

Group Theory in Quantum Mechanics

Lecture 19 (3.30.17)

Octahedral-tetrahedral $O \sim T_d$ symmetries

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)

(PSDS - Ch. 4)

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

Octahedral $O_h \supset O$: Inversion (g&u) parity

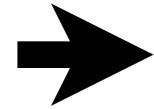
Octahedral $O_h \supset O \supset C_{\infty}$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

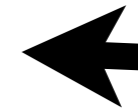


Octahedral-cubic O symmetry and group operations

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Octahedral $O_h \supset O$ subgroup correlations

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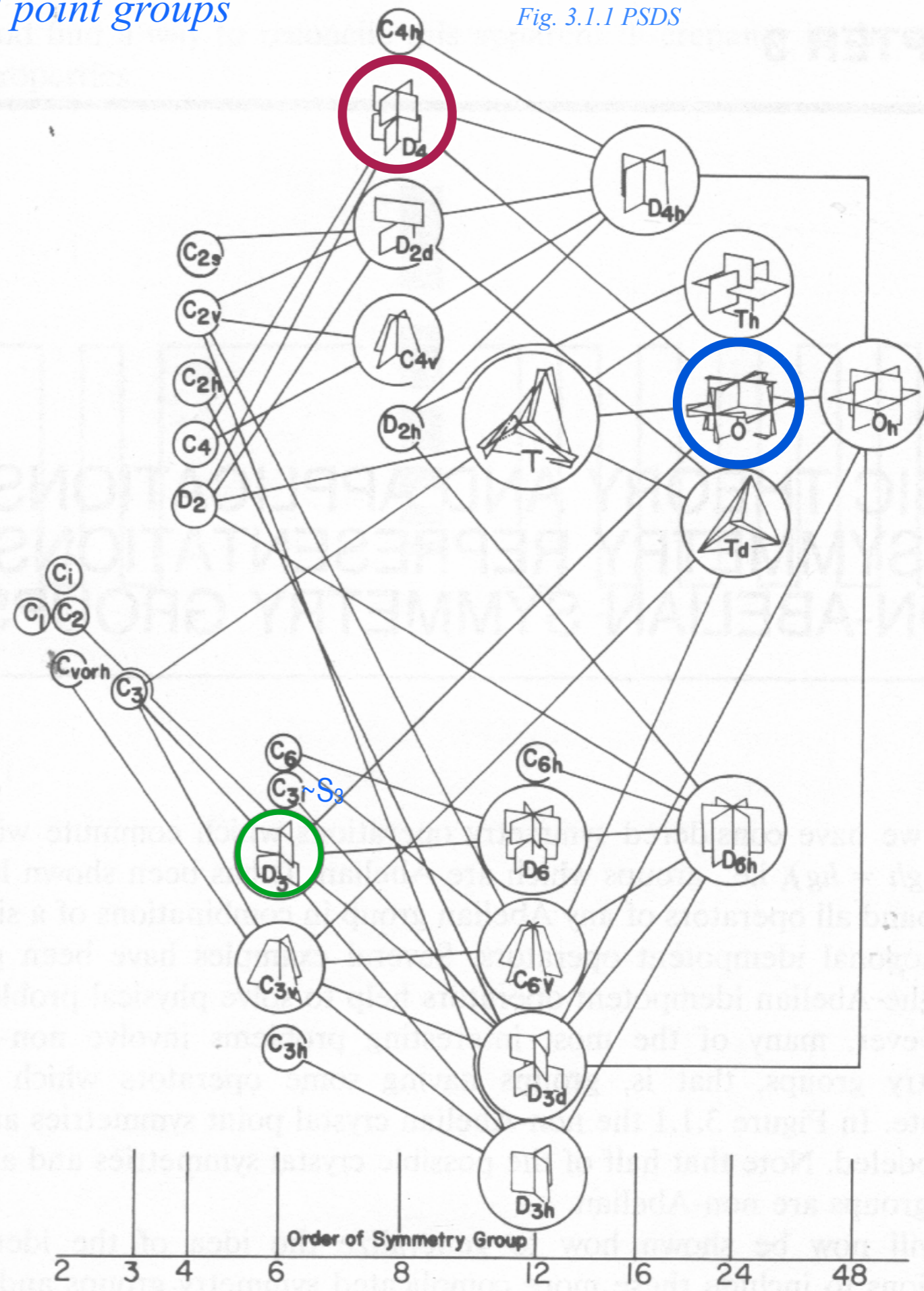
$O_h \supset O \supset D_4$ subgroup correlations

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Preview of applications to high resolution spectroscopy

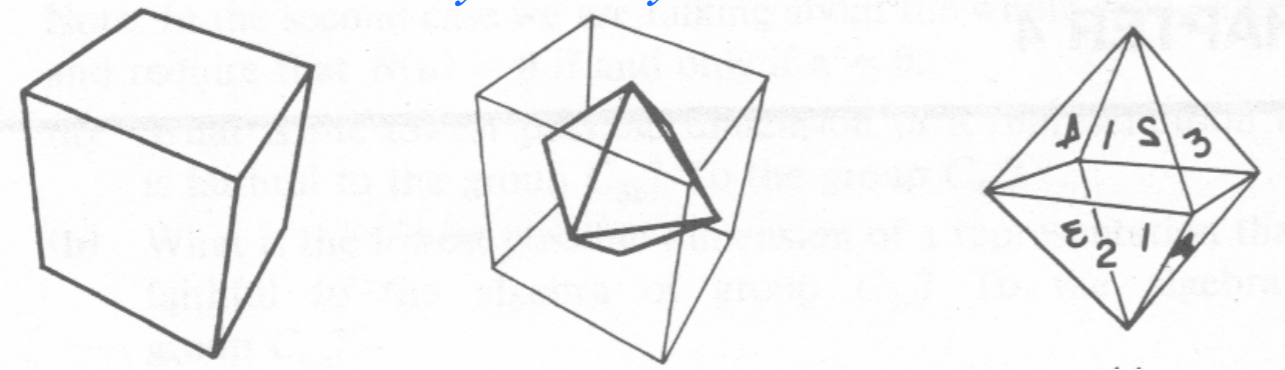
Three groups: O , D_4 , and D_3 let you “do”
 Most of the other 32 crystal point groups

Fig. 3.1.1 PSDS



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

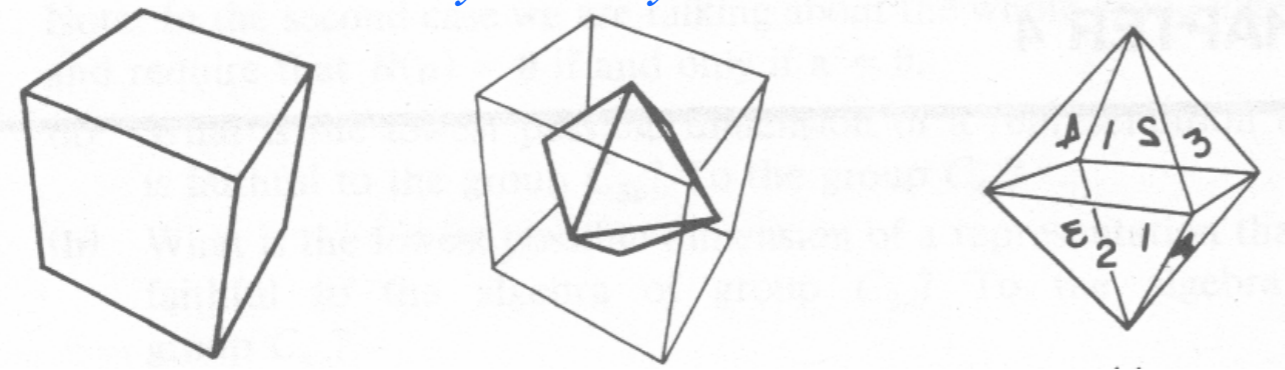
Octahedral-cubic O symmetry



*Order $^{\circ}O = 6$ hexahedron squares $\cdot 4$ pts = 24
= 8 octahedron triangles $\cdot 3$ pts = 24
= 12 lines $\cdot 2$ pts = 24 positions*

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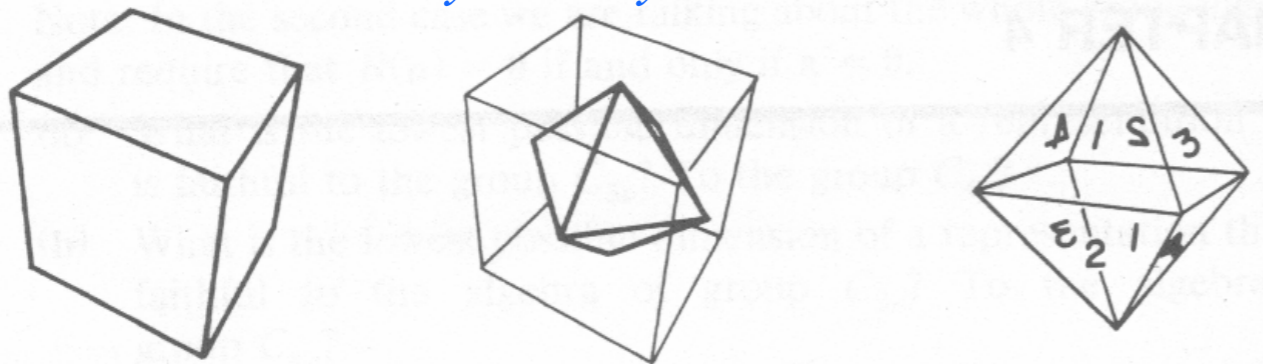
Octahedral group O operations

*Class of 1: **1***



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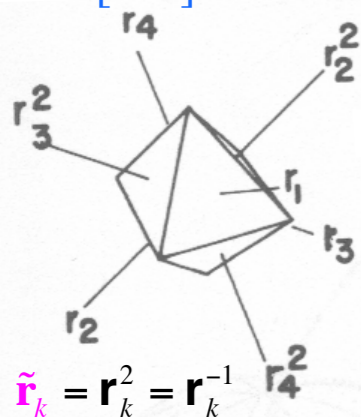
Octahedral group O operations

Class of 1: $\mathbf{1}$

$\mathbf{r}_k = \mathbf{r}_k$

Class of 8:

*120° rotations
 on [111] axes*

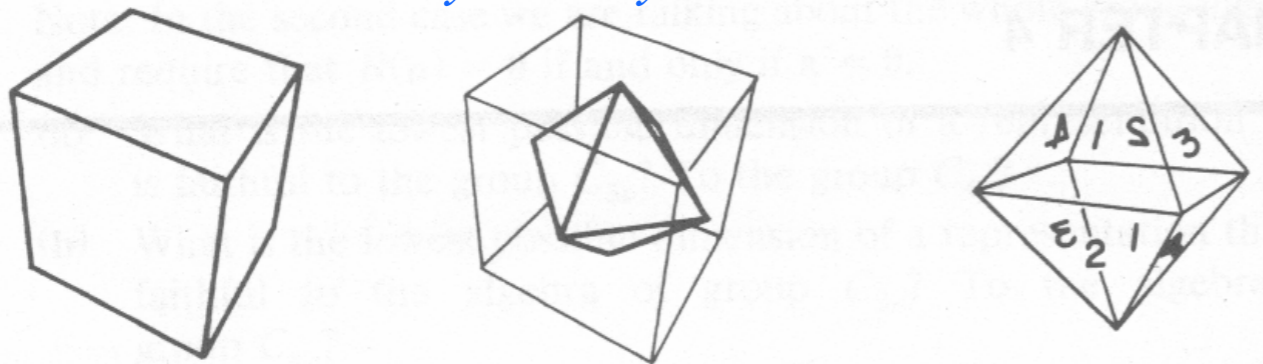


$\tilde{\mathbf{r}}_k = \mathbf{r}_k^2 = \mathbf{r}_k^{-1}$



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Octahedral-cubic O symmetry



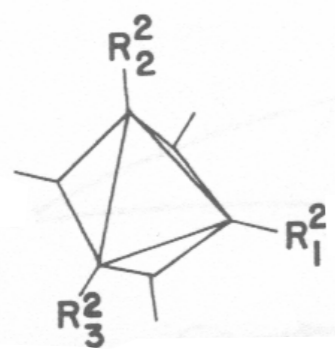
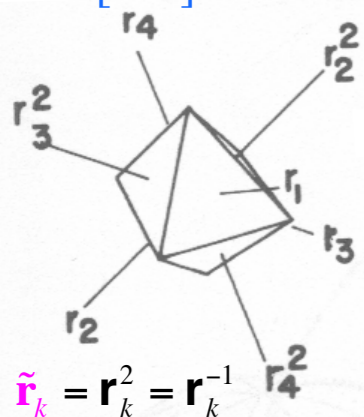
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Octahedral group O operations

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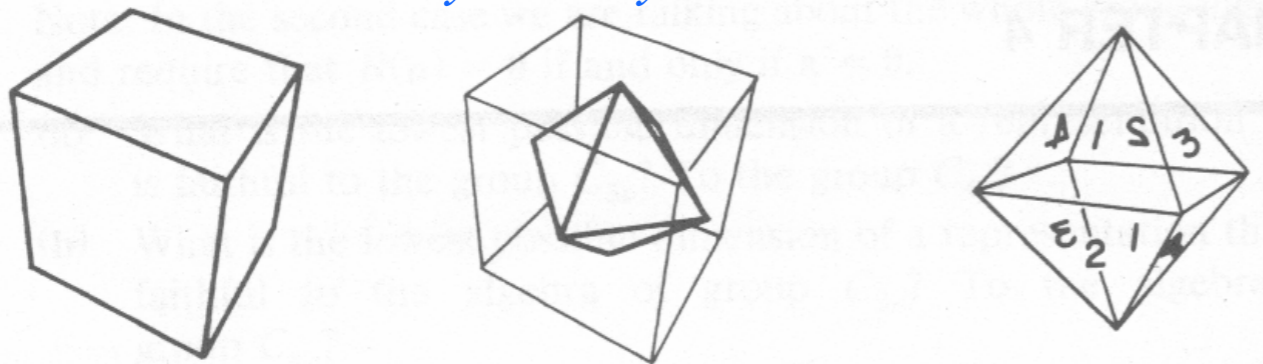
Class of 3:
 180° rotations
 on $[100]$ axes

$\rho_{x,y,z} = R_{1,2,3}^2$



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Octahedral-cubic O symmetry



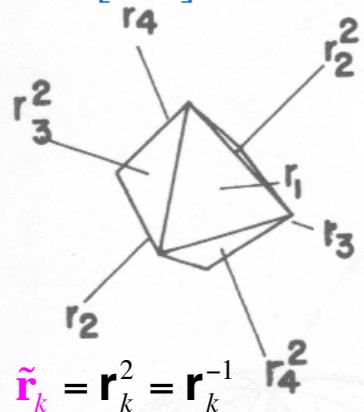
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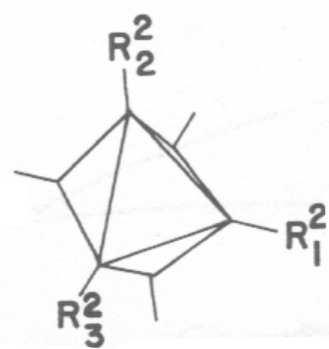
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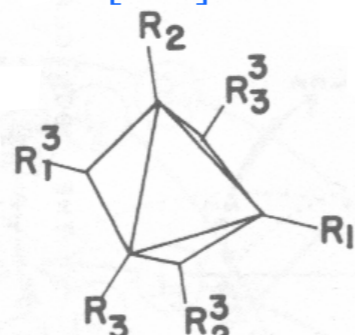


Class of 3:
 180° rotations
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$$\rho_{x,y,z} = \mathbf{R}_{1,2,3}^2$$

$$\mathbf{R}_{x,y,z} = \mathbf{R}_{1,2,3}$$

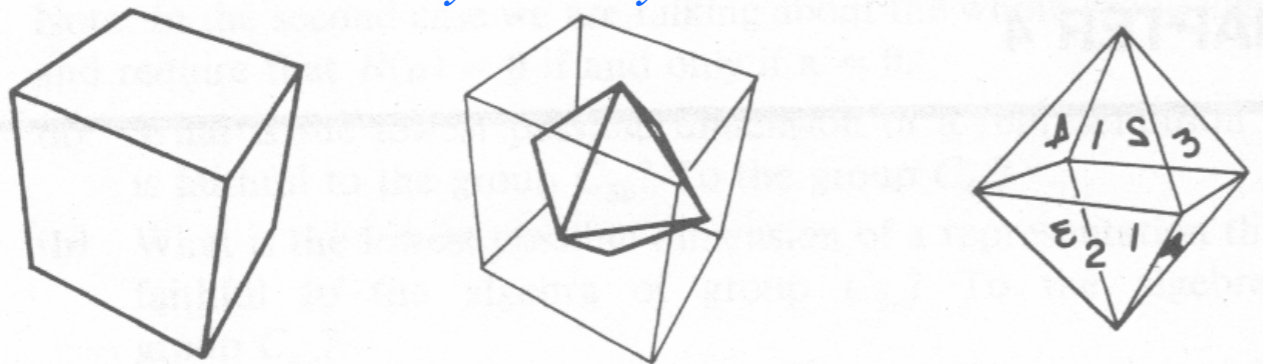
Class of 6:
 $\pm 90^\circ$ rotations
 on $[100]$ axes



$$\tilde{\mathbf{R}}_{x,y,z} = \mathbf{R}_{1,2,3}^3 = \mathbf{R}_{1,2,3}^{-1}$$

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Octahedral-cubic O symmetry



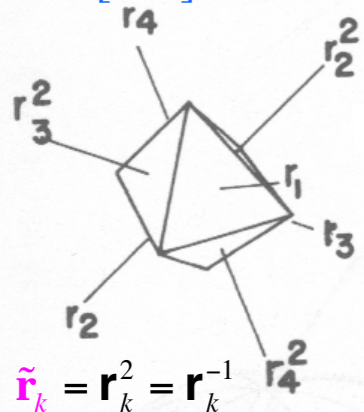
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Octahedral group O operations

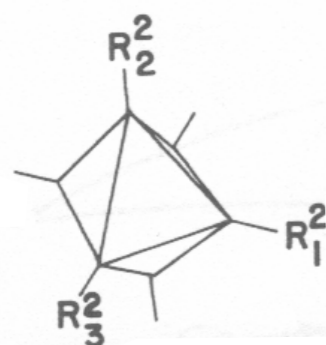
Class of 1: $\mathbf{1}$

$$\mathbf{r}_k = \mathbf{r}_k$$

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$$\tilde{\mathbf{r}}_k = \mathbf{r}_k^2 = \mathbf{r}_k^{-1}$$

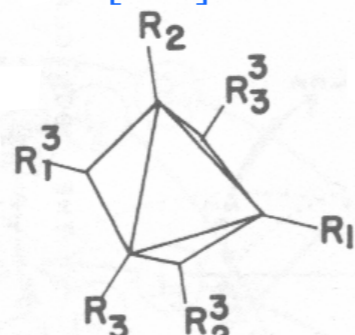


Class of 3:
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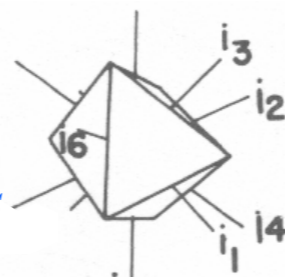
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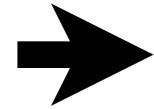
$$\tilde{\mathbf{R}}_{x,y,z} = \mathbf{R}_{1,2,3}^3 = \mathbf{R}_{1,2,3}^{-1}$$



Class of 6:
 180° rotations
 on $[110]$ diagonals

$$\mathbf{i}_k = \mathbf{i}_k$$

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

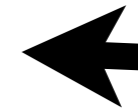


Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$



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Tetrahedral T class minimal equations

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Octahedral $O_h \supset O$ subgroup correlations

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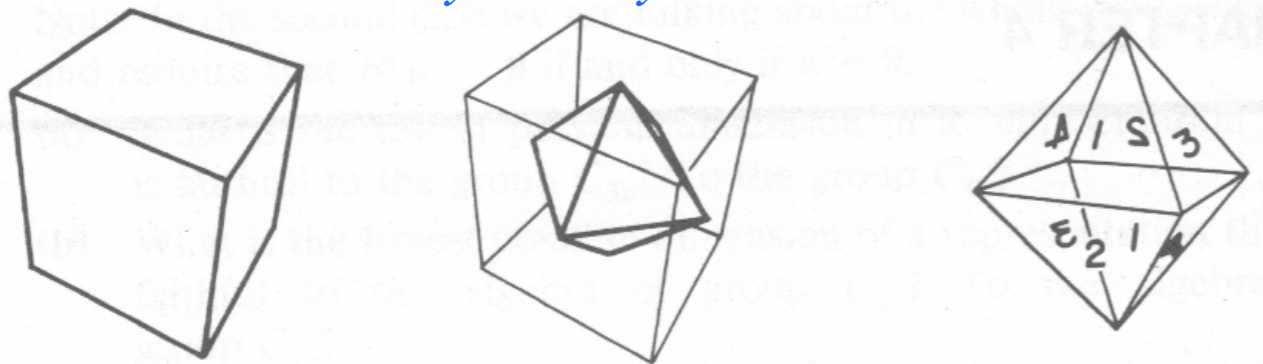
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Preview of applications to high resolution spectroscopy

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Octahedral-cubic O symmetry



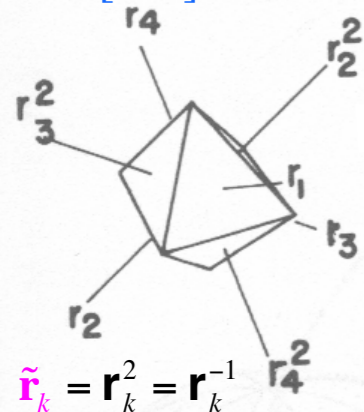
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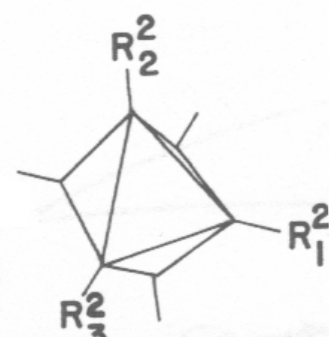
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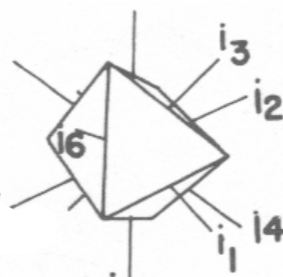


Class of 3:
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$$\rho_{x,y,z} = \mathbf{R}_{1,2,3}^2$$

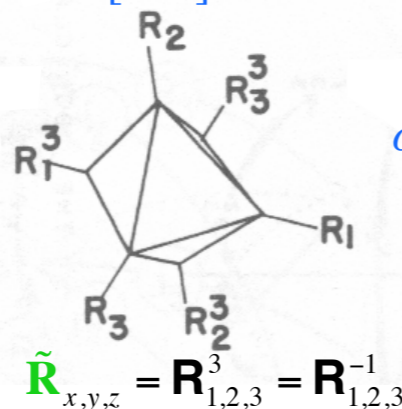
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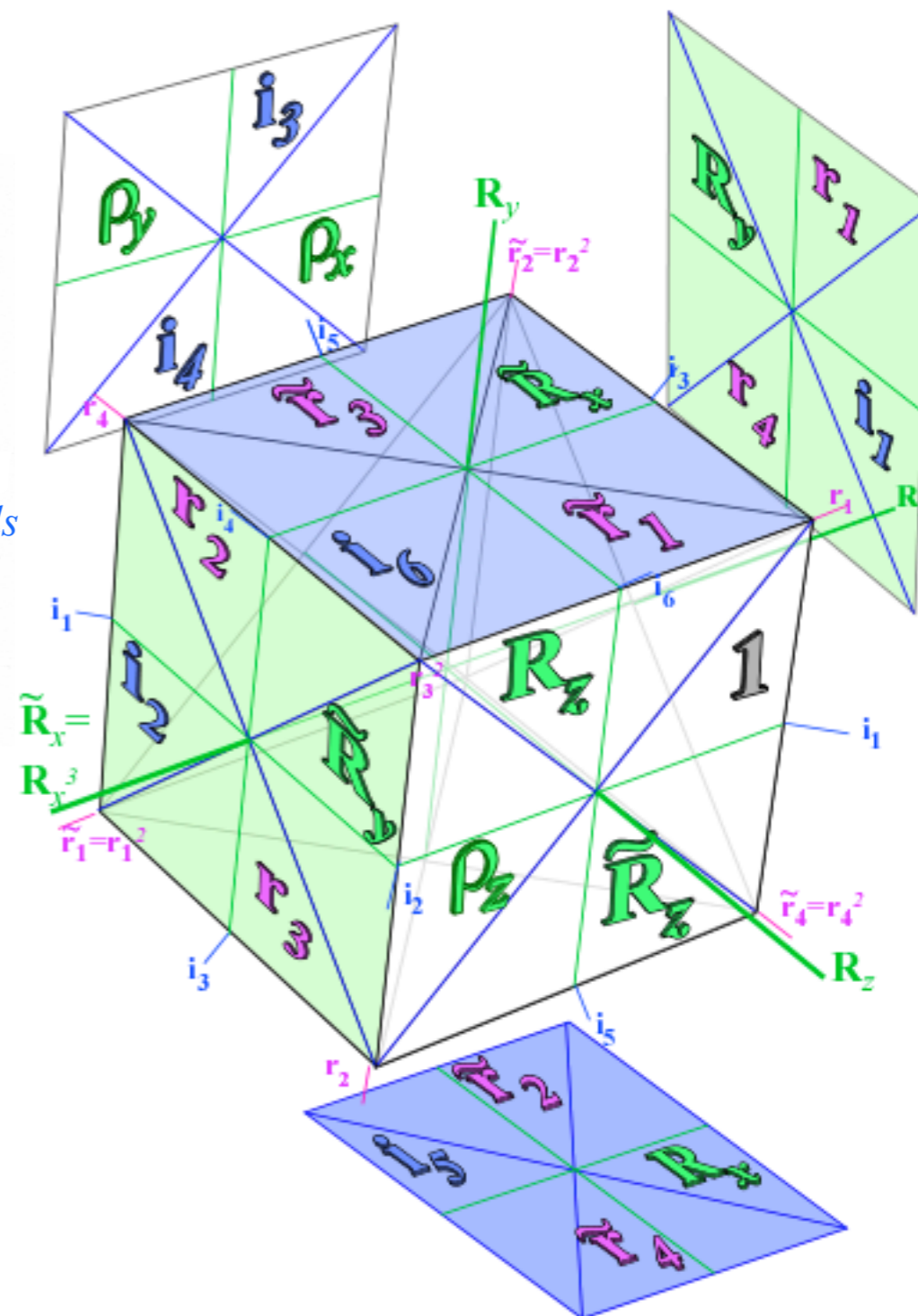


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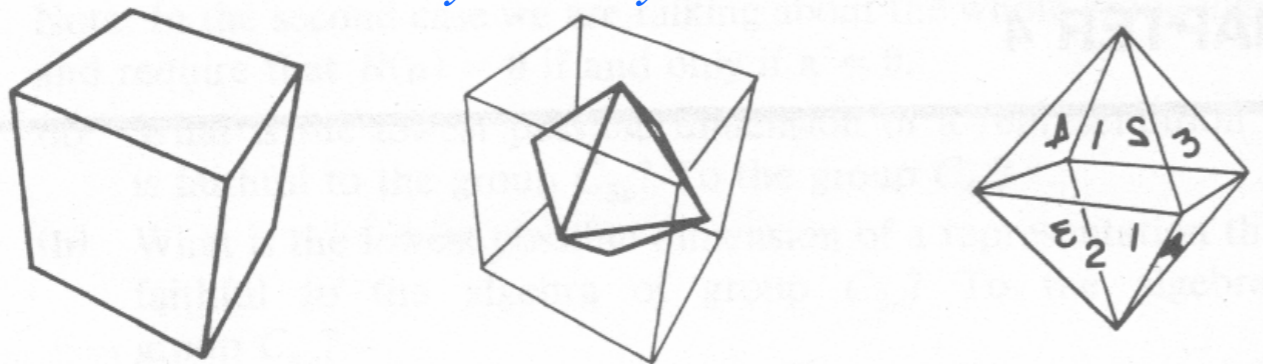


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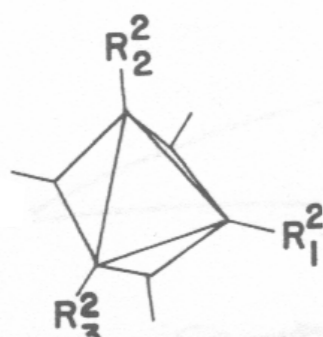
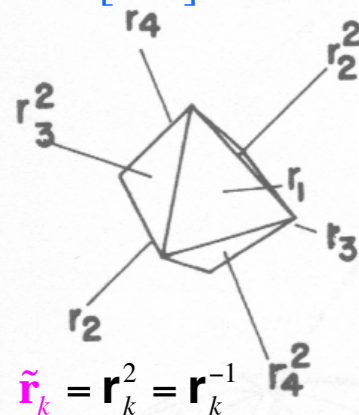
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Octahedral group O operations

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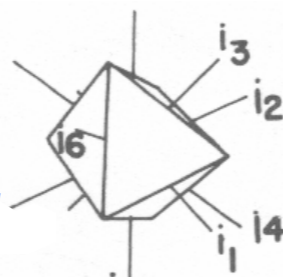
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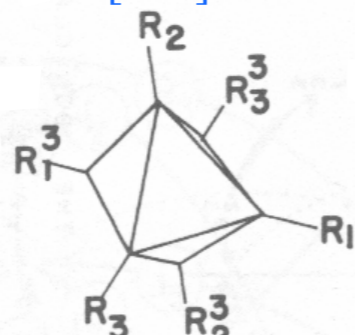


Class of 6:
 180° rotations
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$$\mathbf{i}_k = \mathbf{i}_k$$

Class of 3:
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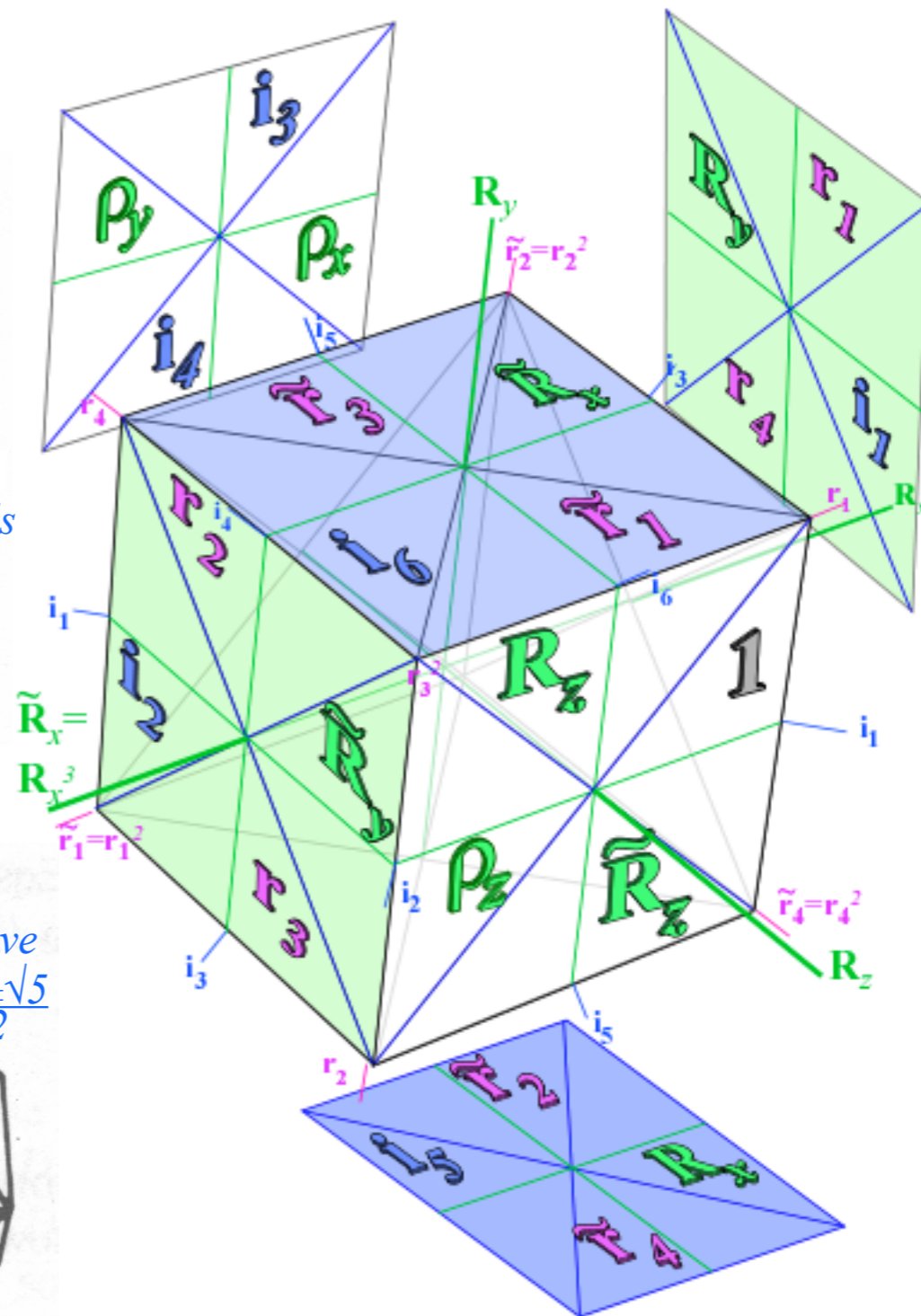
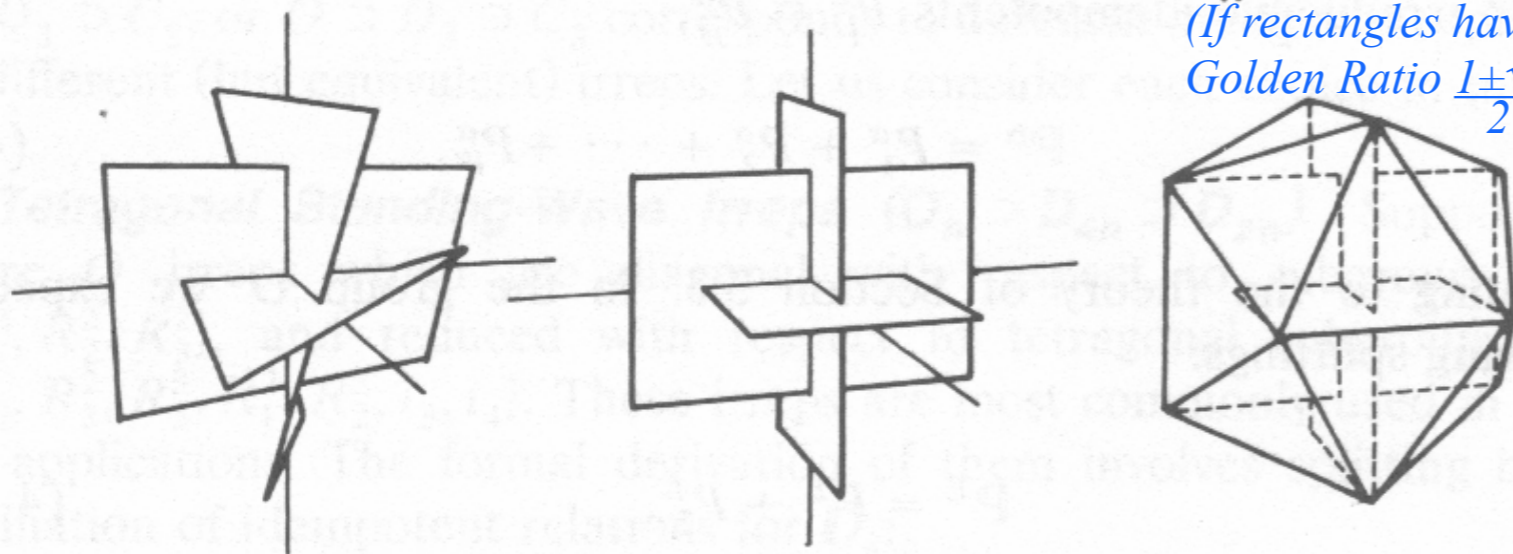
Tetrahedral symmetry becomes Icosahedral

T symmetry

T_h symmetry

I_h symmetry

(If rectangles have
 Golden Ratio $\frac{1 \pm \sqrt{5}}{2}$)



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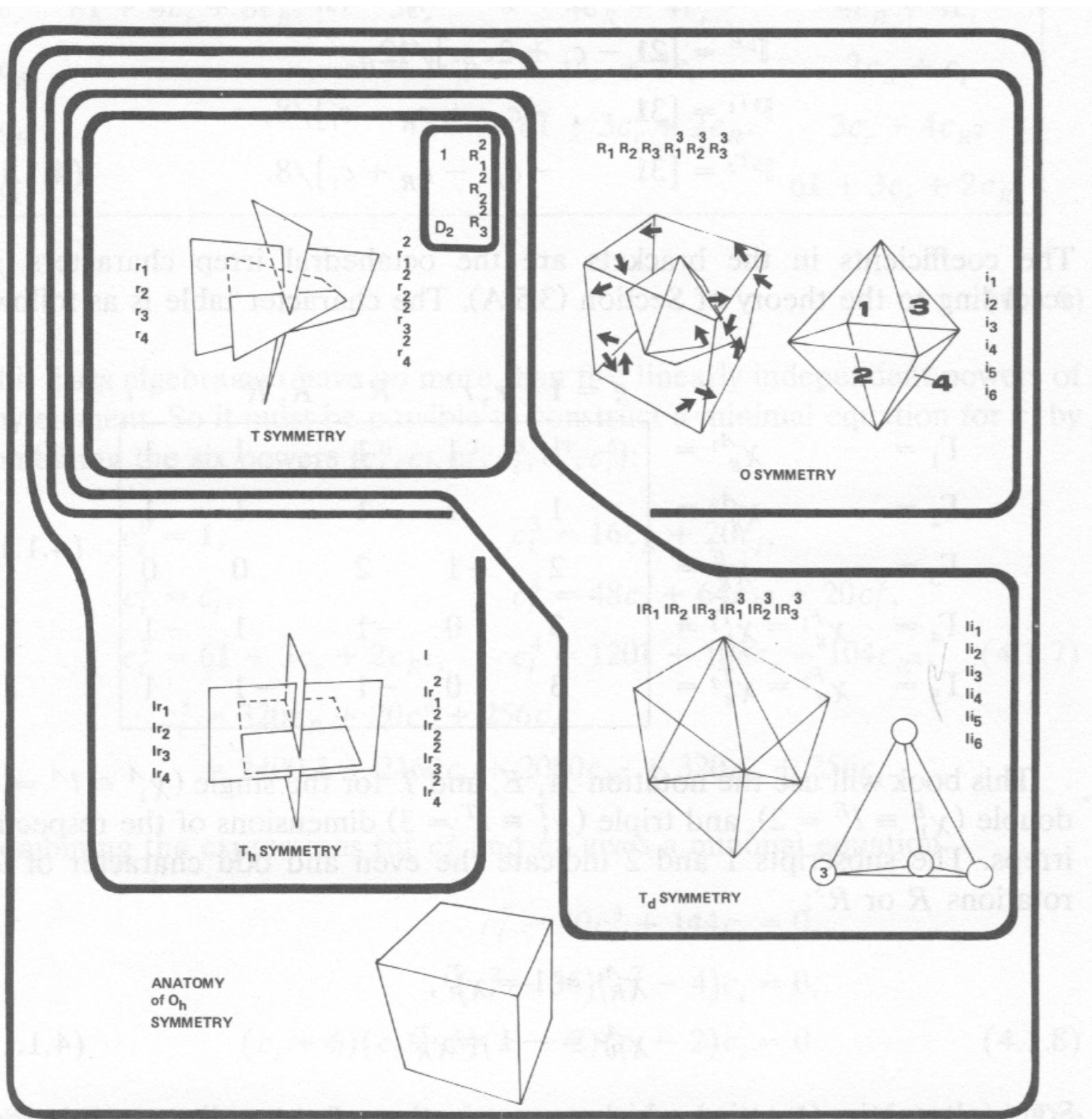


Figure 4.1.5 The full octahedral group (O_h) and four non-Abelian subgroups T , T_h , T_d , and O . The Abelian D_2 subgroup of T is indicated also.

Introduction to octahedral tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral groups $O_h \supset O \sim T_d \supset T$

Fig. 3.1.1 PSDS

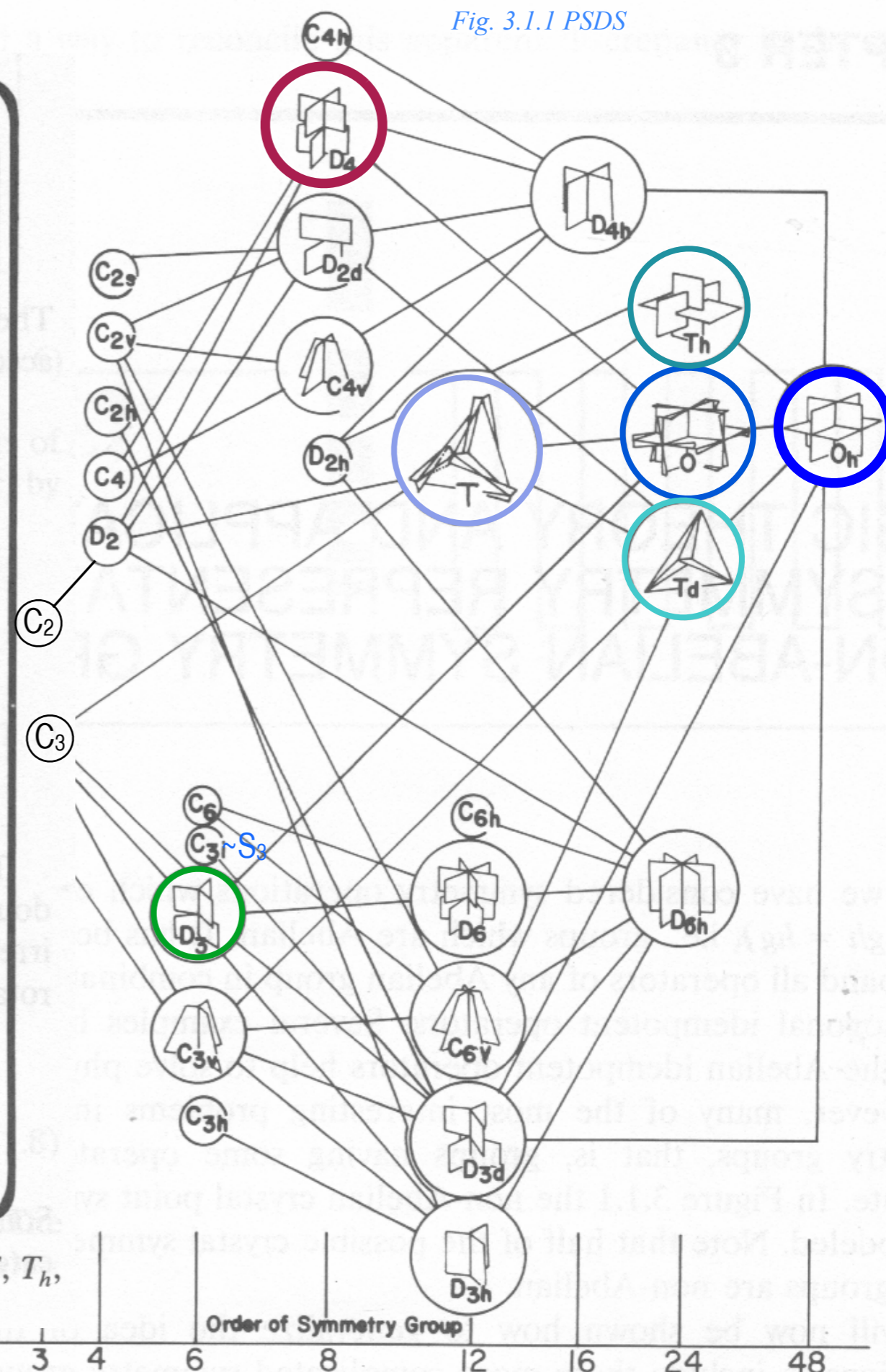
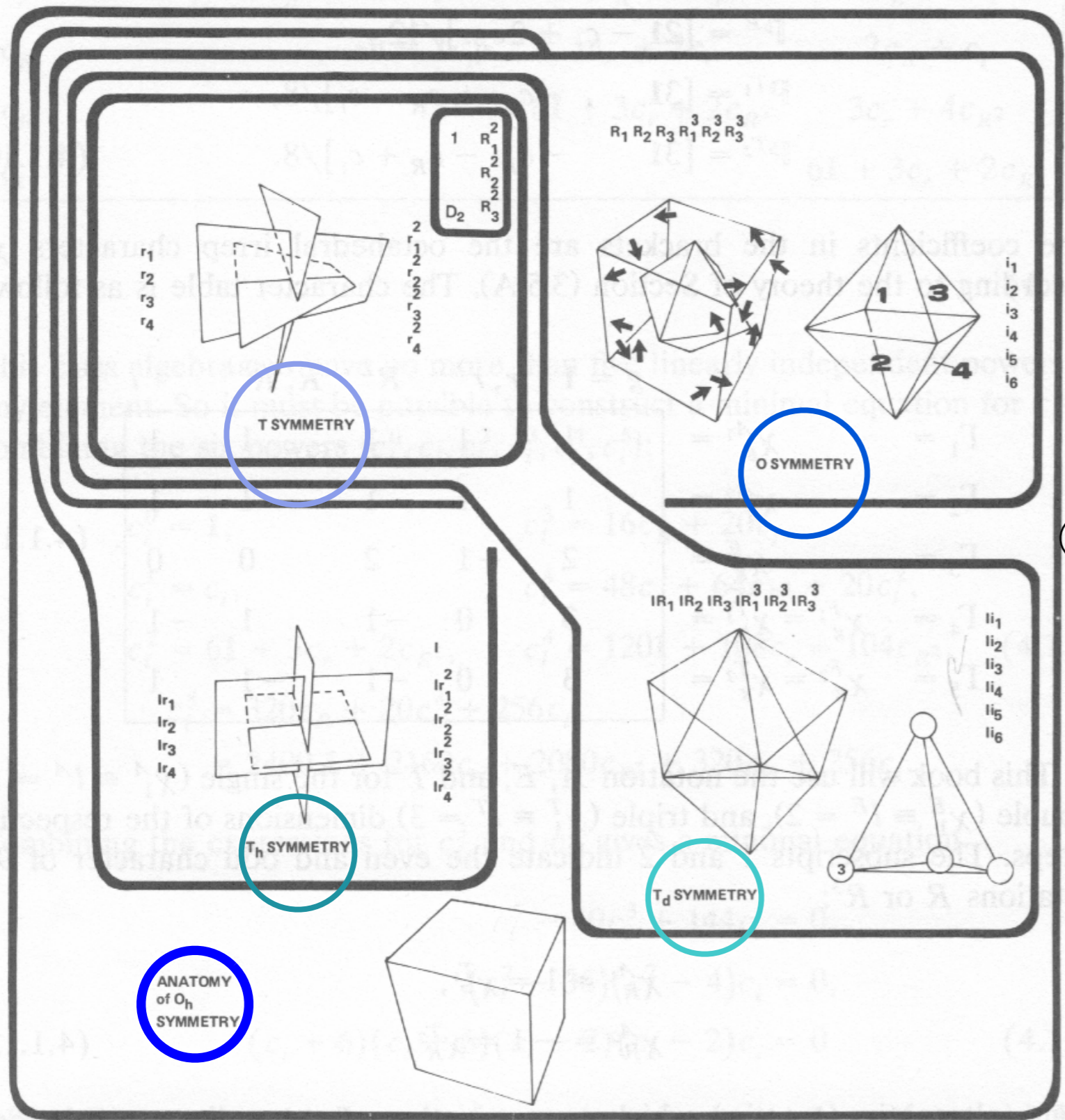


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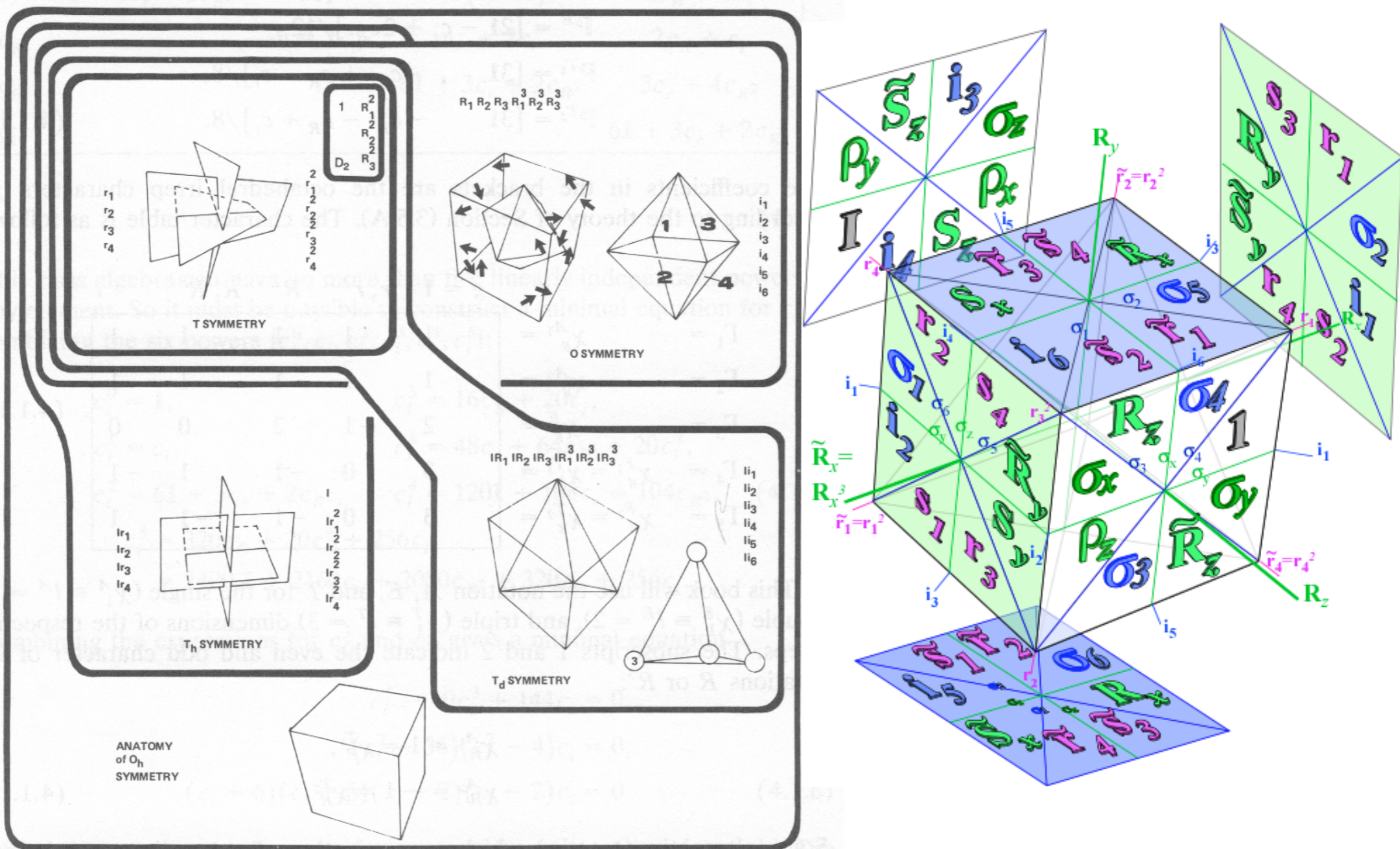


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Octahedral groups $O_h \supset O \sim T_d$ and $O_h \supset T_h \supset T$

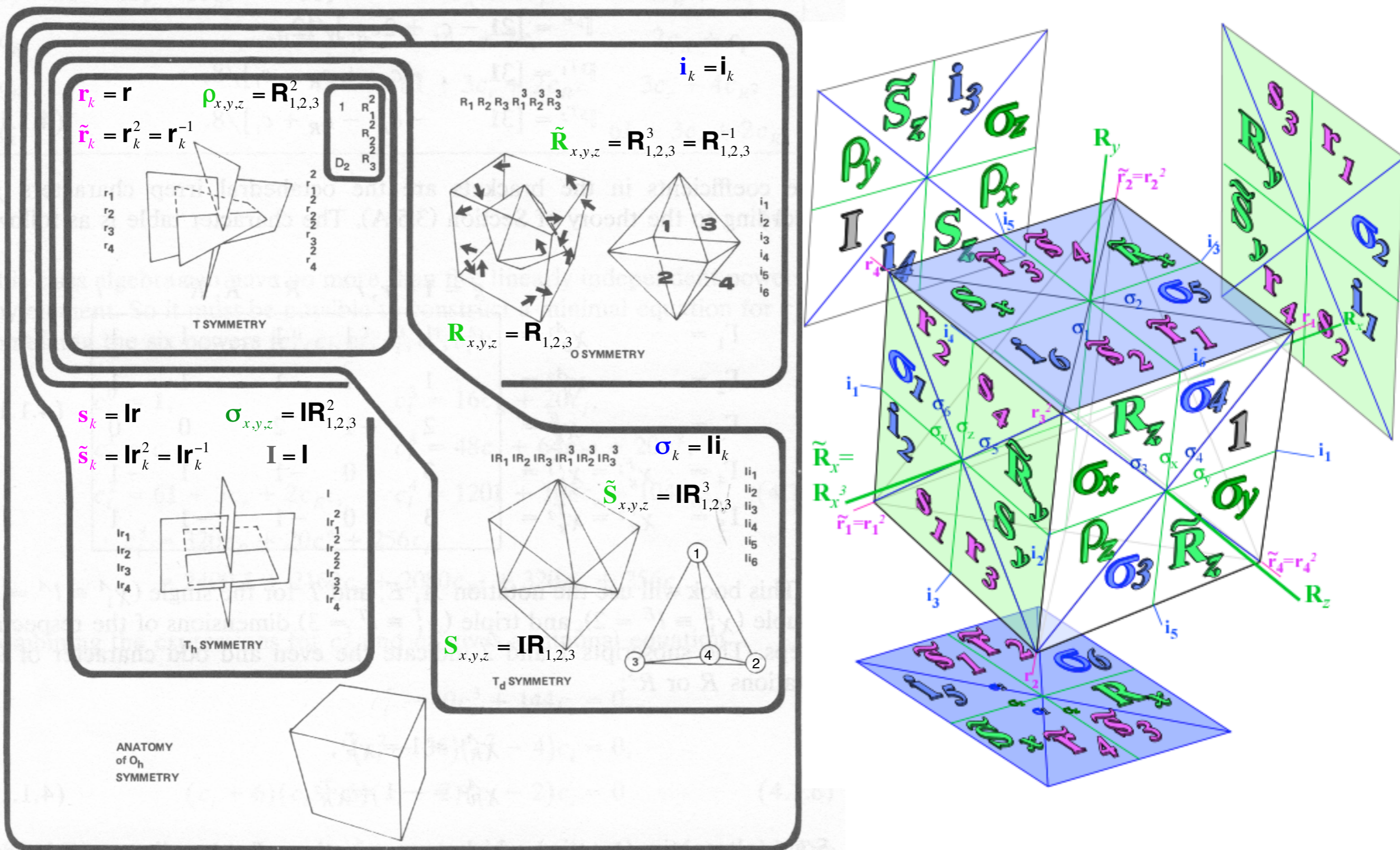


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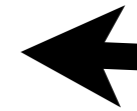
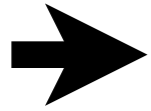
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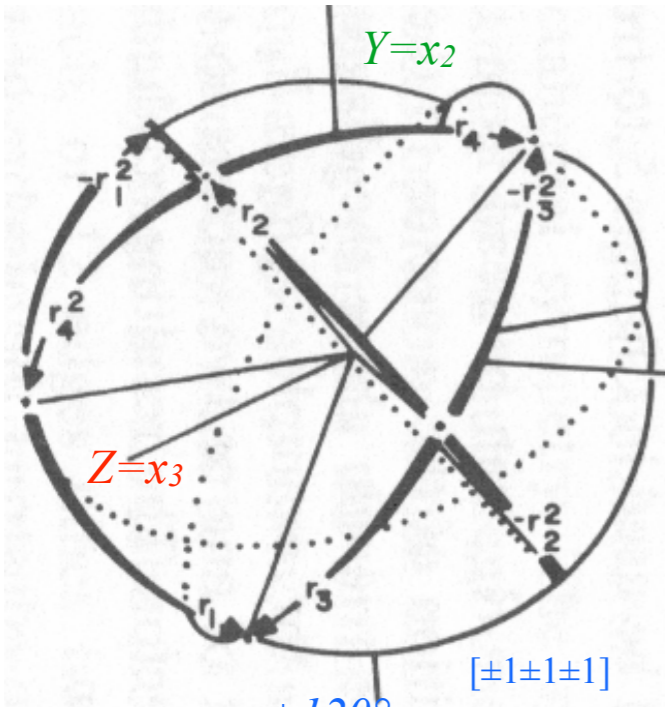
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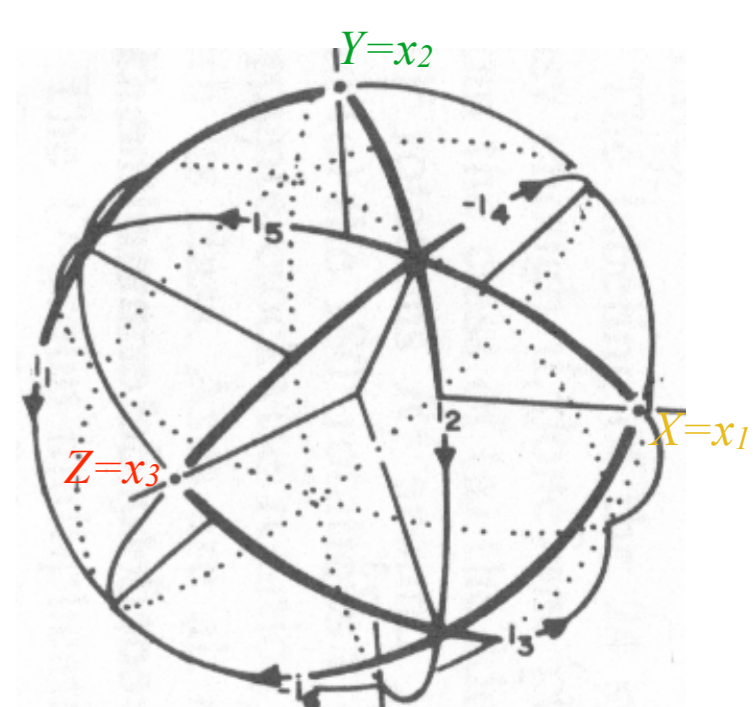
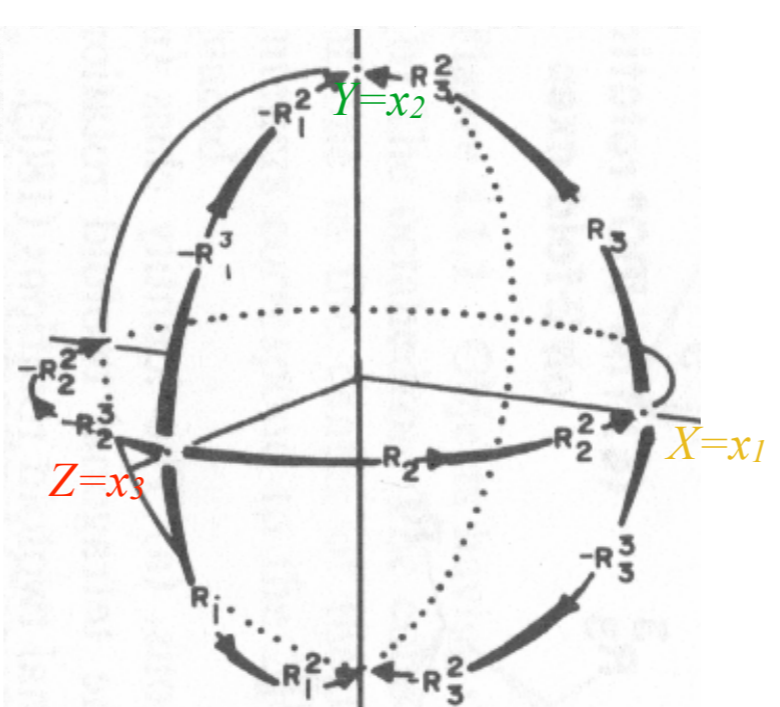
$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy



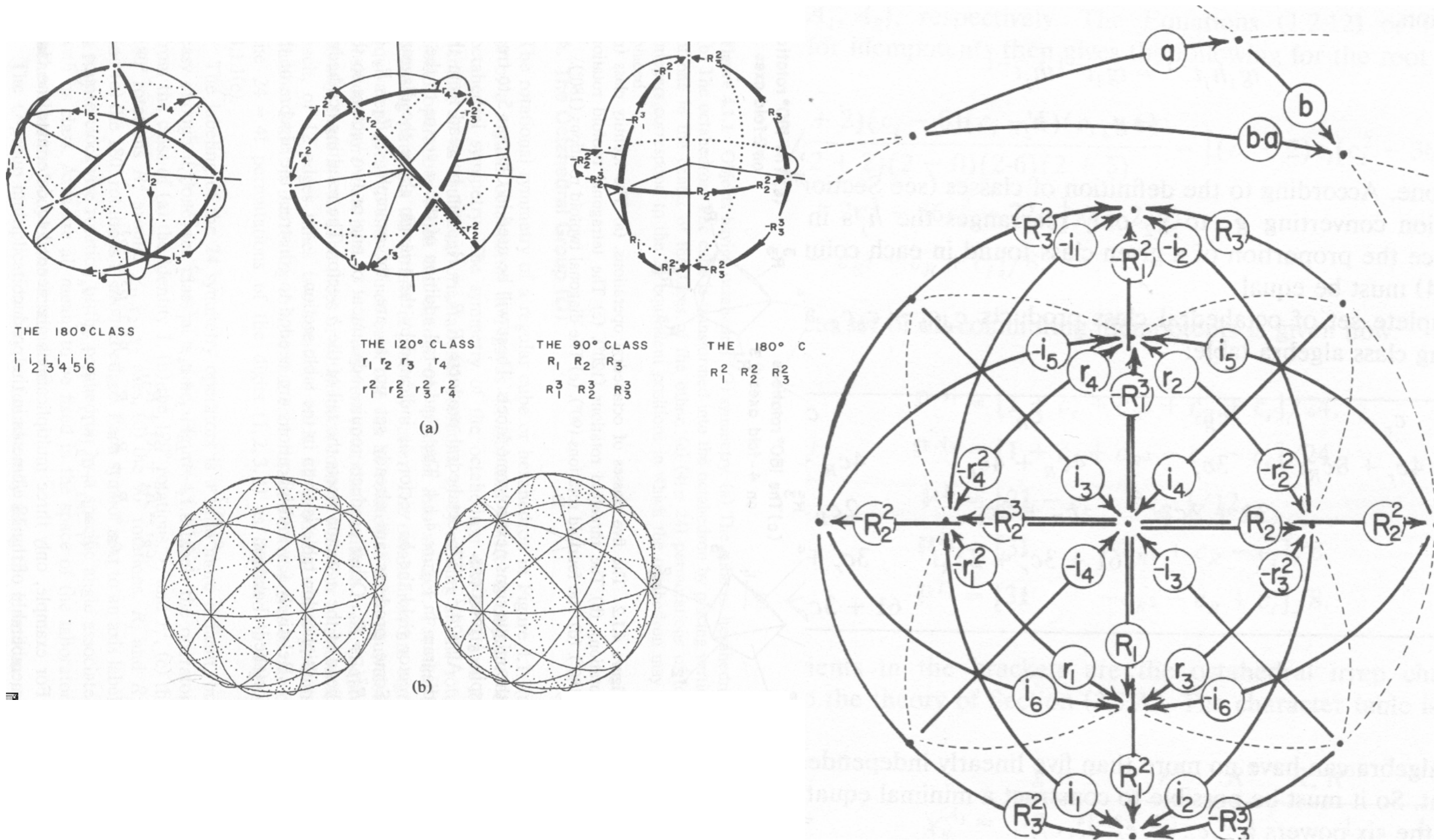
1E 180° CLASS
1 2 3 4 5 6



$\underbrace{[111] \quad [1\bar{1}\bar{1}] \quad [1\bar{1}\bar{1}] \quad [\bar{1}\bar{1}\bar{1}]}_{+120^\circ}$
 $\underbrace{[\bar{1}\bar{1}\bar{1}] \quad [11\bar{1}] \quad [\bar{1}11] \quad [1\bar{1}\bar{1}]}_{-120^\circ}$
 $\underbrace{[100] \quad [010] \quad [001]}_{\pm 180^\circ XYZ}$
 $\underbrace{[100] \quad [010] \quad [001]}_{+90^\circ XYZ}$
 $\underbrace{[\bar{1}00] \quad [0\bar{1}0] \quad [00\bar{1}]}_{-90^\circ XYZ}$
 $\underbrace{[101] \quad [10\bar{1}] \quad [110] \quad [1\bar{1}0] \quad [01\bar{1}] \quad [011]}_{\pm 180^\circ i_k}$

| 1 | r_1 | r_2 | r_3 | r_4 | r_1^2 | r_2^2 | r_3^2 | r_4^2 | R_1^2 | R_2^2 | R_3^2 | R_1 | R_2 | R_3 | R_1^3 | R_2^3 | R_3^3 | i_1 | i_2 | i_3 | i_4 | i_5 | i_6 |
|---------|----------|----------|----------|----------|---------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| r_1 | r_1^2 | $-r_4^2$ | $-r_2^2$ | $-r_3^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_1^2$ | $-r_2$ | $-r_3$ | $-r_4$ | i_3 | i_6 | i_1 | $-R_3$ | $-R_1$ | $-R_2$ | R_1^3 | i_5 | R_2^3 | i_2 | $-i_4$ | R_3^3 |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | R_3 | $-R_1^3$ | i_2 | i_3 | $-i_5$ | R_2^3 | i_6 | $-R_1$ | R_2 | $-i_1$ | R_3^3 | i_4 |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | $-i_4$ | R_1 | $-R_2^3$ | R_3^3 | i_6 | i_2 | i_5 | $-R_1^3$ | i_1 | R_2 | $-i_3$ | R_3 |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_1^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | $-R_3^3$ | $-i_5$ | R_2 | $-i_4$ | R_1^3 | i_1 | R_1 | i_6 | $-i_2$ | R_2^3 | R_3 | i_3 |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | R_2^3 | R_3^3 | R_1^3 | $-i_1$ | $-i_3$ | $-i_6$ | $-R_3$ | $-i_4$ | $-R_1$ | i_5 | $-i_2$ | $-R_2$ |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | i_2 | $-i_3$ | $-R_1$ | R_2 | $-R_3^3$ | $-i_5$ | i_4 | $-R_3$ | $-R_1^3$ | $-i_6$ | R_2^3 | $-i_1$ |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | $-R_2$ | $-i_4$ | $-i_6$ | i_2 | R_3 | $-R_1^3$ | $-i_3$ | $-R_3^3$ | i_5 | R_1 | $-i_1$ | $-R_2^3$ |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | $-i_1$ | $-R_3$ | $-i_5$ | $-R_2^3$ | $-i_4$ | R_1 | $-R_3^3$ | i_3 | $-i_6$ | R_1^3 | R_2 | $-i_2$ |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | R_1^3 | i_1 | $-i_4$ | $-R_1$ | i_2 | $-i_3$ | $-R_2$ | $-R_2^3$ | R_3^3 | R_3 | $-i_6$ | i_5 |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | $-i_5$ | R_2^3 | i_3 | $-i_6$ | $-R_2$ | $-i_4$ | $-i_2$ | i_1 | $-R_3$ | R_3^3 | R_1 | R_1^3 |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | i_6 | i_2 | R_3^3 | $-i_5$ | $-i_1$ | $-R_3$ | R_2^3 | $-R_2$ | i_4 | $-i_3$ | R_1^3 | $-R_1$ |
| R_1 | i_1 | $-R_2^3$ | $-i_2$ | R_2 | R_3^3 | $-i_3$ | $-R_3$ | i_4 | R_1^3 | i_6 | i_5 | R_1^2 | r_1 | $-r_4^2$ | -1 | $-r_3$ | r_2^2 | $-r_4$ | r_2 | r_1^2 | $-r_3^2$ | $-R_2^2$ | R_3^2 |
| R_2 | i_3 | R_3 | $-R_3^3$ | i_4 | R_1^3 | i_5 | $-i_6$ | $-R_1$ | $-i_2$ | R_2^3 | i_1 | $-r_2^2$ | R_2^2 | r_1 | r_3^2 | -1 | $-r_4$ | R_1^2 | R_3^3 | $-r_2$ | $-r_3$ | $-r_4^2$ | r_1^2 |
| R_3 | i_6 | i_5 | R_1 | $-R_1^3$ | R_2^3 | $-R_2$ | $-i_2$ | $-i_1$ | i_3 | i_4 | R_3^3 | r_1 | $-r_3^2$ | R_2^3 | $-r_2$ | r_4^2 | -1 | r_1^2 | r_2^2 | R_2^2 | $-R_1^2$ | $-r_4$ | $-r_3$ |
| R_1^3 | $-R_2$ | $-i_2$ | R_2^3 | i_1 | $-i_3$ | $-R_3^3$ | i_4 | R_3 | $-R_1$ | i_5 | $-i_6$ | -1 | $-r_4$ | r_3^2 | $-R_1^2$ | r_2 | $-r_1^2$ | $-r_1$ | r_3 | r_2^2 | $-r_4^2$ | $-R_3^3$ | $-R_2^2$ |
| R_2^3 | $-R_3$ | i_3 | i_4 | R_3^3 | $-i_6$ | R_1 | $-R_1^3$ | i_5 | $-i_1$ | $-R_2$ | $-i_2$ | r_4^2 | -1 | $-r_2$ | $-r_1^2$ | $-R_2^2$ | r_3 | $-R_3^3$ | R_1^2 | $-r_1$ | $-r_4$ | $-r_2^2$ | r_3^2 |
| R_3^3 | $-R_1$ | R_1^3 | i_6 | i_5 | $-i_1$ | $-i_2$ | R_2 | $-R_2^3$ | i_4 | $-i_3$ | $-R_3$ | $-r_3$ | r_2^2 | -1 | r_4 | $-r_1^2$ | $-R_3^3$ | r_4^2 | r_3^2 | $-R_1^2$ | $-R_2^2$ | $-r_2$ | $-r_1$ |
| i_1 | R_3^3 | $-i_4$ | i_3 | R_3 | $-R_1$ | $-i_6$ | $-i_5$ | $-R_1^3$ | R_2^3 | i_2 | $-R_2$ | r_1^2 | R_3^3 | $-r_4$ | r_4^2 | $-R_1^2$ | $-r_1$ | -1 | $-R_2^2$ | $-r_3$ | r_2 | r_3^2 | r_2^2 |
| i_2 | i_4 | R_3^3 | R_3 | $-i_3$ | $-i_5$ | R_1^3 | R_1 | $-i_6$ | R_2 | $-i_1$ | R_2^3 | $-r_3^2$ | $-R_1^2$ | $-r_3$ | $-r_2^2$ | $-R_3^3$ | $-r_2$ | R_2^2 | -1 | r_4 | $-r_1$ | r_1^2 | r_4^2 |
| i_3 | R_1^3 | R_1 | $-i_5$ | i_6 | $-R_2$ | $-R_2^3$ | $-i_1$ | i_2 | $-R_3$ | R_3^3 | $-i_4$ | $-r_2$ | r_1^2 | R_1^2 | $-r_1$ | r_2^2 | $-R_2^2$ | r_3^2 | $-r_4^2$ | -1 | R_3^3 | r_3 | $-r_4$ |
| i_4 | $-i_5$ | i_6 | $-R_1^3$ | $-R_1$ | $-i_2$ | i_1 | $-R_2^3$ | $-R_2$ | $-R_3^3$ | $-R_3$ | i_3 | r_4 | r_4^2 | R_2^2 | r_3 | r_3^2 | R_1^2 | $-r_2^2$ | r_1^2 | $-R_3^3$ | -1 | r_1 | $-r_2$ |
| i_5 | i_2 | $-R_2$ | i_1 | $-R_2^3$ | i_4 | $-R_3$ | i_3 | $-R_3^3$ | i_6 | $-R_1^3$ | $-R_1$ | R_3^3 | r_2 | r_2^2 | R_2^2 | r_4 | r_4^2 | $-r_3$ | $-r_1$ | $-r_3^2$ | $-r_1^2$ | -1 | $-R_1^2$ |
| i_6 | R_2^3 | i_1 | R_2 | i_2 | $-R_3$ | $-i_4$ | $-R_3^3$ | $-i_3$ | $-i_5$ | $-R_1$ | R_1^3 | R_2^2 | $-r_3$ | r_1^2 | $-R_2^3$ | $-r_1$ | r_3^2 | $-r_2$ | $-r_4$ | r_4^2 | r_2^2 | R_1^2 | -1 |

Octahedral O and spin-OCU(2) rotation product Table F.2.1 from Principles of Symmetry, Dynamics and Spectroscopy



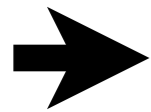
Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

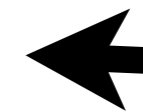
Octahedral O and spin- $O \subset U(2)$



Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters



Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

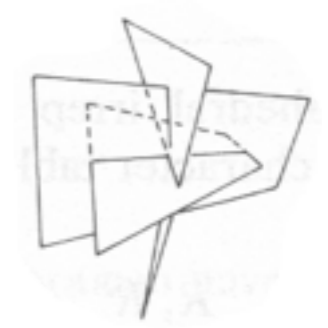
Preview of applications to high resolution spectroscopy

Tetrahedral T class algebra

$$\mathbf{c}_l = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | | |
|---------|--------------|---------------------|---------------------|---------------------|---------------------------|-----------------|-----------------|-----------------|---------------------|----------|----------|---------|
| | $[111]$ | $[\bar{1}\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[\bar{1}1\bar{1}]$ | $[\bar{1}\bar{1}\bar{1}]$ | $[11\bar{1}]$ | $[\bar{1}11]$ | $[1\bar{1}1]$ | $[\bar{1}\bar{1}1]$ | $[100]$ | $[010]$ | $[001]$ |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z | |
| | R_1^2 | R_2^2 | R_3^2 | R_4^2 | r_1^2 | r_2^2 | r_3^2 | r_4^2 | R_1^2 | R_2^2 | R_3^2 | |
| r_1 | r_1^2 | $-r_2^2$ | $-r_3^2$ | $-r_4^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_4^2$ | $-r_2$ | $-r_3$ | $-r_4$ | |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_4^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | |



Tetrahedral T class algebra

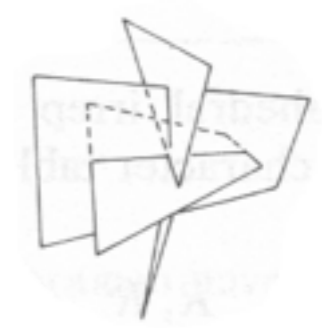
$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2,$$

$$\mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | | |
|---------|--------------|---------------------|---------------------|---------------------|---------------------------|-----------------|-----------------|-----------------|---------------------|----------|----------|---------|
| | $[111]$ | $[\bar{1}\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[\bar{1}1\bar{1}]$ | $[\bar{1}\bar{1}\bar{1}]$ | $[11\bar{1}]$ | $[\bar{1}11]$ | $[1\bar{1}1]$ | $[\bar{1}\bar{1}1]$ | $[100]$ | $[010]$ | $[001]$ |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z | |
| r_1 | r_1^2 | $-r_2^2$ | $-r_2^2$ | $-r_3^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_1^2$ | $-r_2$ | $-r_3$ | $-r_4$ | |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_1^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | |

T class products



| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_r^\dagger | \mathbf{c}_ρ |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| \mathbf{c}_r | $4\mathbf{c}_r^\dagger$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| \mathbf{c}_r^\dagger | | $4\mathbf{c}_r$ | $3\mathbf{c}_r^\dagger$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |

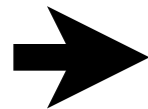
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Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters



Octahedral O class algebra

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Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy

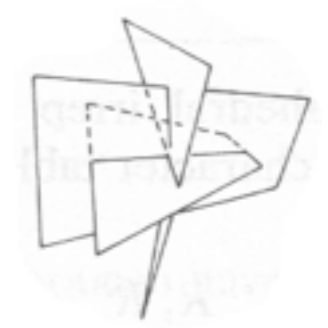
Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | | |
|---------|--------------|---------------------|---------------------|---------------------|---------------------------|-----------------|-----------------|-----------------|---------------------|----------|----------|---------|
| | $[111]$ | $[\bar{1}\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[\bar{1}1\bar{1}]$ | $[\bar{1}\bar{1}\bar{1}]$ | $[11\bar{1}]$ | $[\bar{1}11]$ | $[1\bar{1}1]$ | $[\bar{1}\bar{1}1]$ | $[100]$ | $[010]$ | $[001]$ |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z | |
| | R_1^2 | R_2^2 | R_3^2 | R_4^2 | R_1^2 | R_2^2 | R_3^2 | R_4^2 | R_1^2 | R_2^2 | R_3^2 | |
| r_1 | r_1^2 | $-r_2^2$ | $-r_2^2$ | $-r_3^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_1^2$ | $-r_2$ | $-r_3$ | $-r_4$ | |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_1^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | |

T class products



| | | | |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_r^\dagger | \mathbf{c}_ρ |
| \mathbf{c}_r | $4\mathbf{c}_r^\dagger$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| \mathbf{c}_r^\dagger | | $4\mathbf{c}_r$ | $3\mathbf{c}_r^\dagger$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | | |
|---------|--------------|---------------------|---------------------|---------------------|---------------------------|-----------------|-----------------|-----------------|---------------------|----------|----------|---------|
| | $[111]$ | $[\bar{1}\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[\bar{1}1\bar{1}]$ | $[\bar{1}\bar{1}\bar{1}]$ | $[11\bar{1}]$ | $[\bar{1}11]$ | $[1\bar{1}1]$ | $[\bar{1}\bar{1}1]$ | $[100]$ | $[010]$ | $[001]$ |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z | |
| r_1 | r_1^2 | $-r_2^2$ | $-r_2^2$ | $-r_3^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_1^2$ | $-r_2$ | $-r_3$ | $-r_4$ | |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_1^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | |

T class products



| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_r^\dagger | \mathbf{c}_ρ |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| \mathbf{c}_r | $4\mathbf{c}_r^\dagger$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| \mathbf{c}_r^\dagger | | $4\mathbf{c}_r$ | $3\mathbf{c}_r^\dagger$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

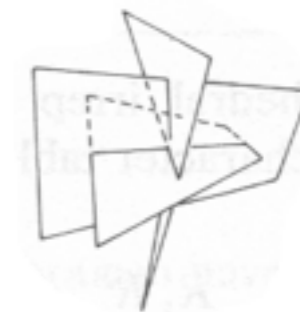
Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | | |
|---------|--------------|---------------------|---------------------|---------------------|---------------------------|-----------------|-----------------|-----------------|---------------------|----------|----------|---------|
| | $[111]$ | $[\bar{1}\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[\bar{1}1\bar{1}]$ | $[\bar{1}\bar{1}\bar{1}]$ | $[11\bar{1}]$ | $[\bar{1}11]$ | $[1\bar{1}1]$ | $[\bar{1}\bar{1}1]$ | $[100]$ | $[010]$ | $[001]$ |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z | |
| | R_1^2 | R_2^2 | R_3^2 | R_4^2 | R_1^2 | R_2^2 | R_3^2 | R_4^2 | R_1^2 | R_2^2 | R_3^2 | |
| r_1 | r_1^2 | $-r_2^2$ | $-r_2^2$ | $-r_3^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_1^2$ | $-r_2$ | $-r_3$ | $-r_4$ | |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_1^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | |

T class products



| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_r^\dagger | \mathbf{c}_ρ |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| \mathbf{c}_r | $4\mathbf{c}_r^\dagger$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| \mathbf{c}_r^\dagger | | $4\mathbf{c}_r$ | $3\mathbf{c}_r^\dagger$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_r

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\cdot\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\cdot\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\cdot\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

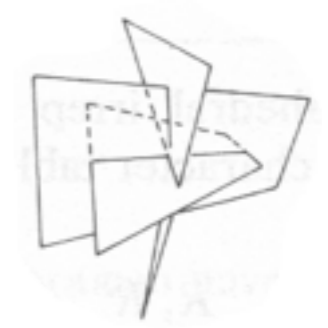
Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | |
|---------|--------------|----------|----------|----------|-----------------|-----------------|-----------------|-----------------|---------------------|----------|----------|
| | [111] | [111] | [111] | [111] | [111] | [111] | [111] | [111] | [100] | [010] | [001] |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z |
| | r_1^2 | r_2^2 | r_3^2 | r_4^2 | R_1^2 | R_2^2 | R_3^2 | R_4^2 | R_1^2 | R_2^2 | R_3^2 |
| r_1 | r_1^2 | $-r_2^2$ | $-r_3^2$ | $-r_4^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_4^2$ | $-r_2$ | $-r_3$ | $-r_4$ |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_4^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 |

T class products



| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_r^\dagger | \mathbf{c}_ρ |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| \mathbf{c}_r | $4\mathbf{c}_r^\dagger$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| \mathbf{c}_r^\dagger | | $4\mathbf{c}_r$ | $3\mathbf{c}_r^\dagger$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_r

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r \mathbf{c}_r = 4(4\mathbf{1} + 4\mathbf{c}_\rho) = 16\mathbf{1} + 16\mathbf{c}_\rho$$

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

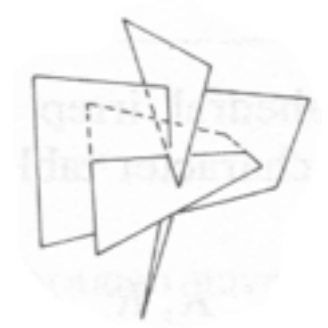
Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | | |
|---------|--------------|---------------------|---------------------|---------------------|---------------------------|-----------------|-----------------|-----------------|---------------------|----------|----------|---------|
| | $[111]$ | $[\bar{1}\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[\bar{1}1\bar{1}]$ | $[\bar{1}\bar{1}\bar{1}]$ | $[11\bar{1}]$ | $[\bar{1}11]$ | $[1\bar{1}1]$ | $[\bar{1}\bar{1}1]$ | $[100]$ | $[010]$ | $[001]$ |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z | |
| | R_1^2 | R_2^2 | R_3^2 | R_4^2 | R_1^2 | R_2^2 | R_3^2 | R_4^2 | R_1^2 | R_2^2 | R_3^2 | |
| r_1 | r_1^2 | $-r_2^2$ | $-r_3^2$ | $-r_4^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_4^2$ | $-r_2$ | $-r_3$ | $-r_4$ | |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_4^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | |

T class products



| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_r^\dagger | \mathbf{c}_ρ |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| \mathbf{c}_r | $4\mathbf{c}_r^\dagger$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| \mathbf{c}_r^\dagger | | $4\mathbf{c}_r$ | $3\mathbf{c}_r^\dagger$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_r

$$\mathbf{c}_r^2 = 4\tilde{\mathbf{c}}_r$$

$$\mathbf{c}_r^3 = 4\tilde{\mathbf{c}}_r \mathbf{c}_r = 4(4\mathbf{1} + 4\mathbf{c}_\rho) = 16\mathbf{1} + 16\mathbf{c}_\rho$$

$$\mathbf{c}_r^4 = 16\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho \mathbf{c}_r = 16\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r)$$

Minimal equation for \mathbf{c}_ρ

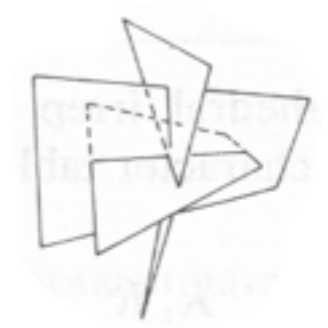
$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$



T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | | |
|---------|--------------|---------------------|---------------------|---------------------|---------------------------|-----------------|-----------------|-----------------|---------------------|----------|----------|---------|
| | $[111]$ | $[\bar{1}\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[\bar{1}1\bar{1}]$ | $[\bar{1}\bar{1}\bar{1}]$ | $[11\bar{1}]$ | $[\bar{1}11]$ | $[1\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[100]$ | $[010]$ | $[001]$ |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z | |
| | R_1^2 | R_2^2 | R_3^2 | R_4^2 | | | | | | | | |
| r_1 | r_1^2 | $-r_2^2$ | $-r_3^2$ | $-r_4^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_4^2$ | $-r_2$ | $-r_3$ | $-r_4$ | |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_1^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | |

T class products

| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_r^\dagger | \mathbf{c}_ρ |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| \mathbf{c}_r | $4\mathbf{c}_r^\dagger$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| \mathbf{c}_r^\dagger | | $4\mathbf{c}_r$ | $3\mathbf{c}_r^\dagger$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_r

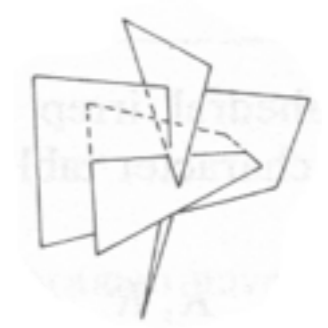
$$\begin{aligned} \mathbf{c}_r^2 &= 4\tilde{\mathbf{c}}_r \\ \mathbf{c}_r^3 &= 4\tilde{\mathbf{c}}_r \mathbf{c}_r = 4(4\mathbf{1} + 4\mathbf{c}_\rho) = 16\mathbf{1} + 16\mathbf{c}_\rho \\ \mathbf{c}_r^4 &= 16\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho \mathbf{c}_r = 16\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r) \\ \mathbf{c}_r^4 - 64\mathbf{c}_r &= (\mathbf{c}_r^3 - 64\mathbf{1})\mathbf{c}_r = \mathbf{0} \end{aligned}$$

Minimal equation for \mathbf{c}_ρ

$$\begin{aligned} \mathbf{c}_\rho^2 &= 3\mathbf{1} + 2\mathbf{c}_\rho \\ \mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} &= \mathbf{0} \\ (\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) &= \mathbf{0} \end{aligned}$$

Tetrahedral T class algebra

$$\mathbf{c}_1 = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4, \quad \mathbf{c}_r^\dagger = \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$



T group products

| | $+120^\circ$ | | | | -120° | | | | $\pm 180^\circ$ XYZ | | | |
|---------|--------------|---------------------|---------------------|---------------------|---------------------------|-----------------|-----------------|-----------------|---------------------|----------|----------|---------|
| | $[111]$ | $[\bar{1}\bar{1}1]$ | $[1\bar{1}\bar{1}]$ | $[\bar{1}1\bar{1}]$ | $[\bar{1}\bar{1}\bar{1}]$ | $[11\bar{1}]$ | $[\bar{1}11]$ | $[1\bar{1}1]$ | $[\bar{1}\bar{1}1]$ | $[100]$ | $[010]$ | $[001]$ |
| 1 | r_1 | r_2 | r_3 | r_4 | \tilde{r}_1^2 | \tilde{r}_2^2 | \tilde{r}_3^2 | \tilde{r}_4^2 | ρ_x | ρ_y | ρ_z | |
| | R_1^2 | R_2^2 | R_3^2 | R_4^2 | | | | | | | | |
| r_1 | r_1^2 | $-r_2^2$ | $-r_3^2$ | $-r_4^2$ | -1 | $-R_2^2$ | $-R_3^2$ | $-R_4^2$ | $-r_2$ | $-r_3$ | $-r_4$ | |
| r_2 | $-r_3^2$ | r_2^2 | $-r_4^2$ | $-r_1^2$ | R_2^2 | -1 | R_1^2 | $-R_3^2$ | r_1 | r_4 | $-r_3$ | |
| r_3 | $-r_4^2$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | R_3^2 | $-R_1^2$ | -1 | R_2^2 | $-r_4$ | r_1 | r_2 | |
| r_4 | $-r_2^2$ | $-r_3^2$ | $-r_1^2$ | r_4^2 | R_4^2 | R_3^2 | $-R_2^2$ | -1 | r_3 | $-r_2$ | r_1 | |
| r_1^2 | -1 | R_1^2 | R_2^2 | R_3^2 | $-r_1$ | r_3 | r_4 | r_2 | r_4^2 | r_2^2 | r_3^2 | |
| r_2^2 | $-R_1^2$ | -1 | R_3^2 | $-R_2^2$ | r_4 | $-r_2$ | r_1 | r_3 | $-r_3^2$ | $-r_1^2$ | r_4^2 | |
| r_3^2 | $-R_2^2$ | $-R_3^2$ | -1 | R_1^2 | r_2 | r_4 | $-r_3$ | r_1 | r_2^2 | $-r_4^2$ | $-r_1^2$ | |
| r_4^2 | $-R_3^2$ | R_2^2 | $-R_1^2$ | -1 | r_3 | r_1 | r_2 | $-r_4$ | $-r_1^2$ | r_3^2 | $-r_2^2$ | |
| R_1^2 | $-r_4$ | r_3 | $-r_2$ | r_1 | r_2^2 | $-r_1^2$ | r_4^2 | $-r_3^2$ | -1 | R_3^2 | $-R_2^2$ | |
| R_2^2 | $-r_2$ | r_1 | r_4 | $-r_3$ | r_3^2 | $-r_4^2$ | $-r_1^2$ | r_2^2 | $-R_3^2$ | -1 | R_1^2 | |
| R_3^2 | $-r_3$ | $-r_4$ | r_1 | r_2 | r_4^2 | r_3^2 | $-r_2^2$ | $-r_1^2$ | R_2^2 | $-R_1^2$ | -1 | |

T class products

| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_r^\dagger | \mathbf{c}_ρ |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| \mathbf{c}_r | $4\mathbf{c}_r^\dagger$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| \mathbf{c}_r^\dagger | | $4\mathbf{c}_r$ | $3\mathbf{c}_r^\dagger$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_r

$$\begin{aligned} \mathbf{c}_r^2 &= 4\tilde{\mathbf{c}}_r \\ \mathbf{c}_r^3 &= 4\tilde{\mathbf{c}}_r \mathbf{c}_r = 4(4\mathbf{1} + 4\mathbf{c}_\rho) = 16\mathbf{1} + 16\mathbf{c}_\rho \\ \mathbf{c}_r^4 &= 16\mathbf{1}\mathbf{c}_r + 16\mathbf{c}_\rho \mathbf{c}_r = 16\mathbf{1}\mathbf{c}_r + 16(3\mathbf{c}_r) \\ \mathbf{c}_r^4 - 64\mathbf{c}_r &= (\mathbf{c}_r^3 - 64\mathbf{1})\mathbf{c}_r = \mathbf{0} \end{aligned}$$

Minimal equation for \mathbf{c}_ρ

$$\begin{aligned} \mathbf{c}_\rho^2 &= 3\mathbf{1} + 2\mathbf{c}_\rho \\ \mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} &= \mathbf{0} \\ (\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) &= \mathbf{0} \end{aligned}$$

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - \mathbf{0}) = \mathbf{0}$$

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

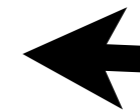
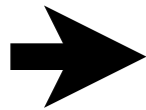
Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters



Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy

Tetrahedral T class projectors

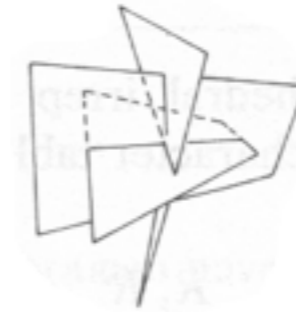
$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

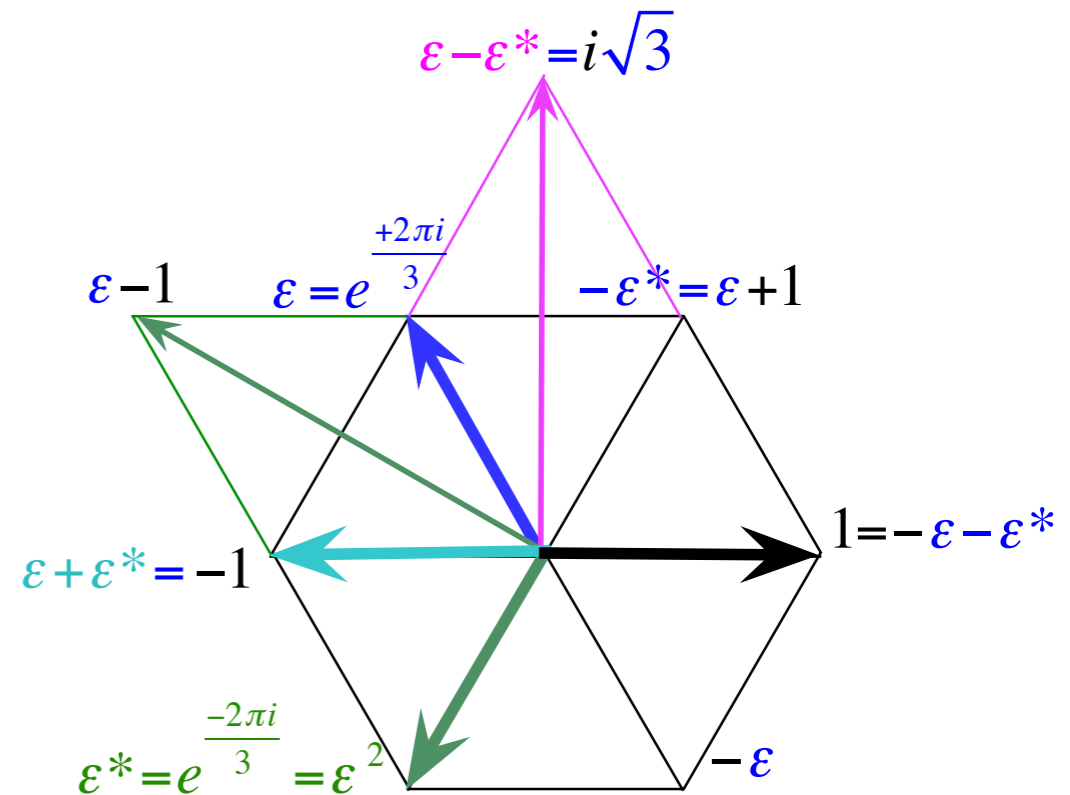
T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

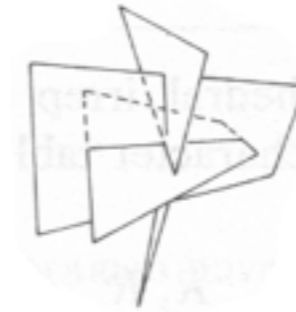
$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

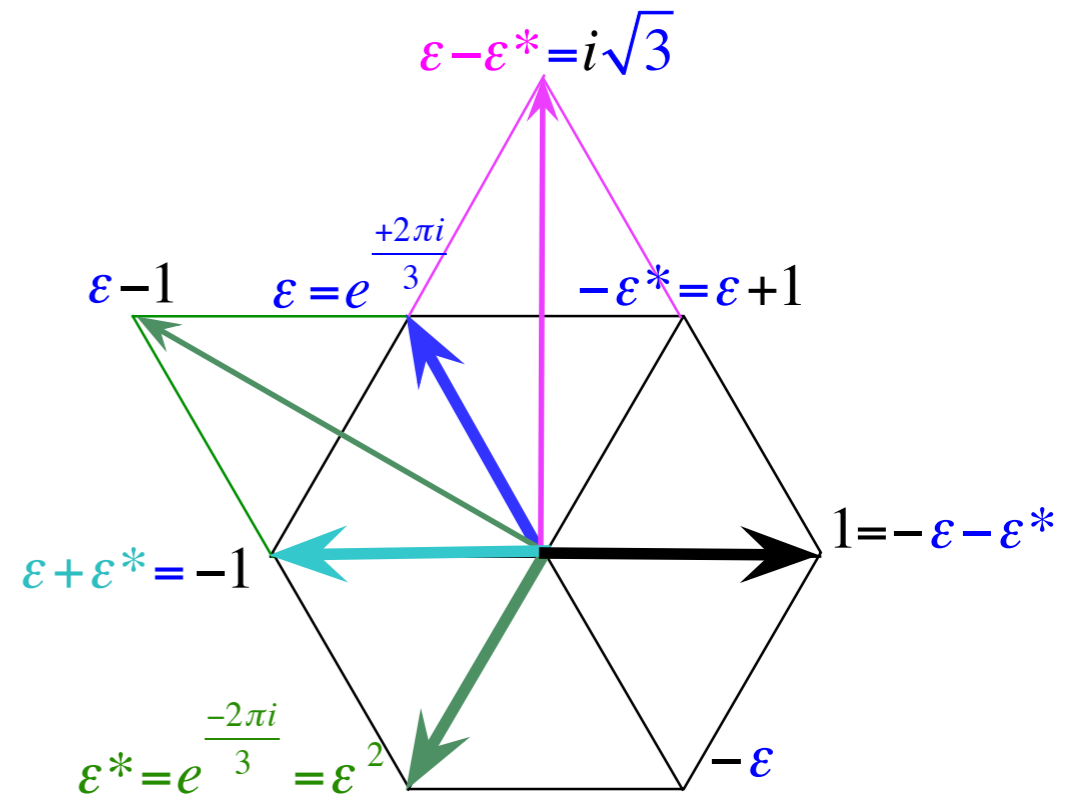
T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = 0$$



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

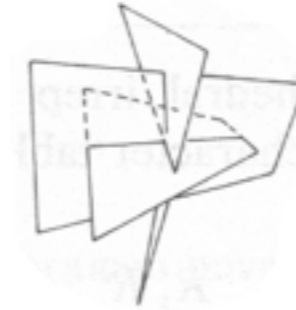
Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

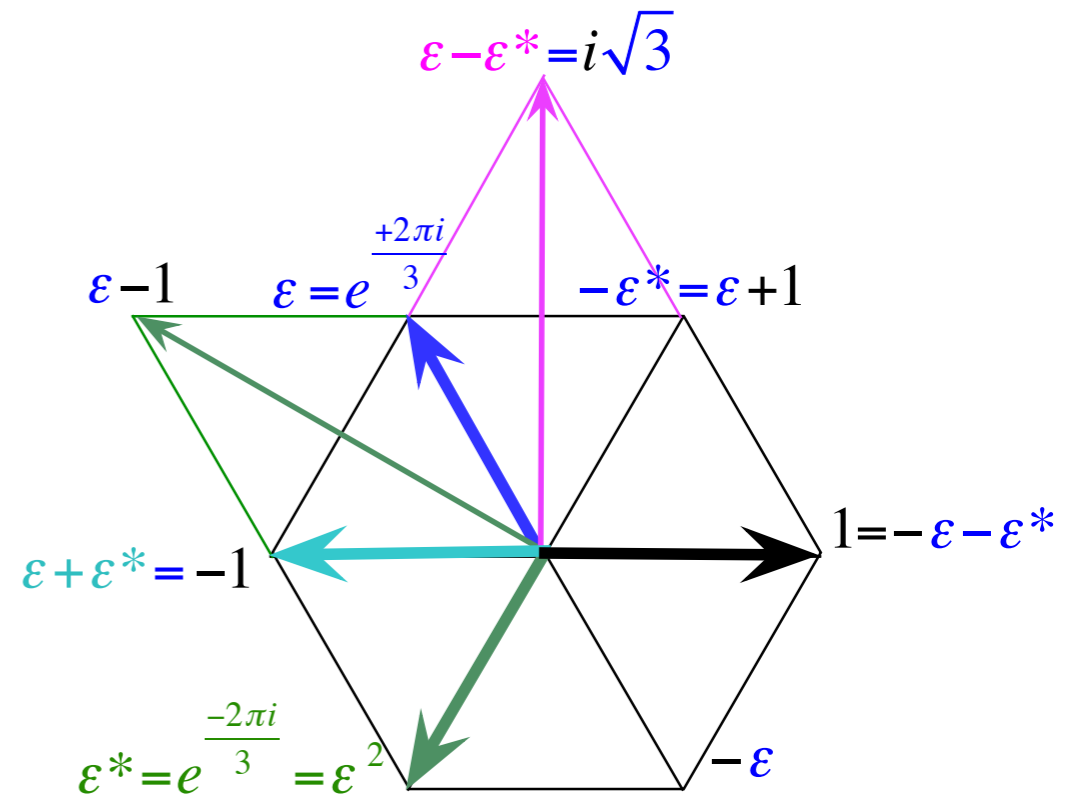
T class products

| | | | |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_ρ |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |



Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = 0$$



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\epsilon)} = \frac{(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon - 4\epsilon^*)(4\epsilon - 4)(4\epsilon - 0)}$$

$$\mathbf{P}^{(4\epsilon^*)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\epsilon^* - 4\epsilon)(4\epsilon^* - 4)(4\epsilon^* - 0)}$$

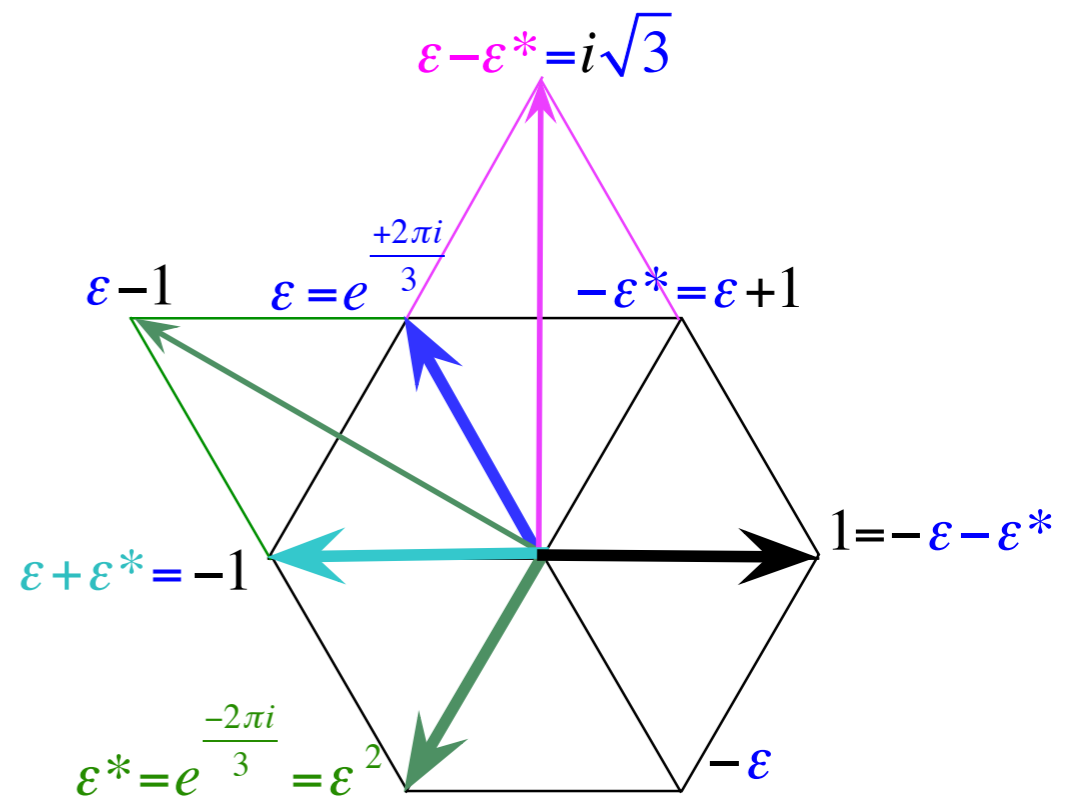
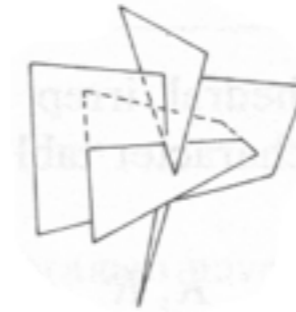
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\epsilon \mathbf{1})(\mathbf{c}_r - 4\epsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\epsilon)(4 - 4\epsilon^*)(4 - 0)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

| | | | |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_ρ |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} c_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

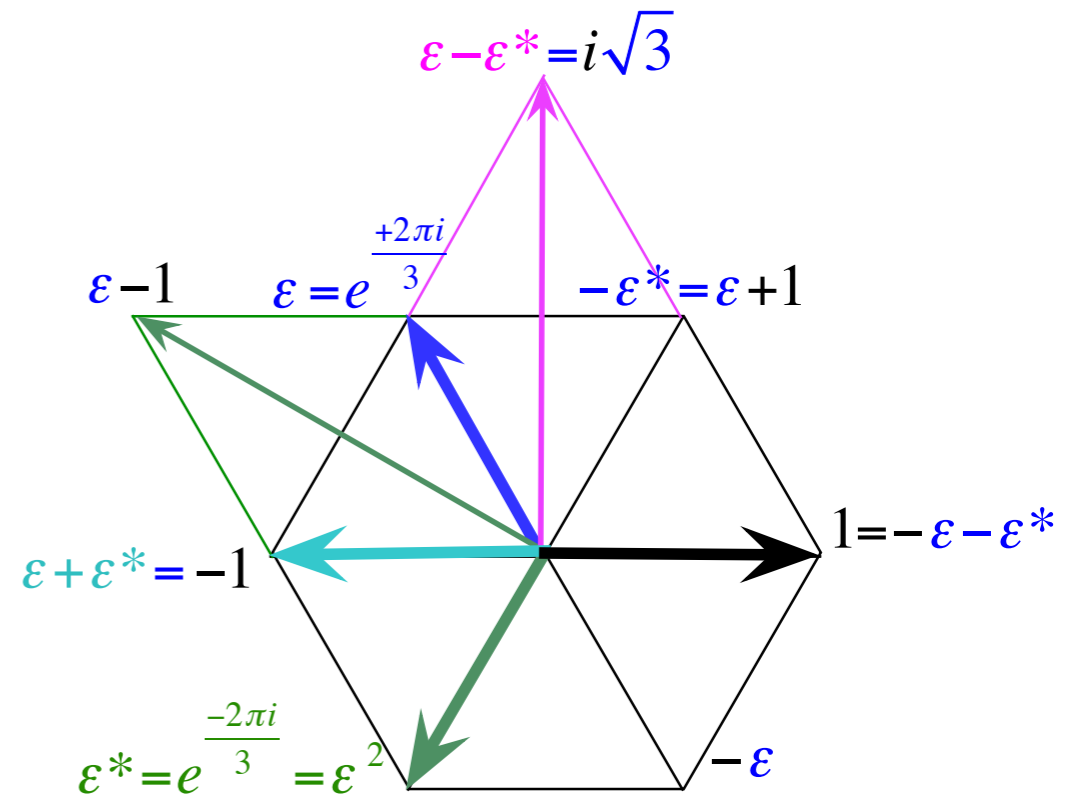
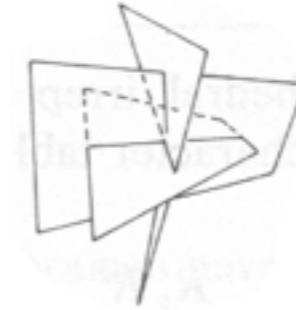
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

| | | | |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_ρ |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r - 16\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

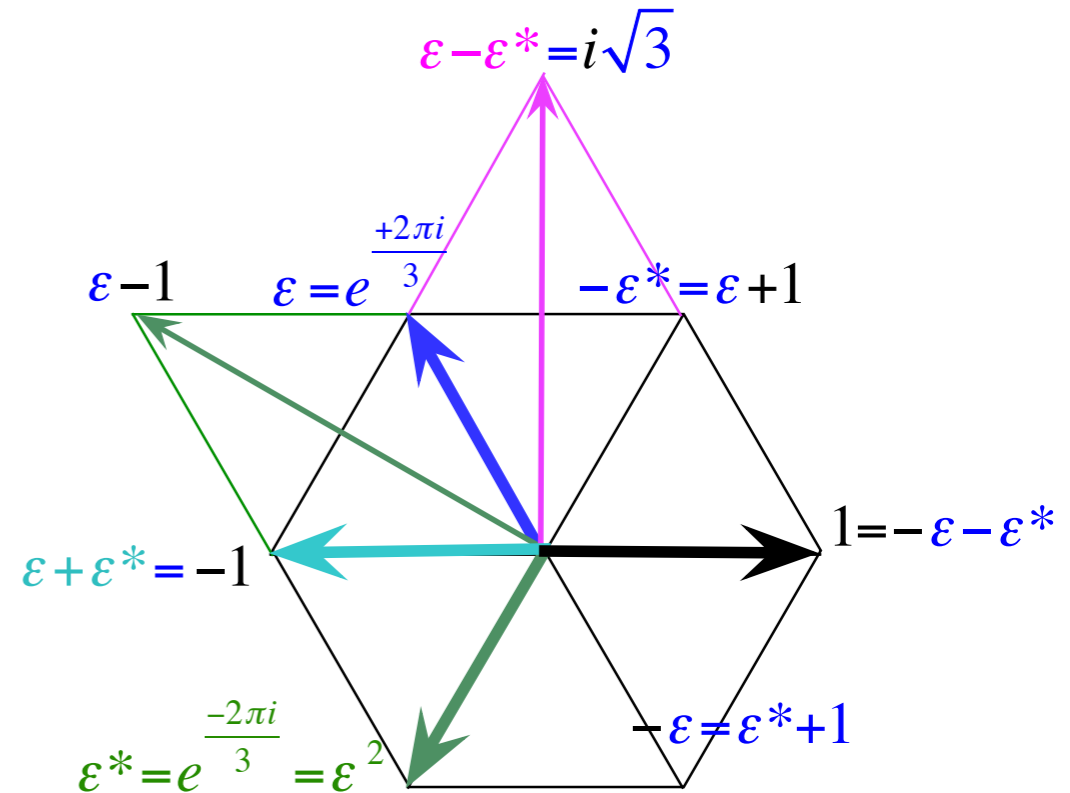
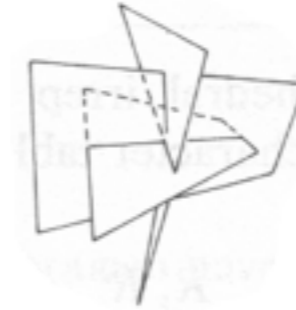
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

| | | | |
|-----------------------------|-------------------------|----------------------------------|----------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_ρ |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_\rho$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_ρ | | | $3\mathbf{1} + 2\mathbf{c}_\rho$ |



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} \mathbf{c}_{g_g}$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16 \cdot \varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

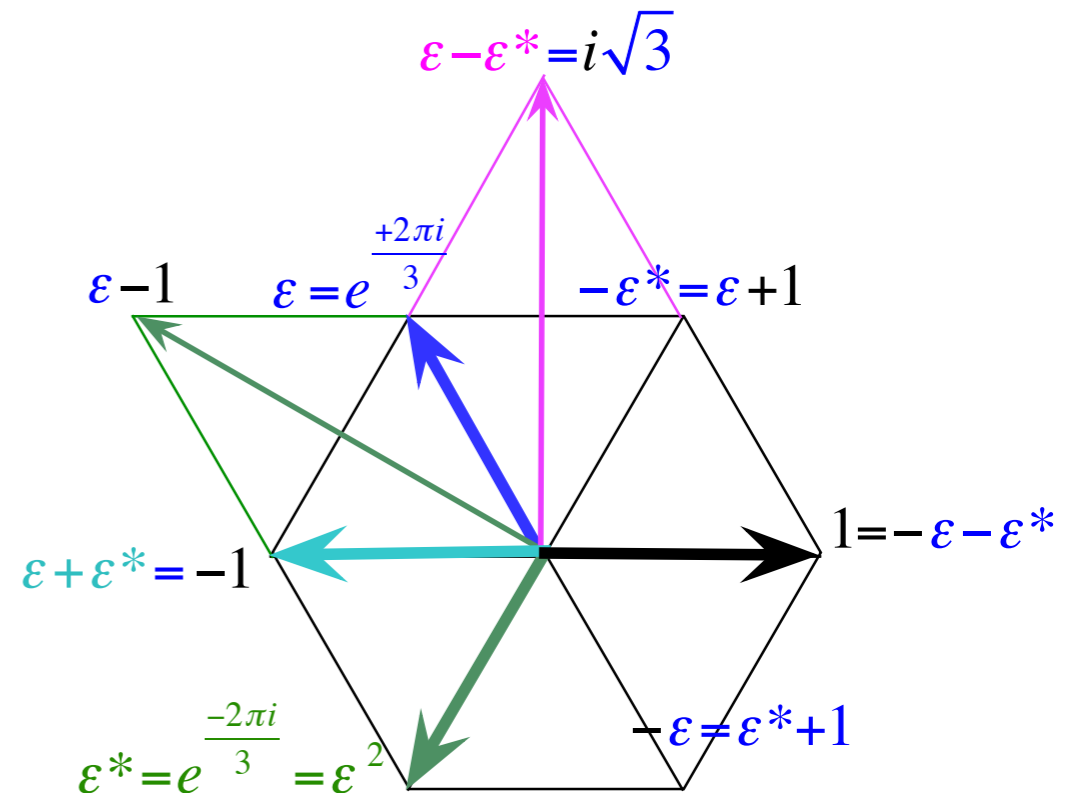
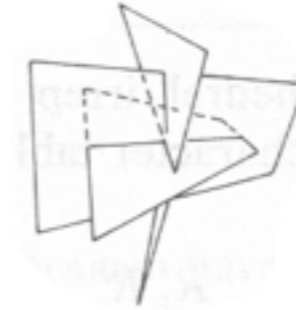
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu}} \chi_g^{\mu*} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

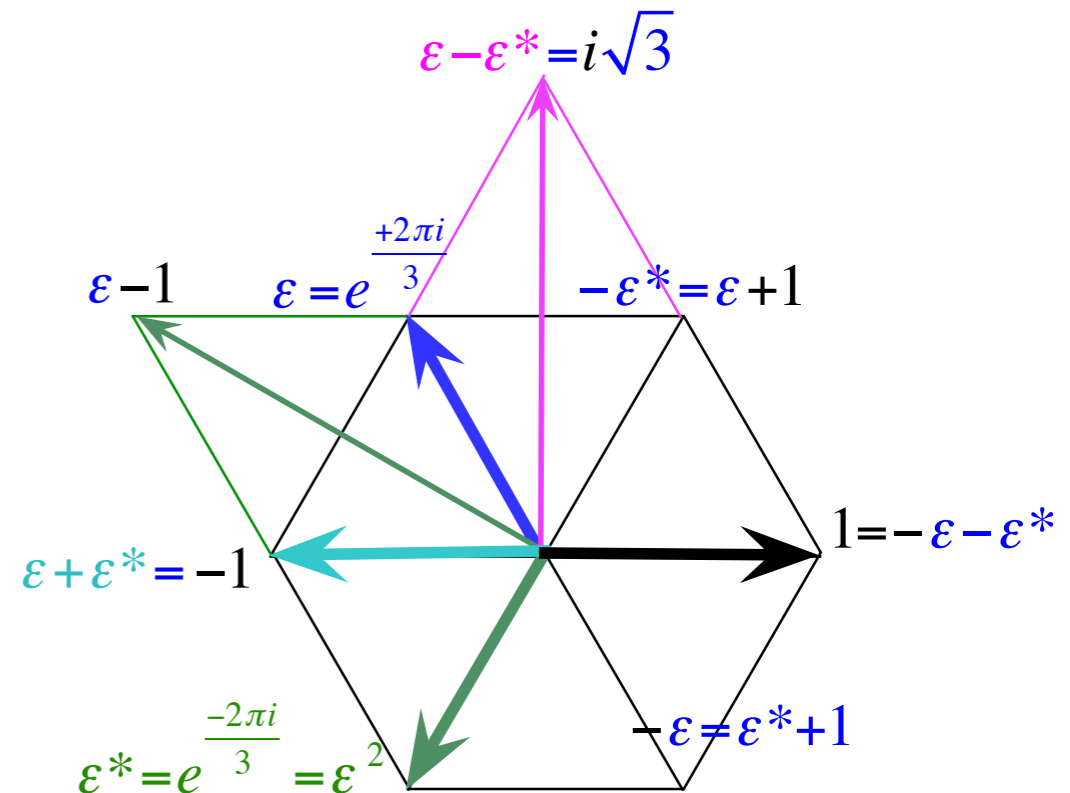
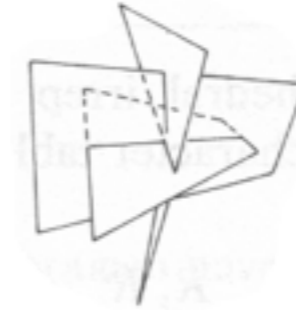
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{\circ c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\circ G} \chi_g^{\mu*} c_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^* \tilde{\mathbf{c}}_r + \varepsilon \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

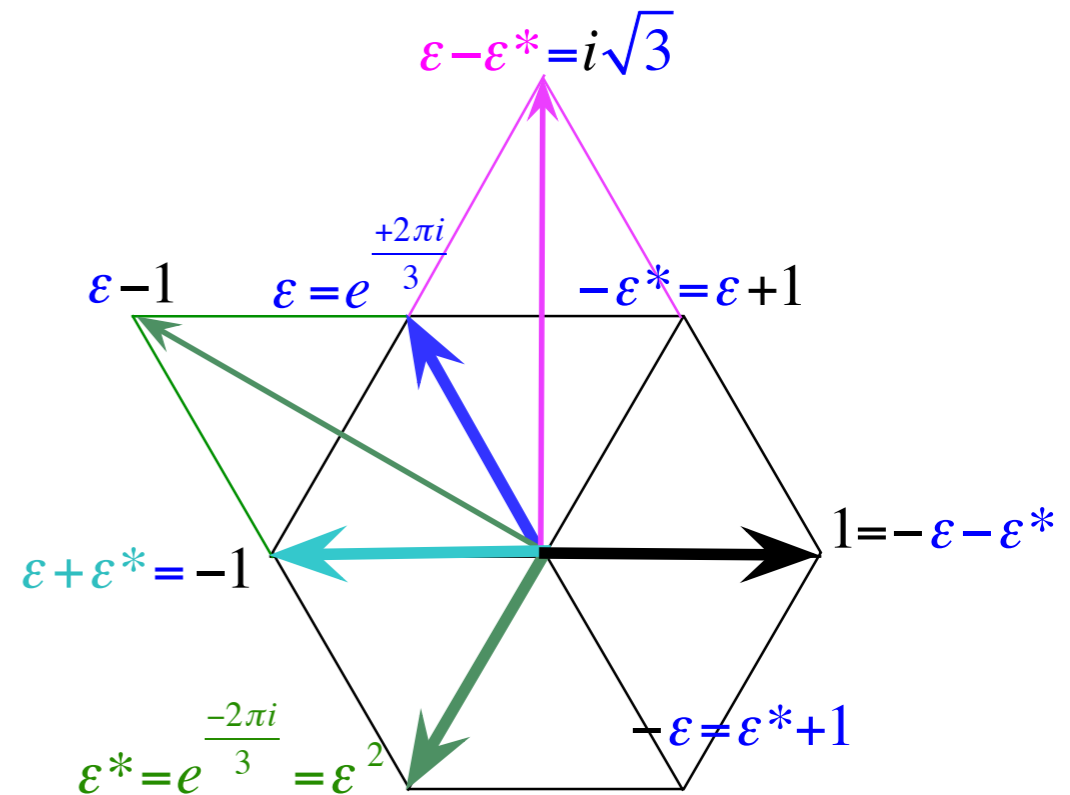
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G \chi_g^{\mu*}} \mathbf{c}_{g_g}$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

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$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^* \tilde{\mathbf{c}}_r + \varepsilon \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)}$$

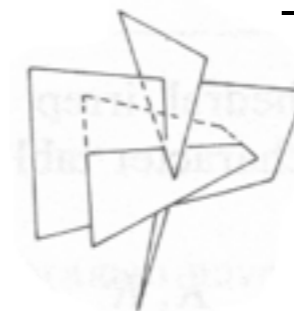
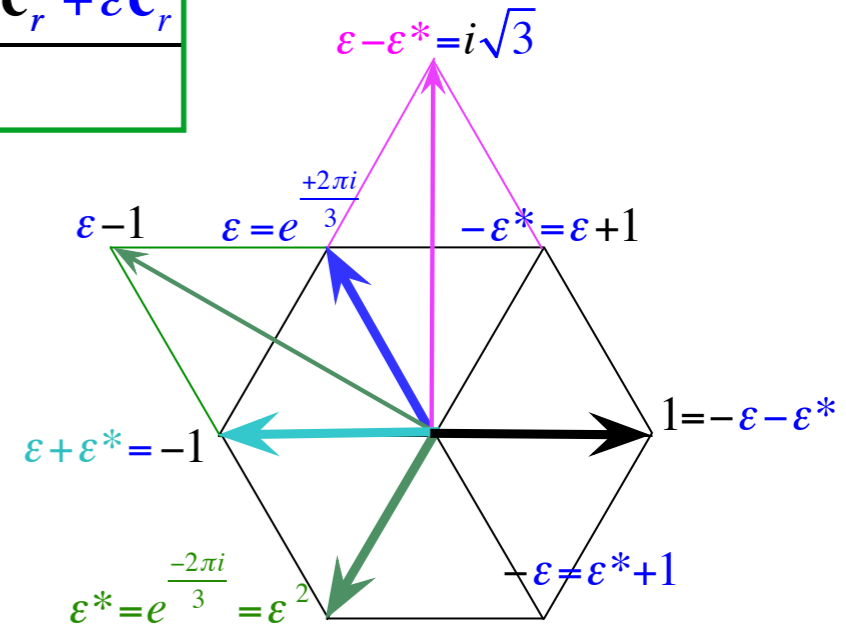
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



| $T : \mathbf{c}_g =$ | \mathbf{c}_1 | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
|----------------------------|----------------|-----------------|------------------------|---------------------|
| \cdot | | | | |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
| $\chi_g^{\varepsilon^*} =$ | 1 | ε | ε^* | 1 |
| \cdot | | | | |

Tetrahedral T class projectors

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$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$

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$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^* \tilde{\mathbf{c}}_r + \varepsilon \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

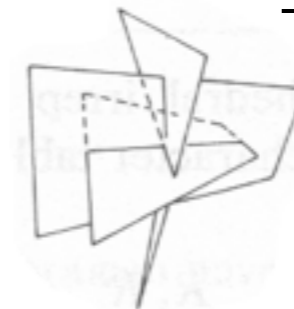
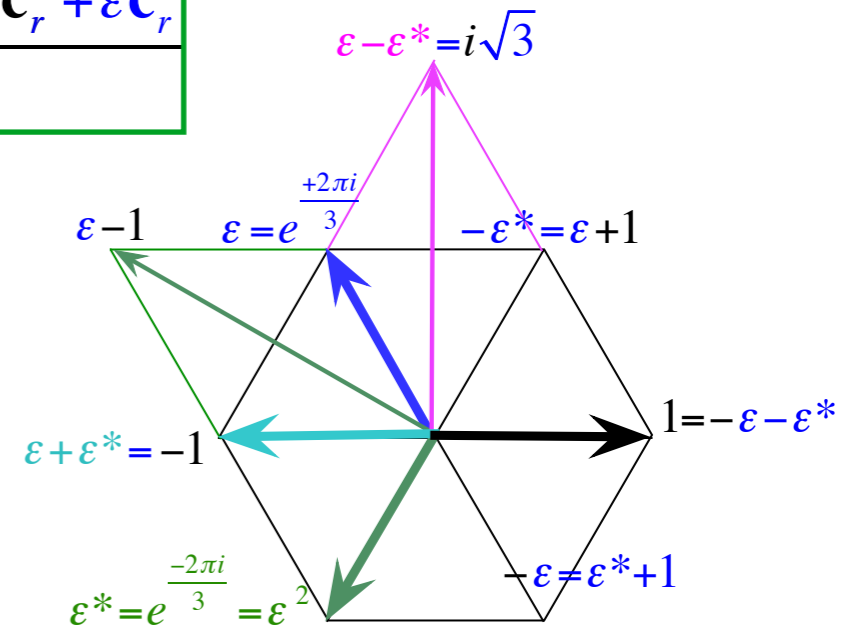
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon \mathbf{1})(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$

T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
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| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



| $T : \mathbf{c}_g =$ | \mathbf{c}_1 | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
|----------------------------|----------------|-----------------|------------------------|---------------------|
| \cdot | | | | |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
| $\chi_g^{\varepsilon^*} =$ | 1 | ε | ε^* | 1 |
| \cdot | | | | |

Tetrahedral T class projectors

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$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^* \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16 \cdot \varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16 \cdot \varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^* \mathbf{c}_r}{12}$$

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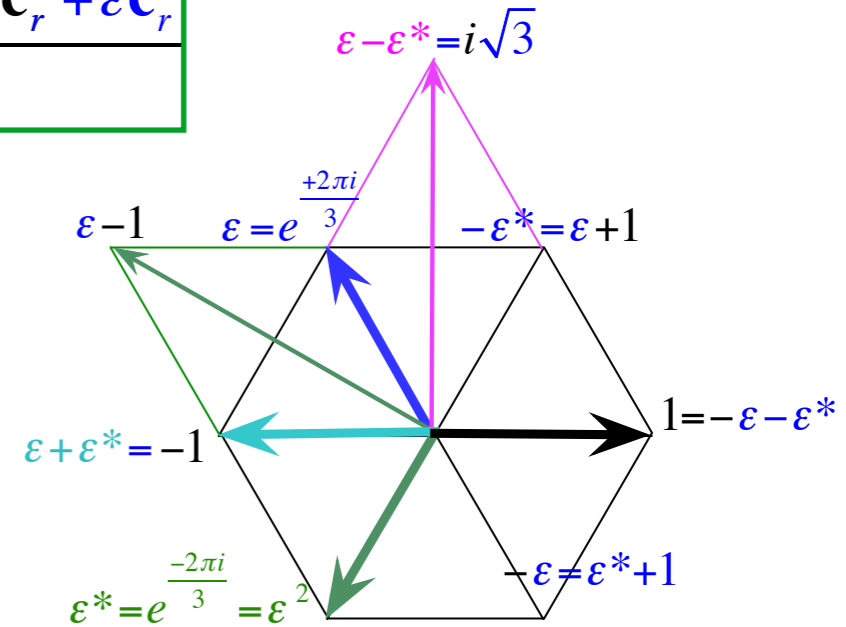
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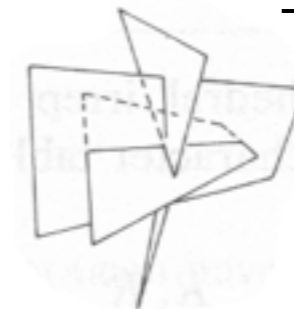
T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
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| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4 \cdot \mathbf{1})(\mathbf{c}_r - 0) = 0$$



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|----------------------------|----------------|-----------------|------------------------|---------------------|
| \cdot | | | | |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
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Tetrahedral T class projectors

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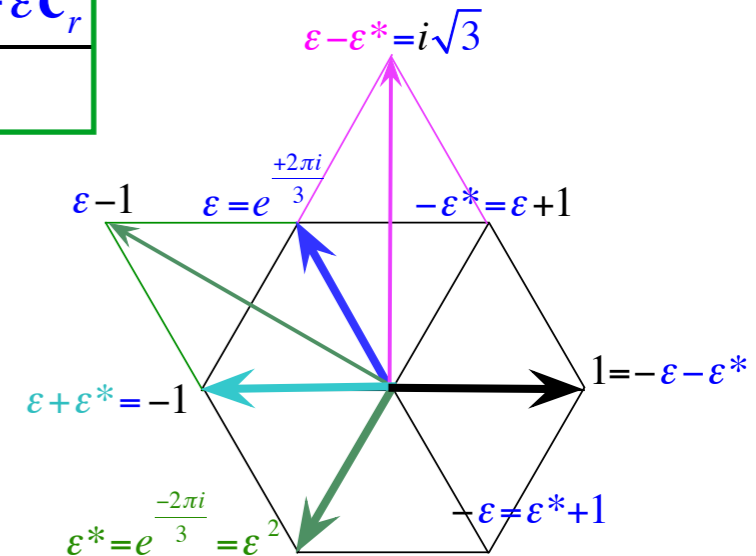
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)}$$

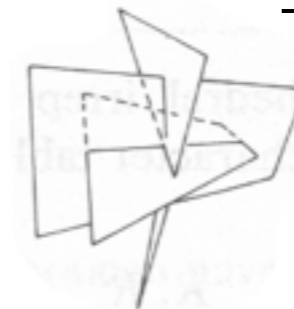
T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$$



| $T : \mathbf{c}_g =$ | \mathbf{c}_1 | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
|----------------------------|----------------|-----------------|------------------------|---------------------|
| $\chi_g^A =$ | 1 | 1 | 1 | 1 |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
| $\chi_g^{\varepsilon^*} =$ | 1 | ε | ε^* | 1 |
| . | | | | |

Tetrahedral T class projectors

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} c_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^* \tilde{\mathbf{c}}_r + \varepsilon \mathbf{c}_r}{12}$$

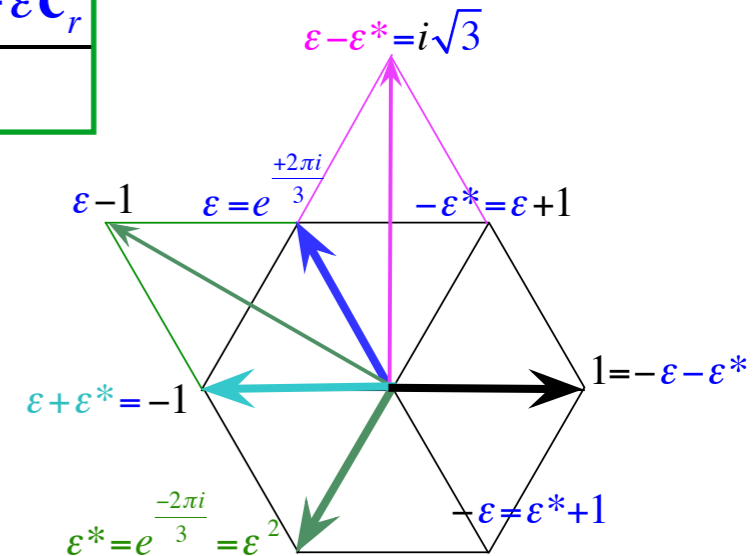
$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

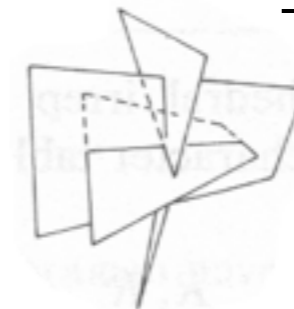
T class products

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}} \mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$$



| $T : \mathbf{c}_g =$ | \mathbf{c}_1 | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
|----------------------------|----------------|-----------------|------------------------|---------------------|
| $\chi_g^A =$ | 1 | 1 | 1 | 1 |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
| $\chi_g^{\varepsilon^*} =$ | 1 | ε | ε^* | 1 |
| . | | | | |

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} c_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^* \mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^* \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^* \tilde{\mathbf{c}}_r + \varepsilon \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r \mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

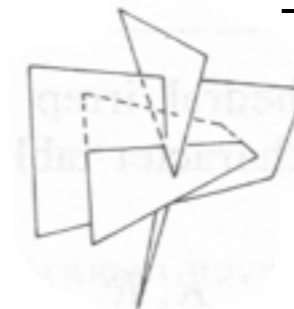
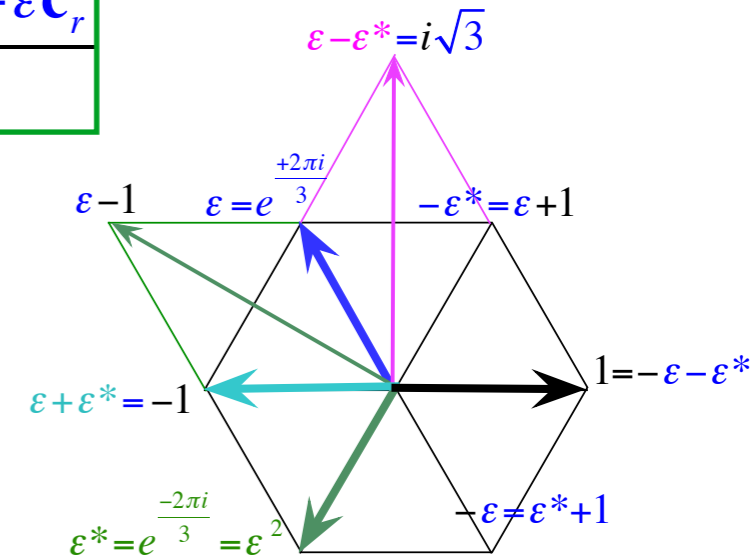
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

Minimal equation for \mathbf{c}_r

$$(\mathbf{c}_r - 4e^{\frac{+2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4e^{\frac{-2\pi i}{3}}\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$$

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



| $T : \mathbf{c}_g =$ | \mathbf{c}_1 | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
|----------------------------|----------------|-----------------|------------------------|---------------------|
| $\chi_g^A =$ | 1 | 1 | 1 | 1 |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
| $\chi_g^{\varepsilon^*} =$ | 1 | ε | ε^* | 1 |
| . | | | | |

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

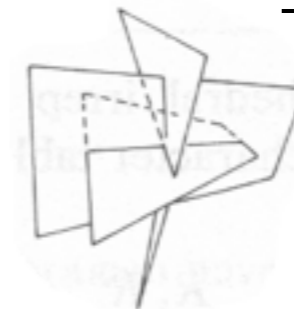
$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64} =$$

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



| $T : \mathbf{c}_g =$ | \mathbf{c}_1 | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
|----------------------------|----------------|-----------------|------------------------|---------------------|
| $\chi_g^A =$ | 1 | 1 | 1 | 1 |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
| $\chi_g^{\varepsilon^*} =$ | 1 | ε | ε^* | 1 |
| . | | | | |

Tetrahedral T class projectors

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G \chi_g^{\mu*}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1 + 1 + 1)} = \frac{\mathbf{1} + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$

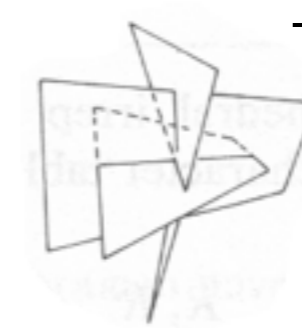
$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64}$$

$$= \frac{4(4\mathbf{1} + 4\mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64}$$

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |



| $T : \mathbf{c}_g =$ | \mathbf{c}_1 | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
|----------------------------|----------------|-----------------|------------------------|---------------------|
| $\chi_g^A =$ | 1 | 1 | 1 | 1 |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
| $\chi_g^{\varepsilon^*} =$ | 1 | ε | ε^* | 1 |
| . | | | | |

Tetrahedral T class characters

T class products

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu} \mathbb{P}^{\mu}}{\ell^{\mu}}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu}} \mathbf{c}_g$$

Minimal equation for \mathbf{c}_r : $(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0) = 0$*

$$\mathbf{P}^{(4\varepsilon)} = \frac{(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon - 4\varepsilon^*)(4\varepsilon - 4)(4\varepsilon - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon^* + 1)\mathbf{c}_r + 16\cdot\varepsilon^*)\mathbf{c}_r}{64(\varepsilon - \varepsilon^*)(\varepsilon - 1)\varepsilon}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4(\varepsilon)4\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(\varepsilon^2 - \varepsilon)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\varepsilon\tilde{\mathbf{c}}_r + 16\cdot\varepsilon^*\mathbf{c}_r}{64i\sqrt{3}(-i\sqrt{3})}$$

$$\mathbf{P}^{(4\varepsilon^*)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})(\mathbf{c}_r - 0)}{(4\varepsilon^* - 4\varepsilon)(4\varepsilon^* - 4)(4\varepsilon^* - 0)}$$

$$\mathbf{P}^{(4)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 0)}{(4 - 4\varepsilon)(4 - 4\varepsilon^*)(4 - 0)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\mathbf{1})\mathbf{c}_r}{64(1 - (\varepsilon + \varepsilon^*) + 1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\cdot 4\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)} = \frac{16(\mathbf{1} + \mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r}{64(1+1+1)}$$

$$\mathbf{P}^{(0)} = \frac{(\mathbf{c}_r - 4\varepsilon\mathbf{1})(\mathbf{c}_r - 4\varepsilon^*\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{(0 - 4\varepsilon)(0 - 4\varepsilon^*)(0 - 4)} = \frac{(\mathbf{c}_r^2 - 4(\varepsilon + \varepsilon^*)\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{64(-\varepsilon)(-\varepsilon^*)(-1)}$$

$$= \frac{4\tilde{\mathbf{c}}_r + 4\mathbf{c}_r + 16\cdot\mathbf{1})(\mathbf{c}_r - 4\cdot\mathbf{1})}{-64}$$

$$= \frac{4\tilde{\mathbf{c}}_r\mathbf{c}_r + 4\mathbf{c}_r^2 + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64}$$

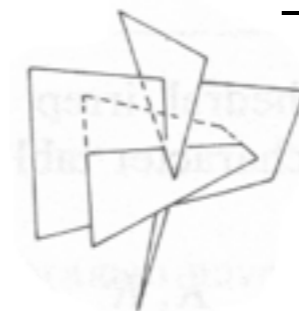
$$= \frac{4(4\mathbf{1} + 4\mathbf{c}_{\rho}) + 16\tilde{\mathbf{c}}_r + 16\mathbf{c}_r - 16\tilde{\mathbf{c}}_r - 16\mathbf{c}_r - 64\cdot\mathbf{1}}{-64} = \frac{-48\cdot\mathbf{1} + 16\mathbf{c}_{\rho}}{-64}$$

| | | | |
|-----------------------------|-------------------------|------------------------------------|------------------------------------|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
| \mathbf{c}_r | $4\tilde{\mathbf{c}}_r$ | $4\mathbf{1} + 4\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ |
| $\tilde{\mathbf{c}}_r$ | | $4\mathbf{c}_r$ | $3\tilde{\mathbf{c}}_r$ |
| \mathbf{c}_{ρ} | | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ |

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon\tilde{\mathbf{c}}_r + \varepsilon^*\mathbf{c}_r}{12}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \varepsilon^*\tilde{\mathbf{c}}_r + \varepsilon\mathbf{c}_r}{12}$$

$$= \frac{\mathbf{1} + \mathbf{c}_{\rho} + \tilde{\mathbf{c}}_r + \mathbf{c}_r}{12}$$



| $T : \mathbf{c}_g =$ | \mathbf{c}_1 | \mathbf{c}_r | $\tilde{\mathbf{c}}_r$ | \mathbf{c}_{ρ} |
|----------------------------|----------------|-----------------|------------------------|---------------------|
| $\chi_g^A =$ | 1 | 1 | 1 | 1 |
| $\chi_g^{\varepsilon} =$ | 1 | ε^* | ε | 1 |
| $\chi_g^{\varepsilon^*} =$ | 1 | ε | ε^* | 1 |
| $\chi_g^T =$ | 3 | 0 | 0 | -1 |

$$= \frac{3}{4}\mathbf{1} - \frac{1}{4}\mathbf{c}_{\rho}$$

Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

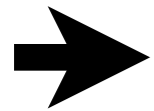
Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

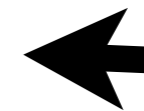
Tetrahedral T class projectors and characters



Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters



Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy

Octahedral O class algebra

$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

$$\mathbf{c}_R = \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_1^3 + \mathbf{R}_2^3 + \mathbf{R}_3^3, \quad \mathbf{c}_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_4 + \mathbf{i}_5 + \mathbf{i}_6$$



O class products

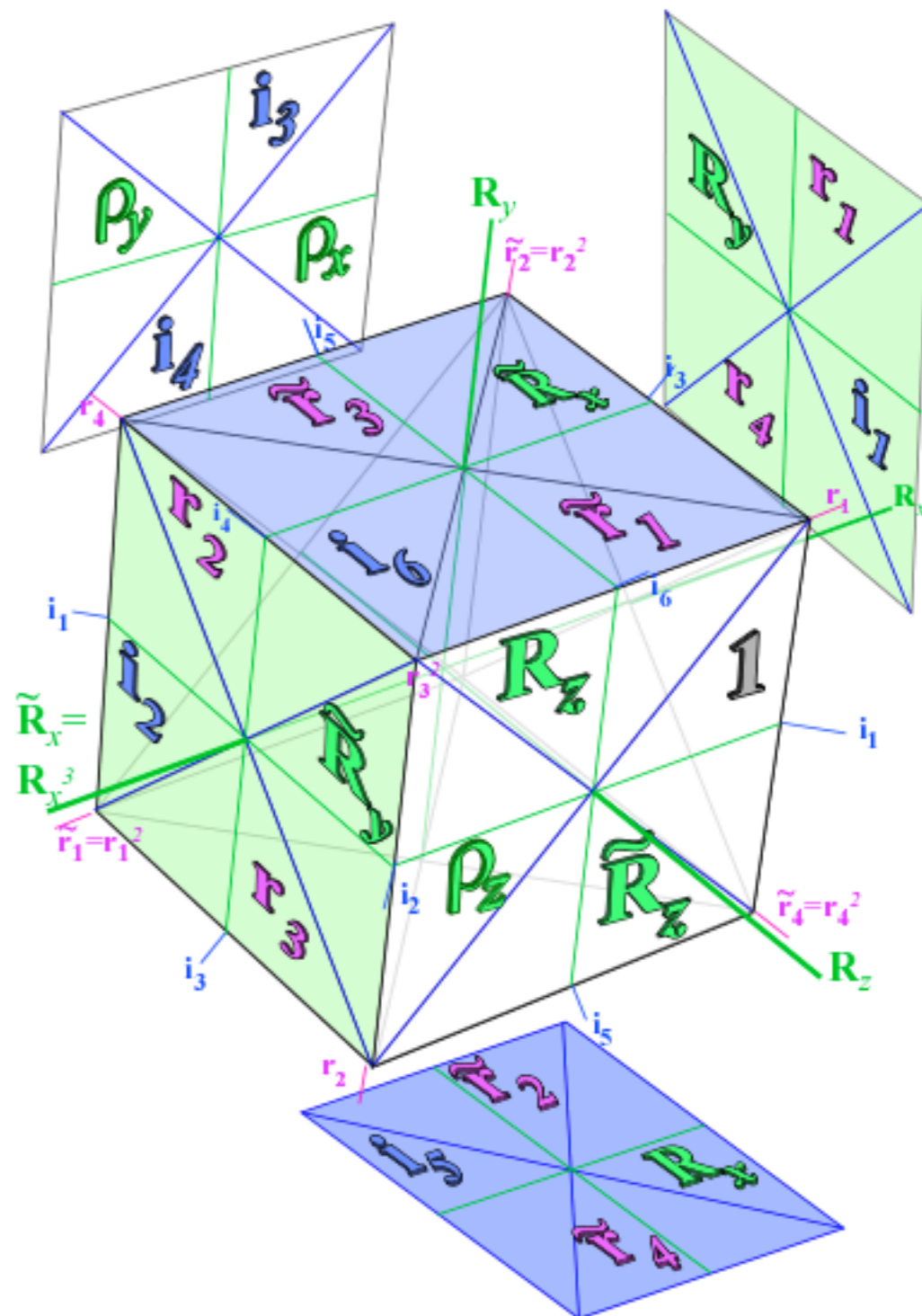
Unnecessary to do $24^2 = 576$ products since each row (or column) of $\mathbf{c}_A \mathbf{c}_B$ has same class proportion

For example:

$$\mathbf{c}_\rho \mathbf{c}_i = \mathbf{R}_1^2 \mathbf{i}_1 + \dots = \mathbf{R}_2 + \dots$$

$$+ \mathbf{R}_2^2 \mathbf{i}_1 + \dots + \mathbf{i}_2 + \dots$$

$$+ \mathbf{R}_3^2 \mathbf{i}_1 + \dots + \mathbf{R}_2^3 + \dots$$



Octahedral O class algebra



$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

$$\mathbf{c}_R = \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_1^3 + \mathbf{R}_2^3 + \mathbf{R}_3^3, \quad \mathbf{c}_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_4 + \mathbf{i}_5 + \mathbf{i}_6$$

O class products

Unnecessary to do $24^2 = 576$ products since each row (or column) of $\mathbf{c}_A \mathbf{c}_B$ has same class proportion

For example:

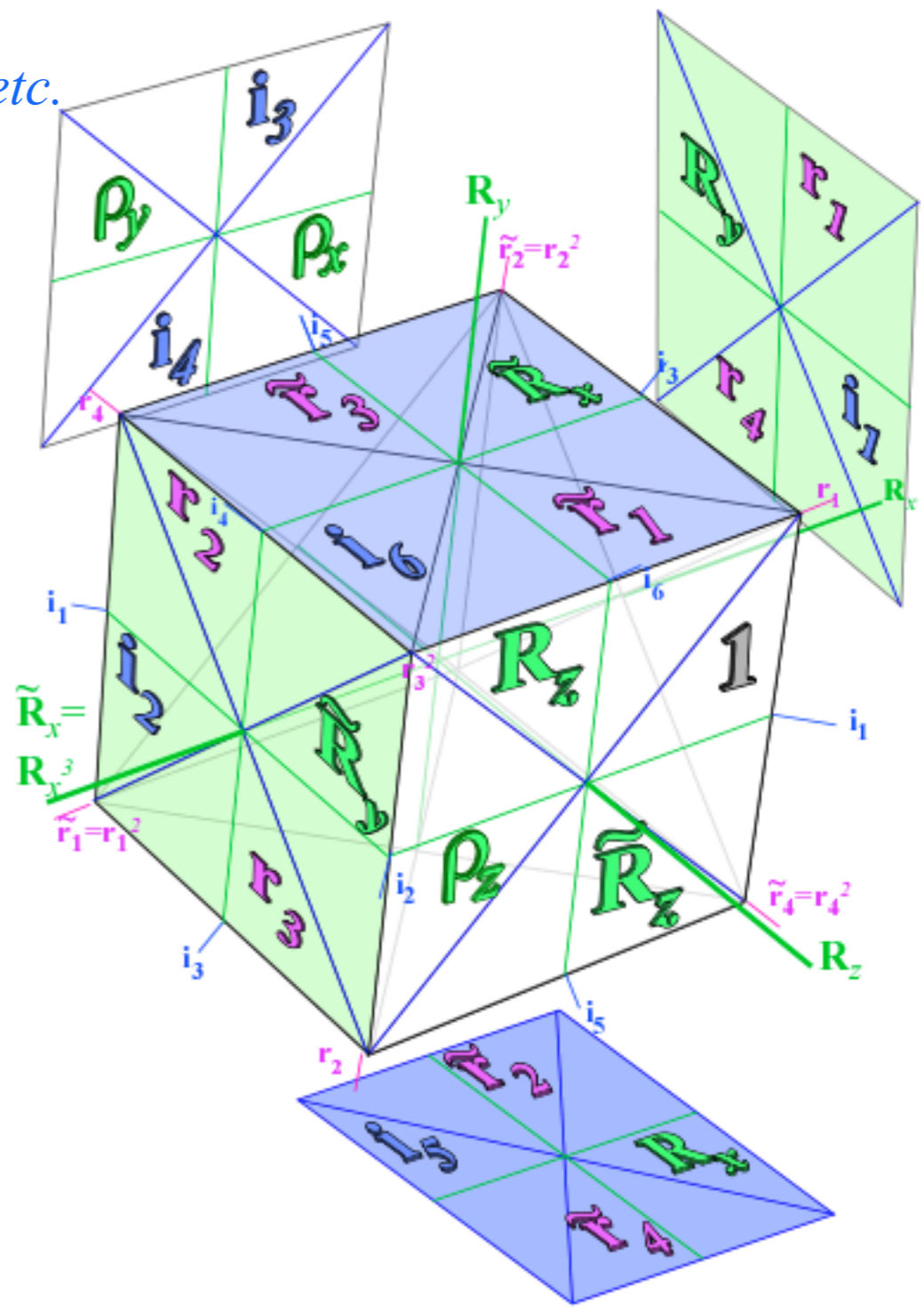
So there are $2\mathbf{c}_R$ for each \mathbf{c}_i :

$$\mathbf{c}_{R^2} \mathbf{c}_i = \mathbf{R}_1^2 \mathbf{i}_1 + \dots = \mathbf{R}_2 + \dots$$

$$+ \mathbf{R}_2^2 \mathbf{i}_1 + \dots + \mathbf{i}_2 + \dots$$

$$+ \mathbf{R}_3^2 \mathbf{i}_1 + \dots + \mathbf{R}_2^3 + \dots$$

$$\mathbf{c}_\rho \mathbf{c}_i = 2\mathbf{c}_R + \mathbf{c}_i \quad \text{or:} \quad 4\mathbf{c}_R + 2\mathbf{c}_i \quad \text{etc.}$$



Octahedral O class algebra



$$\mathbf{c}_I = \mathbf{1}, \quad \mathbf{c}_r = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 + \mathbf{r}_4^2, \quad \mathbf{c}_\rho = \mathbf{R}_1^2 + \mathbf{R}_2^2 + \mathbf{R}_3^2,$$

$$\mathbf{c}_R = \mathbf{R}_1 + \mathbf{R}_2 + \mathbf{R}_3 + \mathbf{R}_1^3 + \mathbf{R}_2^3 + \mathbf{R}_3^3, \quad \mathbf{c}_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_4 + \mathbf{i}_5 + \mathbf{i}_6$$

O class products

Unnecessary to do $24^2 = 576$ products since each row (or column) of $\mathbf{c}_A \mathbf{c}_B$ has same class proportion

For example: $\mathbf{c}_\rho \mathbf{c}_i = ?$ So there are $2\mathbf{c}_R$ for each \mathbf{c}_i in $({}^\circ \mathbf{c}_\rho) \cdot ({}^\circ \mathbf{c}_i) = (3) \cdot (6) = 18$ terms

$$\mathbf{c}_{R^2} \mathbf{c}_i = \mathbf{R}_1^2 \mathbf{i}_1 + \dots = \mathbf{R}_2 + \dots \quad \mathbf{c}_\rho \mathbf{c}_i = 2\mathbf{c}_R + \mathbf{c}_i; \text{ or: } 4\mathbf{c}_R + 2\mathbf{c}_i \text{ etc.}$$

$$+ \mathbf{R}_2^2 \mathbf{i}_1 + \dots + \mathbf{i}_2 + \dots$$

$$+ \mathbf{R}_3^2 \mathbf{i}_1 + \dots + \mathbf{R}_2^3 + \dots$$

$$\text{So: } 2({}^\circ \mathbf{c}_R) + ({}^\circ \mathbf{c}_i) = 2 \cdot 6 + 6 = 18$$

Proof that class proportion cannot vary:

$$\mathbf{c}_g \mathbf{c}_h = \mathbf{g}_1 \mathbf{h}_1 + \mathbf{g}_2 \mathbf{h}_1 + \dots = \mathbf{g}_1 \mathbf{h}_1 + \mathbf{t} \mathbf{g}_1 \mathbf{h}_1 \mathbf{t}^{-1} + \dots = \mathbf{g}_1 \mathbf{h}_1 + \mathbf{t} \mathbf{g}_1 \mathbf{t}^{-1} \mathbf{t} \mathbf{h}_1 \mathbf{t}^{-1} + \dots = \mathbf{g}_1 \mathbf{h}_1 + \mathbf{g}_2 \mathbf{t} \mathbf{h}_1 \mathbf{t}^{-1} + \dots$$

$$+ \mathbf{g}_1 \mathbf{h}_2 + \mathbf{g}_2 \mathbf{h}_2 + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{t} \mathbf{g}_1 \mathbf{h}_2 \mathbf{t}^{-1} + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{t} \mathbf{g}_1 \mathbf{t}^{-1} \mathbf{t} \mathbf{h}_2 \mathbf{t}^{-1} + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{g}_2 \mathbf{t} \mathbf{h}_2 \mathbf{t}^{-1} + \dots$$

$$= \mathbf{g}_1 \mathbf{h}_3 + \mathbf{g}_2 \mathbf{h}_3 + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{t} \mathbf{g}_1 \mathbf{h}_3 \mathbf{t}^{-1} + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{t} \mathbf{g}_1 \mathbf{t}^{-1} \mathbf{t} \mathbf{h}_3 \mathbf{t}^{-1} + \dots + \mathbf{g}_1 \mathbf{h}_2 + \mathbf{g}_2 \mathbf{t} \mathbf{h}_3 \mathbf{t}^{-1} + \dots$$

O class product table

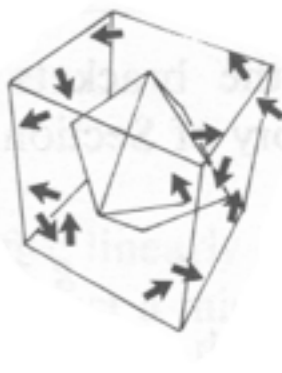
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
|-----------------------------|--|----------------------------------|--|--|
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| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_ρ

$$\mathbf{c}_\rho^2 = 3\mathbf{1} + 2\mathbf{c}_\rho$$

$$\mathbf{c}_\rho^2 - 2\mathbf{c}_\rho - 3\mathbf{1} = \mathbf{0}$$

$$(\mathbf{c}_\rho - 3\mathbf{1})(\mathbf{c}_\rho + \mathbf{1}) = \mathbf{0}$$



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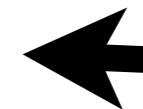
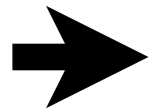
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$O_h \supset O \supset D_4$ subgroup correlations

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Preview of applications to high resolution spectroscopy



Octahedral O class minimal equations



$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} \mathbf{c}_g$$

O class product table

| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_{ρ} | \mathbf{c}_R | \mathbf{c}_i |
|---------------------|--|------------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_{ρ} | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$ | $3\mathbf{c}_r + 4\mathbf{c}_{\rho}$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$ |

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$$

Octahedral O class minimal equations



$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

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Octahedral O class minimal equations



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| \mathbf{c}_{ρ} | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$ | $3\mathbf{c}_r + 4\mathbf{c}_{\rho}$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$ |

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$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

Octahedral O class minimal equations



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Octahedral O class minimal equations



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$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

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$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

$$= 120 \cdot \mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

Octahedral O class minimal equations



$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu} \chi_g^{\mu*}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \chi_g^{\mu} \mathbf{c}_g$$

O class product table

| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
|--------------------|--|----------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

Minimal equation for \mathbf{c}_i

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = 6 \cdot \mathbf{1} \mathbf{c}_i + 3\mathbf{c}_r \mathbf{c}_i + 2\mathbf{c}_\rho \mathbf{c}_i$$

$$= 6\mathbf{c}_i + 3(4\mathbf{c}_R + 4\mathbf{c}_i) + 2(2\mathbf{c}_R + \mathbf{c}_i)$$

$$\mathbf{c}_i^3 = 16\mathbf{c}_R + 20\mathbf{c}_i$$

$$\mathbf{c}_i^4 = 16\mathbf{c}_R \mathbf{c}_i + 20\mathbf{c}_i \mathbf{c}_i$$

$$= 16(3\mathbf{c}_r + 4\mathbf{c}_\rho) + 20(6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho)$$

$$= 48\mathbf{c}_r + 64\mathbf{c}_\rho + 120 \cdot \mathbf{1} + 60\mathbf{c}_r + 40\mathbf{c}_\rho$$

$$= 120 \cdot \mathbf{1} + 108\mathbf{c}_r + 104\mathbf{c}_\rho$$

$$\mathbf{c}_i^5 = 120\mathbf{c}_i + 108\mathbf{c}_r \mathbf{c}_i + 104\mathbf{c}_\rho \mathbf{c}_i$$

$$= 120\mathbf{c}_i + 108(4\mathbf{c}_R + 4\mathbf{c}_i) + 104(2\mathbf{c}_R + \mathbf{c}_i)$$

$$= 640\mathbf{c}_R + 656\mathbf{c}_i$$

Octahedral O class minimal equations



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{|G|} \chi_g^{\mu*} c_g$$

O class product table

| $\mathbf{1} = c_1$ | c_r | c_{ρ} | c_R | c_i |
|--------------------|----------------------------------|---------------------------|----------------------------------|----------------------------------|
| c_r | $8\mathbf{1} + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $3\mathbf{1} + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ |

Minimal equation for c_i

$$c_i^2 = 6 \cdot \mathbf{1} + 3c_r + 2c_{\rho}$$

$$c_i^3 = 6 \cdot \mathbf{1} c_i + 3c_r c_i + 2c_{\rho} c_i$$

$$= 6c_i + 3(4c_R + 4c_i) + 2(2c_R + c_i)$$

$$c_i^3 = 16c_R + 20c_i$$

$$c_i^4 = 16c_R c_i + 20c_i c_i$$

$$= 16(3c_r + 4c_{\rho}) + 20(6 \cdot \mathbf{1} + 3c_r + 2c_{\rho})$$

$$= 48c_r + 64c_{\rho} + 120 \cdot \mathbf{1} + 60c_r + 40c_{\rho}$$

$$= 120 \cdot \mathbf{1} + 108c_r + 104c_{\rho}$$

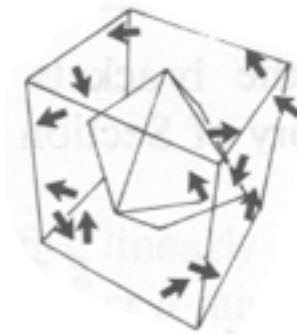
$$c_i^5 = 120c_i + 108c_r c_i + 104c_{\rho} c_i$$

$$= 120c_i + 108(4c_R + 4c_i) + 104(2c_R + c_i)$$

$$= 640c_R + 656c_i$$

$$40c_i^3 = 640c_R + 800c_i$$

Octahedral O class minimal equations



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{|G|} \chi_g^{\mu*} c_g$$

O class product table

| $\mathbf{1} = c_1$ | c_r | c_{ρ} | c_R | c_i |
|--------------------|----------------------------------|---------------------------|----------------------------------|----------------------------------|
| c_r | $8\mathbf{1} + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $3\mathbf{1} + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ |

Minimal equation for c_i

$$c_i^2 = 6 \cdot \mathbf{1} + 3c_r + 2c_{\rho}$$

$$c_i^3 = 6 \cdot \mathbf{1} c_i + 3c_r c_i + 2c_{\rho} c_i$$

$$= 6c_i + 3(4c_R + 4c_i) + 2(2c_R + c_i)$$

$$c_i^3 = 16c_R + 20c_i$$

$$c_i^4 = 16c_R c_i + 20c_i c_i$$

$$= 16(3c_r + 4c_{\rho}) + 20(6 \cdot \mathbf{1} + 3c_r + 2c_{\rho})$$

$$= 48c_r + 64c_{\rho} + 120 \cdot \mathbf{1} + 60c_r + 40c_{\rho}$$

$$= 120 \cdot \mathbf{1} + 108c_r + 104c_{\rho}$$

$$c_i^5 = 120c_i + 108c_r c_i + 104c_{\rho} c_i$$

$$= 120c_i + 108(4c_R + 4c_i) + 104(2c_R + c_i)$$

$$= 640c_R + 656c_i$$

$$40c_i^3 = 640c_R + 800c_i$$

$$c_i^5 - 40c_i^3 + 144c_i = 0$$

800

-656

144

Octahedral O class minimal equations



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{|G|} \chi_g^{\mu*} c_g$$

O class product table

| | | | | |
|--------------------|----------------------------------|---------------------------|----------------------------------|----------------------------------|
| $\mathbf{1} = c_1$ | c_r | c_{ρ} | c_R | c_i |
| c_r | $8\mathbf{1} + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $3\mathbf{1} + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ |

Minimal equation for c_i

$$c_i^2 = 6 \cdot \mathbf{1} + 3c_r + 2c_{\rho}$$

$$c_i^3 = 6 \cdot \mathbf{1} c_i + 3c_r c_i + 2c_{\rho} c_i$$

$$= 6c_i + 3(4c_R + 4c_i) + 2(2c_R + c_i)$$

$$c_i^3 = 16c_R + 20c_i$$

$$c_i^4 = 16c_R c_i + 20c_i c_i$$

$$= 16(3c_r + 4c_{\rho}) + 20(6 \cdot \mathbf{1} + 3c_r + 2c_{\rho})$$

$$= 48c_r + 64c_{\rho} + 120 \cdot \mathbf{1} + 60c_r + 40c_{\rho}$$

$$= 120 \cdot \mathbf{1} + 108c_r + 104c_{\rho}$$

$$c_i^5 = 120c_i + 108c_r c_i + 104c_{\rho} c_i$$

$$= 120c_i + 108(4c_R + 4c_i) + 104(2c_R + c_i)$$

$$= 640c_R + 656c_i$$

$$40c_i^3 = 640c_R + 800c_i \quad \quad \quad 800$$

$$c_i^5 - 40c_i^3 + 144c_i = 0 = (c_i^2 - 36 \cdot \mathbf{1})(c_i^2 - 4 \cdot \mathbf{1})(c_i - 0 \cdot \mathbf{1}) \quad \quad \quad \frac{-656}{144}$$

Octahedral O class minimal equations



$$c_g = \sum_{\mu} \frac{\chi_g^{\mu} \chi_g^{\mu*}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \chi_g^{\mu} c_g$$

O class product table

| | | | | |
|--------------------|----------------------------------|---------------------------|----------------------------------|----------------------------------|
| $\mathbf{1} = c_1$ | c_r | c_{ρ} | c_R | c_i |
| c_r | $8\mathbf{1} + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $3\mathbf{1} + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ |

Minimal equation for c_i

$$c_i^2 = 6 \cdot \mathbf{1} + 3c_r + 2c_{\rho}$$

$$c_i^3 = 6 \cdot \mathbf{1} c_i + 3c_r c_i + 2c_{\rho} c_i$$

$$= 6c_i + 3(4c_R + 4c_i) + 2(2c_R + c_i)$$

$$c_i^3 = 16c_R + 20c_i$$

$$c_i^4 = 16c_R c_i + 20c_i c_i$$

$$= 16(3c_r + 4c_{\rho}) + 20(6 \cdot \mathbf{1} + 3c_r + 2c_{\rho})$$

$$= 48c_r + 64c_{\rho} + 120 \cdot \mathbf{1} + 60c_r + 40c_{\rho}$$

$$= 120 \cdot \mathbf{1} + 108c_r + 104c_{\rho}$$

$$c_i^5 = 120c_i + 108c_r c_i + 104c_{\rho} c_i$$

$$= 120c_i + 108(4c_R + 4c_i) + 104(2c_R + c_i)$$

$$= 640c_R + 656c_i$$

$$40c_i^3 = 640c_R + 800c_i$$

$$c_i^5 - 40c_i^3 + 144c_i = 0 = (c_i^2 - 36 \cdot \mathbf{1})(c_i^2 - 4 \cdot \mathbf{1})(c_i - 0 \cdot \mathbf{1})$$

800

-656

144

Minimal equation for c_i

$$0 = (c_i + 6 \cdot \mathbf{1})(c_i - 6 \cdot \mathbf{1})(c_i + 2 \cdot \mathbf{1})(c_i - 2 \cdot \mathbf{1})(c_i - 0 \cdot \mathbf{1})$$



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

Octahedral $O_h \supset O$ subgroup correlations

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy



Octahedral O projector algebra

Begin with minimal equation:

$$0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$

$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$



$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)}$$

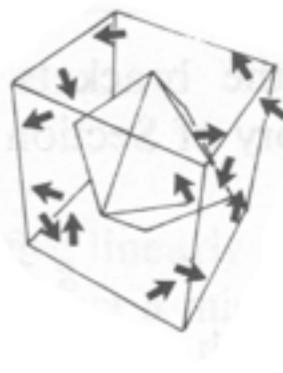
O class product table

| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
|--------------------|--|----------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$



$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256}$$

O class product table

| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_{ρ} | \mathbf{c}_R | \mathbf{c}_i |
|---------------------|--|------------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_{ρ} | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$ | $3\mathbf{c}_r + 4\mathbf{c}_{\rho}$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$



$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2 \cdot \mathbf{c}_i^3 - 36 \mathbf{c}_i^2 - 72 \mathbf{c}_i}{-256}$$

O class product table

| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_{ρ} | \mathbf{c}_R | \mathbf{c}_i |
|---------------------|--|------------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_{\rho}$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_{ρ} | | $3\mathbf{1} + 2\mathbf{c}_{\rho}$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$ | $3\mathbf{c}_r + 4\mathbf{c}_{\rho}$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{|G|} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

$$c_i = \quad + \quad c_i$$

O class product table

| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation: $0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{|G|} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

$$c_i^2 = 6 \cdot 1 + 3c_r + 2c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

O class product table

| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$



$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu} \chi_g^{\mu*}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2 \cdot \mathbf{c}_i^3 - 36 \mathbf{c}_i^2 - 72 \mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$$\begin{aligned} \mathbf{c}_i^3 &= && +16\mathbf{c}_R + 20\mathbf{c}_i \\ \mathbf{c}_i^2 &= 6 \cdot \mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho \\ \mathbf{c}_i &= && + \mathbf{c}_i \end{aligned}$$

O class product table

| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
|-----------------------------|--|----------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

Octahedral O projector algebra

Begin with minimal equation: $0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{|G|} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

O class product table

| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{|G|} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

O class product table

| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation: $0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{c_g} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

$$\begin{aligned} c_i^4 &= 120 \cdot 1 + 108 c_r + 104 c_{\rho} & c_i^4 &= 120 \cdot 1 + 108 c_r + 104 c_{\rho} \\ c_i^3 &= & + 16 c_R + 20 c_i & + 2 c_i^3 = & + 32 c_R + 40 c_i \\ c_i^2 &= 6 \cdot 1 + 3 c_r + 2 c_{\rho} \\ c_i &= & + & c_i \end{aligned}$$

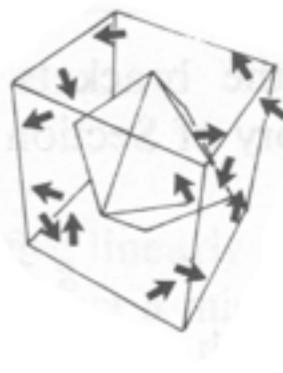
O class product table

| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$$



$$\mathbf{c}_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} \mathbb{P}^{\mu}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2 \cdot \mathbf{c}_i^3 - 36 \mathbf{c}_i^2 - 72 \mathbf{c}_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$$\mathbf{c}_i^4 = 120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = \quad \quad \quad + 16 \mathbf{c}_R + 20 \mathbf{c}_i$$

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3 \mathbf{c}_r + 2 \mathbf{c}_\rho$$

$$\mathbf{c}_i = \quad \quad \quad + \quad \quad \quad \mathbf{c}_i$$

$$\mathbf{c}_i^4 = 120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho$$

$$+ 2 \mathbf{c}_i^3 = \quad \quad \quad + 32 \mathbf{c}_R + 40 \mathbf{c}_i$$

$$- 36 \mathbf{c}_i^2 = -216 \cdot \mathbf{1} - 108 \mathbf{c}_r - 72 \mathbf{c}_\rho$$

O class product table

| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
|--------------------|--|----------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$



$$c_g = \sum_{\mu} \frac{{}^{\circ}c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{{}^{\circ}G} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

| | |
|--|--|
| $c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$ | $c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$ |
| $c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$ | $+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$ |
| $c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$ | $- 36 c_i^2 = - 216 \cdot 1 - 108 c_r - 72 c_{\rho}$ |
| $c_i = \quad \quad \quad + \quad c_i$ | $- 72 c_i = \quad \quad \quad - 72 c_i$ |

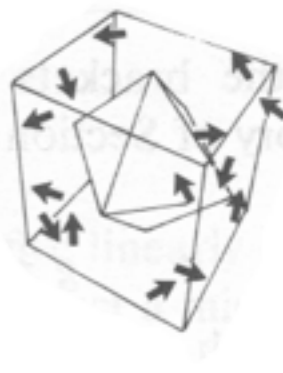
O class product table

| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{c_g} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$$

$$- 36 c_i^2 = -216 \cdot 1 - 108 c_r - 72 c_{\rho}$$

$$- 72 c_i = \quad \quad \quad - 72 c_i$$

$$-256 P^{(2)} = -96 \cdot 1 + 0 c_r + 32 c_{\rho} + 32 c_R - 32 c_i$$

O class product table

| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{|G|} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$$

$$- 36 c_i^2 = -216 \cdot 1 - 108 c_r - 72 c_{\rho}$$

$$- 72 c_i = \quad \quad \quad - 72 c_i$$

$$-256 P^{(2)} = -96 \cdot 1 + 0 c_r + 32 c_{\rho} + 32 c_R - 32 c_i$$

$$P^{(2)} = \frac{3}{8} 1 - \frac{0}{8} c_r + \frac{1}{8} c_{\rho} - \frac{1}{8} c_R + \frac{1}{8} c_i$$

O class product table

| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Octahedral O projector algebra

Begin with minimal equation:

$$0 = (c_i + 2 \cdot 1)(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$$



$$c_g = \sum_{\mu} \frac{c_g \chi_g^{\mu}}{\ell^{\mu}} P^{\mu}$$

$$P^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{c_g} \chi_g^{\mu*} c_g$$

$$P^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $P^{(2)}$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$$

$$- 36 c_i^2 = -216 \cdot 1 - 108 c_r - 72 c_{\rho}$$

$$- 72 c_i = \quad \quad \quad - 72 c_i$$

$$- 256 P^{(2)} = -96 \cdot 1 + 0 c_r + 32 c_{\rho} + 32 c_R - 32 c_i$$

$$P^{(2)} = \frac{3}{8} 1 + \frac{0}{8} c_r - \frac{1}{8} c_{\rho} - \frac{1}{8} c_R + \frac{1}{8} c_i$$

O class product table

| | | | | |
|------------|-------------------------|------------------|-------------------------|-------------------------|
| $1 = c_1$ | c_r | c_{ρ} | c_R | c_i |
| c_r | $81 + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $31 + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $61 + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $61 + 3c_r + 2c_{\rho}$ |

Applying the conventional label T_2 for (2)

| χ_g^{μ} | $g = 1$ | $r_{1...4}$ | ρ_{xyz} | R_{xyz} | $i_{1...6}$ |
|----------------|---------|-------------|--------------|-----------|-------------|
| \cdot | | | | | |
| \cdot | | | | | |
| \cdot | | | | | |
| \cdot | | | | | |
| χ^{T_2} | 3 | 0 | -1 | -1 | 1 |

Octahedral O projector algebra

Begin with minimal equation: $0 = (\cancel{c_i + 2 \cdot 1})(c_i - 2 \cdot 1)(c_i + 6 \cdot 1)(c_i - 6 \cdot 1)(c_i - 0 \cdot 1)$



$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu} \ell^{\mu}}{\ell^{\mu}} \mathbf{P}^{\mu}$$

$$\mathbf{P}^{\mu} = \sum_{\text{classes } c_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \chi_g^{\mu*} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(c_i + 2 \cdot 1)(c_i - 6 \cdot 1)(c_i + 6 \cdot 1)(c_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(c_i + 2 \cdot 1)(c_i^2 - 36 \cdot 1)c_i}{-256} = \frac{c_i^4 + 2 \cdot c_i^3 - 36 c_i^2 - 72 c_i}{-256}$$

Expanding $\mathbf{P}^{(2)}$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$c_i^3 = \quad \quad \quad + 16 c_R + 20 c_i$$

$$c_i^2 = 6 \cdot 1 + 3 c_r + 2 c_{\rho}$$

$$c_i = \quad \quad \quad + \quad \quad c_i$$

$$c_i^4 = 120 \cdot 1 + 108 c_r + 104 c_{\rho}$$

$$+ 2 c_i^3 = \quad \quad \quad + 32 c_R + 40 c_i$$

$$- 36 c_i^2 = -216 \cdot 1 - 108 c_r - 72 c_{\rho}$$

$$- 72 c_i = \quad \quad \quad - 72 c_i$$

$$-256 \mathbf{P}^{(2)} = -96 \cdot 1 + 0 c_r + 32 c_{\rho} + 32 c_R - 32 c_i$$

$$\mathbf{P}^{(2)} = \frac{3}{8} \mathbf{1} + \frac{0}{8} c_r - \frac{1}{8} c_{\rho} - \frac{1}{8} c_R + \frac{1}{8} c_i$$

$$\mathbf{P}^{(-2)} = \frac{3}{8} \mathbf{1} + \frac{0}{8} c_r - \frac{1}{8} c_{\rho} + \frac{1}{8} c_R - \frac{1}{8} c_i$$

Expansion of $\mathbf{P}^{(-2)}$ has (-) sign on last 2 terms...

O class product table

Octahedral O characters

| | | | | |
|--------------------|----------------------------------|---------------------------|----------------------------------|----------------------------------|
| $\mathbf{1} = c_1$ | c_r | c_{ρ} | c_R | c_i |
| c_r | $8\mathbf{1} + 4c_r + 8c_{\rho}$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_{ρ} | | $3\mathbf{1} + 2c_{\rho}$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ | $3c_r + 4c_{\rho}$ |
| c_i | | | | $6\mathbf{1} + 3c_r + 2c_{\rho}$ |

Applying the conventional label T_2 for (2) and T_1 for (-2)

| χ_g^{μ} | $\mathbf{g} = \mathbf{1}$ | $\mathbf{r}_{1...4}$ | $\mathbf{\rho}_{xyz}$ | \mathbf{R}_{xyz} | $\mathbf{i}_{1...6}$ |
|----------------|---------------------------|----------------------|-----------------------|--------------------|----------------------|
| \cdot | | | | | |
| \cdot | | | | | |
| \cdot | | | | | |
| χ^{T_1} | 3 | 0 | -1 | 1 | -1 |
| χ^{T_2} | 3 | 0 | -1 | -1 | 1 |

Octahedral O projector algebra

Begin with minimal equation: $0 = (\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 2 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i - 0 \cdot \mathbf{1})$



$$\mathbf{c}_g = \sum_{\mu} \frac{\chi_g^{\mu} \mathbb{P}^{\mu}}{\ell^{\mu}}$$

$$\mathbb{P}^{\mu} = \sum_{\text{classes } \mathbf{c}_g} \frac{\ell^{\mu}}{\chi_g^{\mu*}} \mathbf{c}_g$$

$$\mathbf{P}^{(2)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i - 6 \cdot \mathbf{1})(\mathbf{c}_i + 6 \cdot \mathbf{1})(\mathbf{c}_i - 0)}{(2 + 2)(2 - 6)(2 + 6)(2 - 0)} = \frac{(\mathbf{c}_i + 2 \cdot \mathbf{1})(\mathbf{c}_i^2 - 36 \cdot \mathbf{1})\mathbf{c}_i}{-256} = \frac{\mathbf{c}_i^4 + 2 \cdot \mathbf{c}_i^3 - 36 \mathbf{c}_i^2 - 72 \mathbf{c}_i}{-256}$$

$$\mathbf{c}_i^4 = 120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho$$

$$\mathbf{c}_i^3 = \quad \quad \quad + 16 \mathbf{c}_R + 20 \mathbf{c}_i$$

$$\mathbf{c}_i^2 = 6 \cdot \mathbf{1} + 3 \mathbf{c}_r + 2 \mathbf{c}_\rho$$

$$\mathbf{c}_i = \quad \quad \quad + \quad \quad \quad \mathbf{c}_i$$

$$\mathbf{c}_i^4 = 120 \cdot \mathbf{1} + 108 \mathbf{c}_r + 104 \mathbf{c}_\rho$$

$$+ 2 \mathbf{c}_i^3 = \quad \quad \quad + 32 \mathbf{c}_R + 40 \mathbf{c}_i$$

$$- 36 \mathbf{c}_i^2 = -216 \cdot \mathbf{1} - 108 \mathbf{c}_r - 72 \mathbf{c}_\rho$$

$$- 72 \mathbf{c}_i = \quad \quad \quad - 72 \mathbf{c}_i$$

$$-256 \mathbf{P}^{(2)} = -96 \cdot \mathbf{1} + 0 \mathbf{c}_r + 32 \mathbf{c}_\rho + 32 \mathbf{c}_R - 32 \mathbf{c}_i$$

$$\mathbf{P}^{(2)} = \frac{3}{8} \mathbf{1} + \frac{0}{8} \mathbf{c}_r - \frac{1}{8} \mathbf{c}_\rho - \frac{1}{8} \mathbf{c}_R + \frac{1}{8} \mathbf{c}_i$$

$$\mathbf{P}^{(-2)} = \frac{3}{8} \mathbf{1} + \frac{0}{8} \mathbf{c}_r - \frac{1}{8} \mathbf{c}_\rho + \frac{1}{8} \mathbf{c}_R - \frac{1}{8} \mathbf{c}_i$$

Expansion of $\mathbf{P}^{(-2)}$ has (-) sign on last 2 terms...

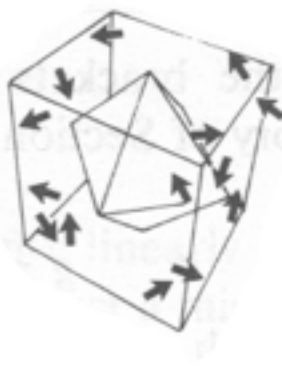
O class product table

Octahedral O characters

| | | | | |
|-----------------------------|--|----------------------------------|--|--|
| $\mathbf{1} = \mathbf{c}_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

(Remaining character derivations left as an exercise)

| χ_g^{μ} | $\mathbf{g} = \mathbf{1}$ | $\mathbf{r}_{1..4}$ | ρ_{xyz} | \mathbf{R}_{xyz} | $\mathbf{i}_{1..6}$ |
|----------------|---------------------------|---------------------|--------------|--------------------|---------------------|
| χ^{A_1} | 1 | 1 | 1 | 1 | 1 |
| χ^{A_2} | 1 | 1 | 1 | -1 | -1 |
| χ^E | 2 | -1 | 2 | 0 | 0 |
| χ^{T_1} | 3 | 0 | -1 | 1 | -1 |
| χ^{T_2} | 3 | 0 | -1 | -1 | 1 |



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

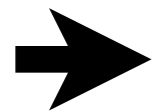
Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters



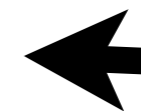
Octahedral $O_h \supset O$: Inversion (g&u) parity

Octahedral $O_h \supset O \supset C_{\infty}$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

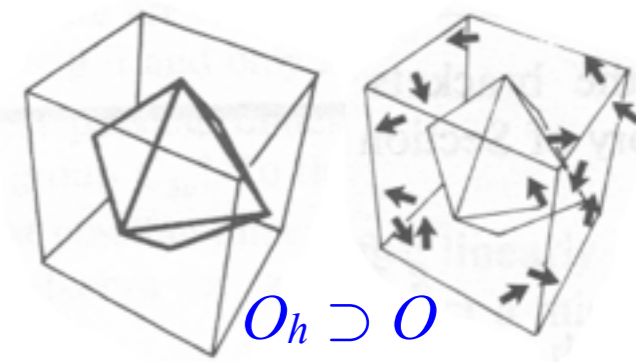
Preview of applications to high resolution spectroscopy



Octahedral $O_h = O \times \{1, \mathbf{I}\}$ characters of $O \times C_I \supset O$

EVEN
parity
(gerade)

| χ_g^μ | $\mathbf{g} = \mathbf{1}$ | $\mathbf{r}_{1..4}$ | ρ_{xyz} | \mathbf{R}_{xyz} | $\mathbf{i}_{1..6}$ |
|-----------------|---------------------------|---------------------|--------------|--------------------|---------------------|
| $\chi^{A_{1g}}$ | 1 | 1 | 1 | 1 | 1 |
| $\chi^{A_{2g}}$ | 1 | 1 | 1 | -1 | -1 |
| χ^{E_g} | 2 | -1 | 2 | 0 | 0 |
| $\chi^{T_{1g}}$ | 3 | 0 | -1 | 1 | -1 |
| $\chi^{T_{2g}}$ | 3 | 0 | -1 | -1 | 1 |



$O_h \supset O$
symmetry

3D – Inversion

$$\langle \mathbf{I} \rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

C_I -symmetry

| | |
|--------------|--------------|
| 1 | \mathbf{I} |
| \mathbf{I} | 1 |

C_I -characters

| C_I | 1 | \mathbf{I} | \pm Parity P (gerade) |
|-------|---|--------------|-------------------------------|
| g | 1 | 1 | (gerade) |
| u | 1 | -1 | (ungerade) |

O class product table

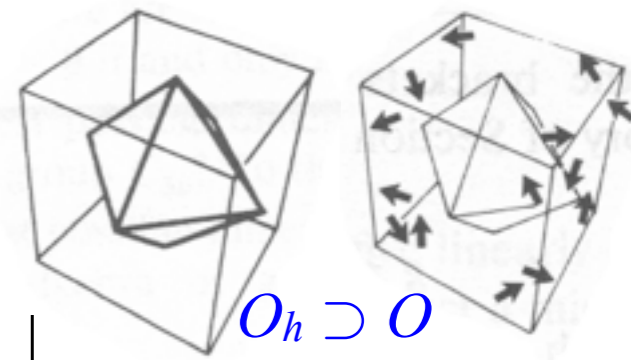
Octahedral O characters

| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
|--------------------|--|----------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

(Remaining
character
derivations
left as an
exercise)

| χ_g^μ | $\mathbf{g} = \mathbf{1}$ | $\mathbf{r}_{1..4}$ | ρ_{xyz} | \mathbf{R}_{xyz} | $\mathbf{i}_{1..6}$ |
|--------------|---------------------------|---------------------|--------------|--------------------|---------------------|
| χ^{A_1} | 1 | 1 | 1 | 1 | 1 |
| χ^{A_2} | 1 | 1 | 1 | -1 | -1 |
| χ^E | 2 | -1 | 2 | 0 | 0 |
| χ^{T_1} | 3 | 0 | -1 | 1 | -1 |
| χ^{T_2} | 3 | 0 | -1 | -1 | 1 |

Octahedral $O_h = O \times \{1, \mathbf{I}\}$ characters of $O \times C_I \supset O$



$O_h \supset O$
symmetry

EVEN
parity
(gerade)

| χ_g^μ | $\mathbf{g} = \mathbf{1}$ | $\mathbf{r}_{1..4}$ | ρ_{xyz} | \mathbf{R}_{xyz} | $\mathbf{i}_{1..6}$ | $\mathbf{g} = \mathbf{I}$ | $\mathbf{I}\mathbf{r}_{1..4}$ | $\mathbf{I}\rho_{xyz}$ | $\mathbf{I}\mathbf{R}_{xyz}$ | $\mathbf{I}\mathbf{i}_{1..6}$ |
|-----------------|---------------------------|---------------------|--------------|--------------------|---------------------|---------------------------|-------------------------------|------------------------|------------------------------|-------------------------------|
| $\chi^{A_{1g}}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^{A_{2g}}$ | 1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | -1 | -1 |
| χ^{E_g} | 2 | -1 | 2 | 0 | 0 | 2 | -1 | 2 | 0 | 0 |
| $\chi^{T_{1g}}$ | 3 | 0 | -1 | 1 | -1 | 3 | 0 | -1 | 1 | -1 |
| $\chi^{T_{2g}}$ | 3 | 0 | -1 | -1 | 1 | 3 | 0 | -1 | -1 | 1 |

3D – Inversion

$$\langle \mathbf{I} \rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

C_I -symmetry

| | |
|--------------|--------------|
| 1 | \mathbf{I} |
| \mathbf{I} | 1 |

C_I -characters

| C_I | 1 | \mathbf{I} | \pm Parity P (gerade) (ungerade) |
|-------|---|--------------|---|
| g | 1 | 1 | (gerade) |
| u | 1 | -1 | (ungerade) |

O class product table

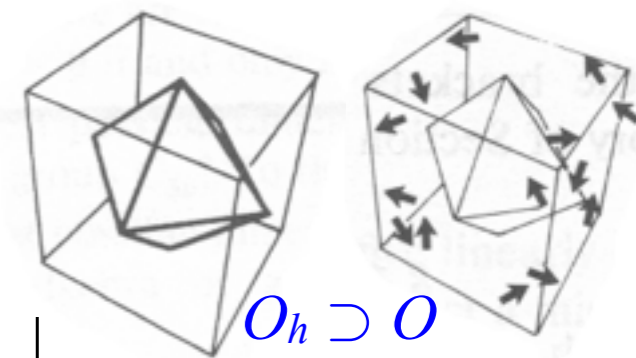
Octahedral O characters

| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
|--------------------|--|----------------------------------|--|--|
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

(Remaining
character
derivations
left as an
exercise)

| χ_g^μ | $\mathbf{g} = \mathbf{1}$ | $\mathbf{r}_{1..4}$ | ρ_{xyz} | \mathbf{R}_{xyz} | $\mathbf{i}_{1..6}$ |
|--------------|---------------------------|---------------------|--------------|--------------------|---------------------|
| χ^{A_1} | 1 | 1 | 1 | 1 | 1 |
| χ^{A_2} | 1 | 1 | 1 | -1 | -1 |
| χ^E | 2 | -1 | 2 | 0 | 0 |
| χ^{T_1} | 3 | 0 | -1 | 1 | -1 |
| χ^{T_2} | 3 | 0 | -1 | -1 | 1 |

Octahedral $O_h = O \times \{1, \mathbf{I}\}$ characters of $O \times C_I \supset O$



$O_h \supset O$
symmetry

3D – Inversion

$$\langle \mathbf{I} \rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

C_I -symmetry

$$\begin{matrix} 1 & \mathbf{I} \\ \mathbf{I} & 1 \end{matrix}$$

C_I -characters

| | | | |
|-------|---|--------------|-------------------|
| C_I | 1 | \mathbf{I} | \pm |
| g | 1 | 1 | Parity P (gerade) |
| u | 1 | -1 | (ungerade) |

EVEN
parity
(gerade)

A_{1g}

| χ_g^μ | $g = 1$ | $r_{1..4}$ | ρ_{xyz} | R_{xyz} | $i_{1..6}$ | $g = \mathbf{I}$ | $\mathbf{I}r_{1..4}$ | $\mathbf{I}\rho_{xyz}$ | $\mathbf{I}R_{xyz}$ | $\mathbf{I}i_{1..6}$ |
|-----------------|---------|------------|--------------|-----------|------------|------------------|----------------------|------------------------|---------------------|----------------------|
| $\chi^{A_{1g}}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^{A_{2g}}$ | 1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | -1 | -1 |
| χ^{E_g} | 2 | -1 | 2 | 0 | 0 | 2 | -1 | 2 | 0 | 0 |
| $\chi^{T_{1g}}$ | 3 | 0 | -1 | 1 | -1 | 3 | 0 | -1 | 1 | -1 |
| $\chi^{T_{2g}}$ | 3 | 0 | -1 | -1 | 1 | 3 | 0 | -1 | -1 | 1 |

ODD
parity
(ungerade)

A_{1u}

| | | | | | | | | | | |
|-----------------|---|----|----|----|----|--|--|--|--|--|
| $\chi^{A_{1u}}$ | 1 | 1 | 1 | 1 | 1 | | | | | |
| $\chi^{A_{2u}}$ | 1 | 1 | 1 | -1 | -1 | | | | | |
| χ^{E_u} | 2 | -1 | 2 | 0 | 0 | | | | | |
| $\chi^{T_{1u}}$ | 3 | 0 | -1 | 1 | -1 | | | | | |
| $\chi^{T_{2u}}$ | 3 | 0 | -1 | -1 | 1 | | | | | |

O class product table

Octahedral O characters

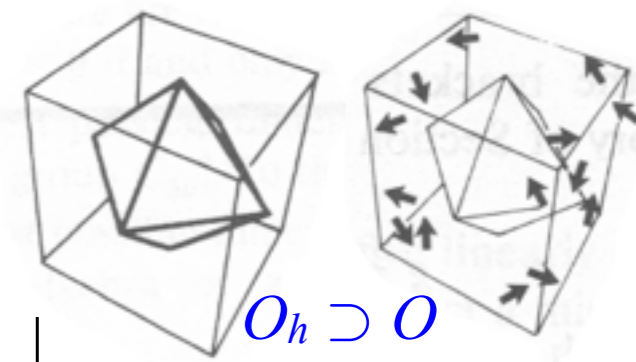
| $\mathbf{1} = c_1$ | c_r | c_ρ | c_R | c_i |
|--------------------|--------------------------------|-------------------------|--------------------------------|--------------------------------|
| c_r | $8\mathbf{1} + 4c_r + 8c_\rho$ | $3c_r$ | $4c_R + 4c_i$ | $4c_R + 4c_i$ |
| c_ρ | | $3\mathbf{1} + 2c_\rho$ | $c_R + 2c_i$ | $2c_R + c_i$ |
| c_R | | | $6\mathbf{1} + 3c_r + 2c_\rho$ | $3c_r + 4c_\rho$ |
| c_i | | | | $6\mathbf{1} + 3c_r + 2c_\rho$ |

(Remaining character derivations left as an exercise)

| χ_g^μ | $g = 1$ | $r_{1..4}$ | ρ_{xyz} | R_{xyz} | $i_{1..6}$ |
|--------------|---------|------------|--------------|-----------|------------|
| χ^{A_1} | 1 | 1 | 1 | 1 | 1 |
| χ^{A_2} | 1 | 1 | 1 | -1 | -1 |
| χ^E | 2 | -1 | 2 | 0 | 0 |
| χ^{T_1} | 3 | 0 | -1 | 1 | -1 |
| χ^{T_2} | 3 | 0 | -1 | -1 | 1 |

Octahedral $O_h = O \times \{1, \mathbf{I}\}$ characters of $O \times C_I \supset O$

O_h easily derived from those of O and C_I !



$O_h \supset O$
symmetry

3D – Inversion

$$\langle \mathbf{I} \rangle = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

C_I -symmetry

$$\begin{pmatrix} 1 & \mathbf{I} \\ \mathbf{I} & 1 \end{pmatrix}$$

C_I -characters

| | | | |
|-------|---|--------------|-------------------|
| C_I | 1 | \mathbf{I} | \pm |
| g | 1 | 1 | Parity P (gerade) |
| u | 1 | -1 | (ungerade) |

EVEN
parity
(gerade)

A_{1g}

$\chi^{A_{1g}}$

1 1 1 1 1

1 1 1 1 1

A_{2g}

$\chi^{A_{2g}}$

1 1 1 -1 -1

1 1 1 -1 -1

E_g

χ^{E_g}

2 -1 2 0 0

2 -1 2 0 0

T_{1g}

$\chi^{T_{1g}}$

3 0 -1 1 -1

3 0 -1 1 -1

T_{2g}

$\chi^{T_{2g}}$

3 0 -1 -1 1

3 0 -1 -1 1

ODD
parity
(ungerade)

A_{1u}

$\chi^{A_{1u}}$

1 1 1 1 1

-1 -1 -1 -1 -1

A_{2u}

$\chi^{A_{2u}}$

1 1 1 -1 -1

-1 -1 -1 +1 +1

E_u

χ^{E_u}

2 -1 2 0 0

-2 +1 -2 0 0

T_{1u}

$\chi^{T_{1u}}$

3 0 -1 1 -1

-3 0 +1 -1 +1

T_{2u}

$\chi^{T_{2u}}$

3 0 -1 -1 1

-3 0 +1 +1 -1

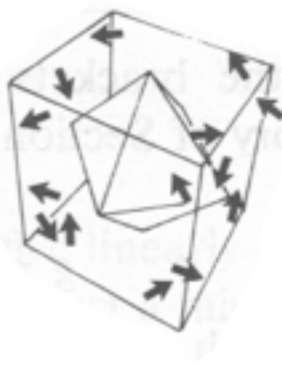
O class product table

Octahedral O characters

| | | | | |
|--------------------|--|----------------------------------|--|--|
| $\mathbf{1} = c_1$ | \mathbf{c}_r | \mathbf{c}_ρ | \mathbf{c}_R | \mathbf{c}_i |
| \mathbf{c}_r | $8\mathbf{1} + 4\mathbf{c}_r + 8\mathbf{c}_\rho$ | $3\mathbf{c}_r$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ | $4\mathbf{c}_R + 4\mathbf{c}_i$ |
| \mathbf{c}_ρ | | $3\mathbf{1} + 2\mathbf{c}_\rho$ | $\mathbf{c}_R + 2\mathbf{c}_i$ | $2\mathbf{c}_R + \mathbf{c}_i$ |
| \mathbf{c}_R | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ | $3\mathbf{c}_r + 4\mathbf{c}_\rho$ |
| \mathbf{c}_i | | | | $6\mathbf{1} + 3\mathbf{c}_r + 2\mathbf{c}_\rho$ |

(Remaining character derivations left as an exercise)

| | | | | | |
|--------------|---------------------------|---------------------|--------------|--------------------|---------------------|
| χ_g^μ | $\mathbf{g} = \mathbf{1}$ | $\mathbf{r}_{1..4}$ | ρ_{xyz} | \mathbf{R}_{xyz} | $\mathbf{i}_{1..6}$ |
| χ^{A_1} | 1 | 1 | 1 | 1 | 1 |
| χ^{A_2} | 1 | 1 | 1 | -1 | -1 |
| χ^E | 2 | -1 | 2 | 0 | 0 |
| χ^{T_1} | 3 | 0 | -1 | 1 | -1 |
| χ^{T_2} | 3 | 0 | -1 | -1 | 1 |



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

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Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

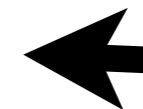
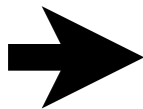
Octahedral $O_h \supset O$: Inversion (g&u) parity

Octahedral $O_h \supset O \supset C_i$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

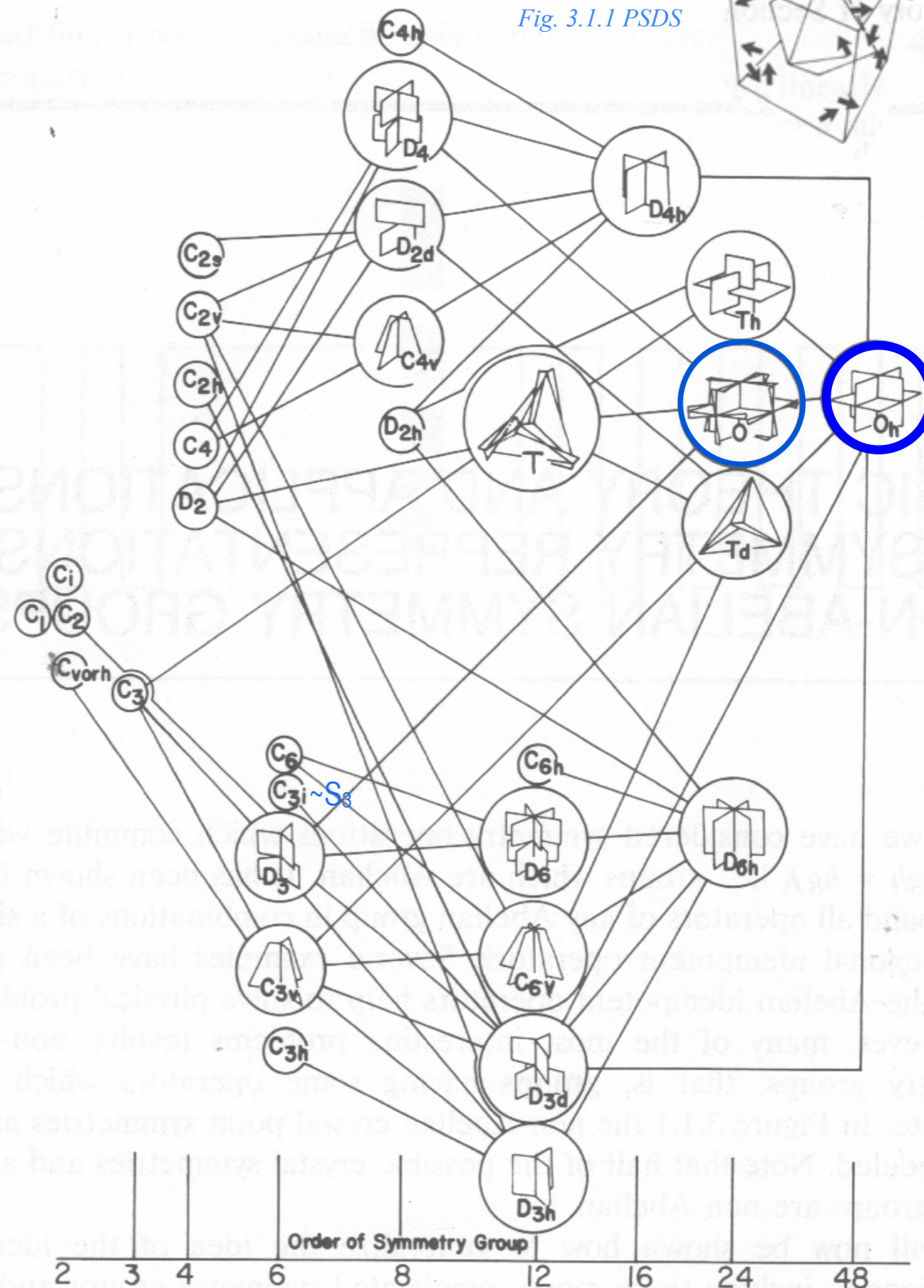
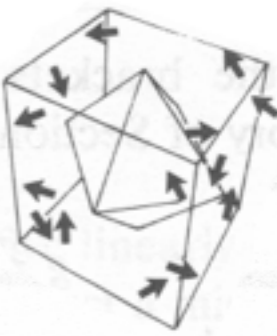
Preview of applications to high resolution spectroscopy



Octahedral $O_h \supset O$ subgroup correlations

| χ_g^μ | $g = 1$ | $r_{1...4}$ | ρ_{xyz} | R_{xyz} | $i_{1...6}$ |
|--------------|---------|-------------|--------------|-----------|-------------|
| χ^{A_1} | 1 | 1 | 1 | 1 | 1 |
| χ^{A_2} | 1 | 1 | 1 | -1 | -1 |
| χ^E | 2 | -1 | 2 | 0 | 0 |
| χ^{T_1} | 3 | 0 | -1 | 1 | -1 |
| χ^{T_2} | 3 | 0 | -1 | -1 | 1 |

Fig. 3.1.1 PSDS

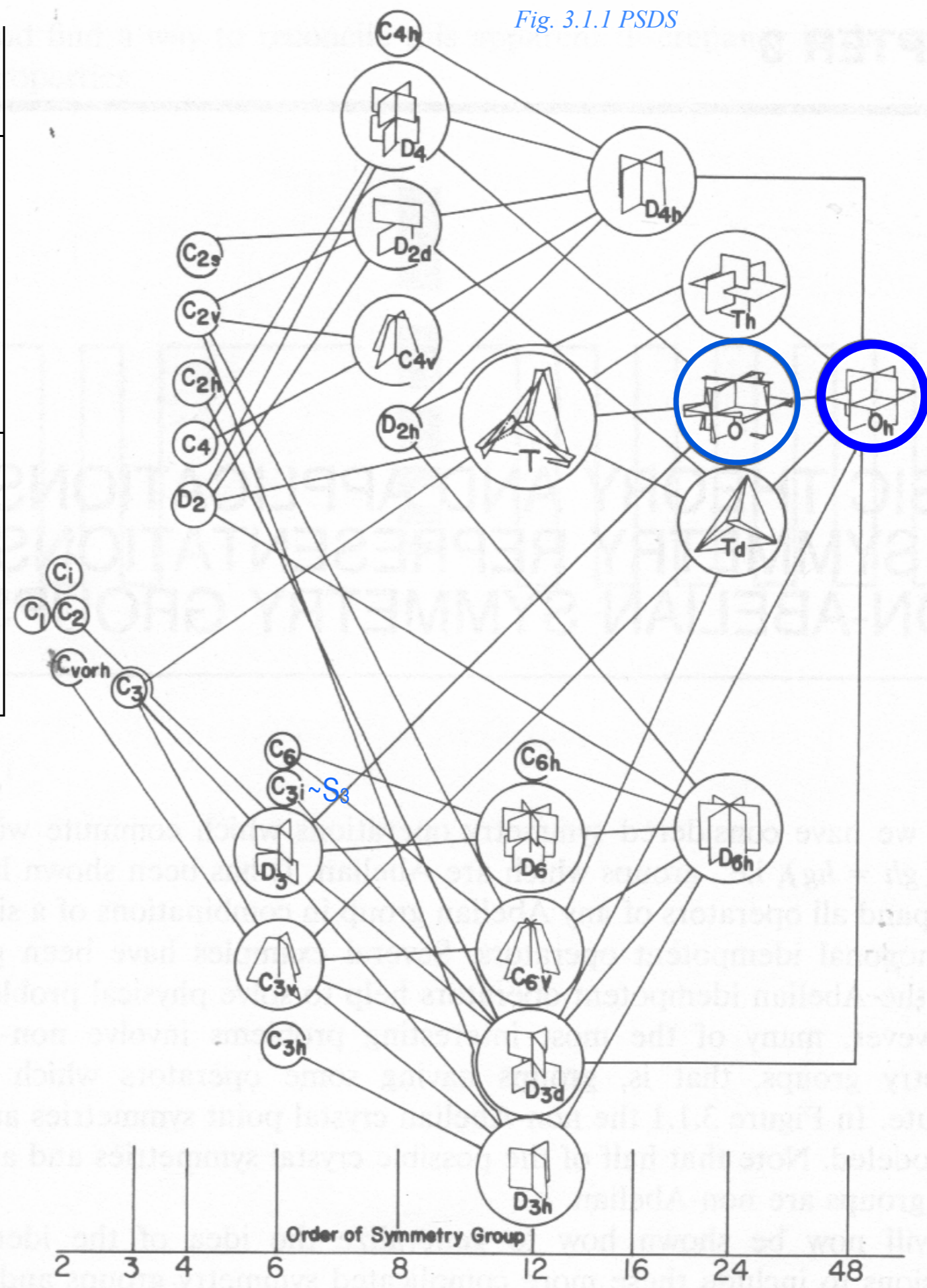


Octahedral $O_h \supset O$ subgroup correlations

Fig. 3.1.1 PSDS

| $\chi_g^{\mu_p}$ | 1 | r _{1...4} | ρ _{xyz} | R _{xyz} | i _{1...6} | I | Ir= s _{1...4} | Ip= σ _{xyz} | IR= S _{xyz} | Ii= σ _{1...6} |
|------------------|----------|---------------------------|-------------------------|-------------------------|---------------------------|----------|----------------------------------|--------------------------------|--------------------------------|----------------------------------|
| $\chi^{A_{1g}}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^{A_{2g}}$ | 1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | -1 | -1 |
| χ^{E_g} | 2 | -1 | 2 | 0 | 0 | 2 | -1 | 2 | 0 | 0 |
| $\chi^{T_{1g}}$ | 3 | 0 | -1 | 1 | -1 | 3 | 0 | -1 | 1 | -1 |
| $\chi^{T_{2g}}$ | 3 | 0 | -1 | -1 | 1 | 3 | 0 | -1 | -1 | 1 |
| $\chi^{A_{1u}}$ | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 |
| $\chi^{A_{2u}}$ | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 |
| χ^{E_u} | 2 | -1 | 2 | 0 | 0 | -2 | 1 | -2 | 0 | 0 |
| $\chi^{T_{1u}}$ | 3 | 0 | -1 | 1 | -1 | -3 | 0 | 1 | -1 | 1 |
| $\chi^{T_{2u}}$ | 3 | 0 | -1 | -1 | 1 | -3 | 0 | 1 | 1 | -1 |

| $O_h \supset O$ | A_1 | A_2 | E | T_1 | T_2 |
|-----------------|-------|-------|-----|-------|-------|
| A_{1g} | 1 | · | · | · | · |
| A_{2g} | · | 1 | · | · | · |
| E_g | · | · | 1 | · | · |
| T_{1g} | · | · | · | 1 | · |
| T_{2g} | · | · | · | · | 1 |
| A_{1u} | 1 | · | · | · | · |
| A_{2u} | · | 1 | · | · | · |
| E_u | · | · | 1 | · | · |
| T_{1u} | · | · | · | 1 | · |
| T_{2u} | · | · | · | · | 1 |



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Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

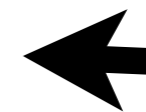
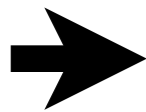
Octahedral $O_h \supset O$ subgroup correlations (Parity)

Octahedral $O_h \supset O$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

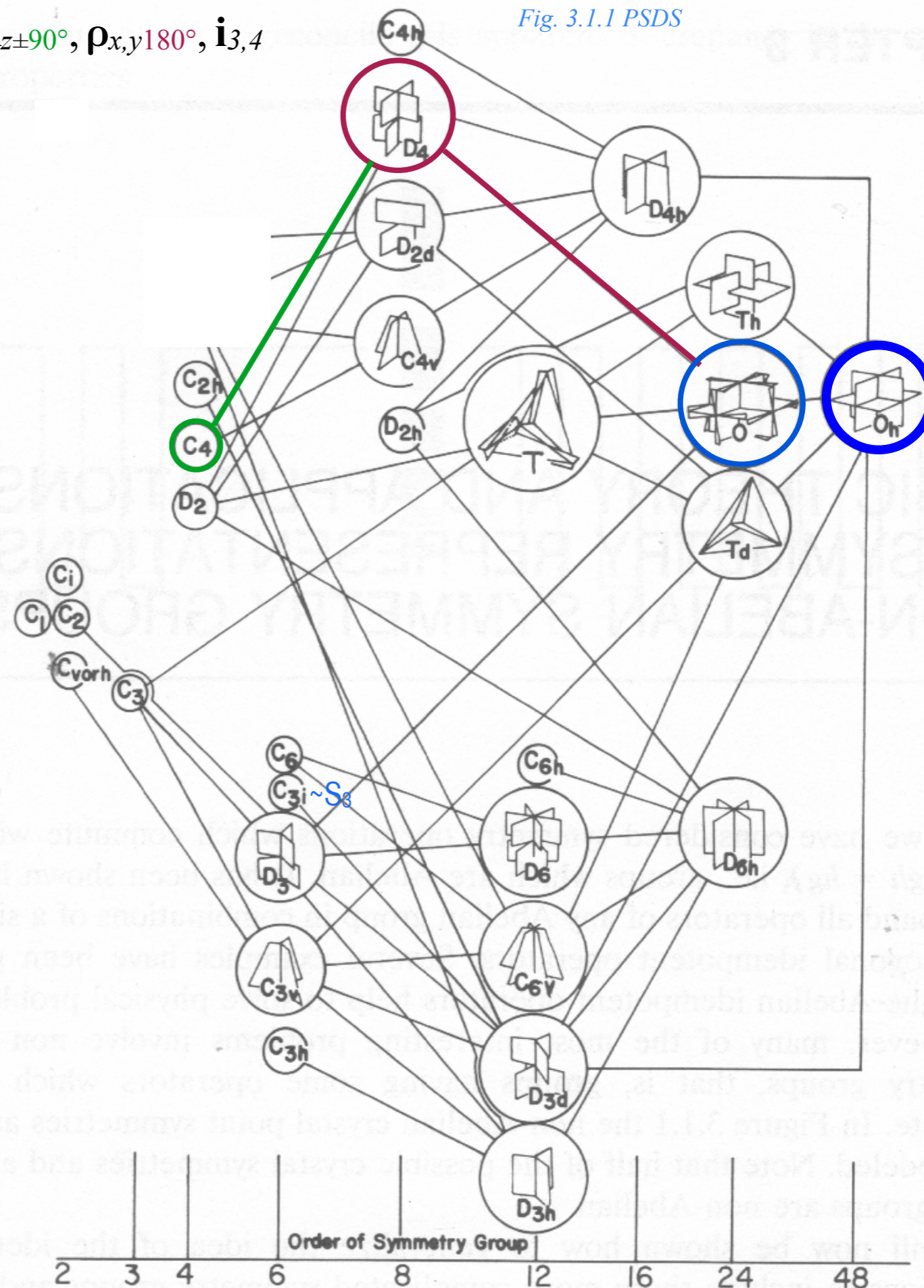
D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

Fig. 3.1.1 PSDS

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

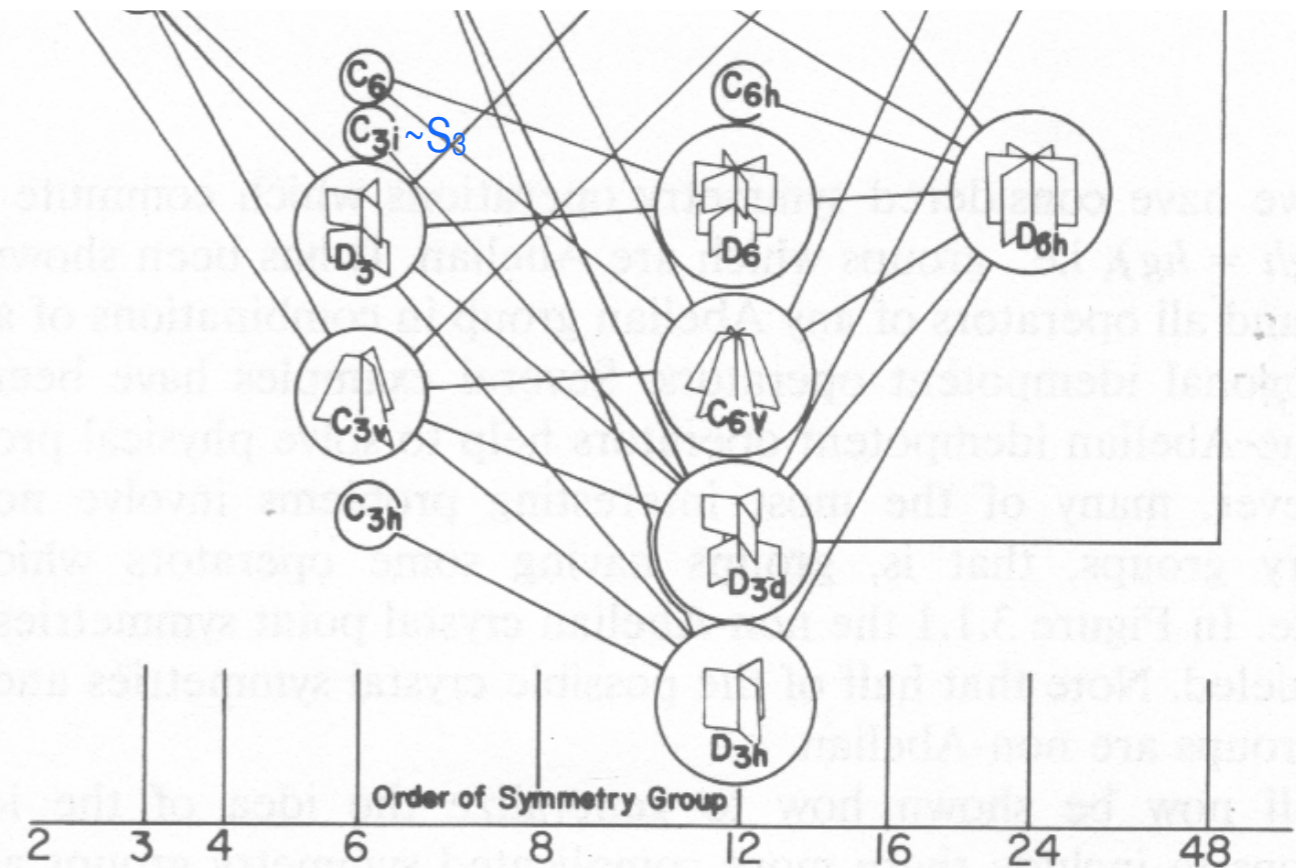
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$.

| $\chi_g^\mu(O)$ | $g=1$ | $r_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

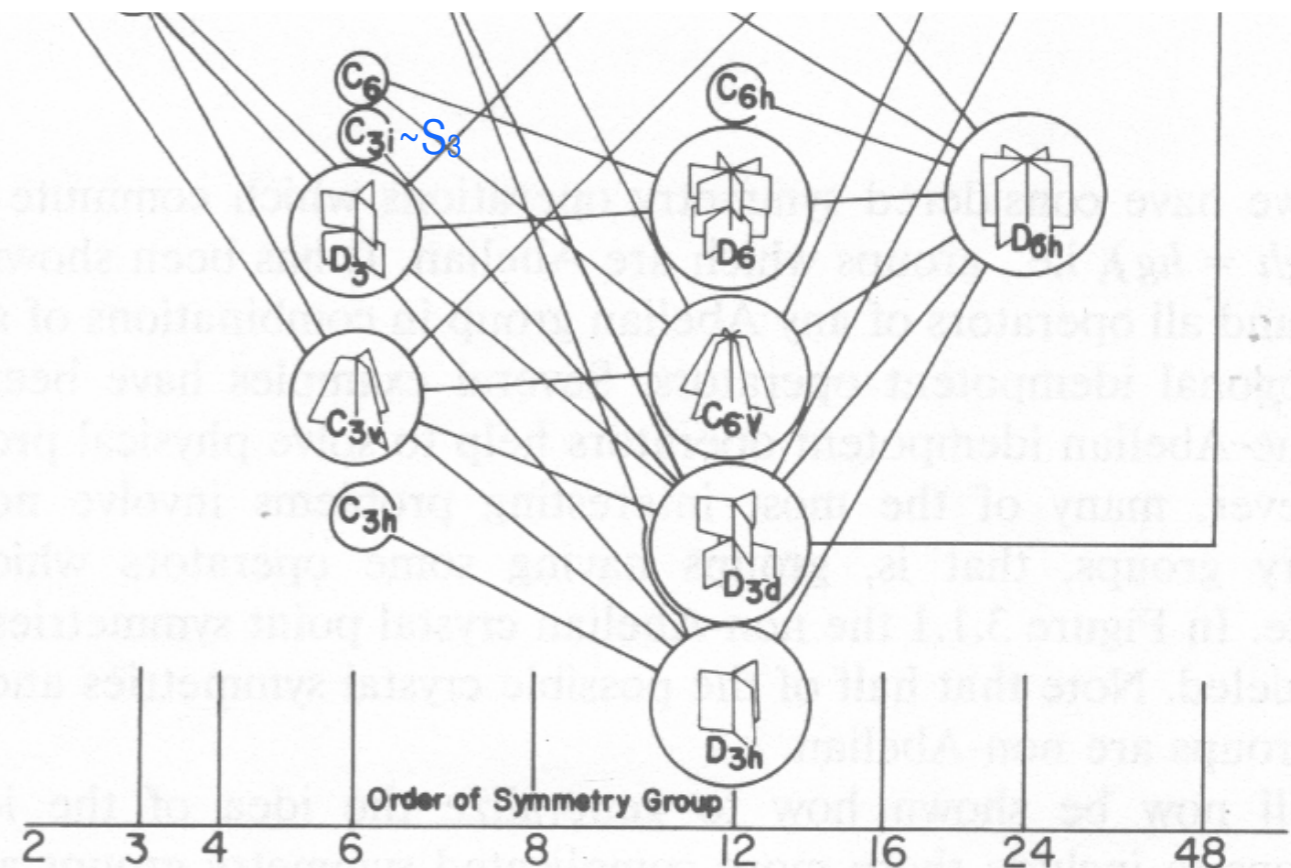
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$

| $\chi_g^\mu(O)$ | $g=1$ | $r_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

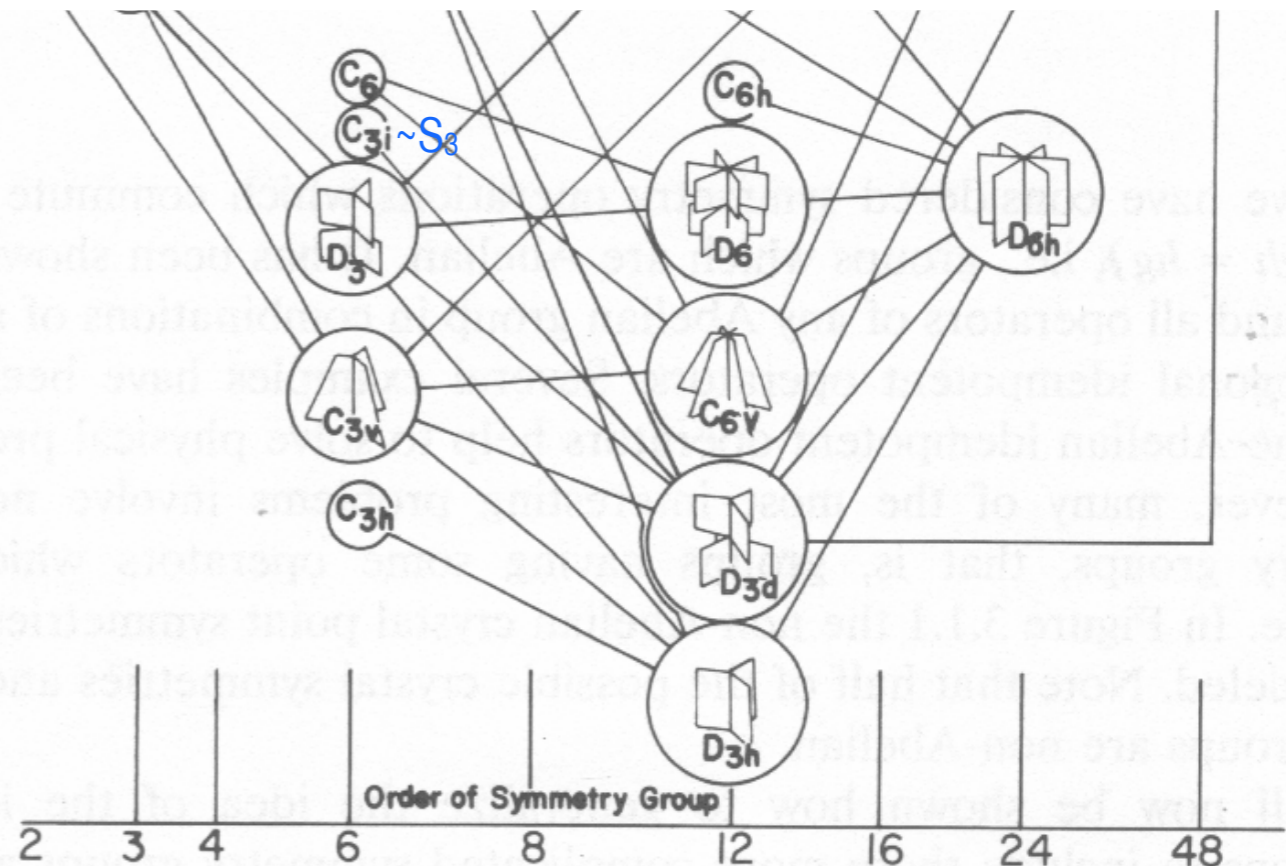
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$.

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

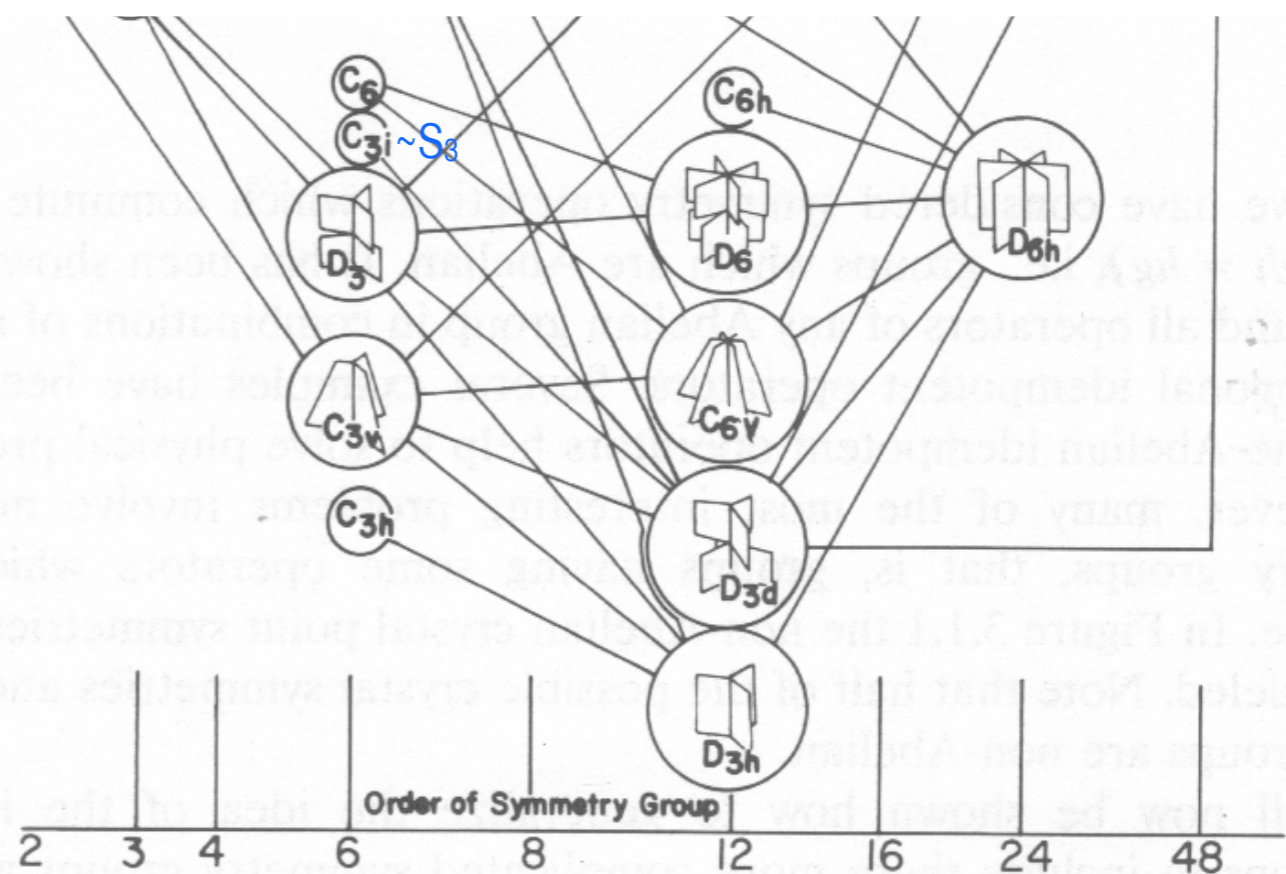
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

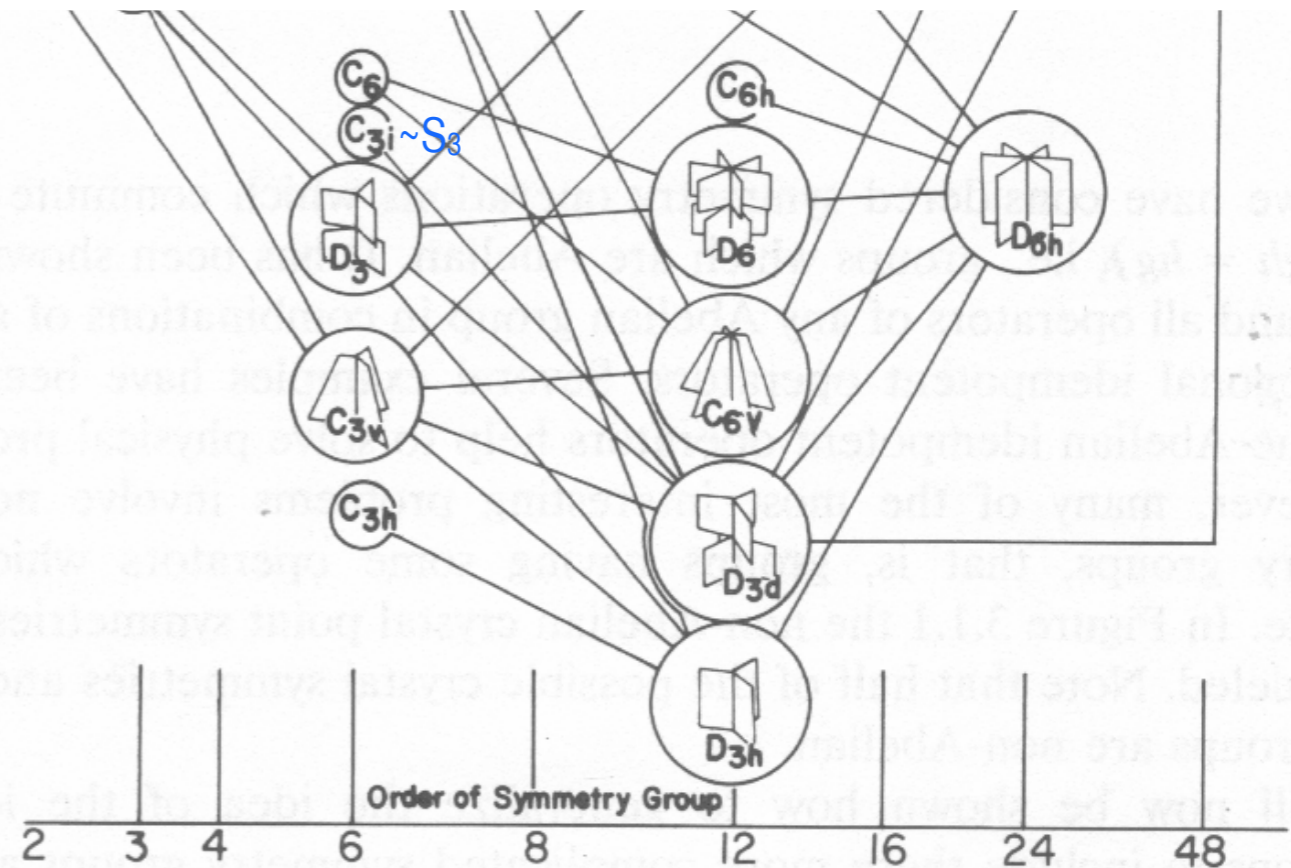
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$.

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

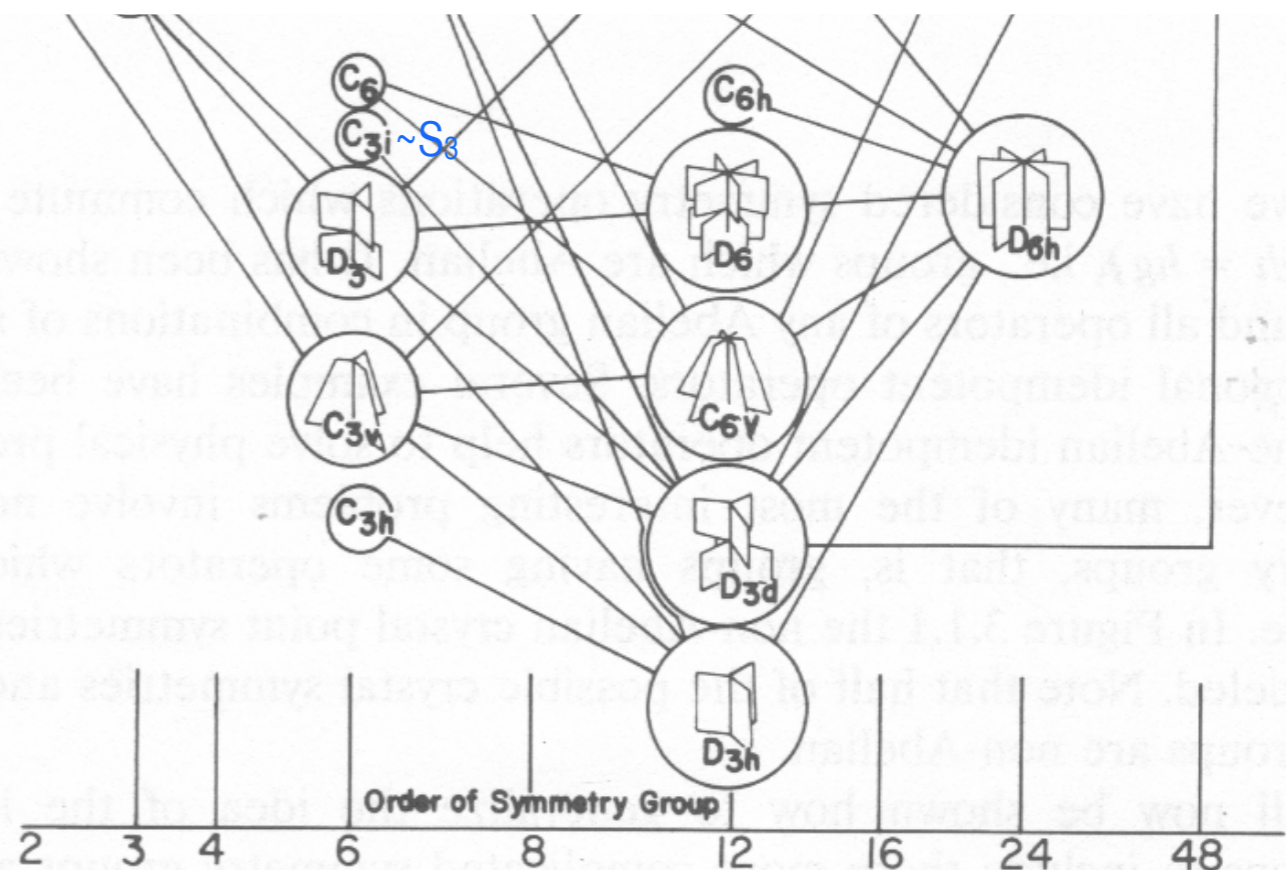
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $r_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 2 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

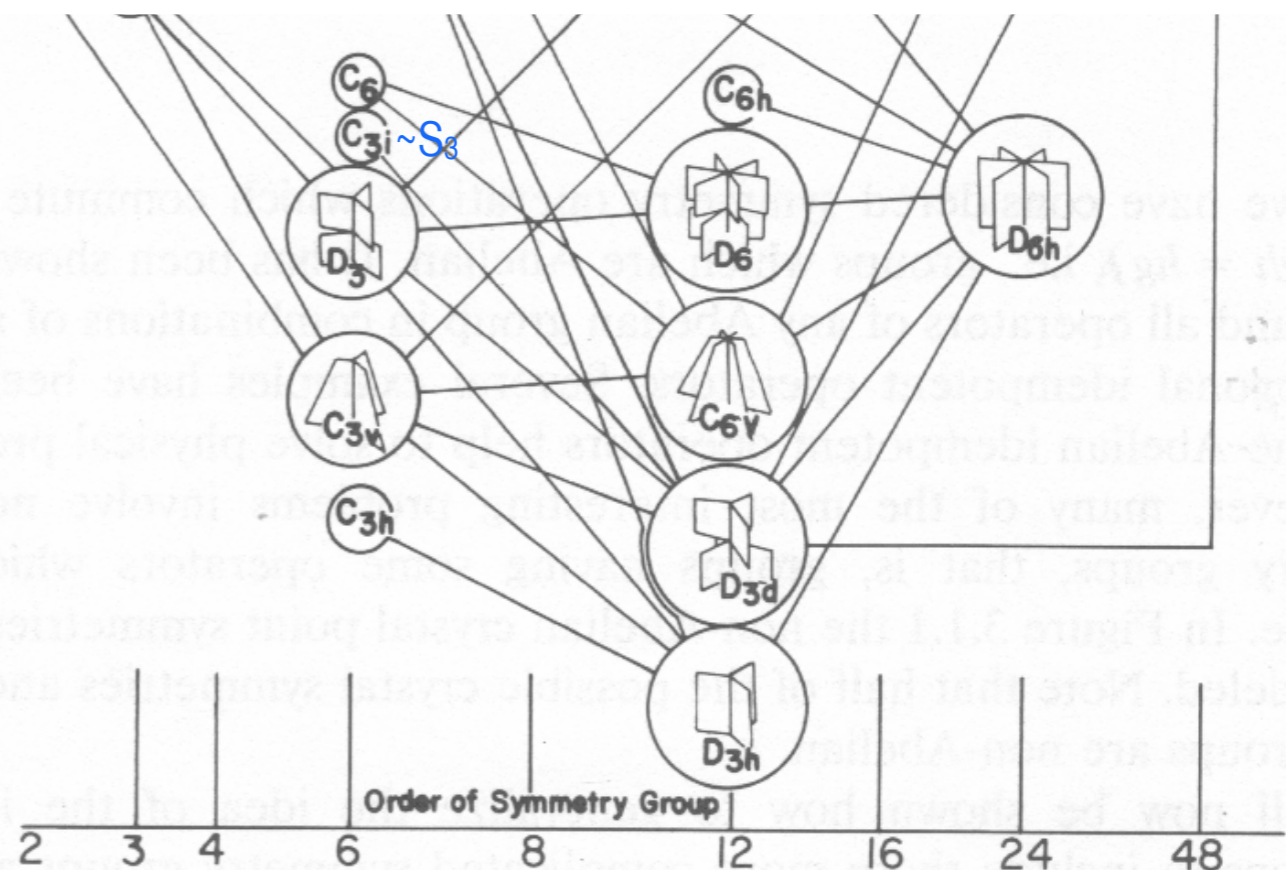
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $r_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 2 | 0 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$.

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

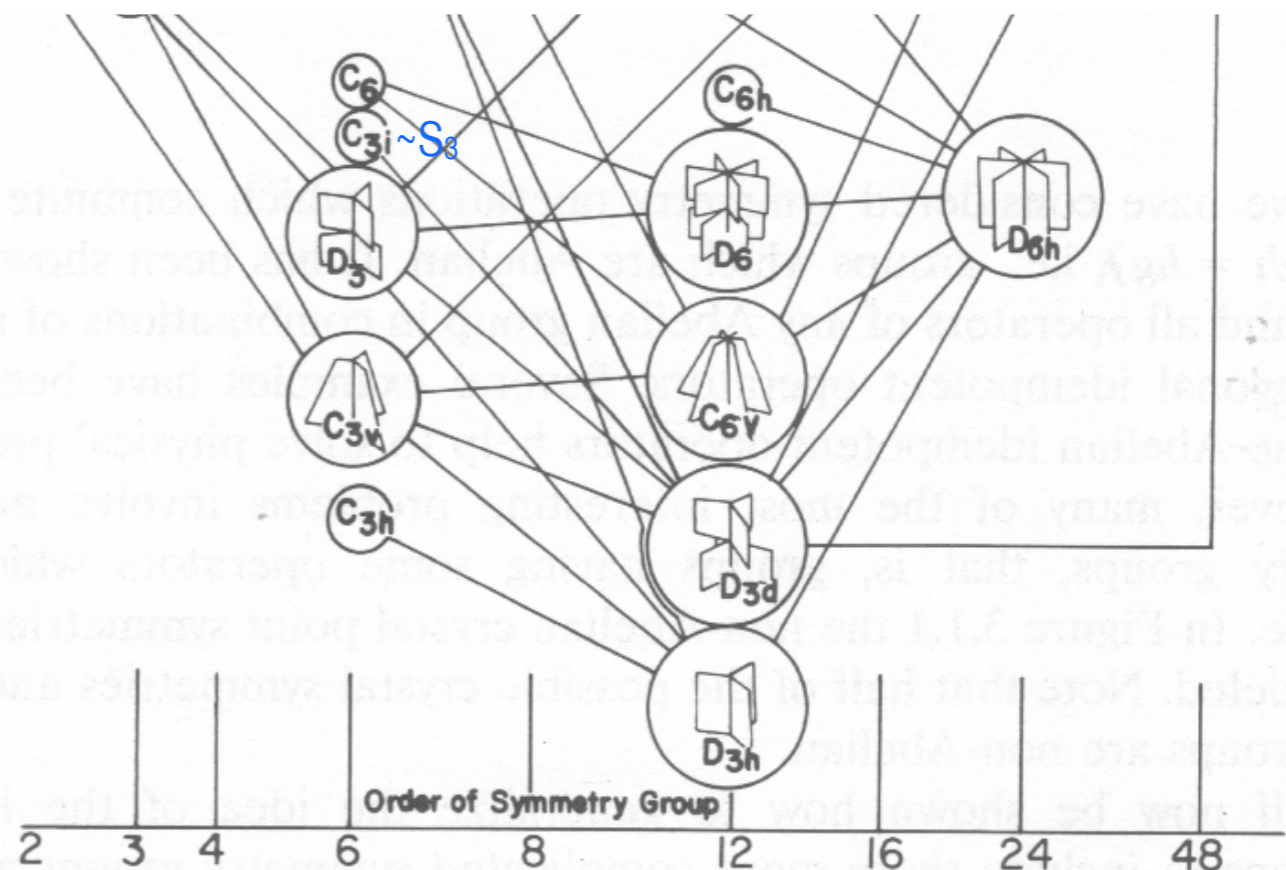
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 2 | 0 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$. $T_1(O) \downarrow D_4 = E \oplus A_2(D_4)$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

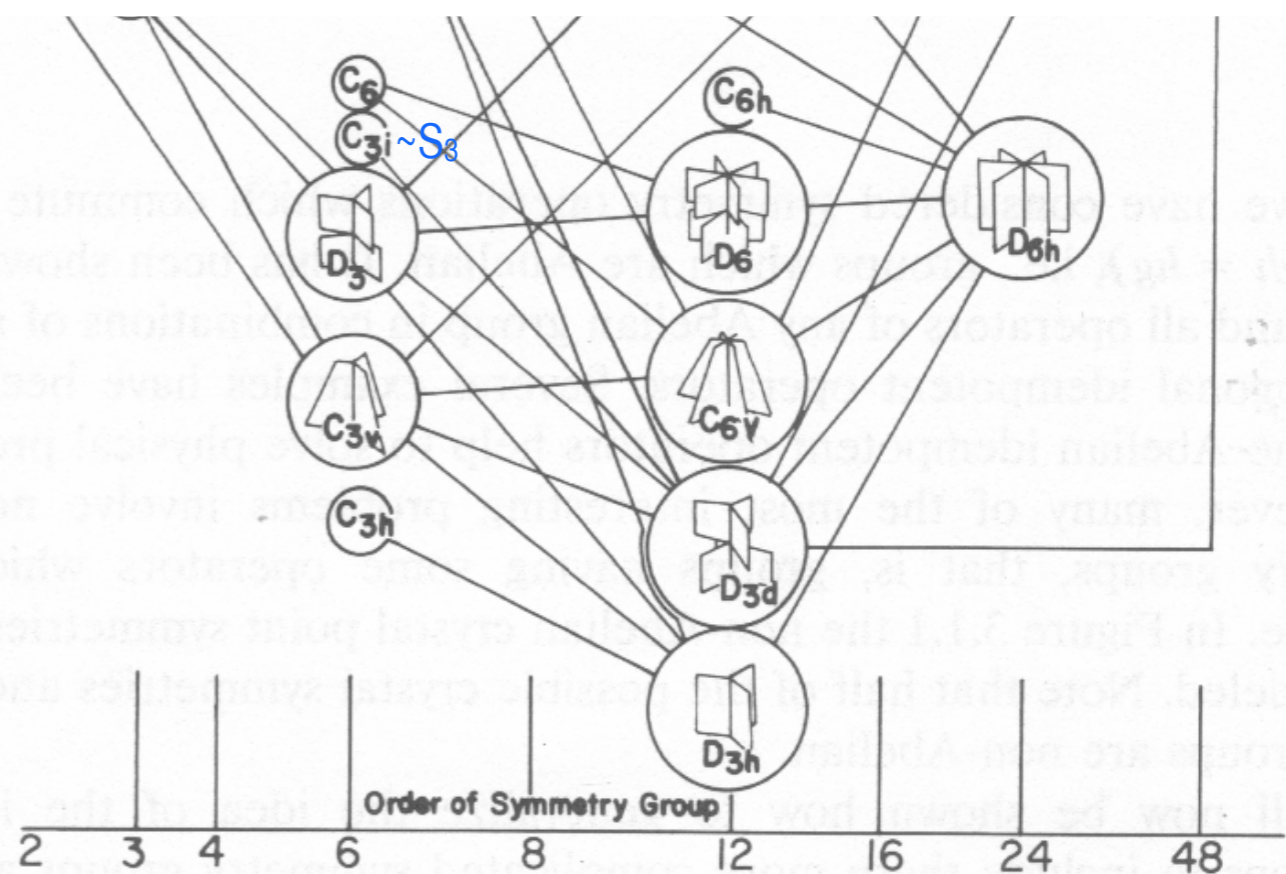
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $r_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$. $T_1(O) \downarrow D_4 = E \oplus A_2(D_4)$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$.

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

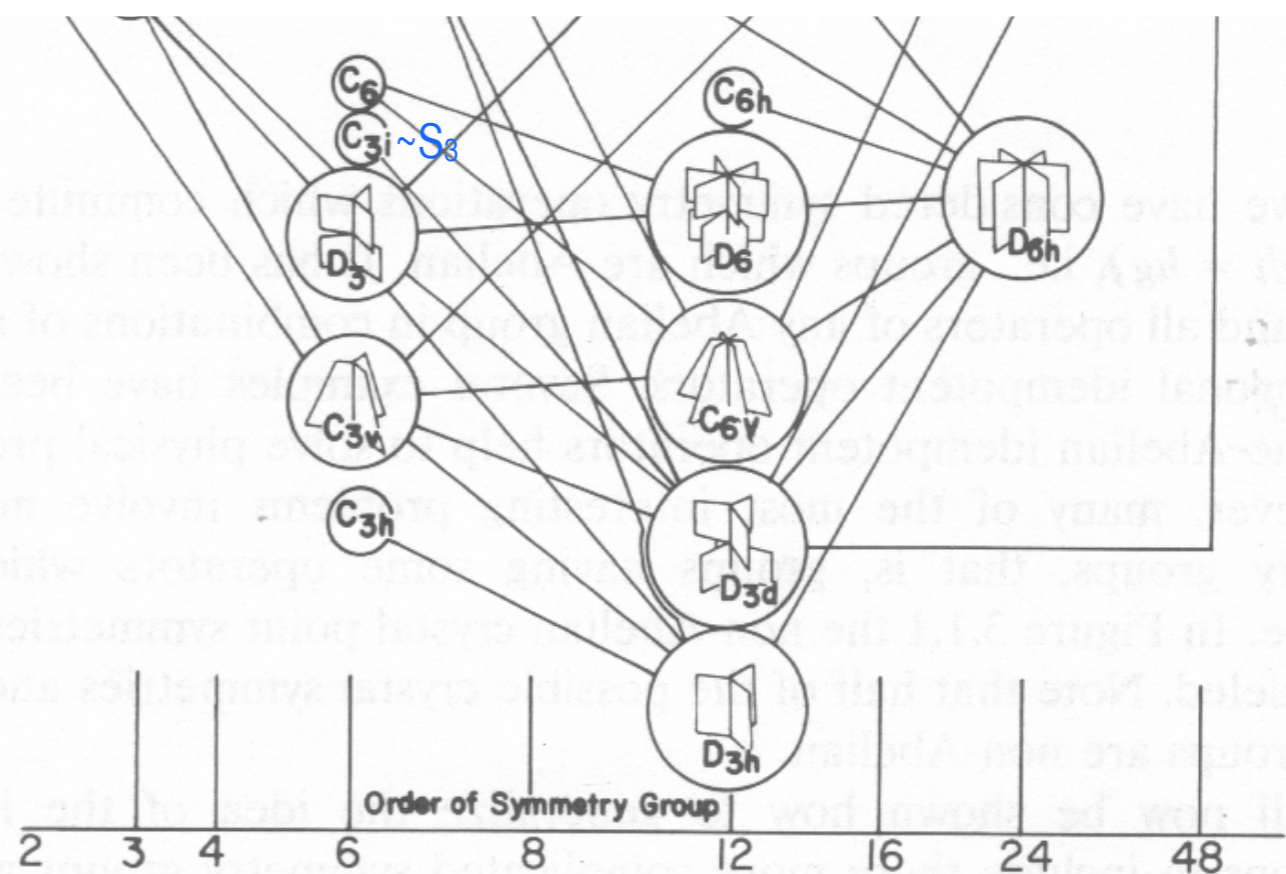
$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $r_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$. $T_1(O) \downarrow D_4 = E \oplus A_2(D_4)$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$. $T_2(O) \downarrow D_4 = E \oplus B_2(D_4)$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

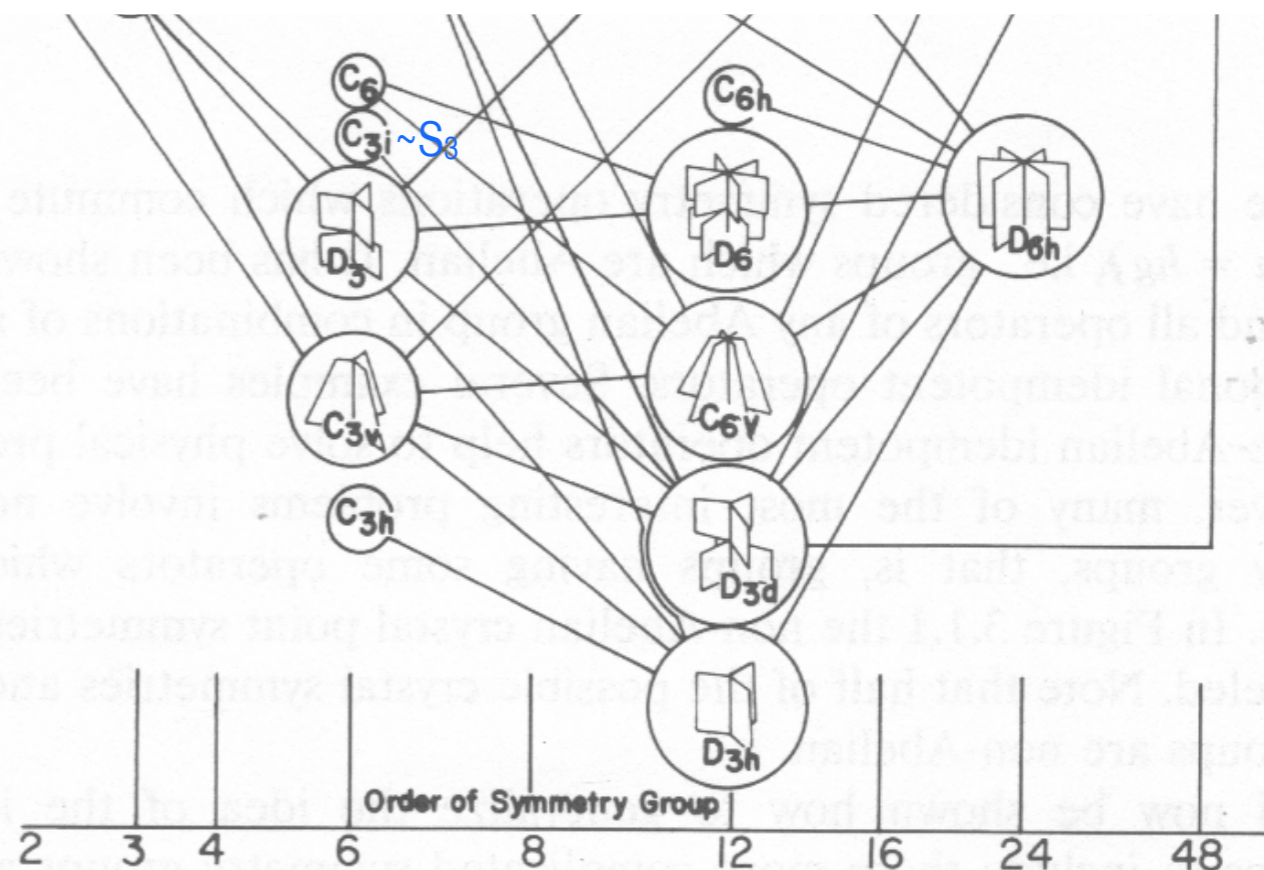
D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$. $A_1(O) \downarrow D_4 = A_1(D_4)$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$. $A_2(O) \downarrow D_4 = B_1(D_4)$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$. $E(O) \downarrow D_4 = A_1 \oplus B_1(D_4)$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$. $T_1(O) \downarrow D_4 = E \oplus A_2(D_4)$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$. $T_2(O) \downarrow D_4 = E \oplus B_2(D_4)$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

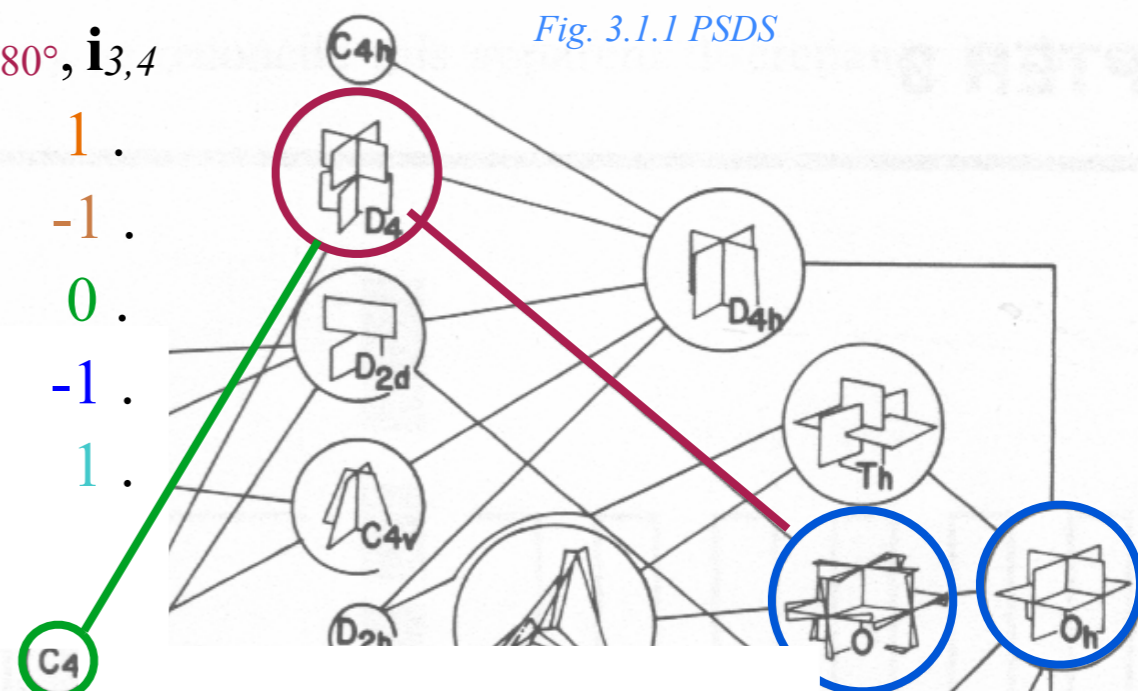
$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

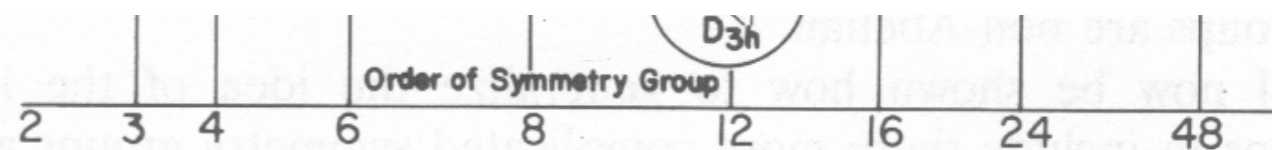
$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

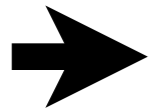
Octahedral $O_h \supset O$: Inversion (g&u) parity

Octahedral $O_h \supset O \supset C_{\infty}$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

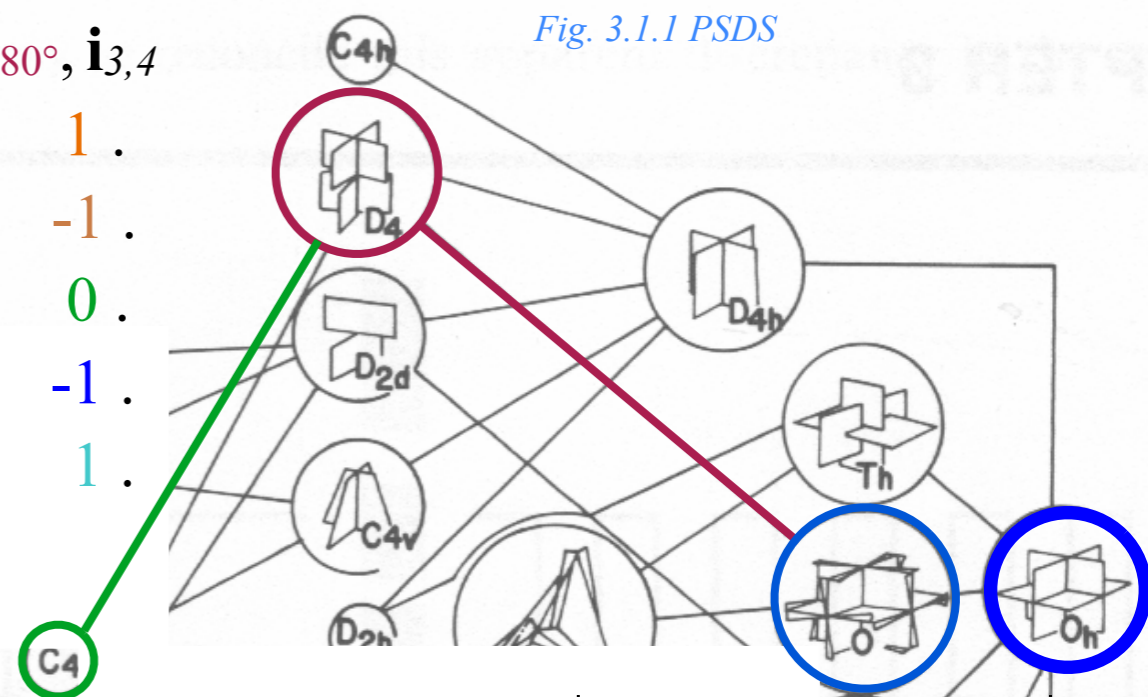
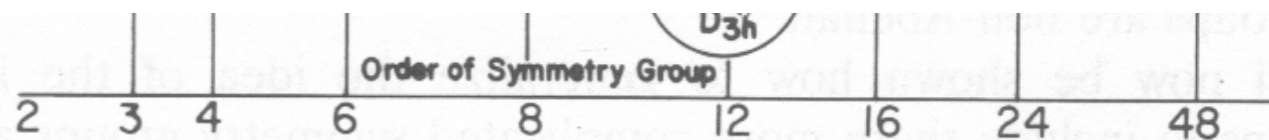


Fig. 3.1.1 PSDS

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $r_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 2 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

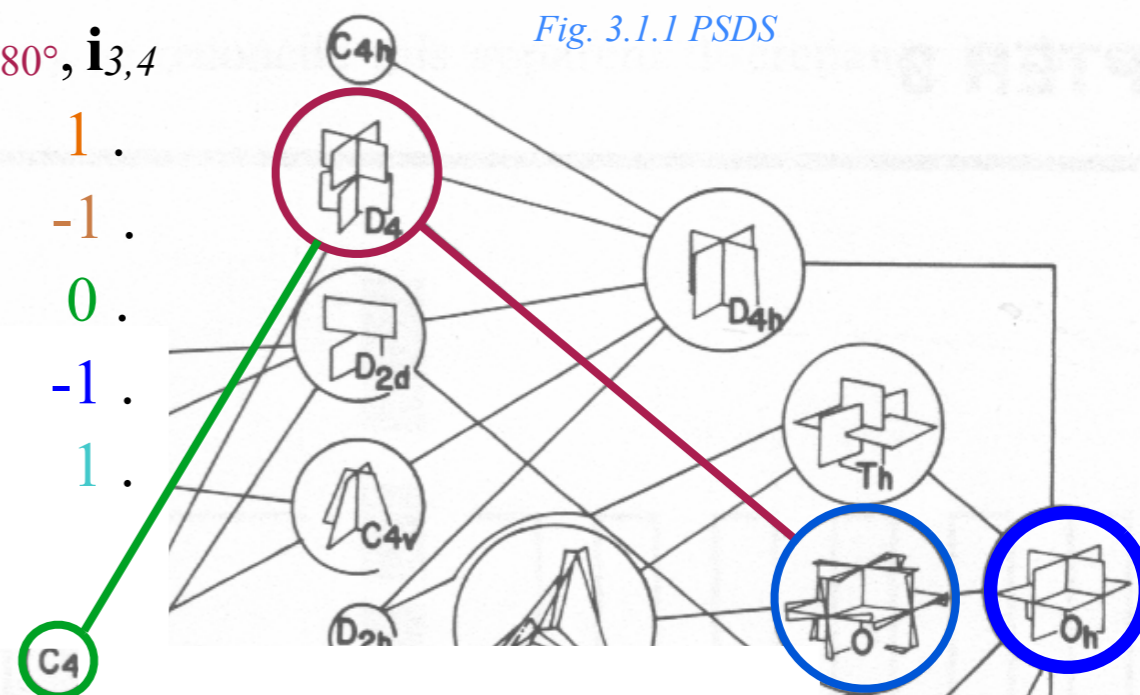
| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

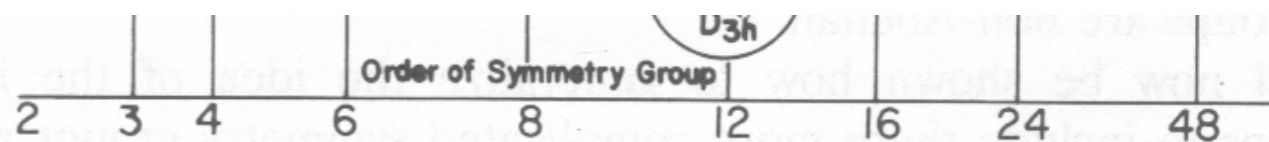
$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1.$

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |



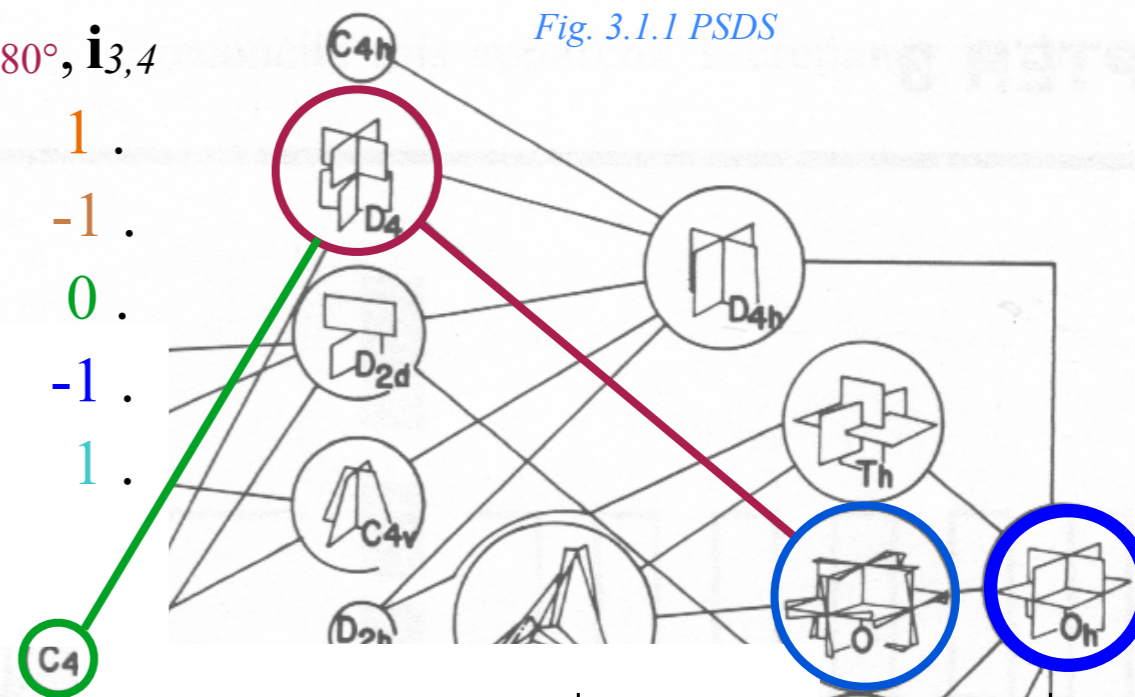
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$



| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

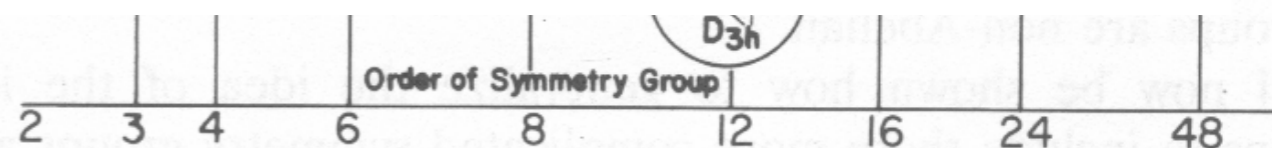
$D_4 \downarrow C_4$ subduction

$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $r_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

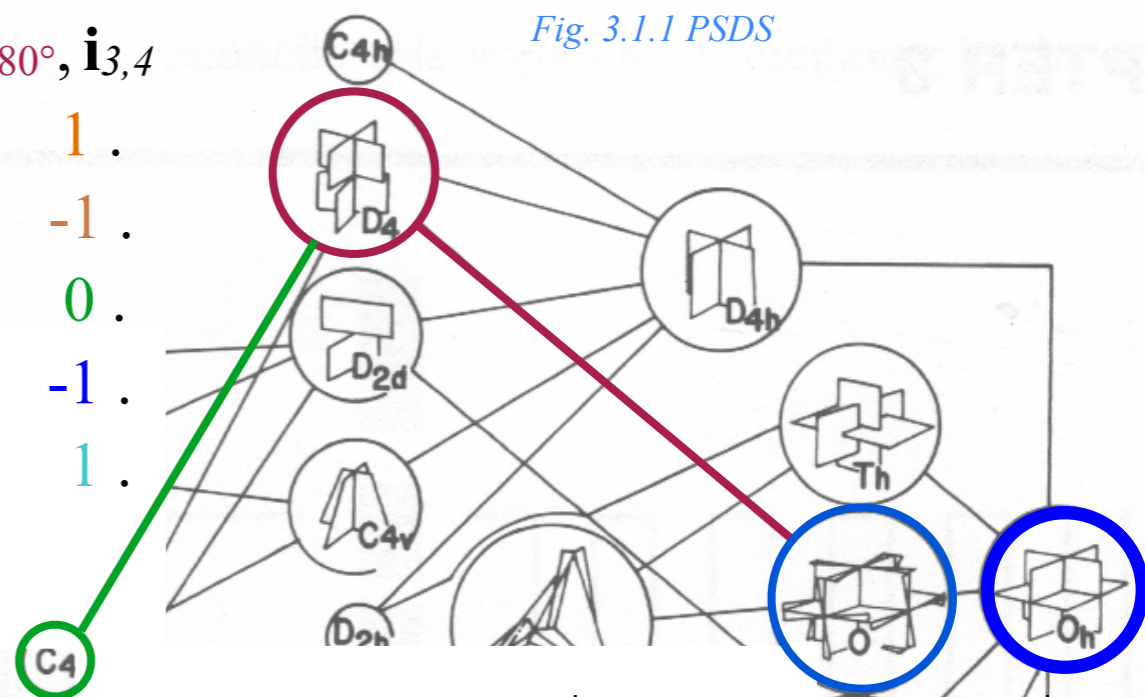
$D_4 \downarrow C_4$ subduction

$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

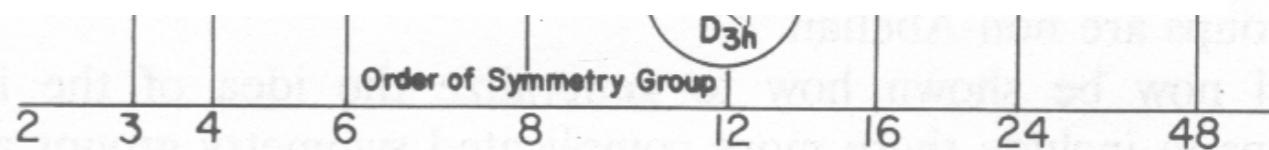
$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1.$

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

Fig. 3.1.1 PSDS



| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | . | . | . | . |
| A_2 | . | 1 | . | . | . |
| E | 1 | 1 | . | . | . |
| T_1 | . | . | 1 | . | 1 |
| T_2 | . | . | . | 1 | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

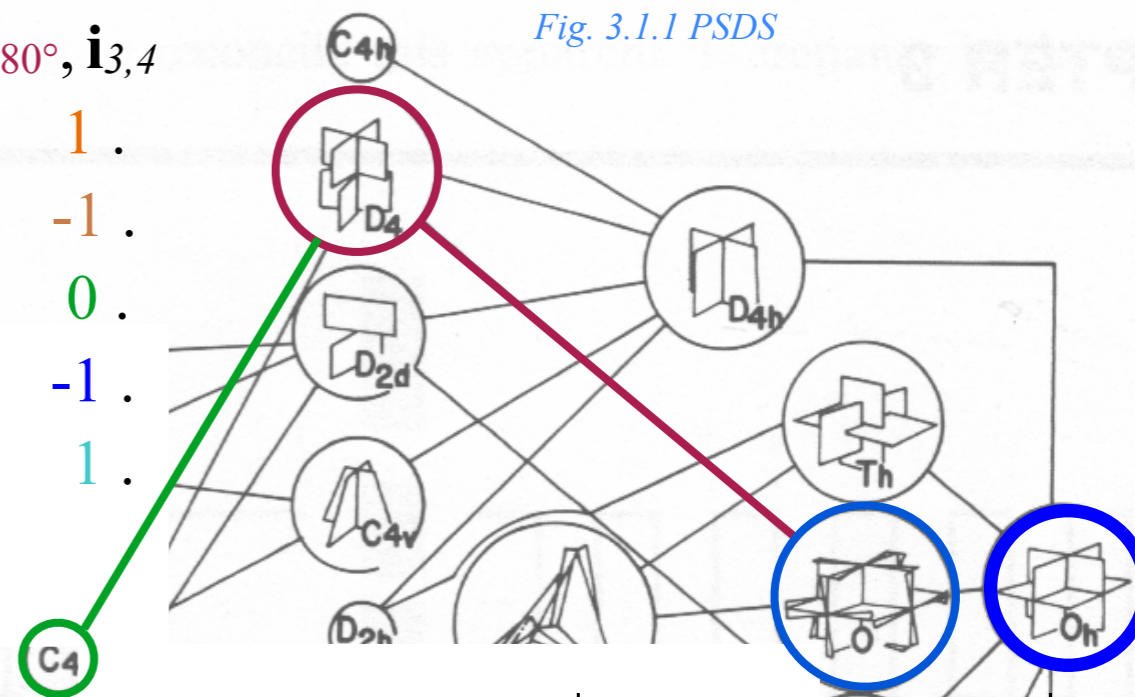
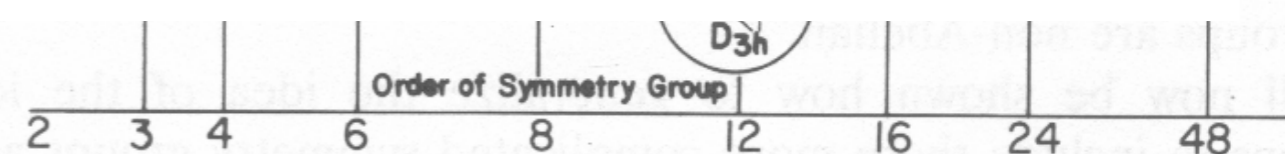


Fig. 3.1.1 PSDS

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | . | . | . | . |
| A_2 | . | 1 | . | . | . |
| E | 1 | 1 | . | . | . |
| T_1 | . | . | 1 | . | 1 |
| T_2 | . | . | . | 1 | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

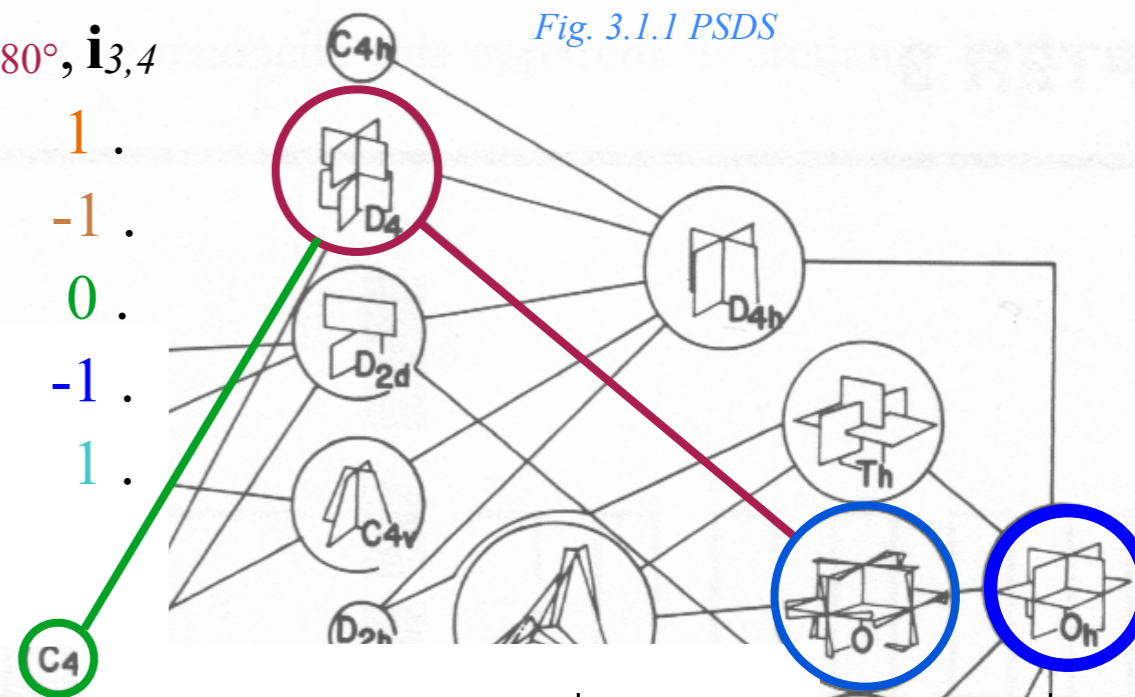


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

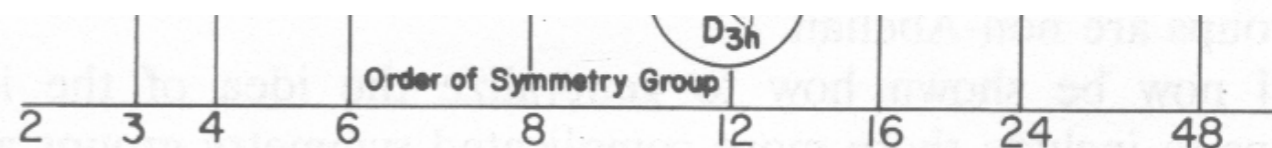
$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1.$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | . | . | . | . |
| A_2 | . | 1 | . | . | . |
| E | 1 | 1 | . | . | . |
| T_1 | . | . | 1 | . | 1 |
| T_2 | . | . | . | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

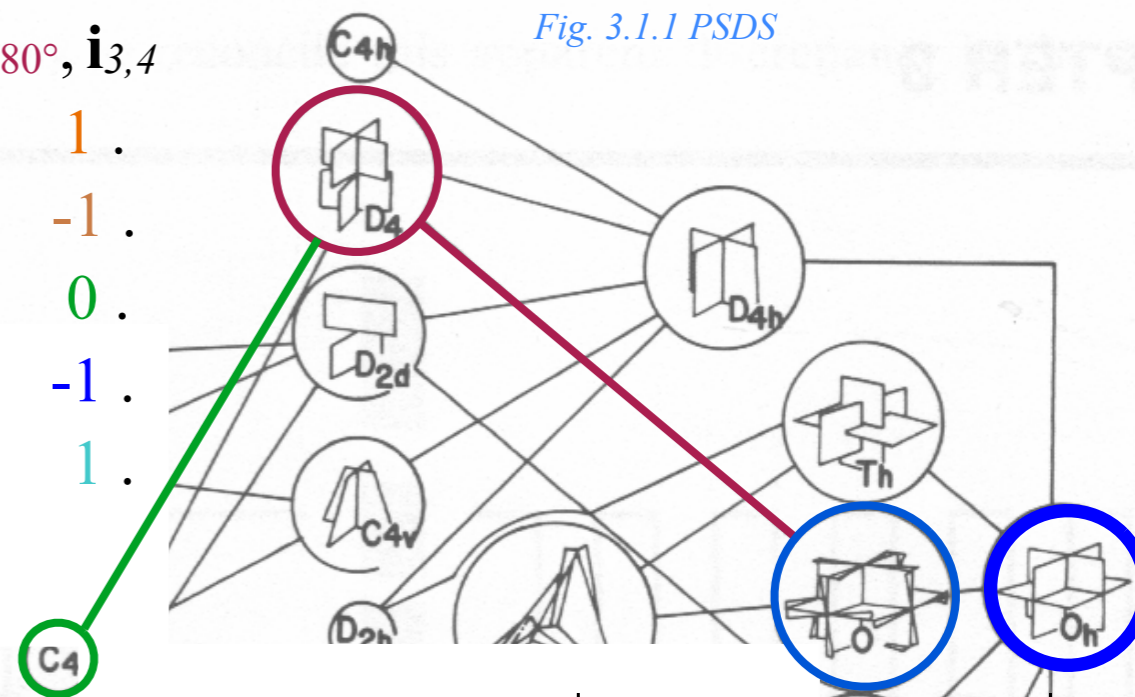


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

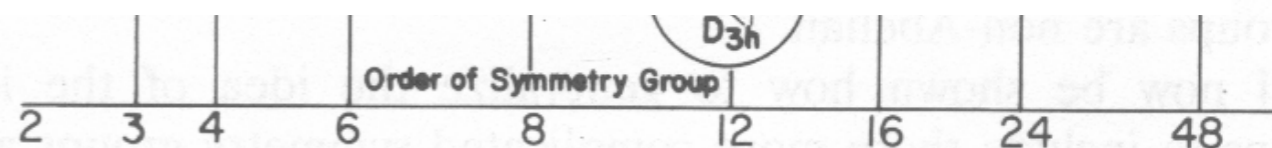
$D_4 \downarrow C_4$ subduction

$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

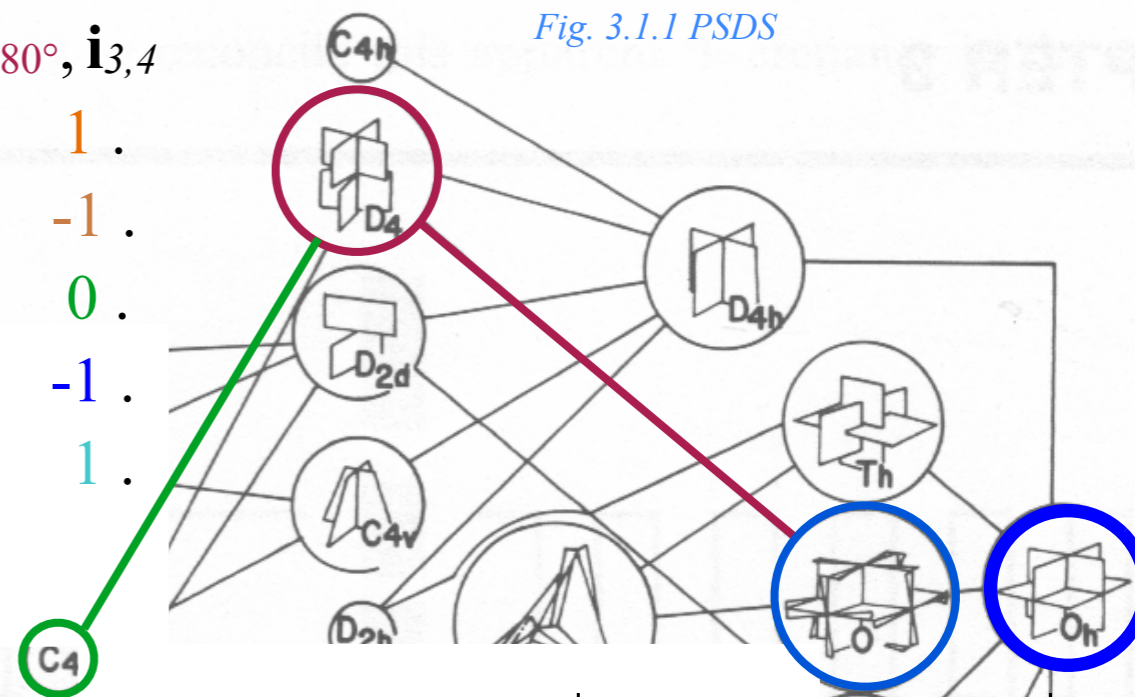


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

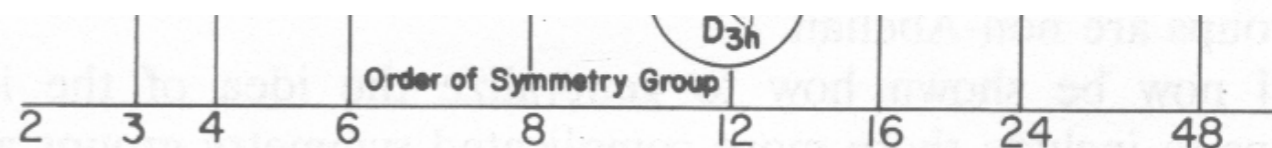
$D_4 \downarrow C_4$ subduction

$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1.$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

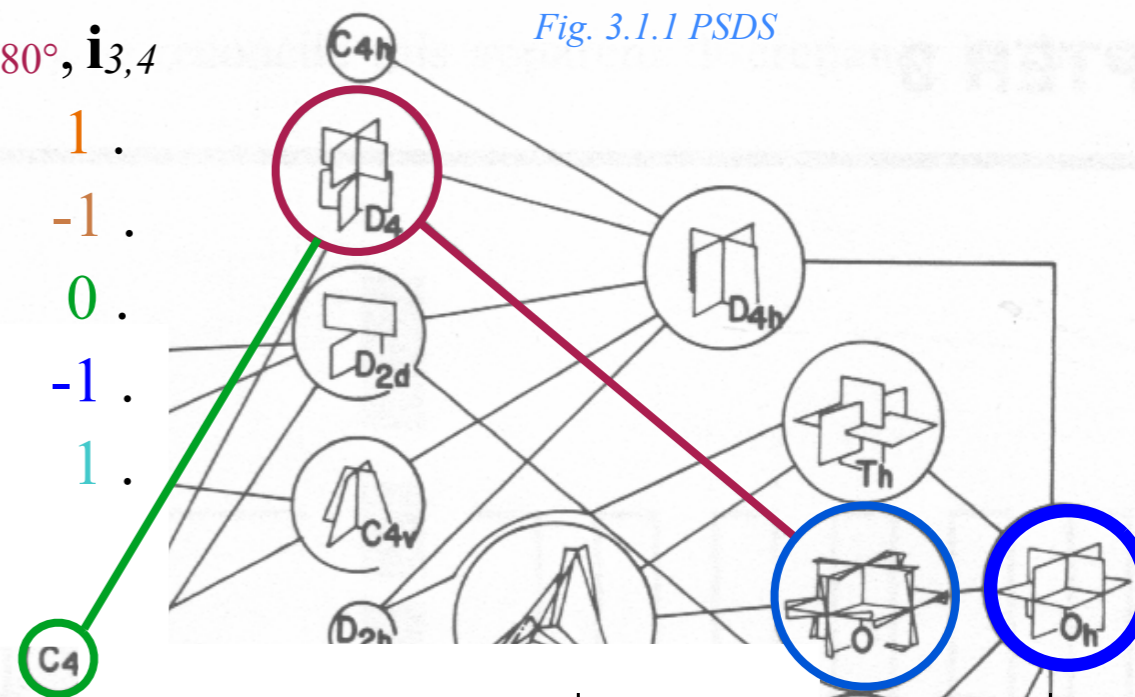


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

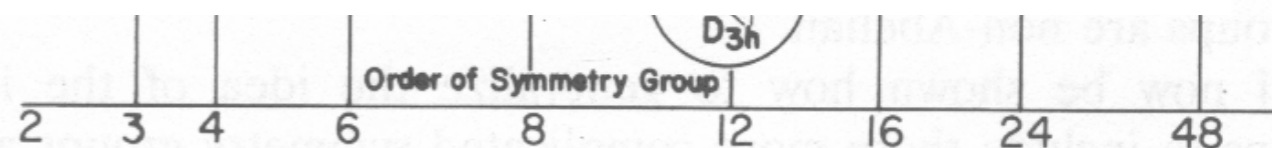
$D_4 \downarrow C_4$ subduction

$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

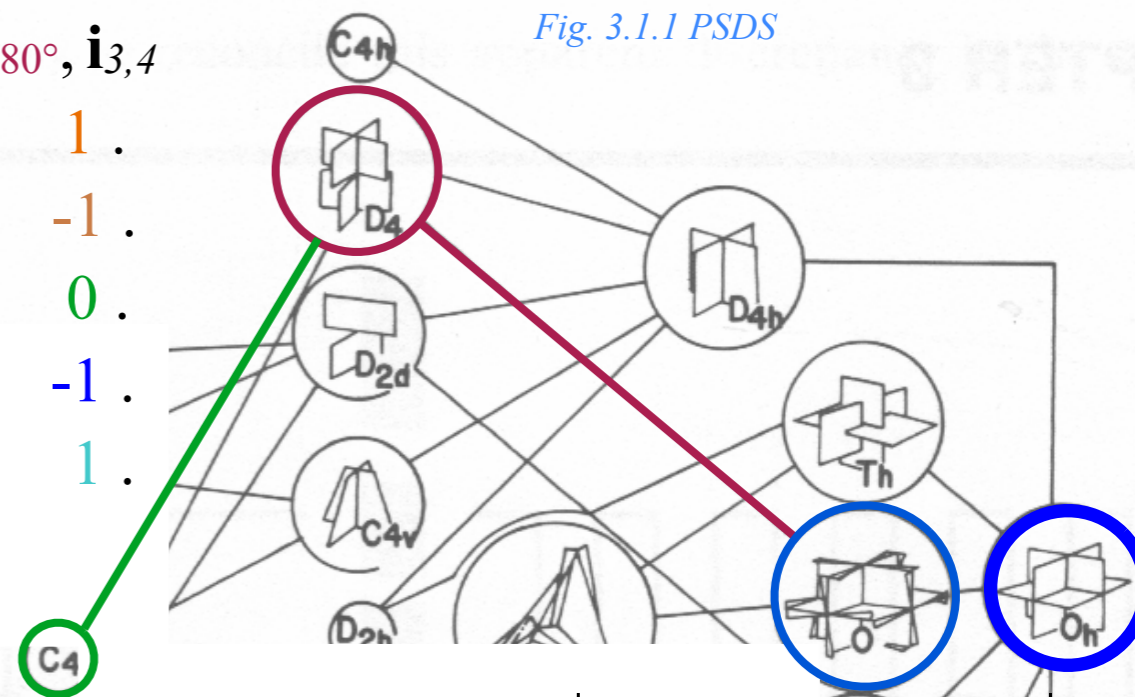


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

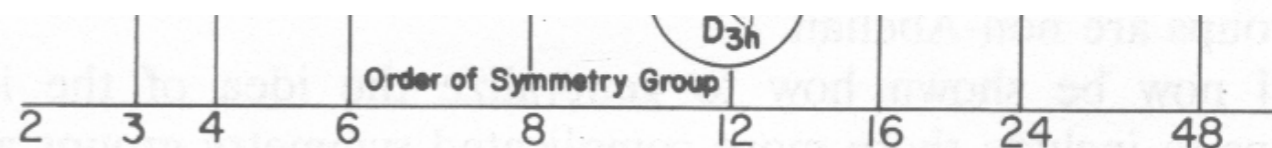
$D_4 \downarrow C_4$ subduction

$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0.$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

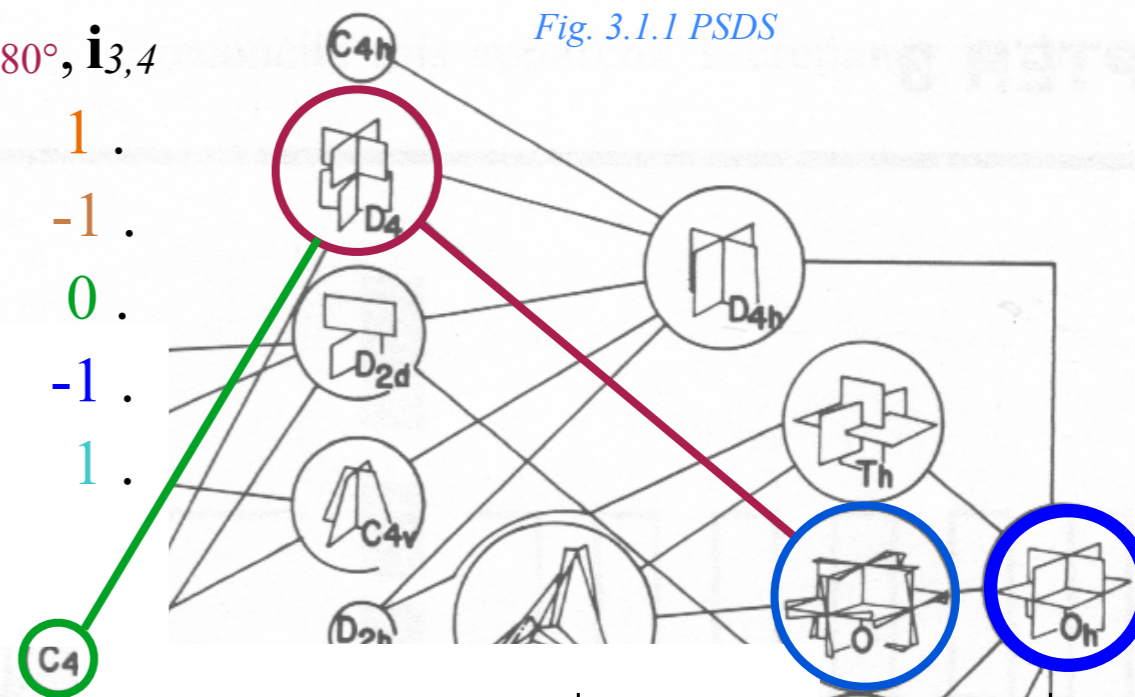


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

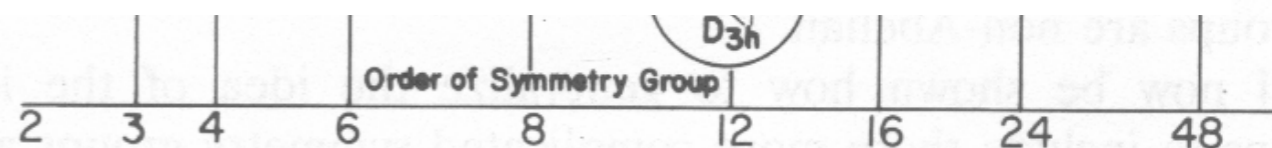
$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0. = (1)_4 \oplus (3)_4$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g = \mathbf{1}$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|------------------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 2 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

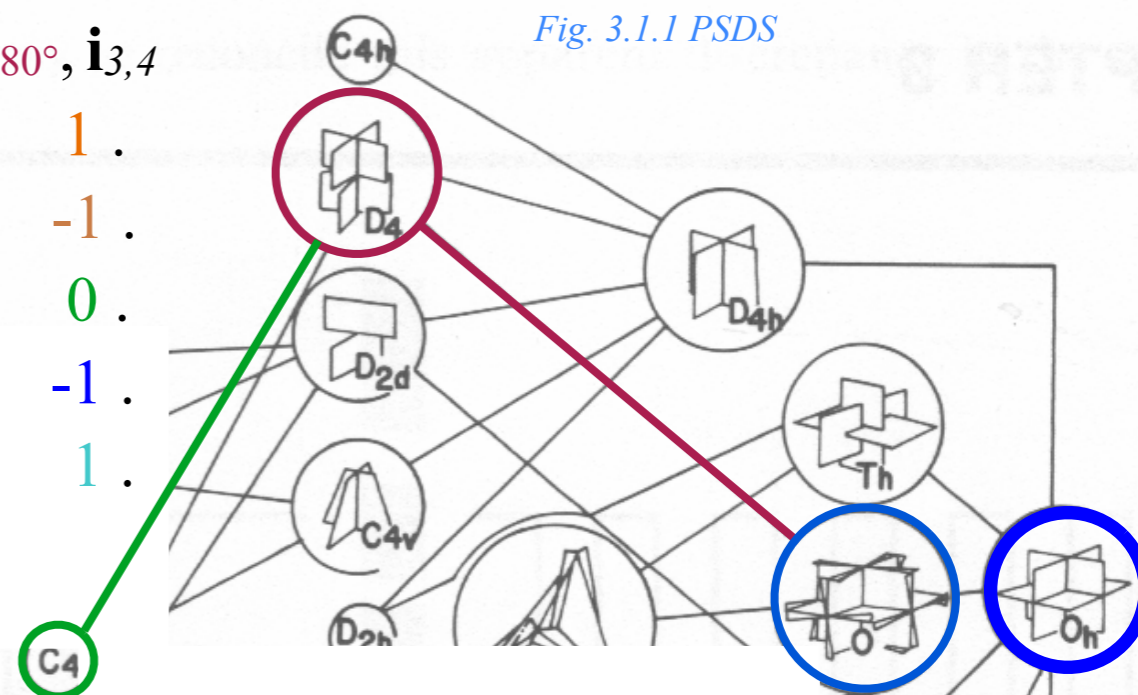


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g = \mathbf{1}$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|------------------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

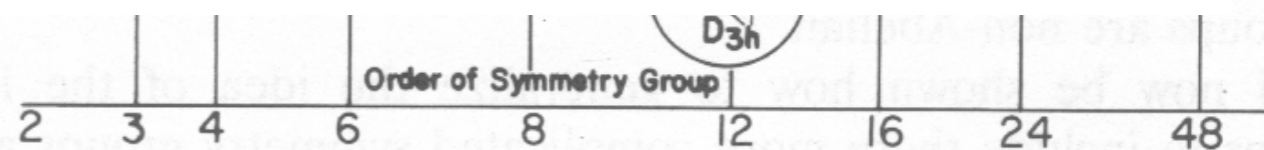
C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g = \mathbf{1}$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|------------------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-------|-------|-------------------|
| A_1 | 1 | · | · | · |
| B_1 | · | · | 1 | · |
| A_2 | 1 | · | · | · |
| B_2 | · | · | 1 | · |
| E | · | 1 | · | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

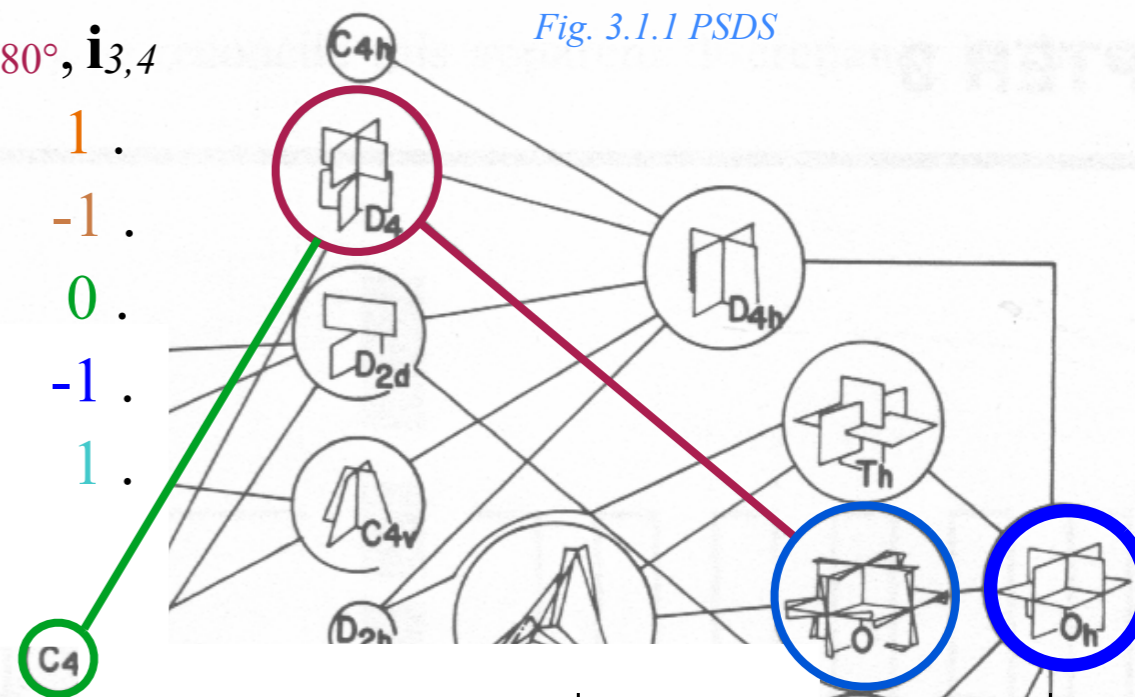


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

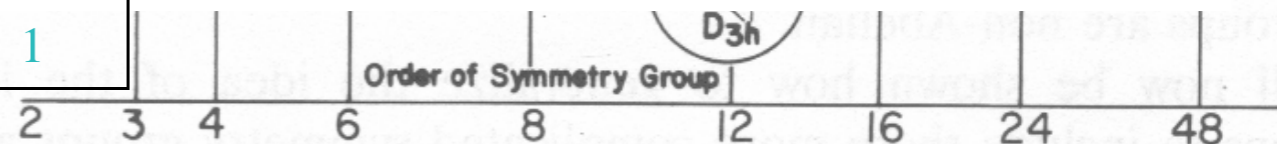
| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

$O \downarrow C_4$ subduction

| $O \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|--------------------|-------|-------|-------|-------------------|
| A_1 | 1 | · | · | · |
| A_2 | · | · | 1 | · |
| E | 1 | · | 1 | · |
| T_1 | 1 | 1 | · | 1 |
| T_2 | · | 1 | 1 | 1 |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-------|-------|-------------------|
| A_1 | 1 | · | · | · |
| B_1 | · | · | 1 | · |
| A_2 | 1 | · | · | · |
| B_2 | · | · | 1 | · |
| E | · | 1 | · | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | 2 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

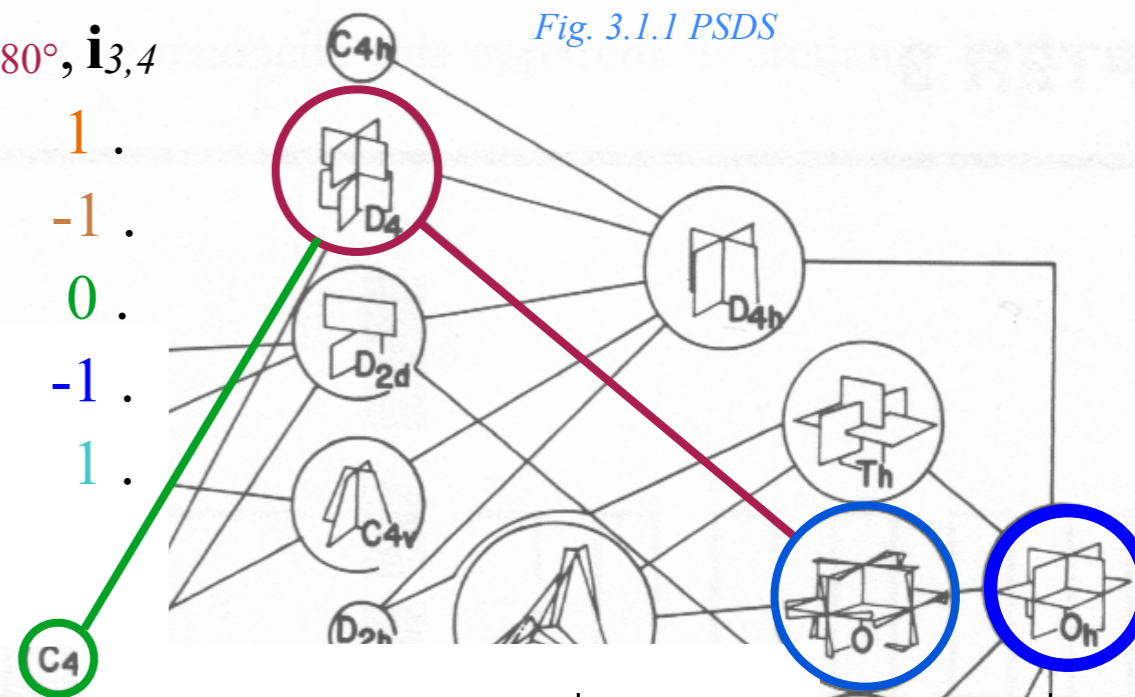


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

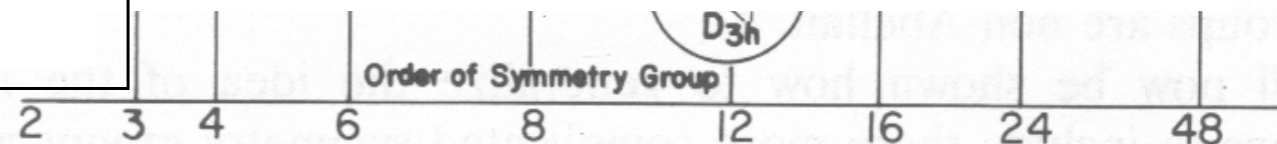
| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

$O \downarrow C_4$ subduction

| $O \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|--------------------|-------|-------|-------|-------------------|
| A_1 | → 1 | · | · | · |
| A_2 | · | · | · | · |
| E | · | · | · | · |
| T_1 | · | · | · | · |
| T_2 | · | · | · | · |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-------|-------|-------------------|
| A_1 | → 1 | · | · | · |
| B_1 | · | · | 1 | · |
| A_2 | 1 | · | · | · |
| B_2 | · | · | 1 | · |
| E | · | 1 | · | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

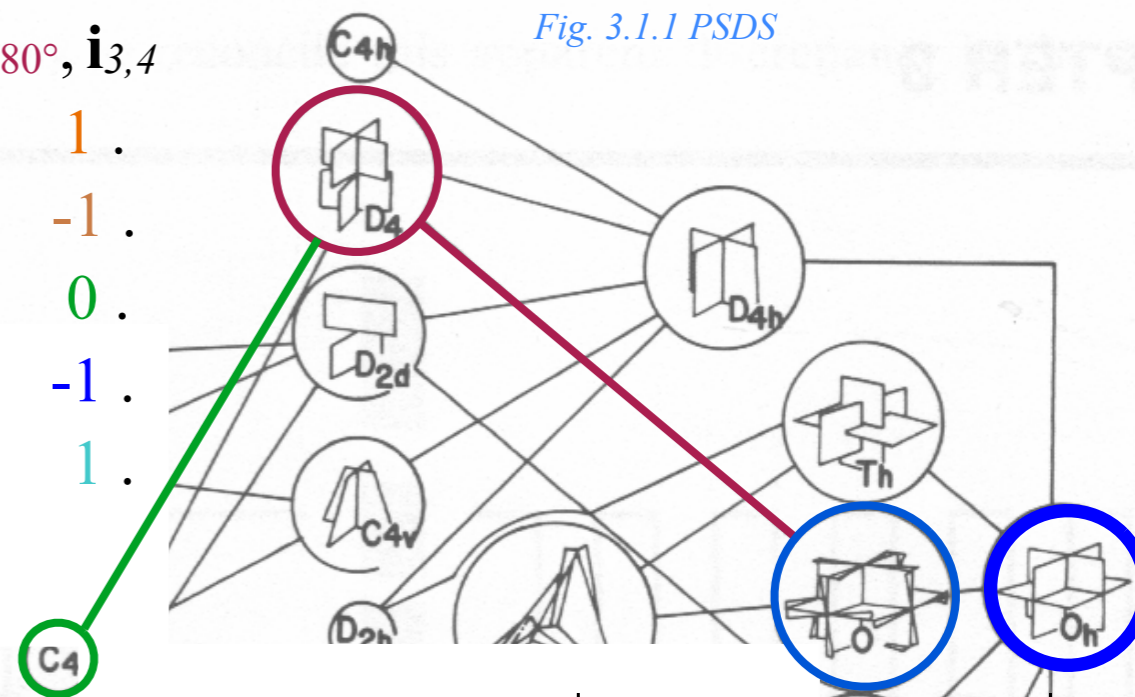


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0. = (1)_4 \oplus (3)_4$

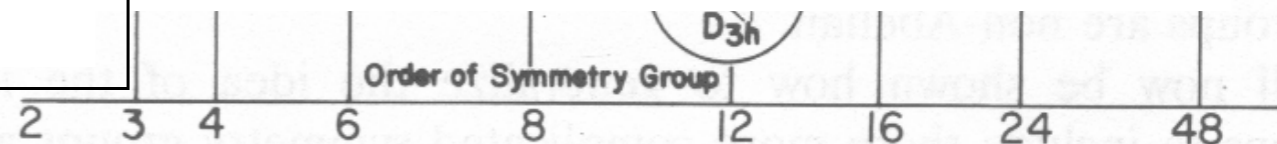
| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | . | . | . | . |
| A_2 | . | 1 | . | . | . |
| E | 1 | 1 | . | . | . |
| T_1 | . | . | 1 | . | 1 |
| T_2 | . | . | . | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

$O \downarrow C_4$ subduction

| $O \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|--------------------|-------|-------|-----------------|-------------------|
| A_1 | 1 | . | . | . |
| A_2 | . | . | $\rightarrow 1$ | . |
| E | | | | |
| T_1 | | | | |
| T_2 | | | | |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-------|-----------------|-------------------|
| A_1 | 1 | . | . | . |
| B_1 | . | . | $\rightarrow 1$ | . |
| A_2 | 1 | . | . | . |
| B_2 | . | . | 1 | . |
| E | . | 1 | . | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

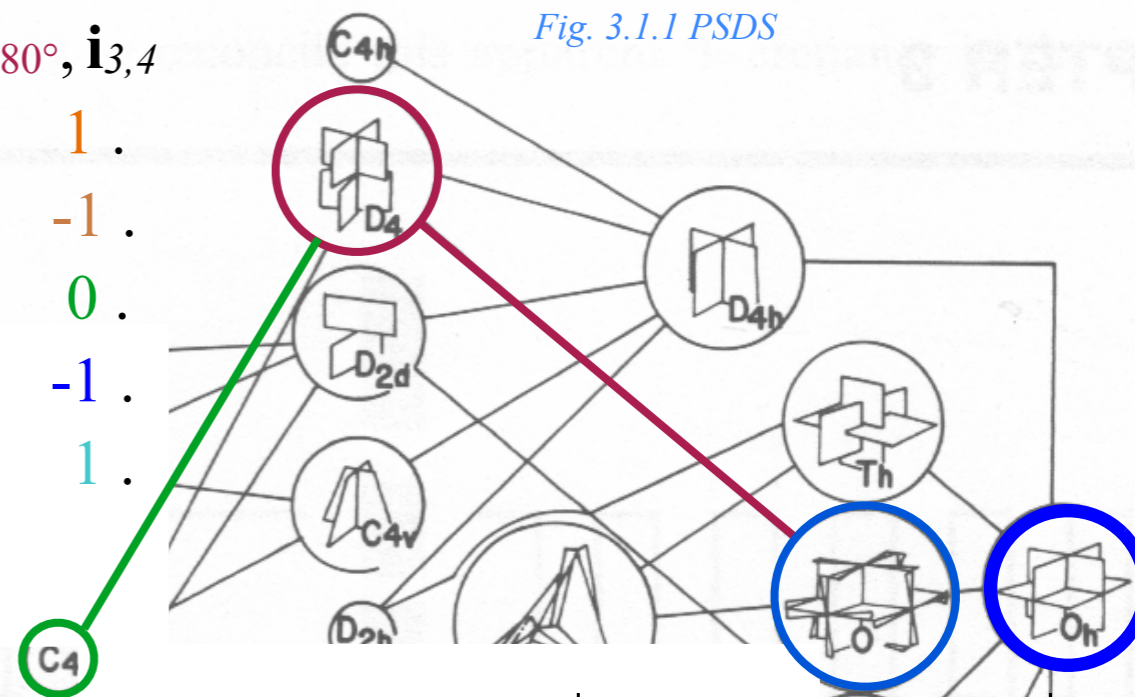


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

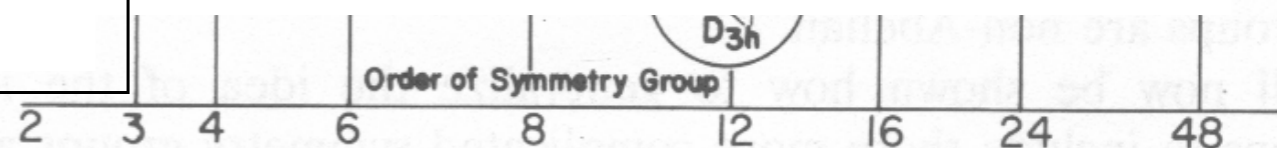
| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

$O \downarrow C_4$ subduction

| $O \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|--------------------|-------|-------|-------|-------------------|
| A_1 | 1 | · | · | · |
| A_2 | · | · | 1 | · |
| E | →1 | · | →1 | · |
| T_1 | · | · | · | · |
| T_2 | · | · | · | · |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-------|-------|-------------------|
| A_1 | →1 | · | · | · |
| B_1 | · | →1 | · | · |
| A_2 | 1 | · | · | · |
| B_2 | · | · | 1 | · |
| E | · | 1 | · | 1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

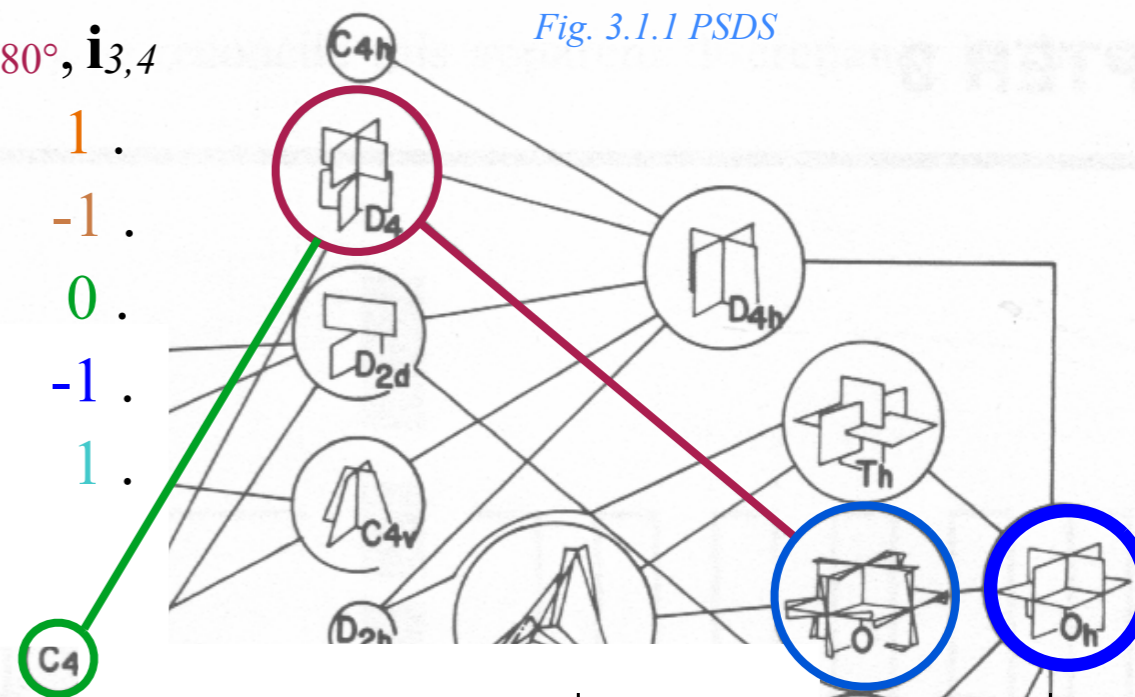


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

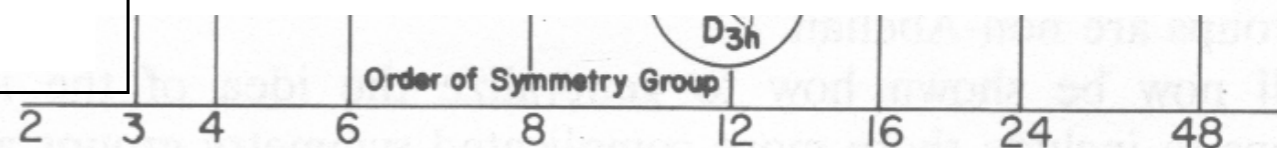
| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

$O \downarrow C_4$ subduction

| $O \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|--------------------|-------|-------|-------|-------------------|
| A_1 | 1 | · | · | · |
| A_2 | · | · | 1 | · |
| E | 1 | · | 1 | · |
| T_1 | →1 | →1 | · | →1 |
| T_2 | | | | |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-------|-------|-------------------|
| A_1 | 1 | · | · | · |
| B_1 | · | · | 1 | · |
| A_2 | →1 | · | · | · |
| B_2 | · | · | 1 | · |
| E | · | →1 | · | →1 |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

D_4 : $\mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

| $\chi_g^\mu(O)$ | $g=1$ | $\mathbf{r}_{1..4}$ | 180° ρ_{xyz} | 90° \mathbf{R}_{xyz} | 180° $\mathbf{i}_{1..6}$ |
|-----------------|-------|---------------------|-----------------------------|----------------------------------|------------------------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| E | 2 | -1 | 2 | 0 | 0 |
| T_1 | 3 | 0 | -1 | 1 | -1 |
| T_2 | 3 | 0 | -1 | -1 | 1 |

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1$
 $A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1$
 $E(O) \downarrow D_4 = 2, 2, 0, 2, 0$
 $T_1(O) \downarrow D_4 = 3, -1, 1, -1, -1$
 $T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1$

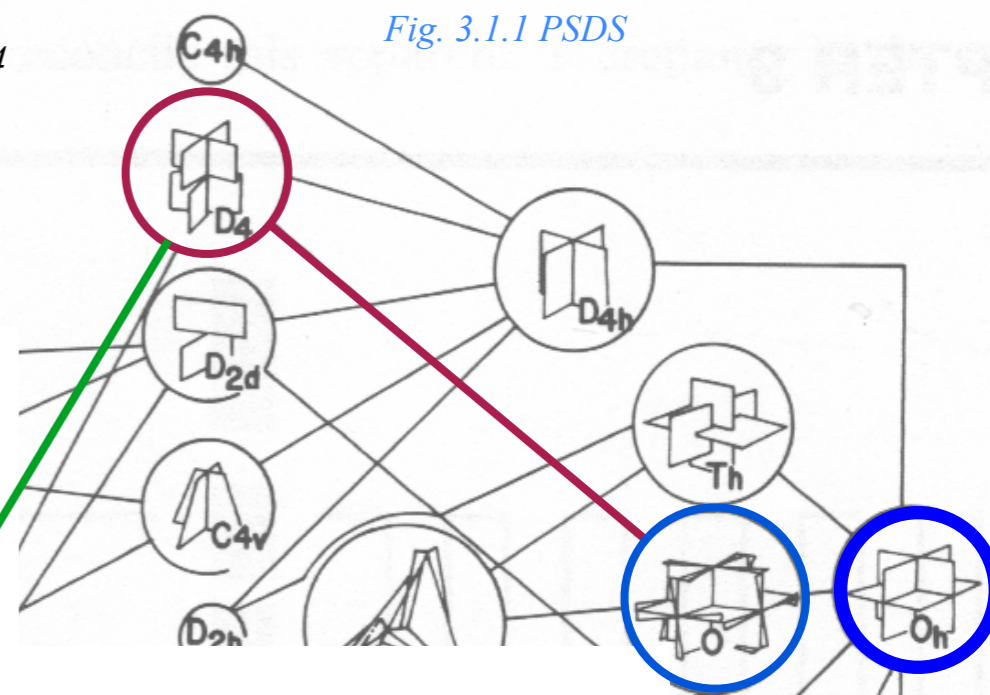


Fig. 3.1.1 PSDS

| $\chi_g^\mu(D_4)$ | $g=1$ | ρ_{z180° | $\mathbf{R}_{z\pm 90^\circ}$ | $\rho_{x,y180^\circ}$ | $\mathbf{i}_{3,4}$ |
|-------------------|-------|---------------------|------------------------------|-----------------------|--------------------|
| A_1 | 1 | 1 | 1 | 1 | 1 |
| B_1 | 1 | 1 | -1 | 1 | -1 |
| A_2 | 1 | 1 | 1 | -1 | -1 |
| B_2 | 1 | 1 | -1 | -1 | 1 |
| E | 2 | -2 | 0 | 0 | 0 |

$D_4 \downarrow C_4$ subduction

C_4 : $\mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_1(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $A_2(D_4) \downarrow C_4 = 1, 1, 1, 1 = (0)_4$
 $B_2(D_4) \downarrow C_4 = 1, -1, 1, -1 = (2)_4$
 $E(D_4) \downarrow C_4 = 2, 0, -2, 0 = (1)_4 \oplus (3)_4$

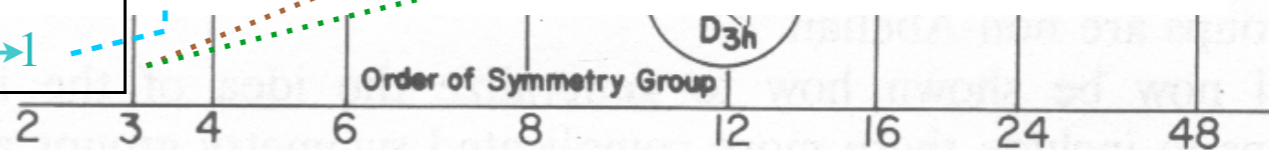
| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

| $\chi_g^\mu(C_4)$ | $g=1$ | \mathbf{R}_{z+90° | \mathbf{R}_{z+180° | \mathbf{R}_{z-90° |
|-------------------|-------|---------------------------|----------------------------|---------------------------|
| $(0)_4$ | 1 | 1 | 1 | 1 |
| $(1)_4$ | 1 | i | -1 | $-i$ |
| $(2)_4$ | 1 | -1 | 1 | -1 |
| $(3)_4$ | 1 | $-i$ | -1 | i |

$O \downarrow C_4$ subduction

| $O \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|--------------------|-------|-----------------|-----------------|-------------------|
| A_1 | 1 | · | · | · |
| A_2 | · | · | 1 | · |
| E | 1 | · | 1 | · |
| T_1 | 1 | 1 | · | 1 |
| T_2 | · | $\rightarrow 1$ | $\rightarrow 1$ | $\rightarrow 1$ |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-----------------|-----------------|-------------------|
| A_1 | 1 | · | · | · |
| B_1 | · | · | 1 | · |
| A_2 | 1 | · | · | · |
| B_2 | · | · | $\rightarrow 1$ | · |
| E | · | $\rightarrow 1$ | · | $\rightarrow 1$ |



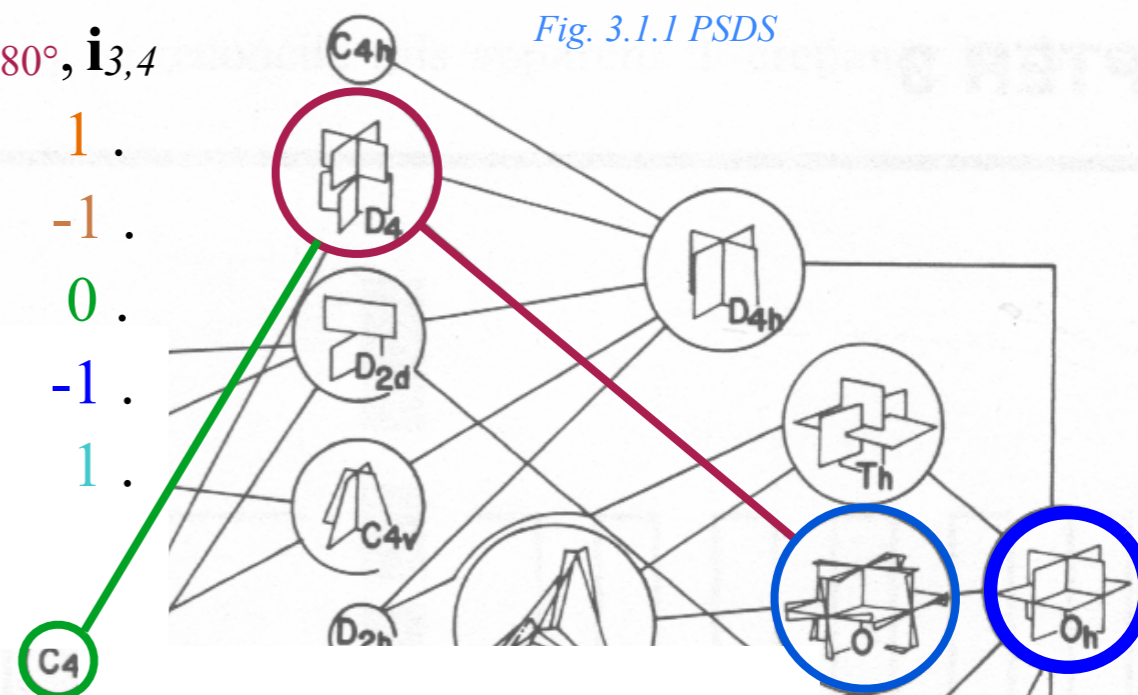
Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$$

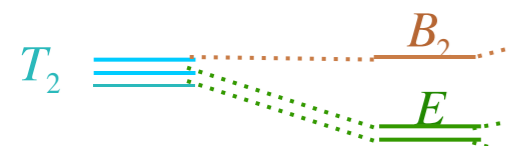
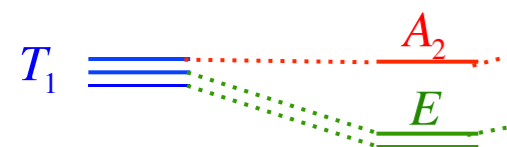
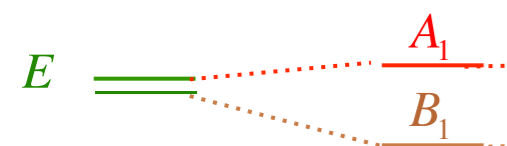
$$\begin{aligned} A_1(O) \downarrow D_4 &= 1, 1, 1, 1, 1. \\ A_2(O) \downarrow D_4 &= 1, 1, -1, 1, -1. \\ E(O) \downarrow D_4 &= 2, 2, 0, 2, 0. \\ T_2(O) \downarrow D_4 &= 3, -1, 1, -1, -1. \\ T_2(O) \downarrow D_4 &= 3, -1, -1, -1, 1. \end{aligned}$$

Fig. 3.1.1 PSDS



$O \supset D_4 \supset C_4$ subgroup and level-splitting/relabeling correlations

O levels \downarrow D_4 levels



$D_4 \downarrow C_4$ subduction

$$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$$

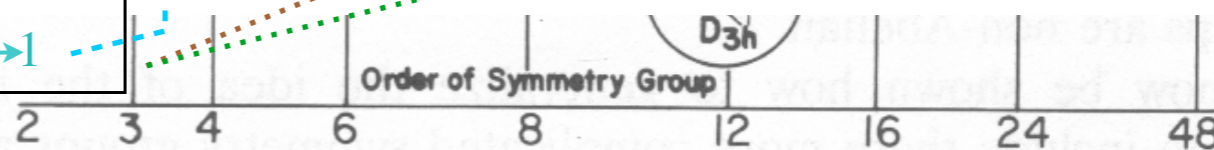
$$\begin{aligned} A_1(D_4) \downarrow C_4 &= 1, 1, 1, 1. = (0)_4 \\ B_1(D_4) \downarrow C_4 &= 1, -1, 1, -1. = (2)_4 \\ A_2(D_4) \downarrow C_4 &= 1, 1, 1, 1. = (0)_4 \\ B_2(D_4) \downarrow C_4 &= 1, -1, 1, -1. = (2)_4 \\ E(D_4) \downarrow C_4 &= 2, 0, -2, 0. = (1)_4 \oplus (3)_4 \end{aligned}$$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

$O \downarrow C_4$ subduction

| $O \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|--------------------|-------|-----------------|-----------------|-------------------|
| A_1 | 1 | · | · | · |
| A_2 | · | · | 1 | · |
| E | 1 | · | 1 | · |
| T_1 | 1 | 1 | · | 1 |
| T_2 | · | $\rightarrow 1$ | $\rightarrow 1$ | $\rightarrow 1$ |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-----------------|-----------------|-------------------|
| A_1 | 1 | · | · | · |
| B_1 | · | · | 1 | · |
| A_2 | 1 | · | · | · |
| B_2 | · | · | $\rightarrow 1$ | · |
| E | · | $\rightarrow 1$ | · | $\rightarrow 1$ |



Octahedral $O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

$O \downarrow D_4$ subduction

$D_4: \mathbf{1}, \rho_{z180^\circ}, \mathbf{R}_{z\pm 90^\circ}, \rho_{x,y180^\circ}, \mathbf{i}_{3,4}$

$A_1(O) \downarrow D_4 = 1, 1, 1, 1, 1.$

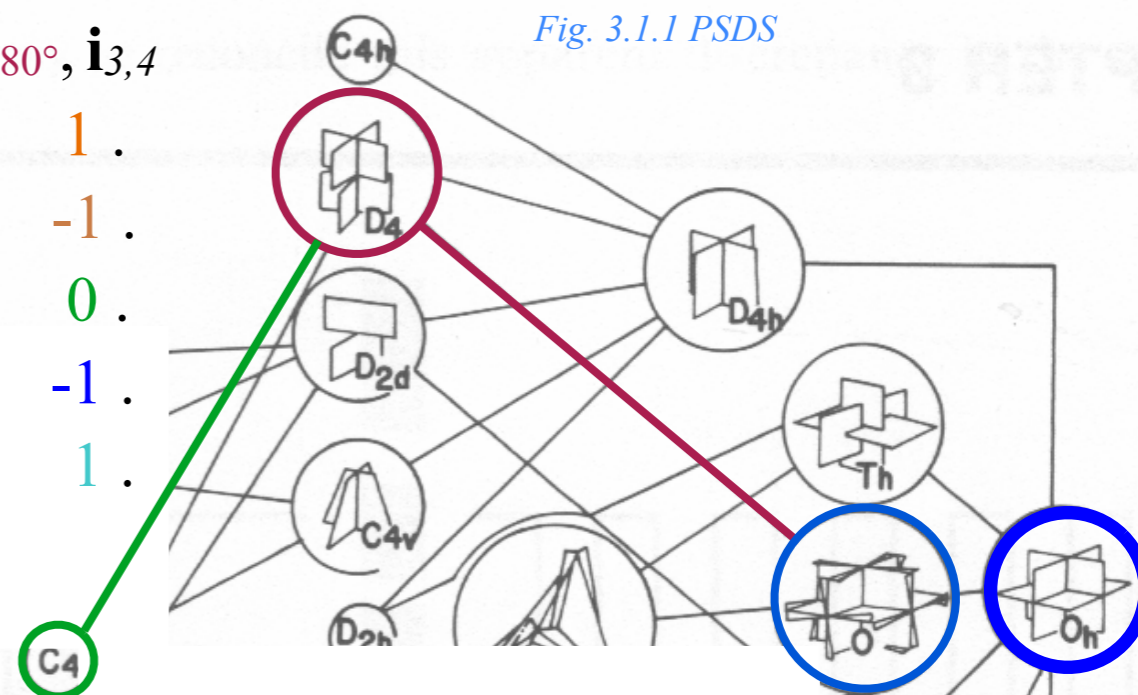
$A_2(O) \downarrow D_4 = 1, 1, -1, 1, -1.$

$E(O) \downarrow D_4 = 2, 2, 0, 2, 0.$

$T_2(O) \downarrow D_4 = 3, -1, 1, -1, -1.$

$T_2(O) \downarrow D_4 = 3, -1, -1, -1, 1.$

Fig. 3.1.1 PSDS



$O \supset D_4 \supset C_4$ subgroup and level-splitting/relabeling correlations

O levels \downarrow D_4 levels \downarrow C_4 levels

$A_1 \xrightarrow{A_1} 0_4$

$A_2 \xrightarrow{B_1} 2_4$

$E \xrightarrow{A_1, B_1} 0_4, 2_4$

$T_1 \xrightarrow{A_2, E} 0_4, 1_4, \bar{1}_4$

$T_2 \xrightarrow{B_2, E} 2_4, 1_4, \bar{1}_4$

$D_4 \downarrow C_4$ subduction

$C_4: \mathbf{1}, \mathbf{R}_{z+90^\circ}, \rho_{z180^\circ}, \mathbf{R}_{z-90^\circ}$

$A_1(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$

$B_1(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$

$A_2(D_4) \downarrow C_4 = 1, 1, 1, 1. = (0)_4$

$B_2(D_4) \downarrow C_4 = 1, -1, 1, -1. = (2)_4$

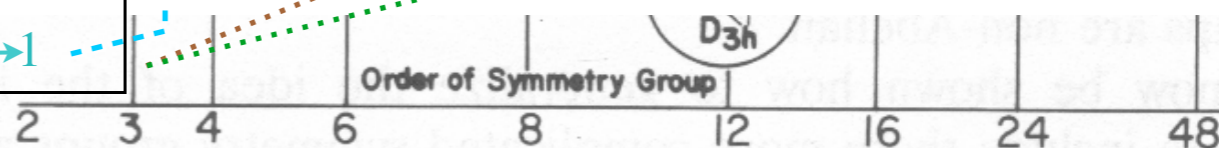
$E(D_4) \downarrow C_4 = 2, 0, -2, 0. = (1)_4 \oplus (3)_4$

| $O \downarrow D_4$ | A_1 | B_1 | A_2 | B_2 | E |
|--------------------|-------|-------|-------|-------|-----|
| A_1 | 1 | · | · | · | · |
| A_2 | · | 1 | · | · | · |
| E | 1 | 1 | · | · | · |
| T_1 | · | · | 1 | · | 1 |
| T_2 | · | · | · | 1 | 1 |

$O \downarrow C_4$ subduction

| $O \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|--------------------|-------|-----------------|-----------------|-------------------|
| A_1 | 1 | · | · | · |
| A_2 | · | · | 1 | · |
| E | 1 | · | 1 | · |
| T_1 | 1 | 1 | · | 1 |
| T_2 | · | $\rightarrow 1$ | $\rightarrow 1$ | $\rightarrow 1$ |

| $D_4 \downarrow C_4$ | 0_4 | 1_4 | 2_4 | $3_4 = \bar{1}_4$ |
|----------------------|-------|-----------------|-----------------|-------------------|
| A_1 | 1 | · | · | · |
| B_1 | · | · | 1 | · |
| A_2 | 1 | · | · | · |
| B_2 | · | · | $\rightarrow 1$ | · |
| E | · | $\rightarrow 1$ | · | $\rightarrow 1$ |



Introduction to octahedral/ tetrahedral symmetry $O_h \supset O \sim T_d \supset T$

Octahedral-cubic O symmetry and group operations

Tetrahedral symmetry becomes Icosahedral

Octahedral groups $O_h \supset O \sim T_d \supset T$

Octahedral O and spin- $O \subset U(2)$

Tetrahedral T class algebra

Tetrahedral T class minimal equations

Tetrahedral T class projectors and characters

Octahedral O class algebra

Octahedral O class minimal equations

Octahedral O class projectors and characters

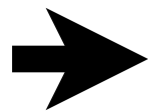
Octahedral $O_h \supset O$: Inversion (g&u) parity

Octahedral $O_h \supset O \supset C_{\infty}$ subgroup correlations

$O_h \supset O \supset D_4$ subgroup correlations

$O_h \supset O \supset D_4 \supset C_4$ subgroup correlations

Preview of applications to high resolution spectroscopy



Summary of some Octahedral symmetry results:

$$l^{A_1} = 1$$

$$l^{A_2} = 1$$

$$l^E = 2$$

$$l^{T_1} = 3$$

$$l^{T_2} = 3$$

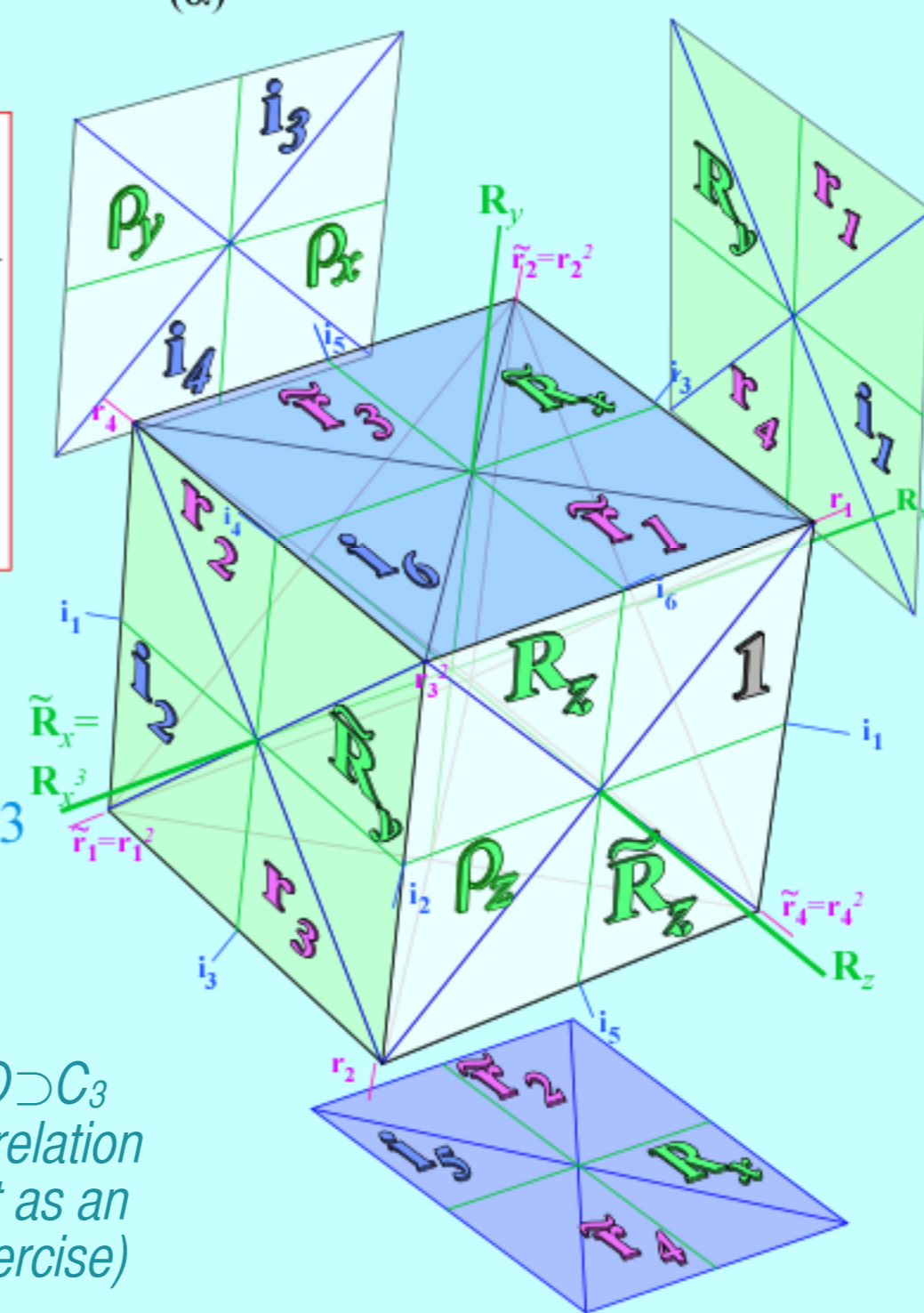
**Cubic-Octahedral
Group O**

Centrum: $\kappa(O) = \sum_{(\alpha)} (l^\alpha)^0 = 1^0 + 1^0 + 2^0 + 3^0 + 3^0 = 5$

Rank: $\rho(O) = \sum_{(\alpha)} (l^\alpha)^1 = 1^1 + 1^1 + 2^1 + 3^1 + 3^1 = 10$

Order: $o(O) = \sum_{(\alpha)} (l^\alpha)^2 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2 = 24$

| O group | $g = 1$ | r_{1-4} | ρ_{xyz} | R_{xyz} | i_{1-6} |
|---|---------|-------------------|--------------|-------------------|-----------|
| $\chi_{\kappa g}^\alpha$ | | \tilde{r}_{1-4} | | \tilde{R}_{xyz} | |
| $\alpha = A_1$ <i>s-orbital r^2</i> | 1 | 1 | 1 | 1 | 1 |
| A_2 <i>d-orbitals</i> | 1 | 1 | 1 | -1 | -1 |
| E $\{x^2+y^2-2z^2, x^2-y^2\}$ <i>d-orbitals</i> | 2 | -1 | 2 | 0 | 0 |
| T_1 $\{x, y, z\}$ <i>p-orbitals</i> | 3 | 0 | -1 | 1 | -1 |
| T_2 $\{xz, yz, xy\}$ <i>d-orbitals</i> | 3 | 0 | -1 | -1 | 1 |



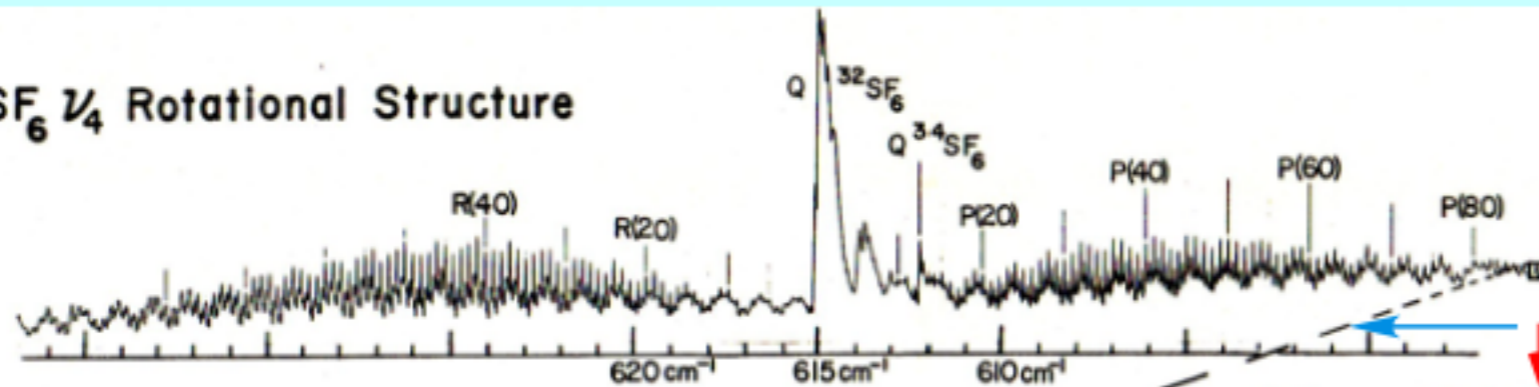
$O \supset C_4$ $(0)_4 (1)_4 (2)_4 (3)_4 = (-1)_4$ $O \supset C_3$ $(0)_3 (1)_3 (2)_3 = (-1)_3$

| | | | | |
|-------|---|---|---|---|
| A_1 | 1 | • | • | • |
| A_2 | • | • | 1 | • |
| E | 1 | • | 1 | • |
| T_1 | 1 | 1 | • | 1 |
| T_2 | • | 1 | 1 | 1 |

| | | | |
|-------|---|---|---|
| A_1 | 1 | • | • |
| A_2 | 1 | • | • |
| E | • | 1 | 1 |
| T_1 | 1 | 1 | 1 |
| T_2 | 1 | 1 | 1 |

*($O \supset C_3$
correlation
left as an
exercise)*

(a) SF₆ ν₄ Rotational Structure



FT IR and Laser Diode Spectra
K.C. Kim, W.B. Person, D. Seitz, and B.J. Krohn
J. Mol. Spectrosc. 76, 322 (1979).

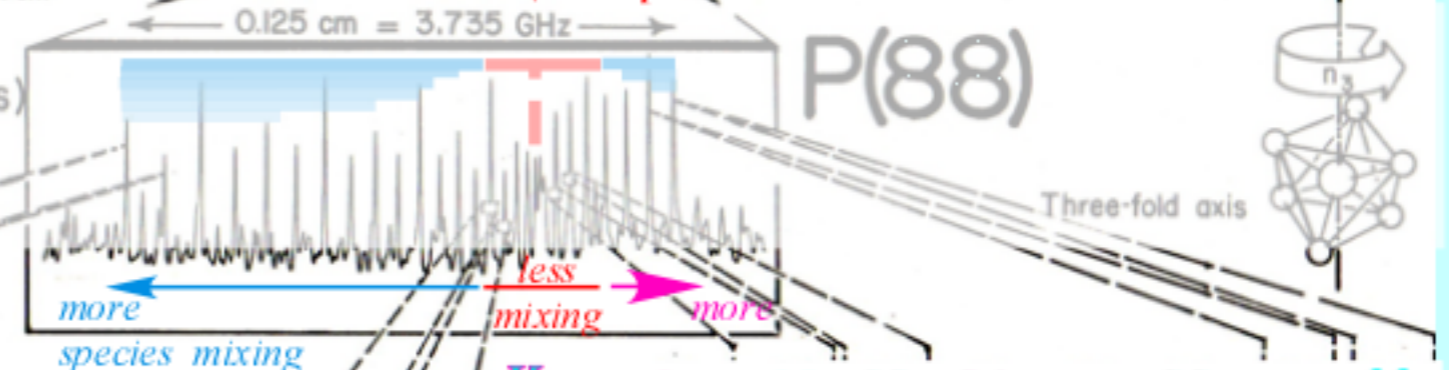
Primary AET species mixing increases with distance from "separatrix"

(b) P(88) Fine Structure (Rotational anisotropy effects)

SF₆ ν₃ P(88) ~ 16m

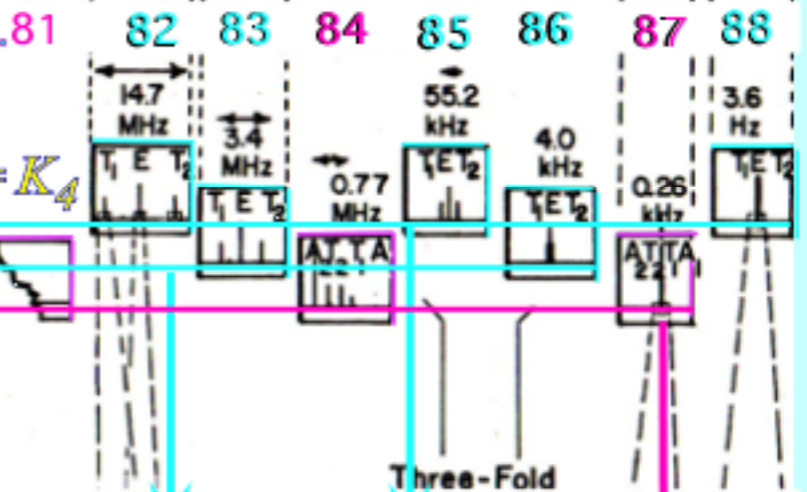
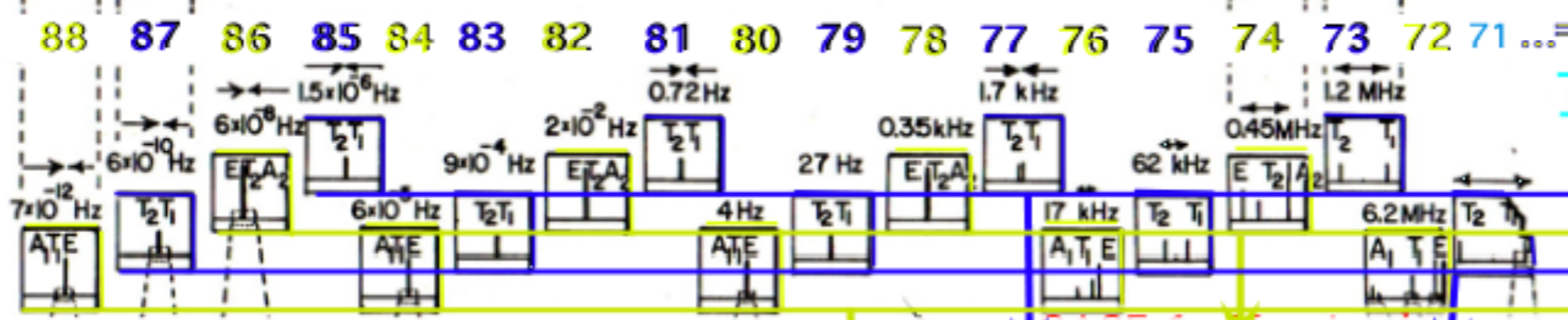


Four fold axis



Three-fold axis

(c) Superfine Structure (Rotational axis tunneling)



Observed repeating sequence(s) .. A₁ T₁ E T₂ T₁ E T₂ A₂ T₂ T₁ A₁ T₁ E T₂ T₁ E T₂ A₂ T₂ T₁ A₁ ..

O=C₄ (0)₄ (1)₄ (2)₄ (3)₄ = (-1)₄

O=C₃ (0)₃ (1)₃ (2)₃ = (-1)₃

| | | | | |
|----------------|---|---|---|---|
| A ₁ | 1 | • | • | • |
| A ₂ | • | • | 1 | • |
| E | 1 | • | 1 | • |
| T ₁ | 1 | 1 | • | 1 |
| T ₂ | • | 1 | 1 | 1 |

| | | | |
|----------------|---|---|---|
| A ₁ | 1 | • | • |
| A ₂ | 1 | • | • |
| E | • | 1 | 1 |
| T ₁ | 1 | 1 | 1 |
| T ₂ | 1 | 1 | 1 |

Local correlations explain clustering...
... but what about spacing and ordering?...

...and physical consequences?

major mixing lowest two LUSTERS

Eigenvalues of $\mathbf{H} = B\mathbf{J}^2 + \cos\phi\mathbf{T}^{[4]} + \sin\phi\mathbf{T}^{[6]}$ vs. mix angle $\phi: 0 < \phi < \pi$

