Group Theory in Quantum Mechanics

Lecture 6 (1.29.15)

Spectral Decomposition of Bi-Cyclic ($C_2 \subset U(2)$) Operators

(Quantum Theory for Computer Age - Ch. 7-9 of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 2)

Review:How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)
- C_2 (Bilateral σ_B reflection) symmetry conditions:
- Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions
- C_2 Mode phase character table
- C_2 Symmetric 2DHO uncoupling and mixed mode projector algebra
- 2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

ANALOGY: $U(2)$ vs $R(3)$: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ vs. Classical 2D-HO: $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$
- Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

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Review: How symmetry groups become eigen-solvers

Suppose you need to diagonalize a complicated operator **K** and knew that **K** commutes with some other operators **G** and **H** for which irreducible projectors are more easily found.

$$\mathbf{KG} = \mathbf{GK} \text{ or } \mathbf{G}^\dagger\mathbf{KG} = \mathbf{K} \text{ or } \mathbf{GKG}^\dagger = \mathbf{K}$$

$$\mathbf{KH} = \mathbf{HK} \text{ or } \mathbf{H}^\dagger\mathbf{KH} = \mathbf{K} \text{ or } \mathbf{HKH}^\dagger = \mathbf{K}$$

(Here assuming *unitary* $\mathbf{G}^\dagger = \mathbf{G}^{-1}$ and $\mathbf{H}^\dagger = \mathbf{H}^{-1}$.)

This means **K** is *invariant* to the transformation by **G** and **H** and all their products $\mathbf{GH}$, $\mathbf{GH}^2$, $\mathbf{G}^2\mathbf{H}$,.. *etc*. and all their inverses $\mathbf{G}^\dagger$,$\mathbf{H}^\dagger$,.. etc.

The group $\mathscr{G}_{\mathbf{K}} = \{\mathbf{1}, \mathbf{G}, \mathbf{H}, ..\}$ so formed by such operators is called a *symmetry group* for **K**.

In certain ideal cases a **K**-matrix $\langle \mathbf{K} \rangle$ is a linear combination of matrices $\langle \mathbf{1} \rangle, \langle \mathbf{G} \rangle, \langle \mathbf{H} \rangle$,... from $\mathscr{G}_{\mathbf{K}}$.
Then spectral resolution of $\{\langle \mathbf{1} \rangle, \langle \mathbf{G} \rangle, \langle \mathbf{H} \rangle, ..\}$ also resolves $\langle \mathbf{K} \rangle$.

We will study ideal cases first. More general cases are built from this idea.

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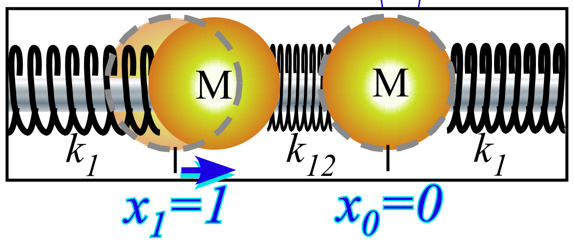
2D HO "binary" bases and coord. $\{x_0,x_1\}$

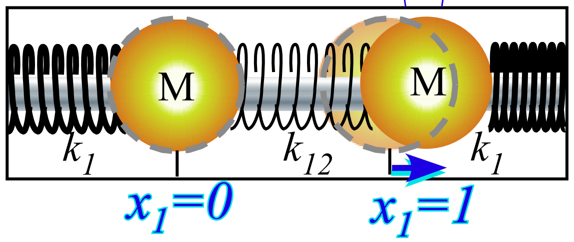
(a) unit base state

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix}1\\0\end{pmatrix}$

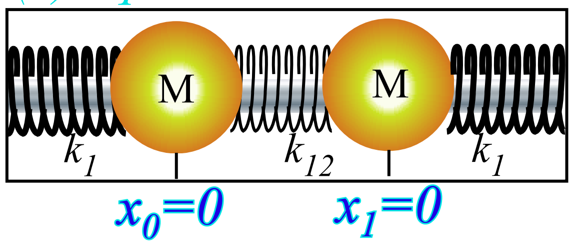


(b) unit base state

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix}0\\1\end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix}0\\0\end{pmatrix}$



2D HO Matrix operator equations

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}\frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M}\\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$=-\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

More conventional coordinate notation

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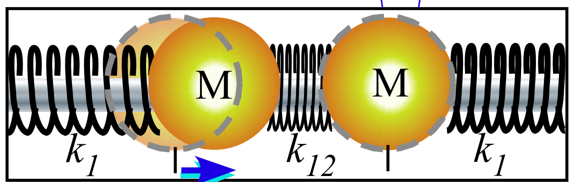

$x_1=1$ $x_0=0$

(b) unit base state

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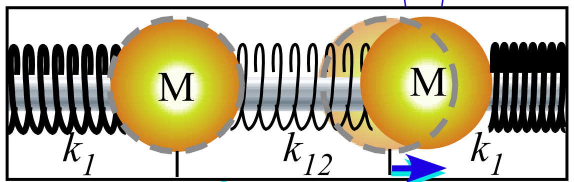

$x_1=0$ $x_1=1$

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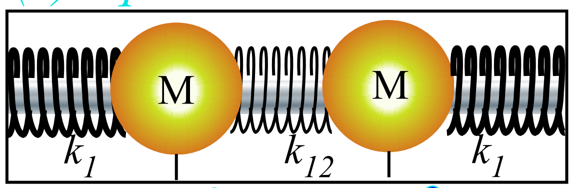

$x_0=0$ $x_1=0$

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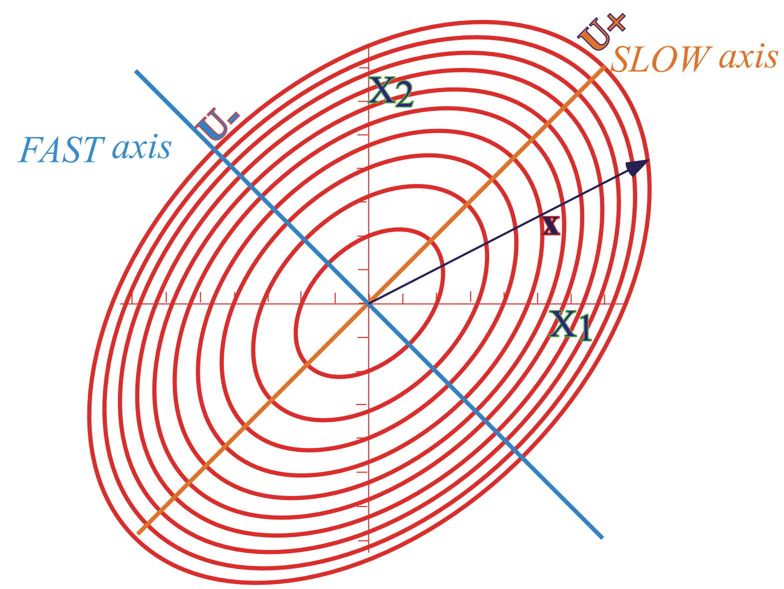
(a) PE Contours



2D HO kinetic energy $T(v_1, v_2)$

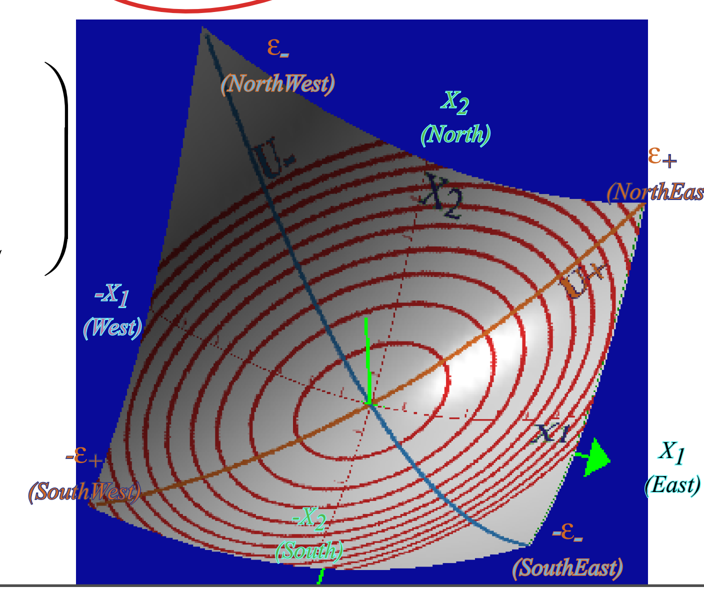
$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1+k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2+k_{12})x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix}$$



$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1+k_{12})x_1 + k_{12}x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12}x_1 - (k_2+k_{12})x_2$$

2D HO Lagrange equations

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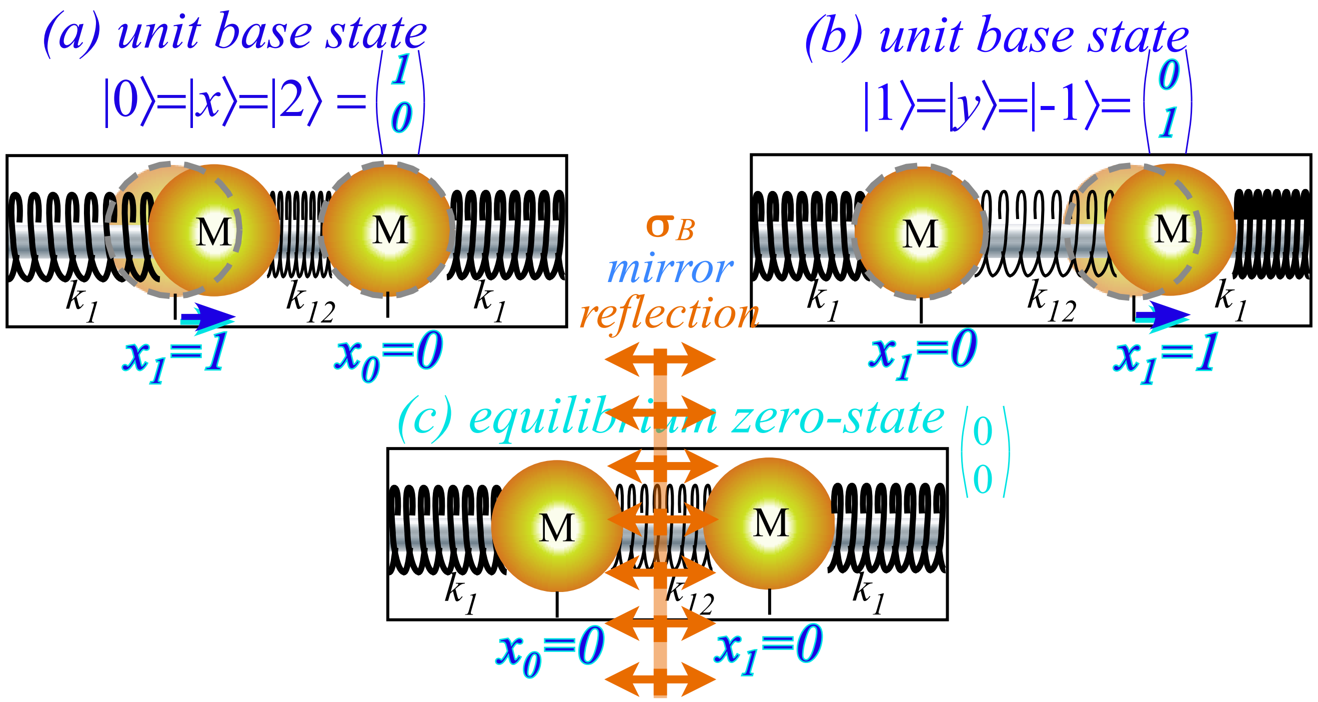

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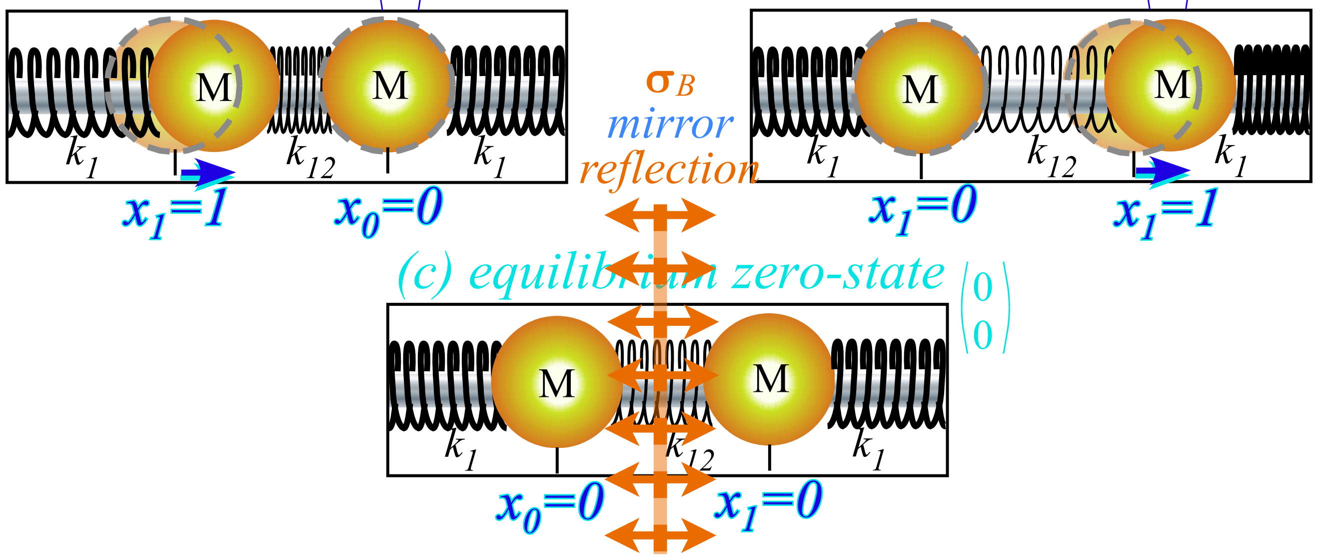

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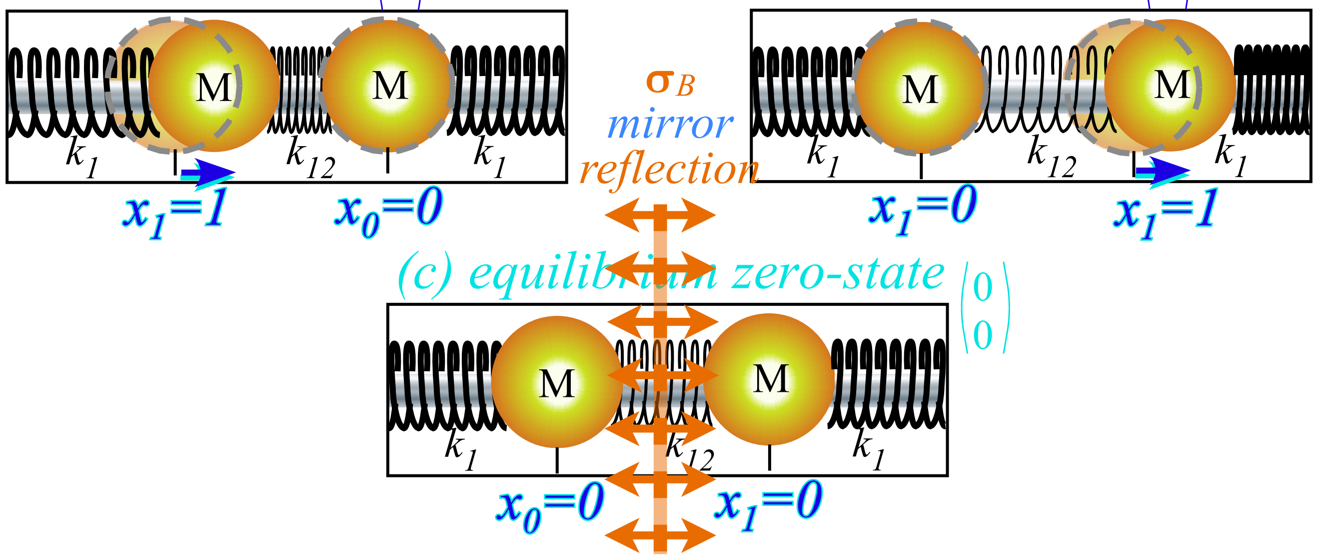

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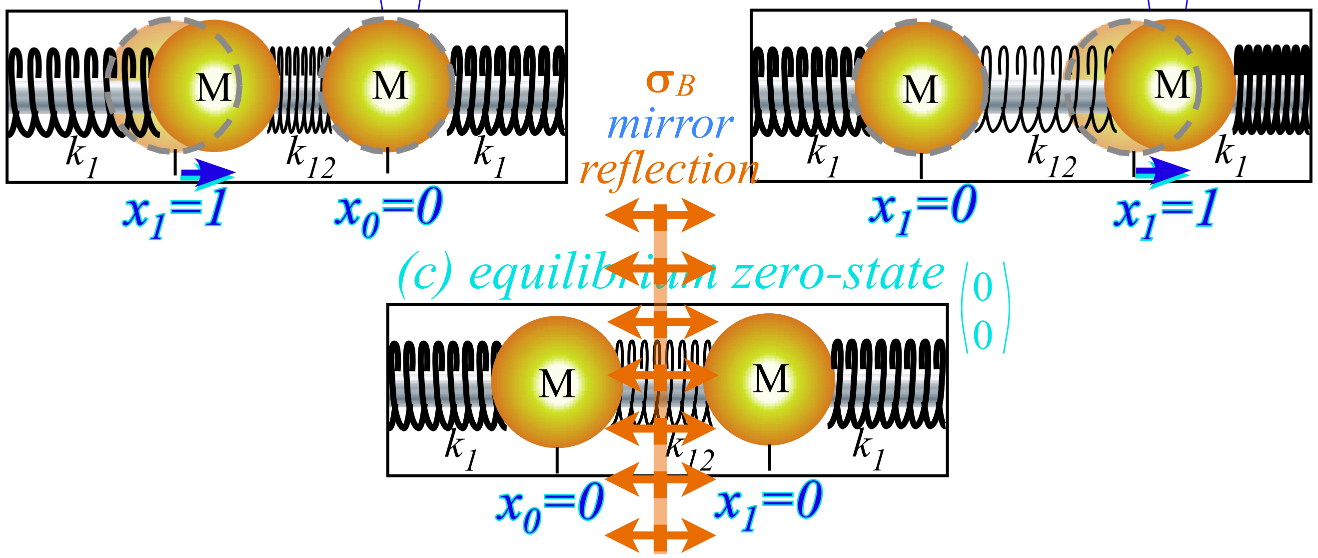

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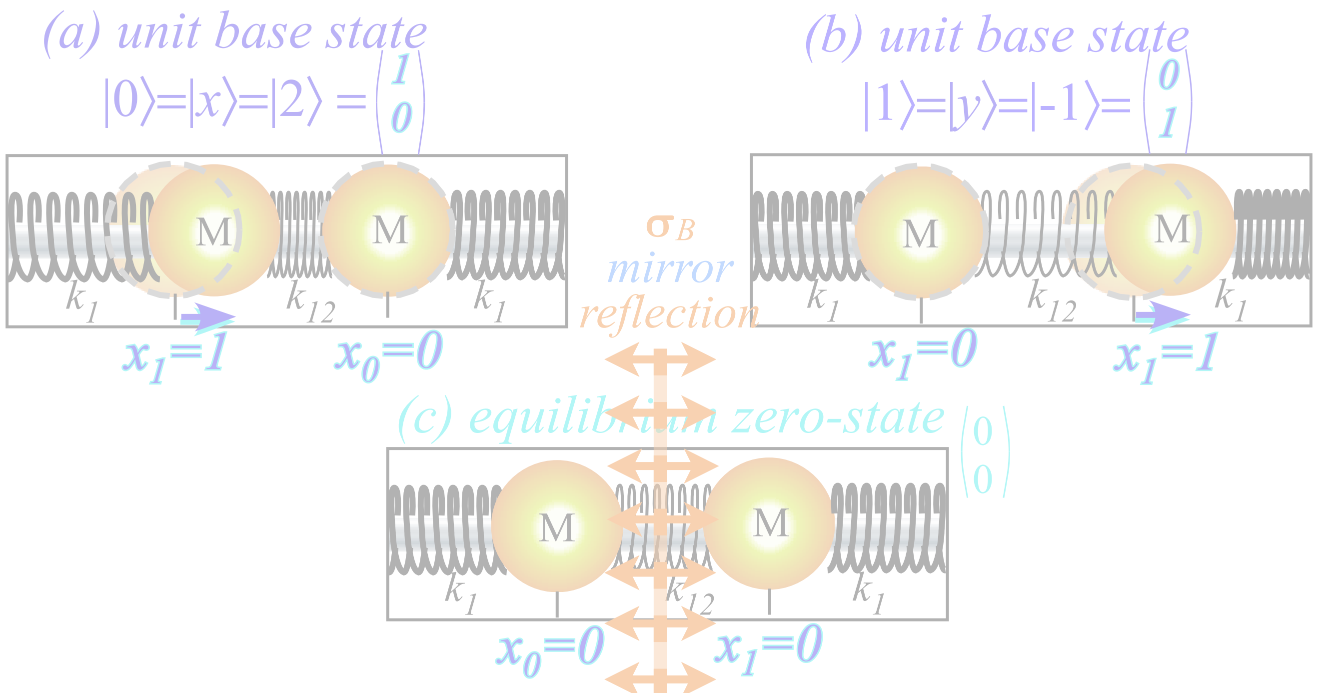

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Minimal equation of σ_B is: $\sigma_B{}^2 = 1$

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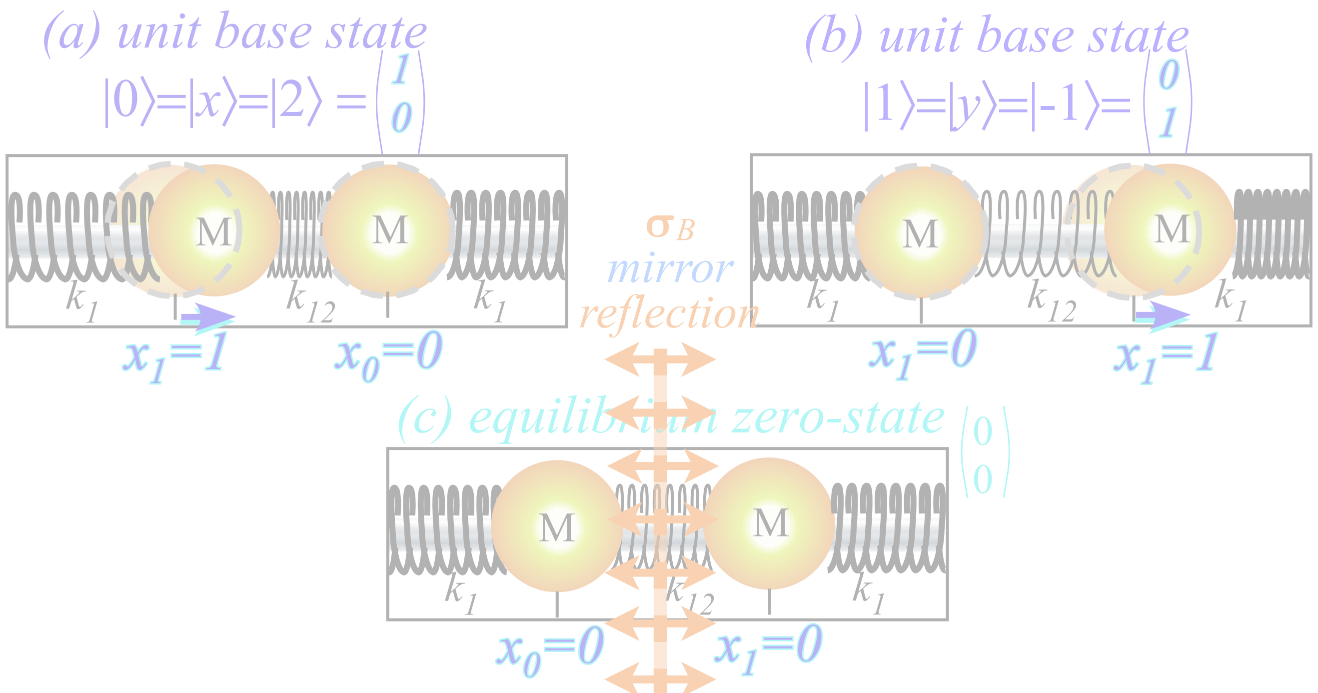

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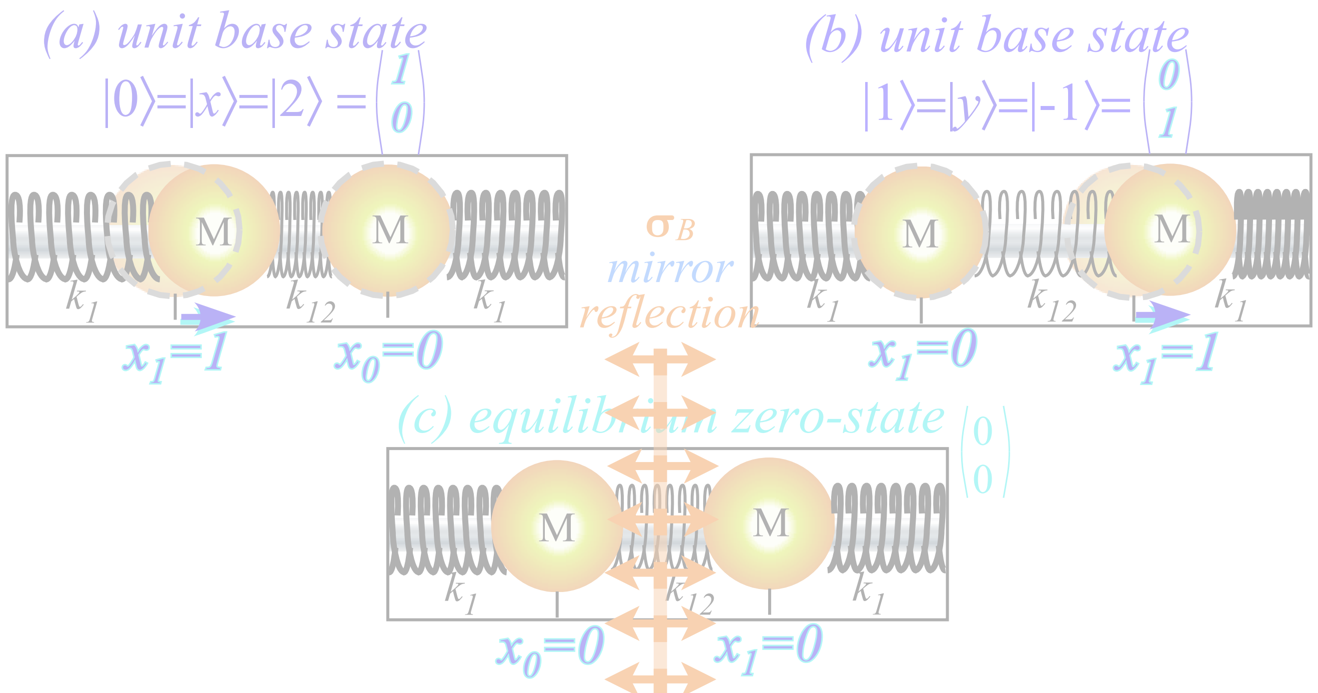
$$\begin{pmatrix} \langle \mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle \mathbf{1}|\mathbf{1}|\sigma_B\rangle \\ \langle \sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle \sigma_B|\mathbf{1}|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \langle \mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle \mathbf{1}|\sigma_B|\sigma_B\rangle \\ \langle \sigma_B|\sigma_B|\mathbf{1}\rangle & \langle \sigma_B|\sigma_B|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2 = 1$

or: $\sigma_B{}^2 - 1 = 0 = (\sigma_B - 1)(\sigma_B + 1)$

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0,x_1\}$



2D HO Matrix operator equations

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}\frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M}\\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$=-\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\,|\mathbf{x}\rangle$$

More conventional coordinate notation $\{x_0,x_1\}\longrightarrow\{x_1,x_2\}$

C_2 (Bilateral σ_B reflection) symmetry conditions:

$K_{11}\equiv K\equiv K_{22}$ and: $K_{12}\equiv k\equiv K_{12}=-k_{12}$ (Let: $M=1$)

$$\begin{pmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{pmatrix}=\begin{pmatrix}K & k\\ k & K\end{pmatrix}=K\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+k\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\mathbf{K}= K\cdot\mathbf{1} + k\cdot\sigma_B$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

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$$\begin{pmatrix}\langle\mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle\mathbf{1}|\mathbf{1}|\sigma_B\rangle\\ \langle\sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle\sigma_B|\mathbf{1}|\sigma_B\rangle\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad \begin{pmatrix}\langle\mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle\mathbf{1}|\sigma_B|\sigma_B\rangle\\ \langle\sigma_B|\sigma_B|\mathbf{1}\rangle & \langle\sigma_B|\sigma_B|\sigma_B\rangle\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

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or: $\sigma_B{}^2-1=0=(\sigma_B-1)(\sigma_B+1)$

with eigenvalues:

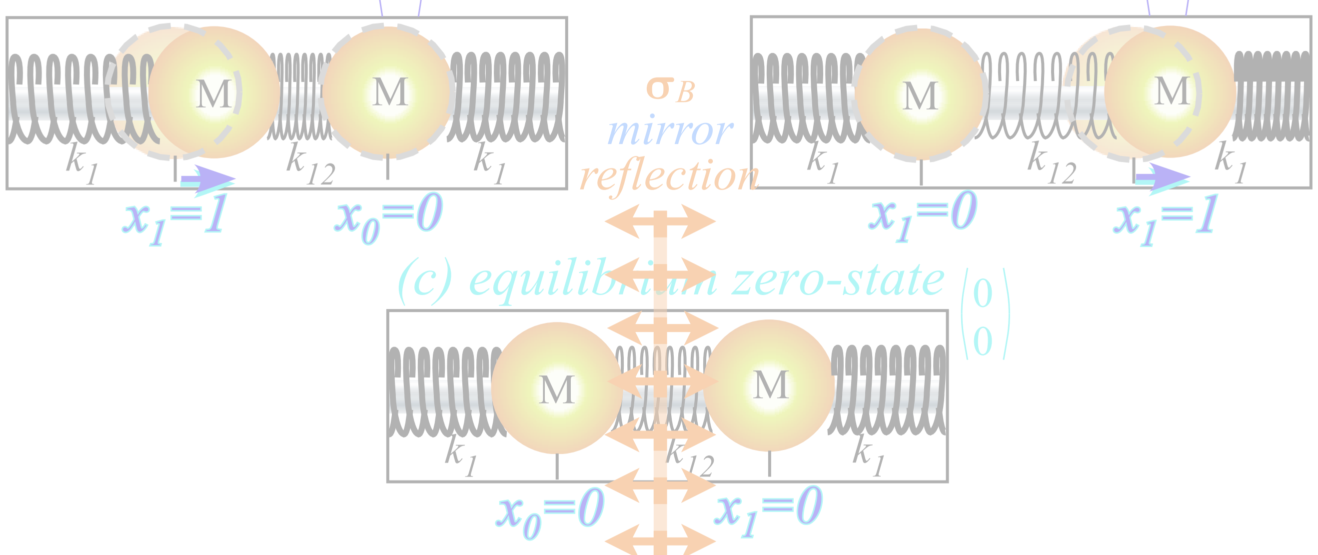
$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0,x_1\}$

(a) unit base state $|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix}1\\0\end{pmatrix}$

(b) unit base state $|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix}0\\1\end{pmatrix}$



2D HO Matrix operator equations

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix}=-\begin{pmatrix}\frac{k_1+k_{12}}{M} & \frac{-k_{12}}{M}\\ \frac{-k_{12}}{M} & \frac{k_1+k_{12}}{M}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

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K-matrix is made of its symmetry operators in

group $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

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$$\begin{pmatrix}\langle\mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle\mathbf{1}|\mathbf{1}|\sigma_B\rangle\\ \langle\sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle\sigma_B|\mathbf{1}|\sigma_B\rangle\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad \begin{pmatrix}\langle\mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle\mathbf{1}|\sigma_B|\sigma_B\rangle\\ \langle\sigma_B|\sigma_B|\mathbf{1}\rangle & \langle\sigma_B|\sigma_B|\sigma_B\rangle\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

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or: $\sigma_B{}^2-1=0=(\sigma_B-1)(\sigma_B+1)$

with eigenvalues:

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

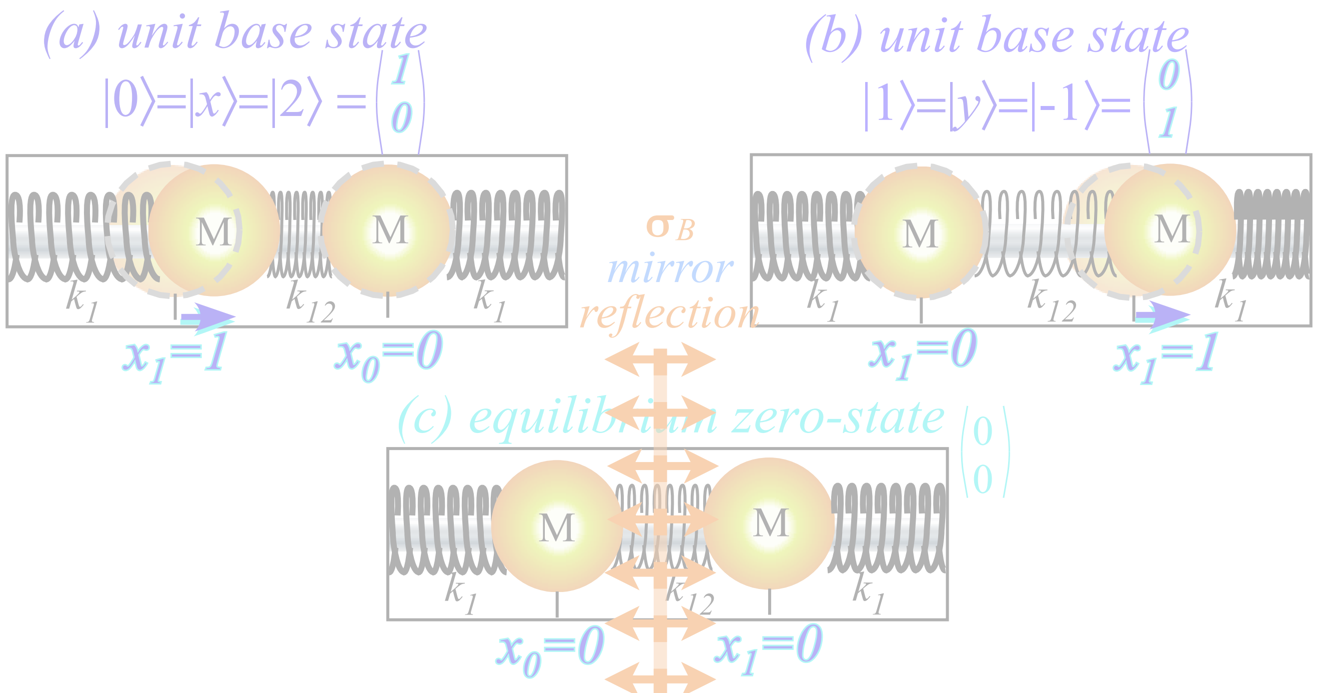
Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+,\mathbf{P}^-\}$

$$\mathbf{1}\;=\mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B=\mathbf{P}^+ - \mathbf{P}^-$$

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

2D HO "binary" bases and coord. $\{x_0, x_1\}$



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$$= -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$\mathbf{K} = K \cdot \mathbf{1} + k \cdot \sigma_B$$

K-matrix is made of its symmetry operators in

group $C_2 = \{\mathbf{1}, \sigma_B\}$ with product table:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

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$$\begin{pmatrix} \langle \mathbf{1}|\mathbf{1}|\mathbf{1}\rangle & \langle \mathbf{1}|\mathbf{1}|\sigma_B\rangle \\ \langle \sigma_B|\mathbf{1}|\mathbf{1}\rangle & \langle \sigma_B|\mathbf{1}|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \langle \mathbf{1}|\sigma_B|\mathbf{1}\rangle & \langle \mathbf{1}|\sigma_B|\sigma_B\rangle \\ \langle \sigma_B|\sigma_B|\mathbf{1}\rangle & \langle \sigma_B|\sigma_B|\sigma_B\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Minimal equation of σ_B is: $\sigma_B{}^2 = 1$

or: $\sigma_B{}^2 - 1 = 0 = (\sigma_B - 1)(\sigma_B + 1)$

with eigenvalues:
$\{\chi^+(\sigma_B) = +1, \; \chi^-(\sigma_B) = -1\}$

Spectral decomposition of $C_2(\sigma_B)$ into $\{\mathbf{P}^+, \mathbf{P}^-\}$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$\mathbf{P}^{\pm}$-projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^+ = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Review:How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \quad = \quad K\cdot\mathbf{1} \quad - \quad k_{12}\cdot\sigma_B$$

$$\begin{pmatrix} K & k \\ k & K \end{pmatrix} = K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \quad = \quad K\cdot\mathbf{1} \quad - \quad k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1} + \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbf{P}^- = \frac{\mathbf{1} - \sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$$\varepsilon^+(\mathbf{K}) = K - k_{12}, \qquad \varepsilon^-(\mathbf{K}) = K + k_{12}$$

$$= k_1 \qquad\qquad = k_1 + 2k_{12}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}\ -\ k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K}) = K - k_{12}$, $\quad \varepsilon^-(\mathbf{K}) = K + k_{12}$

$\quad = k_1 \qquad\qquad = k_1 + 2k_{12}$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \quad = \quad K\cdot\mathbf{1} \quad - \quad k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - k_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2 = \{\mathbf{1}, \sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+, \mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = |+\rangle\langle +|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = |-\rangle\langle -|$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$$

Eigenvalues of $\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$$\varepsilon^+(\mathbf{K}) = K - k_{12}, \qquad \varepsilon^-(\mathbf{K}) = K + k_{12}$$
$$= k_1 \qquad\qquad\qquad = k_1 + 2k_{12}$$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_1+k_{12} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} k_1 & 0 \\ 0 & k_1+2k_{12} \end{pmatrix}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - k_{12}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators

in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

factored projectors

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

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Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}\ -\ k_{12}\cdot\sigma_B$:

$$\varepsilon^+(\mathbf{K}) = K - k_{12},\qquad \varepsilon^-(\mathbf{K}) = K + k_{12}$$
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(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \;=\; K\cdot\mathbf{1} \;-\; k_{12}\cdot\sigma_B$$

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factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

Eigenvalues of σ_B :

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$\qquad = k_1 \qquad\qquad\qquad = k_1+2k_{12}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \;=\; K\cdot\mathbf{1} \;-\; k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix}-k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{P}^+=\frac{\mathbf{1}+\sigma_B}{2}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}=|+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^-=\frac{\mathbf{1}-\sigma_B}{2}=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=|-\rangle\langle-|$$

$$\mathbf{1}=\mathbf{P}^++\mathbf{P}^-$$

$$\sigma_B=\mathbf{P}^+-\mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}\ -\ k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K})=K-k_{12}$, $\quad\varepsilon^-(\mathbf{K})=K+k_{12}$

$=k_1$ $\quad\quad\quad\quad\quad\quad = k_1+2k_{12}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

$$\begin{pmatrix}\langle+|x_1\rangle&\langle+|x_2\rangle\\\langle-|x_1\rangle&\langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}k_1+k_{12}&k_{12}\\k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle+|x_1\rangle&\langle+|x_2\rangle\\\langle-|x_1\rangle&\langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle&\langle x_1|\mathbf{K}|x_2\rangle\\\langle x_2|\mathbf{K}|x_1\rangle&\langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}=\begin{pmatrix}\langle+|\mathbf{K}|+\rangle&\langle+|\mathbf{K}|-\rangle\\\langle-|\mathbf{K}|+\rangle&\langle-|\mathbf{K}|-\rangle\end{pmatrix}$$

Full Dirac notation

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix}-k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}$ - $k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K}) = K - k_{12}$, $\quad \varepsilon^-(\mathbf{K}) = K + k_{12}$

$= k_1 \qquad\qquad = k_1+2k_{12}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle+|x_1\rangle&\langle+|x_2\rangle\\\langle-|x_1\rangle&\langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}k_1+k_{12}&k_{12}\\k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle+|x_1\rangle&\langle+|x_2\rangle\\\langle-|x_1\rangle&\langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle&\langle x_1|\mathbf{K}|x_2\rangle\\\langle x_2|\mathbf{K}|x_1\rangle&\langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}=\begin{pmatrix}\langle+|\mathbf{K}|+\rangle&\langle+|\mathbf{K}|-\rangle\\\langle-|\mathbf{K}|+\rangle&\langle-|\mathbf{K}|-\rangle\end{pmatrix}$$

Full Dirac notation

$$\mathbf{T}(\pm\leftarrow x_j)\cdot\mathbf{K}\cdot\mathbf{T}^\dagger(\pm\leftarrow x_j)$$

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$ projectors:

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac12&\frac12\\ \frac12&\frac12\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{pmatrix} = |+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac12&-\frac12\\ -\frac12&\frac12\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix} = |-\rangle\langle-|$$

Eigenvalues of σ_B :

$$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$:

$$\varepsilon^+(\mathbf{K}) = K - k_{12} = k_1, \qquad \varepsilon^-(\mathbf{K}) = K + k_{12} = k_1+2k_{12}$$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\ -k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix} = \begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle+|x_1\rangle & \langle+|x_2\rangle\\ \langle-|x_1\rangle & \langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}k_1+k_{12}&k_{12}\\ k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix} = \begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

$$\begin{pmatrix}\langle+|x_1\rangle & \langle+|x_2\rangle\\ \langle-|x_1\rangle & \langle-|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|\mathbf{K}|x_1\rangle & \langle x_1|\mathbf{K}|x_2\rangle\\ \langle x_2|\mathbf{K}|x_1\rangle & \langle x_2|\mathbf{K}|x_2\rangle\end{pmatrix}\begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix} = \begin{pmatrix}\langle+|\mathbf{K}|+\rangle & \langle+|\mathbf{K}|-\rangle\\ \langle-|\mathbf{K}|+\rangle & \langle-|\mathbf{K}|-\rangle\end{pmatrix}$$

Full Dirac notation

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\ \frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

$$\mathbf{T}(\pm\leftarrow x_j)|x_1\rangle = |+\rangle$$
$$\mathbf{T}(\pm\leftarrow x_j)|x_2\rangle = |-\rangle$$

$$\mathbf{T}(\pm\leftarrow x_j)\cdot\mathbf{K}\cdot\mathbf{T}^\dagger(\pm\leftarrow x_j)$$

$$|x_1\rangle = \mathbf{T}^\dagger(\pm\leftarrow x_j)|+\rangle$$
$$|x_2\rangle = \mathbf{T}^\dagger(\pm\leftarrow x_j)|-\rangle$$

$$\mathbf{T}^\dagger(\pm\leftarrow x_j) = \mathbf{T}(x_j\leftarrow\pm)$$

Review:How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

C_2 Symmetric 2DHO eigensolutions

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$$K\begin{pmatrix}1&0\\0&1\end{pmatrix}-k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}$$

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factored projectors

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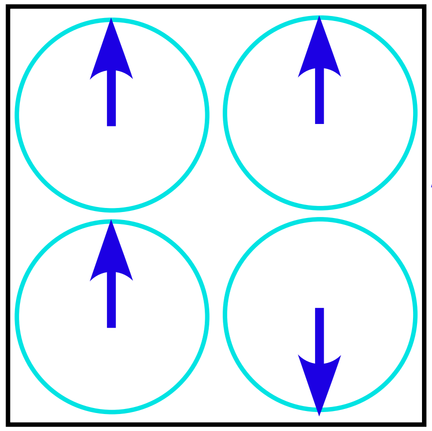
$\varepsilon^+(\mathbf{K})=K-k_{12},\qquad \varepsilon^-(\mathbf{K})=K+k_{12}$

$\quad=k_1 \qquad\qquad\qquad\quad =k_1+2k_{12}$

C_2 mode phase character tables

p is position

	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1



norm: $1/\sqrt{2}$

m is wave-number
or "momentum"

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

C_2 Symmetric 2DHO eigensolutions

$$\mathbf{K} \quad = \quad K\cdot\mathbf{1} \quad - \quad k_{12}\cdot\sigma_B$$

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$$\mathbf{1}=\mathbf{P}^++\mathbf{P}^-$$

factored projectors

$$\sigma_B=\mathbf{P}^+-\mathbf{P}^-$$

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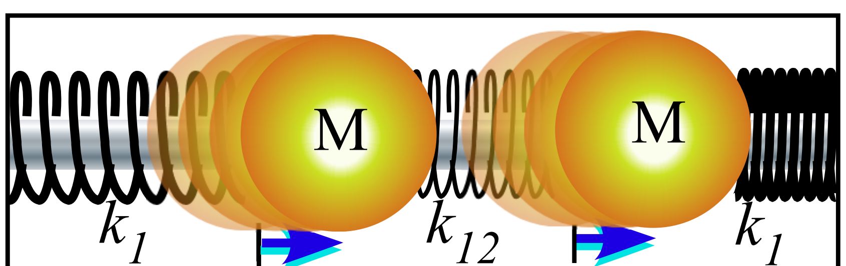
Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}\ -\ k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K})=K-k_{12}, \qquad \varepsilon^-(\mathbf{K})=K+k_{12}$

$\quad = k_1 \qquad\qquad\qquad\quad = k_1+2k_{12}$

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2} \qquad x_1=1/\sqrt{2}$

C_2 mode phase character tables

p is position

	$p=0$	$p=1$
$m=0_2$	1	1
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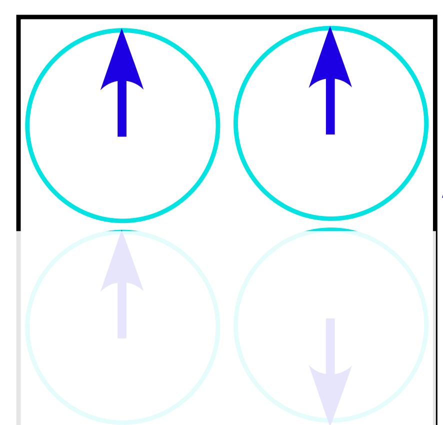

norm: $1/\sqrt{2}$

m is wave-number or "momentum"

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

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$$\mathbf{K} = K\cdot\mathbf{1} - k_{12}\cdot\sigma_B$$

$$K\begin{pmatrix}1&0\\0&1\end{pmatrix} - k_{12}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}$$

K-matrix is made of its symmetry operators

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$$\mathbf{P}^+ = \frac{\mathbf{1}+\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\ \frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix} = |+\rangle\langle+|$$

$$\mathbf{1} = \mathbf{P}^+ + \mathbf{P}^-$$

factored projectors

$$\sigma_B = \mathbf{P}^+ - \mathbf{P}^-$$

$$\mathbf{P}^- = \frac{\mathbf{1}-\sigma_B}{2} = \begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\ -\frac{1}{2}&\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = |-\rangle\langle-|$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B) = +1,\ \chi^-(\sigma_B) = -1\}$

Diagonalizing transformation (D-tran) of K-matrix:

$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12} & -k_{12}\\ -k_{12} & k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}k_1 & 0\\ 0 & k_1+2k_{12}\end{pmatrix}$$

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$\varepsilon^+(\mathbf{K}) = K - k_{12}$, $\quad\varepsilon^-(\mathbf{K}) = K + k_{12}$

$= k_1$ $\quad\quad = k_1+2k_{12}$

(D-tran)

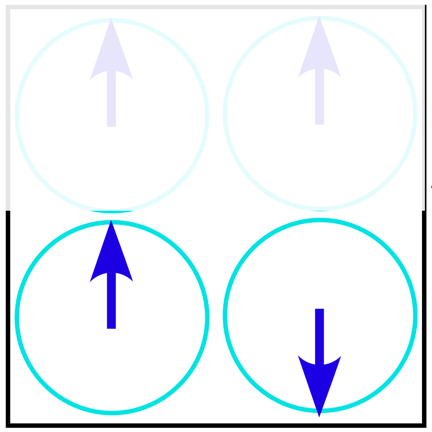
$$\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} = \begin{pmatrix}\langle x_1|+\rangle & \langle x_1|-\rangle\\ \langle x_2|+\rangle & \langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

C_2 *mode phase character tables*

p is position

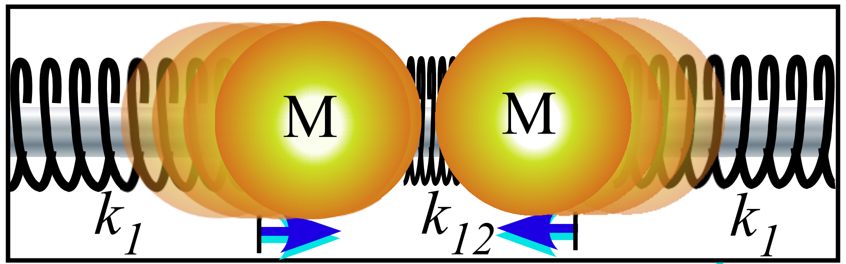
	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1



norm: $1/\sqrt{2}$

m is wave-number or "momentum"

(b) *Odd mode* $|-\rangle = |1_2\rangle = \begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$



$x_0 = 1/\sqrt{2}$ $\quad x_1 = -1/\sqrt{2}$

C_2 Symmetric 2DHO eigensolutions

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in *group* $C_2=\{\mathbf{1},\sigma_B\}$ with *product table*:

$C_2(\sigma_B)$ spectrally decomposed into $\{\mathbf{P}^+,\mathbf{P}^-\}$projectors:

$$\mathbf{P}^+=\frac{\mathbf{1}+\sigma_B}{2}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}=|+\rangle\langle+|$$

factored projectors

$$\mathbf{P}^-=\frac{\mathbf{1}-\sigma_B}{2}=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{-1}{\sqrt{2}}\end{pmatrix}\otimes\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=|-\rangle\langle-|$$

$$\mathbf{1}=\mathbf{P}^++\mathbf{P}^-$$

$$\sigma_B=\mathbf{P}^+-\mathbf{P}^-$$

Eigenvalues of σ_B :

$\{\chi^+(\sigma_B)=+1,\ \chi^-(\sigma_B)=-1\}$

Diagonalizing transformation (D-tran) of K-matrix:

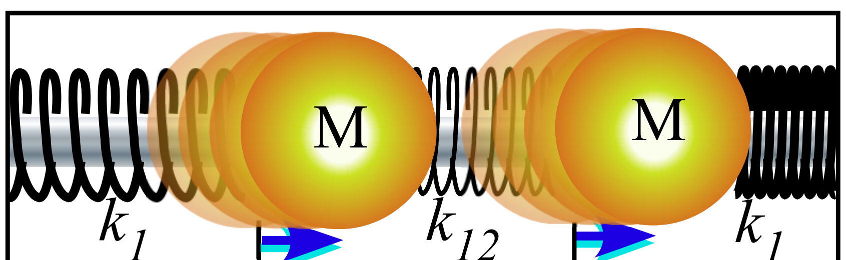
$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}k_1+k_{12}&-k_{12}\\-k_{12}&k_1+k_{12}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}k_1&0\\0&k_1+2k_{12}\end{pmatrix}$$

Eigenvalues of $\mathbf{K}=K\cdot\mathbf{1}\ -\ k_{12}\cdot\sigma_B$:

$\varepsilon^+(\mathbf{K})=K-k_{12},\qquad \varepsilon^-(\mathbf{K})=K+k_{12}$

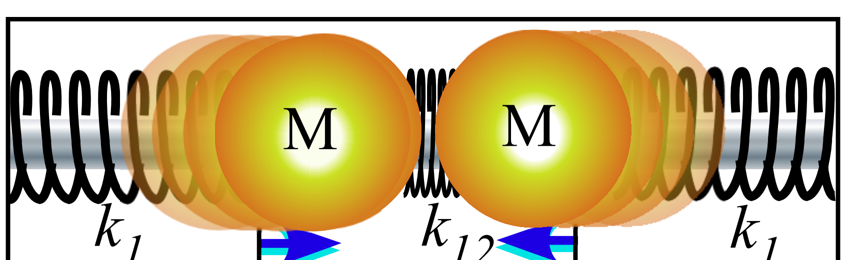
$\quad =k_1 \qquad\qquad\qquad\qquad =k_1+2k_{12}$

Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix}1\\1\end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2} \qquad x_1=1/\sqrt{2}$

Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix}1\\-1\end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2} \qquad x_1=-1/\sqrt{2}$

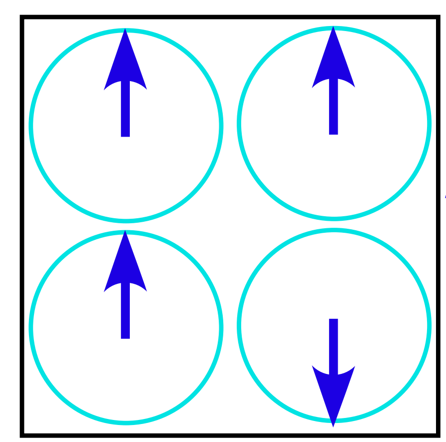
C_2 *mode phase character tables*

(D-tran)

$$\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{2}}\end{pmatrix}=$$

p is position

	p=0	*p=1*
m=0$_2$	1	1
m=1$_2$	1	-1



norm:
$1/\sqrt{2}$

$$\begin{pmatrix}\langle x_1|+\rangle&\langle x_1|-\rangle\\\langle x_2|+\rangle&\langle x_2|-\rangle\end{pmatrix}$$

(D-tran is its own inverse in this case!)

m is wave-number
or "momentum"

Review:How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

C_2 Symmetric 2DHO uncoupling

2D HO Matrix operator equations are coupled in $\{x_1, x_2\}$-basis

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_1 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$|\ddot{\mathbf{x}}\rangle \quad = \quad - \quad \mathbf{K} \quad |\mathbf{x}\rangle$$

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C_2 *Symmetric 2DHO* ***un****coupled dynamics*

$$\begin{pmatrix} M\ddot{x}_+ + (k_1)x_+ \\ M\ddot{x}_- + (k_1+2k_{12})x_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Each mode runs independently

$(+)$-*mode at frequency* $\omega_+ = \sqrt{(k_1/M)}$

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2D HO Matrix operator equations are coupled in $\{x_1,x_2\}$-basis *...but are* ***un****coupled in* $\{+,-\}$-basis

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C_2 Symmetric 2DHO ***un**coupled dynamics*

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Mixed mode dynamics

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$$\text{so: }\mathbf{x}(t)=e^{-i\omega_+ t}\begin{pmatrix}\frac{1}{2}\\ \frac{1}{2}\end{pmatrix}+e^{-i\omega_- t}\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\end{pmatrix}$$

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C_2 Symmetric 2DHO ***un****coupled dynamics*

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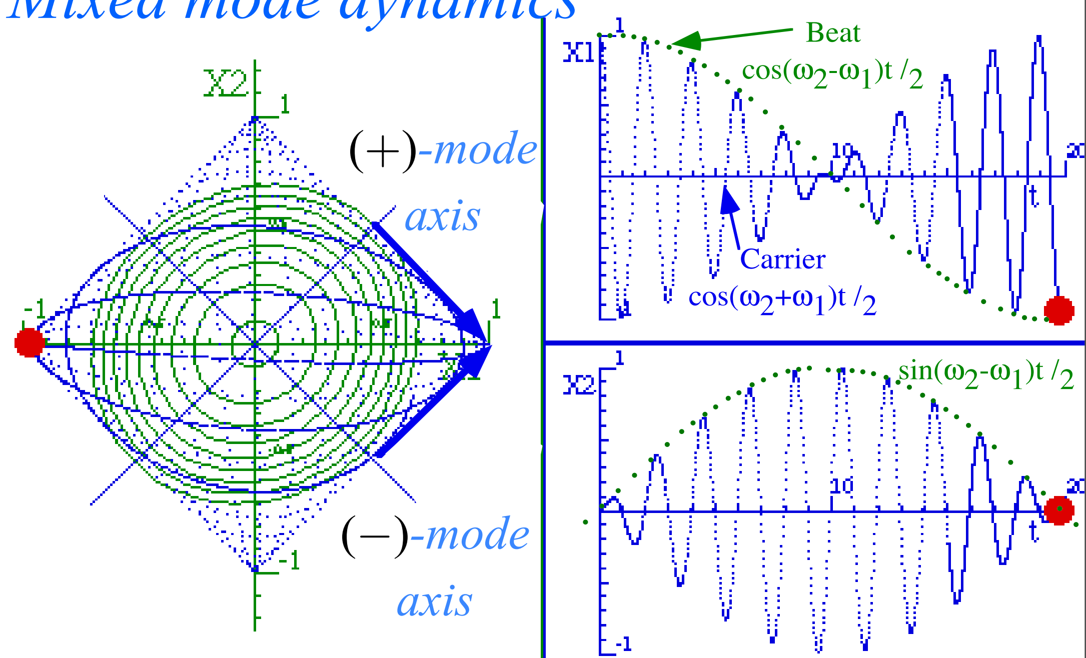

100% AM modulation results

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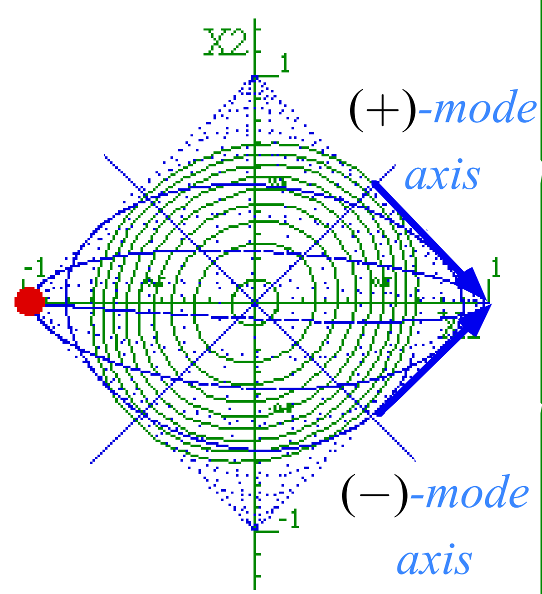

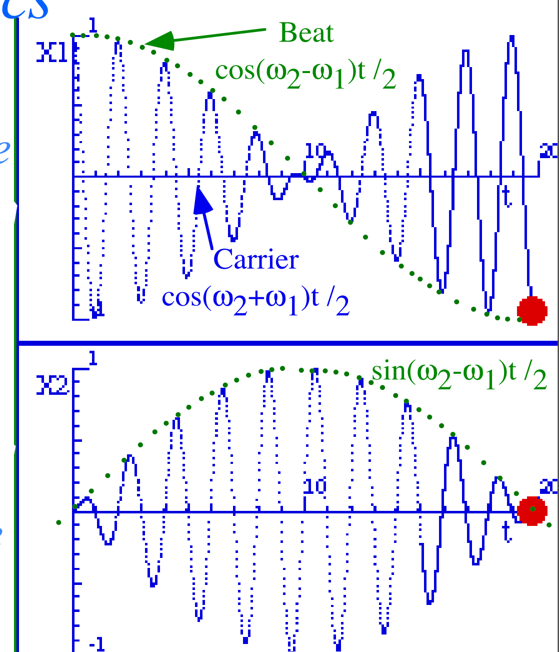

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Note the i *phase*

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How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

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2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

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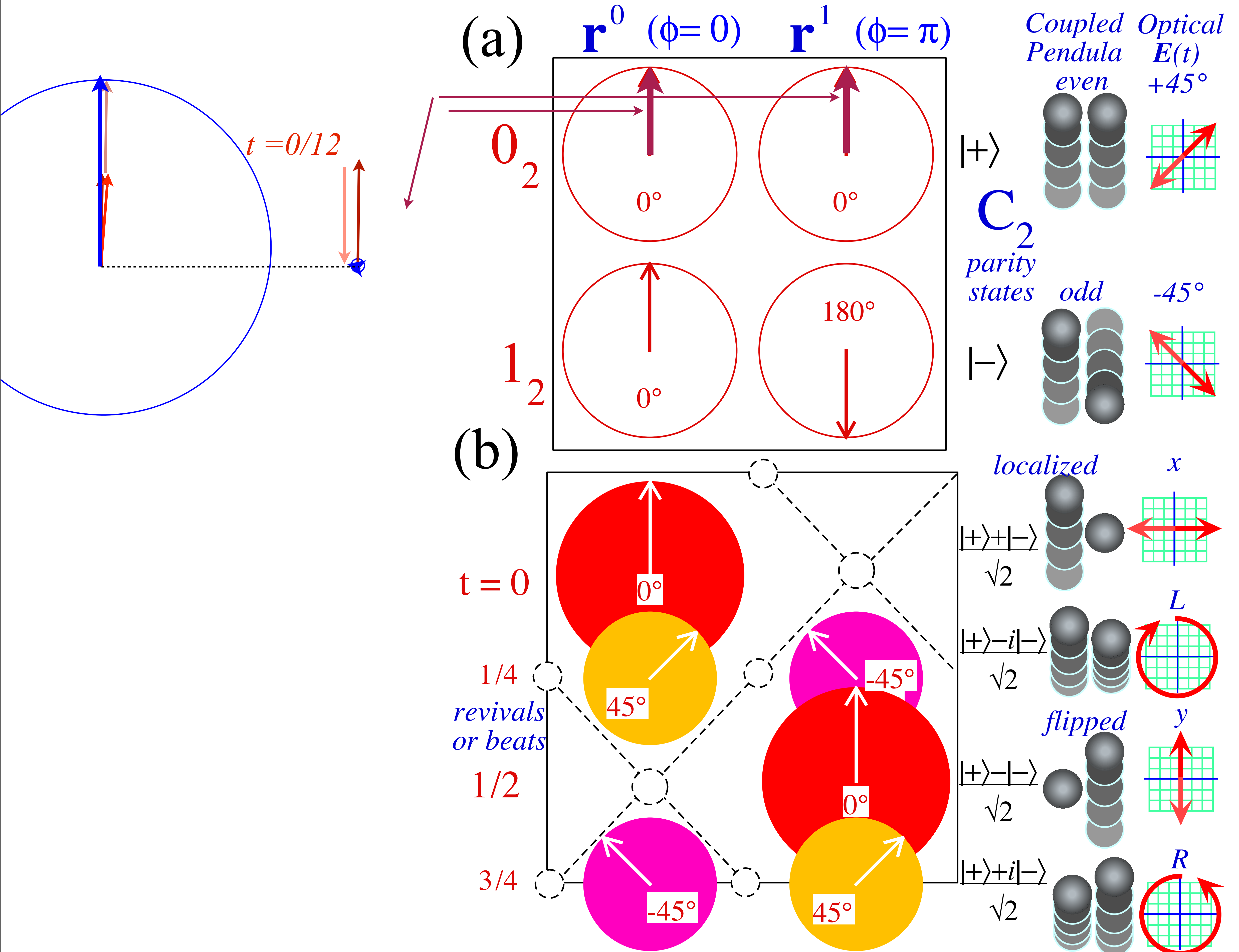
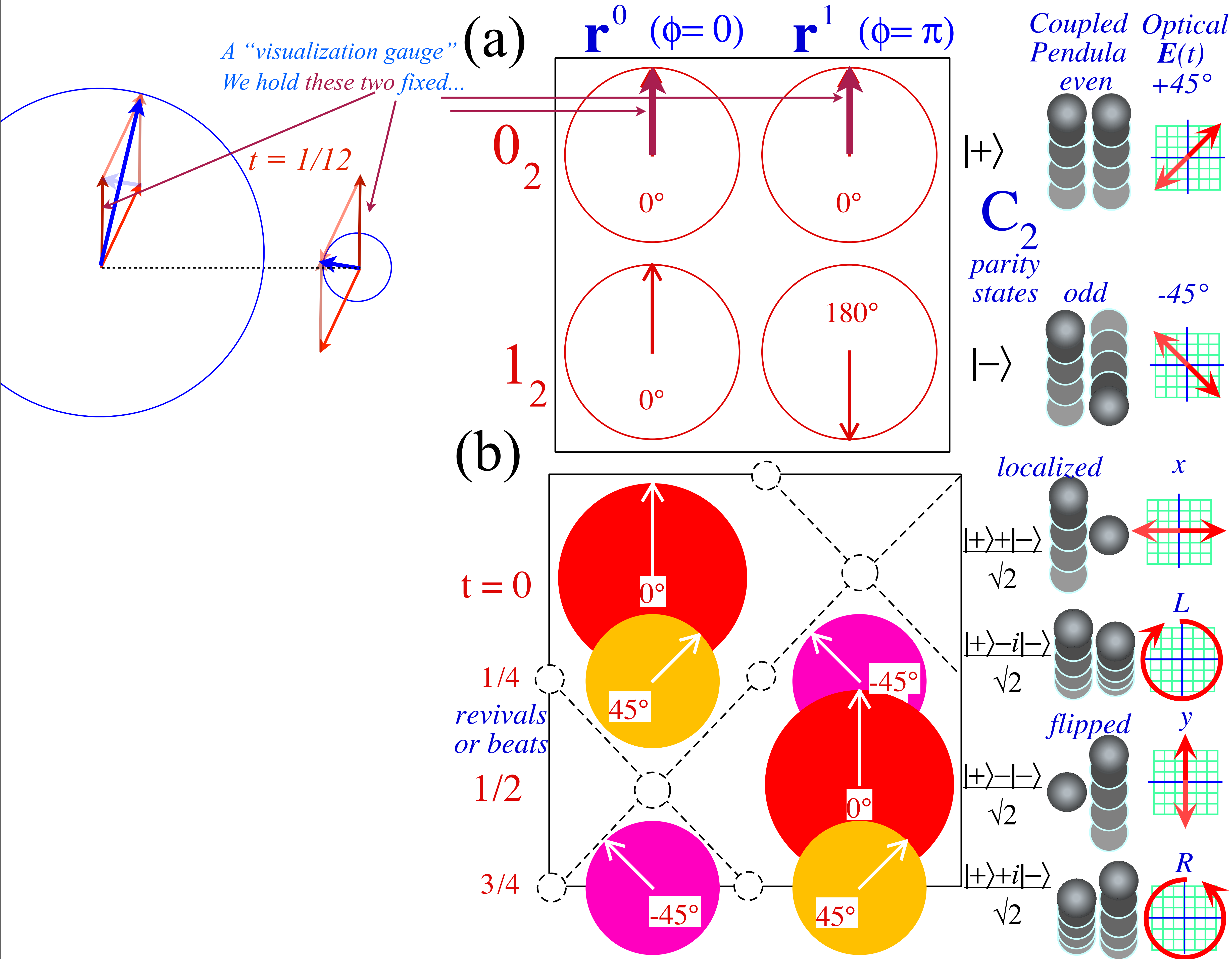

t =0/12
(a)
r^0 (φ= 0)
r^1 (φ= π)
0_2
1_2
0°
0°
0°
180°
Coupled Pendula
Optical E(t)
even
+45°
|+⟩
C_2 parity states
odd
-45°
|−⟩
(b)
t = 0
1/4
revivals or beats
1/2
3/4
0°
45°
-45°
0°
-45°
45°
localized
x
(|+⟩+|−⟩)/√2
L
(|+⟩−i|−⟩)/√2
flipped
y
(|+⟩−|−⟩)/√2
R
(|+⟩+i|−⟩)/√2

2D-HO beats and mixed mode geometry



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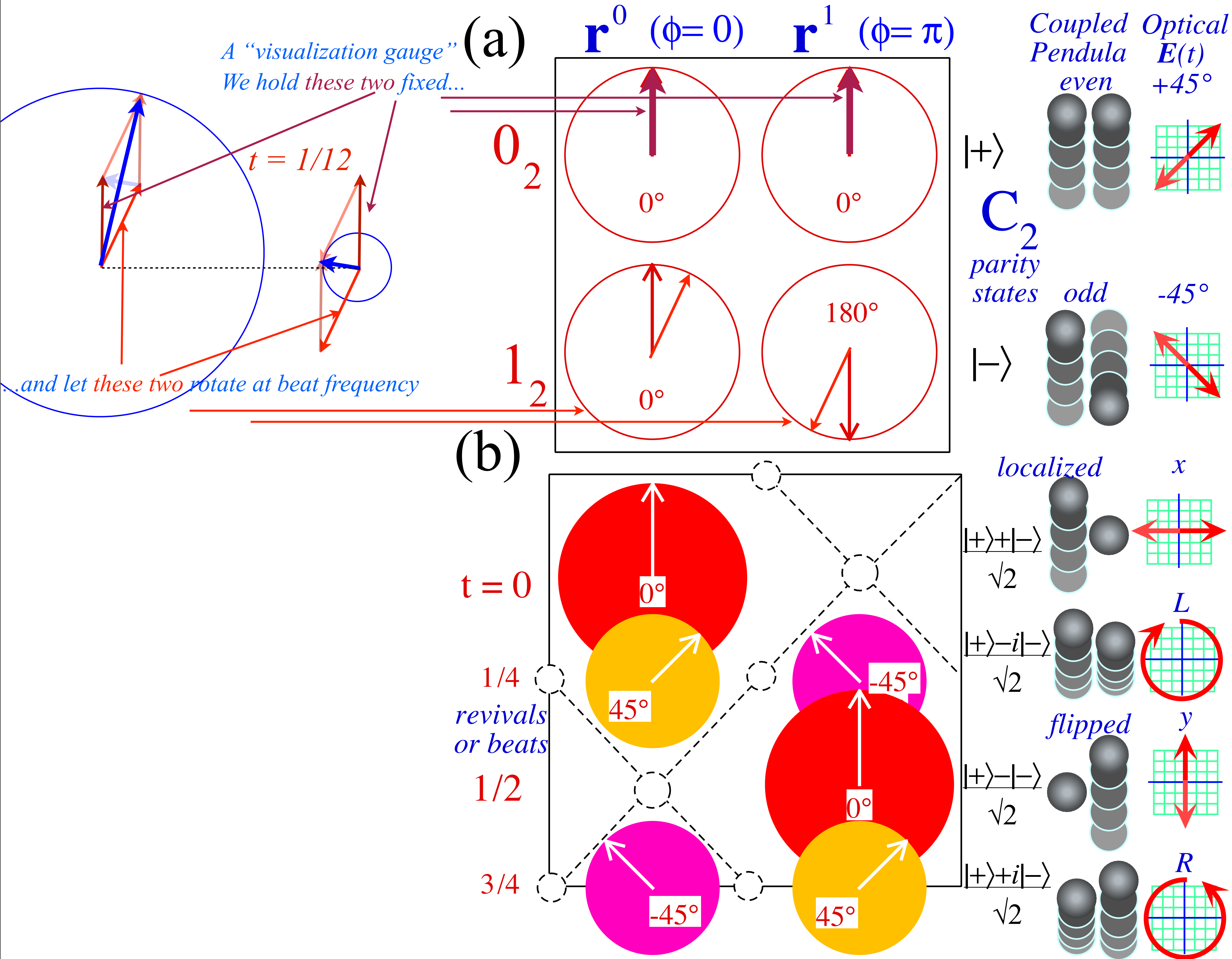

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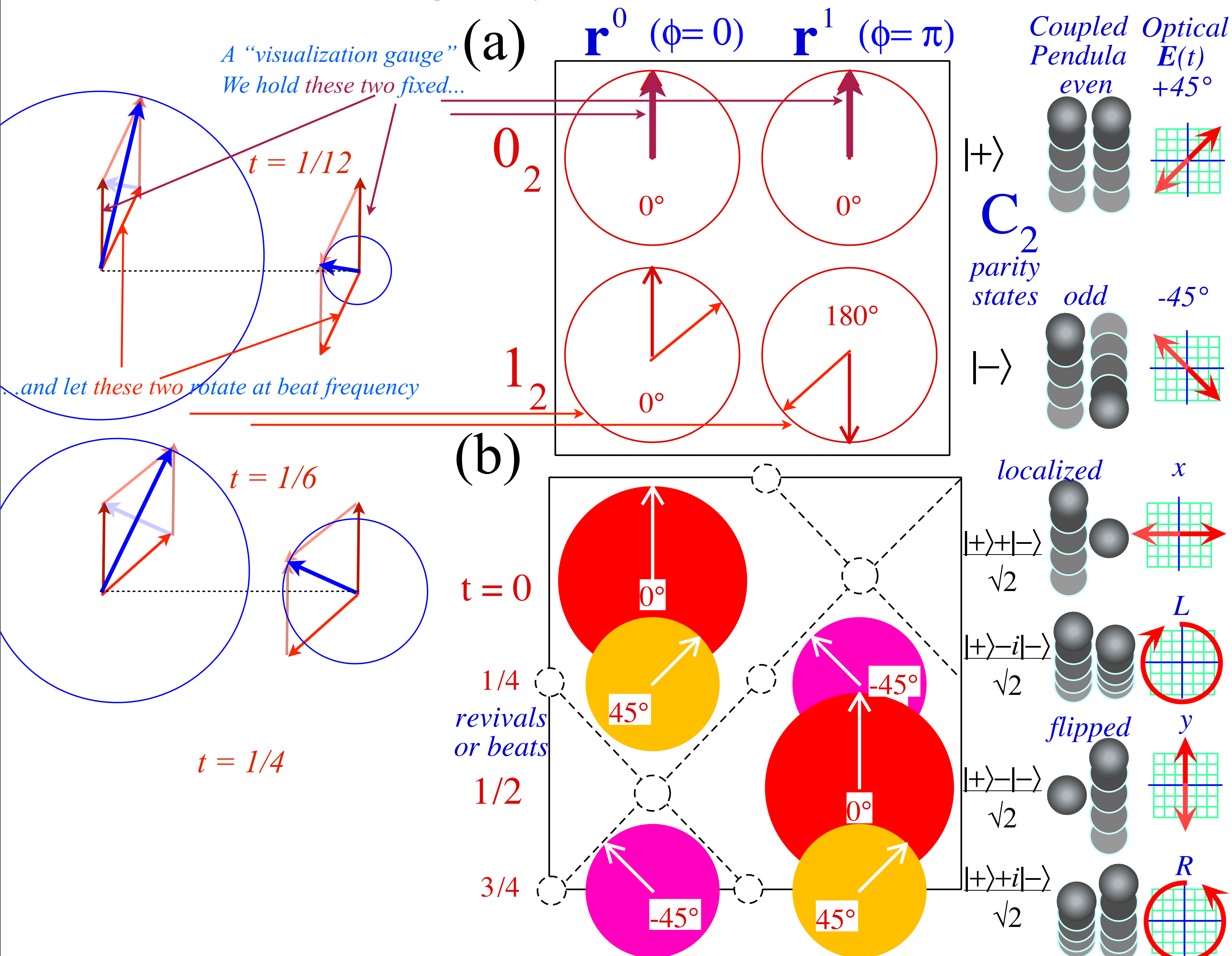

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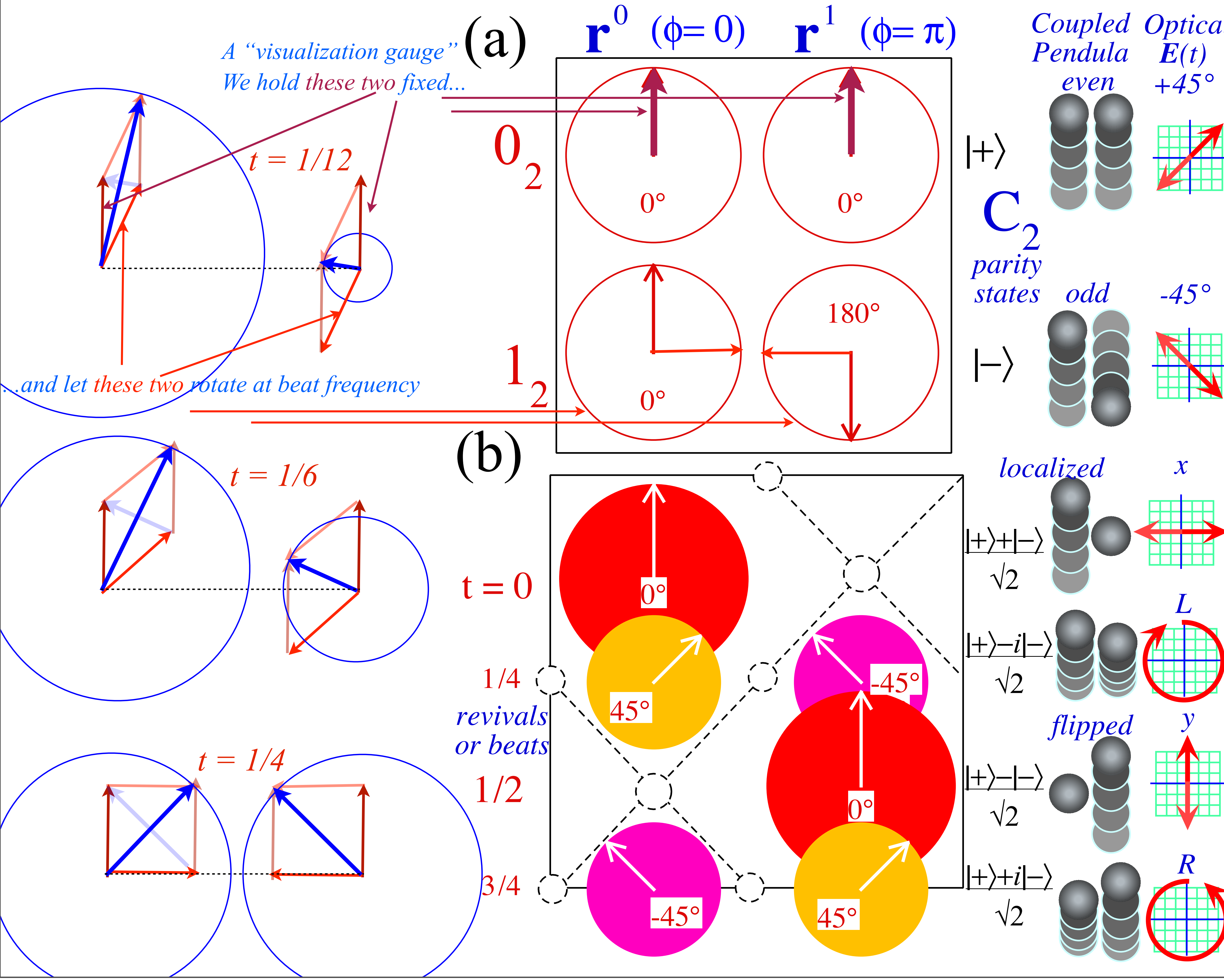

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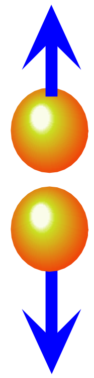
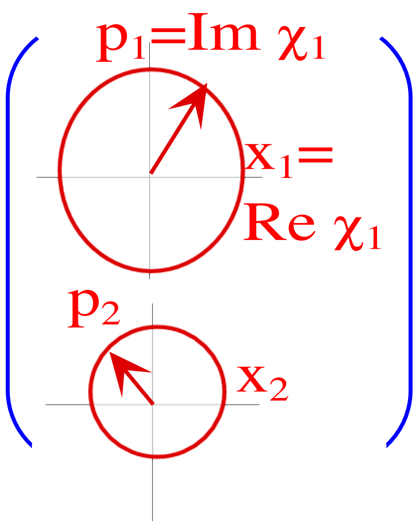
(a) Electron Spin-1/2-Polarization

Spin-up
$|1\rangle = |\uparrow\rangle$



Spin-dn
$|2\rangle = |\downarrow\rangle$

$$|\chi\rangle = \begin{pmatrix} \chi\uparrow \\ \chi\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\chi\rangle \\ \langle\downarrow|\chi\rangle \end{pmatrix} = $$



$$= |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$$

Rabi, Ramsey, and Schwinger 1954
Rev. Mod. Phys. **26** *167 (1954)*

(b) Photon Spin-1-Polarization

Plane-x
$|1\rangle = |x\rangle$

Plane-y
$|2\rangle = |y\rangle$

$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \langle x|\psi\rangle \\ \langle y|\psi\rangle \end{pmatrix} = $$

$$= |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

John Stokes 1862
Proc. Soc. London **11** *547 (1862)*

(c) Ammonia (NH_3) Inversion States

N-UP
$|1\rangle = |UP\rangle$

N-DN
$|2\rangle = |DN\rangle$

$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|\nu\rangle \\ \langle DN|\nu\rangle \end{pmatrix} = $$

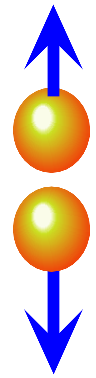
$$= |UP\rangle\langle UP|\nu\rangle + |DN\rangle\langle DN|\nu\rangle$$

Feynman, Vernon, and Hellwarth 1957
J. Appl. Phys. **28** *49 (1957)*

Fig. 10.5.1
QTCA Unit 3 Chapter 10

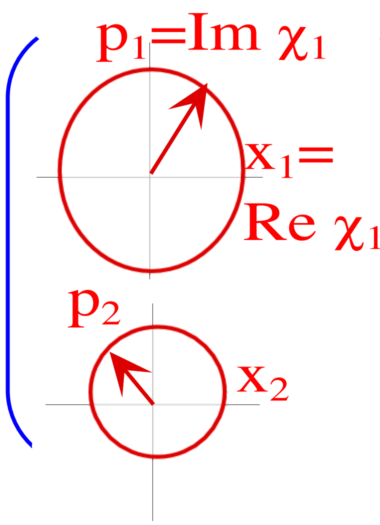
(a) Electron Spin-1/2-Polarization

Spin-up $|1\rangle = |\uparrow\rangle$



Spin-dn $|2\rangle = |\downarrow\rangle$

$$|\chi\rangle = \begin{pmatrix} \chi\uparrow \\ \chi\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\chi\rangle \\ \langle\downarrow|\chi\rangle \end{pmatrix} =$$



$$=|\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$$

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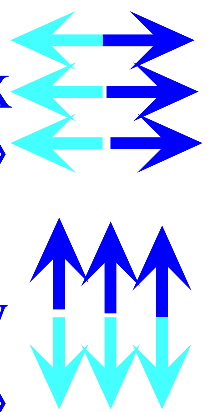
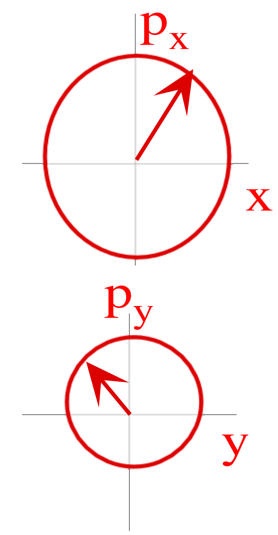
(b) Photon Spin-1-Polarization

Plane-x $|1\rangle = |x\rangle$



Plane-y $|2\rangle = |y\rangle$

$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \langle x|\psi\rangle \\ \langle y|\psi\rangle \end{pmatrix} =$$



$$=|x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

John Stokes 1862
Proc. Soc. London **11** *547 (1862)*

Harter and Dos Santos
Am. J. Phys. **46** *251 (1986)*
J. Chem. Phys. **85** *5560 (1986)*

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N-DN $|2\rangle = |DN\rangle$

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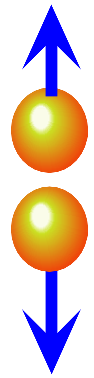
Feynman, Vernon, and Hellwarth 1957
J. Appl. Phys. **28** *49 (1957)*

Fig. 10.5.1
QTCA Unit 3 Chapter 10

Three famous 2-state systems and two-complex-component coordinates

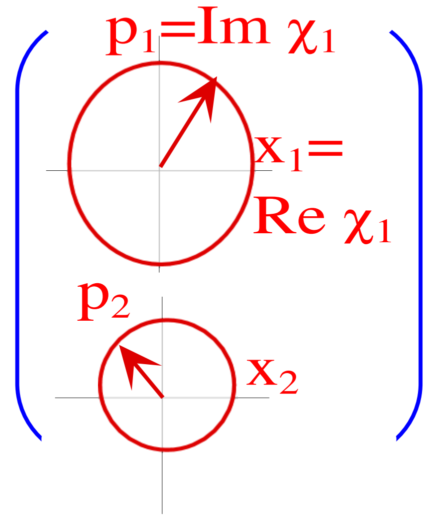
(a) Electron Spin-1/2-Polarization

Spin-up $|1\rangle = |\uparrow\rangle$



Spin-dn $|2\rangle = |\downarrow\rangle$

$$|\chi\rangle = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix} = \begin{pmatrix} \langle\uparrow|\chi\rangle \\ \langle\downarrow|\chi\rangle \end{pmatrix} =$$



$$= |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$$

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Rev. Mod. Phys. **26** *167 (1954)*

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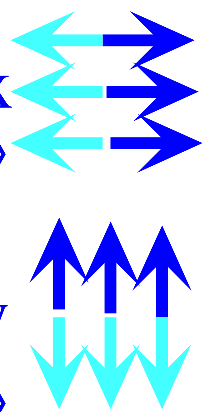
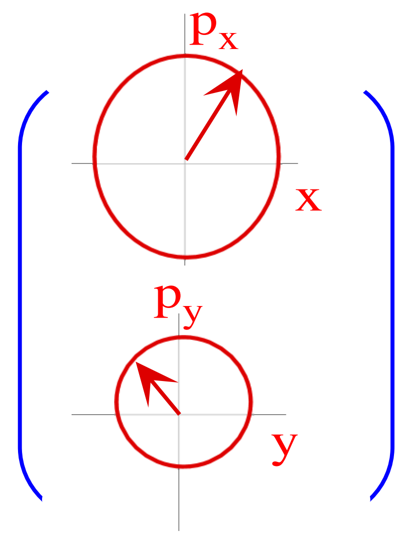


Plane-y $|2\rangle = |y\rangle$

$$|\psi\rangle = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \begin{pmatrix} \langle x|\psi\rangle \\ \langle y|\psi\rangle \end{pmatrix} =$$



$$= |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$$

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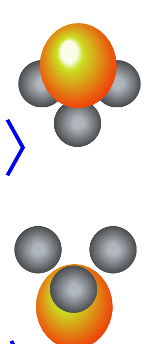
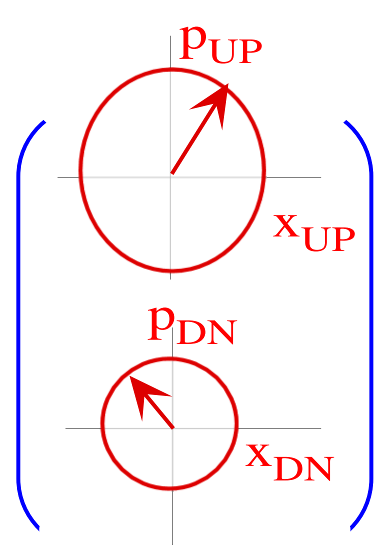


N-DN $|2\rangle = |DN\rangle$

$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|\nu\rangle \\ \langle DN|\nu\rangle \end{pmatrix} =$$



$$= |UP\rangle\langle UP|\nu\rangle + |DN\rangle\langle DN|\nu\rangle$$

Feynman, Vernon, and Hellwarth 1957
J. Appl. Phys. **28** *49 (1957)*

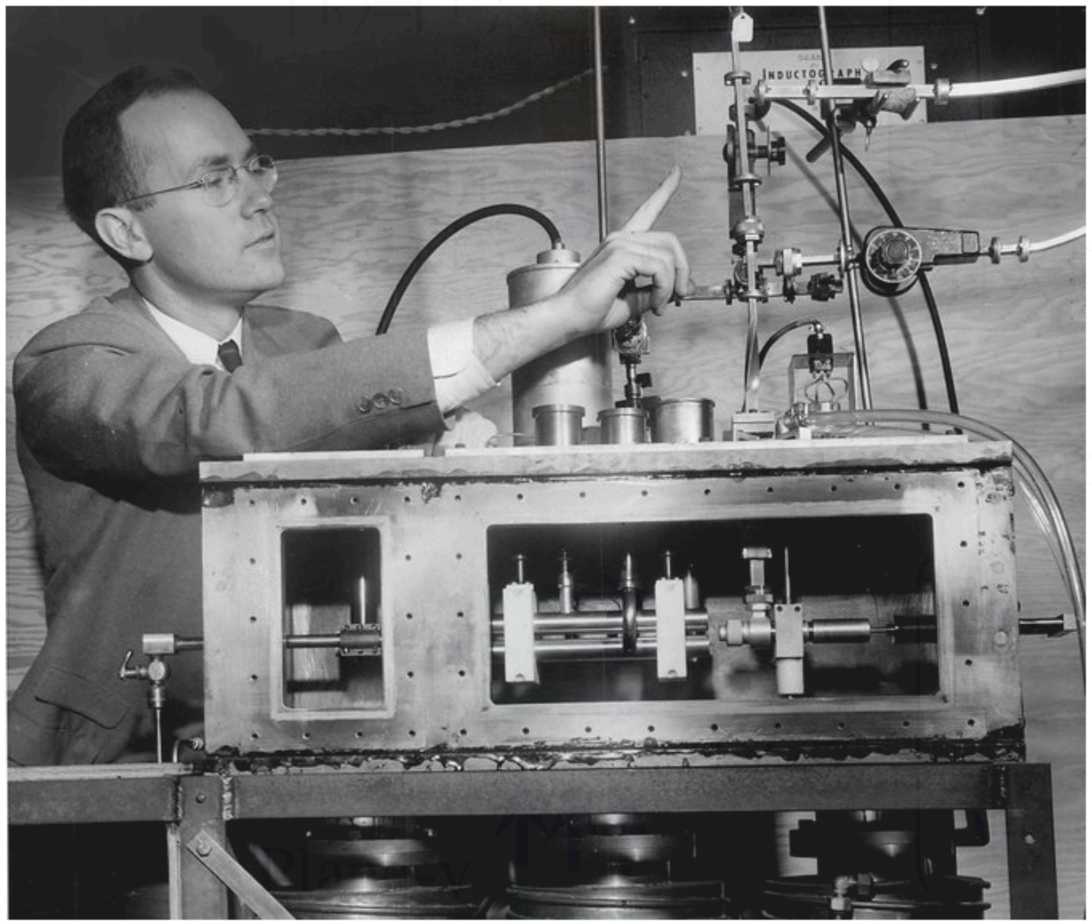
Fig. 10.5.1
QTCA Unit 3 Chapter 10

Three famous 2-state systems and two-complex-component coordinates

(a) Electron Spin-1/2-Polarization

Charles H. Townes, Who Paved Way for the Laser in Daily Life, Dies at 99

By ROBERT D. McFADDEN JAN. 28, 2015



Charles Townes in 1955. Eddie Hausner/The New York Times

The New York Times

Today's Headlines

Thursday, January 29, 2015

He had an "a-ha!" moment. Sitting on a park bench in Washington one April morning in 1951, pondering how to stimulate molecular energy to create shorter wavelengths, he conceived of a device he called a maser, for microwave amplification by stimulated emission of radiation. It would use molecules to nudge other molecules, and amplify their thrust by getting them to resonate like tuning forks and line up in a powerful beam.

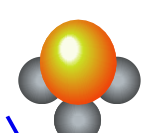
He and two graduate students, James P. Gordon and H. J. Zeigler, built his maser in 1953 and patented their creation. It was the first device operating on the principles of the laser, although it amplified microwave radiation rather than infrared or visible light radiation.

Five years later, Dr. Townes and Dr. Schawlow, who was his brother-in-law and would win the 1981 Nobel Prize in Physics for work on laser spectroscopy, drew a blueprint for a laser. They called it an optical maser, a term that never caught on, and through Bell Laboratories they secured the first laser patent in 1959, a year before Dr. Maiman's first working model.

(c) Ammonia (NH_3) Inversion States

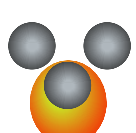
N-UP

$|1\rangle = |UP\rangle$

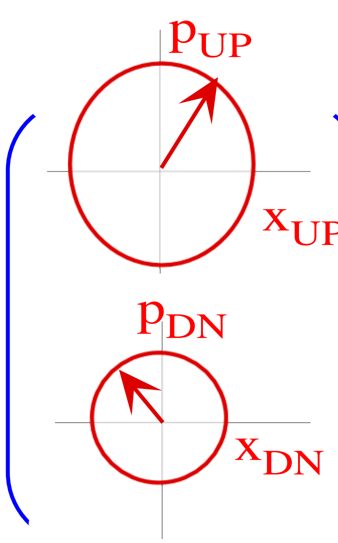


N-DN

$|2\rangle = |DN\rangle$



$$|\nu\rangle = \begin{pmatrix} \nu_{UP} \\ \nu_{DN} \end{pmatrix} = \begin{pmatrix} \langle UP|\nu\rangle \\ \langle DN|\nu\rangle \end{pmatrix} =$$



$$= |UP\rangle\langle UP|\nu\rangle + |DN\rangle\langle DN|\nu\rangle$$

Feynman, Vernon, and Hellwarth 1957

J. Appl. Phys. **28** *49 (1957)*

Fig. 10.5.1

QTCA Unit 3 Chapter 10

2D harmonic oscillator equations

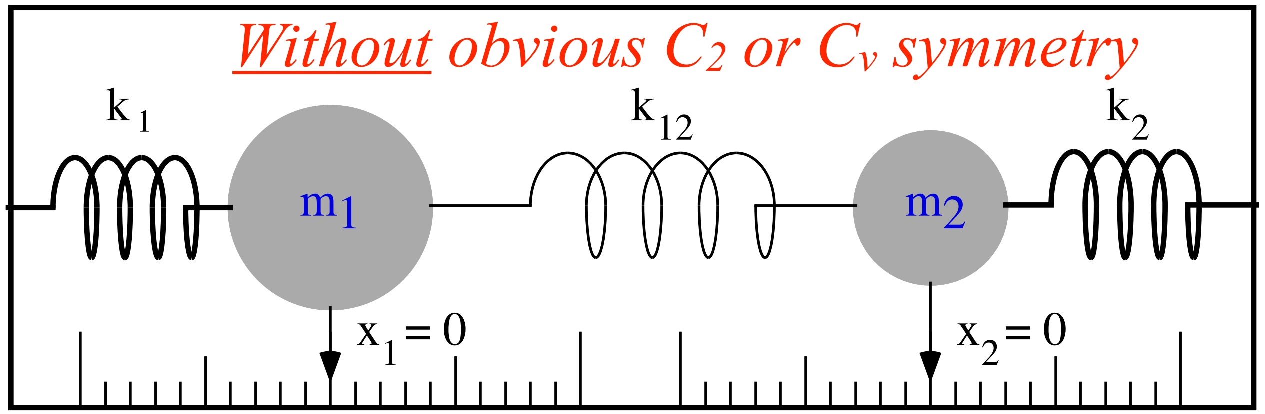
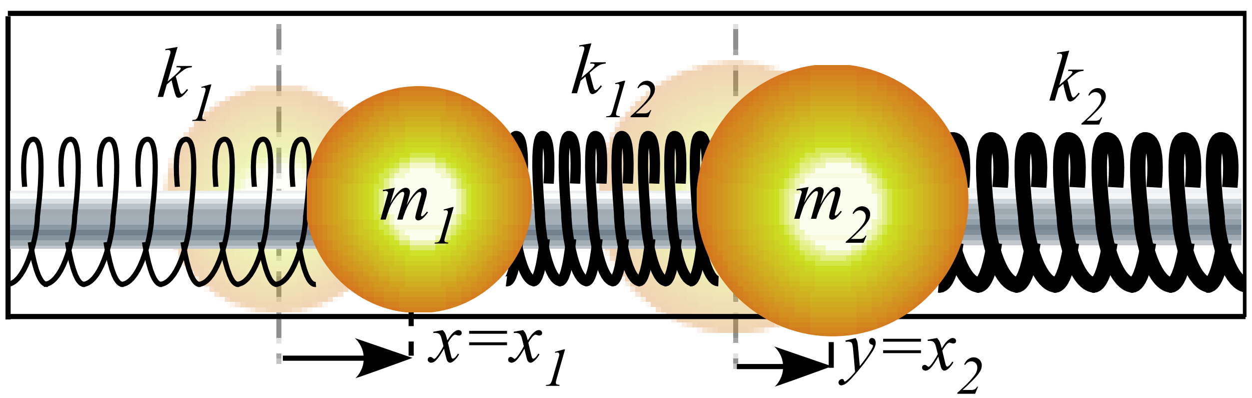


Fig. 3.3.1 Two 1-dimensional coupled oscillators



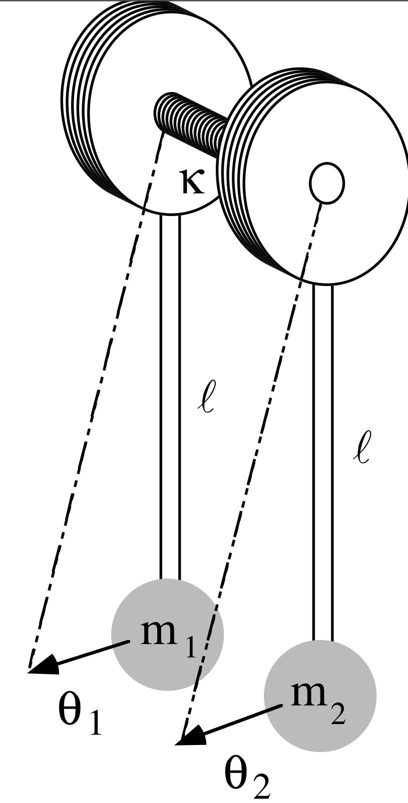


Fig. 3.3.2 Coupled pendulums

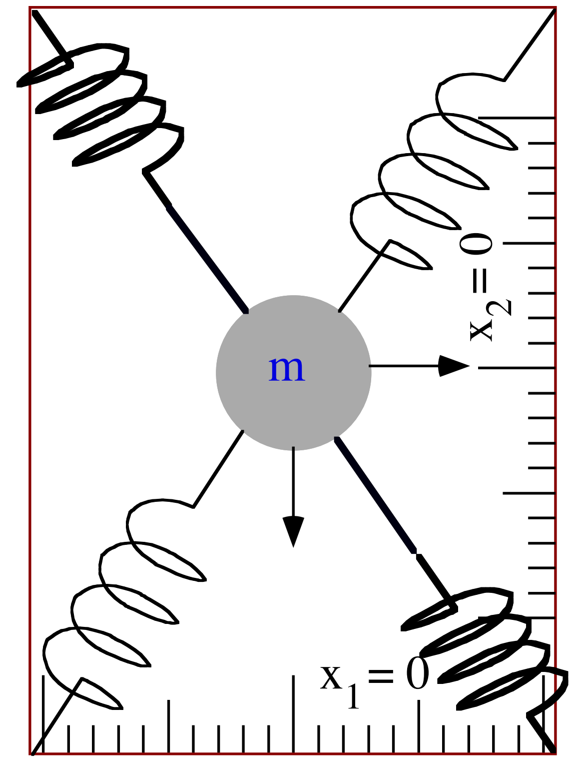


Fig. 3.3.3 One 2-dimensional coupled oscillator

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$= \frac{1}{2} \langle \dot{\mathbf{x}} | \mathbf{M} | \dot{\mathbf{x}} \rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1 + k_{12}) x_1^2 - k_{12} x_1 x_2 + \frac{1}{2}(k_2 + k_{12}) x_2^2$$

$$= \frac{1}{2} \langle \mathbf{x} | \mathbf{K} | \mathbf{x} \rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12}) x_1 + k_{12} x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12} x_1 - (k_2 + k_{12}) x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M} \cdot |\ddot{\mathbf{x}}\rangle \quad = \quad - \quad \mathbf{K} \cdot |\mathbf{x}\rangle$$

2D harmonic oscillator equation solutions

Without obvious C_2 or C_v symmetry

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2+k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

Note eigenvalue is square of eigenfrequency

To introduce eigensolutions we take a simple case of unit masses *(m_1=1=m_2)*

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

$$\boxed{\text{Eigenvectors } |\mathbf{x}\rangle = |\mathbf{e}_n\rangle \text{ are in special directions where } |\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle \text{ is in same direction as } |\mathbf{x}\rangle}$$

Review:How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$-expansion in ABCD-Types) $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

$$i\hbar|\dot{\Psi}(t)\rangle = \mathbf{H}|\Psi(t)\rangle \qquad |\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H}^\dagger$$

H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

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that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Both have 4 parameters
($2^2 = 2+2$)

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

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Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1st-order equation $i\partial_t\Psi = \mathbf{H}\Psi$ into pairs of <u>*real*</u> 1st-order differential equations.

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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$$\dot{x}_1 = Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-Ax_1-Bx_2-Cp_2$$
$$\dot{x}_2 = Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-Bx_1-Dx_2+Cp_1$$

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} i\dot{x}_1-\dot{p}_1 \\ i\dot{x}_2-\dot{p}_2 \end{pmatrix}=\begin{pmatrix} Ax_1+Bx_2+Cp_2+iAp_1+iBp_2-iCx_2 \\ Bx_1+Dx_2-Cp_1+iBp_1+iDp_2+iCx_1 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

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$$\dot{x}_1 = Ap_1 + Bp_2 - Cx_2 \qquad \dot{p}_1 = -Ax_1 - Bx_2 - Cp_2$$

$$\dot{x}_2 = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -Bx_1 - Dx_2 + Cp_1$$

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c = \frac{A}{2}\left(p_1^2 + x_1^2\right) + B\left(x_1x_2 + p_1p_2\right) + C\left(x_1p_2 - x_2p_1\right) + \frac{D}{2}\left(p_2^2 + x_2^2\right)$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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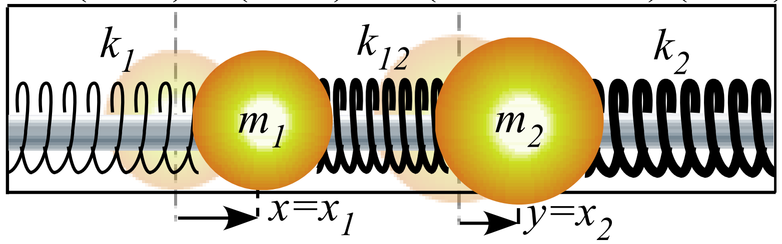

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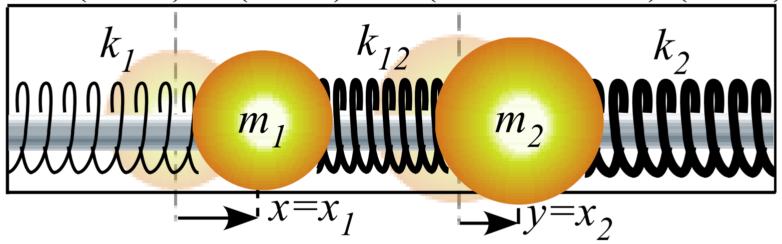

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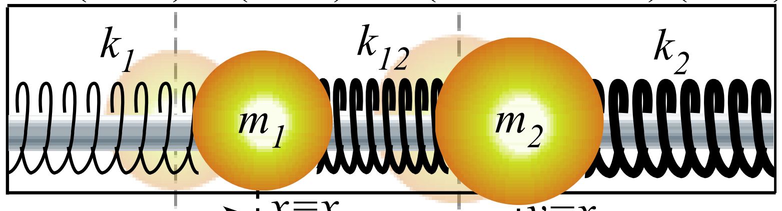

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t = \mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \Rightarrow \left(i\frac{\partial}{\partial t}\right)^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^2 \Rightarrow -\frac{\partial^2}{\partial t^2} = \begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}$$

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$$m_1K_{11}=A^2+B^2=k_1+k_{12}, \qquad m_1K_{12}=AB+BD=-k_{12},$$
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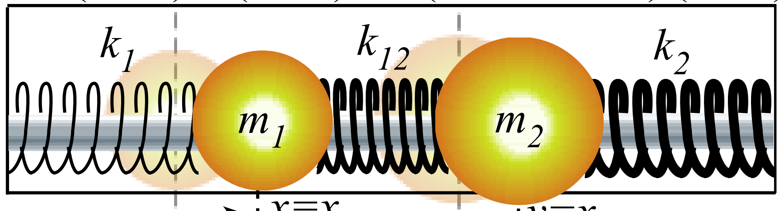

Here is an operator view of the QM-Classical connection: Take Schrodinger operator $i\partial_t=\mathbf{H}$ (with $C=0$) and square it!

$$i\frac{\partial}{\partial t}=\begin{pmatrix}A & B\\ B & D\end{pmatrix}\Rightarrow\left(i\frac{\partial}{\partial t}\right)^2=\begin{pmatrix}A & B\\ B & D\end{pmatrix}^2\Rightarrow-\frac{\partial^2}{\partial t^2}=\begin{pmatrix}A^2+B^2 & AB+BD\\ AB+BD & B^2+D^2\end{pmatrix}$$

Conclusion: 2-state Schro-equation $i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ is like "square-root" of Newton-Hooke. $\sqrt{\left|\ddot{\mathbf{x}}\right\rangle=-\mathbf{K}\bullet\left|\mathbf{x}\right\rangle}$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

$$i\hbar\left|\dot{\Psi}(t)\right\rangle=\mathbf{H}\left|\Psi(t)\right\rangle \qquad \left|\ddot{\mathbf{x}}\right\rangle=-\mathbf{K}\bullet\left|\mathbf{x}\right\rangle$$

First start with 2-by-2 Hermitian (self-conjugate) matrix

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\mathbf{H}^\dagger$$

that operates on 2-D complex Dirac ket vector $|\Psi\rangle$.

$$|\Psi\rangle=\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}=\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Separate real x_k and imaginary p_k parts of Ψ_k amplitudes to convert the complex 1^{st}-order equation $i\partial_t\Psi=\mathbf{H}\Psi$ into pairs of ***real*** 1^{st}-order differential equations.

$$\dot{x}_1=Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-Ax_1-Bx_2-Cp_2$$
$$\dot{x}_2=Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-Bx_1-Dx_2+Cp_1$$

QM vs. Classical Equations are identical

Then start with classical Hamiltonian. (Designed to give same result.)

$$H_c=\frac{A}{2}\left(p_1^2+x_1^2\right)+B\left(x_1x_2+p_1p_2\right)+C\left(x_1p_2-x_2p_1\right)+\frac{D}{2}\left(p_2^2+x_2^2\right)$$

Then Hamilton's equations of motion are the following.

$$\dot{x}_1=\frac{\partial H_c}{\partial p_1}=Ap_1+Bp_2-Cx_2 \qquad \dot{p}_1=-\frac{\partial H_c}{\partial x_1}=-\left(Ax_1+Bx_2+Cp_2\right)$$
$$\dot{x}_2=\frac{\partial H_c}{\partial p_2}=Bp_1+Dp_2+Cx_1 \qquad \dot{p}_2=-\frac{\partial H_c}{\partial x_2}=-\left(Bx_1+Dx_2-Cp_1\right)$$

Finally a 2^{nd} time derivative (Assume *constant* A, B, D, and *let* $C=0$) gives 2^{nd}-order classical Newton-Hooke-like equation: $\left|\ddot{\mathbf{x}}\right\rangle=-\mathbf{K}\bullet\left|\mathbf{x}\right\rangle$

$$\begin{aligned}\ddot{x}_1&=A\dot{p}_1+B\dot{p}_2-C\dot{x}_2\\&=-A\left(Ax_1+Bx_2+Cp_2\right)-B\left(Bx_1+Dx_2-Cp_1\right)-C\left(Bp_1+Dp_2+Cx_1\right)\\&=-\left(A^2+B^2+C^2\right)x_1-\left(AB+BD\right)x_2-C\left(A+D\right)p_2\end{aligned}$$

$$\begin{aligned}\ddot{x}_2&=B\dot{p}_1+D\dot{p}_2+C\dot{x}_1\\&=-B\left(Ax_1+Bx_2+Cp_2\right)-D\left(Bx_1+Dx_2-Cp_1\right)+C\left(Ap_1+Bp_2-Cx_2\right)\\&=-\left(AB+BD\right)x_1-\left(B^2+D^2+C^2\right)x_2+C\left(A+D\right)p_1\end{aligned}$$

For constant A, B, C, and D

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}=-\begin{pmatrix} A^2+B^2 & AB+BD \\ AB+BD & B^2+D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For $C=0$ Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\equiv\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}=-\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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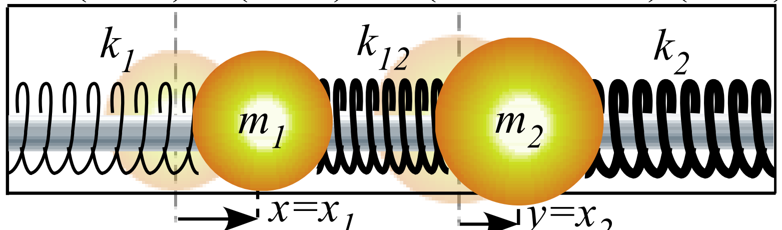

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Review:How symmetry groups become eigen-solvers

How C_2 (Bilateral σ_B reflection) symmetry is eigen-solver

C_2 Symmetric two-dimensional harmonic oscillators (2DHO)

C_2 (Bilateral σ_B reflection) symmetry conditions:

Minimal equation of σ_B and spectral decomposition of $C_2(\sigma_B)$

C_2 Symmetric 2DHO eigensolutions

C_2 Mode phase character table

C_2 Symmetric 2DHO uncoupling

2D-HO beats and mixed mode geometry

Three famous 2-state systems and two-complex-component coordinates

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*Hamilton-Pauli spinor symmetry (**σ**-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

$ABCD$ *Symmetry operator analysis and U(2) spinors*

Decompose the Hamiltonian operator **H** into four $ABCD$ *symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

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$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (complex, circular, chiral, cyclotron, ...*

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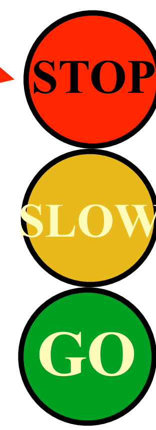
Standing waves

Motivation for coloring scheme:
The Traffic Signal



Moving waves

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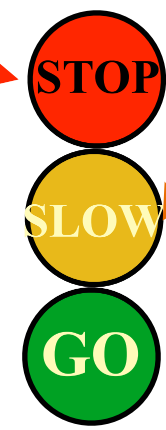

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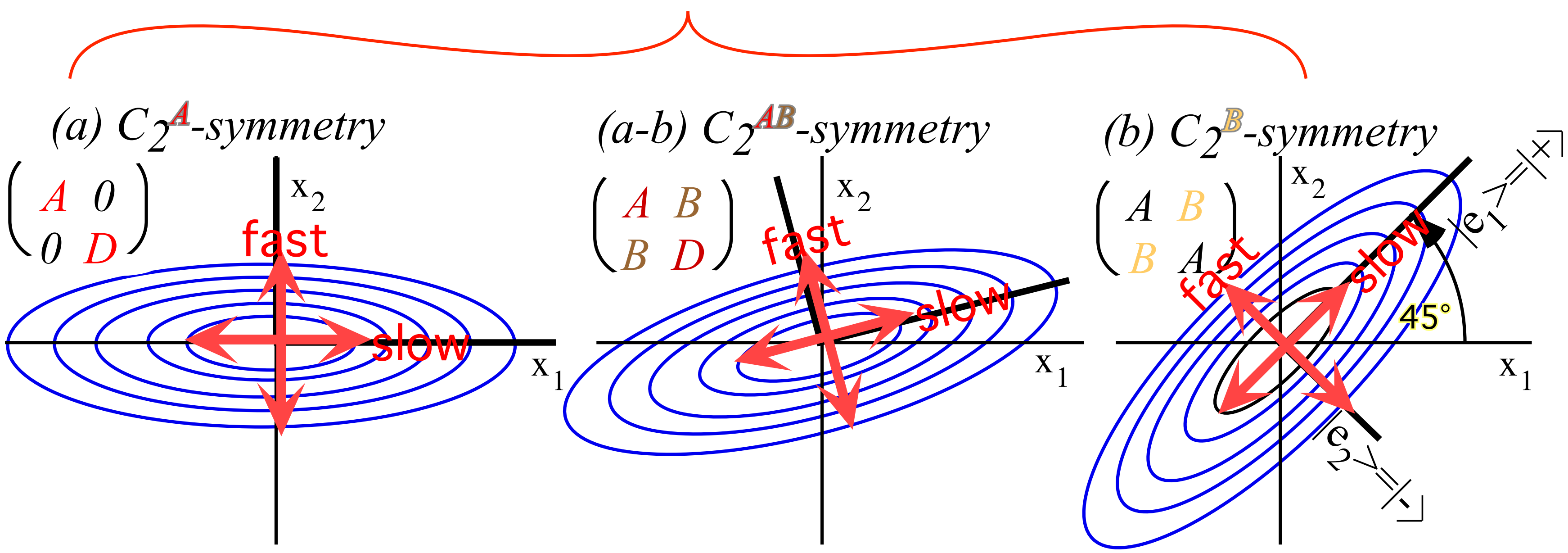

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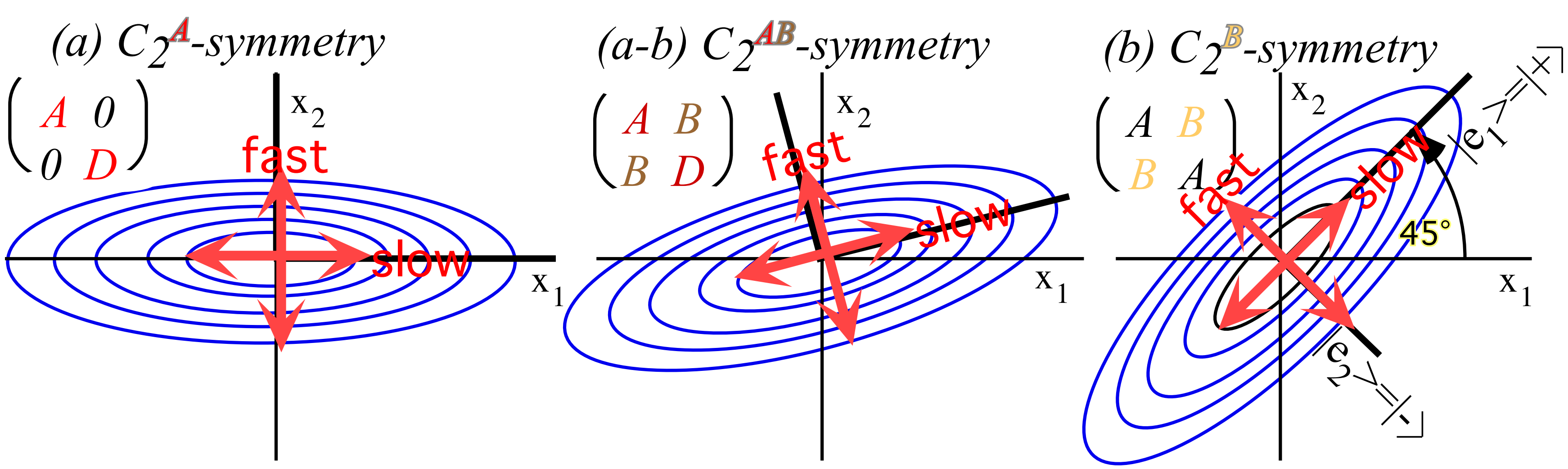
Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623



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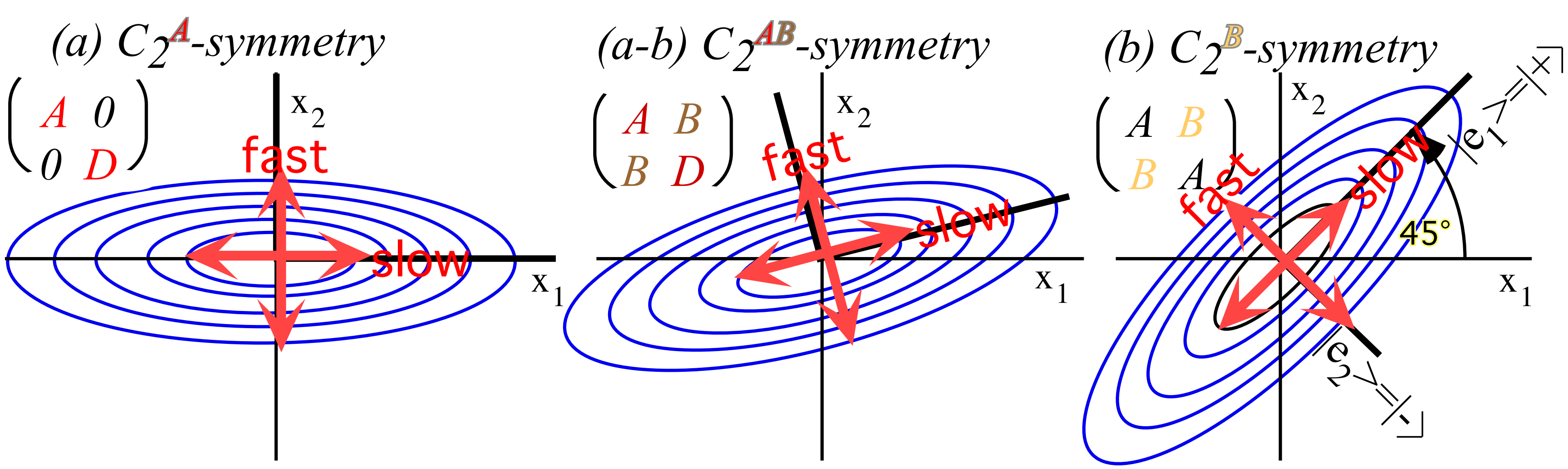

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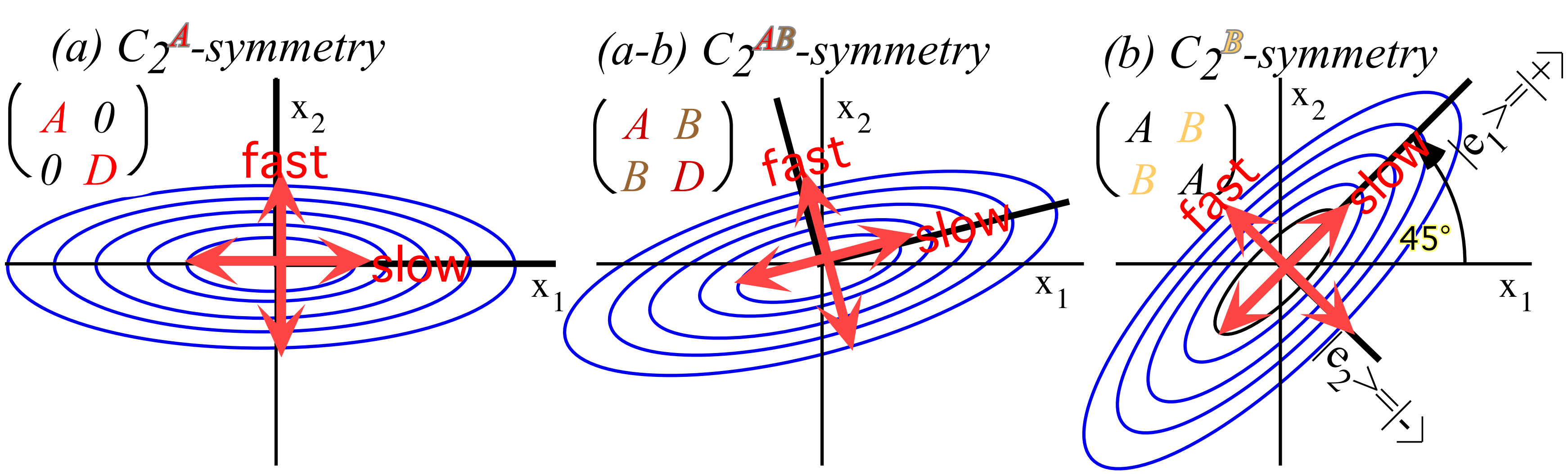

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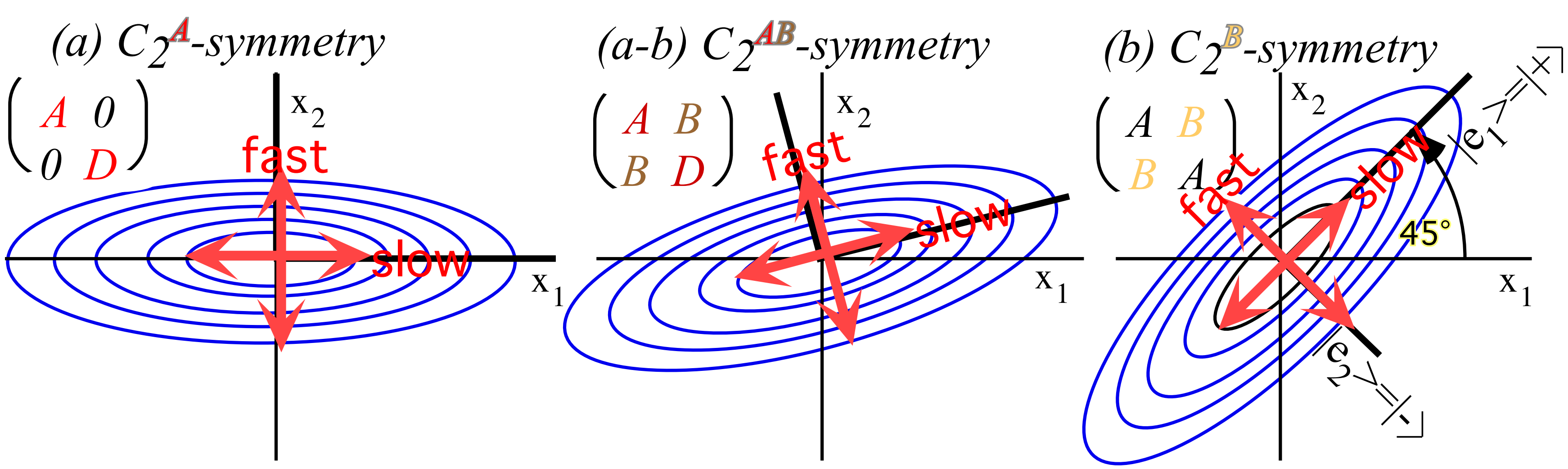
Each Pauli σ_μ squares to *positive*-**1** ($\sigma_X{}^2=\sigma_Y{}^2=\sigma_Z{}^2=+\mathbf{1}$) (Each makes a cyclic C_2 group $C_2{}^A=\{\mathbf{1},\sigma_A\}$, $C_2{}^B=\{\mathbf{1},\sigma_B\}$, or $C_2{}^C=\{\mathbf{1},\sigma_C\}$.)



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