Group Theory in Quantum Mechanics

Lecture 5 (1.31.17)

Spectral Decomposition with Repeated Eigenvalues

(Quantum Theory for Computer Age - Ch. 3 of Unit 1)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 1)

Review: matrix **eigen**states ("**own**states) and **Idempotent projectors** (*Non-degeneracy case*)
- Operator orthonormality, completeness, and spectral decomposition(Non-degenerate e-values)

(Preparing for: Degenerate eigenvalues)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)
- Secular→ Hamilton-Cayley→Minimal equations
- Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, and: $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
- Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$
- Secular equation by minor expansion
- Example of minimal equation projection

Orthonormalization of degenerate eigensolutions
- Projection $\mathbf{P}_j$-matrix anatomy (Gramian matrices)
- Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ and: $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$
- The old "$\mathbf{1}=\mathbf{1}\cdot\mathbf{1}$ trick"-Spectral decomposition by projector splitting
- Irreducible projectors and representations (Trace checks)
- Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$
- How symmetry groups become eigen-solvers

Review: matrix eigenstates ("ownstates) and Idempotent projectors (Non-degeneracy case)

Operator orthonormality, completeness, and spectral decomposition(Non-degenerate e-values)

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Orthonormalization of commuting eigensolutions. Examples: **G=** *and:* **H=**

*The old "***1=1·1** *trick"-Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

How symmetry groups become eigen-solvers

Unitary operators and matrices that change state vectors...

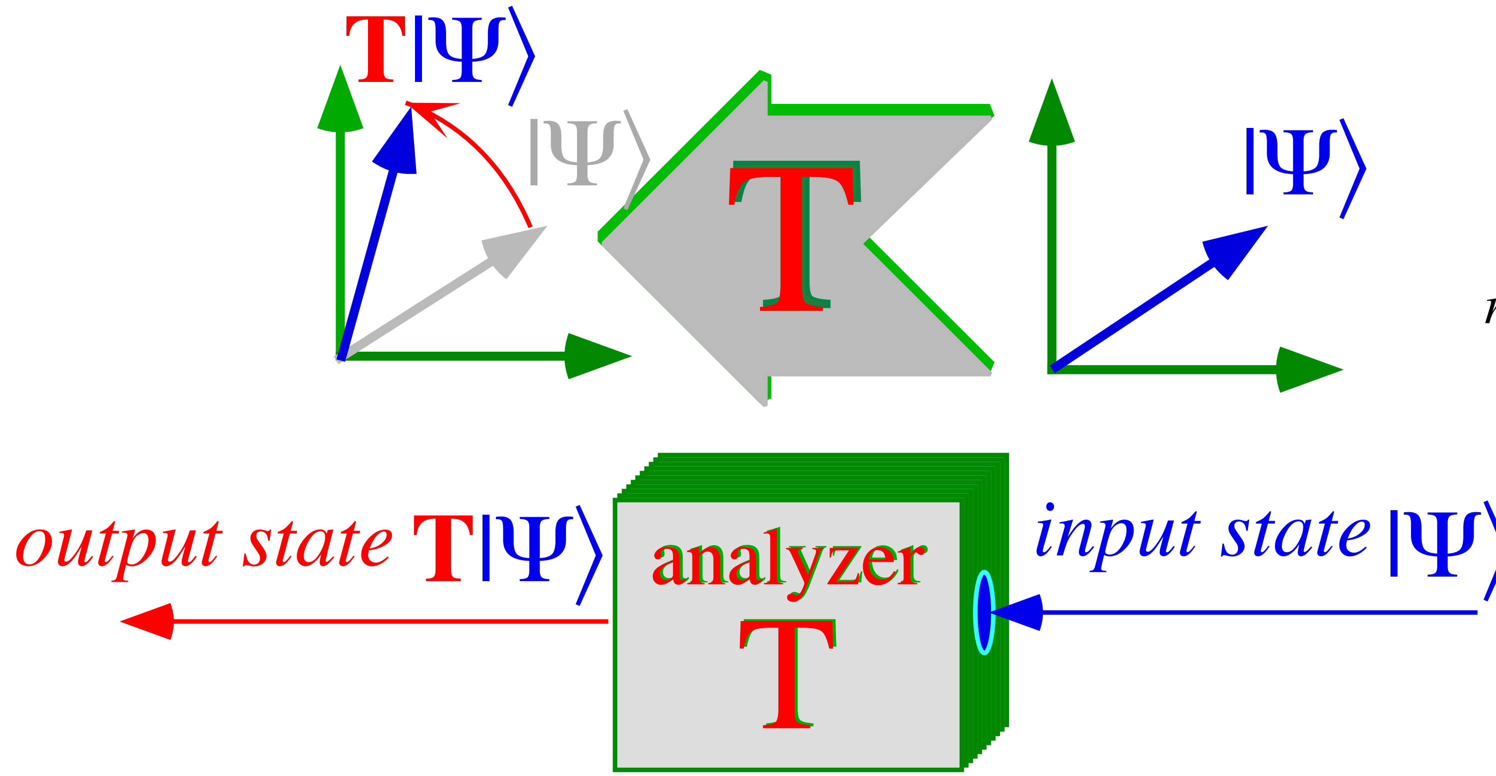

Fig. 3.1.1 Effect of analyzer represented by ket vector transformation of $|\Psi\rangle$ *to new ket vector* $\mathbf{T}|\Psi\rangle$.

*...and **eigen**states ("**own**states) that are mostly immune to* **T***...*

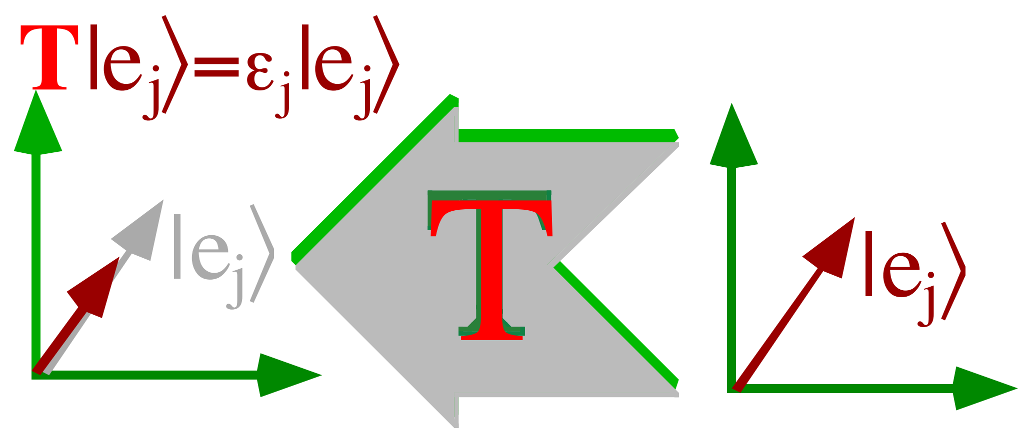

Fig. 3.1.2 Effect of analyzer on eigenket $|\varepsilon_j\rangle$ *is only to multiply by eigenvalue* ε_j *(* $\mathbf{T}|\varepsilon_j\rangle = \varepsilon_j|\varepsilon_j\rangle$ *).*

For Unitary operators $\mathbf{T}=\mathbf{U}$*, the eigenvalues must be phase factors* $\varepsilon_k=e^{i\alpha_k}$

Operator ortho-completeness, and spectral decomposition

(For: Non-Degenerate eigenvalues *)*

Eigen-Operator-Projectors $\mathbf{P}_k$:

$$\mathbf{M}\mathbf{P}_k = \varepsilon_k \mathbf{P}_k = \mathbf{P}_k \mathbf{M}$$

$$\mathbf{P}_k = \frac{\prod_{m \neq k} \left(\mathbf{M} - \varepsilon_m \mathbf{1} \right)}{\prod_{m \neq k} \left(\varepsilon_k - \varepsilon_m \right)}$$

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Dirac notation form:

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Eigen-Operator-$\mathbf{P}_k$-Orthonormality Relations

$$\mathbf{P}_j \mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases}$$

Dirac notation form:

$$|\varepsilon_j\rangle\langle\varepsilon_j|\cdot|\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk}|\varepsilon_k\rangle\langle\varepsilon_k|$$

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Eigen-Operator-$\mathbf{P}_j$ *-Completeness Relations*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n$$

Dirac notation form:

$$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$$

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of operator $\mathbf{M} = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_N\mathbf{P}_N$

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(Preparing for: Degenerate eigenvalues *)*

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Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

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(For: Non-Degenerate eigenvalues *)*

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of operator $\mathbf{M} = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_N\mathbf{P}_N$

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To be discussed in this lecture.

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(For: Non-Degenerate eigenvalues *)*

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Eigen-Operator-$\mathbf{P}_k$*-Orthonormality Relations*

$$\mathbf{P}_j\mathbf{P}_k = \delta_{jk}\mathbf{P}_k = \begin{cases} \mathbf{0} & if : j \neq k \\ \mathbf{P}_k & if : j = k \end{cases} \longrightarrow \mathbf{P}_{\varepsilon_j}\mathbf{P}_{\varepsilon_k} = \delta_{\varepsilon_j \varepsilon_k}\mathbf{P}_{\varepsilon_k} = \begin{cases} \mathbf{0} & if : \varepsilon_j \neq \varepsilon_k \\ \mathbf{P}_{\varepsilon_k} & if : \varepsilon_j = \varepsilon_k \end{cases}$$

Dirac notation form:

$|\varepsilon_j\rangle\langle\varepsilon_j| \cdot |\varepsilon_k\rangle\langle\varepsilon_k| = \delta_{jk} |\varepsilon_k\rangle\langle\varepsilon_k|$ $\longrightarrow$ (Dirac notation form is more complicated.)
To be discussed in this lecture.

Eigen-Operator-$\mathbf{P}_j$ *-Completeness Relations*

$$\mathbf{1} = \mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_n \longrightarrow \mathbf{1} = \mathbf{P}_{\varepsilon_1} + \mathbf{P}_{\varepsilon_2} + \ldots + \mathbf{P}_{\varepsilon_n}$$

Dirac notation form:

$\mathbf{1} = |\varepsilon_1\rangle\langle\varepsilon_1| + |\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + |\varepsilon_n\rangle\langle\varepsilon_n|$ $\longrightarrow$ (Dirac notation form is more complicated.)
To be discussed in this lecture.

Eigen-operators have *Spectral Decomposition*
of operator $\mathbf{M} = \varepsilon_1\mathbf{P}_1 + \varepsilon_2\mathbf{P}_2 + \ldots + \varepsilon_N\mathbf{P}_N \longrightarrow \mathbf{M} = \varepsilon_1\mathbf{P}_{\varepsilon_1} + \varepsilon_2\mathbf{P}_{\varepsilon_2} + \ldots + \varepsilon_n\mathbf{P}_{\varepsilon_n}$

Dirac notation form:

$\mathbf{M} = \varepsilon_1|\varepsilon_1\rangle\langle\varepsilon_1| + \varepsilon_2|\varepsilon_2\rangle\langle\varepsilon_2| + \ldots + \varepsilon_n|\varepsilon_n\rangle\langle\varepsilon_n|$ $\longrightarrow$ (Dirac notation form is more complicated.)

...and operator *Functional Spectral Decomposition*
of a function $f(\mathbf{M}) = f(\varepsilon_1)\mathbf{P}_1 + f(\varepsilon_2)\mathbf{P}_2 + \ldots + f(\varepsilon_N)\mathbf{P}_N \longrightarrow f(\mathbf{M}) = f(\varepsilon_1)\mathbf{P}_{\varepsilon_1} + f(\varepsilon_2)\mathbf{P}_{\varepsilon_2} + \ldots + f(\varepsilon_n)\mathbf{P}_{\varepsilon_n}$

Dirac notation form:

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(Preparing for: Degenerate eigenvalues)

Review: matrix *eigen*states (“*own*states) and *Idempotent projectors* (*Degeneracy case*)
Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)
Secular→ Hamilton-Cayley→Minimal equations
Diagonalizability criterion

Nilpotents and “Bad degeneracy” examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
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The old "$\mathbf{1=1\cdot 1}$ trick"-Spectral decomposition by projector splitting
Irreducible projectors and representations (Trace checks)
Minimal equation for projector $\mathbf{P=P}^2$
How symmetry groups become eigen-solvers

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

What if *secular equation* ($det|\mathrm{M}$-$\varepsilon_j\mathbf{1}|$−0) of *N-by-N* matrix **H** has ℓ-repeated ε_1-roots $\{\varepsilon_{1_1}, \varepsilon_{1_2} ... \varepsilon_{1_\ell}\}$?

If so, it's possible **H** can't be completely diagonalized, though this is rarely the case.

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A *diagonalizability criterion* has just been proved:

In general, matrix $\mathbf{H}$ *can make an ortho-complete set of* $\mathbf{P}_{\varepsilon_j}$ *if and only if, the* $\mathbf{H}$ *minimal equation has no repeated factors. Then and only then is matrix* $\mathbf{H}$ *fully diagonalizable.*

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Repeated minimal equation factors means you will not get an ortho-complete set of $\mathbf{P}_j$.

Even *one repeat* is fatal...

(like this ↓)

$$\mathbf{0} = \boxed{\left(\mathbf{H} - \varepsilon_1 \mathbf{1}\right)^2} \left(\mathbf{H} - \varepsilon_2 \mathbf{1}\right)^1 \ldots,$$

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Then squaring $\mathbf{N}$ puts back the missing $(\mathbf{H}-\varepsilon_1\mathbf{1})$-factor that completes the zero minimal equation.

$$\mathbf{N}^2=\left(\mathbf{H}-\varepsilon_1\mathbf{1}\right)^2\left(\mathbf{H}-\varepsilon_2\mathbf{1}\right)^2 \ldots\ldots = \mathbf{0}$$

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(Preparing for: Degenerate eigenvalues *)*

*Review: matrix **eigen**states ("**own**states) and Idempotent projectors (Degeneracy case)*

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values *)*

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

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Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$

Secular equation by minor expansion

Example of minimal equation projection

Orthonormalization of degenerate eigensolutions

Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*

Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$

*The old "**1=1·1** trick"-Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

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$\mathbf{N}$ and its partners comprise a 4-dimensional *U(2) unit tensor operator space*

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They form an *elementary matrix algebra* $\mathbf{e}_{ij}\,\mathbf{e}_{km} = \delta_{jk}\,\mathbf{e}_{im}$ of unit tensor operators.
The non-diagonal ones are non-diagonalizable *nilpotent* operators

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The non-diagonal ones are non-diagonalizable *nilpotent* operators

Their ∞-Dimensional cousins are the *creation-destruction* $\mathbf{a}_i^\dagger \mathbf{a}_j$ operators.

(Preparing for: Degenerate eigenvalues)

*Review: matrix **eigen**states (“**own**states) and Idempotent projectors (Degeneracy case)*

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

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ε has a 4th degree *Secular Equation (SEq)*

$\varepsilon^4 -$ ($\sum$1x1 diag of **G**) $\varepsilon^3 +$ ($\sum$2x2 diag minors of **G**) $\varepsilon^2 -$ ($\sum$3x3 diag minors of **G**) $\varepsilon^1 +$ (4x4 determinant of **G**)

$\varepsilon^1 = 0$

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$$\det\mathbf{G}=$$
$$=(-1)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$
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$$=(-1)(1)(-1)$$
$$=+1$$

$$\begin{array}{cccc} + & - & + & - \end{array}$$
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Each of these projectors contains two linearly independent ket or bra vectors:

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$$\mathbf{G}=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \textit{SEq:} \qquad S(\varepsilon)=\det|\mathbf{G}-\varepsilon\mathbf{1}|=\det\begin{vmatrix} -\varepsilon & 0 & 0 & 1 \\ 0 & -\varepsilon & 1 & 0 \\ 0 & 1 & -\varepsilon & 0 \\ 1 & 0 & 0 & -\varepsilon \end{vmatrix}$$

ε has a 4th degree *Secular Equation (SEq)* with repeat pairs of degenerate roots ($\varepsilon_k=\pm 1$)

$$S(\varepsilon)=0=\varepsilon^4-2\varepsilon^2+1=(\varepsilon-1)^2(\varepsilon+1)^2$$

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Yet **G** satisfies *Minimal Equation (MinEq)* of only 2nd degree with no repeats. So $\mathbf{P}_{\varepsilon_k}$ formulae work!

$$\mathbf{0}=(\mathbf{G}-\mathbf{1})(\mathbf{G}+\mathbf{1})$$

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$$\mathbf{P}^G_{+1}=\frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)}=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{P}^G_{-1}=\frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)}=\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Each of these projectors contains two linearly independent ket or bra vectors:

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If projector $\mathbf{P}_j$ is idempotent $(\mathbf{P}_j \mathbf{P}_j = \mathbf{P}_j)$, all matrix elements $(\mathbf{P}_j)_{bk}$ are row$_b$-column$_k$-•-products $(j_b|j_k)$

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$$
\begin{array}{ccccc}
(\mathbf{P}_j) & \cdot & (\mathbf{P}_j) & = & (\mathbf{P}_j) \\
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
& \cdot &
\left(\begin{array}{ccc|c|cc}
\cdot & \cdot & \cdot & k_1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_5 & \cdot & \cdot \\
\cdot & \cdot & \cdot & k_6 & \cdot & \cdot
\end{array}\right)
& = &
\left(\begin{array}{ccc|c|cc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & (b|k) & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
\end{array}
$$

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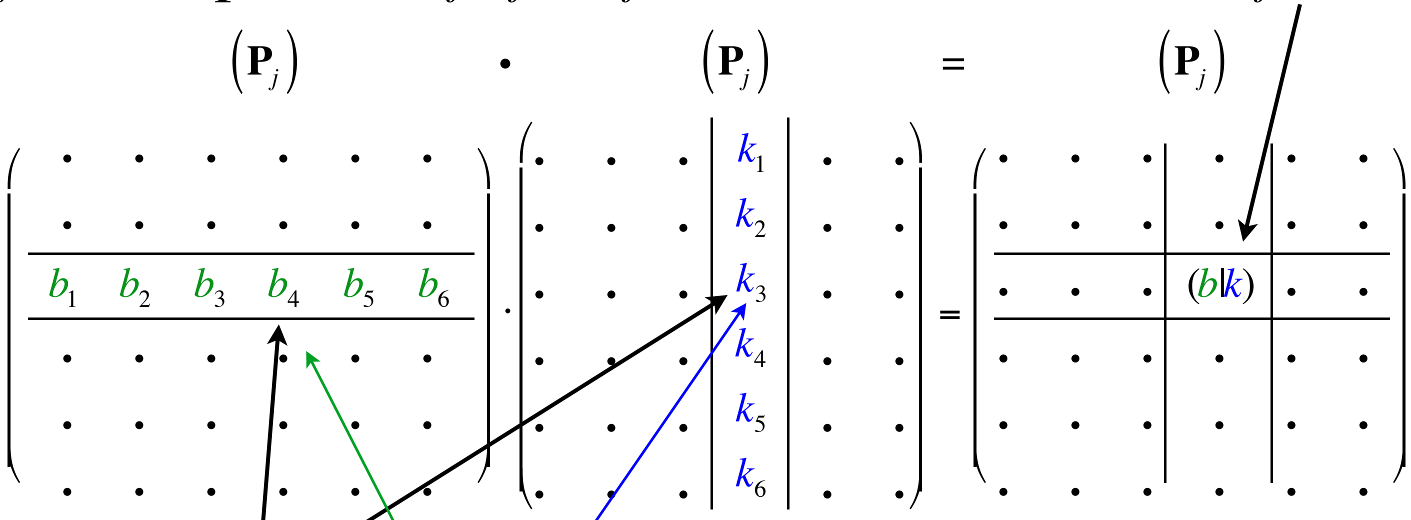

$(\mathbf{P}_j)_{34} = b_4 = k_3 = (j_3|j_4) =$

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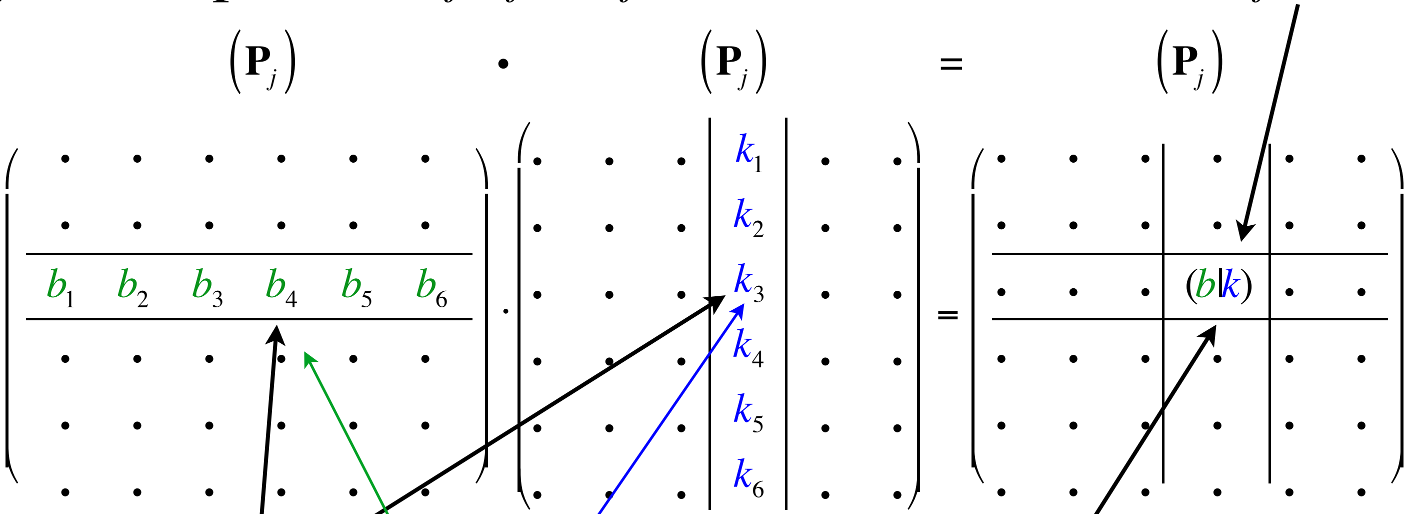

$$(\mathbf{P}_j)_{34} = b_4 = k_3 = (j_3|j_4) = (b|k) = \mathbf{b}\bullet\mathbf{k} = b_1k_1+b_2k_2+b_3k_3+b_4k_4+b_5k_5+b_6k_6$$

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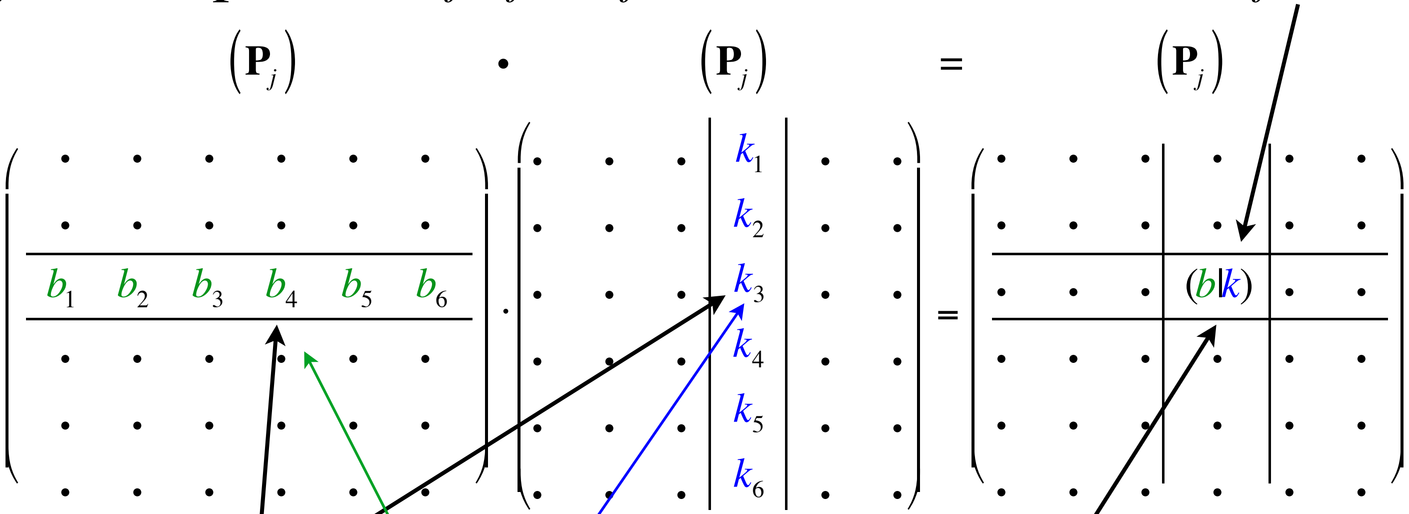

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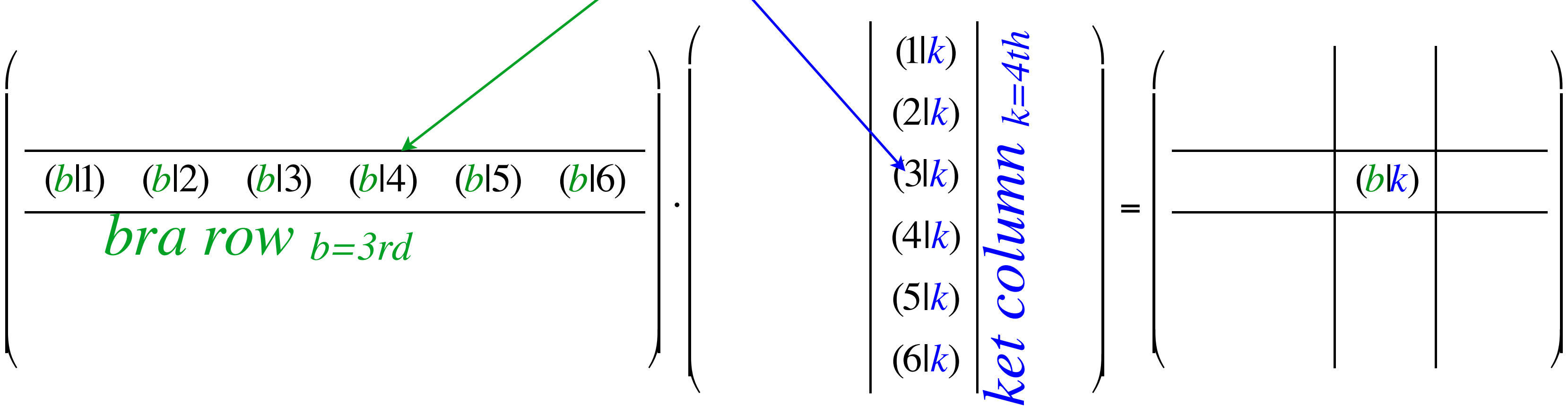

Quasi-Dirac notation shows vector relations

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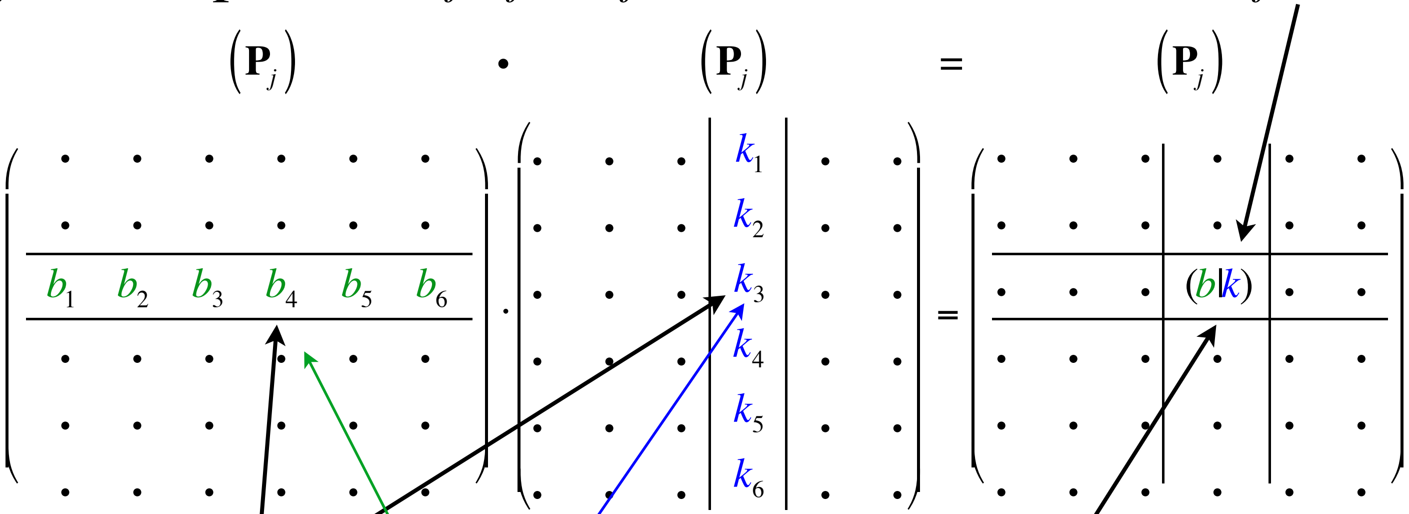

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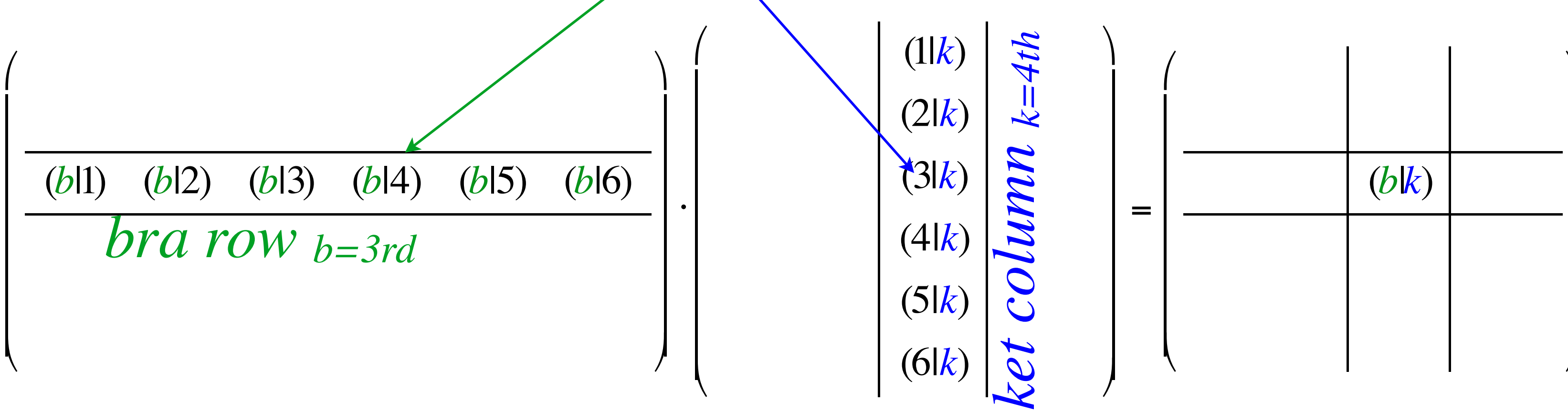

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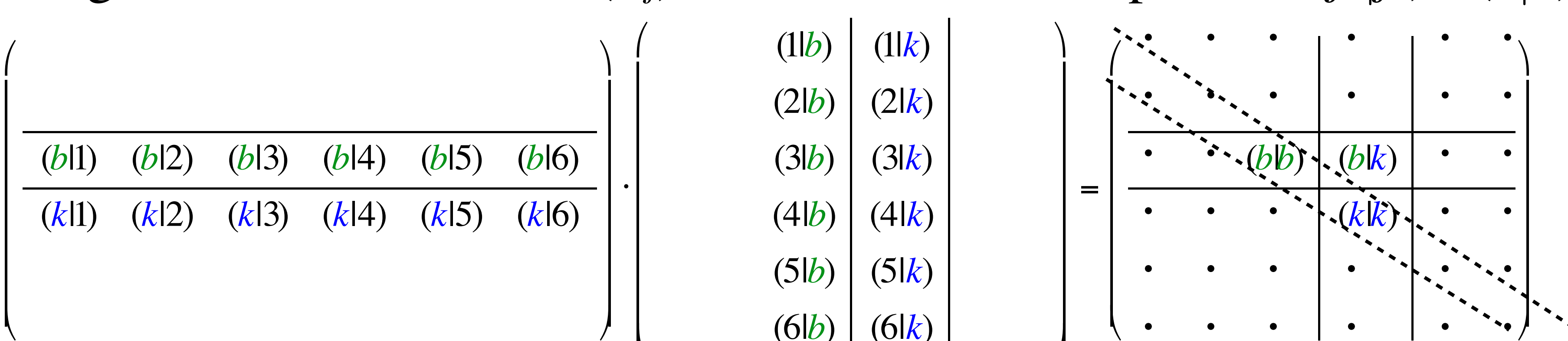
Diagonal matrix elements $(\mathbf{P}_j)_{kk}$ = row$_k$-column$_k$-•-product $(j_k|j_k) = (k|k)$ is k^{th}*-norm value* (usually real)



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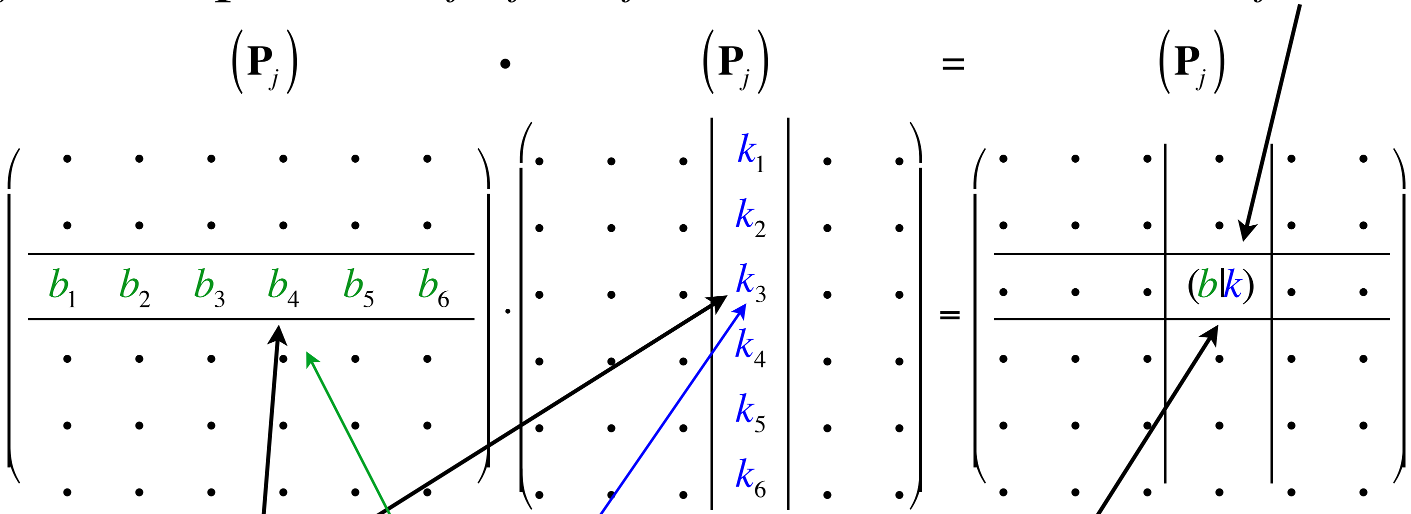

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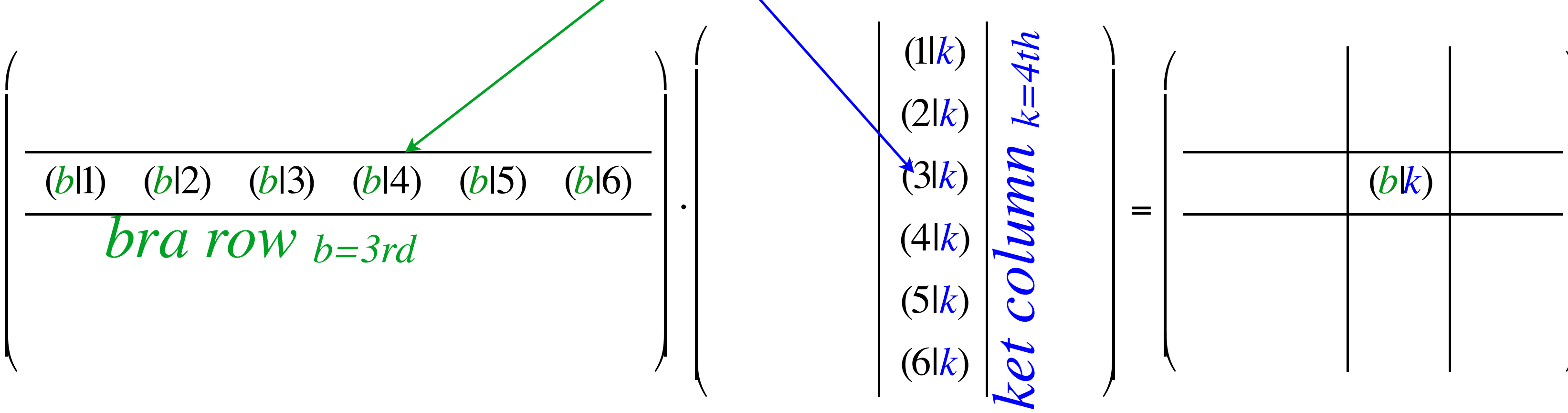

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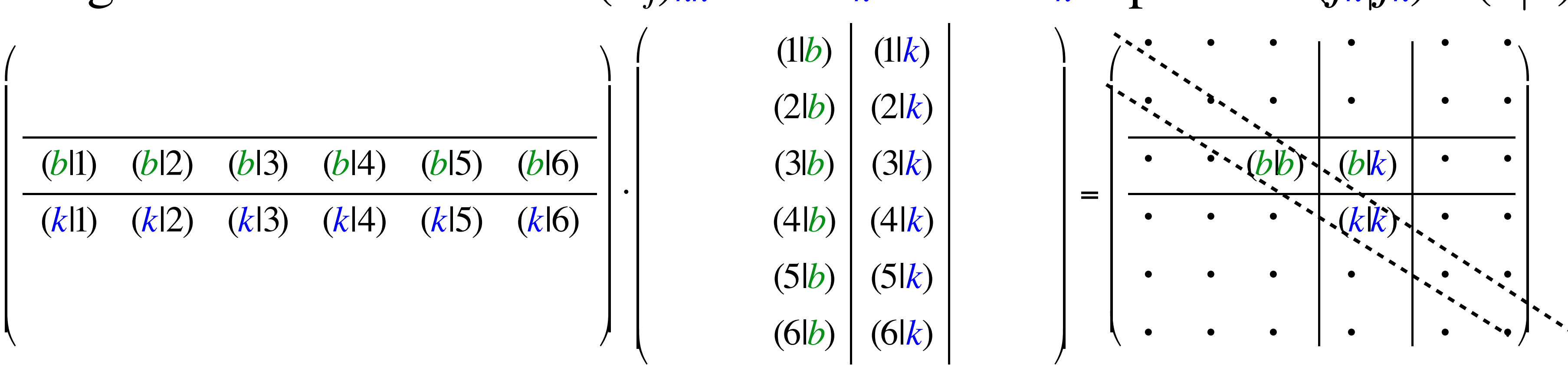

k^{th} ***normalized vectors***

ket= $|j_k\rangle=|j_k)/\sqrt{(k|k)}$

bra= $\langle j_k|=(j_k|/\sqrt{(k|k)}$

so: $\langle j_k|j_k\rangle=1$

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Solve these by substituting: $\quad N_1 = - N_2\,(j_1|j_2)/(j_1|j_1)$

to give: $\quad 1= N_2^2\,(j_1|j_2)^2/(j_1|j_1) - N_2^2[(j_1|j_2)+ (j_2|j_1)](j_1|j_2)/(j_1|j_1) + N_2^2(j_2|j_2)$

$$1/N_2^2 = (j_2|j_2) + \cancel{(j_1|j_2)^2/(j_1|j_1)} - \cancel{(j_1|j_2)^2/(j_1|j_1)} - (j_2|j_1)(j_1|j_2)/(j_1|j_1)$$

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So the new orthonormal pair is:

$$|j_1\rangle = \frac{|j_1)}{\sqrt{(j_1|j_1)}}$$

$$|j_2\rangle = N_1|j_1) + N_2|j_2) = -\frac{N_2(j_1|j_2)}{(j_1|j_1)}|j_1) + N_2|j_2)$$

$$= N_2\left(|j_2) - \frac{(j_1|j_2)}{(j_1|j_1)}|j_1)\right) = \sqrt{\frac{1}{(j_2|j_2) - \frac{(j_2|j_1)(j_1|j_2)}{(j_1|j_1)}}}\left(|j_2) - \frac{(j_1|j_2)}{(j_1|j_1)}|j_1)\right)$$

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Solve these by substituting: $\quad N_1 =- N_2\,(j_1|j_2)/(j_1|j_1)$

to give: $\quad 1= N_2^2\,(j_1|j_2)^2/(j_1|j_1) - N_2^2[(j_1|j_2)+ (j_2|j_1)](j_1|j_2)/(j_1|j_1) + N_2^2(j_2|j_2)$

$$1/N_2^2 = (j_2|j_2) + \cancel{(j_1|j_2)^2/(j_1|j_1)} - \cancel{(j_1|j_2)^2/(j_1|j_1)} - (j_2|j_1)(j_1|j_2)/(j_1|j_1)$$

$$1/N_2^2 = (j_2|j_2) - (j_2|j_1)(j_1|j_2)/(j_1|j_1)$$

So the new orthonormal pair is:

$$|j_1\rangle = \frac{|j_1)}{\sqrt{(j_1|j_1)}}$$

$$|j_2\rangle = N_1|j_1) + N_2|j_2) = -\frac{N_2(j_1|j_2)}{(j_1|j_1)}|j_1) + N_2|j_2)$$

$$= N_2\left(|j_2) - \frac{(j_1|j_2)}{(j_1|j_1)}|j_1)\right) = \sqrt{\frac{1}{(j_2|j_2) - \frac{(j_2|j_1)(j_1|j_2)}{(j_1|j_1)}}}\left(|j_2) - \frac{(j_1|j_2)}{(j_1|j_1)}|j_1)\right)$$

OK. That's for 2 vectors. Like to try for 3?

Orthonormalization of degenerate eigensolutions

The **G** example is unusually convenient since components $(\mathbf{P}_j)_{12}$ of projectors $\mathbf{P}_j$ happen to be zero, and this means row-*1* vector $(j_1|$ is already orthogonal to row-*2* vector $|j_2)$: $(j_1|j_2) = 0$

Gram-Schmidt procedure

Suppose a non-zero scalar product $(j_1|j_2)\neq 0$. Replace vector $|j_2)$ with a vector $|j_2\rangle=|j_{\perp 1})$ normal to $(j_1|$?

Define: $|j_2\rangle= N_1|j_1) + N_2|j_2)$ such that: $(j_1|j_2\rangle=0= N_1\,(j_1|j_1) + N_2\,(j_1|j_2)$

...and normalized so that: $\langle j_2|j_2\rangle=1= N_1^2(j_1|j_1)+ N_1 N_2[(j_1|j_2)+ (j_2|j_1)]+ N_2^2(j_2|j_2)$

Solve these by substituting: $N_1 =- N_2\,(j_1|j_2)/(j_1|j_1)$

to give: $1= N_2^2\,(j_1|j_2)^2/(j_1|j_1) - N_2^2[(j_1|j_2)+ (j_2|j_1)](j_1|j_2)/(j_1|j_1) + N_2^2(j_2|j_2)$

$$1/N_2^2 = (j_2|j_2) + \cancel{(j_1|j_2)^2/(j_1|j_1)} - \cancel{(j_1|j_2)^2/(j_1|j_1)} - (j_2|j_1)(j_1|j_2)/(j_1|j_1)$$

$$1/N_2^2 = (j_2|j_2) - (j_2|j_1)(j_1|j_2)/(j_1|j_1)$$

So the new orthonormal pair is:

$$|j_1\rangle = \frac{|j_1)}{\sqrt{(j_1|j_1)}}$$

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OK. That's for 2 vectors. Like to try for 3?

Instead, let' try another way to "orthogonalize" that might be more *elegante*.

(Preparing for: Degenerate eigenvalues *)*

Review: matrix eigenstates ("ownstates) and Idempotent projectors (*Degeneracy case* *)*

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values *)*

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$

Secular equation by minor expansion

Example of minimal equation projection

Orthonormalization of degenerate eigensolutions

Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*

Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$

The old "$\mathbf{1=1\cdot 1}$ *trick*"-*Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P=P^2}$

How symmetry groups become eigen-solvers

Orthonormalization by commuting projector splitting

The **G** projectors and eigenvectors were derived several pages back: *(And, we got a lucky orthogonality)*

$$\mathbf{P}^{G}_{+1}=\frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)}=\frac{1}{2}\begin{pmatrix}1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix}\qquad \mathbf{P}^{G}_{-1}=\frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)}=\frac{1}{2}\begin{pmatrix}1&0&0&-1\\0&1&-1&0\\0&-1&1&0\\-1&0&0&1\end{pmatrix}$$

$$\left|1_1\right\rangle=\frac{\left|1_1\right)}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\quad \left|1_2\right\rangle=\frac{\left|1_2\right)}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\quad \left|-1_1\right\rangle=\frac{\left|-1_1\right)}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\quad \left|-1_2\right\rangle=\frac{\left|-1_2\right)}{\sqrt{2}}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

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Dirac notation for **G**-*split completeness relation* using eigenvectors is the following:

$$\begin{aligned} 1 = \mathbf{P}^{\mathbf{G}}_{1} + \mathbf{P}^{\mathbf{G}}_{-1} &= \left|1_1\right\rangle\left\langle 1_1\right| + \left|1_2\right\rangle\left\langle 1_2\right| + \left|-1_1\right\rangle\left\langle -1_1\right| + \left|-1_2\right\rangle\left\langle -1_2\right| \\ &= \mathbf{P}_{1_1} + \mathbf{P}_{1_2} + \mathbf{P}_{-1_1} + \mathbf{P}_{-1_2} \end{aligned}$$

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$$\mathbf{P}^{G}_{+1} = \frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{P}^{G}_{-1} = \frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

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Each of the original $\mathbf{G}$ projectors are split in two parts with one ket-bra in each.

$$\begin{aligned} \mathbf{P}^{\mathbf{G}}_{1} = \mathbf{P}_{1_1} + \mathbf{P}_{1_2} &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \left|1_1\right\rangle\left\langle 1_1\right| + \left|1_2\right\rangle\left\langle 1_2\right| \end{aligned}$$

$$\begin{aligned} \mathbf{P}^{\mathbf{G}}_{-1} = \mathbf{P}_{-1_1} + \mathbf{P}_{-1_2} &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \left|-1_1\right\rangle\left\langle -1_1\right| + \left|-1_2\right\rangle\left\langle -1_2\right| \end{aligned}$$

Orthonormalization by commuting projector splitting

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Each of the original **G** projectors are split in two parts with one ket-bra in each.

$$\begin{aligned} \mathbf{P}^{\mathbf{G}}_{1} = \mathbf{P}_{1_1} + \mathbf{P}_{1_2} &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \left|1_1\right\rangle\left\langle 1_1\right| + \left|1_2\right\rangle\left\langle 1_2\right| \end{aligned}$$

$$\begin{aligned} \mathbf{P}^{\mathbf{G}}_{-1} = \mathbf{P}_{-1_1} + \mathbf{P}_{-1_2} &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \left|-1_1\right\rangle\left\langle -1_1\right| + \left|-1_2\right\rangle\left\langle -1_2\right| \end{aligned}$$

There are ∞-*ly* many ways to split **G** projectors. Now we let another operator **H** do the final splitting.

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH**=**HG**

the **G=** $\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator **H=** $\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

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(First, it is important to verify that they do, in fact, commute.)

$$\mathbf{GH} = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} = \mathbf{HG}$$

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Problem:

Find an ortho-complete projector set that spectrally resolves <u>both</u> **G** <u>and</u> **H**.

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Previous completeness for **G**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \mathbf{P}^{G}_{+1} = \frac{\mathbf{G}-(-1)\mathbf{1}}{+1-(-1)} \quad + \mathbf{P}^{G}_{-1} = \frac{\mathbf{G}-(1)\mathbf{1}}{-1-(1)}$$

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Current completeness for **H**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

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(Left as an exercise)

(Preparing for: Degenerate eigenvalues *)*

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The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

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Problem:

Find an ortho-complete projector set that spectrally resolves both **G** and **H**.

Previous completeness for **G**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Current completeness for **H**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Solution:

*The old "**1=1·1** trick"-Spectral decomposition by projector splitting*

Multiplying **G** and **H** completeness relations

$$\mathbf{1}=\mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}\right)\left(\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}\right) = \mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{-2}\right)$$

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$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Current completeness for **H**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Solution:

The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*

Multiplying **G** and **H** completeness relations gives a set of projectors

$$\mathbf{1}=\mathbf{1}\cdot\mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}\right)\left(\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}\right) = \mathbf{1} = \left(\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{-2}\right)$$

$$\mathbf{P}^{\mathbf{GH}}_{+1,+2} \equiv \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} =$$

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH**=**HG**

the $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

Problem:

Find an ortho-complete projector set that spectrally resolves <u>both</u> **G** <u>and</u> **H**.

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$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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(Left as an exercise)

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(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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$$\mathbf{P}^{\mathbf{GH}}_{+1,-2} \equiv \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2} = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\mathbf{P}^{\mathbf{GH}}_{-1,+2} \equiv \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

Orthonormalization of commuting eigensolutions.

Suppose we have two mutually commuting matrix operators: **GH=HG**

the $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

Problem:

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$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Current completeness for **H**:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

(Left as an exercise)

Solution:

*The old "**1=1·1** trick"-Spectral decomposition by projector splitting*

Multiplying **G** and **H** completeness relations gives a set of projectors

$$\mathbf{1}=\mathbf{1}\cdot\mathbf{1}=\left(\mathbf{P}^{\mathbf{G}}_{+1}+\mathbf{P}^{\mathbf{G}}_{-1}\right)\left(\mathbf{P}^{\mathbf{H}}_{+2}+\mathbf{P}^{\mathbf{H}}_{-2}\right)=\mathbf{1}=\left(\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2}+\mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2}+\mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2}+\mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{-2}\right)$$

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Suppose we have two mutually commuting matrix operators: $\mathbf{GH}=\mathbf{HG}$

the $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

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$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Current completeness for $\mathbf{H}$:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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$$\boxed{\begin{aligned} \mathbf{G}\mathbf{P}^{\mathbf{GH}}_{g,h} &= \mathbf{G}\mathbf{P}^{\mathbf{G}}_{g}\mathbf{P}^{\mathbf{H}}_{h} = \varepsilon^{\mathbf{G}}_{g}\mathbf{P}^{\mathbf{GH}}_{g,h} \\ \mathbf{H}\mathbf{P}^{\mathbf{GH}}_{g,h} &= \mathbf{H}\mathbf{P}^{\mathbf{G}}_{g}\mathbf{P}^{\mathbf{H}}_{h} = \mathbf{P}^{\mathbf{G}}_{g}\mathbf{H}\mathbf{P}^{\mathbf{H}}_{h} = \varepsilon^{\mathbf{H}}_{h}\mathbf{P}^{\mathbf{GH}}_{g,h} \end{aligned}}$$

$$\mathbf{P}^{\mathbf{GH}}_{+1,+2} \equiv \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

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Suppose we have two mutually commuting matrix operators: $\mathbf{GH}=\mathbf{HG}$

the $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ from before, and new operator $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$.

Problem:

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Current completeness for $\mathbf{H}$:

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2}$$

(Left as an exercise)

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...and a the same $\mathbf{P}^{\mathbf{GH}}_{g,h}$ projectors spectrally resolve <u>both</u> $\mathbf{G}$ <u>and</u> $\mathbf{H}$.

$$\boxed{\mathbf{G} = (+1)\mathbf{P}^{\mathbf{GH}}_{+1,+2} + (+1)\mathbf{P}^{\mathbf{GH}}_{+1,-2} + (-1)\mathbf{P}^{\mathbf{GH}}_{-1,+2} + (-1)\mathbf{P}^{\mathbf{GH}}_{-1,-2}}$$

$$\boxed{\mathbf{H} = (+2)\mathbf{P}^{\mathbf{GH}}_{+1,+2} + (-2)\mathbf{P}^{\mathbf{GH}}_{+1,-2} + (+2)\mathbf{P}^{\mathbf{GH}}_{-1,+2} + (-2)\mathbf{P}^{\mathbf{GH}}_{-1,-2}}$$

(Preparing for: Degenerate eigenvalues)

Review: matrix eigenstates ("ownstates) and Idempotent projectors (Degeneracy case)

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values)

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

Nilpotents and "Bad degeneracy" examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$

Secular equation by minor expansion

Example of minimal equation projection

Orthonormalization of degenerate eigensolutions

Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*

Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$

*The old "***1=1·1** *trick"-Spectral decomposition by projector splitting*

→ *Irreducible projectors and representations (Trace checks)*

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

How symmetry groups become eigen-solvers

Another Problem: How do you tell when a Projector $\mathbf{P}_g^{\mathbf{G}}$ or $\mathbf{P}_{g,h}^{\mathbf{GH}}$ is 'splittable' (Correct term is *reducible*.)

$$\mathbf{1} = \mathbf{P}_{+1}^{\mathbf{G}} + \mathbf{P}_{-1}^{\mathbf{G}} = \frac{1}{2}\begin{pmatrix} 1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1&0&0&-1\\0&1&-1&0\\0&-1&1&0\\-1&0&0&1\end{pmatrix}$$

$$\mathbf{1} = \mathbf{P}_{+2}^{\mathbf{H}} + \mathbf{P}_{-2}^{\mathbf{H}} = \frac{1}{2}\begin{pmatrix} 1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}$$

(Left as an exercise)

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Multiplying **G** and **H** completeness relations gives a set of projectors and eigen-relations for both:

$$\mathbf{1}=\mathbf{1}\cdot\mathbf{1}=\left(\mathbf{P}_{+1}^{\mathbf{G}}+\mathbf{P}_{-1}^{\mathbf{G}}\right)\left(\mathbf{P}_{+2}^{\mathbf{H}}+\mathbf{P}_{-2}^{\mathbf{H}}\right)=\mathbf{1}=\left(\mathbf{P}_{+1}^{\mathbf{G}}\mathbf{P}_{+2}^{\mathbf{H}}+\mathbf{P}_{+1}^{\mathbf{G}}\mathbf{P}_{-2}^{\mathbf{H}}+\mathbf{P}_{-1}^{\mathbf{G}}\mathbf{P}_{+2}^{\mathbf{H}}+\mathbf{P}_{-1}^{\mathbf{G}}\mathbf{P}_{-2}^{\mathbf{H}}\right)$$

$$\mathbf{G}\mathbf{P}_{g,h}^{\mathbf{GH}} = \mathbf{G}\mathbf{P}_{g}^{\mathbf{G}}\mathbf{P}_{h}^{\mathbf{H}} = \varepsilon_g^{\mathbf{G}}\mathbf{P}_{g,h}^{\mathbf{GH}}$$

$$\mathbf{H}\mathbf{P}_{g,h}^{\mathbf{GH}} = \mathbf{H}\mathbf{P}_{g}^{\mathbf{G}}\mathbf{P}_{h}^{\mathbf{H}} = \mathbf{P}_{g}^{\mathbf{G}}\mathbf{H}\mathbf{P}_{h}^{\mathbf{H}} = \varepsilon_h^{\mathbf{H}}\mathbf{P}_{g,h}^{\mathbf{GH}}$$

$$\mathbf{P}_{+1,+2}^{\mathbf{GH}} \equiv \mathbf{P}_{+1}^{\mathbf{G}}\mathbf{P}_{+2}^{\mathbf{H}} = \frac{1}{4}\begin{pmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$$

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...and a the same $\mathbf{P}_{g,h}^{\mathbf{GH}}$ projectors spectrally resolve both **G** and **H**.

$$\boxed{\mathbf{G} = (+1)\mathbf{P}_{+1,+2}^{\mathbf{GH}} + (+1)\mathbf{P}_{+1,-2}^{\mathbf{GH}} + (-1)\mathbf{P}_{-1,+2}^{\mathbf{GH}} + (-1)\mathbf{P}_{-1,-2}^{\mathbf{GH}}}$$

$$\boxed{\mathbf{H} = (+2)\mathbf{P}_{+1,+2}^{\mathbf{GH}} + (-2)\mathbf{P}_{+1,-2}^{\mathbf{GH}} + (+2)\mathbf{P}_{-1,+2}^{\mathbf{GH}} + (-2)\mathbf{P}_{-1,-2}^{\mathbf{GH}}}$$

Another Problem: How do you tell when a Projector $\mathbf{P}_g^{\mathbf{G}}$ or $\mathbf{P}_{g,h}^{\mathbf{GH}}$ is 'splittable' (Correct term is *reducible*.)

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(Left as an exercise)

*The old "**1=1·1** trick"*

Multiplying **G** and **H** completeness relations gives a set of projectors and eigen-relations for both:

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$\mathbf{P}_{+1}^{\mathbf{G}}$

$$\mathbf{P}_{+1}^{\mathbf{G}} = \mathbf{P}_{+1,+2}^{\mathbf{GH}} + \mathbf{P}_{+1,-2}^{\mathbf{GH}}$$

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$$\mathbf{P}_{-1,+2}^{\mathbf{GH}} \equiv \mathbf{P}_{-1}^{\mathbf{G}}\mathbf{P}_{+2}^{\mathbf{H}} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \qquad \mathbf{P}_{-1,-2}^{\mathbf{GH}} \equiv \mathbf{P}_{-1}^{\mathbf{G}}\mathbf{P}_{-2}^{\mathbf{H}} = \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

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Another Problem: How do you tell when a Projector $\mathbf{P}^{\mathbf{G}}_{g}$ or $\mathbf{P}^{\mathbf{GH}}_{g,h}$ is 'splittable' (Correct term is *reducible*.)

Solution: It's all in the matrix Trace = sum of its diagonal elements.

Trace $(\mathbf{P}^{\mathbf{G}}_{+1})=2$ so that projector is *reducible* to 2 irreducible projectors. (In this case: $\mathbf{P}^{\mathbf{G}}_{+1} = \mathbf{P}^{\mathbf{GH}}_{+1,+2} + \mathbf{P}^{\mathbf{GH}}_{+1,-2}$)

Trace $(\mathbf{P}^{\mathbf{GH}}_{+1,+2})=1$ so that projector is *irreducible*.

$$\mathbf{1} = \mathbf{P}^{\mathbf{G}}_{+1} + \mathbf{P}^{\mathbf{G}}_{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{1} = \mathbf{P}^{\mathbf{H}}_{+2} + \mathbf{P}^{\mathbf{H}}_{-2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

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$$\mathbf{G}\mathbf{P}^{\mathbf{GH}}_{g,h} = \mathbf{G}\mathbf{P}^{\mathbf{G}}_{g}\mathbf{P}^{\mathbf{H}}_{h} = \varepsilon^{\mathbf{G}}_{g}\mathbf{P}^{\mathbf{GH}}_{g,h}$$

$$\mathbf{H}\mathbf{P}^{\mathbf{GH}}_{g,h} = \mathbf{H}\mathbf{P}^{\mathbf{G}}_{g}\mathbf{P}^{\mathbf{H}}_{h} = \mathbf{P}^{\mathbf{G}}_{g}\mathbf{H}\mathbf{P}^{\mathbf{H}}_{h} = \varepsilon^{\mathbf{H}}_{h}\mathbf{P}^{\mathbf{GH}}_{g,h}$$

$$\mathbf{P}^{\mathbf{GH}}_{+1,+2} \equiv \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{+2} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{P}^{\mathbf{GH}}_{+1,-2} \equiv \mathbf{P}^{\mathbf{G}}_{+1}\mathbf{P}^{\mathbf{H}}_{-2} = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\mathbf{P}^{\mathbf{GH}}_{-1,+2} \equiv \mathbf{P}^{\mathbf{G}}_{-1}\mathbf{P}^{\mathbf{H}}_{+2} = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

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$$\mathbf{H} = (+2)\mathbf{P}^{\mathbf{GH}}_{+1,+2} + (-2)\mathbf{P}^{\mathbf{GH}}_{+1,-2} + (+2)\mathbf{P}^{\mathbf{GH}}_{-1,+2} + (-2)\mathbf{P}^{\mathbf{GH}}_{-1,-2}$$

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Trace ($\mathbf{P}_{+1}^{\mathbf{G}}$)=*2* so that projector is *reducible* to 2 irreducible projectors. (In this case: $\mathbf{P}_{+1}^{\mathbf{G}} = \mathbf{P}_{+1,+2}^{\mathbf{GH}} + \mathbf{P}_{+1,-2}^{\mathbf{GH}}$)

Trace ($\mathbf{P}_{+1,+2}^{\mathbf{GH}}$)=*1* so that projector is *irreducible*.

Trace (**1**)=*4* so that is *reducible* to 4 irreducible projectors.

$$\mathbf{1} = \mathbf{P}_{+1}^{\mathbf{G}} + \mathbf{P}_{-1}^{\mathbf{G}} = \frac{1}{2}\begin{pmatrix} 1&0&0&1\\ 0&1&1&0\\ 0&1&1&0\\ 1&0&0&1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1&0&0&-1\\ 0&1&-1&0\\ 0&-1&1&0\\ -1&0&0&1 \end{pmatrix}$$

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$$\mathbf{P}_{-1,+2}^{\mathbf{GH}} \equiv \mathbf{P}_{-1}^{\mathbf{G}}\mathbf{P}_{+2}^{\mathbf{H}} = \frac{1}{4}\begin{pmatrix} 1&-1&1&-1\\ -1&1&-1&1\\ 1&-1&1&-1\\ -1&1&-1&1 \end{pmatrix} \qquad \mathbf{P}_{-1,-2}^{\mathbf{GH}} \equiv \mathbf{P}_{-1}^{\mathbf{G}}\mathbf{P}_{-2}^{\mathbf{H}} = \frac{1}{4}\begin{pmatrix} 1&1&-1&-1\\ 1&1&-1&-1\\ -1&-1&1&1\\ -1&-1&1&1 \end{pmatrix}$$

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(Preparing for: Degenerate eigenvalues *)*

Review: matrix ***eigen****states (“****own****states) and* ***Idempotent projectors*** *(Degeneracy case)*

Operator orthonormality, completeness, and spectral decomposition(Degenerate e-values *)*

Eigensolutions with degenerate eigenvalues (Possible?... or not?)

Secular→ Hamilton-Cayley→Minimal equations

Diagonalizability criterion

Nilpotents and “Bad degeneracy” examples: $\mathbf{B}=\begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$, *and:* $\mathbf{N}=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Applications of Nilpotent operators later on

Idempotents and “Good degeneracy” example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$

Secular equation by minor expansion

Example of minimal equation projection

Orthonormalization of degenerate eigensolutions

Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*

Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$

*The old "***1=1·1** *trick"-Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

→ *Minimal equation for projector* $\mathbf{P}=\mathbf{P}^2$

How symmetry groups become eigen-solvers

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Trace ($\mathbf{P}^{\mathbf{GH}}_{+1,+2}$)=*1* so that projector is *irreducible*.

Trace ($\mathbf{1}$)=*4* so that is *reducible* to 4 irreducible projectors.

Minimal equation for an idempotent projector is: $\mathbf{P}^2=\mathbf{P}$ or: $\mathbf{P}^2-\mathbf{P} = (\mathbf{P}-0\cdot\mathbf{1})(\mathbf{P}-1\cdot\mathbf{1}) = \mathbf{0}$
So projector eigenvalues are limited to repeated *0*'s and *1*'s. Trace counts the latter.

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Applications of Nilpotent operators later on

Idempotents and "Good degeneracy" example: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$

Secular equation by minor expansion

Example of minimal equation projection

Orthonormalization of degenerate eigensolutions

Projection $\mathbf{P}_j$*-matrix anatomy (Gramian matrices)*

Gram-Schmidt procedure

Orthonormalization of commuting eigensolutions. Examples: $\mathbf{G}=\begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}$ *and:* $\mathbf{H}=\begin{pmatrix} \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \\ 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{pmatrix}$

The old "**1=1·1** *trick*"-*Spectral decomposition by projector splitting*

Irreducible projectors and representations (Trace checks)

Minimal equation for projector $\mathbf{P}=\mathbf{P}^2$

How symmetry groups become eigen-solvers

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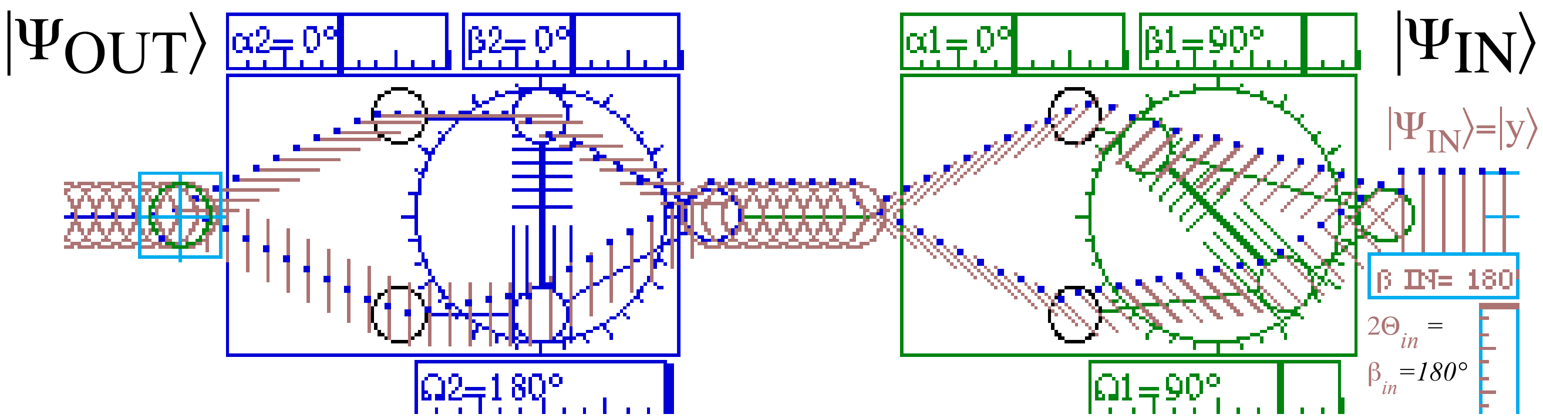
We will study ideal cases first. More general cases are built from these.

Eigensolutions for active analyzers

Matrix products and eigensolutions for active analyzers

Consider a *45°* tilted ($\theta_1=\beta_1/2=\pi/4$ or $\beta_1=90°$) analyzer followed by a untilted ($\beta_2=0$) analyzer.
Active analyzers have both paths open and a phase shift $e^{-i\Omega}$ between each path.
Here the first analyzer has $\Omega_1=90°$. The second has $\Omega_2=180°$.



The transfer matrix for each analyzer is a sum of projection operators for each open path multiplied by the phase factor that is active at that path. Apply phase factor $e^{-i\Omega_1}=e^{-i\pi/2}$ to top path in the first analyzer and the factor $e^{-i\Omega_2}=e^{-i\pi}$ to the top path in the second analyzer.

$$T(2)=e^{-i\pi}|x\rangle\langle x|+|y\rangle\langle y|=\begin{pmatrix} e^{-i\pi} & 0 \\ 0 & 1\end{pmatrix} \qquad T(1)=e^{-i\pi/2}|x'\rangle\langle x'|+|y'\rangle\langle y'|=e^{-i\pi/2}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}+\begin{pmatrix}\frac{1}{2} & \frac{-1}{2}\\ \frac{-1}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}\frac{1-i}{2} & \frac{-1-i}{2}\\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}$$

The matrix product $T(total)=T(2)T(1)$ relates input states $|\Psi_{IN}\rangle$ to output states: $|\Psi_{OUT}\rangle=T(total)|\Psi_{IN}\rangle$

$$T(total)=T(2)T(1)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1-i}{2} & \frac{-1-i}{2}\\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}=\begin{pmatrix}\frac{-1+i}{2} & \frac{1+i}{2}\\ \frac{-1-i}{2} & \frac{1-i}{2}\end{pmatrix}=e^{-i\pi/4}\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\sim\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

We drop the overall phase $e^{-i\pi/4}$ since we can re-attach it later. *T(total)* yields two eigenvalues and projectors.

$\lambda^2-0\lambda-1=0$, or: $\lambda=+1,\ -1$, gives projectors

$$P_{+1}=\frac{\begin{pmatrix}\frac{-1}{\sqrt{2}}+1 & \frac{i}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}}+1\end{pmatrix}}{1-(-1)}=\frac{\begin{pmatrix}-1+\sqrt{2} & i\\ -i & 1+\sqrt{2}\end{pmatrix}}{2\sqrt{2}},\quad P_{-1}=\frac{\begin{pmatrix}1+\sqrt{2} & -i\\ i & -1+\sqrt{2}\end{pmatrix}}{2\sqrt{2}}$$