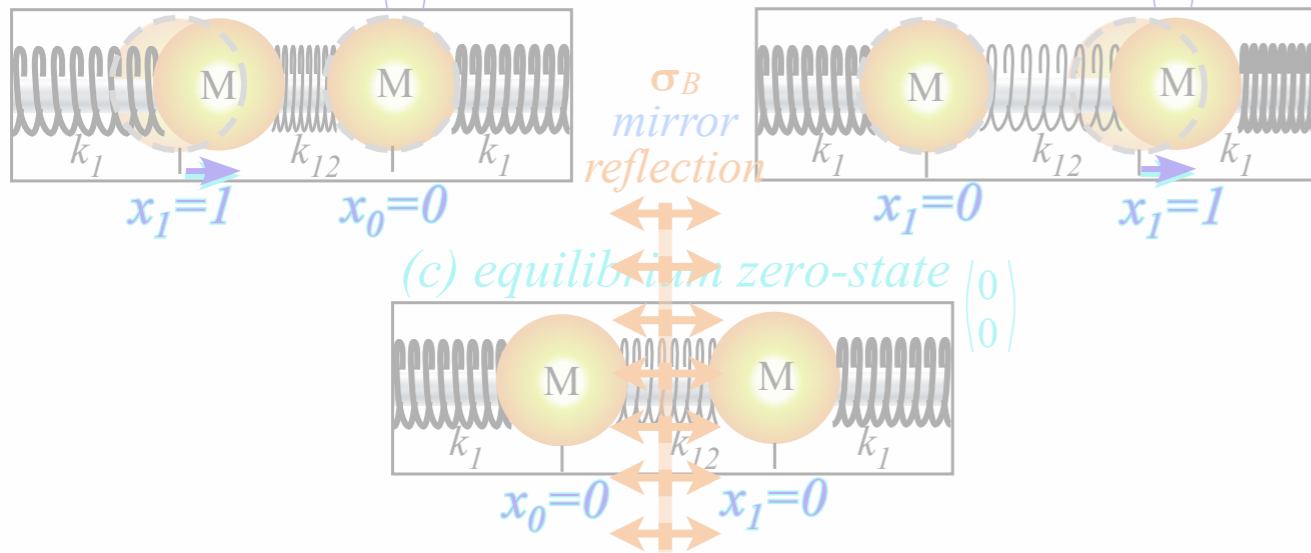
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\mathbf{P}^\pm -projectors:

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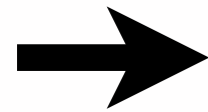
Review: How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$



C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$

Hamilton-Pauli spinor symmetry (σ -expansion in $ABCD$ -Types) $\mathbf{H} = \omega_\mu\sigma_\mu$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$$

$$\begin{pmatrix} K & k \\ k & K \end{pmatrix} = K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

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factored projectors

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Diagonalizing transformation (D-tran) of K -matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

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$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$$

$$K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

K -matrix is made of its symmetry operators

in group $C_2 = \{\mathbf{1}, \sigma_B\}$ with product table:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1, \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$:

$$\begin{aligned} \varepsilon^+(\mathbf{K}) &= K - k_{12}, & \varepsilon^-(\mathbf{K}) &= K + k_{12} \\ &= k_1 & &= k_1 + 2k_{12} \end{aligned}$$

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle +|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle -|$$

Diagonalizing transformation (D-tran) of K -matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

(D-tran is its own inverse in this case!)

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$$

K -matrix is made of its symmetry operators

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$$K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

in group $C_2 = \{\mathbf{1}, \sigma_B\}$ with product table:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1, \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$:

$$\begin{aligned} \varepsilon^+(\mathbf{K}) &= K - k_{12}, & \varepsilon^-(\mathbf{K}) &= K + k_{12} \\ &= k_1 & &= k_1 + 2k_{12} \end{aligned}$$

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle +|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle -|$$

Diagonalizing transformation (D-tran) of K -matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

(D-tran is its own inverse in this case!)

$$\begin{pmatrix} \langle + | x_1 \rangle & \langle + | x_2 \rangle \\ \langle - | x_1 \rangle & \langle - | x_2 \rangle \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & k_{12} \\ k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

$$\begin{pmatrix} \langle + | x_1 \rangle & \langle + | x_2 \rangle \\ \langle - | x_1 \rangle & \langle - | x_2 \rangle \end{pmatrix} \begin{pmatrix} \langle x_1 | \mathbf{K} | x_1 \rangle & \langle x_1 | \mathbf{K} | x_2 \rangle \\ \langle x_2 | \mathbf{K} | x_1 \rangle & \langle x_2 | \mathbf{K} | x_2 \rangle \end{pmatrix} \begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix} = \begin{pmatrix} \langle + | \mathbf{K} | + \rangle & \langle + | \mathbf{K} | - \rangle \\ \langle - | \mathbf{K} | + \rangle & \langle - | \mathbf{K} | - \rangle \end{pmatrix}$$

Full Dirac notation

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$$

K -matrix is made of its symmetry operators

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$$K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

in group $C_2 = \{\mathbf{1}, \sigma_B\}$ with product table:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1, \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$:

$$\begin{aligned} \varepsilon^+(\mathbf{K}) &= K - k_{12}, & \varepsilon^-(\mathbf{K}) &= K + k_{12} \\ &= k_1 & &= k_1 + 2k_{12} \end{aligned}$$

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle +|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle -|$$

Diagonalizing transformation (D-tran) of K -matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

(D-tran is its own inverse in this case!)

$$\begin{pmatrix} \langle + | x_1 \rangle & \langle + | x_2 \rangle \\ \langle - | x_1 \rangle & \langle - | x_2 \rangle \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & k_{12} \\ k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

$$\begin{pmatrix} \langle + | x_1 \rangle & \langle + | x_2 \rangle \\ \langle - | x_1 \rangle & \langle - | x_2 \rangle \end{pmatrix} \begin{pmatrix} \langle x_1 | \mathbf{K} | x_1 \rangle & \langle x_1 | \mathbf{K} | x_2 \rangle \\ \langle x_2 | \mathbf{K} | x_1 \rangle & \langle x_2 | \mathbf{K} | x_2 \rangle \end{pmatrix} \begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix} = \begin{pmatrix} \langle + | \mathbf{K} | + \rangle & \langle + | \mathbf{K} | - \rangle \\ \langle - | \mathbf{K} | + \rangle & \langle - | \mathbf{K} | - \rangle \end{pmatrix}$$

Full Dirac notation

$$\mathbf{T}(\pm \leftarrow x_j) \cdot \mathbf{K} \cdot \mathbf{T}^\dagger(\pm \leftarrow x_j)$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$$

K -matrix is made of its symmetry operators

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$$K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

in group $C_2 = \{\mathbf{1}, \sigma_B\}$ with product table:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1, \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$:

$$\begin{aligned} \varepsilon^+(\mathbf{K}) &= K - k_{12}, & \varepsilon^-(\mathbf{K}) &= K + k_{12} \\ &= k_1 & &= k_1 + 2k_{12} \end{aligned}$$

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle +|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle -|$$

Diagonalizing transformation (D-tran) of \mathbf{K} -matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

(D-tran is its own inverse in this case!)

$$\begin{pmatrix} \langle + | x_1 \rangle & \langle + | x_2 \rangle \\ \langle - | x_1 \rangle & \langle - | x_2 \rangle \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & k_{12} \\ k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

$$\begin{pmatrix} \langle + | x_1 \rangle & \langle + | x_2 \rangle \\ \langle - | x_1 \rangle & \langle - | x_2 \rangle \end{pmatrix} \begin{pmatrix} \langle x_1 | \mathbf{K} | x_1 \rangle & \langle x_1 | \mathbf{K} | x_2 \rangle \\ \langle x_2 | \mathbf{K} | x_1 \rangle & \langle x_2 | \mathbf{K} | x_2 \rangle \end{pmatrix} \begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix} = \begin{pmatrix} \langle + | \mathbf{K} | + \rangle & \langle + | \mathbf{K} | - \rangle \\ \langle - | \mathbf{K} | + \rangle & \langle - | \mathbf{K} | - \rangle \end{pmatrix}$$

Full Dirac notation

$$\mathbf{T}(\pm \leftarrow x_j) |x_1\rangle = |+\rangle$$

$$\mathbf{T}(\pm \leftarrow x_j) |x_2\rangle = |-\rangle$$

$$\mathbf{T}(\pm \leftarrow x_j) \cdot \mathbf{K} \cdot \mathbf{T}^\dagger(\pm \leftarrow x_j)$$

$$|x_1\rangle = \mathbf{T}^\dagger(\pm \leftarrow x_j) |+\rangle$$

$$|x_2\rangle = \mathbf{T}^\dagger(\pm \leftarrow x_j) |-\rangle$$

$$\mathbf{T}^\dagger(\pm \leftarrow x_j) = \mathbf{T}(x_j \leftarrow \pm)$$

Review: How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions



C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$

Hamilton-Pauli spinor symmetry (σ -expansion in ABCD-Types) $\mathbf{H} = \omega_\mu\sigma_\mu$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$$

K -matrix is made of its symmetry operators

$$K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

in group $C_2 = \{\mathbf{1}, \sigma_B\}$ with product table:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle +|$$

factored projectors

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1, \chi^-(\sigma_B) = -1\}$$

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle -|$$

Diagonalizing transformation (D-tran) of K -matrix:

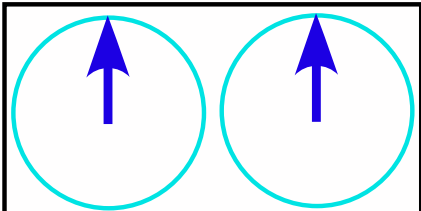
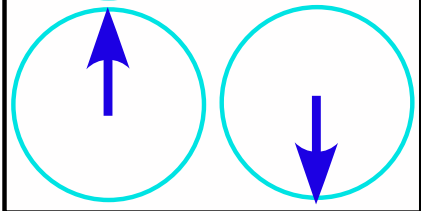
Eigenvalues of $\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$:

$$\begin{aligned} \varepsilon^+(\mathbf{K}) &= K - k_{12}, & \varepsilon^-(\mathbf{K}) &= K + k_{12} \\ &= k_1, & &= k_1 + 2k_{12} \end{aligned}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

C_2 mode phase character tables

p is position
 $p=0$ $p=1$

$m=0$	$\frac{1}{2}$	$\frac{1}{2}$	
$m=1$	$\frac{1}{2}$	$-\frac{1}{2}$	

m is wave-number
 or "momentum"

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

norm: $1/\sqrt{2}$

(D-tran is its own inverse in this case!)

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$$

K -matrix is made of its symmetry operators

$$K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

in group $C_2 = \{\mathbf{1}, \sigma_B\}$ with product table:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle +|$$

factored projectors

Eigenvalues of σ_B :

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\{\chi^+(\sigma_B) = +1, \chi^-(\sigma_B) = -1\}$$

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle -|$$

Diagonalizing transformation (D-tran) of K -matrix:

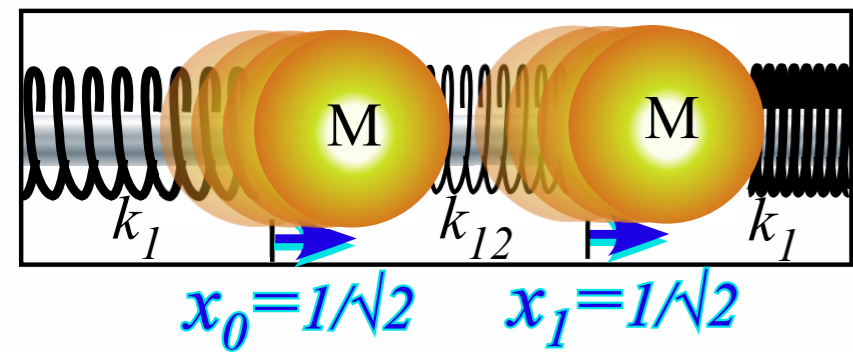
Eigenvalues of $\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$:

$$\varepsilon^+(\mathbf{K}) = K - k_{12}, \quad \varepsilon^-(\mathbf{K}) = K + k_{12}$$

$$= k_1 \qquad \qquad \qquad = k_1 + 2k_{12}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

(a) Even mode $|+\rangle = |0_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2}$



C_2 mode phase character tables

p is position
 $p=0 \quad p=1$

$m=0$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	
$m=1$	$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$	

norm: $1/\sqrt{2}$

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

(D-tran is its own inverse in this case!)

m is wave-number or "momentum"

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$$

K -matrix is made of its symmetry operators

$$K \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

in group $C_2 = \{\mathbf{1}, \sigma_B\}$ with product table:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1, \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K} = K \cdot \mathbf{1} - k_{12} \cdot \sigma_B$:

$$\begin{aligned} \varepsilon^+(\mathbf{K}) &= K - k_{12}, & \varepsilon^-(\mathbf{K}) &= K + k_{12} \\ &= k_1 & &= k_1 + 2k_{12} \end{aligned}$$

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle +|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle -|$$

Diagonalizing transformation (D-tran) of K -matrix:

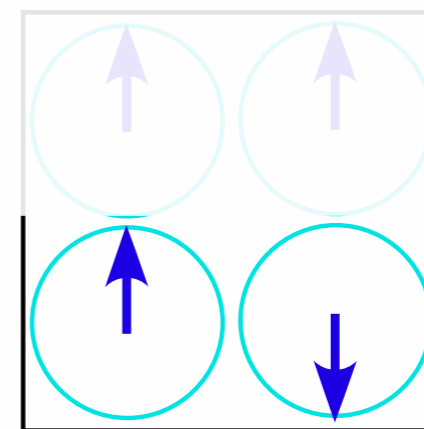
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$

C_2 mode phase character tables

p is position
 $p=0$ $p=1$

$m=0$

1	1
1	-1



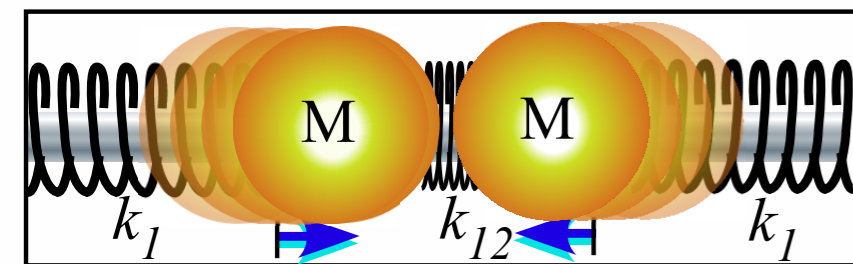
norm:
 $1/\sqrt{2}$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

(D-tran is its own inverse in this case!)

(b) Odd mode $|-\rangle = |1_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix} / \sqrt{2}$



$$x_0 = 1/\sqrt{2} \quad x_1 = -1/\sqrt{2}$$

m is wave-number or "momentum"

C_2 Symmetric 2DHO eigensolutions

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$$\begin{aligned} \varepsilon^+(\mathbf{K}) &= K - k_{12}, & \varepsilon^-(\mathbf{K}) &= K + k_{12} \\ &= k_1 & &= k_1 + 2k_{12} \end{aligned}$$

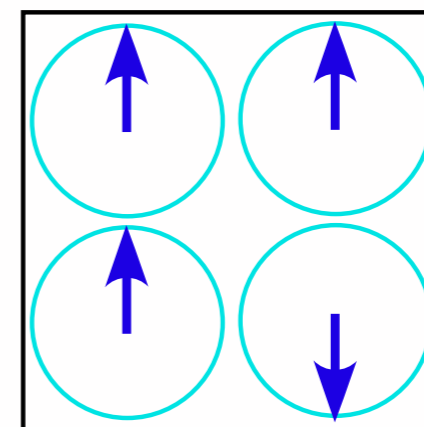
Even mode $|+\rangle = |0_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2}$

C_2 mode phase character tables

p is position
 $p=0$ $p=1$

$m=0$

$m=0$	1	1
$m=1$	1	-1



norm: $1/\sqrt{2}$

(D-tran)

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \langle x_1 | + \rangle & \langle x_1 | - \rangle \\ \langle x_2 | + \rangle & \langle x_2 | - \rangle \end{pmatrix}$$

(D-tran is its own inverse in this case!)

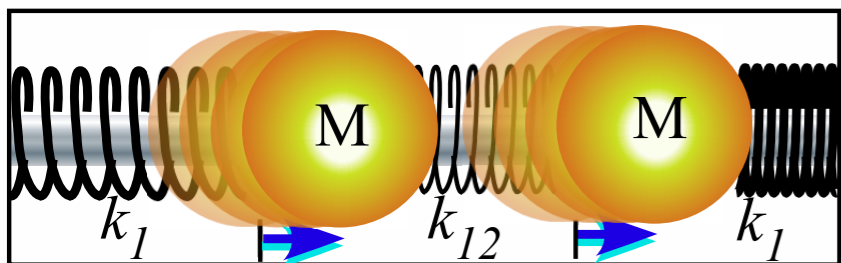
$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle \langle +|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = |-\rangle \langle -|$$

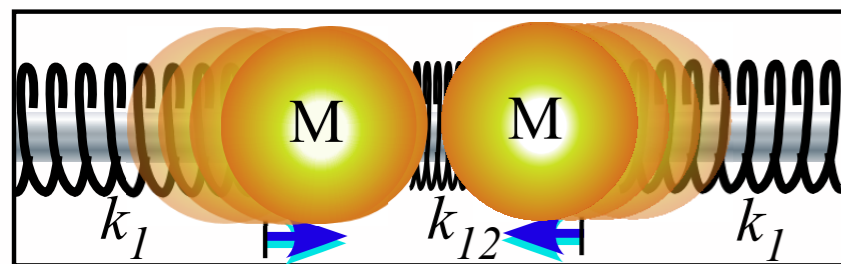
Diagonalizing transformation (D-tran) of K -matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix}$$



$$x_0 = 1/\sqrt{2} \quad x_1 = 1/\sqrt{2}$$

Odd mode $|-\rangle = |1_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix} / \sqrt{2}$



$$x_0 = 1/\sqrt{2} \quad x_1 = -1/\sqrt{2}$$

m is wave-number or "momentum"

Review: How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

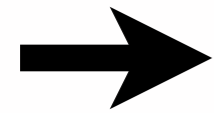
C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table



C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$

Hamilton-Pauli spinor symmetry (σ -expansion in ABCD-Types) $\mathbf{H} = \omega_\mu\sigma_\mu$

C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1, x_2\}$ -basis

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$|\ddot{\mathbf{x}}\rangle = - \mathbf{K} |\mathbf{x}\rangle$$
$$\begin{pmatrix} \langle x_1 | \ddot{\mathbf{x}} \rangle \\ \langle x_2 | \ddot{\mathbf{x}} \rangle \end{pmatrix} = - \begin{pmatrix} \langle x_1 | \mathbf{K} | x_1 \rangle & \langle x_1 | \mathbf{K} | x_2 \rangle \\ \langle x_2 | \mathbf{K} | x_1 \rangle & \langle x_2 | \mathbf{K} | x_2 \rangle \end{pmatrix} \begin{pmatrix} \langle x_1 | \mathbf{x} \rangle \\ \langle x_2 | \mathbf{x} \rangle \end{pmatrix}$$

C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1, x_2\}$ -basis ...but are **uncoupled** in $\{+, -\}$ -basis

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$$|\ddot{\mathbf{x}}\rangle = - \mathbf{K} |\mathbf{x}\rangle$$

$$\begin{pmatrix} \langle x_1 | \ddot{\mathbf{x}} \rangle \\ \langle x_2 | \ddot{\mathbf{x}} \rangle \end{pmatrix} = - \begin{pmatrix} \langle x_1 | \mathbf{K} | x_1 \rangle & \langle x_1 | \mathbf{K} | x_2 \rangle \\ \langle x_2 | \mathbf{K} | x_1 \rangle & \langle x_2 | \mathbf{K} | x_2 \rangle \end{pmatrix} \begin{pmatrix} \langle x_1 | \mathbf{x} \rangle \\ \langle x_2 | \mathbf{x} \rangle \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_+ \\ \ddot{x}_- \end{pmatrix} = - \begin{pmatrix} k_1 & 0 \\ 0 & k_1 + 2k_{12} \end{pmatrix} \begin{pmatrix} x_+ \\ x_- \end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = - \mathbf{K} |\mathbf{x}\rangle$$

$$\begin{pmatrix} \langle + | \ddot{\mathbf{x}} \rangle \\ \langle - | \ddot{\mathbf{x}} \rangle \end{pmatrix} = - \begin{pmatrix} \langle + | \mathbf{K} | + \rangle & \langle + | \mathbf{K} | - \rangle \\ \langle - | \mathbf{K} | + \rangle & \langle - | \mathbf{K} | - \rangle \end{pmatrix} \begin{pmatrix} \langle + | \mathbf{x} \rangle \\ \langle - | \mathbf{x} \rangle \end{pmatrix}$$

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Eigenbra vectors: $\langle + | = \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right)$, $\langle - | = \left(\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \right)$

Eigenket vectors: $|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

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C_2 Symmetric 2DHO **uncoupled** dynamics

Eigenbra vectors: $\langle + | = \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right)$, $\langle - | = \left(\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \right)$

$$\begin{pmatrix} M \ddot{x}_+ + (k_1) x_+ \\ M \ddot{x}_- + (k_1 + 2k_{12}) x_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Each mode runs independently

(+)-mode at frequency $\omega_+ = \sqrt{(k_1/M)}$

(-)-mode at frequency $\omega_- = \sqrt{(k_1 + 2k_{12})/M}$

Eigenket vectors: $|+\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $|-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

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Spectral decomposition of initial state $\mathbf{x}(0) = (x_1 \ x_2) = (1, 0)$:

$$\mathbf{1} \cdot \mathbf{x}(0) = (\mathbf{P}_+ + \mathbf{P}_-) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

C_2 Symmetric 2DHO uncoupling

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$$\begin{pmatrix} \langle + | \ddot{\mathbf{x}} \rangle \\ \langle - | \ddot{\mathbf{x}} \rangle \end{pmatrix} = - \begin{pmatrix} \langle + | \mathbf{K} | + \rangle & \langle + | \mathbf{K} | - \rangle \\ \langle - | \mathbf{K} | + \rangle & \langle - | \mathbf{K} | - \rangle \end{pmatrix} \begin{pmatrix} \langle + | \mathbf{x} \rangle \\ \langle - | \mathbf{x} \rangle \end{pmatrix}$$

C_2 Symmetric 2DHO **uncoupled** dynamics

Eigenbra vectors: $\langle + | = \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right)$, $\langle - | = \left(\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}} \right)$

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$$\mathbf{x}(0) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \left(\frac{1}{\sqrt{2}} \right) + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \left(\frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

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C_2 Symmetric 2DHO **uncoupled** dynamics

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(-)-mode at frequency $\omega_- = \sqrt{(k_1 + 2k_{12})/M}$

Spectral decomposition of initial state $\mathbf{x}(0) = (x_1 \ x_2) = (1, 0)$:

$$\mathbf{1} \cdot \mathbf{x}(0) = (\mathbf{P}_+ + \mathbf{P}_-) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{x}(0) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \left(\frac{1}{\sqrt{2}} \right) + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \left(\frac{1}{\sqrt{2}} \right) = \frac{1}{2} |+\rangle + \frac{1}{2} |-\rangle = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\text{so: } \mathbf{x}(t) = e^{-i\omega_+ t} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + e^{-i\omega_- t} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

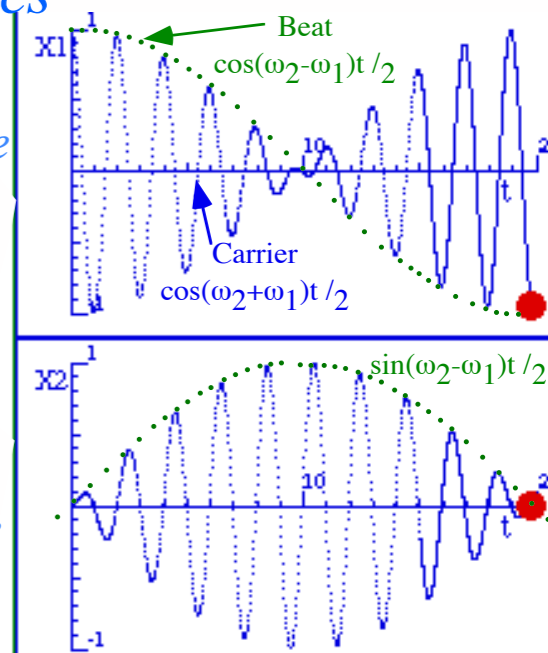
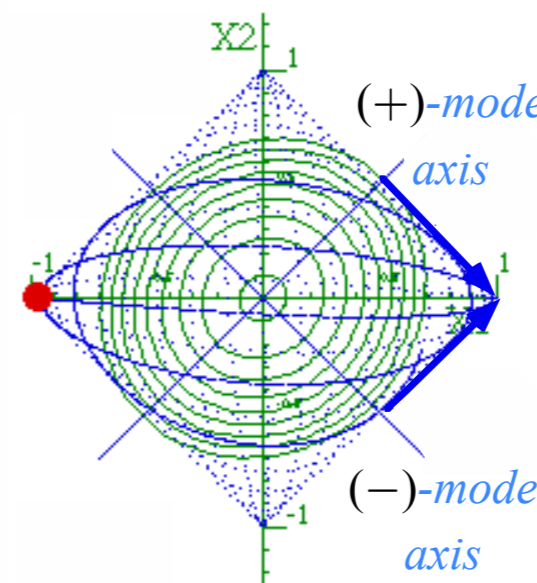
100% AM modulation results

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} = e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2}\right)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{e^{-i\omega_+ t} + e^{-i\omega_- t}}{2} \\ \frac{e^{-i\omega_+ t} - e^{-i\omega_- t}}{2} \end{pmatrix} = \frac{e^{-i\frac{(\omega_+ + \omega_-)t}}{2}}}{2} \begin{pmatrix} e^{-i\frac{(\omega_+ - \omega_-)t}}{2} + e^{i\frac{(\omega_+ - \omega_-)t}}{2} \\ e^{-i\frac{(\omega_+ - \omega_-)t}}{2} - e^{i\frac{(\omega_+ - \omega_-)t}}{2} \end{pmatrix} = e^{-i\frac{(\omega_+ + \omega_-)t}{2}} \begin{pmatrix} \cos\frac{(\omega_- - \omega_+)t}{2} \\ i \sin\frac{(\omega_- - \omega_+)t}{2} \end{pmatrix}$$

Note i phase

Mixed mode dynamics

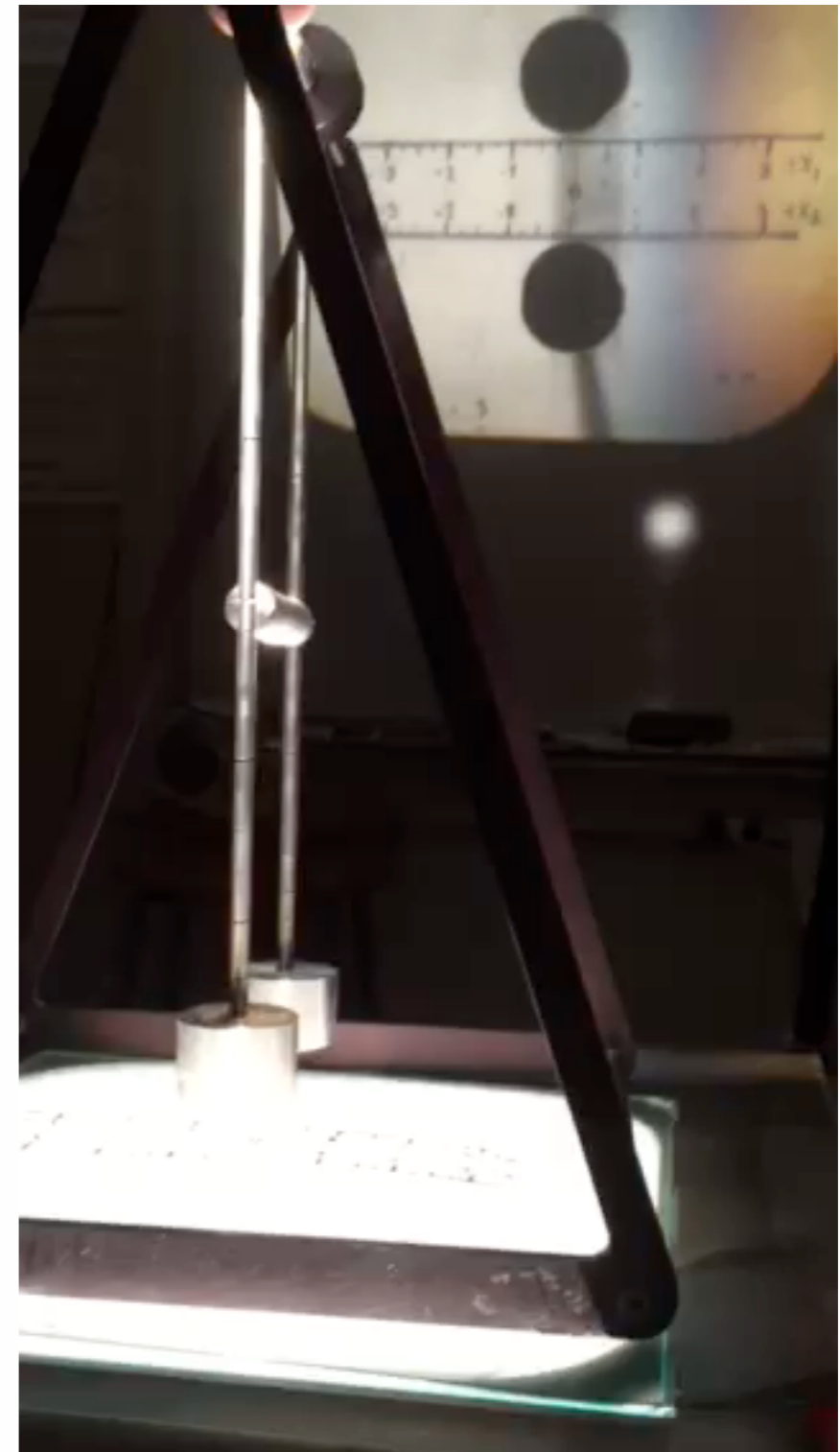


BoxIt (Beating) Simulation

Videos of Coupled Pendula aided by Overhead Projector



[View on YouTube](#) 



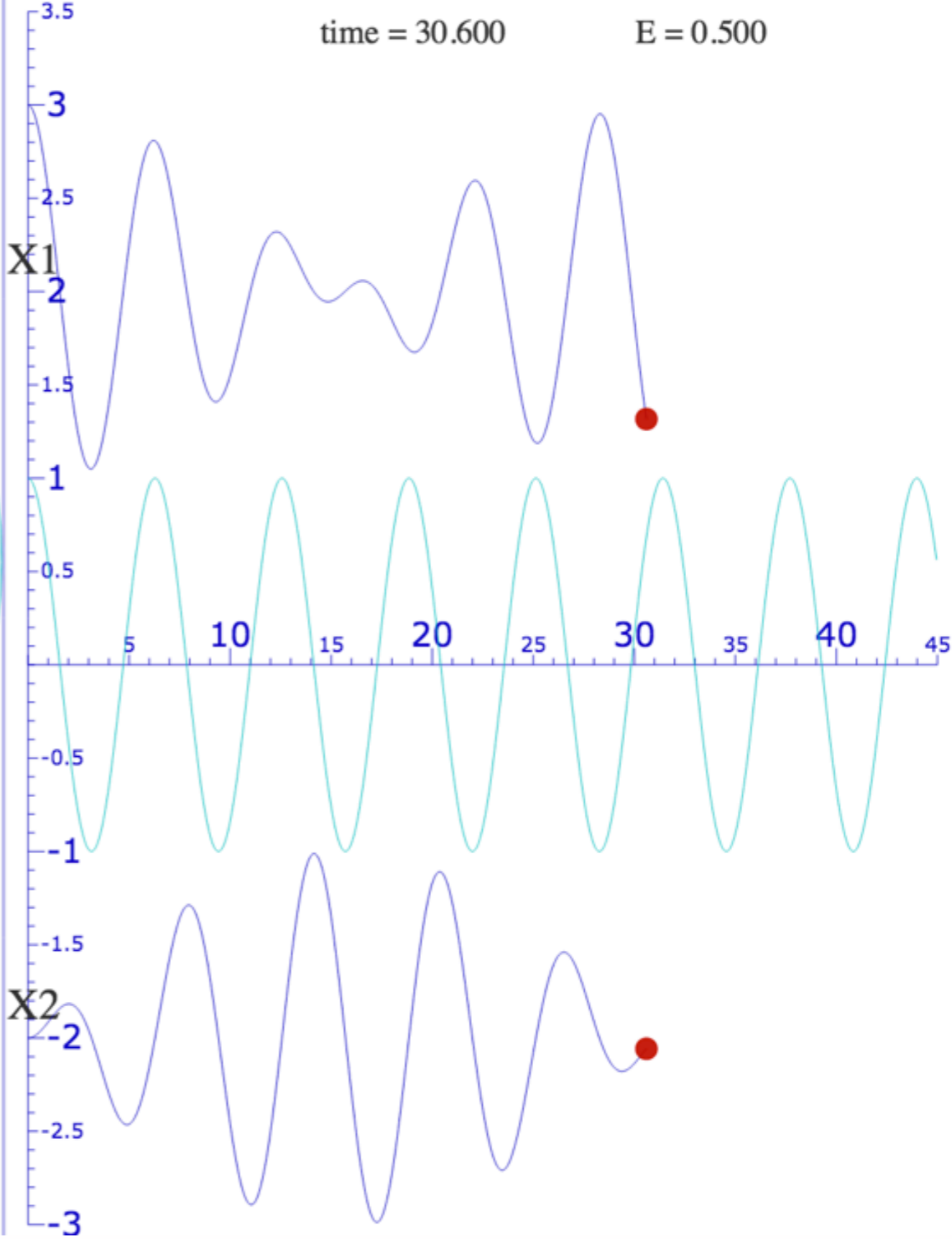
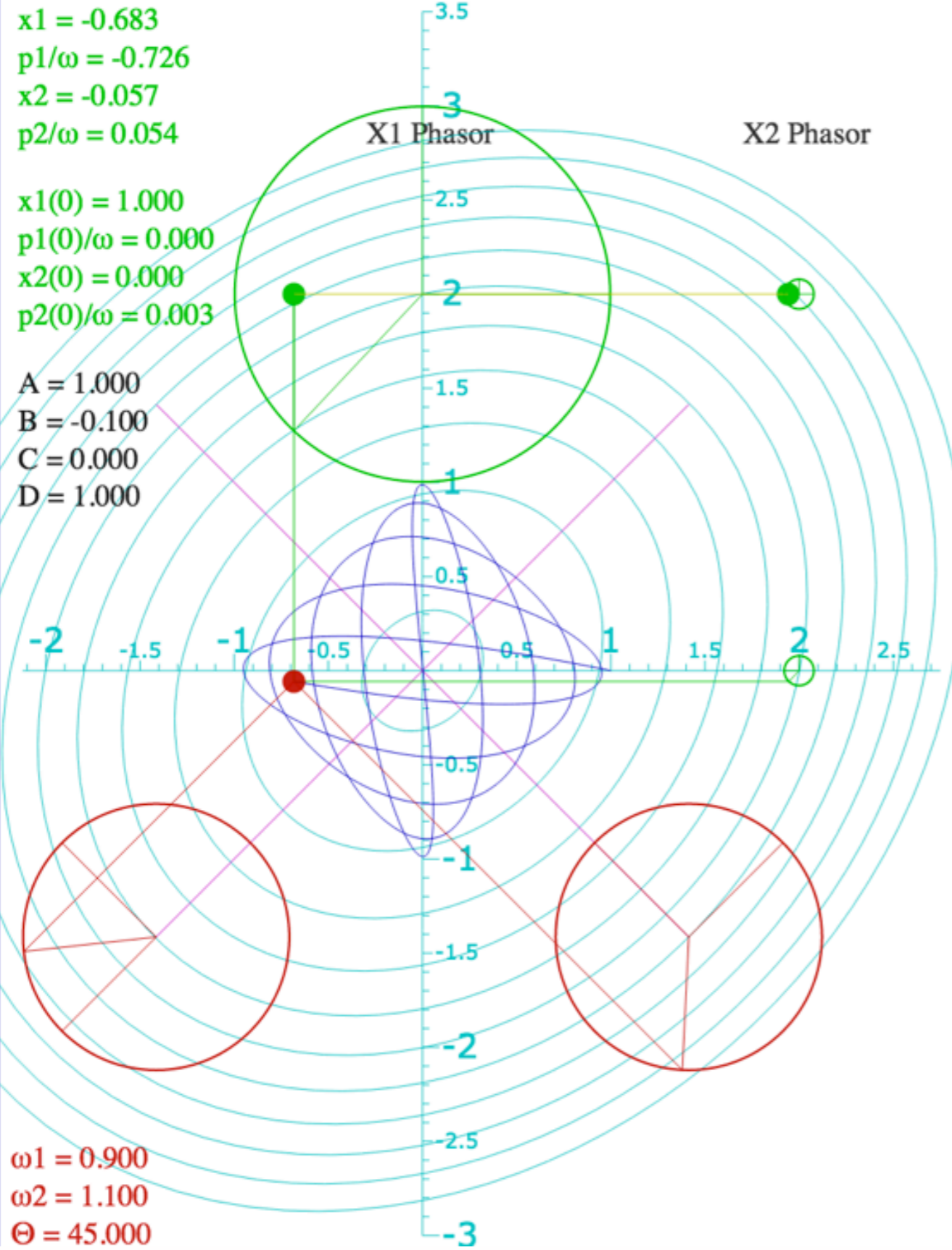
[View on YouTube](#) 

*Launch embedded videos
using your browser/App
or*

⇐ view on YouTube ⇒

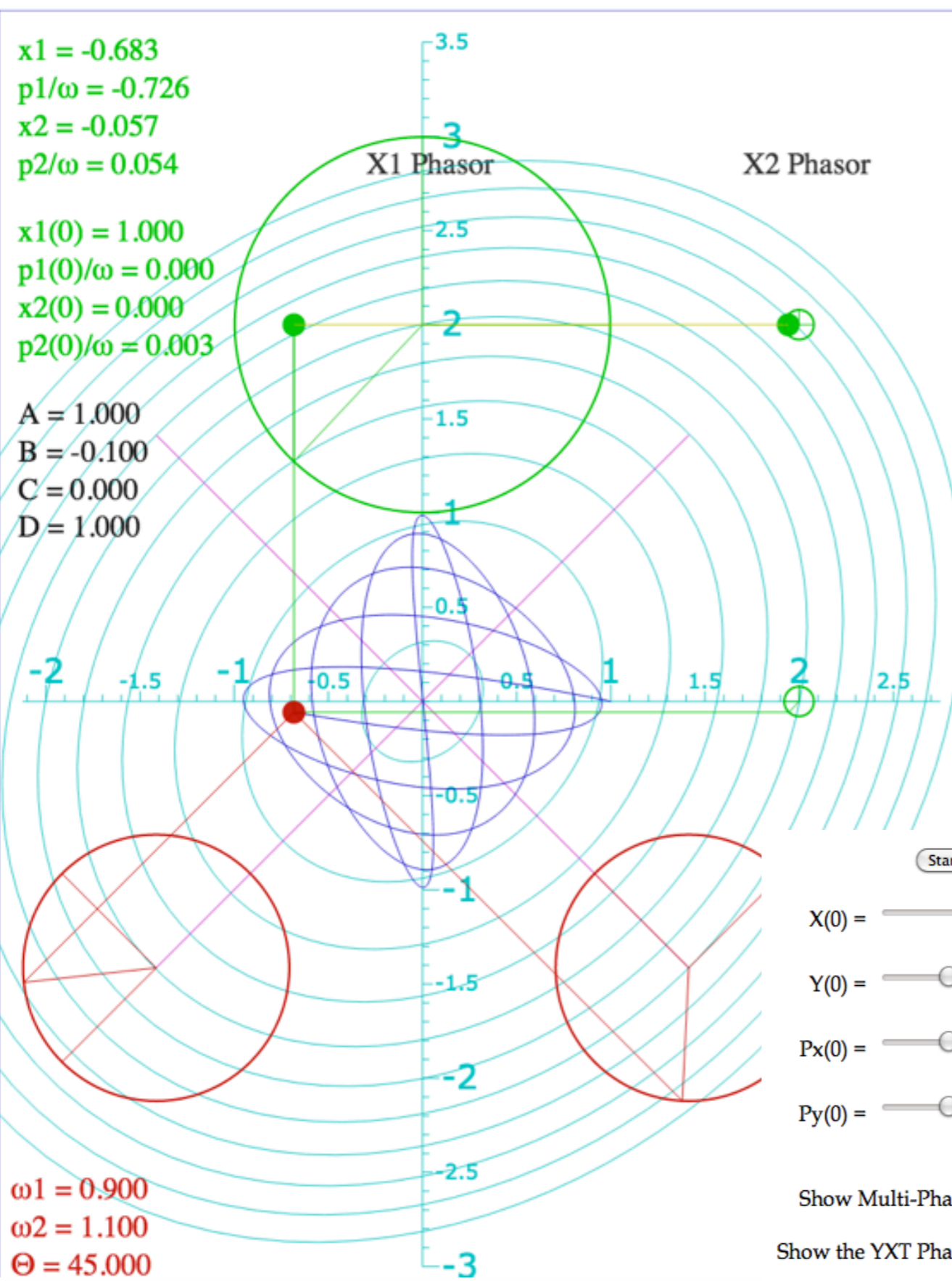
Stronger coupling on the right, illustrated indirectly by a darker looking spring on screen

$x1 = -0.683$
 $p1/\omega = -0.726$
 $x2 = -0.057$
 $p2/\omega = 0.054$
 $x1(0) = 1.000$
 $p1(0)/\omega = 0.000$
 $x2(0) = 0.000$
 $p2(0)/\omega = 0.003$
 $A = 1.000$
 $B = -0.100$
 $C = 0.000$
 $D = 1.000$

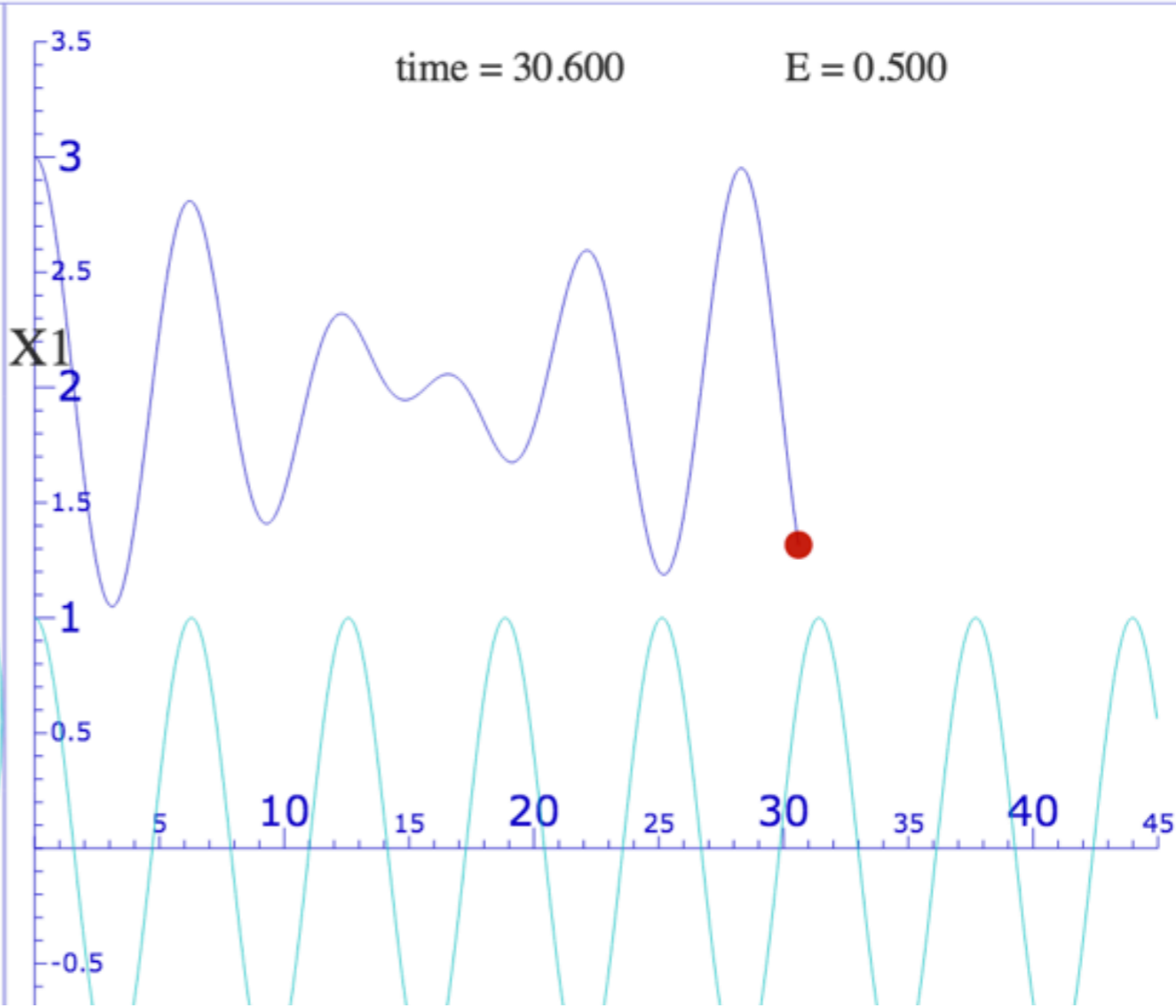


[BoxIt \(Beating\) Web Simulation](#)
(A=1, B=-0.1, C=0, D=1)

$x1 = -0.683$
 $p1/\omega = -0.726$
 $x2 = -0.057$
 $p2/\omega = 0.054$
 $x1(0) = 1.000$
 $p1(0)/\omega = 0.000$
 $x2(0) = 0.000$
 $p2(0)/\omega = 0.003$
 $A = 1.000$
 $B = -0.100$
 $C = 0.000$
 $D = 1.000$



$\omega1 = 0.900$
 $\omega2 = 1.100$
 $\Theta = 45.000$



$X(0) =$ $A =$ $\text{Number of Derivatives} =$
 $Y(0) =$ $B =$
 $Px(0) =$ $C =$
 $Py(0) =$ $D =$

- Show Multi-Phasor View
- Show the YXT Phasor View
- Draw Main Phasors
- Draw Vector Heads
- wantVectorHeads, wantTimeRateTangents
- Draw PE Levels
- Draw Box Lines
- Draw Modal Phasors
- Draw Time Rate Tangents
- Left Phasor Rides on Right Phasor
- Left Phasor Rides on Right Phasor
- Normalize Phasors
- Print $\omega1:\omega2$ fractions

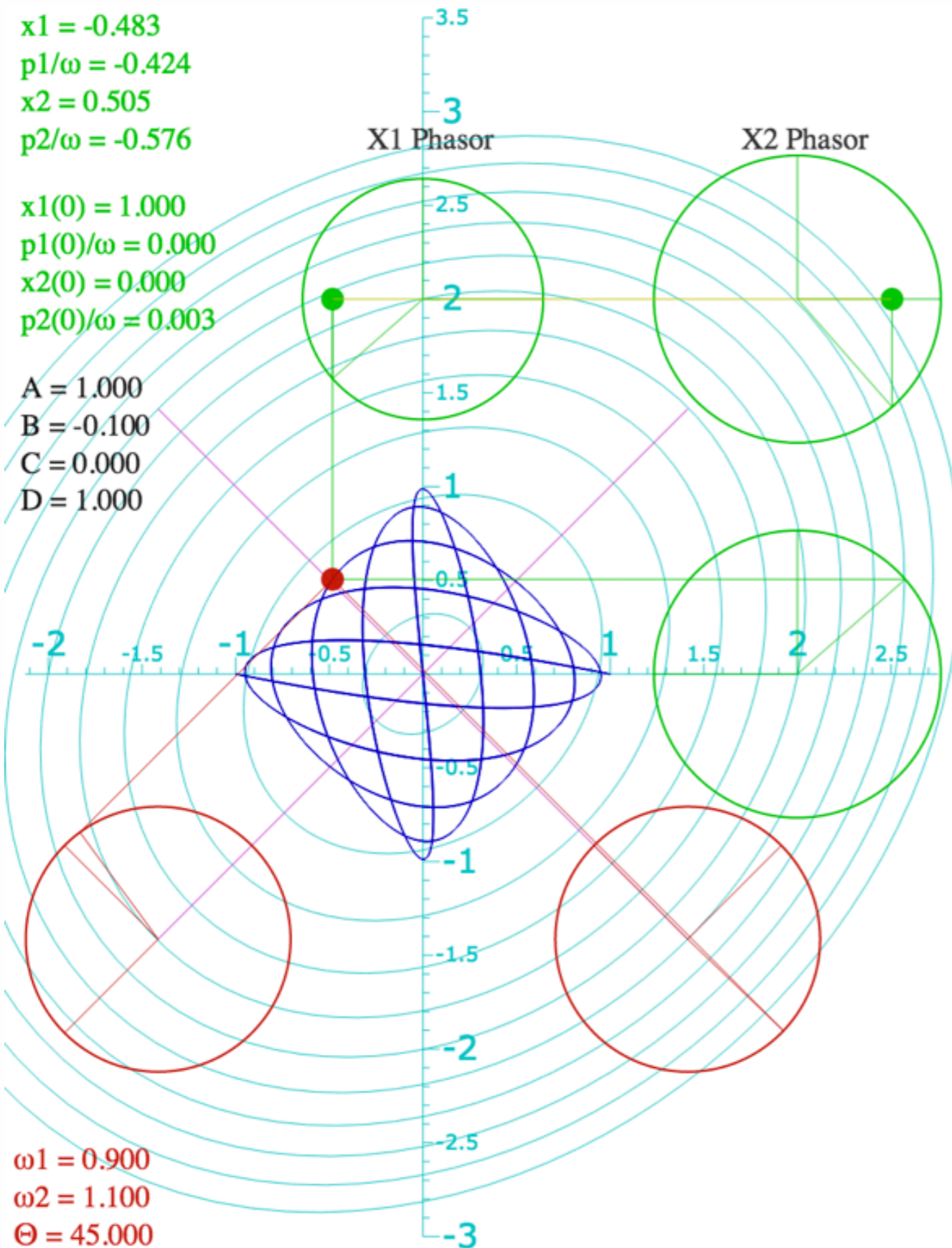
[BoxIt \(Beating\) Web Simulation](#)
(A=1, B=-0.1, C=0, D=1)

$x_1 = -0.483$
 $p_1/\omega = -0.424$
 $x_2 = 0.505$
 $p_2/\omega = -0.576$

$x_1(0) = 1.000$
 $p_1(0)/\omega = 0.000$
 $x_2(0) = 0.000$
 $p_2(0)/\omega = 0.003$

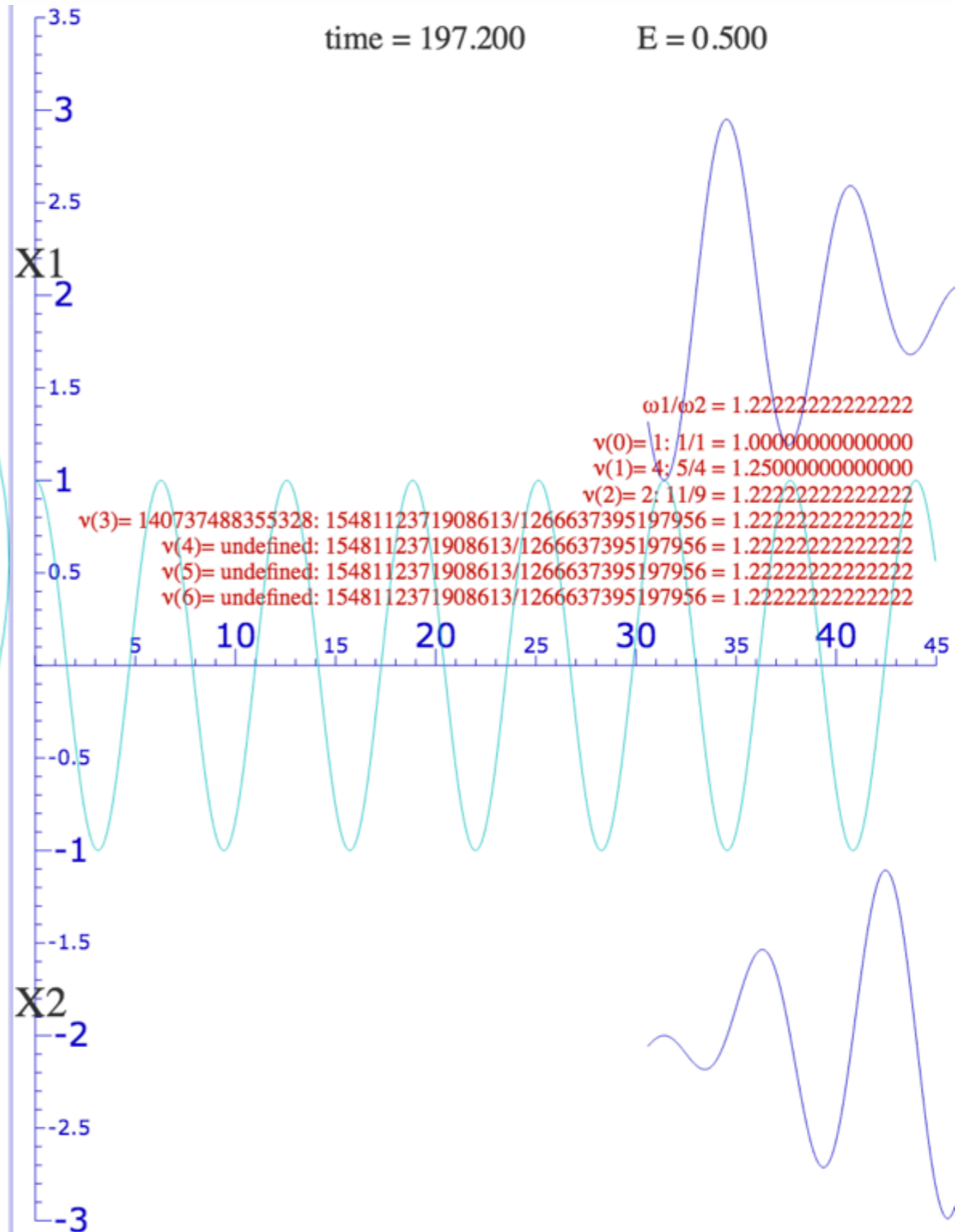
$A = 1.000$
 $B = -0.100$
 $C = 0.000$
 $D = 1.000$

$\omega_1 = 0.900$
 $\omega_2 = 1.100$
 $\Theta = 45.000$



time = 197.200

E = 0.500



[BoxIt \(Beating\) Web Simulation \(\$A=1, B=-0.1, C=0, D=1\$ \) with frequency ratios](#)

Review: How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)


C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

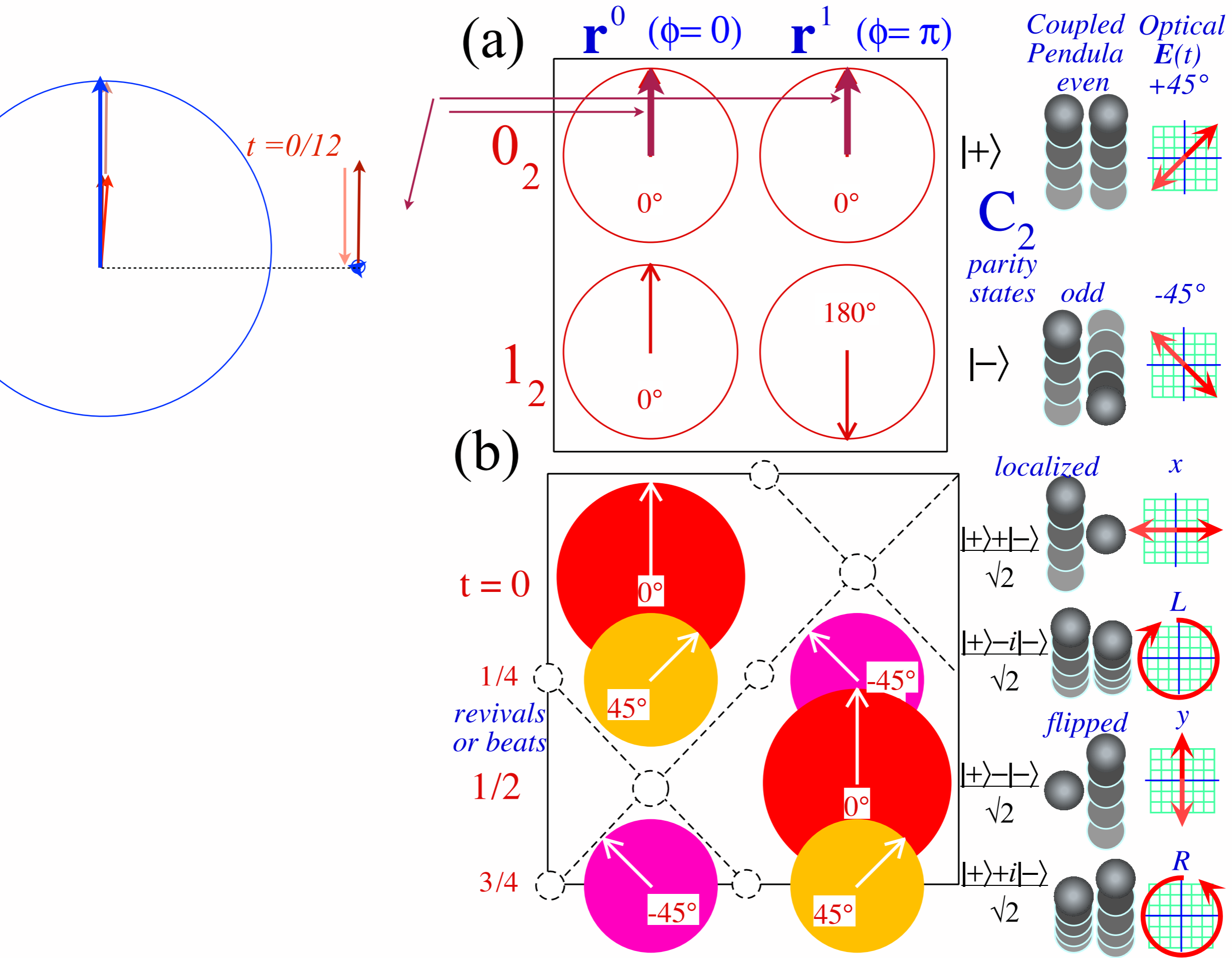
 *2D-HO beats and mixed mode geometry*

Three famous 2-state systems and two-complex-component coordinates

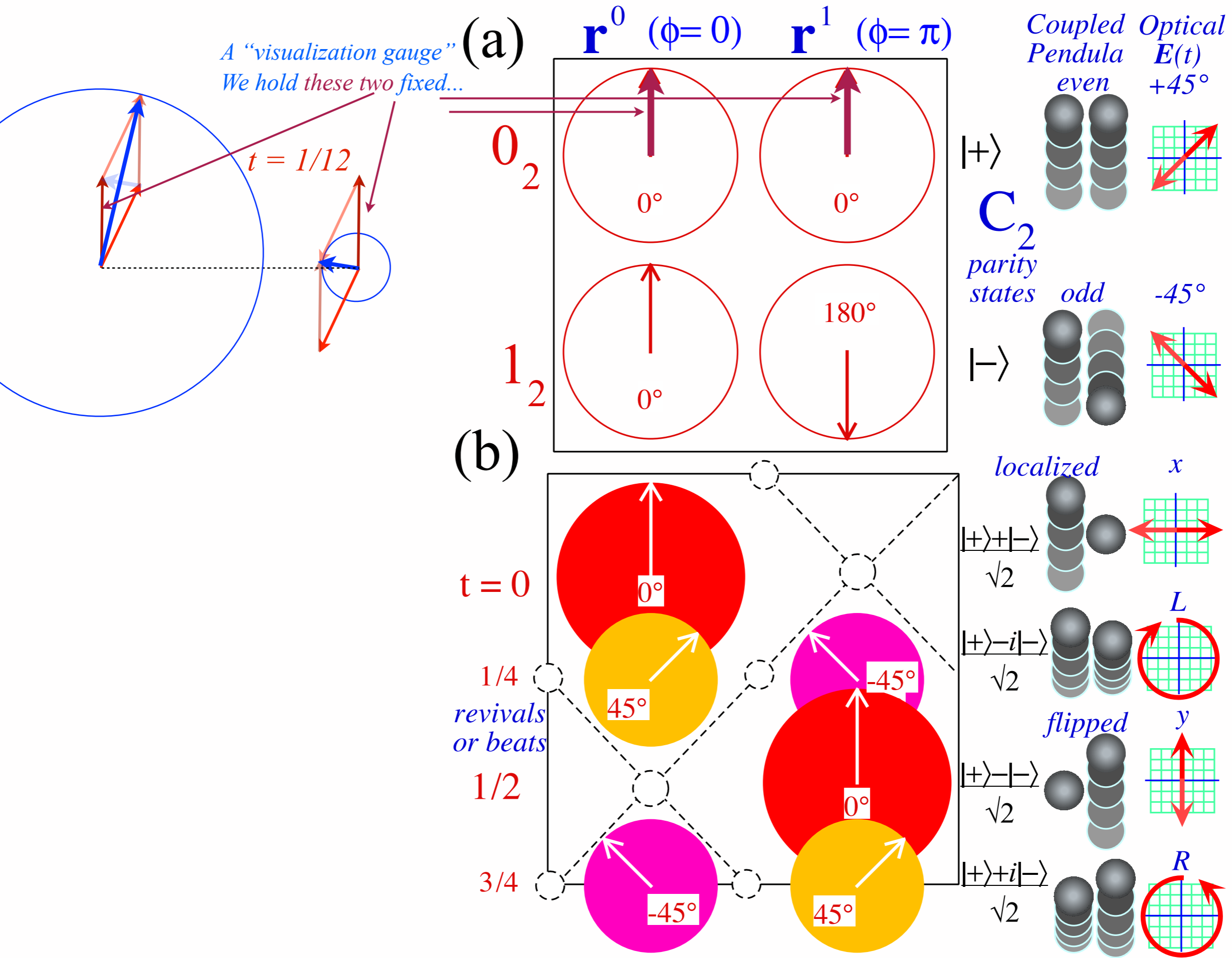
ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$

Hamilton-Pauli spinor symmetry (σ -expansion in ABCD-Types) $\mathbf{H} = \omega_\mu \sigma_\mu$

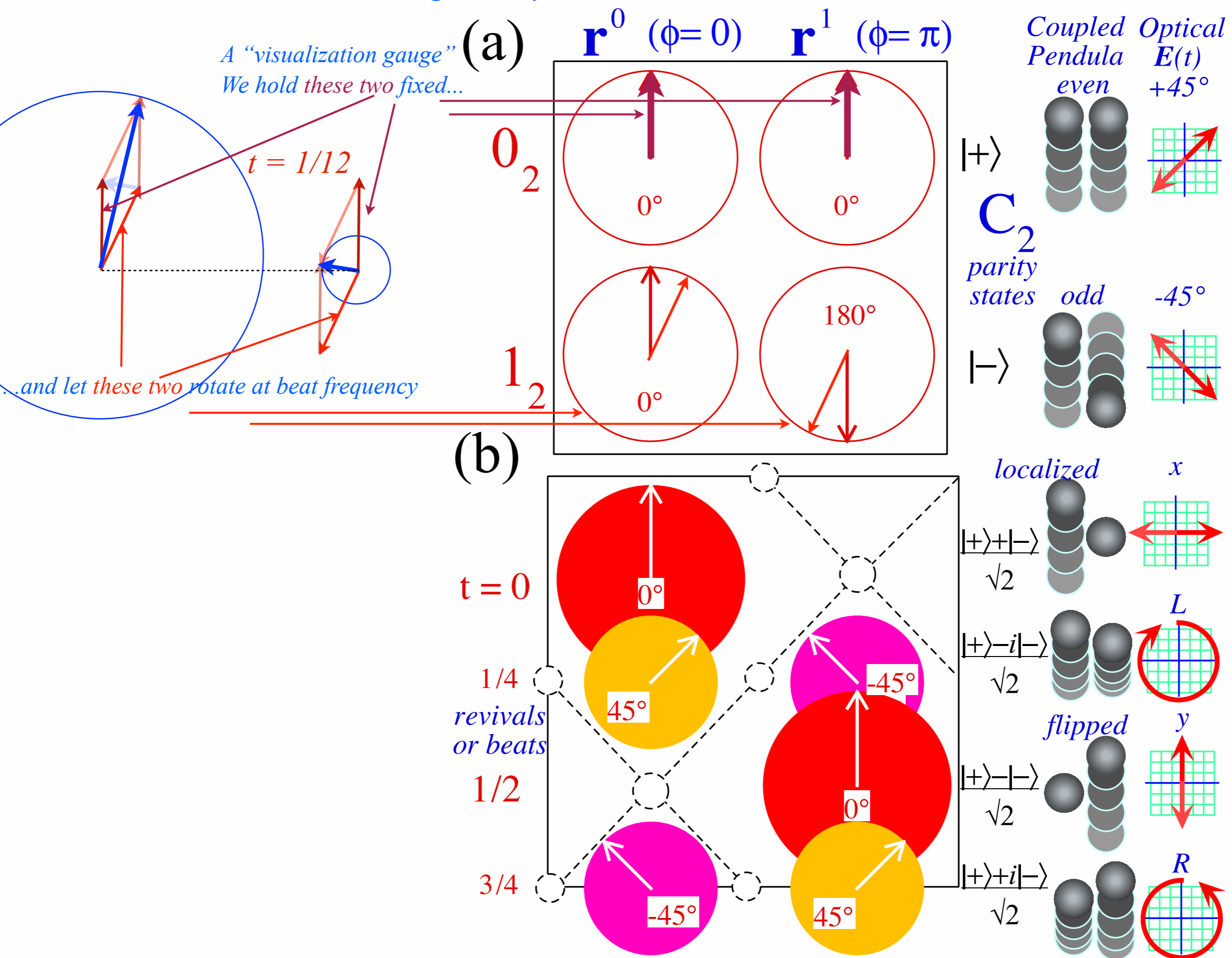
2D-HO beats and mixed mode geometry



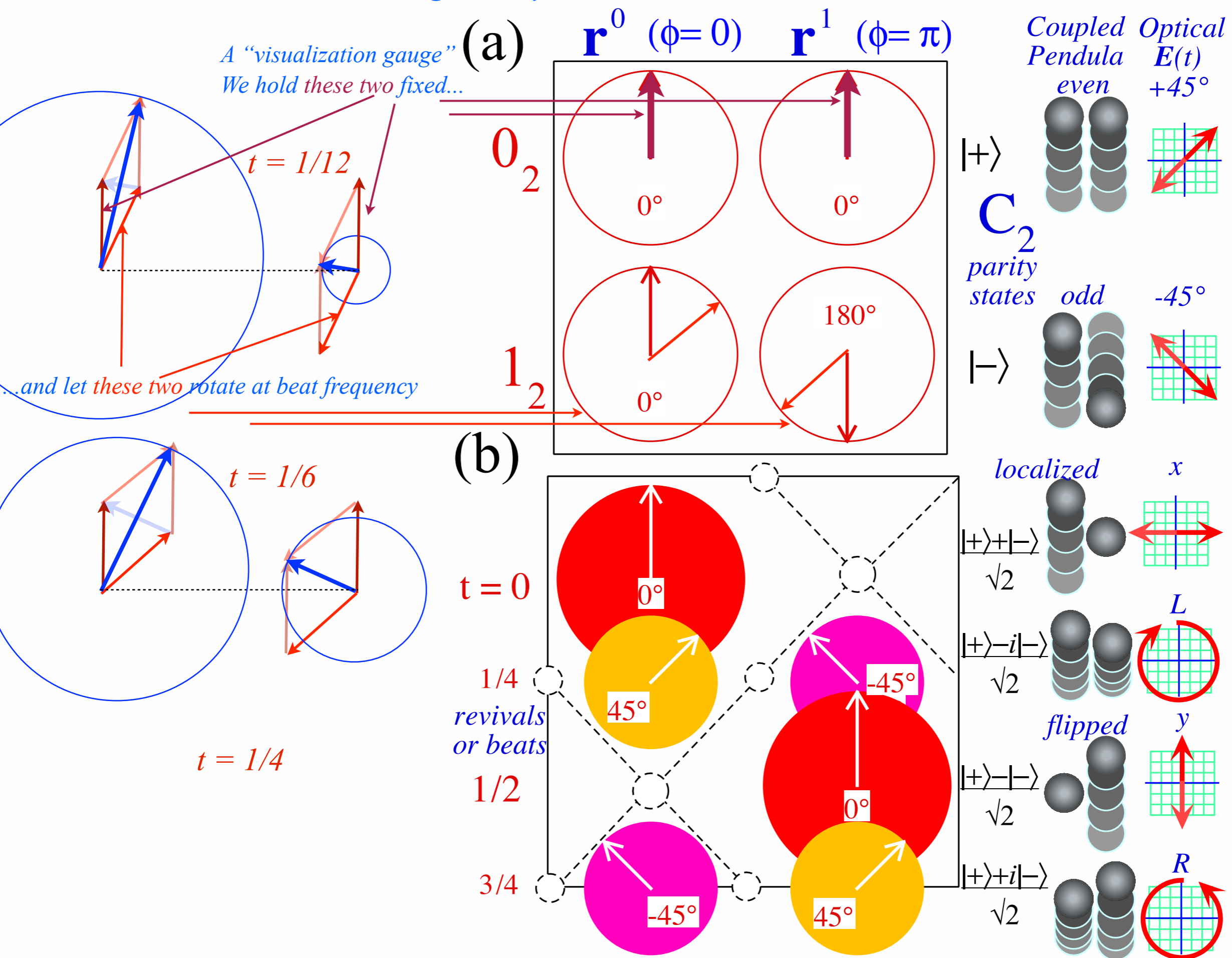
2D-HO beats and mixed mode geometry



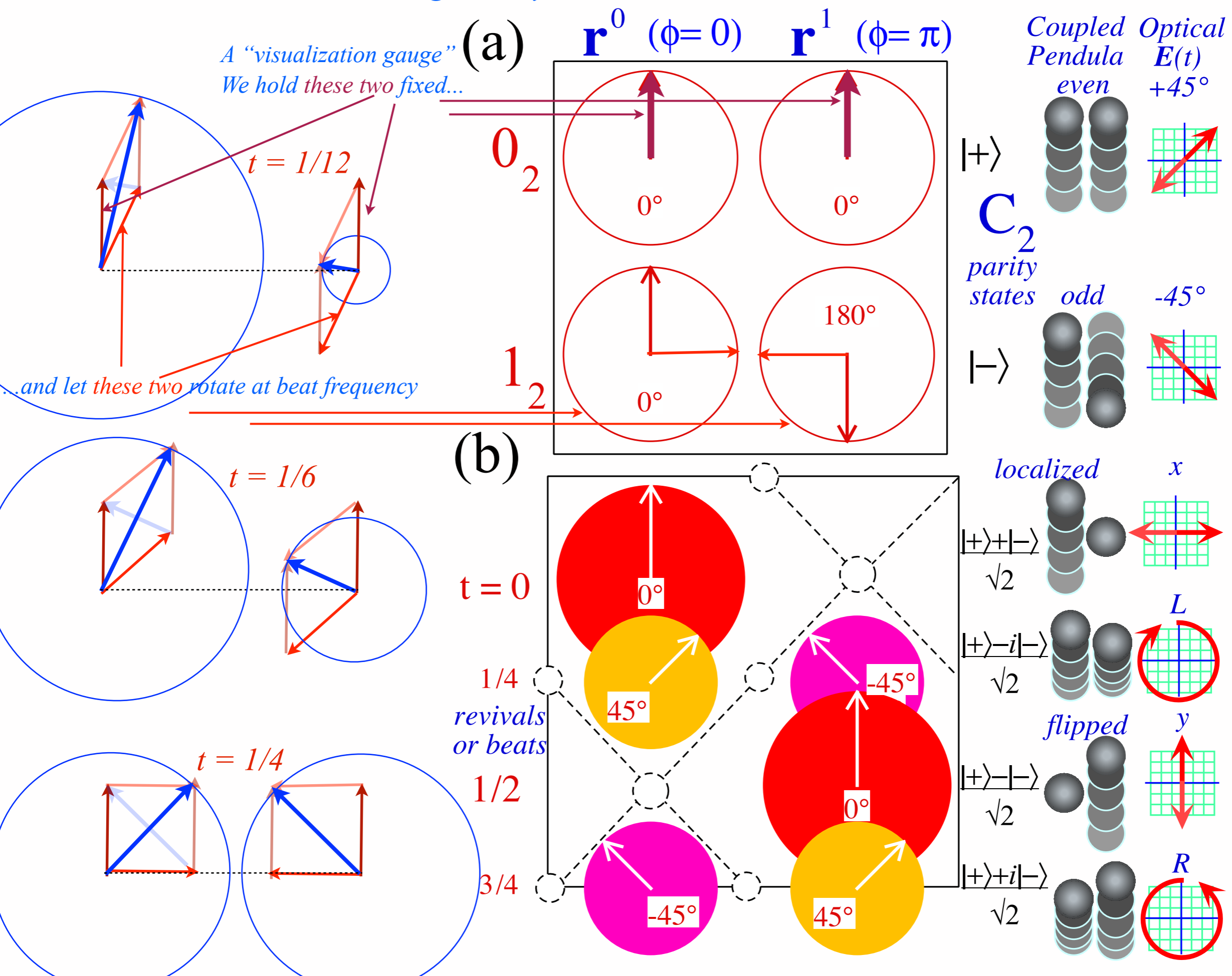
2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry



2D-HO beats and mixed mode geometry



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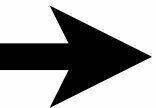
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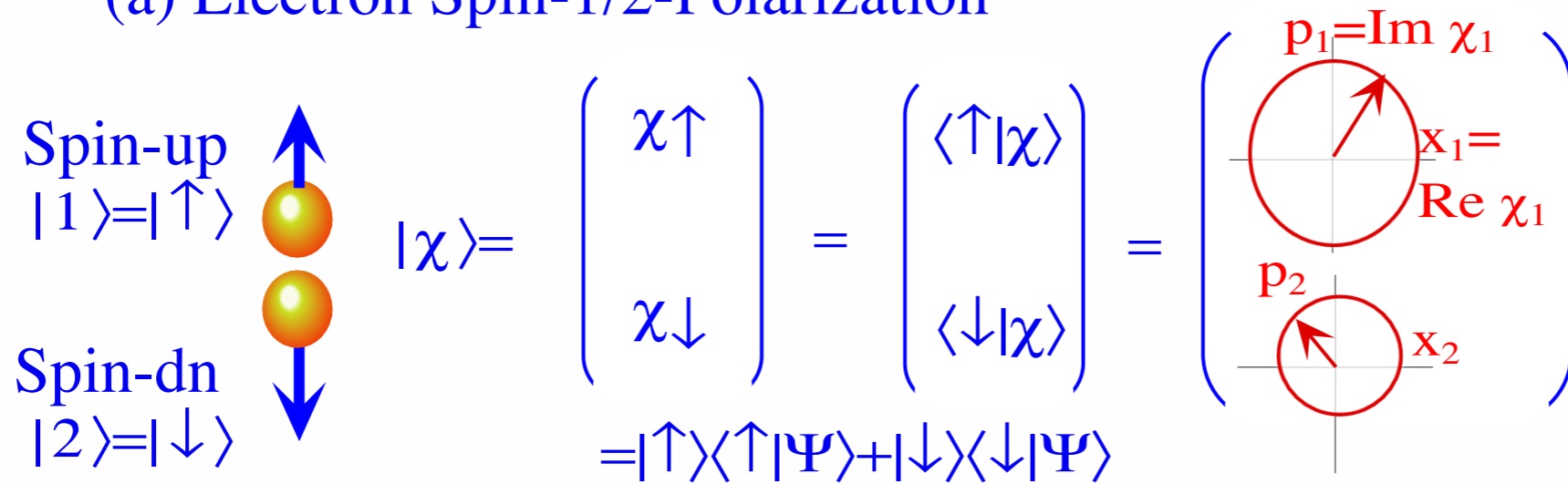
Three famous 2-state systems and two-complex-component coordinates

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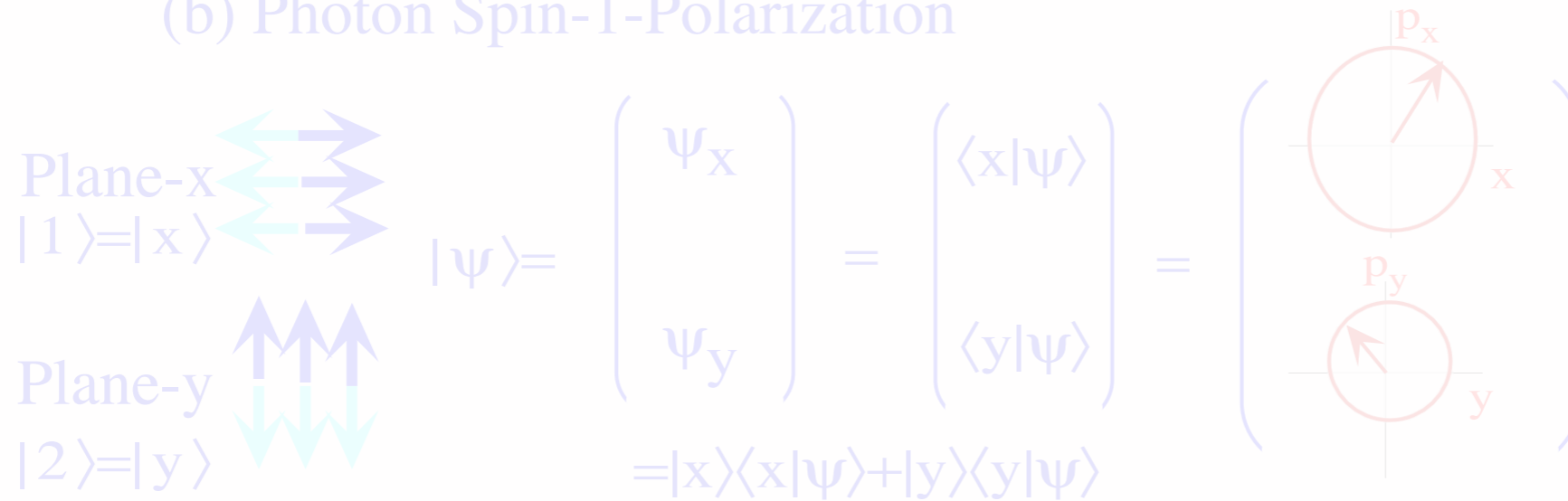
Three famous 2-state systems and two-complex-component coordinates

(a) Electron Spin-1/2-Polarization



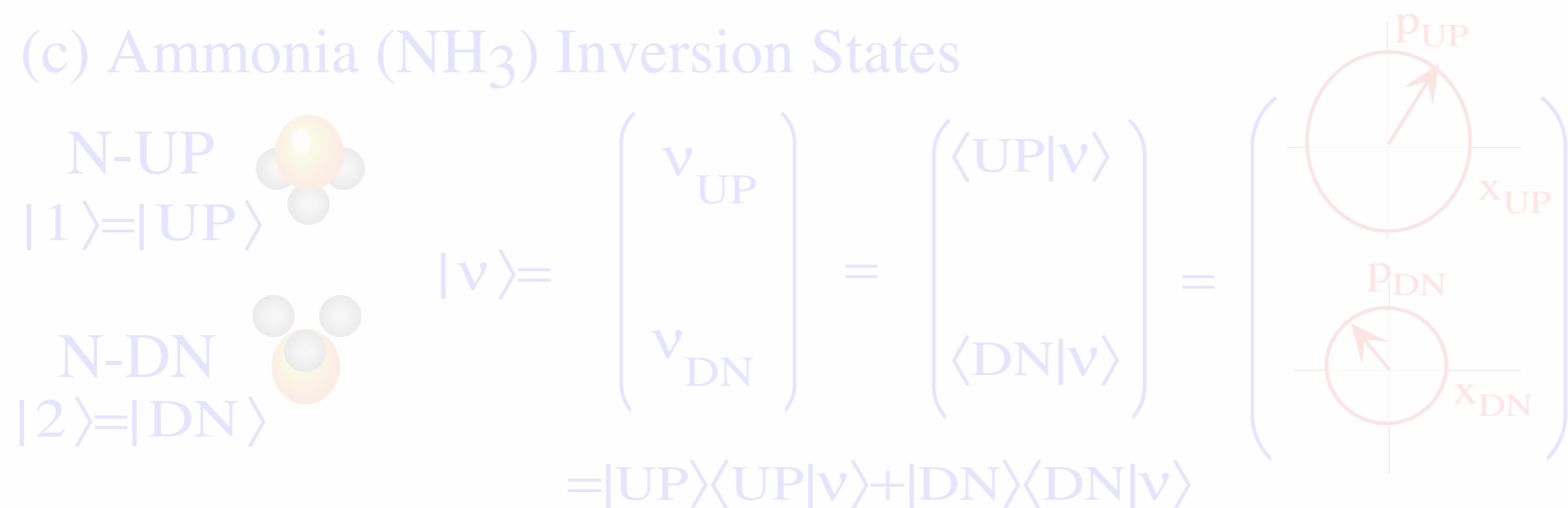
Rabi, Ramsey, and Schwinger 1954
Rev. Mod. Phys. 26 167 (1954)

(b) Photon Spin-1-Polarization



John Stokes 1862
Proc. Soc. London 11 547 (1862)

(c) Ammonia (NH₃) Inversion States







Feynman, Vernon, and Hellwarth 1957
J. Appl. Phys. 28 49 (1957)

Fig. 10.5.1
QTCA Unit 3 Chapter 10

Three famous 2-state systems and two-complex-component coordinates

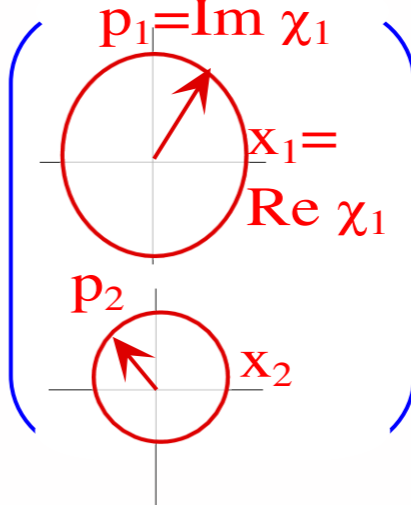
(a) Electron Spin-1/2-Polarization

Spin-up $|1\rangle=|\uparrow\rangle$  

Spin-dn $|2\rangle=|\downarrow\rangle$  



$$|\chi\rangle = \begin{pmatrix} \chi_{\uparrow} \\ \chi_{\downarrow} \end{pmatrix} = \begin{pmatrix} \langle \uparrow | \chi \rangle \\ \langle \downarrow | \chi \rangle \end{pmatrix} = \begin{pmatrix} \text{p}_1 = \text{Im } \chi_1 \\ \text{p}_2 \end{pmatrix} \begin{pmatrix} \text{x}_1 = \text{Re } \chi_1 \\ \text{x}_2 \end{pmatrix}$$



$= |\uparrow\rangle\langle \uparrow | \Psi \rangle + |\downarrow\rangle\langle \downarrow | \Psi \rangle$



Rabi, Ramsey, and Schwinger 1954
Rev. Mod. Phys. 26 167 (1954)

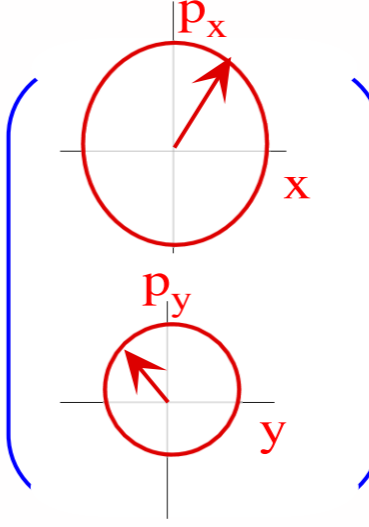
(b) Photon Spin-1-Polarization

Plane-x $|1\rangle=|x\rangle$  

Plane-y $|2\rangle=|y\rangle$  

$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \langle x | \psi \rangle \\ \langle y | \psi \rangle \end{pmatrix} = \begin{pmatrix} \text{p}_x \\ \text{p}_y \end{pmatrix} \begin{pmatrix} \text{x} \\ \text{y} \end{pmatrix}$$


$= |x\rangle\langle x | \psi \rangle + |y\rangle\langle y | \psi \rangle$




John Stokes 1862
Proc. Soc. London 11 547 (1862)

Harter and Dos Santos
Am. J. Phys. 46 251 (1986)
J. Chem. Phys. 85 5560 (1986)

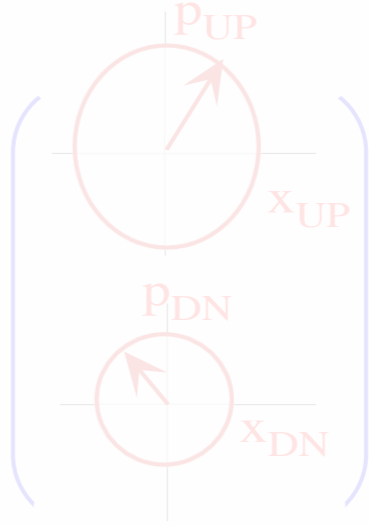
(c) Ammonia (NH₃) Inversion States

N-UP $|1\rangle=|UP\rangle$ 

N-DN $|2\rangle=|DN\rangle$ 

$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP | \nu \rangle \\ \langle DN | \nu \rangle \end{pmatrix} = \begin{pmatrix} \text{p}_{UP} \\ \text{p}_{DN} \end{pmatrix} \begin{pmatrix} \text{x}_{UP} \\ \text{x}_{DN} \end{pmatrix}$$

$= |UP\rangle\langle UP | \nu \rangle + |DN\rangle\langle DN | \nu \rangle$

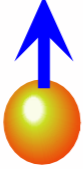


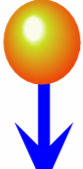
Feynman, Vernon, and Hellwarth 1957
J. Appl. Phys. 28 49 (1957)

Fig. 10.5.1
QTCA Unit 3 Chapter 10

Three famous 2-state systems and two-complex-component coordinates

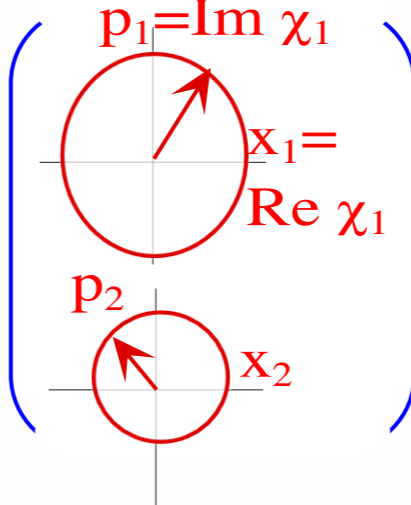
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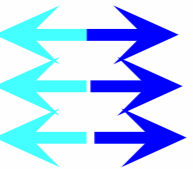
$$|\chi\rangle = \begin{pmatrix} \chi_{\uparrow} \\ \chi_{\downarrow} \end{pmatrix} = \begin{pmatrix} \langle \uparrow | \chi \rangle \\ \langle \downarrow | \chi \rangle \end{pmatrix} = \begin{pmatrix} \text{Polarization Blobs} \end{pmatrix}$$


$= |\uparrow\rangle\langle \uparrow | \Psi \rangle + |\downarrow\rangle\langle \downarrow | \Psi \rangle$



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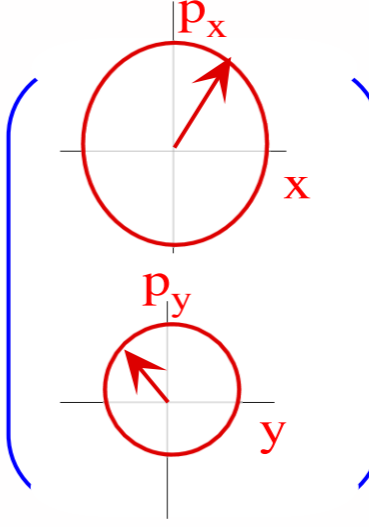
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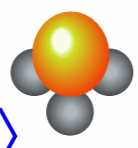
$= |x\rangle\langle x | \psi \rangle + |y\rangle\langle y | \psi \rangle$

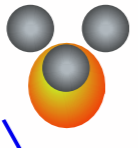


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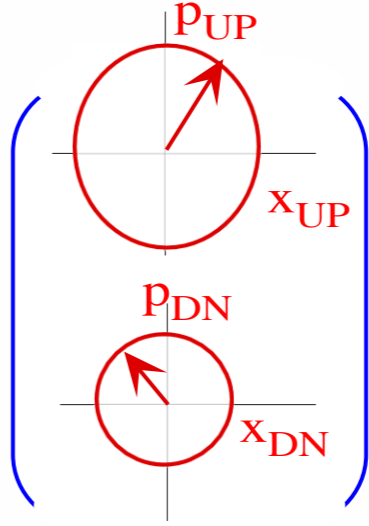
(c) Ammonia (NH₃) Inversion States

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$= |UP\rangle\langle UP | \nu \rangle + |DN\rangle\langle DN | \nu \rangle$



Feynman, Vernon, and Hellwarth 1957
J. Appl. Phys. 28 49 (1957)

Fig. 10.5.1
 QTCA Unit 3 Chapter 10

Three famous 2-state systems and two-complex-component coordinates

(a) Electron Spin-1/2-Polarization

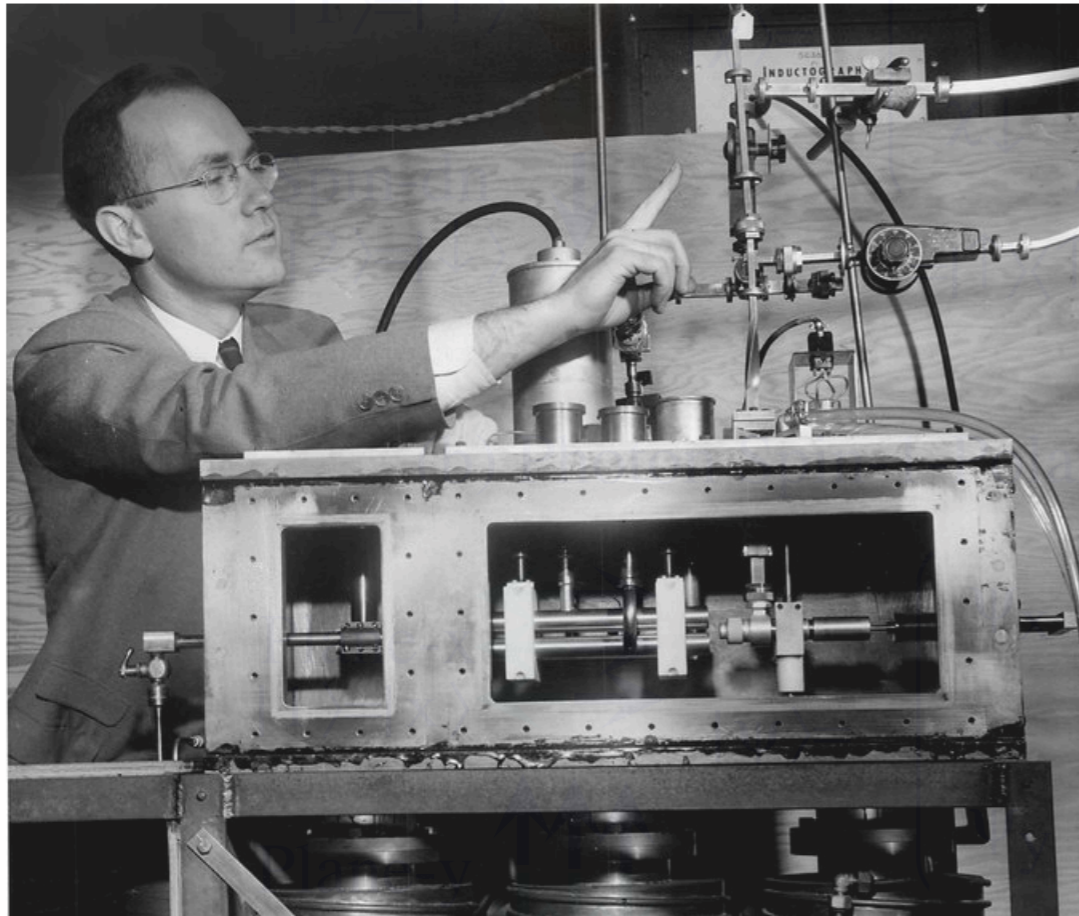
Charles H. Townes, Who Paved Way for the Laser in Daily Life, Dies at 99

$$p_1 = \text{Im } \chi_1$$

Rabi Ramson and
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The New York Times

By ROBERT D. McFADDEN JAN. 28, 2015



Charles Townes in 1955. Eddie Hausner/The New York Times

Today's Headlines

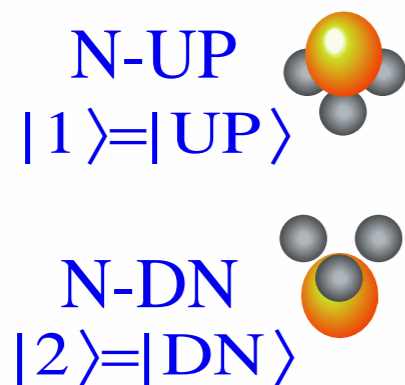
Thursday, January 29, 2015

He had an "a-ha!" moment. Sitting on a park bench in Washington one April morning in 1951, pondering how to stimulate molecular energy to create shorter wavelengths, he conceived of a device he called a maser, for microwave amplification by stimulated emission of radiation. It would use molecules to nudge other molecules, and amplify their thrust by getting them to resonate like tuning forks and line up in a powerful beam.

He and two graduate students, [James P. Gordon](#) and H. J. Zeigler, built his maser in 1953 and patented their creation. It was the first device operating on the principles of the laser, although it amplified microwave radiation rather than infrared or visible light radiation.

Five years later, Dr. Townes and Dr. Schawlow, who was his brother-in-law and would [win the 1981 Nobel Prize in Physics](#) for work on laser spectroscopy, drew a blueprint for a laser. They called it an optical maser, a term that never caught on, and through Bell Laboratories they secured the first laser patent in 1959, a year before Dr. Maiman's first working model.

(c) Ammonia (NH₃) Inversion States



$$|v\rangle = \begin{pmatrix} v_{UP} \\ v_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|v\rangle \\ \langle DN|v\rangle \end{pmatrix} = \begin{pmatrix} \text{PUP} \\ \text{PDN} \end{pmatrix} \begin{pmatrix} x_{UP} \\ x_{DN} \end{pmatrix}$$

$$= |UP\rangle \langle UP|v\rangle + |DN\rangle \langle DN|v\rangle$$

[Feynman, Vernon, and Hellwarth 1957](#)
J. Appl. Phys. **28** 49 (1957)

Fig. 10.5.1
QTCA Unit 3 Chapter 10

2D harmonic oscillator equations

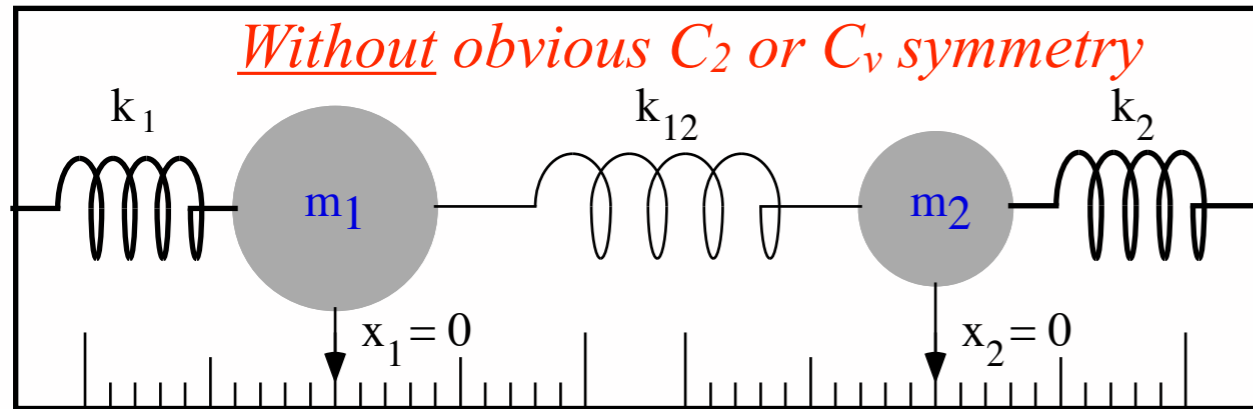


Fig. 3.3.1 Two 1-dimensional coupled oscillators

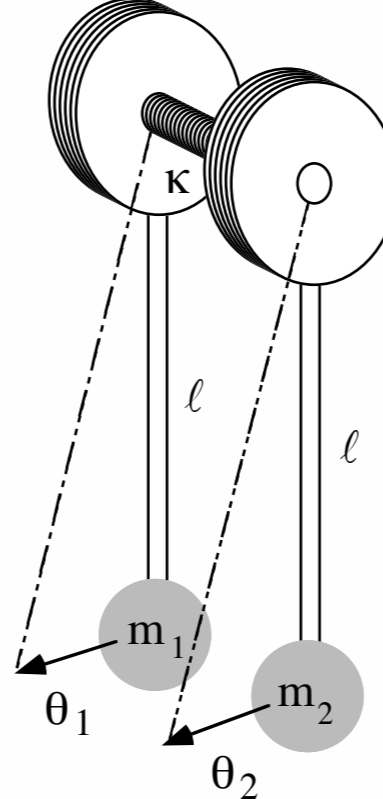
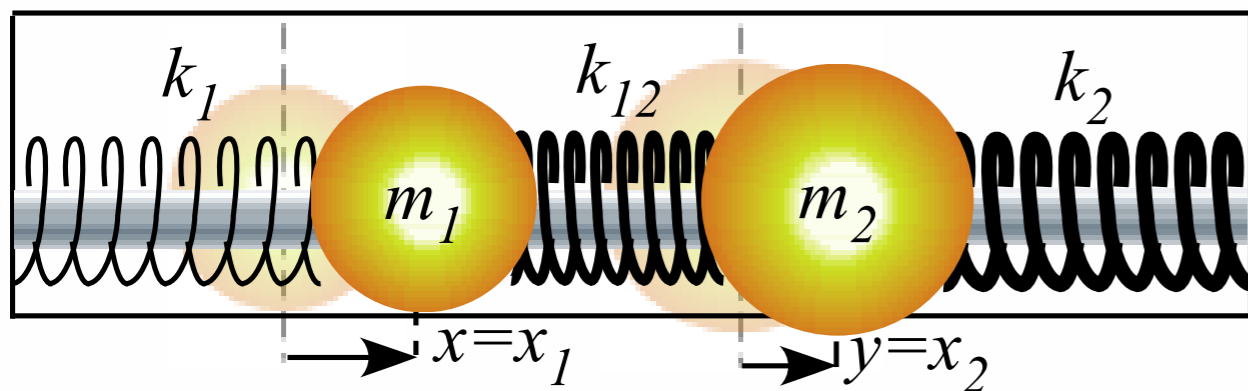


Fig. 3.3.2 Coupled pendulums

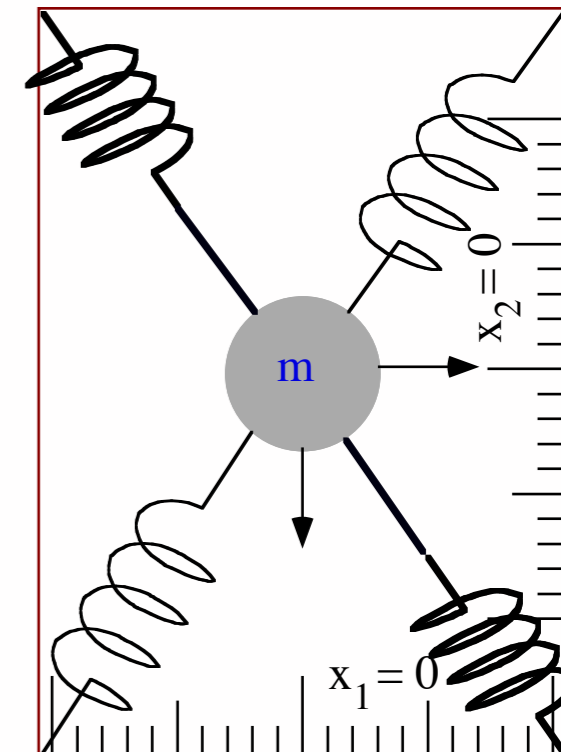


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2} (k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2} (k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle$$

where: $\mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_1} \right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}_2} \right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \cdot | \ddot{\mathbf{x}} \rangle = - \mathbf{K} \cdot | \mathbf{x} \rangle$$

2D harmonic oscillator equation solutions

Without obvious C_2 or C_v symmetry

1. May rewrite equation $\mathbf{M} \cdot |\ddot{\mathbf{x}}\rangle = -\mathbf{K} \cdot |\mathbf{x}\rangle$ in acceleration matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1} \cdot \mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = - \begin{pmatrix} \frac{k_1 + k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2 + k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \dots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue* and ω_n is an *eigenfrequency*

Note eigenvalue is square of eigenfrequency

To introduce eigensolutions we take a simple case of unit masses ($m_1=1=m_2$)

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

Review: How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

 **ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial_t^2\mathbf{x} = -\mathbf{K}\cdot\mathbf{x}$**

*Hamilton-Pauli spinor symmetry (σ -expansion in **ABCD**-Types) $\mathbf{H} = \omega_\mu \sigma_\mu$*

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that operates on 2-D complex Dirac ket vector $|\Psi\rangle$. *Both have 4 parameters*

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad (2^2 = 2+2)$$

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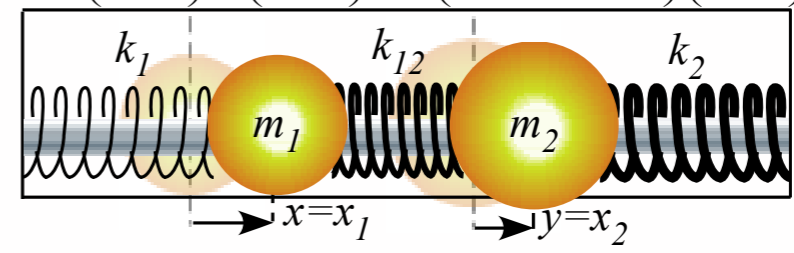
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For C=0 Is form of 2D Hooke harmonic oscillator

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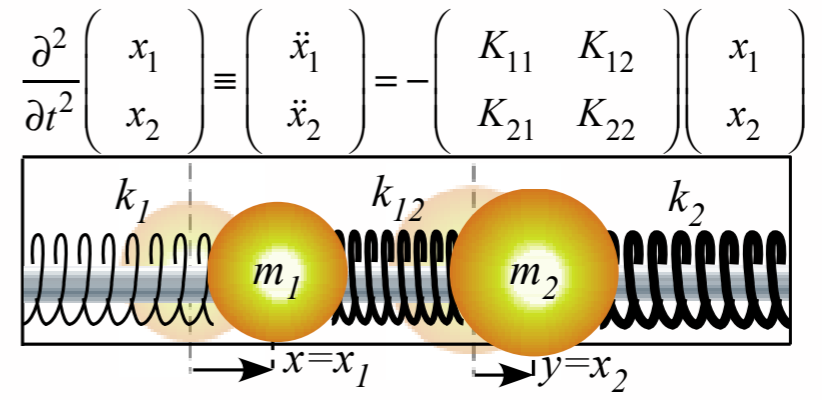
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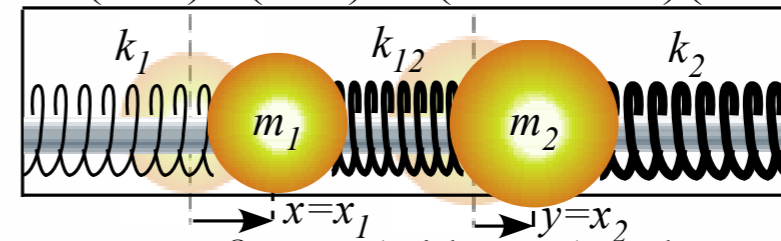
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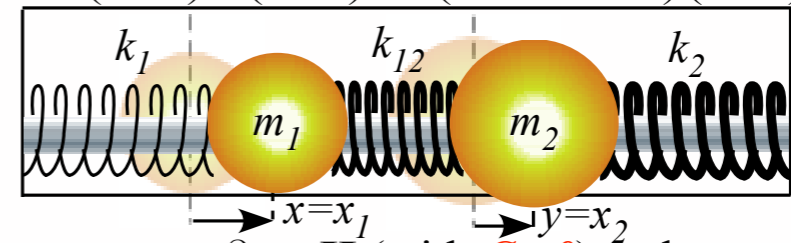
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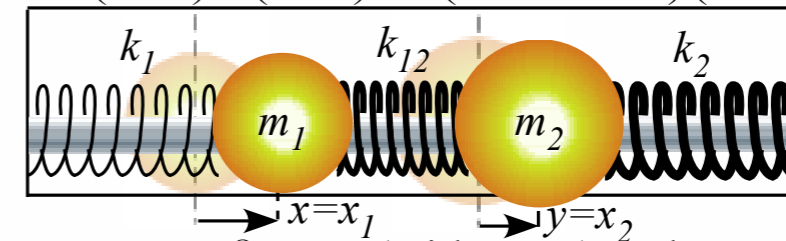
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Review: How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

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*Hamilton-Pauli spinor symmetry (σ -expansion in **ABCD**-Types) $\mathbf{H} = \omega_\mu \sigma_\mu$*

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator \mathbf{H} into four *ABCD symmetry operators*

(Labeled to provide dynamic mnemonics as well as colorful analogies)

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Standing waves

Motivation for coloring scheme:
The Traffic Signal



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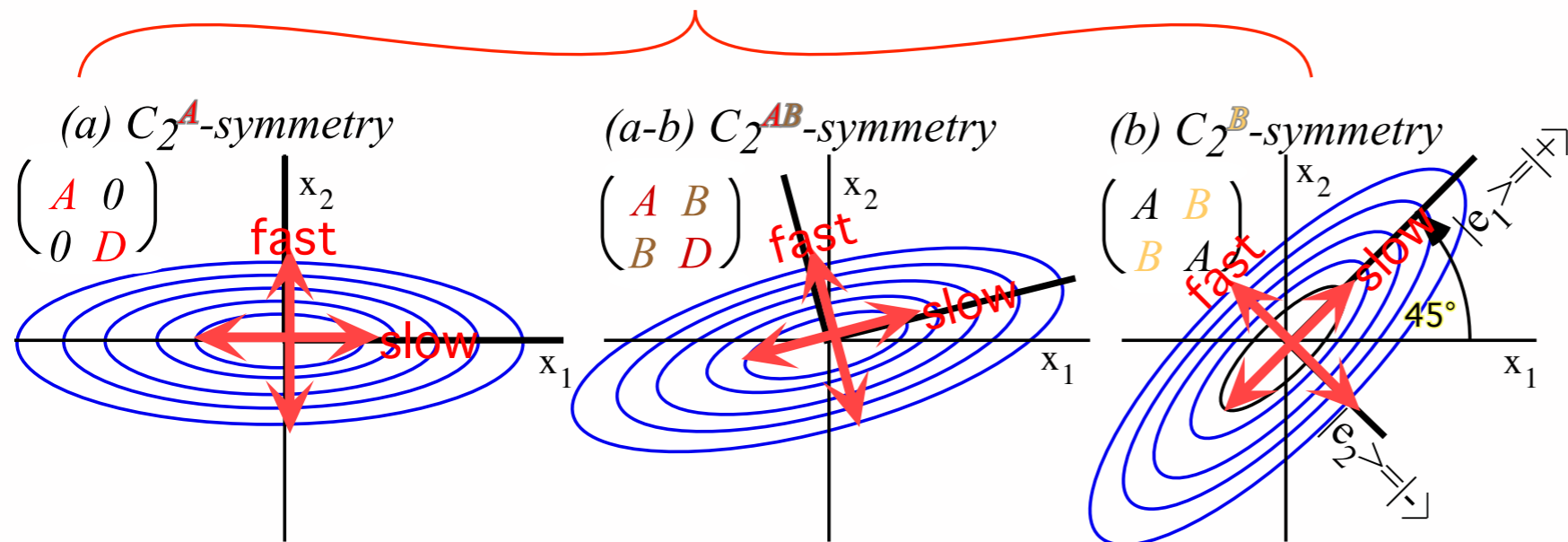


Fig. 10.1.2 Potentials for (a) C_2^A -asymmetric-diagonal, (ab) C_2^{AB} -mixed, (b) C_2^B -bilateral U(2)system.

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 The $\{\boldsymbol{\sigma}_I, \boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B, \boldsymbol{\sigma}_C\}$ are best known as *Pauli-spin operators* $\{\boldsymbol{\sigma}_I = \boldsymbol{\sigma}_0, \boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X, \boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y, \boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z\}$ developed in 1927.

Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623

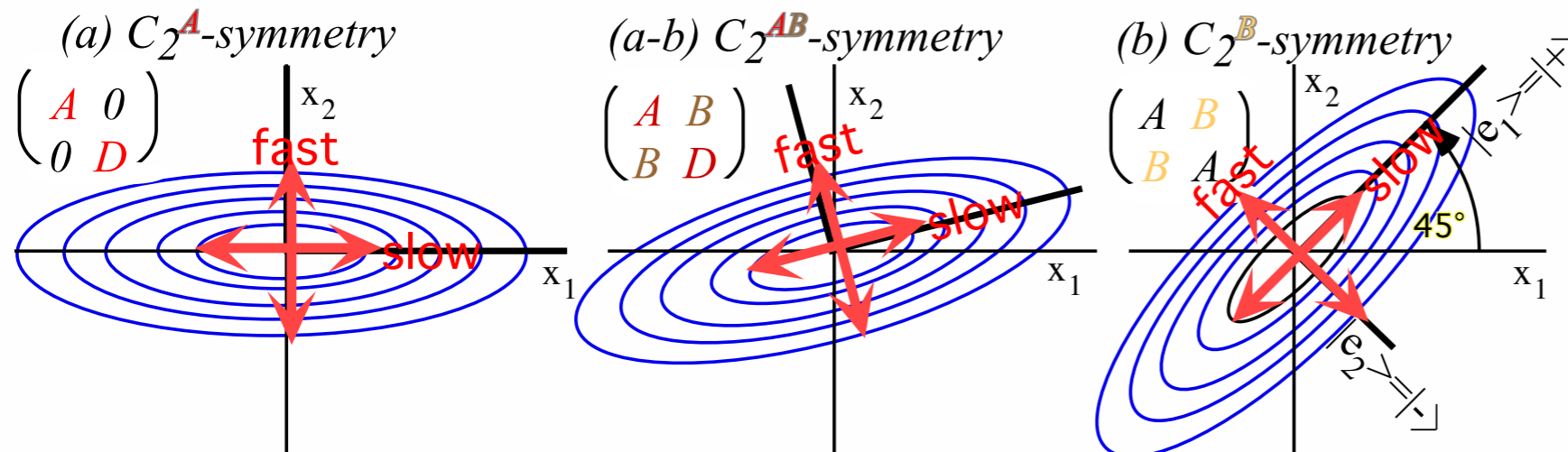


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Symmetry archetypes: *A (Asymmetric-diagonal)* | *B (Bilateral-balanced)* | *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*
 The $\{\boldsymbol{\sigma}_I, \boldsymbol{\sigma}_A, \boldsymbol{\sigma}_B, \boldsymbol{\sigma}_C\}$ are best known as *Pauli-spin operators* $\{\boldsymbol{\sigma}_I = \boldsymbol{\sigma}_0, \boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X, \boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y, \boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z\}$ developed in 1927.

Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623

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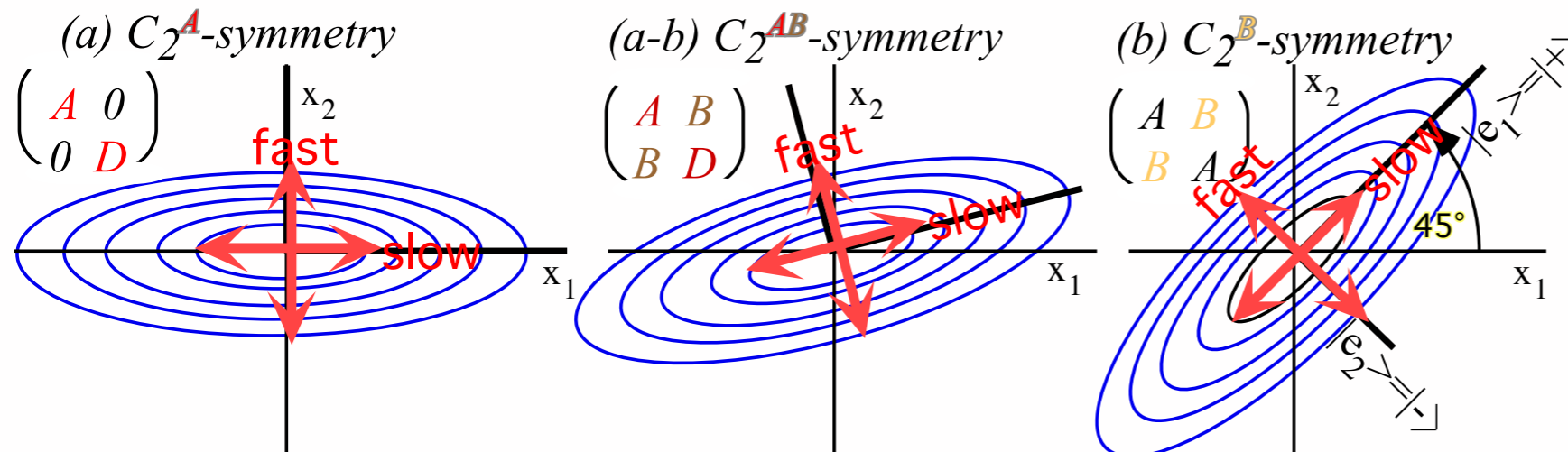


Fig. 10.1.2 Potentials for (a) *C2^A-asymmetric-diagonal*, (ab) *C2^AB-mixed*, (b) *C2^B-bilateral U(2)system*.

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator \mathbf{H} into four *ABCD symmetry operators*
 (Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{aligned} \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} &= A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22} \\ &= \frac{A-D}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \mathbf{H} &= \frac{A-D}{2} \boldsymbol{\sigma}_A + B \boldsymbol{\sigma}_B + C \boldsymbol{\sigma}_C + \frac{A+D}{2} \boldsymbol{\sigma}_0 \end{aligned}$$

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Each Hamilton quaternion squares to *negative-1* ($\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$) like imaginary number $i^2 = -1$. (They make up the Quaternion group.)

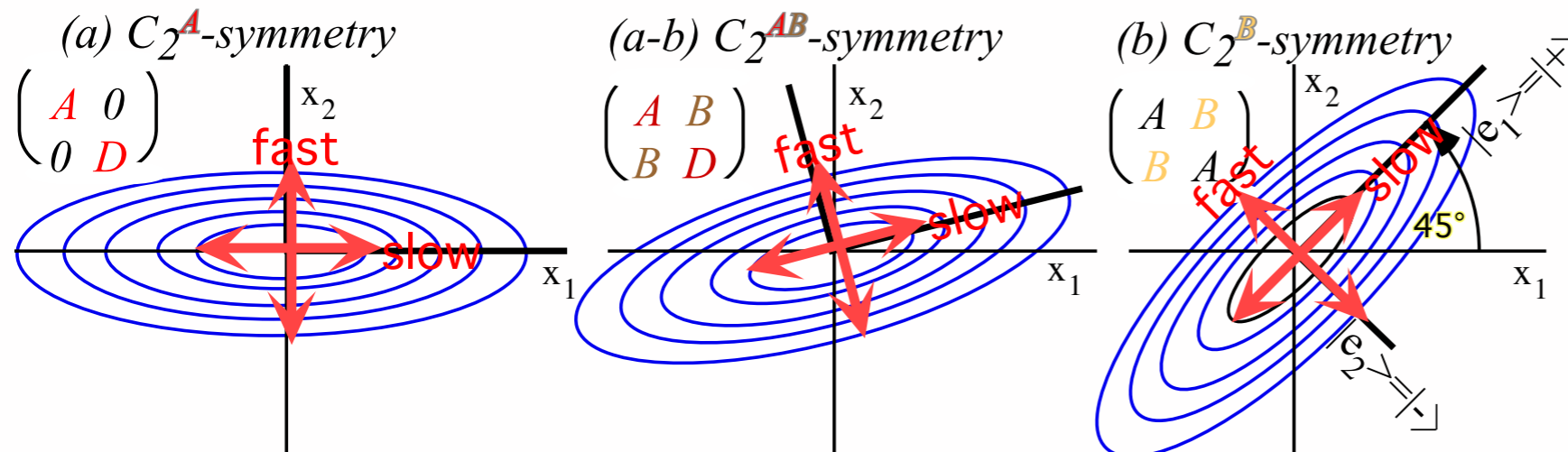


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Symmetry archetypes: *A* (Asymmetric-diagonal) | *B* (Bilateral-balanced) | *C* (Chiral-circular-complex-Coriolis-cyclotron-curly...)

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Each Hamilton quaternion squares to *negative-1* ($\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$) like imaginary number $i^2 = -1$. (They make up the Quaternion group.)

Each Pauli $\boldsymbol{\sigma}_\mu$ squares to *positive-1* ($\boldsymbol{\sigma}_X^2 = \boldsymbol{\sigma}_Y^2 = \boldsymbol{\sigma}_Z^2 = +1$) (Each makes a cyclic C_2 group $C_2^A = \{\mathbf{1}, \boldsymbol{\sigma}_A\}$, $C_2^B = \{\mathbf{1}, \boldsymbol{\sigma}_B\}$, or $C_2^C = \{\mathbf{1}, \boldsymbol{\sigma}_C\}$.)

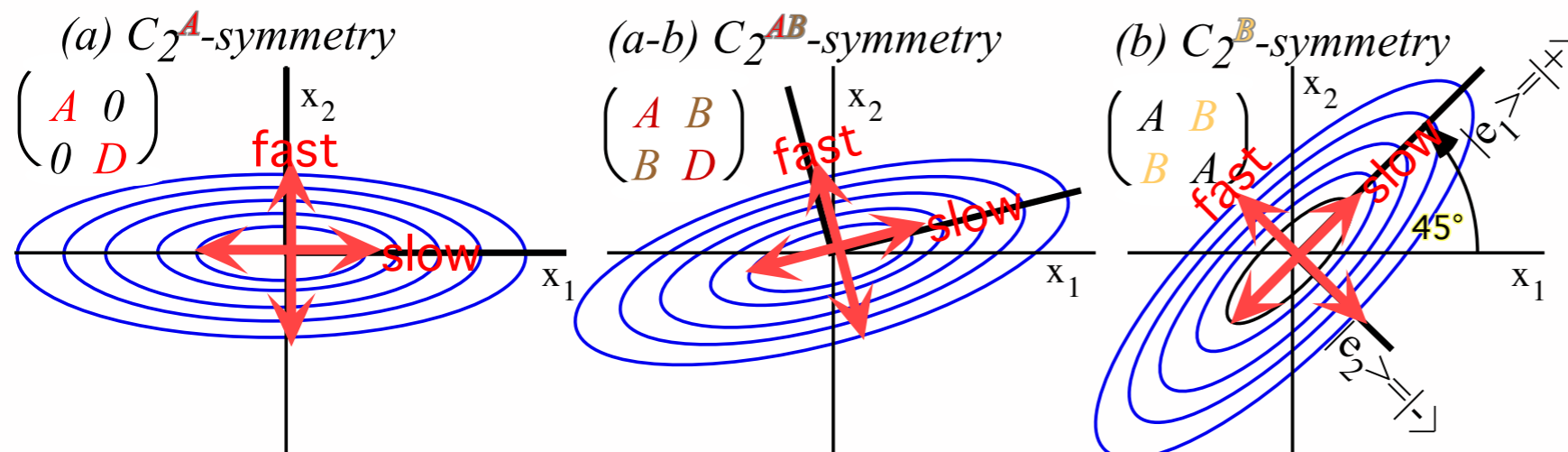


Fig. 10.1.2 Potentials for (a) C_2^A -asymmetric-diagonal, (ab) C_2^{AB} -mixed, (b) C_2^B -bilateral U(2)system.