Group Theory in Quantum Mechanics

Lecture 22 (4.14.15)

Harmonic oscillator symmetry $U(1) \subset U(2) \subset U(3)$...

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 7 Ch. 20-22)
(PSDS - Ch. 8)

1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations
- Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra
- Eigenstate creationism (and destruction)
  - Vacuum state
  - 1st excited state
- Normal ordering for matrix calculation
  - Commutator derivative identities
  - Binomial expansion identities
- Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ calculations
  - Number operator and Hamiltonian operator
  - Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$
  - Harmonic oscillator beat dynamics of mixed states
- Oscillator coherent states ("Shoved" and "kicked" states)
  - Translation operators vs. boost operators
  - Applying boost-translation combinations
  - Time evolution of coherent state
  - Properties of coherent state and "squeezed" states

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

1-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

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1-D **a†a** *algebra of U(1) representations*

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Proof:

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...so make symmetric factors.

Recall *commutator* [**x** , **p**] *relation*: $[\mathbf{x},\mathbf{p}] \equiv \mathbf{xp} - \mathbf{px} = \hbar i\mathbf{1}$ (They only miss commuting by 0.00000000000000... !)

Proof: *QED:*

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Define

$$\mathbf{a} = \frac{(\mathbf{X} + i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Destruction operator

and

$$\mathbf{a}^\dagger = \frac{(\mathbf{X} - i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Creation Operator

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] \equiv \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right) - \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)$$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define $$\mathbf{a} = \frac{(\mathbf{X} + i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$ *Destruction operator*

and $$\mathbf{a}^\dagger = \frac{(\mathbf{X} - i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$ *Creation Operator*

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$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \frac{2i}{2\hbar}(\mathbf{p}\mathbf{x} - \mathbf{x}\mathbf{p}) = \frac{-i}{\hbar}[\mathbf{x},\mathbf{p}]$$

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$$\mathbf{a}^\dagger = \frac{(\mathbf{X}-i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Creation Operator

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]\equiv\mathbf{a}\mathbf{a}^\dagger-\mathbf{a}^\dagger\mathbf{a}=\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)-\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]=\frac{2i}{2\hbar}(\mathbf{px}-\mathbf{xp})=\frac{-i}{\hbar}\left[\mathbf{x},\mathbf{p}\right]=\mathbf{1}$$

Recall *commutator* **[x , p]** *relation*: $\left[\mathbf{x},\mathbf{p}\right]\equiv\mathbf{xp}-\mathbf{px}=\hbar i\,\mathbf{1}$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define

$$\mathbf{a} = \frac{(\mathbf{X}+i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Destruction operator

and

$$\mathbf{a}^\dagger = \frac{(\mathbf{X}-i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Creation Operator

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]\equiv\mathbf{a}\mathbf{a}^\dagger-\mathbf{a}^\dagger\mathbf{a}=\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)-\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]=\frac{2i}{2\hbar}(\mathbf{px}-\mathbf{xp})=\frac{-i}{\hbar}[\mathbf{x},\mathbf{p}]=\mathbf{1}$$

$$\boxed{\left[\mathbf{a},\mathbf{a}^\dagger\right]=\mathbf{1}} \quad \text{or} \quad \boxed{\mathbf{a}\mathbf{a}^\dagger=\mathbf{a}^\dagger\mathbf{a}+\mathbf{1}}$$

Recall *commutator* **[x , p]** *relation*: $[\mathbf{x},\mathbf{p}]\equiv\mathbf{xp}-\mathbf{px}=\hbar i\,\mathbf{1}$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define $\mathbf{a} = \frac{(\mathbf{X}+i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$ *Destruction operator* and $\mathbf{a}^\dagger = \frac{(\mathbf{X}-i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$ *Creation Operator*

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]\equiv\mathbf{a}\mathbf{a}^\dagger\text{-}\mathbf{a}^\dagger\mathbf{a}=\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)-\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]=\frac{2i}{2\hbar}(\mathbf{px}-\mathbf{xp})=\frac{-i}{\hbar}[\mathbf{x},\mathbf{p}]=\mathbf{1}$$

$\left[\mathbf{a},\mathbf{a}^\dagger\right]=\mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger=\mathbf{a}^\dagger\mathbf{a}+\mathbf{1}$

1D-HO Hamiltonian in terms of $\mathbf{a}^\dagger\mathbf{a}$ *operator*

Recall: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger)/2$

Recall *commutator* [**x** , **p**] *relation*: $[\mathbf{x},\mathbf{p}]\equiv\mathbf{xp}-\mathbf{px}=\hbar i\,\mathbf{1}$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define

$$\mathbf{a}=\frac{(\mathbf{X}+i\mathbf{P})}{\sqrt{\hbar\omega}}=\frac{\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Destruction operator

and

$$\mathbf{a}^\dagger=\frac{(\mathbf{X}-i\mathbf{P})}{\sqrt{\hbar\omega}}=\frac{\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Creation Operator

Commutation relations between $\mathbf{a}=(\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger=(\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X}\equiv\sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P}\equiv\mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]\equiv\mathbf{a}\mathbf{a}^\dagger-\mathbf{a}^\dagger\mathbf{a}=\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)-\frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x}-i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x}+i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]=\frac{2i}{2\hbar}(\mathbf{px}-\mathbf{xp})=\frac{-i}{\hbar}[\mathbf{x},\mathbf{p}]=\mathbf{1}$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right]=\mathbf{1}$$

or

$$\mathbf{a}\mathbf{a}^\dagger=\mathbf{a}^\dagger\mathbf{a}+\mathbf{1}$$

1D-HO Hamiltonian in terms of $\mathbf{a}^\dagger\mathbf{a}$ *operator*

$$\mathbf{H}(\mathbf{x},\mathbf{p})=\hbar\omega\,(\mathbf{a}^\dagger\mathbf{a}+\mathbf{a}\mathbf{a}^\dagger)/2\quad=\hbar\omega\,(\mathbf{a}^\dagger\mathbf{a}+\mathbf{a}^\dagger\mathbf{a}+\mathbf{1})/2$$

Recall *commutator* [**x** , **p**] *relation*: $[\mathbf{x},\mathbf{p}]\equiv\mathbf{xp}-\mathbf{px}=\hbar i\,\mathbf{1}$

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Define

$$\mathbf{a} = \frac{(\mathbf{X} + i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Destruction operator

and

$$\mathbf{a}^\dagger = \frac{(\mathbf{X} - i\mathbf{P})}{\sqrt{\hbar\omega}} = \frac{\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)}{\sqrt{2\hbar}}$$

Creation Operator

Commutation relations between $\mathbf{a} = (\mathbf{X}+i\mathbf{P})/2$ and $\mathbf{a}^\dagger = (\mathbf{X}-i\mathbf{P})/2$ with $\mathbf{X} \equiv \sqrt{M\omega}\mathbf{x}/\sqrt{2}$ and $\mathbf{P} \equiv \mathbf{p}/\sqrt{2M}$:

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] \equiv \mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right) - \frac{1}{2\hbar}\left(\sqrt{M\omega}\,\mathbf{x} - i\mathbf{p}/\sqrt{M\omega}\right)\left(\sqrt{M\omega}\,\mathbf{x} + i\mathbf{p}/\sqrt{M\omega}\right)$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \frac{2i}{2\hbar}(\mathbf{p}\mathbf{x} - \mathbf{x}\mathbf{p}) = \frac{-i}{\hbar}[\mathbf{x},\mathbf{p}] = \mathbf{1}$$

$$\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}$$

or

$$\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$$

1D-HO Hamiltonian in terms of $\mathbf{a}^\dagger\mathbf{a}$ *operator*

$$\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}\mathbf{a}^\dagger)/2 \quad = \hbar\omega\,(\mathbf{a}^\dagger\mathbf{a} + \mathbf{a}^\dagger\mathbf{a} + \mathbf{1})/2 \quad = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a} + \mathbf{1}\hbar\omega/2$$

Recall *commutator* [**x** , **p**] *relation*: $[\mathbf{x},\mathbf{p}] \equiv \mathbf{x}\mathbf{p} - \mathbf{p}\mathbf{x} = \hbar i\,\mathbf{1}$

1-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(1) representations*

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ *algebra*

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculations*

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $\mathbf{a}^\dagger\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

Eigenstate creationism (and destruction)

Given1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $\left[\mathbf{a}, \mathbf{a}^\dagger\right] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}$

Define *ground state* $|0\rangle$ as the eigenstate of $\mathbf{H}(\mathbf{x},\mathbf{p})$ with the *zero point eigenvalue* $E_0 = \hbar\omega/2$.

Eigenstate creationism (and destruction)

Given1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

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$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|0\rangle = \hbar\omega/2\,|0\rangle \qquad \langle 0|\,\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega/2\,\langle 0|$$

Given 1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $\left[\mathbf{a}, \mathbf{a}^\dagger\right] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}$

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Action by $\mathbf{a}$ on ground ket $|0\rangle$ (or $\mathbf{a}^\dagger$ on ground bra $\langle 0|$) gives *nothing* (zero vectors $\boldsymbol{0}$).

$$\mathbf{a}\,|0\rangle = \boldsymbol{0} \qquad \langle 0|\,\mathbf{a}^\dagger = \boldsymbol{0}$$

Eigenstate creationism (and destruction)

Given1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

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But, $\mathbf{a}^\dagger$ acts on ground ket to give $|1\rangle = \mathbf{a}^\dagger|0\rangle$ with $\mathbf{H}$ eigenvalue $E_1 = \hbar\omega + E_0$. ($|1\rangle = \mathbf{a}^\dagger|0\rangle$, $\langle 0|\mathbf{a} = \langle 1|$.)

Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle \qquad + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

Eigenstate creationism (and destruction)

Given1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

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Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger(\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

Eigenstate creationism (and destruction)

Given 1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $[\mathbf{a},\mathbf{a}^\dagger] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

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Eigenstate creationism (and destruction)

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Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

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$$= \hbar\omega\,\mathbf{a}^\dagger|0\rangle + \boldsymbol{0} + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

QED:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|1\rangle = (\hbar\omega + \hbar\omega/2)\,|1\rangle = E_1|1\rangle$$ where: $E_1 = \hbar\omega + E_0$

Eigenstate creationism (and destruction)

Given1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

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Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger(\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

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QED:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|1\rangle = (\hbar\omega + \hbar\omega/2)\,|1\rangle = E_1|1\rangle \text{ where: } E_1 = \hbar\omega + E_0$$

One-quantum or *1st excited eigenket* $|1\rangle = \mathbf{a}^\dagger|0\rangle$

Eigenstate creationism (and destruction)

Given 1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $[\mathbf{a},\mathbf{a}^\dagger] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

Define *ground state* $|0\rangle$ as the eigenstate of $\mathbf{H}(\mathbf{x},\mathbf{p})$ with the *zero point eigenvalue* $E_0 = \hbar\omega/2$.

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|0\rangle = \hbar\omega/2\,|0\rangle \qquad \langle 0|\,\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega/2\,\langle 0|$$

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Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,\mathbf{a}^\dagger|0\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,\mathbf{a}^\dagger|0\rangle + \hbar\omega/2\,\mathbf{a}^\dagger|0\rangle$$

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QED:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\,|1\rangle = (\hbar\omega + \hbar\omega/2)\,|1\rangle = E_1|1\rangle$$ where: $E_1 = \hbar\omega + E_0$

One-quantum or *1st excited eigenket* $|1\rangle = \mathbf{a}^\dagger|0\rangle$

For kets, $\mathbf{a}^\dagger$ is *creation operator* while $\mathbf{a}$ is *destruction operator*.

$$\mathbf{a}|1\rangle = \mathbf{a}\mathbf{a}^\dagger|0\rangle = (\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle = |0\rangle$$

Eigenstate creationism (and destruction)

Given 1D-HO Hamiltonian: $\mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a} + \mathbf{1}\hbar\omega/2$ *and commutation:* $\left[\mathbf{a},\mathbf{a}^\dagger\right] = \mathbf{1}$ or $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} + \mathbf{1}$

Define *ground state* $|0\rangle$ as the eigenstate of $\mathbf{H}(\mathbf{x},\mathbf{p})$ with the *zero point eigenvalue* $E_0 = \hbar\omega/2$.

$$\mathbf{H}(\mathbf{x},\mathbf{p})\, |0\rangle = \hbar\omega/2\, |0\rangle \qquad \langle 0|\, \mathbf{H}(\mathbf{x},\mathbf{p}) = \hbar\omega/2\, \langle 0|$$

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Proof:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\, \mathbf{a}^\dagger|0\rangle = \hbar\omega\, \mathbf{a}^\dagger\mathbf{a}\, \mathbf{a}^\dagger|0\rangle + \hbar\omega/2\, \mathbf{a}^\dagger|0\rangle$$

$$\mathbf{H}(\mathbf{x},\mathbf{p})\, \mathbf{a}^\dagger|0\rangle = \hbar\omega\, \mathbf{a}^\dagger(\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle + \hbar\omega/2\, \mathbf{a}^\dagger|0\rangle$$

$$= \hbar\omega\, \mathbf{a}^\dagger|0\rangle + \mathbf{0} + \hbar\omega/2\, \mathbf{a}^\dagger|0\rangle$$

QED:

$$\mathbf{H}(\mathbf{x},\mathbf{p})\, |1\rangle = (\hbar\omega + \hbar\omega/2)\, |1\rangle = E_1|1\rangle$$ where: $E_1 = \hbar\omega + E_0$

One-quantum or *1st excited eigenket* $|1\rangle = \mathbf{a}^\dagger|0\rangle$

For kets, $\mathbf{a}^\dagger$ is *creation operator* while $\mathbf{a}$ is *destruction operator*.

$$\mathbf{a}|1\rangle = \mathbf{a}\mathbf{a}^\dagger|0\rangle = (\mathbf{a}^\dagger\mathbf{a} + \mathbf{1})|0\rangle = |0\rangle$$

For bras, $\mathbf{a}^\dagger$ is *destruction operator* while $\mathbf{a}$ is *creation operator*.

$$\langle 1|\mathbf{a}^\dagger = \langle 0|\mathbf{a}\mathbf{a}^\dagger = \langle 0|(\mathbf{a}^\dagger\mathbf{a} + \mathbf{1}) = \langle 0|$$

1-D $\mathbf{a}^{\dagger}\mathbf{a}$ *algebra of U(1) representations*

Creation-Destruction $\mathbf{a}^{\dagger}\mathbf{a}$ *algebra*

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

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2-D $\mathbf{a}^{\dagger}\mathbf{a}$ *algebra of U(2) representations and R(3) angular momentum operators*

Wavefunction creationism (Vacuum state)

Coordinate representation of the "nothing" equation $\langle x|\mathbf{a}|0\rangle = \boldsymbol{0}$ with: $\mathbf{p} = \hbar\mathbf{k} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$$\langle x|\mathbf{a}|0\rangle = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,\langle x|\mathbf{x}|0\rangle + i\langle x|\mathbf{p}|0\rangle / \sqrt{M\omega}\right) = 0$$

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$$\int \frac{d\psi}{\psi} = \int \frac{-M\omega}{\hbar}xdx\,, \quad \ln\psi + \ln const. = \frac{-M\omega}{\hbar}\frac{x^2}{2}, \quad \psi = \frac{e^{-M\omega x^2/2\hbar}}{const.}$$

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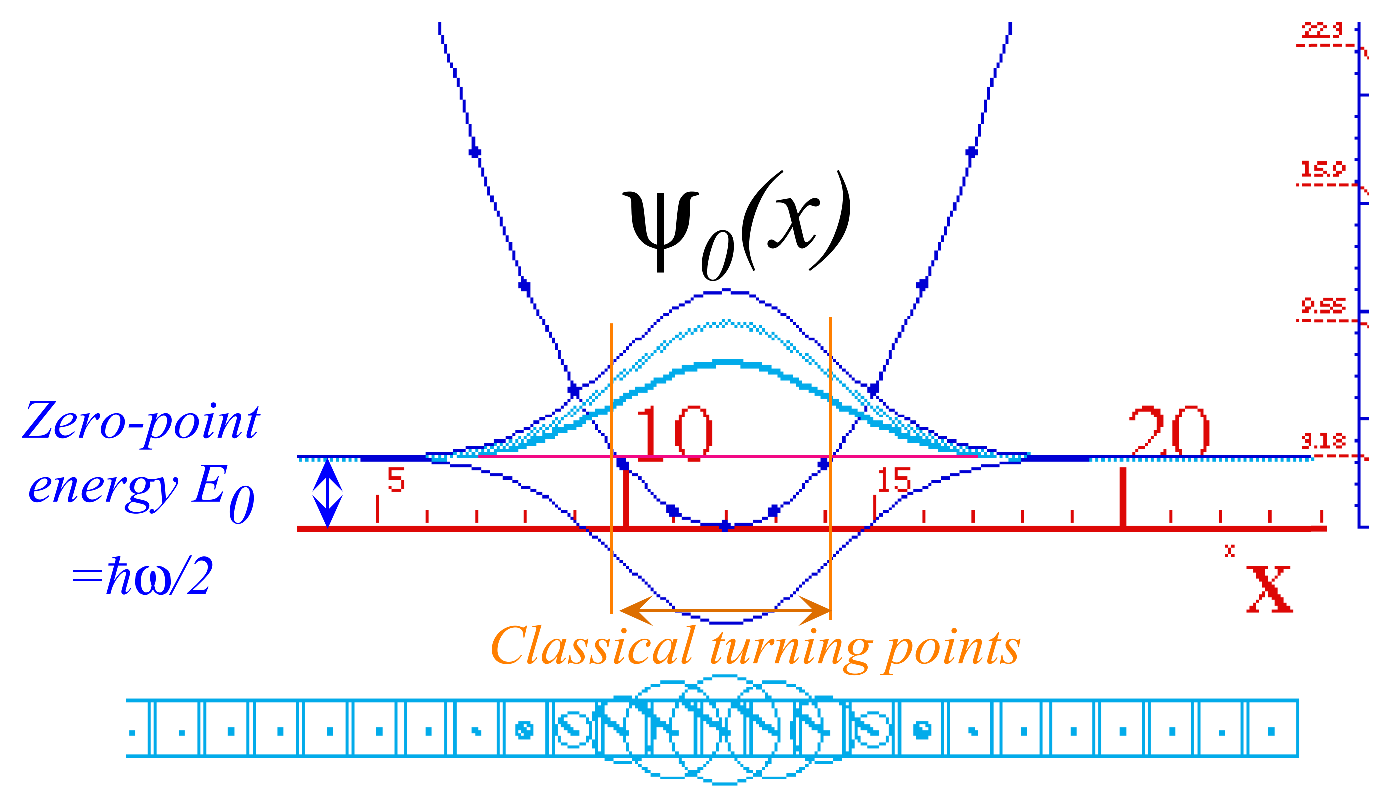

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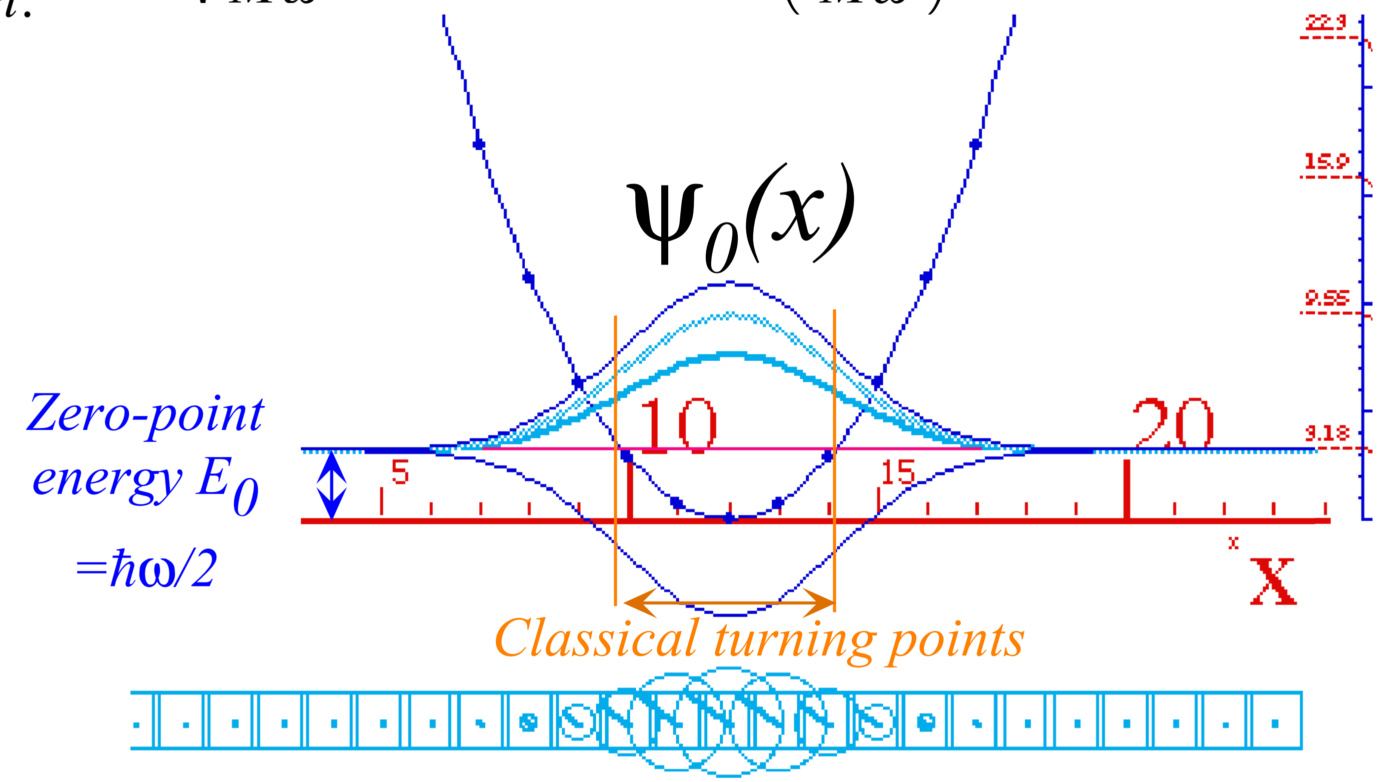
The normalization *const.* is evaluated using a standard Gaussian integral: $\int_{-\infty}^{\infty}dx\,e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$

$$\langle\psi_0|\psi_0\rangle = 1 = \int_{-\infty}^{\infty}dx\,\frac{e^{-M\omega x^2 2/2\hbar}}{const.^2} = \sqrt{\frac{\pi\hbar}{M\omega}}/const.^2 \Rightarrow const. = \left(\frac{\pi\hbar}{M\omega}\right)^{1/4}$$



1-D **a**†**a** *algebra of U(1) representations*

Creation-Destruction **a**†**a** *algebra*

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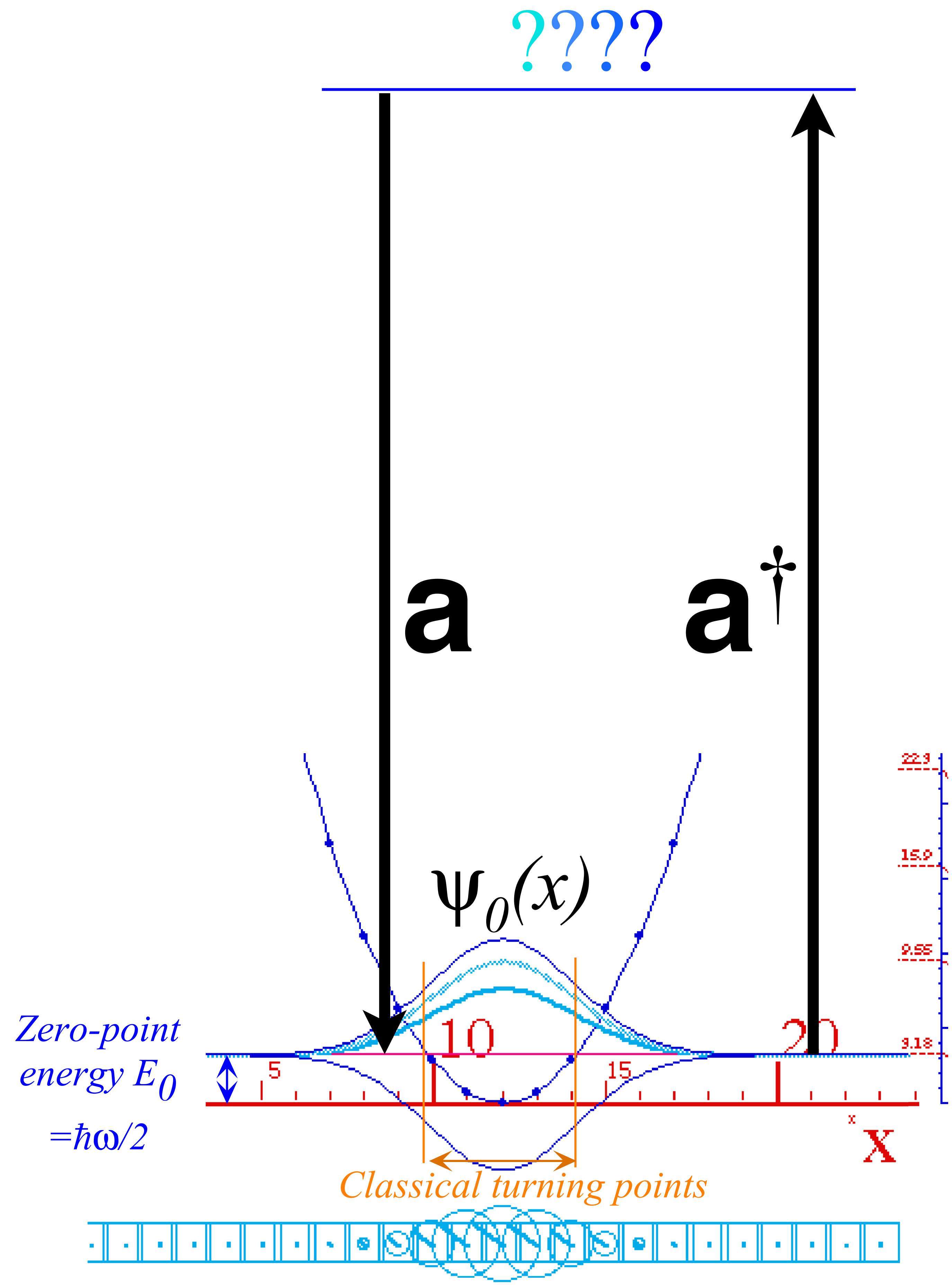
2-D **a**†**a** *algebra of U(2) representations and R(3) angular momentum operators*

Wavefunction creationism (1st Excited state)

1st excited state wavefunction $\psi_1(x) = \langle x \,|1\rangle$

$$\langle x\,|\mathbf{a}^\dagger|0\rangle = \langle x\,|1\rangle = \psi_1(x)$$



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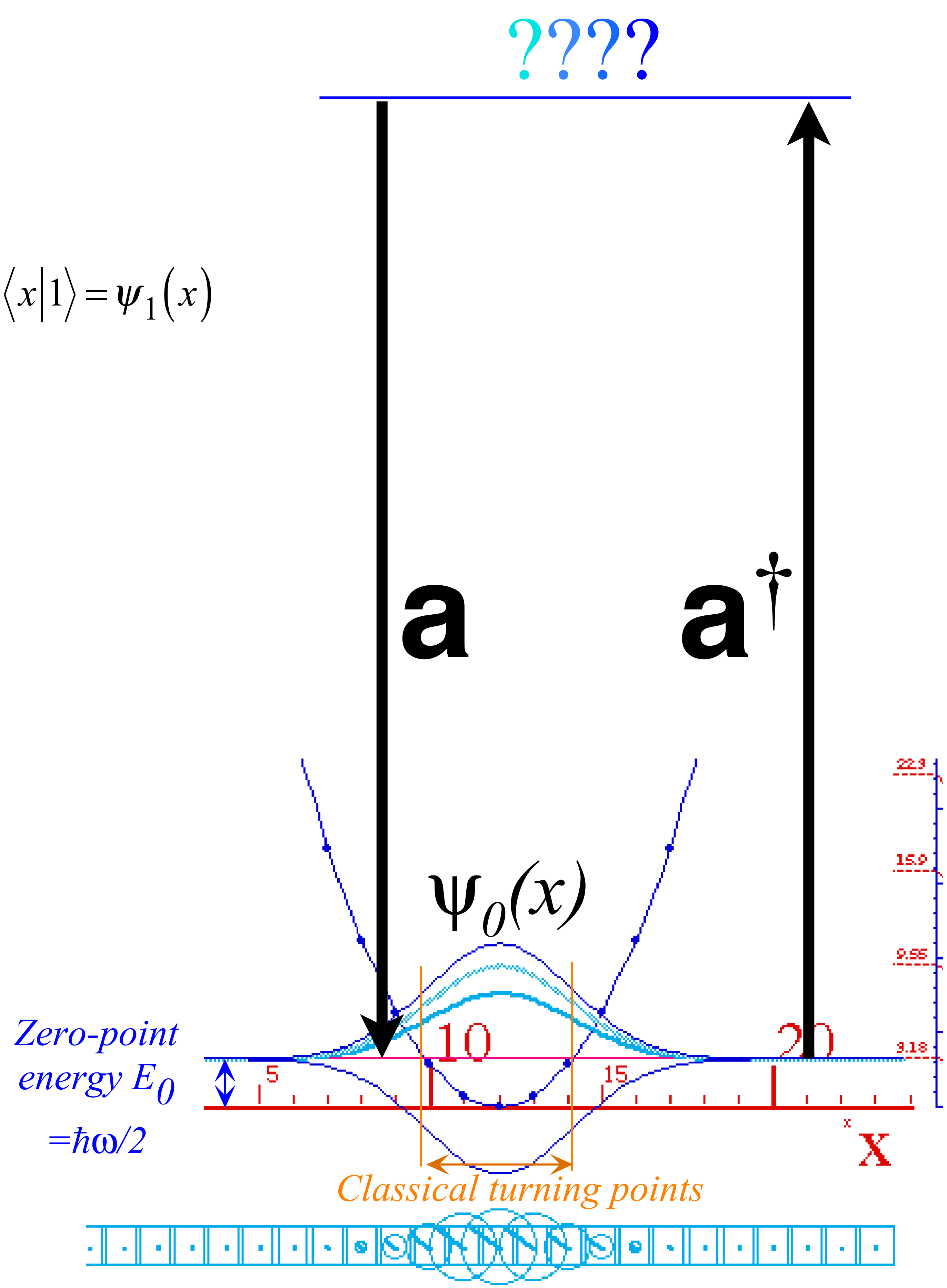
Expanding the creation operator

$$\langle x | \mathbf{a}^\dagger | 0 \rangle = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{M\omega} \langle x | \mathbf{x} | 0 \rangle - i \langle x | \mathbf{p} | 0 \rangle / \sqrt{M\omega} \right) = \langle x | 1 \rangle = \psi_1(x)$$



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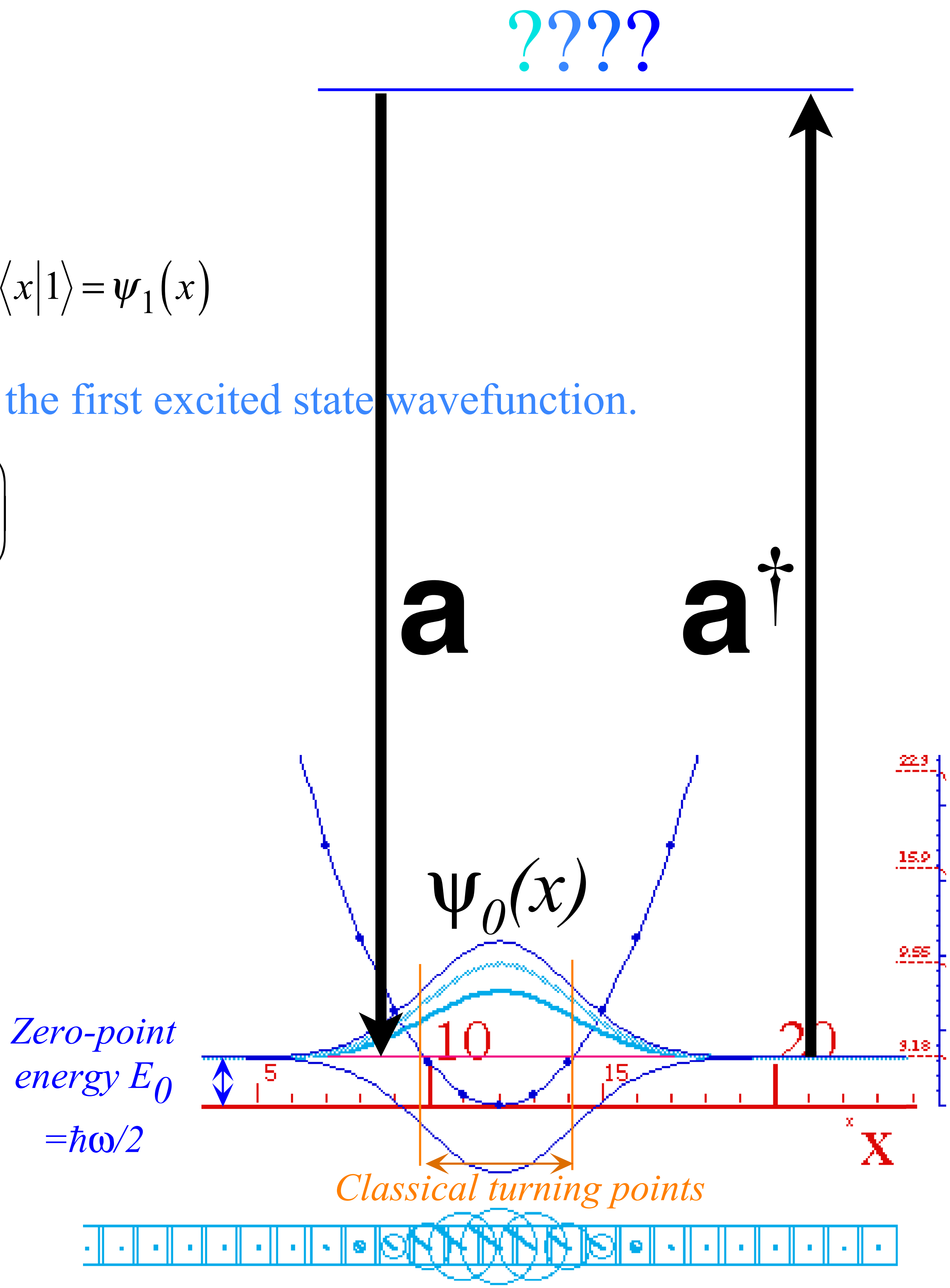
The operator coordinate representations generate the first excited state wavefunction.

$$\langle x|1\rangle = \psi_1(x) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{M\omega}\,x\psi_0(x) - i\frac{\hbar}{i}\frac{\partial\psi_0(x)}{\partial x}/\sqrt{M\omega}\right)$$



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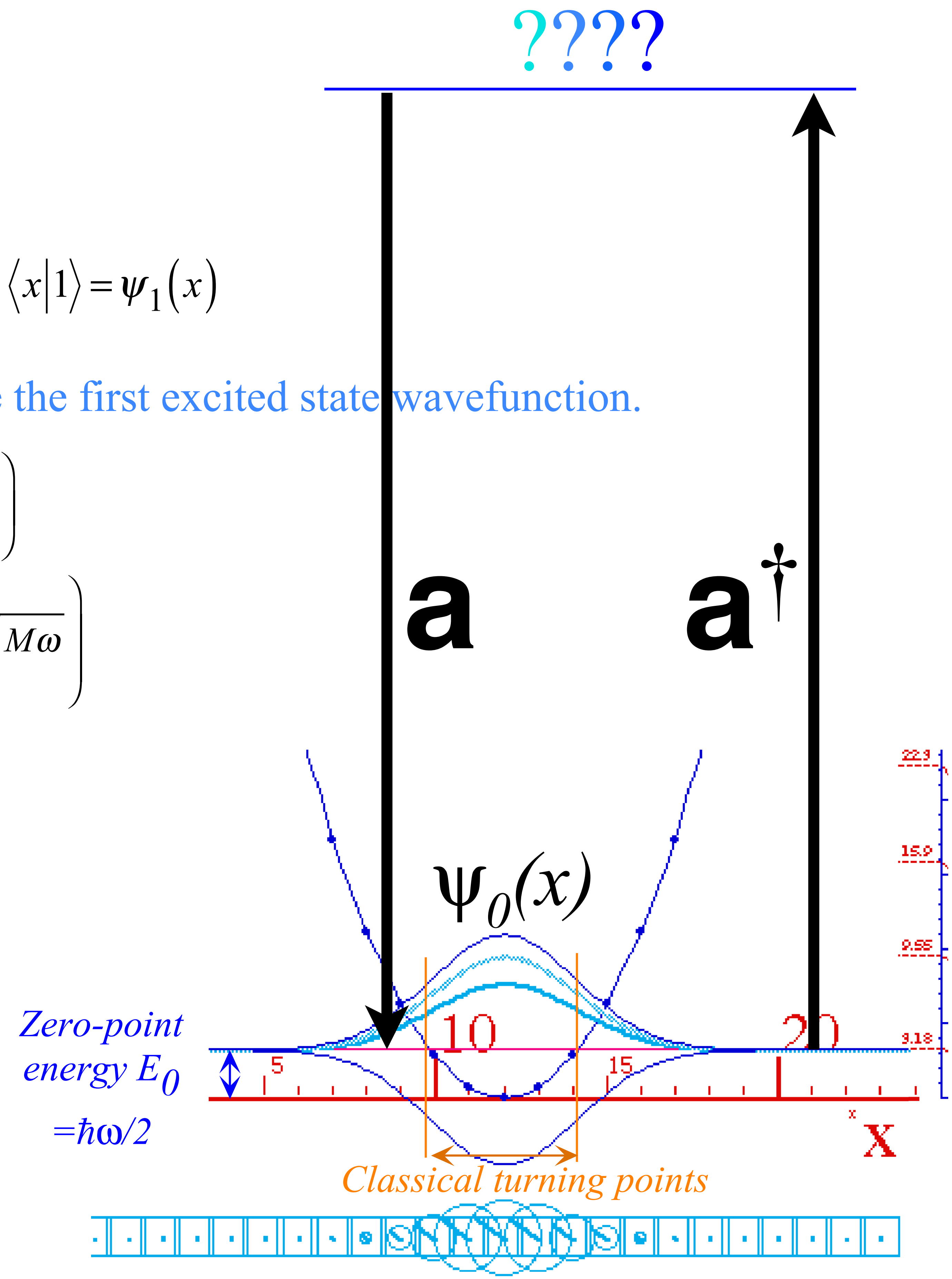

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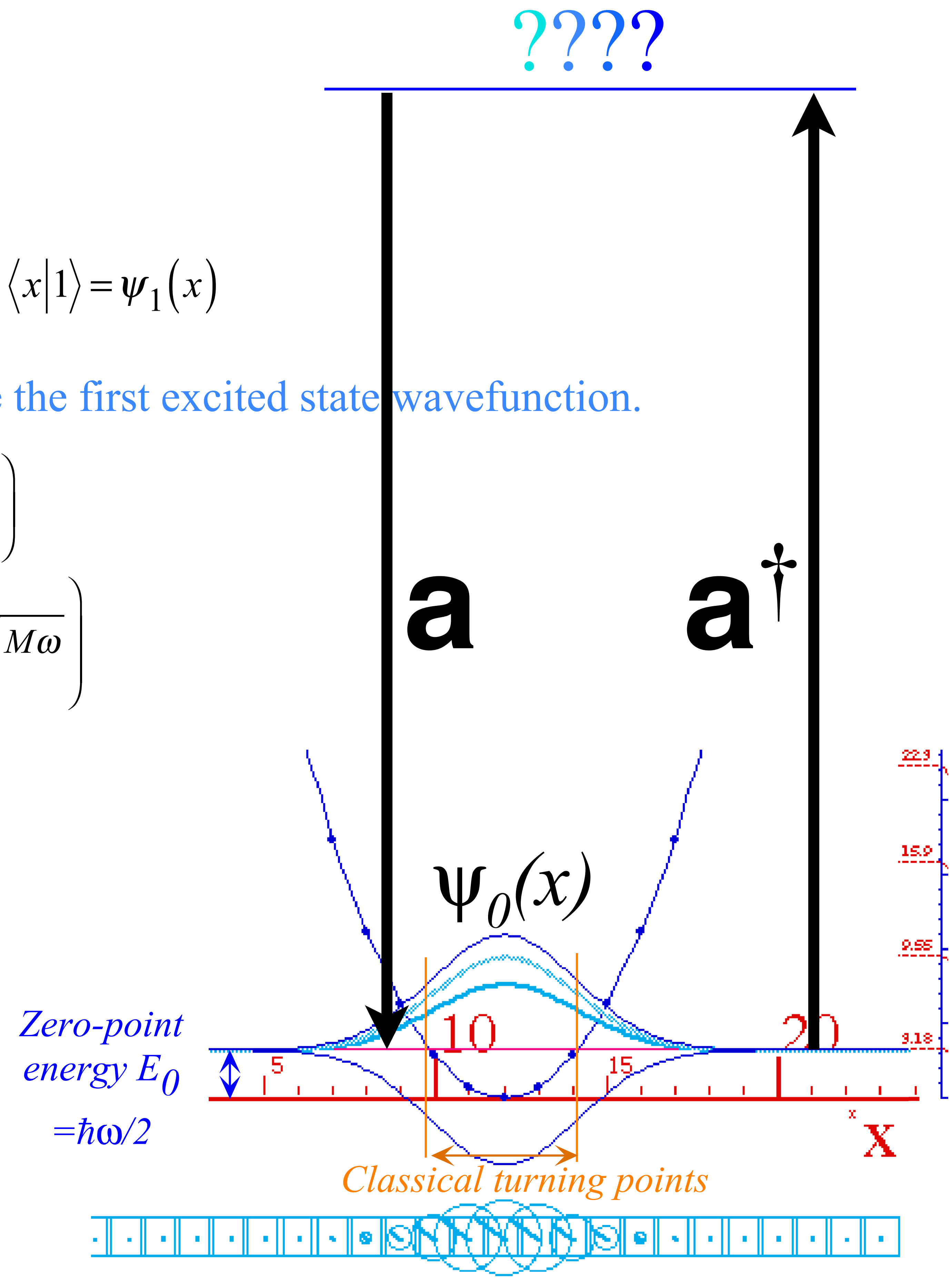

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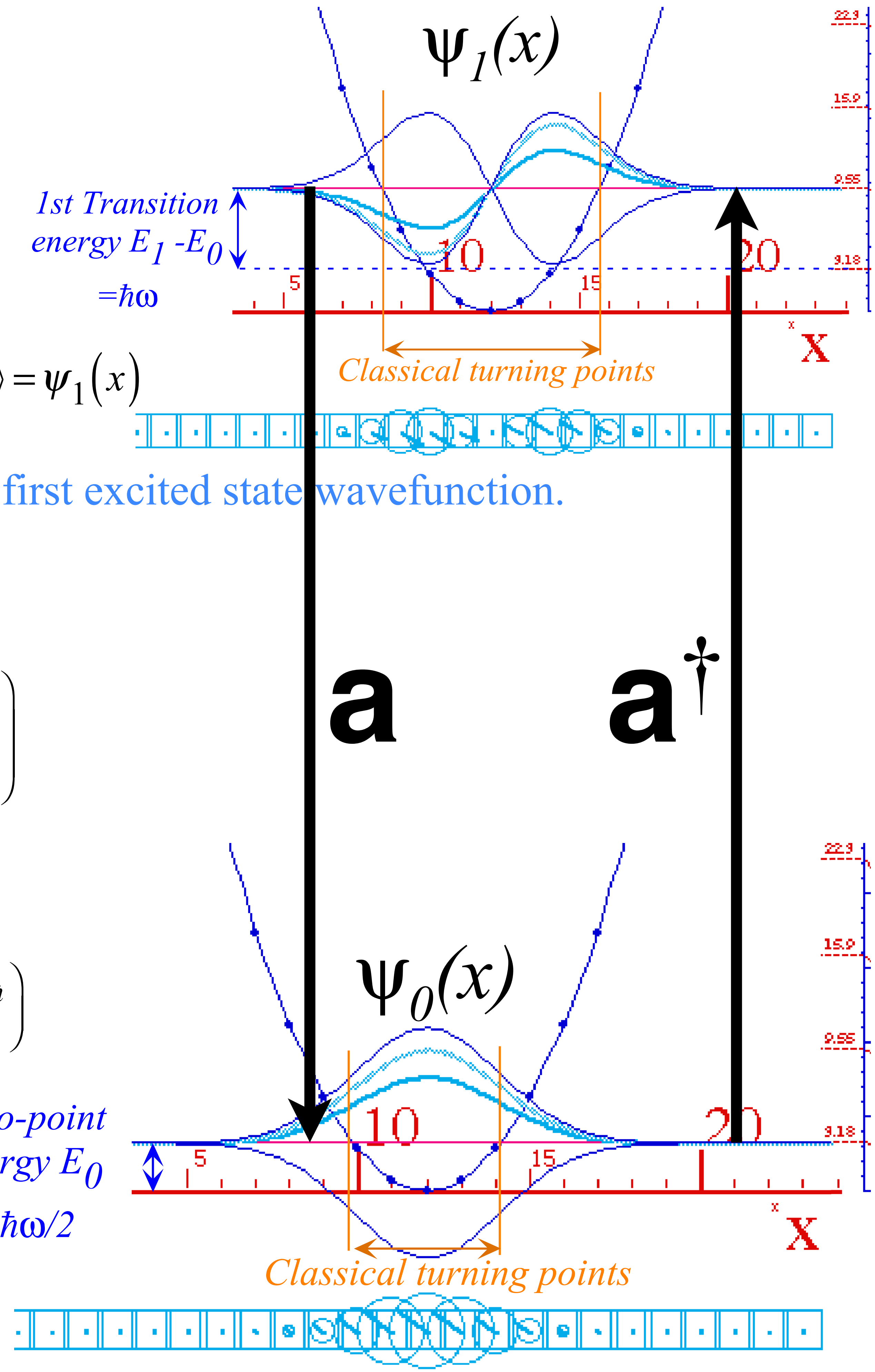
$$= \frac{\sqrt{M\omega}}{\sqrt{2\hbar}}\frac{e^{-M\omega x^2/2\hbar}}{const.}(2x) = \left(\frac{M\omega}{\pi\hbar}\right)^{3/4}\sqrt{2\pi}\left(x e^{-M\omega x^2/2\hbar}\right)$$



1-D $a^\dagger a$ algebra of U(1) representations

Creation-Destruction $a^\dagger a$ algebra

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Normal ordering: move destructive **a** operators to the right of creation $\mathbf{a}^\dagger$ to zero out on vacuum $|0\rangle$.

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(Power-law derivative-like relations)

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$[\mathbf{A}, \mathbf{BC}] = \mathbf{ABC} - \mathbf{BCA} = [\mathbf{A}, \mathbf{B}]\mathbf{C} + \mathbf{BAC} - \mathbf{BCA}$
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$$= n(n-1)\mathbf{a}^{\dagger n-2} + n\mathbf{a}^{\dagger n-1}\mathbf{a} + n\mathbf{a}^{\dagger n-1}\mathbf{a} + \mathbf{a}^{\dagger n}\mathbf{a}^2$$
$$= n(n-1)\mathbf{a}^{\dagger n-2} + 2n\mathbf{a}^{\dagger n-1}\mathbf{a} + \mathbf{a}^{\dagger n}\mathbf{a}^2$$

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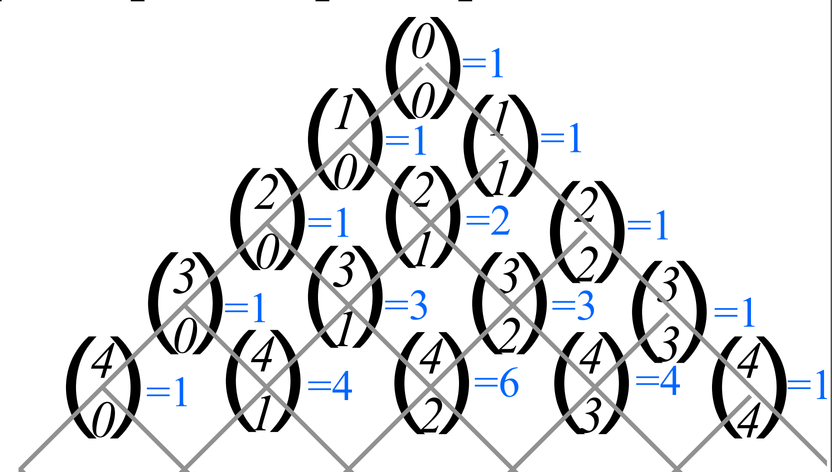

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Creation-Destruction $a^\dagger a$ algebra

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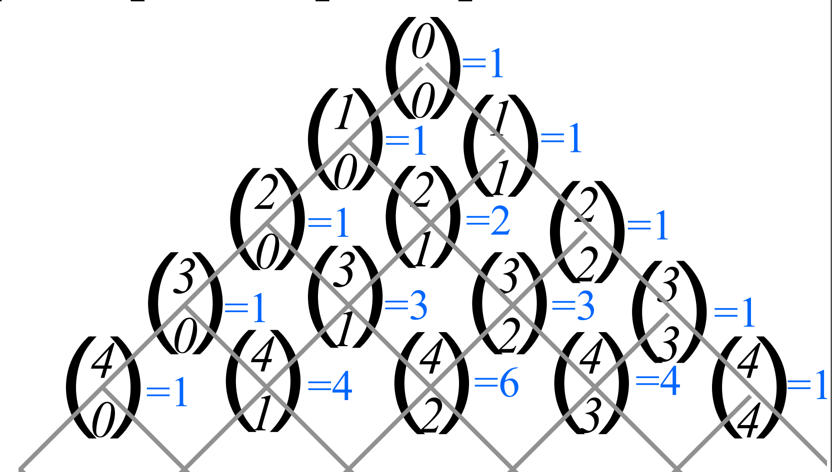

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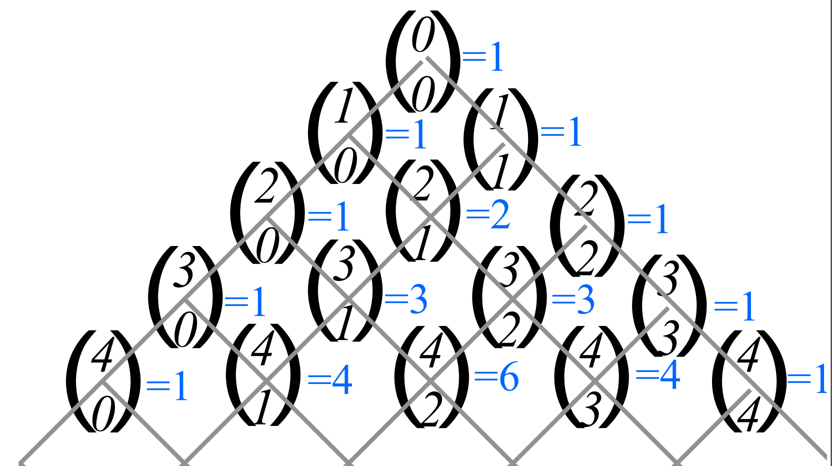

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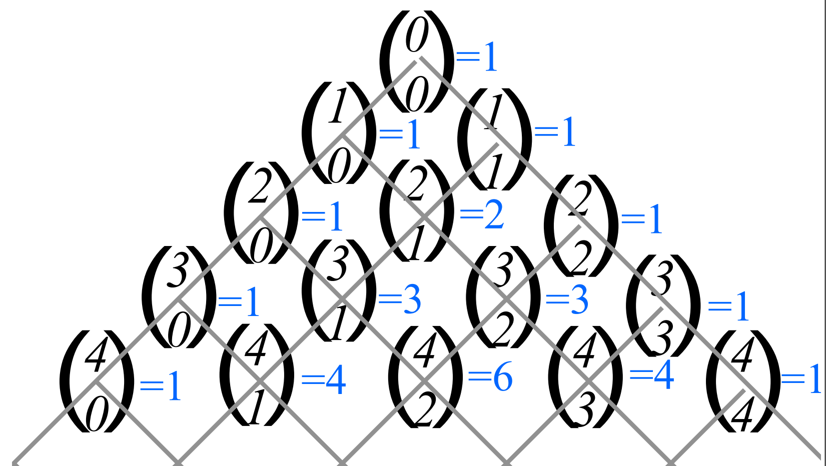

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$\mathbf{a}^n\mathbf{a}^{\dagger n}$ *to* $\mathbf{a}^{\dagger r}\mathbf{a}^r$ *case*

$$\mathbf{a}^n\mathbf{a}^{\dagger n} = \sum_{r=0}^{n}\binom{n}{r}\frac{n!}{r!}\mathbf{a}^{\dagger r}\mathbf{a}^r = n!\left(\mathbf{1} + n\mathbf{a}^\dagger\mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \frac{n(n-1)(n-3)}{3!\cdot 3!}\mathbf{a}^{\dagger 3}\mathbf{a}^3 + \ldots\right)$$

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So: $(const.)^2 = n!$

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Root-factorial normalization

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger\left|n\right\rangle = \frac{\mathbf{a}^{\dagger n+1}\left|0\right\rangle}{\sqrt{n!}}$$

Apply destruction $\mathbf{a}$:

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Root-factorial normalization

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^{\dagger}$:

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Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

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Feynman's mnemonic rule: Larger of two quanta goes in radical factor

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(Welcome to ∞-dimensional... quantum space!)

1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1^{st} excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ calculations

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

Use: $\mathbf{a}^n \mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\,\mathbf{a}^\dagger \mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots\right)$

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Number operator and Hamiltonian operator

Number operator $\mathbf{N}=\mathbf{a}^\dagger\mathbf{a}$ counts quanta.

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

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Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

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Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Number operator and Hamiltonian operator

Number operator $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a}$ counts quanta.

$$\mathbf{a}^\dagger \mathbf{a}|n\rangle = \frac{\mathbf{a}^\dagger \mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^\dagger \mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{n!}} = n\frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = n|n\rangle$$

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ *calculation*

Derive normalization for n^{th} state obtained by $(\mathbf{a}^\dagger)^n$ operator:

Use: $\mathbf{a}^n \mathbf{a}^{\dagger n} = n!\left(\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + \frac{n(n-1)}{2!\cdot 2!}\mathbf{a}^{\dagger 2}\mathbf{a}^2 + \ldots\right)$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{const.}, \quad \text{where:} \quad 1 = \langle n|n\rangle = \frac{\langle 0|\mathbf{a}^n \mathbf{a}^{\dagger n}|0\rangle}{(const.)^2} = n!\frac{\langle 0|\mathbf{1} + n\mathbf{a}^\dagger \mathbf{a} + ..|0\rangle}{(const.)^2} = \frac{n!}{(const.)^2}$$

$$|n\rangle = \frac{\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}}$$ *Root-factorial normalization*

Use: $\mathbf{a}\mathbf{a}^{\dagger n} = n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a}$

Apply creation $\mathbf{a}^\dagger$:

$$\mathbf{a}^\dagger|n\rangle = \frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{n!}} = \sqrt{n+1}\frac{\mathbf{a}^{\dagger n+1}|0\rangle}{\sqrt{(n+1)!}}$$

Apply destruction $\mathbf{a}$:

$$\mathbf{a}|n\rangle = \frac{\mathbf{a}\mathbf{a}^{\dagger n}|0\rangle}{\sqrt{n!}} = \frac{(n\mathbf{a}^{\dagger n-1} + \mathbf{a}^{\dagger n}\mathbf{a})|0\rangle}{\sqrt{n!}} = \sqrt{n}\frac{\mathbf{a}^{\dagger n-1}|0\rangle}{\sqrt{(n-1)!}}$$

$$\mathbf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle$$

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Hamiltonian operator

$$\mathbf{H}\,|n\rangle = \hbar\omega\,\mathbf{a}^\dagger\mathbf{a}\,|n\rangle + \hbar\omega/2\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

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$$\mathbf{H}\,|n\rangle = \hbar\omega\, \mathbf{a}^\dagger \mathbf{a}\,|n\rangle + \hbar\omega/2\mathbf{1}\,|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

$$\langle \mathbf{H} \rangle = \hbar\omega\langle \mathbf{a}^\dagger \mathbf{a} + \tfrac{1}{2}\mathbf{1}\rangle = \hbar\omega \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} + \hbar\omega \begin{pmatrix} 1/2 & & & & \\ & 1/2 & & & \\ & & 1/2 & & \\ & & & 1/2 & \\ & & & & \ddots \end{pmatrix}$$

Hamiltonian operator is $\hbar\omega$ $\mathbf{N}$ plus zero-point energy $\mathbf{1}\hbar\omega/2$.

1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ calculations

Number operator and Hamiltonian operator

➡ *Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$* ⬅

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Operator for position **x**: $\sqrt{\frac{M\omega}{2\hbar}}\mathbf{x} = \frac{\mathbf{a}+\mathbf{a}^\dagger}{2}$

Operator for momentum **p**: $\sqrt{\frac{1}{2\hbar M\omega}}\mathbf{p} = \frac{\mathbf{a}-\mathbf{a}^\dagger}{2i}$

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$$\bar{\mathbf{x}}\big|_n = \langle n|\mathbf{x}|n\rangle = \sqrt{\frac{\hbar}{2M\omega}}\langle n|\left(\mathbf{a}+\mathbf{a}^\dagger\right)|n\rangle = 0$$

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$$\overline{\mathbf{x}^2}\big|_n = \langle n|\mathbf{x}^2|n\rangle = \frac{\hbar}{2M\omega}\langle n|\left(\mathbf{a}+\mathbf{a}^\dagger\right)^2|n\rangle$$

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Uncertainty or ***standard deviation*** Δq of a statistical quantity q is its root mean-square difference.

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Heisenberg uncertainty product for the n-quantum eigenstate $|n\rangle$

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$$\boxed{(\Delta x\cdot\Delta p)|_n = \hbar\left(n+\frac{1}{2}\right)}$$

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expectation for (position)^2 $\langle \mathbf{x}^2 \rangle$:

$$\overline{\mathbf{x}^2}\,|_n = \langle n|\mathbf{x}^2|n\rangle = \frac{\hbar}{2M\omega}\langle n|\left(\mathbf{a}+\mathbf{a}^\dagger\right)^2|n\rangle$$

$$= \frac{\hbar}{2M\omega}\langle n|\left(\mathbf{a}^2+\mathbf{a}^\dagger\mathbf{a}+\mathbf{a}\mathbf{a}^\dagger+\mathbf{a}^{\dagger 2}\right)|n\rangle$$

$$= \frac{\hbar}{2M\omega}\quad(2n+1)$$

Use: $\mathbf{a}\mathbf{a}^\dagger = \mathbf{1}+\mathbf{a}^\dagger\mathbf{a}$

Operator for momentum $\mathbf{p}$: $\sqrt{\frac{1}{2\hbar M\omega}}\mathbf{p} = \frac{\mathbf{a}-\mathbf{a}^\dagger}{2i}$

expectation for momentum $\langle \mathbf{p} \rangle$:

$$\bar{\mathbf{p}}\,|_n = \langle n|\mathbf{p}|n\rangle = i\sqrt{\frac{\hbar M\omega}{2}}\langle n|\left(\mathbf{a}^\dagger-\mathbf{a}\right)|n\rangle = 0$$

expectation for (momentum)^2 $\langle \mathbf{p}^2 \rangle$:

$$\overline{\mathbf{p}^2}\,|_n = \langle n|\mathbf{p}^2|n\rangle = i^2\frac{\hbar M\omega}{2}\langle n|\left(\mathbf{a}^\dagger-\mathbf{a}\right)^2|n\rangle$$

$$= -\frac{\hbar M\omega}{2}\langle n|\left(\mathbf{a}^{\dagger 2}-\mathbf{a}^\dagger\mathbf{a}-\mathbf{a}\mathbf{a}^\dagger+\mathbf{a}^2\right)|n\rangle$$

$$= \frac{\hbar M\omega}{2}\quad(2n+1)$$

Uncertainty or *standard deviation* Δq of a statistical quantity q is its root mean-square difference.

$$(\Delta q)^2 = \overline{(q-\bar{q})^2} \qquad \text{or:}\quad \Delta q = \sqrt{\overline{(q-\bar{q})^2}}$$

$$\Delta x|_n = \sqrt{\overline{\mathbf{x}^2}} = \sqrt{\frac{\hbar(2n+1)}{2M\omega}} \qquad \Delta p|_n = \sqrt{\overline{\mathbf{p}^2}} = \sqrt{\frac{\hbar M\omega(2n+1)}{2}}$$

Heisenberg uncertainty product for the n-quantum eigenstate $|n\rangle$

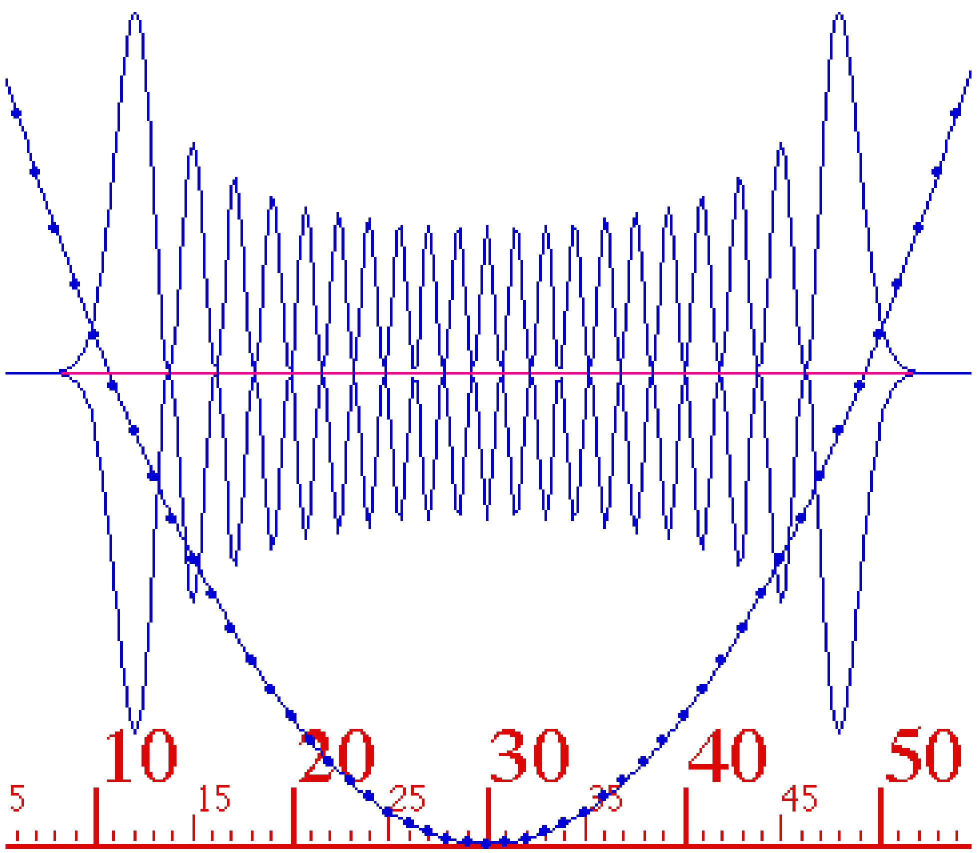
$$(\Delta x\cdot\Delta p)|_n = \sqrt{\overline{\mathbf{x}^2}}\sqrt{\overline{\mathbf{p}^2}} = \sqrt{\frac{\hbar(2n+1)}{2M\omega}}\sqrt{\frac{\hbar M\omega(2n+1)}{2}}$$

$$\boxed{(\Delta x\cdot\Delta p)|_n = \hbar\left(n+\frac{1}{2}\right)}$$

Heisenberg minimum uncertainty product occurs for the *0*-quantum (ground) eigenstate.

$$\boxed{(\Delta x\cdot\Delta p)|_0 = \frac{\hbar}{2}}$$

We pause for sobering considerations of the quantum world *vs.* the classical one.
Consider a "high"-quantum (*n=20*) eigenstate wavefunction:



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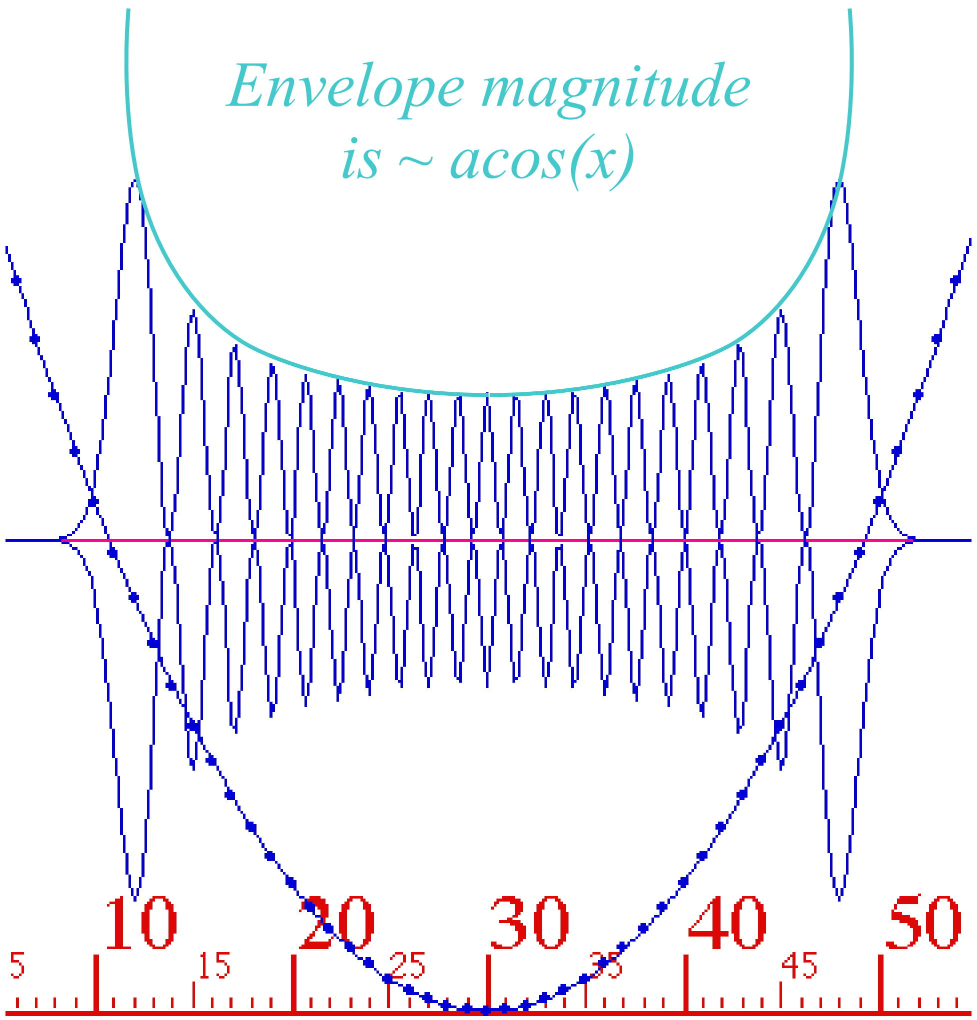

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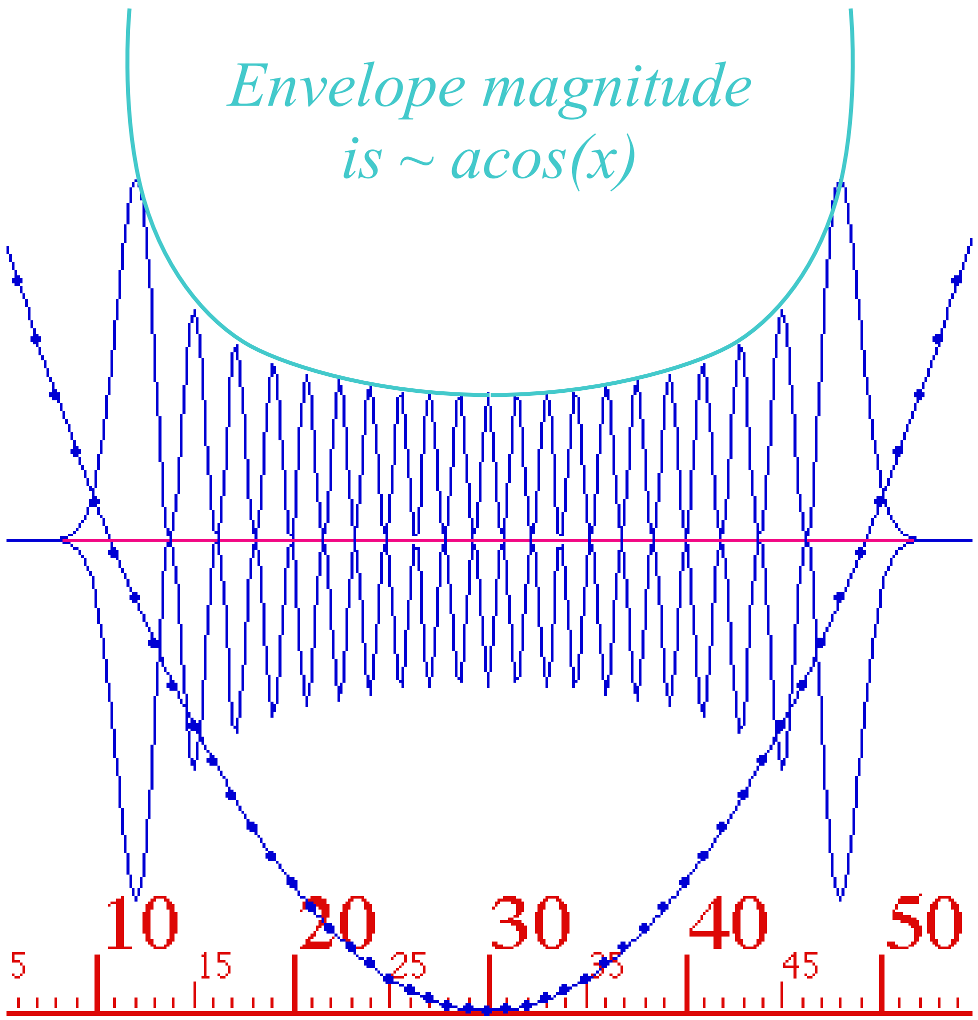

n=20 wave is still a long way from a classical energy value of *1 Joule*.
For a *1 Hz* oscillator, *1 Joule* would take a quantum number of roughly
$n = 100{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000 = 10^{35}$

1-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(1) representations

Creation-Destruction $\mathbf{a}^\dagger\mathbf{a}$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

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Matrix $\langle \mathbf{a}^n \mathbf{a}^{\dagger n} \rangle$ calculations

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Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

➡ *Harmonic oscillator beat dynamics of mixed states* ⬅

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

Applying boost-translation combinations

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Harmonic oscillator beat dynamics of mixed states

$$|\Psi\rangle = |0\rangle\langle 0|\Psi\rangle + |1\rangle\langle 1|\Psi\rangle = |0\rangle\Psi_0 + |1\rangle\Psi_1$$

$$\Psi(x) = \langle x|\Psi\rangle = \langle x|0\rangle\langle 0|\Psi\rangle + \langle x|1\rangle\langle 1|\Psi\rangle = \psi_0(x)\,\Psi 0 + \psi_1(x)\,\Psi 1$$

The time dependence $\Psi(x,t)$ of the mixed wave is then

$$\Psi(x,t) = \psi_0(x)\,e^{-i\omega_0 t}\,\Psi_0 + \psi_1(x)\,e^{-i\omega_1 t}\,\Psi_1 = (\psi_0(x)\,e^{-i\omega_0 t} + \psi_1(x)\,e^{-i\omega_1 t})/\sqrt{2}$$

$$\left|\Psi(x,t)\right| = \sqrt{\Psi^*\Psi} = \sqrt{\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)^*\left(e^{-i\omega_0 t}\psi_0(x) + e^{-i\omega_1 t}\psi_1(x)\right)/2}$$

$$= \sqrt{\left(\left|\psi_0(x)\right|^2 + \left|\psi_1(x)\right|^2 + \psi_0(x)\psi_1(x)\left(e^{i(\omega_1-\omega_0)t} + e^{-i(\omega_1-\omega_0)t}\right)\right)/2}$$

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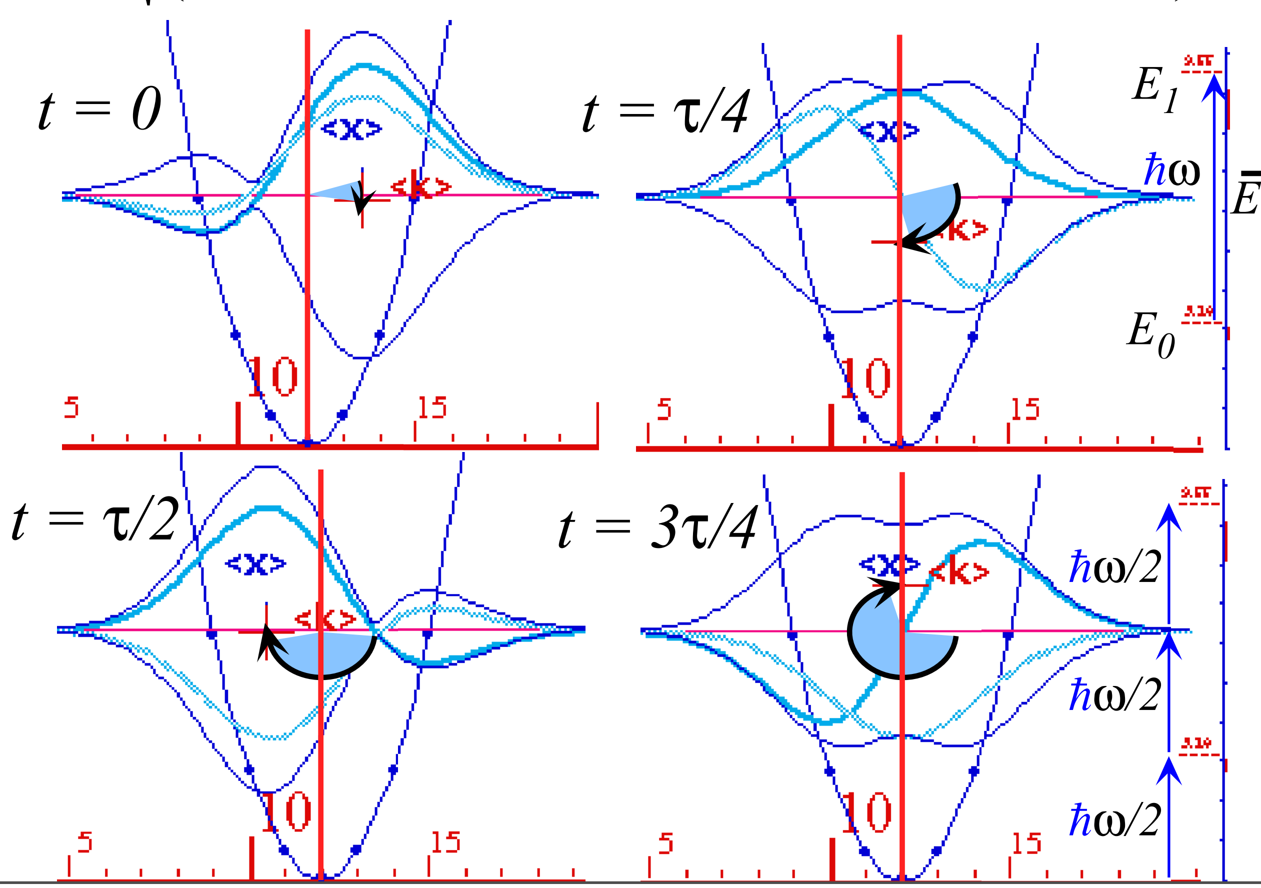

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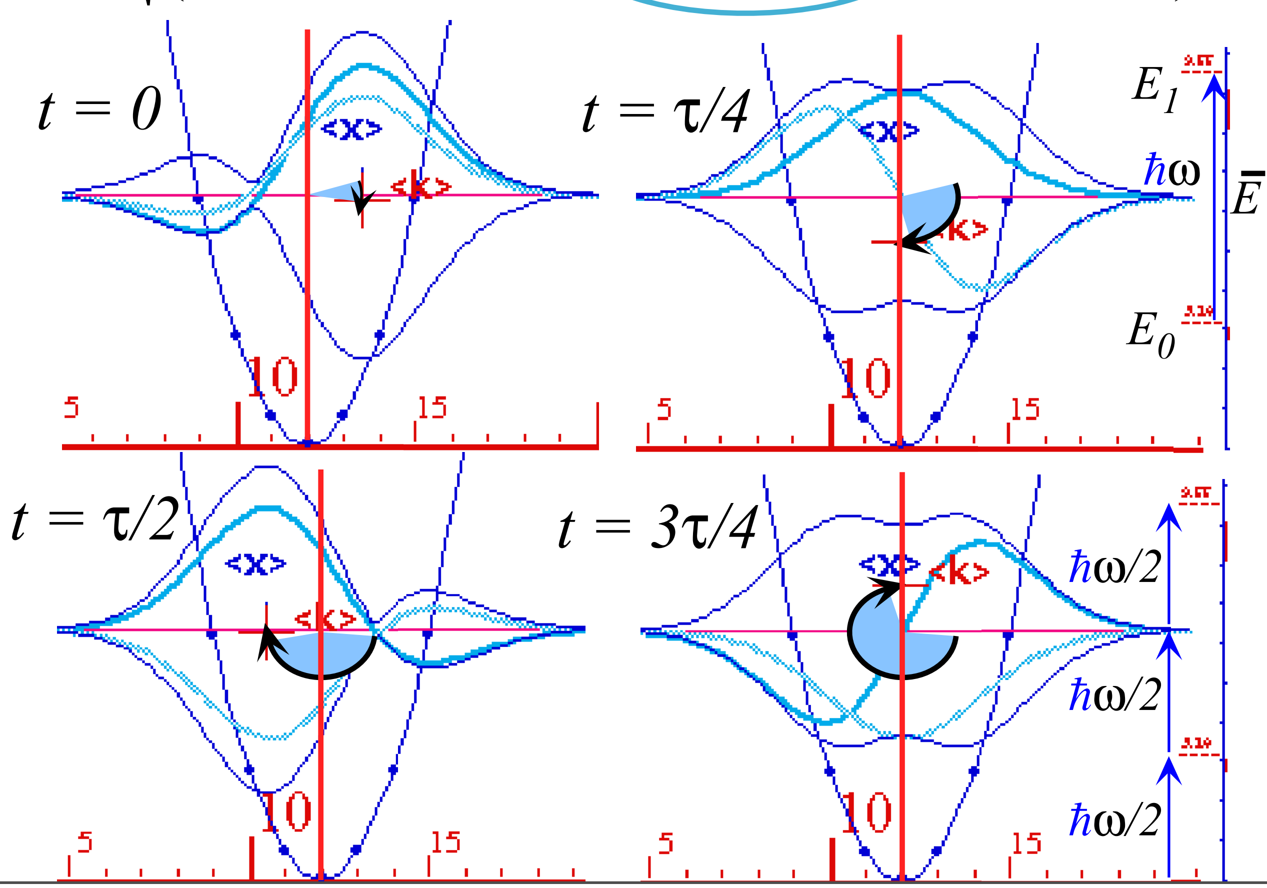
Need some *overlap* somewhere to get some *wiggle*



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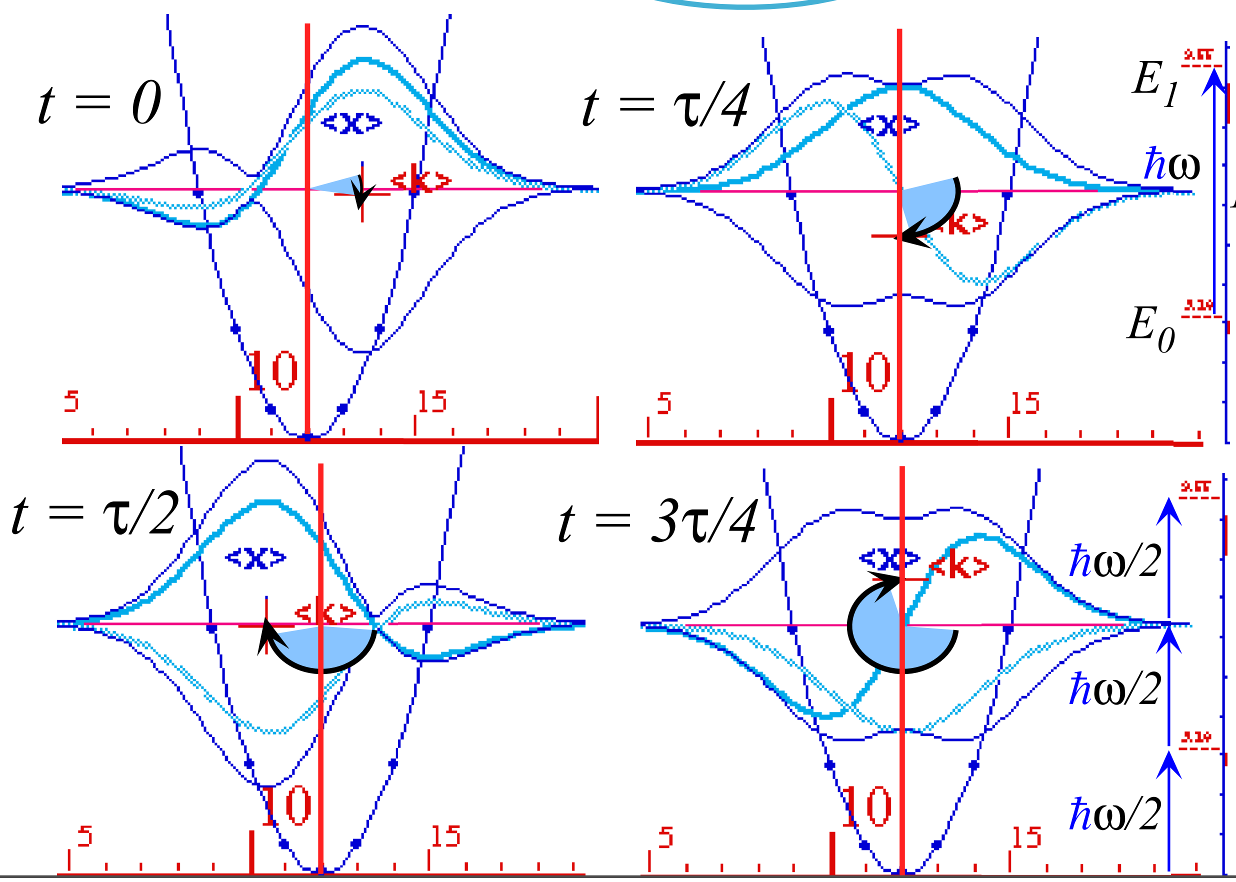

Beat frequency is eigenfrequency difference

$$\omega_{beat} = \omega_1 - \omega_0 = \omega$$

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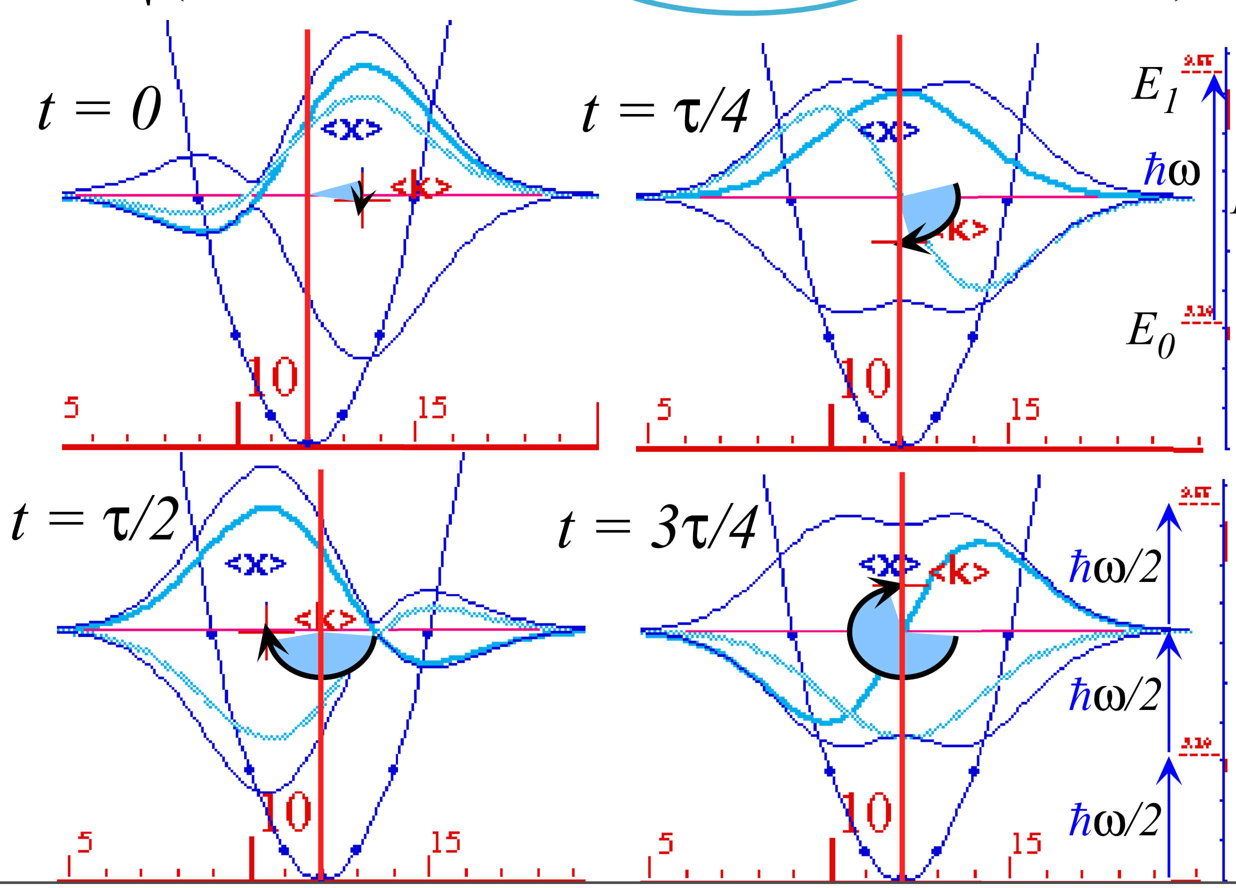

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Beat frequency ω = *Transition frequency* ω

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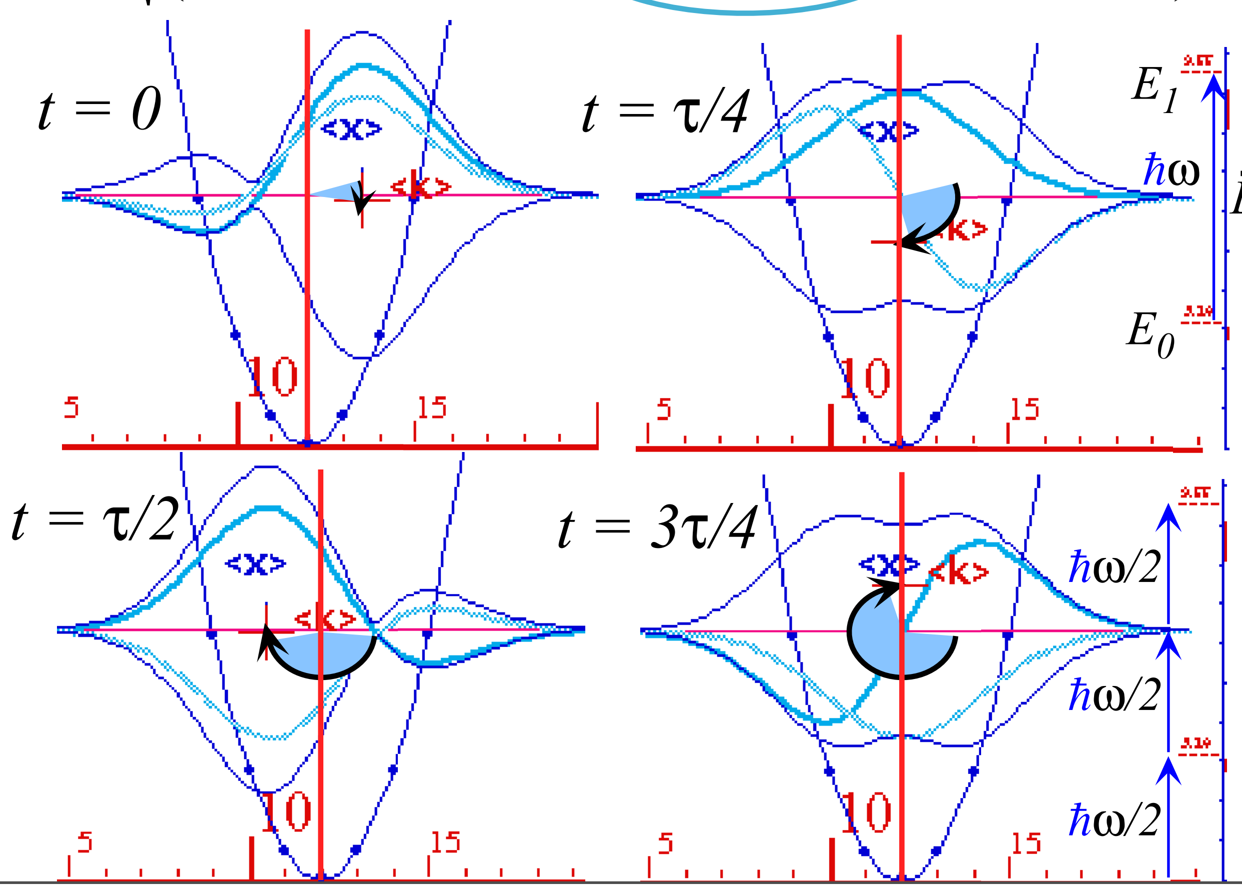

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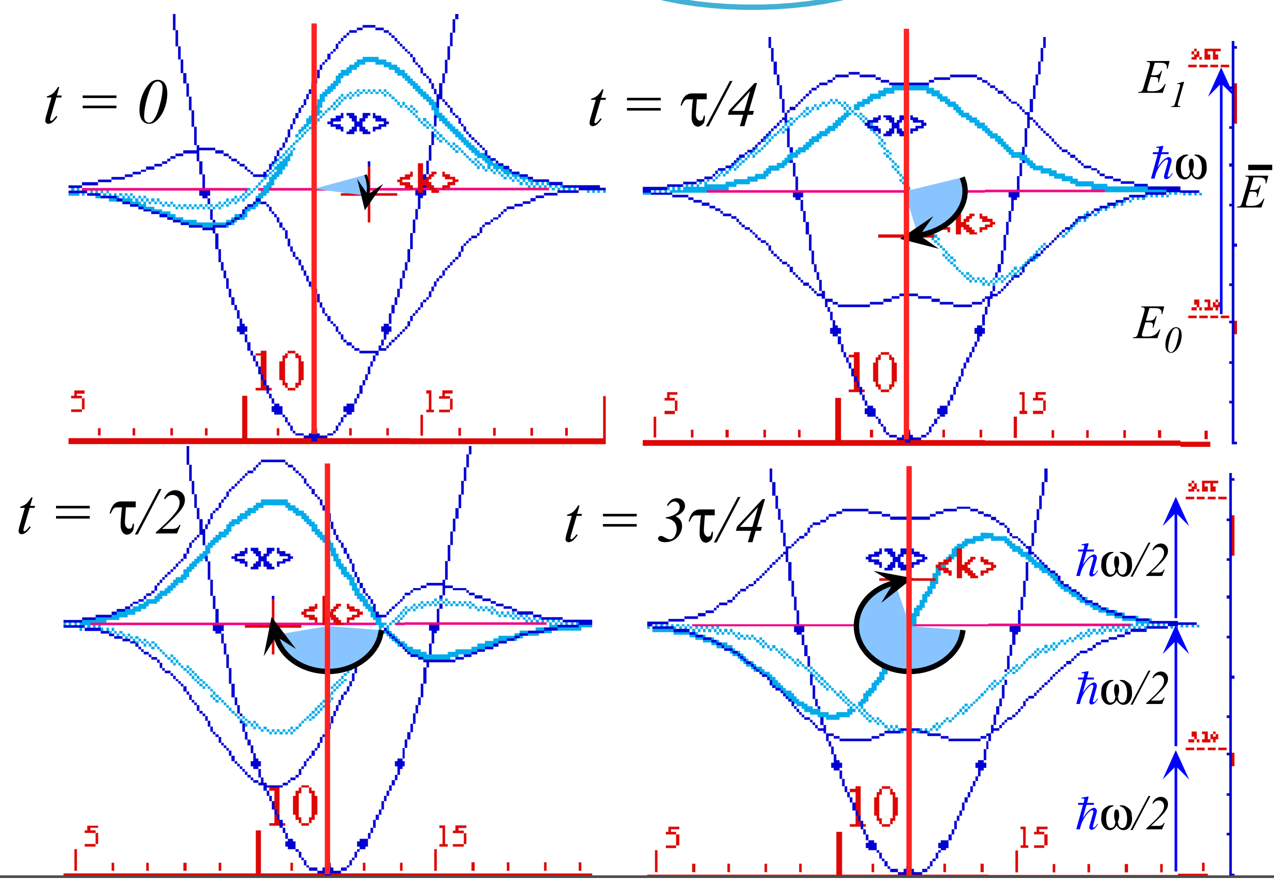

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$\Delta E = E_{1\leftarrow 0}$ *transition* $= E_1 - E_0 = \hbar\omega$

ω is frequency of radiating antenna
of a transmitter or of a receiver, i.e.,
of an emitter or an absorber
(Usually of a dipole symmetry)

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Translation operators vs. boost operators

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Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators and generators: (A "shove")

Translation operator $\mathbf{T}(a)$ shoves x-wavefunctions

$$\mathbf{T}(a)\cdot\psi(x) = \psi(x-a) = \langle x|\mathbf{T}(a)|\psi\rangle = \langle x-a\,|\psi\rangle$$

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Increases momentum of ket-state by b units

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Check $\mathbf{T}(a)$ on plane-wave with $p=\hbar k$

$$\mathbf{T}(a)e^{ikx} = e^{-ia\mathbf{p}/\hbar}e^{ikx} = e^{-iak}e^{ikx} = e^{ik(x-a)}$$

Bottom Line

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1-D $a^\dagger a$ algebra of U(1) representations

Creation-Destruction $a^\dagger a$ algebra

Eigenstate creationism (and destruction)

Vacuum state

1st excited state

Normal ordering for matrix calculation

Commutator derivative identities

Binomial expansion identities

Matrix $\langle a^n a^{\dagger n} \rangle$ calculations

Number operator and Hamiltonian operator

Expectation values of position, momentum, and uncertainty for eigenstate $|n\rangle$

Harmonic oscillator beat dynamics of mixed states

Oscillator coherent states ("Shoved" and "kicked" states)

Translation operators vs. boost operators

➡ *Applying boost-translation combinations* ⬅

Time evolution of coherent state

Properties of coherent state and "squeezed" states

2-D $a^\dagger a$ algebra of U(2) representations and R(3) angular momentum operators

Applying boost-translation combinations

T*(a)* and **B***(b)* operations do not commute. Q. Which should come first? ??

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Applying boost-translation combinations

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→ *Time evolution of coherent state* ←

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Time evolution operator for constant **H** has general form : $\mathbf{U}(t,0)=\mathrm{e}^{-i\mathbf{H}t/\hbar}$

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$$\mathbf{U}(t,0)\left|n\right\rangle = e^{-i\mathbf{H}t/\hbar}\left|n\right\rangle = e^{-i(n+1/2)\omega t}\left|n\right\rangle$$

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Evolution simplifies to a variable-α_0 coherent state with a *time dependent phasor coordinate* α_t:

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$$\left[x_t + i\frac{p_t}{M\omega}\right] = e^{-i\omega t}\left[x_0 + i\frac{p_0}{M\omega}\right]$$

(x_t, p_t) mimics classical oscillator

$$x_t = x_0\cos\omega t + \frac{p_0}{M\omega}\sin\omega t$$

$$\frac{p_t}{M\omega} = -x_0\sin\omega t + \frac{p_0}{M\omega}\cos\omega t$$

Real and imaginary parts (x_t and $p_t/M\omega$) of α_t go clockwise on phasor circle

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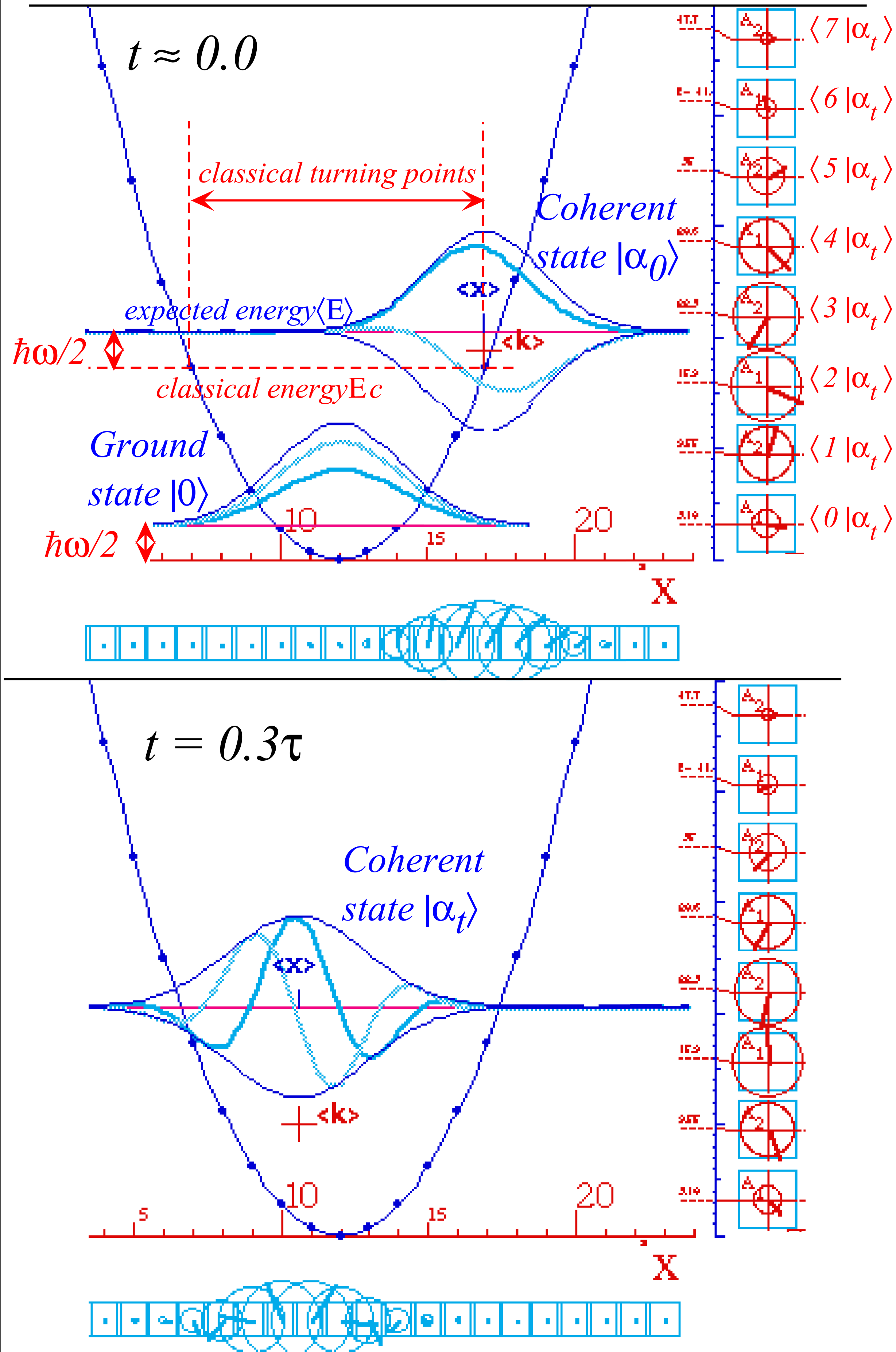
➡ *Properties of coherent state and "squeezed" states* ⬅

2-D $\mathbf{a}^\dagger\mathbf{a}$ algebra of U(2) representations and R(3) angular momentum operators

Properties of coherent state



Coherent ket $|\alpha(x0,p0)\rangle$ is eigenvector of destruct-op. **a**.

$$\mathbf{a}\left|\alpha_0(x_0,p_0)\right\rangle = e^{-|\alpha_0|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha_0)^n}{\sqrt{n!}}\mathbf{a}|n\rangle$$

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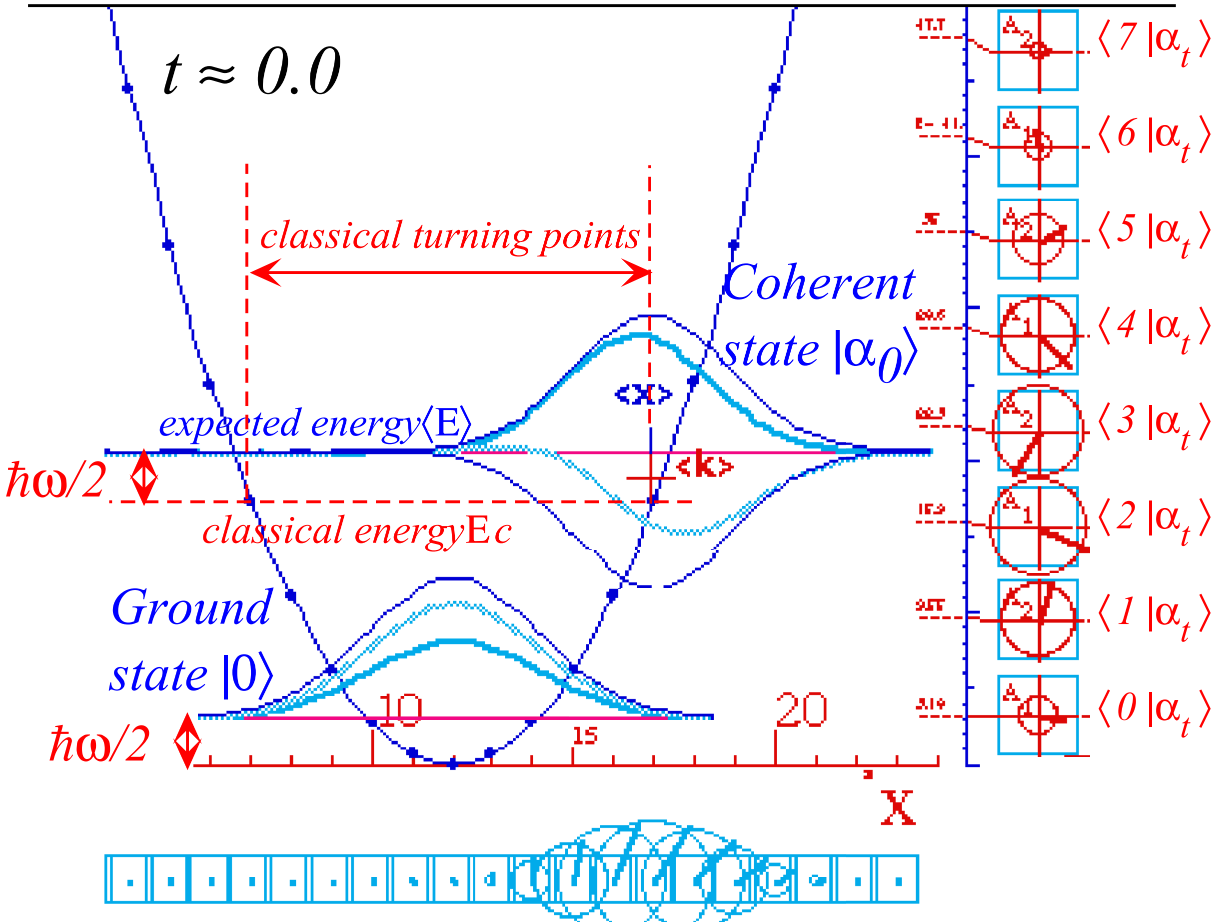

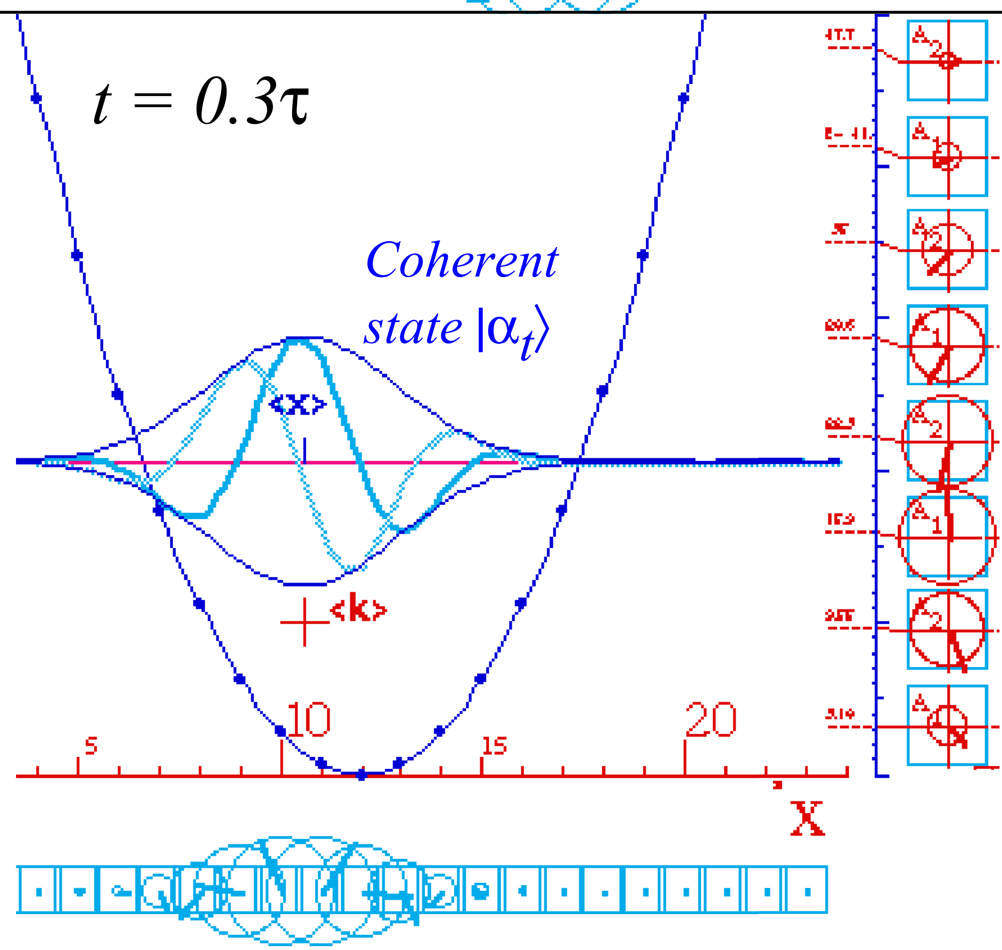

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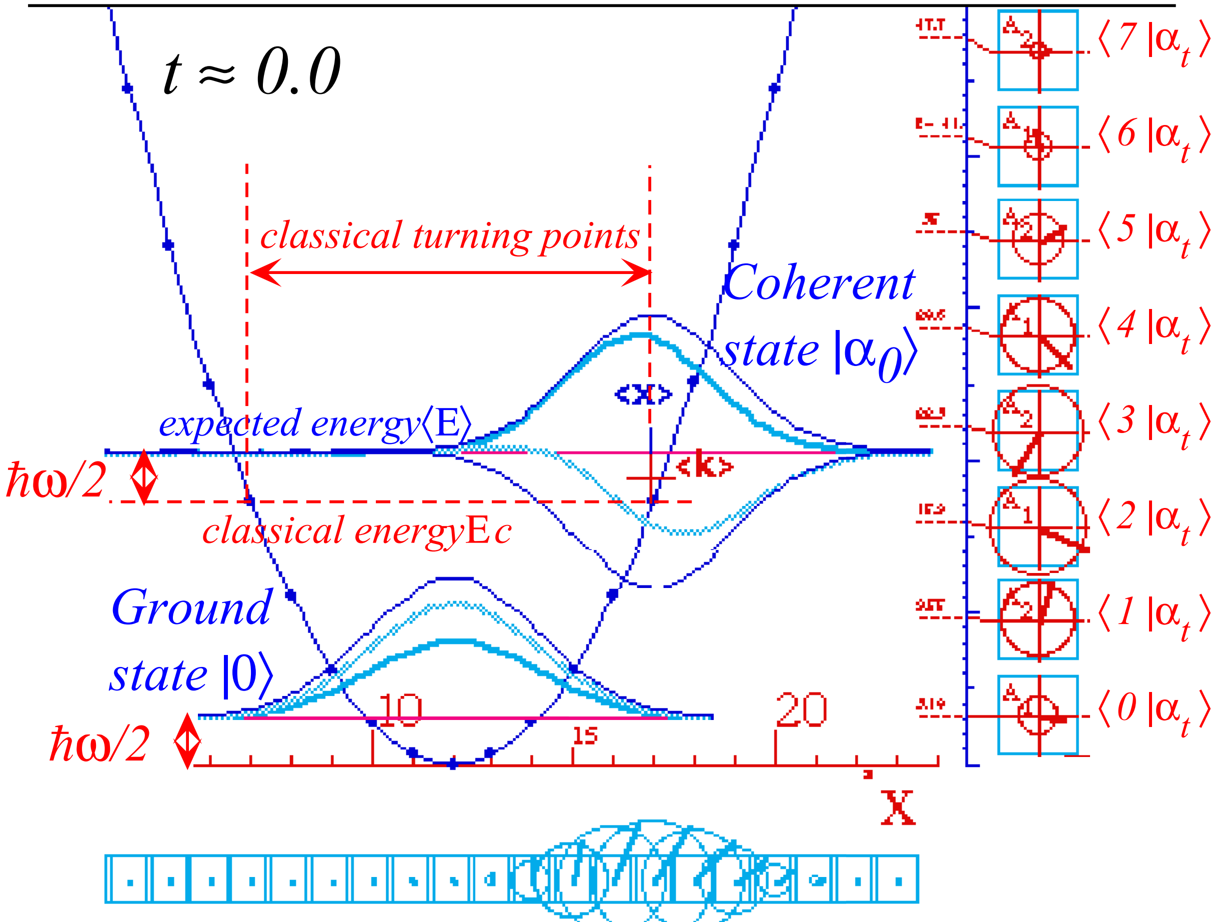

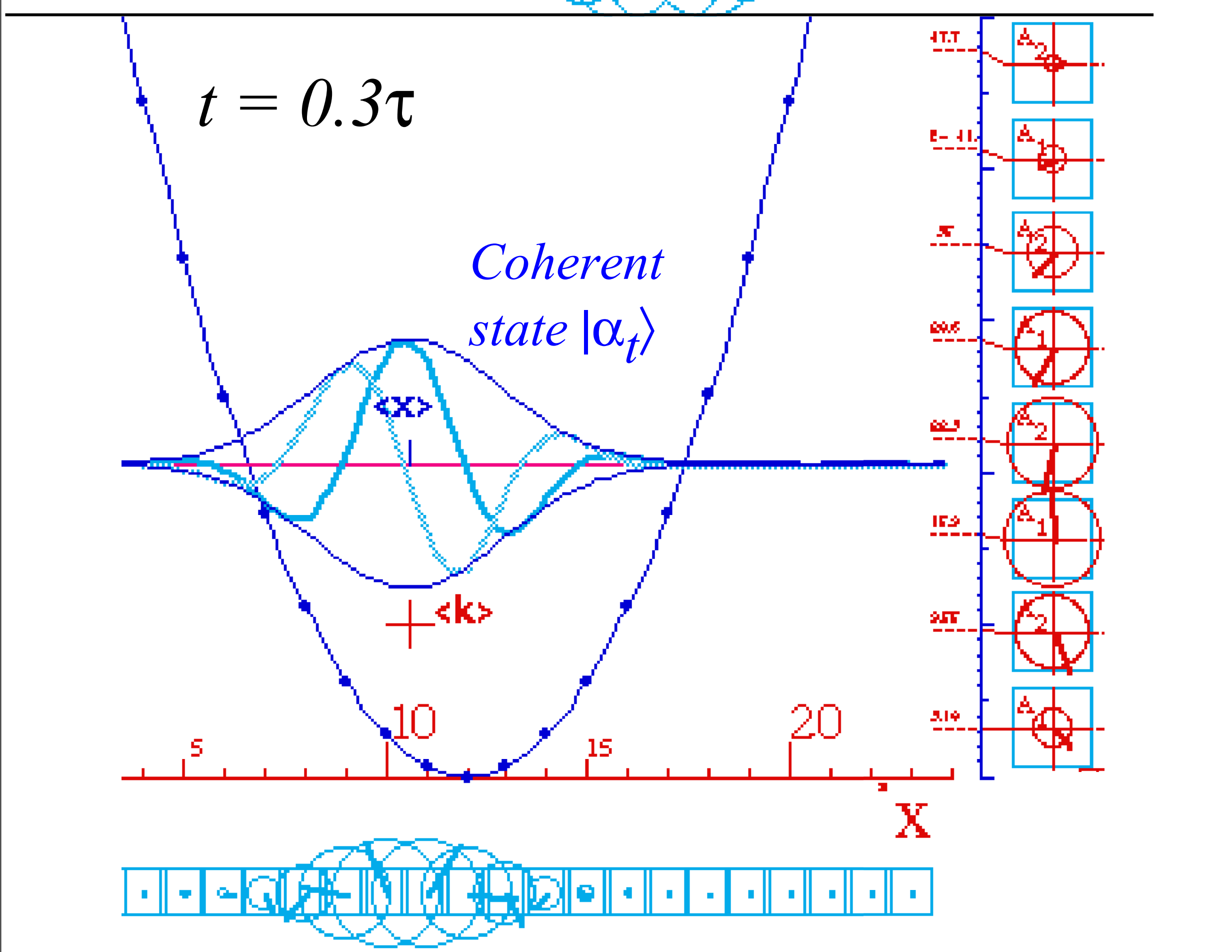

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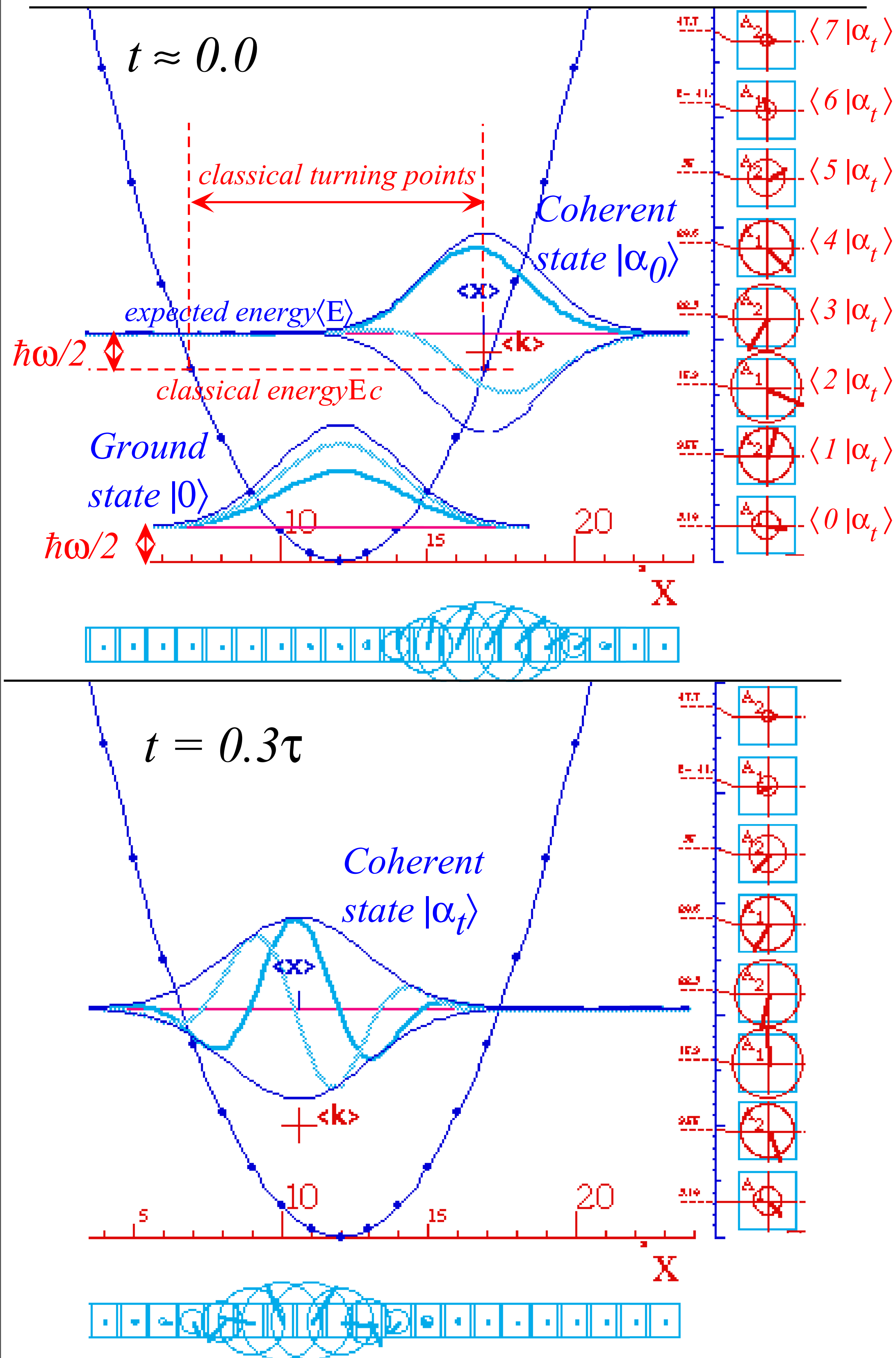

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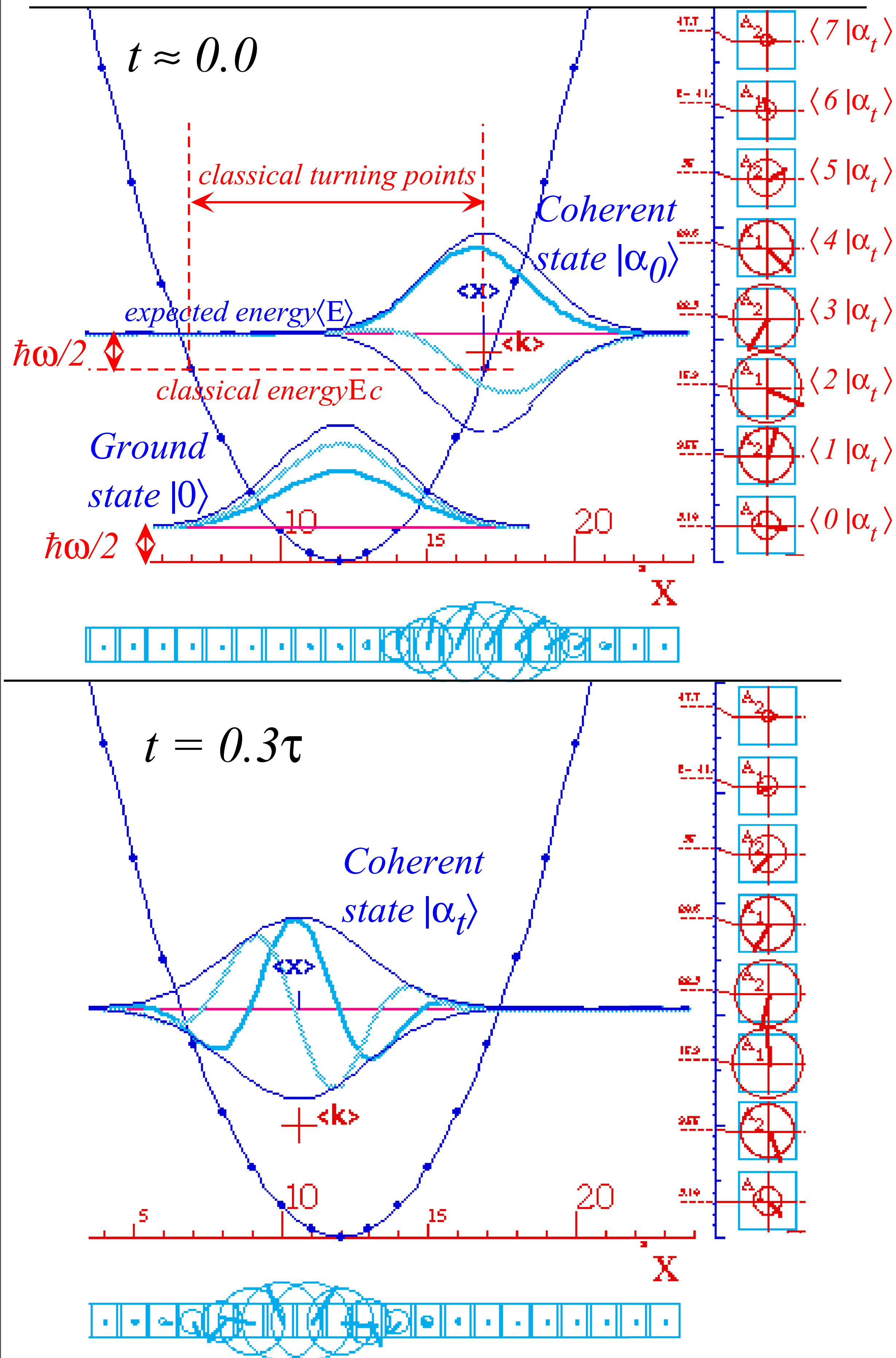

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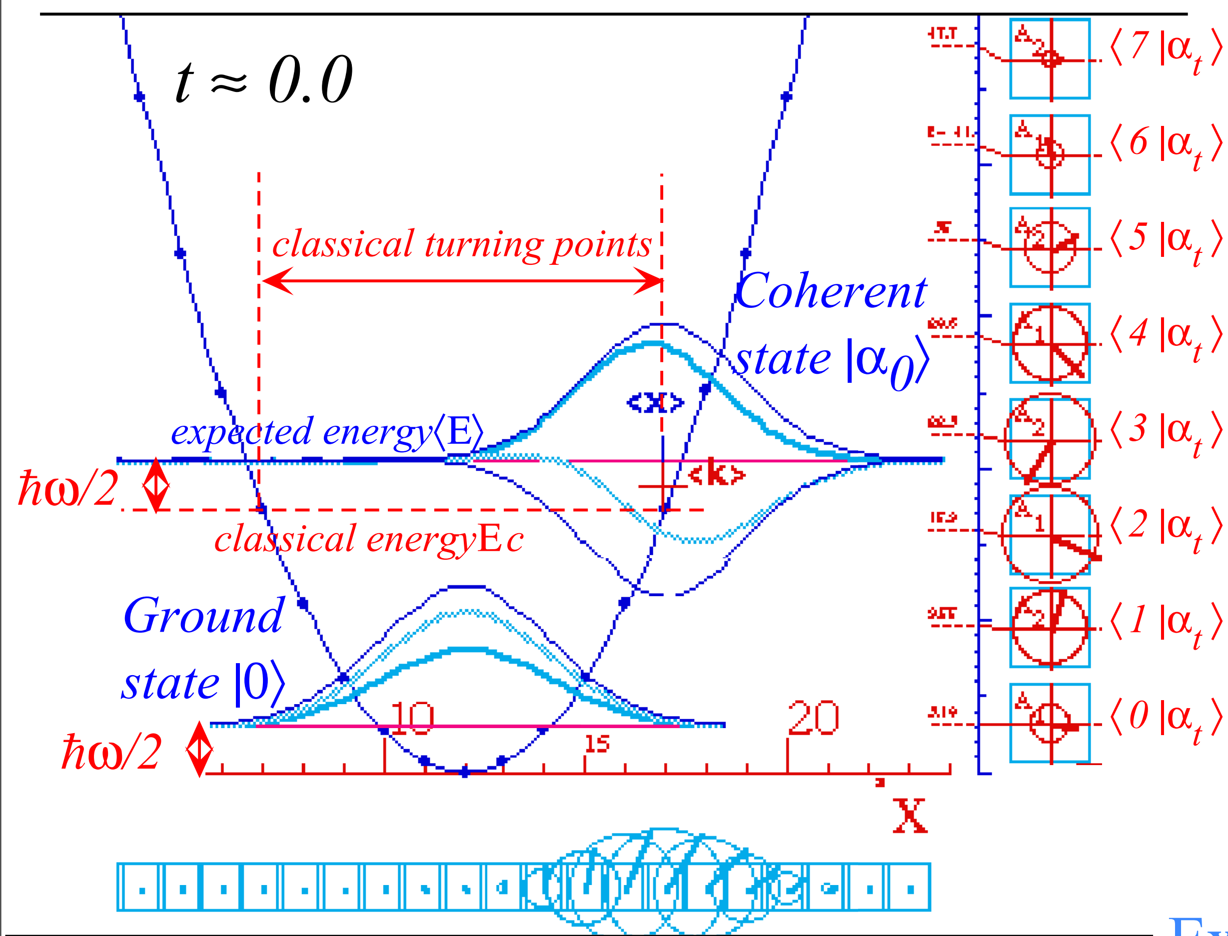
Properties of coherent state



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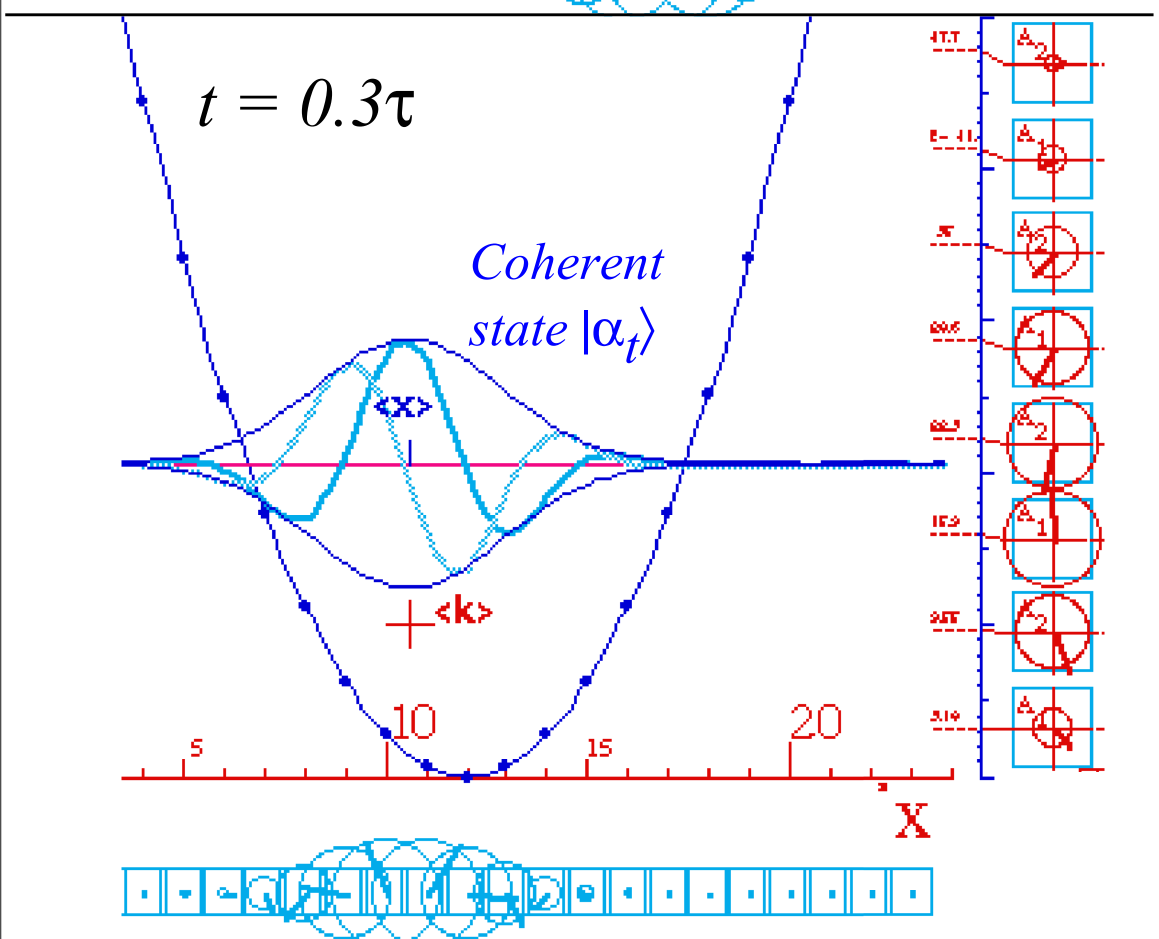


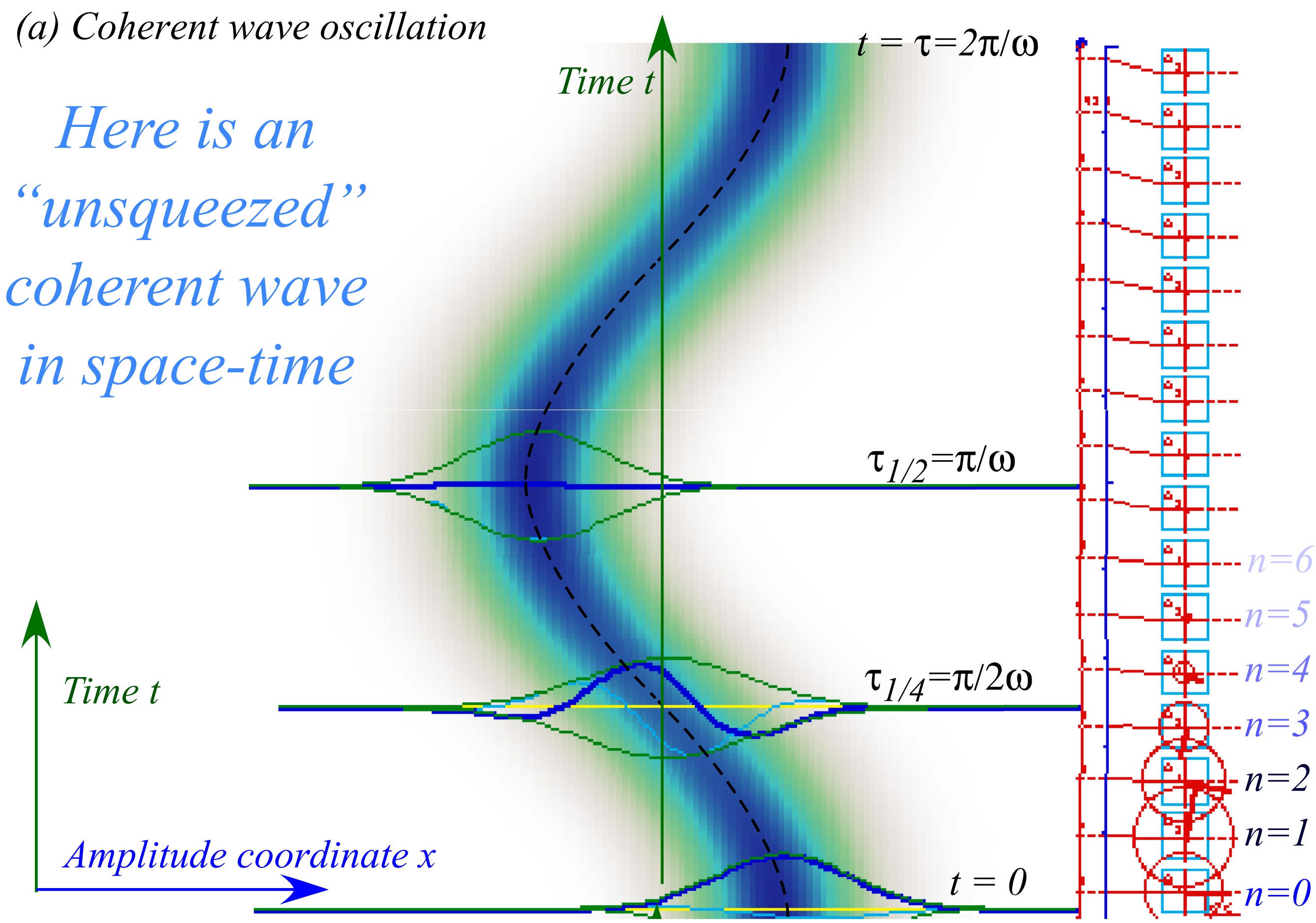
Expected quantum energy has simple time independent form.

$$\langle E\rangle\Big|_{\alpha_0} = \left\langle\alpha_0(x_0,p_0)\right|\mathbf{H}\left|\alpha_0(x_0,p_0)\right\rangle$$

$$= \left\langle\alpha_0(x_0,p_0)\right|\left(\hbar\omega\mathbf{a}^\dagger\mathbf{a} + \frac{\hbar\omega}{2}\mathbf{1}\right)\left|\alpha_0(x_0,p_0)\right\rangle$$

$$= \hbar\omega\alpha_0^*\alpha_0 + \frac{\hbar\omega}{2}$$

Properties of "squeezed" coherent states



Yeah! Cosine trajectory!

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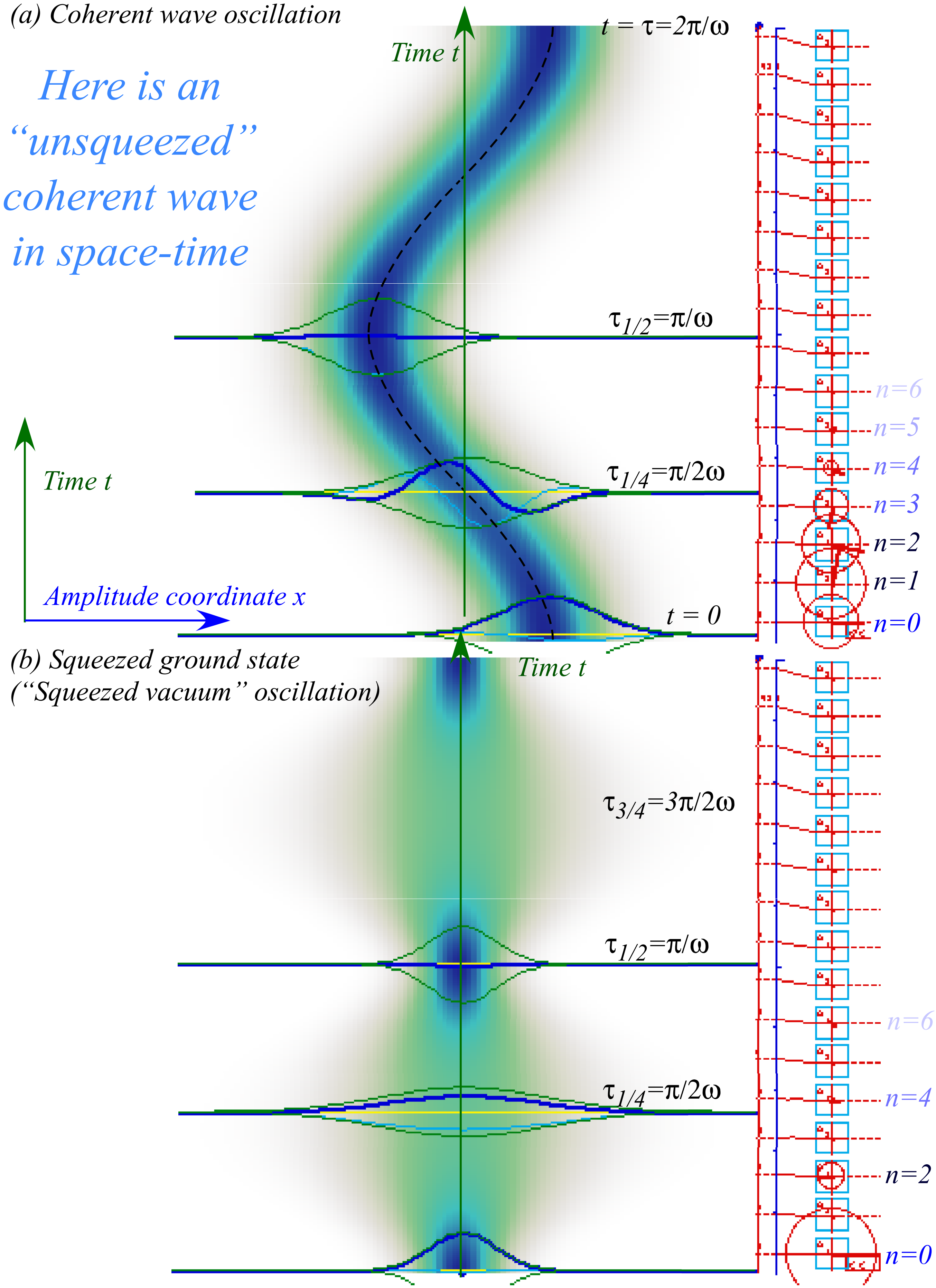

Here is an "unsqueezed" coherent wave in space-time

Yeah! Cosine trajectory!

what happens if you apply operators with non-linear "tensor" exponents exp(s$\mathbf{x}^2$), exp(f $\mathbf{p}^2$), etc.

Properties of "squeezed" coherent states