# *Group Theory in Quantum Mechanics*
## *Lecture 23* *(4.20.17)*
# *Harmonic oscillator symmetry U(1)⊂U(2)⊂U(3)...*

*(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 21-22 )*
*(PSDS - Ch. 8 )*

Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

*2-D* **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
- *2D-Oscillator basic states and operations*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - *2D-Oscillator states and related 3D angular momentum multiplets*
  - *ND multiplets*

Mostly Notation and Bookkeeping :

*R(3) Angular momentum generators by U(2) analysis*
*Angular momentum raise-n-lower operators* $\mathbf{s}_+$ and $\mathbf{s}_-$
*SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

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Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

Define
$$\mathbf{a} = \frac{(\mathbf{X} + i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$
*Destruction operator*

and
$$\mathbf{a}^\dagger = \frac{(\mathbf{X} - i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$
*Creation Operator*

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$ :

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] \equiv \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right) - \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \frac{2i}{2\hbar}\left(\mathbf{p}\mathbf{x} - \mathbf{x}\mathbf{p}\right) = \frac{-i}{\hbar}\left[\mathbf{x},\mathbf{p}\right] = \mathbf{1}$$

$$\boxed{\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}}$$ or $$\boxed{\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}}$$

$$\left[\mathbf{x},\mathbf{p}\right] \equiv \mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x} = \hbar i\,\mathbf{1}$$

Review : *1-D* $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

1st excited state wavefunction $\psi_1(x) = \langle x|1\rangle$

$$\langle x|\mathbf{a}^\dagger|0\rangle = \langle x|1\rangle = \psi_1(x)$$

Expanding the creation operator

$$\langle x|\mathbf{a}^\dagger|0\rangle = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,\langle x|\mathbf{x}|0\rangle - i\langle x|\mathbf{p}|0\rangle/\sqrt{M\omega}\right) = \langle x|1\rangle = \psi_1(x)$$

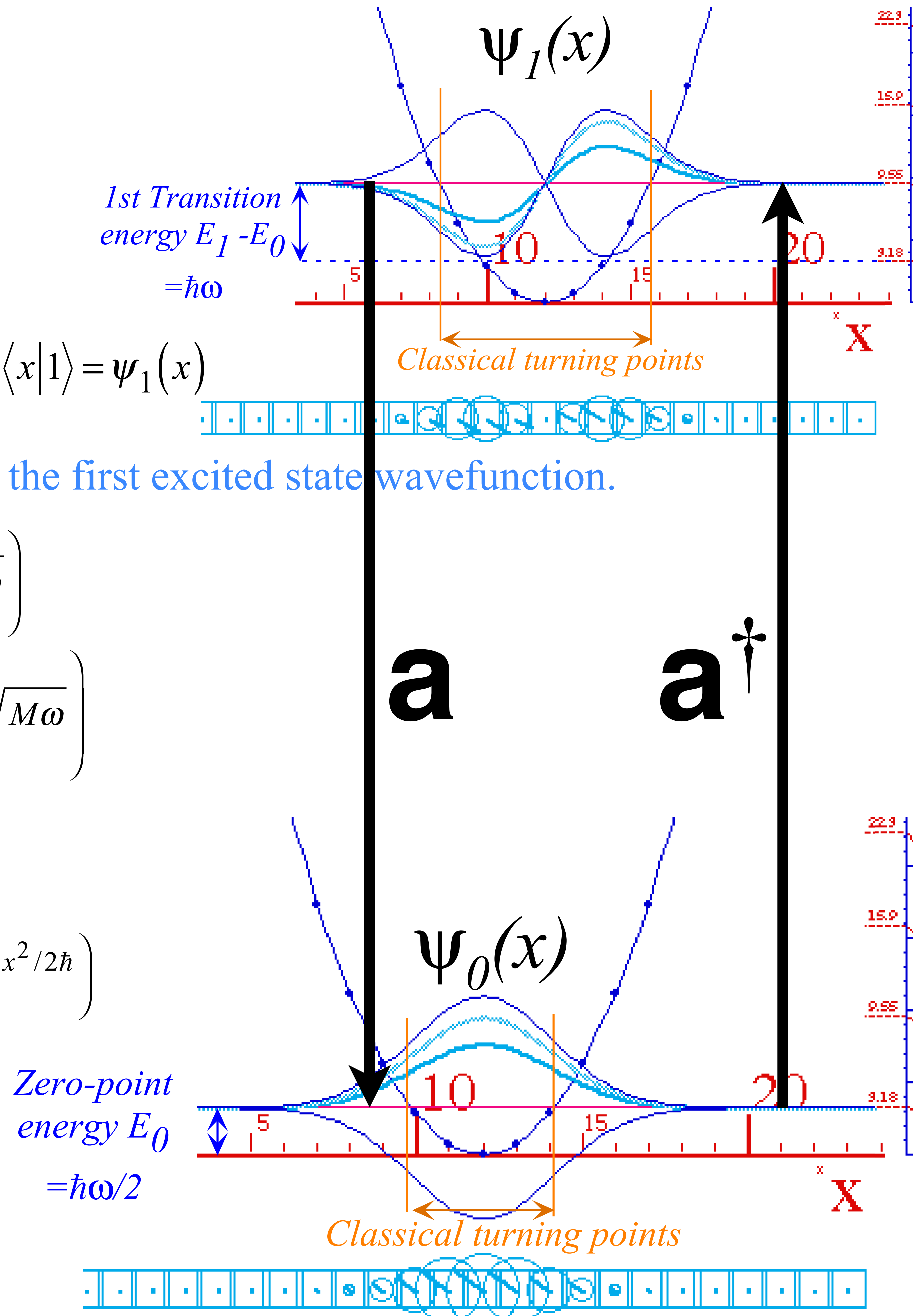
The operator coordinate representations generate the first excited state wavefunction.

$$\langle x|1\rangle = \psi_1(x) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,x\psi_0(x) - i\frac{\hbar}{i}\frac{\partial\psi_0(x)}{\partial x}/\sqrt{M\omega}\right)$$

$$= \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,x\frac{e^{-M\omega x^2/2\hbar}}{const.} - i\frac{\hbar}{i}\frac{\partial}{\partial x}\frac{e^{-M\omega x^2/2\hbar}}{const.}/\sqrt{M\omega}\right)$$

$$= \frac{1}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}\left(\sqrt{M\omega}\,x + i\frac{\hbar}{i}\frac{M\omega x}{\hbar}/\sqrt{M\omega}\right)$$

$$= \frac{\sqrt{M\omega}}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}(2x) = \left(\frac{M\omega}{\pi\hbar}\right)^{3/4}\sqrt{2\pi}\left(x\,e^{-M\omega x^2/2\hbar}\right)$$



Review : *1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations*

Derive normalization for $n^{th}$ state obtained by $(\mathbf{a}^\dagger)^n$ operator:

Use: $\mathbf{a}^n\mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger\mathbf{a} + \frac{n(n-1)}{2!\,2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots\right)$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{const.}, \quad \text{where:} \quad 1 = \langle n|n\rangle = \frac{\langle 0|\mathbf{a}^n\mathbf{a}^{\dagger n}|0\rangle}{(const.)^2} = n!\frac{\langle 0|\mathbf{1} + n\mathbf{a}^\dagger\mathbf{a} + ..|0\rangle}{(const.)^2} = \frac{n!}{(const.)^2}$$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} \quad \textit{Root-factorial normalization}$$

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

Apply destruction $\mathbf{a}$:

$$\mathbf{a}|n\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = \frac{(n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a})|0\rangle}{\sqrt{n!}} = \sqrt{n}\frac{\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{(n-1)!}}$$

$$\mathbf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle$$

*Feynman's mnemonic rule:* Larger of two quanta goes in radical factor

$$\langle\mathbf{a}^\dagger\rangle = \begin{pmatrix} \cdot & & & & & \\ 1 & \cdot & & & & \\ & \sqrt{2} & \cdot & & & \\ & & \sqrt{3} & \cdot & & \\ & & & \sqrt{4} & \cdot & \\ & & & & \ddots & \cdot \end{pmatrix} \qquad \langle\mathbf{a}\rangle = \begin{pmatrix} \cdot & 1 & & & & \\ & \cdot & \sqrt{2} & & & \\ & & \cdot & \sqrt{3} & & \\ & & & \cdot & \sqrt{4} & \\ & & & & \cdot & \ddots \\ & & & & & \cdot \end{pmatrix}$$

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

*Number operator and Hamiltonian operator*

*Number operator* $\mathbf{N}=\mathbf{a}^\dagger\mathbf{a}$ counts quanta.

$$\mathbf{a}^\dagger\mathbf{a}|n\rangle = \frac{\mathbf{a}^\dagger\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^\dagger\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n|n\rangle$$

*Hamiltonian operator*

$$\mathbf{H}\,|n\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,|n\rangle + \hbar\omega/2\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

$$\langle\mathbf{H}\rangle = \hbar\omega\langle\mathbf{a}^\dagger\mathbf{a} + \tfrac{1}{2}\mathbf{1}\rangle = \hbar\omega\begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} + \hbar\omega\begin{pmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix}$$

*Hamiltonian operator* is $\hbar\omega$ $\mathbf{N}$ plus zero-point energy $\mathbf{1}\hbar\omega/2$ .

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$\mathbf{T}(a)$ and $\mathbf{B}(b)$ operations do not commute. $\mathbf{T}(a) = e^{-ia\mathbf{p}/\hbar}$ *or* $\mathbf{B}(b) = e^{ib\mathbf{x}/\hbar}$

Define a *combined boost-translation operation:* $\mathbf{C}(a,b) = e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}$

Use *Baker-Campbell-Hausdorf identity* since $[\mathbf{x},\mathbf{p}]=i\hbar\mathbf{1}$ and $[[\mathbf{x},\mathbf{p}],\mathbf{x}]=[[\mathbf{x},\mathbf{p}],\mathbf{p}]=\mathbf{0}$.

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}e^{-[\mathbf{A},\mathbf{B}]/2} = e^{\mathbf{B}}e^{\mathbf{A}}e^{[\mathbf{A},\mathbf{B}]/2} \text{ , where: } \left[\mathbf{A},\left[\mathbf{A},\mathbf{B}\right]\right] = \mathbf{0} = \left[\mathbf{B},\left[\mathbf{A},\mathbf{B}\right]\right]$$

$$\mathbf{C}(a,b) = e^{i(b\mathbf{x}-a\mathbf{p})/\hbar} = e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-ab[\mathbf{x},\mathbf{p}]/2\hbar^2} = e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-iab/2\hbar}$$

$$\mathbf{C}(a,b) = \mathbf{B}(b)\mathbf{T}(a)e^{-iab/2\hbar} = \mathbf{T}(a)\mathbf{B}(b)e^{iab/2\hbar}$$

 *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*

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$$\mathbf{C}(a,b) = \mathbf{B}(b)\mathbf{T}(a)e^{-iab/2\hbar} = \mathbf{T}(a)\mathbf{B}(b)e^{iab/2\hbar}$$

Complex *phasor coordinate* α*(a,b)* defined by: $\alpha(a,b)$

$$= a\sqrt{M\omega/2\hbar} + ib/\sqrt{2\hbar M\omega}$$

$$= \left[a + i\frac{b}{M\omega}\right]\sqrt{M\omega/2\hbar}$$

Reordering only affects the overall phase.

$$\mathbf{C}(a,b) = e^{i(b\mathbf{x}-a\mathbf{p})/\hbar} = e^{ib(\mathbf{a}^\dagger+\mathbf{a})/\sqrt{2\hbar M\omega}+a(\mathbf{a}^\dagger-\mathbf{a})\sqrt{M\omega/2\hbar}}$$

$$= e^{\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}} = e^{-|\alpha|^2/2}e^{\alpha\mathbf{a}^\dagger}e^{-\alpha^*\mathbf{a}} = e^{|\alpha|^2/2}e^{-\alpha^*\mathbf{a}}e^{\alpha\mathbf{a}^\dagger}$$

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$$\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}=e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-ab[\mathbf{x},\mathbf{p}]/2\hbar^2}=e^{ib\mathbf{x}/\hbar}e^{-ia\mathbf{p}/\hbar}e^{-iab/2\hbar}$$

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Complex *phasor coordinate* $\alpha(a,b)$ defined by: $\alpha(a,b)$

$$=a\sqrt{M\omega/2\hbar}+ib/\sqrt{2\hbar M\omega}$$

$$=\left[a+i\frac{b}{M\omega}\right]\sqrt{M\omega/2\hbar}$$

Reordering only affects the overall phase.

$$\mathbf{C}(a,b)=e^{i(b\mathbf{x}-a\mathbf{p})/\hbar}=e^{ib(\mathbf{a}^\dagger+\mathbf{a})/\sqrt{2\hbar M\omega}+a(\mathbf{a}^\dagger-\mathbf{a})\sqrt{M\omega/2\hbar}}$$

$$=e^{\alpha\mathbf{a}^\dagger-\alpha^*\mathbf{a}}=e^{-|\alpha|^2/2}e^{\alpha\mathbf{a}^\dagger}e^{-\alpha^*\mathbf{a}}=e^{|\alpha|^2/2}e^{-\alpha^*\mathbf{a}}e^{\alpha\mathbf{a}^\dagger}$$

*Coherent wavepacket state* $|\alpha(x_0,p_0)\rangle$: $|\alpha_0(x_0,p_0)\rangle=\mathbf{C}(x_0,p_0)|0\rangle=e^{i(x_0\mathbf{x}-p_0\mathbf{p})/\hbar}|0\rangle$

$$=e^{-|\alpha_0|^2/2}e^{\alpha_0\mathbf{a}^\dagger}e^{-\alpha_0^*\mathbf{a}}|0\rangle$$

$$=e^{-|\alpha_0|^2/2}e^{\alpha_0\mathbf{a}^\dagger}|0\rangle$$

$$=e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\left(\alpha_0\mathbf{a}^\dagger\right)^n|0\rangle/n!$$

$$=e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}|n\rangle\ ,\quad \text{where: } |n\rangle=\frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$

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Review : *Time evolution of coherent state (and "squeezed" states)*

$$\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\left|n\right\rangle$$

Time evolution operator for constant $\mathbf{H}$ has general form : $\mathbf{U}(t,0)=\mathrm{e}^{-\mathrm{i}\mathbf{H}t/\hbar}$

Oscillator eigenstate time evolution is simply determined by harmonic phases.

$$\mathbf{U}(t,0)\left|n\right\rangle = e^{-i\mathbf{H}t/\hbar}\left|n\right\rangle = e^{-i(n+1/2)\omega t}\left|n\right\rangle$$

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Coherent state evolution results.

$$\mathbf{U}(t,0)\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\mathbf{U}(t,0)\left|n\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}e^{-i(n+1/2)\omega t}\left|n\right\rangle$$

$$= e^{-i\omega t/2}e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha_0 e^{-i\omega t}\right)^n}{\sqrt{n!}}\left|n\right\rangle$$

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$$= e^{-i\omega t/2}e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha_0 e^{-i\omega t}\right)^n}{\sqrt{n!}}\left|n\right\rangle$$

Evolution simplifies to a variable-$\alpha_0$ coherent state with a *time dependent phasor coordinate* $\alpha_t$:

$$\mathbf{U}(t,0)\left|\alpha_0(x_0,p_0)\right\rangle = e^{-i\omega t/2}\left|\alpha_t(x_t,p_t)\right\rangle$$ where:

$$\alpha_t(x_t,p_t) = e^{-i\omega t}\,\alpha_0(x_0,p_0)$$

$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

Review : *Time evolution of coherent state (and "squeezed" states)* $\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\left|n\right\rangle$

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$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

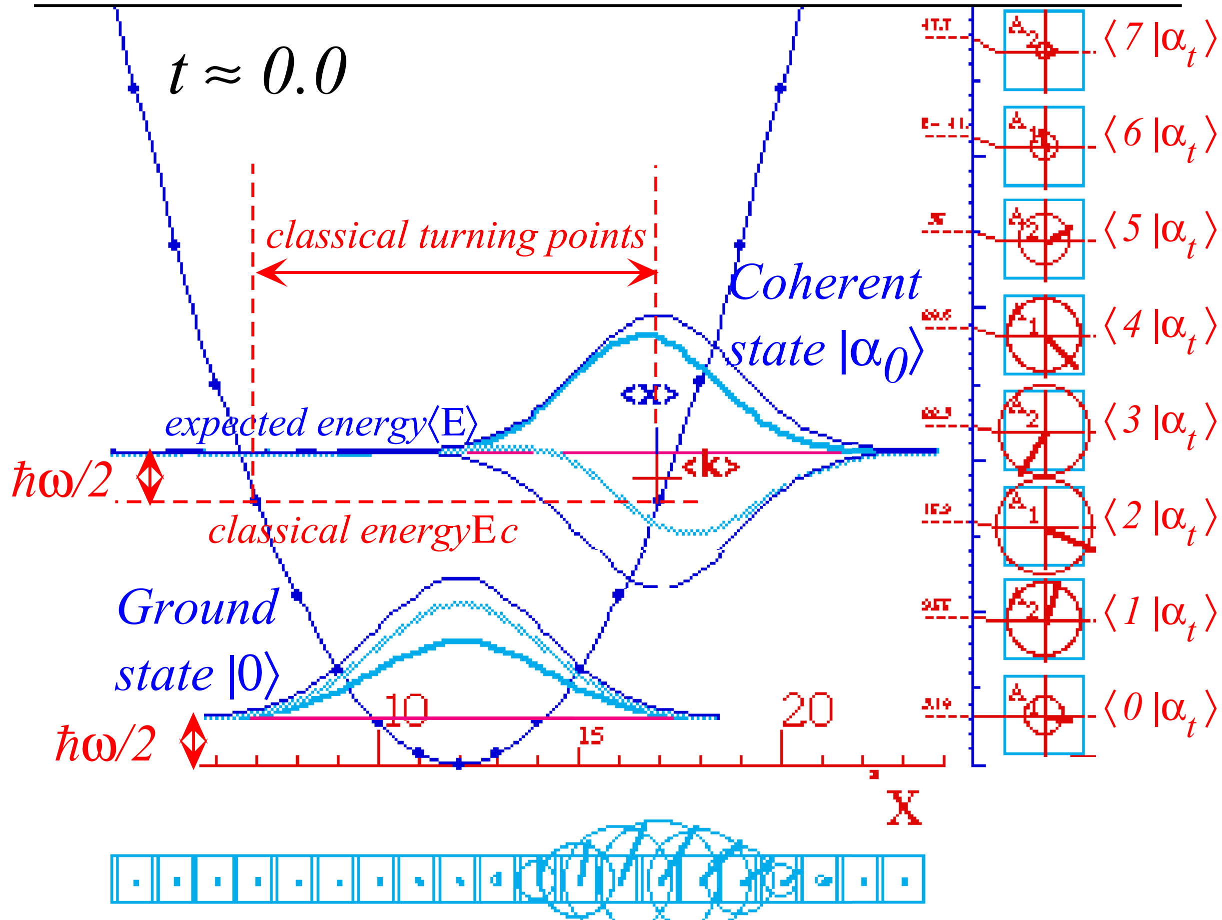
*$(x_t,p_t)$* mimics classical oscillator

$$x_t = x_0\cos\omega t + \frac{p_0}{M\omega}\sin\omega t$$

$$\frac{p_t}{M\omega} = -x_0\sin\omega t + \frac{p_0}{M\omega}\cos\omega t$$

(Real and imaginary parts ($x_t$ and $p_t/M\omega$) of $\alpha_t$ go clockwise on phasor circle.)

 *Time evolution of coherent state (and "squeezed" states)*




Coherent ket $|\alpha(x0,p0)\rangle$ is eigenvector of destruct-op. **a**.

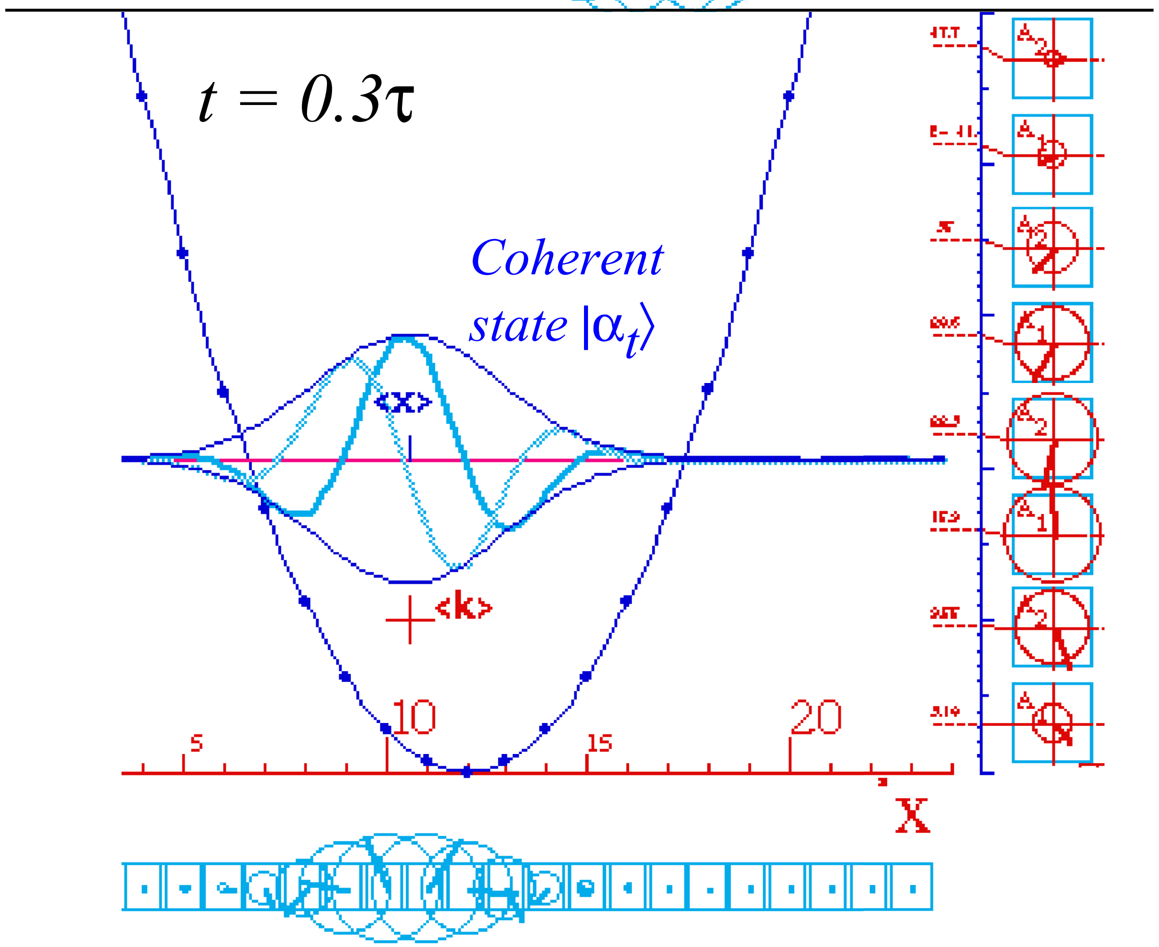
$$\mathbf{a}|\alpha_0(x_0,p_0)\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\mathbf{a}|n\rangle$$

$$= e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle$$

$$= \alpha_0|\alpha_0(x_0,p_0)\rangle \quad \text{with eigenvalue } \alpha_0$$

Coherent bra $\langle\alpha(x0,p0)|$ is eigenvector of create-op. $\mathbf{a}^\dagger$.

$$\langle\alpha_0(x_0,p_0)|\mathbf{a}^\dagger = \langle\alpha_0(x_0,p_0)|\alpha_0^*$$



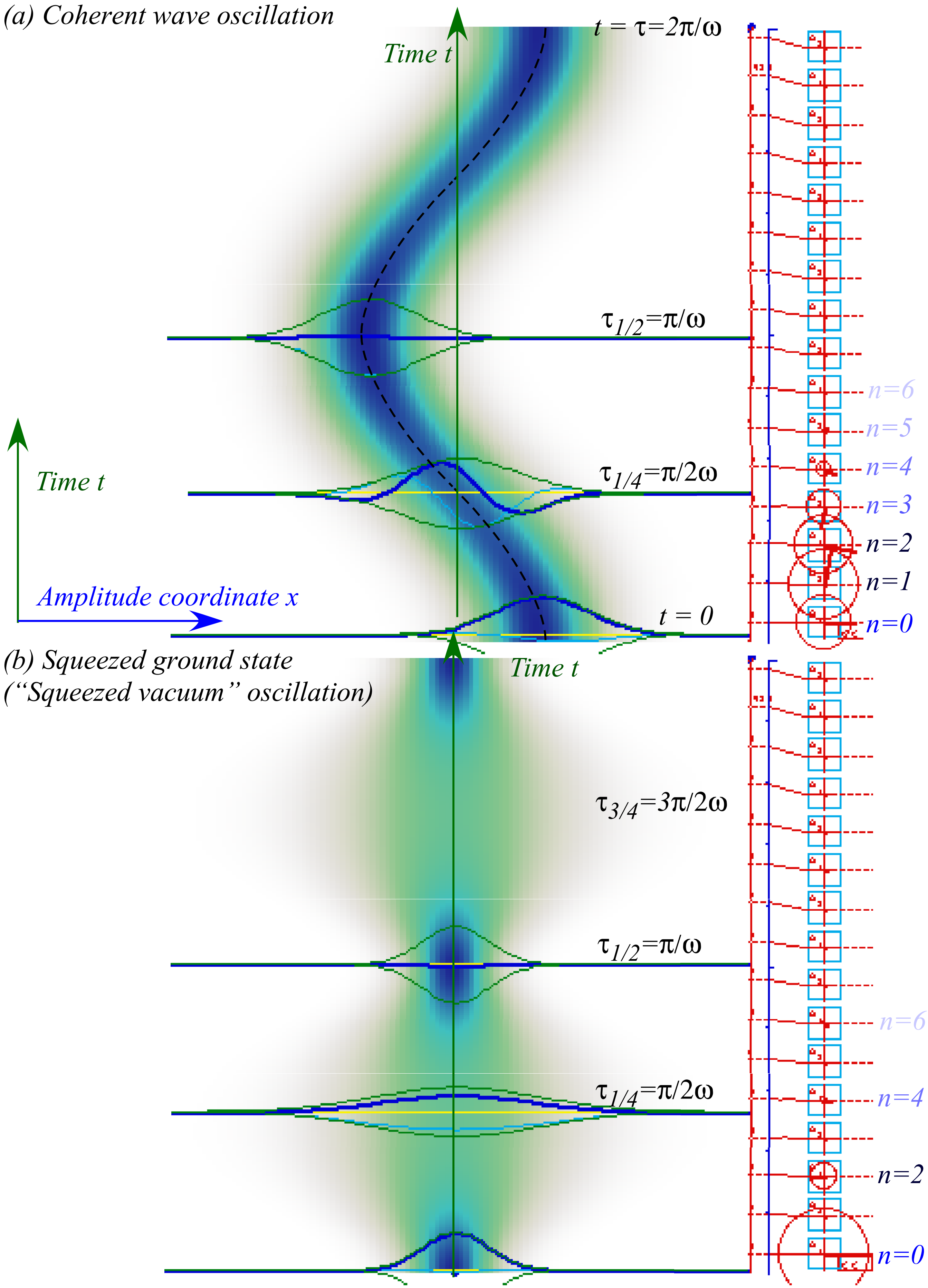

Expected quantum energy has simple time independent form.

$$\langle E\rangle\big|_{\alpha_0} = \langle\alpha_0(x_0,p_0)|\mathbf{H}|\alpha_0(x_0,p_0)\rangle$$

$$= \langle\alpha_0(x_0,p_0)|\left(\hbar\omega\mathbf{a}^\dagger\mathbf{a} + \frac{\hbar\omega}{2}\mathbf{1}\right)|\alpha_0(x_0,p_0)\rangle$$

$$= \hbar\omega\alpha_0^*\alpha_0 + \frac{\hbar\omega}{2}$$

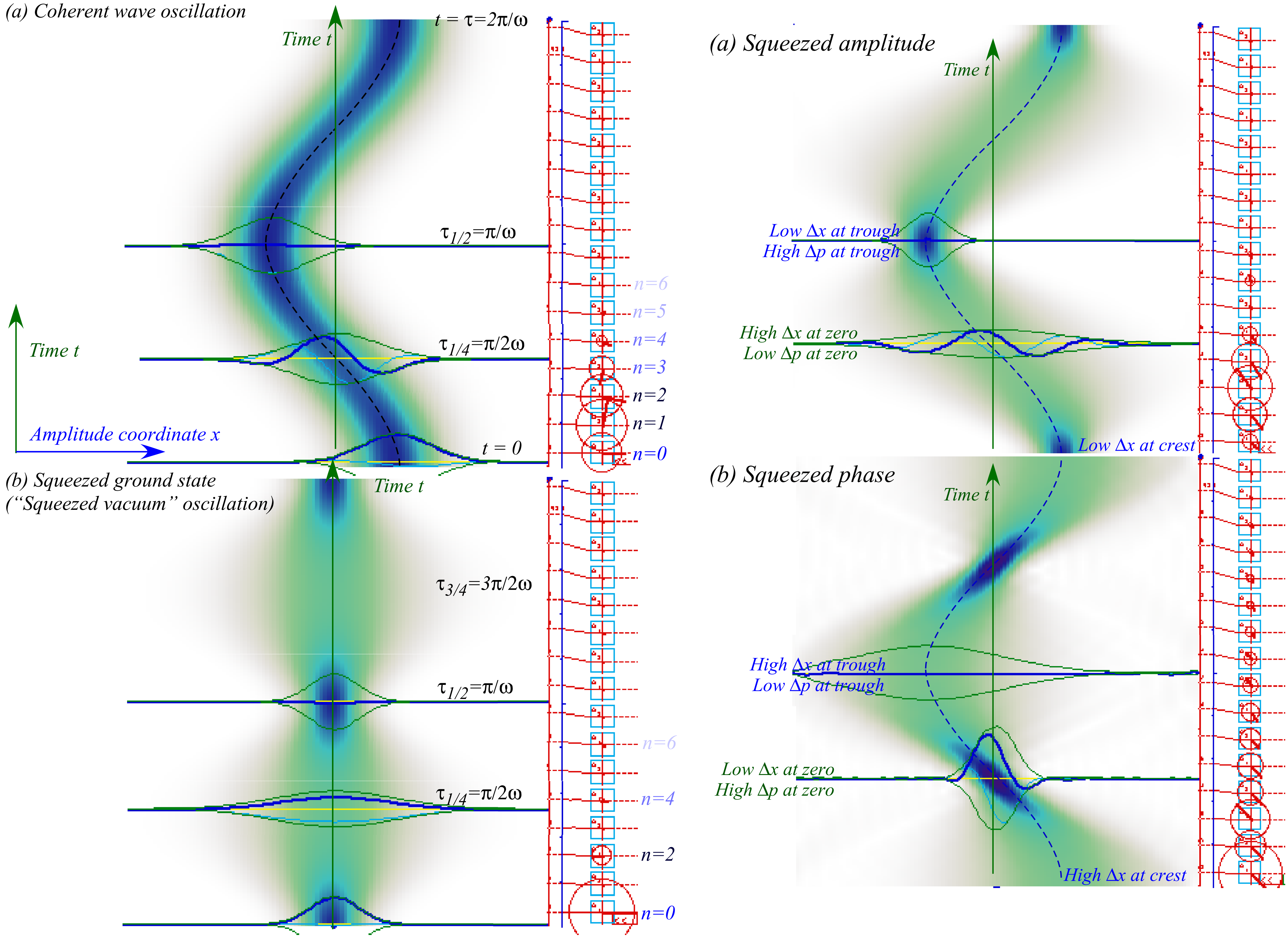
# *Properties of "squeezed" coherent states*




*Yay! Classical Cosine trajectory!*

*what happens if you apply operators with non-linear "tensor" exponents* $\exp(s\mathbf{x}^2)$, $\exp(f\,\mathbf{p}^2)$, *etc.*

# Properties of “squeezed” coherent states



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  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
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$$\mathbf{H} = H_{11}\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + H_{12}\mathbf{a}_1^\dagger\mathbf{a}_2 + H_{21}\mathbf{a}_2^\dagger\mathbf{a}_1 + H_{22}\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$= A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{1}/2\right) + (B - iC)\mathbf{a}_1^\dagger\mathbf{a}_2 + (B + iC)\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

# *2D-Oscillator basic states and operations - Commutattion*

First rewrite a classical 2-D Hamiltonian (Lecture. 6-9) with a thick-tip pen! (They're **operators** now!)

$$\mathbf{H} = \frac{A}{2}\left(\mathbf{p}_1^2 + \mathbf{x}_1^2\right) + B\left(\mathbf{x}_1\mathbf{x}_2 + \mathbf{p}_1\mathbf{p}_2\right) + C\left(\mathbf{x}_1\mathbf{p}_2 - \mathbf{x}_2\mathbf{p}_1\right) + \frac{D}{2}\left(\mathbf{p}_2^2 + \mathbf{x}_2^2\right)$$

(Mass factors $\sqrt{M}$, spring constants $K_{ij}$, and Planck $\hbar$ constants are absorbed into $A$, $B$, $C$, and $D$ constants used in Lectures 6-9.)

Define **a** and **a**$^\dagger$ **operators**

$\mathbf{a}_1 = (\mathbf{x}_1 + i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger_1 = (\mathbf{x}_1 - i\,\mathbf{p}_1)/\sqrt{2}$ $\quad$ $\mathbf{a}_2 = (\mathbf{x}_2 + i\,\mathbf{p}_2)/\sqrt{2}$ $\quad$ $\mathbf{a}^\dagger_2 = (\mathbf{x}_2 - i\,\mathbf{p}_2)/\sqrt{2}$

$\mathbf{x}_1 = (\mathbf{a}^\dagger_1 + \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{p}_1 = i\,(\mathbf{a}^\dagger_1 - \mathbf{a}_1)/\sqrt{2}$ $\quad$ $\mathbf{x}_2 = (\mathbf{a}^\dagger_2 + \mathbf{a}_2)/\sqrt{2}$ $\quad$ $\mathbf{p}_2 = i\,(\mathbf{a}^\dagger_2 - \mathbf{a}_2)/\sqrt{2}$

Each system dimension $\mathbf{x}_1$ and $\mathbf{x}_2$ is assumed orthogonal, neither being constrained by the other. This includes an axiom of *inter-dimensional commutivity*.

$$[\,\mathbf{x}_1, \mathbf{p}_2] = \mathbf{0} = [\,\mathbf{x}_2, \mathbf{p}_1], \quad [\,\mathbf{a}_1, \mathbf{a}^\dagger_2] = \mathbf{0} = [\,\mathbf{a}_2, \mathbf{a}^\dagger_1]$$

Commutation relations within space-1 or space-2 space are those of a 1D-oscillator.

$$[\,\mathbf{a}_1, \mathbf{a}^\dagger_1] = \mathbf{1}, \quad [\,\mathbf{a}_2, \mathbf{a}^\dagger_2] = \mathbf{1}$$

This applies in general to $N$-dimensional oscillator problems.

$$[\,\mathbf{a}_m, \mathbf{a}_n] = \mathbf{a}_m\mathbf{a}_n - \mathbf{a}_n\mathbf{a}_m = \mathbf{0} \qquad [\,\mathbf{a}_m, \mathbf{a}^\dagger_n] = \mathbf{a}_m\mathbf{a}^\dagger_n - \mathbf{a}^\dagger_n\mathbf{a}_m = \delta_{mn}\mathbf{1} \qquad [\,\mathbf{a}^\dagger_m, \mathbf{a}^\dagger_n] = \mathbf{a}^\dagger_m\mathbf{a}^\dagger_n - \mathbf{a}^\dagger_n\mathbf{a}^\dagger_m = \mathbf{0}$$

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$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

Both are elementary "place-holders" for parameters $H_{mn}$ or $A$, $B \pm iC$, and $D$.

$$|m\rangle\langle n| \rightarrow \left(\mathbf{a}_m^\dagger\mathbf{a}_n + \mathbf{a}_n\mathbf{a}_m^\dagger\right)/2 = \mathbf{a}_m^\dagger\mathbf{a}_n + \delta_{m,n}\mathbf{1}/2$$

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*Fermi operators* ($\mathbf{c}_m$,$\mathbf{c}_n$) are defined to create *Fermions* and use *anti-commutators* $\{\mathbf{A},\mathbf{B}\} = \mathbf{AB}+\mathbf{BA}$.

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Fermi $\mathbf{c}^\dagger{}_n$ has a rigid birth-control policy; they are allowed just one Fermion or else, none at all.

Creating two Fermions of the same type is punished by death. This is because $x=-x$ implies $x=0$.

$$\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = -\,\mathbf{c}^\dagger{}_m\mathbf{c}^\dagger{}_m\,|0\rangle = \mathbf{0}$$

That no two indistinguishable Fermions can be in the same state, is called the *Pauli exclusion principle*.

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Quantum numbers of $n=0$ and $n=1$ are the only allowed eigenvalues of the number operator $\mathbf{c}^\dagger_m\mathbf{c}_m$.

$$\mathbf{c}^\dagger_m\mathbf{c}_m|0\rangle = \mathbf{0}\ ,\ \mathbf{c}^\dagger_m\mathbf{c}_m|1\rangle = |1\rangle\ ,\ \mathbf{c}^\dagger_m\mathbf{c}_m|n\rangle = \mathbf{0}\ \text{ for: } n>1$$

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A state for a particle in two-dimensions (or two one-dimensional particles) is a *"ket-ket"* $|n_1\rangle|n_2\rangle$

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Must ask a perennial modern question: *"How are these structures stored in a computer program?"*
The usual answer is in *outer product* or *tensor arrays*. Next pages show sketches of these objects.

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*2-D* **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
*2D-Oscillator basic states and operations*
*Commutation relations*
*Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
*Anti-commutation relations*
*Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
Mostly Notation and Bookkeeping : → *Outer product arrays* ←
*Entangled 2-particle states*
*Two-particle (or 2-dimensional) matrix operators*
*U(2) Hamiltonian and irreducible representations*
*2D-Oscillator states and related 3D angular momentum multiplets*
*R(3) Angular momentum generators by U(2) analysis*
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*SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

# *Outer product arrays*

Start with an elementary ket basis for each dimension or particle type-1 and type-2.

$$\begin{array}{lll} Type-1 & Type-2 & \cdots \end{array}$$

$$|0_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |1_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, |2_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \cdots \qquad |0_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |1_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, |2_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \cdots$$

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Outer products are constructed for the states that might have non-negligible amplitudes.

$$|0_1\rangle|0_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, |0_1\rangle|1_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \cdots|1_1\rangle|0_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \hline 1 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \cdots|1_1\rangle|2_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 1 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

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$$\left|\Psi_1\right\rangle\left|\Psi_2\right\rangle=\begin{pmatrix}\langle 0|\Psi_1\rangle\\\langle 1|\Psi_1\rangle\\\langle 2|\Psi_1\rangle\\\vdots\end{pmatrix}\otimes\begin{pmatrix}\langle 0|\Psi_2\rangle\\\langle 1|\Psi_2\rangle\\\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\langle 0|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline\langle 1|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline\langle 2|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\langle 0_10_2|\Psi_1\Psi_2\rangle\\\langle 0_11_2|\Psi_1\Psi_2\rangle\\\langle 0_12_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline\langle 1_10_2|\Psi_1\Psi_2\rangle\\\langle 1_11_2|\Psi_1\Psi_2\rangle\\\langle 1_12_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline\langle 2_10_2|\Psi_1\Psi_2\rangle\\\langle 2_11_2|\Psi_1\Psi_2\rangle\\\langle 2_12_2|\Psi_1\Psi_2\rangle\\\vdots\end{pmatrix}$$

"Little-Endian" indexing *(...01,02,03..10,11,12,13 ... 20,21,22,23,...)*

Least significant digit at (right) END

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"Little-Endian" indexing *(...01,02,03..10,11,12,13 ... 20,21,22,23,...)*

Least significant digit at (right) END

$$\left|\Psi_1\right\rangle\left|\Psi_2\right\rangle=\begin{pmatrix}\langle 0|\Psi_1\rangle\\\langle 1|\Psi_1\rangle\\\langle 2|\Psi_1\rangle\\\vdots\end{pmatrix}\otimes\begin{pmatrix}\langle 0|\Psi_2\rangle\\\langle 1|\Psi_2\rangle\\\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\langle 0|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 0|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline\langle 1|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 1|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\\\hline\langle 2|\Psi_1\rangle\langle 0|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 1|\Psi_2\rangle\\\langle 2|\Psi_1\rangle\langle 2|\Psi_2\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\langle 0_1 0_2|\Psi_1\Psi_2\rangle\\\langle 0_1 1_2|\Psi_1\Psi_2\rangle\\\langle 0_1 2_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline\langle 1_1 0_2|\Psi_1\Psi_2\rangle\\\langle 1_1 1_2|\Psi_1\Psi_2\rangle\\\langle 1_1 2_2|\Psi_1\Psi_2\rangle\\\vdots\\\hline\langle 2_1 0_2|\Psi_1\Psi_2\rangle\\\langle 2_1 1_2|\Psi_1\Psi_2\rangle\\\langle 2_1 2_2|\Psi_1\Psi_2\rangle\\\vdots\end{pmatrix}$$

or anti-lexicographic *(00, 10, 20, ...01, 11, 21,..., 02, 12, 22, ..)* array indexing

"Big-Endian" indexing *(...00,10,20..01,11,21,31 ... 02,12,22,32...)*

Most significant digit at (right) END

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shorthand *big-bra-big-ket* notation

$$\left|\Psi\right\rangle=\begin{pmatrix}\langle 0_10_2|\Psi\rangle\\\langle 0_11_2|\Psi\rangle\\\langle 0_12_2|\Psi\rangle\\\vdots\\\hline\langle 1_10_2|\Psi\rangle\\\langle 1_11_2|\Psi\rangle\\\langle 1_12_2|\Psi\rangle\\\vdots\\\hline\langle 2_10_2|\Psi\rangle\\\langle 2_11_2|\Psi\rangle\\\langle 2_12_2|\Psi\rangle\\\vdots\end{pmatrix}=\begin{pmatrix}\Psi_{00}\\\Psi_{01}\\\Psi_{02}\\\vdots\\\hline\Psi_{10}\\\Psi_{11}\\\Psi_{12}\\\vdots\\\hline\Psi_{20}\\\Psi_{21}\\\Psi_{22}\\\vdots\end{pmatrix}$$

"Little-Endian" indexing

*(...01,02,03..10,11,12,13 ... 20,21,22,23,...)*

Least significant digit at (right) END

Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
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- *2D-Oscillator basic states and operations*
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## *Entangled 2-particle states*

A matrix operator **M** is rarely a single nilpotent operator $|1\rangle\langle 2|$ or idempotent $|1\rangle\langle 1|$.

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A two-particle state $|\Psi\rangle$ is rarely a single outer product $|\Psi_1\rangle|\Psi_2\rangle$ of 1-particle states $|\Psi_1\rangle$ and $|\Psi_2\rangle$.
(Even rarer is $|\Psi_1\rangle|\Psi_1\rangle$.)

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ANALOGY:

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Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

*2-D* **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
*2D-Oscillator basic states and operations*
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*Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
*Anti-commutation relations*
*Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
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| | $\|00\rangle$ | $\|01\rangle$ | $\|02\rangle$ | $\cdots$ | $\|10\rangle$ | $\|11\rangle$ | $\|12\rangle$ | $\cdots$ | $\|20\rangle$ | $\|21\rangle$ | $\|22\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\|$ | 0 | | | $\cdots$ | · | | | $\cdots$ | | | | |
| $\langle 01\|$ | | $D$ | | $\cdots$ | $B+iC$ | · | | $\cdots$ | | | | |
| $\langle 02\|$ | | | $2D$ | $\cdots$ | | $\sqrt{2}(B+iC)$ | · | $\cdots$ | | | | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | | | | $\cdots$ |
| $\langle 10\|$ | · | $B-iC$ | | $\cdots$ | $A$ | | | $\cdots$ | · | | | $\cdots$ |
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|---|---|---|---|---|---|---|---|---|---|---|---|---|
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| $\langle 02\vert$ | | | $2D$ | $\cdots$ | | $\sqrt{2}(B+iC)$ | $\cdot$ | $\cdots$ | | | | $\cdots$ |
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| $\langle 22\rvert$ | | | | | | | $\cdot$ | $\cdots$ | | | $2A+2D$ | $\cdots$ |
| $\vdots$ | | | | | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

"Little-Endian" indexing (...*01,02,03..10,11,12,13 ... 20,21,22,23,...*)

Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T**(*a*) *and/or Boost* **B**(*b*) *to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

*2-D* **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
*2D-Oscillator basic states and operations*
*Commutation relations*
*Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
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*Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
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*Angular momentum raise-n-lower operators* $\mathbf{s}_+$ and $\mathbf{s}_-$
*SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

# *U(2)-2D-HO Hamiltonian and irreducible representations*

"Little-Endian" indexing
*(...01,02,03..10,11,12,13 ... 20,21,22,23,...)*

$$\mathbf{H}=$$

$$A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+\left(B-iC\right)\mathbf{a}_1^\dagger\mathbf{a}_2$$

$$+\left(B+iC\right)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$$\langle\mathbf{H}\rangle=A(\mathbf{1}/2)+D(\mathbf{1}/2)+$$

| | $\vert 00\rangle$ | $\vert 01\rangle$ | $\vert 02\rangle$ | $\cdots$ | $\vert 10\rangle$ | $\vert 11\rangle$ | $\vert 12\rangle$ | $\cdots$ | $\vert 20\rangle$ | $\vert 21\rangle$ | $\vert 22\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\vert$ | 0 | | | $\cdots$ | $\cdot$ | | | $\cdots$ | | | | $\cdots$ |
| $\langle 01\vert$ | | $D$ | | $\cdots$ | $B+iC$ | $\cdot$ | | $\cdots$ | | | | $\cdots$ |
| $\langle 02\vert$ | | | $2D$ | $\cdots$ | | $\sqrt{2}(B+iC)$ | $\cdot$ | $\cdots$ | | | | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | | | | $\cdots$ |
| $\langle 10\vert$ | $\cdot$ | $B-iC$ | | $\cdots$ | $A$ | | | $\cdots$ | $\cdot$ | | | $\cdots$ |
| $\langle 11\vert$ | | $\cdot$ | $\sqrt{2}(B-iC)$ | $\cdots$ | | $A+D$ | | $\cdots$ | $\sqrt{2}(B+iC)$ | $\cdot$ | | $\cdots$ |
| $\langle 12\vert$ | | *Example:* | $\cdot$ | $\cdots$ | | | $A+2D$ | $\cdots$ | | $\sqrt{4}(B+iC)$ | $\cdot$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
| $\langle 20\vert$ | $\mathbf{a}_1^\dagger\mathbf{a}_2\vert 02\rangle=\sqrt{0+1}\sqrt{2}\vert 0+1\ 2-1\rangle=\sqrt{2}\vert 1\,1\rangle$ | | | | $\cdot$ | $\sqrt{2}(B-iC)$ | | $\cdots$ | $2A$ | | | $\cdots$ |
| $\langle 21\vert$ | $\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\ n_2-1\rangle$ | | | | | $\cdot$ | $\sqrt{4}(B-iC)$ | $\cdots$ | | $2A+D$ | | $\cdots$ |
| $\langle 22\vert$ | | | | | | | $\cdot$ | $\cdots$ | | | $2A+2D$ | $\cdots$ |
| $\vdots$ | | | | | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

$$\mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=n_1\left|n_1 n_2\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\ n_2+1\right\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\ n_2-1\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=n_2\left|n_1 n_2\right\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

# U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ...
20,21,22,23,...)

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1+\mathbf{1}/2\right)+(B-iC)\mathbf{a}_1^\dagger\mathbf{a}_2+(B+iC)\mathbf{a}_2^\dagger\mathbf{a}_1+D\left(\mathbf{a}_2^\dagger\mathbf{a}_2+\mathbf{1}/2\right)$$

$$\langle\mathbf{H}\rangle = A(\mathbf{1}/2)+D(\mathbf{1}/2)+$$

| | $\vert 00\rangle$ | $\vert 01\rangle$ | $\vert 02\rangle$ | $\cdots$ | $\vert 10\rangle$ | $\vert 11\rangle$ | $\vert 12\rangle$ | $\cdots$ | $\vert 20\rangle$ | $\vert 21\rangle$ | $\vert 22\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\vert$ | 0 | | | $\cdots$ | $\cdot$ | | | $\cdots$ | | | | $\cdots$ |
| $\langle 01\vert$ | | $D$ | | $\cdots$ | $B+iC$ | $\cdot$ | | $\cdots$ | | | | $\cdots$ |
| $\langle 02\vert$ | | | $2D$ | $\cdots$ | | $\sqrt{2}(B+iC)$ | $\cdot$ | $\cdots$ | | | | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | | | | $\cdots$ |
| $\langle 10\vert$ | $\cdot$ | $B-iC$ | | $\cdots$ | $A$ | | | $\cdots$ | $\cdot$ | | | $\cdots$ |
| $\langle 11\vert$ | | $\cdot$ | $\sqrt{2}(B-iC)$ | $\cdots$ | | $A+D$ | | $\cdots$ | $\sqrt{2}(B+iC)$ | $\cdot$ | | $\cdots$ |
| $\langle 12\vert$ | | *Example:* | $\cdot$ | $\cdots$ | | | $A+2D$ | $\cdots$ | | $\sqrt{4}(B+iC)$ | $\cdot$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
| $\langle 20\vert$ | $\mathbf{a}_1^\dagger\mathbf{a}_2\vert 02\rangle=\sqrt{0+1}\sqrt{2}\vert 0+1\ 2-1\rangle=\sqrt{2}\vert 1\,1\rangle$ | | | | $\cdot$ | $\sqrt{2}(B-iC)$ | | $\cdots$ | $2A$ | | | $\cdots$ |
| $\langle 21\vert$ | $\mathbf{a}_1^\dagger\mathbf{a}_2\vert n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}\vert n_1+1\ n_2-1\rangle$ | | | | | $\cdot$ | $\sqrt{4}(B-iC)$ | $\cdots$ | | $2A+D$ | | $\cdots$ |
| $\langle 22\vert$ | | | | | | | $\cdot$ | $\cdots$ | | | $2A+2D$ | $\cdots$ |
| $\vdots$ | | | | | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

$$\mathbf{a}_1^\dagger\mathbf{a}_1|n_1n_2\rangle=n_1|n_1n_2\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}|n_1-1\,n_2+1\rangle$$

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}|n_1+1\,n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_2|n_1n_2\rangle=n_2|n_1n_2\rangle$$

Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $|n_1\rangle|n_2\rangle$ with the same *total quantum number* $\nu = n_1 + n_2$ define each block.

# U(2)-2D-HO Hamiltonian and irreducible representations

"Little-Endian" indexing
(...01,02,03..10,11,12,13 ...
20,21,22,23,...)

$$\mathbf{H} = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + 1/2\right) + (B - iC)\,\mathbf{a}_1^\dagger\mathbf{a}_2 + (B + iC)\,\mathbf{a}_2^\dagger\mathbf{a}_1 + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + 1/2\right)$$

$$\langle\mathbf{H}\rangle = A(1/2) + D(1/2) +$$

| | $\vert 00\rangle$ | $\vert 01\rangle$ | $\vert 02\rangle$ | $\cdots$ | $\vert 10\rangle$ | $\vert 11\rangle$ | $\vert 12\rangle$ | $\cdots$ | $\vert 20\rangle$ | $\vert 21\rangle$ | $\vert 22\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\vert$ | 0 | | | $\cdots$ | $\cdot$ | | | $\cdots$ | | | | $\cdots$ |
| $\langle 01\vert$ | | $D$ | | $\cdots$ | $B+iC$ | $\cdot$ | | $\cdots$ | | | | $\cdots$ |
| $\langle 02\vert$ | | | $2D$ | $\cdots$ | | $\sqrt{2}(B+iC)$ | $\cdot$ | $\cdots$ | | | | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | | | | $\cdots$ |
| $\langle 10\vert$ | $\cdot$ | $B-iC$ | | $\cdots$ | $A$ | | | $\cdots$ | $\cdot$ | | | $\cdots$ |
| $\langle 11\vert$ | | $\cdot$ | $\sqrt{2}(B-iC)$ | $\cdots$ | | $A+D$ | | $\cdots$ | $\sqrt{2}(B+iC)$ | $\cdot$ | | $\cdots$ |
| $\langle 12\vert$ | | Example: | $\cdot$ | $\cdots$ | | | $A+2D$ | $\cdots$ | | $\sqrt{4}(B+iC)$ | $\cdot$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |
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| $\langle 22\vert$ | | | | | | | $\cdot$ | $\cdots$ | | | $2A+2D$ | $\cdots$ |
| $\vdots$ | | | | | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

$$\mathbf{a}_1^\dagger\mathbf{a}_1\vert n_1 n_2\rangle = n_1\vert n_1 n_2\rangle$$

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Rearrangement of rows and columns brings the matrix to a block-diagonal form.

Base states $\vert n_1\rangle\vert n_2\rangle$ with the same *total quantum number* $\upsilon = n_1 + n_2$ define each block.

Group reorganized
"Little-Endian" indexing
(...01,02,03..10,11,12,13 ...
20,21,22,23,...)

$$\langle\mathbf{H}\rangle = A(1/2) + D(1/2) +$$

| | $\vert 00\rangle$ | $\vert 01\rangle$ | $\vert 10\rangle$ | $\vert 02\rangle$ | $\vert 11\rangle$ | $\vert 20\rangle$ | $\vert 03\rangle$ | $\vert 12\rangle$ | $\vert 21\rangle$ | $\vert 30\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\vert$ | 0 | *Vacuum* (ν=0) | | | | | | | | | |
| $\langle 01\vert$ | | $D$ | $B+iC$ | *Fundamental* (ν=1) | | | | | | | |
| $\langle 10\vert$ | | $B-iC$ | $A$ | vibrational sub-space | | | | | | | |
| $\langle 02\vert$ | | | | $2D$ | $\sqrt{2}(B+iC)$ | | | | | | |
| $\langle 11\vert$ | | | | $\sqrt{2}(B-iC)$ | $A+D$ | $\sqrt{2}(B+iC)$ | *Overtone* (ν=2) | | | | |
| $\langle 20\vert$ | | | | | $\sqrt{2}(B-iC)$ | $2A$ | vibrational sub-space | | | | |
| $\langle 03\vert$ | | | | | | | $3D$ | $\sqrt{3}(B+iC)$ | | | |
| $\langle 12\vert$ | | | | | | | $\sqrt{3}(B-iC)$ | $A+2D$ | $\sqrt{4}(B+iC)$ | | *Overtone* (ν=3) |
| $\langle 21\vert$ | | | | | | | | $\sqrt{4}(B-iC)$ | $2A+D$ | $\sqrt{3}(B+iC)$ | vibrational sub-space |
| $\langle 30\vert$ | | | | | | | | | $\sqrt{3}(B-iC)$ | $3A$ | |
| $\vdots$ | | | | | | | | | | | |

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + 1/2\right) + D\left(\mathbf{a}_2^\dagger\mathbf{a}_2 + 1/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$

Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

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*SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

# *2D-Oscillator states* *and related 3D angular momentum multiplets*

## *Fundamental eigenstates*

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing

(...*00*,*10*,*20*..*01*,*11*,*21*,*31* ...*02*,*12*,*22*,*32*...)

# *2D-Oscillator states* *and related 3D angular momentum multiplets*

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| $\langle 0,1\vert$ | $B+iC$ | $D$ |

$+\frac{A+D}{2}\mathbf{1}$

Group reorganized "Big-Endian" indexing

(...*00*,*10*,*20*..*01*,*11*,*21*,*31* ...*02*,*12*,*22*,*32*...)

Recall decomposition of **H** ( Lectures 6-10 )

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

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$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}+\frac{A+D}{2}\mathbf{1}=(A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2}+2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2}+(A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H}=\Omega_0\mathbf{1}+\mathbf{\Omega}\bullet\vec{\mathbf{S}}=\Omega_0\mathbf{1}+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C+\Omega_A\mathbf{S}_A \quad \text{(ABC Optical vector notation)}$$

$$=\Omega_0\mathbf{1}+\Omega_X\mathbf{S}_X+\Omega_Y\mathbf{S}_Y+\Omega_Z\mathbf{S}_Z \quad \text{(XYZ Electron spin notation)}$$

# *2D-Oscillator states* *and related 3D angular momentum multiplets*

*Fundamental eigenstates*

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1, n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

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in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0 \mathbf{1} + \mathbf{\Omega} \bullet \vec{\mathbf{S}} = \Omega_0 \mathbf{1} + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C + \Omega_A \mathbf{S}_A \quad \text{(ABC Optical vector notation)}$$

$$= \Omega_0 \mathbf{1} + \Omega_X \mathbf{S}_X + \Omega_Y \mathbf{S}_Y + \Omega_Z \mathbf{S}_Z \quad \text{(XYZ Electron spin notation)}$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}\text{-}\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$ :

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

# *2D-Oscillator states and related 3D angular momentum multiplets*

*Fundamental eigenstates*

The first step is to diagonalize the fundamental 2-by-2 matrix .

$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} =$

| $n_1, n_2$ | $\|1,0\rangle$ | $\|0,1\rangle$ |
| --- | --- | --- |
| $\langle 1,0\|$ | $A$ | $B-iC$ |
| $\langle 0,1\|$ | $B+iC$ | $D$ |

$+\frac{A+D}{2}\mathbf{1}$

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$$\mathbf{H} = \Omega_0\mathbf{1} + \boldsymbol{\Omega}\bullet\vec{\mathbf{S}} = \Omega_0\mathbf{1} + \Omega_B\mathbf{S}_B + \Omega_C\mathbf{S}_C + \Omega_A\mathbf{S}_A \quad \textit{(ABC Optical vector notation)}$$

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Polar angles $(\varphi,\vartheta)$ of $+\boldsymbol{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\boldsymbol{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where: } \begin{cases} \cos\vartheta = \frac{A\text{-}D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

# 2D-Oscillator states *and related 3D angular momentum multiplets*

*Fundamental eigenstates*

The first step is to diagonalize the fundamental 2-by-2 matrix .

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$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad \textit{(XYZ Electron spin notation)}$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}\text{-}\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$ :

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

Polar angles $(\varphi,\vartheta)$ of $+\boldsymbol{\Omega}$-vector (or polar angles $(\varphi,\vartheta\pm\pi)$ of $-\boldsymbol{\Omega}$-vector) gives **H** eigenvectors.

$$|\omega_+\rangle = \begin{pmatrix} e^{-i\varphi/2}\cos\frac{\vartheta}{2} \\ e^{i\varphi/2}\sin\frac{\vartheta}{2} \end{pmatrix}, \quad |\omega_-\rangle = \begin{pmatrix} -e^{-i\varphi/2}\sin\frac{\vartheta}{2} \\ e^{i\varphi/2}\cos\frac{\vartheta}{2} \end{pmatrix} \quad \text{where:} \begin{cases} \cos\vartheta = \frac{A\text{-}D}{\Omega} \\ \tan\varphi = \frac{C}{B} \end{cases}$$

More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

# 2D-Oscillator states *and related 3D angular momentum multiplets*

## *Fundamental eigenstates*

The first step is to diagonalize the fundamental 2-by-2 matrix .

$$\langle \mathbf{H} \rangle_{\upsilon=1}^{Fundamental} = \begin{array}{c|cc} n_1,n_2 & |1,0\rangle & |0,1\rangle \\ \hline \langle 1,0| & A & B-iC \\ \langle 0,1| & B+iC & D \end{array} + \frac{A+D}{2}\mathbf{1}$$

Group reorganized "Big-Endian" indexing
(...00,10,20..01,11,21,31 ...02,12,22,32...)

Recall decomposition of **H** ( Lectures 6-10 )

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} + \frac{A+D}{2}\mathbf{1} = (A+D)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{2} + 2C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{2} + (A-D)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}$$

in terms of Jordan-Pauli spin operators.

$$\mathbf{H} = \Omega_0\mathbf{1} + \boldsymbol{\Omega}\bullet\vec{\mathbf{S}} = \Omega_0\mathbf{1} + \Omega_B\mathbf{S}_B + \Omega_C\mathbf{S}_C + \Omega_A\mathbf{S}_A \quad (ABC \textit{ Optical vector notation})$$

$$= \Omega_0\mathbf{1} + \Omega_X\mathbf{S}_X + \Omega_Y\mathbf{S}_Y + \Omega_Z\mathbf{S}_Z \quad (XYZ \textit{ Electron spin notation})$$

Frequency eigenvalues $\omega_\pm$ of $\mathbf{H}$-$\Omega_0\mathbf{1}/2$ and *fundamental transition frequency* $\Omega = \omega_+ - \omega_-$ :

$$\omega_\pm = \frac{\Omega_0 \pm \Omega}{2} = \frac{A+D \pm \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}}{2} = \frac{A+D}{2} \pm \sqrt{\left(\frac{A-D}{2}\right)^2 + B^2 + C^2}$$

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More important for the general solution, are the *eigen-creation operators* $\mathbf{a}^\dagger_+$ *and* $\mathbf{a}^\dagger_-$ defined by

$$\mathbf{a}_+^\dagger = e^{-i\varphi/2}\left(\cos\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\sin\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right), \quad \mathbf{a}_-^\dagger = e^{-i\varphi/2}\left(-\sin\frac{\vartheta}{2}\mathbf{a}_1^\dagger + e^{i\varphi}\cos\frac{\vartheta}{2}\mathbf{a}_2^\dagger\right)$$

$\mathbf{a}_\pm^\dagger$ create **H** eigenstates directly from the ground state.

$$\mathbf{a}_+^\dagger|0\rangle = |\omega_+\rangle, \quad \mathbf{a}_-^\dagger|0\rangle = |\omega_-\rangle$$

# *2D-Oscillator states* *and related 3D angular momentum multiplets*

Setting *(B=0=C)* and *(A=ω₊ )* and *(D=ω₋ )* gives diagonal block matrices.

Group reorganized "Little-Endian" indexing (*...01,02,03..10,11,12,13 ... 20,21,22,23,...*)

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

| | $\vert 00\rangle$ | $\vert 01\rangle$ | $\vert 10\rangle$ | $\vert 02\rangle$ | $\vert 11\rangle$ | $\vert 20\rangle$ | $\vert 03\rangle$ | $\vert 12\rangle$ | $\vert 21\rangle$ | $\vert 30\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\vert$ | 0 | | | | | | | | | | |
| $\langle 01\vert$ | | $\omega_-$ | | | | | | | | | |
| $\langle 10\vert$ | | | $\omega_+$ | | | | | | | | |
| $\langle 02\vert$ | | | | $2\omega_-$ | | | | | | | |
| $\langle 11\vert$ | | | | | $\omega_+ + \omega_-$ | | | | | | |
| $\langle 20\vert$ | | | | | | $2\omega_+$ | | | | | |
| $\langle 03\vert$ | | | | | | | $3\omega_-$ | | | | |
| $\langle 12\vert$ | | | | | | | | $\omega_+ + 2\omega_-$ | | | |
| $\langle 21\vert$ | | | | | | | | | $2\omega_+ + \omega_-$ | | |
| $\langle 30\vert$ | | | | | | | | | | $3\omega_+$ | |
| $\vdots$ | | | | | | | | | | | |

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

# 2D-Oscillator states and related 3D angular momentum multiplets

Group reorganized "Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

Setting (*B*=0=*C*) and (*A*=ω₊ ) and (*D*=ω₋ ) gives diagonal block matrices.

$\langle \mathbf{H}\rangle = A(\mathbf{1}/2)+D(\mathbf{1}/2)+$

| | $\|00\rangle$ | $\|01\rangle$ | $\|10\rangle$ | $\|02\rangle$ | $\|11\rangle$ | $\|20\rangle$ | $\|03\rangle$ | $\|12\rangle$ | $\|21\rangle$ | $\|30\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\|$ | 0 | | | | | | | | | | |
| $\langle 01\|$ | | $\omega_-$ | | | | | | | | | |
| $\langle 10\|$ | | | $\omega_+$ | | | | | | | | |
| $\langle 02\|$ | | | | $2\omega_-$ | | | | | | | |
| $\langle 11\|$ | | | | | $\omega_+ + \omega_-$ | | | | | | |
| $\langle 20\|$ | | | | | | $2\omega_+$ | | | | | |
| $\langle 03\|$ | | | | | | | $3\omega_-$ | | | | |
| $\langle 12\|$ | | | | | | | | $\omega_+ + 2\omega_-$ | | | |
| $\langle 21\|$ | | | | | | | | | $2\omega_+ + \omega_-$ | | |
| $\langle 30\|$ | | | | | | | | | | $3\omega_+$ | |
| $\vdots$ | | | | | | | | | | | |

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}\left(n_1 + n_2 + 1\right) + \frac{A-D}{2}\left(n_1 - n_2\right)$$

# 2D-Oscillator states and related 3D angular momentum multiplets

Setting ($B$=0=$C$) and ($A$=$\omega_+$) and ($D$=$\omega_-$) gives diagonal block matrices.

Group reorganized "Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

| | $\vert 00\rangle$ | $\vert 01\rangle$ | $\vert 10\rangle$ | $\vert 02\rangle$ | $\vert 11\rangle$ | $\vert 20\rangle$ | $\vert 03\rangle$ | $\vert 12\rangle$ | $\vert 21\rangle$ | $\vert 30\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\vert$ | 0 | | | | | | | | | | |
| $\langle 01\vert$ | | $\omega_-$ | | | | | | | | | |
| $\langle 10\vert$ | | | $\omega_+$ | | | | | | | | |
| $\langle 02\vert$ | | | | $2\omega_-$ | | | | | | | |
| $\langle 11\vert$ | | | | | $\omega_+ + \omega_-$ | | | | | | |
| $\langle 20\vert$ | | | | | | $2\omega_+$ | | | | | |
| $\langle 03\vert$ | | | | | | | $3\omega_-$ | | | | |
| $\langle 12\vert$ | | | | | | | | $\omega_+ + 2\omega_-$ | | | |
| $\langle 21\vert$ | | | | | | | | | $2\omega_+ + \omega_-$ | | |
| $\langle 30\vert$ | | | | | | | | | | $3\omega_+$ | |
| $\vdots$ | | | | | | | | | | | |

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
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$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

$$\varepsilon^A_{n_1 n_2} = A\left(n_1 + \frac{1}{2}\right) + D\left(n_2 + \frac{1}{2}\right) = \frac{A+D}{2}(n_1 + n_2 + 1) + \frac{A-D}{2}(n_1 - n_2)$$
$$= \Omega_0 (n_1 + n_2 + 1) + \frac{\Omega}{2}(n_1 - n_2) = \Omega_0 (\upsilon + 1) + \Omega\, m$$

# 2D-Oscillator states and related 3D angular momentum multiplets

Setting ($B=0=C$) and ($A=\omega_+$) and ($D=\omega_-$) gives diagonal block matrices.

Group reorganized
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(...01,02,03..10,11,12,13 ...
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$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

| | $\lvert 00\rangle$ | $\lvert 01\rangle$ | $\lvert 10\rangle$ | $\lvert 02\rangle$ | $\lvert 11\rangle$ | $\lvert 20\rangle$ | $\lvert 03\rangle$ | $\lvert 12\rangle$ | $\lvert 21\rangle$ | $\lvert 30\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\rvert$ | $0$ | | | | | | | | | | |
| $\langle 01\rvert$ | | $\omega_-$ | | | | | | | | | |
| $\langle 10\rvert$ | | | $\omega_+$ | | | | | | | | |
| $\langle 02\rvert$ | | | | $2\omega_-$ | | | | | | | |
| $\langle 11\rvert$ | | | | | $\omega_+ + \omega_-$ | | | | | | |
| $\langle 20\rvert$ | | | | | | $2\omega_+$ | | | | | |
| $\langle 03\rvert$ | | | | | | | $3\omega_-$ | | | | |
| $\langle 12\rvert$ | | | | | | | | $\omega_+ + 2\omega_-$ | | | |
| $\langle 21\rvert$ | | | | | | | | | $2\omega_+ + \omega_-$ | | |
| $\langle 30\rvert$ | | | | | | | | | | $3\omega_+$ | |
| $\vdots$ | | | | | | | | | | | |

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Define *total quantum number* $\upsilon=2j$ and half-difference or *asymmetry quantum number m*

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1 + n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

# 2D-Oscillator states and related 3D angular momentum multiplets

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

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| | $\lvert 00\rangle$ | $\lvert 01\rangle$ | $\lvert 10\rangle$ | $\lvert 02\rangle$ | $\lvert 11\rangle$ | $\lvert 20\rangle$ | $\lvert 03\rangle$ | $\lvert 12\rangle$ | $\lvert 21\rangle$ | $\lvert 30\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\rvert$ | 0 | | | | | | | | | | |
| $\langle 01\rvert$ | | $\omega_-$ | | | | | | | | | |
| $\langle 10\rvert$ | | | $\omega_+$ | | | | | | | | |
| $\langle 02\rvert$ | | | | $2\omega_-$ | | | | | | | |
| $\langle 11\rvert$ | | | | | $\omega_+ + \omega_-$ | | | | | | |
| $\langle 20\rvert$ | | | | | | $2\omega_+$ | | | | | |
| $\langle 03\rvert$ | | | | | | | $3\omega_-$ | | | | |
| $\langle 12\rvert$ | | | | | | | | $\omega_+ + 2\omega_-$ | | | |
| $\langle 21\rvert$ | | | | | | | | | $2\omega_+ + \omega_-$ | | |
| $\langle 30\rvert$ | | | | | | | | | | $3\omega_+$ | |
| $\vdots$ | | | | | | | | | | | |

$$\omega_+ - \omega_- = \Omega$$
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$$\mathbf{H}^A = A\left(\mathbf{a}_1^\dagger \mathbf{a}_1 + \mathbf{1}/2\right) + D\left(\mathbf{a}_2^\dagger \mathbf{a}_2 + \mathbf{1}/2\right)$$

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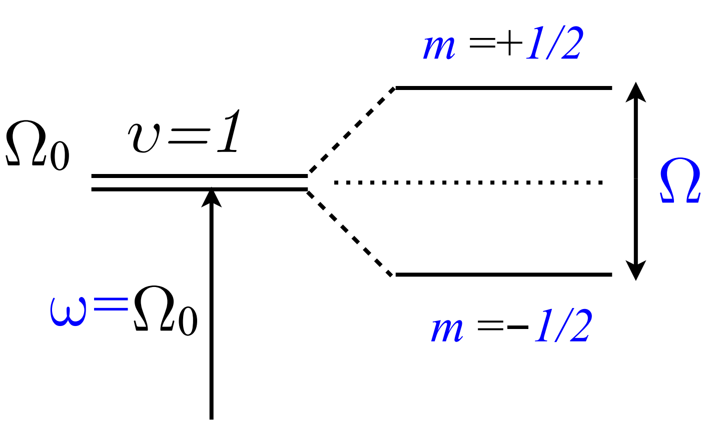
$$= \Omega_0 (n_1+n_2+1) + \frac{\Omega}{2}(n_1-n_2) = \Omega_0(\upsilon+1) + \Omega\, m$$

Define *total quantum number* $\upsilon=2j$ and half-difference or *asymmetry quantum number* $m$

$$\upsilon = n_1 + n_2 = 2j \qquad j = \frac{n_1+n_2}{2} = \frac{\upsilon}{2} \qquad m = \frac{n_1 - n_2}{2}$$

$\upsilon+1=2j+1$ multiplies *base frequency* $\omega=\Omega_0$

$m$ multiplies *beat frequency* $\Omega$




$$\omega_+ = \Omega_0 + \Omega\left(+\tfrac{1}{2}\right)$$
$$\omega_- = \Omega_0 + \Omega\left(-\tfrac{1}{2}\right)$$

Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

*2-D* **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
- *2D-Oscillator basic states and operations*
  - *Commutation relations*
  - *Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
    - *Anti-commutation relations*
  - *Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
    - *Outer product arrays*
    - *Entangled 2-particle states*
- *Two-particle (or 2-dimensional) matrix operators*
  - *U(2) Hamiltonian and irreducible representations*
  - ➡ *2D-Oscillator states and related 3D angular momentum multiplets* ⬅
    - *ND multiplets*

*R(3) Angular momentum generators by U(2) analysis*
*Angular momentum raise-n-lower operators* $\mathbf{s}_+$ and $\mathbf{s}_-$
*SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

# 2D-Oscillator states and related 3D angular momentum multiplets

Setting $(B=0=C)$ and $(A=\omega_+)$ and $(D=\omega_-)$ gives diagonal block matrices.

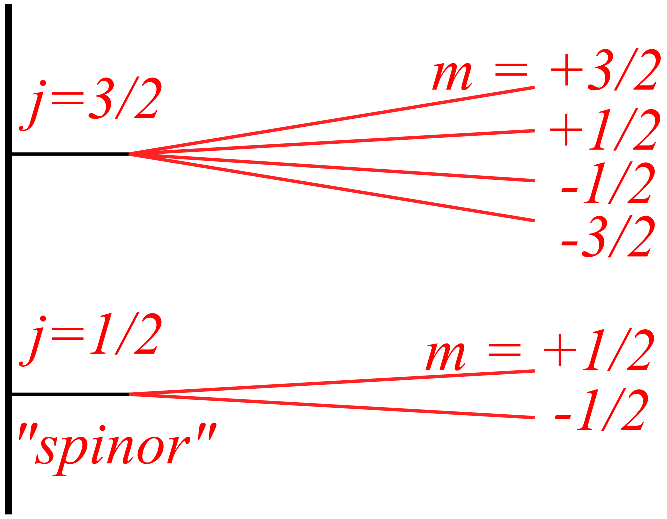
Group reorganized "Little-Endian" indexing (...01,02,03..10,11,12,13 ... 20,21,22,23,...)

$\langle \mathbf{H} \rangle = A(1/2) + D(1/2) +$

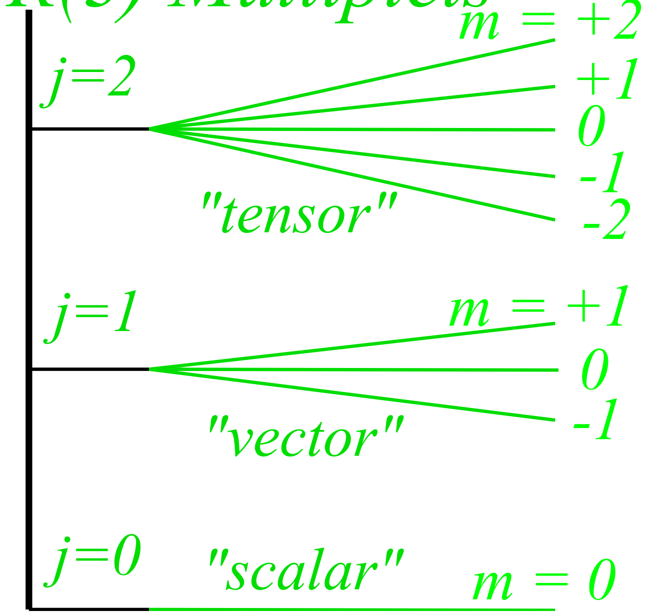
| | $\|00\rangle$ | $\|01\rangle$ | $\|10\rangle$ | $\|02\rangle$ | $\|11\rangle$ | $\|20\rangle$ | $\|03\rangle$ | $\|12\rangle$ | $\|21\rangle$ | $\|30\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\|$ | 0 | | | | | | | | | | |
| $\langle 01\|$ | | $\omega_-$ | | | | | | | | | |
| $\langle 10\|$ | | | $\omega_+$ | | | | | | | | |
| $\langle 02\|$ | | | | $2\omega_-$ | | | | | | | |
| $\langle 11\|$ | | | | | $\omega_+ + \omega_-$ | | | | | | |
| $\langle 20\|$ | | | | | | $2\omega_+$ | | | | | |
| $\langle 03\|$ | | | | | | | $3\omega_-$ | | | | |
| $\langle 12\|$ | | | | | | | | $\omega_+ + 2\omega_-$ | | | |
| $\langle 21\|$ | | | | | | | | | $2\omega_+ + \omega_-$ | | |
| $\langle 30\|$ | | | | | | | | | | $3\omega_+$ | |
| $\vdots$ | | | | | | | | | | | |

$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

## SU(2) Multiplets




## R(3) Multiplets



# 2D-Oscillator states and related 3D angular momentum multiplets

Setting ($B$=0=$C$) and ($A$=$\omega_+$) and ($D$=$\omega_-$) gives diagonal block matrices.

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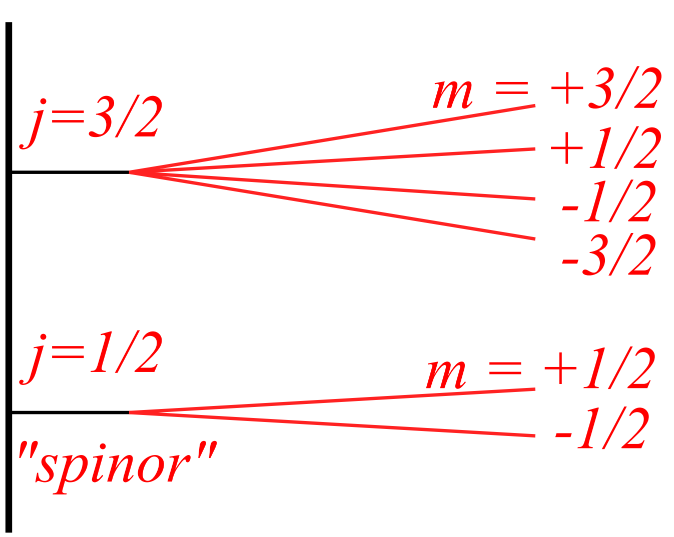
$\langle \mathbf{H} \rangle = A(\mathbf{1}/2) + D(\mathbf{1}/2) +$

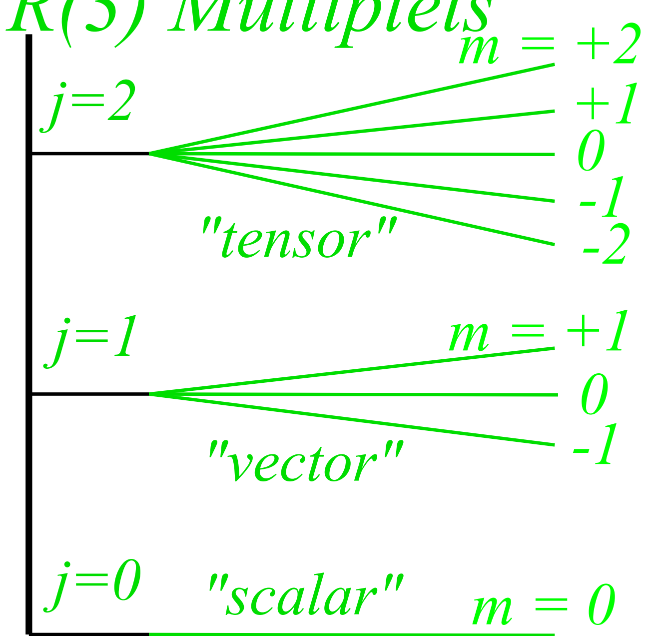
| | $\|00\rangle$ | $\|01\rangle$ | $\|10\rangle$ | $\|02\rangle$ | $\|11\rangle$ | $\|20\rangle$ | $\|03\rangle$ | $\|12\rangle$ | $\|21\rangle$ | $\|30\rangle$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\langle 00\|$ | 0 | | | | | | | | | | |
| $\langle 01\|$ | | $\omega_-$ | | | | | | | | | |
| $\langle 10\|$ | | | $\omega_+$ | | | | | | | | |
| $\langle 02\|$ | | | | $2\omega_-$ | | | | | | | |
| $\langle 11\|$ | | | | | $\omega_+ + \omega_-$ | | | | | | |
| $\langle 20\|$ | | | | | | $2\omega_+$ | | | | | |
| $\langle 03\|$ | | | | | | | $3\omega_-$ | | | | |
| $\langle 12\|$ | | | | | | | | $\omega_+ + 2\omega_-$ | | | |
| $\langle 21\|$ | | | | | | | | | $2\omega_+ + \omega_-$ | | |
| $\langle 30\|$ | | | | | | | | | | $3\omega_+$ | |
| $\vdots$ | | | | | | | | | | | |

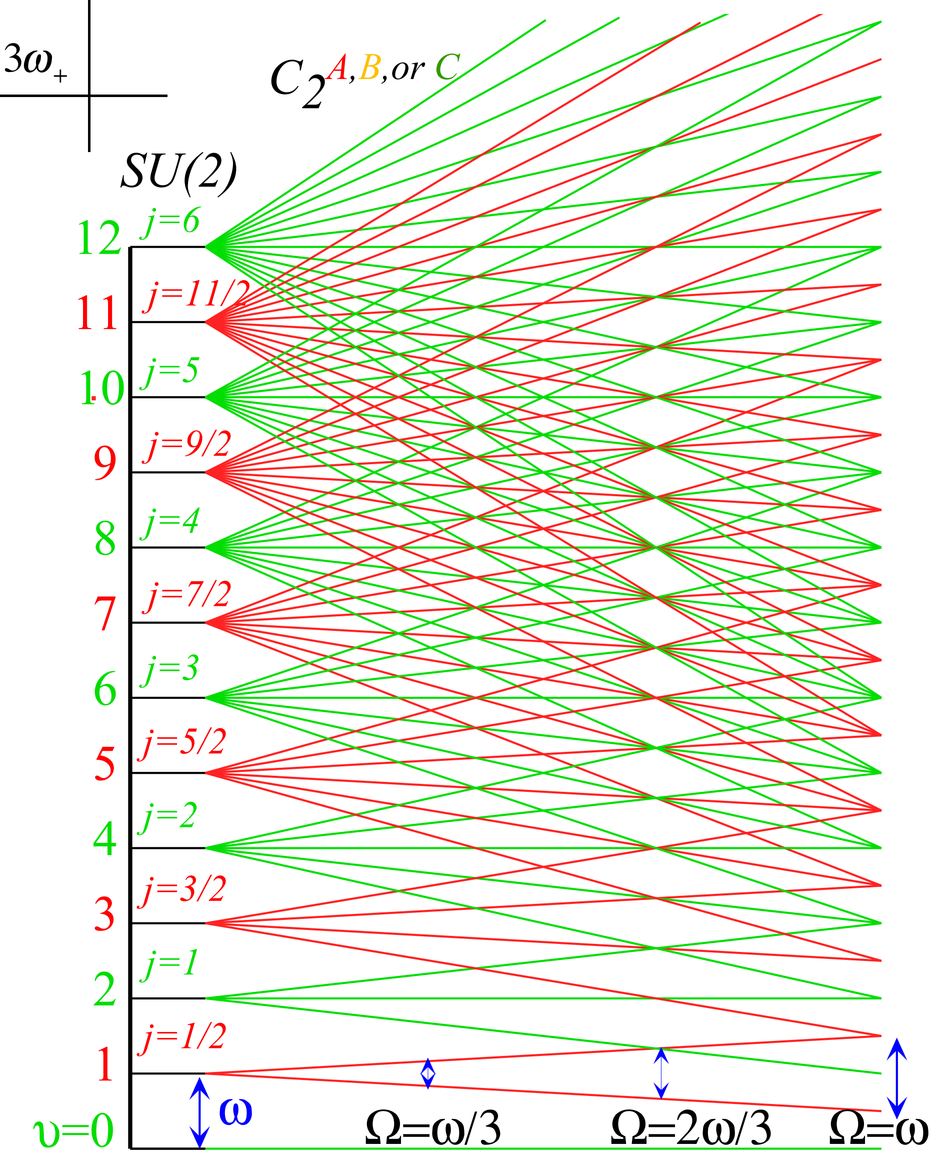
$$\omega_+ - \omega_- = \Omega$$
$$= \sqrt{(2B)^2 + (2C)^2 + (A-D)^2}$$
$$= A - D$$

## SU(2) Multiplets




## R(3) Multiplets



# 2D-Oscillator states and related 3D angular momentum multiplets

## Structure of U(2)

$$\left|{j \atop m}\right\rangle = \left|n_1 n_2\right\rangle$$

$j=0$: $\left|{0 \atop 0}\right\rangle = |00\rangle$ "scalar"

$j=\frac{1}{2}$: $\left|{1/2 \atop 1/2}\right\rangle = |10\rangle = |\uparrow\rangle$, $\left|{1/2 \atop -1/2}\right\rangle = |01\rangle = |\downarrow\rangle$ "spinor"

$j=1$: $\left|{1 \atop 1}\right\rangle = |20\rangle$, $\left|{1 \atop 0}\right\rangle = |11\rangle$, $\left|{1 \atop -1}\right\rangle = |02\rangle$ "3-vector"

$j=\frac{3}{2}$: $\left|{3/2 \atop 1/2}\right\rangle = |30\rangle$, $\left|{3/2 \atop 1/2}\right\rangle = |21\rangle$, $\left|{3/2 \atop -1/2}\right\rangle = |12\rangle$, $\left|{3/2 \atop -3/2}\right\rangle = |03\rangle$ "4-spinor"

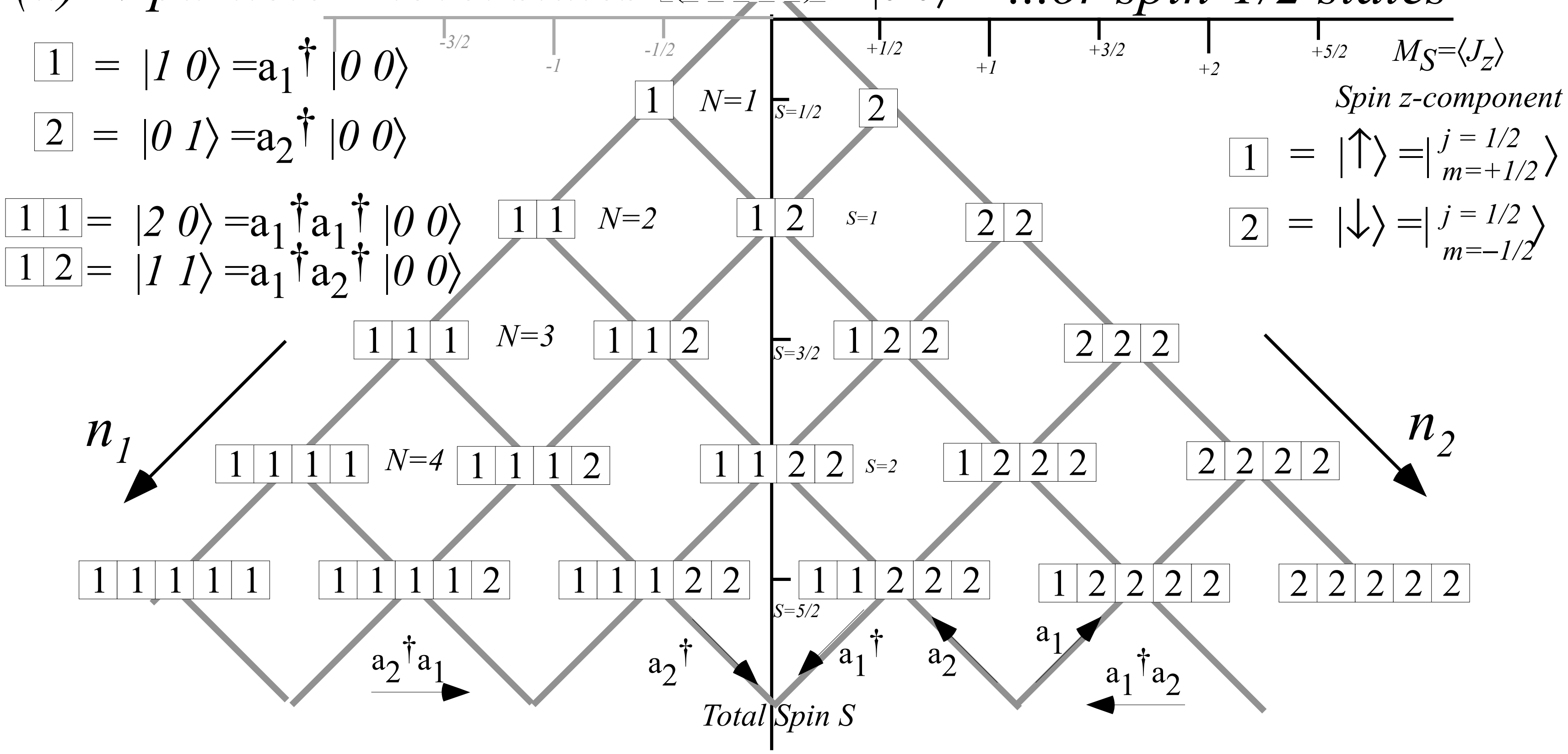
$j=2$: $\left|{2 \atop 2}\right\rangle = |40\rangle$, $\left|{2 \atop 1}\right\rangle = |31\rangle$, $\left|{2 \atop 0}\right\rangle = |22\rangle$, $\left|{2 \atop -1}\right\rangle = |13\rangle$, $\left|{2 \atop -2}\right\rangle = |04\rangle$ "tensor"

$\vdots$

$$j = \frac{\upsilon}{2} = \frac{n_1 + n_2}{2} \qquad n_1 = j + m = 2\upsilon + m$$

$$m = \frac{n_1 - n_2}{2} \qquad n_2 = j - m = 2\upsilon - m$$

(a) *N-particle 2-level states* (vacuum) $= |0\ 0\rangle$ ...*or spin-1/2 states*



Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

*2-D* **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
*2D-Oscillator basic states and operations*
*Commutation relations*
*Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
*Anti-commutation relations*
*Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
*Outer product arrays*
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*Two-particle (or 2-dimensional) matrix operators*
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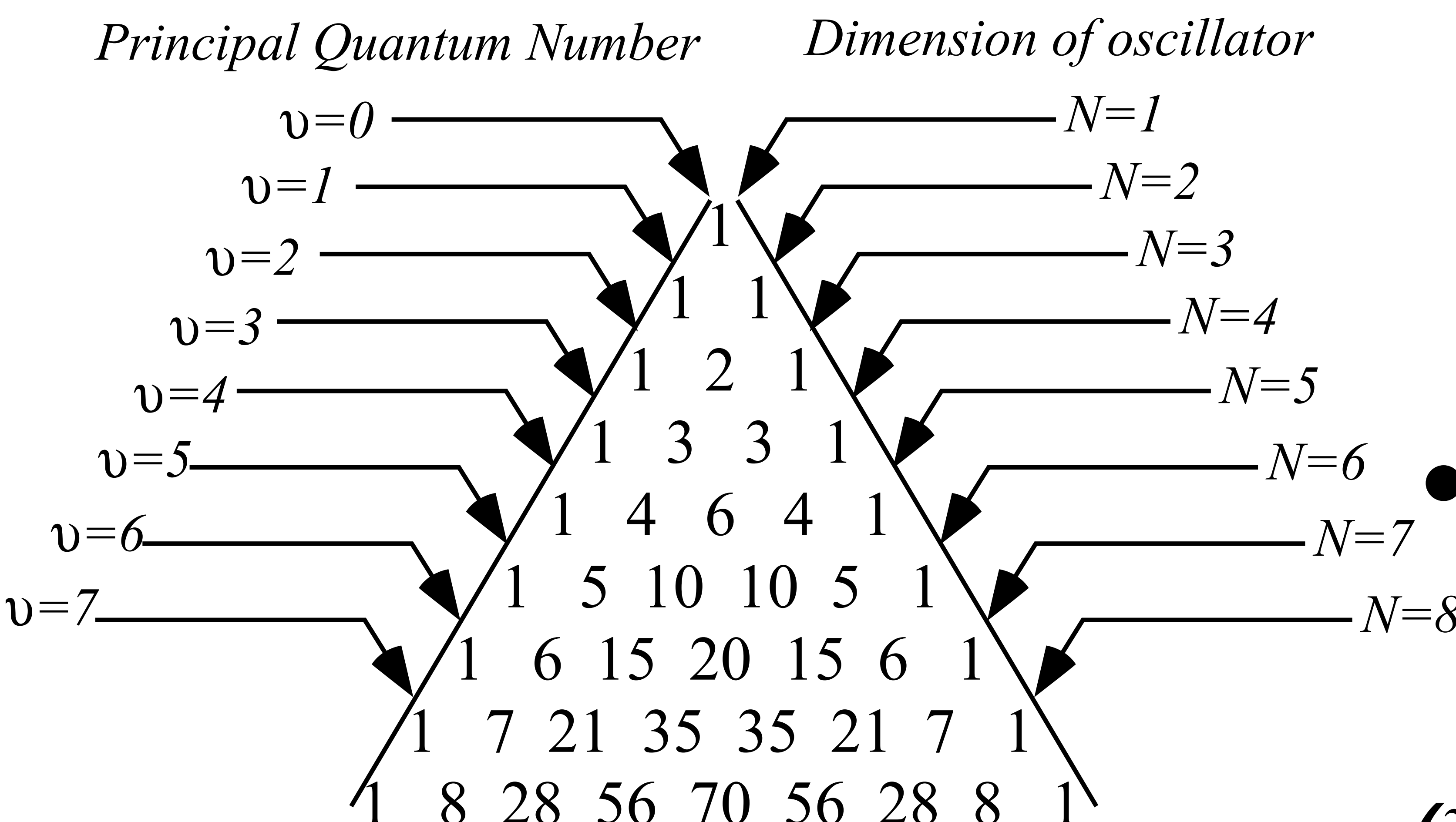
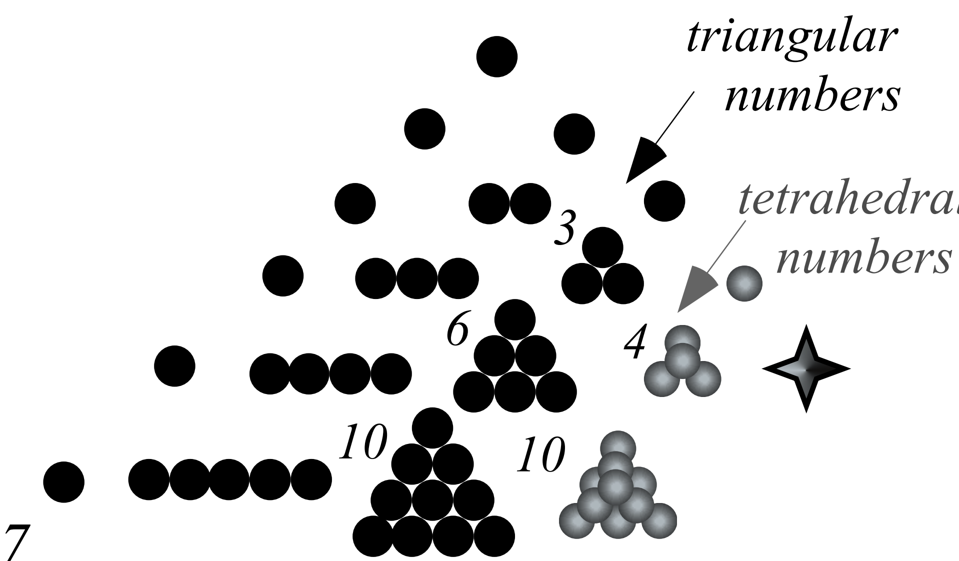
*ND multiplets*
*R(3) Angular momentum generators by U(2) analysis*
*Angular momentum raise-n-lower operators* $\mathbf{s}_+$ and $\mathbf{s}_-$
*SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

# Introducing U(N)

*(a) N-D Oscillator Degeneracy ℓ of quamtum level υ*

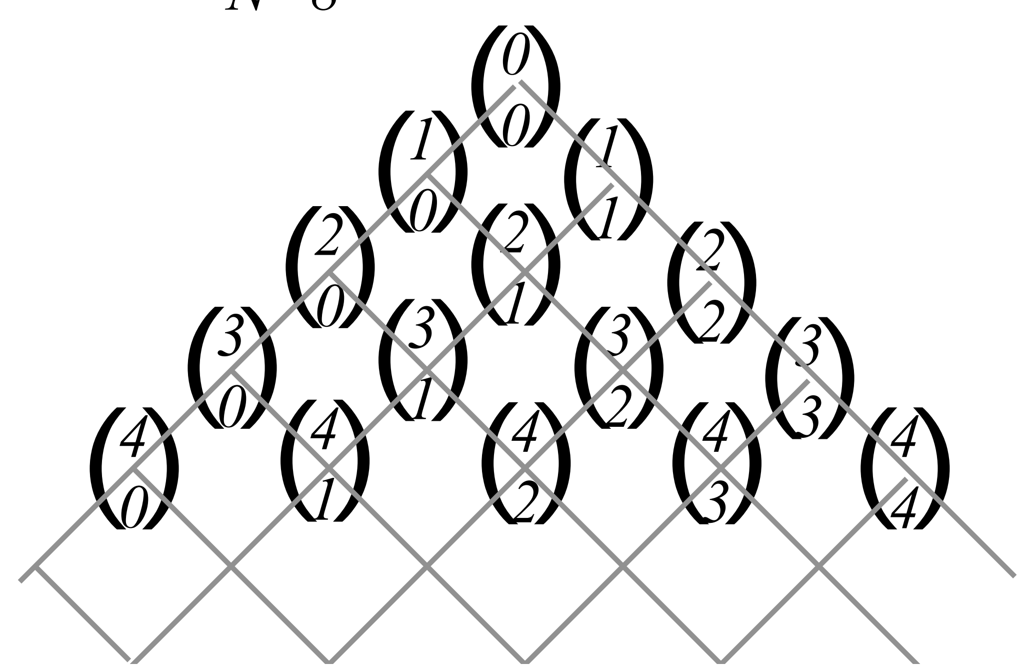



*(b) Stacking numbers*




*(c) Binomial coefficients*

$$\frac{(N\text{-}1+\upsilon)!}{(N\text{-}1)!\,\upsilon!} = \binom{N\text{-}1+\upsilon}{\upsilon} = \binom{N\text{-}1+\upsilon}{N\text{-}1}$$



# Introducing U(3)

## (b) N-particle 3-level states ...or spin-1 states

$\boxed{1} = |1\ 0\ 0\rangle = a_1^\dagger\ |0\ 0\ 0\rangle$

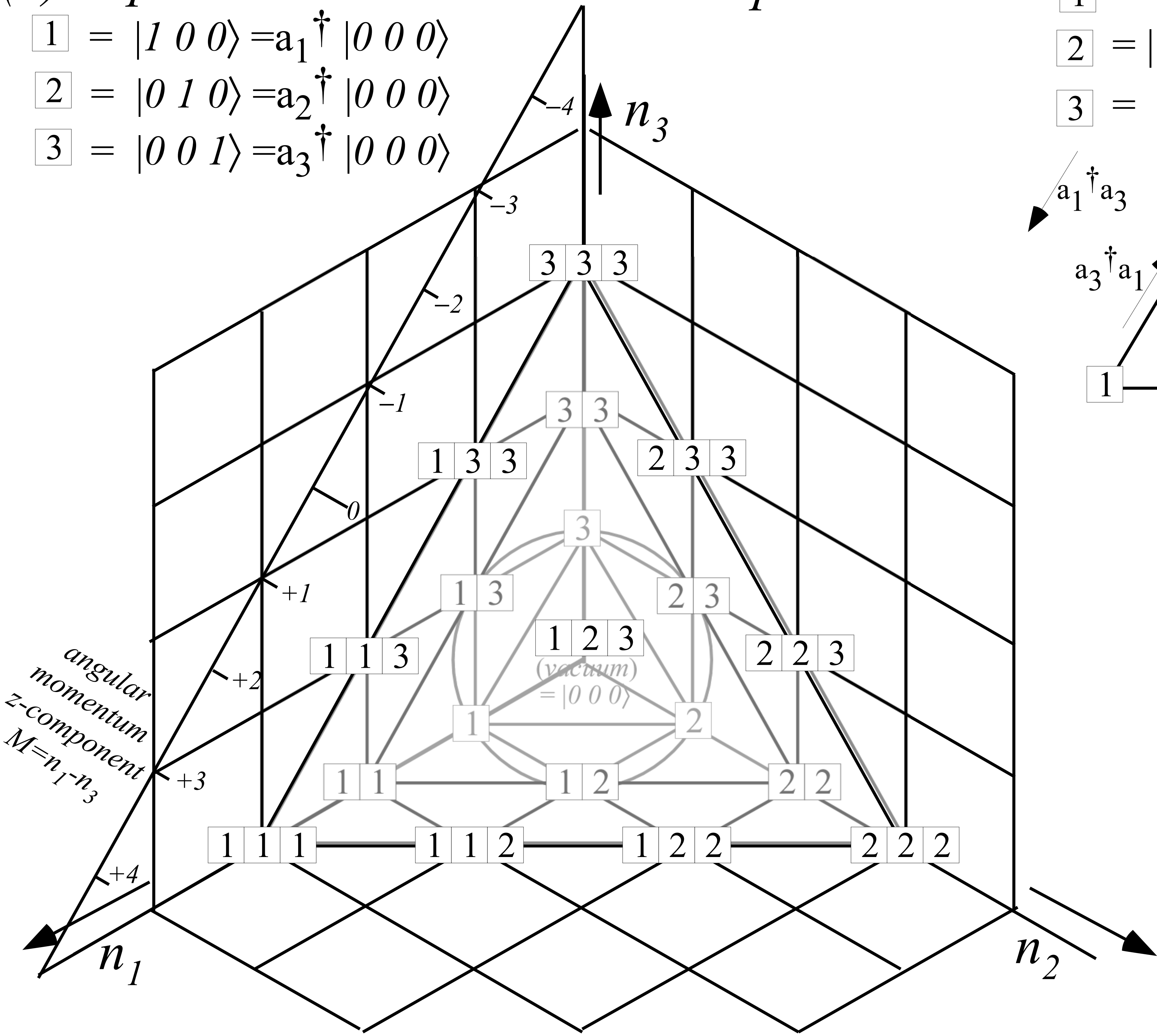
$\boxed{2} = |0\ 1\ 0\rangle = a_2^\dagger\ |0\ 0\ 0\rangle$

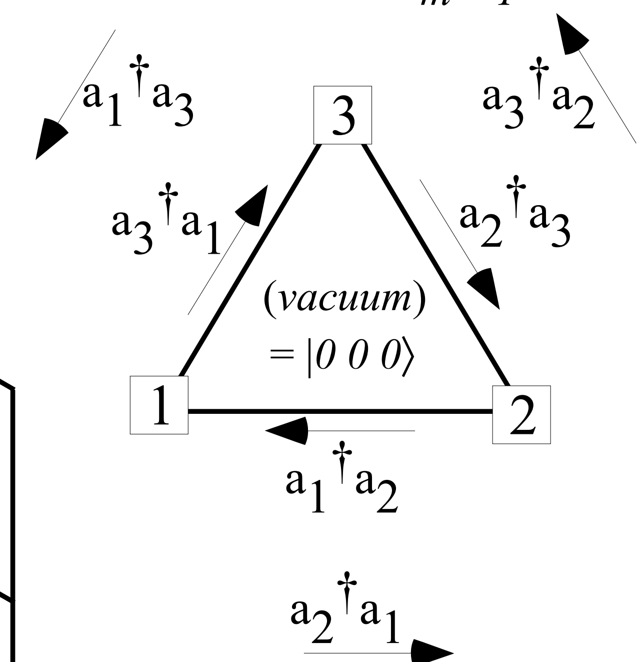
$\boxed{3} = |0\ 0\ 1\rangle = a_3^\dagger\ |0\ 0\ 0\rangle$

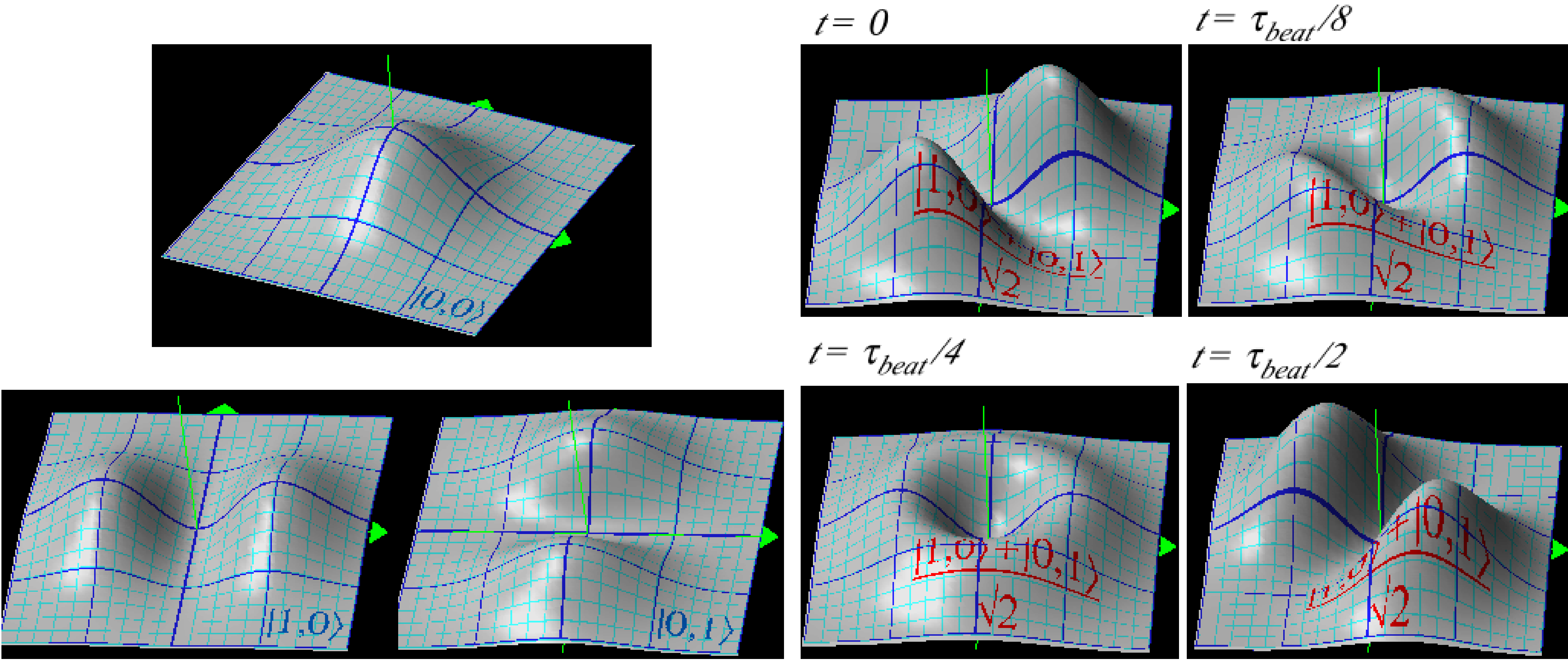
$\boxed{1} = |\uparrow\rangle = \left|{}^{j=1}_{m=+1}\right\rangle$

$\boxed{2} = |\leftrightarrow\rangle = \left|{}^{j=1}_{m=0}\right\rangle$

$\boxed{3} = |\downarrow\rangle = \left|{}^{j=1}_{m=-1}\right\rangle$






$$\Psi(x_1,x_2,t)=\frac{1}{2}\left|\psi_{10}(x_1,x_2)e^{-i\omega_{10}t}+\psi_{01}(x_1,x_2)e^{-i\omega_{01}t}\right|^2 e^{-\left(x_1^2+x_2^2\right)}=\frac{e^{-\left(x_1^2+x_2^2\right)}}{2\pi}\left|\sqrt{2}x_1e^{-i\omega_{10}t}+\sqrt{2}x_1e^{-i\omega_{01}t}\right|^2$$

$$=\frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\left(x_1^2+x_2^2+2x_1x_2\cos(\omega_{10}-\omega_{01})t\right)=\frac{e^{-\left(x_1^2+x_2^2\right)}}{\pi}\begin{cases}\left|x_1+x_2\right|^2 & for\text{: } t=0\\ x_1^2+x_2^2 & for\text{: } t=\tau_{\text{beat}}/4\\ \left|x_1-x_2\right|^2 & for\text{: } t=\tau_{\text{beat}}/2\end{cases}\qquad(21.1.30)$$

Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T***(a) and/or Boost* **B***(b) to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

*2-D* **a†a** *algebra of U(2) representations and R(3) angular momentum operators*

*2D-Oscillator basic states and operations*

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➡ *R(3) Angular momentum generators by U(2) analysis* ⬅

*Angular momentum raise-n-lower operators* $\mathbf{s}_+$ and $\mathbf{s}_-$

*SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

*R(3) Angular momentum generators by U(2) analysis*

Group reorganized "Big-Endian" indexing

(...00,10,20..01,11,21,31 ...02,12,22,32...)

(...00,10, 01, 20,11, 02, 30, 21, 12, 03, 40, 31,22,...)

($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0=\mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + 2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} + (A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

## *R(3) Angular momentum generators by U(2) analysis*

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(...00,10,20..01,11,21,31 ...02,12,22,32...)
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($\upsilon$=*2*) or ($j$=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

# *R(3) Angular momentum generators by U(2) analysis*

Group reorganized "Big-Endian" indexing
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## ($\upsilon$=*1*) or (*j*=*1/2*) block **H** matrices of U(2) oscillator

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## ($\upsilon$=*2*) or (*j*=*1*) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

## ($\upsilon$=*3*) or (*j*=*3/2*) 4-by-4 block uses Dirac spinor irreps.

$$\begin{pmatrix} 3A & \sqrt{3}(B-iC) & & \\ \sqrt{3}(B+iC) & 2A+D & \sqrt{4}(B-iC) & \\ & \sqrt{4}(B+iC) & A+2D & \sqrt{3}(B-iC) \\ & & \sqrt{3}(B+iC) & 3D \end{pmatrix} = \frac{3(A+D)}{2}\begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{3}}{2} & \cdot & \cdot \\ \frac{\sqrt{3}}{2} & \cdot & \frac{\sqrt{4}}{2} & \cdot \\ \cdot & \frac{\sqrt{4}}{2} & \cdot & \frac{\sqrt{3}}{2} \\ \cdot & \cdot & \frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{3}}{2} & \cdot & \cdot \\ i\frac{\sqrt{3}}{2} & \cdot & -i\frac{\sqrt{4}}{2} & \cdot \\ \cdot & i\frac{\sqrt{4}}{2} & \cdot & -i\frac{\sqrt{3}}{2} \\ \cdot & \cdot & i\frac{\sqrt{3}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

# *R(3) Angular momentum generators by U(2) analysis*

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## ($\upsilon=2$) or ($j=1$) 3-by-3 block uses their vector irreps.

$$\begin{pmatrix} 2A & \sqrt{2}(B-iC) & \cdot \\ \sqrt{2}(B+iC) & A+D & \sqrt{2}(B-iC) \\ \cdot & \sqrt{2}(B+iC) & 2D \end{pmatrix} = (A+D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} + 2B\begin{pmatrix} \cdot & \frac{\sqrt{2}}{2} & \cdot \\ \frac{\sqrt{2}}{2} & \cdot & \frac{\sqrt{2}}{2} \\ \cdot & \frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + 2C\begin{pmatrix} \cdot & -i\frac{\sqrt{2}}{2} & \cdot \\ i\frac{\sqrt{2}}{2} & \cdot & -i\frac{\sqrt{2}}{2} \\ \cdot & i\frac{\sqrt{2}}{2} & \cdot \end{pmatrix} + (A-D)\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

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## ($\upsilon=2j$) or ($2j+1$)-by-($2j+1$) block uses $D^{(j)}(\mathbf{s}_\mu)$ irreps of U(2) or R(3).

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

# *R(3) Angular momentum generators by U(2) analysis*

Group reorganized "Big-Endian" indexing
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## ($\upsilon=1$) or ($j=1/2$) block **H** matrices of U(2) oscillator

Use irreps of unit operator $\mathbf{S}_0 = \mathbf{1}$ and spin operators $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{S}_Z\}$. (also known as: $\{\mathbf{S}_B, \mathbf{S}_C, \mathbf{S}_A\}$)

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$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \Omega_X \langle \mathbf{s}_X \rangle^j + \Omega_Y \langle \mathbf{s}_Y \rangle^j + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

All j-block matrix operators factor into *raise-n-lower* operators $\mathbf{s}_\pm = \mathbf{s}_X \pm i\mathbf{s}_Y$ plus the diagonal $\mathbf{s}_Z$

$$\langle \mathbf{H} \rangle^{j-block} = 2j\Omega_0 \langle \mathbf{1} \rangle^j + \left[ (\Omega_X - i\Omega_Y)\langle \mathbf{s}_X + i\mathbf{s}_Y \rangle^j + (\Omega_X + i\Omega_Y)\langle \mathbf{s}_X - i\mathbf{s}_Y \rangle^j \right]/2 + \Omega_Z \langle \mathbf{s}_Z \rangle^j$$

Review : *1-D* **a†a** *algebra of U(1) representations*
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*Angular momentum raise-n-lower operators* $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = \left(\mathbf{a}_1^\dagger \mathbf{a}_2\right)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

*Angular momentum raise-n-lower operators* $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\boxed{\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y} \quad \text{and} \quad \boxed{\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger}$$

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Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{\left(\frac{1}{2}\right)} = D^{\left(\frac{1}{2}\right)}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

## *Angular momentum raise-n-lower operators* $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

Starting with *j=1/2* we see that $\mathbf{S}_+$ is an *elementary projection operator* $\mathbf{e}_{12} = |1\rangle\langle 2| = \mathbf{P}_{12}$

$$\langle \mathbf{s}_+ \rangle^{\frac{1}{2}} = D^{\frac{1}{2}}(\mathbf{s}_+) = D^{\frac{1}{2}}(\mathbf{s}_X + i\mathbf{s}_Y) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{P}_{12}$$

Such operators can be upgraded to *creation-destruction operator* combinations $\mathbf{a}^\dagger\mathbf{a}$

$$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow \quad , \qquad \mathbf{s}_- = (\mathbf{a}_1^\dagger \mathbf{a}_2)^\dagger = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$$

Hamilton-Pauli-Jordan representation of $\mathbf{s}_Z$ is: $\langle \mathbf{s}_Z \rangle^{(\frac{1}{2})} = D^{(\frac{1}{2})}(\mathbf{s}_Z) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

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$$\mathbf{s}_Z = \tfrac{1}{2}(\mathbf{a}_1^\dagger \mathbf{a}_1 - \mathbf{a}_2^\dagger \mathbf{a}_2) = \tfrac{1}{2}(\mathbf{a}_\uparrow^\dagger \mathbf{a}_\uparrow - \mathbf{a}_\downarrow^\dagger \mathbf{a}_\downarrow)$$

*Angular momentum raise-n-lower operators* **S**$_+$ and **S**$_-$

$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y$ and $\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$

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## *Angular momentum raise-n-lower operators* $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\boxed{\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y} \quad \text{and} \quad \boxed{\mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger}$$

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Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin $\downarrow$

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger|0\rangle = \mathbf{a}_\downarrow^\dagger|0\rangle$$

*Angular momentum raise-n-lower operators* $\mathbf{S}_+$ and $\mathbf{S}_-$

$$\mathbf{s}_+ = \mathbf{s}_X + i\mathbf{s}_Y \quad \text{and} \quad \mathbf{s}_- = \mathbf{s}_X - i\mathbf{s}_Y = \mathbf{s}_+^\dagger$$

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$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin ↓

creates up-spin ↑

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

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$\mathbf{s}_+ = \mathbf{a}_1^\dagger \mathbf{a}_2 = \mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow$ destroys dn-spin ↓

creates up-spin ↑

to raise angular momentum by one $\hbar$ unit

$$\mathbf{a}_\uparrow^\dagger \mathbf{a}_\downarrow |\downarrow\rangle = |\uparrow\rangle \quad \text{or:} \quad \mathbf{a}_1^\dagger \mathbf{a}_2 |2\rangle = |1\rangle$$

Let $\mathbf{a}_2^\dagger = \mathbf{a}_\downarrow^\dagger$ create dn-spin ↓

$$|2\rangle = |\downarrow\rangle = \left| \begin{matrix} 1/2 \\ -1/2 \end{matrix} \right\rangle = \mathbf{a}_2^\dagger |0\rangle = \mathbf{a}_\downarrow^\dagger |0\rangle$$

$\mathbf{s}_- = \mathbf{a}_2^\dagger \mathbf{a}_1 = \mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow$ destroys up-spin ↑

creates dn-spin ↓

to lower angular momentum by one $\hbar$ unit

$$\mathbf{a}_\downarrow^\dagger \mathbf{a}_\uparrow |\uparrow\rangle = |\downarrow\rangle \quad \text{or:} \quad \mathbf{a}_2^\dagger \mathbf{a}_1 |1\rangle = |2\rangle$$

Review : *1-D* **a†a** *algebra of U(1) representations*
Review : *Translate* **T**(*a*) *and/or Boost* **B**(*b*) *to construct coherent state*
Review : *Time evolution of coherent state (and "squeezed" states)*

*2-D* **a†a** *algebra of U(2) representations and R(3) angular momentum operators*
*2D-Oscillator basic states and operations*
*Commutation relations*
*Bose-Einstein symmetry vs Pauli-Fermi-Dirac (anti)symmetry*
*Anti-commutation relations*
*Two-dimensional (or 2-particle) base states: ket-kets and bra-bras*
*Outer product arrays*
*Entangled 2-particle states*
*Two-particle (or 2-dimensional) matrix operators*
*U(2) Hamiltonian and irreducible representations*
*2D-Oscillator states and related 3D angular momentum multiplets*
*ND multiplets*
*R(3) Angular momentum generators by U(2) analysis*
*Angular momentum raise-n-lower operators* $\mathbf{s}_+$ and $\mathbf{s}_-$
➡ *SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors* ⬅

## *SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

*U(2)* boson oscillator states $|n_1,n_2\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$

$$|n_1 n_2\rangle = \frac{\left(\mathbf{a}_1^\dagger\right)^{n_1}\left(\mathbf{a}_2^\dagger\right)^{n_2}}{\sqrt{n_1!\,n_2!}}|0\,0\rangle$$

## *SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

*U(2)* boson oscillator states $|n_1,n_2\rangle$ = R(3) spin or rotor states $\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$   Rotor total momenta: $j=\upsilon/2$

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*U(2)* boson oscillator states $|n_1,n_2\rangle$= R(3) spin or rotor states $\left|{j \atop m}\right\rangle$

Oscillator total quanta: $\upsilon=(n_1+n_2)$   Rotor total momenta: $j=\upsilon/2$ and z-momenta: $m=(n_1-n_2)/2$

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Oscillator $\mathbf{a}^\dagger\mathbf{a}$...

$$\mathbf{a}_1^\dagger\mathbf{a}_2|n_1 n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}|n_1+1\; n_2-1\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1|n_1 n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}|n_1-1\; n_2+1\rangle$$

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Oscillator $\mathbf{a}^\dagger\mathbf{a}$ give $\mathbf{s}_+$ matrices.

$$\mathbf{a}_1^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle=\sqrt{n_1+1}\sqrt{n_2}\left|n_1+1\ n_2-1\right\rangle \Rightarrow \mathbf{s}_+\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m+1}\sqrt{j-m}\left|\begin{smallmatrix} j \\ m+1 \end{smallmatrix}\right\rangle$$

$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\ n_2+1\right\rangle$$

## $SU(2)\subset U(2)$ oscillators vs. $R(3)\subset O(3)$ rotors

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$U(2)$ boson oscillator states = $U(2)$ spinor states

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$$\mathbf{a}_2^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle=\sqrt{n_1}\sqrt{n_2+1}\left|n_1-1\; n_2+1\right\rangle \Rightarrow \mathbf{s}_-\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\sqrt{j+m}\sqrt{j-m+1}\left|\begin{smallmatrix} j \\ m-1 \end{smallmatrix}\right\rangle$$

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1/2-difference of number-ops is $\mathbf{s}_Z$ eigenvalue.

$$\left.\begin{aligned} \mathbf{a}_1^\dagger\mathbf{a}_1\left|n_1 n_2\right\rangle&=n_1\left|n_1 n_2\right\rangle \\ \mathbf{a}_2^\dagger\mathbf{a}_2\left|n_1 n_2\right\rangle&=n_2\left|n_1 n_2\right\rangle \end{aligned}\right\}$$

## *SU(2)⊂U(2) oscillators vs. R(3)⊂O(3) rotors*

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*j=1 vector* $\mathbf{s}_+$

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*...and* $\mathbf{s}_Z$

$$D^1(\mathbf{s}_Z)=\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

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*j=3/2 spinor* $\mathbf{s}_+$ *...and* $\mathbf{s}_Z$

$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 0 & \cdot & \sqrt{4} & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger,\qquad D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

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$$\left.\begin{matrix}\mathbf{a}_1^\dagger\mathbf{a}_1|n_1 n_2\rangle=n_1|n_1 n_2\rangle \\ \mathbf{a}_2^\dagger\mathbf{a}_2|n_1 n_2\rangle=n_2|n_1 n_2\rangle\end{matrix}\right\} \mathbf{s}_Z\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\tfrac{1}{2}(\mathbf{a}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\mathbf{a}_2)\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=\frac{n_1-n_2}{2}\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle=m\left|\begin{smallmatrix} j \\ m \end{smallmatrix}\right\rangle$$

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*j=3/2 spinor* $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^{\frac{3}{2}}(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ 0 & \cdot & \sqrt{4} & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} \\ \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^{\frac{3}{2}}(\mathbf{s}_-)\right)^\dagger, \quad D^{\frac{3}{2}}(\mathbf{s}_Z)=\begin{pmatrix} \frac{3}{2} & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & -\frac{3}{2} \end{pmatrix}$$

*j=2 tensor* $\mathbf{s}_+$ ...*and* $\mathbf{s}_Z$

$$D^2(\mathbf{s}_+)=\begin{pmatrix} \cdot & \sqrt{4} & \cdot & \cdot & \cdot \\ 0 & \cdot & \sqrt{3} & \cdot & \cdot \\ \cdot & 0 & \cdot & \sqrt{3} & \cdot \\ \cdot & \cdot & 0 & \cdot & \sqrt{4} \\ \cdot & \cdot & \cdot & 0 & \cdot \end{pmatrix}=\left(D^2(\mathbf{s}_-)\right)^\dagger, \quad D^2(\mathbf{s}_Z)=\begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 \end{pmatrix}$$