Group Theory in Quantum Mechanics

Lecture 19 (4.2.13)

Octahedral-tetrahedral $O \sim T_d$ symmetries

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)
(PSDS - Ch. 4)

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
- Octahedral-cubic O symmetry and group operations
- Tetrahedral symmetry becomes Icosahedral
- Octahedral groups $O_h \supset O \sim T_d \supset T$
  - Octahedral O and spin-$O \subset U(2)$

Tetrahedral T class algebra
- Tetrahedral T class minimal equations
- Tetrahedral T class projectors and characters

Octahedral O class algebra
- Octahedral O class minimal equations
- Octahedral O class projectors and characters

Octahedral $O_h \supset O$:Inversion (g&u) parity
- Octahedral $O_h \supset O \supset C_I$ subgroup correlations
- $O_h \supset O \supset D_4$ subgroup correlations
- $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
- Preview of applications to high resolution spectroscopy

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Three groups: O, D_4, *and* D_3 *let you "do" all the other 32 crystal point groups*

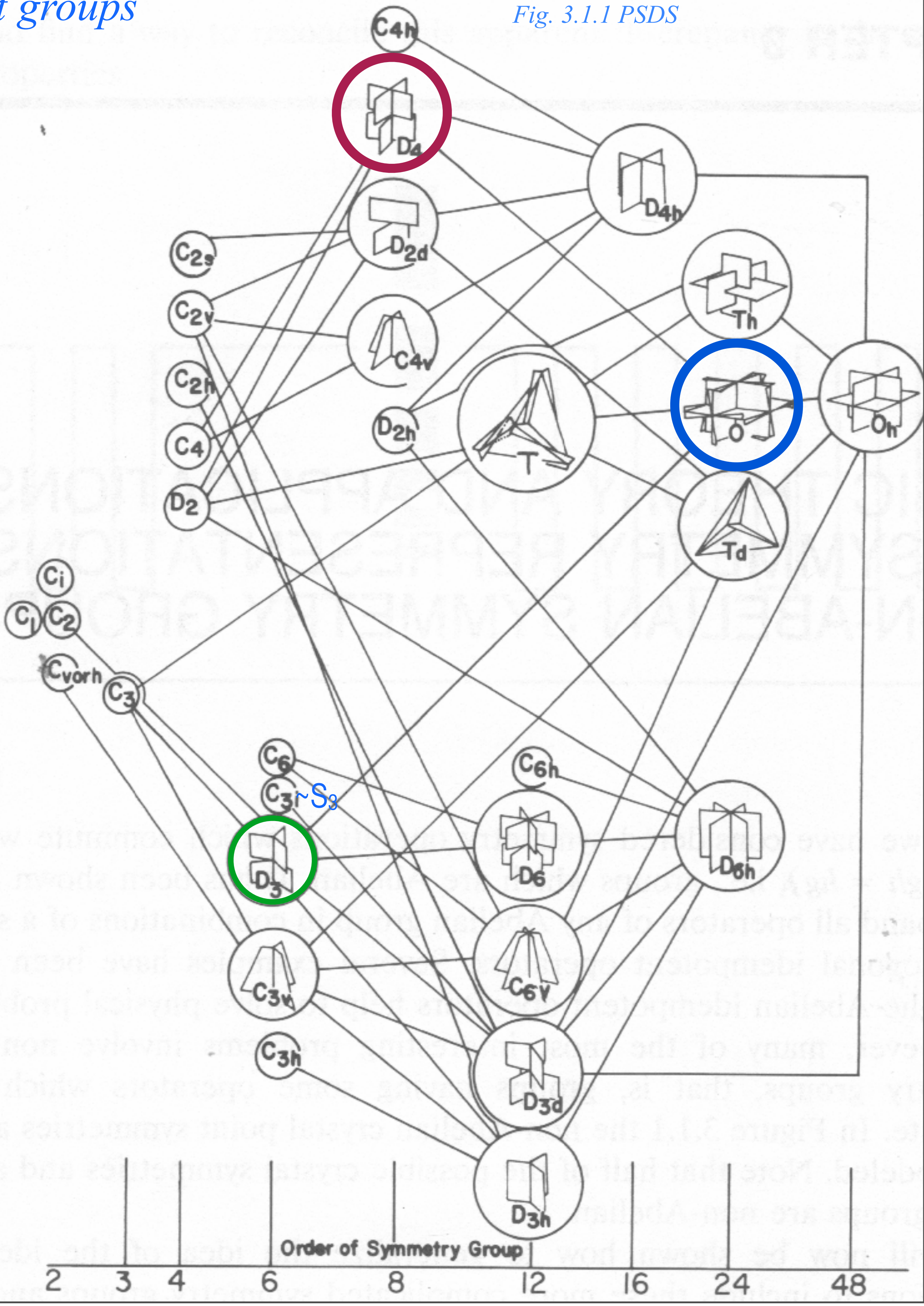
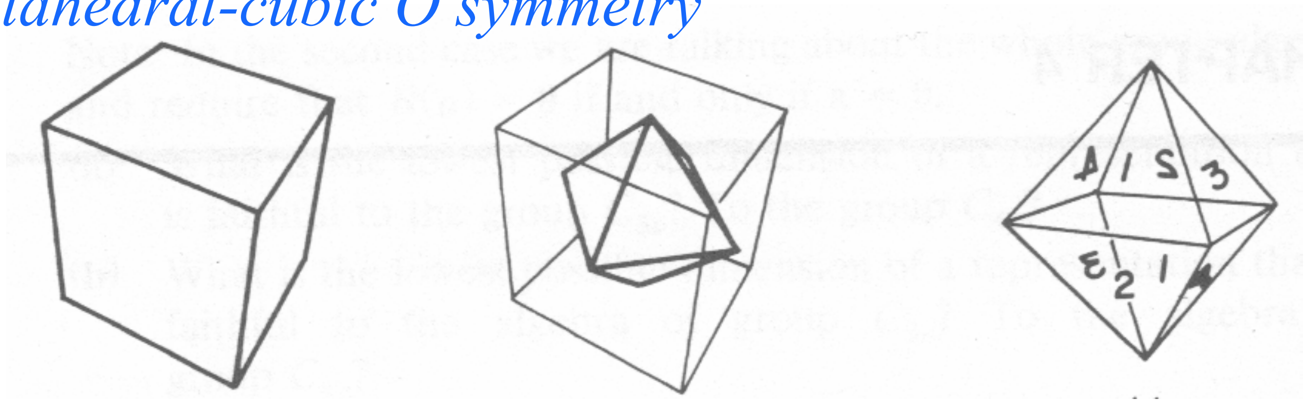


Fig. 3.1.1 PSDS

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry



Order °O=6 hexahedron squares · 4 pts =24
=8 octahedron triangles · 3 pts =24
=12 lines · 2 pts =24 positions

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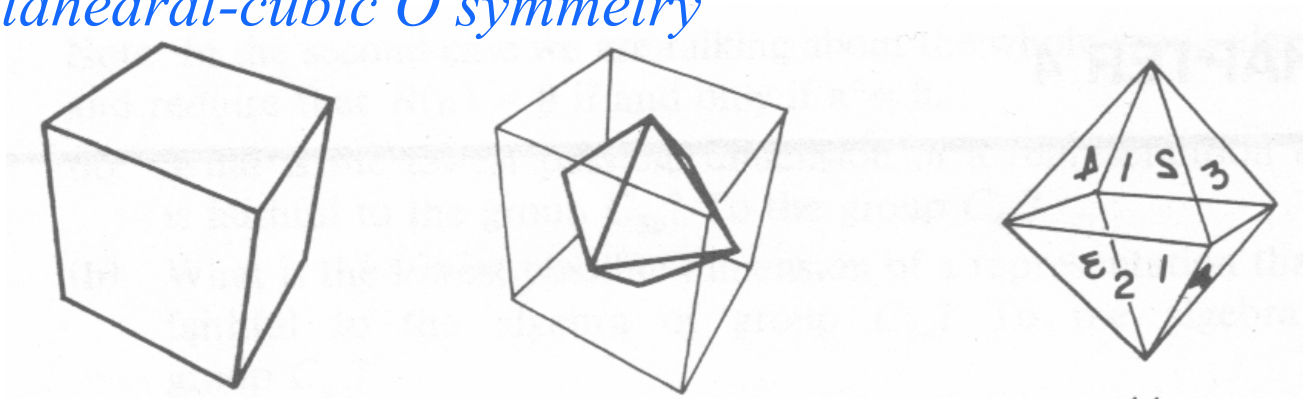

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Octahedral group O operations

Class of 1: **1**

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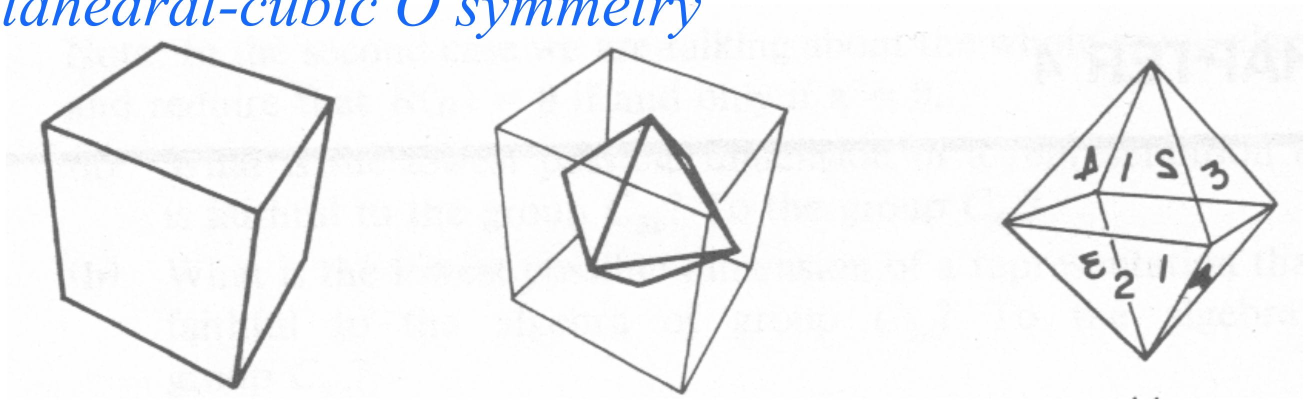
Octahedral-cubic O symmetry



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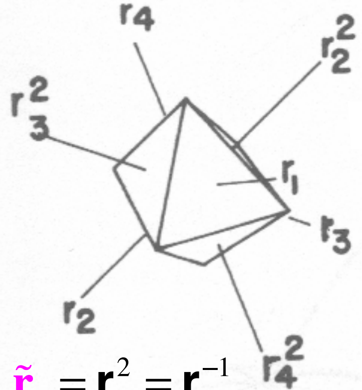
Octahedral group O operations

Class of 1: **1**

$\mathbf{r}_k=\mathbf{r}_k$

Class of 8:
120° rotations
on [111] axes



$\tilde{\mathbf{r}}_k=\mathbf{r}_k^2=\mathbf{r}_k^{-1}$

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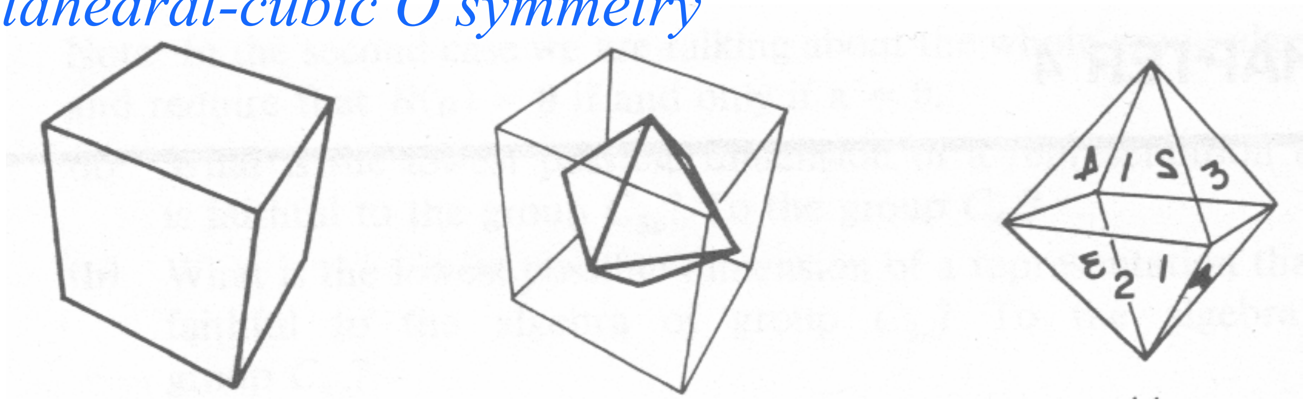
Octahedral-cubic O symmetry



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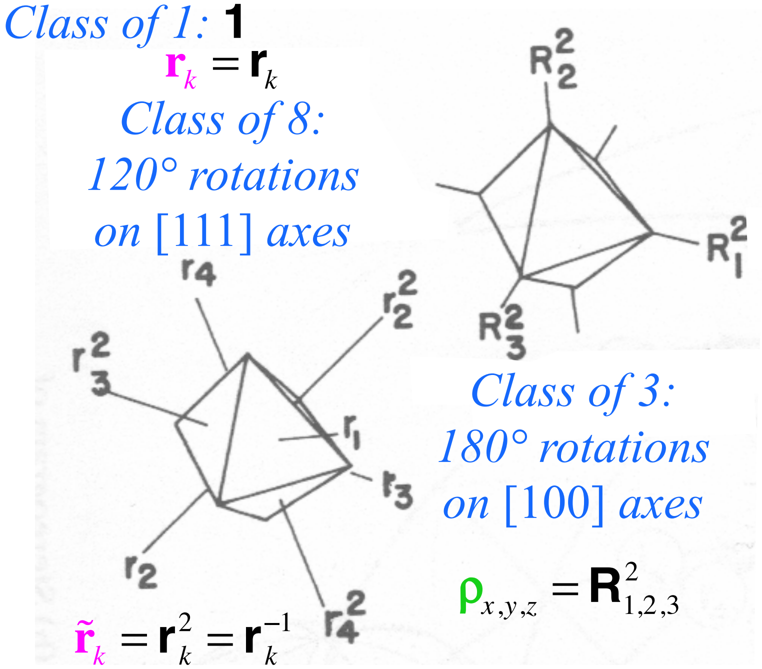
Octahedral group O operations



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180° rotations
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$\rho_{x,y,z} = \mathbf{R}^2_{1,2,3}$

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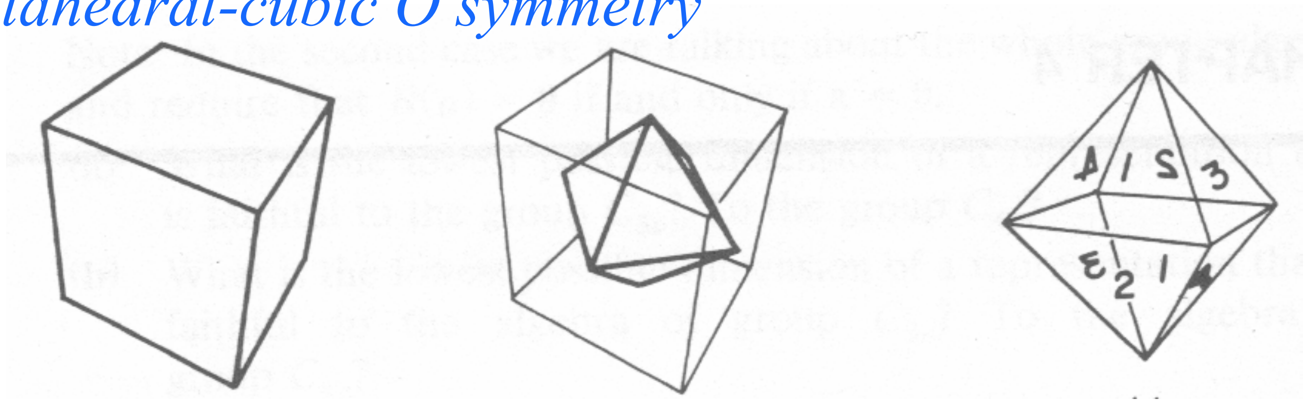

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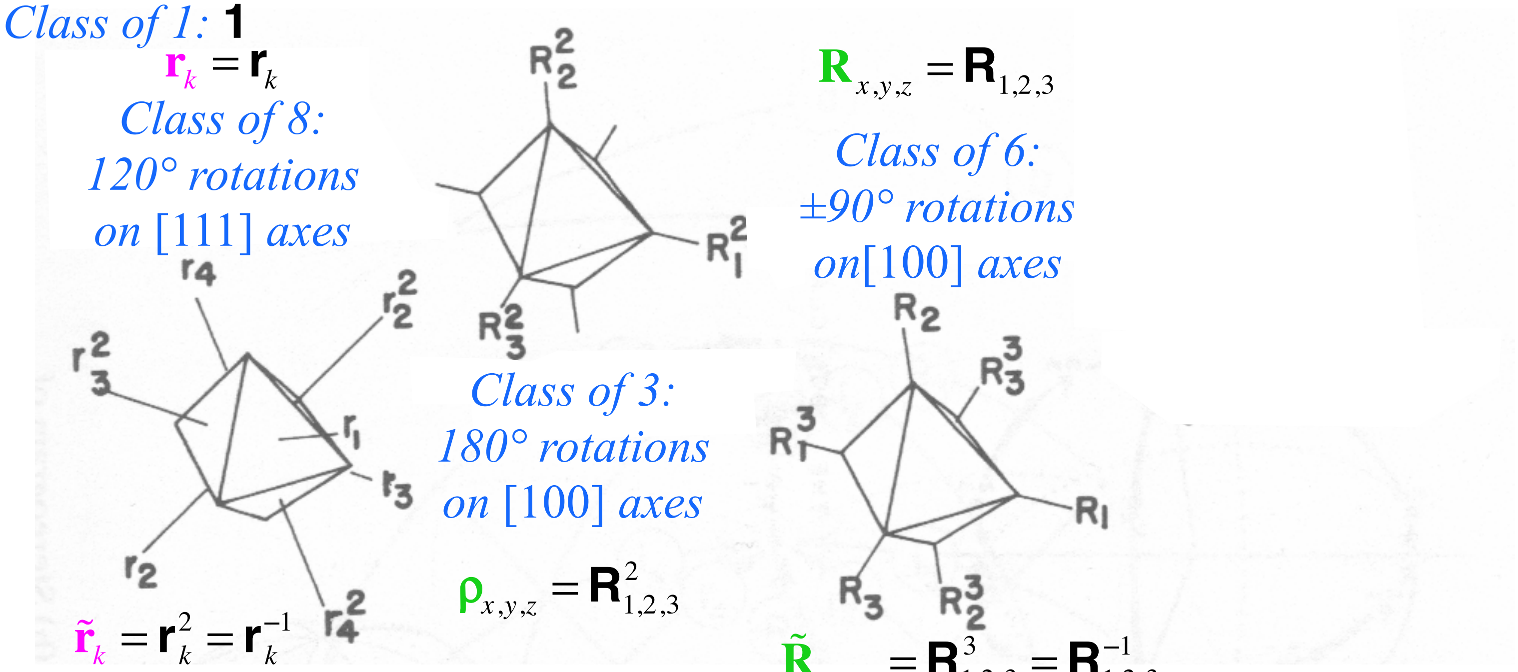

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Class of 6: ±90° rotations on[100] axes

$\tilde{\mathbf{R}}_{x,y,z} = \mathbf{R}^3_{1,2,3} = \mathbf{R}^{-1}_{1,2,3}$

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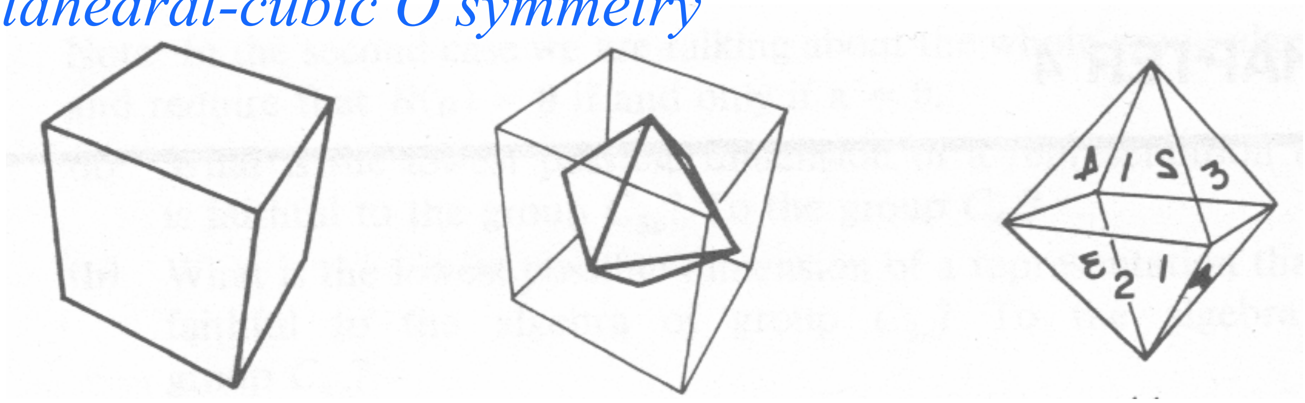

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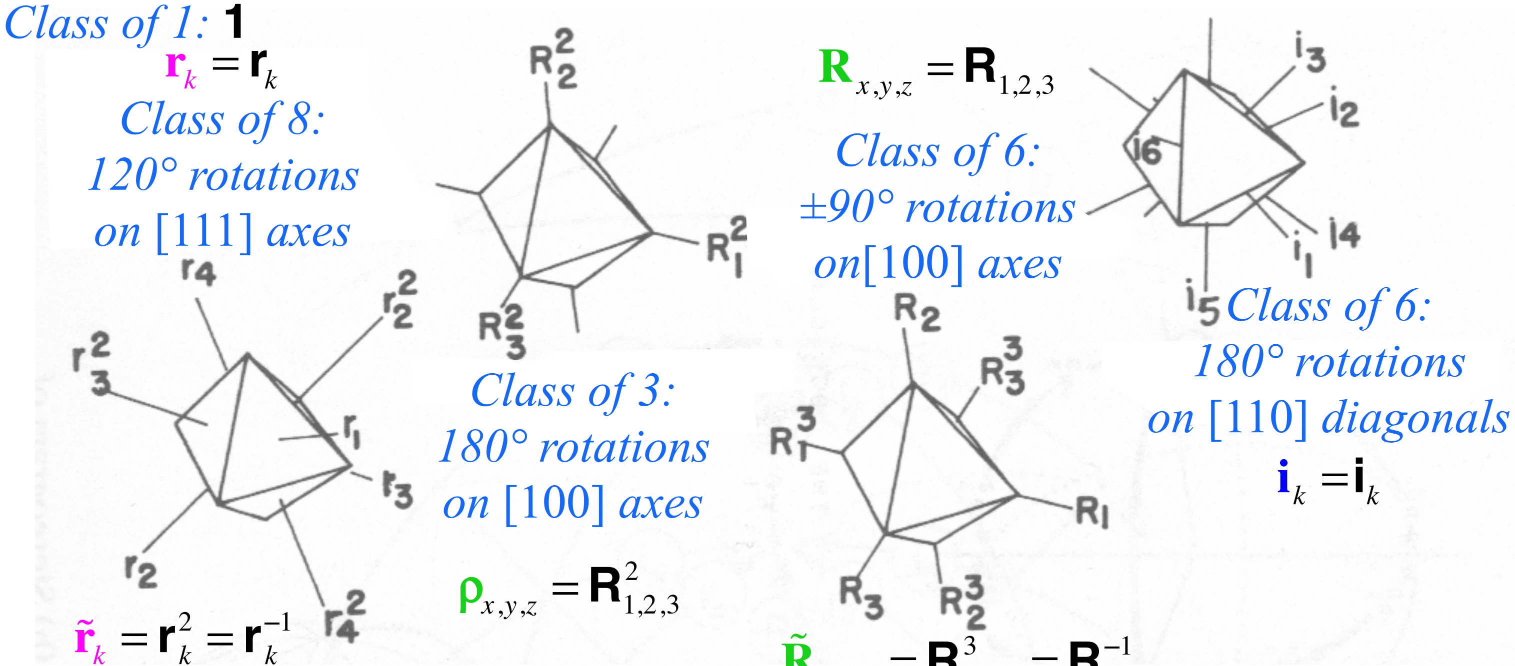

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Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

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- *Tetrahedral symmetry becomes Icosahedral*
- *Octahedral groups $O_h \supset O \sim T_d \supset T$*
  - *Octahedral O and spin-$O \subset U(2)$*

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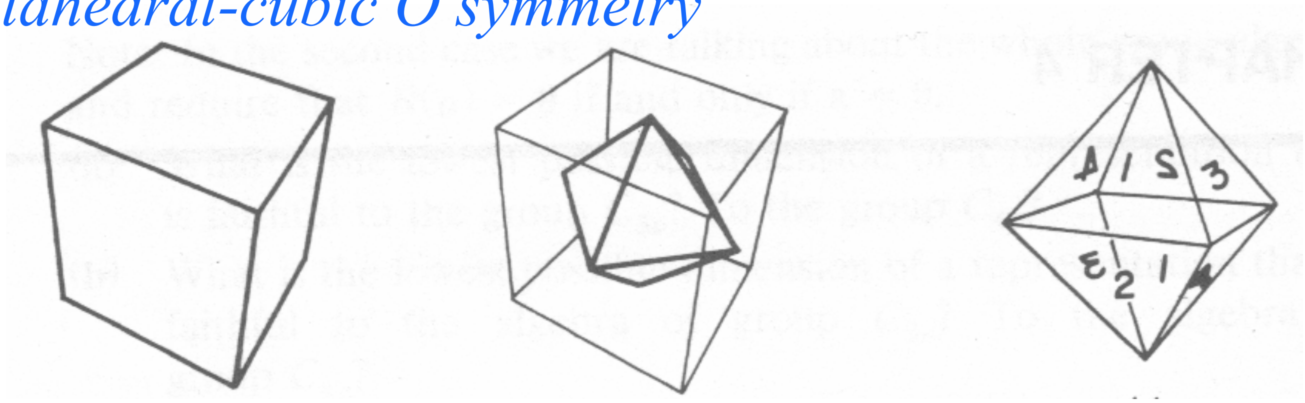
Octahedral-cubic O symmetry



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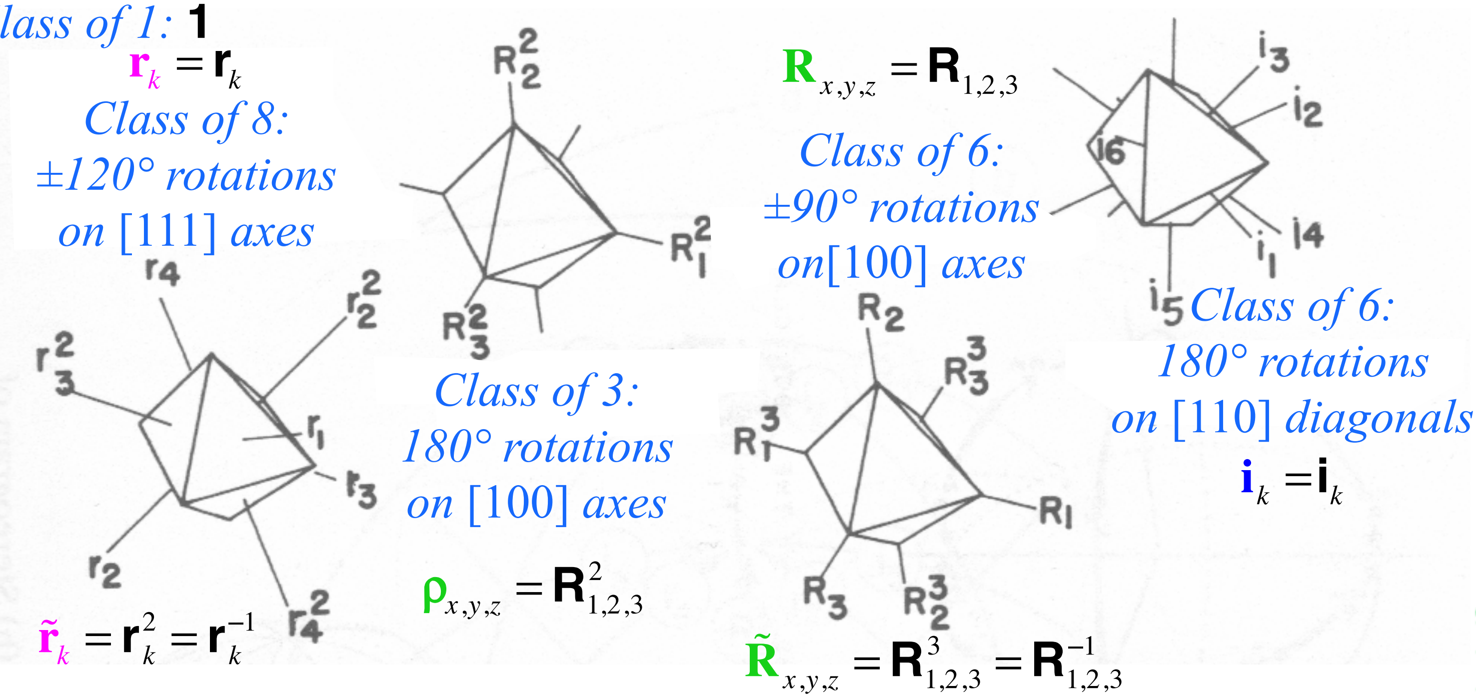

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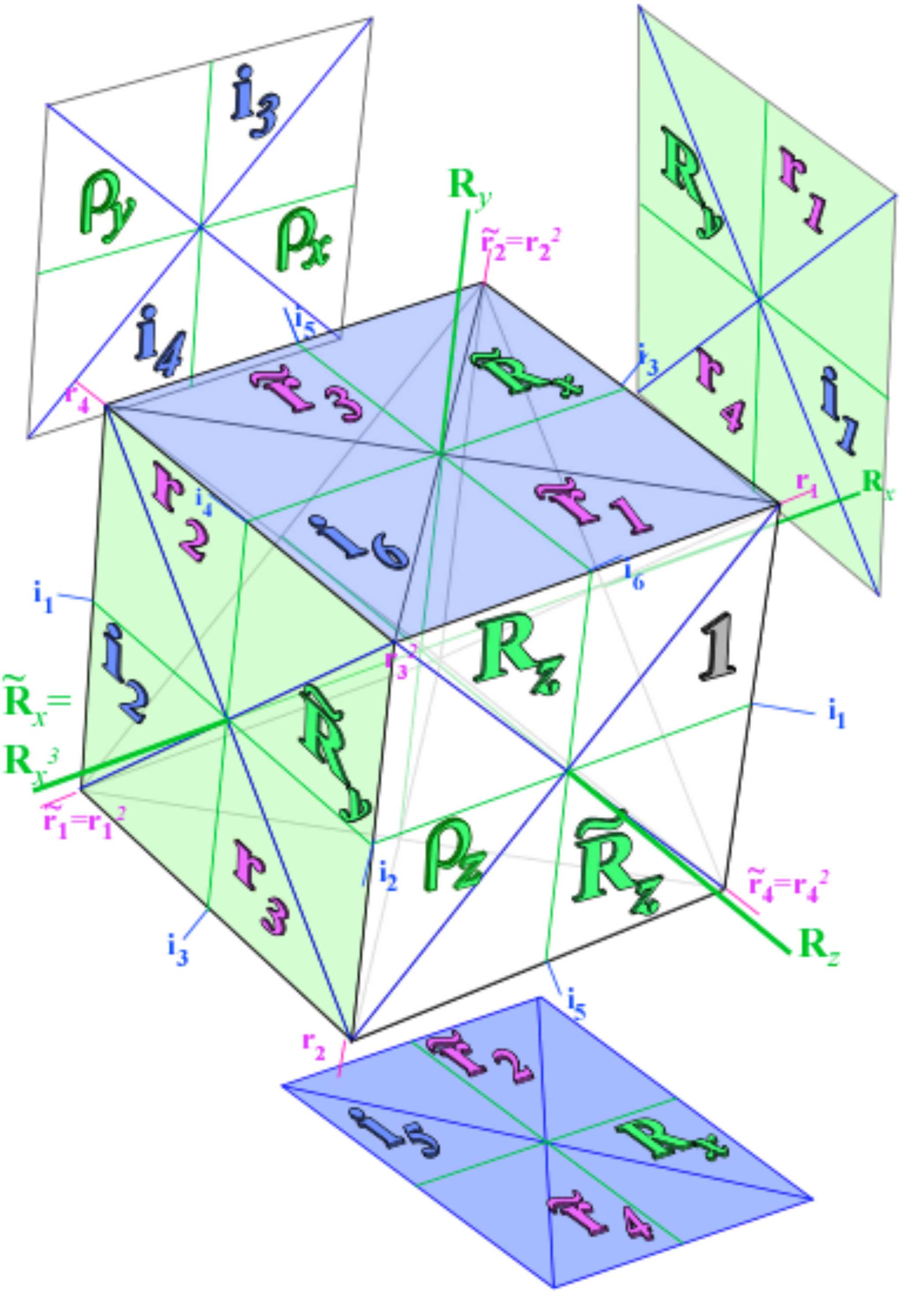

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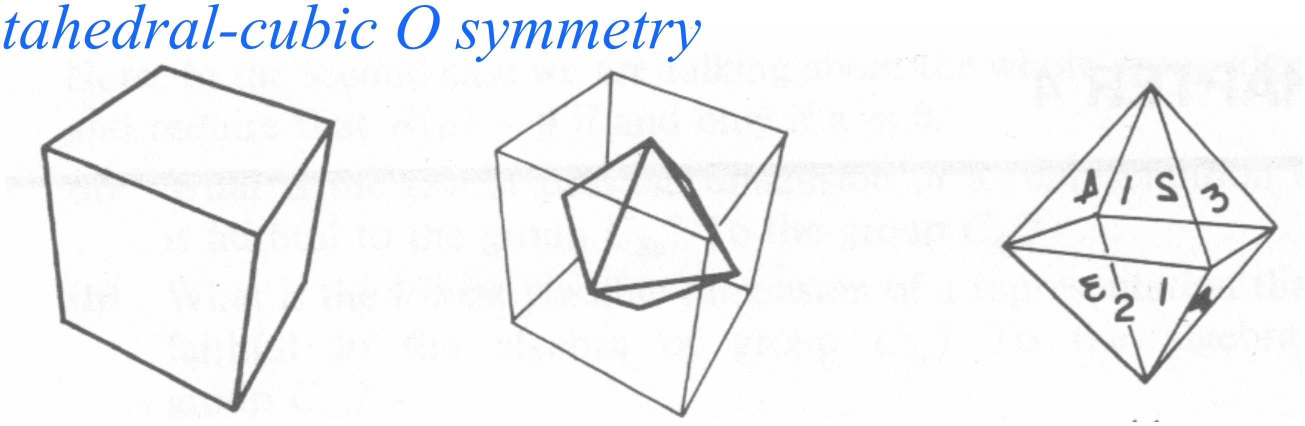

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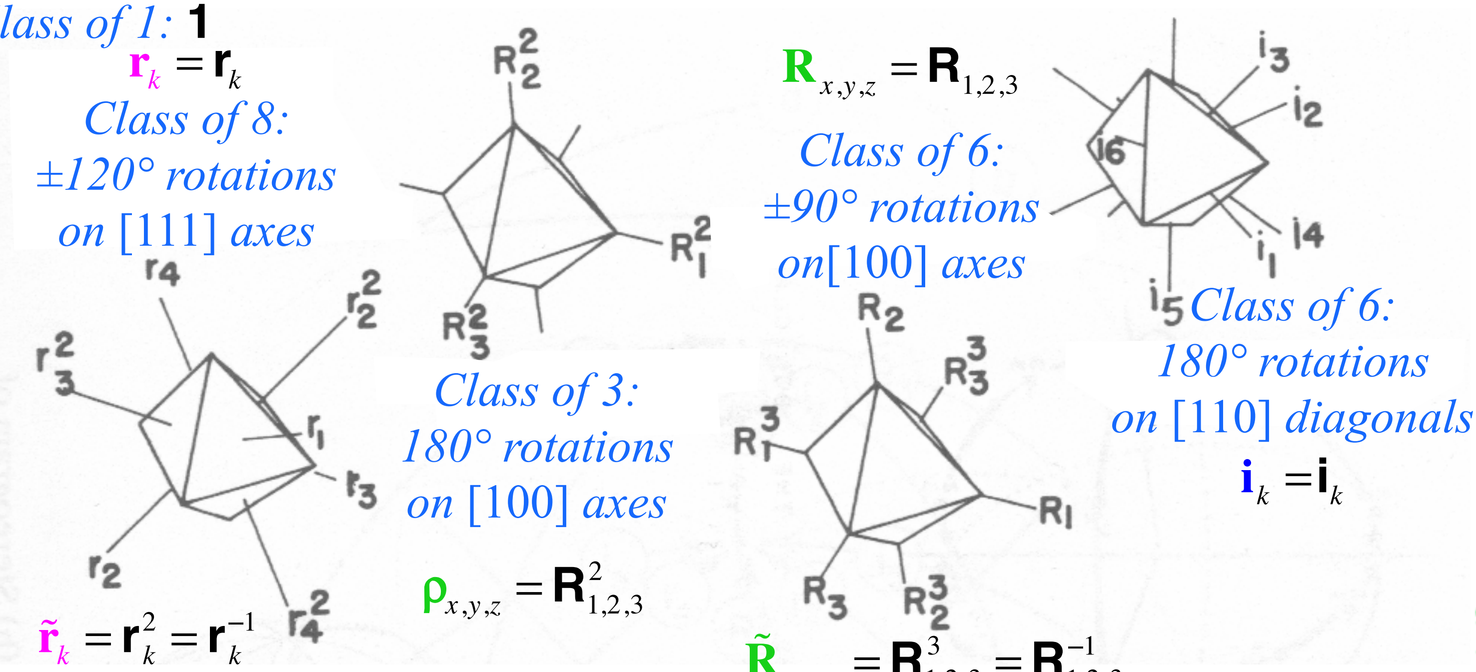

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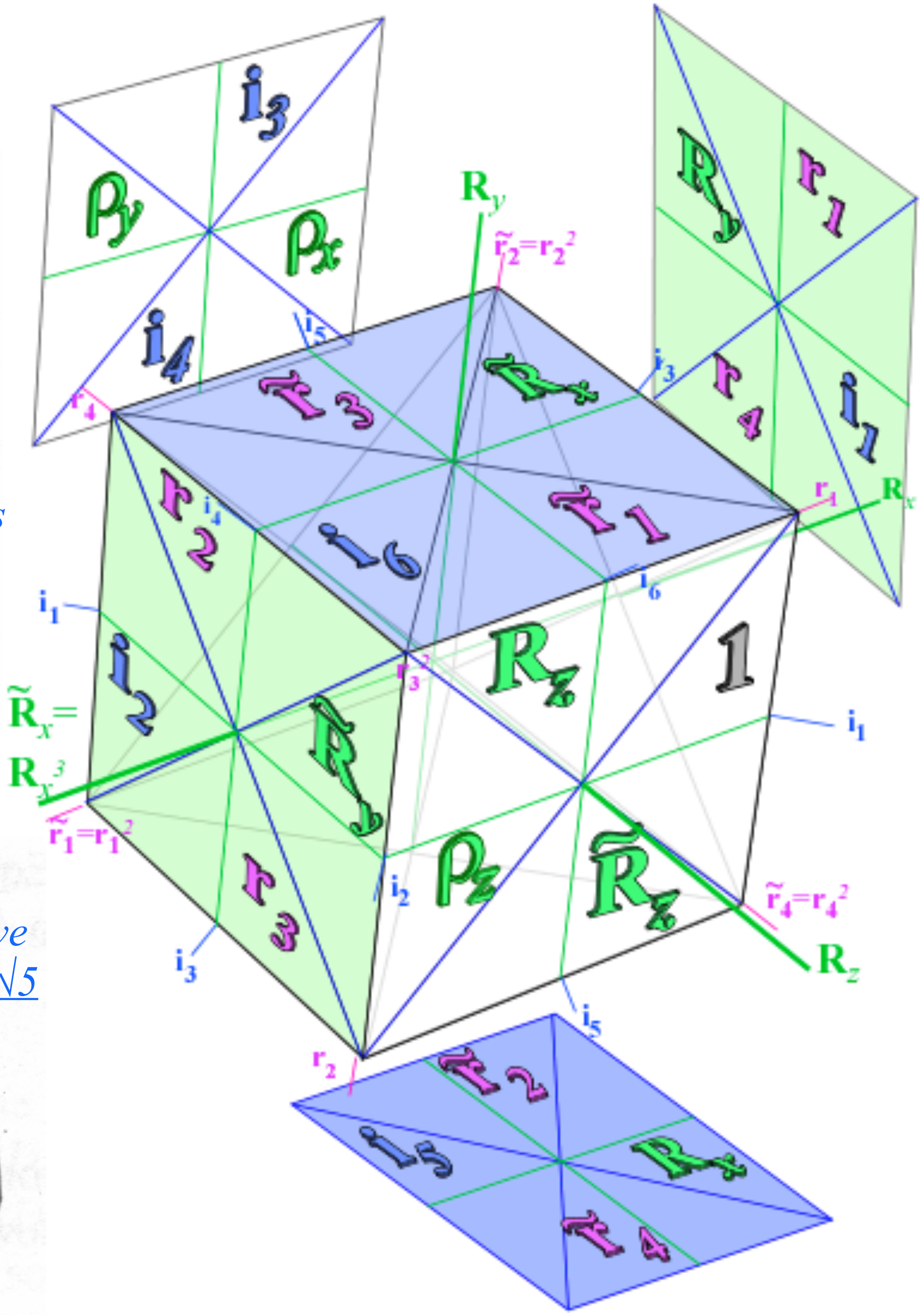
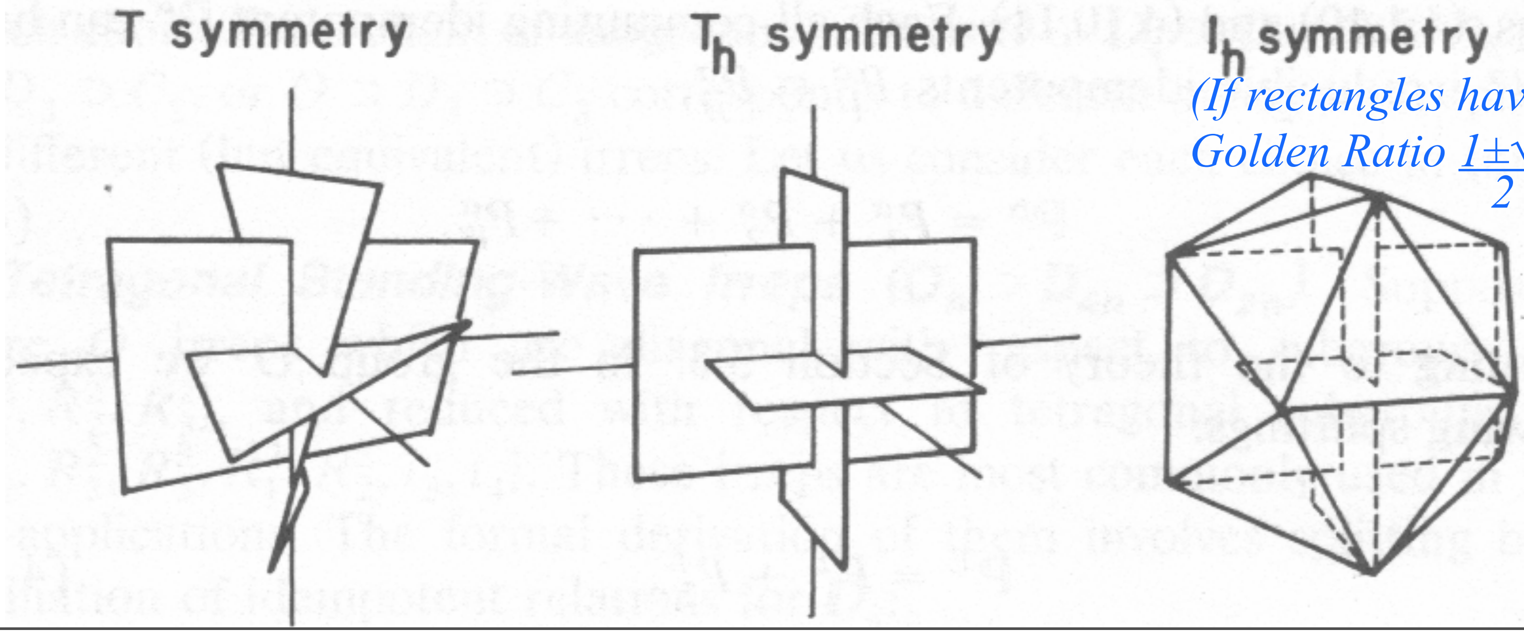


Tetrahedral symmetry becomes Icosahedral



(If rectangles have
Golden Ratio $\frac{1 \pm \sqrt{5}}{2}$

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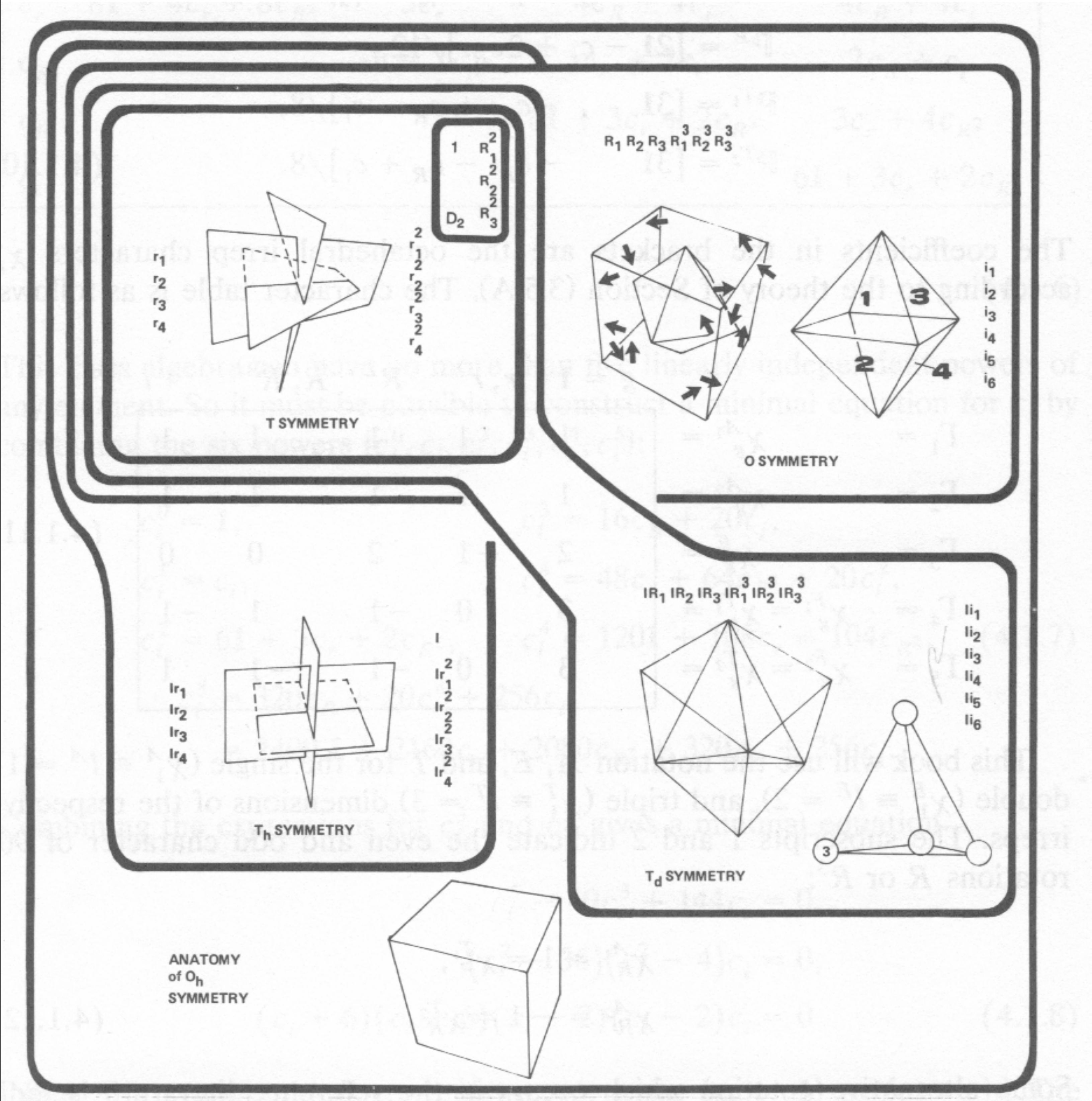

Figure 4.1.5 The full octahedral group (O_h) and four non-Abelian subgroups T, T_h, T_d, and O. The Abelian D_2 subgroup of T is indicated also.

Introduction to octahedral tetrahedral symmetry$O_h \supset O \sim T_d \supset T$

Octahedral groups $O_h \supset O \sim T_d \supset T$

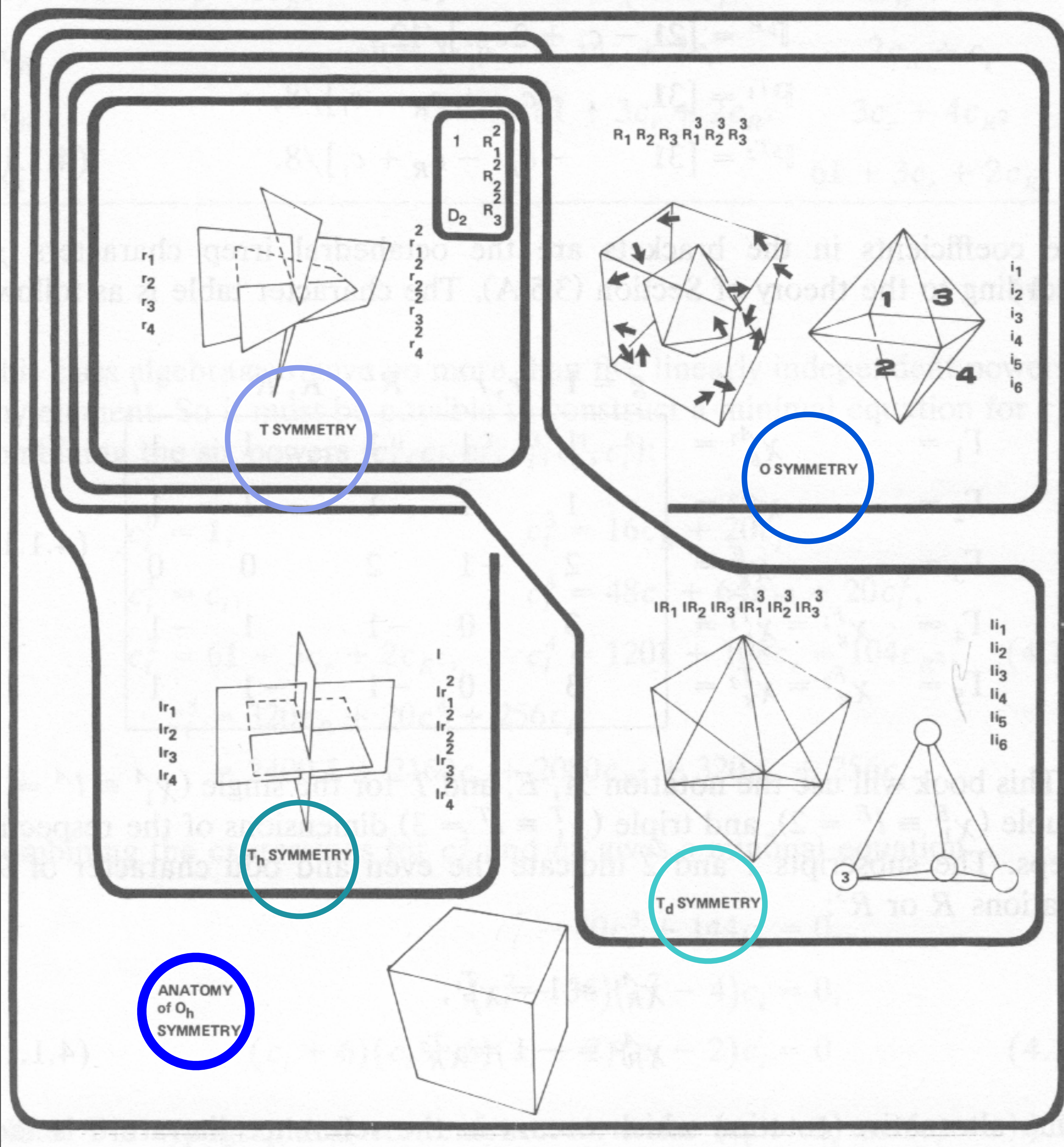

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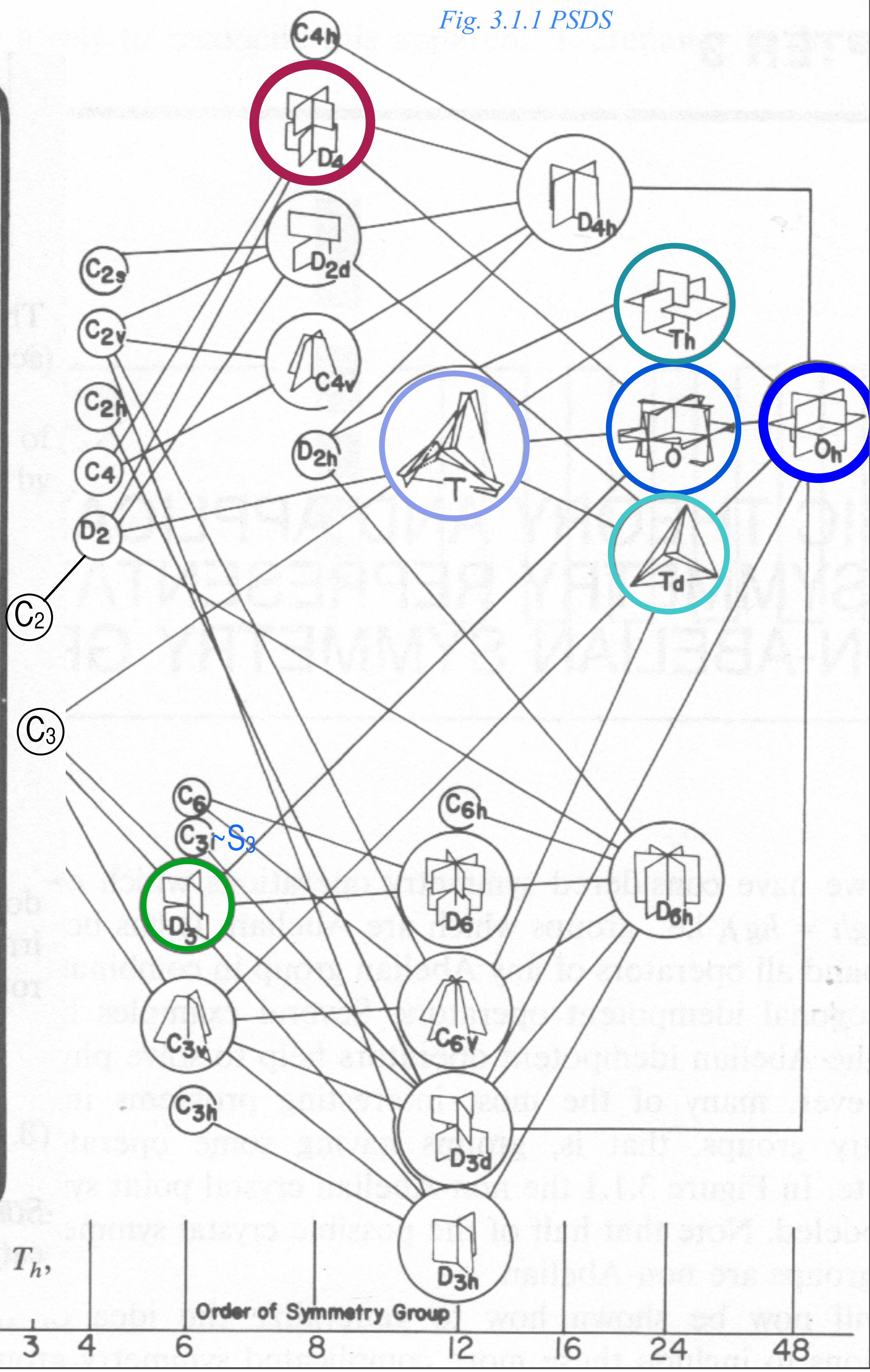
Fig. 3.1.1 PSDS



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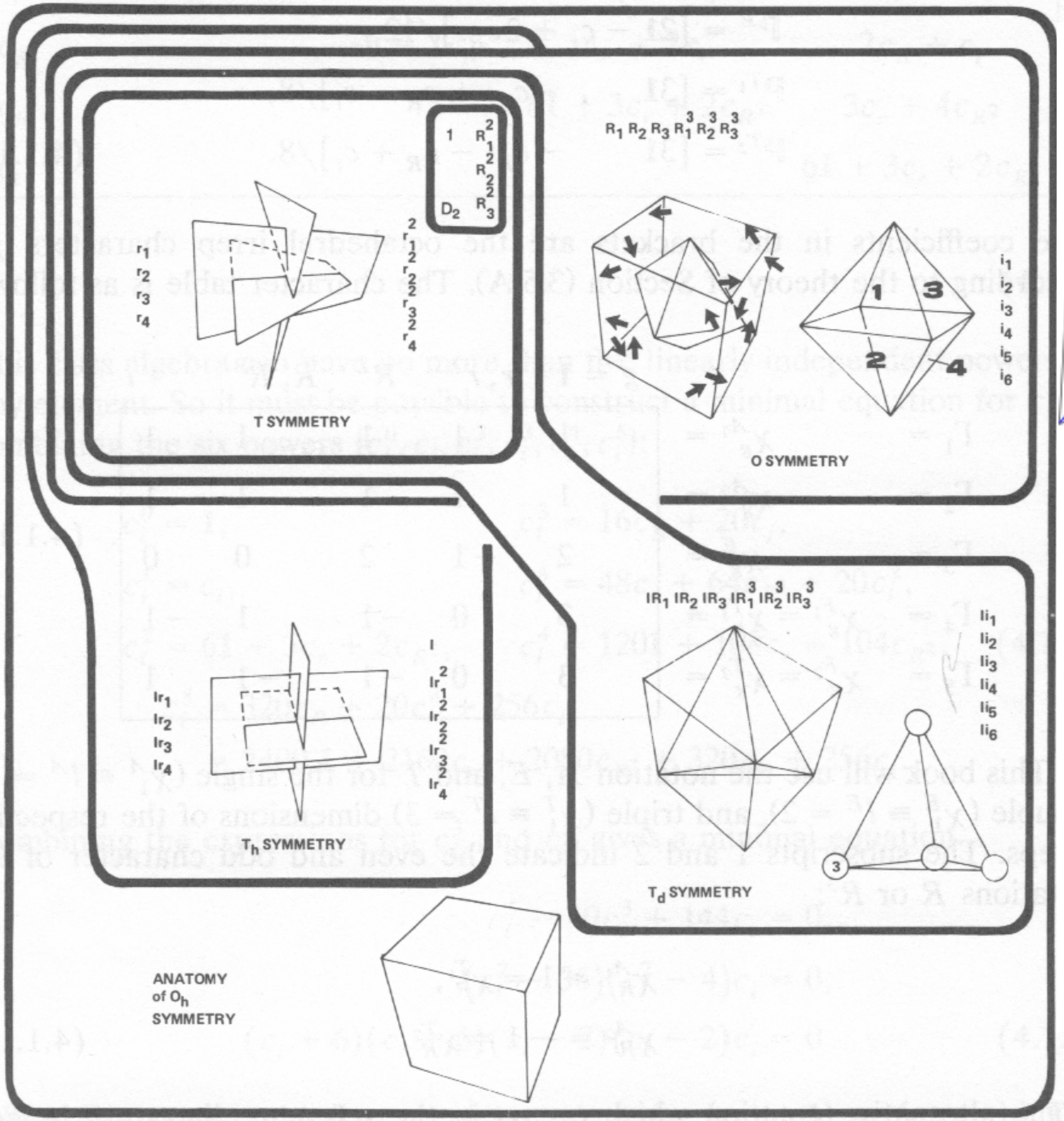

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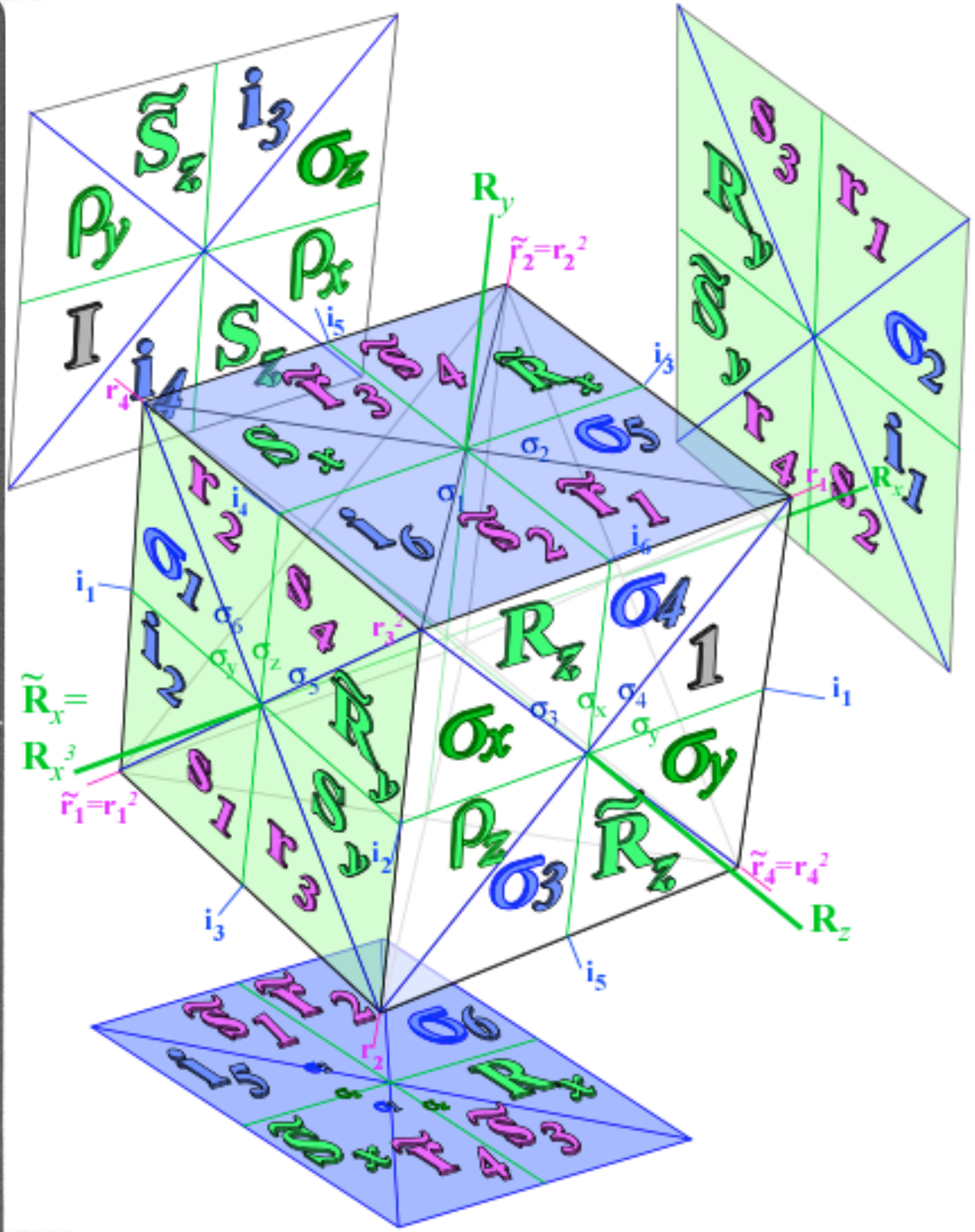

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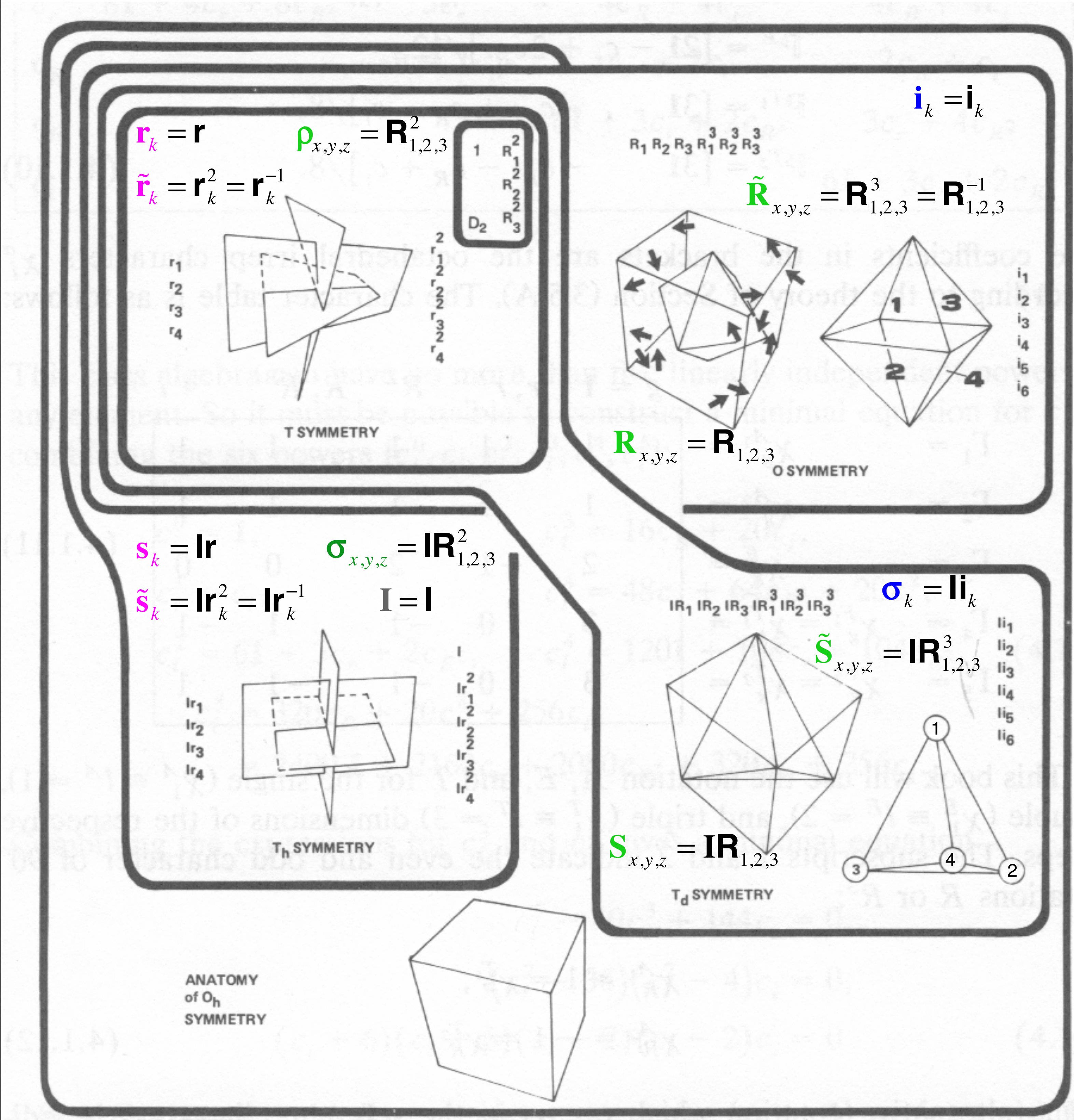
Octahedral groups $O_h \supset O \sim T_d$ and $O_h \supset T_h \supset T$



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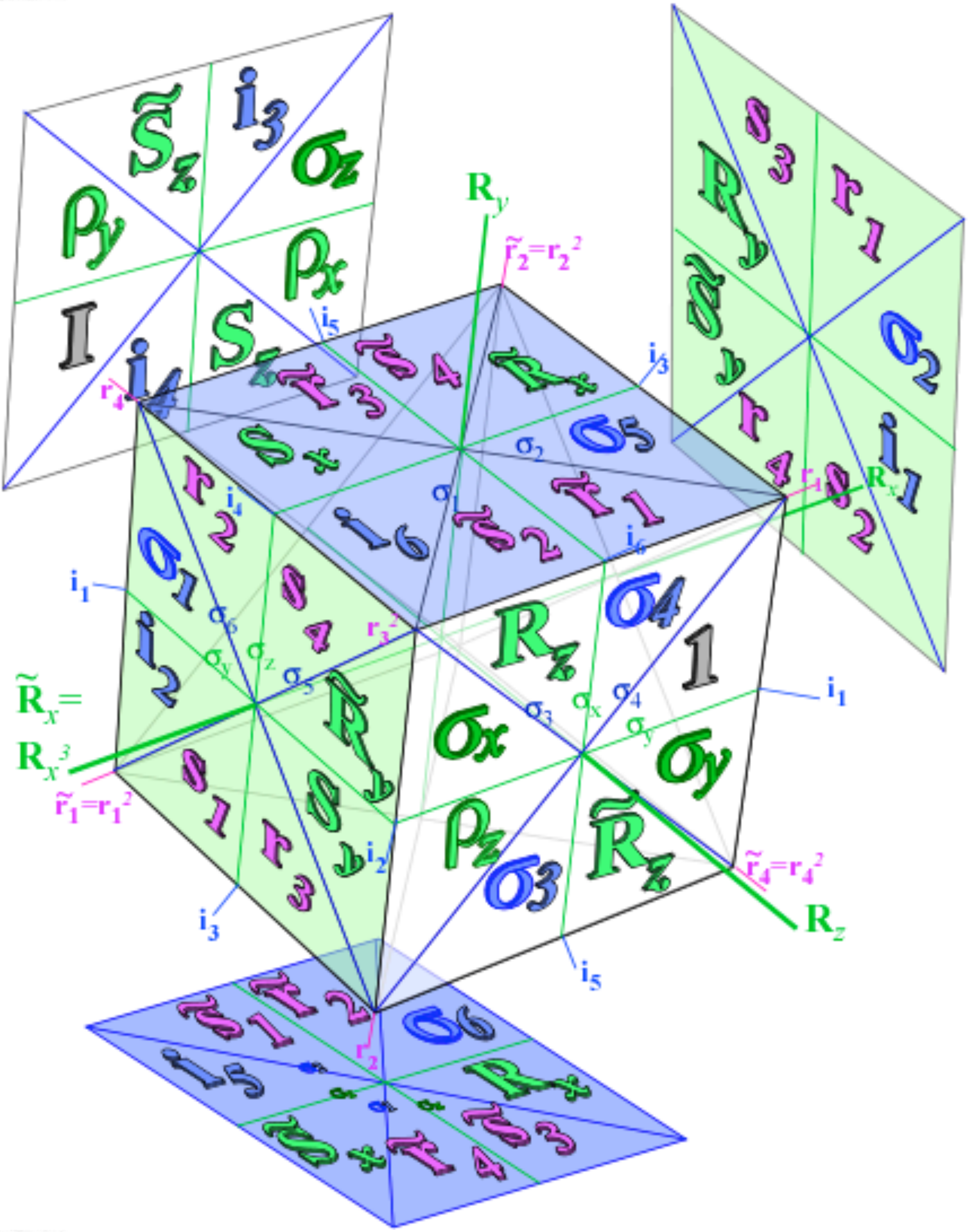

Fig. 4.1.5 from Principles of Symmetry, Dynamics and Spectroscopy

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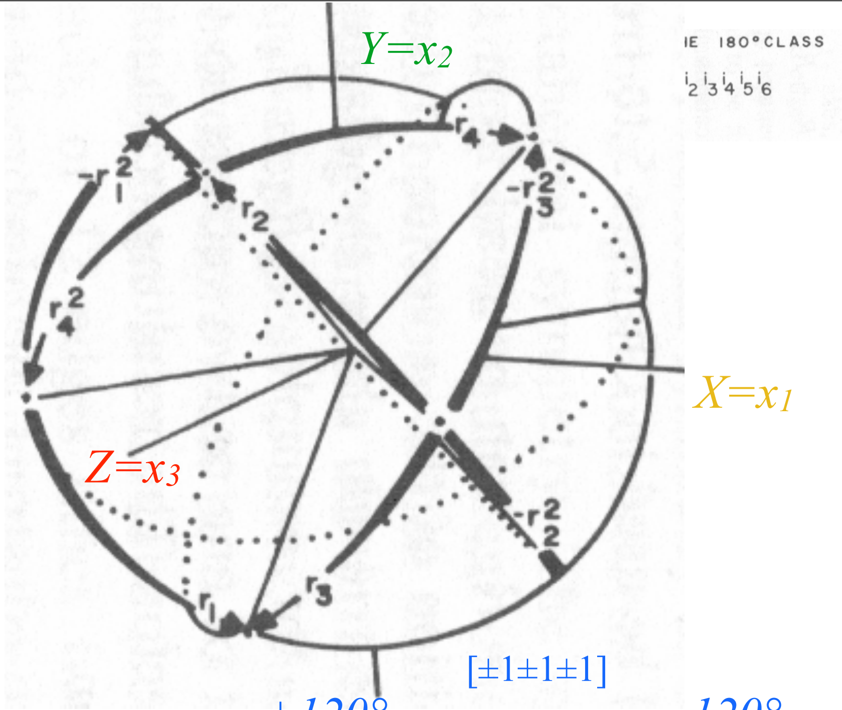
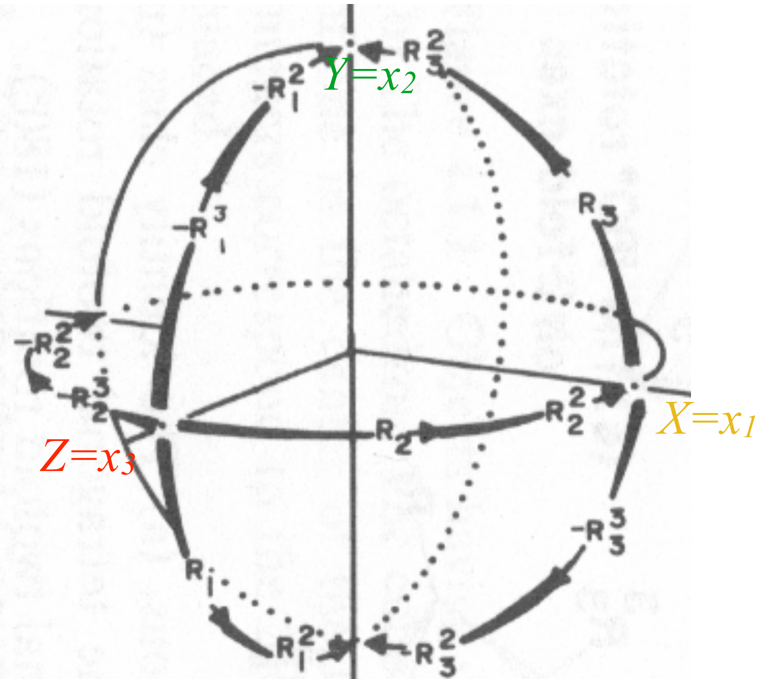
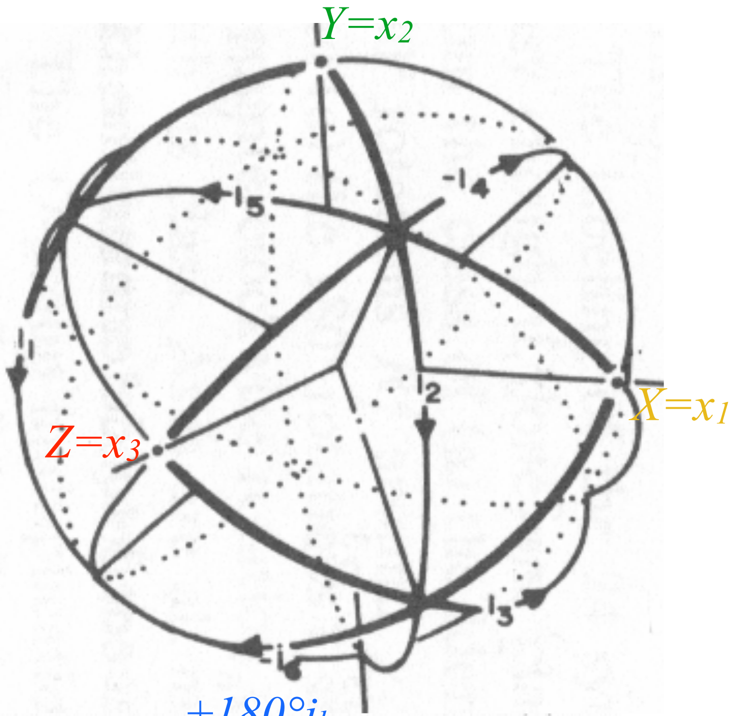




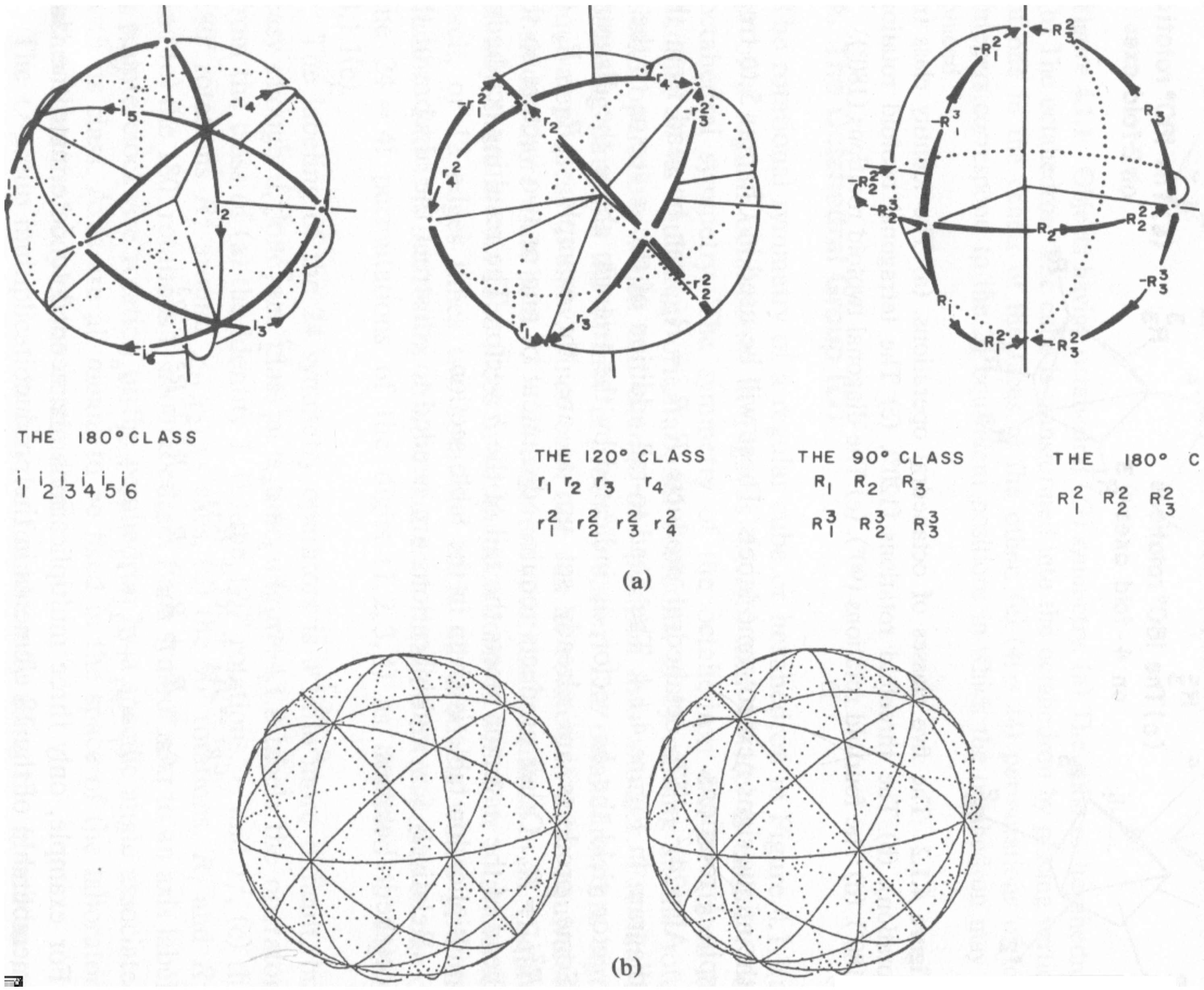


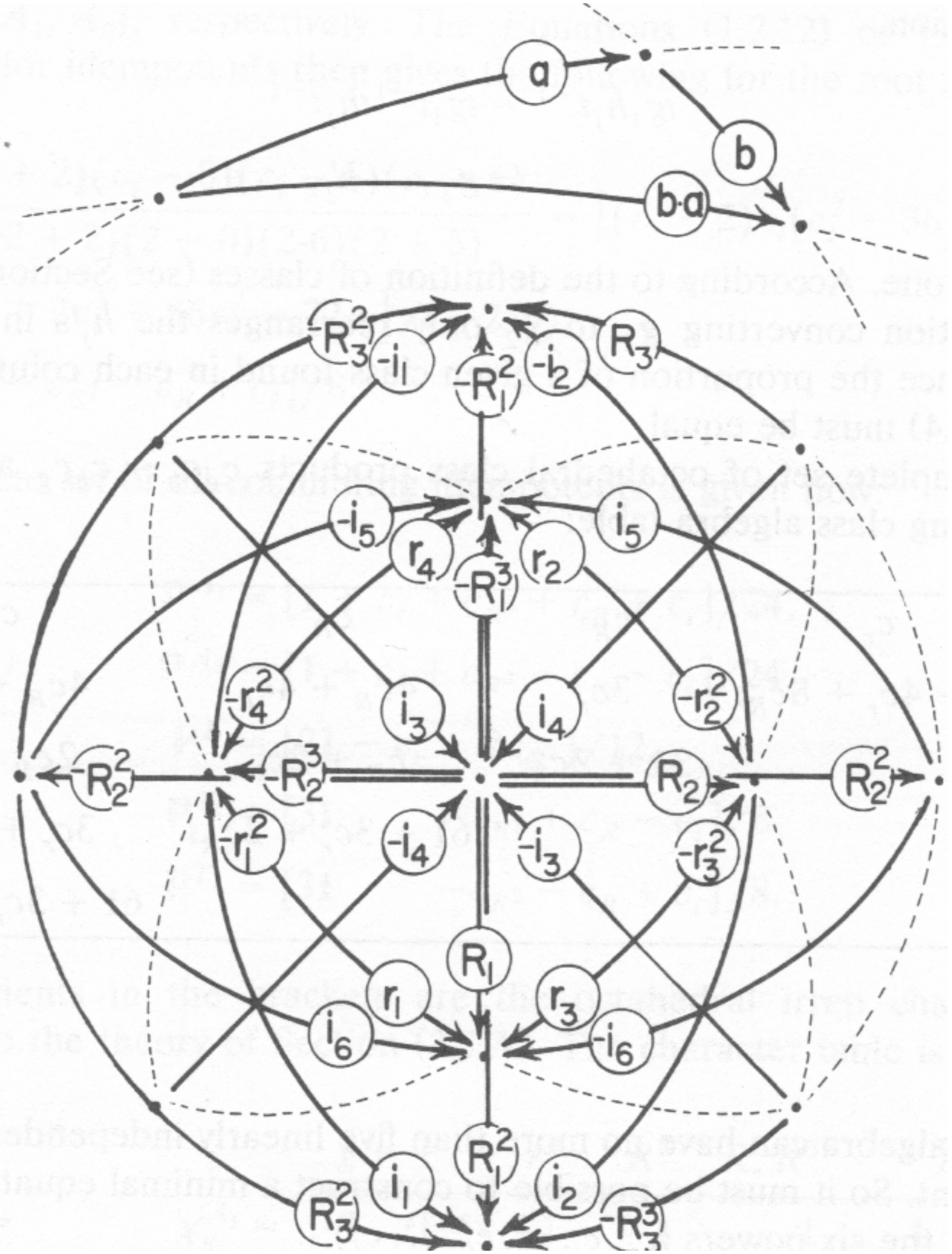
+120° −120° ±180° XYZ +90° XYZ −90° XYZ ±180° i_k

	$[1\,1\,1]$	$[\bar{1}\,\bar{1}\,1]$	$[1\,\bar{1}\,\bar{1}]$	$[\bar{1}\,1\,\bar{1}]$	$[\bar{1}\,\bar{1}\,\bar{1}]$	$[1\,1\,\bar{1}]$	$[\bar{1}\,1\,1]$	$[1\,\bar{1}\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$	$[\bar{1}\,0\,0]$	$[0\,\bar{1}\,0]$	$[0\,0\,\bar{1}]$	$[1\,0\,1]$	$[1\,0\,\bar{1}]$	$[1\,1\,0]$	$[1\,\bar{1}\,0]$	$[0\,1\,\bar{1}]$	$[0\,1\,1]$
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_1	R_2	R_3	R_1^3	R_2^3	R_3^3	i_1	i_2	i_3	i_4	i_5	i_6
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$	i_3	i_6	i_1	$-R_3$	$-R_1$	$-R_2$	R_1^3	i_5	R_2^3	i_2	$-i_4$	R_3^3
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$	R_3	$-R_1^3$	i_2	i_3	$-i_5$	R_2^3	i_6	$-R_1$	R_2	$-i_1$	R_3^3	i_4
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2	$-i_4$	R_1	$-R_2^3$	R_3^3	i_6	i_2	i_5	$-R_1^3$	i_1	R_2	$-i_3$	R_3
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1	$-R_3^3$	$-i_5$	R_2	$-i_4$	R_1^3	i_1	R_1	i_6	$-i_2$	R_2^3	R_3	i_3
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	R_2^3	R_3^3	R_1^3	$-i_1$	$-i_3$	$-i_6$	$-R_3$	$-i_4$	$-R_1$	i_5	$-i_2$	$-R_2$
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2	i_2	$-i_3$	$-R_1$	R_2	$-R_3^3$	$-i_5$	i_4	$-R_3$	$-R_1^3$	$-i_6$	R_2^3	$-i_1$
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$	$-R_2$	$-i_4$	$-i_6$	i_2	R_3	$-R_1^3$	$-i_3$	$-R_3^3$	i_5	R_1	$-i_1$	$-R_2^3$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$	$-i_1$	$-R_3$	$-i_5$	$-R_2^3$	$-i_4$	R_1	$-R_3^3$	i_3	$-i_6$	R_1^3	R_2	$-i_2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$	R_1^3	i_1	$-i_4$	$-R_1$	i_2	$-i_3$	$-R_2$	$-R_2^3$	R_3^3	R_3	$-i_6$	i_5
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2	$-i_5$	R_2^3	i_3	$-i_6$	$-R_2$	$-i_4$	$-i_2$	i_1	$-R_3$	R_3^3	R_1	R_1^3
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1	i_6	i_2	R_3^3	$-i_5$	$-i_1$	$-R_3$	R_2^3	$-R_2$	i_4	$-i_3$	R_1^3	$-R_1$
R_1	i_1	$-R_2^3$	$-i_2$	R_2	R_3^3	$-i_3$	$-R_3$	i_4	R_1^3	i_6	i_5	R_1^2	r_1	$-r_4^2$	-1	$-r_3$	r_2^2	$-r_4$	r_2	r_1^2	$-r_3^2$	$-R_2^2$	R_3^2
R_2	i_3	R_3	$-R_3^3$	i_4	R_1^3	i_5	$-i_6$	$-R_1$	$-i_2$	R_2^3	i_1	$-r_2^2$	R_2^2	r_1	r_3^2	-1	$-r_4$	R_1^2	R_3^2	$-r_2$	$-r_3$	$-r_4^2$	r_1^2
R_3	i_6	i_5	R_1	$-R_1^3$	R_2^3	$-R_2$	$-i_2$	$-i_1$	i_3	i_4	R_3^3	r_1	$-r_3^2$	R_3^2	$-r_2$	r_4^2	-1	r_1^2	r_2^2	R_2^2	$-R_1^2$	$-r_4$	$-r_3$
R_1^3	$-R_2$	$-i_2$	R_2^3	i_1	$-i_3$	$-R_3^3$	i_4	R_3	$-R_1$	i_5	$-i_6$	-1	$-r_4$	r_3^2	$-R_1^2$	r_2	$-r_1^2$	$-r_1$	r_3	r_2^2	$-r_4^2$	$-R_3^2$	$-R_2^2$
R_2^3	$-R_3$	i_3	i_4	R_3^3	$-i_6$	R_1	$-R_1^3$	i_5	$-i_1$	$-R_2$	$-i_2$	r_4^2	-1	$-r_2$	$-r_1^2$	$-R_2^2$	r_3	$-R_3^2$	R_1^2	$-r_1$	$-r_4$	$-r_2^2$	r_3^2
R_3^3	$-R_1$	R_1^3	i_6	i_5	$-i_1$	$-i_2$	R_2	$-R_2^3$	i_4	$-i_3$	$-R_3$	$-r_3$	r_2^2	-1	r_4	$-r_1^2$	$-R_3^2$	r_4^2	r_3^2	$-R_1^2$	$-R_2^2$	$-r_2$	$-r_1$
i_1	R_3^3	$-i_4$	i_3	R_3	$-R_1$	$-i_6$	$-i_5$	$-R_1^3$	R_2^3	i_2	$-R_2$	r_1^2	R_3^2	$-r_4$	r_4^2	$-R_1^2$	$-r_1$	-1	$-R_2^2$	$-r_3$	r_2	r_3^2	r_2^2
i_2	i_4	R_3^3	R_3	$-i_3$	$-i_5$	R_1^3	R_1	$-i_6$	R_2	$-i_1$	R_2^3	$-r_3^2$	$-R_1^2$	$-r_3$	$-r_2^2$	$-R_3^2$	$-r_2$	R_2^2	-1	r_4	$-r_1$	r_1^2	r_4^2
i_3	R_1^3	R_1	$-i_5$	i_6	$-R_2$	$-R_2^3$	$-i_1$	i_2	$-R_3$	R_3^3	$-i_4$	$-r_2$	r_1^2	R_1^2	$-r_1$	r_2^2	$-R_2^2$	r_3^2	$-r_4^2$	-1	R_3^2	r_3	$-r_4$
i_4	$-i_5$	i_6	$-R_1^3$	$-R_1$	$-i_2$	i_1	$-R_2^3$	$-R_2$	$-R_3^3$	$-R_3$	i_3	r_4	r_4^2	R_2^2	r_3	r_3^2	R_1^2	$-r_2^2$	r_1^2	$-R_3^3$	-1	r_1	$-r_2$
i_5	i_2	$-R_2$	i_1	$-R_2^3$	i_4	$-R_3$	i_3	$-R_3^3$	i_6	$-R_1^3$	$-R_1$	R_3^2	r_2	r_2^2	R_2^2	r_4	r_4^2	$-r_3$	$-r_1$	$-r_3^2$	$-r_1^2$	-1	$-R_1^2$
i_6	R_2^3	i_1	R_2	i_2	$-R_3$	$-i_4$	$-R_3^3$	$-i_3$	$-i_5$	$-R_1$	R_1^3	R_2^2	$-r_3$	r_1^2	$-R_3^2$	$-r_1$	r_3^2	$-r_2$	$-r_4$	r_4^2	r_2^2	R_1^2	-1

Octahedral O and spin-O⊂U(2) rotation product Table F.2.1 from Principles of Symmetry, Dynamics and Spectroscopy

Octahedral O and spin-O⊂U(2) rotation nomogram from Fig. 4.1.3-4 *Principles of Symmetry, Dynamics and Spectroscopy*





Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

- Octahedral-cubic O symmetry and group operations
- Tetrahedral symmetry becomes Icosahedral
- Octahedral groups $O_h \supset O \sim T_d \supset T$
  - Octahedral O and spin-$O \subset U(2)$

➡ Tetrahedral T class algebra ⬅

- Tetrahedral T class minimal equations
- Tetrahedral T class projectors and characters

Octahedral O class algebra

- Octahedral O class minimal equations
- Octahedral O class projectors and characters

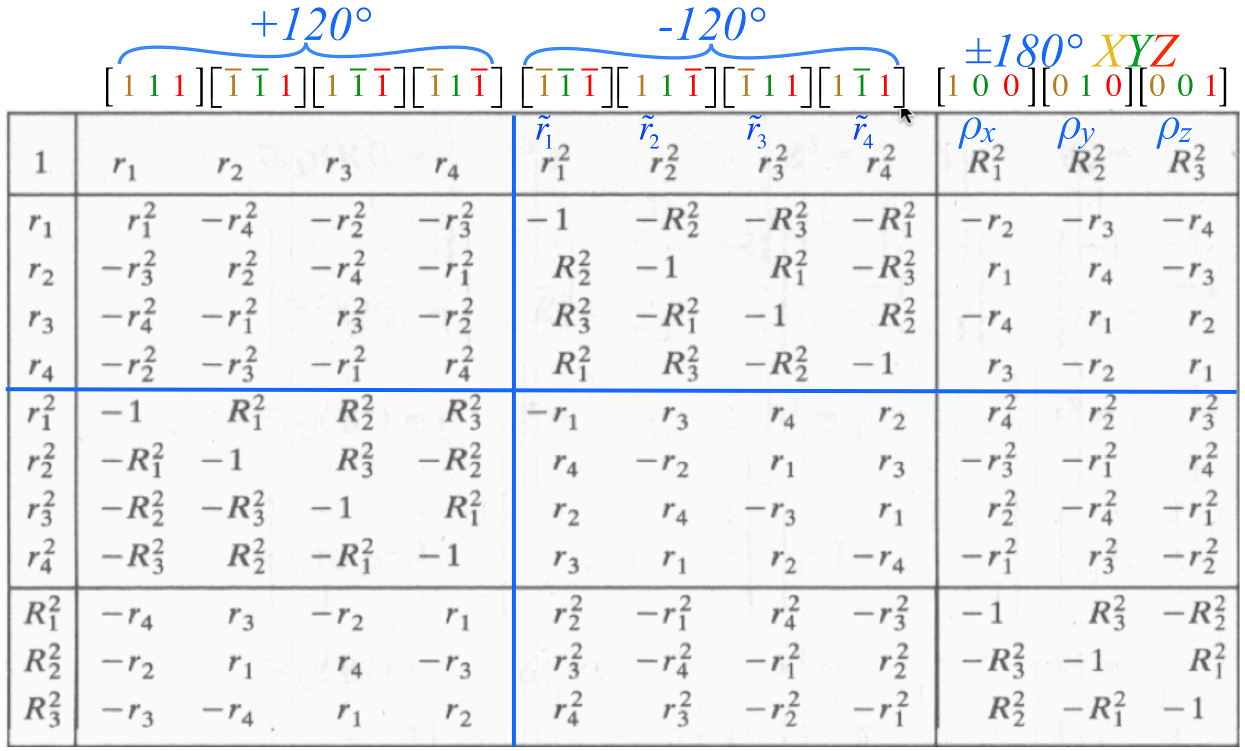
Octahedral $O_h \supset O$ subgroup correlations

- Octahedral $O_h \supset O$ subgroup correlations
- $O_h \supset O \supset D_4$ subgroup correlations
- $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
- Preview of applications to high resolution spectroscopy

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products

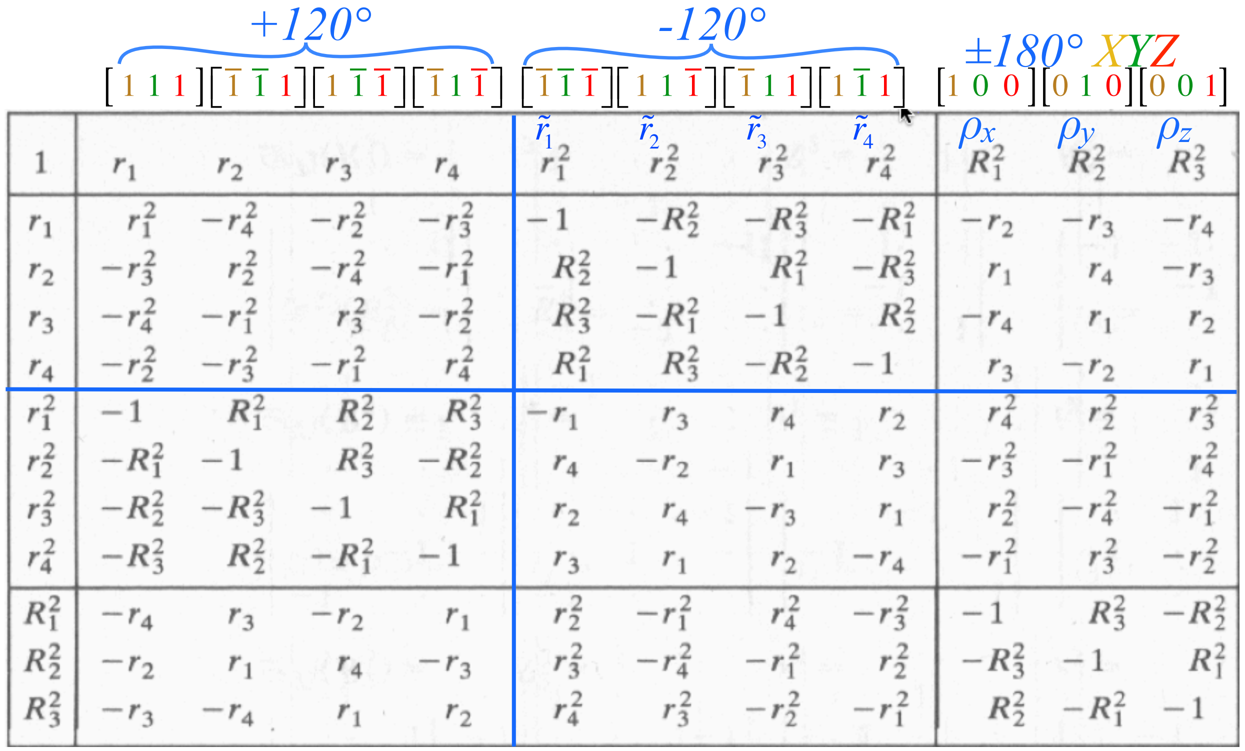


	+120°				-120°				±180° XYZ		
	$[1\ 1\ 1]$	$[\bar{1}\ \bar{1}\ 1]$	$[1\ \bar{1}\ \bar{1}]$	$[\bar{1}\ 1\ \bar{1}]$	$[\bar{1}\ \bar{1}\ \bar{1}]$	$[1\ 1\ \bar{1}]$	$[\bar{1}\ 1\ 1]$	$[1\ \bar{1}\ 1]$	$[1\ 0\ 0]$	$[0\ 1\ 0]$	$[0\ 0\ 1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2$, $\mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2$,

T group products



	$+120°$				$-120°$				$\pm180°$ XYZ		
	$[1\,1\,1]$	$[\bar{1}\,\bar{1}\,1]$	$[1\,\bar{1}\,\bar{1}]$	$[\bar{1}\,1\,\bar{1}]$	$[\bar{1}\,\bar{1}\,\bar{1}]$	$[1\,1\,\bar{1}]$	$[\bar{1}\,1\,1]$	$[1\,\bar{1}\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

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Tetrahedral T class algebra

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T group products

$\underbrace{[1\ 1\ 1][\bar{1}\ \bar{1}\ 1][1\ \bar{1}\ \bar{1}][\bar{1}\ 1\ \bar{1}]}_{+120°}$ $\underbrace{[\bar{1}\ \bar{1}\ \bar{1}][1\ 1\ \bar{1}][\bar{1}\ 1\ 1][1\ \bar{1}\ 1]}_{-120°}$ $\pm 180°\ XYZ$ $[1\ 0\ 0][0\ 1\ 0][0\ 0\ 1]$

1	r_1	r_2	r_3	r_4	$\tilde{r}_1$ r_1^2	$\tilde{r}_2$ r_2^2	$\tilde{r}_3$ r_3^2	$\tilde{r}_4$ r_4^2	ρ_x R_1^2	ρ_y R_2^2	ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_\rho$

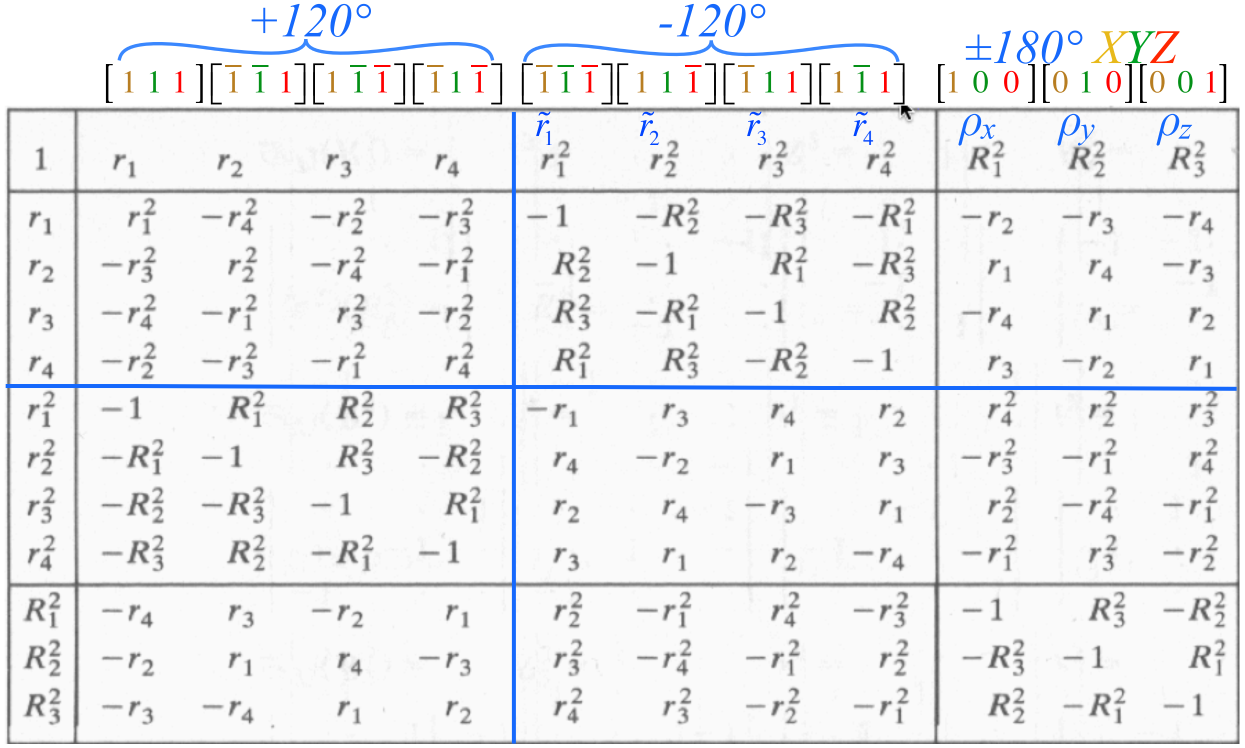
$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\cdot\mathbf{1} = \mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2$, $\mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2$,

T group products



1	r_1	r_2	r_3	r_4	$\tilde r_1$ r_1^2	$\tilde r_2$ r_2^2	$\tilde r_3$ r_3^2	$\tilde r_4$ r_4^2	ρ_x R_1^2	ρ_y R_2^2	ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2=3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\cdot\mathbf{1}=\mathbf{0}$$

$$(\mathbf{c}_\rho-3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1 = \mathbf{1}$, $\mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4$, $\mathbf{c}^\dagger_r = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2$, $\mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2$,

T group products

	+120°				-120°				±180° XYZ		
	$[1\ 1\ 1]$	$[\bar{1}\ \bar{1}\ 1]$	$[1\ \bar{1}\ \bar{1}]$	$[\bar{1}\ 1\ \bar{1}]$	$[\bar{1}\ \bar{1}\ \bar{1}]$	$[1\ 1\ \bar{1}]$	$[\bar{1}\ 1\ 1]$	$[1\ \bar{1}\ 1]$	$[1\ 0\ 0]$	$[0\ 1\ 0]$	$[0\ 0\ 1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1} + 2\mathbf{c}_\rho$$

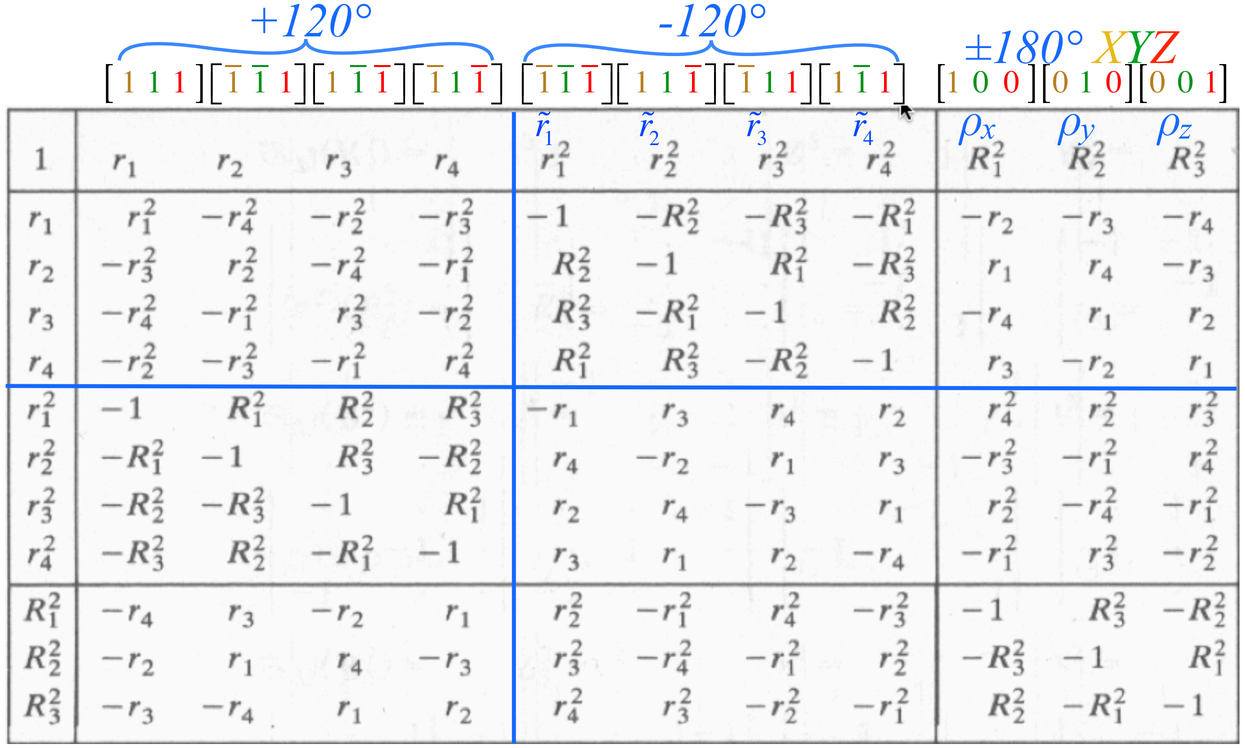
$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\cdot\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\cdot\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products



	+120°				-120°				±180° XYZ		
	$[1\,1\,1]$	$[\bar1\,\bar1\,1]$	$[1\,\bar1\,\bar1]$	$[\bar1\,1\,\bar1]$	$[\bar1\,\bar1\,\bar1]$	$[1\,1\,\bar1]$	$[\bar1\,1\,1]$	$[1\,\bar1\,1]$	$[1\,0\,0]$	$[0\,1\,0]$	$[0\,0\,1]$
					$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_3$	$\tilde{r}_4$	ρ_x	ρ_y	ρ_z
1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2=4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3=4\tilde{\mathbf{c}}_r\mathbf{c}_r=4(4\cdot\mathbf{1}+4\mathbf{c}_\rho)=16\cdot\mathbf{1}+16\mathbf{c}_\rho$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2=3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\cdot\mathbf{1}=\mathbf{0}$$

$$(\mathbf{c}_\rho-3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$$

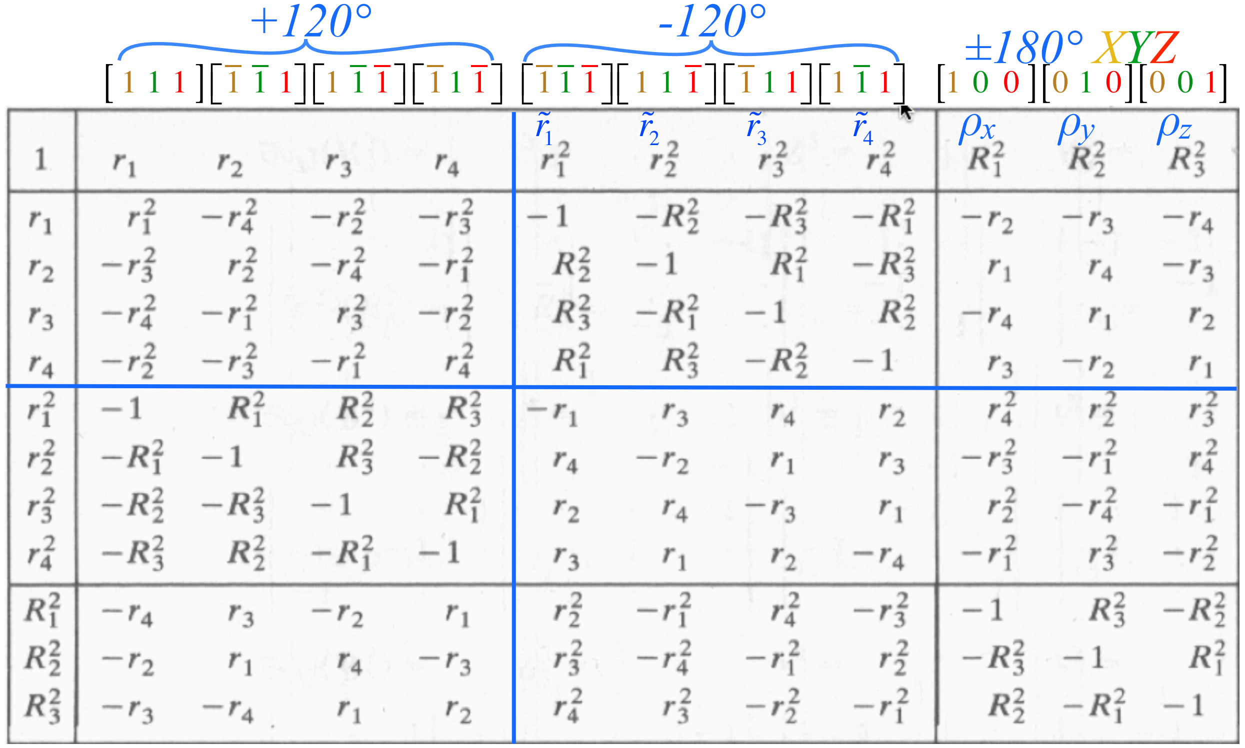
Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2$, $\mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2$,

T group products

+120° -120° ±180° XYZ

$[1\ 1\ 1][\bar{1}\ \bar{1}\ 1][1\ \bar{1}\ \bar{1}][\bar{1}\ 1\ \bar{1}]$ $[\bar{1}\ \bar{1}\ \bar{1}][1\ 1\ \bar{1}][\bar{1}\ 1\ 1][1\ \bar{1}\ 1]$ $[1\ 0\ 0][0\ 1\ 0][0\ 0\ 1]$



1	r_1	r_2	r_3	r_4	$\tilde{r}_1$ r_1^2	$\tilde{r}_2$ r_2^2	$\tilde{r}_3$ r_3^2	$\tilde{r}_4$ r_4^2	ρ_x R_1^2	ρ_y R_2^2	ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r\mathbf{c}_r = 4(4\cdot\mathbf{1}+4\mathbf{c}_\rho) = 16\cdot\mathbf{1}+16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4 = 16\cdot\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho\mathbf{c}_r = 16\cdot\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r)$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

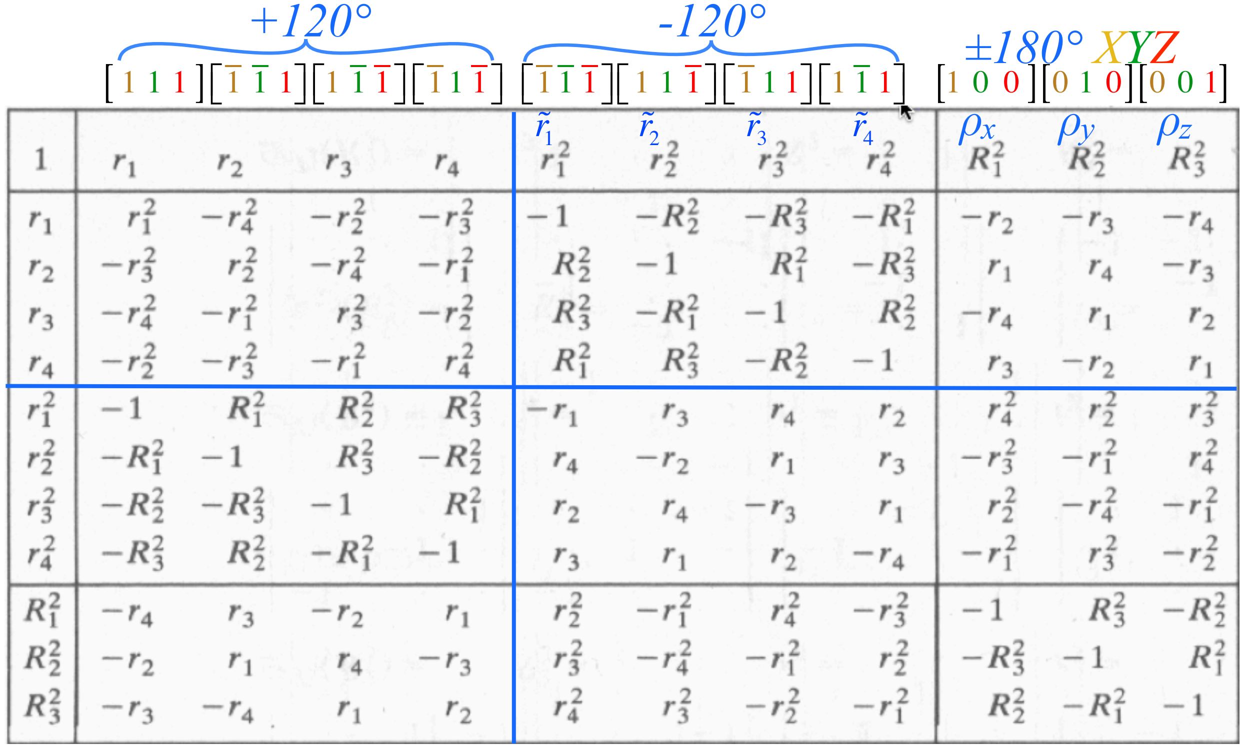
$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\cdot\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

T group products



$\overbrace{[1\,1\,1][\bar1\,\bar1\,1][1\,\bar1\,\bar1][\bar1\,1\,\bar1]}^{+120^\circ}\ \overbrace{[\bar1\,\bar1\,\bar1][1\,1\,\bar1][\bar1\,1\,1][1\,\bar1\,1]}^{-120^\circ}\ \overset{\pm180^\circ\ XYZ}{[1\,0\,0][0\,1\,0][0\,0\,1]}$

1	r_1	r_2	r_3	r_4	$\tilde{r}_1$ r_1^2	$\tilde{r}_2$ r_2^2	$\tilde{r}_3$ r_3^2	$\tilde{r}_4$ r_4^2	ρ_x R_1^2	ρ_y R_2^2	ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r\mathbf{c}_r = 4(4\cdot\mathbf{1}+4\mathbf{c}_\rho) = 16\cdot\mathbf{1}+16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4 = 16\cdot\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho\mathbf{c}_r = 16\cdot\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r)$$

$$\mathbf{c}_r^4 - 64\mathbf{c}_r = (\mathbf{c}_r^3 - 64\cdot\mathbf{1})\mathbf{c}_r = \mathbf{0}$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

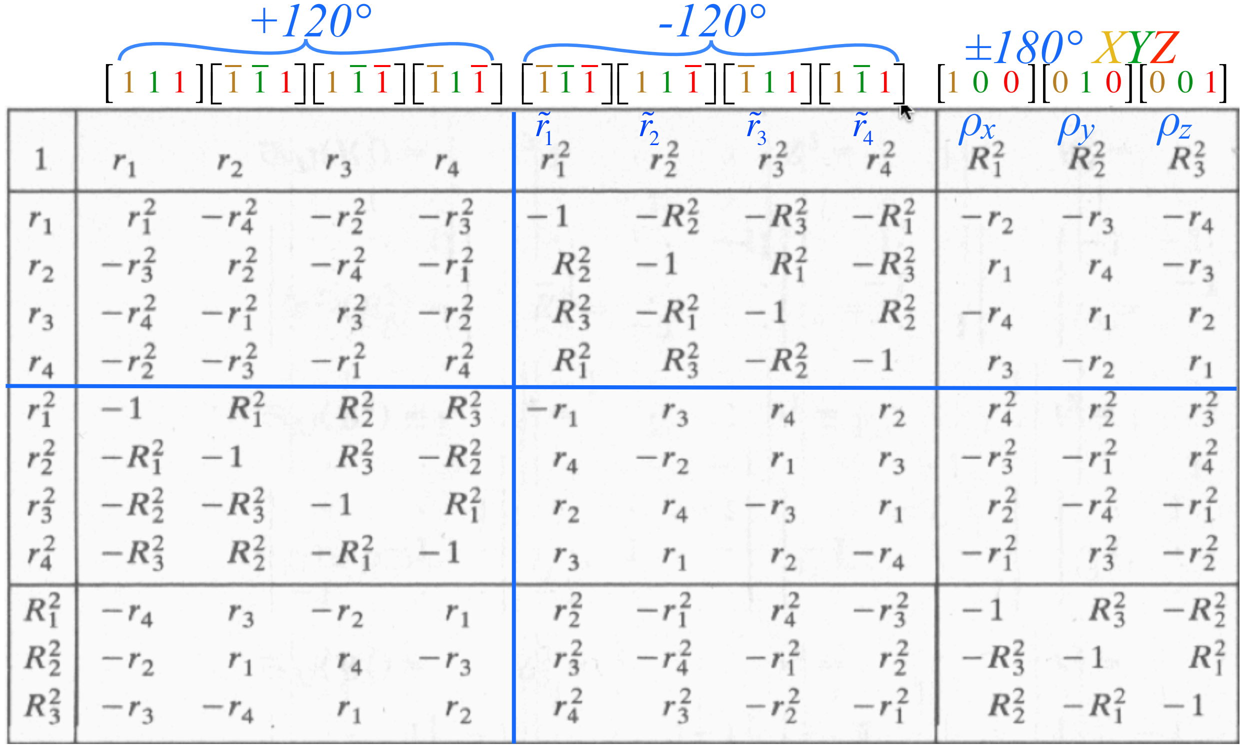
$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\cdot\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4$, $\mathbf{c}^\dagger_r=\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2$, $\mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2$,

T group products



$\overbrace{[1\,1\,1][\bar{1}\,\bar{1}\,1][1\,\bar{1}\,\bar{1}][\bar{1}\,1\,\bar{1}]}^{+120°}$ $\overbrace{[\bar{1}\,\bar{1}\,\bar{1}][1\,1\,\bar{1}][\bar{1}\,1\,1][1\,\bar{1}\,1]}^{-120°}$ $\overbrace{[1\,0\,0][0\,1\,0][0\,0\,1]}^{\pm 180°\ XYZ}$

1	r_1	r_2	r_3	r_4	$\tilde{r}_1$ r_1^2	$\tilde{r}_2$ r_2^2	$\tilde{r}_3$ r_3^2	$\tilde{r}_4$ r_4^2	ρ_x R_1^2	ρ_y R_2^2	ρ_z R_3^2
r_1	r_1^2	$-r_4^2$	$-r_2^2$	$-r_3^2$	-1	$-R_2^2$	$-R_3^2$	$-R_1^2$	$-r_2$	$-r_3$	$-r_4$
r_2	$-r_3^2$	r_2^2	$-r_4^2$	$-r_1^2$	R_2^2	-1	R_1^2	$-R_3^2$	r_1	r_4	$-r_3$
r_3	$-r_4^2$	$-r_1^2$	r_3^2	$-r_2^2$	R_3^2	$-R_1^2$	-1	R_2^2	$-r_4$	r_1	r_2
r_4	$-r_2^2$	$-r_3^2$	$-r_1^2$	r_4^2	R_1^2	R_3^2	$-R_2^2$	-1	r_3	$-r_2$	r_1
r_1^2	-1	R_1^2	R_2^2	R_3^2	$-r_1$	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
r_2^2	$-R_1^2$	-1	R_3^2	$-R_2^2$	r_4	$-r_2$	r_1	r_3	$-r_3^2$	$-r_1^2$	r_4^2
r_3^2	$-R_2^2$	$-R_3^2$	-1	R_1^2	r_2	r_4	$-r_3$	r_1	r_2^2	$-r_4^2$	$-r_1^2$
r_4^2	$-R_3^2$	R_2^2	$-R_1^2$	-1	r_3	r_1	r_2	$-r_4$	$-r_1^2$	r_3^2	$-r_2^2$
R_1^2	$-r_4$	r_3	$-r_2$	r_1	r_2^2	$-r_1^2$	r_4^2	$-r_3^2$	-1	R_3^2	$-R_2^2$
R_2^2	$-r_2$	r_1	r_4	$-r_3$	r_3^2	$-r_4^2$	$-r_1^2$	r_2^2	$-R_3^2$	-1	R_1^2
R_3^2	$-r_3$	$-r_4$	r_1	r_2	r_4^2	r_3^2	$-r_2^2$	$-r_1^2$	R_2^2	$-R_1^2$	-1

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\mathbf{c}_r^\dagger$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\mathbf{c}_r^\dagger$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\mathbf{c}_r^\dagger$		$4\mathbf{c}_r$	$3\mathbf{c}_r^\dagger$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\mathbf{c}_r^2=4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3=4\tilde{\mathbf{c}}_r\mathbf{c}_r=4(4\cdot\mathbf{1}+4\mathbf{c}_\rho)=16\cdot\mathbf{1}+16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4=16\cdot\mathbf{1}\mathbf{c}_r+16\mathbf{c}_\rho\mathbf{c}_r=16\cdot\mathbf{1}\mathbf{c}_r+16(3\mathbf{c}_r)$$

$$\mathbf{c}_r^4-64\mathbf{c}_r=(\mathbf{c}_r^3-64\cdot\mathbf{1})\mathbf{c}_r=\mathbf{0}$$

$$\boxed{(\mathbf{c}_r-4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}}$$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2=3\cdot\mathbf{1}+2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\cdot\mathbf{1}=\mathbf{0}$$

$$(\mathbf{c}_\rho-3\cdot\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$$

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

- Octahedral-cubic O symmetry and group operations
- Tetrahedral symmetry becomes Icosahedral
- Octahedral groups $O_h \supset O \sim T_d \supset T$
  - Octahedral O and spin-$O \subset U(2)$

Tetrahedral T class algebra

- Tetrahedral T class minimal equations
- → Tetrahedral T class projectors and characters ←

Octahedral O class algebra

- Octahedral O class minimal equations
- Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

- Octahedral $O_h \supset O$ subgroup correlations
- $O_h \supset O \supset D_4$ subgroup correlations
- $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
- Preview of applications to high resolution spectroscopy

Tetrahedral T class projectors

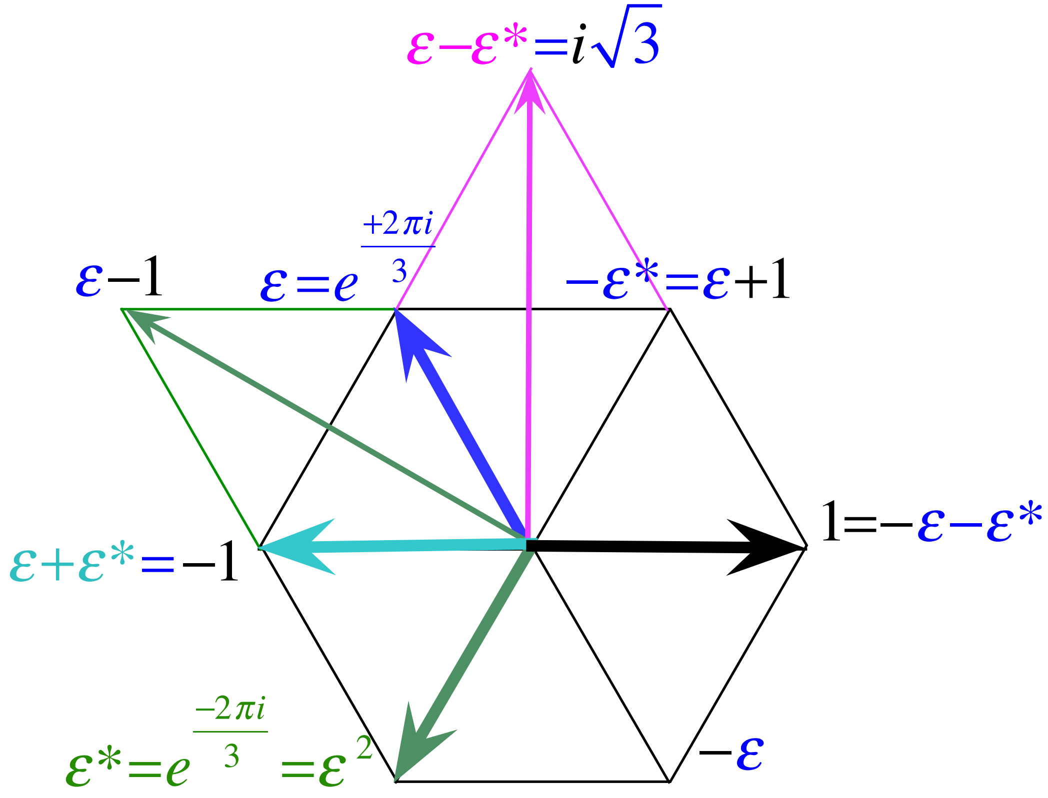
$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}}\,\mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi^{\mu *}_{g}\,\mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

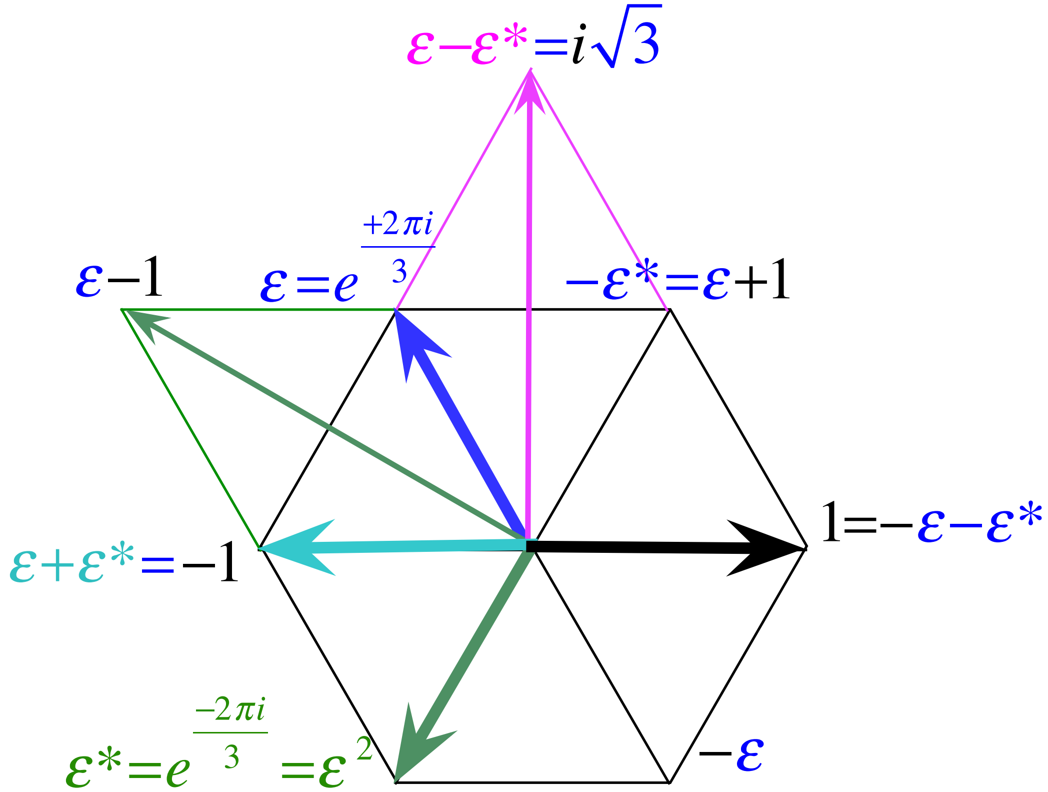
$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$



Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu} \mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G} \chi_g^{\mu *} \mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

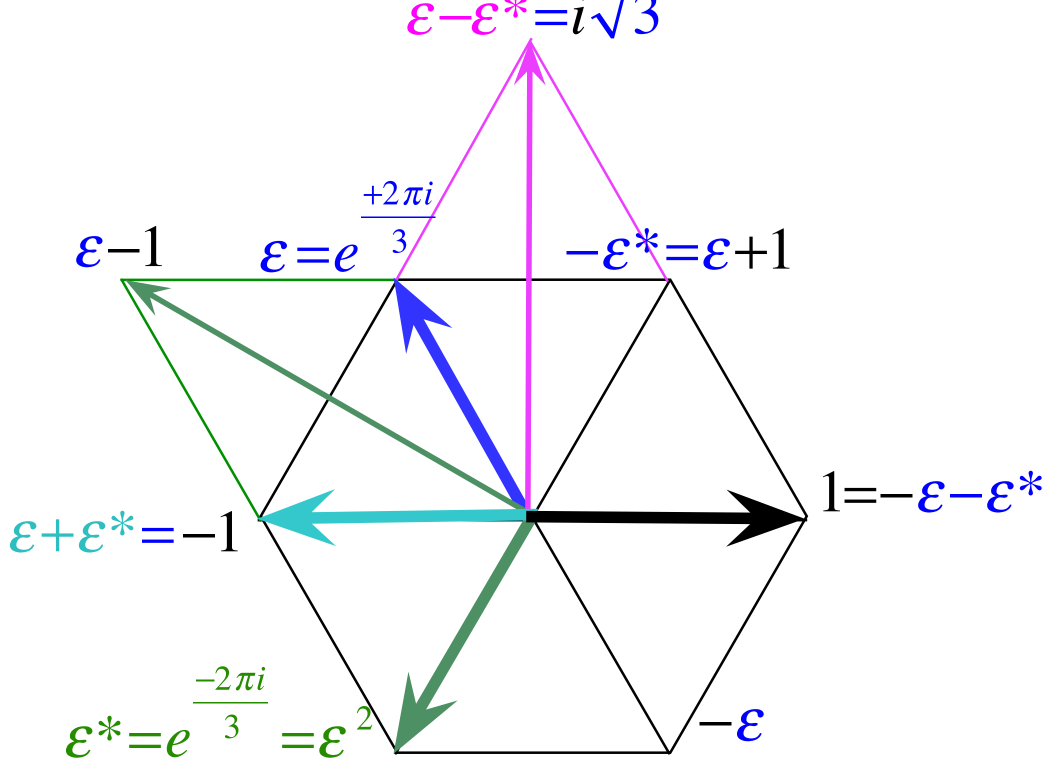
$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$



Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}}$$

T class products

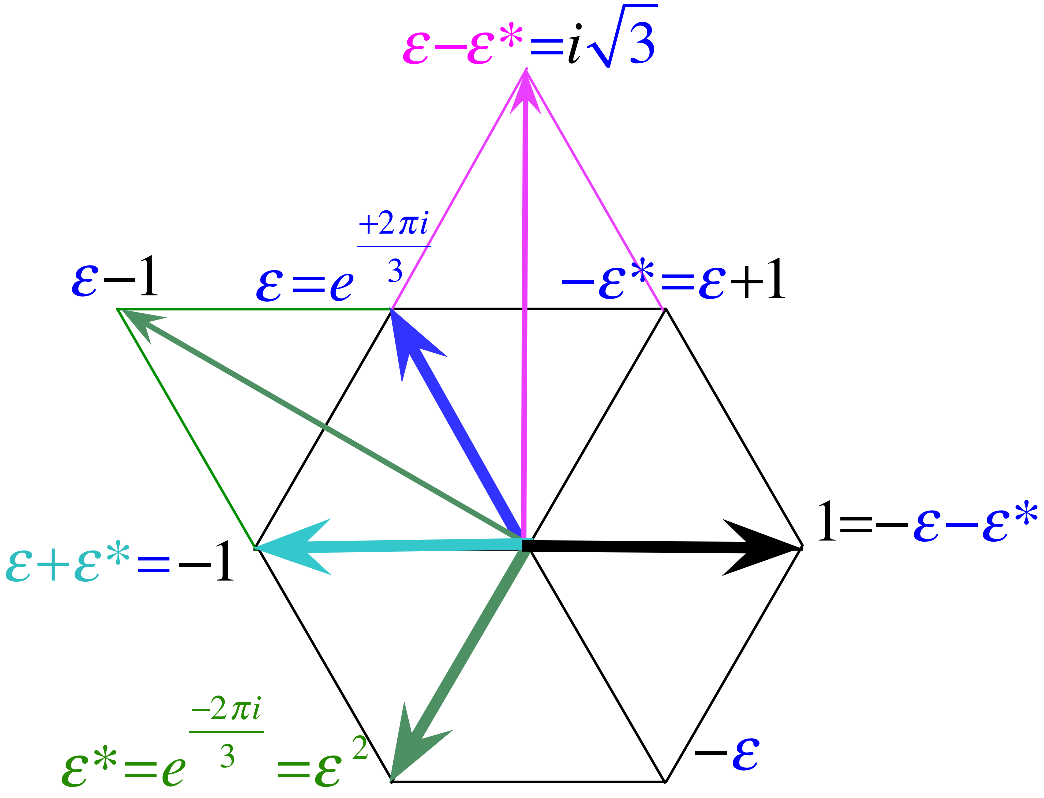
$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

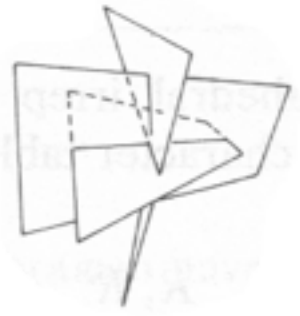
Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{\mathbf{g}}\chi^{\mu}_{\mathbf{g}}}{\ell^{\mu}}\mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi^{\mu *}_{\mathbf{g}}\mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$



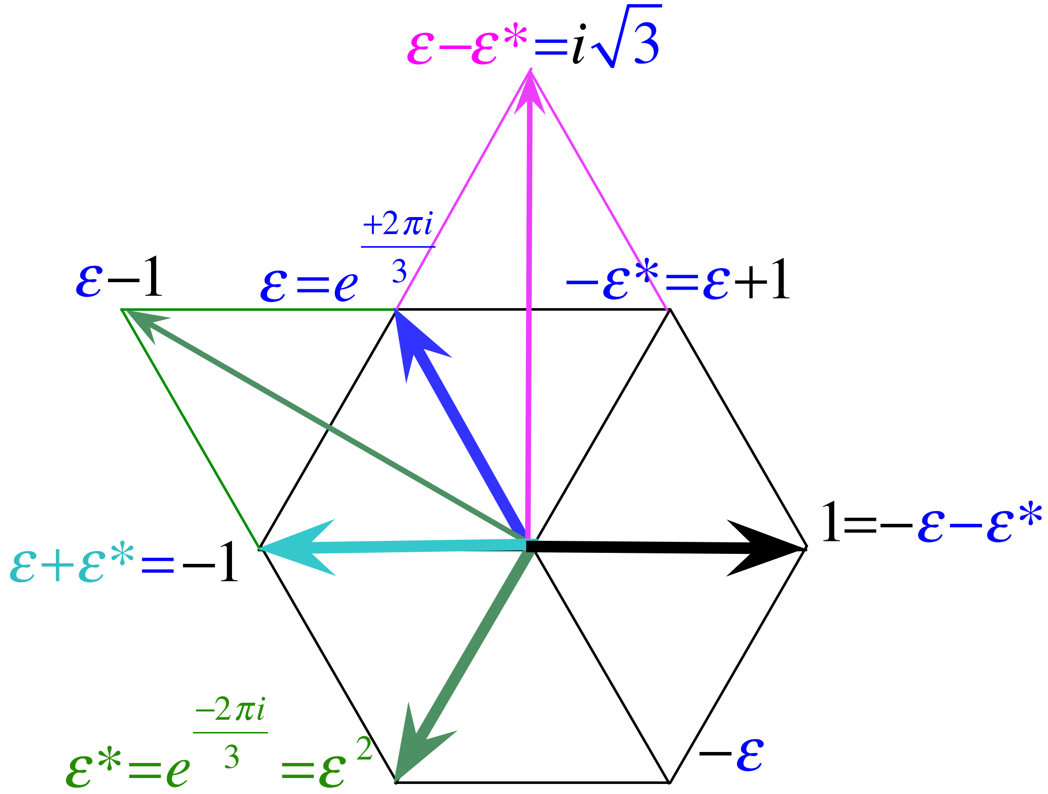
Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

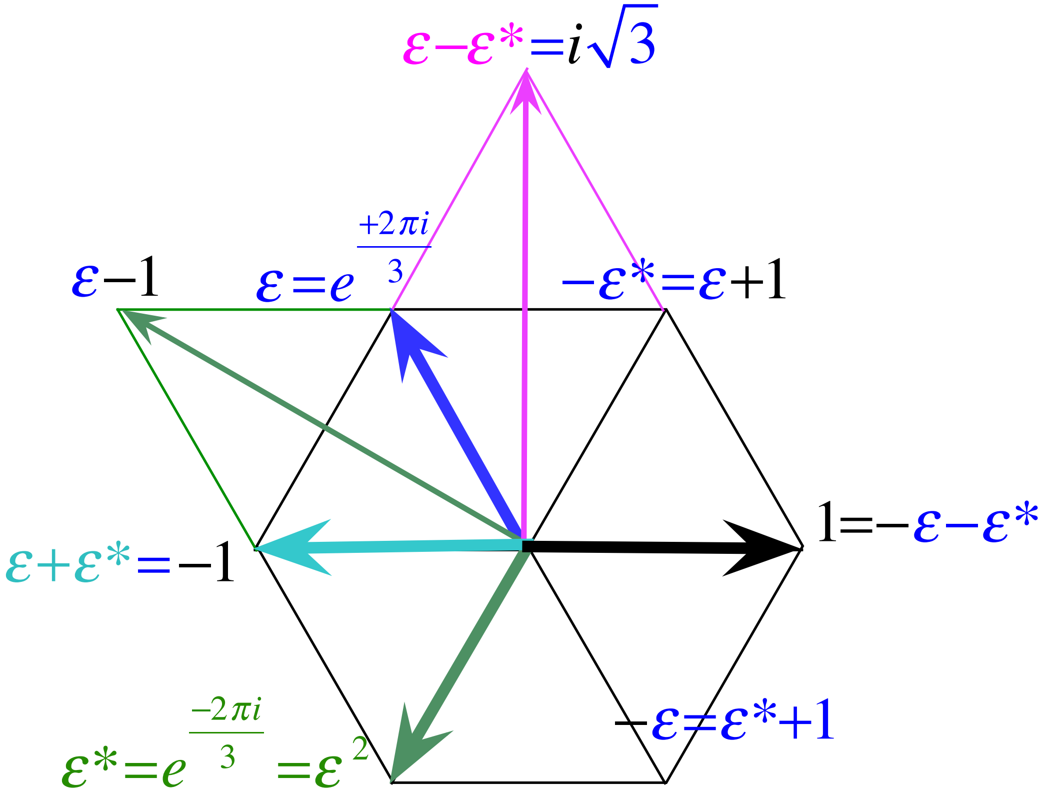
Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r - 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_g}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_\mathbf{g}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_g \mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

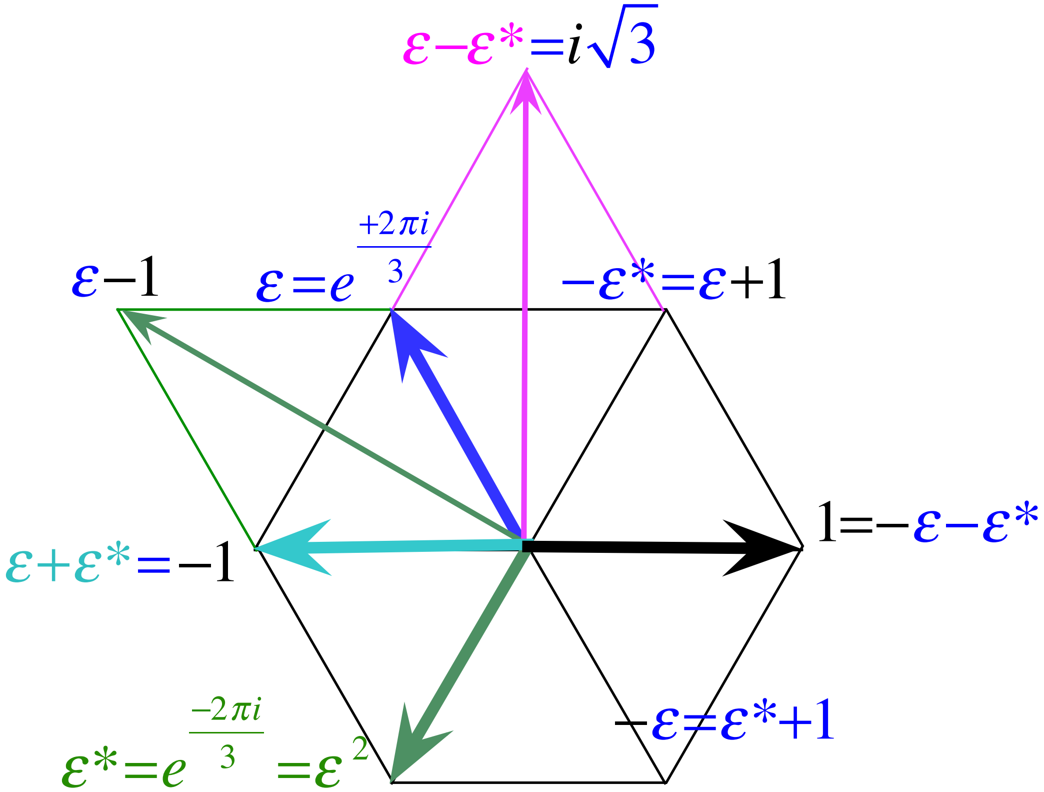
$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^\circ G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

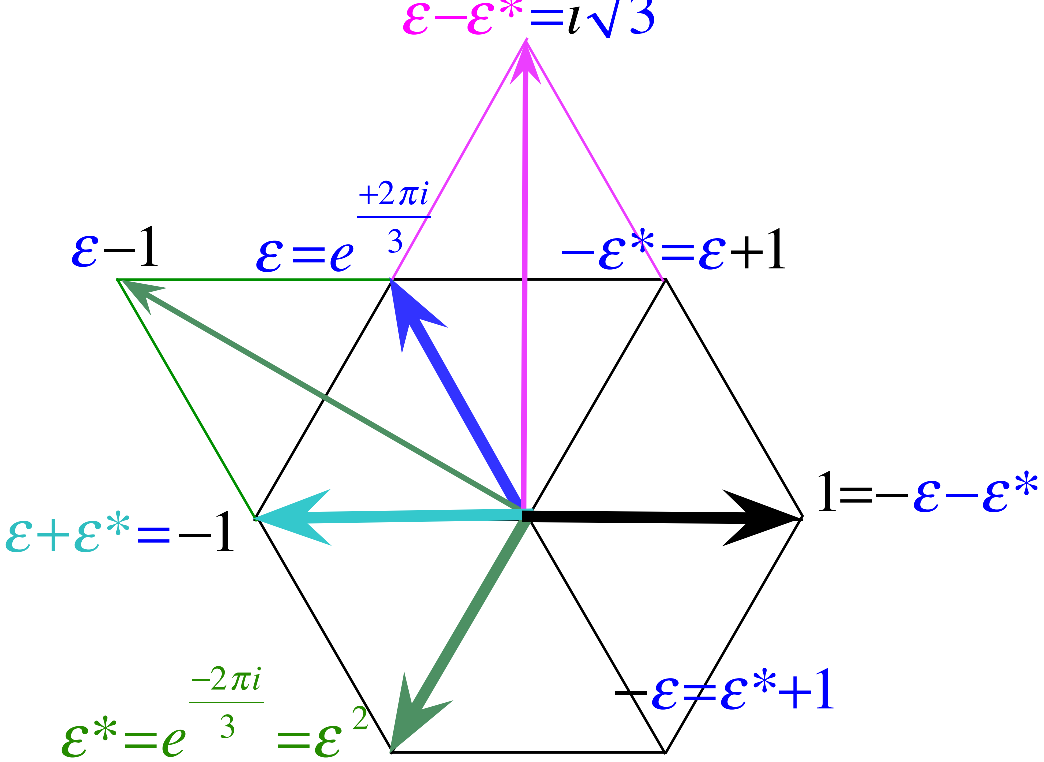
$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi_{g}^{\mu *}\mathbf{c_g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

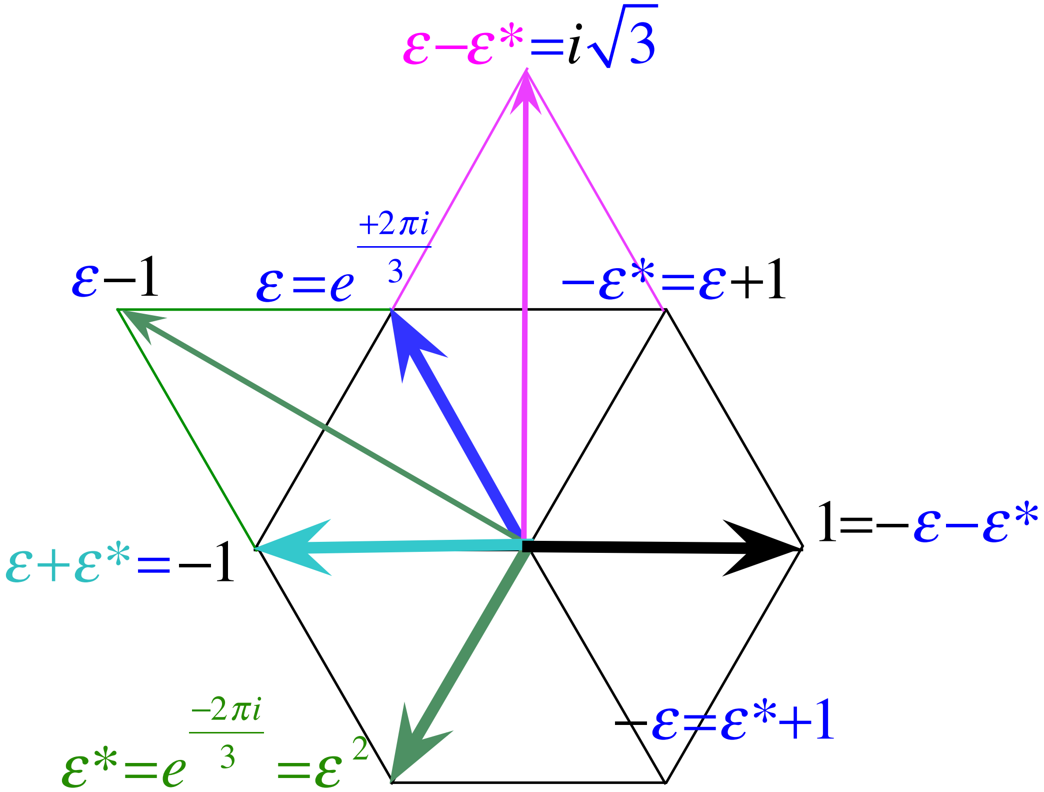
$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^*+1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g}=\sum_{\mu}\frac{{}^\circ c_g\chi^{\mu}_g}{\ell^{\mu}}\mathbb{P}^{\mu}}\qquad\boxed{\mathbb{P}^{\mu}=\sum_{classes\,c_{\mathbf{g}}}\frac{\ell^{\mu}}{{}^\circ G}\chi^{\mu *}_g\mathbf{c_g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r$: $(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}$

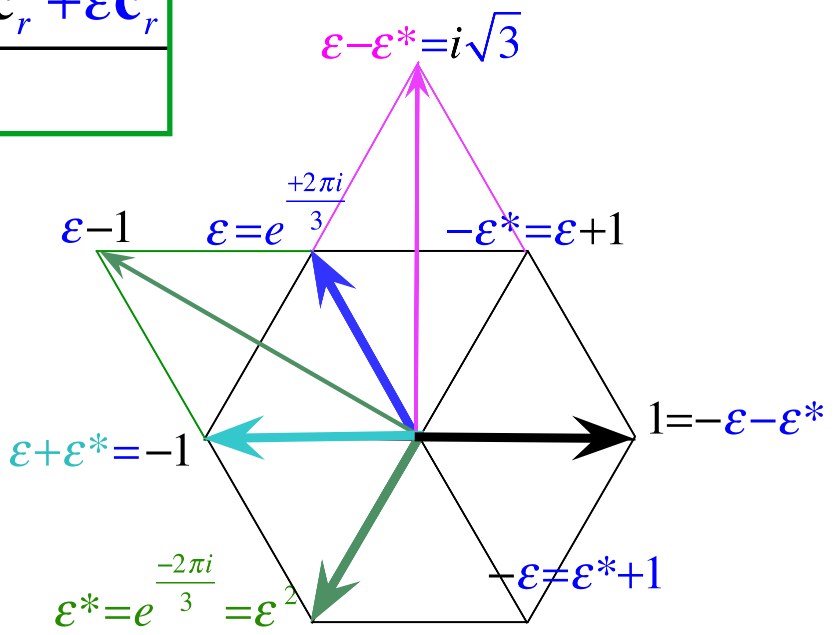
$$\mathbf{P}^{(4\varepsilon)}=\frac{(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon-4\varepsilon^*)(4\varepsilon-4)(4\varepsilon-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon^*+1)\mathbf{c}_r+16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon-\varepsilon^*)(\varepsilon-1)\varepsilon}$$

$$=\frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r+4(\varepsilon)4\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2-\varepsilon)}=\frac{16(\mathbf{1}+\mathbf{c}_\rho)+16\varepsilon\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon\tilde{\mathbf{c}}_r+\varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon^*-4\varepsilon)(4\varepsilon^*-4)(4\varepsilon^*-0)}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon^*\tilde{\mathbf{c}}_r+\varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-0)}{(4-4\varepsilon)(4-4\varepsilon^*)(4-0)}$$

$$\mathbf{P}^{(0)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})}{(0-4\varepsilon)(0-4\varepsilon^*)(0-4)}$$



$T:\mathbf{c}_g=$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
.				
$\chi^{\varepsilon}_g=$	1	ε^*	ε	1
$\chi^{\varepsilon^*}_g=$	1	ε	ε^*	1
.				

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r-4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g}=\sum_{\mu}\frac{{}^{\circ}c_g\chi^{\mu}_g}{\ell^{\mu}}\mathbb{P}^{\mu}}\qquad\boxed{\mathbb{P}^{\mu}=\sum_{classes\,c_\mathbf{g}}\frac{\ell^{\mu}}{{}^{\circ}G}\chi^{\mu*}_g\mathbf{c_g}}$$

T class products

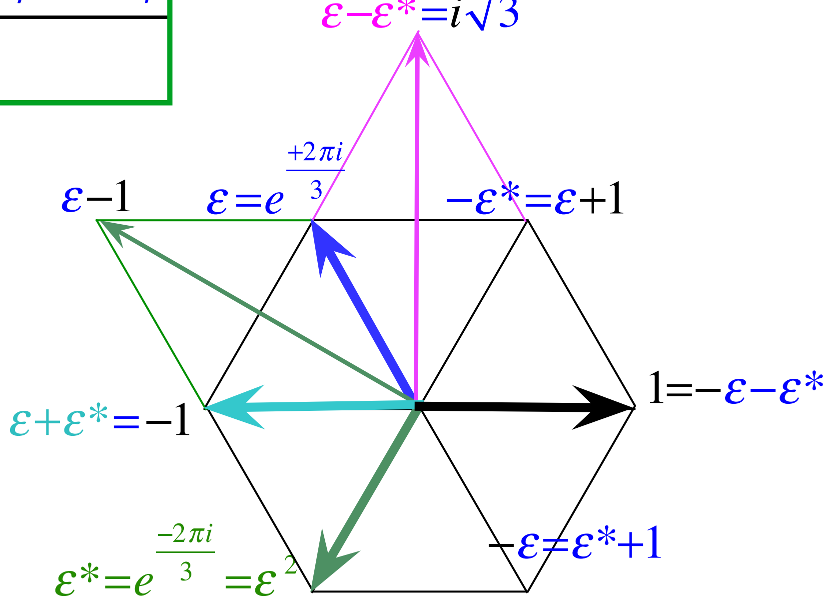
$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r:(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)}=\frac{(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon-4\varepsilon^*)(4\varepsilon-4)(4\varepsilon-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon^*+1)\mathbf{c}_r+16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon-\varepsilon^*)(\varepsilon-1)\varepsilon}$$

$$=\frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r+4(\varepsilon)4\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2-\varepsilon)}=\frac{16(\mathbf{1}+\mathbf{c}_\rho)+16\varepsilon\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon\tilde{\mathbf{c}}_r+\varepsilon^*\mathbf{c}_r}{12}}$$

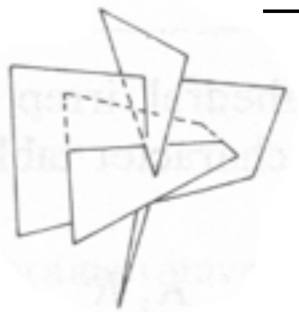
$$\mathbf{P}^{(4\varepsilon^*)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon^*-4\varepsilon)(4\varepsilon^*-4)(4\varepsilon^*-0)}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon^*\tilde{\mathbf{c}}_r+\varepsilon\mathbf{c}_r}{12}}$$



$$\mathbf{P}^{(4)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-0)}{(4-4\varepsilon)(4-4\varepsilon^*)(4-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon+\varepsilon^*)\mathbf{c}_r+16\mathbf{1})\mathbf{c}_r}{64(1-(\varepsilon+\varepsilon^*)+1)}$$

$$\mathbf{P}^{(0)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})}{(0-4\varepsilon)(0-4\varepsilon^*)(0-4)}$$

$T:\mathbf{c}_g=$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
.				
$\chi^{\varepsilon}_g=$	1	ε^*	ε	1
$\chi^{\varepsilon^*}_g=$	1	ε	ε^*	1
.				



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r-4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}}\mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi_{g}^{\mu *}\mathbf{c_g}}$$

T class products

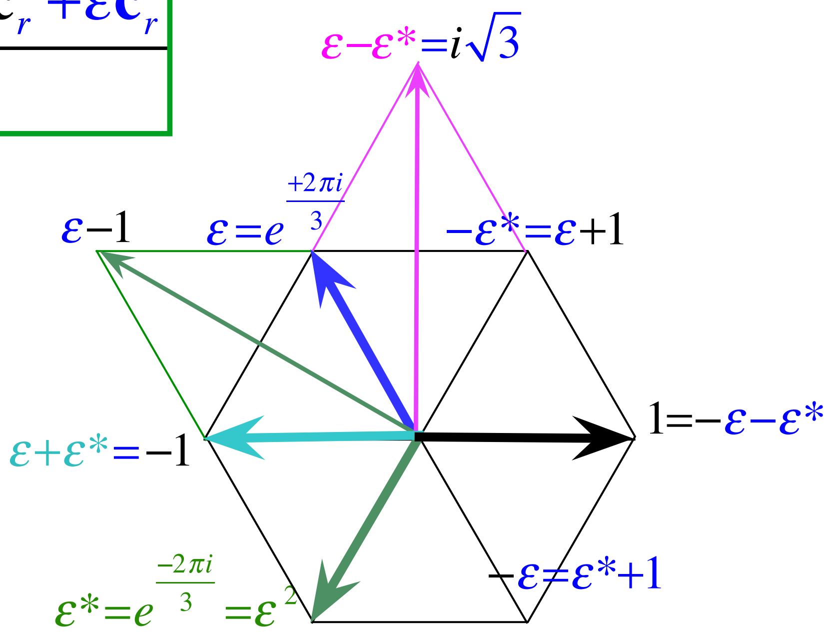
$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

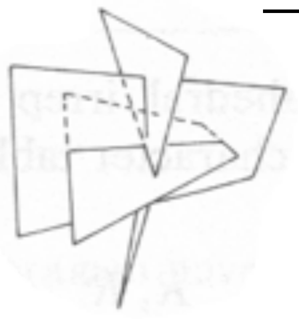


$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
.				
$\chi_g^{\varepsilon} =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
.				



Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

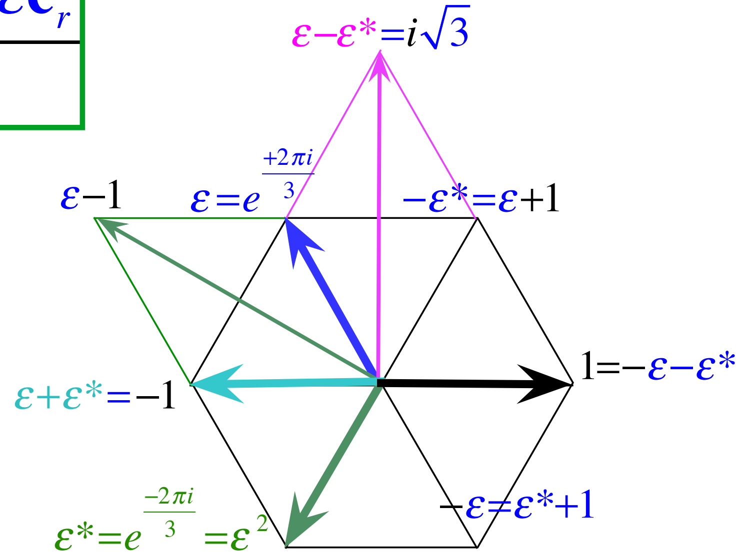
$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^*+1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1}+\mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1}+\mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

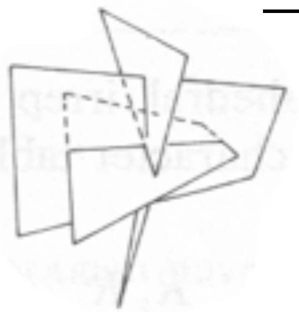
$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1}+\mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$



$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1}+\mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \boxed{\frac{\mathbf{1}+\mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$



$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A =$	1	1	1	1
$\chi_g^\varepsilon =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
.				

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g}=\sum_{\mu}\frac{{}^{\circ}c_g\chi_g^{\mu}}{\ell^{\mu}}\mathbb{P}^{\mu}}\qquad\boxed{\mathbb{P}^{\mu}=\sum_{classes\,c_{\mathbf{g}}}\frac{\ell^{\mu}}{{}^{\circ}G}\chi_g^{\mu*}\mathbf{c_g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
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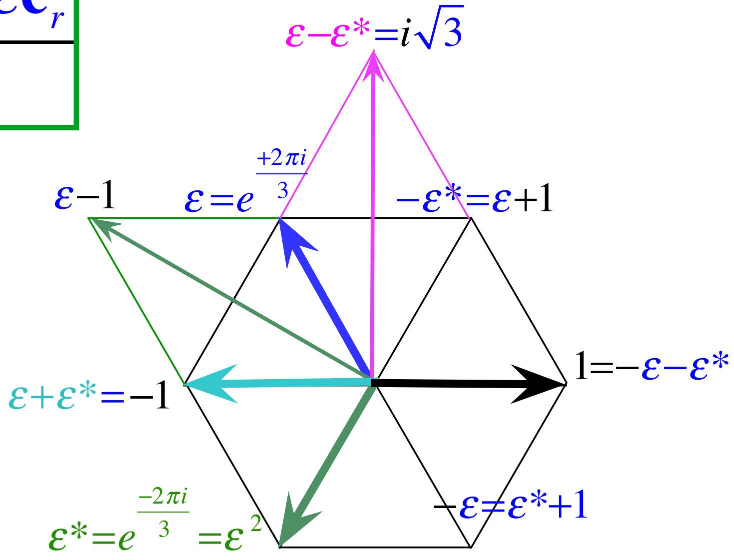
Minimal equation for $\mathbf{c}_r:(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)}=\frac{(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon-4\varepsilon^*)(4\varepsilon-4)(4\varepsilon-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon^*+1)\mathbf{c}_r+16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon-\varepsilon^*)(\varepsilon-1)\varepsilon}$$

$$=\frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r+4(\varepsilon)4\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2-\varepsilon)}=\frac{16(\mathbf{1}+\mathbf{c}_\rho)+16\varepsilon\tilde{\mathbf{c}}_r+16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon\tilde{\mathbf{c}}_r+\varepsilon^*\mathbf{c}_r}{12}}$$

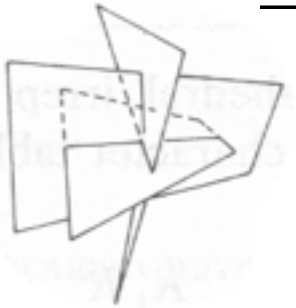
$$\mathbf{P}^{(4\varepsilon^*)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)}{(4\varepsilon^*-4\varepsilon)(4\varepsilon^*-4)(4\varepsilon^*-0)}\qquad=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\varepsilon^*\tilde{\mathbf{c}}_r+\varepsilon\mathbf{c}_r}{12}}$$



$$\mathbf{P}^{(4)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-0)}{(4-4\varepsilon)(4-4\varepsilon^*)(4-0)}=\frac{(\mathbf{c}_r^2-4(\varepsilon+\varepsilon^*)\mathbf{c}_r+16\mathbf{1})\mathbf{c}_r}{64(1-(\varepsilon+\varepsilon^*)+1)}$$

$$=\frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r+4\cdot4\tilde{\mathbf{c}}_r+16\mathbf{c}_r}{64(1+1+1)}=\frac{16(\mathbf{1}+\mathbf{c}_\rho)+16\tilde{\mathbf{c}}_r+16\mathbf{c}_r}{64(1+1+1)}\qquad=\boxed{\frac{\mathbf{1}+\mathbf{c}_\rho+\tilde{\mathbf{c}}_r+\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)}=\frac{(\mathbf{c}_r-4\varepsilon\mathbf{1})(\mathbf{c}_r-4\varepsilon^*\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})}{(0-4\varepsilon)(0-4\varepsilon^*)(0-4)}=\frac{(\mathbf{c}_r^2-4(\varepsilon+\varepsilon^*)\mathbf{c}_r+16\cdot\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$



$T:\mathbf{c}_g=$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A=$	1	1	1	1
$\chi_g^{\varepsilon}=$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*}=$	1	ε	ε^*	1

Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r-4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r-4\cdot\mathbf{1})(\mathbf{c}_r-0)=\mathbf{0}}$$

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
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Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^*+1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1}+\mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1}+\mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

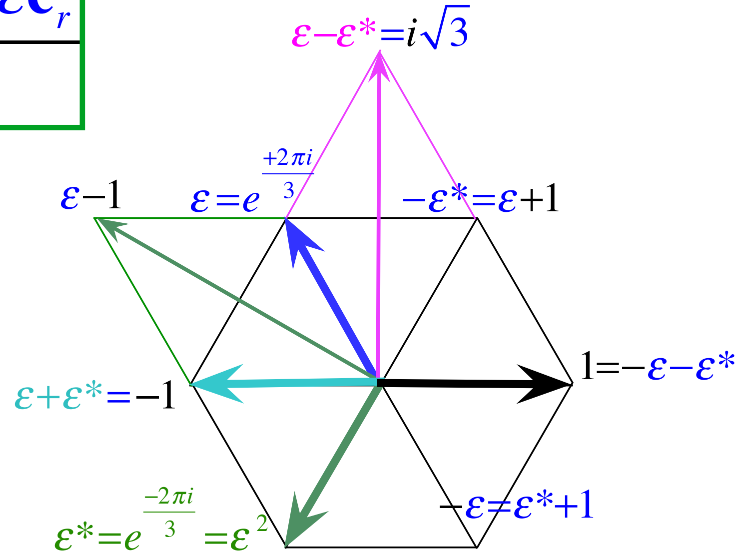
$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1}+\mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

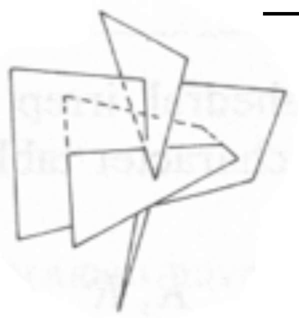
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1}+\mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \boxed{\frac{\mathbf{1}+\mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$





Minimal equation for $\mathbf{c}_r$

$$\boxed{(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}}$$

$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A =$	1	1	1	1
$\chi_g^\varepsilon =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

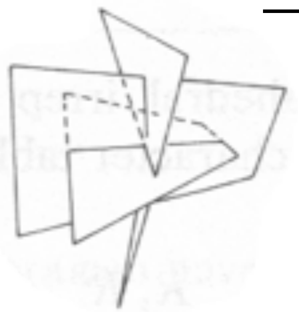
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64} =$$



$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi_g^A =$	1	1	1	1
$\chi_g^\varepsilon =$	1	ε^*	ε	1
$\chi_g^{\varepsilon^*} =$	1	ε	ε^*	1
.				

Tetrahedral T class projectors

$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{\mathbf{g}}\chi^{\mu}_{\mathbf{g}}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi^{\mu*}_{g}\mathbf{c_g}}$$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{\mathbf{1} + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64}$$

$$= \frac{4(4\mathbf{1} + 4\mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64}$$

T class products

$\mathbf{1} = \mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1} + 4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1} + 2\mathbf{c}_\rho$

$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi^{A}_{g} =$	1	1	1	1
$\chi^{\varepsilon}_{g} =$	1	ε^*	ε	1
$\chi^{\varepsilon^*}_{g} =$	1	ε	ε^*	1
.				

Tetrahedral T class characters

$$\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c}_{\mathbf{g}}$$

T class products

$\mathbf{1}=\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\mathbf{c}_r$	$4\tilde{\mathbf{c}}_r$	$4\mathbf{1}+4\mathbf{c}_\rho$	$3\mathbf{c}_r$
$\tilde{\mathbf{c}}_r$		$4\mathbf{c}_r$	$3\tilde{\mathbf{c}}$
$\mathbf{c}_\rho$			$3\mathbf{1}+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_r : (\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = \mathbf{0}$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^*+1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1} + \mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \boxed{\frac{\mathbf{1} + \mathbf{c}_\rho + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64}$$

$$= \frac{4(4\mathbf{1} + 4\mathbf{c}_\rho) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1})}{-64} = \frac{-48\cdot\mathbf{1} + 16\mathbf{c}_\rho}{-64} \quad \boxed{= \frac{3}{4}\mathbf{1} - \frac{1}{4}\mathbf{c}_\rho}$$

$T : \mathbf{c}_g =$	$\mathbf{c}_1$	$\mathbf{c}_r$	$\tilde{\mathbf{c}}_r$	$\mathbf{c}_\rho$
$\chi^{A}_{g} =$	1	1	1	1
$\chi^{\varepsilon}_{g} =$	1	ε^*	ε	1
$\chi^{\varepsilon^*}_{g} =$	1	ε	ε^*	1
$\chi^{T}_{g} =$	3	0	0	-1

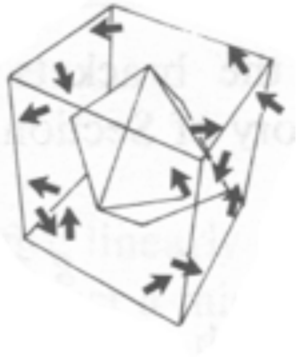
- *Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$*
    - *Octahedral-cubic O symmetry and group operations*
    - *Tetrahedral symmetry becomes Icosahedral*
    - *Octahedral groups $O_h \supset O \sim T_d \supset T$*
        - *Octahedral O and spin-$O \subset U(2)$*

- *Tetrahedral T class algebra*
    - *Tetrahedral T class minimal equations*
    - *Tetrahedral T class projectors and characters*

- *Octahedral O class algebra*
    - *Octahedral O class minimal equations*
    - *Octahedral O class projectors and characters*

- *Octahedral $O_h \supset O$ subgroup correlations*
    - *Octahedral $O_h \supset O$ subgroup correlations*
    - *$O_h \supset O \supset D_4$ subgroup correlations*
    - *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
    - *Preview of applications to high resolution spectroscopy*

Octahedral O class algebra



$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4+\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

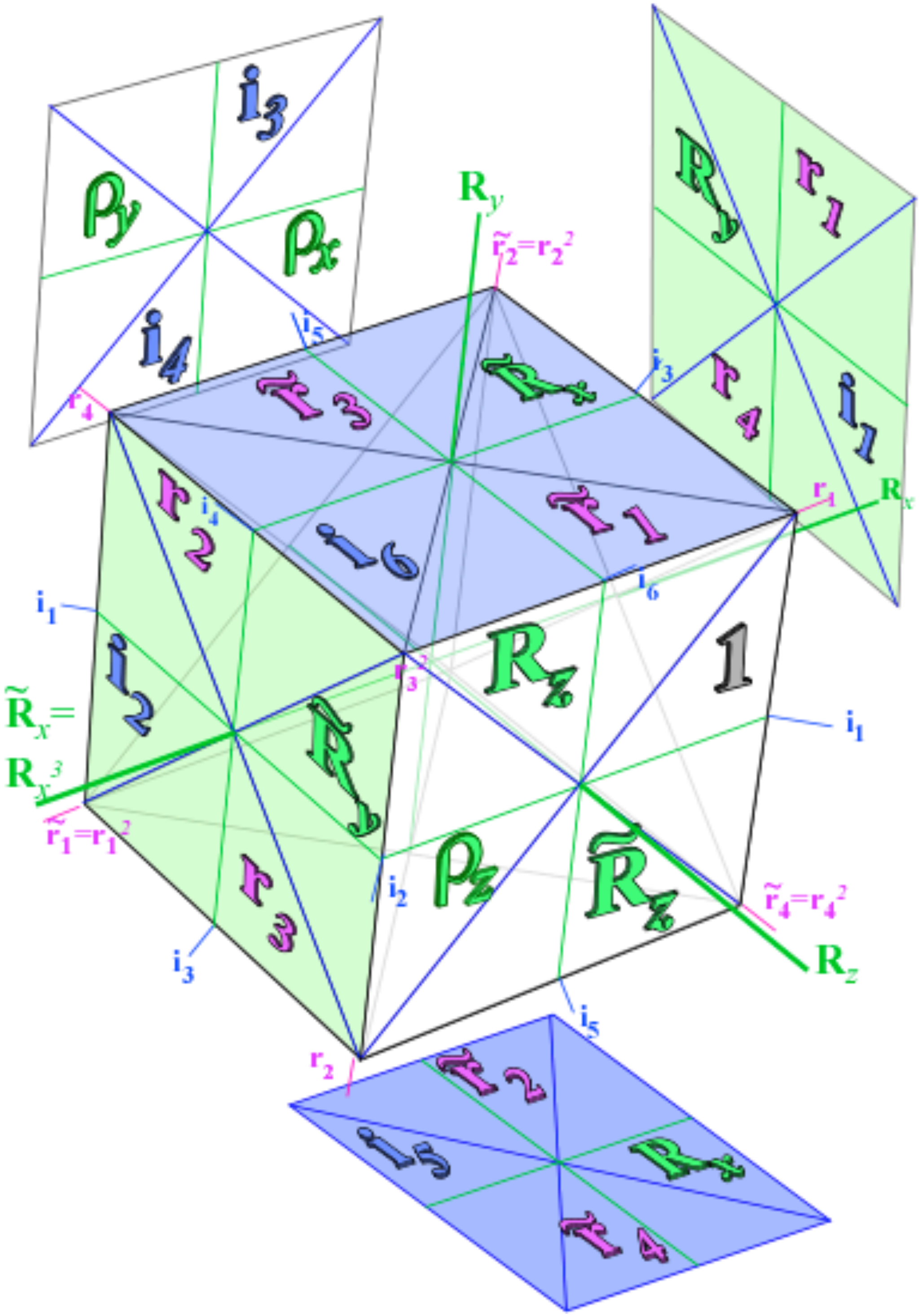
$\mathbf{c}_R=\mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3+\mathbf{R}_1{}^3+\mathbf{R}_2{}^3+\mathbf{R}_3{}^3$, $\mathbf{c}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3+\mathbf{i}_4+\mathbf{i}_5+\mathbf{i}_6$

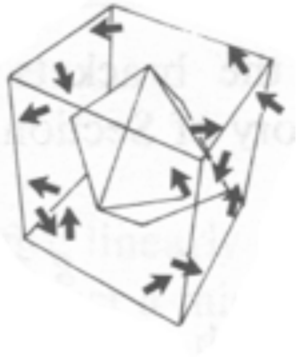
O class products

Unnecessary to do $24^2=576$ products since each row (or column) of $\mathbf{c}_A\mathbf{c}_B$ *has same class proportion*

For example:

$$\mathbf{c}_\rho\mathbf{c}_i = \mathbf{R}_1{}^2\mathbf{i}_1+....= \mathbf{R}_2 +....$$
$$+\mathbf{R}_2{}^2\mathbf{i}_1+.... \quad + \mathbf{i}_2 +....$$
$$+\mathbf{R}_3{}^2\mathbf{i}_1+.... \quad +\mathbf{R}_2{}^3+$$





Octahedral O class algebra

$\mathbf{c}_1=\mathbf{1}$, $\mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4+\mathbf{r}_1{}^2+\mathbf{r}_2{}^2+\mathbf{r}_3{}^2+\mathbf{r}_4{}^2$, $\mathbf{c}_\rho=\mathbf{R}_1{}^2+\mathbf{R}_2{}^2+\mathbf{R}_3{}^2$,

$\mathbf{c}_R=\mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3+\mathbf{R}_1{}^3+\mathbf{R}_2{}^3+\mathbf{R}_3{}^3$, $\mathbf{c}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3+\mathbf{i}_4+\mathbf{i}_5+\mathbf{i}_6$

O class products

Unnecessary to do $24^2=576$ *products since each row (or column) of* $\mathbf{c}_A\mathbf{c}_B$ *has same class proportion*

For example:

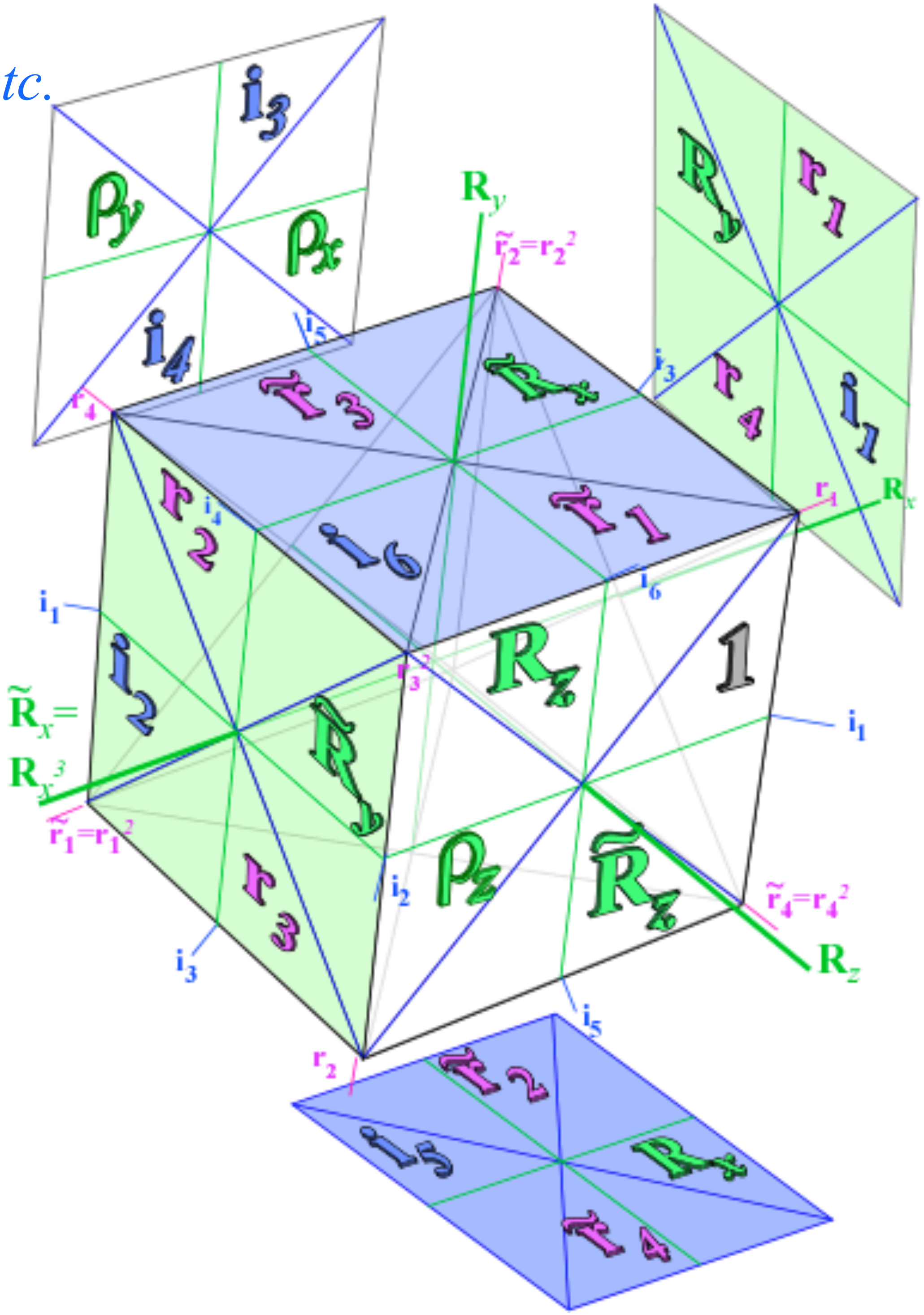
$\mathbf{c}_{R2}\mathbf{c}_i = \mathbf{R}_1{}^2\mathbf{i}_1+\ldots.= \mathbf{R}_2+\ldots.$

$+\mathbf{R}_2{}^2\mathbf{i}_1+\ldots. \quad +\mathbf{i}_2+\ldots.$

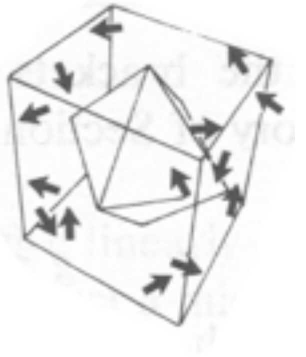
$+\mathbf{R}_3{}^2\mathbf{i}_1+\ldots. \quad +\mathbf{R}_2{}^3+$

So there are $2\mathbf{c}_R$ *for each* $\mathbf{c}_i$ *:*

$\mathbf{c}_\rho\mathbf{c}_i=2\mathbf{c}_R+\mathbf{c}_i$ *or:* $4\mathbf{c}_R+2\mathbf{c}_i$ *etc.*



Octahedral O class algebra



$$\mathbf{c}_1=\mathbf{1},\quad \mathbf{c}_r=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4+\mathbf{r}_1^2+\mathbf{r}_2^2+\mathbf{r}_3^2+\mathbf{r}_4^2,\quad \mathbf{c}_\rho=\mathbf{R}_1^2+\mathbf{R}_2^2+\mathbf{R}_3^2,$$

$$\mathbf{c}_R=\mathbf{R}_1+\mathbf{R}_2+\mathbf{R}_3+\mathbf{R}_1^3+\mathbf{R}_2^3+\mathbf{R}_3^3,\quad \mathbf{c}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3+\mathbf{i}_4+\mathbf{i}_5+\mathbf{i}_6$$

O class products

Unnecessary to do $24^2=576$ products since each row (or column) of $\mathbf{c}_A\mathbf{c}_B$ has same class proportion

For example: $\mathbf{c}_\rho\mathbf{c}_i=?$

$$\mathbf{c}_{R2}\mathbf{c}_i = \mathbf{R}_1^2\mathbf{i}_1+.... = \mathbf{R}_2+....$$

$$+\mathbf{R}_2^2\mathbf{i}_1+.... \quad +\mathbf{i}_2+....$$

$$+\mathbf{R}_3^2\mathbf{i}_1+.... \quad +\mathbf{R}_2^3+$$

So there are 2$\mathbf{c}_R$ *for each* $\mathbf{c}_i$ ....in $(^\circ c_\rho)\cdot(^\circ c_i)=(3)\cdot(6)=18$ terms

$\mathbf{c}_\rho\mathbf{c}_i=2\mathbf{c}_R+\mathbf{c}_i$ *or:* $4\mathbf{c}_R+2\mathbf{c}_i$ *etc.*

So: $2(^\circ c_R)+(^\circ c_i)=2\cdot 6+6=18$

Proof that class proportion cannot vary:

$$\mathbf{c}_g\mathbf{c}_h = \mathbf{g}_1\mathbf{h}_1+\mathbf{g}_2\mathbf{h}_1+....=\mathbf{g}_1\mathbf{h}_1+\mathbf{t}\mathbf{g}_1\mathbf{h}_1\mathbf{t}^{-1}+....=\mathbf{g}_1\mathbf{h}_1+\mathbf{t}\mathbf{g}_1\mathbf{t}^{-1}\mathbf{t}\mathbf{h}_1\mathbf{t}^{-1}+....=\mathbf{g}_1\mathbf{h}_1+\mathbf{g}_2\ \mathbf{t}\mathbf{h}_1\mathbf{t}^{-1}+....$$

$$+\mathbf{g}_1\mathbf{h}_2+\mathbf{g}_2\mathbf{h}_2+.... \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{t}\mathbf{g}_1\mathbf{h}_2\mathbf{t}^{-1}+.... \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{t}\mathbf{g}_1\mathbf{t}^{-1}\mathbf{t}\mathbf{h}_2\mathbf{t}^{-1}+.... \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{g}_2\ \mathbf{t}\mathbf{h}_2\mathbf{t}^{-1}+....$$

$$=\mathbf{g}_1\mathbf{h}_3+\mathbf{g}_2\mathbf{h}_3+.... \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{t}\mathbf{g}_1\mathbf{h}_3\mathbf{t}^{-1}+.... \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{t}\mathbf{g}_1\mathbf{t}^{-1}\mathbf{t}\mathbf{h}_3\mathbf{t}^{-1}+.... \quad +\mathbf{g}_1\mathbf{h}_2+\mathbf{g}_2\ \mathbf{t}\mathbf{h}_3\mathbf{t}^{-1}+....$$

O class product table

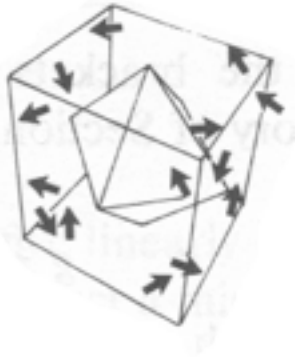
$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_\rho$

$$\mathbf{c}_\rho^2=3\mathbf{1}+2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2-2\mathbf{c}_\rho-3\mathbf{1}=\mathbf{0}$$

$$(\mathbf{c}_\rho-3\mathbf{1})(\mathbf{c}_\rho+\mathbf{1})=\mathbf{0}$$



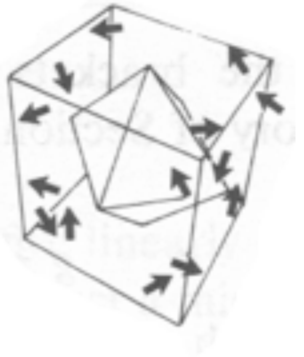
Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
- Octahedral-cubic O symmetry and group operations
- Tetrahedral symmetry becomes Icosahedral
- Octahedral groups $O_h \supset O \sim T_d \supset T$
  - Octahedral O and spin-$O \subset U(2)$

Tetrahedral T class algebra
- Tetrahedral T class minimal equations
- Tetrahedral T class projectors and characters

➡ Octahedral O class algebra ⬅
- Octahedral O class minimal equations
- Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations
- Octahedral $O_h \supset O$ subgroup correlations
- $O_h \supset O \supset D_4$ subgroup correlations
- $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations
- Preview of applications to high resolution spectroscopy

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\,\chi^{\mu}_{g}}{\ell^{\mu}}\,\mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\,\chi^{\mu *}_{g}\,\mathbf{c_g}$$

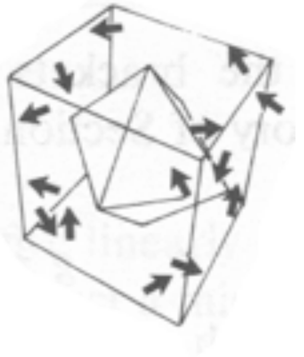
O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}$$

O class product table

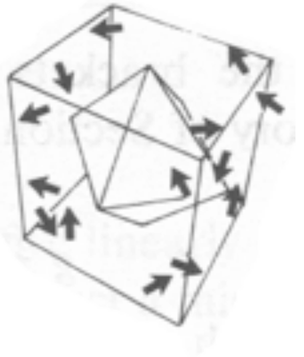
$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

Octahedral O class minimal equations



$$\boxed{\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^\circ c_g \chi^\mu_g}{\ell^\mu} \mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^\mu}{{}^\circ G} \chi^{\mu *}_g \mathbf{c}_{\mathbf{g}}}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

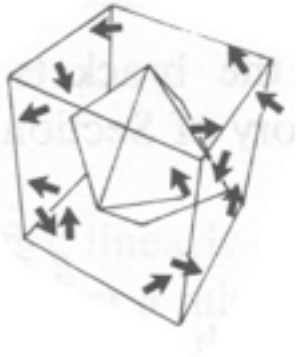
Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2=6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3=6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

Octahedral O class minimal equations



$$\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_{g}^{\mu *} \mathbf{c}_{\mathbf{g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

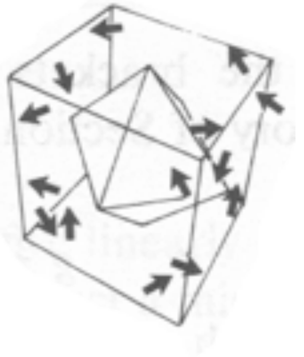
$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes}\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

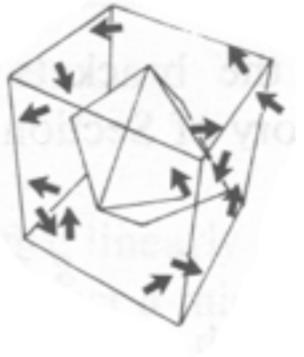
$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad + 20\mathbf{c}_i\mathbf{c}_i$$

Octahedral O class minimal equations



$$\boxed{\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu} \mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G} \chi_g^{\mu *} \mathbf{c}_\mathbf{g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

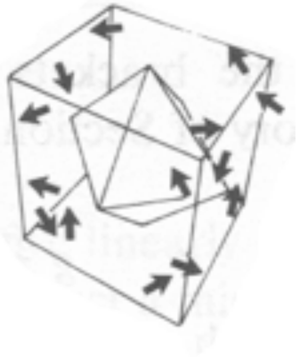
$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad + 20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

Octahedral O class minimal equations



$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

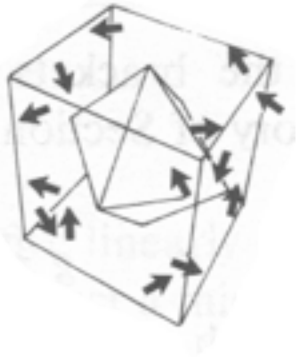
$$\mathbf{c}_i^3 = 16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu *} \mathbf{c_g}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

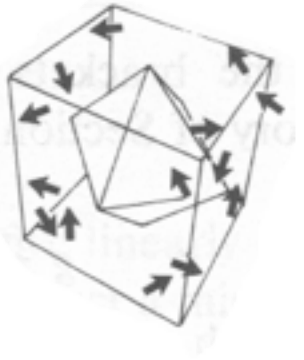
$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad + 20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120\cdot\mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho)$$

$$= 120\cdot\mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

Octahedral O class minimal equations



$$\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi_{g}^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_{g}^{\mu *} \mathbf{c}_{\mathbf{g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad + 20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120\cdot\mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho)$$

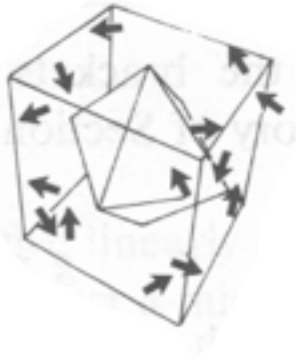
$$= 120\cdot\mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r\mathbf{c}_i \qquad + 104\mathbf{c}_\rho\mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2=6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3=6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i^3=16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4= \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$=16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$=48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

$$=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$$

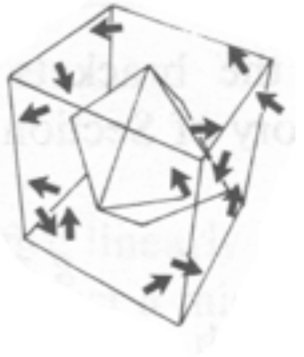
$$\mathbf{c}_i^5=120\mathbf{c}_i+108\mathbf{c}_r\mathbf{c}_i \qquad +104\mathbf{c}_\rho\mathbf{c}_i$$

$$=120\mathbf{c}_i+108(4\mathbf{c}_R+4\mathbf{c}_i)+104(2\mathbf{c}_R+\mathbf{c}_i)$$

$$=640\mathbf{c}_R+656\mathbf{c}_i$$

$$40\mathbf{c}_i^3=640\mathbf{c}_R+800\mathbf{c}_i$$

Octahedral O class minimal equations



$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g} \mathbf{c_g}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = \quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

$$= 120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i+108\mathbf{c}_r\mathbf{c}_i \qquad +104\mathbf{c}_\rho\mathbf{c}_i$$

$$= 120\mathbf{c}_i+108(4\mathbf{c}_R+4\mathbf{c}_i)+104(2\mathbf{c}_R+\mathbf{c}_i)$$

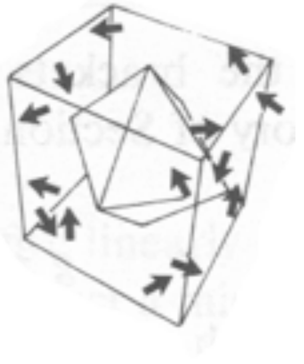
$$= 640\mathbf{c}_R+656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R+800\mathbf{c}_i$$

$$\mathbf{c}_i^5 - 40\mathbf{c}_i^3 + 144\mathbf{c}_i = \mathbf{0}$$

$$\begin{array}{r} 800 \\ -\underline{656} \\ 144 \end{array}$$

Octahedral O class minimal equations



$$\boxed{\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{\mathbf{g}}\chi^{\mu}_{\mathbf{g}}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{\mathbf{g}} \mathbf{c}_{\mathbf{g}}}$$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2 = 6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6\cdot\mathbf{1}\mathbf{c}_i + 3\mathbf{c}_r\mathbf{c}_i + 2\mathbf{c}_\rho\mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R\mathbf{c}_i + 20\mathbf{c}_i\mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6\cdot\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120\cdot\mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho)$$

$$= 120\cdot\mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r\mathbf{c}_i + 104\mathbf{c}_\rho\mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

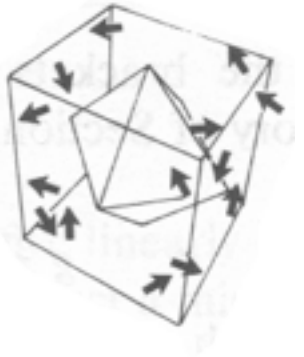
$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

$$40\mathbf{c}_i^3 = 640\mathbf{c}_R + 800\mathbf{c}_i$$

$$\mathbf{c}_i^5 - 40\mathbf{c}_i^3 + 144\mathbf{c}_i = \mathbf{0} = (\mathbf{c}_i^2 - 36\cdot\mathbf{1})(\mathbf{c}_i^2 - 4\cdot\mathbf{1})(\mathbf{c}_i - 0\cdot\mathbf{1})$$

$$\begin{array}{r} 800 \\ \underline{-656} \\ 144 \end{array}$$

Octahedral O class minimal equations



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^\circ c_g \chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

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O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Minimal equation for $\mathbf{c}_i$

$$\mathbf{c}_i^2=6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3=6\cdot\mathbf{1}\mathbf{c}_i+3\mathbf{c}_r\mathbf{c}_i+2\mathbf{c}_\rho\mathbf{c}_i$$

$$=6\mathbf{c}_i+3(4\mathbf{c}_R+4\mathbf{c}_i)+2(2\mathbf{c}_R+\mathbf{c}_i)$$

$$\mathbf{c}_i^3=16\mathbf{c}_R+20\mathbf{c}_i$$

$$\mathbf{c}_i^4=\quad 16\mathbf{c}_R\mathbf{c}_i \quad +20\mathbf{c}_i\mathbf{c}_i$$

$$=16(3\mathbf{c}_r+4\mathbf{c}_\rho)+20(6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho)$$

$$=48\mathbf{c}_r+64\mathbf{c}_\rho+120\cdot\mathbf{1}+60\mathbf{c}_r+40\mathbf{c}_\rho)$$

$$=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5=120\mathbf{c}_i+108\mathbf{c}_r\mathbf{c}_i \qquad +104\mathbf{c}_\rho\mathbf{c}_i$$

$$=120\mathbf{c}_i+108(4\mathbf{c}_R+4\mathbf{c}_i)+104(2\mathbf{c}_R+\mathbf{c}_i)$$

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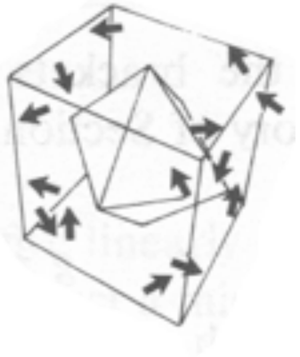
$$\mathbf{c}_i^5-40\mathbf{c}_i^3+144\mathbf{c}_i=\mathbf{0}=(\mathbf{c}_i^2-36\cdot\mathbf{1})(\mathbf{c}_i^2-4\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$$

$$\begin{array}{r} 800 \\ \underline{-656} \\ 144 \end{array}$$

Minimal equation for $\mathbf{c}_i$

$$\boxed{\mathbf{0}=(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-2\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})}$$



- *Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$*
  - *Octahedral-cubic O symmetry and group operations*
  - *Tetrahedral symmetry becomes Icosahedral*
  - *Octahedral groups $O_h \supset O \sim T_d \supset T$*
    - *Octahedral O and spin-$O \subset U(2)$*

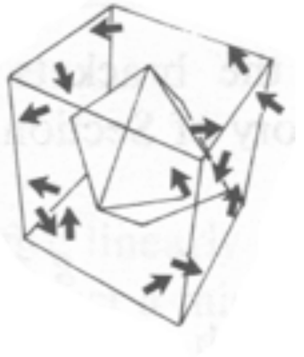
- *Tetrahedral T class algebra*
  - *Tetrahedral T class minimal equations*
  - *Tetrahedral T class projectors and characters*

- *Octahedral O class algebra*
  - *Octahedral O class minimal equations*
  - *Octahedral O class projectors and characters*

- *Octahedral $O_h \supset O$ subgroup correlations*
  - *Octahedral $O_h \supset O$ subgroup correlations*
  - *$O_h \supset O \supset D_4$ subgroup correlations*
  - *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
  - *Preview of applications to high resolution spectroscopy*

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}=(\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c_g}=\sum_{\mu}\frac{{}^{\circ}c_g\chi_g^{\mu}}{\ell^{\mu}}\mathbb{P}^{\mu}$$

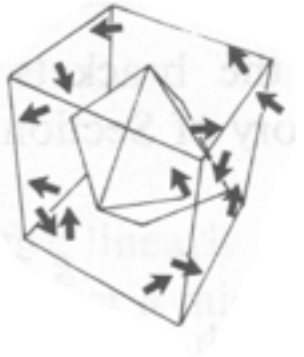
$$\mathbb{P}^{\mu}=\sum_{classes\, c_{\mathbf{g}}}\frac{\ell^{\mu}}{{}^{\circ}G}\chi_g^{\mu *}\mathbf{c_g}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra



Begin with minimal equation: $0= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$

$$\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_g \chi^\mu_g}{\ell^\mu} \mathbb{P}^\mu$$

$$\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G} \chi^{\mu *}_g \mathbf{c}_\mathbf{g}$$

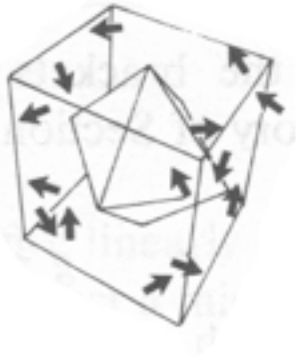
$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})(\cancel{\mathbf{c}_i-2\cdot\mathbf{1}})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

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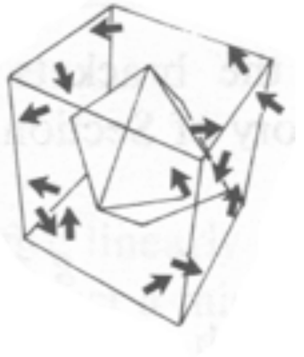
$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\boxed{\mathbf{0} = (\mathbf{c}_i + 2\cdot\mathbf{1})\cancel{(\mathbf{c}_i - 2\cdot\mathbf{1})}(\mathbf{c}_i + 6\cdot\mathbf{1})(\mathbf{c}_i - 6\cdot\mathbf{1})(\mathbf{c}_i - 0\cdot\mathbf{1})}$



$$\boxed{\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_{g}\chi^{\mu}_{g}}{\ell^{\mu}}\mathbb{P}^{\mu}}$$

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$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}_i - 6\cdot\mathbf{1})(\mathbf{c}_i + 6\cdot\mathbf{1})(\mathbf{c}_i - 0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i + 2\cdot\mathbf{1})(\mathbf{c}^2{}_i - 36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}^4{}_i + 2\cdot\mathbf{c}^3{}_i - 36\,\mathbf{c}^2{}_i - 72\,\mathbf{c}\,{}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

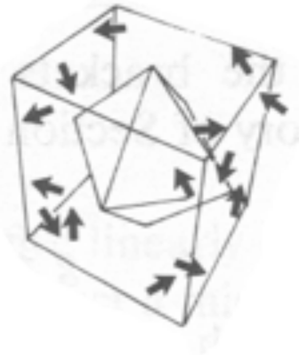
$\mathbf{c}_i =$ $+$ $\mathbf{c}_i$

O class product table

$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})(\cancel{\mathbf{c}_i-2\cdot\mathbf{1}})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



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$$\boxed{\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G} \chi_g^{\mu *} \mathbf{c}_\mathbf{g}}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^2 = \quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

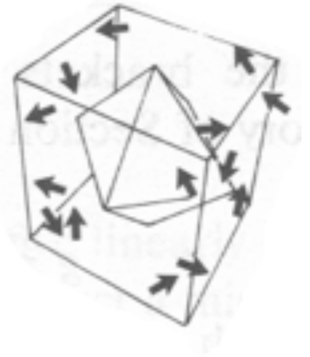
$\mathbf{c}_i = \qquad\qquad\qquad + \quad \mathbf{c}_i$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_g \chi^\mu_g}{\ell^\mu}\,\mathbb{P}^\mu$$

$$\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi^{\mu *}_g\,\mathbf{c}_\mathbf{g}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^3 = \qquad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i^2 = \quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

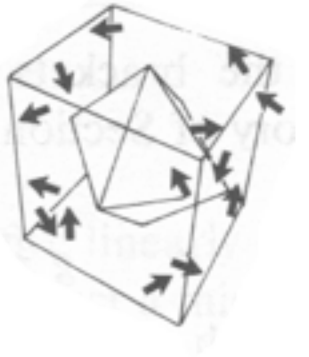
$\mathbf{c}_i = \qquad + \quad \mathbf{c}_i$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c}_\mathbf{g}=\sum_\mu \frac{{}^\circ c_g \chi^\mu_g}{\ell^\mu}\mathbb{P}^\mu$$

$$\mathbb{P}^\mu=\sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi^{\mu *}_g \mathbf{c}_\mathbf{g}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}=\frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i{}^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i{}^3=\qquad\qquad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i{}^2=\quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

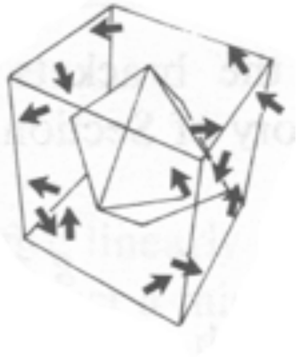
$\mathbf{c}_i\ =\qquad\qquad +\quad \mathbf{c}_i$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c}_\mathbf{g}=\sum_{\mu}\frac{{}^{\circ}c_g\chi_g^{\mu}}{\ell^{\mu}}\mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu}=\sum_{classes\, c_\mathbf{g}}\frac{\ell^{\mu}}{{}^{\circ}G}\chi_g^{\mu *}\mathbf{c}_\mathbf{g}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}=\frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i^3= \qquad\qquad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i^2= \quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

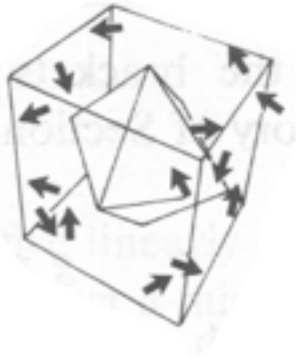
$\mathbf{c}_i = \qquad\qquad + \quad \mathbf{c}_i$

$\mathbf{c}_i^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra



Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$

$$\mathbf{c_g}=\sum_{\mu}\frac{{}^{\circ}c_g\chi_g^{\mu}}{\ell^{\mu}}\mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu}=\sum_{classes\, c_{\mathbf{g}}}\frac{\ell^{\mu}}{{}^{\circ}G}\chi_g^{\mu *}\mathbf{c_g}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}=\frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i{}^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i{}^3=\quad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i{}^2=\quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

$\mathbf{c}_i\ =\quad +\quad \mathbf{c}_i$

$\mathbf{c}_i{}^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

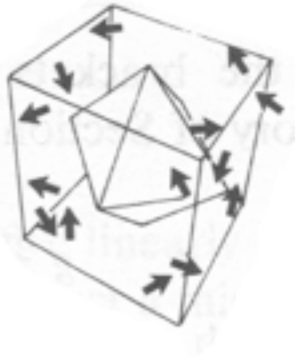
$+2\mathbf{c}_i{}^3=\quad +32\mathbf{c}_R+40\mathbf{c}_i$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_g \chi^\mu_g}{\ell^\mu}\mathbb{P}^\mu$$

$$\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi^{\mu *}_g \mathbf{c}_\mathbf{g}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^4 = 120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i^3 = \qquad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i^2 = \quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

$\mathbf{c}_i = \qquad + \quad \mathbf{c}_i$

$\mathbf{c}_i^4 = 120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$+2\mathbf{c}_i^3 = \qquad +32\mathbf{c}_R+40\mathbf{c}_i$

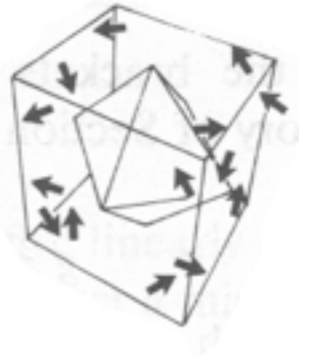
$-36\mathbf{c}_i^2 = -216\cdot\mathbf{1}-108\mathbf{c}_r \quad -72\mathbf{c}_\rho$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c_g} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_g}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_\mathbf{g}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_g \mathbf{c_g}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\mathbf{1})(\mathbf{c}_i-6\mathbf{1})(\mathbf{c}_i+6\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\mathbf{1})(\mathbf{c}_i^2-36\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4+2\mathbf{c}_i^3-36\mathbf{c}_i^2-72\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^4 = 120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i^3 = \qquad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i^2 = \quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

$\mathbf{c}_i = \qquad + \quad \mathbf{c}_i$

$\mathbf{c}_i^4 = 120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$+2\mathbf{c}_i^3 = \qquad +32\mathbf{c}_R+40\mathbf{c}_i$

$-36\mathbf{c}_i^2 = -216\cdot\mathbf{1}-108\mathbf{c}_r \quad -72\mathbf{c}_\rho$

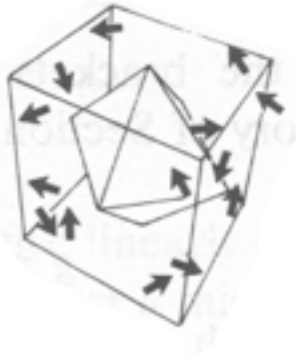
$-72\mathbf{c}_i = \qquad -72\mathbf{c}_i$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}c_g \chi^{\mu}_g}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_g \mathbf{c}_{\mathbf{g}}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i^3= \quad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i^2= \quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

$\mathbf{c}_i = \quad + \quad \mathbf{c}_i$

$\mathbf{c}_i^4 =120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$+2\mathbf{c}_i^3 = \quad +32\mathbf{c}_R+40\mathbf{c}_i$

$-36\mathbf{c}_i^2=-216\cdot\mathbf{1}-108\mathbf{c}_r \quad -72\mathbf{c}_\rho$

$-72\mathbf{c}_i = \quad -72\mathbf{c}_i$

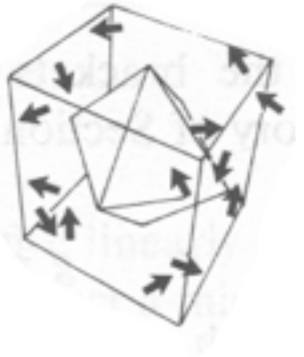
$-256\mathbf{P}^{(2)} =-96\cdot\mathbf{1}+ 0\ \mathbf{c}_r+32\mathbf{c}_\rho +32\mathbf{c}_R - \ 32\mathbf{c}_i$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\boxed{\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})\cancel{(\mathbf{c}_i-2\cdot\mathbf{1})}(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})}$



$$\boxed{\mathbf{c}_\mathbf{g} = \sum_\mu \frac{{}^\circ c_g \chi^\mu_g}{\ell^\mu}\mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi^{\mu*}_g \mathbf{c}_\mathbf{g}}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^4 = 120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i^3 = \quad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i^2 = \quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

$\mathbf{c}_i = \quad + \quad \mathbf{c}_i$

$\mathbf{c}_i^4 = 120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

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$-256\mathbf{P}^{(2)} = -96\cdot\mathbf{1}+ 0\ \mathbf{c}_r+32\mathbf{c}_\rho +32\mathbf{c}_R - \ 32\mathbf{c}_i$

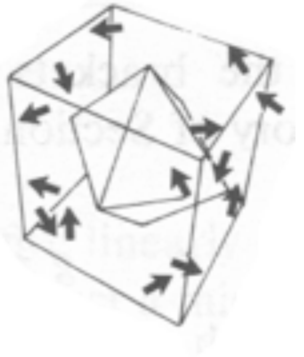
$$\mathbf{P}^{(2)} = \boxed{\frac{3}{8}\mathbf{1}-\frac{0}{8}\mathbf{c}_r+\frac{1}{8}\mathbf{c}_\rho-\frac{1}{8}\mathbf{c}_R+\frac{1}{8}\mathbf{c}_i}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})(\cancel{\mathbf{c}_i-2\cdot\mathbf{1}})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\boxed{\mathbf{c}_\mathbf{g}=\sum_\mu \frac{{}^\circ c_g \chi_g^\mu}{\ell^\mu}\mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu=\sum_{classes\, c_\mathbf{g}} \frac{\ell^\mu}{{}^\circ G}\chi_g^{\mu *}\mathbf{c}_\mathbf{g}}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}=\frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i^3= \qquad +16\mathbf{c}_R+20\mathbf{c}_i$

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$\mathbf{c}_i^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

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$-36\mathbf{c}_i^2=-216\cdot\mathbf{1}-108\mathbf{c}_r \quad -72\mathbf{c}_\rho$

$-72\mathbf{c}_i = \qquad -72\mathbf{c}_i$

$-256\mathbf{P}^{(2)}=-96\cdot\mathbf{1}+0\,\mathbf{c}_r+32\mathbf{c}_\rho+32\mathbf{c}_R-\ 32\mathbf{c}_i$

$$\mathbf{P}^{(2)}=\boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho-\frac{1}{8}\mathbf{c}_R+\frac{1}{8}\mathbf{c}_i}$$

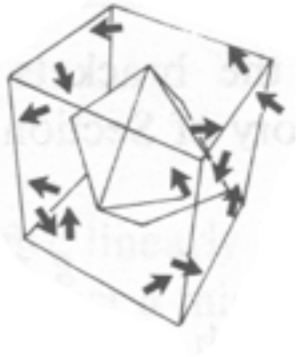
O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Applying the conventional label T_2 for (2)

χ_g^μ	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
·					
·					
·					
·					
χ^{T_2}	3	0	−1	−1	1

Octahedral O projector algebra



Begin with minimal equation: $\mathbf{0}= \cancel{(\mathbf{c}_i+2\cdot\mathbf{1})}(\mathbf{c}_i-2\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i-0\cdot\mathbf{1})$

$$\boxed{\mathbf{c}_{\mathbf{g}} = \sum_{\mu} \frac{{}^\circ c_{g}\chi^{\mu}_{g}}{\ell^{\mu}}\mathbb{P}^{\mu}}$$

$$\boxed{\mathbb{P}^{\mu}= \sum_{classes\, c_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^\circ G}\chi^{\mu *}_{g}\mathbf{c}_{\mathbf{g}}}$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i+2\mathbf{1})(\mathbf{c}_i-6\mathbf{1})(\mathbf{c}_i+6\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)} = \frac{(\mathbf{c}_i+2\mathbf{1})(\mathbf{c}_i^2-36\mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4+2\mathbf{c}_i^3-36\mathbf{c}_i^2-72\mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$\mathbf{c}_i^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i^3= \qquad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i^2= \quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

$\mathbf{c}_i = \qquad + \quad \mathbf{c}_i$

$\mathbf{c}_i^4 =120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$+2\mathbf{c}_i^3 = \qquad +32\mathbf{c}_R+40\mathbf{c}_i$

$-36\mathbf{c}_i^2=-216\cdot\mathbf{1}-108\mathbf{c}_r \quad -72\mathbf{c}_\rho$

$-72\mathbf{c}_i = \qquad -72\mathbf{c}_i$

$-256\mathbf{P}^{(2)} =-96\cdot\mathbf{1}+ 0\,\mathbf{c}_r+32\mathbf{c}_\rho +32\mathbf{c}_R- 32\mathbf{c}_i$

$$\mathbf{P}^{(2)} = \boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho-\frac{1}{8}\mathbf{c}_R+\frac{1}{8}\mathbf{c}_i}$$

Expansion of $\mathbf{P}^{(-2)}$ *has (-)sign on last 2 terms...*

$$\mathbf{P}^{(-2)} = \boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho+\frac{1}{8}\mathbf{c}_R-\frac{1}{8}\mathbf{c}_i}$$

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

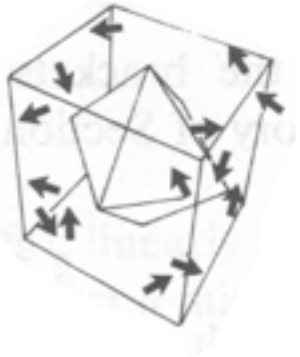
Octahedral O characters

χ^{μ}_{g}	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
·					
·					
·					
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

Applying the conventional label T_2 for (2) and T_1 for (-2)

Octahedral O projector algebra

Begin with minimal equation: $\mathbf{0}= (\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-2\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})\cancel{(\mathbf{c}_i-6\cdot\mathbf{1})}(\mathbf{c}_i-0\cdot\mathbf{1})$



$$\boxed{\mathbf{c}_\mathbf{g}=\sum_\mu \frac{{}^\circ c_g\chi_g^\mu}{\ell^\mu}\mathbb{P}^\mu}$$

$$\boxed{\mathbb{P}^\mu=\sum_{classes\,c_\mathbf{g}}\frac{\ell^\mu}{{}^\circ G}\chi_g^{\mu *}\mathbf{c}_\mathbf{g}}$$

$$\mathbf{P}^{(2)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i-6\cdot\mathbf{1})(\mathbf{c}_i+6\cdot\mathbf{1})(\mathbf{c}_i-0)}{(2+2)(2-6)(2+6)(2-0)}=\frac{(\mathbf{c}_i+2\cdot\mathbf{1})(\mathbf{c}_i^2-36\cdot\mathbf{1})\mathbf{c}_i}{-256}=\frac{\mathbf{c}_i^4+2\cdot\mathbf{c}_i^3-36\,\mathbf{c}_i^2-72\,\mathbf{c}_i}{-256}$$

$\mathbf{c}_i^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$\mathbf{c}_i^3=\qquad +16\mathbf{c}_R+20\mathbf{c}_i$

$\mathbf{c}_i^2=\quad 6\cdot\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

$\mathbf{c}_i\ =\qquad +\quad \mathbf{c}_i$

$\mathbf{c}_i^4=120\cdot\mathbf{1}+108\mathbf{c}_r+104\mathbf{c}_\rho$

$+2\mathbf{c}_i^3=\qquad +32\mathbf{c}_R+40\mathbf{c}_i$

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$$\mathbf{P}^{(2)}=\boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho-\frac{1}{8}\mathbf{c}_R+\frac{1}{8}\mathbf{c}_i}$$

Expansion of $\mathbf{P}^{(-2)}$ *has (-)sign on last 2 terms...*

$$\mathbf{P}^{(-2)}=\boxed{\frac{3}{8}\mathbf{1}+\frac{0}{8}\mathbf{c}_r-\frac{1}{8}\mathbf{c}_\rho+\frac{1}{8}\mathbf{c}_R-\frac{1}{8}\mathbf{c}_i}$$

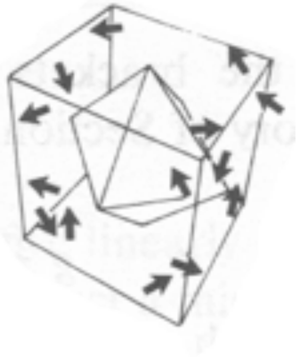
O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O characters

(Remaining character derivations left as an exercise)

χ_g^μ	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

- *Octahedral-cubic O symmetry and group operations*
- *Tetrahedral symmetry becomes Icosahedral*
- *Octahedral groups $O_h \supset O \sim T_d \supset T$*
  - *Octahedral O and spin-$O \subset U(2)$*

Tetrahedral T class algebra

- *Tetrahedral T class minimal equations*
- *Tetrahedral T class projectors and characters*

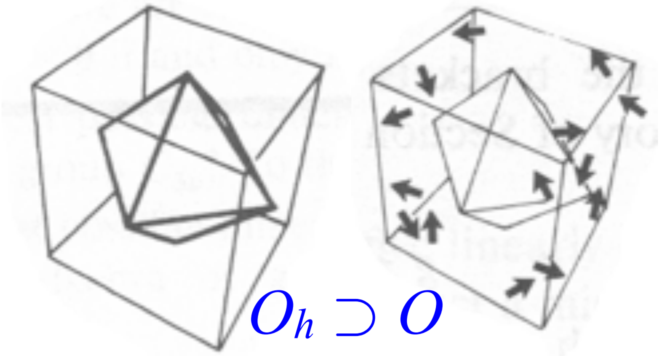
Octahedral O class algebra

- *Octahedral O class minimal equations*
- *Octahedral O class projectors and characters*

➡ *Octahedral $O_h \supset O$:Inversion (g&u) parity* ⬅

- *Octahedral $O_h \supset O \supset C_1$ subgroup correlations*
- *$O_h \supset O \supset D_4$ subgroup correlations*
- *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
- *Preview of applications to high resolution spectroscopy*

Octahedral $O_h = O \times \{\mathbf{1},\mathbf{I}\}$ *characters of* $O \times C_{\mathbf{I}} \supset O$



$O_h \supset O$ *symmetry*

EVEN parity *(gerade)*

	χ^{μ}_{g}	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
A_{1g}	$\chi^{A_{1g}}$	1	1	1	1	1
A_{2g}	$\chi^{A_{2g}}$	1	1	1	−1	−1
E_g	χ^{E_g}	2	−1	2	0	0
T_{1g}	$\chi^{T_{1g}}$	3	0	−1	1	−1
T_{2g}	$\chi^{T_{2g}}$	3	0	−1	−1	1

3D − Inversion

$$\langle \mathbf{I} \rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$C_{\mathbf{I}}$*-symmetry*

1	**I**
I	1

$C_{\mathbf{I}}$*-characters*

$C_{\mathbf{I}}$	1	**I**
g	*1*	*1*
u	*1*	*-1*

± Parity P

(gerade)

(ungerade)

O class product table

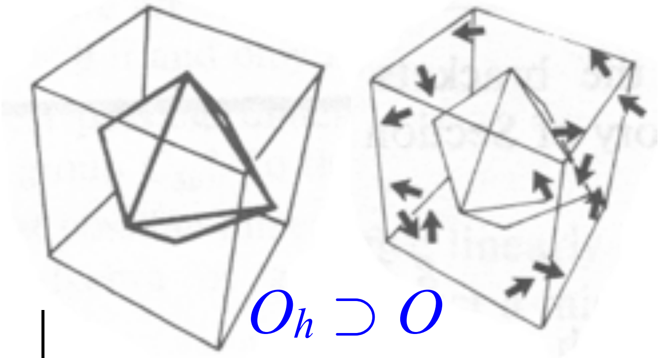
$\mathbf{1} = c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O characters

(Remaining character derivations left as an exercise)

χ^{μ}_{g}	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

Octahedral $O_h = O\times\{\mathbf{1},\mathbf{I}\}$ characters of $O\times C_{\mathbf{I}}\supset O$



$O_h \supset O$ symmetry

EVEN parity (gerade)

	χ^{μ}_{g}	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$	$\mathbf{g=I}$	$\mathbf{Ir}_{1...4}$	$\mathbf{I}\boldsymbol{\rho}_{xyz}$	$\mathbf{IR}_{xyz}$	$\mathbf{Ii}_{1...6}$
A_{1g}	$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
A_{2g}	$\chi^{A_{2g}}$	1	1	1	−1	−1	1	1	1	−1	−1
E_{g}	$\chi^{E_{g}}$	2	−1	2	0	0	2	−1	2	0	0
T_{1g}	$\chi^{T_{1g}}$	3	0	−1	1	−1	3	0	−1	1	−1
T_{2g}	$\chi^{T_{2g}}$	3	0	−1	−1	1	3	0	−1	−1	1

3D – Inversion

$$\langle \mathbf{I}\rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$C_{\mathbf{I}}$-symmetry

$$\boxed{\begin{matrix} 1 & \mathbf{I} \\ \mathbf{I} & 1 \end{matrix}}$$

$C_{\mathbf{I}}$-characters

$C_{\mathbf{I}}$	1	$\mathbf{I}$	± Parity P
g	1	1	(gerade)
u	1	-1	(ungerade)

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

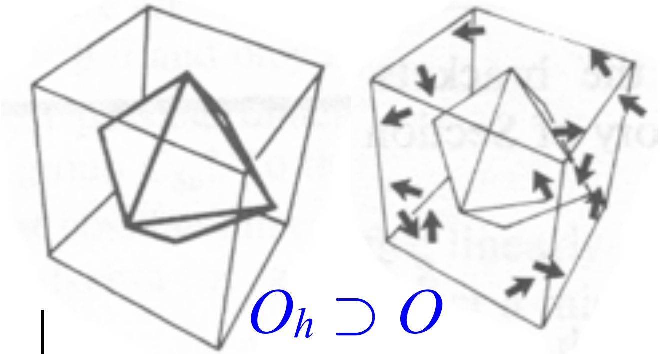
Octahedral O characters

(Remaining character derivations left as an exercise)

χ^{μ}_{g}	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

Octahedral $O_h = O\times\{\mathbf{1},\mathbf{I}\}$ *characters of* $O\times C_{\mathbf{I}}\supset O$

		χ_g^μ	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$	$\mathbf{g=I}$	$\mathbf{Ir}_{1...4}$	$\mathbf{I\rho}_{xyz}$	$\mathbf{IR}_{xyz}$	$\mathbf{Ii}_{1...6}$
EVEN parity *(gerade)*	A_{1g}	$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
	A_{2g}	$\chi^{A_{2g}}$	1	1	1	−1	−1	1	1	1	−1	−1
	E_g	χ^{E_g}	2	−1	2	0	0	2	−1	2	0	0
	T_{1g}	$\chi^{T_{1g}}$	3	0	−1	1	−1	3	0	−1	1	−1
	T_{2g}	$\chi^{T_{2g}}$	3	0	−1	−1	1	3	0	−1	−1	1
ODD parity *(ungerade)*	A_{1u}	$\chi^{A_{1u}}$	1	1	1	1	1					
	A_{2u}	$\chi^{A_{2u}}$	1	1	1	−1	−1					
	E_u	χ^{E_u}	2	−1	2	0	0					
	T_{1u}	$\chi^{T_{1u}}$	3	0	−1	1	−1					
	T_{2u}	$\chi^{T_{2u}}$	3	0	−1	−1	1					



$O_h \supset O$ symmetry

3D – Inversion

$$\langle \mathbf{I}\rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$C_{\mathbf{I}}$-symmetry

$$\boxed{\begin{matrix} 1 & \mathbf{I} \\ \mathbf{I} & 1 \end{matrix}}$$

$C_{\mathbf{I}}$-characters

$C_{\mathbf{I}}$	1	$\mathbf{I}$
g	*1*	*1*
u	*1*	*-1*

± Parity P

(gerade)

(ungerade)

O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

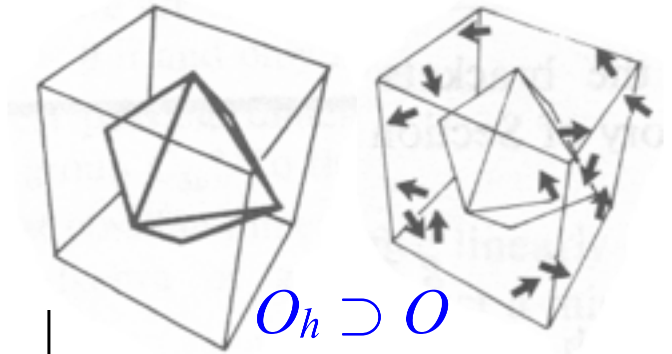
Octahedral O characters

χ_g^μ	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

(Remaining character derivations left as an exercise)

Octahedral $O_h = O \times \{\mathbf{1},\mathbf{I}\}$ *characters of* $O \times C_\mathbf{I} \supset O$

O_h *easily derived from those of* O *and* $C_\mathbf{I}$ *!*



$O_h \supset O$ symmetry

		χ^μ_g	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$	$\mathbf{g=I}$	$\mathbf{Ir}_{1...4}$	$\mathbf{I\rho}_{xyz}$	$\mathbf{IR}_{xyz}$	$\mathbf{Ii}_{1...6}$
EVEN parity *(gerade)*	A_{1g}	$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
	A_{2g}	$\chi^{A_{2g}}$	1	1	1	−1	−1	1	1	1	−1	−1
	E_g	χ^{E_g}	2	−1	2	0	0	2	−1	2	0	0
	T_{1g}	$\chi^{T_{1g}}$	3	0	−1	1	−1	3	0	−1	1	−1
	T_{2g}	$\chi^{T_{2g}}$	3	0	−1	−1	1	3	0	−1	−1	1
ODD parity *(ungerade)*	A_{1u}	$\chi^{A_{1u}}$	1	1	1	1	1	−1	−1	−1	−1	−1
	A_{2u}	$\chi^{A_{2u}}$	1	1	1	−1	−1	−1	−1	−1	+1	+1
	E_u	χ^{E_u}	2	−1	2	0	0	−2	+1	−2	0	0
	T_{1u}	$\chi^{T_{1u}}$	3	0	−1	1	−1	−3	0	+1	−1	+1
	T_{2u}	$\chi^{T_{2u}}$	3	0	−1	−1	1	−3	0	+1	+1	−1

3D – Inversion

$$\langle \mathbf{I} \rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$C_\mathbf{I}$-symmetry

1	**I**
I	1

$C_\mathbf{I}$-characters

$C_\mathbf{I}$	1	**I**
g	*1*	*1*
u	*1*	*-1*

± Parity P

(gerade)

(ungerade)

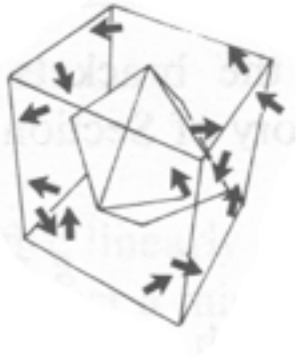
O class product table

$\mathbf{1}=c_1$	$\mathbf{c}_r$	$\mathbf{c}_\rho$	$\mathbf{c}_R$	$\mathbf{c}_i$
$\mathbf{c}_r$	$8\mathbf{1}+4\mathbf{c}_r+8\mathbf{c}_\rho$	$3\mathbf{c}_r$	$4\mathbf{c}_R+4\mathbf{c}_i$	$4\mathbf{c}_R+4\mathbf{c}_i$
$\mathbf{c}_\rho$		$3\mathbf{1}+2\mathbf{c}_\rho$	$\mathbf{c}_R+2\mathbf{c}_i$	$2\mathbf{c}_R+\mathbf{c}_i$
$\mathbf{c}_R$			$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$	$3\mathbf{c}_r+4\mathbf{c}_\rho$
$\mathbf{c}_i$				$6\mathbf{1}+3\mathbf{c}_r+2\mathbf{c}_\rho$

Octahedral O characters

(Remaining character derivations left as an exercise)

χ^μ_g	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1



- *Introduction to octahedral/ tetrahedral symmetry $O_h\supset O\sim T_d\supset T$*
  - *Octahedral-cubic O symmetry and group operations*
  - *Tetrahedral symmetry becomes Icosahedral*
  - *Octahedral groups $O_h\supset O\sim T_d\supset T$*
    - *Octahedral O and spin-$O\subset U(2)$*

- *Tetrahedral T class algebra*
  - *Tetrahedral T class minimal equations*
  - *Tetrahedral T class projectors and characters*

- *Octahedral O class algebra*
  - *Octahedral O class minimal equations*
  - *Octahedral O class projectors and characters*

- *Octahedral $O_h\supset O$:Inversion (g&u) parity*
  - *Octahedral $O_h\supset O\supset C_1$ subgroup correlations*
  - *$O_h\supset O\supset D_4$ subgroup correlations*
  - *$O_h\supset O\supset D_4\supset C_4$ subgroup correlations*
  - *Preview of applications to high resolution spectroscopy*

Octahedral $O_h \supset O$ subgroup correlations

χ_g^μ	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
χ^{A_1}	1	1	1	1	1
χ^{A_2}	1	1	1	−1	−1
χ^{E}	2	−1	2	0	0
χ^{T_1}	3	0	−1	1	−1
χ^{T_2}	3	0	−1	−1	1

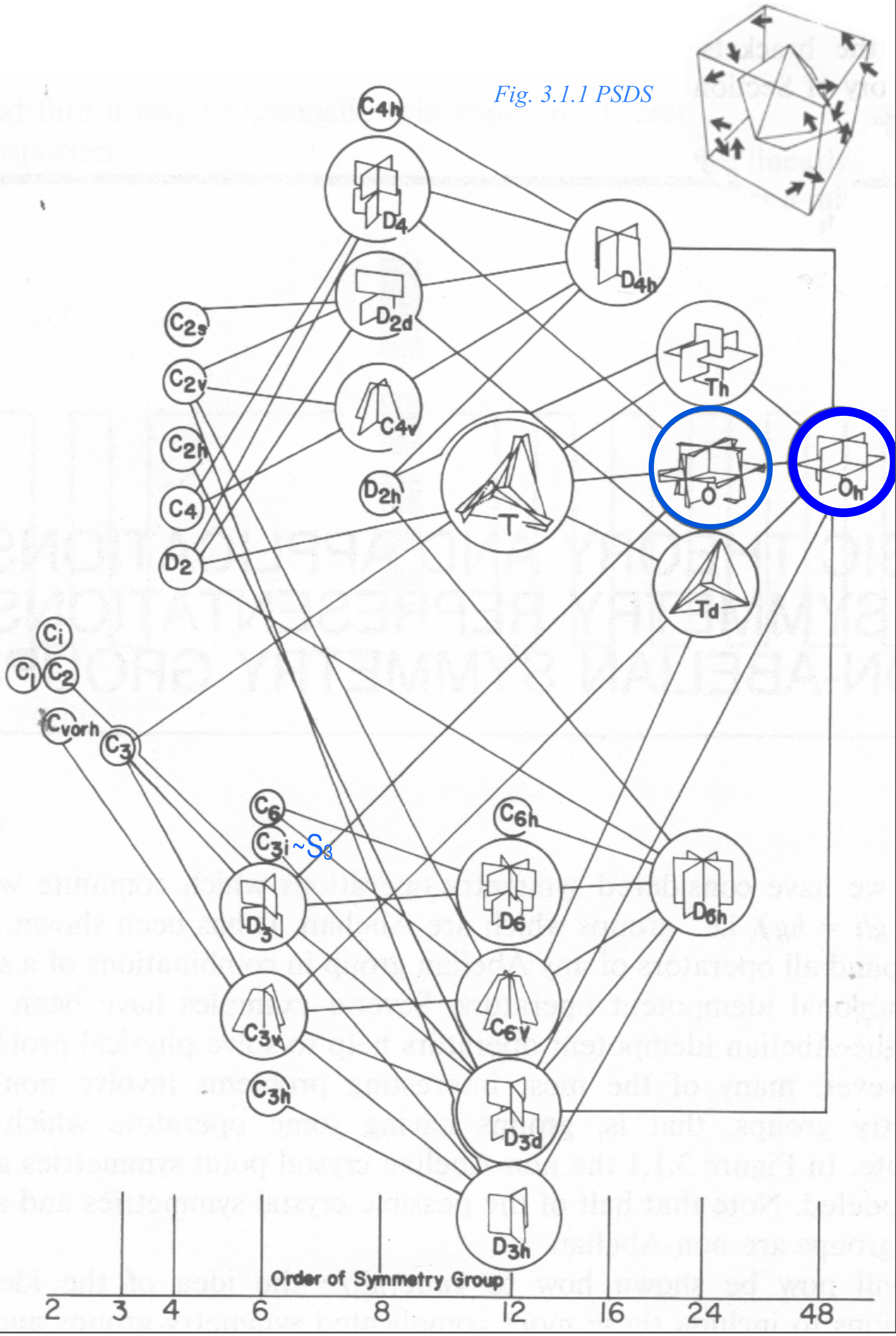


Fig. 3.1.1 PSDS

Octahedral $O_h \supset O$ subgroup correlations

$\chi_g^{\mu_p}$	**1**	$\mathbf{r}_{1\ldots4}$	$\boldsymbol{\rho}_{xyz}$	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$	**I**	**Ir=** $\mathbf{s}_{1\ldots4}$	**Iρ=** $\boldsymbol{\sigma}_{xyz}$	**IR=** $\mathbf{S}_{xyz}$	**Ii=** $\boldsymbol{\sigma}_{1\ldots6}$
$\chi^{A_{1g}}$	1	1	1	1	1	1	1	1	1	1
$\chi^{A_{2g}}$	1	1	1	−1	−1	1	1	1	−1	−1
χ^{E_g}	2	−1	2	0	0	2	−1	2	0	0
$\chi^{T_{1g}}$	3	0	−1	1	−1	3	0	−1	1	−1
$\chi^{T_{2g}}$	3	0	−1	−1	1	3	0	−1	−1	1
$\chi^{A_{1u}}$	1	1	1	1	1	−1	−1	−1	−1	−1
$\chi^{A_{2u}}$	1	1	1	−1	−1	−1	−1	−1	1	1
χ^{E_u}	2	−1	2	0	0	−2	1	−2	0	0
$\chi^{T_{1u}}$	3	0	−1	1	−1	−3	0	1	−1	1
$\chi^{T_{2u}}$	3	0	−1	−1	1	−3	0	1	1	−1

$O_h \supset O$	A_1	A_2	E	T_1	T_2
A_{1g}	1	·	·	·	·
A_{2g}	·	1	·	·	·
E_g	·	·	1	·	·
T_{1g}	·	·	·	1	·
T_{2g}	·	·	·	·	1
A_{1u}	1	·	·	·	·
A_{2u}	·	1	·	·	·
E_u	·	·	1	·	·
T_{1u}	·	·	·	1	·
T_{2u}	·	·	·	·	1

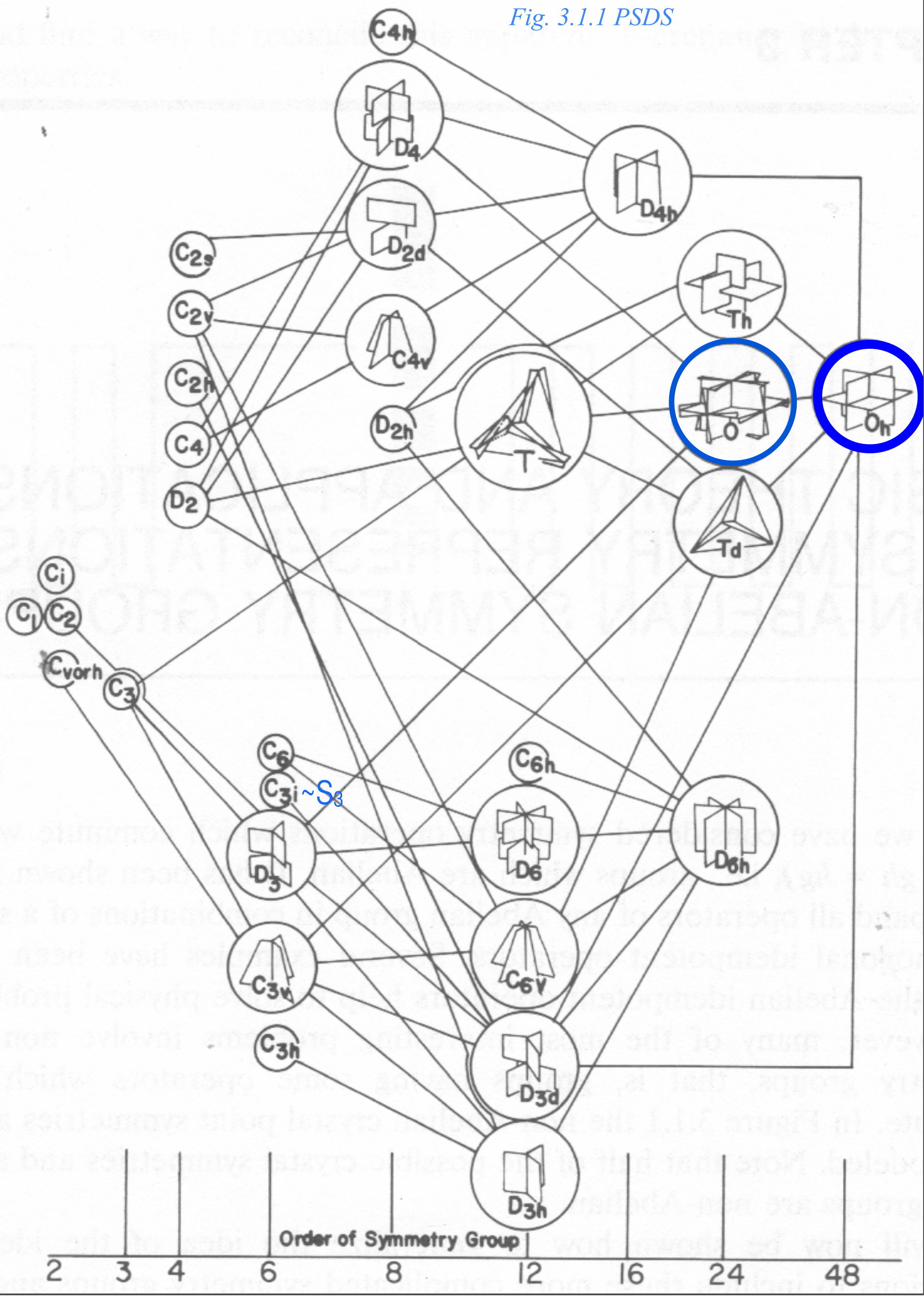


Fig. 3.1.1 PSDS

- *Introduction to octahedral/ tetrahedral symmetry* $O_h \supset O \sim T_d \supset T$
  - *Octahedral-cubic O symmetry and group operations*
  - *Tetrahedral symmetry becomes Icosahedral*
  - *Octahedral groups* $O_h \supset O \sim T_d \supset T$
    - *Octahedral O and spin-*$O \subset U(2)$

- *Tetrahedral T class algebra*
  - *Tetrahedral T class minimal equations*
  - *Tetrahedral T class projectors and characters*

- *Octahedral O class algebra*
  - *Octahedral O class minimal equations*
  - *Octahedral O class projectors and characters*

- *Octahedral* $O_h \supset O$ *subgroup correlations (Parity)*
  - *Octahedral* $O_h \supset O$ *subgroup correlations*
  - ➡ $O_h \supset O \supset D_4$ *subgroup correlations* ⬅
  - $O_h \supset O \supset D_4 \supset C_4$ *subgroup correlations*
  - *Preview of applications to high resolution spectroscopy*

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{z180°}, \mathbf{i}_{3,4}$

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	~~$\mathbf{r}_{1\ldots4}$~~	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

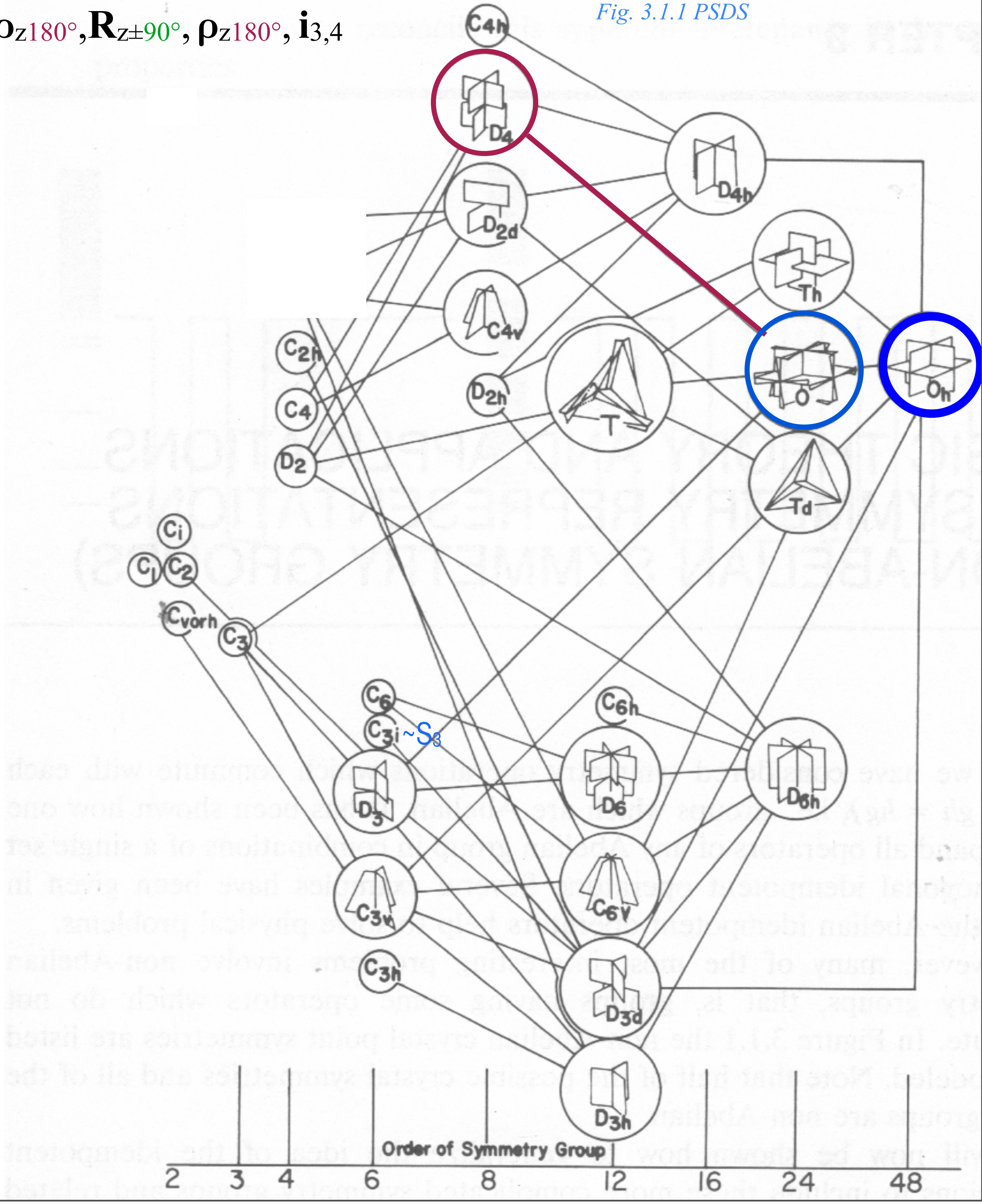


Fig. 3.1.1 PSDS

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$\chi_g^\mu(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

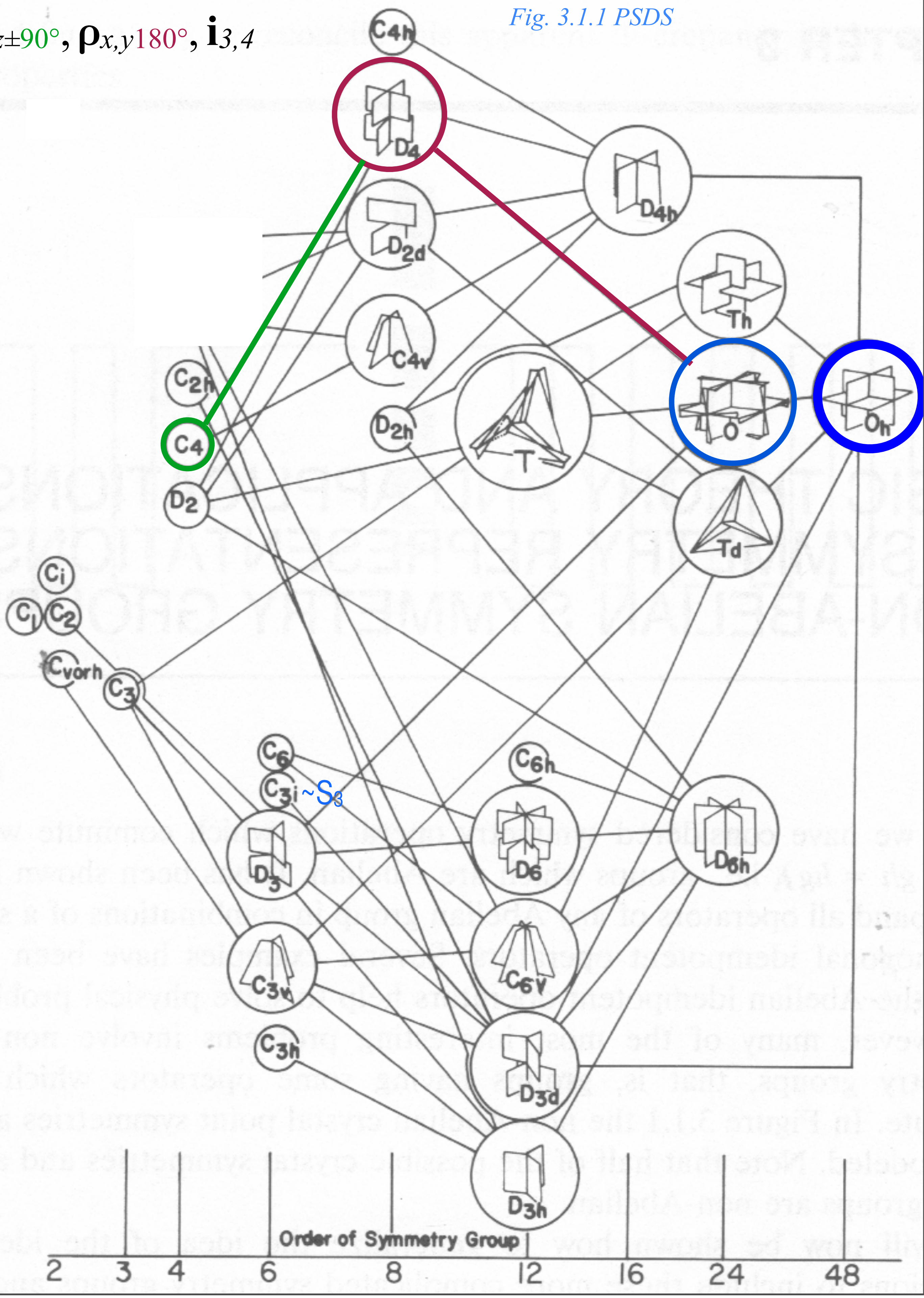


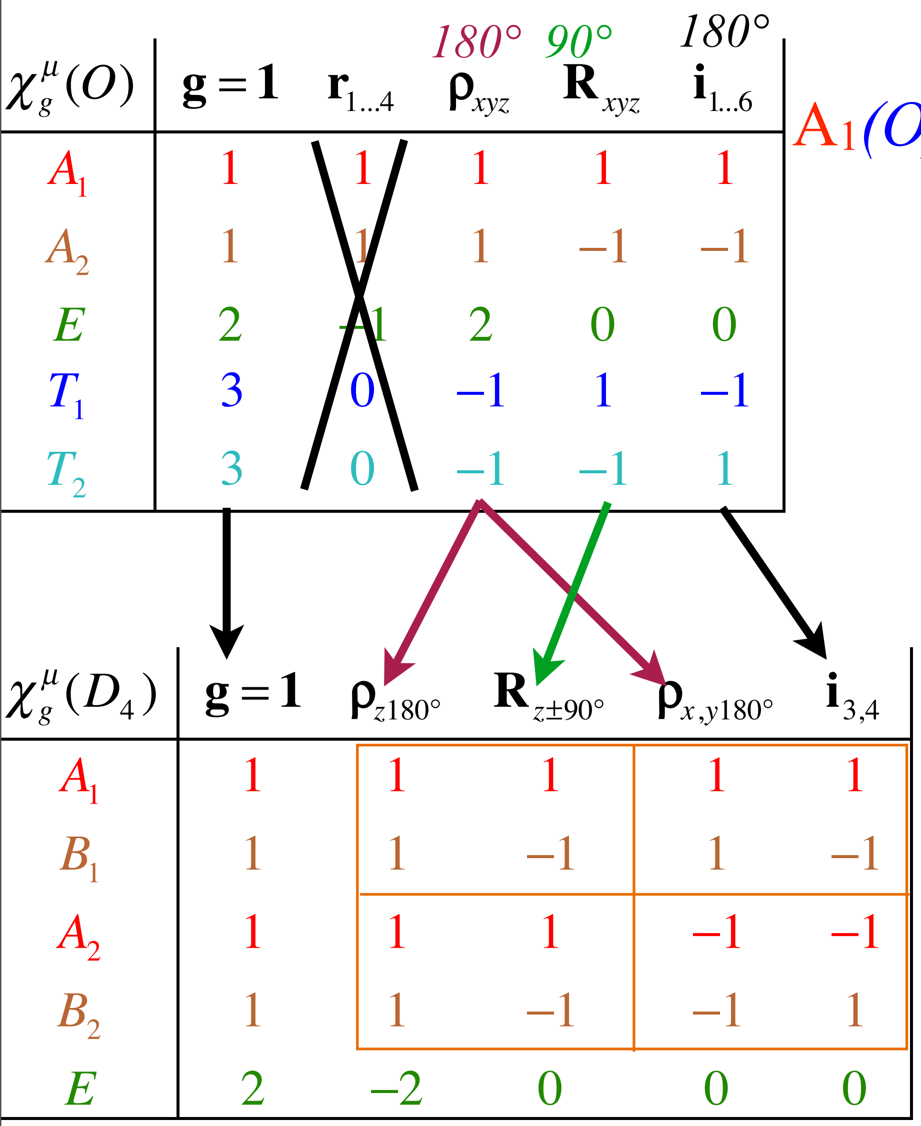
Fig. 3.1.1 PSDS

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

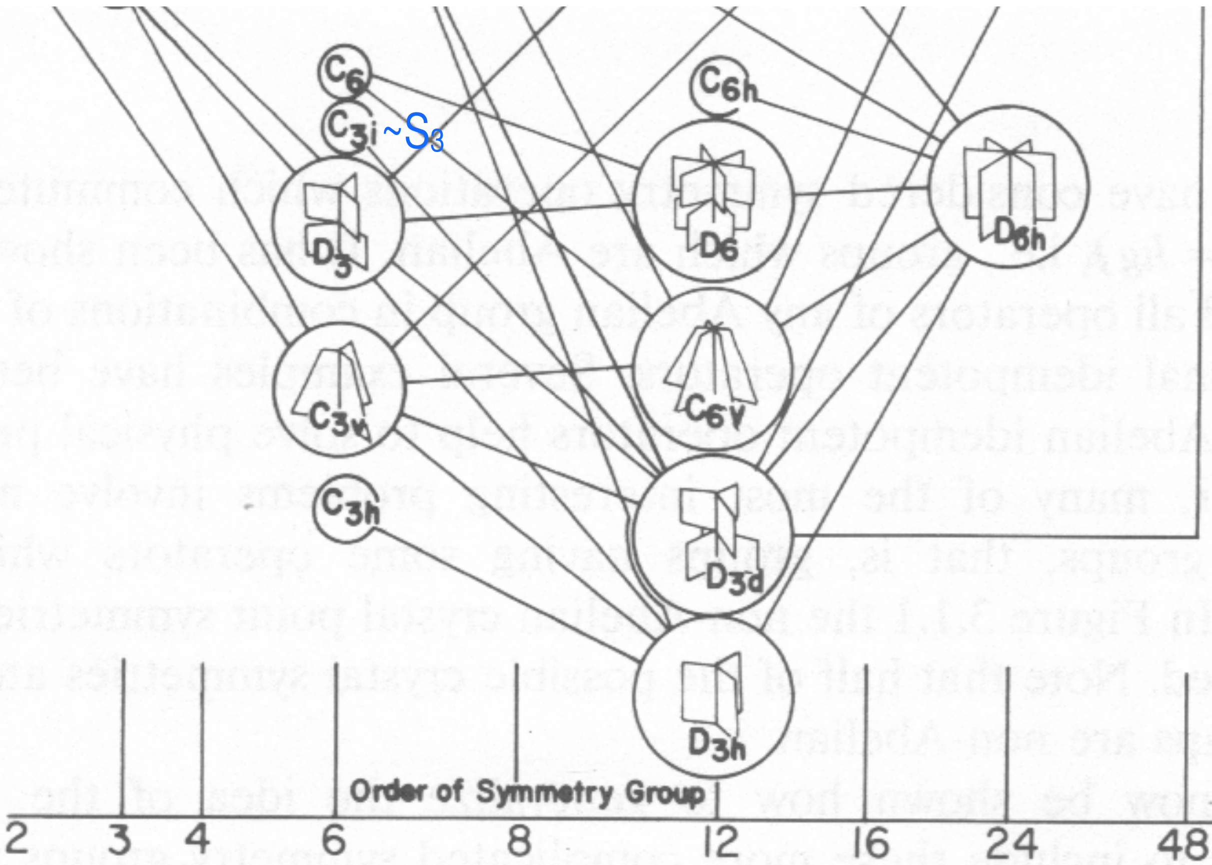
D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O) \downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$



$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

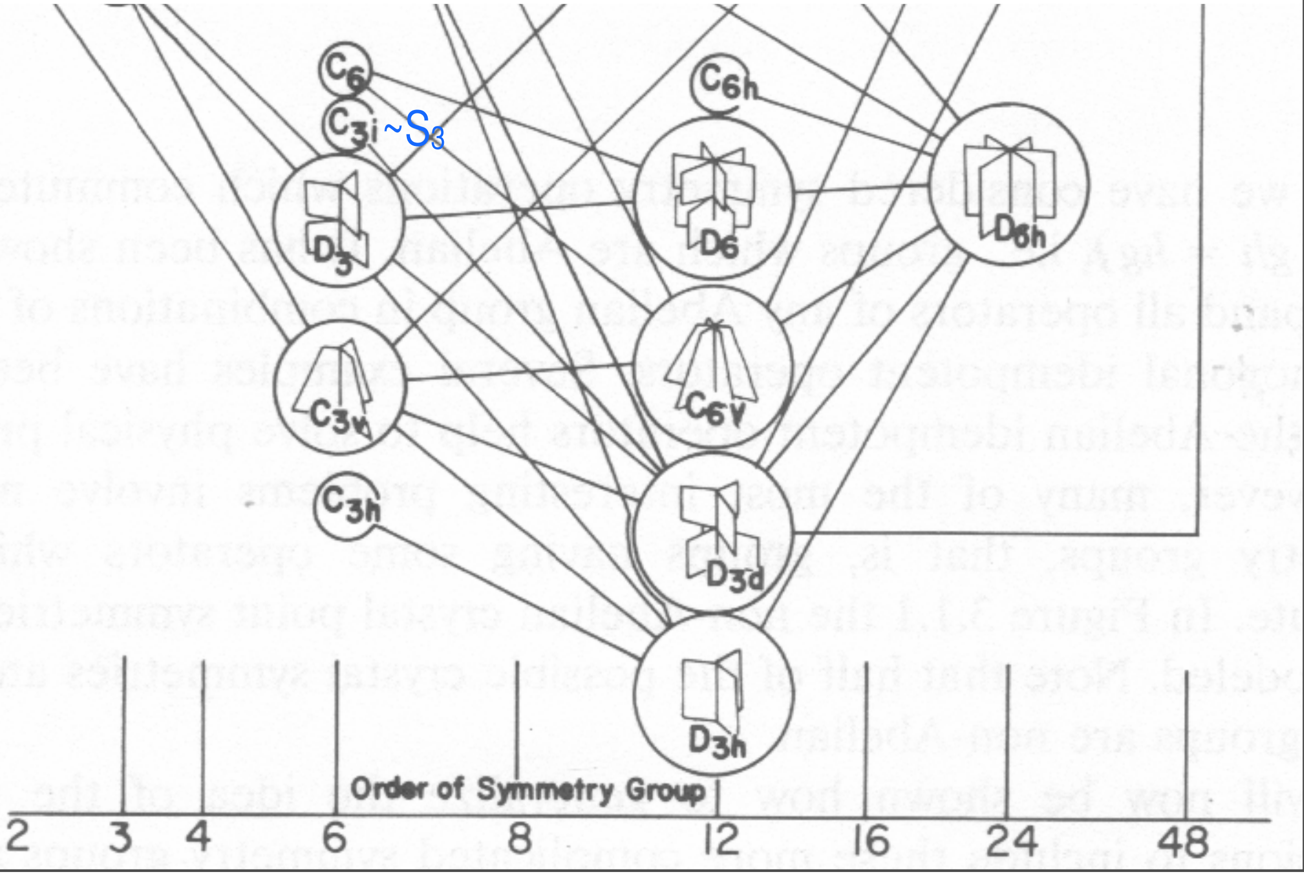
$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



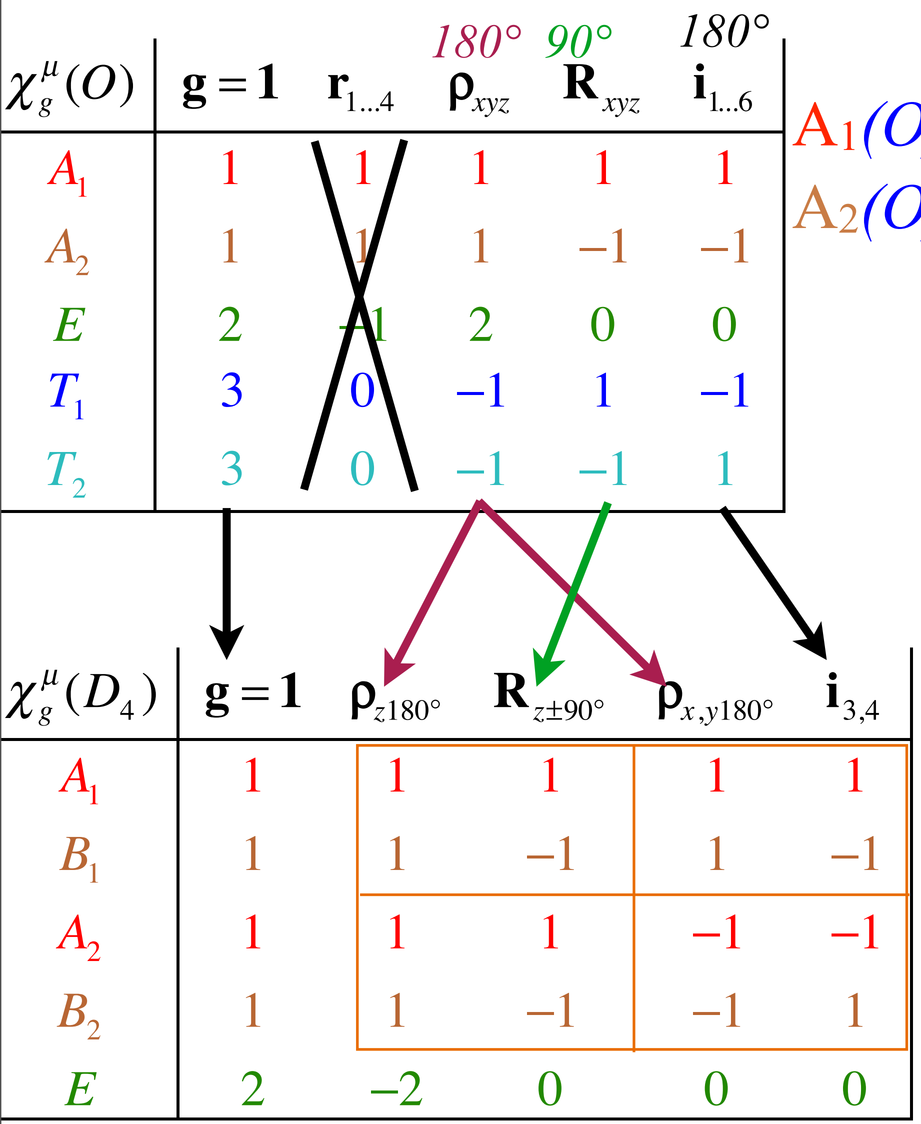
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

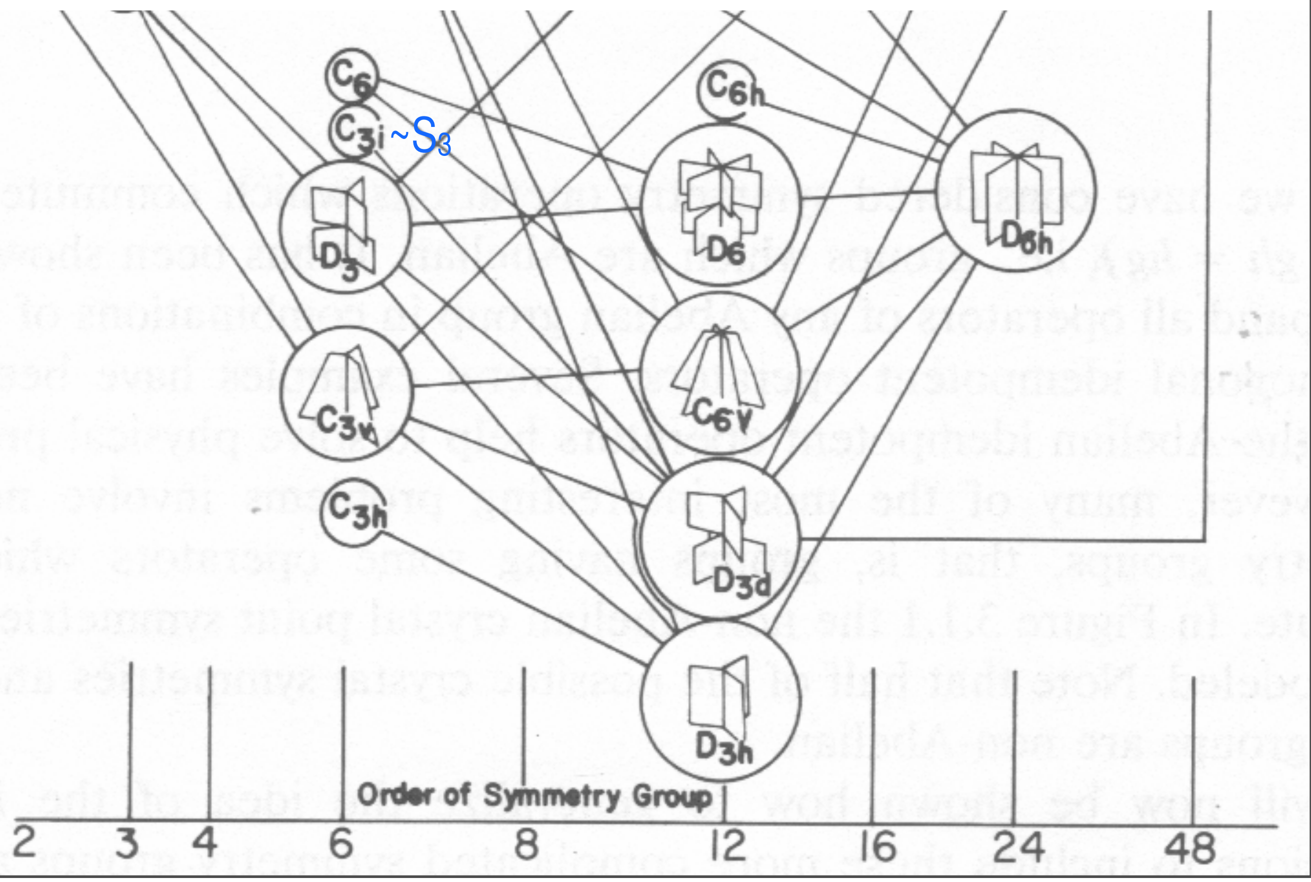
$A_1(O) \downarrow D_4 =$ 1, 1, 1, 1, 1 . $A_1(O) \downarrow D_4 = A_1(D_4)$

$A_2(O) \downarrow D_4 =$ 1, 1, -1, 1, -1 .



$\chi^\mu_g(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^\mu_g(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



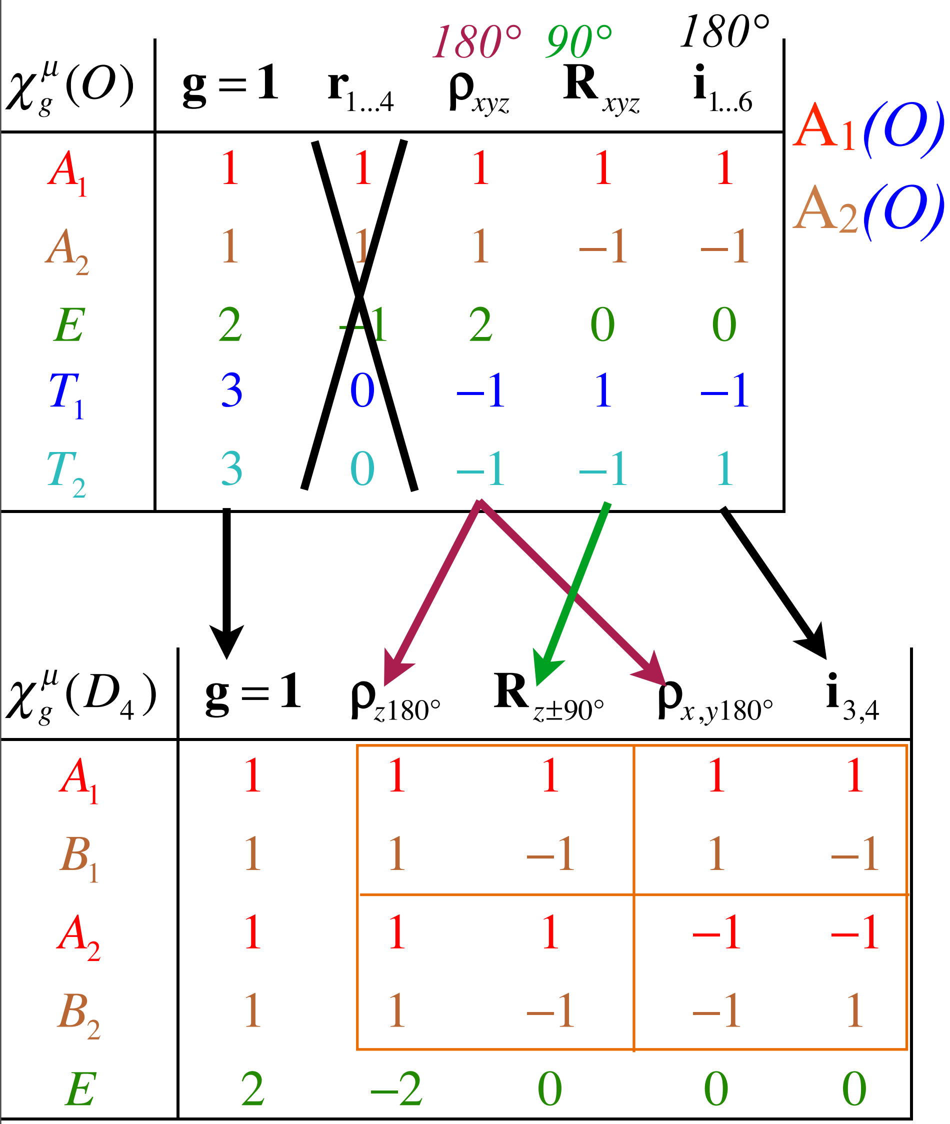
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

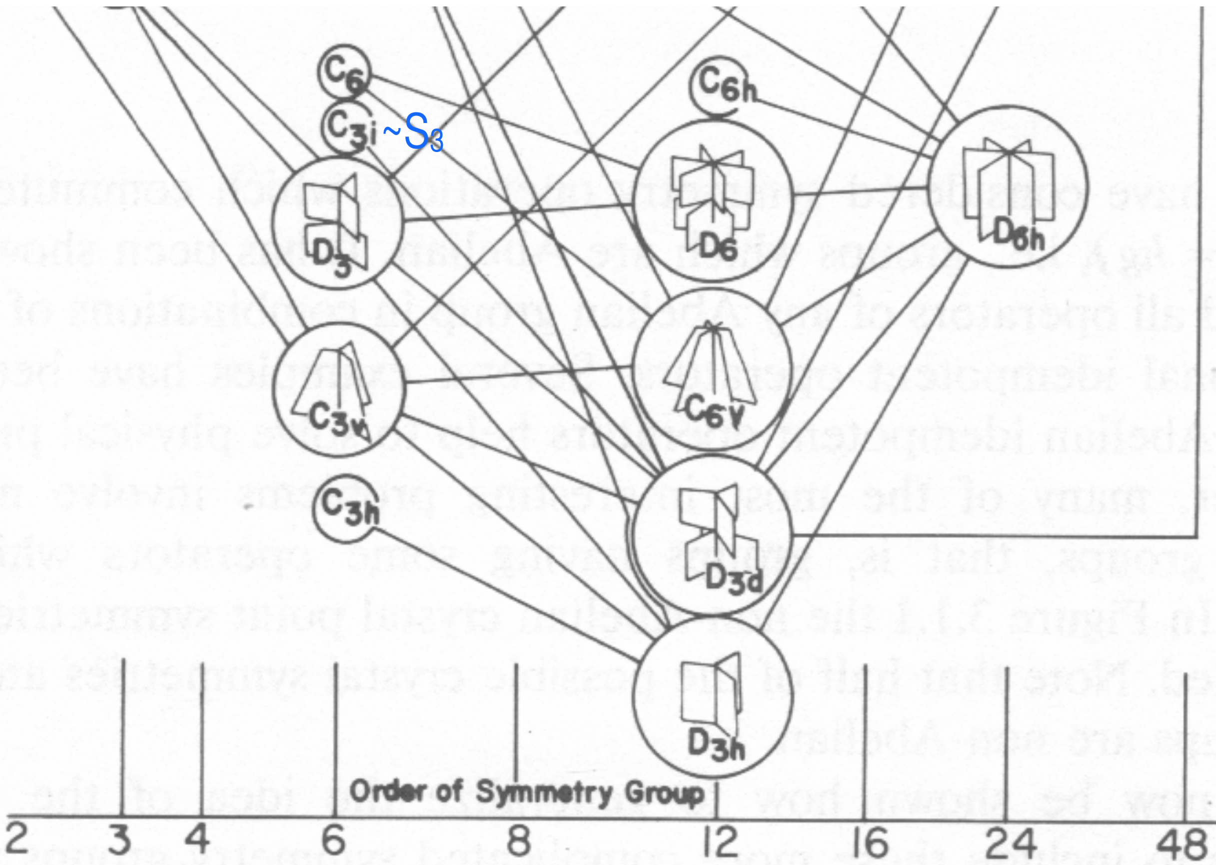
$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1.\quad A_1(O){\downarrow}D_4 = A_1(D_4)$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1.\quad A_2(O){\downarrow}D_4 = B_1(D_4)$



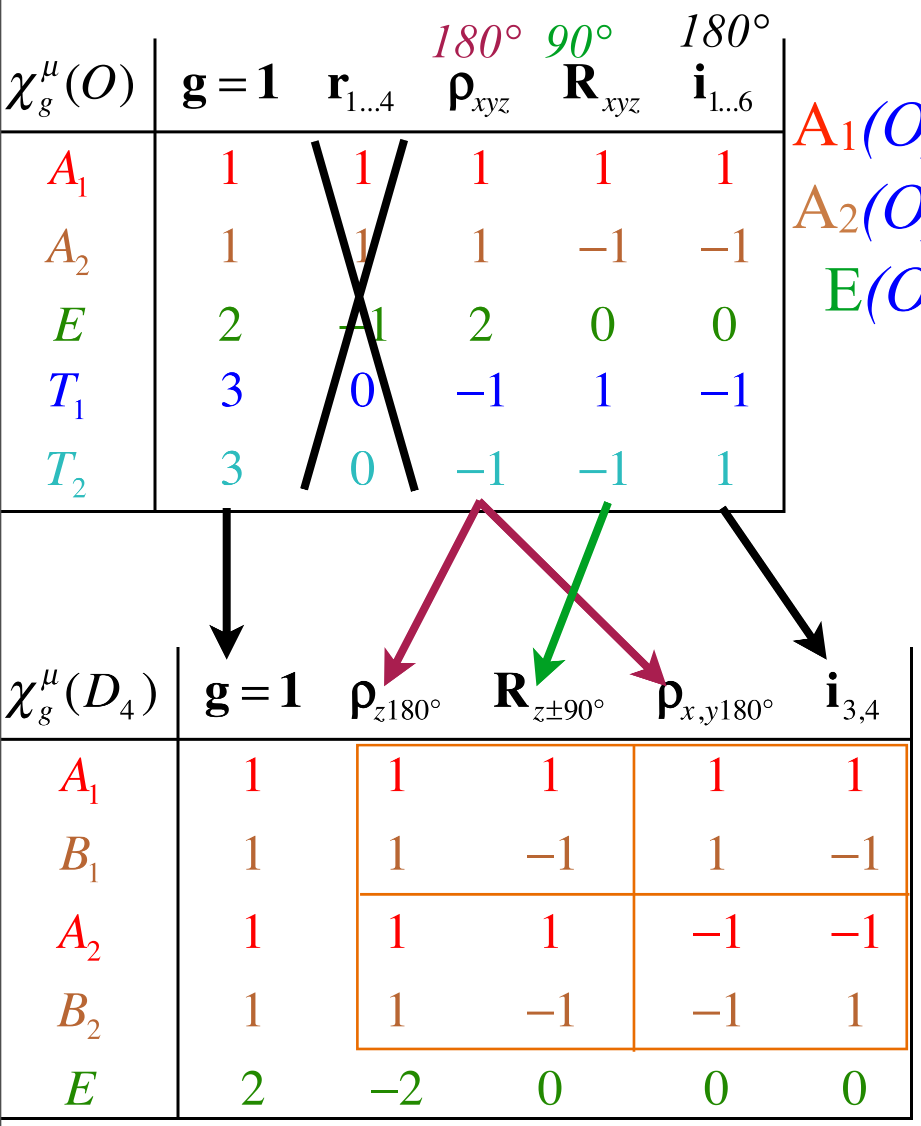
$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction



			180°	90°	180°
$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

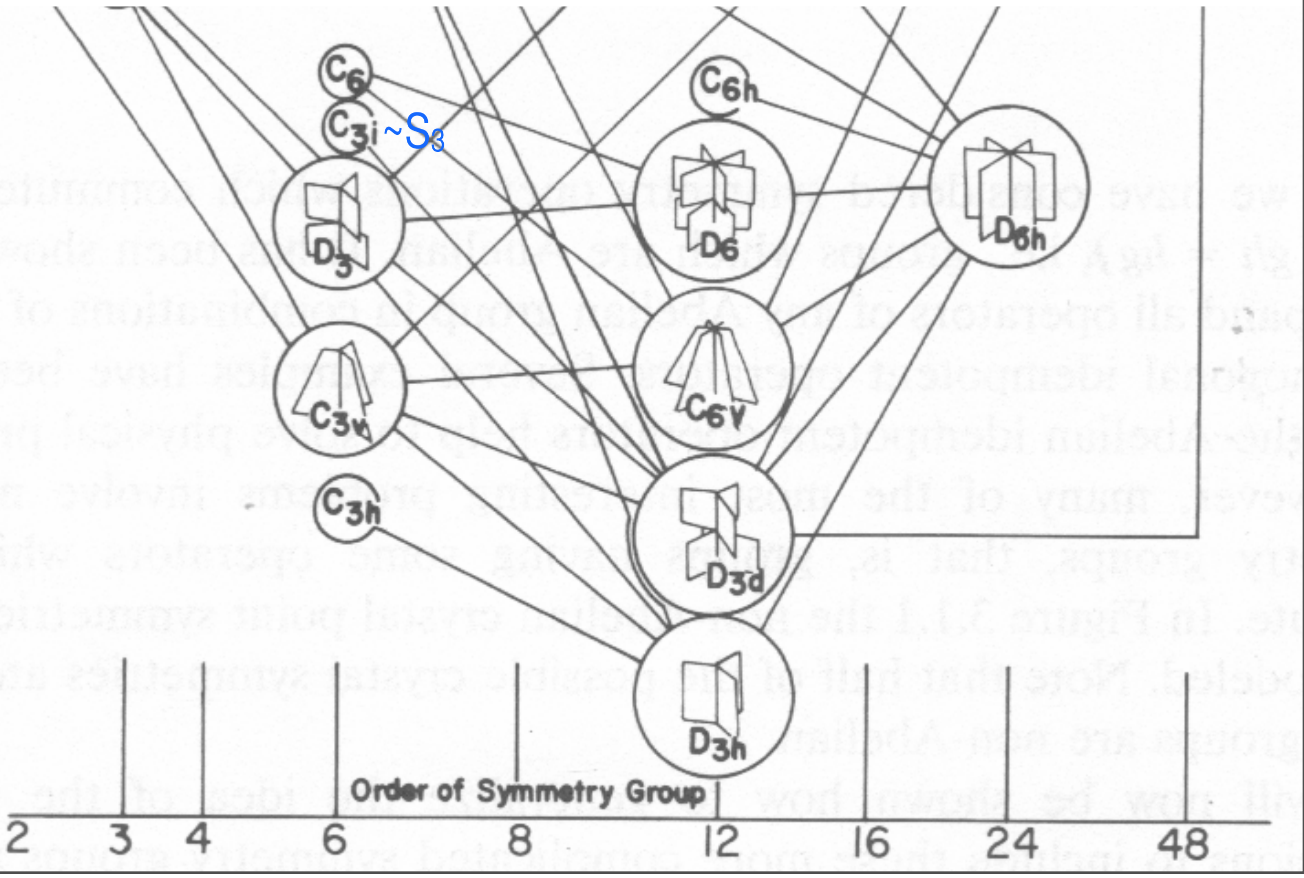
$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O)\downarrow D_4 = B_1(D_4)$

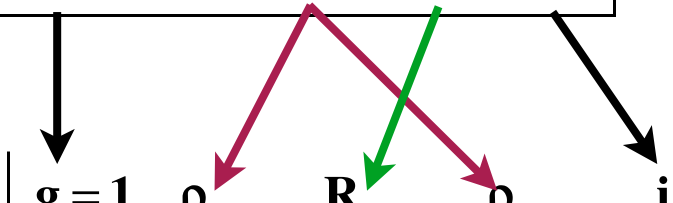
$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1



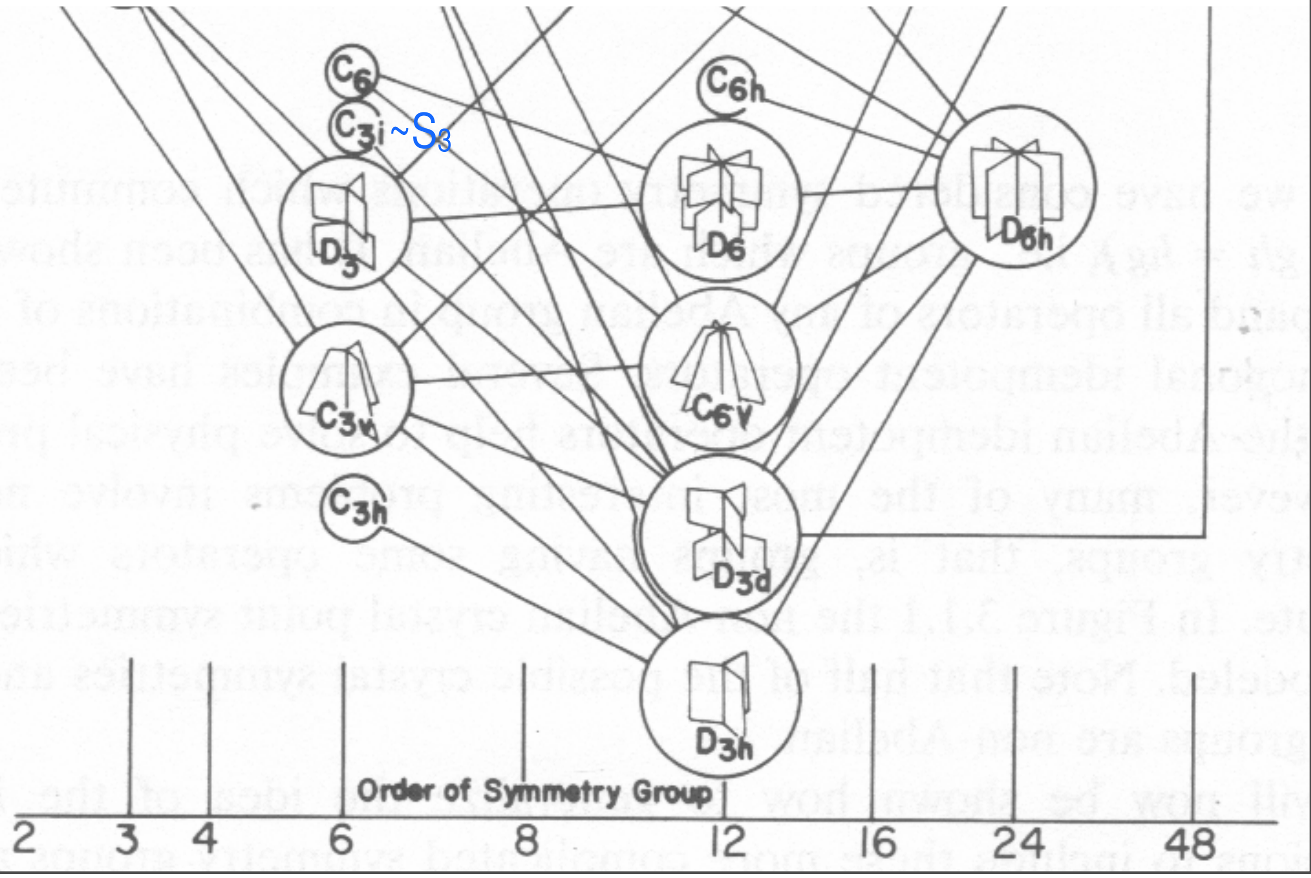
$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O)\downarrow D_4 = B_1(D_4)$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$ $E(O)\downarrow D_4 = A_1 \oplus B_1(D_4)$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$O\downarrow D_4$ subduction

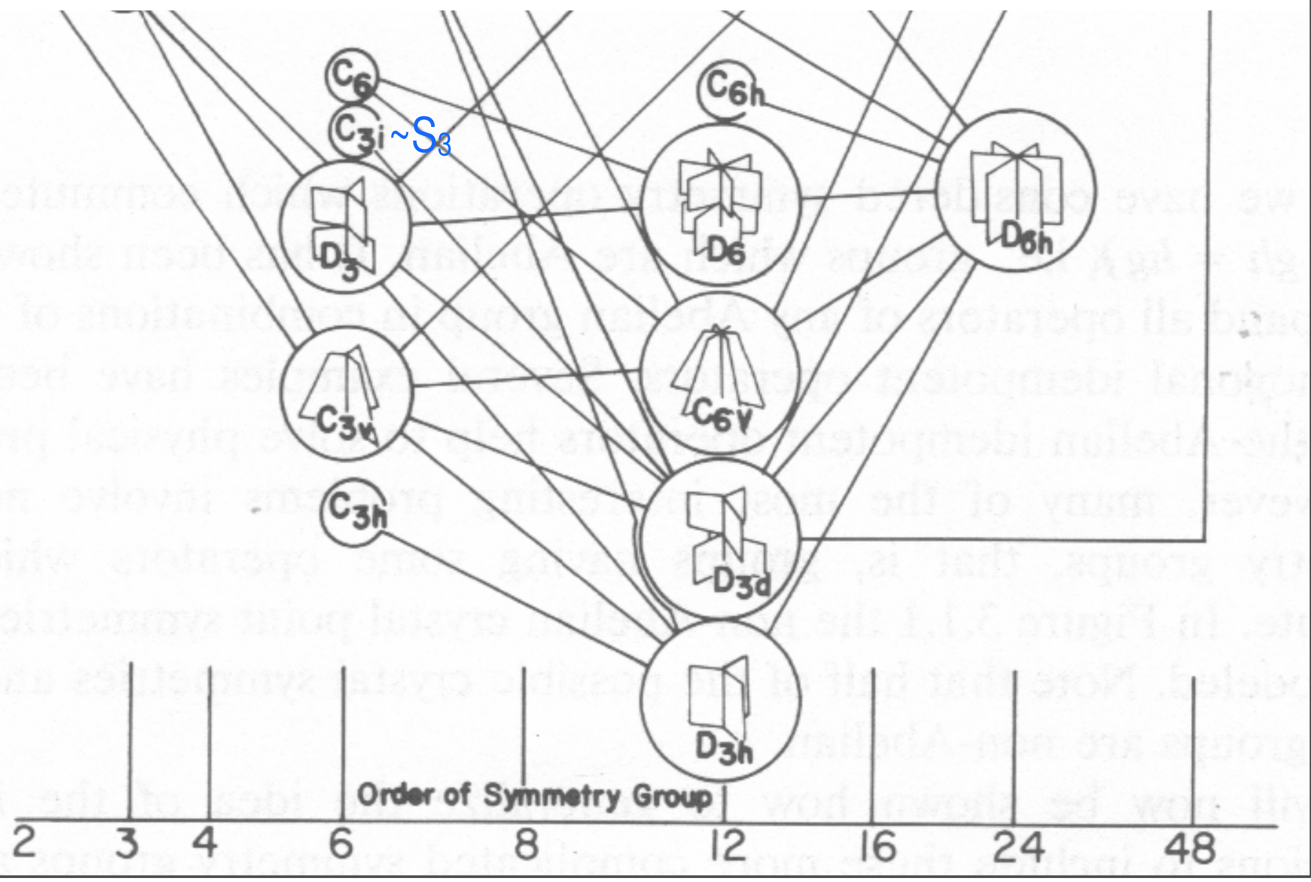
D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1, \;1, \;1, \;1, \;1\,.$ $\quad A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1, \;1, \;-1, \;1, \;-1\,.$ $\quad A_2(O)\downarrow D_4 = B_1(D_4)$

$E(O)\downarrow D_4 = 2, \;2, \;0, \;2, \;0\,.$ $\quad E(O)\downarrow D_4 = A_1 \oplus B_1(D_4)$

$T_1(O)\downarrow D_4 = 3, \;-1, \;1, \;-1, \;-1\,.$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ *subduction*

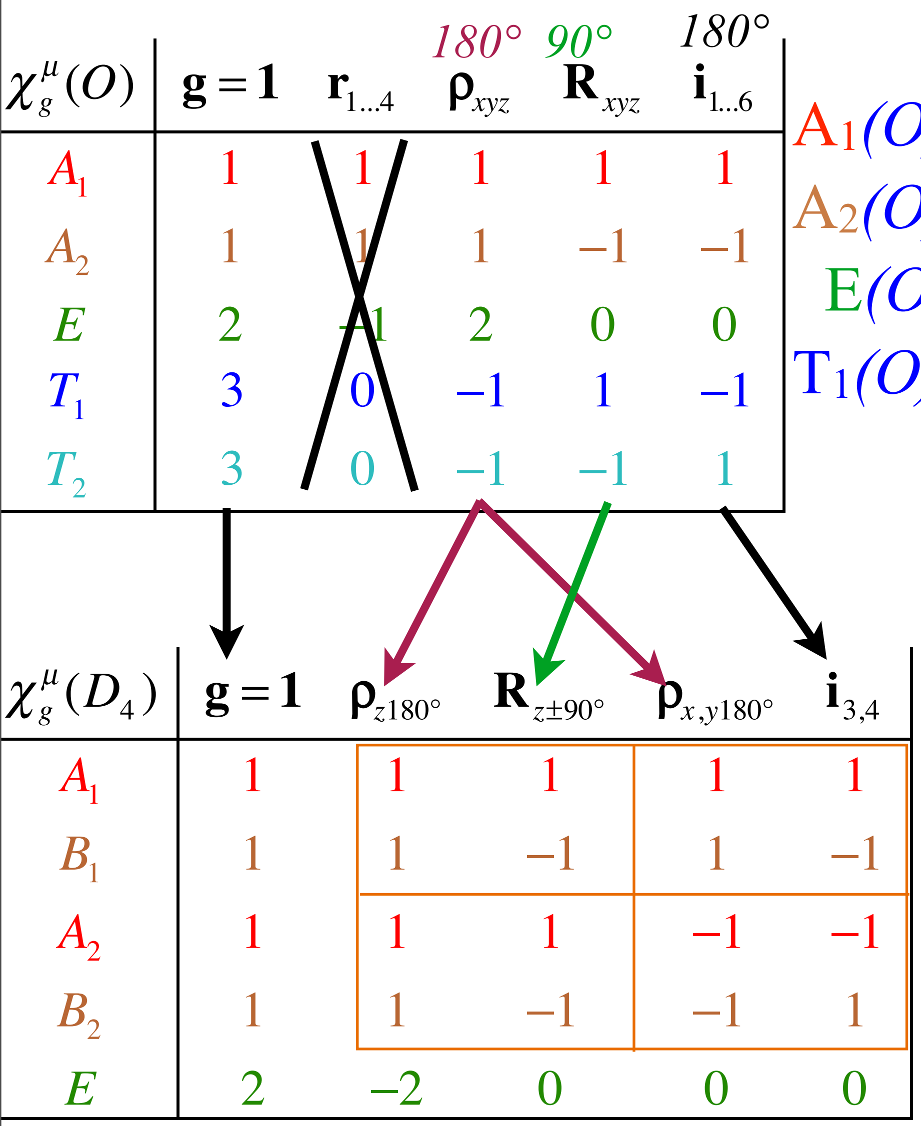
D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O)\downarrow D_4 = B_1(D_4)$

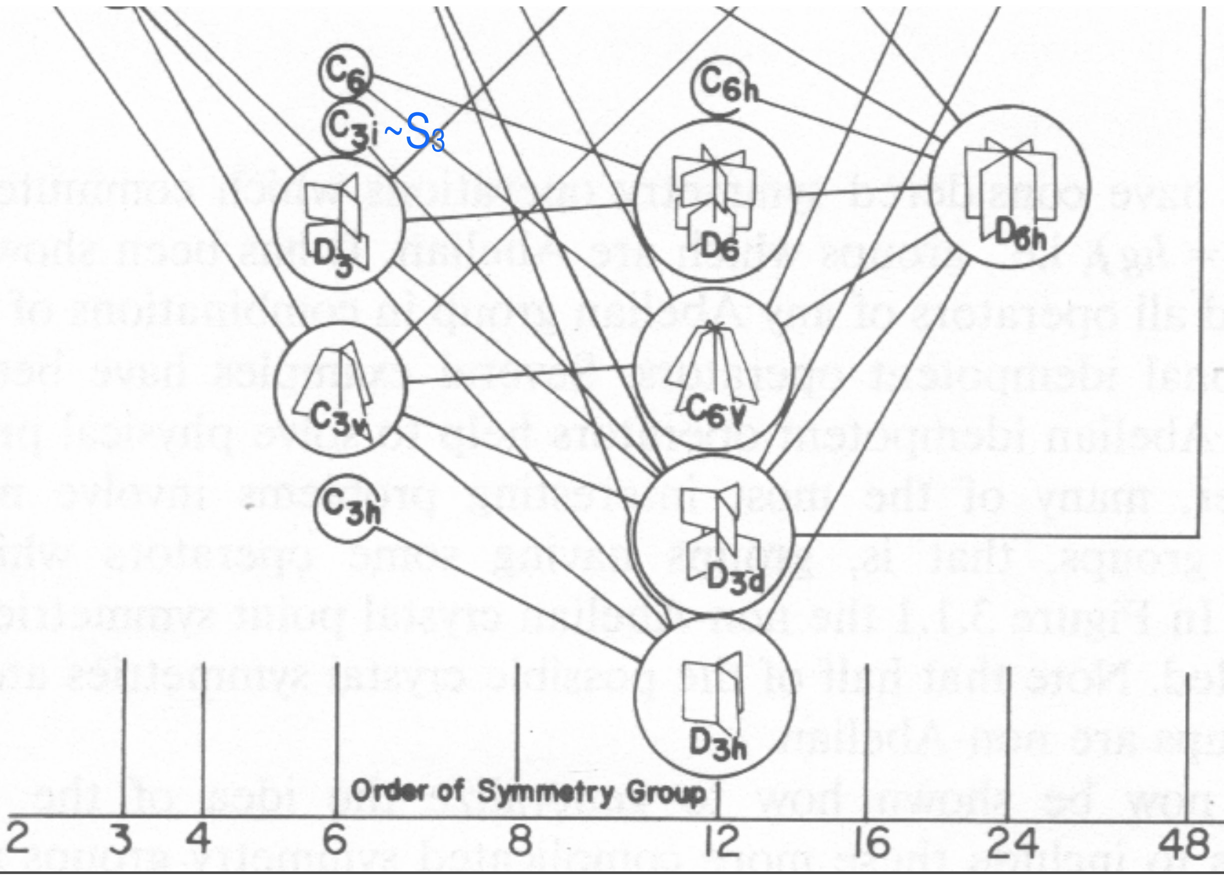
$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$ $E(O)\downarrow D_4 = A_1 \oplus B_1(D_4)$

$T_1(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$ $T_1(O)\downarrow D_4 = E \oplus A_2(D_4)$



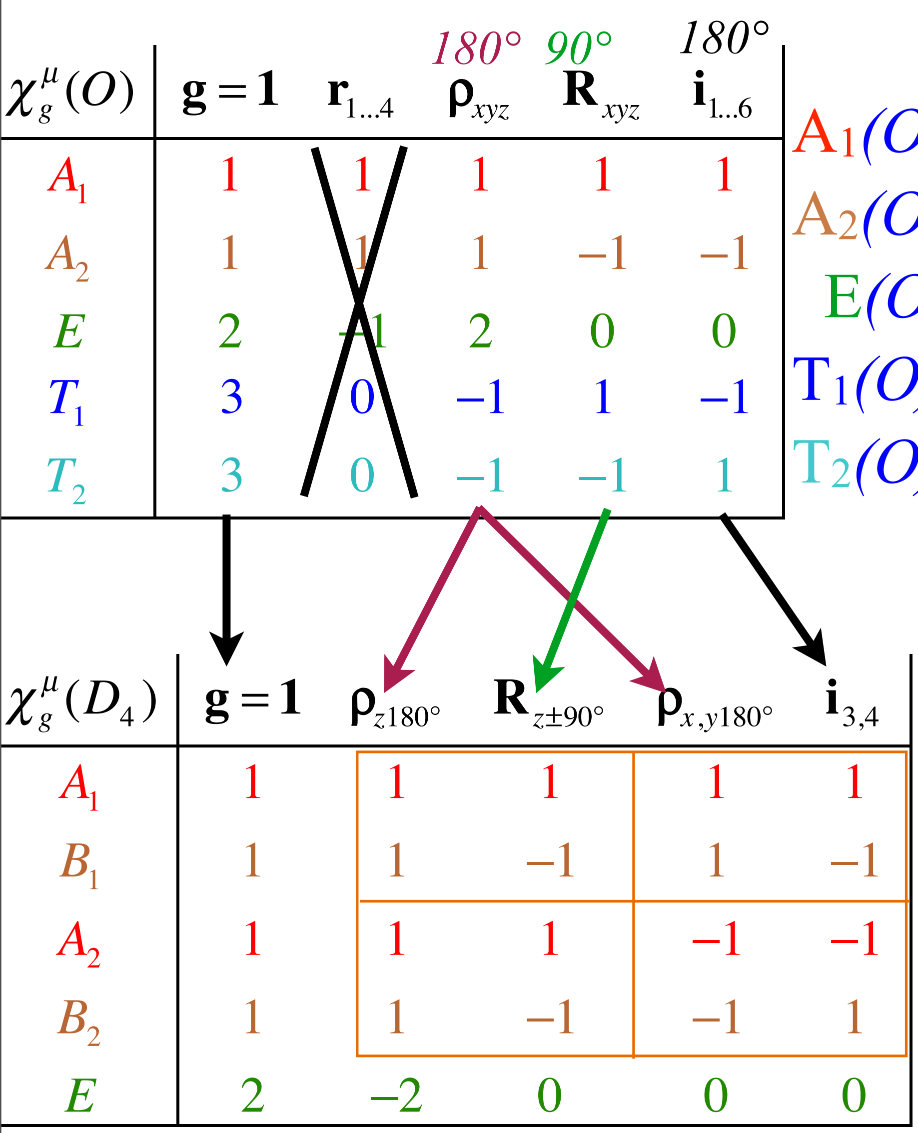
$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction



$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

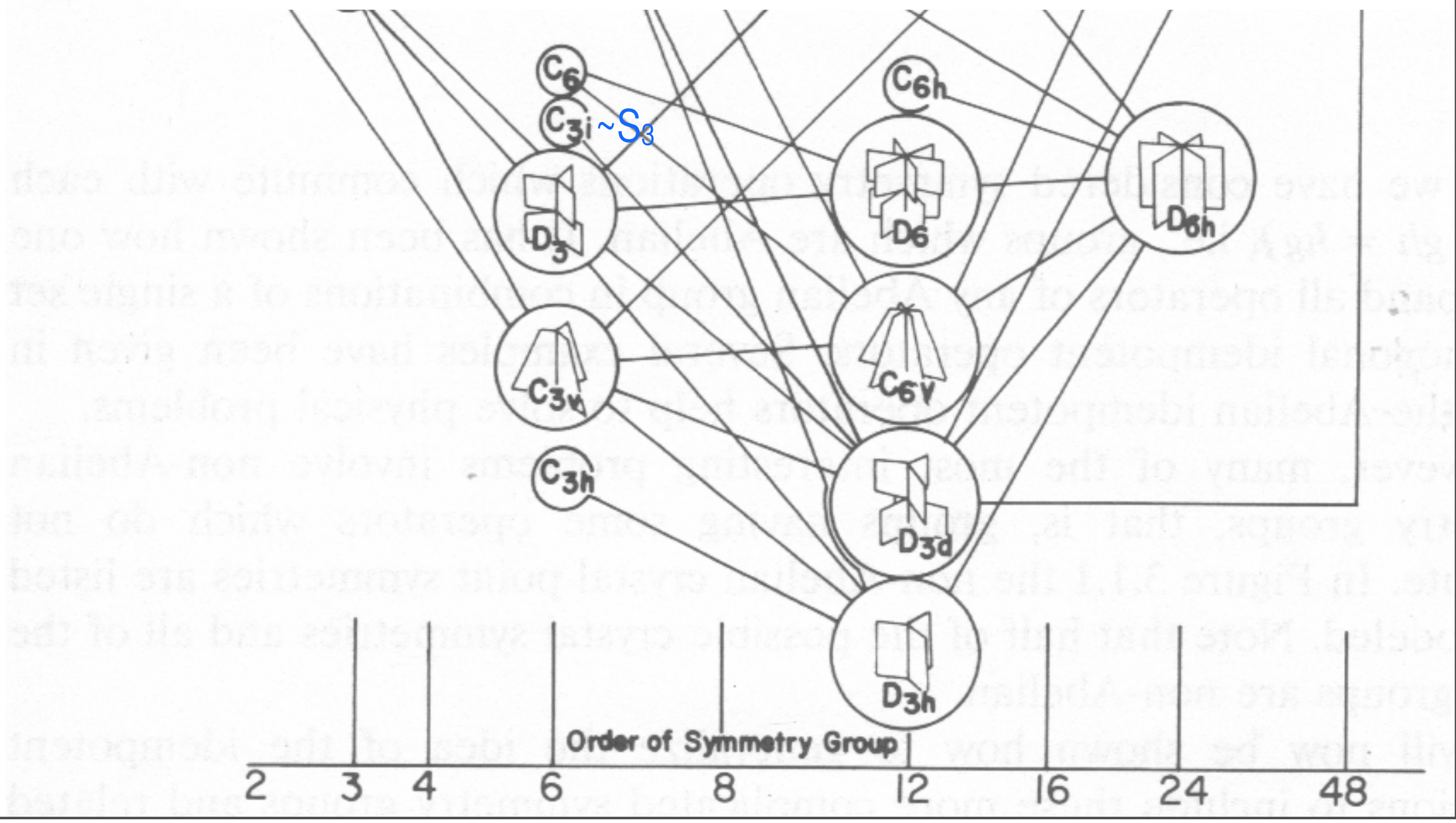
$A_1(O)\downarrow D_4 = 1, \quad 1, \quad 1, \quad 1, \quad 1\ .$ $\quad A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1, \quad 1, \quad -1, \quad 1, \quad -1\ .$ $\quad A_2(O)\downarrow D_4 = B_1(D_4)$

$E(O)\downarrow D_4 = 2, \quad 2, \quad 0, \quad 2, \quad 0\ .$ $\quad E(O)\downarrow D_4 = A_1 \oplus B_1(D_4)$

$T_1(O)\downarrow D_4 = 3, \quad -1, \quad 1, \quad -1, \quad -1\ .$ $\quad T_1(O)\downarrow D_4 = E \oplus A_2(D_4)$

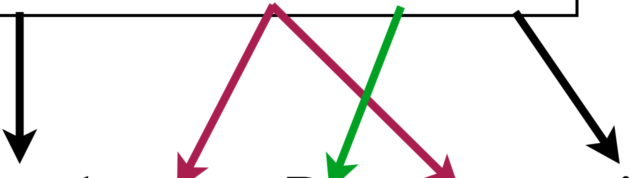
$T_2(O)\downarrow D_4 = 3, \quad -1, \quad -1, \quad -1, \quad 1\ .$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1



$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

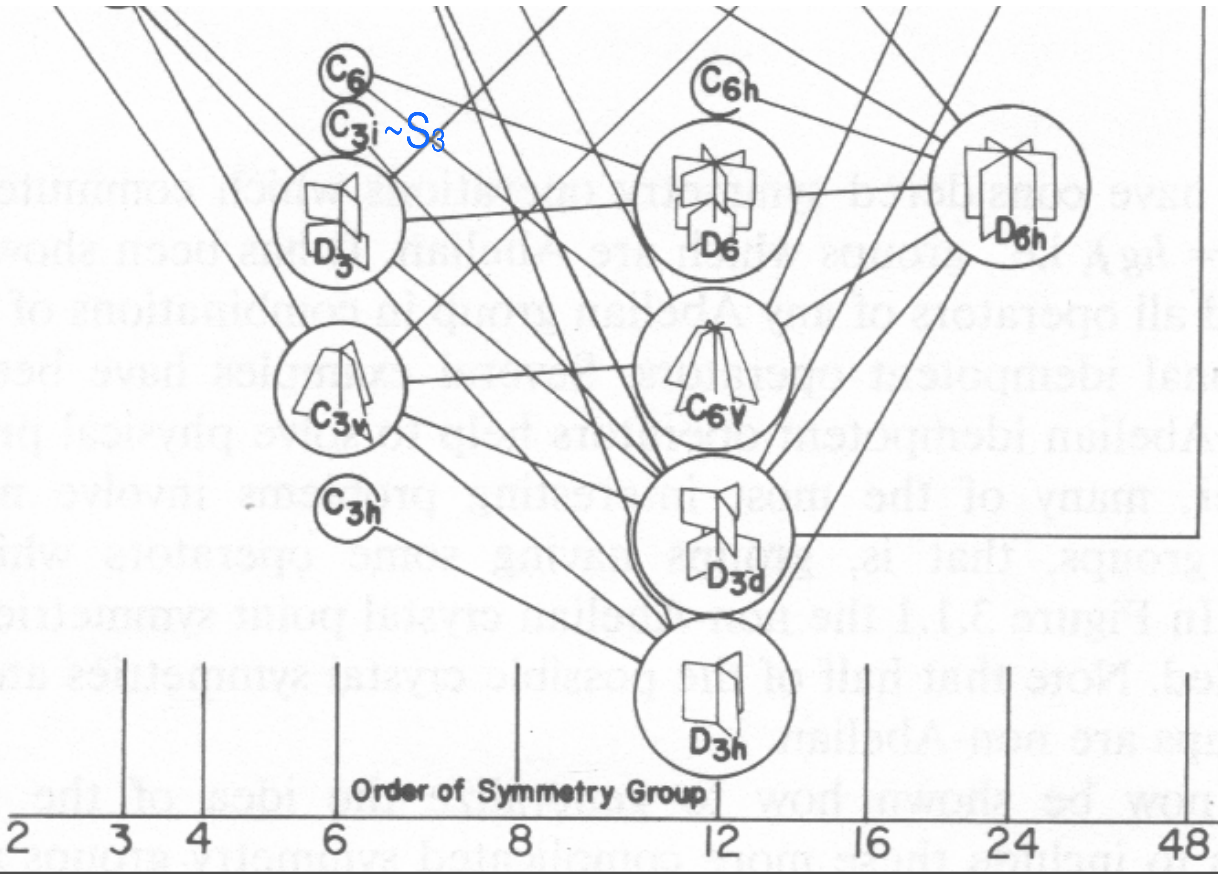
$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O)\downarrow D_4 = B_1(D_4)$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$ $E(O)\downarrow D_4 = A_1 \oplus B_1(D_4)$

$T_1(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$ $T_1(O)\downarrow D_4 = E \oplus A_2(D_4)$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$ $T_2(O)\downarrow D_4 = E \oplus B_2(D_4)$



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$\chi^\mu_g(O)$	$g=1$	$r_{1\ldots4}$	180° ρ_{xyz}	90° R_{xyz}	180° $i_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^\mu_g(D_4)$	$g=1$	$\rho_{z180°}$	$R_{z\pm 90°}$	$\rho_{x,y180°}$	$i_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$ $A_1(O)\downarrow D_4 = A_1(D_4)$

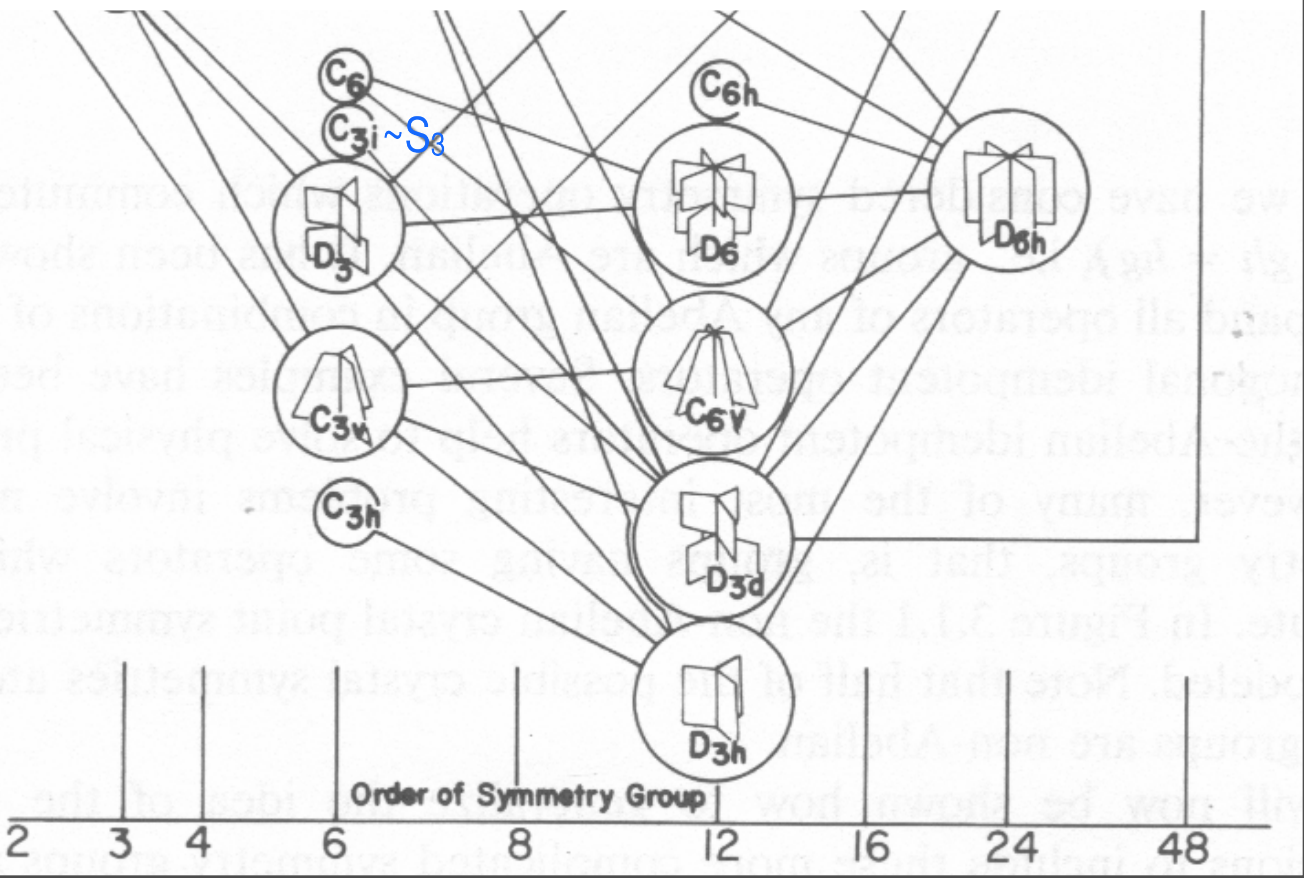
$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$ $A_2(O)\downarrow D_4 = B_1(D_4)$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$ $E(O)\downarrow D_4 = A_1 \oplus B_1(D_4)$

$T_1(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$ $T_1(O)\downarrow D_4 = E \oplus A_2(D_4)$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$ $T_2(O)\downarrow D_4 = E \oplus B_2(D_4)$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi_g^\mu(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	-1	2	0	0
T_1	3	0	-1	1	-1
T_2	3	0	-1	-1	1

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

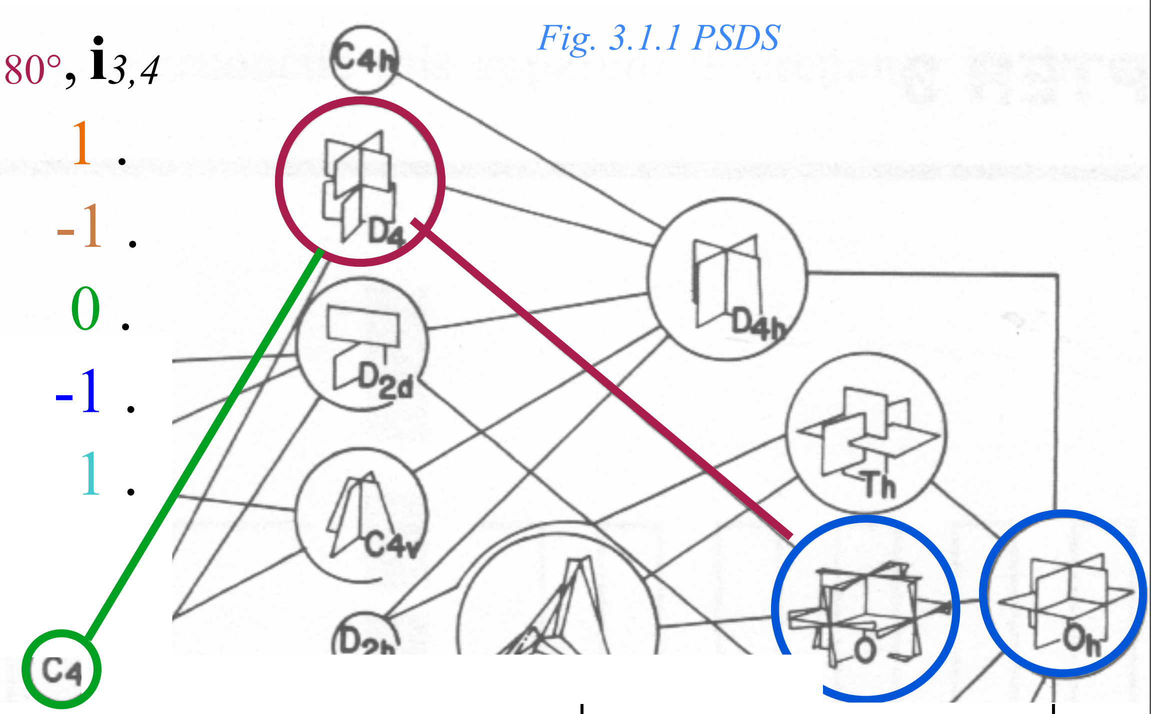
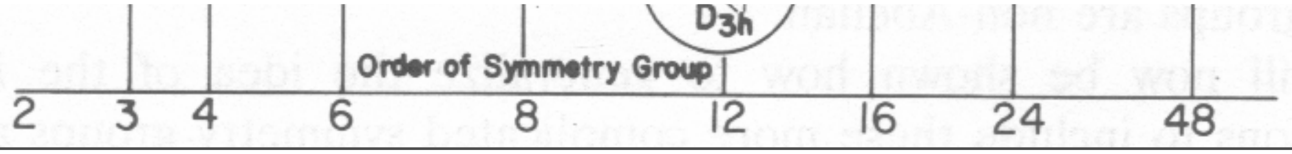


Fig. 3.1.1 PSDS

$\chi_g^\mu(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	-1	1	-1
A_2	1	1	1	-1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi_g^\mu(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	-1	$-i$
$(2)_4$	1	-1	1	-1
$(3)_4$	1	$-i$	-1	i



- *Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$*
  - *Octahedral-cubic O symmetry and group operations*
  - *Tetrahedral symmetry becomes Icosahedral*
  - *Octahedral groups $O_h \supset O \sim T_d \supset T$*
    - *Octahedral O and spin-$O \subset U(2)$*

- *Tetrahedral T class algebra*
  - *Tetrahedral T class minimal equations*
  - *Tetrahedral T class projectors and characters*

- *Octahedral O class algebra*
  - *Octahedral O class minimal equations*
  - *Octahedral O class projectors and characters*

- *Octahedral $O_h \supset O$:Inversion (g&u) parity*
  - *Octahedral $O_h \supset O \supset C_I$ subgroup correlations*
  - *$O_h \supset O \supset D_4$ subgroup correlations*
  - *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
  - *Preview of applications to high resolution spectroscopy*

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

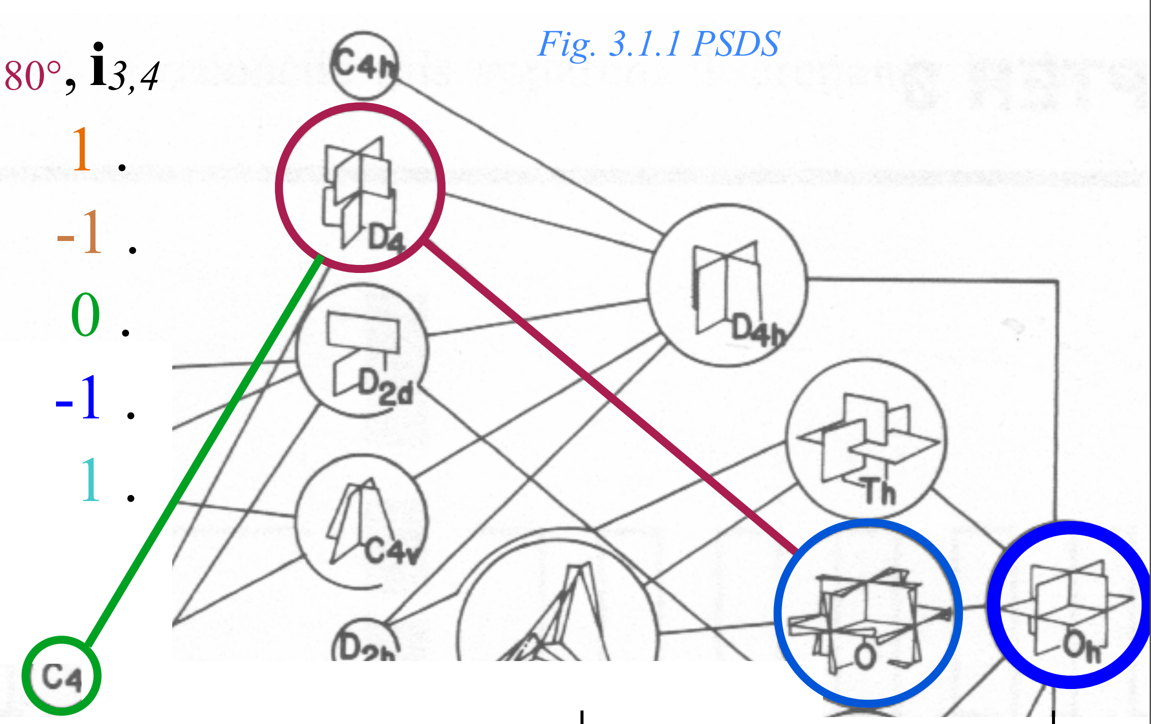
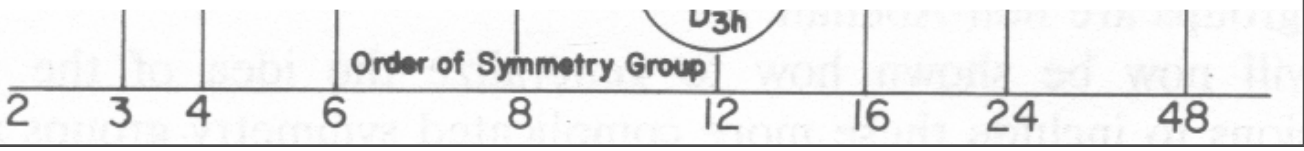


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

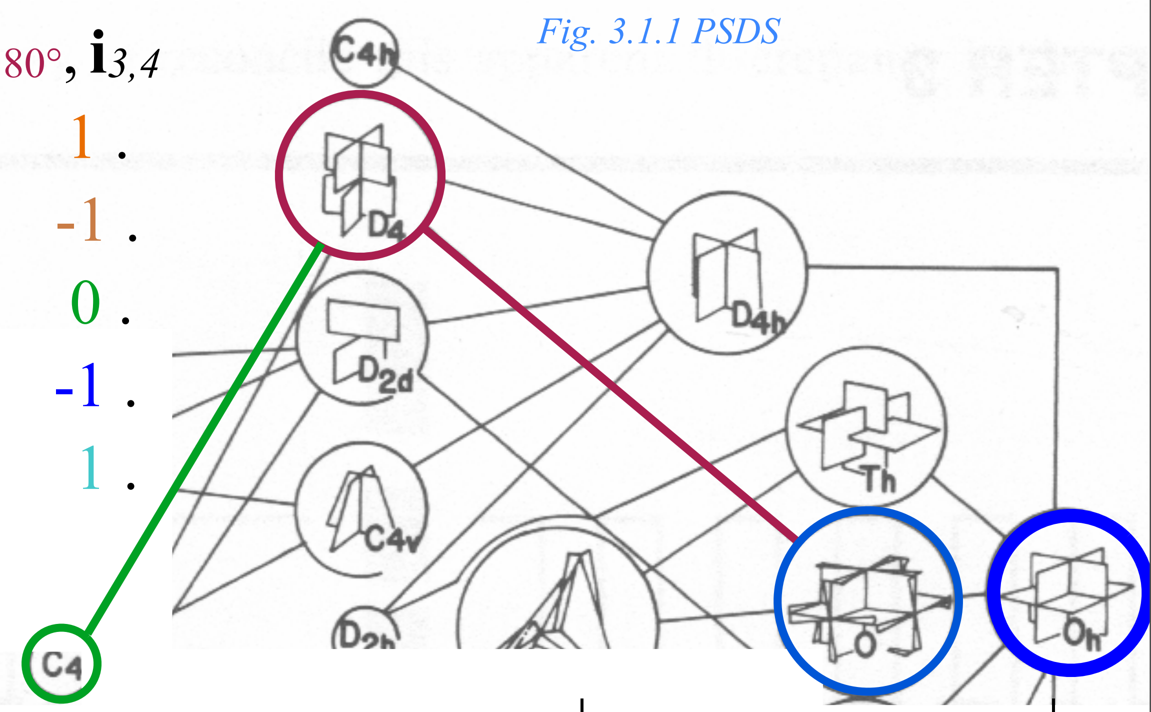


Fig. 3.1.1 PSDS

			180°	90°	180°
$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

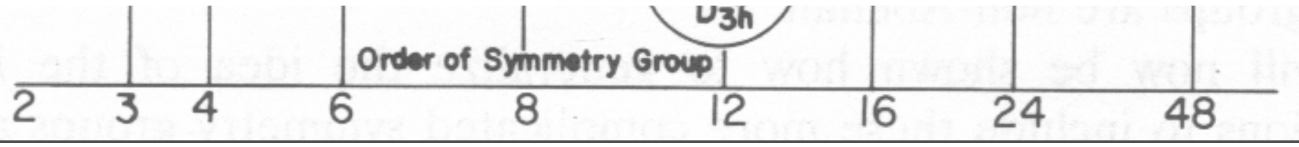
$D_4\downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

$\chi^\mu_g(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^\mu_g(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4{\downarrow}C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$\chi^\mu_g(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

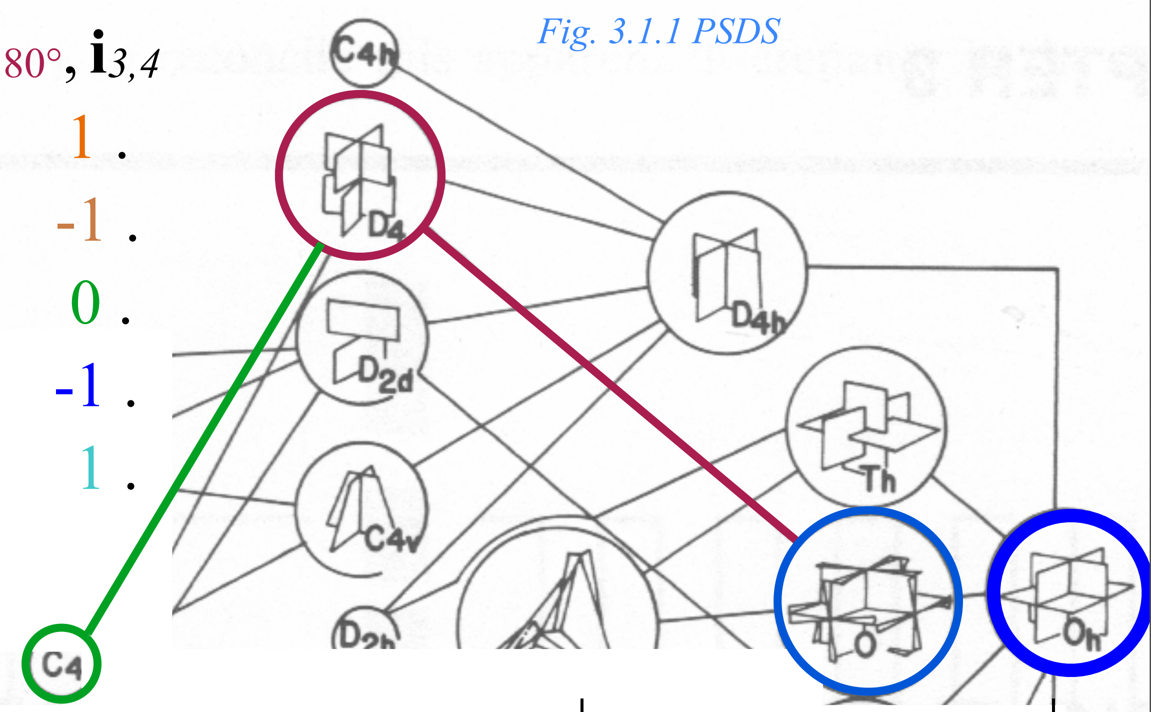
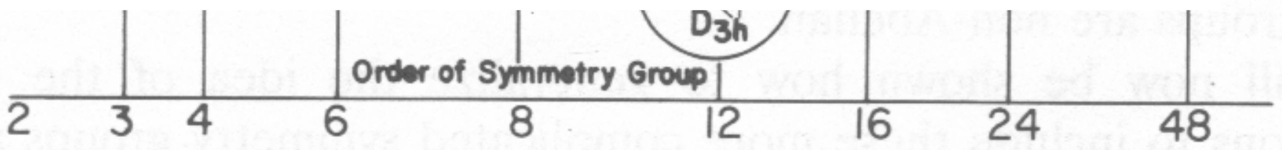


Fig. 3.1.1 PSDS

$O{\downarrow}D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	$\cancel{1}$	1	1	1
A_2	1	$\cancel{1}$	1	−1	−1
E	2	$\cancel{-1}$	2	0	0
T_1	3	$\cancel{0}$	−1	1	−1
T_2	3	$\cancel{0}$	−1	−1	1

$O\downarrow D_4$ *subduction*

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

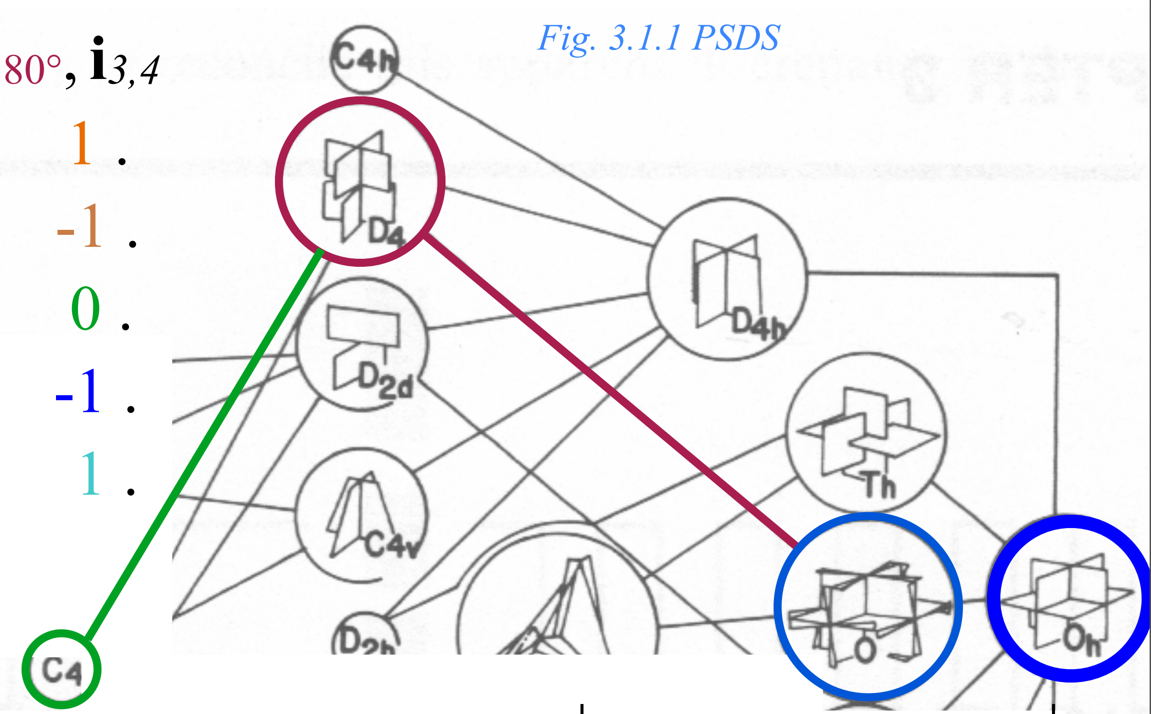


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	$\cancel{1}$	$\cancel{1}$
B_1	1	1	−1	$\cancel{1}$	$\cancel{-1}$
A_2	1	1	1	$\cancel{-1}$	$\cancel{-1}$
B_2	1	1	−1	$\cancel{-1}$	$\cancel{1}$
E	2	−2	0	$\cancel{0}$	$\cancel{0}$

$D_4\downarrow C_4$ *subduction*

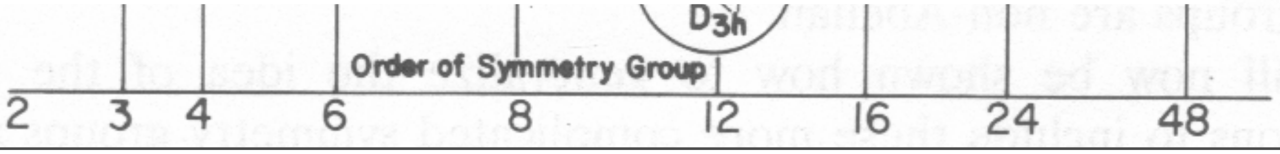
C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1...4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

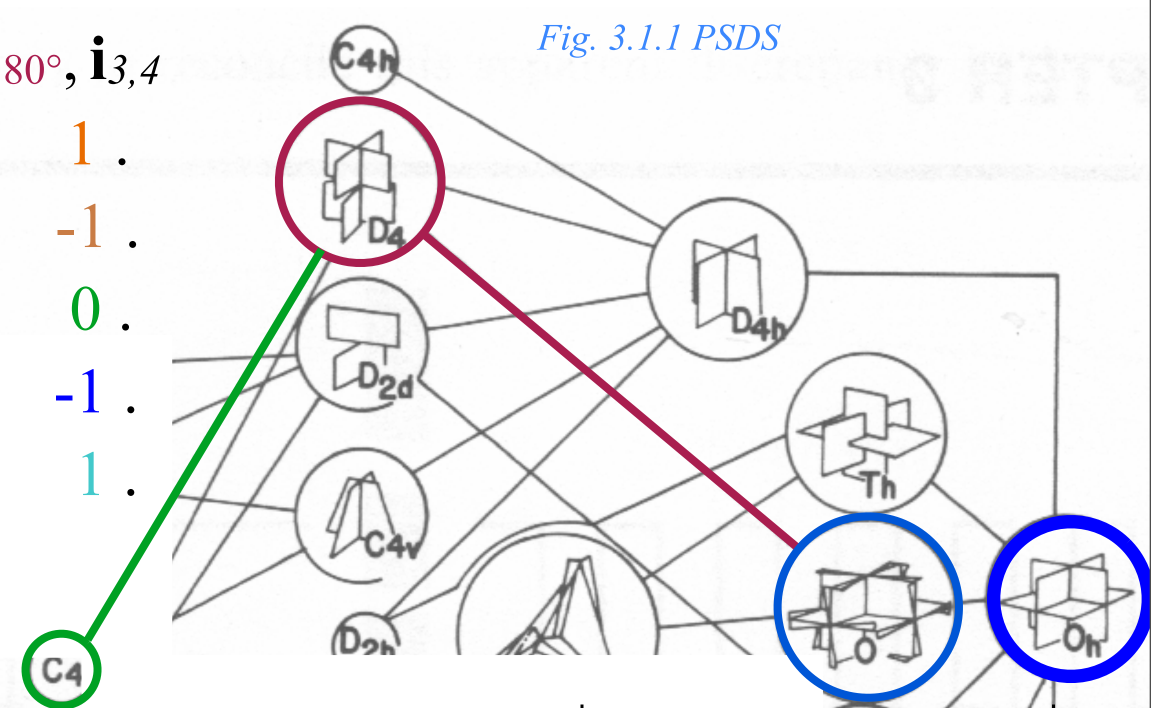


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4 \downarrow C_4$ subduction

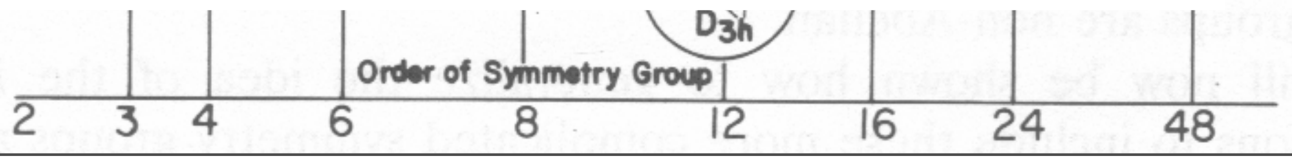
C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	.	.	.	.
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

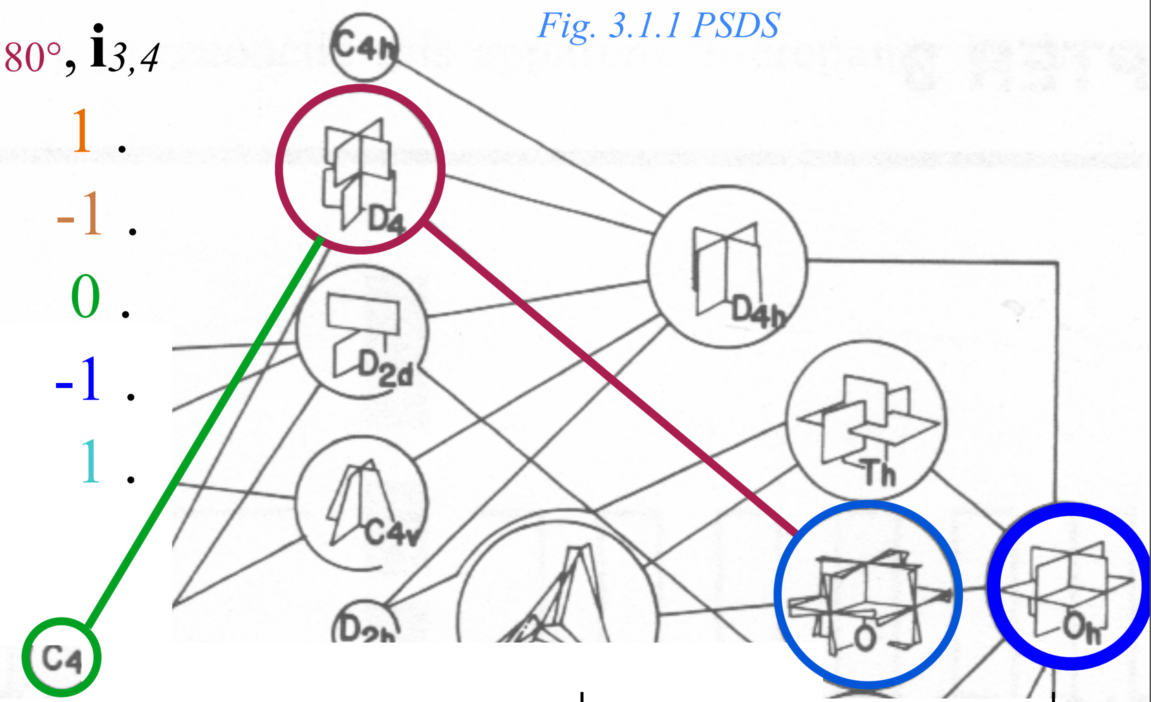


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4 \downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

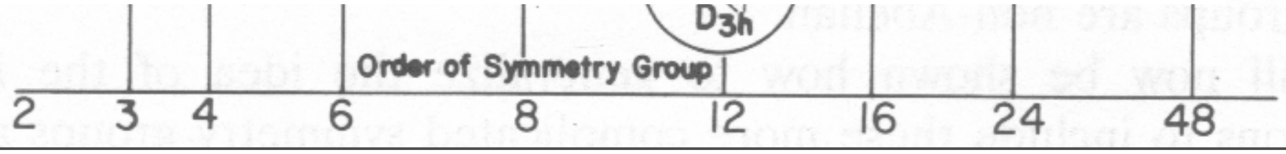
$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	$\cancel{1}$	1	1	1
A_2	1	$\cancel{1}$	1	−1	−1
E	2	$\cancel{-1}$	2	0	0
T_1	3	$\cancel{0}$	−1	1	−1
T_2	3	$\cancel{0}$	−1	−1	1

$O{\downarrow}D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O){\downarrow}D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O){\downarrow}D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O){\downarrow}D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O){\downarrow}D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

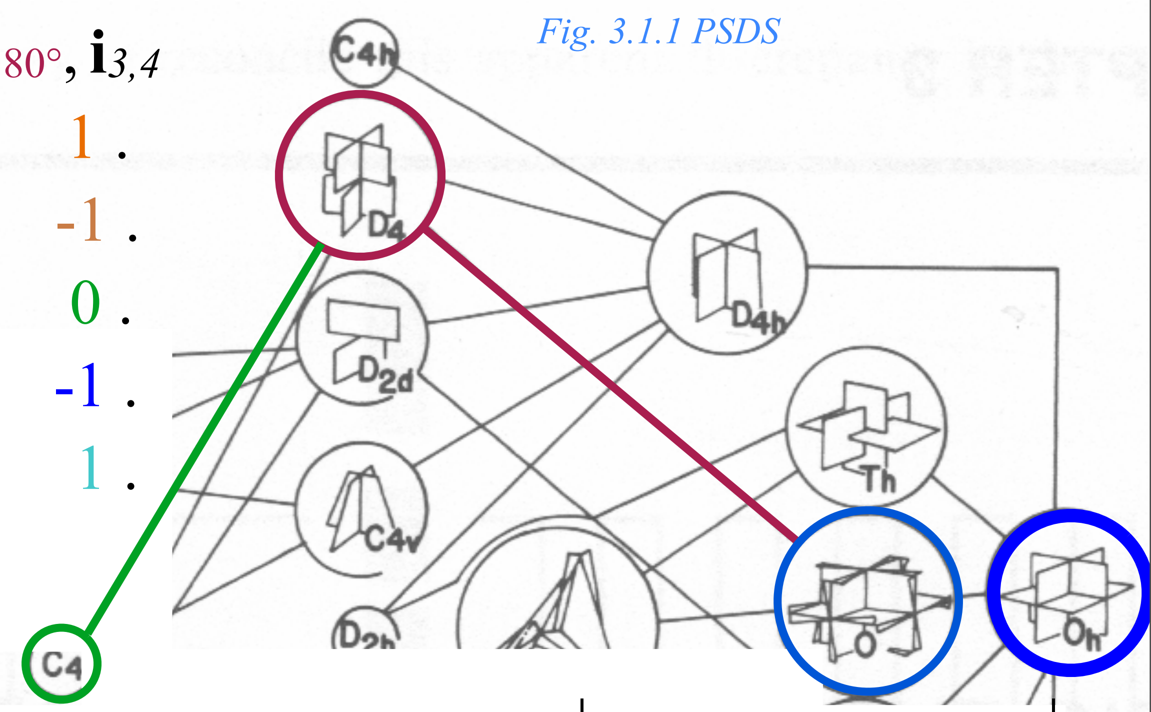


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	$\cancel{1}$	$\cancel{1}$
B_1	1	1	−1	$\cancel{1}$	$\cancel{-1}$
A_2	1	1	1	$\cancel{-1}$	$\cancel{-1}$
B_2	1	1	−1	$\cancel{-1}$	$\cancel{1}$
E	2	−2	0	$\cancel{0}$	$\cancel{0}$

$D_4{\downarrow}C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

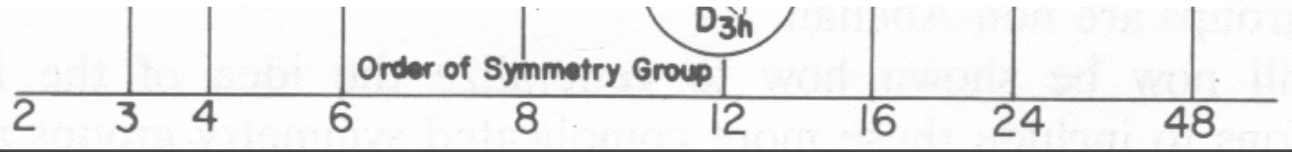
$A_1(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4){\downarrow}C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4){\downarrow}C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$O{\downarrow}D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

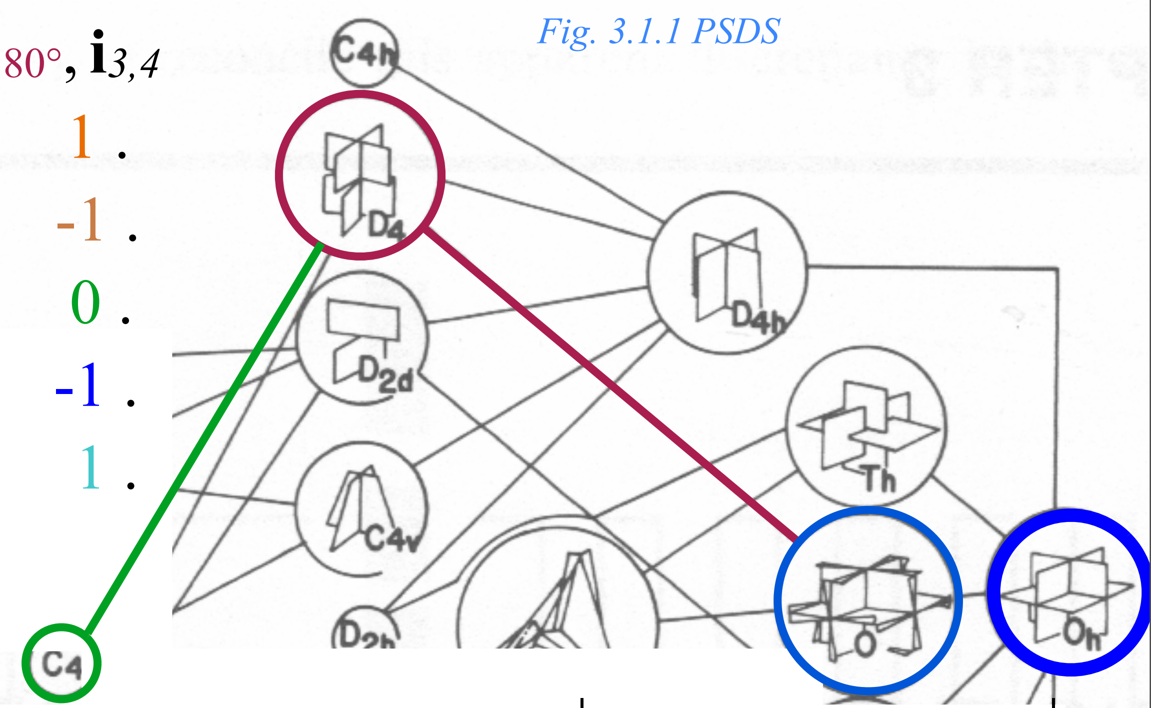


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4 \downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.\quad =(0)_4$

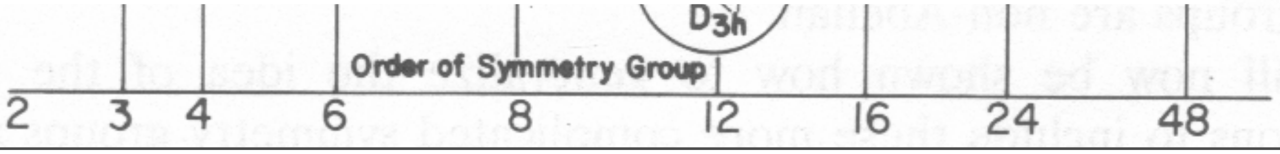
$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.\quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.\quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 =$ 1, 1, 1, 1, 1 .

$A_2(O)\downarrow D_4 =$ 1, 1, -1, 1, -1 .

$E(O)\downarrow D_4 =$ 2, 2, 0, 2, 0 .

$T_2(O)\downarrow D_4 =$ 3, -1, 1, -1, -1 .

$T_2(O)\downarrow D_4 =$ 3, -1, -1, -1, 1 .

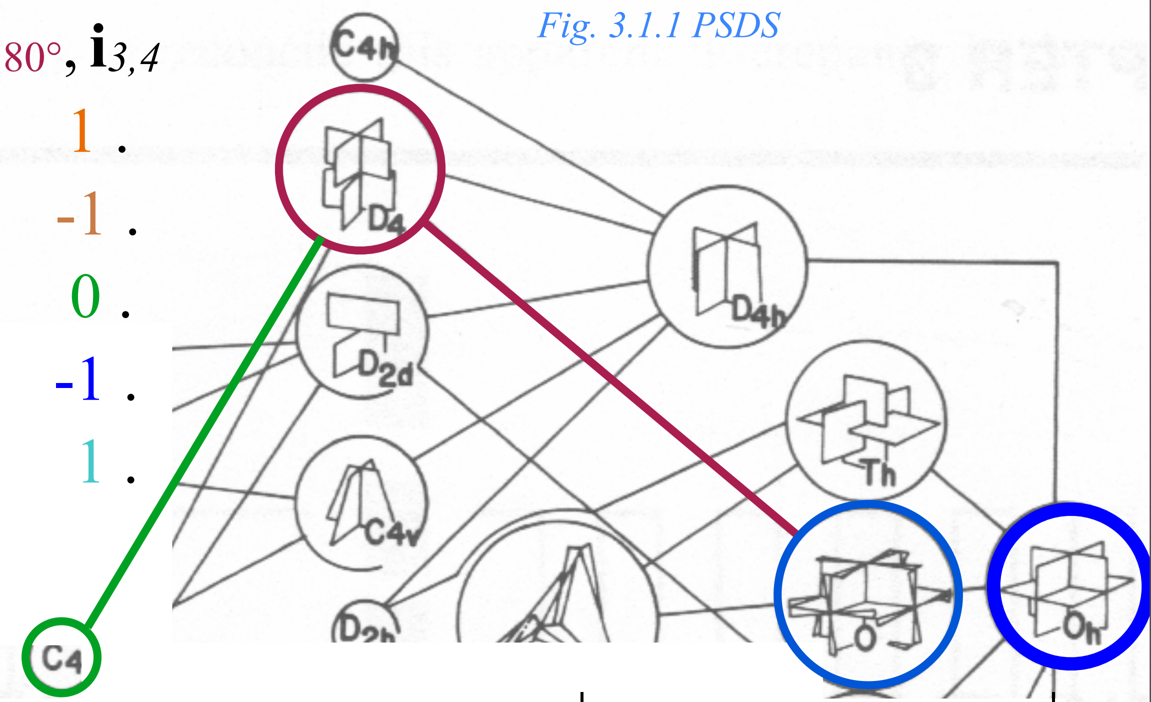


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4 \downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 =$ 1, 1, 1, 1. $=(0)_4$

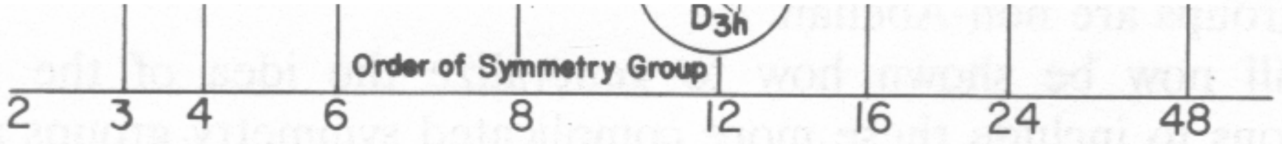
$B_1(D_4)\downarrow C_4 =$ 1, -1, 1, -1. $=(2)_4$

$A_2(D_4)\downarrow C_4 =$ 1, 1, 1, 1. $=(0)_4$

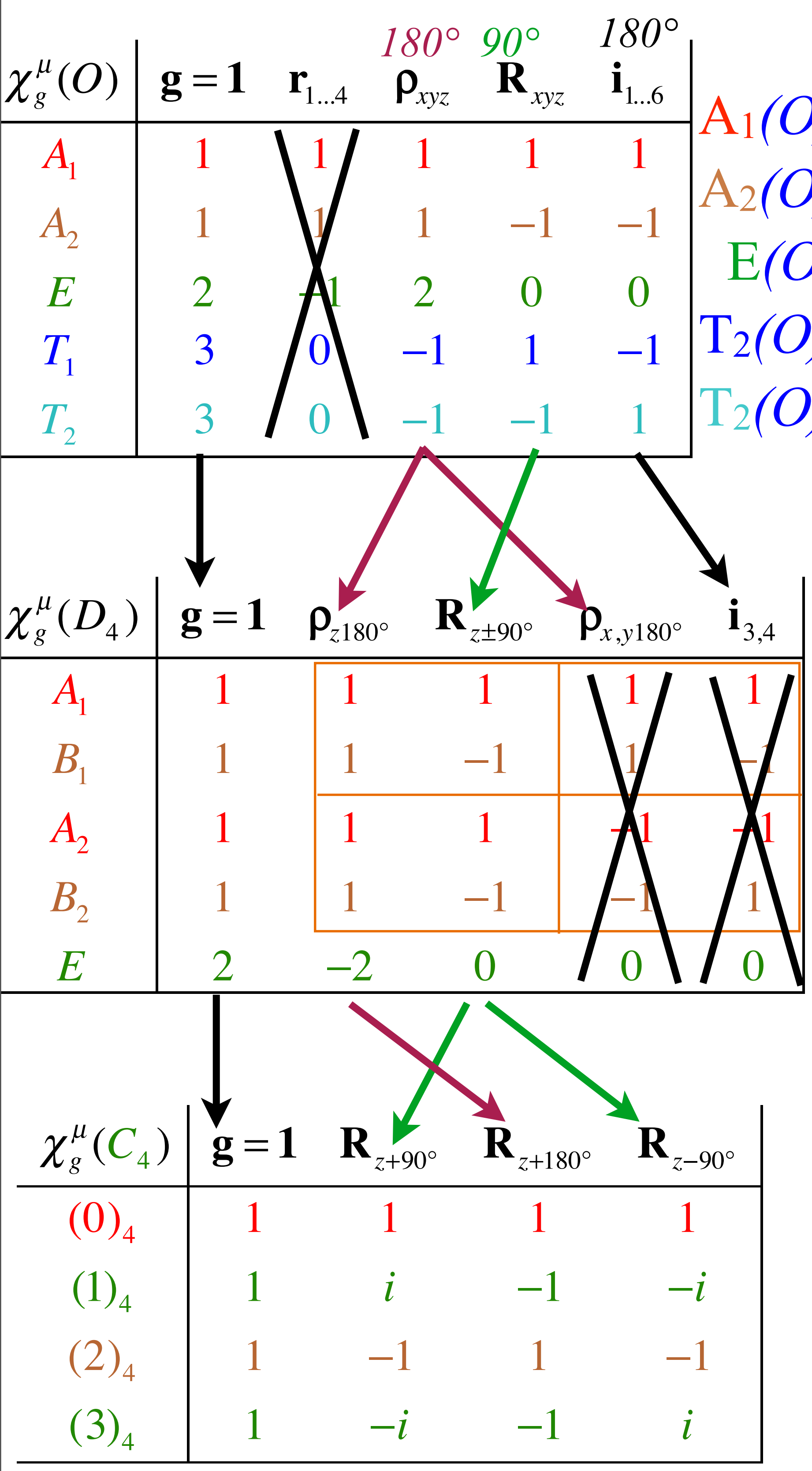
$B_2(D_4)\downarrow C_4 =$ 1, -1, 1, -1. $=(2)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations



$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

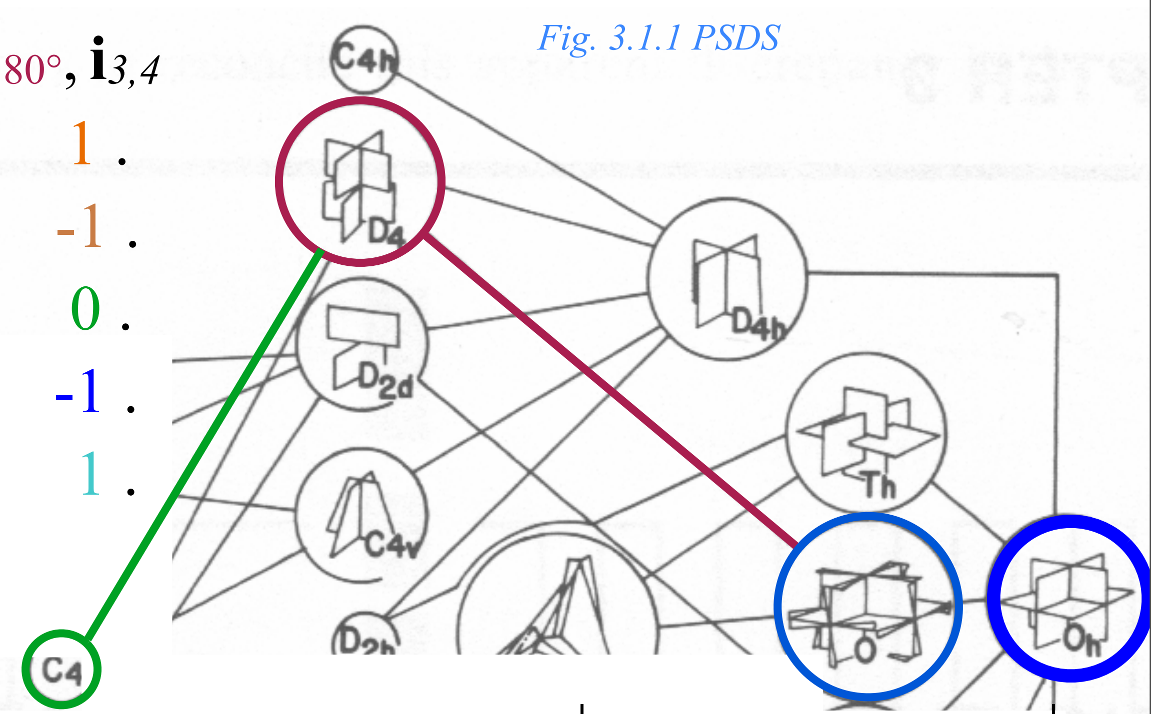


Fig. 3.1.1 PSDS

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$D_4\downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ .$

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1, \ 1, \ 1, \ 1, \ 1 \ .$

$A_2(O)\downarrow D_4 = 1, \ 1, \ -1, \ 1, \ -1 \ .$

$E(O)\downarrow D_4 = 2, \ 2, \ 0, \ 2, \ 0 \ .$

$T_2(O)\downarrow D_4 = 3, \ -1, \ 1, \ -1, \ -1 \ .$

$T_2(O)\downarrow D_4 = 3, \ -1, \ -1, \ -1, \ 1 \ .$

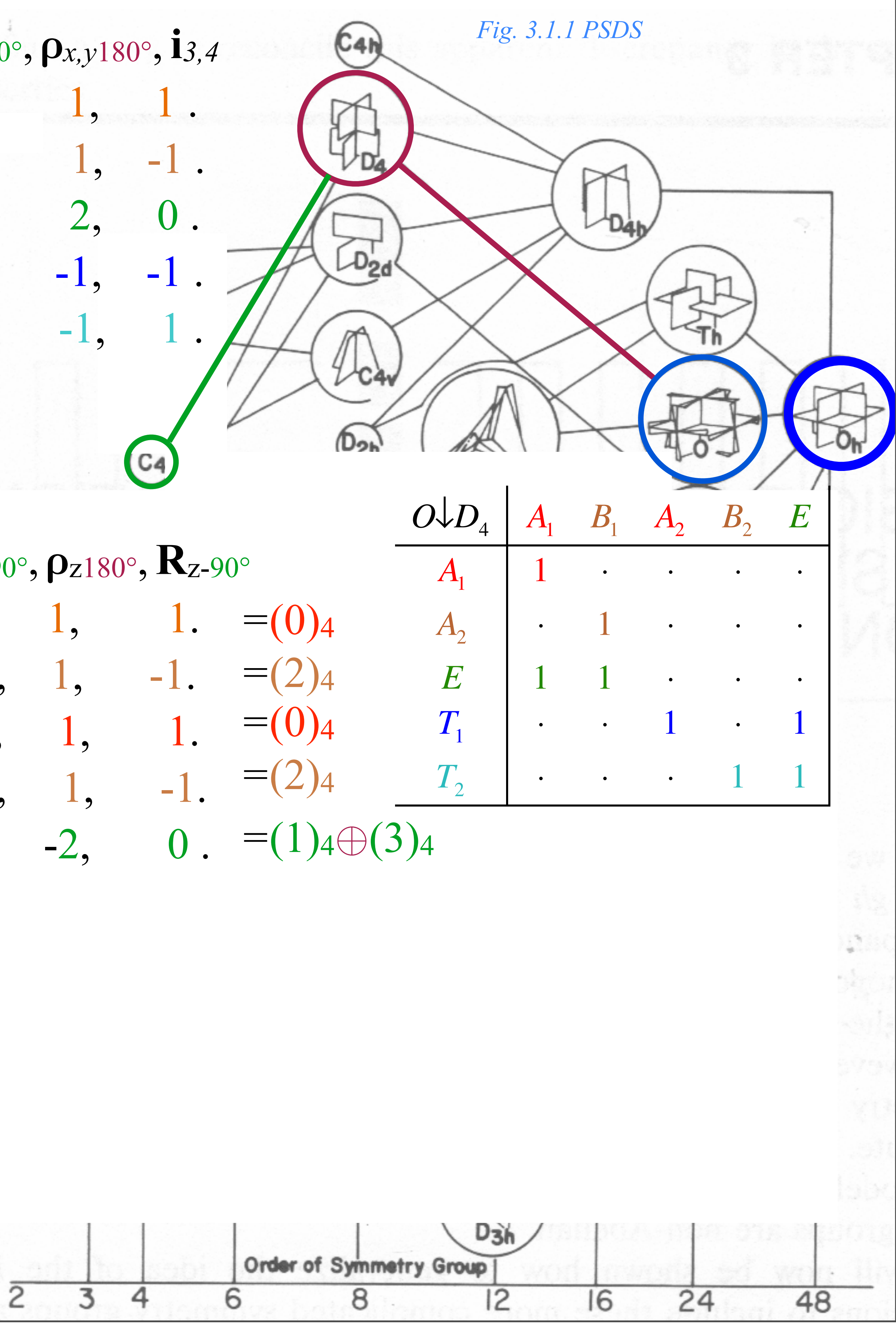


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1, \ 1, \ 1, \ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1, \ -1, \ 1, \ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1, \ 1, \ 1, \ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1, \ -1, \ 1, \ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2, \ 0, \ -2, \ 0 \ . \quad =(1)_4 \oplus (3)_4$

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	.	.	.	.
A_2	.	1	.	.	.
E	1	1	.	.	.
T_1	.	.	1	.	1
T_2	.	.	.	1	1

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

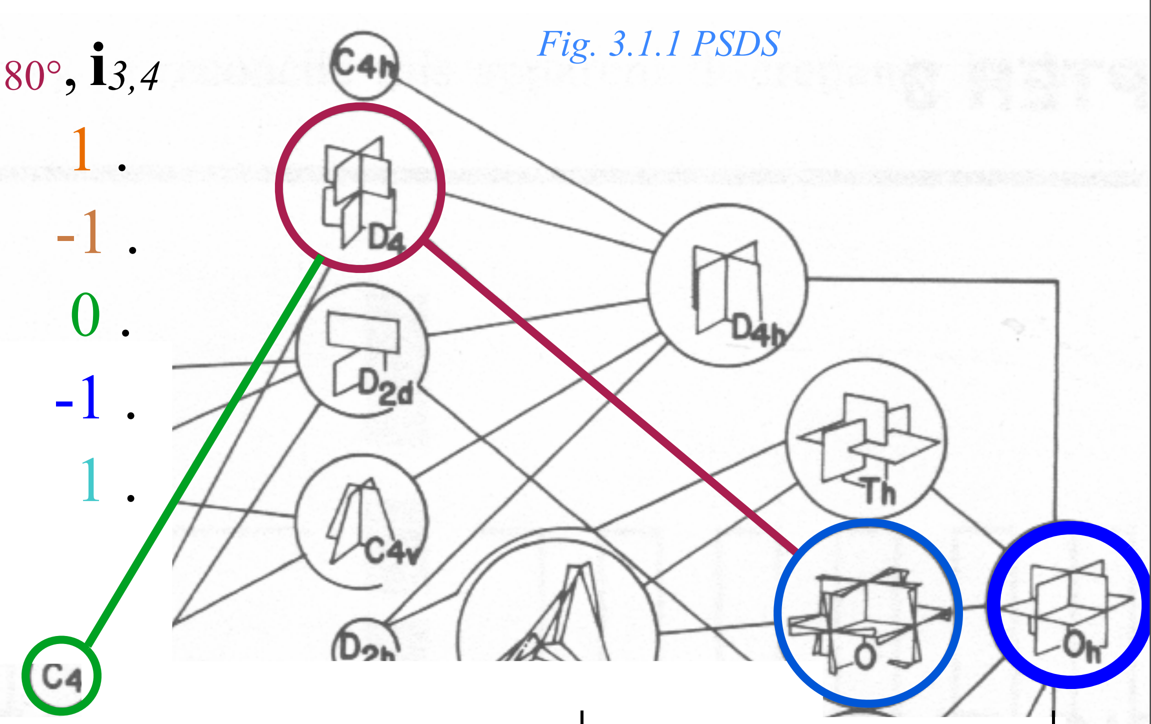


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.\ =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.\ =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1.\ =(0)_4$

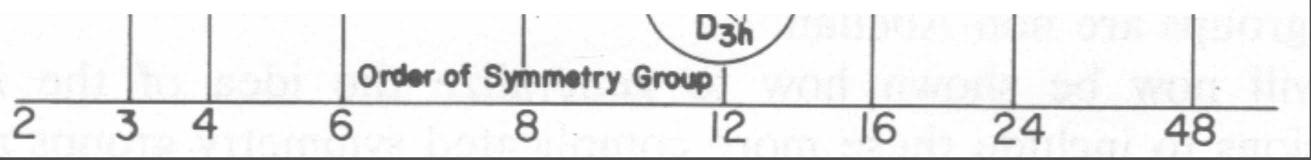
$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1.\ =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ .\ =(1)_4 \oplus (3)_4$

$O \downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$D_4 \downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

O↓D₄ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

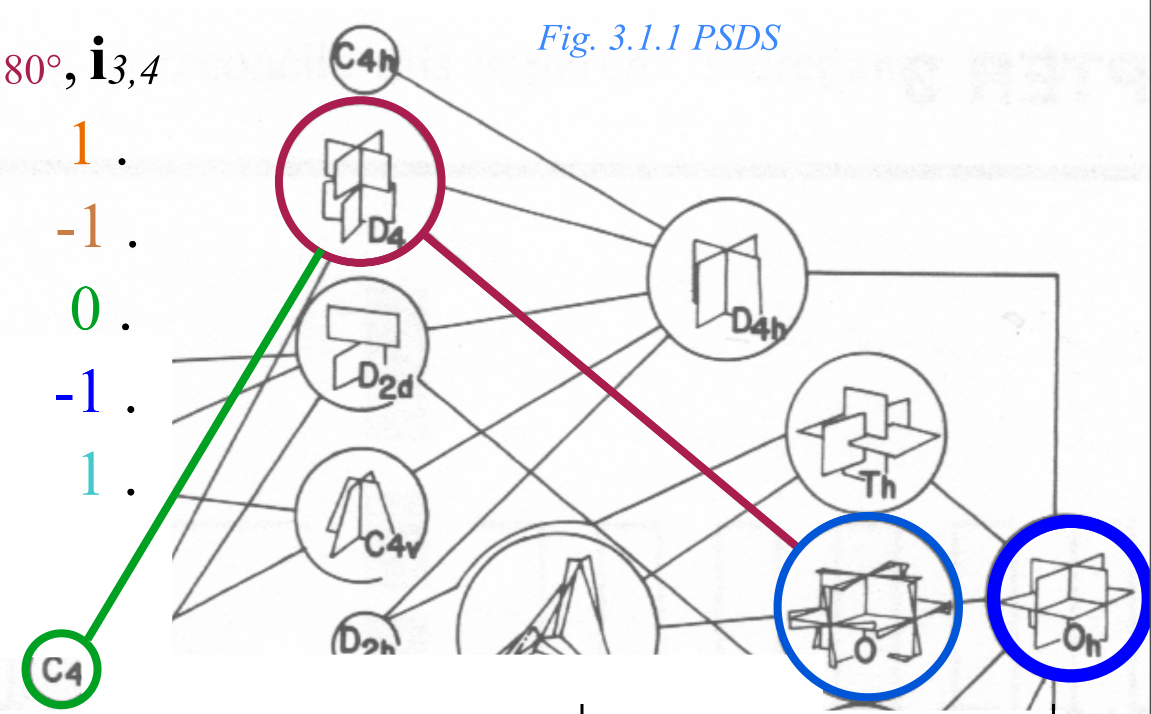


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

D₄↓C₄ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4 \oplus (3)_4$

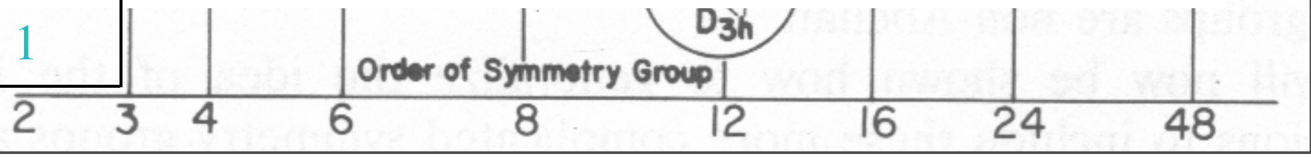
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

O↓C₄ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	1	1	1

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	~~1~~	1	1	1
A_2	1	~~1~~	1	−1	−1
E	2	~~−1~~	2	0	0
T_1	3	~~0~~	−1	1	−1
T_2	3	~~0~~	−1	−1	1

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	~~1~~	~~1~~
B_1	1	1	−1	~~1~~	~~−1~~
A_2	1	1	1	~~−1~~	~~−1~~
B_2	1	1	−1	~~−1~~	~~1~~
E	2	−2	0	~~0~~	~~0~~

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

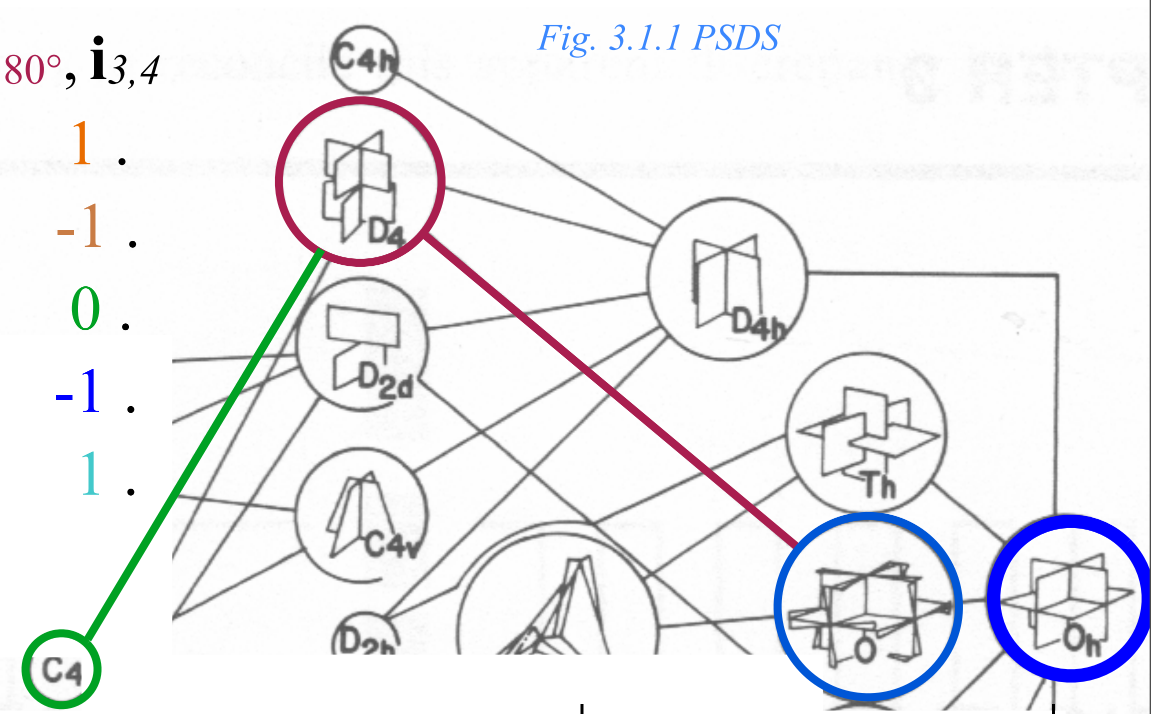


Fig. 3.1.1 PSDS

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$D_4 \downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

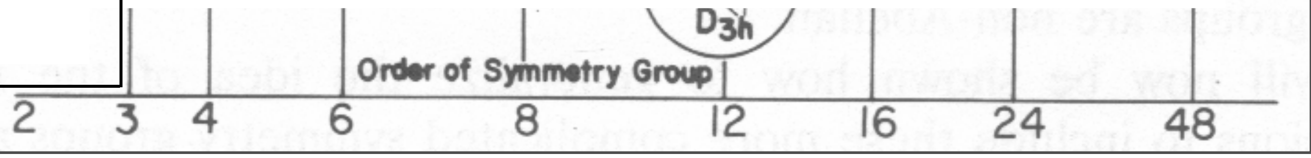
$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4 \oplus (3)_4$

$O \downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	→1	·	·	·
A_2				
E				
T_1				
T_2				

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	→1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$			180°	90°	180°
	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	ρ_{xyz}	$\mathbf{R}_{xyz}$	$\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

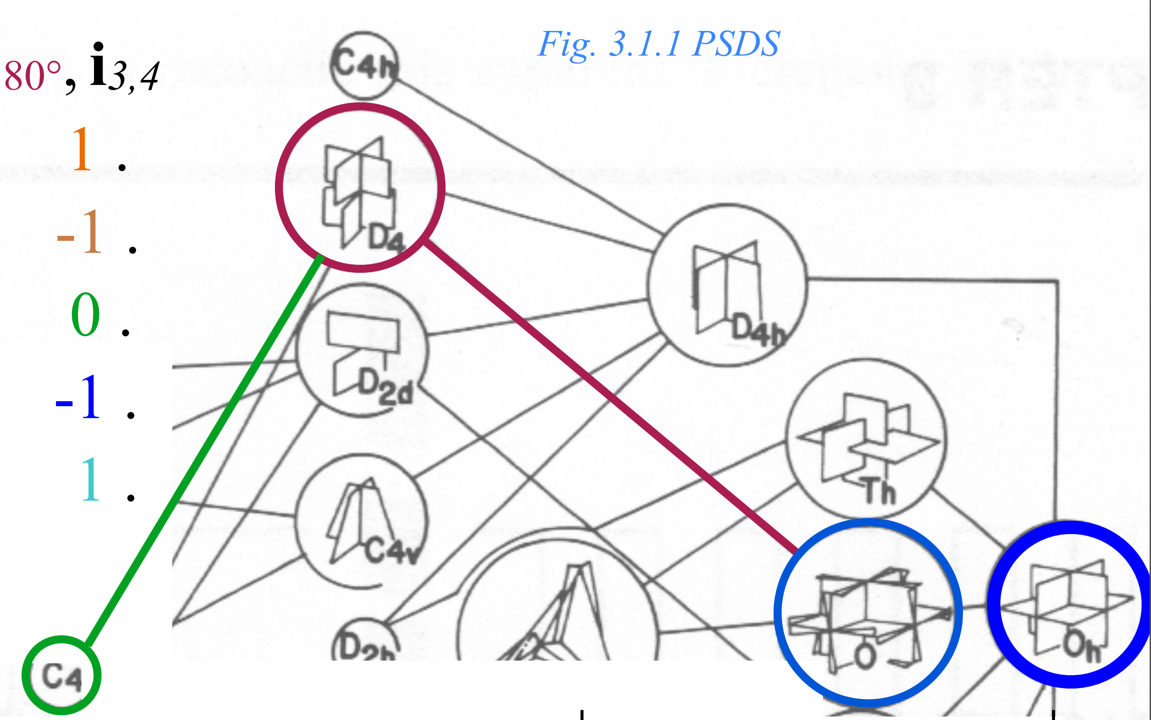


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$D_4\downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4\oplus(3)_4$

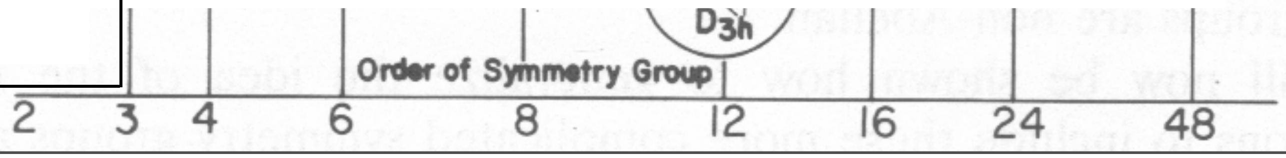
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O\downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
A_2	·	·	→1	·
E				
T_1				
T_2				

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
B_1	·	·	→1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

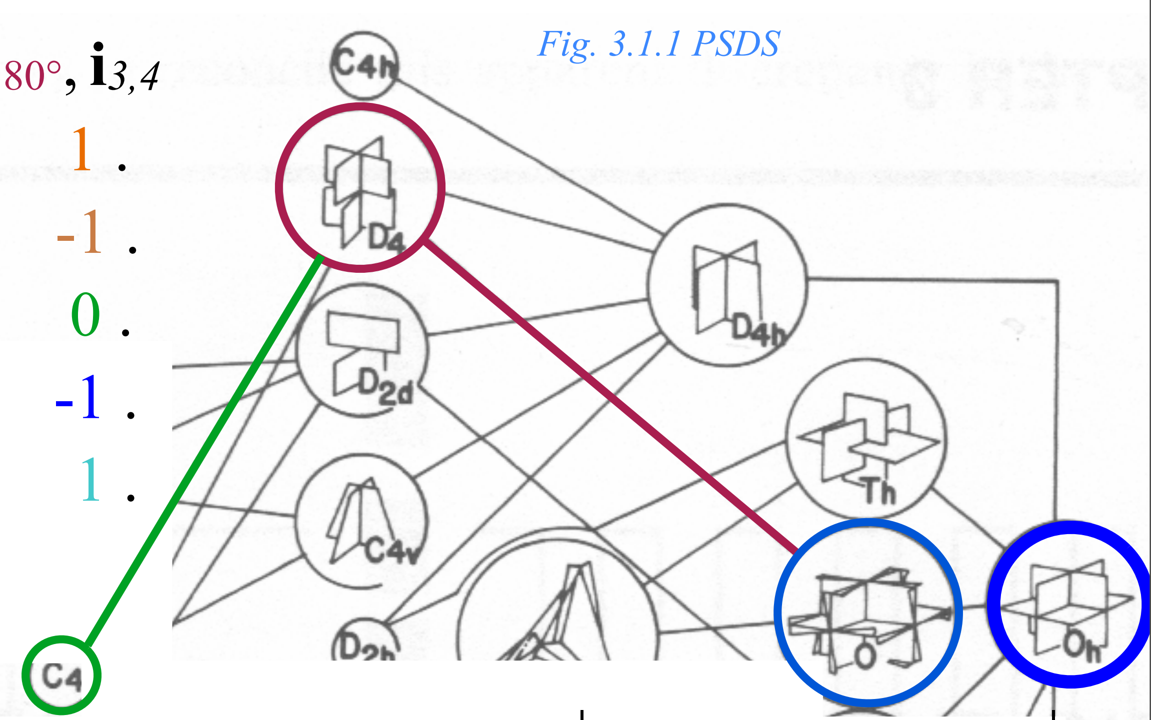


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$D_4\downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4\oplus(3)_4$

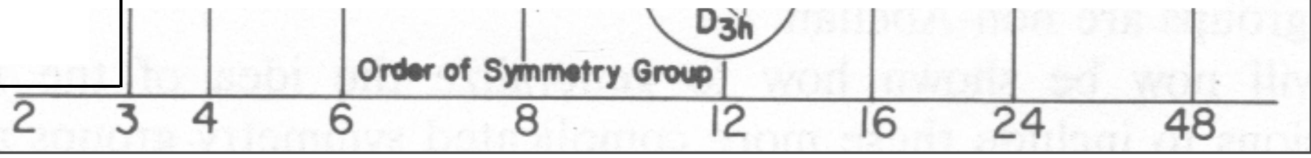
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O\downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	→1	·	→1	·
T_1				
T_2				

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	→1	·	·	·
B_1	·	·	→1	·
A_2	1	·	·	·
B_2	·	·	1	·
E	·	1	·	1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{r}_{1\ldots4}$	180° ρ_{xyz}	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1\ldots6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

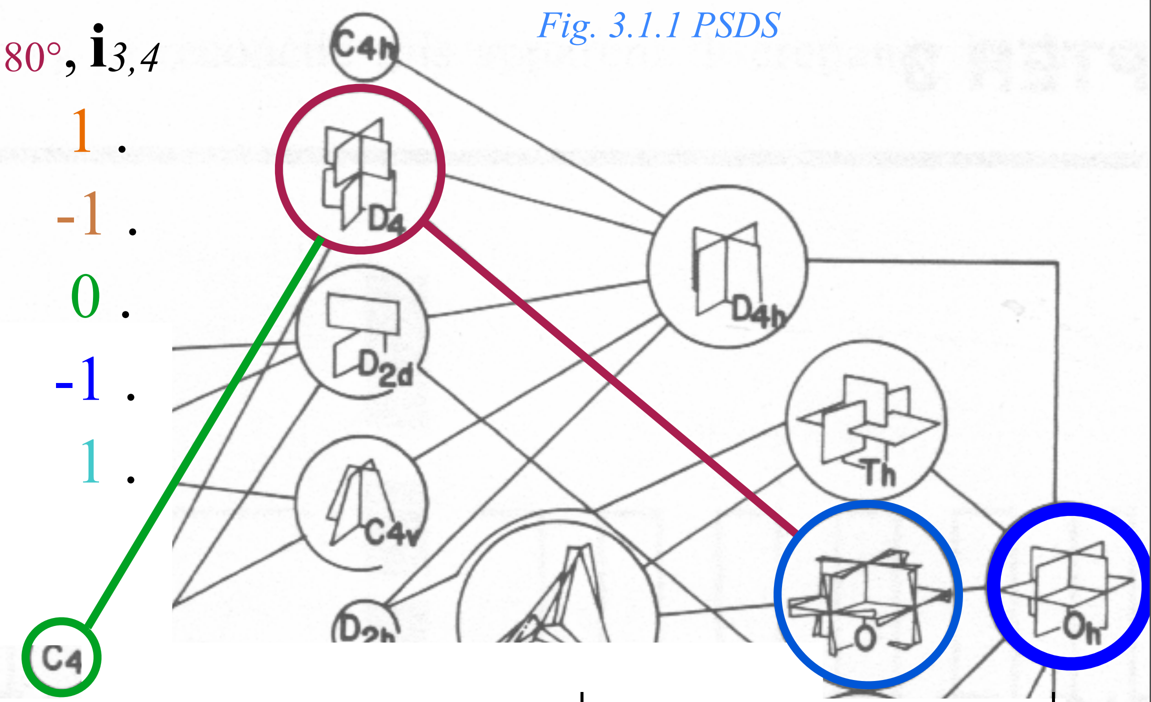


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g}=\mathbf{1}$	$\rho_{z180°}$	$\mathbf{R}_{z\pm90°}$	$\rho_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$D_4\downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4\oplus(3)_4$

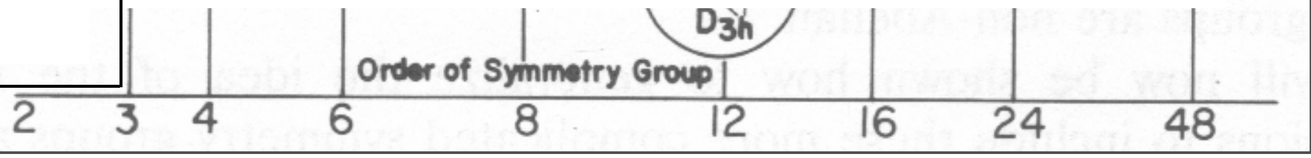
$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g}=\mathbf{1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O\downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	→1	→1	·	→1
T_2				

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	→1	·	·	·
B_2	·	·	1	·
E	·	→1	·	→1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$\chi^{\mu}_{g}(O)$	$\mathbf{g=1}$	$\mathbf{r}_{1...4}$	180° $\boldsymbol{\rho}_{xyz}$	90° $\mathbf{R}_{xyz}$	180° $\mathbf{i}_{1...6}$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
E	2	−1	2	0	0
T_1	3	0	−1	1	−1
T_2	3	0	−1	−1	1

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

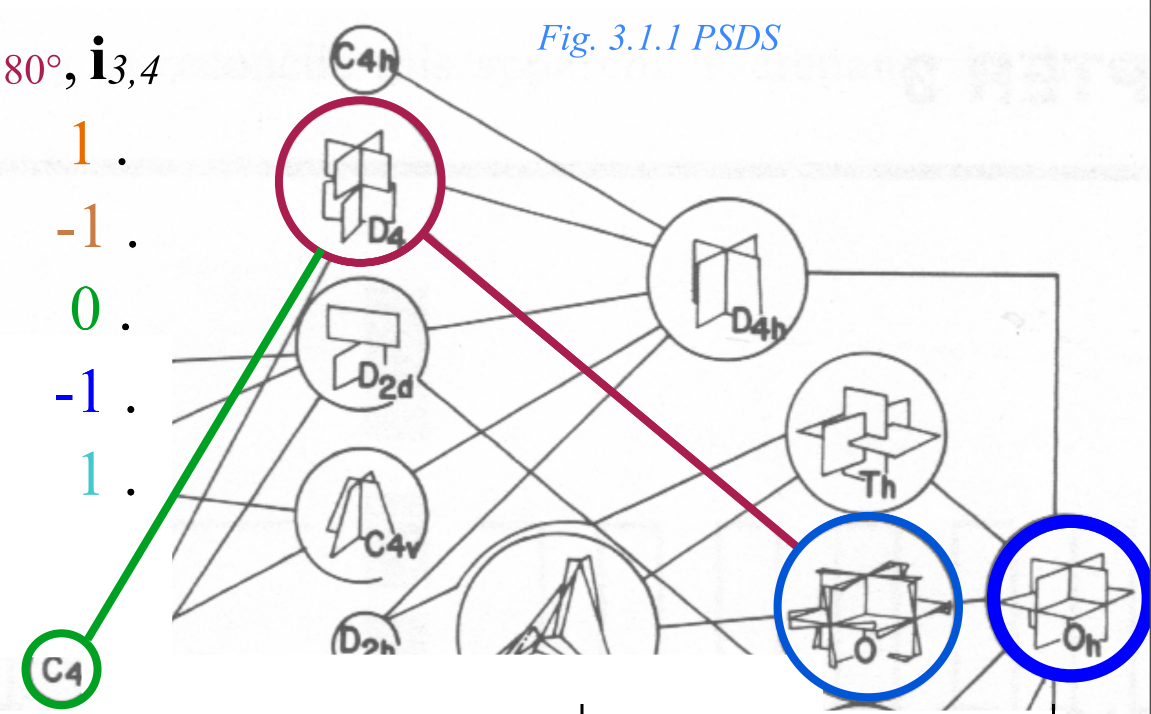


Fig. 3.1.1 PSDS

$\chi^{\mu}_{g}(D_4)$	$\mathbf{g=1}$	$\boldsymbol{\rho}_{z180°}$	$\mathbf{R}_{z\pm 90°}$	$\boldsymbol{\rho}_{x,y180°}$	$\mathbf{i}_{3,4}$
A_1	1	1	1	1	1
B_1	1	1	−1	1	−1
A_2	1	1	1	−1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

$D_4\downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4\oplus(3)_4$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$\chi^{\mu}_{g}(C_4)$	$\mathbf{g=1}$	$\mathbf{R}_{z+90°}$	$\mathbf{R}_{z+180°}$	$\mathbf{R}_{z-90°}$
$(0)_4$	1	1	1	1
$(1)_4$	1	i	−1	$-i$
$(2)_4$	1	−1	1	−1
$(3)_4$	1	$-i$	−1	i

$O\downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	→1	→1	→1

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	→1	·
E	·	→1	·	→1

Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4: $\mathbf{1}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z\pm 90°}, \boldsymbol{\rho}_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1, \quad 1, \quad 1, \quad 1, \quad 1\ .$

$A_2(O)\downarrow D_4 = 1, \quad 1, \quad -1, \quad 1, \quad -1\ .$

$E(O)\downarrow D_4 = 2, \quad 2, \quad 0, \quad 2, \quad 0\ .$

$T_2(O)\downarrow D_4 = 3, \quad -1, \quad 1, \quad -1, \quad -1\ .$

$T_2(O)\downarrow D_4 = 3, \quad -1, \quad -1, \quad -1, \quad 1\ .$

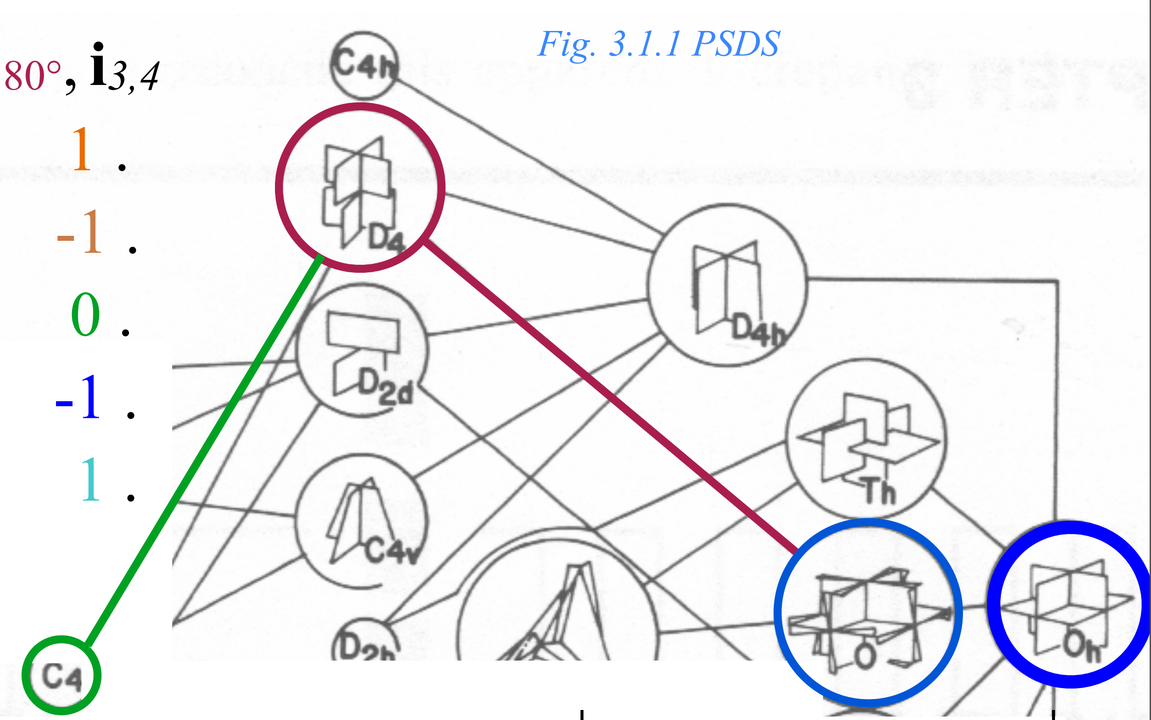
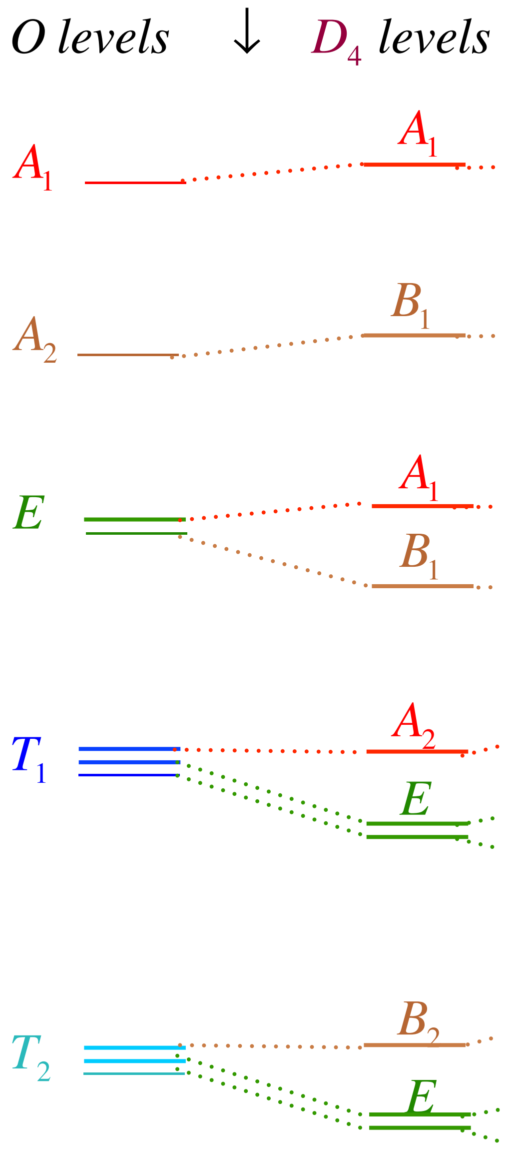


Fig. 3.1.1 PSDS

$O \supset D_4 \supset C_4$ subgroup and level-splitting/relabeling correlations



$D_4 \downarrow C_4$ subduction

C_4: $\mathbf{1}, \mathbf{R}_{z+90°}, \boldsymbol{\rho}_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1, \quad 1, \quad 1, \quad 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1, \quad -1, \quad 1, \quad -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1, \quad 1, \quad 1, \quad 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1, \quad -1, \quad 1, \quad -1. \quad =(2)_4$

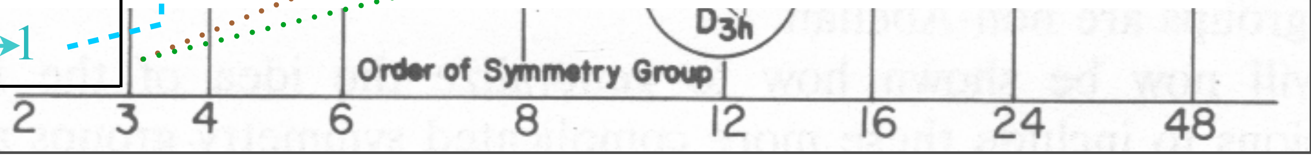
$E(D_4)\downarrow C_4 = 2, \quad 0, \quad -2, \quad 0\ . \quad =(1)_4 \oplus (3)_4$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$O \downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	→1	→1	→1

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4 = \bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	→1	·
E	·	→1	·	→1



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O\downarrow D_4$ subduction

D_4: $\mathbf{1}, \rho_{z180°}, \mathbf{R}_{z\pm 90°}, \rho_{x,y180°}, \mathbf{i}_{3,4}$

$A_1(O)\downarrow D_4 = 1,\ 1,\ 1,\ 1,\ 1\ .$

$A_2(O)\downarrow D_4 = 1,\ 1,\ -1,\ 1,\ -1\ .$

$E(O)\downarrow D_4 = 2,\ 2,\ 0,\ 2,\ 0\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ 1,\ -1,\ -1\ .$

$T_2(O)\downarrow D_4 = 3,\ -1,\ -1,\ -1,\ 1\ .$

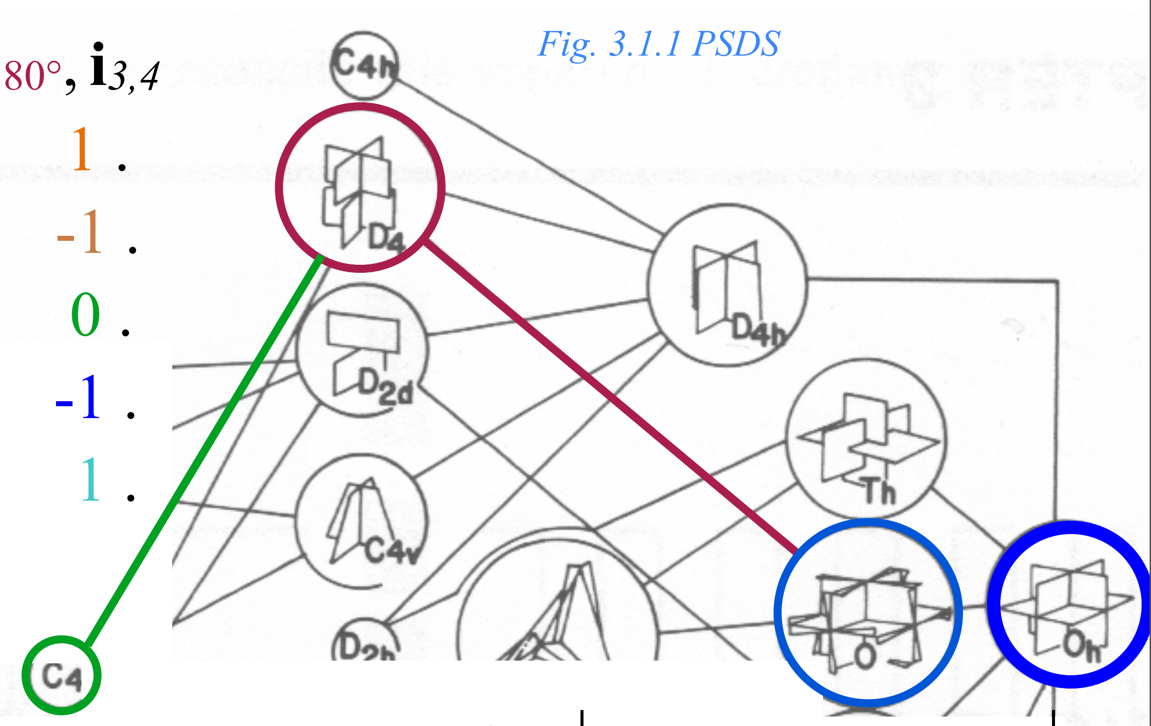
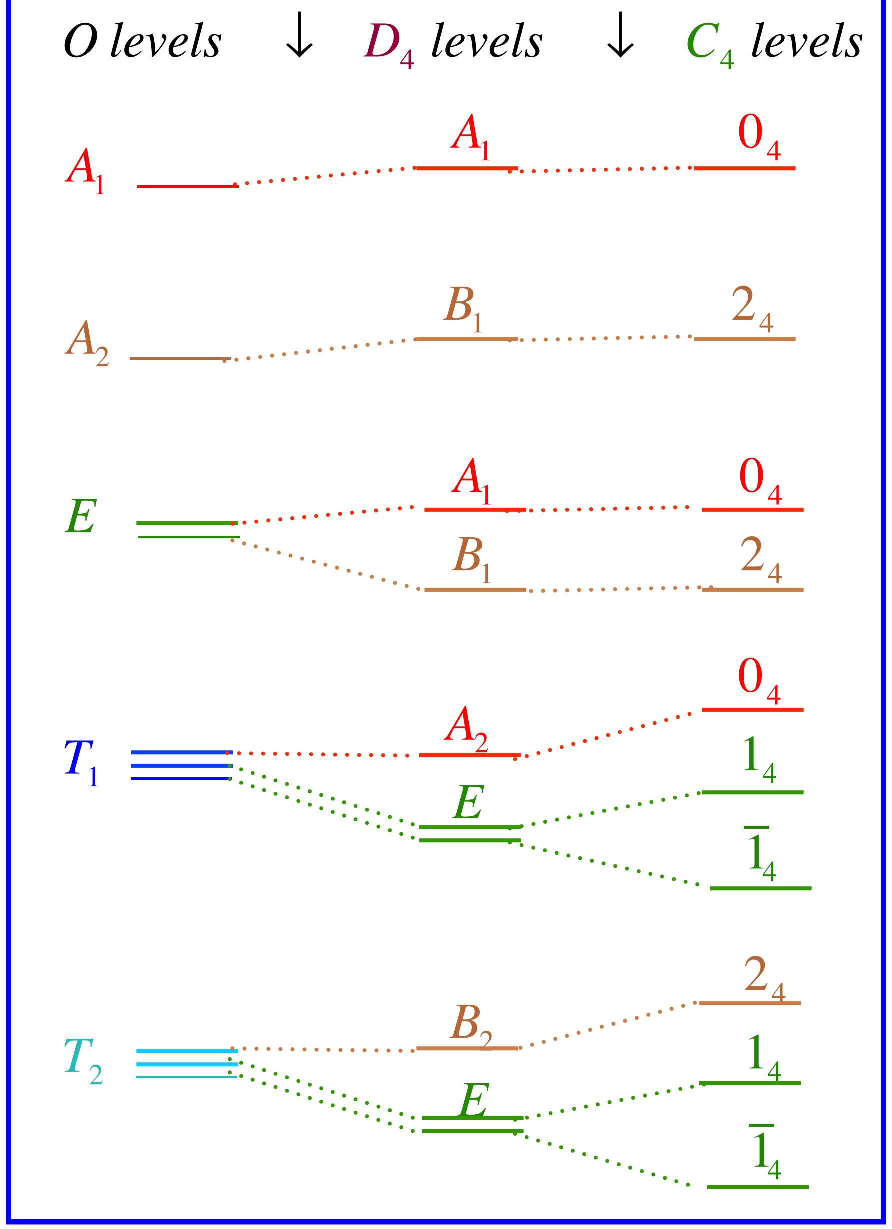


Fig. 3.1.1 PSDS

$O\supset D_4 \supset C_4$ subgroup and level-splitting/relabeling correlations



$D_4\downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90°}, \rho_{z180°}, \mathbf{R}_{z-90°}$

$A_1(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_1(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

$A_2(D_4)\downarrow C_4 = 1,\ 1,\ 1,\ 1. \quad =(0)_4$

$B_2(D_4)\downarrow C_4 = 1,\ -1,\ 1,\ -1. \quad =(2)_4$

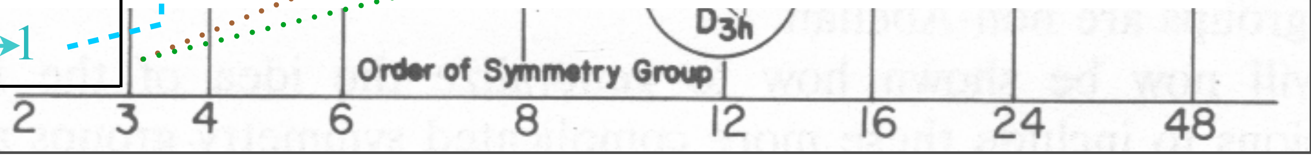
$E(D_4)\downarrow C_4 = 2,\ 0,\ -2,\ 0\ . \quad =(1)_4\oplus(3)_4$

$O\downarrow D_4$	A_1	B_1	A_2	B_2	E
A_1	1	·	·	·	·
A_2	·	1	·	·	·
E	1	1	·	·	·
T_1	·	·	1	·	1
T_2	·	·	·	1	1

$D_4\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
B_1	·	·	1	·
A_2	1	·	·	·
B_2	·	·	→1	·
E	·	→1	·	→1

$O\downarrow C_4$ subduction

$O\downarrow C_4$	0_4	1_4	2_4	$3_4=\bar{1}_4$
A_1	1	·	·	·
A_2	·	·	1	·
E	1	·	1	·
T_1	1	1	·	1
T_2	·	→1	→1	→1



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$
- *Octahedral-cubic O symmetry and group operations*
- *Tetrahedral symmetry becomes Icosahedral*
- *Octahedral groups $O_h \supset O \sim T_d \supset T$*
  - *Octahedral O and spin-$O \subset U(2)$*

Tetrahedral T class algebra
- *Tetrahedral T class minimal equations*
- *Tetrahedral T class projectors and characters*

Octahedral O class algebra
- *Octahedral O class minimal equations*
- *Octahedral O class projectors and characters*

Octahedral $O_h \supset O$:Inversion (g&u) parity
- *Octahedral $O_h \supset O \supset C_{\mathbf{I}}$ subgroup correlations*
- *$O_h \supset O \supset D_4$ subgroup correlations*
- *$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations*
- ➡ *Preview of applications to high resolution spectroscopy* ⬅

Summary of some Octahedral symmetry results:

$\ell^{A_1} = 1$

$\ell^{A_2} = 1$

$\ell^{E} = 2$

$\ell^{T_1} = 3$

$\ell^{T_2} = 3$

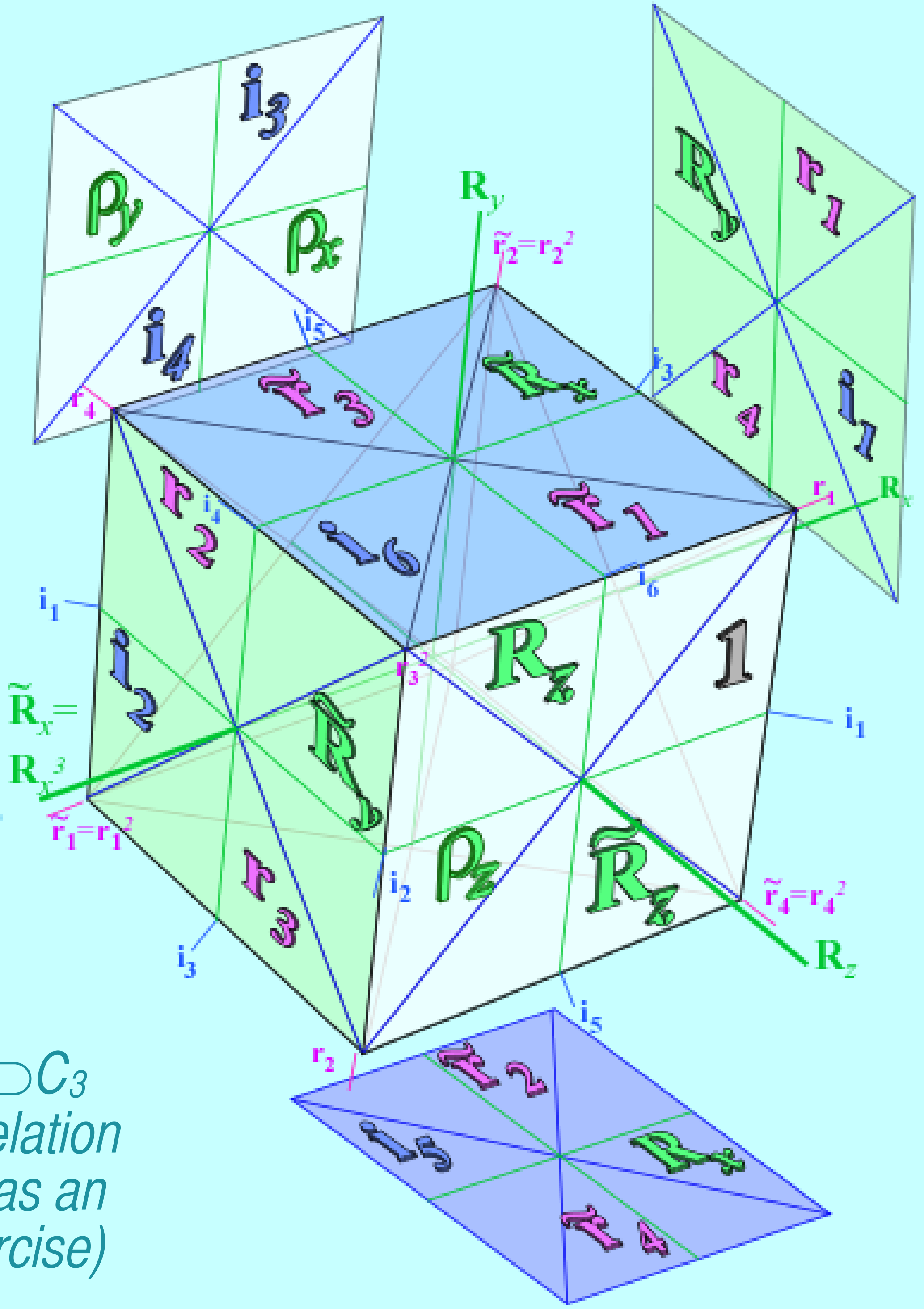
Cubic-Octahedral Group O

Centrum: $\kappa(O)=\Sigma_{(\alpha)} (\ell^{\alpha})^0 = 1^0+1^0+2^0+3^0+3^0=5$

Rank: $\rho(O)=\Sigma_{(\alpha)} (\ell^{\alpha})^1 = 1^1+1^1+2^1+3^1+3^1=10$

Order: $^{o}(O)=\Sigma_{(\alpha)} (\ell^{\alpha})^0 = 1^2+1^2+2^2+3^2+3^2=24$

	O group $\chi^{\alpha}_{\kappa_g}$	$g=1$	r_{1-4} $\tilde{r}_{1-4}$	ρ_{xyz}	R_{xyz} $\tilde{R}_{xyz}$	i_{1-6}
s-orbital r^2	$\alpha = A_1$	1	1	1	1	1
d-orbitals	A_2	1	1	1	-1	-1
$\{x^2+y^2-2z^2, x^2-y^2\}$	E	2	-1	2	0	0
p-orbitals$\{x, y, z\}$	T_1	3	0	-1	1	-1
$\{xz,yz,xy\}$ d-orbitals	T_2	3	0	-1	-1	1



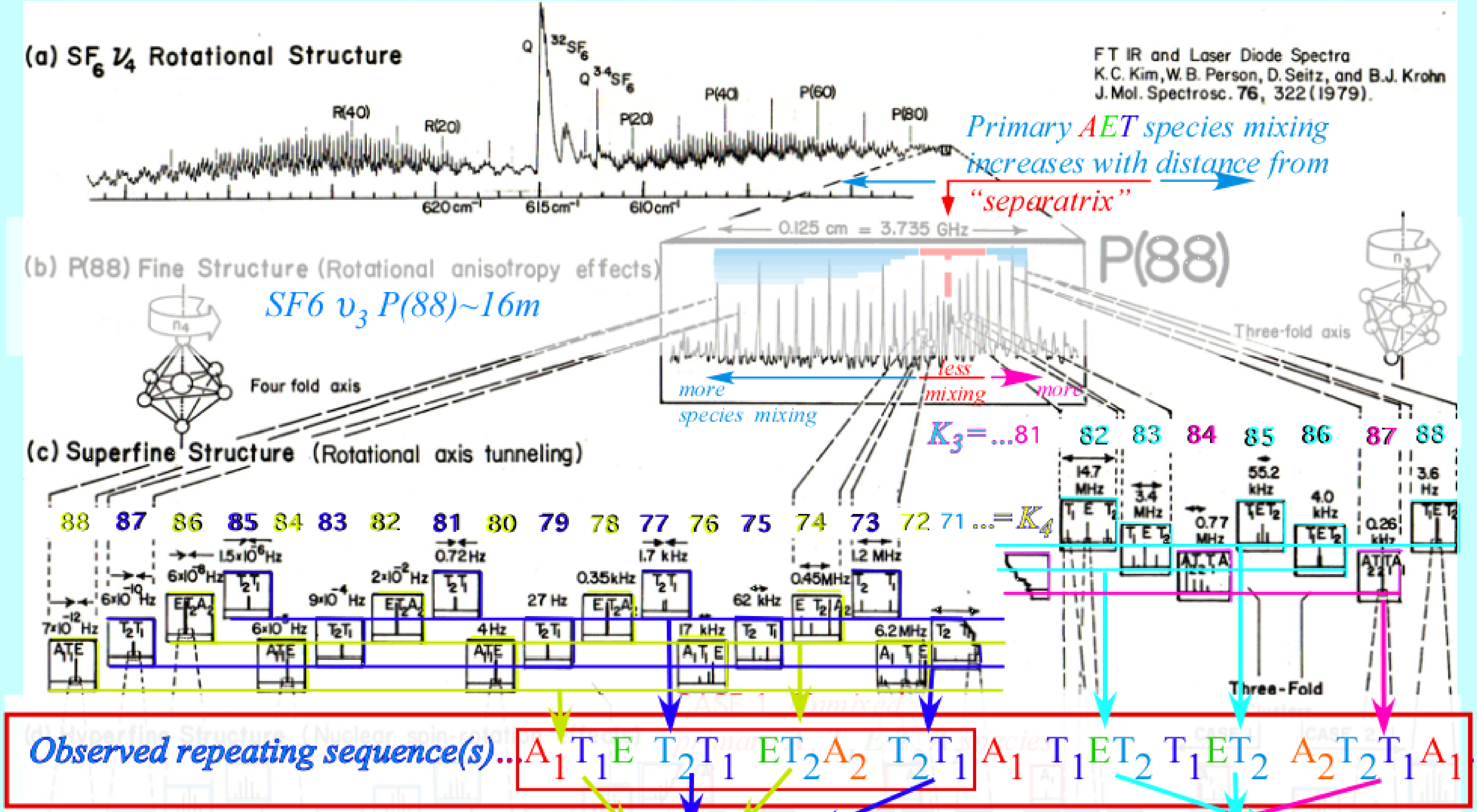
O⊃C₄

	$(0)_4$	$(1)_4$	$(2)_4$	$(3)_4=(-1)_4$
A_1	1	•	•	•
A_2	•	•	1	•
E	1	•	1	•
T_1	1	1	•	1
T_2	•	1	1	1

O⊃C₃

	$(0)_3$	$(1)_3$	$(2)_3=(-1)_3$
A_1	1	•	•
A_2	1	•	•
E	•	1	1
T_1	1	1	1
T_2	1	1	1

(O⊃C₃ correlation left as an exercise)



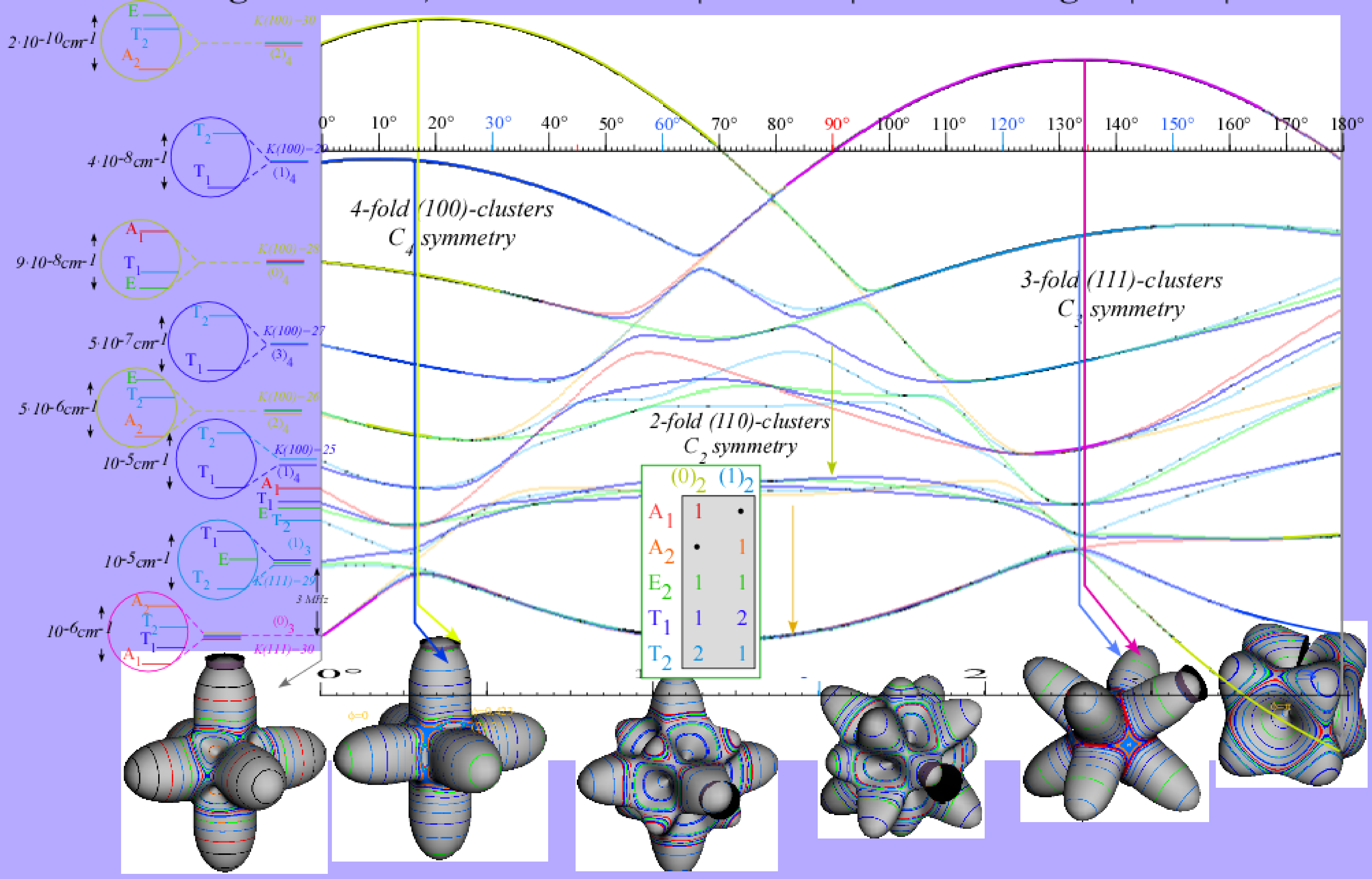
Observed repeating sequence(s)...$A_1T_1E\ T_2T_1\ ET_2A_2\ T_2T_1\ A_1\ T_1ET_2\ T_1ET_2\ A_2T_2T_1A_1$...

$$\begin{array}{c|cccc} O\supset C_4 & (0)_4 & (1)_4 & (2)_4 & (3)_4=(-1)_4 \\ \hline A_1 & 1 & \cdot & \cdot & \cdot \\ A_2 & \cdot & \cdot & 1 & \cdot \\ E & 1 & \cdot & 1 & \cdot \\ T_1 & 1 & 1 & \cdot & 1 \\ T_2 & \cdot & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccc} O\supset C_3 & (0)_3 & (1)_3 & (2)_3=(-1)_3 \\ \hline A_1 & 1 & \cdot & \cdot \\ A_2 & 1 & \cdot & \cdot \\ E & \cdot & 1 & 1 \\ T_1 & 1 & 1 & 1 \\ T_2 & 1 & 1 & 1 \end{array}$$

Local correlations explain clustering...
... but what about spacing and ordering?...

...and physical consequences?

Eigenvalues of $\mathbf{H}=BJ^2+\cos\phi\,\mathbf{T}^{[4]}+\sin\phi\,\mathbf{T}^{[6]}$ *vs. mix angle* ϕ*:* $0<\phi<\pi$



	$(0)_2$	$(1)_2$
A_1	1	•
A_2	•	1
E_2	1	1
T_1	1	2
T_2	2	1