Group Theory in Quantum Mechanics
Lecture 16-DR

(Review of Lectures 15-17 with more detailed and rigorous derivations)

Projector algebra and Hamiltonian local-symmetry eigensolution

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)
(PSDS - Ch. 4)

Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

Review: General formulae for spectral decomposition (D_3 examples)

- *Weyl* **g***-expansion in irep $D^{\mu}_{jk}(g)$ and projectors* $\mathbf{P}^{\mu}_{jk}$
  - $\mathbf{P}^{\mu}_{jk}$ *transforms right-and-left*
  - $\mathbf{P}^{\mu}_{jk}$ *-expansion in* **g***-operators*

$D^{\mu}_{jk}(g)$ orthogonality relations *Details omitted from Lecture 15-17*

Class projector character formulae

- $\mathbb{P}^{\mu}$ *in terms of $\boldsymbol{\kappa}_{\mathbf{g}}$ and $\boldsymbol{\kappa}_{\mathbf{g}}$ in terms of* $\mathbb{P}^{\mu}$

Review: Details of Mock-Mach relativity-duality for D_3 groups and representations

- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis*

Review: Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis

- *Hamiltonian local-symmetry eigensolution*

Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)
- *Weyl* **g***-expansion in irep $D^{\mu}_{jk}(g)$ and projectors $\mathbf{P}^{\mu}_{jk}$*
  - *$\mathbf{P}^{\mu}_{jk}$ transforms right-and-left*
  - *$\mathbf{P}^{\mu}_{jk}$ -expansion in* **g***-operators*
- *$D^{\mu}_{jk}(g)$ orthogonality relations*
- *Class projector character formulae*
  - *$\mathbb{P}^{\mu}$ in terms of $\kappa_{\mathbf{g}}$ and $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$*

Details of Mock-Mach relativity-duality for D_3 groups and representations
- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis*

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis
- *Hamiltonian local-symmetry eigensolution*

Review: Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

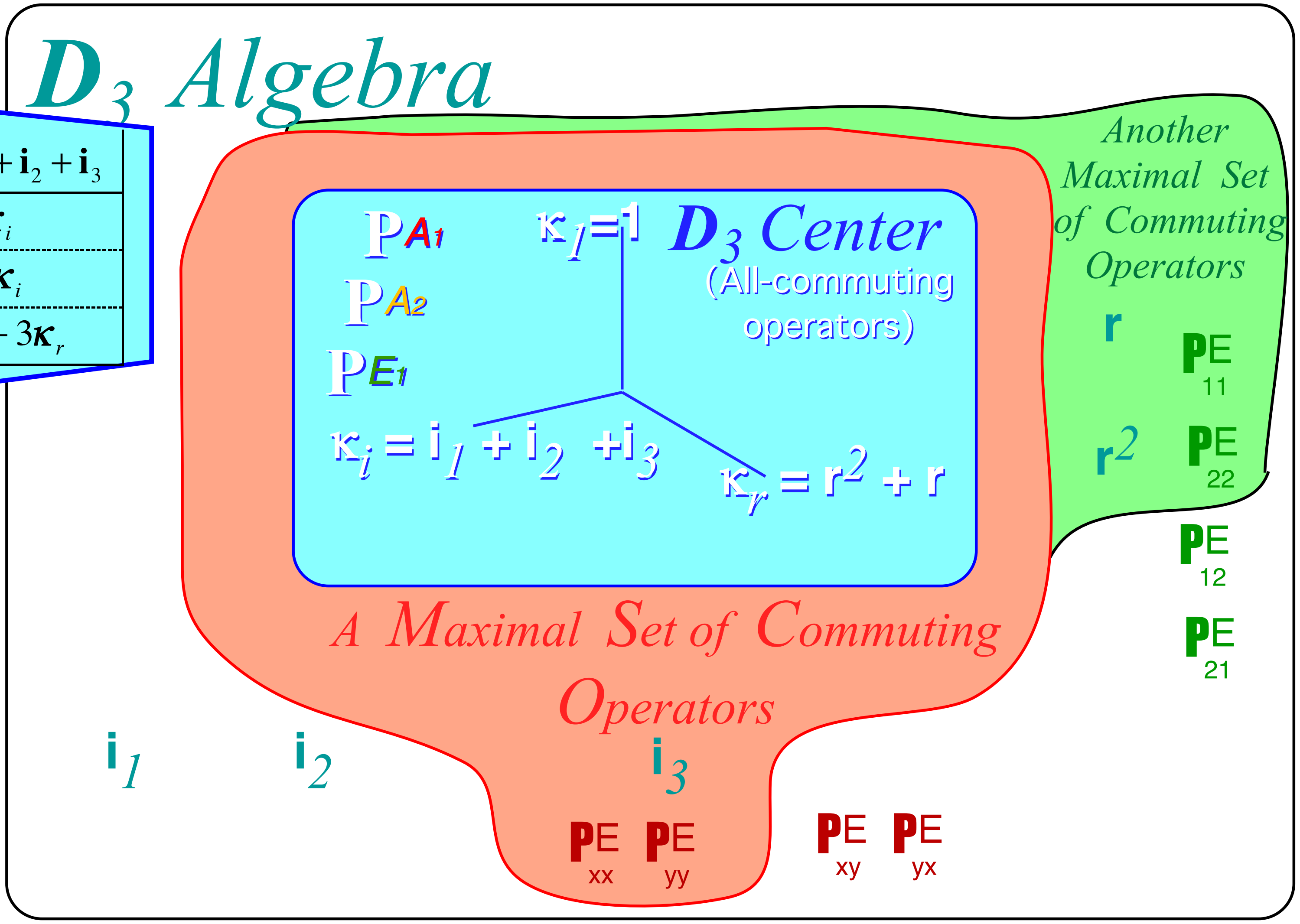
	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$



See Lect.14 p. 2-23

Review: *Spectral resolution of **D_3 Center** (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

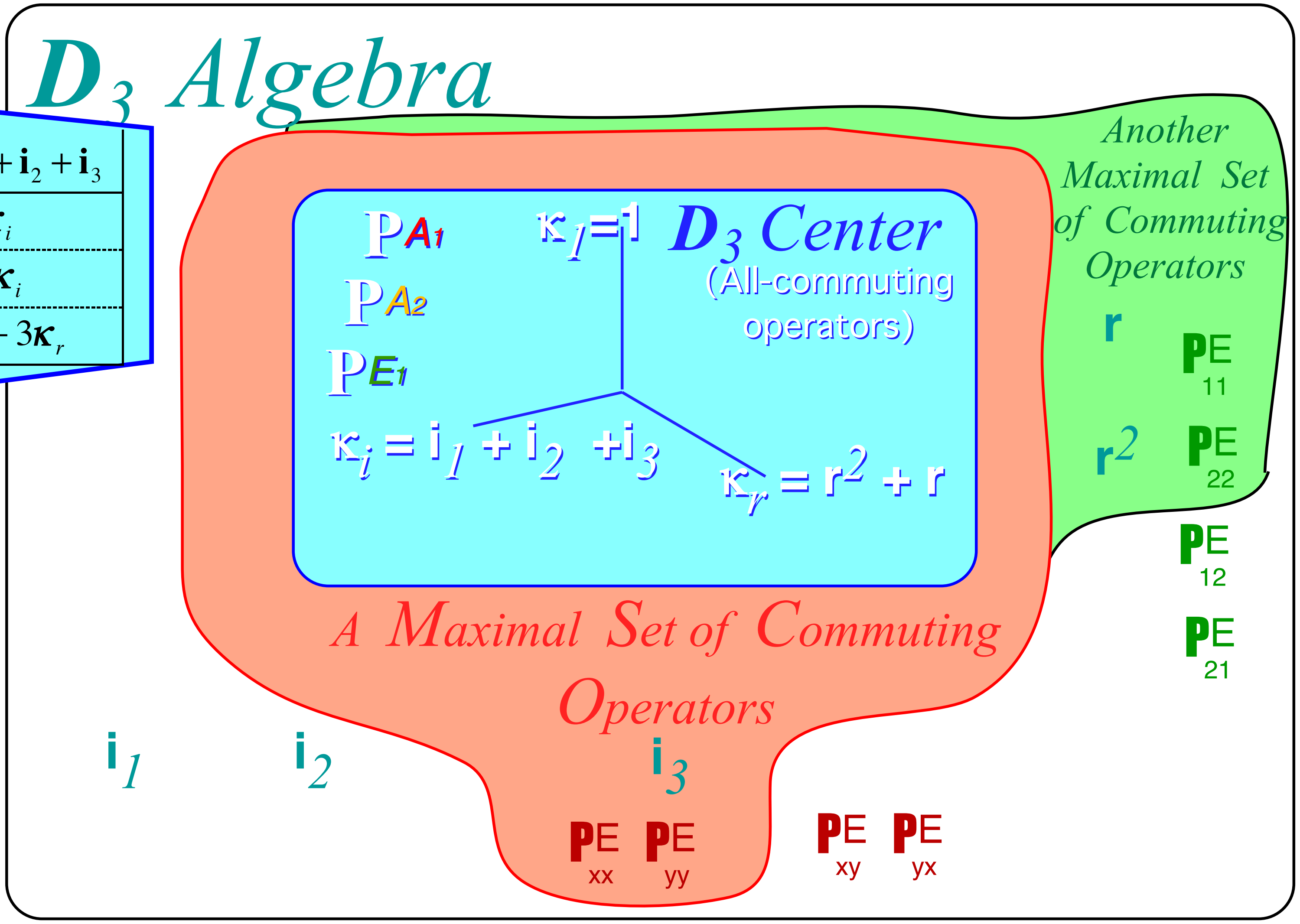
Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$\boldsymbol{\kappa}_1 = 1 \cdot \mathbf{P}^{A_1} + 1 \cdot \mathbf{P}^{A_2} + 1 \cdot \mathbf{P}^{E} = \mathbf{1}$ *(Class completeness)*



See Lect.14 p. 2-23

Review: *Spectral resolution of* ***D₃ Center*** *(Class algebra)*

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

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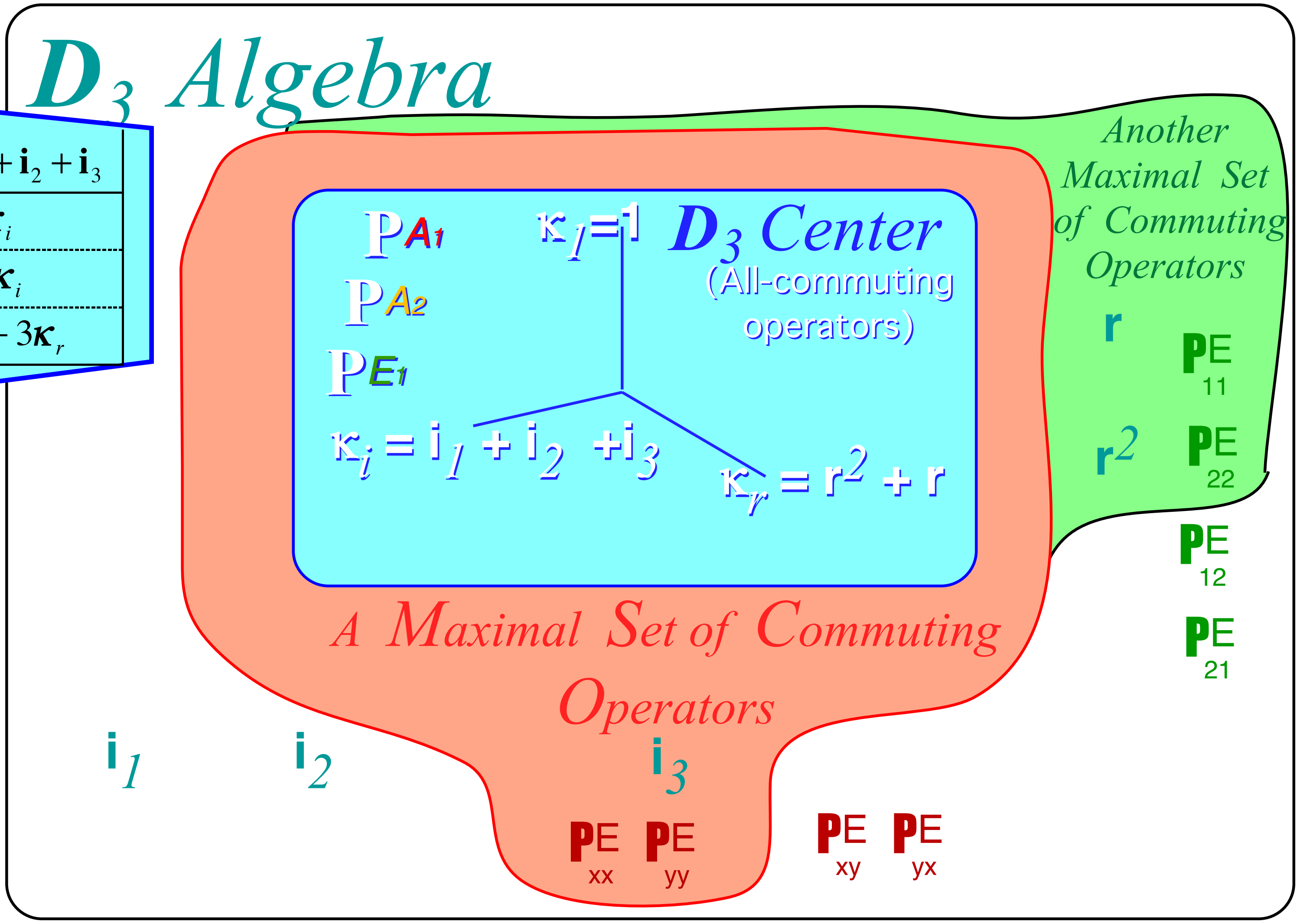
$$\boldsymbol{\kappa}_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E} = \mathbf{1}$$ *(Class completeness)*

$$\boldsymbol{\kappa}_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$



See Lect.14 p. 2-23

Review: Spectral resolution of **D_3 Center** (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

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$$\boldsymbol{\kappa}_r=2\cdot\mathbf{P}^{A_1}+2\cdot\mathbf{P}^{A_2}-1\cdot\mathbf{P}^{E}$$

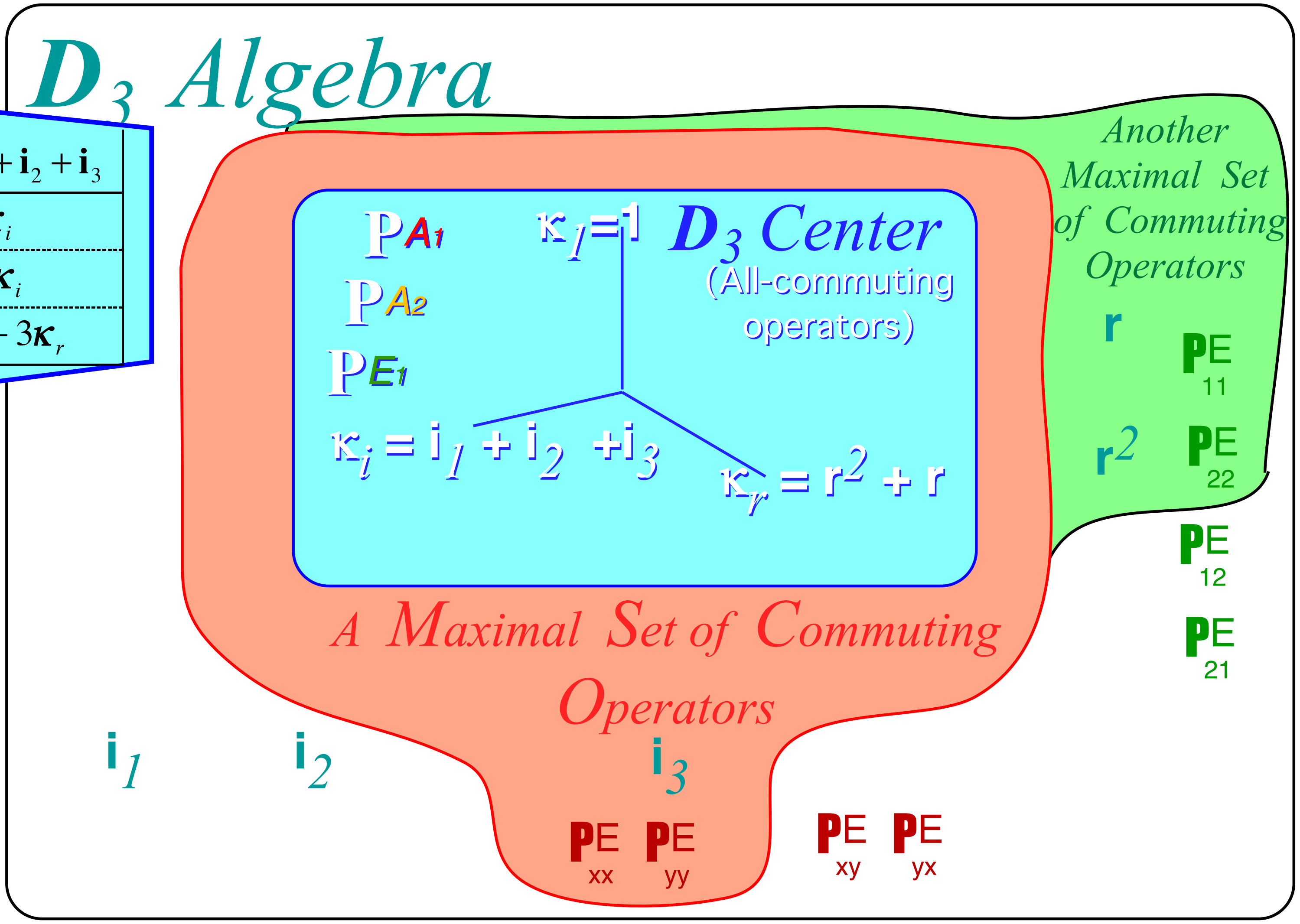
$$\boldsymbol{\kappa}_i=3\cdot\mathbf{P}^{A_1}-3\cdot\mathbf{P}^{A_2}+0\cdot\mathbf{P}^{E}$$

Class projectors:

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r+\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}=(2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_r+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$$



See Lect.14 p. 2-23

Review: *Spectral resolution of* ***D_3 Center*** *(Class algebra)*

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
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$$\boldsymbol{\kappa}_i = 3 \cdot \mathbf{P}^{A_1} - 3 \cdot \mathbf{P}^{A_2} + 0 \cdot \mathbf{P}^{E}$$

Class projectors:

$$\mathbf{P}^{A_1} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_r + \boldsymbol{\kappa}_i)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

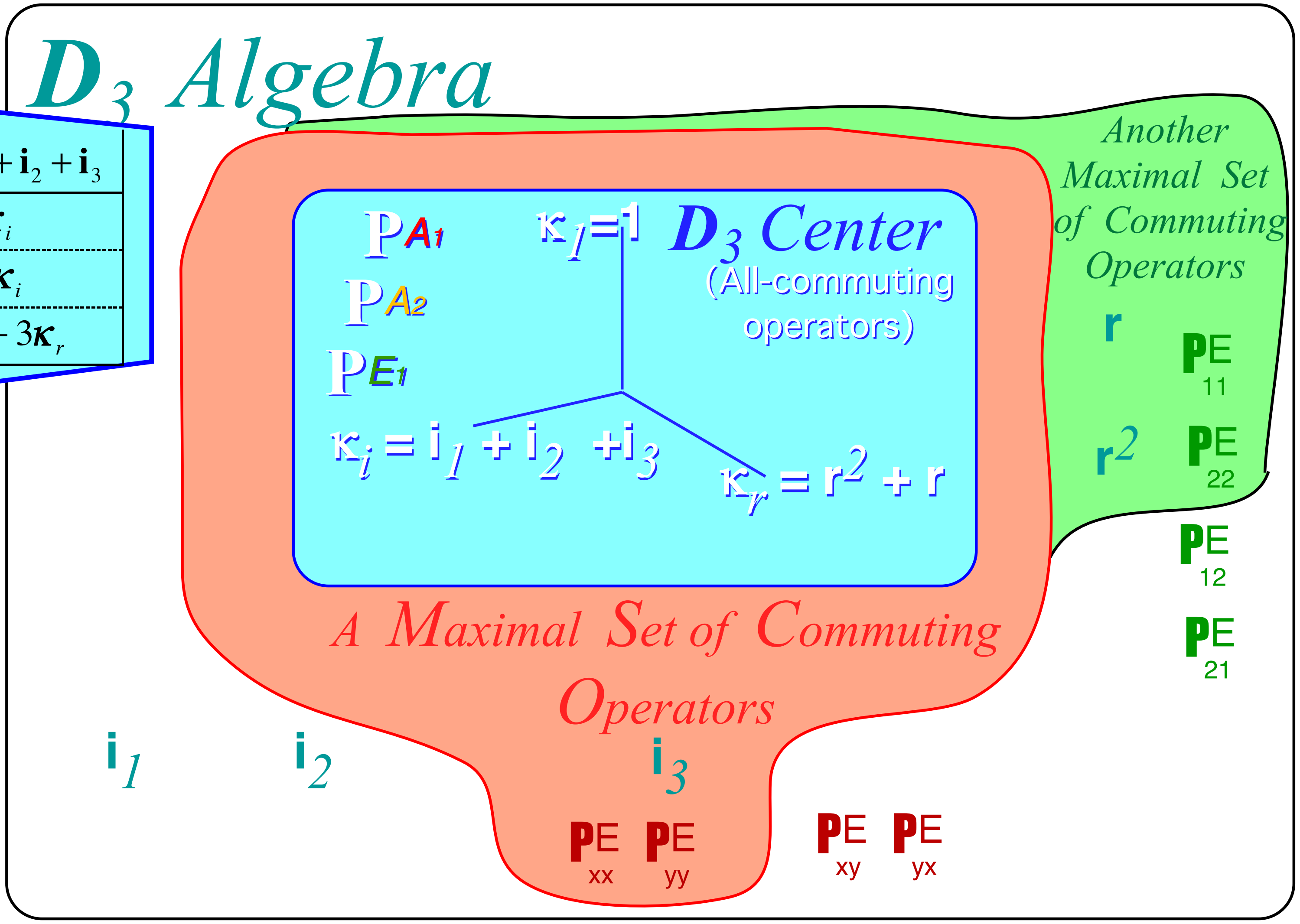
$$\mathbf{P}^{A_2} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_r - \boldsymbol{\kappa}_i)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_r + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

Class characters:

χ_k^α	χ_1^α	χ_r^α	χ_i^α
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	-1
$\alpha = E$	2	-1	0

See Lect.14 p. 2-23



Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)
Weyl **g***-expansion in irep $D^{\mu}_{jk}(g)$ and projectors* $\mathbf{P}^{\mu}_{jk}$
$\mathbf{P}^{\mu}_{jk}$ *transforms right-and-left*
$\mathbf{P}^{\mu}_{jk}$ *-expansion in* **g***-operators*
$D^{\mu}_{jk}(g)$ orthogonality relations
Class projector character formulae
$\mathbb{P}^{\mu}$ *in terms of* $\kappa_{\mathbf{g}}$ *and* $\kappa_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Details of Mock-Mach relativity-duality for D_3 groups and representations
Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)
Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis
Hamiltonian local-symmetry eigensolution

Review: *Spectral resolution of **D_3 Center** (Class algebra) and its subgroup splitting*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\boldsymbol{\kappa}_1=\mathbf{1}$	$\boldsymbol{\kappa}_r=\mathbf{r}+\mathbf{r}^2$	$\boldsymbol{\kappa}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_i$	$\boldsymbol{\kappa}_i$	$2\boldsymbol{\kappa}_i$	$3\boldsymbol{\kappa}_1+3\boldsymbol{\kappa}_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

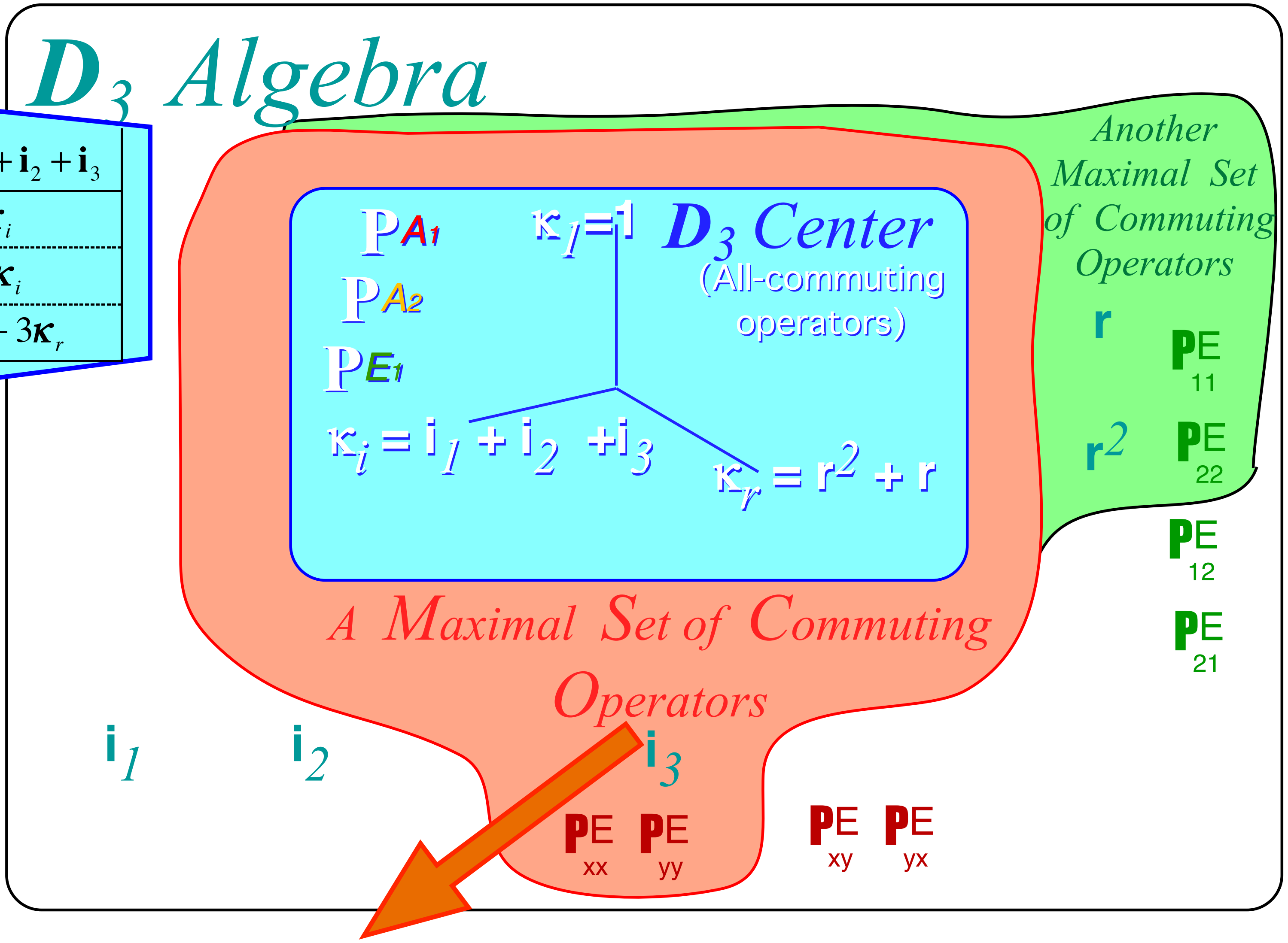
$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\boldsymbol{\kappa}_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E} = \mathbf{1}$$ *(Class completeness)*

$$\boldsymbol{\kappa}_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$



Subgroup $C_2=\{\mathbf{1},\mathbf{i}_3\}$ relabels irreducible class projectors:

Class projectors:

$$\mathbf{P}^{A_1} = (\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r+\boldsymbol{\kappa}_i)/6 = (\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6 \rightarrow \mathbf{P}^{A_1}=\mathbf{P}^{A_1}_{0_2 0_2}$$

$$\mathbf{P}^{A_2} = (\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_i)/6 = (\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6 \rightarrow \mathbf{P}^{A_2}=\mathbf{P}^{A_2}_{1_2 1_2}$$

$$\mathbf{P}^{E} = (2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_r+0)/3 = (2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$$

Class characters:

χ^α_k	χ^α_1	χ^α_r	χ^α_i
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	−1
$\alpha = E$	2	−1	0

See Lect.14 p. 36-54

Review: Spectral resolution of D_3 Center (Class algebra) and its subgroup splitting

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

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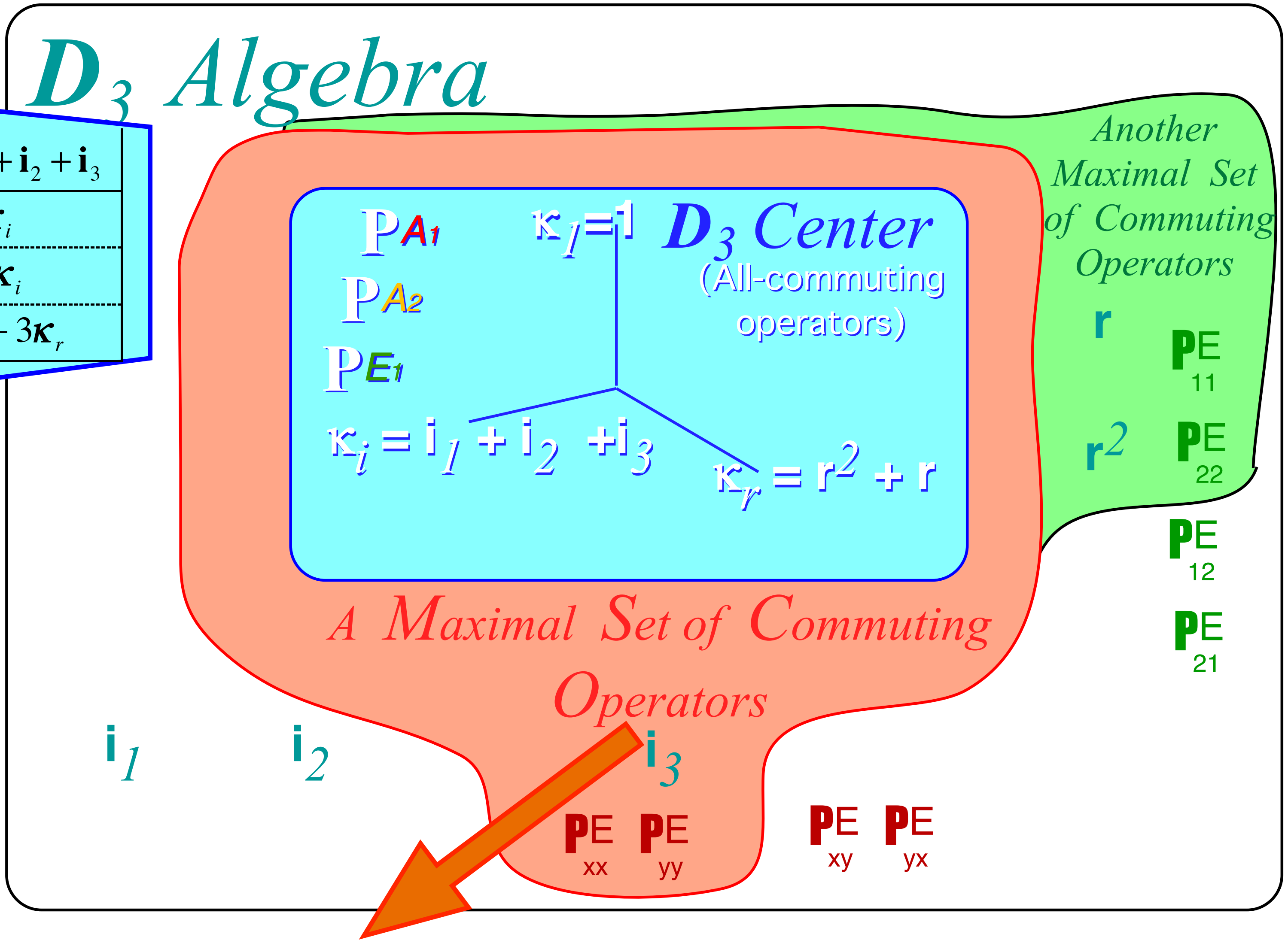
$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\boldsymbol{\kappa}_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E} = \mathbf{1}$$ (Class completeness)

$$\boldsymbol{\kappa}_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$



Subgroup $C_2 = \{\mathbf{1}, \mathbf{i}_3\}$ relabels irreducible class projectors:

Class projectors:

$$\mathbf{P}^{A_1} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_r + \boldsymbol{\kappa}_i)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6 \rightarrow \mathbf{P}^{A_1} = \mathbf{P}^{A_1}_{0_2 0_2}$$

$$\mathbf{P}^{A_2} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_r - \boldsymbol{\kappa}_i)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6 \rightarrow \mathbf{P}^{A_2} = \mathbf{P}^{A_2}_{1_2 1_2}$$

...and splits reducible projector $\mathbf{P}^{E_1} = \mathbf{P}^{E_1}_{0_2 0_2} + \mathbf{P}^{E_1}_{1_2 1_2}$

$$\mathbf{P}^{E} = (2\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_r + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

$$\mathbf{P}^E_{0_2 0_2} = \mathbf{P}^E \mathbf{p}^{0_2} = \mathbf{P}^E \tfrac{1}{2}(\mathbf{1} + \mathbf{i}_3) = \tfrac{1}{6}(2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)$$

$$+\mathbf{P}^E_{1_2 1_2} = \mathbf{P}^E \mathbf{p}^{1_2} = \mathbf{P}^E \tfrac{1}{2}(\mathbf{1} + \mathbf{i}_3) = \tfrac{1}{6}(2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)$$

$$= \tfrac{1}{3}(2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2)$$

Class characters:

χ^α_k	χ^α_1	χ^α_r	χ^α_i
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	−1
$\alpha = E$	2	−1	0

See Lect.14 p. 36-54

Review: *Spectral resolution of **D₃ Center** (Class algebra) and its subgroup splitting*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r}+\mathbf{r}^2$	$\kappa_i = \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

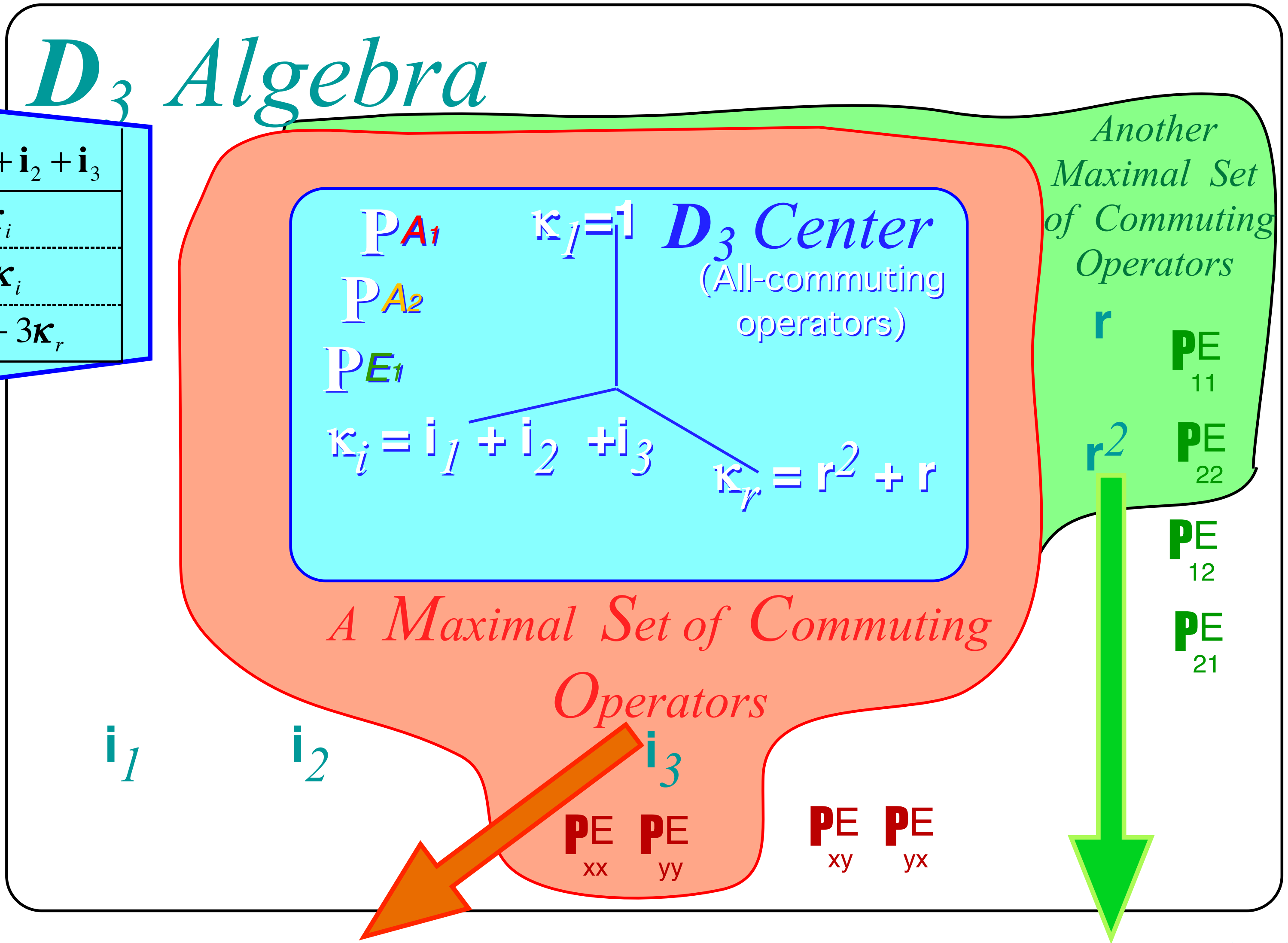
$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$\boldsymbol{\kappa}_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E} = \mathbf{1}$ *(Class completeness)*

$\boldsymbol{\kappa}_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$

$\boldsymbol{\kappa}_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$



Subgroup $C_2=\{\mathbf{1},\mathbf{i}_3\}$ *relabels irreducible class projectors:*

Subgroup $C_3=\{\mathbf{1},\mathbf{r}^1,\mathbf{r}^2\}$ *does similarly:*

Class projectors:

$\mathbf{P}^{A_1} = (\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r+\boldsymbol{\kappa}_i)/6 = (\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6 \rightarrow \mathbf{P}^{A_1}=\mathbf{P}^{A_1}_{0_2 0_2}$

$\mathbf{P}^{A_2} = (\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_i)/6 = (\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6 \rightarrow \mathbf{P}^{A_2}=\mathbf{P}^{A_2}_{1_2 1_2}$

$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}_{0_3 0_3}$

$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}_{0_3 0_3}$

...and splits reducible projector $\mathbf{P}^{E_1}=\mathbf{P}^{E_1}_{0_2 0_2}+\mathbf{P}^{E_1}_{1_2 1_2}$

$\mathbf{P}^{E} = (2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_r+0)/3 = (2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$

$$\mathbf{P}^E_{0_2 0_2} = \mathbf{P}^E\mathbf{p}^{0_2} = \mathbf{P}^E\tfrac{1}{2}(\mathbf{1}+\mathbf{i}_3) = \tfrac{1}{6}(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)$$

$$+\mathbf{P}^E_{1_2 1_2} = \mathbf{P}^E\mathbf{p}^{1_2} = \mathbf{P}^E\tfrac{1}{2}(\mathbf{1}+\mathbf{i}_3) = \tfrac{1}{6}(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)$$

$$= \tfrac{1}{3}(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2)$$

Class characters:

χ^α_k	χ^α_1	χ^α_r	χ^α_i
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	−1
$\alpha = E$	2	−1	0

See Lect.14 p. 36-54

Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

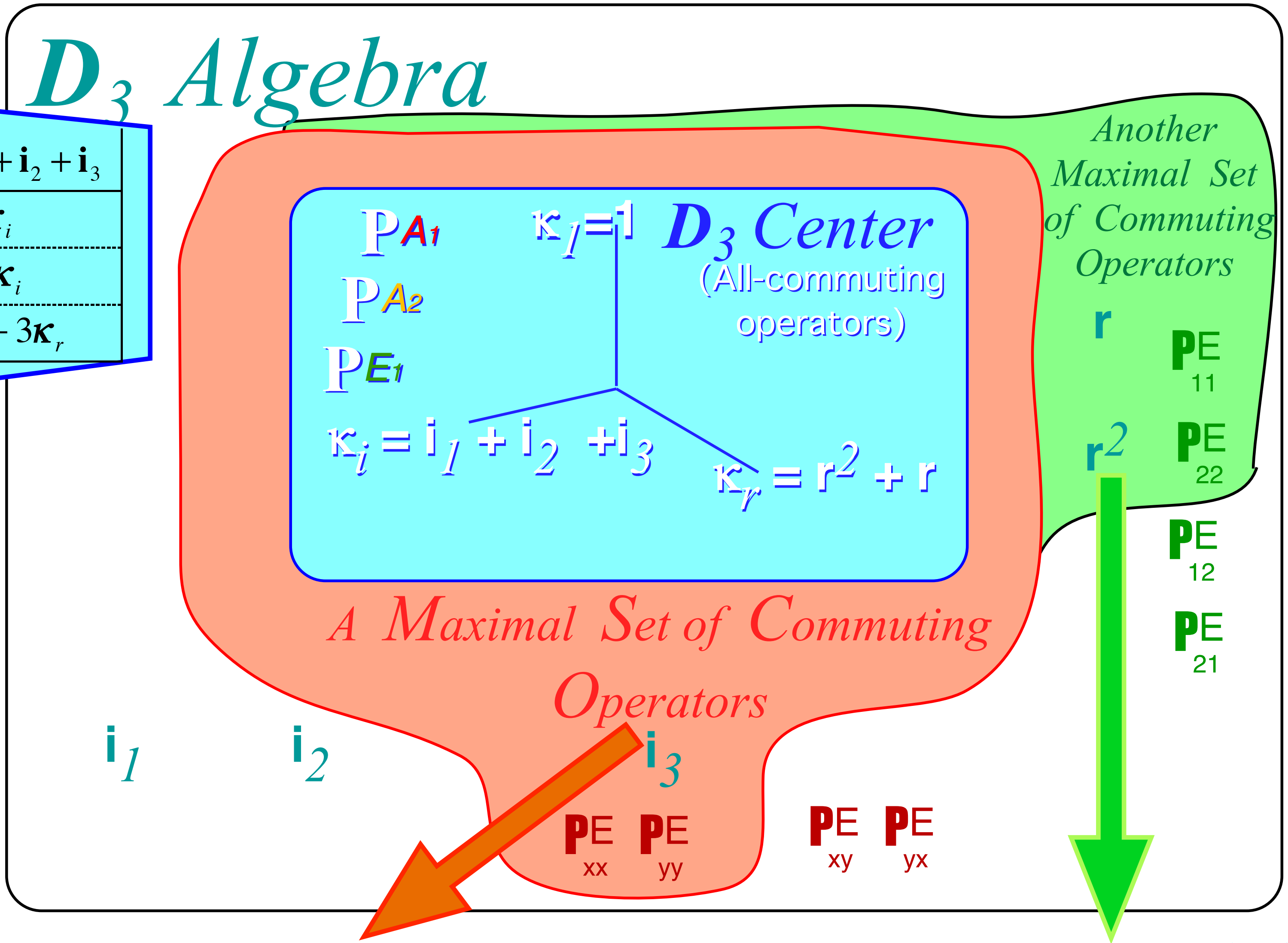
$°G$ = order of group: $(°D_3=6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

$$\boldsymbol{\kappa}_1=1\cdot\mathbf{P}^{A_1}+1\cdot\mathbf{P}^{A_2}+1\cdot\mathbf{P}^{E}=\mathbf{1}$$ *(Class completeness)*

$$\boldsymbol{\kappa}_r=2\cdot\mathbf{P}^{A_1}+2\cdot\mathbf{P}^{A_2}-1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i=3\cdot\mathbf{P}^{A_1}-3\cdot\mathbf{P}^{A_2}+0\cdot\mathbf{P}^{E}$$



Subgroup $C_2=\{\mathbf{1},\mathbf{i}_3\}$ relabels irreducible class projectors:

Class projectors:

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r+\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6\rightarrow\mathbf{P}^{A_1}=\mathbf{P}^{A_1}_{0_20_2}$$

$$\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6\rightarrow\mathbf{P}^{A_2}=\mathbf{P}^{A_2}_{1_21_2}$$

$$\mathbf{P}^{E}=(2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_r+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$$

...and splits reducible projector $\mathbf{P}^{E_1}=\mathbf{P}^{E_1}_{0_20_2}+\mathbf{P}^{E_1}_{1_21_2}$

$$\mathbf{P}^E_{0_20_2}=\mathbf{P}^E\mathbf{p}^{0_2}=\mathbf{P}^E\tfrac{1}{2}(\mathbf{1}+\mathbf{i}_3)=\tfrac{1}{6}(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)$$

$$\underline{+\mathbf{P}^E_{1_21_2}=\mathbf{P}^E\mathbf{p}^{1_2}=\mathbf{P}^E\tfrac{1}{2}(\mathbf{1}+\mathbf{i}_3)=\tfrac{1}{6}(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)}$$

$$=\tfrac{1}{3}(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2)$$

$$\mathbf{P}^E_{1_31_3}=\mathbf{P}^E\mathbf{p}^{1_3}=\mathbf{P}^E\tfrac{1}{3}(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)$$

$$\underline{+\mathbf{P}^E_{2_32_3}=\mathbf{P}^E\mathbf{p}^{2_3}=\mathbf{P}^E\tfrac{1}{3}(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)}$$

$$=\tfrac{1}{3}(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2)$$

Subgroup $C_3=\{\mathbf{1},\mathbf{r}^1,\mathbf{r}^2\}$ does similarly:

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}_{0_30_3}$$

$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}_{0_30_3}$$

...and splits $\mathbf{P}^{E_1}=\mathbf{P}^{E_1}_{0_30_3}+\mathbf{P}^{E_1}_{1_31_3}$ differently

Class characters:

χ^α_k	χ^α_1	χ^α_r	χ^α_i
$\alpha=A_1$	1	1	1
$\alpha=A_2$	1	1	−1
$\alpha=E$	2	−1	0

See Lect.14 p. 36-54

*Review: Spectral resolution of **D_3 Center** (Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)

- *Weyl **g**-expansion in irep $D^{\mu}_{jk}(g)$ and projectors $\mathbf{P}^{\mu}_{jk}$*
  - *$\mathbf{P}^{\mu}_{jk}$ transforms right-and-left*
  - *$\mathbf{P}^{\mu}_{jk}$ -expansion in **g**-operators*
- *$D^{\mu}_{jk}(g)$ orthogonality relations*
- *Class projector character formulae*
  - *$\mathbb{P}^{\mu}$ in terms of $\boldsymbol{\kappa}_{\mathbf{g}}$ and $\boldsymbol{\kappa}_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$*

Details of Mock-Mach relativity-duality for D_3 groups and representations

- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis*

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis

- *Hamiltonian local-symmetry eigensolution*

Weyl expansion of $\mathbf{g}$ *in irep* $D^{\mu}{}_{jk}(g)\mathbf{P}^{\mu}{}_{jk}$

“$\mathbf{g}$*-equals-*$\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$*-trick”*

Irreducible idempotent completeness $\mathbf{1} = \mathbf{P}^{A_1} + \mathbf{P}^{A_2} + \mathbf{P}^{E_1}_{xx} + \mathbf{P}^{E_1}_{yy}$ *completely expands group by* $\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$

$$\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1} = \sum_{\mu}\sum_{m}\sum_{n} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn} = D^{A_1}_{xx}(g)\,\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\,\mathbf{P}^{A_2} + D^{E_1}_{xx}(g)\,\mathbf{P}^{E_1}_{xx} + D^{E_1}_{xy}(g)\,\mathbf{P}^{E_1}_{xy}$$

$$+ D^{E_1}_{yx}(g)\,\mathbf{P}^{E_1}_{yx} + D^{E_1}_{yy}(g)\,\mathbf{P}^{E_1}_{yy}$$

For irreducible class idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are optional*

Previous notation:

$\mathbf{P}^{A_1}_{0_2 0_2} = \mathbf{P}^{A_1}_{xx}$

$\mathbf{P}^{A_2}_{1_2 1_2} = \mathbf{P}^{A_2}_{yy}$

$\mathbf{P}^{E_1}_{0_2 0_2} = \mathbf{P}^{E_1}_{xx}$ $\mathbf{P}^{E_1}_{0_2 1_2} = \mathbf{P}^{E_1}_{xy}$

$\mathbf{P}^{E_1}_{1_2 0_2} = \mathbf{P}^{E_1}_{yx}$ $\mathbf{P}^{E_1}_{1_2 1_2} = \mathbf{P}^{E_1}_{yy}$

Weyl expansion of **g** *in irep* $D^{\mu}{}_{jk}(g)\mathbf{P}^{\mu}{}_{jk}$

*"***g***-equals-***1·g·1***-trick"*

Irreducible idempotent completeness $\mathbf{1}=\mathbf{P}^{A_1}+\mathbf{P}^{A_2}+\mathbf{P}^{E_1}_{xx}+\mathbf{P}^{E_1}_{yy}$ *completely expands group by* $\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$

Previous notation:

$\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}_{xx}$

$\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}_{yy}$

$\mathbf{P}^{E_1}_{0_2 0_2}=\mathbf{P}^{E_1}_{xx}$ $\quad\mathbf{P}^{E_1}_{0_2 1_2}=\mathbf{P}^{E_1}_{xy}$

$\mathbf{P}^{E_1}_{1_2 0_2}=\mathbf{P}^{E_1}_{yx}$ $\quad\mathbf{P}^{E_1}_{1_2 1_2}=\mathbf{P}^{E_1}_{yy}$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=\sum_{\mu}\sum_{m}\sum_{n}D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn}=D^{A_1}_{xx}(g)\,\mathbf{P}^{A_1}+D^{A_2}_{yy}(g)\,\mathbf{P}^{A_2}+D^{E_1}_{xx}(g)\,\mathbf{P}^{E_1}_{xx}+D^{E_1}_{xy}(g)\,\mathbf{P}^{E_1}_{xy}$$

$$+D^{E_1}_{yx}(g)\,\mathbf{P}^{E_1}_{yx}+D^{E_1}_{yy}(g)\,\mathbf{P}^{E_1}_{yy}$$

For irreducible class idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are* *<u>optional</u>*

where:

$$\mathbf{P}^{A_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{xx}=D^{A_1}_{xx}(g)\,\mathbf{P}^{A_1}_{xx},\quad \mathbf{P}^{A_2}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{yy}=D^{A_2}_{yy}(g)\,\mathbf{P}^{A_2}_{yy},\quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx}=D^{E_1}_{xx}(g)\,\mathbf{P}^{E_1}_{xx},$$

For split idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are* *<u>essential</u>*

$$\mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy}=D^{E_1}_{yy}(g)\,\mathbf{P}^{E_1}_{yy}$$

Weyl expansion of **g** *in irep* $D^{\mu}{}_{jk}(g)\mathbf{P}^{\mu}{}_{jk}$

*"***g***-equals-***1·g·1***-trick"*

Irreducible idempotent completeness $\mathbf{1} = \mathbf{P}^{A_1} + \mathbf{P}^{A_2} + \mathbf{P}^{E_1}_{xx} + \mathbf{P}^{E_1}_{yy}$ *completely expands group by* $\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$

$$\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1} = \sum_{\mu}\sum_{m}\sum_{n} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\mathbf{P}^{A_2} + D^{E_1}_{xx}(g)\mathbf{P}^{E_1}_{xx} + D^{E_1}_{xy}(g)\mathbf{P}^{E_1}_{xy}$$

$$+ D^{E_1}_{yx}(g)\mathbf{P}^{E_1}_{yx} + D^{E_1}_{yy}(g)\mathbf{P}^{E_1}_{yy}$$

For irreducible class idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are* *<u>optional</u>*

where:

$$\mathbf{P}^{A_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{xx} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1}_{xx},\quad \mathbf{P}^{A_2}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{yy} = D^{A_2}_{yy}(g)\mathbf{P}^{A_2}_{yy},\quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx} = D^{E_1}_{xx}(g)\mathbf{P}^{E_1}_{xx},$$

For split idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are* *<u>essential</u>*

$$\mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy} = D^{E_1}_{yy}(g)\mathbf{P}^{E_1}_{yy}$$

Previous notation:

$\mathbf{P}^{A_1}_{0202} = \mathbf{P}^{A_1}_{xx}$

$\mathbf{P}^{A_2}_{1212} = \mathbf{P}^{A_2}_{yy}$

$\mathbf{P}^{E_1}_{0202} = \mathbf{P}^{E_1}_{xx}$ $\quad\mathbf{P}^{E_1}_{0212} = \mathbf{P}^{E_1}_{xy}$

$\mathbf{P}^{E_1}_{1202} = \mathbf{P}^{E_1}_{yx}$ $\quad\mathbf{P}^{E_1}_{1212} = \mathbf{P}^{E_1}_{yy}$

Besides four *idempotent* projectors $\mathbf{P}^{A_1}, \mathbf{P}^{A_2}, \mathbf{P}^{E_1}_{xx}$, and $\mathbf{P}^{E_1}_{yy}$

Weyl expansion of **g** *in irep* $D^{\mu}{}_{jk}(g)\mathbf{P}^{\mu}{}_{jk}$

*"***g***-equals-***1**·**g**·**1***-trick"*

Irreducible idempotent completeness $\mathbf{1} = \mathbf{P}^{A_1} + \mathbf{P}^{A_2} + \mathbf{P}^{E_1}_{xx} + \mathbf{P}^{E_1}_{yy}$ *completely expands group by* $\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$

$$\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1} = \sum_{\mu}\sum_{m}\sum_{n} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn} = D^{A_1}_{xx}(g)\,\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\,\mathbf{P}^{A_2} + D^{E_1}_{xx}(g)\,\mathbf{P}^{E_1}_{xx} + D^{E_1}_{xy}(g)\,\mathbf{P}^{E_1}_{xy}$$

$$+ D^{E_1}_{yx}(g)\,\mathbf{P}^{E_1}_{yx} + D^{E_1}_{yy}(g)\,\mathbf{P}^{E_1}_{yy}$$

Previous notation:

$\mathbf{P}^{E_1}_{0_2 0_2} = \mathbf{P}^{E_1}_{xx}$ $\quad\mathbf{P}^{E_1}_{0_2 1_2} = \mathbf{P}^{E_1}_{xy}$

$\mathbf{P}^{E_1}_{1_2 0_2} = \mathbf{P}^{E_1}_{yx}$ $\quad\mathbf{P}^{E_1}_{1_2 1_2} = \mathbf{P}^{E_1}_{yy}$

For irreducible class idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are optional*

where:

$$\mathbf{P}^{A_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{xx} = D^{A_1}_{xx}(g)\,\mathbf{P}^{A_1}_{xx}, \quad \mathbf{P}^{A_2}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{yy} = D^{A_2}_{yy}(g)\,\mathbf{P}^{A_2}_{yy}, \quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx} = D^{E_1}_{xx}(g)\,\mathbf{P}^{E_1}_{xx}, \quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy} = D^{E_1}_{xy}(g)\,\mathbf{P}^{E_1}_{xy}$$

For split idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are essential*

$$, \quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx} = D^{E_1}_{yx}(g)\,\mathbf{P}^{E_1}_{yx}, \quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy} = D^{E_1}_{yy}(g)\,\mathbf{P}^{E_1}_{yy}$$

Besides four *idempotent* projectors $\mathbf{P}^{A_1}, \mathbf{P}^{A_2}, \mathbf{P}^{E_1}_{xx}$, and $\mathbf{P}^{E_1}_{yy}$

there arise two *nilpotent* projectors $\mathbf{P}^{E_1}_{yx}$, and $\mathbf{P}^{E_1}_{xy}$

Weyl expansion of $\mathbf{g}$ *in irep* $D^{\mu}{}_{jk}(g)\mathbf{P}^{\mu}{}_{jk}$

"$\mathbf{g}$*-equals-*$\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$*-trick"*

Irreducible idempotent completeness $\mathbf{1}=\mathbf{P}^{A_1}+\mathbf{P}^{A_2}+\mathbf{P}^{E_1}_{xx}+\mathbf{P}^{E_1}_{yy}$ *completely expands group by* $\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$

Previous notation:

$\mathbf{P}^{E_1}_{0_2 0_2}=\mathbf{P}^{E_1}_{xx}$ $\mathbf{P}^{E_1}_{0_2 1_2}=\mathbf{P}^{E_1}_{xy}$

$\mathbf{P}^{E_1}_{1_2 0_2}=\mathbf{P}^{E_1}_{yx}$ $\mathbf{P}^{E_1}_{1_2 1_2}=\mathbf{P}^{E_1}_{yy}$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=\sum_{\mu}\sum_{m}\sum_{n}D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn}=D^{A_1}_{xx}(g)\mathbf{P}^{A_1}+D^{A_2}_{yy}(g)\mathbf{P}^{A_2}+D^{E_1}_{xx}(g)\mathbf{P}^{E_1}_{xx}+D^{E_1}_{xy}(g)\mathbf{P}^{E_1}_{xy}$$

$$+D^{E_1}_{yx}(g)\mathbf{P}^{E_1}_{yx}+D^{E_1}_{yy}(g)\mathbf{P}^{E_1}_{yy}$$

For irreducible class idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are* *optional*

where:

$$\mathbf{P}^{A_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{xx}=D^{A_1}_{xx}(g)\mathbf{P}^{A_1}_{xx},\quad \mathbf{P}^{A_2}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{yy}=D^{A_2}_{yy}(g)\mathbf{P}^{A_2}_{yy},\quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx}=D^{E_1}_{xx}(g)\mathbf{P}^{E_1}_{xx},\quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy}=D^{E_1}_{xy}(g)\mathbf{P}^{E_1}_{xy}$$

For split idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are* *essential*

$$,\quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx}=D^{E_1}_{yx}(g)\mathbf{P}^{E_1}_{yx},\quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy}=D^{E_1}_{yy}(g)\mathbf{P}^{E_1}_{yy}$$

Besides four *idempotent* projectors $\mathbf{P}^{A_1},\mathbf{P}^{A_2},\mathbf{P}^{E_1}_{xx}$, and $\mathbf{P}^{E_1}_{yy}$

there arise two *nilpotent* projectors $\mathbf{P}^{E_1}_{yx}$, and $\mathbf{P}^{E_1}_{xy}$

Idempotent projector orthogonality... $\boxed{\mathbf{P}_i\,\mathbf{P}_j=\delta_{ij}\,\mathbf{P}_i=\mathbf{P}_j\,\mathbf{P}_i}$

Generalizes...

Weyl expansion of **g** *in irep* $D^{\mu}{}_{jk}(g)\mathbf{P}^{\mu}{}_{jk}$

*"***g***-equals-***1**·**g**·**1***-trick"*

Irreducible idempotent completeness $\mathbf{1} = \mathbf{P}^{A_1} + \mathbf{P}^{A_2} + \mathbf{P}^{E_1}_{xx} + \mathbf{P}^{E_1}_{yy}$ *completely expands group by* $\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$

$$\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1} = \sum_{\mu}\ \sum_{m}\ \sum_{n} D^{\mu}_{mn}\left(g\right)\mathbf{P}^{\mu}_{mn} = D^{A_1}_{xx}\left(g\right)\mathbf{P}^{A_1} + D^{A_2}_{yy}\left(g\right)\mathbf{P}^{A_2} + D^{E_1}_{xx}\left(g\right)\mathbf{P}^{E_1}_{xx} + D^{E_1}_{xy}\left(g\right)\mathbf{P}^{E_1}_{xy}$$

$$+ D^{E_1}_{yx}\left(g\right)\mathbf{P}^{E_1}_{yx} + D^{E_1}_{yy}\left(g\right)\mathbf{P}^{E_1}_{yy}$$

Previous notation:

$\mathbf{P}^{E_1}_{0_2 0_2} = \mathbf{P}^{E_1}_{xx}$ $\mathbf{P}^{E_1}_{0_2 1_2} = \mathbf{P}^{E_1}_{xy}$

$\mathbf{P}^{E_1}_{1_2 0_2} = \mathbf{P}^{E_1}_{yx}$ $\mathbf{P}^{E_1}_{1_2 1_2} = \mathbf{P}^{E_1}_{yy}$

For irreducible class idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are optional*

where:

$$\mathbf{P}^{A_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{xx} = D^{A_1}_{xx}\left(g\right)\mathbf{P}^{A_1}_{xx}, \quad \mathbf{P}^{A_2}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{yy} = D^{A_2}_{yy}\left(g\right)\mathbf{P}^{A_2}_{yy}, \quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx} = D^{E_1}_{xx}\left(g\right)\mathbf{P}^{E_1}_{xx}, \quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy} = D^{E_1}_{xy}\left(g\right)\mathbf{P}^{E_1}_{xy}$$

For split idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are essential*

$$, \quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx} = D^{E_1}_{yx}\left(g\right)\mathbf{P}^{E_1}_{yx}, \quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy} = D^{E_1}_{yy}\left(g\right)\mathbf{P}^{E_1}_{yy}$$

Besides four *idempotent* projectors $\mathbf{P}^{A_1}, \mathbf{P}^{A_2}, \mathbf{P}^{E_1}_{xx}$, and $\mathbf{P}^{E_1}_{yy}$

there arise two *nilpotent* projectors $\mathbf{P}^{E_1}_{yx}$, and $\mathbf{P}^{E_1}_{xy}$

Idempotent projector orthogonality… $\boxed{\mathbf{P}_i\,\mathbf{P}_j = \delta_{ij}\,\mathbf{P}_i = \mathbf{P}_j\,\mathbf{P}_i}$

Generalizes to idempotent/nilpotent orthogonality
known as Simple Matrix Algebra: $\boxed{\mathbf{P}^{\mu}_{jk}\mathbf{P}^{\nu}_{mn} = \delta^{\mu\nu}\delta_{km}\mathbf{P}^{\mu}_{jn}}$

Weyl expansion of $\mathbf{g}$ *in irep* $D^{\mu}{}_{jk}(g)\mathbf{P}^{\mu}{}_{jk}$

"g-equals-1·g·1-trick"

Irreducible idempotent completeness $\mathbf{1}=\mathbf{P}^{A_1}+\mathbf{P}^{A_2}+\mathbf{P}^{E_1}_{xx}+\mathbf{P}^{E_1}_{yy}$ *completely expands group by* $\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=\sum_{\mu}\sum_{m}\sum_{n}D^{\mu}_{mn}(g)\mathbf{P}^{\mu}_{mn}=D^{A_1}_{xx}(g)\mathbf{P}^{A_1}+D^{A_2}_{yy}(g)\mathbf{P}^{A_2}+D^{E_1}_{xx}(g)\mathbf{P}^{E_1}_{xx}+D^{E_1}_{xy}(g)\mathbf{P}^{E_1}_{xy}$$

$$+D^{E_1}_{yx}(g)\mathbf{P}^{E_1}_{yx}+D^{E_1}_{yy}(g)\mathbf{P}^{E_1}_{yy}$$

Previous notation:

$\mathbf{P}^{E_1}_{0202}=\mathbf{P}^{E_1}_{xx}$ $\mathbf{P}^{E_1}_{0212}=\mathbf{P}^{E_1}_{xy}$

$\mathbf{P}^{E_1}_{1202}=\mathbf{P}^{E_1}_{yx}$ $\mathbf{P}^{E_1}_{1212}=\mathbf{P}^{E_1}_{yy}$

For irreducible class idempotents
sub-indices $_{xx}$ *or* $_{yy}$ *are optional*

where:

$$\mathbf{P}^{A_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{xx}=D^{A_1}_{xx}(g)\mathbf{P}^{A_1}_{xx},\quad \mathbf{P}^{A_2}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{yy}=D^{A_2}_{yy}(g)\mathbf{P}^{A_2}_{yy},\quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx}=D^{E_1}_{xx}(g)\mathbf{P}^{E_1}_{xx},\quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy}=D^{E_1}_{xy}(g)\mathbf{P}^{E_1}_{xy}$$

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$$,\quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx}=D^{E_1}_{yx}(g)\mathbf{P}^{E_1}_{yx},\quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy}=D^{E_1}_{yy}(g)\mathbf{P}^{E_1}_{yy}$$

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Group product table boils down to simple projector matrix algebra

	$\mathbf{P}^{A_1}_{xx}$	$\mathbf{P}^{A_2}_{yy}$	$\mathbf{P}^{E_1}_{xx}$	$\mathbf{P}^{E_1}_{xy}$	$\mathbf{P}^{E_1}_{yx}$	$\mathbf{P}^{E_1}_{yy}$
$\mathbf{P}^{A_1}_{xx}$	$\mathbf{P}^{A_1}_{xx}$	.	.	.	.	.
$\mathbf{P}^{A_2}_{yy}$	.	$\mathbf{P}^{A_2}_{yy}$	.	.	.	.
$\mathbf{P}^{E_1}_{xx}$	.	.	$\mathbf{P}^{E_1}_{xx}$	$\mathbf{P}^{E_1}_{xy}$	.	.
$\mathbf{P}^{E_1}_{yx}$	.	.	$\mathbf{P}^{E_1}_{yx}$	$\mathbf{P}^{E_1}_{yy}$	.	.
$\mathbf{P}^{E_1}_{xy}$	.	.	.	.	$\mathbf{P}^{E_1}_{xx}$	$\mathbf{P}^{E_1}_{xy}$
$\mathbf{P}^{E_1}_{yy}$	.	.	.	.	$\mathbf{P}^{E_1}_{yx}$	$\mathbf{P}^{E_1}_{yy}$

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$$+D^{E_1}_{yx}(g)\mathbf{P}^{E_1}_{yx}+D^{E_1}_{yy}(g)\mathbf{P}^{E_1}_{yy}$$

Previous notation:

$\mathbf{P}^{E_1}_{0_20_2}=\mathbf{P}^{E_1}_{xx}$ $\mathbf{P}^{E_1}_{0_21_2}=\mathbf{P}^{E_1}_{xy}$
$\mathbf{P}^{E_1}_{1_20_2}=\mathbf{P}^{E_1}_{yx}$ $\mathbf{P}^{E_1}_{1_21_2}=\mathbf{P}^{E_1}_{yy}$

For irreducible class idempotents sub-indices $_{xx}$ *or* $_{yy}$ *are optional*

where:

$$\mathbf{P}^{A_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{xx}=D^{A_1}_{xx}(g)\mathbf{P}^{A_1}_{xx},\quad \mathbf{P}^{A_2}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{yy}=D^{A_2}_{yy}(g)\mathbf{P}^{A_2}_{yy},\quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx}=D^{E_1}_{xx}(g)\mathbf{P}^{E_1}_{xx},\quad \mathbf{P}^{E_1}_{xx}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy}=D^{E_1}_{xy}(g)\mathbf{P}^{E_1}_{xy}$$

For split idempotents sub-indices $_{xx}$ *or* $_{yy}$ *are essential*

$$,\quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{xx}=D^{E_1}_{yx}(g)\mathbf{P}^{E_1}_{yx},\quad \mathbf{P}^{E_1}_{yy}\cdot\mathbf{g}\cdot\mathbf{P}^{E_1}_{yy}=D^{E_1}_{yy}(g)\mathbf{P}^{E_1}_{yy}$$

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$\mathbf{P}^{A_1}_{xx}$	$\mathbf{P}^{A_1}_{xx}$	·	·	·	·	·
$\mathbf{P}^{A_2}_{yy}$	·	$\mathbf{P}^{A_2}_{yy}$	·	·	·	·
$\mathbf{P}^{E_1}_{xx}$	·	·	$\mathbf{P}^{E_1}_{xx}$	$\mathbf{P}^{E_1}_{xy}$	·	·
$\mathbf{P}^{E_1}_{yx}$	·	·	$\mathbf{P}^{E_1}_{yx}$	$\mathbf{P}^{E_1}_{yy}$	·	·
$\mathbf{P}^{E_1}_{xy}$	·	·	·	·	$\mathbf{P}^{E_1}_{xx}$	$\mathbf{P}^{E_1}_{xy}$
$\mathbf{P}^{E_1}_{yy}$	·	·	·	·	$\mathbf{P}^{E_1}_{yx}$	$\mathbf{P}^{E_1}_{yy}$

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Generalizes to idempotent/nilpotent orthogonality known as Simple Matrix Algebra: $\boxed{\mathbf{P}^{\mu}_{jk}\mathbf{P}^{\nu}_{mn}=\delta^{\mu\nu}\delta_{km}\mathbf{P}^{\mu}_{jn}}$

Coefficients $D^{\mu}_{mn}(g)$ *are irreducible representations (ireps) of* $\mathbf{g}$

$\mathbf{g}=$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$D^{A_1}(\mathbf{g})=$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g})=$	1	1	1	−1	−1	−1
$D^{E_1}_{x,y}(\mathbf{g})=$	$\begin{pmatrix}1&\cdot\\\cdot&1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

Review: Spectral resolution of **D_3 Center** (Class algebra) and its subgroup splitting

General formulae for spectral decomposition (D_3 examples)
- Weyl **g**-expansion in irep $D^{\mu}_{jk}(g)$ and projectors $\mathbf{P}^{\mu}_{jk}$
  - $\mathbf{P}^{\mu}_{jk}$ transforms right-and-left
  - $\mathbf{P}^{\mu}_{jk}$ -expansion in **g**-operators
- $D^{\mu}_{jk}(g)$ orthogonality relations
- Class projector character formulae
  - $\mathbb{P}^{\mu}$ in terms of $\kappa_{\mathbf{g}}$ and $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$

Details of Mock-Mach relativity-duality for D_3 groups and representations
- Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)
- Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis
- Hamiltonian local-symmetry eigensolution

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

Spectral decomposition defines *left and right irep transformation* due to spectrally decomposed **g** acting on left and right side of projector $\mathbf{P}^{\mu}{}_{mn}$.

$$\mathbf{g}\mathbf{P}^{\mu}_{mn} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right) \mathbf{P}^{\mu}_{mn}$$

Use $\mathbf{P}^{\mu}_{mn}$ -orthonormality

$$\mathbf{P}^{\mu'}_{m'n'} \mathbf{P}^{\mu}_{mn} = \delta^{\mu'\mu} \delta_{n'm} \mathbf{P}^{\mu}_{m'n}$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g}=\left(\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}}D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}\right)$$

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$$= \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \delta^{\mu'\mu} \delta_{n'm} \mathbf{P}^{\mu}_{m'n}$$

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Use $\mathbf{P}^{\mu}_{mn}$-orthonormality

$$\mathbf{P}^{\mu'}_{m'n'} \mathbf{P}^{\mu}_{mn} = \delta^{\mu'\mu} \delta_{n'm} \mathbf{P}^{\mu}_{m'n}$$

Left-action transforms irep-ket $\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \dfrac{\mathbf{g}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm.}$

$$\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g) \left|\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right\rangle$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g}=\left(\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}}D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}\right)$$

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A simple irep expression...

$$\left\langle\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right|\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle=D^{\mu}_{m'm}(g)$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

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Use $\mathbf{P}^{\mu}_{mn}$-orthonormality

$$\mathbf{P}^{\mu'}_{m'n'}\mathbf{P}^{\mu}_{mn} = \delta^{\mu'\mu}\delta_{n'm}\mathbf{P}^{\mu}_{m'n}$$

Left-action transforms irep-ket $\mathbf{g}\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \frac{\mathbf{g}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm.}$

$$\mathbf{g}\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g) \left|\begin{matrix}\mu\\ m'n\end{matrix}\right\rangle$$

A simple irep expression...

$$\left\langle\begin{matrix}\mu\\ m'n\end{matrix}\right|\mathbf{g}\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = D^{\mu}_{m'm}(g)$$

...requires proper normalization: $\left\langle\begin{matrix}\mu'\\ m'n'\end{matrix}\middle|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}}{norm.}\frac{\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm*.}$

$$= \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'n}|\mathbf{1}\rangle}{|norm.|^2}$$

$$= \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

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$$= \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'n}|\mathbf{1}\rangle}{|\,norm.|^{2}}$$

$$= \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

$$|\,norm.|^{2} = \langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

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$$\boxed{\begin{array}{l}\text{Use } \mathbf{P}^{\mu}_{mn}\text{-orthonormality} \\ \mathbf{P}^{\mu'}_{m'n'} \mathbf{P}^{\mu}_{mn} = \delta^{\mu'\mu} \delta_{n'm} \mathbf{P}^{\mu}_{m'n}\end{array}}$$

$$\mathbf{P}^{\mu}_{mn}\mathbf{g} = \mathbf{P}^{\mu}_{mn} \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

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$$= \delta^{\mu'\mu} \delta_{m'm} \frac{\left\langle\mathbf{1}\right|\mathbf{P}^{\mu'}_{n'n}\left|\mathbf{1}\right\rangle}{|\,norm.|^2}$$

$$= \delta^{\mu'\mu} \delta_{m'm} \delta_{n'n}$$

$$|\,norm.|^2 = \left\langle\mathbf{1}\right|\mathbf{P}^{\mu}_{nn}\left|\mathbf{1}\right\rangle$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

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$$\mathbf{g}\mathbf{P}^{\mu}_{mn} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right) \mathbf{P}^{\mu}_{mn}$$

$$= \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \delta^{\mu'\mu} \delta_{n'm} \mathbf{P}^{\mu}_{m'n}$$

$$= \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\, \mathbf{P}^{\mu}_{m'n}$$

Use $\mathbf{P}^{\mu}_{mn}$-orthonormality

$$\mathbf{P}^{\mu'}_{m'n'} \mathbf{P}^{\mu}_{mn} = \delta^{\mu'\mu} \delta_{n'm} \mathbf{P}^{\mu}_{m'n}$$

Projector conjugation

$$\left(|m\rangle\langle n| \right)^{\dagger} = |n\rangle\langle m|$$

$$\left(\mathbf{P}^{\mu}_{mn} \right)^{\dagger} = \mathbf{P}^{\mu}_{nm}$$

$$\mathbf{P}^{\mu}_{mn}\mathbf{g} = \mathbf{P}^{\mu}_{mn} \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

$$= \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \delta^{\mu'\mu} \delta_{nm'} \mathbf{P}^{\mu}_{mn'}$$

$$= \sum_{n'}^{\ell^{\mu}} D^{\mu}_{nn'}(g)\, \mathbf{P}^{\mu}_{mn'}$$

Left-action transforms irep-ket $\mathbf{g}\left|\begin{smallmatrix}\mu \\ mn\end{smallmatrix}\right\rangle = \frac{\mathbf{g}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm.}$

Right-action transforms irep-bra $\left\langle\begin{smallmatrix}\mu \\ mn\end{smallmatrix}\right|\mathbf{g}^{\dagger} = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nm}\mathbf{g}^{\dagger}}{norm^{*}}$

$$\mathbf{g}\left|\begin{smallmatrix}\mu \\ mn\end{smallmatrix}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g) \left|\begin{smallmatrix}\mu \\ m'n\end{smallmatrix}\right\rangle$$

A simple irep expression...

$$\left\langle\begin{smallmatrix}\mu \\ m'n\end{smallmatrix}\right|\mathbf{g}\left|\begin{smallmatrix}\mu \\ mn\end{smallmatrix}\right\rangle = D^{\mu}_{m'm}(g)$$

...requires proper normalization: $\left\langle\begin{smallmatrix}\mu' \\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu \\ mn\end{smallmatrix}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}}{norm.} \frac{\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm*.}$

$$= \delta^{\mu'\mu} \delta_{m'm} \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'n}|\mathbf{1}\rangle}{|\,norm.|^{2}}$$

$$= \delta^{\mu'\mu} \delta_{m'm} \delta_{n'n}$$

$$|\,norm.|^{2} = \langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g}=\left(\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}}D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}\right)$$

Spectral decomposition defines *left and right irep transformation* due to spectrally decomposed **g** acting on left and right side of projector $\mathbf{P}^{\mu}{}_{mn}$.

$$\mathbf{g}\mathbf{P}^{\mu}_{mn}=\left(\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}}D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}\right)\mathbf{P}^{\mu}_{mn}$$

$$=\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}}D^{\mu'}_{m'n'}(g)\,\delta^{\mu'\mu}\delta_{n'm}\mathbf{P}^{\mu}_{m'n}$$

$$=\sum_{m'}^{\ell^{\mu}}D^{\mu}_{m'm}(g)\,\mathbf{P}^{\mu}_{m'n}$$

Use $\mathbf{P}^{\mu}_{mn}$-orthonormality

$$\mathbf{P}^{\mu'}_{m'n'}\mathbf{P}^{\mu}_{mn}=\delta^{\mu'\mu}\delta_{n'm}\mathbf{P}^{\mu}_{m'n}$$

Projector conjugation

$$\left(|m\rangle\langle n|\right)^{\dagger}=|n\rangle\langle m|$$

$$\left(\mathbf{P}^{\mu}_{mn}\right)^{\dagger}=\mathbf{P}^{\mu}_{nm}$$

$$\mathbf{P}^{\mu}_{mn}\mathbf{g}=\mathbf{P}^{\mu}_{mn}\left(\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}}D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}\right)$$

$$=\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}}D^{\mu'}_{m'n'}(g)\,\delta^{\mu'\mu}\delta_{nm'}\mathbf{P}^{\mu}_{mn'}$$

$$=\sum_{n'}^{\ell^{\mu}}D^{\mu}_{nn'}(g)\,\mathbf{P}^{\mu}_{mn'}$$

Left-action transforms irep-ket $\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle=\frac{\mathbf{g}\mathbf{P}^{\mu}_{mn}}{norm.}\left|\mathbf{1}\right\rangle$

Right-action transforms irep-bra $\left\langle\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right|\mathbf{g}^{\dagger}=\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nm}}{norm^{*}}\mathbf{g}^{\dagger}$

$$\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle=\sum_{m'}^{\ell^{\mu}}D^{\mu}_{m'm}(g)\left|\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right\rangle$$

$$\left\langle\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right|\mathbf{g}^{\dagger}=\left\langle\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right|\sum_{m'}^{\ell^{\mu}}D^{\mu}_{m'm}\left(\mathbf{g}^{\dagger}\right)$$

A simple irep expression...

$$\left\langle\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right|\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle=D^{\mu}_{m'm}(g)$$

...requires proper normalization: $\left\langle\begin{smallmatrix}\mu'\\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle=\frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}}{norm.}\frac{\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm*.}$

$$=\delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'n}|\mathbf{1}\rangle}{|\,norm.|^{2}}$$

$$=\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

$$|\,norm.|^{2}=\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

Spectral decomposition defines *left and right irep transformation* due to spectrally decomposed $\mathbf{g}$ acting on left and right side of projector $\mathbf{P}^{\mu}{}_{mn}$.

$$\mathbf{g}\mathbf{P}^{\mu}_{mn} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right) \mathbf{P}^{\mu}_{mn}$$

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Left-action transforms irep-ket $\mathbf{g}\left| {}^{\mu}_{mn} \right\rangle = \dfrac{\mathbf{g}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm.}$

$$\mathbf{g}\left| {}^{\mu}_{mn} \right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g) \left| {}^{\mu}_{m'n} \right\rangle$$

A simple irep expression...

$$\left\langle {}^{\mu}_{m'n} \right| \mathbf{g} \left| {}^{\mu}_{mn} \right\rangle = D^{\mu}_{m'm}(g)$$

Right-action transforms irep-bra $\left\langle {}^{\mu}_{mn} \right| \mathbf{g}^{\dagger} = \dfrac{\langle \mathbf{1}|\mathbf{P}^{\mu}_{nm}\mathbf{g}^{\dagger}}{norm^{*}}$

$$\left\langle {}^{\mu}_{mn} \right| \mathbf{g}^{\dagger} = \left\langle {}^{\mu}_{m'n} \right| \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}\left(\mathbf{g}^{\dagger}\right)$$

A less-simple irep expression...

$$\left\langle {}^{\mu}_{mn} \right| \mathbf{g}^{\dagger} \left| {}^{\mu}_{m'n} \right\rangle = D^{\mu}_{m'm}\left(g^{\dagger}\right)$$

...requires proper normalization: $\left\langle {}^{\mu'}_{m'n'} \middle| {}^{\mu}_{mn} \right\rangle = \dfrac{\langle \mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}}{norm.} \dfrac{\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm*.}$

$$= \delta^{\mu'\mu} \delta_{m'm} \frac{\langle \mathbf{1}|\mathbf{P}^{\mu'}_{n'n}|\mathbf{1}\rangle}{|norm.|^2}$$

$$= \delta^{\mu'\mu} \delta_{m'm} \delta_{n'n}$$

$$|norm.|^2 = \langle \mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle$$

$\mathbf{P}^{\mu}{}_{jk}$ *transforms right-and-left*

$$\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

Spectral decomposition defines *left and right irep transformation* due to spectrally decomposed $\mathbf{g}$ acting on left and right side of projector $\mathbf{P}^{\mu}{}_{mn}$.

$$\mathbf{g}\mathbf{P}^{\mu}_{mn} = \left(\sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right) \mathbf{P}^{\mu}_{mn}$$

$$= \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\, \delta^{\mu'\mu}\delta_{n'm}\mathbf{P}^{\mu}_{m'n}$$

$$= \sum_{m'}^{\ell^\mu} D^{\mu}_{m'm}(g)\, \mathbf{P}^{\mu}_{m'n}$$

Use $\mathbf{P}^{\mu}_{mn}$-orthonormality

$$\mathbf{P}^{\mu'}_{m'n'}\mathbf{P}^{\mu}_{mn} = \delta^{\mu'\mu}\delta_{n'm}\mathbf{P}^{\mu}_{m'n}$$

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$$\left(\mathbf{P}^{\mu}_{mn} \right)^\dagger = \mathbf{P}^{\mu}_{nm}$$

$$\mathbf{P}^{\mu}_{mn}\mathbf{g} = \mathbf{P}^{\mu}_{mn} \left(\sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$$

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Left-action transforms irep-ket $\mathbf{g}\left|{\mu \atop mn}\right\rangle = \dfrac{\mathbf{g}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm.}$

$$\mathbf{g}\left|{\mu \atop mn}\right\rangle = \sum_{m'}^{\ell^\mu} D^{\mu}_{m'm}(g) \left|{\mu \atop m'n}\right\rangle$$

A simple irep expression...

$$\left\langle {\mu \atop m'n} \right| \mathbf{g} \left|{\mu \atop mn}\right\rangle = D^{\mu}_{m'm}(g)$$

Right-action transforms irep-bra $\left\langle {\mu \atop mn} \right| \mathbf{g}^\dagger = \dfrac{\langle \mathbf{1}|\mathbf{P}^{\mu}_{nm}\mathbf{g}^\dagger}{norm^*}$

$$\left\langle {\mu \atop mn} \right| \mathbf{g}^\dagger = \left\langle {\mu \atop m'n} \right| \sum_{m'}^{\ell^\mu} D^{\mu}_{m'm}(\mathbf{g}^\dagger)$$

A less-simple irep expression...

$$\left\langle {\mu \atop mn} \right| \mathbf{g}^\dagger \left|{\mu \atop m'n}\right\rangle = D^{\mu}_{m'm}(g^\dagger)$$

$$\left(= D^{\mu *}_{mm'}(g) \right.$$

$$\left. \textit{if } D \textit{ is unitary} \right)$$

...requires proper normalization: $\left\langle {\mu' \atop m'n'} \middle| {\mu \atop mn} \right\rangle = \dfrac{\langle \mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}}{norm.} \dfrac{\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^*.}$

$$= \delta^{\mu'\mu}\delta_{m'm} \frac{\langle \mathbf{1}|\mathbf{P}^{\mu'}_{n'n}|\mathbf{1}\rangle}{|norm.|^2}$$

$$= \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

$$|norm.|^2 = \langle \mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle$$

Review: Spectral resolution of $\mathbf{D_3}$ **Center** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)
- *Weyl* **g***-expansion in irep* $D^{\mu}_{jk}(g)$ *and projectors* $\mathbf{P}^{\mu}_{jk}$
  - $\mathbf{P}^{\mu}_{jk}$ *transforms right-and-left*
  - ➡ $\mathbf{P}^{\mu}_{jk}$ *-expansion in* **g***-operators* ⬅
- $D^{\mu}_{jk}(g)$ *orthogonality relations*
- *Class projector character formulae*
  - $\mathbb{P}^{\mu}$ *in terms of* $\kappa_{\mathbf{g}}$ *and* $\kappa_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Details of Mock-Mach relativity-duality for D_3 groups and representations
- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local* $|\mathbf{g}\rangle$*-basis and Global vs Local* $|\mathbf{P}^{(\mu)}\rangle$*-basis*

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$*-basis*
- *Hamiltonian local-symmetry eigensolution*

$\mathbf{P}^{\mu}{}_{jk}$ *-expansion in* **g***-operators*

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

Derive coefficients $p^{\mu}_{mn}(g)$ of inverse Weyl expansion: $\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{\circ G} p^{\mu}_{mn}(g)\, \mathbf{g}$

$\mathbf{P}^{\mu}{}_{jk}$ -expansion in g-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

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Left action by operator **f** in group $G = \{\mathbf{1}, \ldots, \mathbf{f}, \mathbf{g}, \mathbf{h}, \ldots\}$:

$$\mathbf{f} \cdot \mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{\circ G} p^{\mu}_{mn}(g)\, \mathbf{f} \cdot \mathbf{g}$$

$\mathbf{P}^{\mu}{}_{jk}$ -expansion in **g**-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

Derive coefficients $p^{\mu}_{mn}(g)$ of inverse Weyl expansion: $\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{\circ G} p^{\mu}_{mn}(g)\,\mathbf{g}$

Left action by operator **f** in group $G = \{\mathbf{1},\ldots,\mathbf{f},\mathbf{g},\mathbf{h},\ldots\}$:

$$\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{\circ G} p^{\mu}_{mn}(g)\,\mathbf{f}\cdot\mathbf{g} = \sum_{\mathbf{h}}^{\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right)\mathbf{h} \quad \text{, where: } \mathbf{h} = \mathbf{f}\cdot\mathbf{g}, \text{ or: } \mathbf{g} = \mathbf{f}^{-1}\mathbf{h},$$

$\mathbf{P}^{\mu}{}_{jk}$ *-expansion in* **g***-operators*

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Regular representation *TraceR(**h**)* is zero except for *TraceR(**1**)*= $^{\circ}G$

Regular representation of $D_3 \sim C_{3v}$

$$R^G(\mathbf{1}) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \enclose{circle}{1} \\ \cdot & \cdot & \cdot & \enclose{circle}{1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \enclose{circle}{1} & \cdot \\ \cdot & \enclose{circle}{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \enclose{circle}{1} & \cdot & \cdot & \cdot \\ \enclose{circle}{1} & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\enclose{circle}{\mathbf{i}_3}$
r	**1**	$\mathbf{r}^2$	$\enclose{circle}{\mathbf{i}_3}$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\enclose{circle}{\mathbf{i}_3}$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\enclose{circle}{\mathbf{i}_3}$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\enclose{circle}{\mathbf{i}_3}$	$\mathbf{r}^2$	**1**	**r**
$\enclose{circle}{\mathbf{i}_3}$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

$\mathbf{P}^{\mu}{}_{jk}$ *-expansion in* $\mathbf{g}$*-operators*

Need inverse of Weyl form: $\mathbf{g}=\left(\sum_{\mu'}\sum_{m'}^{\ell^\mu}\sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}\right)$

Derive coefficients $p^{\mu}_{mn}(g)$ of inverse Weyl expansion: $\mathbf{P}^{\mu}_{mn}=\sum_{\mathbf{g}}^{{}^\circ G} p^{\mu}_{mn}(g)\,\mathbf{g}$

Left action by operator $\mathbf{f}$ in group $G=\{\mathbf{1},\ldots,\mathbf{f},\mathbf{g},\mathbf{h},\ldots\}$:

$$\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn}=\sum_{\mathbf{g}}^{{}^\circ G} p^{\mu}_{mn}(g)\,\mathbf{f}\cdot\mathbf{g}=\sum_{\mathbf{h}}^{{}^\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right)\mathbf{h}\quad,\text{ where: }\mathbf{h}=\mathbf{f}\cdot\mathbf{g},\text{ or: }\mathbf{g}=\mathbf{f}^{-1}\mathbf{h},$$

Regular representation *TraceR*(**h**) is zero except for *TraceR*(**1**)= $^\circ G$

$$Trace\,R\left(\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn}\right)=\sum_{\mathbf{h}}^{{}^\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right)TraceR\left(\mathbf{h}\right)$$

Regular representation of $D_3 \sim C_{3v}$

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\textcircled{1}\\\cdot&\cdot&\cdot&\textcircled{1}&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\textcircled{1}&\cdot\\\cdot&\textcircled{1}&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\textcircled{1}&\cdot&\cdot&\cdot\\\textcircled{1}&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\textcircled{\mathbf{i}_3}$
r	**1**	$\mathbf{r}^2$	$\textcircled{\mathbf{i}_3}$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\textcircled{\mathbf{i}_3}$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\textcircled{\mathbf{i}_3}$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\textcircled{\mathbf{i}_3}$	$\mathbf{r}^2$	**1**	**r**
$\textcircled{\mathbf{i}_3}$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

$\mathbf{P}^{\mu}{}_{jk}$ -expansion in g-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

Derive coefficients $p^{\mu}_{mn}(g)$ of inverse Weyl expansion: $\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{\circ G} p^{\mu}_{mn}(g)\, \mathbf{g}$

Left action by operator $\mathbf{f}$ in group $G = \{\mathbf{1}, \ldots, \mathbf{f}, \mathbf{g}, \mathbf{h}, \ldots\}$:

$$\mathbf{f} \cdot \mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{\circ G} p^{\mu}_{mn}(g)\, \mathbf{f} \cdot \mathbf{g} = \sum_{\mathbf{h}}^{\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) \mathbf{h} \quad \text{, where: } \mathbf{h} = \mathbf{f} \cdot \mathbf{g}, \text{ or: } \mathbf{g} = \mathbf{f}^{-1}\mathbf{h},$$

Regular representation *TraceR(**h**)* is zero except for *TraceR(**1**)*= $^{\circ}G$

$$Trace\, R\left(\mathbf{f} \cdot \mathbf{P}^{\mu}_{mn}\right) = \sum_{\mathbf{h}}^{\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) TraceR\left(\mathbf{h}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{1}\right) TraceR\left(\mathbf{1}\right)$$

Regular representation of $D_3 \sim C_{3v}$

$$R^G(\mathbf{1}) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \textcircled{1} \\ \cdot & \cdot & \cdot & \textcircled{1} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \textcircled{1} & \cdot \\ \cdot & \textcircled{1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \textcircled{1} & \cdot & \cdot & \cdot \\ \textcircled{1} & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\mathbf{P}^{\mu}{}_{jk}$ *-expansion in* $\mathbf{g}$*-operators*

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

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Regular representation *TraceR(*$\mathbf{h}$*)* is zero except for *TraceR(*$\mathbf{1}$*)*$= {}^{\circ}G$

$$Trace\ R\left(\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn}\right) = \sum_{\mathbf{h}}^{\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) TraceR\left(\mathbf{h}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{1}\right) TraceR\left(\mathbf{1}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\right) {}^{\circ}G$$

Regular representation of $D_3 \sim C_{3v}$

$$R^G(\mathbf{1}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\textcircled{1}\\ \cdot&\cdot&\cdot&\textcircled{1}&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\textcircled{1}&\cdot\\ \cdot&\textcircled{1}&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\textcircled{1}&\cdot&\cdot&\cdot\\ \textcircled{1}&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\mathbf{P}^{\mu}{}_{jk}$ -expansion in g-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

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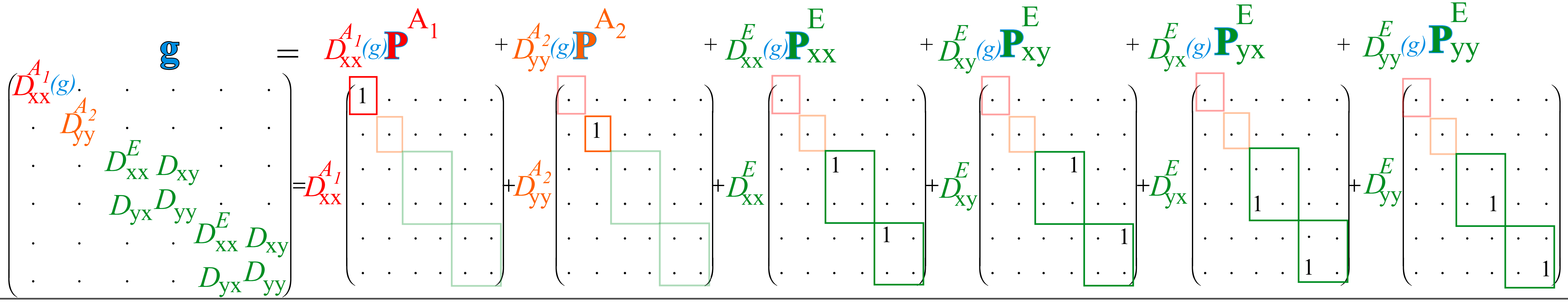
Regular representation *TraceR(***h***)* is zero except for *TraceR(***1***)*= $\circ G$

$$Trace\ R\left(\mathbf{f} \cdot \mathbf{P}^{\mu}_{mn}\right) = \sum_{\mathbf{h}}^{\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) TraceR\left(\mathbf{h}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{1}\right) TraceR\left(\mathbf{1}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\right) \circ G$$

Regular representation *TraceR(*$\mathbf{P}^{\mu}_{mn}$*)* is irep dimension $\ell^{(\mu)}$ for diagonal $\mathbf{P}^{\mu}_{mm}$ or zero otherwise:



$\mathbf{P}^{\mu}{}_{jk}$ *-expansion in* **g***-operators*

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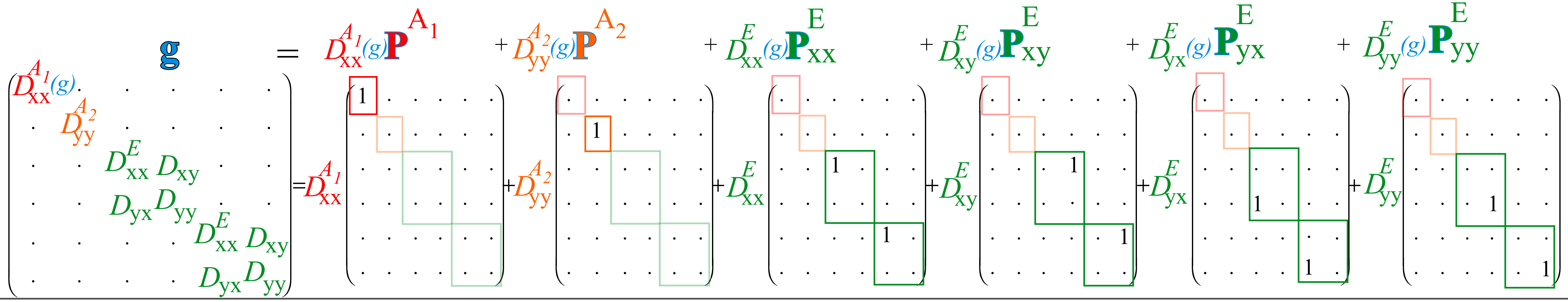
Regular representation *TraceR(*$\mathbf{P}^{\mu}_{mn}$*)* is irep dimension $\ell^{(\mu)}$ for diagonal $\mathbf{P}^{\mu}_{mm}$ or zero otherwise:

$$Trace\ R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\, \ell^{(\mu)}$$



$$\mathbf{g} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\mathbf{P}^{A_2} + D^{E}_{xx}(g)\mathbf{P}^{E}_{xx} + D^{E}_{xy}(g)\mathbf{P}^{E}_{xy} + D^{E}_{yx}(g)\mathbf{P}^{E}_{yx} + D^{E}_{yy}(g)\mathbf{P}^{E}_{yy}$$

$\mathbf{P}^{\mu}{}_{jk}$ *-expansion in* **g***-operators*

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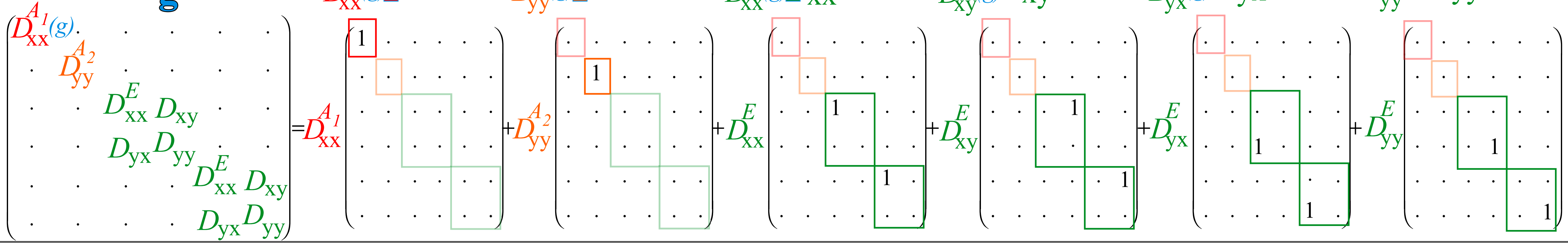
$$Trace\ R(\mathbf{P}^{\mu}_{mn})=\delta_{mn}\ell^{(\mu)}$$

Solving for $p^{\mu}_{mn}(g)$: $\boxed{p^{\mu}_{mn}\left(\mathbf{f}\right)=\frac{1}{\circ G}Trace\ R\left(\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn}\right)}$

$$\mathbf{g}=D^{A_1}_{xx}(g)\mathbf{P}^{A_1}+D^{A_2}_{yy}(g)\mathbf{P}^{A_2}+D^{E}_{xx}(g)\mathbf{P}^{E}_{xx}+D^{E}_{xy}(g)\mathbf{P}^{E}_{xy}+D^{E}_{yx}(g)\mathbf{P}^{E}_{yx}+D^{E}_{yy}(g)\mathbf{P}^{E}_{yy}$$



$\mathbf{P}^{\mu}{}_{jk}$ -expansion in g-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'} \right)$

Derive coefficients $p^{\mu}_{mn}(g)$ of inverse Weyl expansion: $\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{{}^{\circ}G} p^{\mu}_{mn}(g)\,\mathbf{g}$

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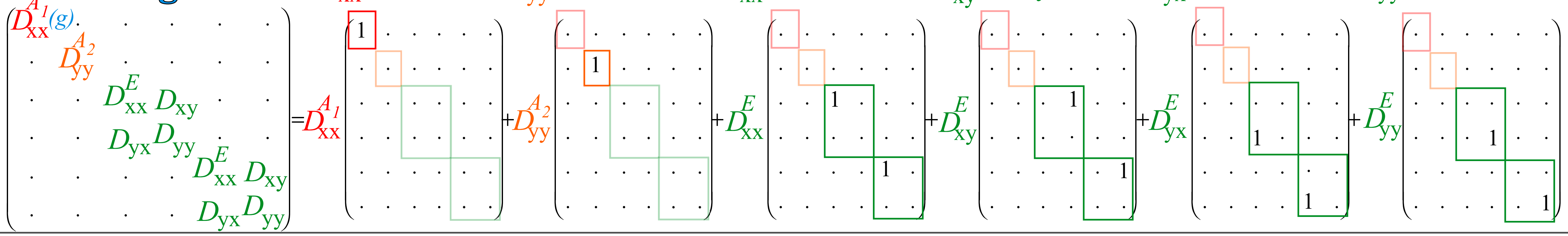
Regular representation *TraceR*($\mathbf{P}^{\mu}_{mn}$) is irep dimension $\ell^{(\mu)}$ for diagonal $\mathbf{P}^{\mu}_{mm}$ or zero otherwise:

$$Trace\ R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\ell^{(\mu)}$$

Solving for $p^{\mu}_{mn}(g)$: $\boxed{p^{\mu}_{mn}\left(\mathbf{f}\right) = \frac{1}{{}^{\circ}G} Trace\ R\left(\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn}\right)}$

Use left-action: $\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right)\mathbf{P}^{\mu}_{m'n}$

$$\mathbf{g} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\mathbf{P}^{A_2} + D^{E}_{xx}(g)\mathbf{P}^{E}_{xx} + D^{E}_{xy}(g)\mathbf{P}^{E}_{xy} + D^{E}_{yx}(g)\mathbf{P}^{E}_{yx} + D^{E}_{yy}(g)\mathbf{P}^{E}_{yy}$$



$\mathbf{P}^{\mu}{}_{jk}$ -expansion in $\mathbf{g}$-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

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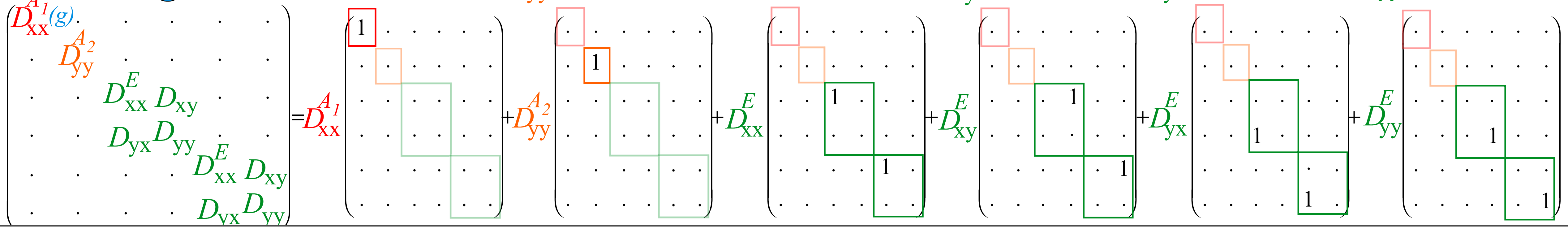
Solving for $p^{\mu}_{mn}(g)$: $p^{\mu}_{mn}(\mathbf{f}) = \frac{1}{{}^\circ G} Trace\ R\left(\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn}\right)$

Use left-action: $\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right)\mathbf{P}^{\mu}_{m'n}$

$$= \frac{1}{{}^\circ G} \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right) Trace\ R\left(\mathbf{P}^{\mu}_{m'n}\right)$$

$$\mathbf{g} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\mathbf{P}^{A_2} + D^{E}_{xx}(g)\mathbf{P}^{E}_{xx} + D^{E}_{xy}(g)\mathbf{P}^{E}_{xy} + D^{E}_{yx}(g)\mathbf{P}^{E}_{yx} + D^{E}_{yy}(g)\mathbf{P}^{E}_{yy}$$



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Regular representation *TraceR(***h***)* is zero except for *TraceR(***1***)*= $°G$

$$Trace\ R\left(\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn}\right) = \sum_{\mathbf{h}}^{°G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) TraceR\left(\mathbf{h}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{1}\right) TraceR\left(\mathbf{1}\right) = \boxed{p^{\mu}_{mn}\left(\mathbf{f}^{-1}\right)°G}$$

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Solving for $p^{\mu}_{mn}(g)$: $\boxed{p^{\mu}_{mn}\left(\mathbf{f}\right) = \frac{1}{°G} Trace\ R\left(\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn}\right)}$ Use left-action: $\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right)\mathbf{P}^{\mu}_{m'n}$

$$= \frac{1}{°G} \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right) Trace\ R\left(\mathbf{P}^{\mu}_{m'n}\right)$$

Use: $Trace\ R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\ell^{(\mu)}$

$$\mathbf{g} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\mathbf{P}^{A_2} + D^{E}_{xx}(g)\mathbf{P}^{E}_{xx} + D^{E}_{xy}(g)\mathbf{P}^{E}_{xy} + D^{E}_{yx}(g)\mathbf{P}^{E}_{yx} + D^{E}_{yy}(g)\mathbf{P}^{E}_{yy}$$

$$\begin{pmatrix} D^{A_1}_{xx}(g) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}_{yy} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^{E}_{xx} & D_{xy} & \cdot & \cdot \\ \cdot & \cdot & D_{yx} & D_{yy} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & D^{E}_{xx} & D_{xy} \\ \cdot & \cdot & \cdot & \cdot & D_{yx} & D_{yy} \end{pmatrix} = D^{A_1}_{xx} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{A_2}_{yy} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xx} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xy} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{yx} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + D^{E}_{yy} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$\mathbf{P}^{\mu}{}_{jk}$ -expansion in g-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

Derive coefficients $p^{\mu}_{mn}(g)$ of inverse Weyl expansion: $\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{{}^\circ G} p^{\mu}_{mn}(g)\, \mathbf{g}$

Left action by operator $\mathbf{f}$ in group $G = \{\mathbf{1}, \ldots, \mathbf{f}, \mathbf{g}, \mathbf{h}, \ldots\}$:

$$\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{{}^\circ G} p^{\mu}_{mn}(g)\, \mathbf{f}\cdot\mathbf{g} = \sum_{\mathbf{h}}^{{}^\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) \mathbf{h} \quad \text{, where: } \mathbf{h} = \mathbf{f}\cdot\mathbf{g}, \text{ or: } \mathbf{g} = \mathbf{f}^{-1}\mathbf{h},$$

Regular representation *TraceR(***h***)* is zero except for *TraceR(***1***)*= ${}^\circ G$

$$Trace\, R\left(\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn}\right) = \sum_{\mathbf{h}}^{{}^\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) TraceR\left(\mathbf{h}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{1}\right) TraceR\left(\mathbf{1}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\right){}^\circ G$$

Regular representation *TraceR(*$\mathbf{P}^{\mu}_{mn}$*)* is irep dimension $\ell^{(\mu)}$ for diagonal $\mathbf{P}^{\mu}_{mm}$ or zero otherwise:

$$Trace\, R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\ell^{(\mu)}$$

Solving for $p^{\mu}_{mn}(g)$: $p^{\mu}_{mn}\left(\mathbf{f}\right) = \frac{1}{{}^\circ G} Trace\, R\left(\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn}\right)$

Use left-action: $\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right)\mathbf{P}^{\mu}_{m'n}$

$$= \frac{1}{{}^\circ G} \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right) Trace\, R\left(\mathbf{P}^{\mu}_{m'n}\right)$$

Use: $Trace\, R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\ell^{(\mu)}$

$$= \frac{\ell^{(\mu)}}{{}^\circ G} D^{\mu}_{nm}\left(\mathbf{f}^{-1}\right)$$

$\mathbf{P}^{\mu}{}_{jk}$ -expansion in $\mathbf{g}$-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

Derive coefficients $p^{\mu}_{mn}(g)$ of inverse Weyl expansion: $\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{\circ G} p^{\mu}_{mn}(g)\, \mathbf{g}$

Left action by operator $\mathbf{f}$ in group $G = \{\mathbf{1}, \ldots, \mathbf{f}, \mathbf{g}, \mathbf{h}, \ldots\}$:

$$\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{\circ G} p^{\mu}_{mn}(g)\, \mathbf{f}\cdot\mathbf{g} = \sum_{\mathbf{h}}^{\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right)\mathbf{h} \quad \text{, where: } \mathbf{h} = \mathbf{f}\cdot\mathbf{g}, \text{ or: } \mathbf{g} = \mathbf{f}^{-1}\mathbf{h},$$

Regular representation *TraceR(**h**)* is zero except for *TraceR(**1**)*= $^{\circ}G$

$$\mathit{Trace}\ R\left(\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn}\right) = \sum_{\mathbf{h}}^{\circ G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right)\mathit{TraceR}\left(\mathbf{h}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{1}\right)\mathit{TraceR}\left(\mathbf{1}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\right){}^{\circ}G$$

Regular representation *TraceR*($\mathbf{P}^{\mu}_{mn}$) is irep dimension $\ell^{(\mu)}$ for diagonal $\mathbf{P}^{\mu}_{mm}$ or zero otherwise:

$$\mathit{Trace}\ R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\ell^{(\mu)}$$

Solving for $p^{\mu}_{mn}(g)$: $p^{\mu}_{mn}\left(\mathbf{f}\right) = \frac{1}{^{\circ}G}\mathit{Trace}\ R\left(\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn}\right)$

Use left-action: $\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right)\mathbf{P}^{\mu}_{m'n}$

$$= \frac{1}{^{\circ}G}\sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right)\mathit{Trace}\ R\left(\mathbf{P}^{\mu}_{m'n}\right)$$

Use: $\mathit{Trace}\ R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\ell^{(\mu)}$

$$= \frac{\ell^{(\mu)}}{^{\circ}G} D^{\mu}_{nm}\left(\mathbf{f}^{-1}\right)$$

$$\boxed{\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{^{\circ}G}\sum_{\mathbf{g}}^{\circ G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}}$$

$\mathbf{P}^{\mu}{}_{jk}$ -expansion in g-operators

Need inverse of Weyl form: $\mathbf{g} = \left(\sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'} \right)$

Derive coefficients $p^{\mu}_{mn}(g)$ of inverse Weyl expansion: $\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{°G} p^{\mu}_{mn}(g)\, \mathbf{g}$

Left action by operator **f** in group $G = \{\mathbf{1}, \ldots, \mathbf{f}, \mathbf{g}, \mathbf{h}, \ldots\}$:

$$\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{\mathbf{g}}^{°G} p^{\mu}_{mn}(g)\, \mathbf{f}\cdot\mathbf{g} = \sum_{\mathbf{h}}^{°G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) \mathbf{h} \quad \text{, where: } \mathbf{h} = \mathbf{f}\cdot\mathbf{g}, \text{ or: } \mathbf{g} = \mathbf{f}^{-1}\mathbf{h},$$

Regular representation *TraceR(***h***)* is zero except for *TraceR(***1***)*= $°G$

$$Trace\ R\left(\mathbf{f}\cdot\mathbf{P}^{\mu}_{mn}\right) = \sum_{\mathbf{h}}^{°G} p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{h}\right) TraceR\left(\mathbf{h}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\mathbf{1}\right) TraceR\left(\mathbf{1}\right) = p^{\mu}_{mn}\left(\mathbf{f}^{-1}\right)°G$$

Regular representation *TraceR(*$\mathbf{P}^{\mu}_{mn}$*)* is irep dimension $\ell^{(\mu)}$ for diagonal $\mathbf{P}^{\mu}_{mm}$ or *0* for off-diagonal $\mathbf{P}^{\mu}_{mn}$

$$Trace\ R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\ell^{(\mu)}$$

Solving for $p^{\mu}_{mn}(g)$: $p^{\mu}_{mn}(\mathbf{f}) = \frac{1}{°G} Trace\ R\left(\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn}\right)$

Use left-action: $\mathbf{f}^{-1}\cdot\mathbf{P}^{\mu}_{mn} = \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right)\mathbf{P}^{\mu}_{m'n}$

$$= \frac{1}{°G} \sum_{m'}^{\ell^{(\mu)}} D^{\mu}_{m'm}\left(\mathbf{f}^{-1}\right) Trace\ R\left(\mathbf{P}^{\mu}_{m'n}\right)$$

Use: $Trace\ R(\mathbf{P}^{\mu}_{mn}) = \delta_{mn}\ell^{(\mu)}$

$$= \frac{\ell^{(\mu)}}{°G} D^{\mu}_{nm}\left(\mathbf{f}^{-1}\right) \qquad \left(= \frac{\ell^{(\mu)}}{°G} D^{\mu *}_{mn}(\mathbf{f}) \quad \text{for unitary } D^{\mu}_{nm} \right)$$

$$\boxed{\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{°G} \sum_{\mathbf{g}}^{°G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g} \qquad \left(\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{°G} \sum_{\mathbf{g}}^{°G} D^{\mu *}_{mn}(g)\,\mathbf{g} \quad \text{for unitary } D^{\mu}_{nm} \right)}$$

Review: Spectral resolution of $\mathbf{D_3}$ **Center** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)

- *Weyl* **g***-expansion in irep* $D^{\mu}_{jk}(g)$ *and projectors* $\mathbf{P}^{\mu}_{jk}$
  - $\mathbf{P}^{\mu}_{jk}$ *transforms right-and-left*
  - $\mathbf{P}^{\mu}_{jk}$ *-expansion in* **g***-operators*
- $D^{\mu}_{jk}(g)$ *orthogonality relations*
- *Class projector character formulae*
  - $\mathbb{P}^{\mu}$ *in terms of* $\kappa_{\mathbf{g}}$ *and* $\kappa_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Details of Mock-Mach relativity-duality for D_3 groups and representations

- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local* $|\mathbf{g}\rangle$*-basis and Global vs Local* $|\mathbf{P}^{(\mu)}\rangle$*-basis*

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$*-basis*

- *Hamiltonian local-symmetry eigensolution*

$D^{\mu}{}_{jk}$-*orthogonality relations*

$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'}$ is a valid expansion of any combination of $\mathbf{g}$ including $\mathbf{P}$.

$D^{\mu}{}_{jk}$-*orthogonality relations*

$$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}\left(g\right)\mathbf{P}^{\mu'}_{m'n'}$$ is a valid expansion of any combination of **g** including **P**.

Simply substitute **P** for **g:**

$$\mathbf{P}^{\mu}_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right)\mathbf{P}^{\mu'}_{m'n'}$$

$D^{\mu}{}_{jk}$-*orthogonality relations*

$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}$ is a valid expansion of any combination of **g** including **P**.

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Useful identity for later

$D^{\mu}{}_{jk}$-*orthogonality relations*

$$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}$$ is a valid expansion of any combination of **g** including **P**.

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Useful identity for later

Then put in **g**-expansion of $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}$

$D^{\mu}{}_{jk}$-*orthogonality relations*

$$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'}$$ is a valid expansion of any combination of **g** including **P**.

Simply substitute **P** for **g**:

$$\mathbf{P}^{\mu}_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) \mathbf{P}^{\mu'}_{m'n'} \quad \Rightarrow \quad \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in **g**-expansion of $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right) \mathbf{g}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}(g)\, \mathbf{g}$$

$\left(\text{for unitary } D^{\mu}_{nm}\right)$

$D^{\mu}{}_{jk}$-*orthogonality relations*

$$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}$$ is a valid expansion of any combination of **g** including **P**.

Simply substitute **P** for **g:**

$$\mathbf{P}^{\mu}_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right)\mathbf{P}^{\mu'}_{m'n'} \quad \Rightarrow \quad \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in **g**-expansion of $\underbrace{\mathbf{P}^{\mu}_{mn}}$ $= \dfrac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g}$$

(for unitary D^{μ}_{nm})

$$D^{\mu'}_{m'n'}\left(\underbrace{\mathbf{P}^{\mu}_{mn}}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = D^{\mu'}_{m'n'}\left(\overbrace{\frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}}\right)$$

$D^{\mu}{}_{jk}$*-orthogonality relations*

$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'}$ is a valid expansion of any combination of **g** including **P**.

Simply substitute **P** for **g**:

$$\mathbf{P}^{\mu}_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) \mathbf{P}^{\mu'}_{m'n'} \quad \Rightarrow \quad \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in **g**-expansion of $\underbrace{\mathbf{P}^{\mu}_{mn}} = \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu}_{nm}\left(g^{-1}\right) \mathbf{g}$

$$D^{\mu'}_{m'n'}\left(\underbrace{\mathbf{P}^{\mu}_{mn}}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = D^{\mu'}_{m'n'}\left(\overbrace{\frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu}_{nm}\left(g^{-1}\right) \mathbf{g}} \right)$$

$$\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu}_{nm}\left(g^{-1}\right) D^{\mu'}_{m'n'}(\,\mathbf{g})$$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\, \mathbf{g}$$

(for unitary D^{μ}_{nm})

$D^{\mu}{}_{jk}$-*orthogonality relations*

$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\, \mathbf{P}^{\mu'}_{m'n'}$ is a valid expansion of any combination of **g** including **P**.

Simply substitute **P** for **g**:

$$\mathbf{P}^{\mu}_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) \mathbf{P}^{\mu'}_{m'n'} \quad \Rightarrow \quad \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in **g**-expansion of $\underbrace{\mathbf{P}^{\mu}_{mn}}_{} = \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu *}_{mn}\left(g\right)\mathbf{g}$$

$$D^{\mu'}_{m'n'}\left(\underbrace{\mathbf{P}^{\mu}_{mn}}_{}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = D^{\mu'}_{m'n'}\left(\overbrace{\frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}}^{}\right)$$

$$\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu}_{nm}\left(g^{-1}\right) D^{\mu'}_{m'n'}\left(\mathbf{g}\right)$$

or:

$$\boxed{\begin{gathered}\left(\text{for unitary } D^{\mu}_{nm}\right) \\ \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu *}_{mn}\left(g\right) D^{\mu'}_{m'n'}\left(\mathbf{g}\right) \\ \textit{Famous } D^{\mu} \textit{ orthogonality relation}\end{gathered}}$$

$D^{\mu}{}_{jk}$-*orthogonality relations*

$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}\left(g\right) \mathbf{P}^{\mu'}_{m'n'}$ is a valid expansion of any combination of **g** including **P**.

Simply substitute **P** for **g**:

$$\mathbf{P}^{\mu}_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) \mathbf{P}^{\mu'}_{m'n'} \quad \Rightarrow \quad \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in **g**-expansion of $\underbrace{\mathbf{P}^{\mu}_{mn}} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \underbrace{\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right) \mathbf{g}}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}\left(g\right) \mathbf{g}$$

$$D^{\mu'}_{m'n'}\underbrace{\left(\mathbf{P}^{\mu}_{mn}\right)} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = D^{\mu'}_{m'n'}\left(\frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right) \mathbf{g}\right)$$

$$\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right) D^{\mu'}_{m'n'}\left(\mathbf{g}\right)$$

or:

$$\boxed{\begin{gathered}\left(\text{for unitary } D^{\mu}_{nm}\right)\\ \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}\left(g\right) D^{\mu'}_{m'n'}\left(\mathbf{g}\right)\\ \textit{Famous } D^{\mu} \textit{ orthogonality relation}\end{gathered}}$$

Put **g′**-expansion of **P** into **P**-expansion of $\mathbf{g} = \sum_{\mu} \sum_{m}^{\ell^{\mu}} \sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}\left(g\right) \mathbf{P}^{\mu}_{mn}$

(Begin search for much less famous D^μ completeness relation)

$$\overbrace{\underbrace{\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}'}^{{}^{\circ}G} D^{\mu}_{nm}\left(g'^{-1}\right) \mathbf{g}'}}$$

$D^{\mu}{}_{jk}$-*orthogonality relations*

$\mathbf{g}=\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}$ is a valid expansion of any combination of **g** including **P**.

Simply substitute **P** for **g**:

$$\mathbf{P}^{\mu}_{mn}=\sum_{\mu'}\sum_{m'}^{\ell^{\mu}}\sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right)\mathbf{P}^{\mu'}_{m'n'} \quad\Rightarrow\quad \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right)=\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in **g**-expansion of $\underbrace{\mathbf{P}^{\mu}_{mn}}=\underbrace{\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}}$

$$\mathbf{P}^{\mu}_{mn}=\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}\left(g\right)\mathbf{g}$$

$$D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right)=\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}=D^{\mu'}_{m'n'}\left(\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}\right)$$

$$\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}=\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)D^{\mu'}_{m'n'}\left(\mathbf{g}\right)$$

or:

$$\boxed{\begin{gathered}\left(\text{for unitary } D^{\mu}_{nm}\right)\\ \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}=\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}\left(g\right)D^{\mu'}_{m'n'}\left(\mathbf{g}\right)\\ \textit{Famous } D^{\mu} \textit{ orthogonality relation}\end{gathered}}$$

Put **g′**-expansion of **P** into **P**-expansion of $\mathbf{g}=\sum_{\mu}\sum_{m}^{\ell^{\mu}}\sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}\left(g\right)\mathbf{P}^{\mu}_{mn}$

$$\overbrace{\underbrace{\mathbf{P}^{\mu}_{mn}=\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}'}^{{}^{\circ}G} D^{\mu}_{nm}\left(g'^{-1}\right)\mathbf{g}'}}$$

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(Begin search for much less famous D^{μ} completeness relation)

$D^{\mu}{}_{jk}$-orthogonality relations

$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}$ is a valid expansion of any combination of **g** including **P**.

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$$\mathbf{P}^{\mu}_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right)\mathbf{P}^{\mu'}_{m'n'} \Rightarrow \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in **g**-expansion of $\underbrace{\mathbf{P}^{\mu}_{mn}} = \underbrace{\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}}$

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(Begin search for much less famous D^{μ} completeness relation)

$D^{\mu}{}_{jk}$-*orthogonality relations*

$$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^{\mu}} \sum_{n'}^{\ell^{\mu}} D^{\mu'}_{m'n'}\left(g\right) \mathbf{P}^{\mu'}_{m'n'}$$

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$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^\circ G} \sum_{\mu} \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{m}^{\ell^{\mu}} \sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}\left(g\right) D^{\mu}_{nm}\left(g'^{-1}\right) \mathbf{g}'$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^\circ G} \sum_{\mu} \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{m}^{\ell^{\mu}} D^{\mu}_{mm}\left(gg'^{-1}\right) \mathbf{g}'$$

(Begin search for much less famous D^μ completeness relation)

$D^{\mu}{}_{jk}$-*orthogonality relations*

$$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}$$

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$$\boxed{\begin{gathered}\left(\text{for unitary } D^{\mu}_{nm}\right)\\ \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}\left(g\right)D^{\mu'}_{m'n'}\left(\mathbf{g}\right)\\ \textit{Famous } D^{\mu} \textit{ orthogonality relation}\end{gathered}}$$

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(Begin search for much less famous D

$$\mathbf{g} = \sum_{\mu}\sum_{m}^{\ell^\mu}\sum_{n}^{\ell^\mu} D^{\mu}_{mn}\left(g\right)\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}'}^{{}^\circ G} D^{\mu}_{nm}\left(g'^{-1}\right)\mathbf{g}'$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^\circ G}\sum_{\mu}\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{m}^{\ell^\mu}\sum_{n}^{\ell^\mu} D^{\mu}_{mn}\left(g\right)D^{\mu}_{nm}\left(g'^{-1}\right)\mathbf{g}'$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^\circ G}\sum_{\mu}\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{m}^{\ell^\mu} D^{\mu}_{mm}\left(gg'^{-1}\right)\mathbf{g}'$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^\circ G}\sum_{\mu}\frac{\ell^{(\mu)}}{{}^\circ G}\chi^{\mu}\left(gg'^{-1}\right)\mathbf{g}'$$

$D^{\mu}{}_{jk}$-orthogonality relations

$\mathbf{g}=\sum_{\mu'}\sum_{m'}^{\ell^\mu}\sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}\left(g\right)\mathbf{P}^{\mu'}_{m'n'}$ is a valid expansion of any combination of **g** including **P**.

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Then put in **g**-expansion of $\underbrace{\mathbf{P}^{\mu}_{mn}}=\frac{\ell^{(\mu)}}{{}^\circ G}\underbrace{\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}}$

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or:

$\left(\text{for unitary } D^{\mu}_{nm}\right)$

$$\boxed{\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}=\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}\left(g\right)D^{\mu'}_{m'n'}\left(\mathbf{g}\right)}$$

Famous D$^{\mu}$ orthogonality relation

Put **g′**-expansion of **P** into **P**-expansion of $\mathbf{g}=\sum_{\mu}\sum_{m}^{\ell^\mu}\sum_{n}^{\ell^\mu} D^{\mu}_{mn}\left(g\right)\mathbf{P}^{\mu}_{mn}$

(Begin search for much less famous D$^{\mu}$ completeness relation)

$$\overbrace{\underbrace{\mathbf{P}^{\mu}_{mn}=\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}'}^{{}^\circ G} D^{\mu}_{nm}\left(g'^{-1}\right)\mathbf{g}'}}$$

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$$\mathbf{g}=\sum_{\mathbf{g}'}^{{}^\circ G}\sum_{\mu}\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{m}^{\ell^\mu} D^{\mu}_{mm}\left(gg'^{-1}\right)\mathbf{g}'$$

$$\mathbf{g}=\sum_{\mathbf{g}'}^{{}^\circ G}\sum_{\mu}\frac{\ell^{(\mu)}}{{}^\circ G}\chi^{\mu}\left(gg'^{-1}\right)\mathbf{g}' \Rightarrow$$

Interesting character sum-rule

$$\boxed{\sum_{\mu}\frac{\ell^{(\mu)}}{{}^\circ G}\chi^{\mu}\left(gg'^{-1}\right)=\delta^{g}_{g'}{}^{-1}}$$

$D^{\mu}{}_{jk}$-orthogonality relations

$$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}$$ is a valid expansion of any combination of **g** including **P**.

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$$\mathbf{P}^{\mu}_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right)\mathbf{P}^{\mu'}_{m'n'} \quad \Rightarrow \quad \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in **g**-expansion of $\underbrace{\mathbf{P}^{\mu}_{mn}}$ $= \underbrace{\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g}$$

$$\underbrace{D^{\mu'}_{m'n'}\left(\mathbf{P}^{\mu}_{mn}\right)} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = D^{\mu'}_{m'n'}\left(\underbrace{\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)\mathbf{g}}\right)$$

$$\left(\text{for unitary } D^{\mu}_{nm}\right)$$

$$\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu}_{nm}\left(g^{-1}\right)D^{\mu'}_{m'n'}(\mathbf{g}) \qquad \text{or:} \qquad \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)D^{\mu'}_{m'n'}(\mathbf{g})$$

Famous D^{μ} orthogonality relation

Put **g′**-expansion of **P** into **P**-expansion of $\mathbf{g} = \sum_{\mu}\sum_{m}^{\ell^\mu}\sum_{n}^{\ell^\mu} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn}$

(Begin search for much less famous D^{μ} completeness relation)

$$\underbrace{\overbrace{\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}'}^{{}^{\circ}G} D^{\mu}_{nm}\left(g'^{-1}\right)\mathbf{g}'}}$$

$$\mathbf{g} = \sum_{\mu}\sum_{m}^{\ell^\mu}\sum_{n}^{\ell^\mu} D^{\mu}_{mn}(g)\underbrace{\overbrace{\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}'}^{{}^{\circ}G} D^{\mu}_{nm}\left(g'^{-1}\right)\mathbf{g}'}}$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^{\circ}G}\sum_{\mu}\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{m}^{\ell^\mu}\sum_{n}^{\ell^\mu} D^{\mu}_{mn}(g)D^{\mu}_{nm}\left(g'^{-1}\right)\mathbf{g}'$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^{\circ}G}\sum_{\mu}\frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{m}^{\ell^\mu} D^{\mu}_{mm}\left(gg'^{-1}\right)\mathbf{g}'$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^{\circ}G}\sum_{\mu}\frac{\ell^{(\mu)}}{{}^{\circ}G}\chi^{\mu}\left(gg'^{-1}\right)\mathbf{g}' \quad \Rightarrow$$

$\chi^{\mu}_{k}(D_3)$	$\chi^{\mu}_{\mathbf{1}}$	$\chi^{\mu}_{\mathbf{r}}$	$\chi^{\mu}_{\mathbf{i}}$
$\mu = A_1$	$\ell^{A_1}=1$	1	1
$\mu = A_2$	$\ell^{A_2}=1$	1	−1
$\mu = E_1$	$\ell^{E_1}=2$	−1	0

Interesting character sum-rule

$$\sum_{\mu}\frac{\ell^{(\mu)}}{{}^{\circ}G}\chi^{\mu}\left(gg'^{-1}\right) = \delta^{g}_{g'}{}^{-1}$$

$D^\mu{}_{jk}$-orthogonality relations

$$\mathbf{g} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}(g)\,\mathbf{P}^{\mu'}_{m'n'}$$ is a valid expansion of any combination of $\mathbf{g}$ including $\mathbf{P}$.

Simply substitute $\mathbf{P}$ for $\mathbf{g}$:

$$\mathbf{P}^\mu_{mn} = \sum_{\mu'} \sum_{m'}^{\ell^\mu} \sum_{n'}^{\ell^\mu} D^{\mu'}_{m'n'}\left(\mathbf{P}^\mu_{mn}\right)\mathbf{P}^{\mu'}_{m'n'} \quad \Rightarrow \quad \boxed{D^{\mu'}_{m'n'}\left(\mathbf{P}^\mu_{mn}\right) = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} \qquad \textit{Useful identity for later}}$$

Then put in $\mathbf{g}$-expansion of $\underbrace{\mathbf{P}^\mu_{mn}} = \underbrace{\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^\mu_{nm}\left(g^{-1}\right)\mathbf{g}}$

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$$\overbrace{D^{\mu'}_{m'n'}\left(\mathbf{P}^\mu_{mn}\right)} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = D^{\mu'}_{m'n'}\left(\overbrace{\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^\mu_{nm}\left(g^{-1}\right)\mathbf{g}}\right)$$

$$\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^\mu_{nm}\left(g^{-1}\right)D^{\mu'}_{m'n'}(\mathbf{g})$$

or:

(for unitary D^μ_{nm})

$$\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n} = \frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\,D^{\mu'}_{m'n'}(\mathbf{g})$$

Famous D^μ orthogonality relation

Put $\mathbf{g}'$-expansion of $\mathbf{P}$ into $\mathbf{P}$-expansion of $\mathbf{g} = \sum_\mu \sum_m^{\ell^\mu} \sum_n^{\ell^\mu} D^\mu_{mn}(g)\,\mathbf{P}^\mu_{mn}$

$$\overbrace{\underbrace{\mathbf{P}^\mu_{mn} = \frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}'}^{{}^\circ G} D^\mu_{nm}\left(g'^{-1}\right)\mathbf{g}'}}$$

(Begin search for much less famous D^μ completeness relation)

$$\mathbf{g} = \sum_\mu \sum_m^{\ell^\mu} \sum_n^{\ell^\mu} D^\mu_{mn}(g)\overbrace{\underbrace{\frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}'}^{{}^\circ G} D^\mu_{nm}\left(g'^{-1}\right)\mathbf{g}'}}$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^\circ G}\sum_\mu \frac{\ell^{(\mu)}}{{}^\circ G}\sum_m^{\ell^\mu}\sum_n^{\ell^\mu} D^\mu_{mn}(g)\,D^\mu_{nm}\left(g'^{-1}\right)\mathbf{g}'$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^\circ G}\sum_\mu \frac{\ell^{(\mu)}}{{}^\circ G}\sum_m^{\ell^\mu} D^\mu_{mm}\left(gg'^{-1}\right)\mathbf{g}'$$

$$\mathbf{g} = \sum_{\mathbf{g}'}^{{}^\circ G}\sum_\mu \frac{\ell^{(\mu)}}{{}^\circ G}\chi^\mu\left(gg'^{-1}\right)\mathbf{g}' \quad \Rightarrow$$

$\chi^\mu_k(D_3)$	χ^μ_1	χ^μ_r	χ^μ_i
$\mu = A_1$	$\ell^{A_1}=1$	1	1
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Interesting character sum-rule

$$\sum_\mu \frac{\ell^{(\mu)}}{{}^\circ G}\chi^\mu\left(gg'^{-1}\right) = \delta^{\,g}_{\,g'}{}^{-1}$$

Character sum-rule becomes Diophantine relation if $\mathbf{g}' = \mathbf{g}^{-1}$

$$\sum_\mu \frac{\left(\ell^{(\mu)}\right)^2}{{}^\circ G} = 1$$

Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)

- *Weyl* **g***-expansion in irep $D^{\mu}_{jk}(g)$ and projectors $\mathbf{P}^{\mu}_{jk}$*
  - *$\mathbf{P}^{\mu}_{jk}$ transforms right-and-left*
  - *$\mathbf{P}^{\mu}_{jk}$ -expansion in* **g***-operators*
- *$D^{\mu}_{jk}(g)$ orthogonality relations*
- *Class projector character formulae* *And review of all-commuting class sums*
  - *$\mathbb{P}^{\mu}$ in terms of $\kappa_{\mathbf{g}}$ and $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$*

Details of Mock-Mach relativity-duality for D_3 groups and representations

- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis*

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis

- *Hamiltonian local-symmetry eigensolution*

Class projector and character formulae

Review of all-commuting class sums (Recall Lagrange coset relations in Lect.14 p.14)

Total-G-transformation $\Sigma_{\mathbf{h}\in G}\mathbf{hgh}^{-1}$ *of* $\mathbf{g}$ *repeats its class-sum* $\boldsymbol{\kappa}_g$ *an integer number* $°n_g = {}^{°G}/_{°\kappa_g}$ *of times.*

$$\sum_{\mathbf{h}=1}^{°G}\mathbf{hgh}^{-1} = °n_g\boldsymbol{\kappa}_g\ , \qquad \text{where: } °n_g = \frac{°G}{°\kappa_g} = \text{order of } \mathbf{g}\text{-}\textit{self-symmetry group } \{\mathbf{n} \text{ such that } \mathbf{ngn}^{-1}=\mathbf{g}\}$$

Suppose *all-commuting operator* $\mathbb{C} = \sum_{\mathbf{g}=1}^{°G} C_g\mathbf{g}$ commutes with all $\mathbf{h}$ in group G so $\mathbf{h}\mathbb{C}=\mathbb{C}\mathbf{h}$ or $\mathbf{h}\mathbb{C}\mathbf{h}^{-1}=\mathbb{C}$.

Class projector and character formulae

Review of all-commuting class sums (Recall Lagrange coset relations in Lect.14 p.14)

Total-G-transformation $\Sigma_{\mathbf{h}\in G}\mathbf{hgh}^{-1}$ *of* **g** *repeats its class-sum* κ_g *an integer number* $°n_g = {}^{°G}/_{°\kappa_g}$ *of times.*

$$\sum_{\mathbf{h}=1}^{°G} \mathbf{hgh}^{-1} = °n_g \boldsymbol{\kappa}_g \ , \quad \text{where: } °n_g = \frac{°G}{°\kappa_g} = \text{order of } \mathbf{g}\text{-}\textit{self-symmetry group } \{\mathbf{n} \text{ such that } \mathbf{ngn}^{-1} = \mathbf{g}\}$$

Suppose *all-commuting operator* $\mathbb{C} = \sum_{\mathbf{g}=1}^{°G} C_g \mathbf{g}$ commutes with all **h** in group *G* so $\mathbf{h}\mathbb{C} = \mathbb{C}\mathbf{h}$ or $\mathbf{h}\mathbb{C}\mathbf{h}^{-1} = \mathbb{C}$.

Then $\mathbb{C}$ must be the following linear combination of *class-sums* $\boldsymbol{\kappa}_g$.

$$\mathbb{C} = \sum_{\mathbf{g}=1}^{°G} C_g \mathbf{g} = \frac{1}{°G} \sum_{\mathbf{h}=1}^{°G} \mathbf{h}\,\mathbb{C}\,\mathbf{h}^{-1} \longleftarrow \mathbb{C} = \frac{1}{°G} \sum_{\mathbf{h}=1}^{°G} \mathbb{C} \quad (\textit{Trivial assumption})$$

Class projector and character formulae

Review of all-commuting class sums (Recall Lagrange coset relations in Lect.14 p.14)

Total-G-transformation $\Sigma_{\mathbf{h}\in G}\mathbf{hgh}^{-1}$ *of* **g** *repeats its class-sum* $\boldsymbol{\kappa}_g$ *an integer number* $°n_g = {}^{°G}/_{°\kappa_g}$ *of times.*

$$\sum_{\mathbf{h}=1}^{°G}\mathbf{hgh}^{-1} = °n_g\boldsymbol{\kappa}_g\ , \qquad \text{where: } °n_g = \frac{°G}{°\kappa_g} = \text{order of } \mathbf{g}\text{-}\textit{self-symmetry group } \{\mathbf{n} \text{ such that } \mathbf{ngn}^{-1}=\mathbf{g}\}$$

Suppose *all-commuting operator* $\mathbb{C} = \sum_{\mathbf{g}=1}^{°G} C_g\mathbf{g}$ commutes with all **h** in group G so $\mathbf{h}\mathbb{C} = \mathbb{C}\mathbf{h}$ or $\mathbf{h}\mathbb{C}\mathbf{h}^{-1} = \mathbb{C}$.

Then $\mathbb{C}$ must be the following linear combination of *class-sums* $\boldsymbol{\kappa}_g$.

$$\mathbb{C} = \sum_{\mathbf{g}=1}^{°G} C_g\mathbf{g} = \frac{1}{°G}\sum_{\mathbf{h}=1}^{°G}\mathbf{h}\mathbb{C}\mathbf{h}^{-1} \longleftarrow \mathbb{C} = \frac{1}{°G}\sum_{\mathbf{h}=1}^{°G}\mathbb{C}$$

(Trivial assumption)

$$= \frac{1}{°G}\sum_{\mathbf{h}=1}^{°G}\mathbf{h}\left(\sum_{\mathbf{g}=1}^{°G} C_g\mathbf{g}\right)\mathbf{h}^{-1}$$

Class projector and character formulae

Review of all-commuting class sums (Recall Lagrange coset relations in Lect.14 p.14)

Total-G-transformation $\Sigma_{\mathbf{h}\in G}\mathbf{hgh}^{-1}$ *of* $\mathbf{g}$ *repeats its class-sum* κ_g *an integer number* ${}^\circ n_g = {}^\circ G/{}^\circ\kappa_g$ *of times.*

$$\sum_{\mathbf{h}=1}^{{}^\circ G}\mathbf{hgh}^{-1} = {}^\circ n_g \kappa_g \;, \qquad \text{where: } {}^\circ n_g = \frac{{}^\circ G}{{}^\circ \kappa_g} = \text{order of } \mathbf{g}\text{-}\textit{self-symmetry group } \{\mathbf{n} \text{ such that } \mathbf{ngn}^{-1}=\mathbf{g}\}$$

Suppose *all-commuting operator* $\mathbb{C} = \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \mathbf{g}$ commutes with all $\mathbf{h}$ in group G so $\mathbf{h}\mathbb{C} = \mathbb{C}\mathbf{h}$ or $\mathbf{h}\mathbb{C}\mathbf{h}^{-1} = \mathbb{C}$.

Then $\mathbb{C}$ must be the following linear combination of *class-sums* κ_g.

$$\mathbb{C} = \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \mathbf{g} = \frac{1}{{}^\circ G}\sum_{\mathbf{h}=1}^{{}^\circ G} \mathbf{h}\mathbb{C}\mathbf{h}^{-1}$$

$$\mathbb{C} = \frac{1}{{}^\circ G}\sum_{\mathbf{h}=1}^{{}^\circ G} \mathbb{C}$$ *(Trivial assumption)*

$$= \frac{1}{{}^\circ G}\sum_{\mathbf{h}=1}^{{}^\circ G} \mathbf{h}\left(\sum_{\mathbf{g}=1}^{{}^\circ G} C_g \mathbf{g}\right)\mathbf{h}^{-1}$$

$$= \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \frac{1}{{}^\circ G}\sum_{\mathbf{h}=1}^{{}^\circ G} \mathbf{hgh}^{-1}$$

Class projector and character formulae

Review of all-commuting class sums (Recall Lagrange coset relations in Lect.14 p.14)

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$$\sum_{\mathbf{h}=1}^{^{\circ}G} \mathbf{hgh}^{-1} = {}^{\circ}n_g \boldsymbol{\kappa}_g \ , \qquad \text{where: } {}^{\circ}n_g = \frac{^{\circ}G}{^{\circ}\kappa_g} = \text{order of } \mathbf{g}\text{-}\textit{self-symmetry group } \{\mathbf{n} \text{ such that } \mathbf{ngn}^{-1}=\mathbf{g}\}$$

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Then $\mathbb{C}$ must be the following linear combination of *class-sums* $\boldsymbol{\kappa}_g$.

$$\mathbb{C} = \sum_{\mathbf{g}=1}^{^{\circ}G} C_g \mathbf{g} = \frac{1}{^{\circ}G} \sum_{\mathbf{h}=1}^{^{\circ}G} \mathbf{h} \mathbb{C} \mathbf{h}^{-1}$$

$$\mathbb{C} = \frac{1}{^{\circ}G} \sum_{\mathbf{h}=1}^{^{\circ}G} \mathbb{C}$$

(Trivial assumption)

$$= \frac{1}{^{\circ}G} \sum_{\mathbf{h}=1}^{^{\circ}G} \mathbf{h} \left(\sum_{\mathbf{g}=1}^{^{\circ}G} C_g \mathbf{g} \right) \mathbf{h}^{-1}$$

$$= \sum_{\mathbf{g}=1}^{^{\circ}G} C_g \frac{1}{^{\circ}G} \sum_{\mathbf{h}=1}^{^{\circ}G} \mathbf{hgh}^{-1}$$

$$= \sum_{\mathbf{g}=1}^{^{\circ}G} C_g \frac{^{\circ}n_g}{^{\circ}G} \boldsymbol{\kappa}_g$$

Class projector and character formulae

Review of all-commuting class sums (Recall Lagrange coset relations in Lect.14 p.14)

Total-G-transformation $\Sigma_{\mathbf{h}\in G}\mathbf{hgh}^{-1}$ *of* **g** *repeats its class-sum* κ_g *an integer number* ${}^\circ n_g = {}^\circ G/{}^\circ\kappa_g$ *of times.*

$$\sum_{\mathbf{h}=1}^{{}^\circ G}\mathbf{hgh}^{-1} = {}^\circ n_g \kappa_g \,, \qquad \text{where: } {}^\circ n_g = \frac{{}^\circ G}{{}^\circ\kappa_g} = \text{order of } \mathbf{g}\text{-}\textit{self-symmetry group } \{\mathbf{n} \text{ such that } \mathbf{ngn}^{-1}=\mathbf{g}\}$$

Suppose *all-commuting operator* $\mathbb{C} = \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \mathbf{g}$ commutes with all **h** in group *G* so $\mathbf{h}\mathbb{C}=\mathbb{C}\mathbf{h}$ or $\mathbf{h}\mathbb{C}\mathbf{h}^{-1}=\mathbb{C}$.

Then $\mathbb{C}$ must be the following linear combination of *class-sums* κ_g.

$$\mathbb{C} = \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \mathbf{g} = \frac{1}{{}^\circ G}\sum_{\mathbf{h}=1}^{{}^\circ G}\mathbf{h}\,\mathbb{C}\,\mathbf{h}^{-1}$$

$$\mathbb{C} = \frac{1}{{}^\circ G}\sum_{\mathbf{h}=1}^{{}^\circ G}\mathbb{C} \qquad (\textit{Trivial assumption})$$

$$= \frac{1}{{}^\circ G}\sum_{\mathbf{h}=1}^{{}^\circ G}\mathbf{h}\left(\sum_{\mathbf{g}=1}^{{}^\circ G} C_g \mathbf{g}\right)\mathbf{h}^{-1}$$

$$= \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \frac{1}{{}^\circ G}\sum_{\mathbf{h}=1}^{{}^\circ G}\mathbf{hgh}^{-1}$$

$$= \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \frac{{}^\circ n_g}{{}^\circ G}\kappa_g$$

Precise combination of *class-sums* κ_g.

$$\mathbb{C} = \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \mathbf{g} = \sum_{\mathbf{g}=1}^{{}^\circ G} C_g \frac{\kappa_g}{{}^\circ\kappa_g}$$

Class projector and character formulae

Review of all-commuting class sums (Recall Lagrange coset relations in Lect.14 p.14)

Total-G-transformation $\Sigma_{\mathbf{h}\in G}\mathbf{hgh}^{-1}$ *of* **g** *repeats its class-sum* κ_g *an integer number* $°n_g = {}^{°G}/_{°\kappa_g}$ *of times.*

$$\sum_{\mathbf{h}=1}^{°G} \mathbf{hgh}^{-1} = °n_g \kappa_g \ , \qquad \text{where: } °n_g = \frac{°G}{°\kappa_g} = \text{order of } \mathbf{g}\text{-}\textit{self-symmetry group } \{\mathbf{n} \text{ such that } \mathbf{ngn}^{-1}=\mathbf{g}\}$$

Suppose *all-commuting operator* $\mathbb{C} = \sum_{\mathbf{g}=1}^{°G} C_g \mathbf{g}$ commutes with all **h** in group G so $\mathbf{h}\mathbb{C}=\mathbb{C}\mathbf{h}$ or $\mathbf{h}\mathbb{C}\mathbf{h}^{-1}=\mathbb{C}$.

Then $\mathbb{C}$ must be the following linear combination of *class-sums* κ_g.

$$\begin{aligned}
\mathbb{C} = \sum_{\mathbf{g}=1}^{°G} C_g \mathbf{g} &= \frac{1}{°G}\sum_{\mathbf{h}=1}^{°G} \mathbf{h}\mathbb{C}\mathbf{h}^{-1} \\
&= \frac{1}{°G}\sum_{\mathbf{h}=1}^{°G} \mathbf{h}\left(\sum_{\mathbf{g}=1}^{°G} C_g \mathbf{g}\right)\mathbf{h}^{-1} \\
&= \sum_{\mathbf{g}=1}^{°G} C_g \frac{1}{°G}\sum_{\mathbf{h}=1}^{°G} \mathbf{hgh}^{-1} \\
&= \sum_{\mathbf{g}=1}^{°G} C_g \frac{°n_g}{°G}\kappa_g
\end{aligned}$$

$$\mathbb{C} = \frac{1}{°G}\sum_{\mathbf{h}=1}^{°G}\mathbb{C}$$

(Trivial assumption)

Precise combination of *class-sums* κ_g.

$$\mathbb{C} = \sum_{\mathbf{g}=1}^{°G} C_g \mathbf{g} = \sum_{\mathbf{g}=1}^{°G} C_g \frac{\kappa_g}{°\kappa_g}$$

(Simple D_3 example)

$\mathbb{C}=8\mathbf{r}^1+8\mathbf{r}^2$

$=8(\mathbf{r}^1+\mathbf{r}^2)/2+8(\mathbf{r}^1+\mathbf{r}^2)/2$

$=8(\kappa_\mathbf{r})/2+8(\kappa_\mathbf{r})/2$

$=8\kappa_\mathbf{r}$

Review: Spectral resolution of **D_3 Center** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)

- *Weyl* **g***-expansion in irep $D^{\mu}_{jk}(g)$ and projectors* $\mathbf{P}^{\mu}_{jk}$
    - $\mathbf{P}^{\mu}_{jk}$ *transforms right-and-left*
    - $\mathbf{P}^{\mu}_{jk}$ *-expansion in* **g***-operators*
- *$D^{\mu}_{jk}(g)$ orthogonality relations*
- *Class projector character formulae*
- ➡ $\mathbb{P}^{\mu}$ *in terms of* $\boldsymbol{\kappa}_{\mathbf{g}}$ *and* $\boldsymbol{\kappa}_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$ ⬅

Details of Mock-Mach relativity-duality for D_3 groups and representations

- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis*

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis

- *Hamiltonian local-symmetry eigensolution*

$\mathbb{P}^\mu$ *in terms of* $\boldsymbol{\kappa}_{\mathbf{g}}$

$\boldsymbol{\kappa}_{\mathbf{g}}$ *in terms of* $\mathbb{P}^\mu$

$\mathbb{P}^{\mu}$ *in terms of* $\boldsymbol{\kappa}_{\mathbf{g}}$

$(\mu)^{\text{th}}$ *irep characters* $\chi^{(\mu)}(\mathbf{g})$ given by trace definition: $\chi^{\mu}(\mathbf{g}) \equiv Trace\ D^{\mu}(\mathbf{g}) = \sum_{m=1}^{\ell^{\mu}} D^{\mu}_{mm}(\mathbf{g})$

$\boldsymbol{\kappa}_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

$\mathbb{P}^{\mu}$ *in terms of* $\kappa_{\mathbf{g}}$

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$(\mu)^{th}$ *all-commuting class projector* given by sum $\mathbb{P}^{\mu} = \mathbf{P}^{\mu}_{11} + \mathbf{P}^{\mu}_{22} + \ldots + \mathbf{P}^{\mu}_{\ell^{\mu}\ell^{\mu}}$ of

irep projectors vs. $\mathbf{g}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}(g)\,\mathbf{g}$$

for unitary D^{μ}_{nm}

$$D^{\mu^{*}}_{mn}(g) = D^{\mu}_{nm}(g^{-1})$$

$\kappa_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

$\mathbb{P}^{\mu}$ *in terms of* $\boldsymbol{\kappa}_{\mathbf{g}}$

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$\boldsymbol{\kappa}_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

$\mathbb{P}^{\mu}$ *in terms of* $\boldsymbol{\kappa}_{\mathbf{g}}$

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$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\,\boldsymbol{\kappa}_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu*}_{g}\, \boldsymbol{\kappa}_{\mathbf{g}}}\ , \quad \text{where: } \chi^{\mu}_{g} = \chi^{\mu}(\mathbf{g}) = \chi^{\mu}(\mathbf{hgh}^{-1})$$

$\boldsymbol{\kappa}_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

$\mathbb{P}^{\mu}$ *in terms of* $\kappa_{\mathbf{g}}$

(μ)th *irep characters* $\chi^{(\mu)}(\mathbf{g})$ given by trace definition: $\chi^{\mu}(\mathbf{g}) \equiv Trace\, D^{\mu}(\mathbf{g}) = \sum_{m=1}^{\ell^{\mu}} D^{\mu}_{m\,m}(\mathbf{g})$

(μ)th *all-commuting class projector* given by sum $\mathbb{P}^{\mu}=\mathbf{P}^{\mu}_{11}+\mathbf{P}^{\mu}_{22}+\ldots+\mathbf{P}^{\mu}_{\ell^{\mu}\ell^{\mu}}$ of

irep projectors vs. **g**

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$$\mathbb{P}^{\mu} = \sum_{m=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{m\,m} = \frac{\ell^{\mu}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G}\sum_{m=1}^{\ell^{\mu}} D^{\mu*}_{m\,m}(g)\,\mathbf{g} = \frac{\ell^{\mu}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G}\chi^{\mu*}(g)\,\mathbf{g}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\,\kappa_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi^{\mu*}_{g}\,\kappa_{\mathbf{g}}}$$

, where: $\chi^{\mu}_{g} = \chi^{\mu}(\mathbf{g}) = \chi^{\mu}(\mathbf{hgh}^{-1})$

$\kappa_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Find *all-commuting class* $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$ given **g** vs. *irep projectors* $\mathbf{P}^{\mu}_{mn}$:

$$\boxed{\mathbf{g} = \sum_{\mu}\sum_{m}^{\ell^{\mu}}\sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn}}$$

$\mathbb{P}^{\mu}$ *in terms of* $\kappa_{\mathbf{g}}$

(μ)th *irep characters* $\chi^{(\mu)}(\mathbf{g})$ given by trace definition: $\chi^{\mu}(\mathbf{g}) \equiv Trace\, D^{\mu}(\mathbf{g}) = \sum_{m=1}^{\ell^{\mu}} D^{\mu}_{mm}(\mathbf{g})$

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irep projectors vs. **g**

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for unitary D^{μ}_{nm}

$$D^{\mu^{*}}_{mn}(g) = D^{\mu}_{nm}(g^{-1})$$

$$\mathbb{P}^{\mu} = \sum_{m=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mm} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} \sum_{m=1}^{\ell^{\mu}} D^{\mu*}_{mm}(g)\,\mathbf{g} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} \chi^{\mu*}(g)\,\mathbf{g}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\,\kappa_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu*}_{g}\, \kappa_{\mathbf{g}}}\;, \quad \text{where: } \chi^{\mu}_{g} = \chi^{\mu}(\mathbf{g}) = \chi^{\mu}(\mathbf{hgh}^{-1})$$

$\kappa_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Find *all-commuting class* $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$ given **g** vs. *irep projectors* $\mathbf{P}^{\mu}_{mn}$:

$$\boxed{\mathbf{g} = \sum_{\mu} \sum_{m}^{\ell^{\mu}} \sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn}}$$

$D^{\mu}_{mn}(\kappa_{\mathbf{g}})$ commutes with $D^{\mu}_{mn}(\mathbf{P}^{\mu}_{pr}) = \delta_{mp}\delta_{nr}$ for *all p and r* :

$\mathbb{P}^{\mu}$ *in terms of* $\boldsymbol{\kappa}_{\mathbf{g}}$

$(\mu)^{th}$ *irep characters* $\chi^{(\mu)}(\mathbf{g})$ given by trace definition: $\chi^{\mu}(\mathbf{g}) \equiv Trace\, D^{\mu}(\mathbf{g}) = \sum_{m=1}^{\ell^{\mu}} D^{\mu}_{mm}(\mathbf{g})$

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irep projectors vs. $\mathbf{g}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}(g)\,\mathbf{g}$$

for unitary D^{μ}_{nm}

$$D^{\mu^{*}}_{mn}(g) = D^{\mu}_{nm}\left(g^{-1}\right)$$

$$\mathbb{P}^{\mu} = \sum_{m=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mm} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} \sum_{m=1}^{\ell^{\mu}} D^{\mu^{*}}_{mm}(g)\,\mathbf{g} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} \chi^{\mu^{*}}(g)\,\mathbf{g}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\,\boldsymbol{\kappa}_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu^{*}}_{g} \boldsymbol{\kappa}_{\mathbf{g}}}\ , \quad \text{where: } \chi^{\mu}_{g} = \chi^{\mu}(\mathbf{g}) = \chi^{\mu}\left(\mathbf{hgh}^{-1}\right)$$

$\boldsymbol{\kappa}_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Find *all-commuting class* $\boldsymbol{\kappa}_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$ given $\mathbf{g}$ vs. *irep projectors* $\mathbf{P}^{\mu}_{mn}$*:*

$$\boxed{\mathbf{g} = \sum_{\mu} \sum_{m}^{\ell^{\mu}} \sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn}}$$

$D^{\mu}_{mn}(\boldsymbol{\kappa}_{\mathbf{g}})$ commutes with $D^{\mu}_{mn}(\mathbf{P}^{\mu}_{pr}) = \delta_{mp}\delta_{nr}$ for *all p and r* :

$$\sum_{b=1}^{\ell^{\mu}} D^{\mu}_{ab}(\boldsymbol{\kappa}_{\mathbf{g}})\, D^{\mu}_{bc}(\mathbf{P}^{\mu}_{pr}) = \sum_{d=1}^{\ell^{\mu}} D^{\mu}_{ad}(\mathbf{P}^{\mu}_{pr})\, D^{\mu}_{dc}(\boldsymbol{\kappa}_{\mathbf{g}})$$

$\mathbb{P}^{\mu}$ *in terms of* $\kappa_{\mathbf{g}}$

$(\mu)^{th}$ *irep characters* $\chi^{(\mu)}(\mathbf{g})$ given by trace definition: $\chi^{\mu}(\mathbf{g}) \equiv Trace\, D^{\mu}(\mathbf{g}) = \sum_{m=1}^{\ell^{\mu}} D^{\mu}_{mm}(\mathbf{g})$

$(\mu)^{th}$ *all-commuting class projector* given by sum $\mathbb{P}^{\mu} = \mathbf{P}^{\mu}_{11} + \mathbf{P}^{\mu}_{22} + \ldots + \mathbf{P}^{\mu}_{\ell^{\mu}\ell^{\mu}}$ of

irep projectors vs. $\mathbf{g}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}(g)\,\mathbf{g}$$

for unitary D^{μ}_{nm}

$$D^{\mu^{*}}_{mn}(g) = D^{\mu}_{nm}(g^{-1})$$

$$\mathbb{P}^{\mu} = \sum_{m=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mm} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} \sum_{m=1}^{\ell^{\mu}} D^{\mu *}_{mm}(g)\,\mathbf{g} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} \chi^{\mu *}(g)\,\mathbf{g}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\,\kappa_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu *}_{g}\, \kappa_{\mathbf{g}}}\text{ , where: } \chi^{\mu}_{g} = \chi^{\mu}(\mathbf{g}) = \chi^{\mu}(\mathbf{h}\mathbf{g}\mathbf{h}^{-1})$$

$\kappa_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Find *all-commuting class* $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$ given $\mathbf{g}$ vs. *irep projectors* $\mathbf{P}^{\mu}_{mn}$:

$$\boxed{\mathbf{g} = \sum_{\mu} \sum_{m}^{\ell^{\mu}} \sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn}}$$

$D^{\mu}_{mn}(\kappa_{\mathbf{g}})$ commutes with $D^{\mu}_{mn}(\mathbf{P}^{\mu}_{pr}) = \delta_{mp}\delta_{nr}$ for *all p and r* :

$$\sum_{b=1}^{\ell^{\mu}} D^{\mu}_{ab}(\kappa_{\mathbf{g}}) D^{\mu}_{bc}(\mathbf{P}^{\mu}_{pr}) = \sum_{d=1}^{\ell^{\mu}} D^{\mu}_{ad}(\mathbf{P}^{\mu}_{pr}) D^{\mu}_{dc}(\kappa_{\mathbf{g}})$$

$$\sum_{b=1}^{\ell^{\mu}} D^{\mu}_{ab}(\kappa_{\mathbf{g}})\ \delta_{bp}\delta_{cr} = \sum_{d=1}^{\ell^{\mu}}\ \delta_{ap}\delta_{dr}\ D^{\mu}_{dc}(\kappa_{\mathbf{g}})$$

$\mathbb{P}^{\mu}$ *in terms of* $\kappa_{\mathbf{g}}$

(μ)th *irep characters* $\chi^{(\mu)}(\mathbf{g})$ given by trace definition: $\chi^{\mu}(\mathbf{g}) \equiv Trace\; D^{\mu}(\mathbf{g}) = \sum_{m=1}^{\ell^{\mu}} D^{\mu}_{mm}(\mathbf{g})$

(μ)th *all-commuting class projector* given by sum $\mathbb{P}^{\mu} = \mathbf{P}^{\mu}_{11} + \mathbf{P}^{\mu}_{22} + \ldots + \mathbf{P}^{\mu}_{\ell^{\mu}\ell^{\mu}}$ of

$$\boxed{\begin{array}{c} \textit{irep projectors}\text{ vs. }\mathbf{g} \\ \mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^{*}}_{mn}(g)\,\mathbf{g} \\ \left(\begin{array}{c} \text{for unitary } D^{\mu}_{nm} \\ D^{\mu^{*}}_{mn}(g) = D^{\mu}_{nm}\left(g^{-1}\right) \end{array}\right) \end{array}}$$

$$\mathbb{P}^{\mu} = \sum_{m=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mm} = \frac{\ell^{\mu}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G}\sum_{m=1}^{\ell^{\mu}} D^{\mu *}_{mm}(g)\,\mathbf{g} = \frac{\ell^{\mu}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G}\chi^{\mu *}(g)\,\mathbf{g}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\,\kappa_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G}\chi^{\mu *}_{g}\kappa_{\mathbf{g}}}\;,\quad \text{where: }\; \chi^{\mu}_{g} = \chi^{\mu}(\mathbf{g}) = \chi^{\mu}\left(\mathbf{hgh}^{-1}\right)$$

$\kappa_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Find *all-commuting class* $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$ given $\mathbf{g}$ vs. *irep projectors* $\mathbf{P}^{\mu}_{mn}$: $\boxed{\mathbf{g} = \sum_{\mu}\sum_{m}^{\ell^{\mu}}\sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}(g)\,\mathbf{P}^{\mu}_{mn}}$

$D^{\mu}_{mn}(\kappa_{\mathbf{g}})$ commutes with $D^{\mu}_{mn}\left(\mathbf{P}^{\mu}_{pr}\right) = \delta_{mp}\delta_{nr}$ for *all p and r* :

$$\sum_{b=1}^{\ell^{\mu}} D^{\mu}_{ab}(\kappa_{\mathbf{g}})\,D^{\mu}_{bc}\left(\mathbf{P}^{\mu}_{pr}\right) = \sum_{d=1}^{\ell^{\mu}} D^{\mu}_{ad}\left(\mathbf{P}^{\mu}_{pr}\right)D^{\mu}_{dc}(\kappa_{\mathbf{g}})$$

$$\sum_{b=1}^{\ell^{\mu}} D^{\mu}_{ab}(\kappa_{\mathbf{g}})\;\delta_{bp}\delta_{cr} = \sum_{d=1}^{\ell^{\mu}}\;\delta_{ap}\delta_{dr}\;D^{\mu}_{dc}(\kappa_{\mathbf{g}})$$

$$D^{\mu}_{ap}(\kappa_{\mathbf{g}})\;\delta_{cr} = \delta_{ap}\;D^{\mu}_{rc}(\kappa_{\mathbf{g}})$$

$\mathbb{P}^\mu$ *in terms of* κ_g

(μ)th *irep characters* $\chi^{(\mu)}(\mathbf{g})$ given by trace definition: $\chi^\mu(\mathbf{g}) \equiv Trace\, D^\mu(\mathbf{g}) = \sum_{m=1}^{\ell^\mu} D^\mu_{mm}(\mathbf{g})$

(μ)th *all-commuting class projector* given by sum $\mathbb{P}^\mu = \mathbf{P}^\mu_{11} + \mathbf{P}^\mu_{22} + \ldots + \mathbf{P}^\mu_{\ell^\mu \ell^\mu}$ of

irep projectors vs. **g**

$$\mathbf{P}^\mu_{mn} = \frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\,\mathbf{g}$$

for unitary D^μ_{nm}

$$D^{\mu^*}_{mn}(g) = D^\mu_{nm}(g^{-1})$$

$$\mathbb{P}^\mu = \sum_{m=1}^{\ell^\mu}\mathbf{P}^\mu_{mm} = \frac{\ell^\mu}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G}\sum_{m=1}^{\ell^\mu} D^{\mu*}_{mm}(g)\,\mathbf{g} = \frac{\ell^\mu}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G}\chi^{\mu*}(g)\,\mathbf{g}$$

$$\boxed{\mathbb{P}^\mu = \sum_{classes\,\kappa_{\mathbf{g}}}\frac{\ell^\mu}{{}^\circ G}\chi^{\mu*}_g\,\kappa_{\mathbf{g}}}$$

, where: $\chi^\mu_g = \chi^\mu(\mathbf{g}) = \chi^\mu(\mathbf{hgh}^{-1})$

κ_g *in terms of* $\mathbb{P}^\mu$

Find *all-commuting class* $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^\mu$ given **g** vs. *irep projectors* $\mathbf{P}^\mu_{mn}$:

$$\boxed{\mathbf{g} = \sum_\mu\sum_m^{\ell^\mu}\sum_n^{\ell^\mu} D^\mu_{mn}(g)\,\mathbf{P}^\mu_{mn}}$$

$D^\mu_{mn}(\kappa_{\mathbf{g}})$ commutes with $D^\mu_{mn}(\mathbf{P}^\mu_{pr}) = \delta_{mp}\delta_{nr}$ for *all p and r* :

$$\sum_{b=1}^{\ell^\mu} D^\mu_{ab}(\kappa_{\mathbf{g}})\,D^\mu_{bc}(\mathbf{P}^\mu_{pr}) = \sum_{d=1}^{\ell^\mu} D^\mu_{ad}(\mathbf{P}^\mu_{pr})\,D^\mu_{dc}(\kappa_{\mathbf{g}})$$

$$\sum_{b=1}^{\ell^\mu} D^\mu_{ab}(\kappa_{\mathbf{g}})\quad \delta_{bp}\delta_{cr} \quad = \sum_{d=1}^{\ell^\mu} \quad \delta_{ap}\delta_{dr} \quad D^\mu_{dc}(\kappa_{\mathbf{g}})$$

$$D^\mu_{ap}(\kappa_{\mathbf{g}}) \qquad \delta_{cr} \qquad = \qquad \delta_{ap} \qquad D^\mu_{rc}(\kappa_{\mathbf{g}})$$

So: $D^\mu_{mn}(\kappa_{\mathbf{g}})$ is multiple of ℓ^μ-by-ℓ^μ unit matrix:

$$D^\mu_{mn}(\kappa_{\mathbf{g}}) = \delta_{mn}\frac{\chi^\mu(\kappa_{\mathbf{g}})}{\ell^\mu} = \delta_{mn}\frac{{}^\circ\kappa_g\,\chi^\mu_g}{\ell^\mu}$$

$\mathbb{P}^{\mu}$ *in terms of* $\boldsymbol{\kappa}_{\mathbf{g}}$

(μ)th *irep characters* $\chi^{(\mu)}(\mathbf{g})$ given by trace definition: $\chi^{\mu}(\mathbf{g}) \equiv Trace\, D^{\mu}(\mathbf{g}) = \sum_{m=1}^{\ell^{\mu}} D^{\mu}_{mm}(\mathbf{g})$

(μ)th *all-commuting class projector* given by sum $\mathbb{P}^{\mu} = \mathbf{P}^{\mu}_{11} + \mathbf{P}^{\mu}_{22} + \ldots + \mathbf{P}^{\mu}_{\ell^{\mu}\ell^{\mu}}$ of

irep projectors vs. $\mathbf{g}$

$$\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\, \mathbf{g}$$

for unitary D^{μ}_{nm}

$$D^{\mu^*}_{mn}(g) = D^{\mu}_{nm}(g^{-1})$$

$$\mathbb{P}^{\mu} = \sum_{m=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mm} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} \sum_{m=1}^{\ell^{\mu}} D^{\mu*}_{mm}(g)\, \mathbf{g} = \frac{\ell^{\mu}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} \chi^{\mu*}(g)\, \mathbf{g}$$

$$\boxed{\mathbb{P}^{\mu} = \sum_{classes\, \boldsymbol{\kappa}_{\mathbf{g}}} \frac{\ell^{\mu}}{{}^{\circ}G} \chi^{\mu*}_{g} \boldsymbol{\kappa}_{\mathbf{g}}}\quad , \text{ where: } \chi^{\mu}_{g} = \chi^{\mu}(\mathbf{g}) = \chi^{\mu}(\mathbf{hgh}^{-1})$$

$\boldsymbol{\kappa}_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Find *all-commuting class* $\boldsymbol{\kappa}_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$ given $\mathbf{g}$ vs. *irep projectors* $\mathbf{P}^{\mu}_{mn}$:

$$\boxed{\mathbf{g} = \sum_{\mu} \sum_{m}^{\ell^{\mu}} \sum_{n}^{\ell^{\mu}} D^{\mu}_{mn}(g)\, \mathbf{P}^{\mu}_{mn}}$$

$D^{\mu}_{mn}(\boldsymbol{\kappa}_{\mathbf{g}})$ commutes with $D^{\mu}_{mn}(\mathbf{P}^{\mu}_{pr}) = \delta_{mp}\delta_{nr}$ for *all p and r* :

$$\sum_{b=1}^{\ell^{\mu}} D^{\mu}_{ab}(\boldsymbol{\kappa}_{\mathbf{g}}) D^{\mu}_{bc}(\mathbf{P}^{\mu}_{pr}) = \sum_{d=1}^{\ell^{\mu}} D^{\mu}_{ad}(\mathbf{P}^{\mu}_{pr}) D^{\mu}_{dc}(\boldsymbol{\kappa}_{\mathbf{g}})$$

$$\sum_{b=1}^{\ell^{\mu}} D^{\mu}_{ab}(\boldsymbol{\kappa}_{\mathbf{g}})\ \delta_{bp}\delta_{cr} = \sum_{d=1}^{\ell^{\mu}} \delta_{ap}\delta_{dr}\ D^{\mu}_{dc}(\boldsymbol{\kappa}_{\mathbf{g}})$$

$$D^{\mu}_{ap}(\boldsymbol{\kappa}_{\mathbf{g}})\ \delta_{cr} = \delta_{ap}\ D^{\mu}_{rc}(\boldsymbol{\kappa}_{\mathbf{g}})$$

So: $D^{\mu}_{mn}(\boldsymbol{\kappa}_{\mathbf{g}})$ is multiple of ℓ^{μ}-by-ℓ^{μ} unit matrix:

$$D^{\mu}_{mn}(\boldsymbol{\kappa}_{\mathbf{g}}) = \delta_{mn} \frac{\chi^{\mu}(\boldsymbol{\kappa}_{\mathbf{g}})}{\ell^{\mu}} = \delta_{mn} \frac{{}^{\circ}\kappa_{g}\chi^{\mu}_{g}}{\ell^{\mu}}$$

$$\boxed{\boldsymbol{\kappa}_{\mathbf{g}} = \sum_{\mu} \frac{{}^{\circ}\kappa_{g}\chi^{\mu}_{g}}{\ell^{\mu}} \mathbb{P}^{\mu}}$$

Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)

Weyl **g***-expansion in irep $D^{\mu}_{jk}(g)$ and projectors $\mathbf{P}^{\mu}_{jk}$*

$\mathbf{P}^{\mu}_{jk}$ transforms right-and-left

$\mathbf{P}^{\mu}_{jk}$ -expansion in **g***-operators*

$D^{\mu}_{jk}(g)$ orthogonality relations

Class projector character formulae

$\mathbb{P}^{\mu}$ in terms of κ_g and κ_g in terms of $\mathbb{P}^{\mu}$

➡ *Details of Mock-Mach relativity-duality for D_3 groups and representations* ⬅

Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)

Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis

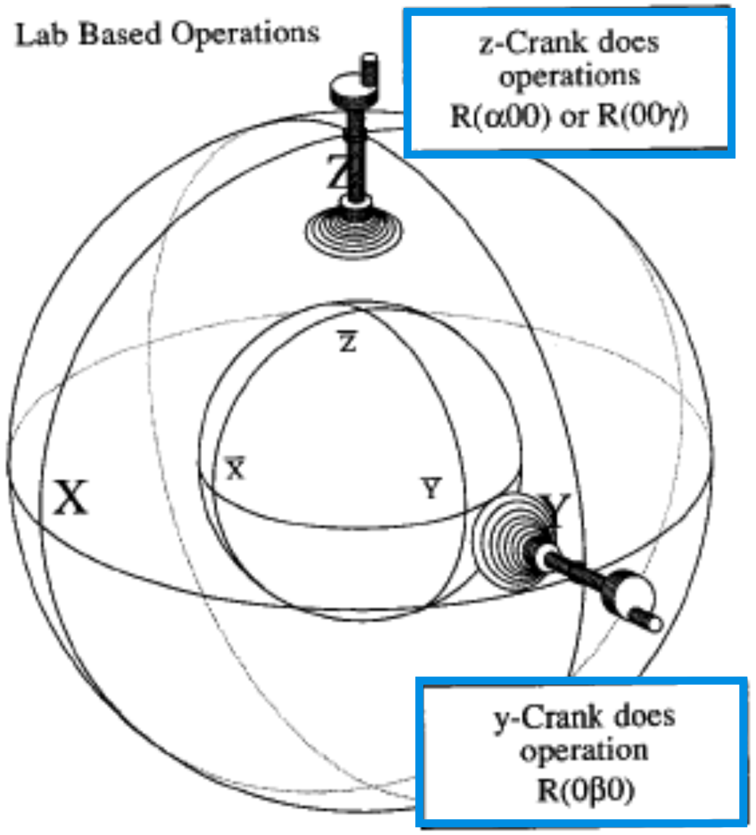
Hamiltonian local-symmetry eigensolution

Details of Mock-Mach relativity-duality for D_3 groups and representations

> *"Give me a place to stand...*
> *and I will move the Earth"*
> Archimedes 287-212 B.C.E

Ideas of duality/relativity go *way* back (...VanVleck, Casimir..., Mach, Newton, Archimedes...)

Lab-fixed(Extrinsic-Global)**R**,**S**,..*vs.* Body-fixed (Intrinsic-Local)$\bar{\mathbf{R}}$,$\bar{\mathbf{S}}$,..



all **R**,**S**,..
commute with
all $\bar{\mathbf{R}}$,$\bar{\mathbf{S}}$,..

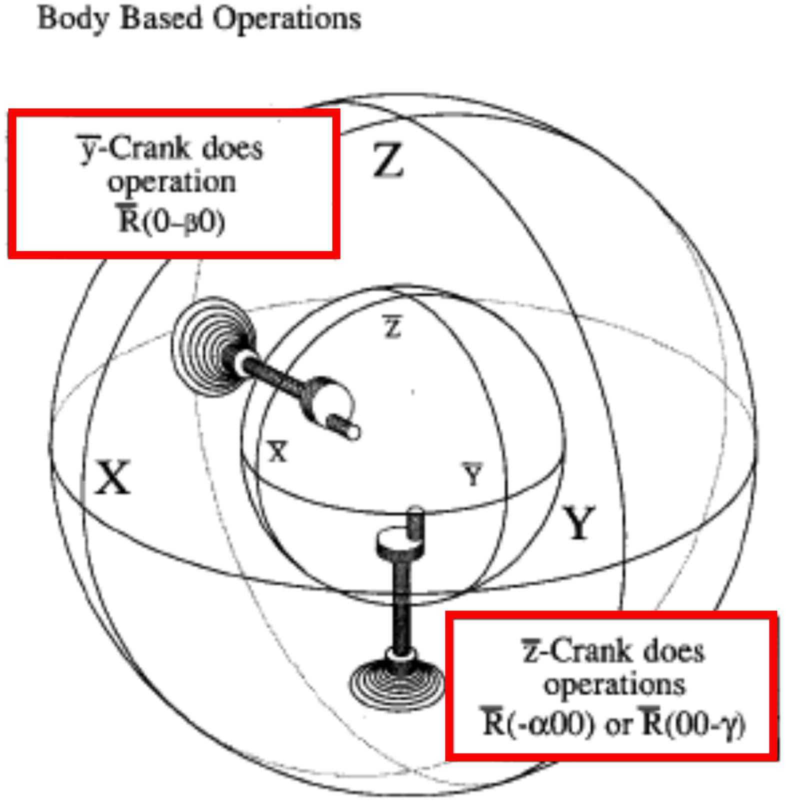
"Mock-Mach"
relativity principles

$$\mathbf{R}|1\rangle=\bar{\mathbf{R}}^{-1}|1\rangle$$

$$\mathbf{S}|1\rangle=\bar{\mathbf{S}}^{-1}|1\rangle$$

$$\vdots$$

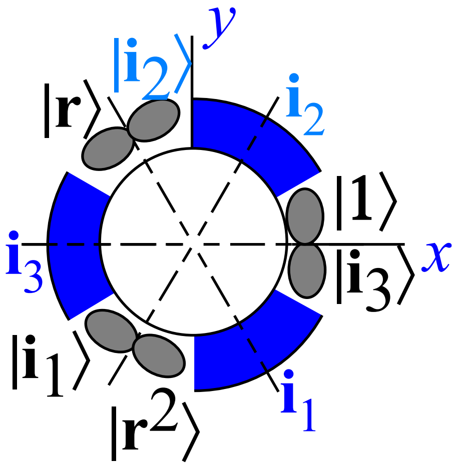
...for *one* state $|1\rangle$ *only!*

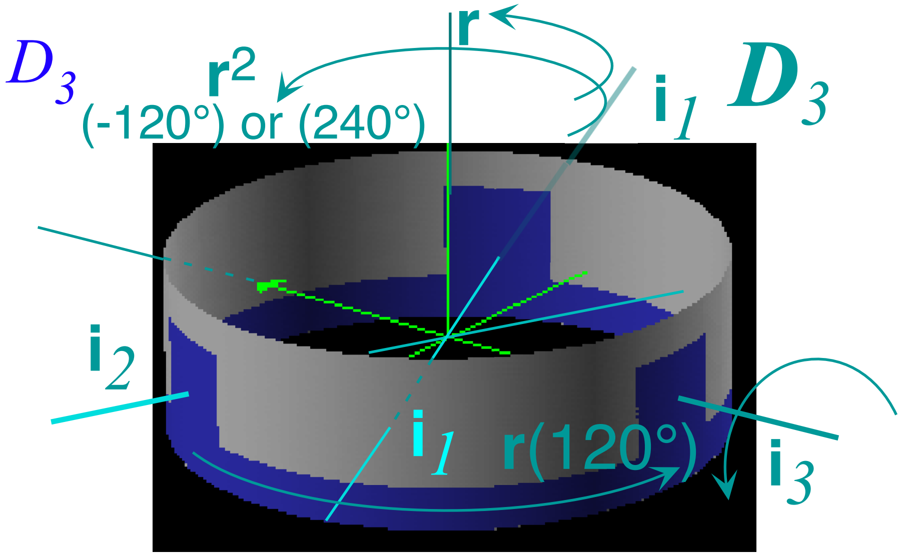


...But *how* do you actually *make* the **R** and $\bar{\mathbf{R}}$ operations?

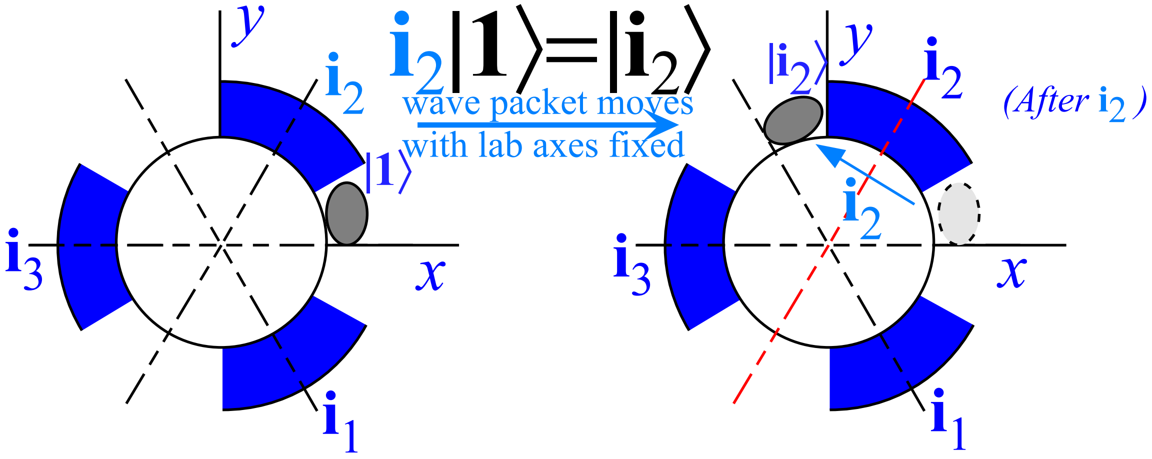
Details of RELATIVITY-DUALITY for D_3

D_3-defined local-wave bases





Lab-fixed (Extrinsic-Global) operations&axes fixed

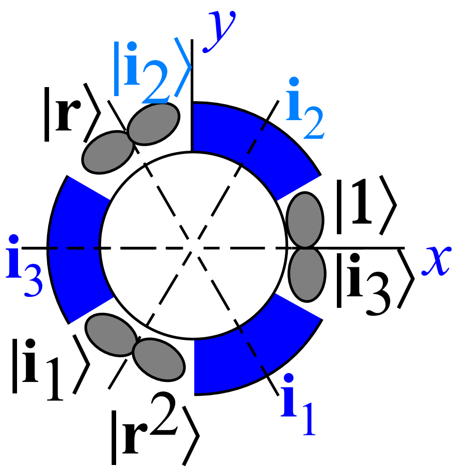

$\mathbf{i}_2|1\rangle=|\mathbf{i}_2\rangle$

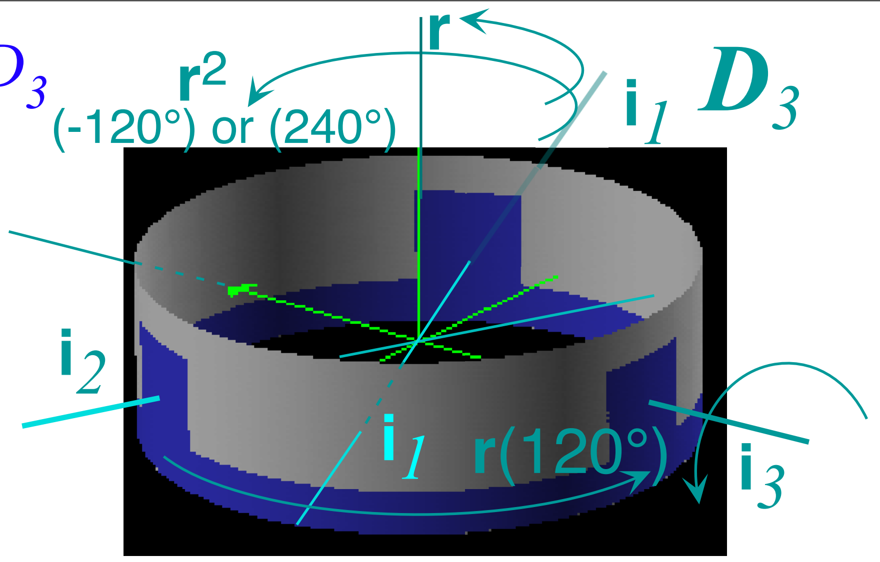
wave packet moves with lab axes fixed

(After $\mathbf{i}_2$)

Details of RELATIVITY-DUALITY for D_3

D_3-defined local-wave bases

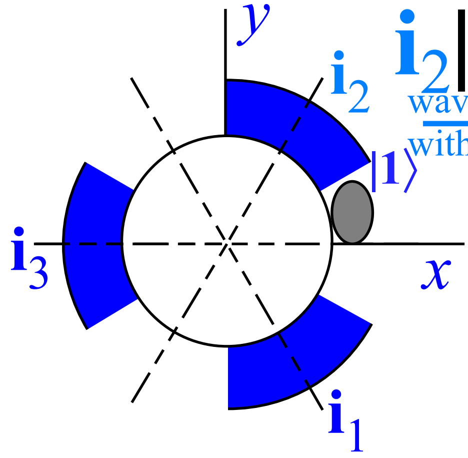


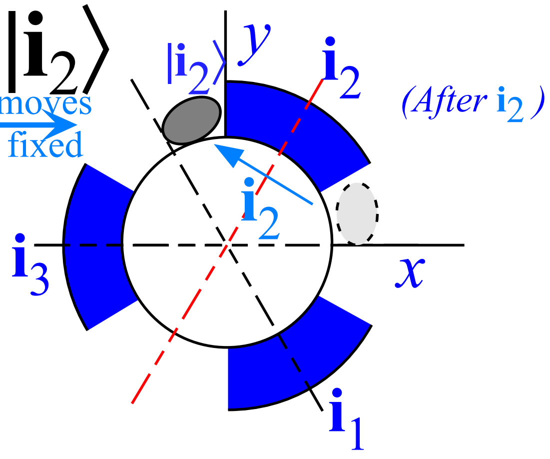


Lab-fixed (Extrinsic-Global) operations&axes fixed

$\mathbf{i}_2|1\rangle=|\mathbf{i}_2\rangle$

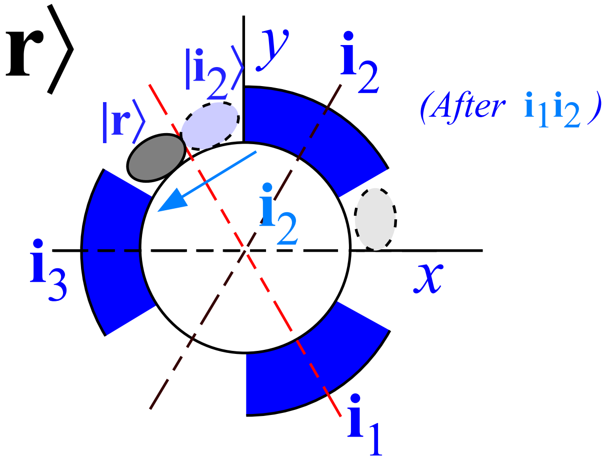
wave packet moves with lab axes fixed





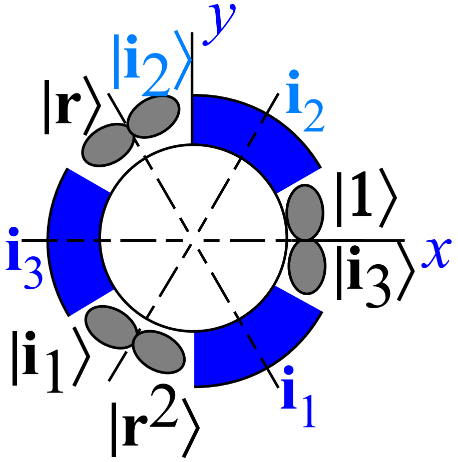
$\mathbf{i}_1\mathbf{i}_2|1\rangle=\mathbf{r}|1\rangle=|\mathbf{r}\rangle$

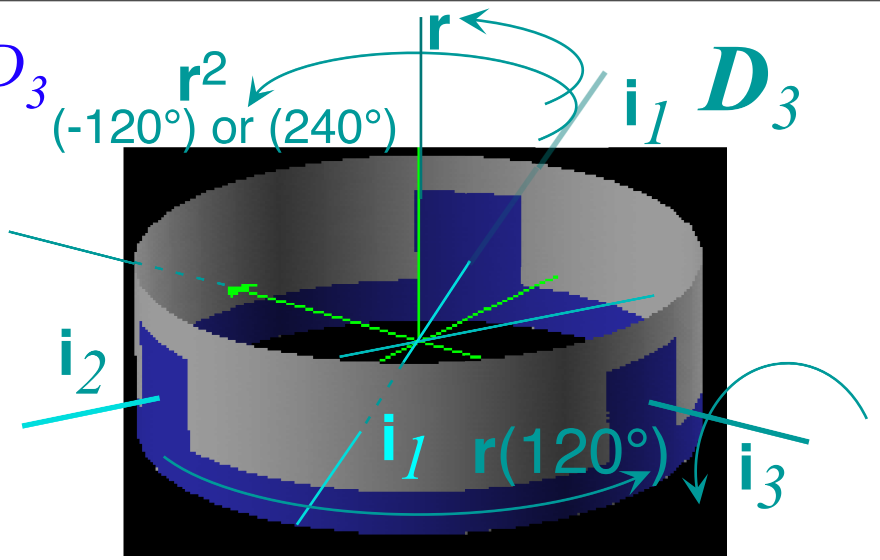
wave packet moves with lab axes fixed



Details of RELATIVITY-DUALITY for D_3

D_3-defined local-wave bases

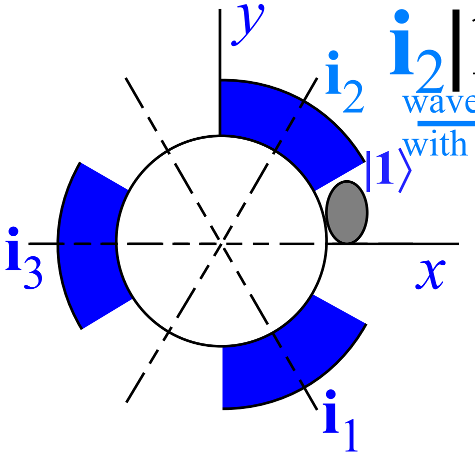




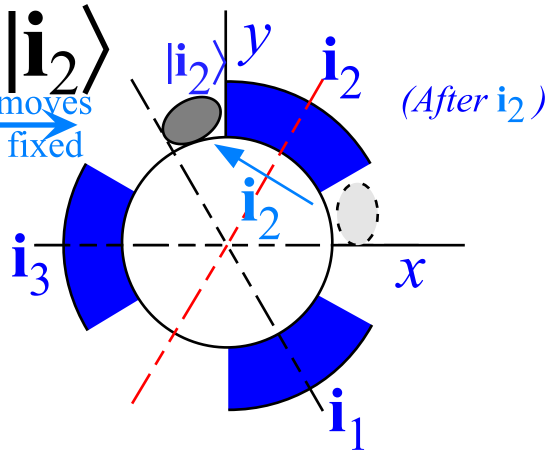
1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

Lab-fixed (Extrinsic-Global) operations&axes fixed

$\mathbf{i}_2|1\rangle=|\mathbf{i}_2\rangle$

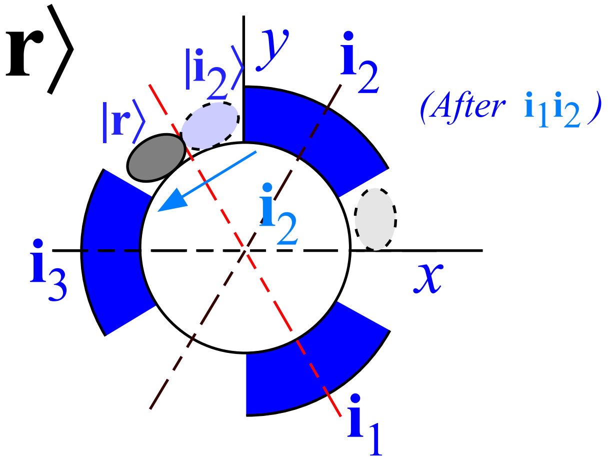


wave packet moves
with lab axes fixed



$\mathbf{i}_1\mathbf{i}_2|1\rangle=\mathbf{r}|1\rangle=|\mathbf{r}\rangle$

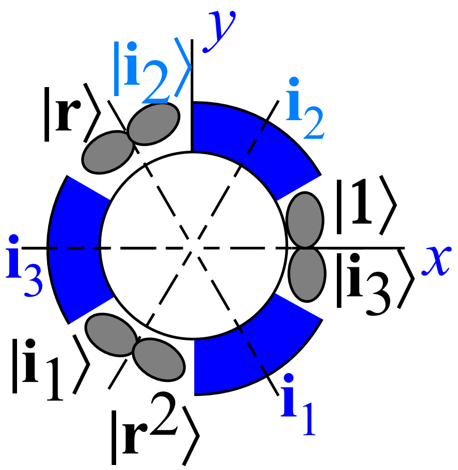
wave packet moves
with lab axes fixed

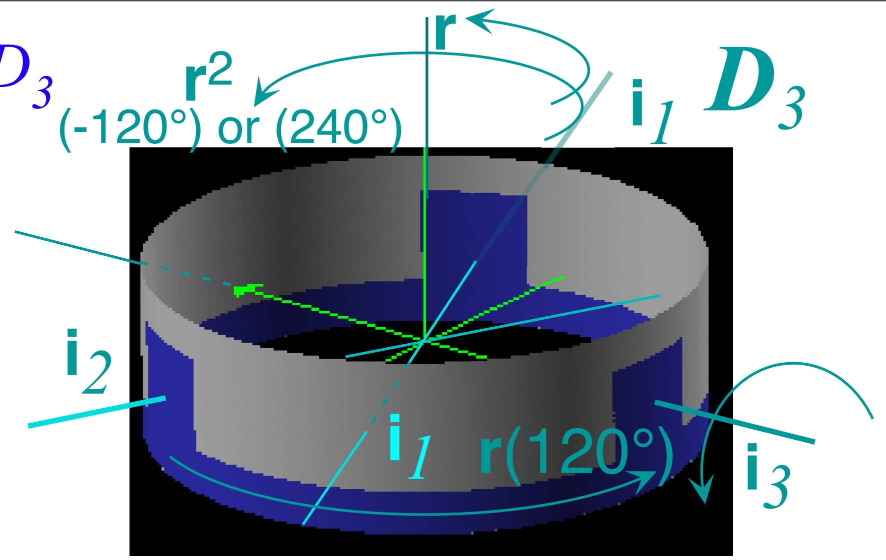


$\mathbf{i}_1\mathbf{i}_2 = \mathbf{r}$

Details of RELATIVITY-DUALITY for D_3

D_3-defined local-wave bases

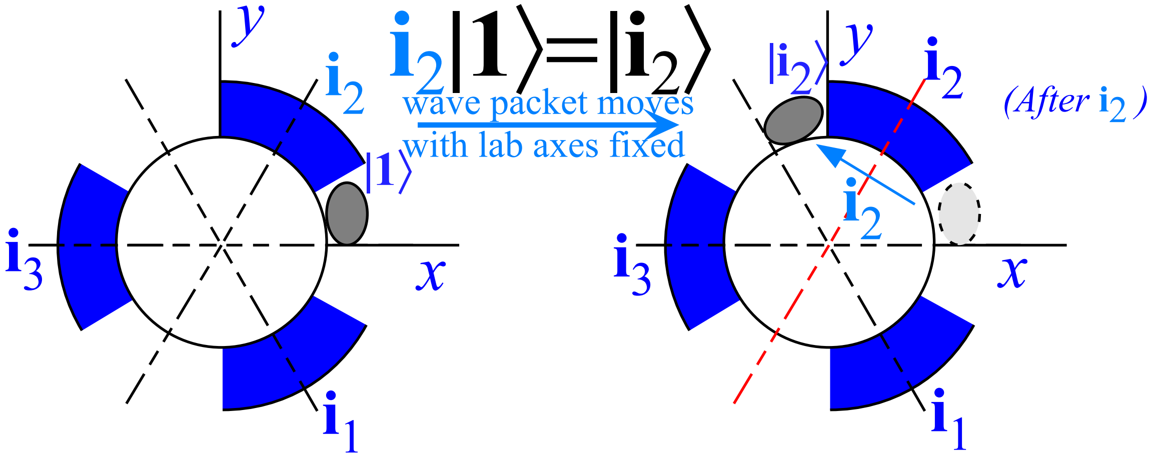




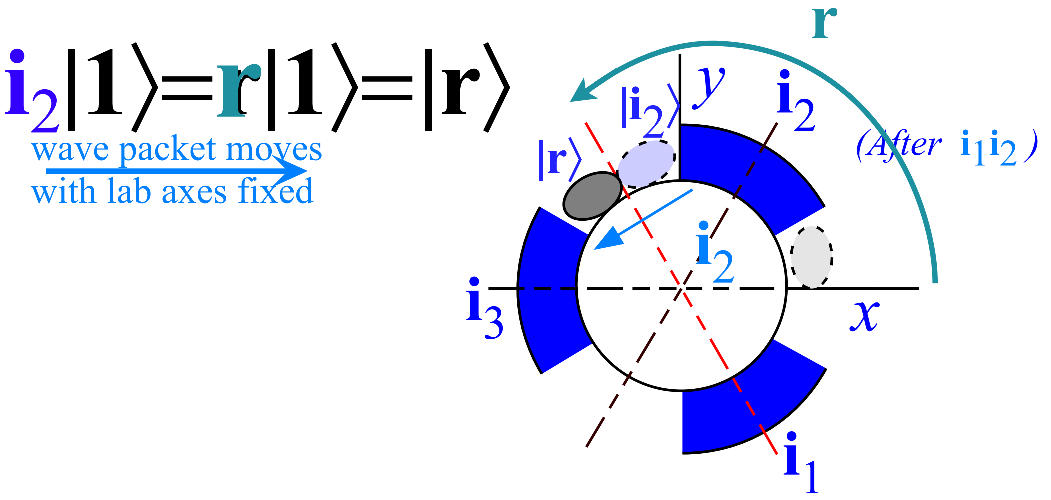
1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

Lab-fixed (Extrinsic-Global) operations&axes fixed

$$\mathbf{i}_2|1\rangle=|\mathbf{i}_2\rangle$$



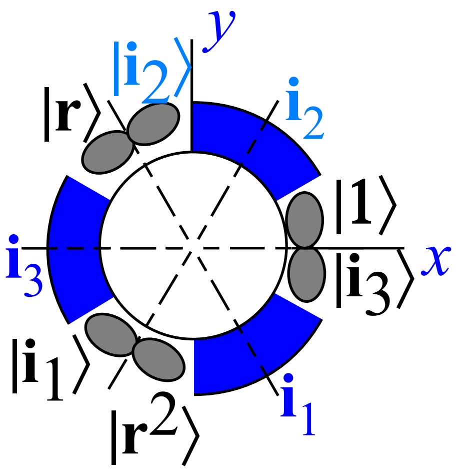
$$\mathbf{i}_1\mathbf{i}_2|1\rangle=\mathbf{r}|1\rangle=|\mathbf{r}\rangle$$

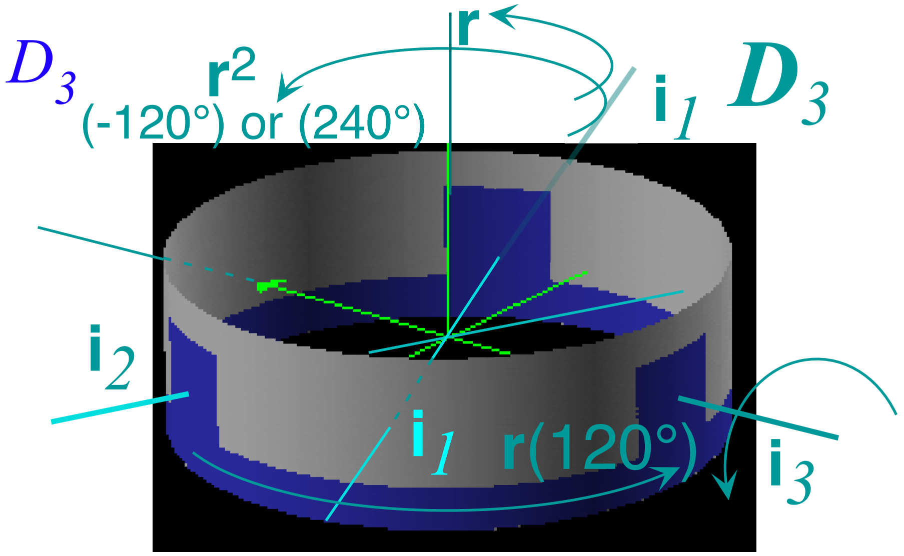


$i_1 i_2 = r$

Details of RELATIVITY-DUALITY for D_3

D_3-defined local-wave bases

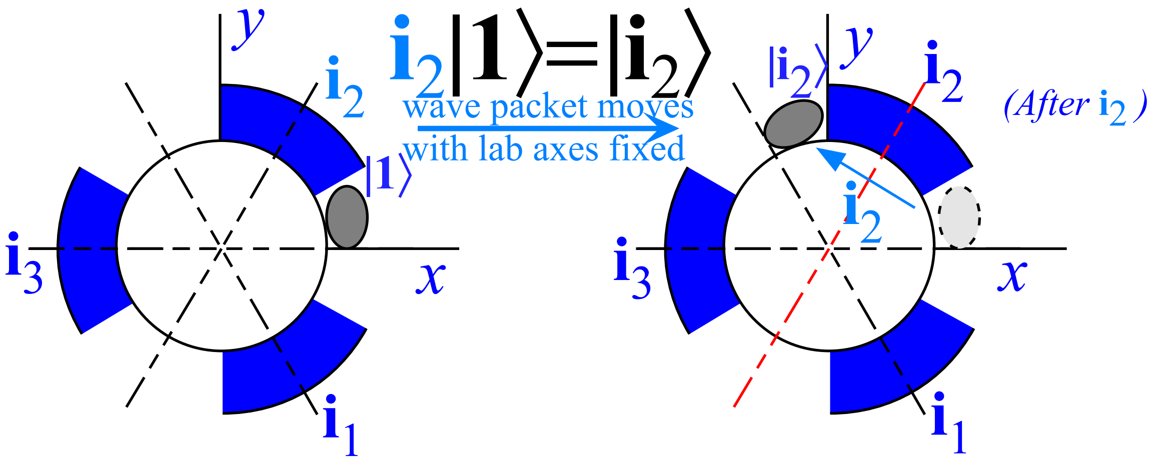




1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

$i_1 i_2 = r$

Lab-fixed (Extrinsic-Global) operations&axes fixed

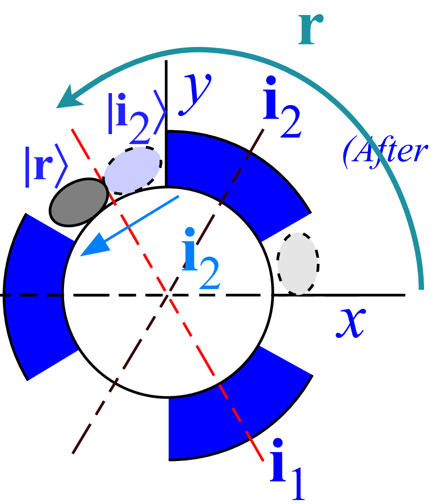


$\mathbf{i}_2|\mathbf{1}\rangle=|\mathbf{i}_2\rangle$

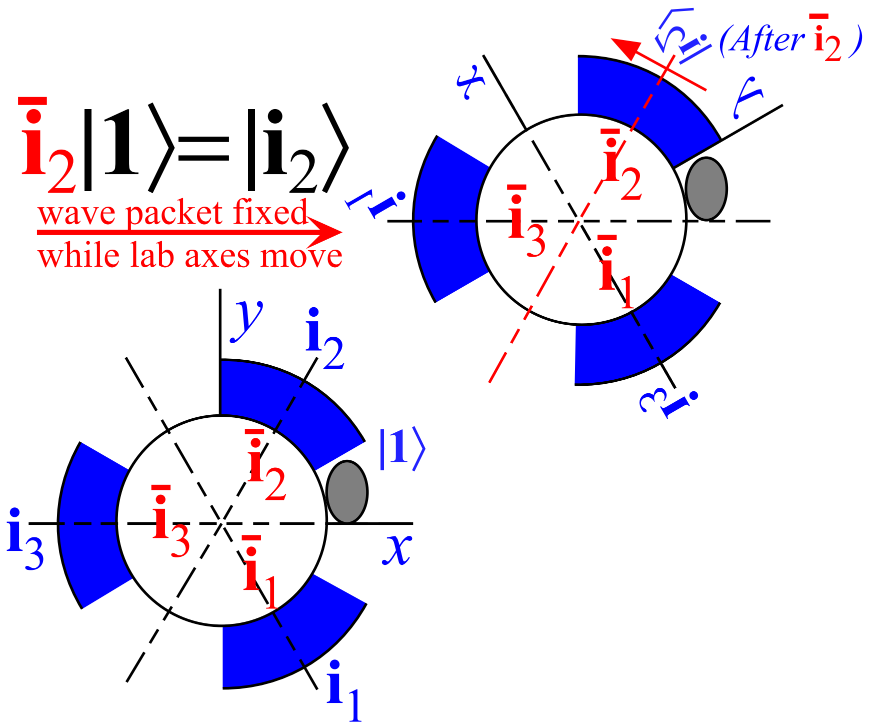
wave packet moves with lab axes fixed

$\mathbf{i}_1\mathbf{i}_2|\mathbf{1}\rangle=\mathbf{r}|\mathbf{1}\rangle=|\mathbf{r}\rangle$

wave packet moves with lab axes fixed



Body-fixed (Intrinsic-Local) operations appear to move their rotation axes (relative to lab)

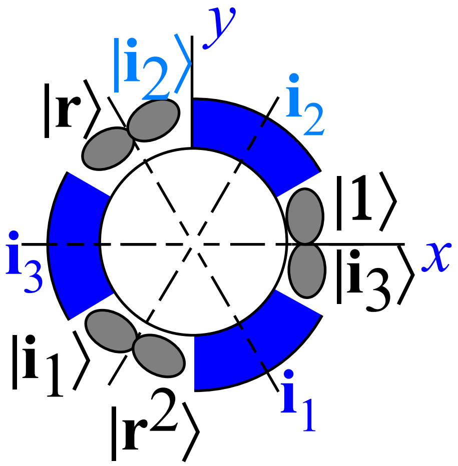


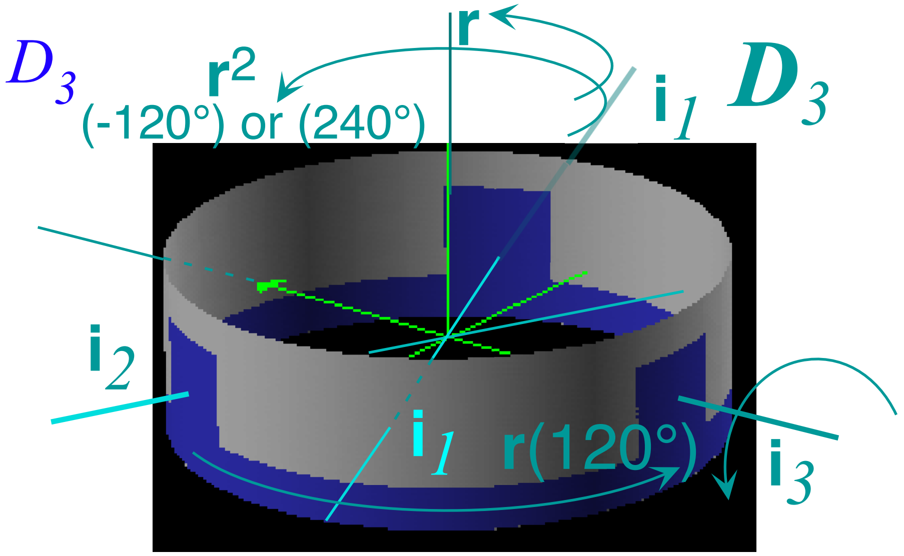
$\bar{\mathbf{i}}_2|\mathbf{1}\rangle=|\mathbf{i}_2\rangle$

wave packet fixed while lab axes move

Details of RELATIVITY-DUALITY for D_3

D_3-defined local-wave bases





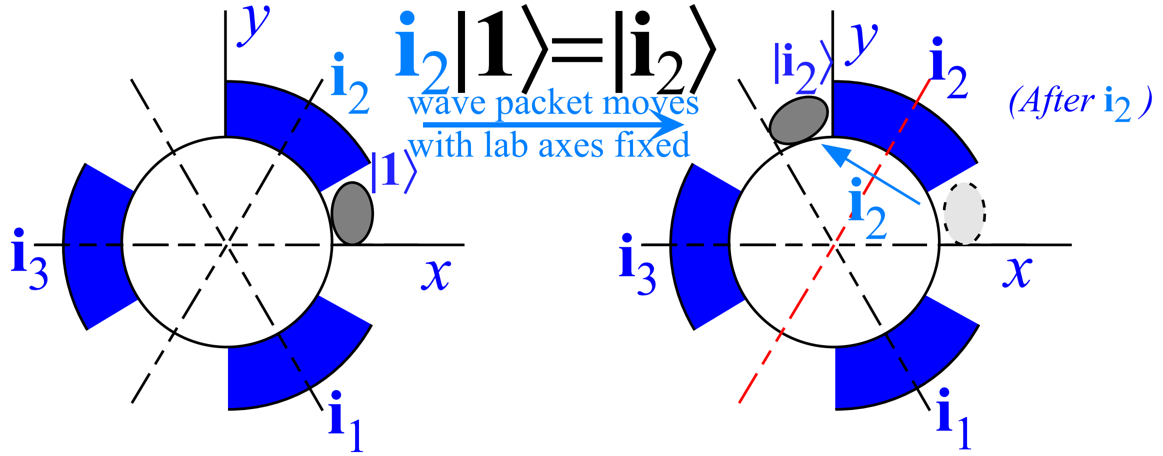
1	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	1	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	1	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	1	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	1	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	1

$\mathbf{i}_1\mathbf{i}_2 = \mathbf{r}$

Lab-fixed (Extrinsic-Global) operations&axes fixed

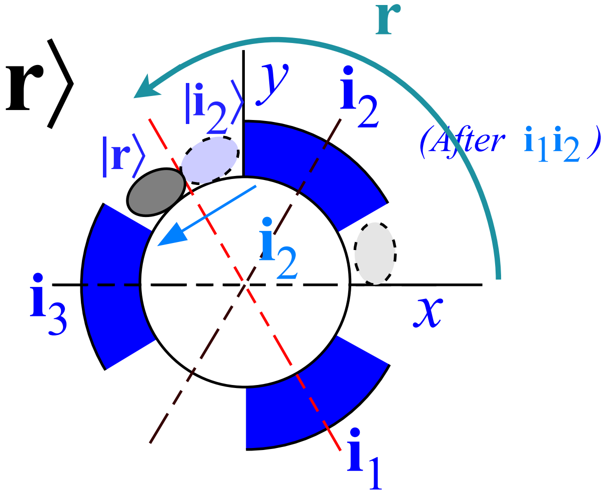
$\mathbf{i}_2|1\rangle=|\mathbf{i}_2\rangle$

wave packet moves with lab axes fixed



$\mathbf{i}_1\mathbf{i}_2|1\rangle=\mathbf{r}|1\rangle=|\mathbf{r}\rangle$

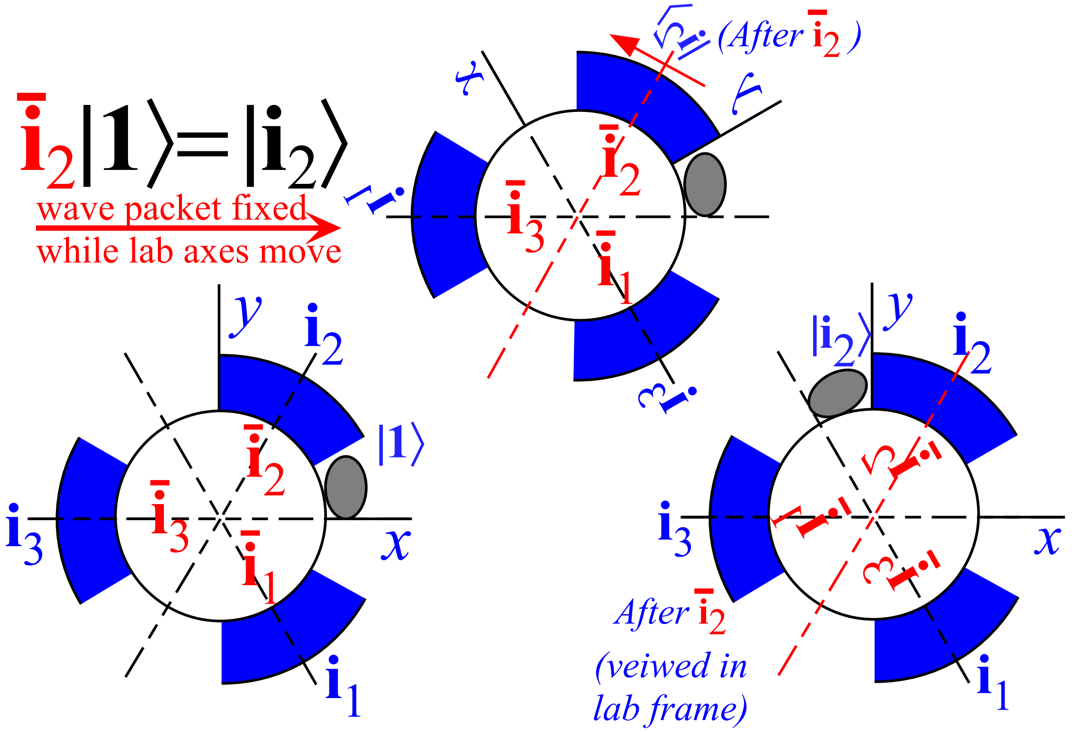
wave packet moves with lab axes fixed



Body-fixed (Intrinsic-Local) operations appear to move their rotation axes (relative to lab)

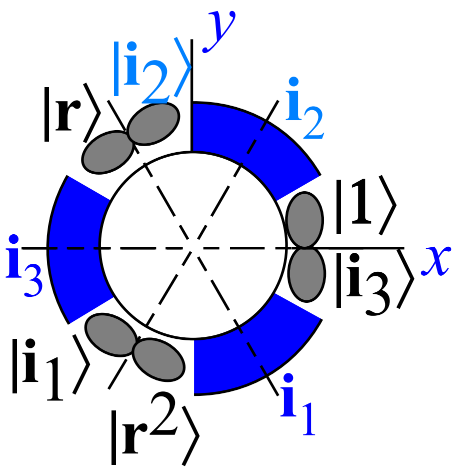
$\bar{\mathbf{i}}_2|1\rangle=|\mathbf{i}_2\rangle$

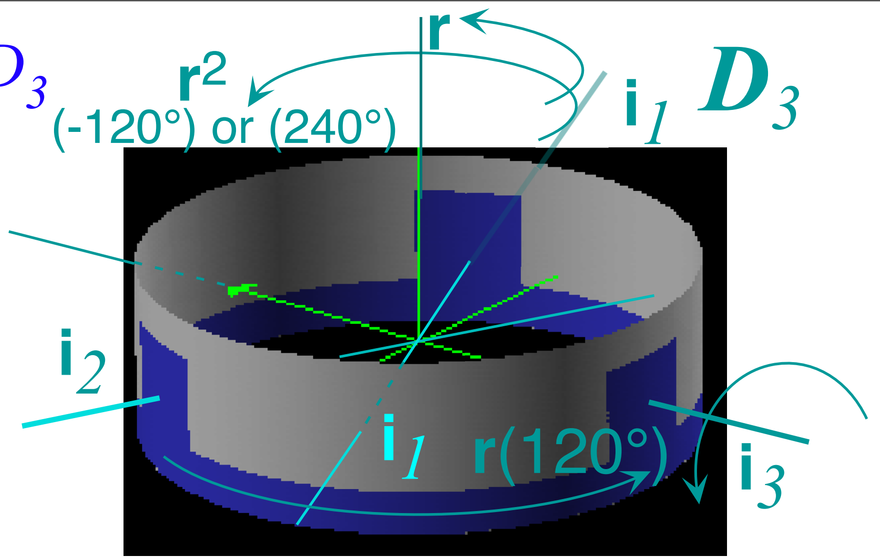
wave packet fixed while lab axes move



Details of RELATIVITY-DUALITY for D_3

D_3-defined local-wave bases

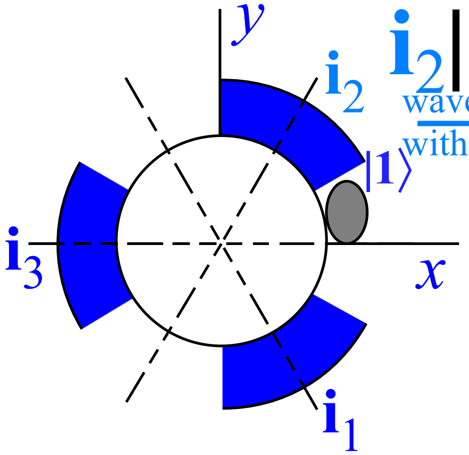




$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

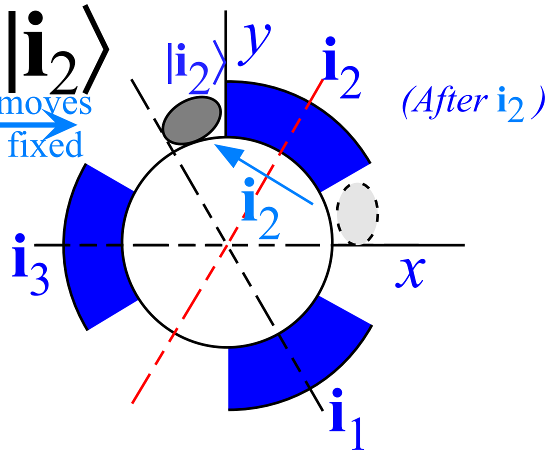
$\mathbf{i}_1\mathbf{i}_2 = \mathbf{r}$

Lab-fixed (Extrinsic-Global) operations&axes fixed



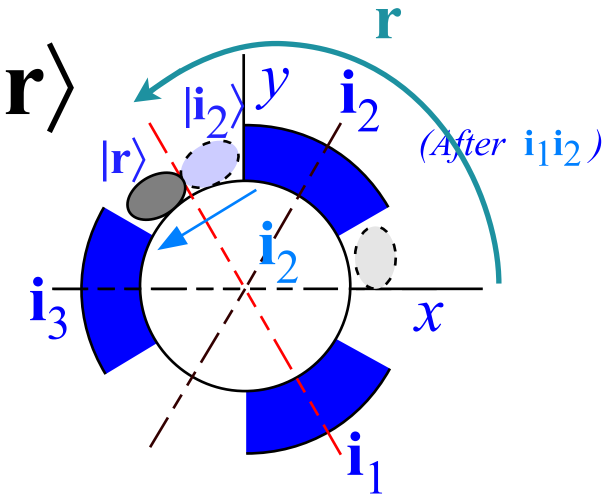
$\mathbf{i}_2|1\rangle=|\mathbf{i}_2\rangle$

wave packet moves with lab axes fixed



$\mathbf{i}_1\mathbf{i}_2|1\rangle=\mathbf{r}|1\rangle=|\mathbf{r}\rangle$

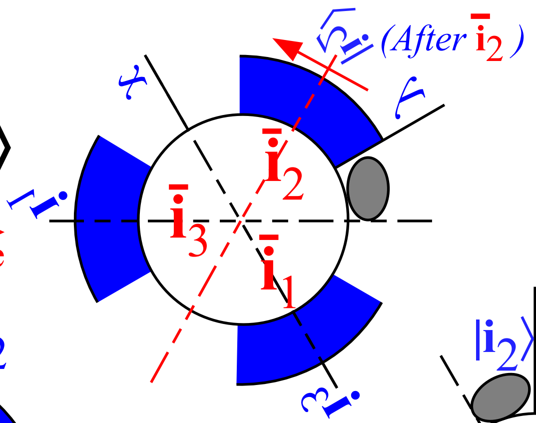
wave packet moves with lab axes fixed

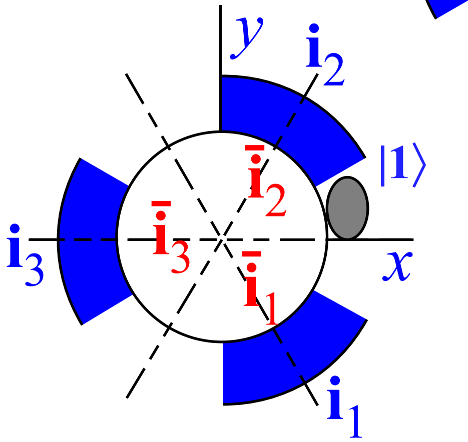


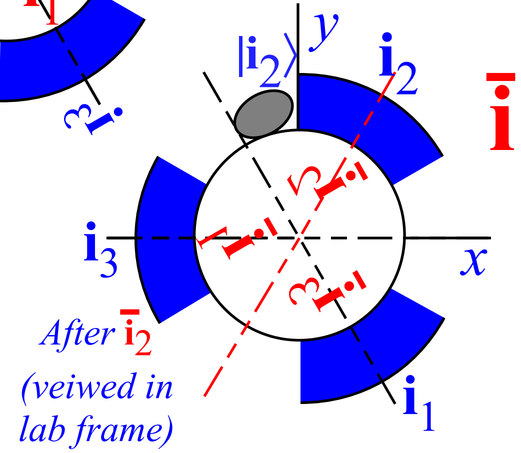
Body-fixed (Intrinsic-Local) operations appear to move their rotation axes (relative to lab)

$\bar{\mathbf{i}}_2|1\rangle=|\mathbf{i}_2\rangle$

wave packet fixed while lab axes move



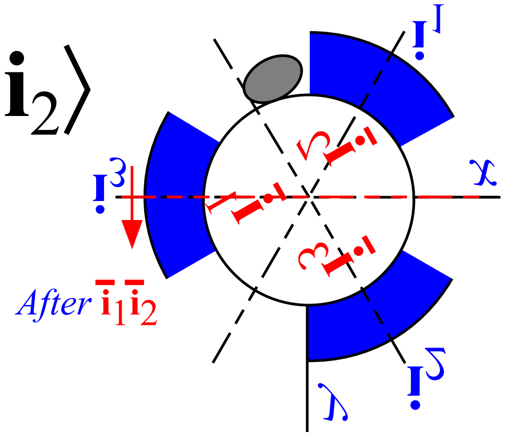




After $\bar{\mathbf{i}}_2$ (veiwed in lab frame)

$\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2|1\rangle=\bar{\mathbf{i}}_1|\mathbf{i}_2\rangle$

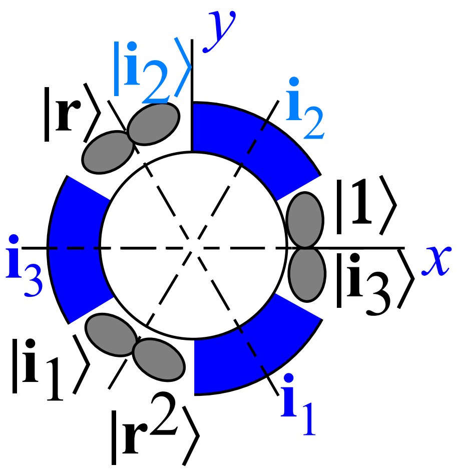
wave packet fixed while lab axes move

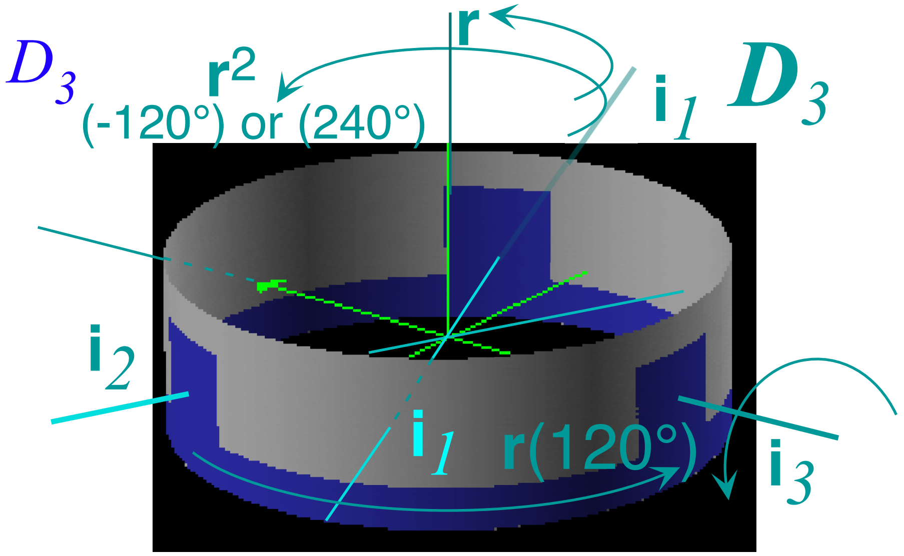


After $\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2$

Details of RELATIVITY-DUALITY for D_3

D_3-defined local-wave bases



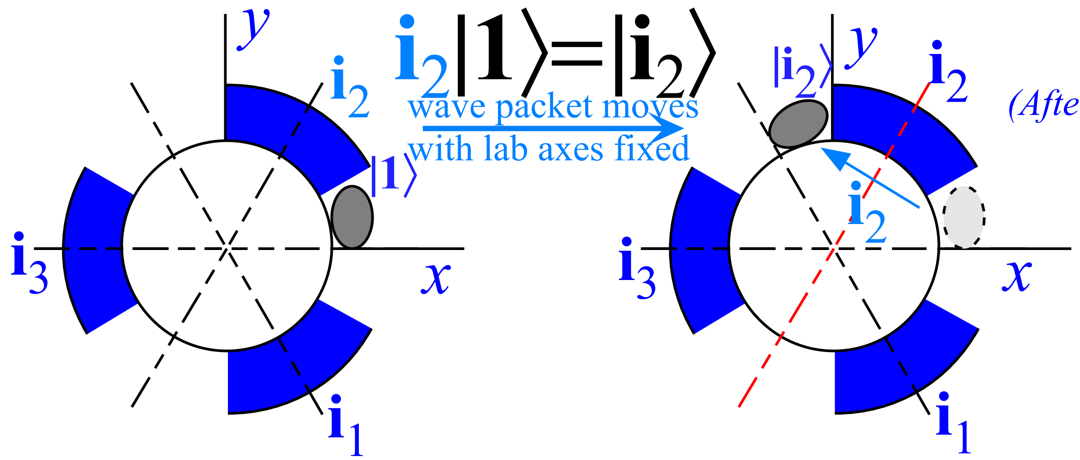


1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

Lab-fixed (Extrinsic-Global) operations&axes fixed

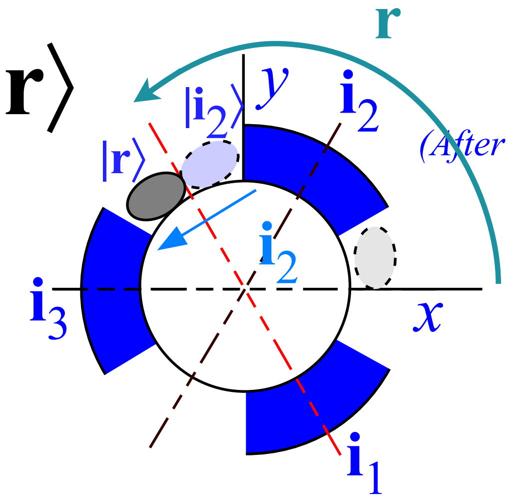
$\mathbf{i}_2|1\rangle=|\mathbf{i}_2\rangle$

wave packet moves with lab axes fixed



$\mathbf{i}_1\mathbf{i}_2|1\rangle=\mathbf{r}|1\rangle=|\mathbf{r}\rangle$

wave packet moves with lab axes fixed



Body-fixed (Intrinsic-Local) operations appear to move their rotation axes (relative to lab)

...but, THEY OBEY THE SAME GROUP TABLE.

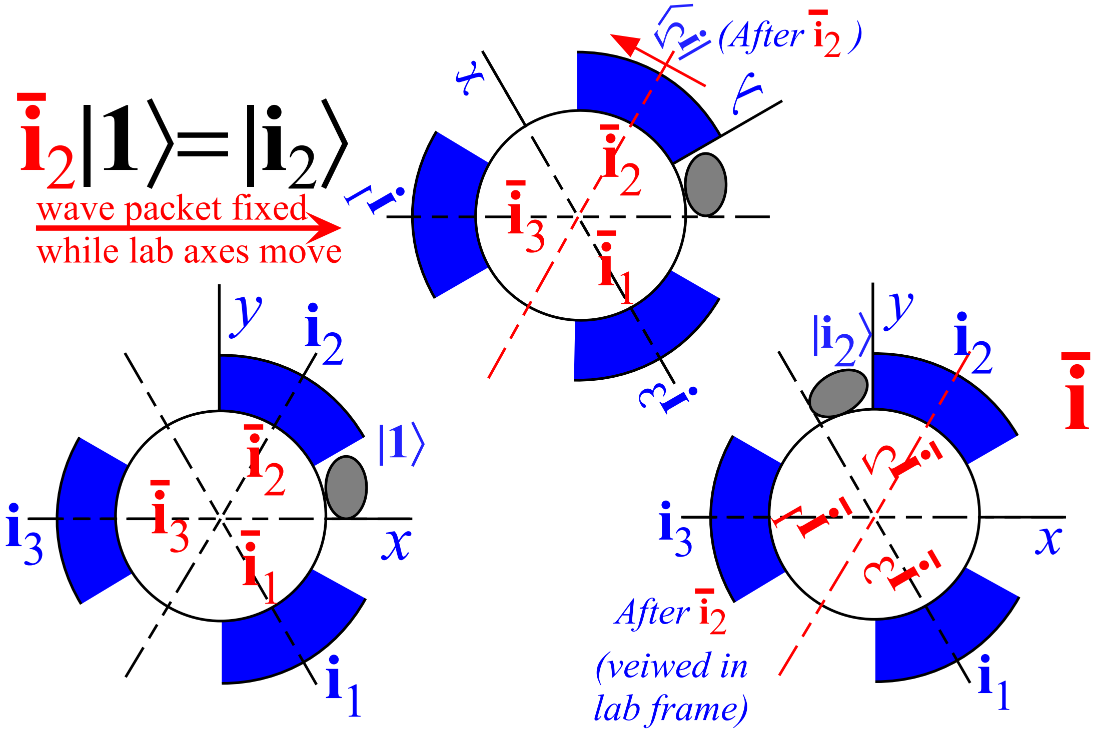
$\mathbf{i}_1\mathbf{i}_2=\mathbf{r}$

implies:

$\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2=\bar{\mathbf{r}}$

$\bar{\mathbf{i}}_2|1\rangle=|\mathbf{i}_2\rangle$

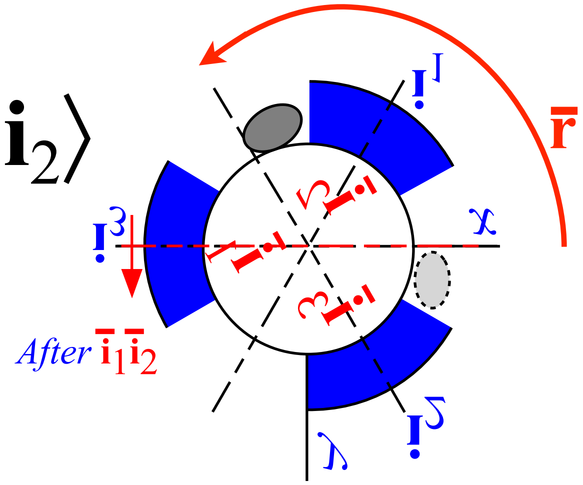
wave packet fixed while lab axes move



$\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2|1\rangle=\bar{\mathbf{i}}_1|\mathbf{i}_2\rangle$

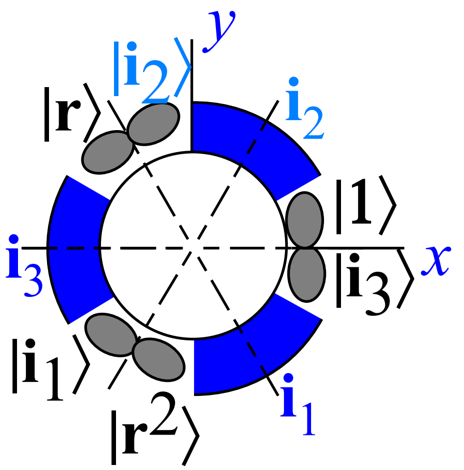
wave packet fixed while lab axes move

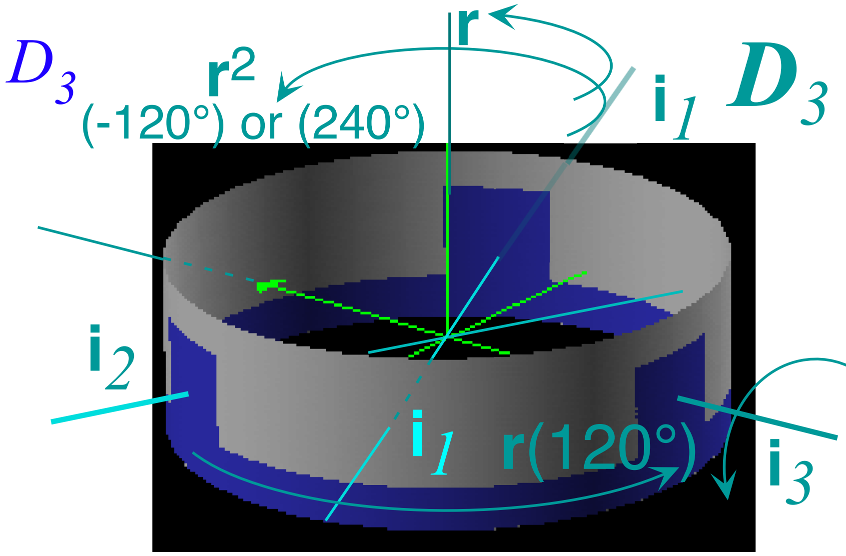
$\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2|1\rangle=\bar{\mathbf{i}}_1|\mathbf{i}_2\rangle=\bar{\mathbf{r}}|1\rangle$



Details of RELATIVITY-DUALITY for D_3

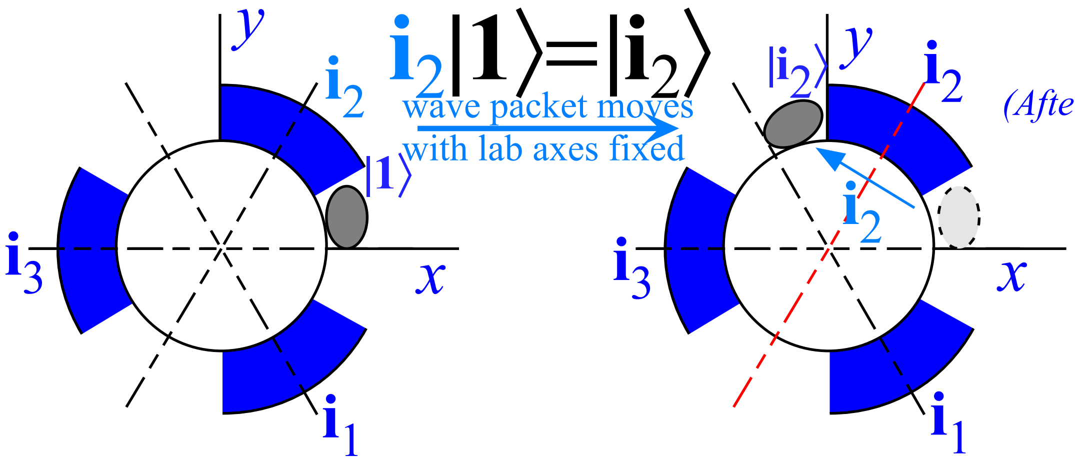
D_3-defined local-wave bases





1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

Lab-fixed (Extrinsic-Global) operations&axes fixed



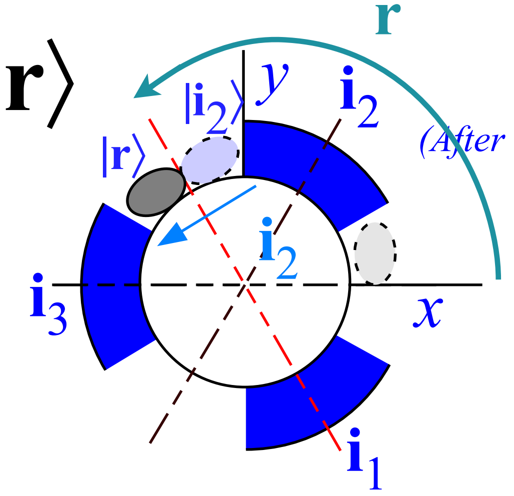
$\mathbf{i}_2|\mathbf{1}\rangle=|\mathbf{i}_2\rangle$

wave packet moves
with lab axes fixed

(After i_2)

$\mathbf{i}_1\mathbf{i}_2|\mathbf{1}\rangle=\mathbf{r}|\mathbf{1}\rangle=|\mathbf{r}\rangle$

wave packet moves
with lab axes fixed



(After i_1i_2)

Body-fixed (Intrinsic-Local) operations appear to move their rotation axes (relative to lab)

...but, THEY OBEY THE SAME GROUP TABLE.

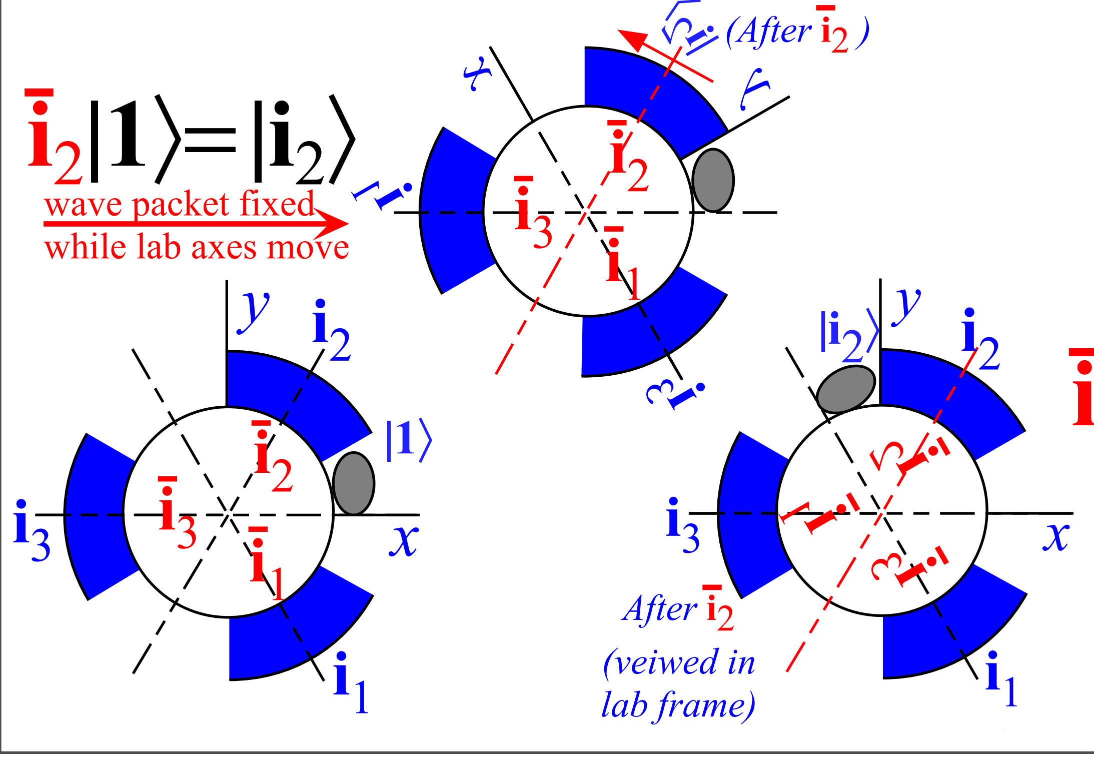
$\mathbf{i}_1\mathbf{i}_2 = \mathbf{r}$

implies:

$\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2 = \bar{\mathbf{r}}$

...and Mock-Mach principle $\bar{\mathbf{g}}|\mathbf{1}\rangle=\mathbf{g}^{-1}|\mathbf{1}\rangle$

$$\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2|\mathbf{1}\rangle=\bar{\mathbf{i}}_1|\mathbf{i}_2\rangle=\boxed{\bar{\mathbf{r}}|\mathbf{1}\rangle=\mathbf{r}^2|\mathbf{1}\rangle}$$



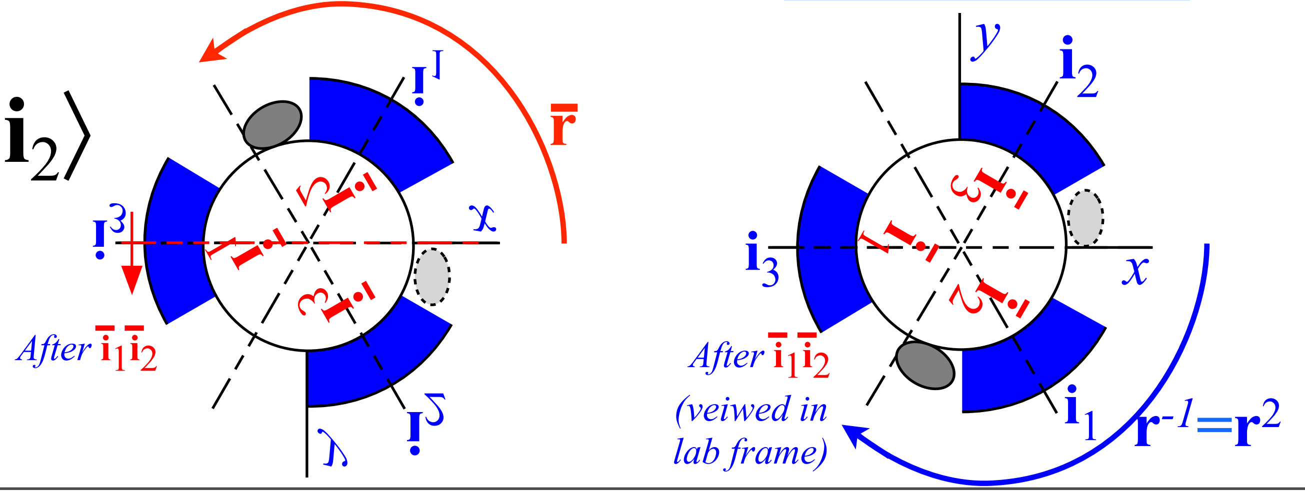
$\bar{\mathbf{i}}_2|\mathbf{1}\rangle=|\mathbf{i}_2\rangle$

wave packet fixed
while lab axes move

After $\bar{i}_2$
(veiwed in lab frame)

$\bar{\mathbf{i}}_1\bar{\mathbf{i}}_2|\mathbf{1}\rangle=\bar{\mathbf{i}}_1|\mathbf{i}_2\rangle$

wave packet fixed
while lab axes move



After $\bar{i}_1\bar{i}_2$

After $\bar{i}_1\bar{i}_2$
(veiwed in lab frame)

Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)

- *Weyl* **g**-*expansion in irep $D^{\mu}_{jk}(g)$ and projectors* $\mathbf{P}^{\mu}_{jk}$
  - $\mathbf{P}^{\mu}_{jk}$ *transforms right-and-left*
  - $\mathbf{P}^{\mu}_{jk}$ *-expansion in* **g**-*operators*
- *$D^{\mu}_{jk}(g)$ orthogonality relations*
- *Class projector character formulae*
  - $\mathbb{P}^{\mu}$ *in terms of* $\boldsymbol{\kappa}_{\mathbf{g}}$ *and* $\boldsymbol{\kappa}_{\mathbf{g}}$ *in terms of* $\mathbb{P}^{\mu}$

Details of Mock-Mach relativity-duality for D_3 groups and representations

- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local* $|\mathbf{g}\rangle$*-basis and Global vs Local* $|\mathbf{P}^{(\mu)}\rangle$*-basis*

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$*-basis*

- *Hamiltonian local-symmetry eigensolution*

Compare Global vs Local $|\mathbf{g}\rangle$*-basis vs. Global vs Local* $|\mathbf{P}^{(\mu)}\rangle$*-basis*

D_3 global group product table

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

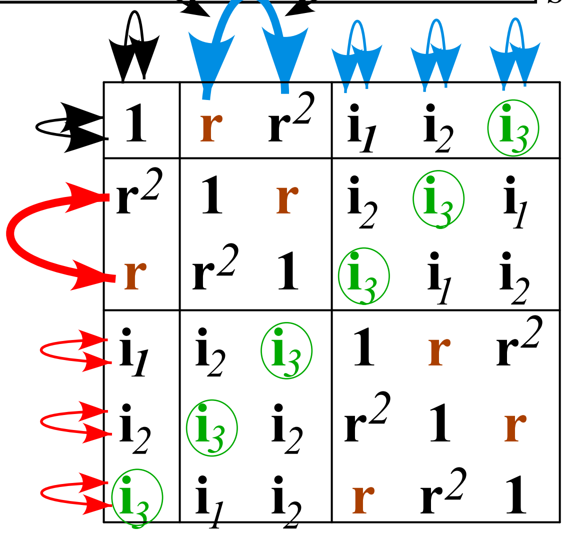
Change Global to Local by switching

...column-g with column-$g^\dagger$

....and row-g with row-$g^\dagger$

Just switch r with $r^\dagger = r^2$.

(all others are self-conjugate)



D_3 local group table

1	r	r^2	i_1	i_2	i_3
r^2	1	r	i_2	i_3	i_1
r	r^2	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r	r^2
i_2	i_3	i_2	r^2	1	r
i_3	i_1	i_2	r	r^2	1

Compare Global vs Local $|\mathbf{g}\rangle$-basis vs. Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis

D_3 global
group
product
table

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

D_3 global
projector
product
table

D_3	$P^{A_1}_{xx}$	$P^{A_2}_{yy}$	P^E_{xx}	P^E_{xy}	P^E_{yx}	P^E_{yy}
$P^{A_1}_{xx}$	$P^{A_1}_{xx}$	·	·	·	·	·
$P^{A_2}_{yy}$	·	$P^{A_2}_{yy}$	·	·	·	·
P^E_{xx}	·	·	P^E_{xx}	P^E_{xy}	·	·
P^E_{yx}	·	·	P^E_{yx}	P^E_{yy}	·	·
P^E_{xy}	·	·	·	·	P^E_{xx}	P^E_{xy}
P^E_{y}	·	·	·	·	P^E_{y}	P^E_{y}

$$\mathbf{P}^{(m)}_{ab}\mathbf{P}^{(n)}_{cd} = \delta^{mn}\delta_{bc}\,\mathbf{P}^{(m)}_{ad}$$

Change Global to Local by switching

...column-P with column-P†

....and row-P with row-P†

(Just switch P^E_{yx} with $P^{E\dagger}_{yx}=P^E_{xy}$.)

Just switch r with $r^\dagger=r^2$.

(all others are self-conjugate)

D_3 local
group
product
table

1	r	r^2	i_1	i_2	i_3
r^2	1	r	i_2	i_3	i_1
r	r^2	1	i_3	i_1	i_2
i_1	i_2	i_3	1	r	r^2
i_2	i_3	i_2	r^2	1	r
i_3	i_1	i_2	r	r^2	1

D_3 local
projector
product
table

	$P^{A_1}_{xx}$	$P^{A_2}_{yy}$	P^E_{xx}	P^E_{yx}	P^E_{xy}	P^E_{yy}
$P^{A_1}_{xx}$	$P^{A_1}_{xx}$	·	·	·	·	·
$P^{A_2}_{yy}$	·	$P^{A_2}_{yy}$	·	·	·	·
P^E_{xx}	·	·	P^E_{xx}	0	P^E_{xy}	0
P^E_{xy}	·	·	0	P^E_{xx}	0	P^E_{xy}
P^E_{yx}	·	·	P^E_{yx}	0	P^E_{yy}	0
P^E_{yy}	·	·	0	P^E_{yx}	0	P^E_{yy}

$$\overline{\mathbf{P}}^{(m)}_{ab}\overline{\mathbf{P}}^{(n)}_{cd} = \delta^{mn}\delta_{bc}\,\overline{\mathbf{P}}^{(m)}_{ad}$$

Compare Global vs Local $|\mathbf{g}\rangle$*-basis*

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

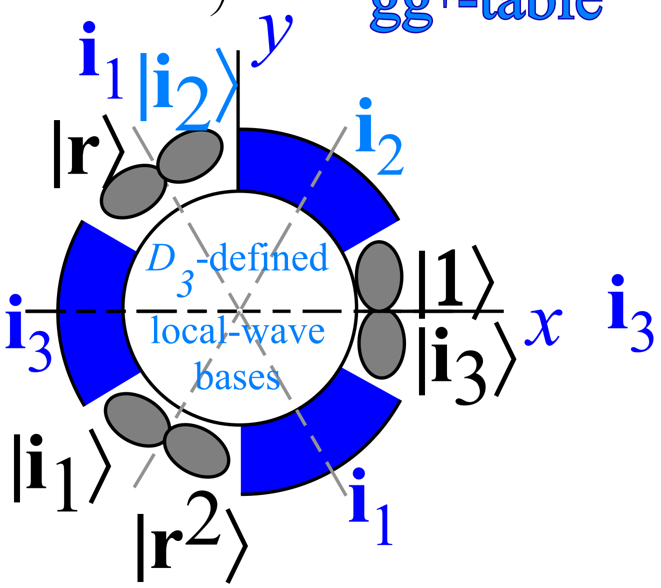
$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\textcircled{1}\\\cdot&\cdot&\cdot&\textcircled{1}&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\textcircled{1}&\cdot\\\cdot&\textcircled{1}&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\textcircled{1}&\cdot&\cdot&\cdot\\\textcircled{1}&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\textcircled{\mathbf{i}_3}$
r	**1**	$\mathbf{r}^2$	$\textcircled{\mathbf{i}_3}$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\textcircled{\mathbf{i}_3}$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\textcircled{\mathbf{i}_3}$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\textcircled{\mathbf{i}_3}$	$\mathbf{r}^2$	**1**	**r**
$\textcircled{\mathbf{i}_3}$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

D_3 global

$\mathbf{gg}^\dagger$-table



Compare Global vs Local $|\mathbf{g}\rangle$-basis

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \rightleftarrows \mathbf{g}^\dagger$ on top of group table

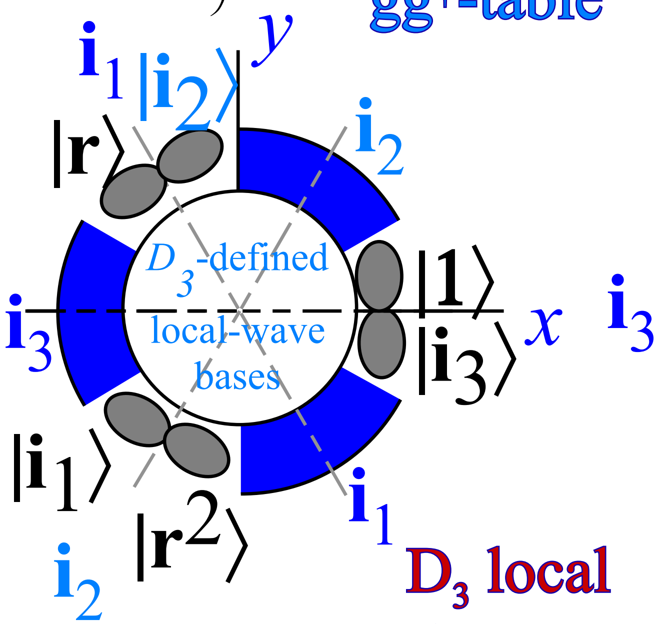
1	r²	r	i₁	i₂	i₃
r	1	r²	i₃	i₁	i₂
r²	r	1	i₂	i₃	i₁
i₁	i₃	i₂	1	r	r²
i₂	i₁	i₃	r²	1	r
i₃	i₂	i₁	r	r²	1

D_3 global

$gg^\dagger$-table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}, \quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix}, \quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix}, \quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}, \quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$



RESULT:

Any $R(\mathbf{T})$ *commute* (Even if $\mathbf{T}$ and $\mathbf{U}$ do not...)

with any $R(\bar{\mathbf{U}})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

D_3 local

$g^\dagger g$-table

To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$ switch $\mathbf{g} \rightleftarrows \mathbf{g}^\dagger$ on side of group table

1	r	r²	i₁	i₂	i₃
r²	1	r	i₂	i₃	i₁
r	r²	1	i₃	i₁	i₂
i₁	i₂	i₃	1	r	r²
i₂	i₃	i₂	r²	1	r
i₃	i₁	i₂	r	r²	1

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}, \quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix}, \quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

Compare Global $|\mathbf{P}^{(\mu)}\rangle$-basis vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis

Matrix "Placeholders" $\mathbf{P}^{(m)}_{ab}$ for GLOBAL $\mathbf{g}$ operators in D_3

$$\mathbf{g} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\mathbf{P}^{A_2} + D^{E}_{xx}(g)\mathbf{P}^{E}_{xx} + D^{E}_{xy}(g)\mathbf{P}^{E}_{xy} + D^{E}_{yx}(g)\mathbf{P}^{E}_{yx} + D^{E}_{yy}(g)\mathbf{P}^{E}_{yy}$$

$$\begin{pmatrix} D^{A_1}_{xx}(g) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}_{yy} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^{E}_{xx} & D_{xy} & \cdot & \cdot \\ \cdot & \cdot & D_{yx} & D_{yy} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & D^{E}_{xx} & D_{xy} \\ \cdot & \cdot & \cdot & \cdot & D_{yx} & D_{yy} \end{pmatrix} = D^{A_1}_{xx}\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{A_2}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{yx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + D^{E}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Compare Global $|\mathbf{P}^{(\mu)}\rangle$-basis vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis

Matrix "Placeholders" $\mathbf{P}^{(m)}_{ab}$ *for GLOBAL* $\mathbf{g}$ *operators in* D_3

$$\mathbf{g} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\mathbf{P}^{A_2} + D^{E}_{xx}(g)\mathbf{P}^{E}_{xx} + D^{E}_{xy}(g)\mathbf{P}^{E}_{xy} + D^{E}_{yx}(g)\mathbf{P}^{E}_{yx} + D^{E}_{yy}(g)\mathbf{P}^{E}_{yy}$$

$$\begin{pmatrix} D^{A_1}_{xx}(g) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}_{yy} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^{E}_{xx} & D_{xy} & \cdot & \cdot \\ \cdot & \cdot & D_{yx} & D_{yy} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & D^{E}_{xx} & D_{xy} \\ \cdot & \cdot & \cdot & \cdot & D_{yx} & D_{yy} \end{pmatrix} = D^{A_1}_{xx}\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{A_2}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{yx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + D^{E}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$\overline{\mathbf{P}}^{(m)}_{ab}$ *...for LOCAL* $\overline{\mathbf{g}}$ *operators in* $\overline{D}_3$

$$\overline{\mathbf{g}} = D^{A_1}_{xx}(g)\overline{\mathbf{P}}^{A_1} + D^{A_2}_{yy}(g)\overline{\mathbf{P}}^{A_2} + D^{E}_{xx}(g)\overline{\mathbf{P}}^{E}_{xx} + D^{E}_{xy}(g)\overline{\mathbf{P}}^{E}_{xy} + D^{E}_{yx}(g)\overline{\mathbf{P}}^{E}_{yx} + D^{E}_{yy}(g)\overline{\mathbf{P}}^{E}_{yy}$$

$$\begin{pmatrix} D^{A_1}_{xx}(g) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}_{yy} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^{E}_{xx} & \cdot & D^{E}_{xy} & \cdot \\ \cdot & \cdot & \cdot & D_{xx} & \cdot & D_{xy} \\ \cdot & \cdot & D^{E}_{yx} & \cdot & D^{E}_{yy} & \cdot \\ \cdot & \cdot & \cdot & D_{yx} & \cdot & D_{yy} \end{pmatrix} = D^{A_1}_{xx}\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{A_2}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{yx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + D^{E}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Compare Global $|\mathbf{P}^{(\mu)}\rangle$-basis vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis

Matrix "Placeholders" $\mathbf{P}^{(m)}_{ab}$ for GLOBAL $\mathbf{g}$ operators in D_3

$$\mathbf{g} = D^{A_1}_{xx}(g)\mathbf{P}^{A_1} + D^{A_2}_{yy}(g)\mathbf{P}^{A_2} + D^{E}_{xx}(g)\mathbf{P}^{E}_{xx} + D^{E}_{xy}(g)\mathbf{P}^{E}_{xy} + D^{E}_{yx}(g)\mathbf{P}^{E}_{yx} + D^{E}_{yy}(g)\mathbf{P}^{E}_{yy}$$

$$\begin{pmatrix} D^{A_1}_{xx}(g) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}_{yy} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^{E}_{xx} & D_{xy} & \cdot & \cdot \\ \cdot & \cdot & D_{yx} & D_{yy} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & D^{E}_{xx} & D_{xy} \\ \cdot & \cdot & \cdot & \cdot & D_{yx} & D_{yy} \end{pmatrix} = D^{A_1}_{xx}\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{A_2}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{yx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix} + D^{E}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$\bar{\mathbf{P}}^{(m)}_{ab}$...for LOCAL $\bar{\mathbf{g}}$ operators in $\bar{D}_3$

$$\bar{\mathbf{g}} = D^{A_1}_{xx}(g)\bar{\mathbf{P}}^{A_1} + D^{A_2}_{yy}(g)\bar{\mathbf{P}}^{A_2} + D^{E}_{xx}(g)\bar{\mathbf{P}}^{E}_{xx} + D^{E}_{xy}(g)\bar{\mathbf{P}}^{E}_{xy} + D^{E}_{yx}(g)\bar{\mathbf{P}}^{E}_{yx} + D^{E}_{yy}(g)\bar{\mathbf{P}}^{E}_{yy}$$

$$\begin{pmatrix} D^{A_1}_{xx}(g) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}_{yy} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^{E}_{xx} & \cdot & D^{E}_{xy} & \cdot \\ \cdot & \cdot & \cdot & D_{xx} & \cdot & D_{xy} \\ \cdot & \cdot & D^{E}_{yx} & \cdot & D^{E}_{yy} & \cdot \\ \cdot & \cdot & \cdot & D_{yx} & \cdot & D_{yy} \end{pmatrix} = D^{A_1}_{xx}\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{A_2}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{xy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} + D^{E}_{yx}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix} + D^{E}_{yy}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Note how any global g-matrix commutes with any local g-matrix

$$\left|\begin{array}{cc:cc} a & b & \cdot & \cdot \\ c & d & \cdot & \cdot \\ \hdashline \cdot & \cdot & a & b \\ \cdot & \cdot & c & d \end{array}\right| \cdot \left|\begin{array}{cc:cc} A & \cdot & B & \cdot \\ \cdot & A & \cdot & B \\ \hdashline C & & D & \\ & C & & D \end{array}\right| = \left|\begin{array}{cc:cc} A & \cdot & B & \cdot \\ \cdot & A & \cdot & B \\ \hdashline C & & D & \\ & C & & D \end{array}\right| \cdot \left|\begin{array}{cc:cc} a & b & \cdot & \cdot \\ c & d & \cdot & \cdot \\ \hdashline \cdot & \cdot & a & b \\ \cdot & \cdot & c & d \end{array}\right|$$

$$\left|\begin{array}{cc:cc} aA & bA & aB & bB \\ cA & dA & cB & dB \\ \hdashline aC & bC & aD & bD \\ cC & dC & cD & dD \end{array}\right| = \left|\begin{array}{cc:cc} Aa & Ab & Ba & Bb \\ Ac & Ad & Bc & Bd \\ \hdashline Ca & Cb & Da & Db \\ Cc & Cd & Dc & Dd \end{array}\right|$$

Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)

- *Weyl* **g***-expansion in irep $D^{\mu}_{jk}(g)$ and projectors $\mathbf{P}^{\mu}_{jk}$*
  - *$\mathbf{P}^{\mu}_{jk}$ transforms right-and-left*
  - *$\mathbf{P}^{\mu}_{jk}$ -expansion in* **g***-operators*
- *$D^{\mu}_{jk}(g)$ orthogonality relations*
- *Class projector character formulae*
  - *$\mathbb{P}^{\mu}$ in terms of $\kappa_{\mathbf{g}}$ and $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$*

Details of Mock-Mach relativity-duality for D_3 groups and representations

- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis*

→ *Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis* ←

- *Hamiltonian local-symmetry eigensolution*

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$*-basis are projected by* $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^\circ G} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket* $|\mathbf{1}\rangle$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{array}{c}\mu\\ mn\end{array}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm}$$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\overline{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{matrix}\mu \\ mn\end{matrix}\right\rangle = \mathbf{P}^{\mu}_{mn}\left|\mathbf{1}\right\rangle \frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^{\circ}G \cdot norm} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\left|\mathbf{g}\right\rangle$$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \mathbf{P}^{\mu}_{mn}\left|\mathbf{1}\right\rangle \frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^{\circ}G \cdot norm} \sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\left|\mathbf{g}\right\rangle$$ *subject to normalization:*

$$\left\langle\begin{smallmatrix}\mu'\\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \frac{\left\langle\mathbf{1}\middle|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}\middle|\mathbf{1}\right\rangle}{norm^2}$$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{array}{c}\mu\\ mn\end{array}\right\rangle = \mathbf{P}^{\mu}_{mn}\left|\mathbf{1}\right\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^{\circ}G\cdot norm}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\left|\mathbf{g}\right\rangle$$

subject to normalization:

$$\left\langle\begin{array}{c}\mu'\\ m'n'\end{array}\middle|\begin{array}{c}\mu\\ mn\end{array}\right\rangle = \frac{\left\langle\mathbf{1}\middle|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}\middle|\mathbf{1}\right\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\left\langle\mathbf{1}\middle|\mathbf{P}^{\mu}_{n'n}\middle|\mathbf{1}\right\rangle}{norm^2}$$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{^\circ G} \sum_{\mathbf{g}}^{^\circ G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{^\circ G\cdot norm}\sum_{\mathbf{g}}^{^\circ G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle\begin{smallmatrix}\mu'\\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{^\circ G}}$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

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$$\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle=\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm}=\frac{\ell^{(\mu)}}{{}^\circ G\cdot norm}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle\begin{smallmatrix}\mu'\\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle=\frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2}=\delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2}=\delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm=\sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle}=\sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$

Left-action of global $\mathbf{g}$ on irep-ket $\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle$

$$\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle=\sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right\rangle$$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{°G} \sum_{\mathbf{g}}^{°G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{°G\cdot norm}\sum_{\mathbf{g}}^{°G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle\begin{smallmatrix}\mu'\\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{°G}}$

Left-action of global $\mathbf{g}$ on irep-ket $\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle$

$$\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\middle|\mathbf{g}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = D^{\mu}_{m'm}(g)$$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|{}^{\mu}_{mn}\right\rangle = \mathbf{P}^{\mu}_{mn}\left|\mathbf{1}\right\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G\cdot norm}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\left|\mathbf{g}\right\rangle$$

subject to normalization:

$$\left\langle {}^{\mu'}_{m'n'}\middle|{}^{\mu}_{mn}\right\rangle = \frac{\left\langle\mathbf{1}\middle|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}\middle|\mathbf{1}\right\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\left\langle\mathbf{1}\middle|\mathbf{P}^{\mu}_{n'n}\middle|\mathbf{1}\right\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\left\langle\mathbf{1}\middle|\mathbf{P}^{\mu}_{nn}\middle|\mathbf{1}\right\rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$

Left-action of global $\mathbf{g}$ on irep-ket $\left|{}^{\mu}_{mn}\right\rangle$

$$\mathbf{g}\left|{}^{\mu}_{mn}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|{}^{\mu}_{m'n}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle{}^{\mu}_{m'n}\middle|\mathbf{g}\middle|{}^{\mu}_{mn}\right\rangle = D^{\mu}_{m'm}(g)$$

Left-action of local $\bar{\mathbf{g}}$ on irep-ket $\left|{}^{\mu}_{mn}\right\rangle$ is quite different

$$\bar{\mathbf{g}}\left|{}^{\mu}_{mn}\right\rangle = \bar{\mathbf{g}}\mathbf{P}^{\mu}_{mn}\left|\mathbf{1}\right\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

$$= \mathbf{P}^{\mu}_{mn}\bar{\mathbf{g}}\left|\mathbf{1}\right\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

Use
Mock-Mach
commutation
and

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^{\circ}G\cdot norm}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle\begin{smallmatrix}\mu'\\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^{\circ}G}}$

Left-action of global $\mathbf{g}$ on irep-ket $\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle$

$$\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\middle|\mathbf{g}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = D^{\mu}_{m'm}(g)$$

Left-action of local $\bar{\mathbf{g}}$ on irep-ket $\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle$ is quite different

$$\bar{\mathbf{g}}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \bar{\mathbf{g}}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}$$

$$= \mathbf{P}^{\mu}_{mn}\bar{\mathbf{g}}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}$$

$$= \mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}$$

Use Mock-Mach commutation and inverse

Hamiltonian and D_3 **global-$\mathbf{g}$** and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ acting on original ket $|\mathbf{1}\rangle$ to give:

$$\left|{\mu \atop mn}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G\cdot norm}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle{\mu' \atop m'n'}\middle|{\mu \atop mn}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$

Left-action of **global $\mathbf{g}$** on irep-ket $\left|{\mu \atop mn}\right\rangle$

$$\mathbf{g}\left|{\mu \atop mn}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|{\mu \atop m'n}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle{\mu \atop m'n}\middle|\mathbf{g}\middle|{\mu \atop mn}\right\rangle = D^{\mu}_{m'm}(g)$$

compute $\mathbf{g}^{-1}$ right action

$$\boxed{\mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1} = \sum_{m'=1}^{\ell^{\mu}}\sum_{n'=1}^{\ell^{\mu}}\mathbf{P}^{\mu}_{mn}\mathbf{P}^{\mu}_{m'n'}D^{\mu}_{m'n'}(g^{-1})}$$

Left-action of local $\bar{\mathbf{g}}$ on irep-ket $\left|{\mu \atop mn}\right\rangle$ is quite different

$$\bar{\mathbf{g}}\left|{\mu \atop mn}\right\rangle = \bar{\mathbf{g}}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

$$= \mathbf{P}^{\mu}_{mn}\bar{\mathbf{g}}|\mathbf{1}\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

$$= \mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1}|\mathbf{1}\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

Use Mock-Mach commutation

and inverse

Hamiltonian and D₃ global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

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$$\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{°G\cdot norm}\sum_{\mathbf{g}}^{°G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle\begin{matrix}\mu'\\ m'n'\end{matrix}\right|\left.\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{°G}}$

Left-action of global $\mathbf{g}$ on irep-ket $\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle$

$$\mathbf{g}\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|\begin{matrix}\mu\\ m'n\end{matrix}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle\begin{matrix}\mu\\ m'n\end{matrix}\right|\mathbf{g}\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = D^{\mu}_{m'm}(g)$$

compute $\mathbf{g}^{-1}$ right action

$$\boxed{\begin{aligned}\mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1} &= \sum_{m'=1}^{\ell^{\mu}}\sum_{n'=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mn}\mathbf{P}^{\mu}_{m'n'}D^{\mu}_{m'n'}(g^{-1})\\ &= \sum_{n'=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mn'}\,D^{\mu}_{nn'}(g^{-1})\end{aligned}}$$

Left-action of local $\bar{\mathbf{g}}$ on irep-ket $\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle$ *is quite different*

$$\begin{aligned}\bar{\mathbf{g}}\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle &= \bar{\mathbf{g}}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\sqrt{\frac{°G}{\ell^{(\mu)}}}\\ &= \mathbf{P}^{\mu}_{mn}\bar{\mathbf{g}}|\mathbf{1}\rangle\sqrt{\frac{°G}{\ell^{(\mu)}}}\\ &= \mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1}|\mathbf{1}\rangle\sqrt{\frac{°G}{\ell^{(\mu)}}}\end{aligned}$$

Use Mock-Mach commutation and inverse

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^\circ G}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \mathbf{P}^{\mu}_{mn}\left|\mathbf{1}\right\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G\cdot norm}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)\left|\mathbf{g}\right\rangle$$

subject to normalization:

$$\left\langle\begin{smallmatrix}\mu'\\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \frac{\left\langle\mathbf{1}\middle|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}\middle|\mathbf{1}\right\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\left\langle\mathbf{1}\middle|\mathbf{P}^{\mu}_{n'n}\middle|\mathbf{1}\right\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

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Left-action of global $\mathbf{g}$ on irep-ket $\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle$

$$\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \sum_{m'}^{\ell^\mu} D^{\mu}_{m'm}(g)\left|\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\middle|\mathbf{g}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = D^{\mu}_{m'm}(g)$$

Left-action of local $\bar{\mathbf{g}}$ on irep-ket $\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle$ is quite different

$$\bar{\mathbf{g}}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \bar{\mathbf{g}}\mathbf{P}^{\mu}_{mn}\left|\mathbf{1}\right\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

$$= \mathbf{P}^{\mu}_{mn}\bar{\mathbf{g}}\left|\mathbf{1}\right\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

Use Mock-Mach commutation

$$= \mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1}\left|\mathbf{1}\right\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

and inverse

$$= \sum_{n'=1}^{\ell^\mu} D^{\mu}_{nn'}(g^{-1})\mathbf{P}^{\mu}_{mn'}\left|\mathbf{1}\right\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

compute $\mathbf{g}^{-1}$ right action

$$\boxed{\begin{aligned}\mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1} &= \sum_{m'=1}^{\ell^\mu}\sum_{n'=1}^{\ell^\mu}\mathbf{P}^{\mu}_{mn}\mathbf{P}^{\mu}_{m'n'}D^{\mu}_{m'n'}(g^{-1})\\ &= \sum_{n'=1}^{\ell^\mu}\mathbf{P}^{\mu}_{mn'}\,D^{\mu}_{nn'}(g^{-1})\end{aligned}}$$

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^{\circ}G\cdot norm}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle\begin{smallmatrix}\mu'\\ m'n'\end{smallmatrix}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^{\circ}G}}$

Left-action of global $\mathbf{g}$ on irep-ket $\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle$

$$\mathbf{g}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle\begin{smallmatrix}\mu\\ m'n\end{smallmatrix}\middle|\mathbf{g}\middle|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle = D^{\mu}_{m'm}(g)$$

compute $\mathbf{g}^{-1}$ right action

$$\boxed{\begin{aligned}\mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1} &= \sum_{m'=1}^{\ell^{\mu}}\sum_{n'=1}^{\ell^{\mu}}\mathbf{P}^{\mu}_{mn}\mathbf{P}^{\mu}_{m'n'}D^{\mu}_{m'n'}(g^{-1})\\ &= \sum_{n'=1}^{\ell^{\mu}}\mathbf{P}^{\mu}_{mn'}\;D^{\mu}_{nn'}(g^{-1})\end{aligned}}$$

Left-action of local $\bar{\mathbf{g}}$ on irep-ket $\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle$ is quite different

$$\begin{aligned}\bar{\mathbf{g}}\left|\begin{smallmatrix}\mu\\ mn\end{smallmatrix}\right\rangle &= \bar{\mathbf{g}}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}\\ &= \mathbf{P}^{\mu}_{mn}\bar{\mathbf{g}}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}\\ &= \mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}\\ &= \sum_{n'=1}^{\ell^{\mu}} D^{\mu}_{nn'}(g^{-1})\mathbf{P}^{\mu}_{mn'}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}\\ &= \sum_{n'=1}^{\ell^{\mu}} D^{\mu}_{nn'}(g^{-1})\left|\begin{smallmatrix}\mu\\ mn'\end{smallmatrix}\right\rangle\end{aligned}$$

Use Mock-Mach commutation

and

inverse

Hamiltonian and D_3 global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

For unitary $D^{(\mu)}$: (p.33)

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$$\left|{\mu \atop mn}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G\cdot norm}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle{\mu' \atop m'n'}\middle|{\mu \atop mn}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$

Left-action of global $\mathbf{g}$ on irep-ket $\left|{\mu \atop mn}\right\rangle$

$$\mathbf{g}\left|{\mu \atop mn}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|{\mu \atop m'n}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle{\mu \atop m'n}\middle|\mathbf{g}\middle|{\mu \atop mn}\right\rangle = D^{\mu}_{m'm}(g)$$

Left-action of local $\bar{\mathbf{g}}$ on irep-ket $\left|{\mu \atop mn}\right\rangle$ *is quite different*

$$\bar{\mathbf{g}}\left|{\mu \atop mn}\right\rangle = \bar{\mathbf{g}}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

$$= \mathbf{P}^{\mu}_{mn}\bar{\mathbf{g}}|\mathbf{1}\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

Use Mock-Mach commutation and inverse

$$= \mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1}|\mathbf{1}\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

$$= \sum_{n'=1}^{\ell^{\mu}} D^{\mu}_{nn'}(g^{-1})\mathbf{P}^{\mu}_{mn'}|\mathbf{1}\rangle\sqrt{\frac{{}^\circ G}{\ell^{(\mu)}}}$$

$$= \sum_{n'=1}^{\ell^{\mu}} D^{\mu}_{nn'}(g^{-1})\left|{\mu \atop mn'}\right\rangle$$

compute $\mathbf{g}^{-1}$ right action

$$\boxed{\begin{aligned}\mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1} &= \sum_{m'=1}^{\ell^{\mu}}\sum_{n'=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mn}\mathbf{P}^{\mu}_{m'n'}D^{\mu}_{m'n'}(g^{-1})\\ &= \sum_{n'=1}^{\ell^{\mu}} \mathbf{P}^{\mu}_{mn'}\,D^{\mu}_{nn'}(g^{-1})\end{aligned}}$$

***Local** $\bar{\mathbf{g}}$-matrix component*

$$\boxed{\left\langle{\mu \atop mn'}\middle|\bar{\mathbf{g}}\middle|{\mu \atop mn}\right\rangle = D^{\mu}_{nn'}(g^{-1}) = D^{\mu^*}_{n'n}(g)}$$

Hamiltonian and D₃ global-$\mathbf{g}$ and local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

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$|\mathbf{P}^{(\mu)}\rangle$-basis are projected by $\mathbf{P}^{\mu}_{mn} = \frac{\ell^{(\mu)}}{{}^{\circ}G}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)\,\mathbf{g} = \mathbf{P}^{\mu\dagger}_{nm}$ *acting on original ket $|\mathbf{1}\rangle$ to give:*

$$\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^{\circ}G\cdot norm}\sum_{\mathbf{g}}^{{}^{\circ}G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization:

$$\left\langle\begin{matrix}\mu'\\ m'n'\end{matrix}\middle|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \frac{\langle\mathbf{1}|\mathbf{P}^{\mu'}_{n'm'}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\frac{\langle\mathbf{1}|\mathbf{P}^{\mu}_{n'n}|\mathbf{1}\rangle}{norm^2} = \delta^{\mu'\mu}\delta_{m'm}\delta_{n'n}$$

where: $norm = \sqrt{\langle\mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^{\circ}G}}$

Left-action of global $\mathbf{g}$ on irep-ket $\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle$

$$\mathbf{g}\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \sum_{m'}^{\ell^{\mu}} D^{\mu}_{m'm}(g)\left|\begin{matrix}\mu\\ m'n\end{matrix}\right\rangle$$

Matrix is same as given on p.23-28

$$\left\langle\begin{matrix}\mu\\ m'n\end{matrix}\middle|\mathbf{g}\middle|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = D^{\mu}_{m'm}(g)$$

Left-action of local $\bar{\mathbf{g}}$ on irep-ket $\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle$ is quite different

$$\bar{\mathbf{g}}\left|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = \bar{\mathbf{g}}\mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}$$

$$= \mathbf{P}^{\mu}_{mn}\bar{\mathbf{g}}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}$$

Use Mock-Mach commutation and inverse

$$= \mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}$$

compute $\mathbf{g}^{-1}$ right action

$$\boxed{\begin{aligned}\mathbf{P}^{\mu}_{mn}\mathbf{g}^{-1} &= \sum_{m'=1}^{\ell^{\mu}}\sum_{n'=1}^{\ell^{\mu}}\mathbf{P}^{\mu}_{mn}\mathbf{P}^{\mu}_{m'n'}D^{\mu}_{m'n'}(g^{-1})\\ &= \sum_{n'=1}^{\ell^{\mu}}\mathbf{P}^{\mu}_{mn'}\,D^{\mu}_{nn'}(g^{-1})\end{aligned}}$$

$$= \sum_{n'=1}^{\ell^{\mu}} D^{\mu}_{nn'}(g^{-1})\mathbf{P}^{\mu}_{mn'}|\mathbf{1}\rangle\sqrt{\frac{{}^{\circ}G}{\ell^{(\mu)}}}$$

$$= \sum_{n'=1}^{\ell^{\mu}} D^{\mu}_{nn'}(g^{-1})\left|\begin{matrix}\mu\\ mn'\end{matrix}\right\rangle$$

***Global** $\mathbf{g}$-matrix component*

$$\boxed{\left\langle\begin{matrix}\mu\\ m'n\end{matrix}\middle|\mathbf{g}\middle|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = D^{\mu}_{m'm}(g)}$$

***Local** $\bar{\mathbf{g}}$-matrix component*

$$\boxed{\left\langle\begin{matrix}\mu\\ mn'\end{matrix}\middle|\bar{\mathbf{g}}\middle|\begin{matrix}\mu\\ mn\end{matrix}\right\rangle = D^{\mu}_{nn'}(g^{-1}) = D^{\mu*}_{n'n}(g)}$$

D_3 global-$\mathbf{g}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

D_3 local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$$R^P(\mathbf{g}) = TR^G(\mathbf{g})T^\dagger =$$

$$\begin{array}{c|c|cc|cc}
\left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & D^{E_1}_{xy} & \cdot & \cdot \\
\cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & D^{E_1}_{yy} & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & D^{E_1}_{xy} \\
\cdot & \cdot & \cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & D^{E_1}_{yy}
\end{array}$$

$|\mathbf{P}^{(\mu)}\rangle$-base ordering to concentrate global-$\mathbf{g}$ D-matrices

Global $\mathbf{g}$-matrix component

$$\boxed{\left\langle {}^{\mu}_{m'n} \middle| \mathbf{g} \middle| {}^{\mu}_{mn} \right\rangle = D^{\mu}_{m'm}(g)}$$

Local $\bar{\mathbf{g}}$-matrix component

$$\boxed{\left\langle {}^{\mu}_{mn'} \middle| \bar{\mathbf{g}} \middle| {}^{\mu}_{mn} \right\rangle = D^{\mu}_{nn'}(g^{-1}) = D^{\mu *}_{n'n}(g)}$$

D_3 global-$\mathbf{g}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$$R^P(\mathbf{g}) = TR^G(\mathbf{g})T^\dagger =$$

$$\begin{array}{cccccc}
|\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \\
\left(\begin{array}{c|c|cc|cc}
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & D^{E_1}_{xy} & \cdot & \cdot \\
\cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & D^{E_1}_{yy} & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & D^{E_1}_{xy} \\
\cdot & \cdot & \cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & D^{E_1}_{yy}
\end{array}\right)
\end{array}$$

$|\mathbf{P}^{(\mu)}\rangle$-base ordering to concentrate global-$\mathbf{g}$ D-matrices ←

D_3 local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$$R^P(\bar{\mathbf{g}}) = TR^G(\bar{\mathbf{g}})T^\dagger =$$

$$\begin{array}{cccccc}
|\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \\
\left(\begin{array}{c|c|cc|cc}
D^{A_1*}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2*}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1*}_{xx}(\mathbf{g}) & \cdot & D^{E_1*}_{xy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1*}_{xx}(\mathbf{g}) & \cdot & D^{E_1*}_{xy}(\mathbf{g}) \\ \hline
\cdot & \cdot & D^{E_1*}_{yx}(\mathbf{g}) & \cdot & D^{E_1*}_{yy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1*}_{yx}(\mathbf{g}) & \cdot & D^{E_1*}_{yy}(\mathbf{g})
\end{array}\right)
\end{array}$$

↑

here

Local $\bar{\mathbf{g}}$-matrix

is not concentrated

Global **g**-*matrix component*

$$\boxed{\left\langle {\mu \atop m'n} \middle| \mathbf{g} \middle| {\mu \atop mn} \right\rangle = D^{\mu}_{m'm}(g)}$$

Local $\bar{\mathbf{g}}$-*matrix component*

$$\boxed{\left\langle {\mu \atop mn'} \middle| \bar{\mathbf{g}} \middle| {\mu \atop mn} \right\rangle = D^{\mu}_{nn'}(g^{-1}) = D^{\mu *}_{n'n}(g)}$$

D_3 global-$\mathbf{g}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$$R^P(\mathbf{g}) = T R^G(\mathbf{g}) T^\dagger =$$

$$\begin{array}{cc|c|cc|cc}
\left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\
\end{array}$$

$$\left(\begin{array}{c|c|cc|cc}
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & D^{E_1}_{xy} & \cdot & \cdot \\
\cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & D^{E_1}_{yy} & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & D^{E_1}_{xy} \\
\cdot & \cdot & \cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & D^{E_1}_{yy}
\end{array}\right)$$

$|\mathbf{P}^{(\mu)}\rangle$-base ordering to concentrate global-$\mathbf{g}$ D-matrices

D_3 local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$$R^P(\bar{\mathbf{g}}) = T R^G(\bar{\mathbf{g}}) T^\dagger =$$

$$\begin{array}{cccccc}
\left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\
\end{array}$$

$$\left(\begin{array}{c|c|cc|cc}
D^{A_1{}^*}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2{}^*}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1{}^*}_{xx}(\mathbf{g}) & \cdot & D^{E_1{}^*}_{xy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1{}^*}_{xx}(\mathbf{g}) & \cdot & D^{E_1{}^*}_{xy}(\mathbf{g}) \\ \hline
\cdot & \cdot & D^{E_1{}^*}_{yx}(\mathbf{g}) & \cdot & D^{E_1{}^*}_{yy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1{}^*}_{yx}(\mathbf{g}) & \cdot & D^{E_1{}^*}_{yy}(\mathbf{g})
\end{array}\right)$$

here

Local $\bar{\mathbf{g}}$-matrix

is not concentrated

$$\bar{R}^P(\mathbf{g}) = \bar{T} R^G(\mathbf{g}) \bar{T}^\dagger =$$

$$\begin{array}{cccccc}
\left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\
\end{array}$$

$$\left(\begin{array}{c|c|cc|cc}
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & \cdot & D^{E_1}_{xy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1}_{xx} & \cdot & D^{E_1}_{xy} \\ \hline
\cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & \cdot & D^{E_1}_{yy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1}_{yx} & \cdot & D^{E_1}_{yy}
\end{array}\right)$$

here

global $\mathbf{g}$-matrix

is not concentrated

Global $\mathbf{g}$-*matrix component*

$$\boxed{\left\langle \begin{matrix}\mu\\ m'n\end{matrix} \right| \mathbf{g} \left| \begin{matrix}\mu\\ mn\end{matrix} \right\rangle = D^{\mu}_{m'm}(g)}$$

Local $\bar{\mathbf{g}}$-*matrix component*

$$\boxed{\left\langle \begin{matrix}\mu\\ mn'\end{matrix} \right| \bar{\mathbf{g}} \left| \begin{matrix}\mu\\ mn\end{matrix} \right\rangle = D^{\mu}_{nn'}(g^{-1}) = D^{\mu *}_{n'n}(g)}$$

D_3 global-$\mathbf{g}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

D_3 local-$\bar{\mathbf{g}}$ group matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$$R^P(\mathbf{g}) = TR^G(\mathbf{g})T^\dagger =$$

$$\begin{array}{c|c|cc|cc}
\left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & D^{E_1}_{xy} & \cdot & \cdot \\
\cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & D^{E_1}_{yy} & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & D^{E_1}_{xy} \\
\cdot & \cdot & \cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & D^{E_1}_{yy}
\end{array}$$

$|\mathbf{P}^{(\mu)}\rangle$-base ordering to concentrate global-$\mathbf{g}$ D-matrices ←

$$R^P(\bar{\mathbf{g}}) = TR^G(\bar{\mathbf{g}})T^\dagger =$$

$$\begin{array}{c|c|cc|cc}
\left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\
D^{A_1 *}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2 *}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1 *}_{xx}(\mathbf{g}) & \cdot & D^{E_1 *}_{xy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1 *}_{xx}(\mathbf{g}) & \cdot & D^{E_1 *}_{xy}(\mathbf{g}) \\ \hline
\cdot & \cdot & D^{E_1 *}_{yx}(\mathbf{g}) & \cdot & D^{E_1 *}_{yy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1 *}_{yx}(\mathbf{g}) & \cdot & D^{E_1 *}_{yy}(\mathbf{g})
\end{array}$$

$$\bar{R}^P(\mathbf{g}) = \bar{T}R^G(\mathbf{g})\bar{T}^\dagger =$$

$$\begin{array}{c|c|cc|cc}
\left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1}_{xx}(\mathbf{g}) & \cdot & D^{E_1}_{xy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1}_{xx} & \cdot & D^{E_1}_{xy} \\ \hline
\cdot & \cdot & D^{E_1}_{yx}(\mathbf{g}) & \cdot & D^{E_1}_{yy}(\mathbf{g}) & \cdot \\
\cdot & \cdot & \cdot & D^{E_1}_{yx} & \cdot & D^{E_1}_{yy}
\end{array}$$

*$|\mathbf{P}^{(\mu)}\rangle$-base ordering to concentrate local-$\bar{\mathbf{g}}$ D-matrices and **H**-matrices* →

$$\bar{R}^P(\bar{\mathbf{g}}) = \bar{T}R^G(\bar{\mathbf{g}})\bar{T}^\dagger =$$

$$\begin{array}{c|c|cc|cc}
\left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\
D^{A_1 *}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & D^{A_2 *}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \cdot & D^{E_1 *}_{xx}(\mathbf{g}) & D^{E_1 *}_{xy}(\mathbf{g}) & \cdot & \cdot \\
\cdot & \cdot & D^{E_1 *}_{yx}(\mathbf{g}) & D^{E_1 *}_{yy}(\mathbf{g}) & \cdot & \cdot \\ \hline
\cdot & \cdot & \cdot & \cdot & D^{E_1 *}_{xx}(\mathbf{g}) & D^{E_1 *}_{xy}(\mathbf{g}) \\
\cdot & \cdot & \cdot & \cdot & D^{E_1 *}_{yx}(\mathbf{g}) & D^{E_1 *}_{yy}(\mathbf{g})
\end{array}$$

***Global** $\mathbf{g}$-matrix component*

$$\boxed{\left\langle {}^{\mu}_{m'n} \middle| \mathbf{g} \middle| {}^{\mu}_{mn} \right\rangle = D^{\mu}_{m'm}(g)}$$

***Local** $\bar{\mathbf{g}}$-matrix component*

$$\boxed{\left\langle {}^{\mu}_{mn'} \middle| \bar{\mathbf{g}} \middle| {}^{\mu}_{mn} \right\rangle = D^{\mu}_{nn'}(g^{-1}) = D^{\mu *}_{n'n}(g)}$$

Review: Spectral resolution of ***D_3 Center*** *(Class algebra) and its subgroup splitting*

General formulae for spectral decomposition (D_3 examples)

- *Weyl* **g***-expansion in irep $D^{\mu}_{jk}(g)$ and projectors $\mathbf{P}^{\mu}_{jk}$*
  - *$\mathbf{P}^{\mu}_{jk}$ transforms right-and-left*
  - *$\mathbf{P}^{\mu}_{jk}$ -expansion in* **g***-operators*
- *$D^{\mu}_{jk}(g)$ orthogonality relations*
- *Class projector character formulae*
  - *$\mathbb{P}^{\mu}$ in terms of $\kappa_{\mathbf{g}}$ and $\kappa_{\mathbf{g}}$ in terms of $\mathbb{P}^{\mu}$*

Details of Mock-Mach relativity-duality for D_3 groups and representations

- *Lab-fixed(Extrinsic-Global) vs. Body-fixed (Intrinsic-Local)*
- *Compare Global vs Local $|\mathbf{g}\rangle$-basis and Global vs Local $|\mathbf{P}^{(\mu)}\rangle$-basis*

Hamiltonian and D_3 group matrices in global and local $|\mathbf{P}^{(\mu)}\rangle$-basis

- ➡ *Hamiltonian local-symmetry eigensolution* ⬅

D_3 Hamiltonian *local*- **H** *matrices in* $|\mathbf{P}^{(\mu)}\rangle$*-basis*

H *matrix in*
$|\mathbf{g}\rangle$*-basis:*

$$\left(\mathbf{H}\right)_G = \sum_{g=1}^{{}^{o}G} r_g \bar{\mathbf{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H *matrix in*
$|\mathbf{P}^{(\mu)}\rangle$*-basis:*

D_3 Hamiltonian local- $\mathbf{H}$ matrices in $\left|\mathbf{P}^{(\mu)}\right\rangle$-basis

H *matrix in* $|\mathbf{g}\rangle$*-basis:*

$$(\mathbf{H})_G = \sum_{g=1}^{o^G} r_g \overline{\mathbf{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H *matrix in* $\left|\mathbf{P}^{(\mu)}\right\rangle$*-basis:*

$$(\mathbf{H})_P = \overline{T}(\mathbf{H})_G \overline{T}^\dagger = \begin{array}{c|c|cc|cc} \left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\ H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \left\langle \mathbf{P}^{\mu}_{ma} \middle| \mathbf{H} \middle| \mathbf{P}^{\mu}_{nb} \right\rangle$$

D_3 Hamiltonian local- **H** matrices in $\left|\mathbf{P}^{(\mu)}\right\rangle$-basis

H matrix in $|\mathbf{g}\rangle$-basis:

$$(\mathbf{H})_G = \sum_{g=1}^{{}^oG} r_g \mathbf{g} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H matrix in $\left|\mathbf{P}^{(\mu)}\right\rangle$-basis:

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c|c|cc|cc} \left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\ H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \left\langle \mathbf{P}^{\mu}_{ma} \middle| \mathbf{H} \middle| \mathbf{P}^{\mu}_{nb} \right\rangle$$

Let: $\left|{\mu \atop mn}\right\rangle \equiv \left|\mathbf{P}^{\mu}_{mn}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle \frac{1}{norm}$

$$\left|{\mu \atop mn}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle \frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G \cdot norm} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization (from p. 116-122):

$$norm = \sqrt{\left\langle \mathbf{1} \middle| \mathbf{P}^{\mu}_{nn} \middle| \mathbf{1} \right\rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$$

(which will cancel out)

So, fuggettabout it!

D_3 Hamiltonian local- **H** matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H matrix in $|\mathbf{g}\rangle$-basis:

$$(\mathbf{H})_G = \sum_{g=1}^{{}^oG} r_g \mathbf{g} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H matrix in $|\mathbf{P}^{(\mu)}\rangle$-basis:

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c|c|cc|cc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \\ \hline H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \langle \mathbf{P}^{\mu}_{ma} | \mathbf{H} | \mathbf{P}^{\mu}_{nb} \rangle = \frac{\langle \mathbf{1} | \mathbf{P}^{\mu}_{am} \mathbf{H} \mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}{(norm)^2}$$

Projector conjugation p.31

$$\left(|m\rangle\langle n|\right)^\dagger = |n\rangle\langle m|$$

$$\left(\mathbf{P}^{\mu}_{mn}\right)^\dagger = \mathbf{P}^{\mu}_{nm}$$

$$\left|{}^{\mu}_{mn}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle \frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G \cdot norm} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization (from p. 116-122):

$$norm = \sqrt{\langle \mathbf{1} | \mathbf{P}^{\mu}_{nn} | \mathbf{1} \rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$$

(which will cancel out)

So, fuggettabout it!

D_3 Hamiltonian *local*- **H** matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H matrix in $|\mathbf{g}\rangle$-basis:

$$(\mathbf{H})_G = \sum_{g=1}^{{}^oG} r_g \mathbf{g} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H matrix in $|\mathbf{P}^{(\mu)}\rangle$-basis:

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c|c|cc|cc} \left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\ H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \left\langle \mathbf{P}^{\mu}_{ma} \middle| \mathbf{H} \middle| \mathbf{P}^{\mu}_{nb} \right\rangle = \frac{\langle \mathbf{1}|\mathbf{P}^{\mu}_{am}\mathbf{H}\mathbf{P}^{\mu}_{nb}|\mathbf{1}\rangle}{(norm)^2} = \frac{\langle \mathbf{1}|\mathbf{H}\mathbf{P}^{\mu}_{am}\mathbf{P}^{\mu}_{nb}|\mathbf{1}\rangle}{(norm)^2}$$

Mock-Mach
commutation

$$\mathbf{r}\,\bar{\mathbf{r}} = \bar{\mathbf{r}}\,\mathbf{r}$$

(p.89)

$$\left|\begin{matrix}\mu \\ mn\end{matrix}\right\rangle = \mathbf{P}^{\mu}_{mn}|\mathbf{1}\rangle\frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G \cdot norm}\sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g)|\mathbf{g}\rangle$$

subject to normalization (from p. 116-122):

$$norm = \sqrt{\langle \mathbf{1}|\mathbf{P}^{\mu}_{nn}|\mathbf{1}\rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$$

(which will cancel out)

So, fuggettabout it!

D_3 Hamiltonian local- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$\mathbf{H}$ *matrix in* $|\mathbf{g}\rangle$*-basis:*

$$(\mathbf{H})_G = \sum_{g=1}^{{}^oG} r_g \mathbf{\bar{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

$\mathbf{H}$ *matrix in* $|\mathbf{P}^{(\mu)}\rangle$*-basis:*

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c|c|cc|cc} \left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\ \hline H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \left\langle \mathbf{P}^{\mu}_{ma} \middle| \mathbf{H} \middle| \mathbf{P}^{\mu}_{nb} \right\rangle = \underset{(norm)^2}{\underline{\langle \mathbf{1} | \mathbf{P}^{\mu}_{am} \mathbf{H} \mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}} = \langle \mathbf{1} | \mathbf{H} \mathbf{P}^{\mu}_{am} \underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}} = \delta_{mn} \langle \mathbf{1} | \mathbf{H}\, \underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{ab} | \mathbf{1} \rangle}} = \sum_{g=1}^{{}^\circ G} \langle \mathbf{1} | \mathbf{H} | \mathbf{g} \rangle D^{\alpha^*}_{ab}(g)$$

Use $\mathbf{P}^{\mu}_{mn}$-orthonormality

$$\mathbf{P}^{\mu'}_{m'n'} \mathbf{P}^{\mu}_{mn} = \delta^{\mu'\mu} \delta_{n'm} \mathbf{P}^{\mu}_{m'n}$$

(p.18)

D_3 Hamiltonian *local*- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$\mathbf{H}$ *matrix in* $|\mathbf{g}\rangle$*-basis:*

$$(\mathbf{H})_G = \sum_{g=1}^{{}^oG} r_g \overline{\mathbf{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

$\mathbf{H}$ *matrix in* $|\mathbf{P}^{(\mu)}\rangle$*-basis:*

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c} \begin{array}{cccccc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \end{array} \\ \left(\begin{array}{c|c|cc|cc} H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}\right) \end{array}$$

$$H^{\alpha}_{ab} = \langle \mathbf{P}^\mu_{ma} | \mathbf{H} | \mathbf{P}^\mu_{nb} \rangle = \frac{\langle \mathbf{1} | \mathbf{P}^\mu_{am} \mathbf{H} \mathbf{P}^\mu_{nb} | \mathbf{1} \rangle}{(norm)^2} = \frac{\langle \mathbf{1} | \mathbf{H} \mathbf{P}^\mu_{am} \mathbf{P}^\mu_{nb} | \mathbf{1} \rangle}{(norm)^2} = \delta_{mn} \frac{\langle \mathbf{1} | \mathbf{H}\, \mathbf{P}^\mu_{ab} | \mathbf{1} \rangle}{(norm)^2} = \sum_{g=1}^{{}^\circ G} \langle \mathbf{1} | \mathbf{H} | \mathbf{g} \rangle D^{\mu^*}_{ab}(g)$$

$$\left| \begin{matrix} \mu \\ mn \end{matrix} \right\rangle = \mathbf{P}^\mu_{mn} |\mathbf{1}\rangle \frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G \cdot norm} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g) |\mathbf{g}\rangle$$

subject to normalization (from p. 116-122):

$$norm = \sqrt{\langle \mathbf{1} | \mathbf{P}^\mu_{nn} | \mathbf{1} \rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$$

(which will cancel out)

So, fuggettabout it!

Coefficients $D^\mu_{mn}(g)$ *are irreducible representations (ireps) of* $\mathbf{g}$

$\mathbf{g} =$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$D^{A_1}(\mathbf{g}) =$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g}) =$	1	1	1	-1	-1	-1
$D^{E_1}_{x,y}(\mathbf{g}) =$	$\begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

D_3 Hamiltonian *local*- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H *matrix in* $|\mathbf{g}\rangle$*-basis:*

$$(\mathbf{H})_G = \sum_{g=1}^{{}^oG} r_g \bar{\mathbf{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H *matrix in* $|\mathbf{P}^{(\mu)}\rangle$*-basis:*

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{cccccc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \\ \hline H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \langle \mathbf{P}^{\mu}_{ma} | \mathbf{H} | \mathbf{P}^{\mu}_{nb} \rangle = \underbrace{\langle \mathbf{1} | \mathbf{P}^{\mu}_{am}}_{(norm)^2} \mathbf{H} \mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle = \langle \mathbf{1} | \mathbf{H} \mathbf{P}^{\mu}_{am} \underbrace{\mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}_{(norm)^2} = \delta_{mn} \langle \mathbf{1} | \mathbf{H}\, \underbrace{\mathbf{P}^{\mu}_{ab} | \mathbf{1} \rangle}_{(norm)^2} = \sum_{g=1}^{{}^\circ G} \langle \mathbf{1} | \mathbf{H} | \mathbf{g} \rangle D^{\mu^*}_{ab}(g) = \sum_{g=1}^{{}^\circ G} r_g D^{\mu^*}_{ab}(g)$$

$$\left| {}^{\mu}_{mn} \right\rangle = \mathbf{P}^{\mu}_{mn} |\mathbf{1}\rangle \frac{1}{norm} = \frac{\ell^{(\mu)}}{{}^\circ G \cdot norm} \sum_{\mathbf{g}}^{{}^\circ G} D^{\mu^*}_{mn}(g) |\mathbf{g}\rangle$$

subject to normalization (from p. 116-122):

$$norm = \sqrt{\langle \mathbf{1} | \mathbf{P}^{\mu}_{nn} | \mathbf{1} \rangle} = \sqrt{\frac{\ell^{(\mu)}}{{}^\circ G}}$$

(which will cancel out)

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Coefficients $D^{\mu}_{mn}(g)$ *are irreducible representations (ireps) of* $\mathbf{g}$

$\mathbf{g} =$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$D^{A_1}(\mathbf{g}) =$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g}) =$	1	1	1	−1	−1	−1
$D^{E_1}_{x,y}(\mathbf{g}) =$	$\begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

D_3 Hamiltonian *local*- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H *matrix in* $|\mathbf{g}\rangle$*-basis:*

$$(\mathbf{H})_G = \sum_{g=1}^{^oG} r_g \mathbf{\bar{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H *matrix in* $|\mathbf{P}^{(\mu)}\rangle$*-basis:*

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c|c|cc|cc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \\ H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \langle \mathbf{P}^\mu_{ma} | \mathbf{H} | \mathbf{P}^\mu_{nb} \rangle = \underbrace{\langle \mathbf{1} | \mathbf{P}^\mu_{am} \mathbf{H} \mathbf{P}^\mu_{nb} | \mathbf{1} \rangle}_{(norm)^2} = \underbrace{\langle \mathbf{1} | \mathbf{H} \mathbf{P}^\mu_{am} \mathbf{P}^\mu_{nb} | \mathbf{1} \rangle}_{(norm)^2} = \delta_{mn} \underbrace{\langle \mathbf{1} | \mathbf{H}\, \mathbf{P}^\mu_{ab} | \mathbf{1} \rangle}_{(norm)^2} = \sum_{g=1}^{^\circ G} \langle \mathbf{1} | \mathbf{H} | \mathbf{g} \rangle D^{\alpha *}_{ab}(g) = \sum_{g=1}^{^\circ G} r_g D^{\alpha *}_{ab}(g)$$

$$H^{A_1} = r_0 D^{A_1 *}(1) + r_1 D^{A_1 *}(r^1) + r_1^* D^{A_1 *}(r^2) + i_1 D^{A_1 *}(i_1) + i_2 D^{A_1 *}(i_2) + i_3 D^{A_1 *}(i_3) = r_0 + r_1 + r_1^* + i_1 + i_2 + i_3$$

Coefficients $D^\mu_{mn}(g)$ *are irreducible representations (ireps) of* $\mathbf{g}$

$\mathbf{g} =$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$D^{A_1}(\mathbf{g}) =$	1	1	1	1	1	1
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$D^{E_1}_{x,y}(\mathbf{g}) =$	$\begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

D_3 Hamiltonian local- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H *matrix in* $|\mathbf{g}\rangle$*-basis:*

$$(\mathbf{H})_G = \sum_{g=1}^{^oG} r_g \overline{\mathbf{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H *matrix in* $|\mathbf{P}^{(\mu)}\rangle$*-basis:*

$$(\mathbf{H})_P = \overline{T}(\mathbf{H})_G \overline{T}^\dagger = \begin{array}{c} \begin{array}{cccccc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \end{array} \\ \left(\begin{array}{c|c|cc|cc} H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}\right) \end{array}$$

$$H^{\alpha}_{ab} = \langle \mathbf{P}^{\mu}_{ma} | \mathbf{H} | \mathbf{P}^{\mu}_{nb} \rangle = \underset{(norm)^2}{\underline{\langle \mathbf{1} | \mathbf{P}^{\mu}_{am}}} \mathbf{H}\mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle = \langle \mathbf{1} | \mathbf{H}\mathbf{P}^{\mu}_{am} \underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}} = \delta_{mn} \langle \mathbf{1} | \mathbf{H} \underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{ab} | \mathbf{1} \rangle}} = \sum_{g=1}^{^\circ G} \langle \mathbf{1} | \mathbf{H} | \mathbf{g} \rangle D^{\alpha *}_{ab}(g) = \sum_{g=1}^{^\circ G} r_g D^{\alpha *}_{ab}(g)$$

$$H^{A_1} = r_0 D^{A_1 *}(1) + r_1 D^{A_1 *}(r^1) + r_1^* D^{A_1 *}(r^2) + i_1 D^{A_1 *}(i_1) + i_2 D^{A_1 *}(i_2) + i_3 D^{A_1 *}(i_3) \ = r_0 + r_1 + r_1^* + i_1 + i_2 + i_3$$

$$H^{A_2} = r_0 D^{A_2 *}(1) + r_1 D^{A_2 *}(r^1) + r_1^* D^{A_2 *}(r^2) + i_1 D^{A_2 *}(i_1) + i_2 D^{A_2 *}(i_2) + i_3 D^{A_2 *}(i_3) = r_0 + r_1 + r_1^* - i_1 - i_2 - i_3$$

Coefficients $D^{\mu}_{mn}(g)$ *are irreducible representations (ireps) of* $\mathbf{g}$

$\mathbf{g} =$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$D^{A_1}(\mathbf{g}) =$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g}) =$	1	1	1	−1	−1	−1
$D^{E_1}_{x,y}(\mathbf{g}) =$	$\begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

D_3 Hamiltonian local- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H matrix in $|\mathbf{g}\rangle$-basis:

$$(\mathbf{H})_G = \sum_{g=1}^{{}^oG} r_g \mathbf{\bar{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H matrix in $|\mathbf{P}^{(\mu)}\rangle$-basis:

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{cccccc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \\ \hline H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \langle \mathbf{P}^{\mu}_{ma} | \mathbf{H} | \mathbf{P}^{\mu}_{nb} \rangle = \underset{(norm)^2}{\langle \mathbf{1} | \mathbf{P}^{\mu}_{am}} \mathbf{H} \mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle = \langle \mathbf{1} | \mathbf{H} \mathbf{P}^{\mu}_{am} \underset{(norm)^2}{\mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle} = \delta_{mn} \langle \mathbf{1} | \mathbf{H} \underset{(norm)^2}{\mathbf{P}^{\mu}_{ab} | \mathbf{1} \rangle} = \sum_{g=1}^{{}^\circ G} \langle \mathbf{1} | \mathbf{H} | \mathbf{g} \rangle D^{\alpha *}_{ab}(g) = \sum_{g=1}^{{}^\circ G} r_g D^{\alpha *}_{ab}(g)$$

$$H^{A_1} = r_0 D^{A_1 *}(1) + r_1 D^{A_1 *}(r^1) + r_1^* D^{A_1 *}(r^2) + i_1 D^{A_1 *}(i_1) + i_2 D^{A_1 *}(i_2) + i_3 D^{A_1 *}(i_3) \; = r_0 + r_1 + r_1^* + i_1 + i_2 + i_3$$

$$H^{A_2} = r_0 D^{A_2 *}(1) + r_1 D^{A_2 *}(r^1) + r_1^* D^{A_2 *}(r^2) + i_1 D^{A_2 *}(i_1) + i_2 D^{A_2 *}(i_2) + i_3 D^{A_2 *}(i_3) = r_0 + r_1 + r_1^* - i_1 - i_2 - i_3$$

$$H^{E_1}_{xx} = r_0 D^{E*}_{xx}(1) + r_1 D^{E*}_{xx}(r^1) + r_1^* D^{E*}_{xx}(r^2) + i_1 D^{E*}_{xx}(i_1) + i_2 D^{E*}_{xx}(i_2) + i_3 D^{E*}_{xx}(i_3) \; = (2r_0 - r_1 - r_1^* - i_1 - i_2 + 2i_3)/2$$

Coefficients $D^{\mu}_{mn}(g)$ are irreducible representations (ireps) of $\mathbf{g}$

$\mathbf{g} =$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$D^{A_1}(\mathbf{g}) =$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g}) =$	1	1	1	-1	-1	-1
$D^{E_1}_{x,y}(\mathbf{g}) =$	$\begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

D_3 Hamiltonian local- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

$\mathbf{H}$ matrix in $|\mathbf{g}\rangle$-basis:

$$(\mathbf{H})_G = \sum_{g=1}^{^oG} r_g \mathbf{g} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

$\mathbf{H}$ matrix in $|\mathbf{P}^{(\mu)}\rangle$-basis:

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c} \begin{array}{cccccc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \end{array} \\ \left(\begin{array}{c|c|cc|cc} H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}\right) \end{array}$$

$$H^{\alpha}_{ab} = \langle \mathbf{P}^{\mu}_{ma} | \mathbf{H} | \mathbf{P}^{\mu}_{nb} \rangle = \frac{\langle \mathbf{1} | \mathbf{P}^{\mu}_{am} \mathbf{H} \mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}{(norm)^2} = \frac{\langle \mathbf{1} | \mathbf{H} \mathbf{P}^{\mu}_{am} \mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}{(norm)^2} = \delta_{mn} \frac{\langle \mathbf{1} | \mathbf{H}\, \mathbf{P}^{\mu}_{ab} | \mathbf{1} \rangle}{(norm)^2} = \sum_{g=1}^{^\circ G} \langle \mathbf{1} | \mathbf{H} | \mathbf{g} \rangle D^{\alpha *}_{ab}(g) = \sum_{g=1}^{^\circ G} r_g D^{\alpha *}_{ab}(g)$$

$$H^{A_1} = r_0 D^{A_1 *}(1) + r_1 D^{A_1 *}(r^1) + r_1^* D^{A_1 *}(r^2) + i_1 D^{A_1 *}(i_1) + i_2 D^{A_1 *}(i_2) + i_3 D^{A_1 *}(i_3) \;\; = r_0 + r_1 + r_1^* + i_1 + i_2 + i_3$$

$$H^{A_2} = r_0 D^{A_2 *}(1) + r_1 D^{A_2 *}(r^1) + r_1^* D^{A_2 *}(r^2) + i_1 D^{A_2 *}(i_1) + i_2 D^{A_2 *}(i_2) + i_3 D^{A_2 *}(i_3) = r_0 + r_1 + r_1^* - i_1 - i_2 - i_3$$

$$H^{E_1}_{xx} = r_0 D^{E*}_{xx}(1) + r_1 D^{E*}_{xx}(r^1) + r_1^* D^{E*}_{xx}(r^2) + i_1 D^{E*}_{xx}(i_1) + i_2 D^{E*}_{xx}(i_2) + i_3 D^{E*}_{xx}(i_3) \;\; = (2r_0 - r_1 - r_1^* - i_1 - i_2 + 2i_3)/2$$

$$H^{E_1}_{xy} = r_0 D^{E*}_{xy}(1) + r_1 D^{E*}_{xy}(r^1) + r_1^* D^{E*}_{xy}(r^2) + i_1 D^{E*}_{xy}(i_1) + i_2 D^{E*}_{xy}(i_2) + i_3 D^{E*}_{xy}(i_3) \;\; = \sqrt{3}(-r_1 + r_1^* - i_1 + i_2)/2 = H^{E*}_{yx}$$

Coefficients $D^{\mu}_{mn}(g)$ are irreducible representations (ireps) of $\mathbf{g}$

$\mathbf{g} =$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$D^{A_1}(\mathbf{g}) =$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g}) =$	1	1	1	−1	−1	−1
$D^{E_1}_{x,y}(\mathbf{g}) =$	$\begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

D_3 Hamiltonian local- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H matrix in $|\mathbf{g}\rangle$-basis:

$$(\mathbf{H})_G=\sum_{g=1}^{^oG} r_g\mathbf{g}=\begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3\\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2\\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1\\ i_i & i_3 & i_2 & r_0 & r_1 & r_2\\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1\\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H matrix in $|\mathbf{P}^{(\mu)}\rangle$-basis:

$$(\mathbf{H})_P=\bar{T}(\mathbf{H})_G\bar{T}^\dagger=\begin{array}{c} \begin{array}{cccccc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \end{array}\\ \left(\begin{array}{c|c|cc|cc} H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot\\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot\\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot\\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot\\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy}\\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}\right)\end{array}$$

$$H^{\alpha}_{ab}=\langle\mathbf{P}^{\mu}_{ma}|\mathbf{H}|\mathbf{P}^{\mu}_{nb}\rangle=\underset{(norm)^2}{\underline{\langle\mathbf{1}|\mathbf{P}^{\mu}}}_{am}\mathbf{H}\mathbf{P}^{\mu}_{nb}|\mathbf{1}\rangle=\langle\mathbf{1}|\mathbf{H}\mathbf{P}^{\mu}_{am}\underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{nb}|\mathbf{1}\rangle}}=\delta_{mn}\langle\mathbf{1}|\mathbf{H}\,\underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{ab}|\mathbf{1}\rangle}}=\sum_{g=1}^{^\circ G}\langle\mathbf{1}|\mathbf{H}|\mathbf{g}\rangle D^{\alpha *}_{ab}(g)=\sum_{g=1}^{^\circ G} r_g D^{\alpha *}_{ab}(g)$$

$$H^{A_1}=r_0D^{A_1*}(1)+r_1D^{A_1*}(r^1)+r_1^*D^{A_1*}(r^2)+i_1D^{A_1*}(i_1)+i_2D^{A_1*}(i_2)+i_3D^{A_1*}(i_3)\;=r_0+r_1+r_1^*+i_1+i_2+i_3$$

$$H^{A_2}=r_0D^{A_2*}(1)+r_1D^{A_2*}(r^1)+r_1^*D^{A_2*}(r^2)+i_1D^{A_2*}(i_1)+i_2D^{A_2*}(i_2)+i_3D^{A_2*}(i_3)\;=r_0+r_1+r_1^*-i_1-i_2-i_3$$

$$H^{E_1}_{xx}=r_0D^{E*}_{xx}(1)+r_1D^{E*}_{xx}(r^1)+r_1^*D^{E*}_{xx}(r^2)+i_1D^{E*}_{xx}(i_1)+i_2D^{E*}_{xx}(i_2)+i_3D^{E*}_{xx}(i_3)\;=(2r_0-r_1-r_1^*-i_1-i_2+2i_3)/2$$

$$H^{E_1}_{xy}=r_0D^{E*}_{xy}(1)+r_1D^{E*}_{xy}(r^1)+r_1^*D^{E*}_{xy}(r^2)+i_1D^{E*}_{xy}(i_1)+i_2D^{E*}_{xy}(i_2)+i_3D^{E*}_{xy}(i_3)\;=\sqrt{3}(-r_1+r_1^*-i_1+i_2)/2\;=H^{E*}_{yx}$$

$$H^{E_1}_{yy}=r_0D^{E*}_{yy}(1)+r_1D^{E*}_{yy}(r^1)+r_1^*D^{E*}_{yy}(r^2)+i_1D^{E*}_{yy}(i_1)+i_2D^{E*}_{yy}(i_2)+i_3D^{E*}_{yy}(i_3)\;=(2r_0-r_1-r_1^*+i_1+i_2-2i_3)/2$$

Coefficients $D^{\mu}_{mn}(g)$ are irreducible representations (ireps) of **g**

$\mathbf{g}=$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$D^{A_1}(\mathbf{g})=$	1	1	1	1	1	1
$D^{A_2}(\mathbf{g})=$	1	1	1	-1	-1	-1
$D^{E_1}_{x,y}(\mathbf{g})=$	$\begin{pmatrix}1&\cdot\\ \cdot&1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}1&0\\ 0&-1\end{pmatrix}$

D₃ Hamiltonian local- **H** *matrices in* $|\mathbf{P}^{(\mu)}\rangle$*-basis*

H *matrix in* $|\mathbf{g}\rangle$*-basis:*

$$(\mathbf{H})_G = \sum_{g=1}^{{}^oG} r_g \mathbf{g} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H *matrix in* $|\mathbf{P}^{(\mu)}\rangle$*-basis:*

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c|c|cc|cc} \left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\ \hline H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}$$

$$H^{\alpha}_{ab} = \left\langle \mathbf{P}^{\mu}_{ma} \middle| \mathbf{H} \middle| \mathbf{P}^{\mu}_{nb} \right\rangle = \underset{(norm)^2}{\underline{\langle \mathbf{1}|\mathbf{P}^{\mu}_{am}}} \mathbf{H}\mathbf{P}^{\mu}_{nb}|\mathbf{1}\rangle = \langle \mathbf{1}|\mathbf{H}\mathbf{P}^{\mu}_{am} \underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{nb}|\mathbf{1}\rangle}} = \delta_{mn}\langle \mathbf{1}|\mathbf{H}\, \underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{ab}|\mathbf{1}\rangle}} = \sum_{g=1}^{{}^\circ G} \langle \mathbf{1}|\mathbf{H}|\mathbf{g}\rangle D^{\alpha^*}_{ab}(g) = \sum_{g=1}^{{}^\circ G} r_g D^{\alpha^*}_{ab}(g)$$

$$H^{A_1} = r_0 D^{A_1 *}(1) + r_1 D^{A_1 *}(r^1) + r_1^* D^{A_1 *}(r^2) + i_1 D^{A_1 *}(i_1) + i_2 D^{A_1 *}(i_2) + i_3 D^{A_1 *}(i_3) \; = r_0 + r_1 + r_1^* + i_1 + i_2 + i_3$$

$$H^{A_2} = r_0 D^{A_2 *}(1) + r_1 D^{A_2 *}(r^1) + r_1^* D^{A_2 *}(r^2) + i_1 D^{A_2 *}(i_1) + i_2 D^{A_2 *}(i_2) + i_3 D^{A_2 *}(i_3) = r_0 + r_1 + r_1^* - i_1 - i_2 - i_3$$

$$H^{E_1}_{xx} = r_0 D^{E^*}_{xx}(1) + r_1 D^{E^*}_{xx}(r^1) + r_1^* D^{E^*}_{xx}(r^2) + i_1 D^{E^*}_{xx}(i_1) + i_2 D^{E^*}_{xx}(i_2) + i_3 D^{E^*}_{xx}(i_3) \; = (2r_0 - r_1 - r_1^* - i_1 - i_2 + 2i_3)/2$$

$$H^{E_1}_{xy} = r_0 D^{E^*}_{xy}(1) + r_1 D^{E^*}_{xy}(r^1) + r_1^* D^{E^*}_{xy}(r^2) + i_1 D^{E^*}_{xy}(i_1) + i_2 D^{E^*}_{xy}(i_2) + i_3 D^{E^*}_{xy}(i_3) \; = \sqrt{3}(-r_1 + r_1^* - i_1 + i_2)/2 = H^{E^*}_{yx}$$

$$H^{E_1}_{yy} = r_0 D^{E^*}_{yy}(1) + r_1 D^{E^*}_{yy}(r^1) + r_1^* D^{E^*}_{yy}(r^2) + i_1 D^{E^*}_{yy}(i_1) + i_2 D^{E^*}_{yy}(i_2) + i_3 D^{E^*}_{yy}(i_3) \; = (2r_0 - r_1 - r_1^* + i_1 + i_2 - 2i_3)/2$$

$$\begin{pmatrix} H^{E_1}_{xx} & H^{E_1}_{xy} \\ H^{E_1}_{yx} & H^{E_1}_{yy} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2r_0 - r_1 - r_1^* - i_1 - i_2 + 2i_3 & \sqrt{3}(-r_1 + r_1^* - i_1 + i_2) \\ \sqrt{3}(-r_1^* + r_1 - i_1 + i_2) & 2r_0 - r_1 - r_1^* + i_1 + i_2 - 2i_3 \end{pmatrix}$$

D_3 Hamiltonian local- $\mathbf{H}$ matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H *matrix in* $|\mathbf{g}\rangle$*-basis:*

$$(\mathbf{H})_G = \sum_{g=1}^{^oG} r_g \bar{\mathbf{g}} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H *matrix in* $|\mathbf{P}^{(\mu)}\rangle$*-basis:*

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{cccccc} \left|\mathbf{P}^{A_1}_{xx}\right\rangle & \left|\mathbf{P}^{A_2}_{yy}\right\rangle & \left|\mathbf{P}^{E_1}_{xx}\right\rangle & \left|\mathbf{P}^{E_1}_{xy}\right\rangle & \left|\mathbf{P}^{E_1}_{yx}\right\rangle & \left|\mathbf{P}^{E_1}_{yy}\right\rangle \\ \left(\begin{array}{c|c|cc|cc} H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}\right) & & & & & \end{array}$$

$$H^{\alpha}_{ab} = \left\langle \mathbf{P}^{\mu}_{ma} \middle| \mathbf{H} \middle| \mathbf{P}^{\mu}_{nb} \right\rangle = \underset{(norm)^2}{\underline{\langle \mathbf{1}|\mathbf{P}^{\mu}_{am}}}\mathbf{H}\mathbf{P}^{\mu}_{nb}|\mathbf{1}\rangle = \langle \mathbf{1}|\mathbf{H}\mathbf{P}^{\mu}_{am}\underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{nb}|\mathbf{1}\rangle}} = \delta_{mn}\langle \mathbf{1}|\mathbf{H}\,\underset{(norm)^2}{\underline{\mathbf{P}^{\mu}_{ab}|\mathbf{1}\rangle}} = \sum_{g=1}^{^\circ G} \langle \mathbf{1}|\mathbf{H}|\mathbf{g}\rangle D^{\alpha *}_{ab}(g) = \sum_{g=1}^{^\circ G} r_g D^{\alpha *}_{ab}(g)$$

$$H^{A_1} = r_0 D^{A_1 *}(1) + r_1 D^{A_1 *}(r^1) + r_1^* D^{A_1 *}(r^2) + i_1 D^{A_1 *}(i_1) + i_2 D^{A_1 *}(i_2) + i_3 D^{A_1 *}(i_3) \;= r_0 + r_1 + r_1^* + i_1 + i_2 + i_3 \qquad = r_0 + 2r_1 + 2i_{12} + i_3$$

$$H^{A_2} = r_0 D^{A_2 *}(1) + r_1 D^{A_2 *}(r^1) + r_1^* D^{A_2 *}(r^2) + i_1 D^{A_2 *}(i_1) + i_2 D^{A_2 *}(i_2) + i_3 D^{A_2 *}(i_3) = r_0 + r_1 + r_1^* - i_1 - i_2 - i_3 \qquad = r_0 + 2r_1 \; -2i_{12} \; -i_3$$

$$H^{E_1}_{xx} = r_0 D^{E*}_{xx}(1) + r_1 D^{E*}_{xx}(r^1) + r_1^* D^{E*}_{xx}(r^2) + i_1 D^{E*}_{xx}(i_1) + i_2 D^{E*}_{xx}(i_2) + i_3 D^{E*}_{xx}(i_3) \;= (2r_0 - r_1 - r_1^* - i_1 - i_2 + 2i_3)/2 \qquad = r_0 \; -r_1 \quad -i_{12} + i_3$$

$$H^{E_1}_{xy} = r_0 D^{E*}_{xy}(1) + r_1 D^{E*}_{xy}(r^1) + r_1^* D^{E*}_{xy}(r^2) + i_1 D^{E*}_{xy}(i_1) + i_2 D^{E*}_{xy}(i_2) + i_3 D^{E*}_{xy}(i_3) \;= \sqrt{3}(-r_1 + r_1^* - i_1 + i_2)/2 = H^{E*}_{yx} \qquad = 0$$

$$H^{E_1}_{yy} = r_0 D^{E*}_{yy}(1) + r_1 D^{E*}_{yy}(r^1) + r_1^* D^{E*}_{yy}(r^2) + i_1 D^{E*}_{yy}(i_1) + i_2 D^{E*}_{yy}(i_2) + i_3 D^{E*}_{yy}(i_3) \;= (2r_0 - r_1 - r_1^* + i_1 + i_2 - 2i_3)/2 \qquad = r_0 \; -r_1 \quad +i_{12} \; -i_3$$

$$\begin{pmatrix} H^{E_1}_{xx} & H^{E_1}_{xy} \\ H^{E_1}_{yx} & H^{E_1}_{yy} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2r_0 - r_1 - r_1^* - i_1 - i_2 + 2i_3 & \sqrt{3}(-r_1 + r_1^* - i_1 + i_2) \\ \sqrt{3}(-r_1^* + r_1 - i_1 + i_2) & 2r_0 - r_1 - r_1^* + i_1 + i_2 - 2i_3 \end{pmatrix}$$

$$= \begin{pmatrix} r_0 - r_1 - i_{12} + i_3 & 0 \\ 0 & r_0 - r_1 - i_{12} - i_3 \end{pmatrix}_{For:\, r_1 = r_1^* \, and:\, i_1 = i_2}$$

Choosing local $C_2 = \{\mathbf{1}, \mathbf{i_3}\}$ *symmetry with local constraints* $r_1 = r_1{}^* = r_2$ *and* $i_1 = i_2$

D_3 Hamiltonian local- **H** matrices in $|\mathbf{P}^{(\mu)}\rangle$-basis

H matrix in $|\mathbf{g}\rangle$-basis:

$$(\mathbf{H})_G = \sum_{g=1}^{^oG} r_g \mathbf{g} = \begin{pmatrix} r_0 & r_2 & r_1 & i_1 & i_2 & i_3 \\ r_1 & r_0 & r_1 & i_3 & i_1 & i_2 \\ r_2 & r_1 & r_0 & i_2 & i_3 & i_1 \\ i_i & i_3 & i_2 & r_0 & r_1 & r_2 \\ i_2 & i_1 & i_3 & r_2 & r_0 & r_1 \\ i_3 & i_2 & i_1 & r_1 & r_2 & r_0 \end{pmatrix}$$

H matrix in $|\mathbf{P}^{(\mu)}\rangle$-basis:

$$(\mathbf{H})_P = \bar{T}(\mathbf{H})_G \bar{T}^\dagger = \begin{array}{c} \begin{array}{cccccc} |\mathbf{P}^{A_1}_{xx}\rangle & |\mathbf{P}^{A_2}_{yy}\rangle & |\mathbf{P}^{E_1}_{xx}\rangle & |\mathbf{P}^{E_1}_{xy}\rangle & |\mathbf{P}^{E_1}_{yx}\rangle & |\mathbf{P}^{E_1}_{yy}\rangle \end{array} \\ \left(\begin{array}{c|c|cc|cc} H^{A_1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & H^{A_2} & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} & \cdot & \cdot \\ \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & H^{E_1}_{xx} & H^{E_1}_{xy} \\ \cdot & \cdot & \cdot & \cdot & H^{E_1}_{yx} & H^{E_1}_{yy} \end{array}\right) \end{array}$$

$$H^{\alpha}_{ab} = \langle \mathbf{P}^{\mu}_{ma} | \mathbf{H} | \mathbf{P}^{\mu}_{nb} \rangle = \frac{\langle \mathbf{1} | \mathbf{P}^{\mu}_{am} \mathbf{H} \mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}{(norm)^2} = \frac{\langle \mathbf{1} | \mathbf{H} \mathbf{P}^{\mu}_{am} \mathbf{P}^{\mu}_{nb} | \mathbf{1} \rangle}{(norm)^2} = \delta_{mn} \frac{\langle \mathbf{1} | \mathbf{H}\, \mathbf{P}^{\mu}_{ab} | \mathbf{1} \rangle}{(norm)^2} = \sum_{g=1}^{^\circ G} \langle \mathbf{1} | \mathbf{H} | \mathbf{g} \rangle D^{\alpha *}_{ab}(g) = \sum_{g=1}^{^\circ G} r_g D^{\alpha *}_{ab}(g)$$

$$H^{A_1} = r_0 D^{A_1 *}(1) + r_1 D^{A_1 *}(r^1) + r_1^* D^{A_1 *}(r^2) + i_1 D^{A_1 *}(i_1) + i_2 D^{A_1 *}(i_2) + i_3 D^{A_1 *}(i_3) = r_0 + r_1 + r_1^* + i_1 + i_2 + i_3 \quad = r_0 + 2r_1 + 2i_{12} + i_3$$

$$H^{A_2} = r_0 D^{A_2 *}(1) + r_1 D^{A_2 *}(r^1) + r_1^* D^{A_2 *}(r^2) + i_1 D^{A_2 *}(i_1) + i_2 D^{A_2 *}(i_2) + i_3 D^{A_2 *}(i_3) = r_0 + r_1 + r_1^* - i_1 - i_2 - i_3 \quad = r_0 + 2r_1 - 2i_{12} - i_3$$

$$H^{E_1}_{xx} = r_0 D^{E*}_{xx}(1) + r_1 D^{E*}_{xx}(r^1) + r_1^* D^{E*}_{xx}(r^2) + i_1 D^{E*}_{xx}(i_1) + i_2 D^{E*}_{xx}(i_2) + i_3 D^{E*}_{xx}(i_3) = (2r_0 - r_1 - r_1^* - i_1 - i_2 + 2i_3)/2 \quad = r_0 - r_1 - i_{12} + i_3$$

$$H^{E_1}_{xy} = r_0 D^{E*}_{xy}(1) + r_1 D^{E*}_{xy}(r^1) + r_1^* D^{E*}_{xy}(r^2) + i_1 D^{E*}_{xy}(i_1) + i_2 D^{E*}_{xy}(i_2) + i_3 D^{E*}_{xy}(i_3) = \sqrt{3}(-r_1 + r_1^* - i_1 + i_2)/2 = H^{E*}_{yx} \quad = 0$$

$$H^{E_1}_{yy} = r_0 D^{E*}_{yy}(1) + r_1 D^{E*}_{yy}(r^1) + r_1^* D^{E*}_{yy}(r^2) + i_1 D^{E*}_{yy}(i_1) + i_2 D^{E*}_{yy}(i_2) + i_3 D^{E*}_{yy}(i_3) = (2r_0 - r_1 - r_1^* + i_1 + i_2 - 2i_3)/2 \quad = r_0 - r_1 + i_{12} - i_3$$

$C_2 = \{\mathbf{1}, \mathbf{i_3}\}$

Local symmetry determines all levels and eigenvectors with just 4 real parameters

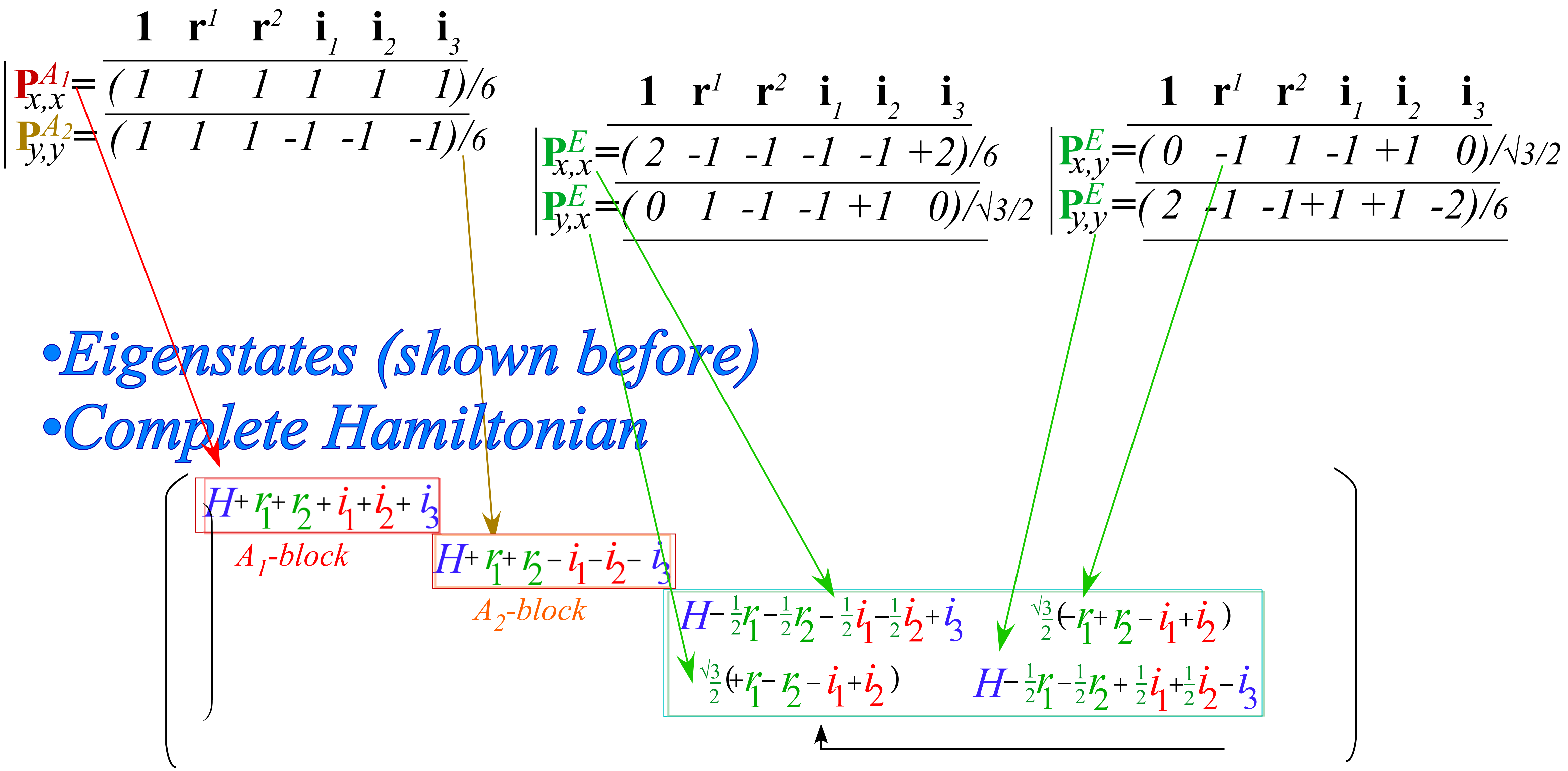
$$\begin{pmatrix} H^{E_1}_{xx} & H^{E_1}_{xy} \\ H^{E_1}_{yx} & H^{E_1}_{yy} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2r_0 - r_1 - r_1^* - i_1 - i_2 + 2i_3 & \sqrt{3}(-r_1 + r_1^* - i_1 + i_2) \\ \sqrt{3}(-r_1^* + r_1 - i_1 + i_2) & 2r_0 - r_1 - r_1^* + i_1 + i_2 - 2i_3 \end{pmatrix}$$

$$= \begin{pmatrix} r_0 - r_1 - i_{12} + i_3 & 0 \\ 0 & r_0 - r_1 - i_{12} - i_3 \end{pmatrix}_{For: r_1 = r_1^* and: i_1 = i_2}$$

Choosing local $C_2 = \{\mathbf{1}, \mathbf{i_3}\}$ symmetry with local constraints $r_1 = r_1^* = r_2$ and $i_1 = i_2$

$$\mathbf{P}^{(\mu)}_{mn}=\frac{\ell^{(\mu)}}{{}^\circ G}\Sigma_{\mathbf{g}}D^{(\mu)*}_{mn}(\mathbf{g})\mathbf{g}$$

Spectral Efficiency: Same $D(a)_{mn}$ projectors give a lot!



$$\begin{array}{r|cccccc} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \hline \mathbf{P}^{A_1}_{x,x}= & (1 & 1 & 1 & 1 & 1 & 1)/6 \\ \mathbf{P}^{A_2}_{y,y}= & (1 & 1 & 1 & -1 & -1 & -1)/6 \end{array}$$

$$\begin{array}{r|cccccc} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \hline \mathbf{P}^{E}_{x,x}= & (2 & -1 & -1 & -1 & -1 & +2)/6 \\ \mathbf{P}^{E}_{y,x}= & (0 & 1 & -1 & -1 & +1 & 0)/\sqrt{3/2} \end{array}$$

$$\begin{array}{r|cccccc} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \hline \mathbf{P}^{E}_{x,y}= & (0 & -1 & 1 & -1 & +1 & 0)/\sqrt{3/2} \\ \mathbf{P}^{E}_{y,y}= & (2 & -1 & -1 & +1 & +1 & -2)/6 \end{array}$$

- Eigenstates (shown before)
- Complete Hamiltonian

$$\left(\begin{array}{cccc} \boxed{H+r_1+r_2+i_1+i_2+i_3} & & & \\ A_1\text{-block} & \boxed{H+r_1+r_2-i_1-i_2-i_3} & & \\ & A_2\text{-block} & H-\frac{1}{2}r_1-\frac{1}{2}r_2-\frac{1}{2}i_1-\frac{1}{2}i_2+i_3 & \frac{\sqrt{3}}{2}(-r_1+r_2-i_1+i_2) \\ & & \frac{\sqrt{3}}{2}(+r_1-r_2-i_1+i_2) & H-\frac{1}{2}r_1-\frac{1}{2}r_2+\frac{1}{2}i_1+\frac{1}{2}i_2-i_3 \end{array}\right)$$

- Local symmetery eigenvalue formulae (L.S.=> off-diagonal zero.)

$r_1=r_2=r_1{}^*=r,\quad i_1=i_2=i_1{}^*=i$

gives:

A_1-level: $H+2r+2i+i_3$

A_1-level: $H+2r-2i-i_3$

E_x-level: $H-r-i+i_3$

E_y-level: $H-r+i-i_3$

$D_3 > C_2$ $\mathbf{i}_3$ projector states

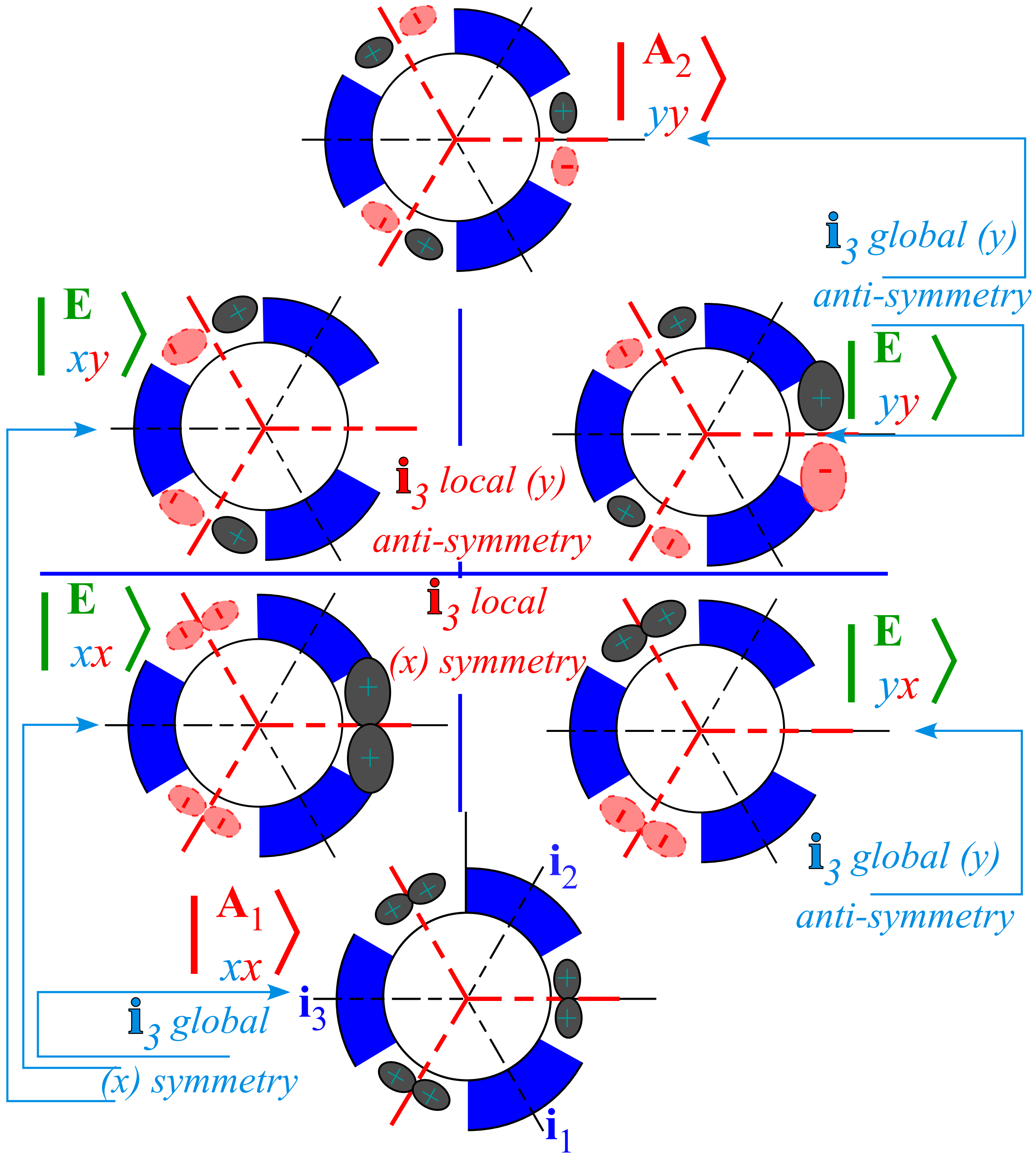
$$|^{(m)}_{eb}\rangle = \mathbf{P}^{(m)}_{eb}|1\rangle$$

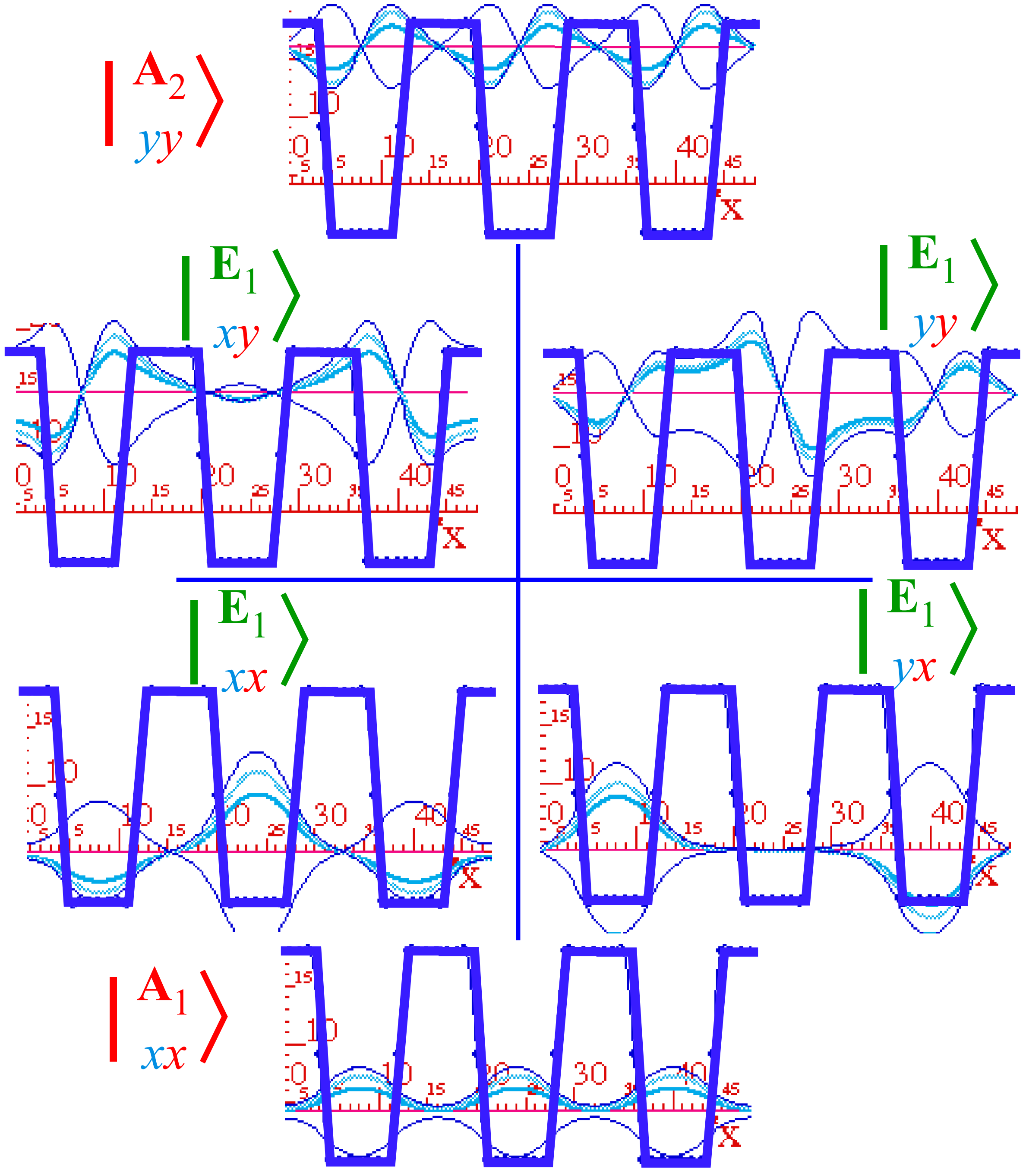
Global (LAB) symmetry

$$\mathbf{i}_3|^{(m)}_{eb}\rangle = \mathbf{i}_3\mathbf{P}^{(m)}_{eb}|1\rangle$$
$$= (-1)^e\,|^{(m)}\rangle$$

Local (BOD) symmetry

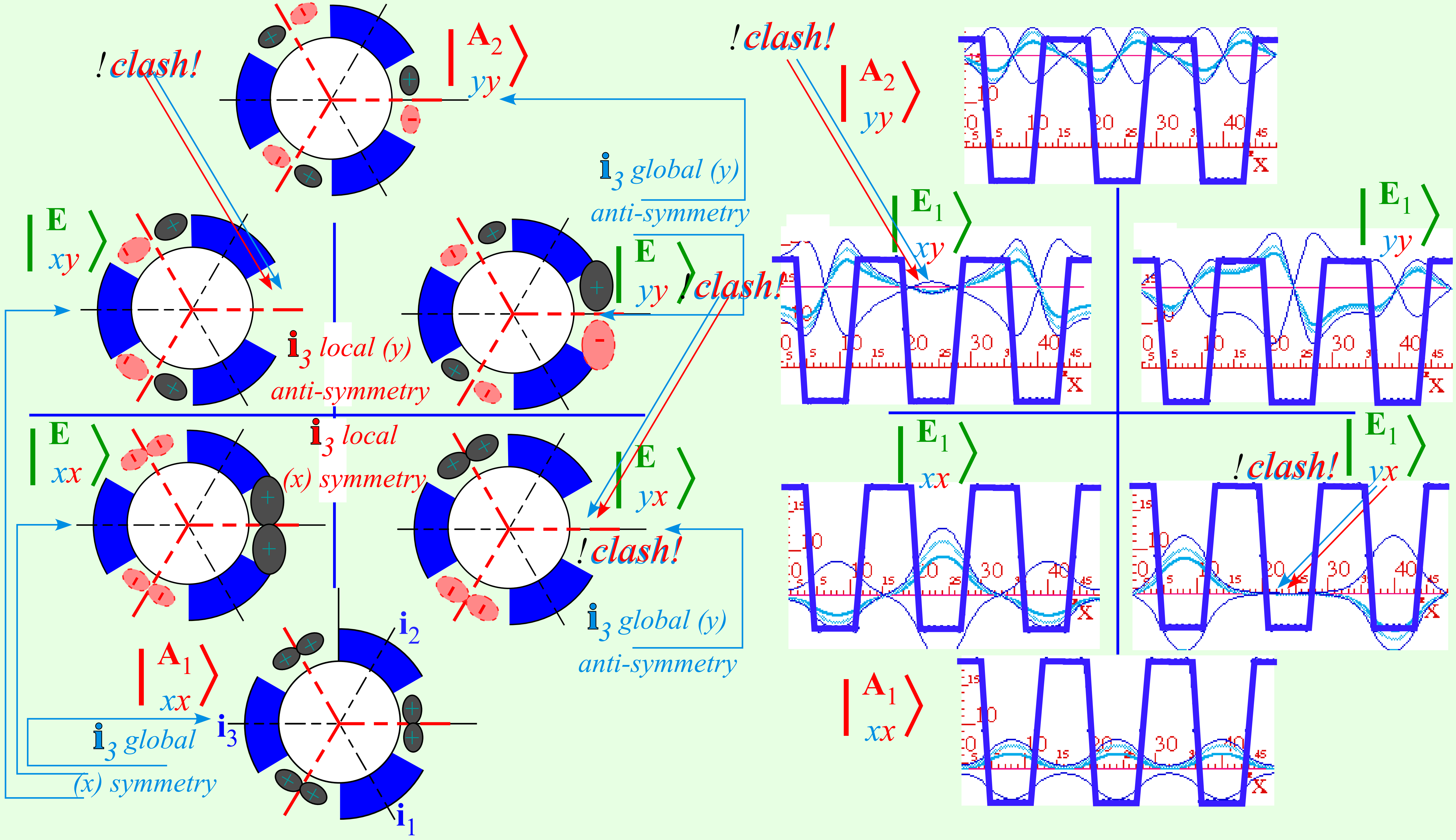
$$\bar{\mathbf{i}}_3|^{(m)}_{eb}\rangle = \bar{\mathbf{i}}_3\mathbf{P}^{(m)}_{eb}|1\rangle = \mathbf{P}^{(m)}_{eb}\bar{\mathbf{i}}_3|1\rangle$$
$$= \mathbf{P}^{(m)}_{eb}\mathbf{i}_3^{\dagger}|1\rangle = (-1)^b\,|^{(m)}\rangle$$



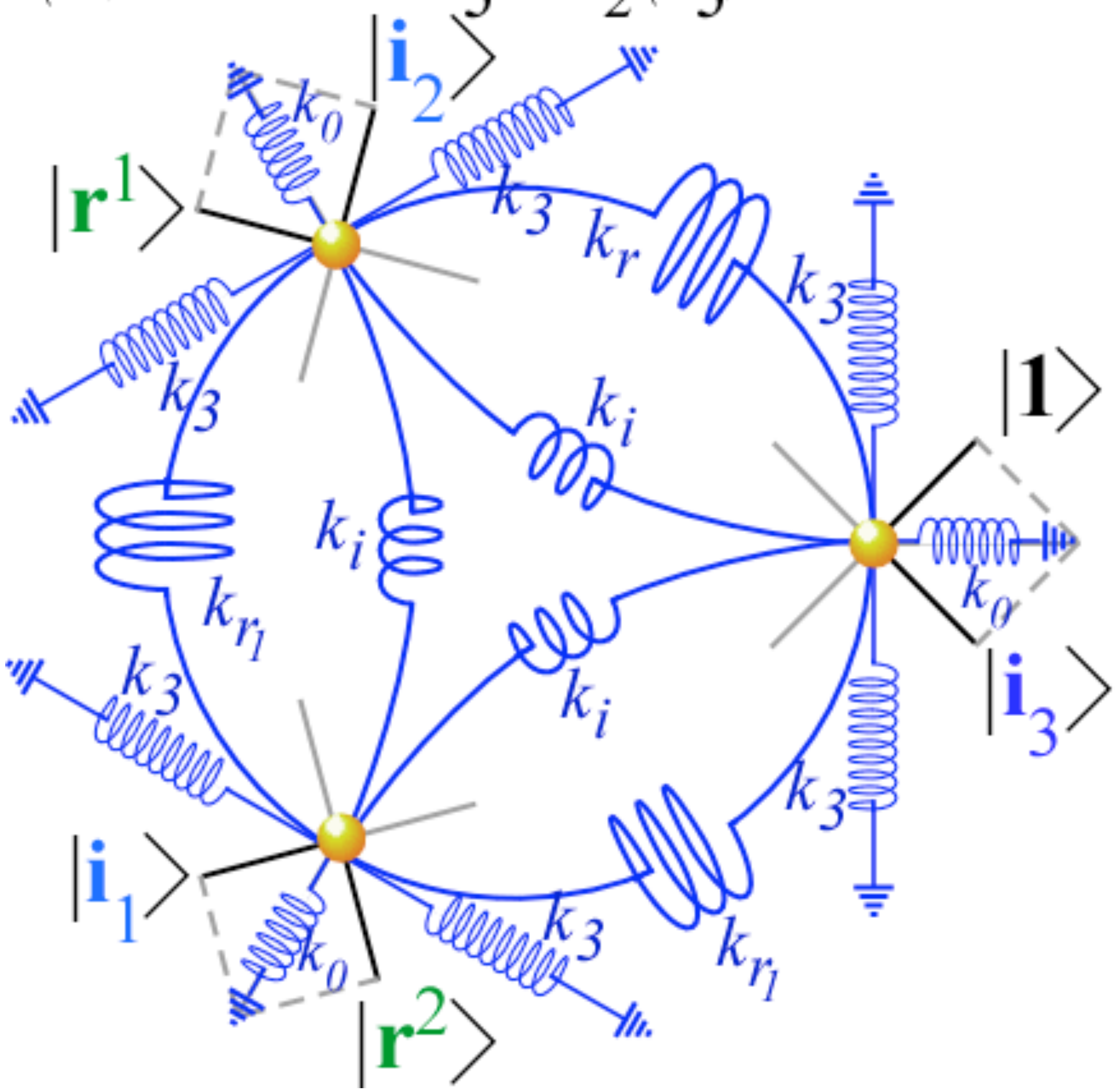


When there is no there, there...

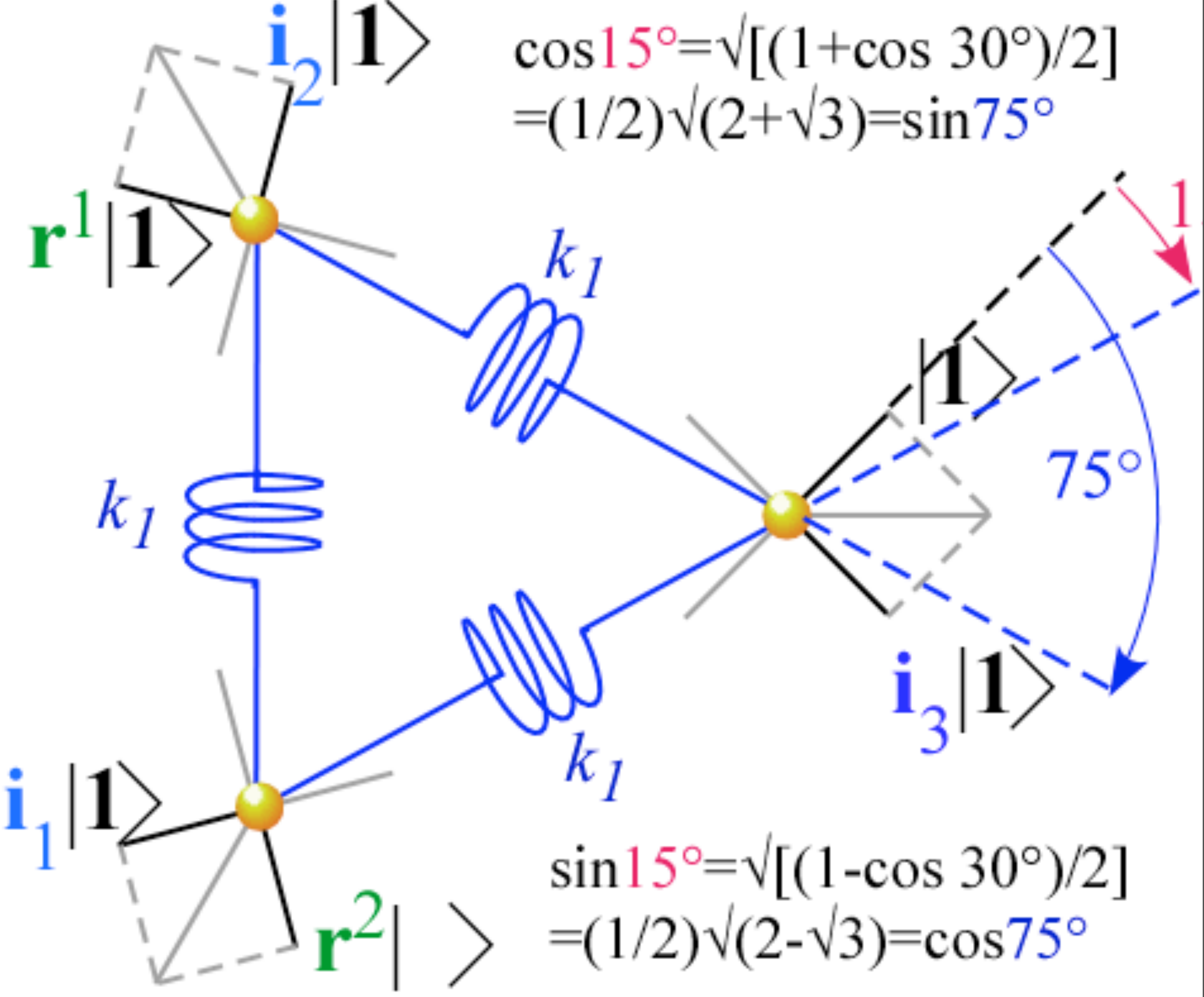
Nobody Home
where LOCAL
and GLOBAL



(a) Local $D_3 \supset C_2(i_3)$ model

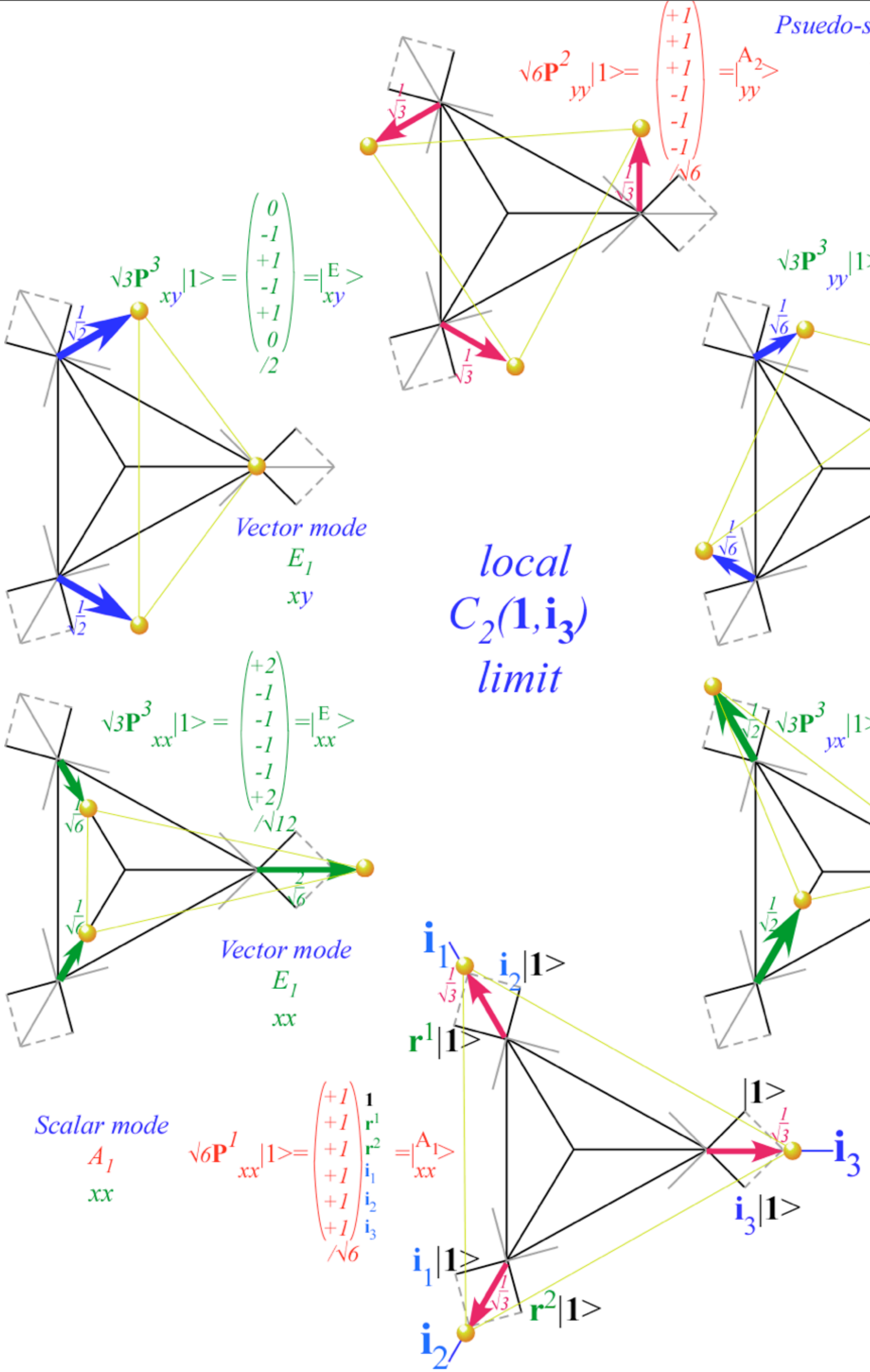


(b) Mixed local symmetry D_3 model



cos15°=√[(1+cos 30°)/2]
=(1/2)√(2+√3)=sin75°

sin15°=√[(1-cos 30°)/2]
=(1/2)√(2-√3)=cos75°



Psuedo-scalar mode
A_2
yy

$$\sqrt{6}\mathbf{P}^2_{yy}|1\rangle = \begin{pmatrix}+1\\+1\\+1\\-1\\-1\\-1\end{pmatrix}/\sqrt6 = |^{A_2}_{yy}\rangle$$

Vector mode
E_1
xy

$$\sqrt{3}\mathbf{P}^3_{xy}|1\rangle = \begin{pmatrix}0\\-1\\+1\\-1\\+1\\0\end{pmatrix}/2 = |^{E}_{xy}\rangle$$

Vector mode
E_1
yy

$$\sqrt{3}\mathbf{P}^3_{yy}|1\rangle = \begin{pmatrix}+2\\-1\\-1\\+1\\+1\\-2\end{pmatrix}/\sqrt{12} = |^{E}_{yy}\rangle$$

local
$C_2(\mathbf{1},\mathbf{i}_3)$
limit

Vector mode
E_1
xx

$$\sqrt{3}\mathbf{P}^3_{xx}|1\rangle = \begin{pmatrix}+2\\-1\\-1\\-1\\-1\\+2\end{pmatrix}/\sqrt{12} = |^{E}_{xx}\rangle$$

Vector mode
E_1
yx

$$\sqrt{3}\mathbf{P}^3_{yx}|1\rangle = \begin{pmatrix}0\\+1\\-1\\-1\\+1\\0\end{pmatrix}/2 = |^{E}_{yx}\rangle$$

Scalar mode
A_1
xx

$$\sqrt{6}\mathbf{P}^1_{xx}|1\rangle = \begin{array}{c|c}\begin{pmatrix}+1\\+1\\+1\\+1\\+1\\+1\end{pmatrix}/\sqrt6 & \begin{matrix}\mathbf{1}\\ \mathbf{r}^1\\ \mathbf{r}^2\\ \mathbf{i}_1\\ \mathbf{i}_2\\ \mathbf{i}_3\end{matrix}\end{array} = |^{A_1}_{xx}\rangle$$

(a) Local $D_3 \supset C_2(i_3)$ model