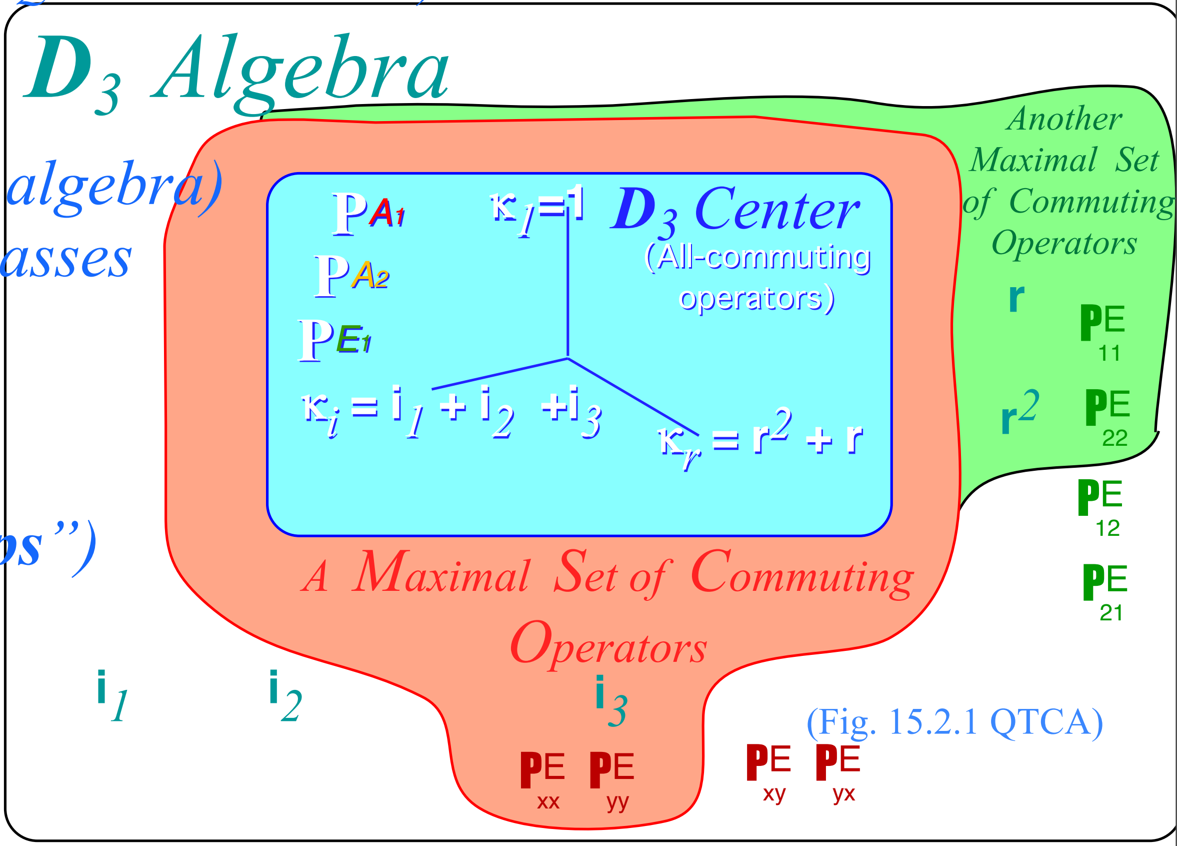
Group Theory in Quantum Mechanics

Lecture 15 *(3.12.15)*

revised(4.1.15)

Spectral decomposition of groups $D_3 \sim C_{3v}$

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)
(PSDS - Ch. 3)



(Fig. 15.2.1 QTCA)

*Review: 1st-Stage Spectral resolution of **D_3 Center** (Class algebra)*
- *Group theory of equivalence transformations and classes*
  - *Lagrange theorems*
- *All-commuting class projectors*

D_3-invariant character ortho-completeness

*Spectral resolution to **irreducible representations** (“**irreps**”) foretold by **characters** or <u>traces</u>*
- *Subgroup splitting or correlation frequency formula:*
  - *$f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*
  - *Atomic ℓ-level or $2\ell+1$-multiplet splitting*
    - *D_3 examples for ℓ=1-6*
- *Group invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3
- *Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*
  - *Splitting classes*

*3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions*
- *Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

Review:Spectral resolution of ***D_3 Center*** *(Class algebra)*

- *Group theory of equivalence transformations and classes*
    - *Lagrange theorems*
- *All-commuting class projectors and D_3-invariant character ortho-completeness*

Spectral resolution to ***irreducible representations*** *(or "****irreps****") foretold by* ***characters*** *or traces*

- *Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*
- *Atomic ℓ-level or $2\ell+1$-multiplet splitting*
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2nd-Stage spectral decompositions of global/local D_3

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3rd-stage spectral resolution to ***irreducible representations*** *(ireps) and Hamiltonian eigensolutions*

- *Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

Review: 1st-Stage Spectral resolution of D_3 Center (Class algebra)

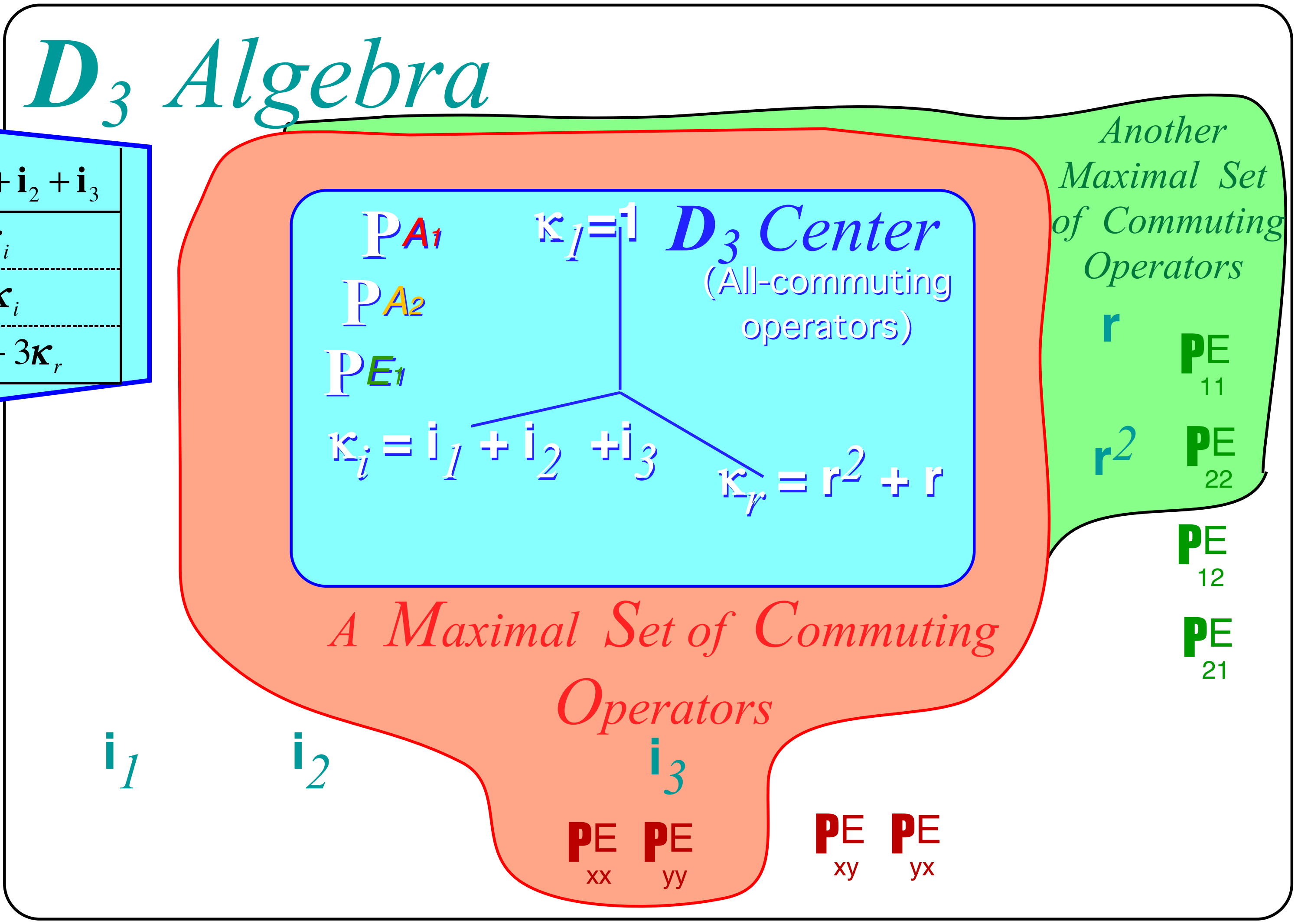
1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

→

	$\kappa_1 = 1$	$\kappa_r = r + r^2$	$\kappa_i = i_1 + i_2 + i_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum κ_k commutes with all g_t

Class-sum κ_k invariance: $g_t \kappa_k = \kappa_k g_t$



*Review: 1st-Stage Spectral resolution of **D3 Center** (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

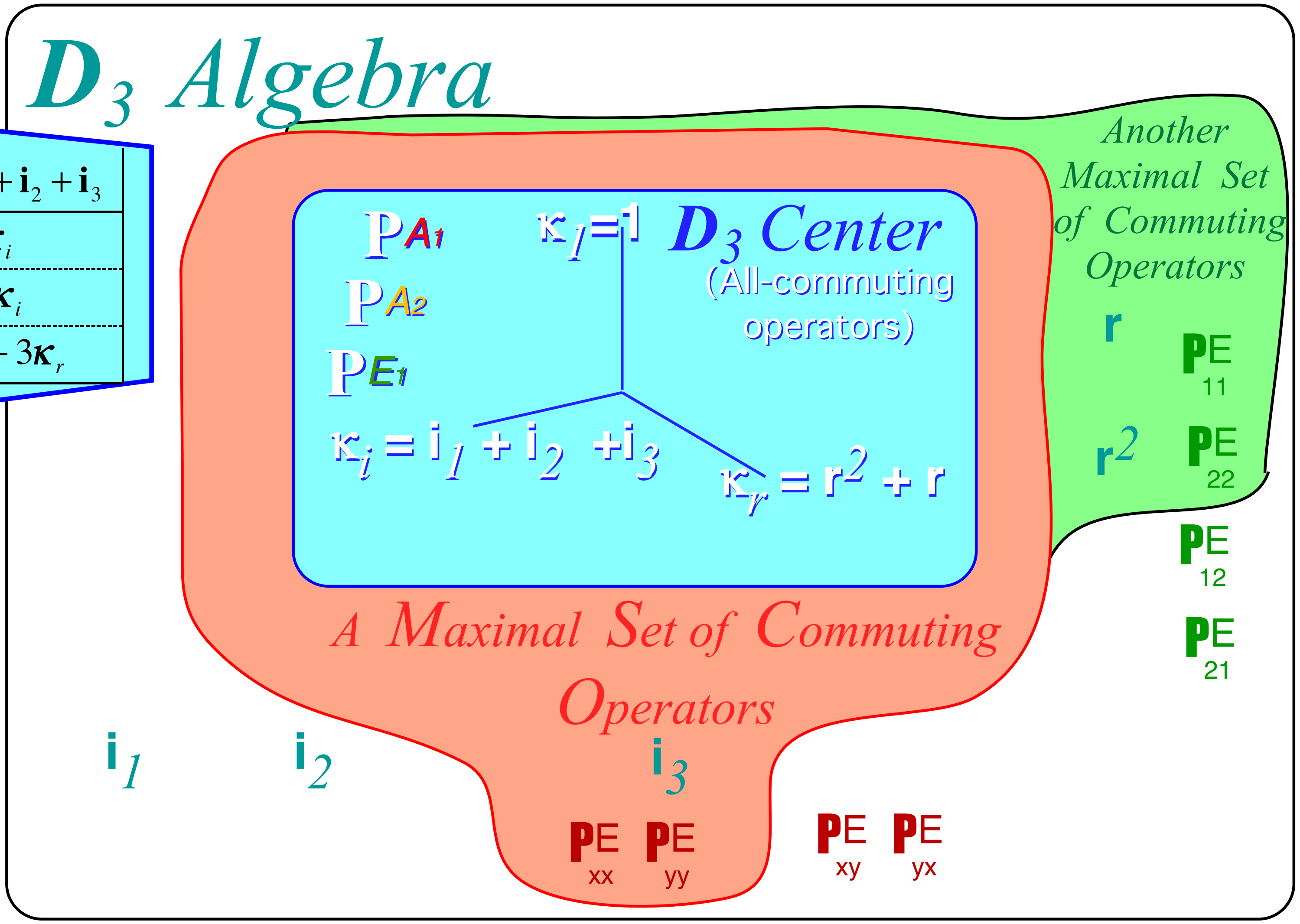
	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G =$ order of group: $(°D_3 = 6)$

$°\kappa_k =$ order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$



Review: 1^{st}-Stage Spectral resolution of **D_3 Center** (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

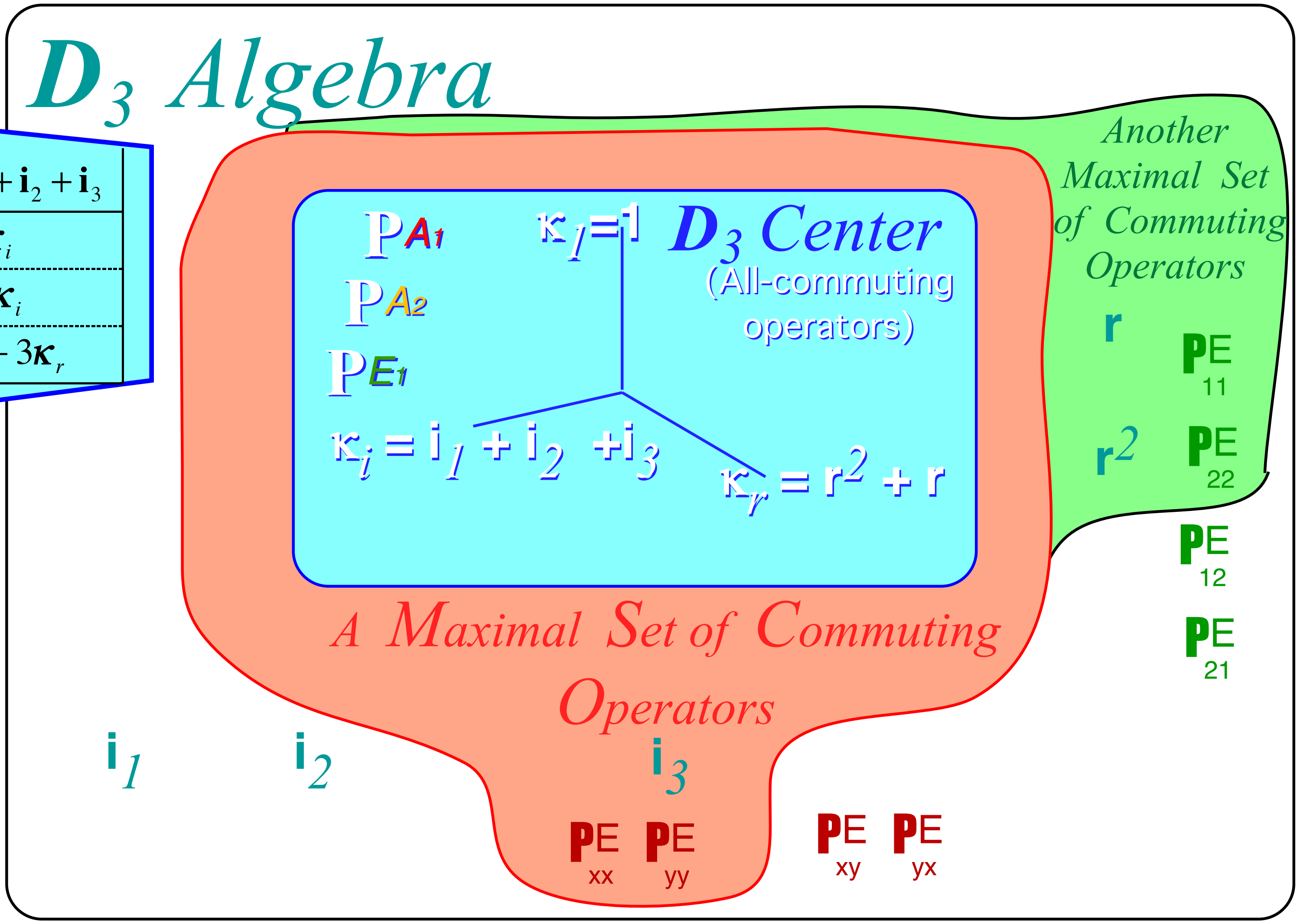
Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$^\circ G =$ order of group: $(^\circ D_3 = 6)$

$^\circ \kappa_k =$ order of class κ_k: $(^\circ \kappa_1 = 1,\ ^\circ \kappa_r = 2,\ ^\circ \kappa_i = 3)$

$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j={}^\circ\kappa_k} \mathbf{g}_j$$



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$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\boldsymbol{\kappa}_1=\mathbf{1}$	$\boldsymbol{\kappa}_r=\mathbf{r}+\mathbf{r}^2$	$\boldsymbol{\kappa}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_i$	$\boldsymbol{\kappa}_i$	$2\boldsymbol{\kappa}_i$	$3\boldsymbol{\kappa}_1+3\boldsymbol{\kappa}_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

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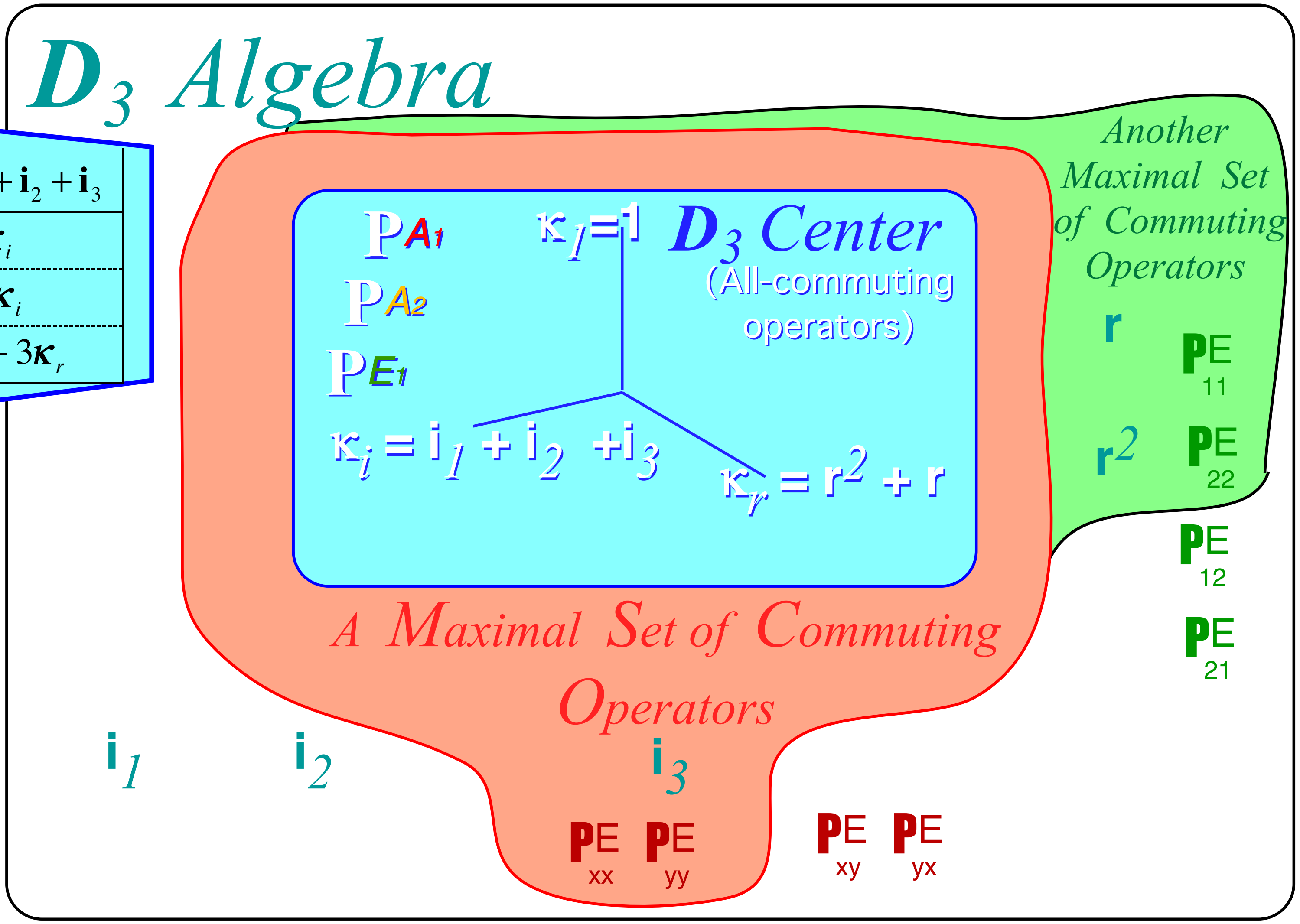
$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j={}^\circ\kappa_k}\mathbf{g}_j = \frac{1}{{}^\circ s_k}\sum_{t=1}^{t={}^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

${}^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $({}^\circ s_1 = 6,\ {}^\circ s_r = 3,\ {}^\circ s_i = 2)$



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$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

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κ_1	κ_1	κ_r	κ_i
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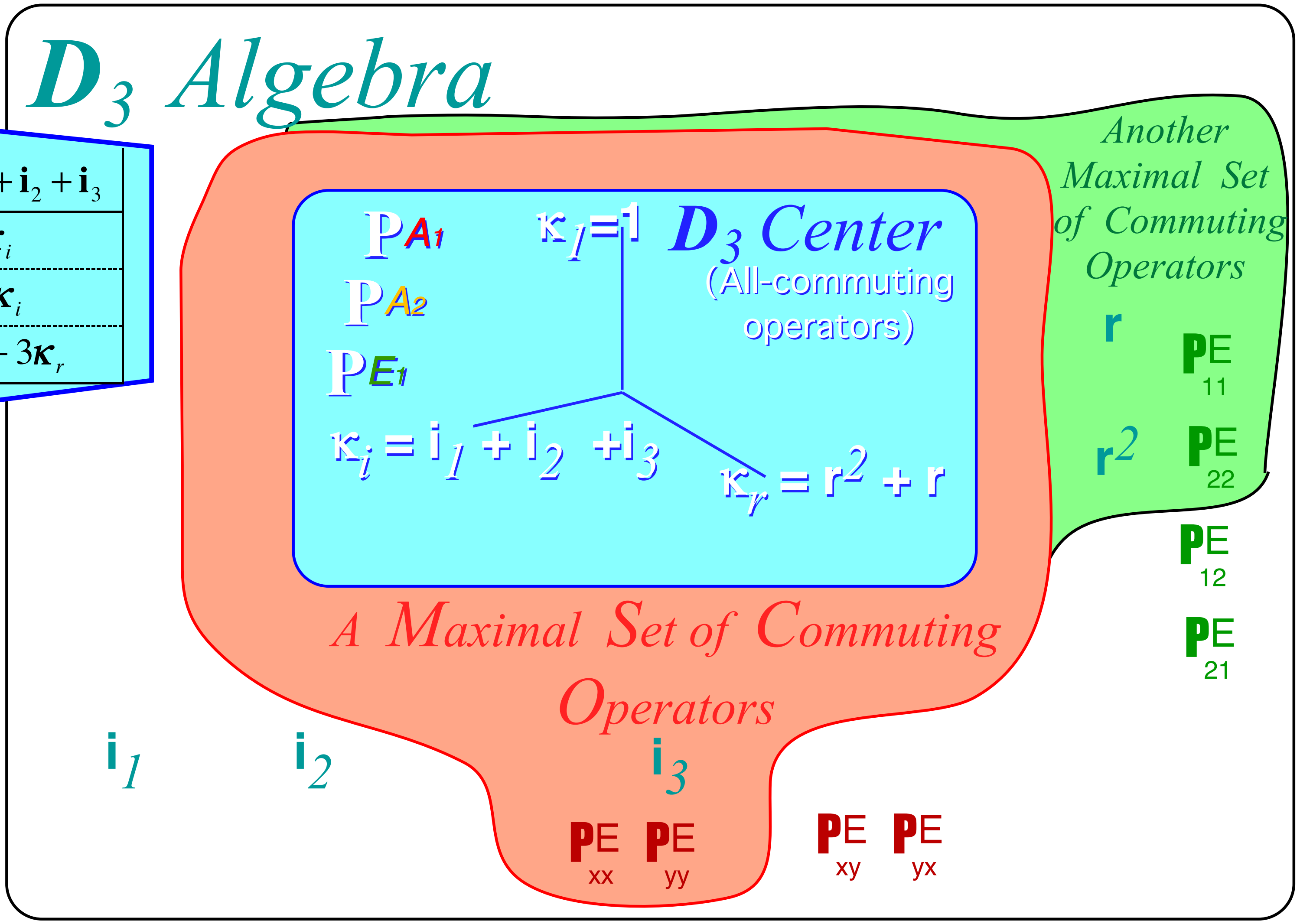

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Group theory of equivalence transformations and classes

➔ *Lagrange theorems* ←

All-commuting class projectors and D_3-invariant character ortho-completeness

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$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

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$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_i$	$\boldsymbol{\kappa}_i$	$2\boldsymbol{\kappa}_i$	$3\boldsymbol{\kappa}_1 + 3\boldsymbol{\kappa}_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

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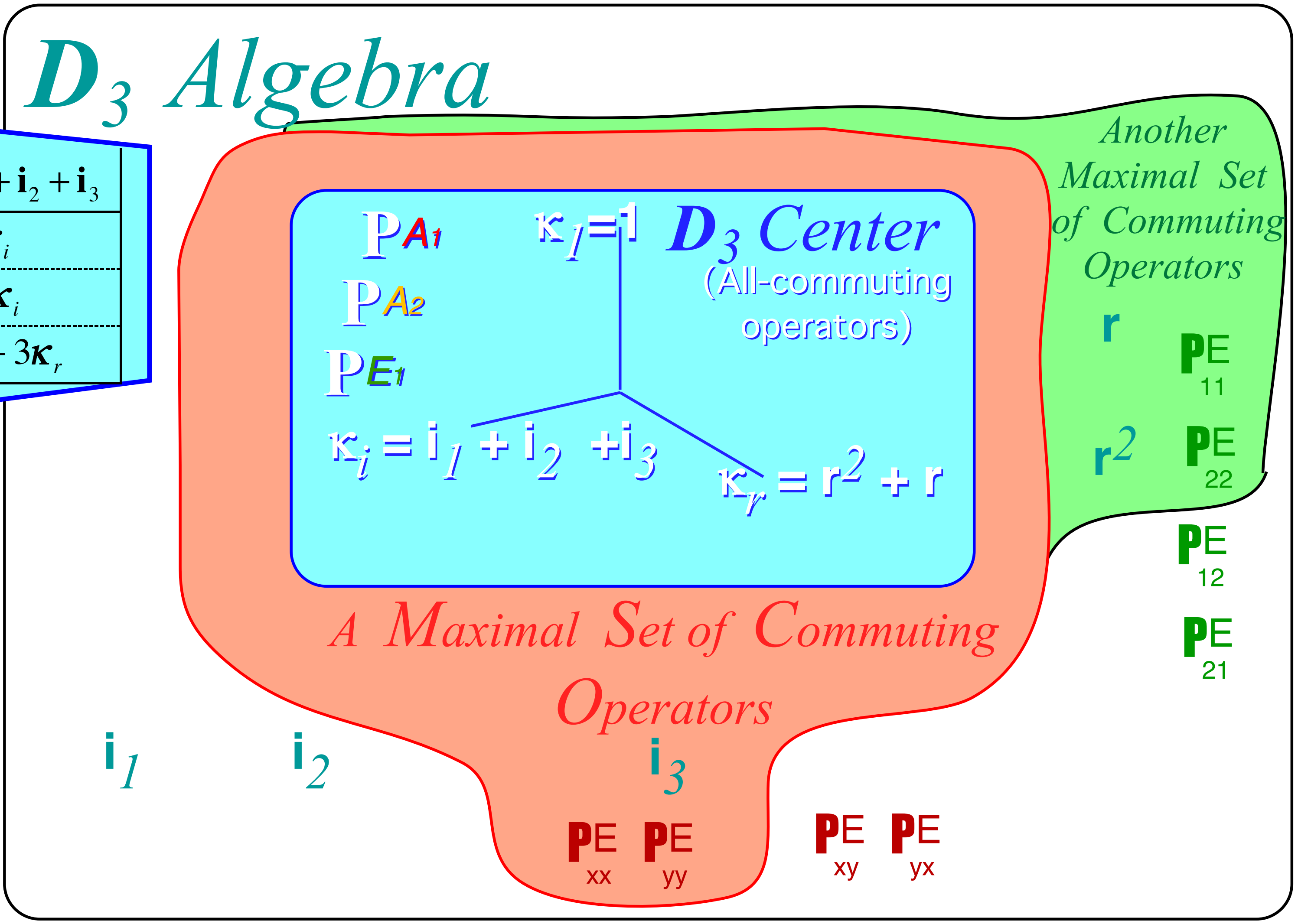
$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry:$(^\circ s_1 = 6,\ ^\circ s_r = 3,\ ^\circ s_i = 2)$

$^\circ s_k = {^\circ G} / {^\circ\kappa_k}$ *$^\circ s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*



Review: 1st-Stage Spectral resolution of ***D₃ Center*** *(Lagrange subgroup/class theorems)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum κ_k *commutes with all* $\mathbf{g}_t$

Class-sum κ_k invariance: $\mathbf{g}_t\kappa_k = \kappa_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

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$$\mathbf{g}_t\kappa_k\mathbf{g}_t^{-1} = \kappa_k \quad \text{where: } \kappa_k = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k}\sum_{t=1}^{t=°G} \mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

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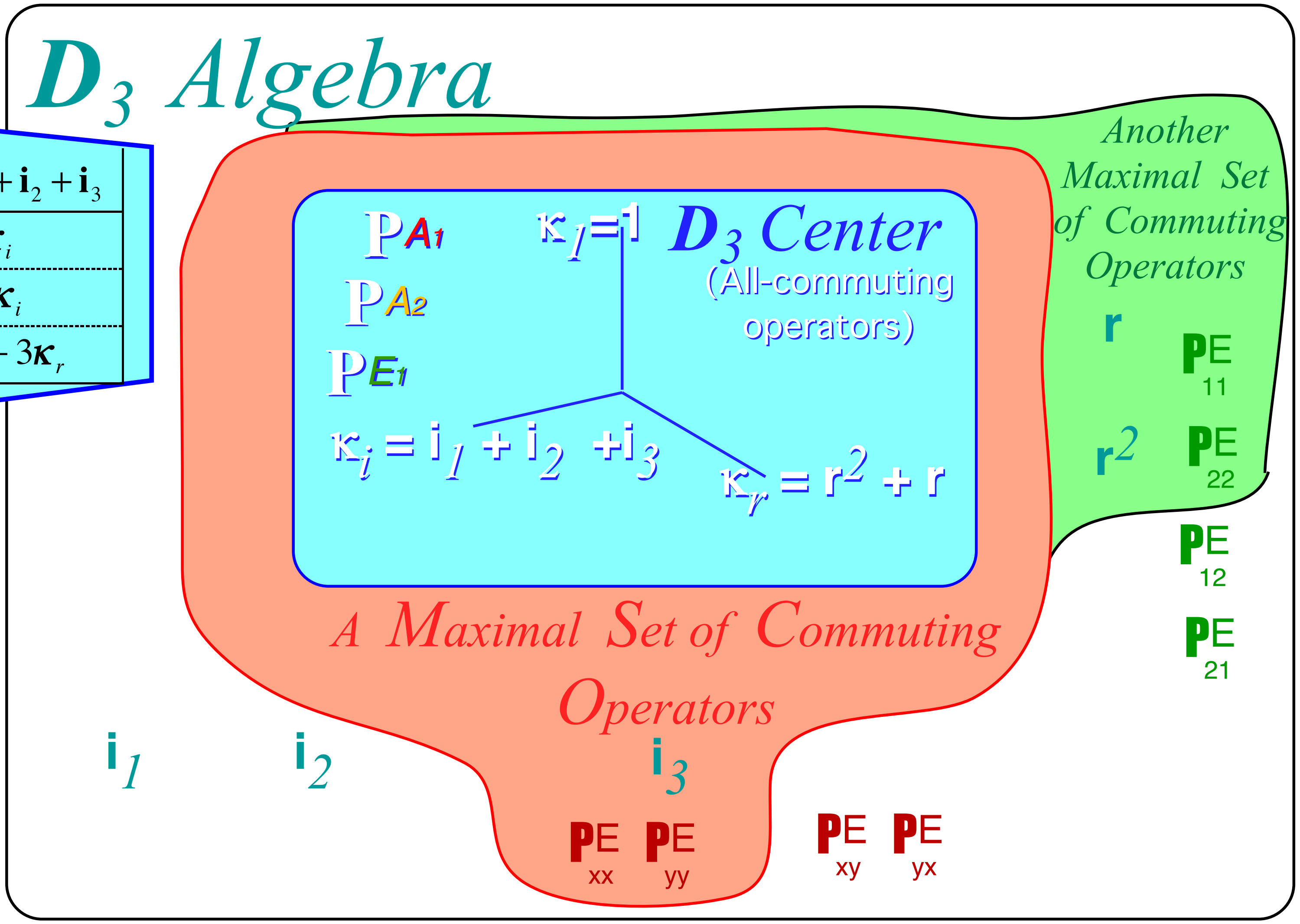
These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k . Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1} = \mathbf{g}_k$



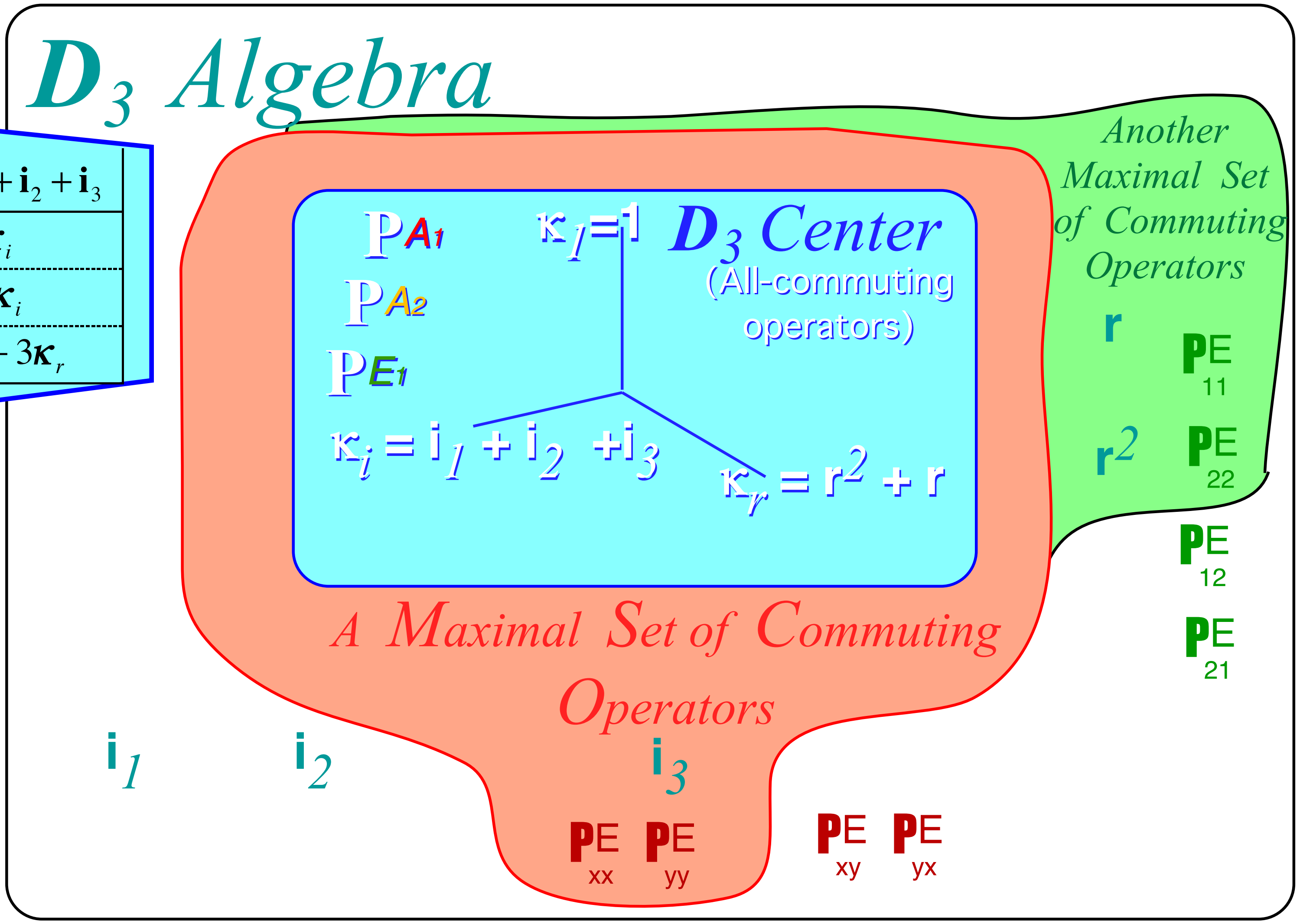
Review: *1^{st}-Stage Spectral resolution of* ***D_3 Center*** *(Lagrange subgroup/class theorems)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r}+\mathbf{r}^2$	$\kappa_i = \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$



Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

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$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j = \frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

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These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k . Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.
that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,

⋮

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$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
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$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

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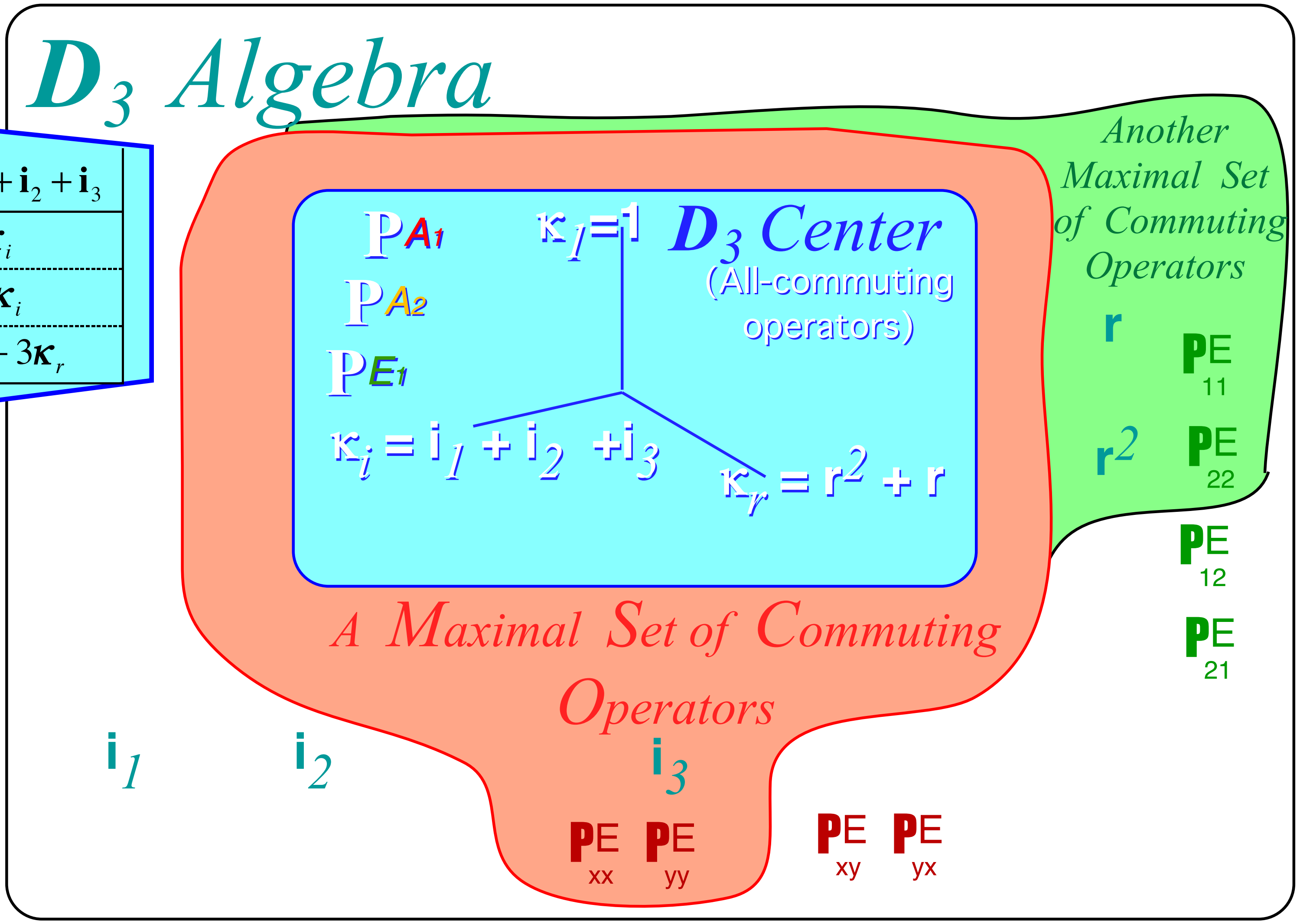
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum κ_k commutes with all $\mathbf{g}_t$



Class-sum κ_k invariance: $\mathbf{g}_t\kappa_k = \kappa_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t\kappa_k\mathbf{g}_t^{-1} = \kappa_k \quad \text{where: } \kappa_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j = \frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry:$(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G\,/\,°\kappa_k$ *$°s_k$ is an integer count of D₃ operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k . Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.

that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,

*Subgroup $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$ has $\ell=(°\kappa_k\text{-}1)$ **Left Cosets*** (one coset for each member of class κ_k).

Review: *1^{st}-Stage Spectral resolution of **D₃ Center** (Lagrange subgroup/class theorems)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum κ_k commutes with all $\mathbf{g}_t$

Class-sum κ_k invariance: $\mathbf{g}_t\kappa_k=\kappa_k\mathbf{g}_t$

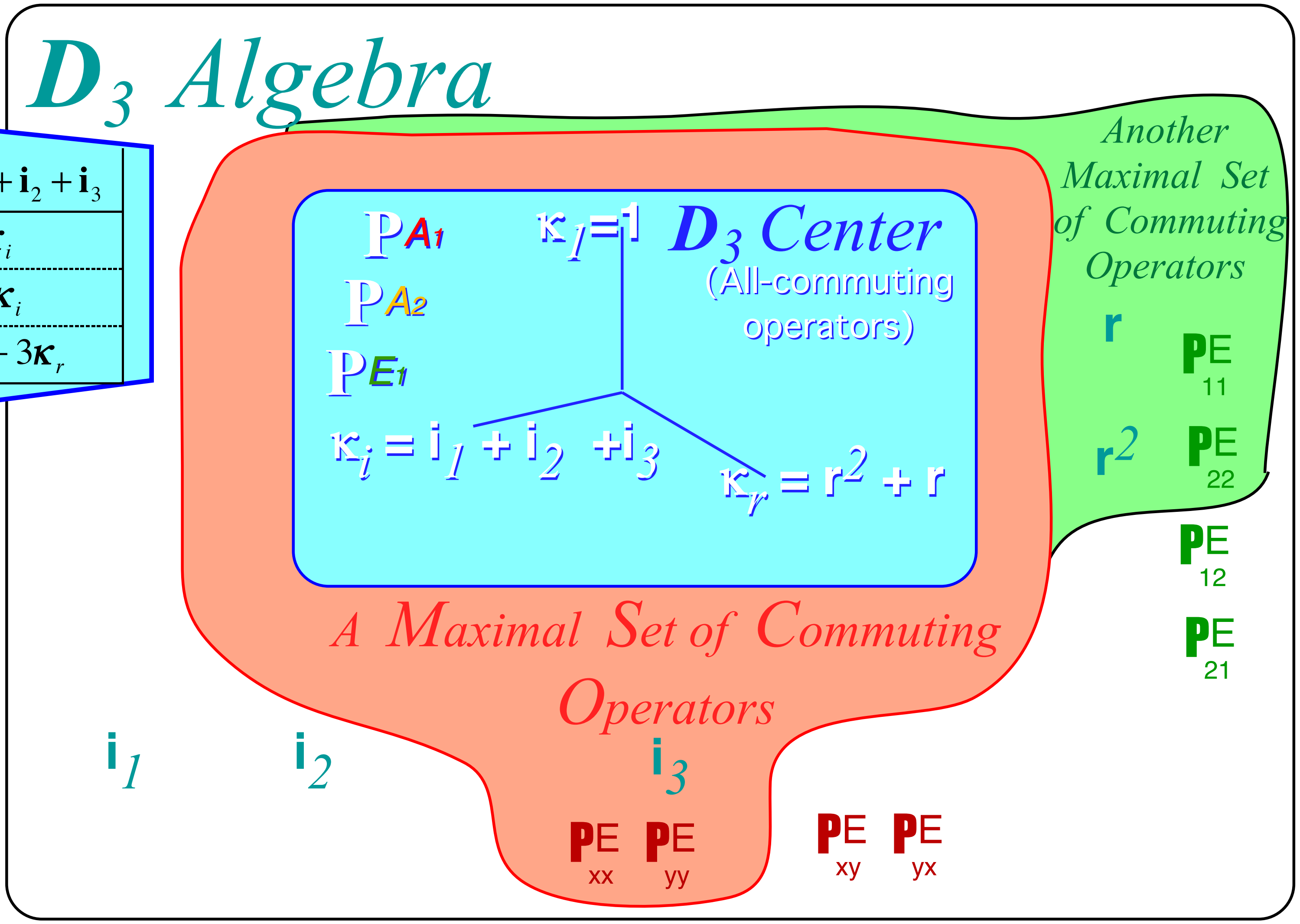
$^\circ G$ = order of group: $(^\circ D_3=6)$

$^\circ\kappa_k$ = order of class κ_k: $(^\circ\kappa_1=1,\ ^\circ\kappa_r=2,\ ^\circ\kappa_i=3)$

$$\mathbf{g}_t\kappa_k\mathbf{g}_t^{-1}=\kappa_k \quad \text{where: } \kappa_\mathbf{k}=\sum_{j=1}^{j=^\circ\kappa_k}\mathbf{g}_j=\frac{1}{^\circ s_k}\sum_{t=1}^{t=^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry:$(^\circ s_1=6,\ ^\circ s_r=3,\ ^\circ s_i=2)$

$^\circ s_k = {^\circ G}/{^\circ\kappa_k}$ *$^\circ s_k$ is an integer count of D₃ operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*



These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k . Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.

that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,

Subgroup $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$ has $\ell=(^\circ\kappa_k\text{-}1)$ ***Left Cosets*** (one coset for each member of class κ_k).

$$^\circ\kappa_k\begin{cases}\mathbf{g}_1\, s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\},\\ \mathbf{g}_2\, s_k=\mathbf{g}_2\underbrace{\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}}_{^\circ s_k},\ldots\end{cases}$$

They will divide the group of order $^\circ D_3={^\circ\kappa_k}\cdot{^\circ s_k}$ evenly into $^\circ\kappa_k$ subsets each of order $^\circ s_k$.

*Review: 1st-Stage Spectral resolution of **D_3 Center** (Lagrange subgroup/class theorems)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\boxed{\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry:$(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$\boxed{°s_k = °G / °\kappa_k}$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k . Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.

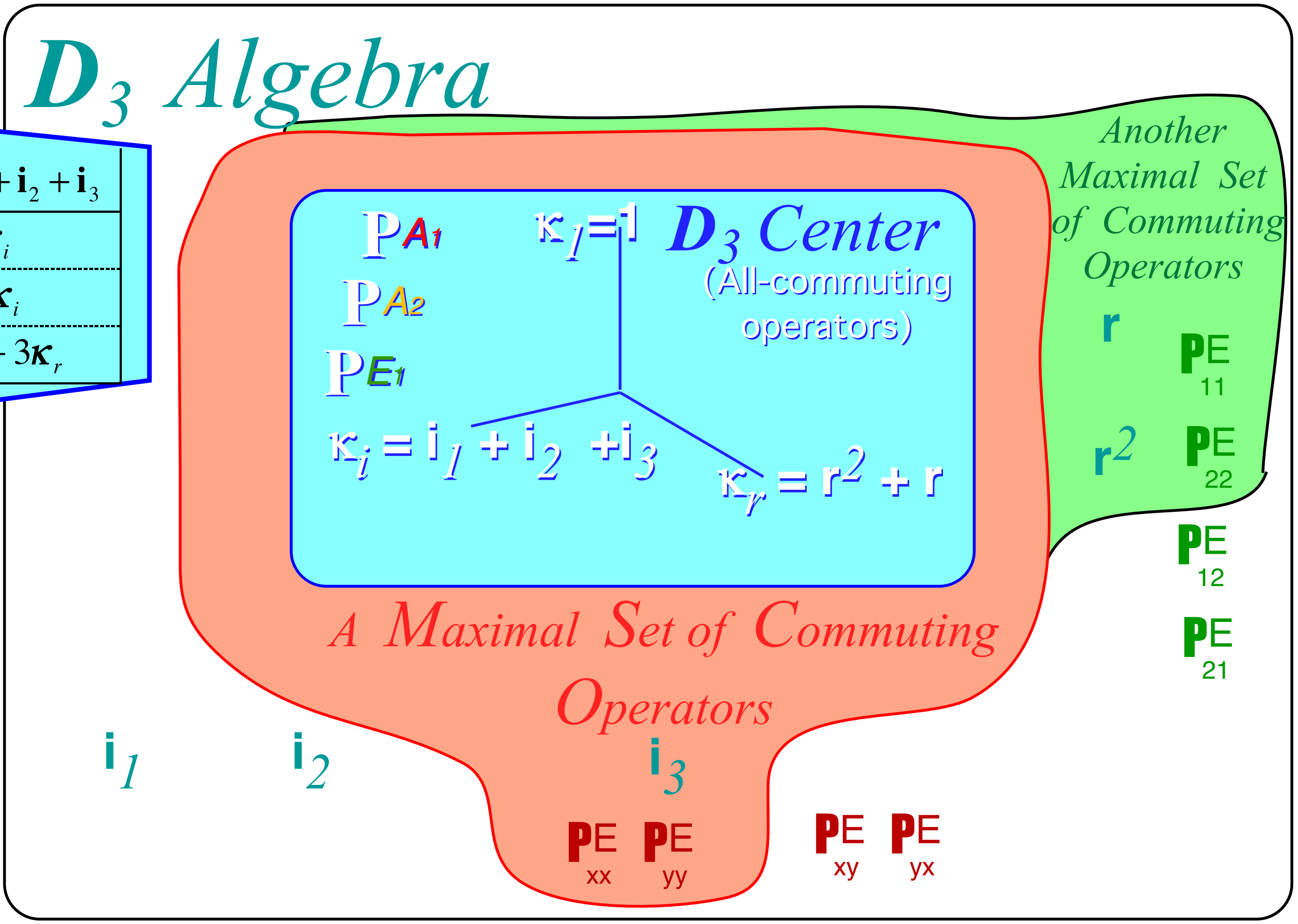
that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,

*Subgroup $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$ has $\ell=(°\kappa_k\text{-}1)$ **Left Cosets*** (one coset for each member of class κ_k).

$$°\kappa_k \left\{ \begin{array}{l} \mathbf{g}_1\, s_k = \mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}, \\ \mathbf{g}_2\, s_k = \mathbf{g}_2\{\underbrace{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots}_{°s_k}\},\ldots \\ \vdots \end{array} \right.$$

*These results are known as **Lagrange's Coset Theorem(s)***

They will divide the group of order $\boxed{°D_3 = °\kappa_k \cdot °s_k}$ evenly into $°\kappa_k$ subsets each of order $°s_k$.



Review:Spectral resolution of ***D_3 Center*** *(Class algebra)*
Group theory of equivalence transformations and classes
Lagrange theorems
All-commuting class projectors and D_3-invariant character ortho-completeness
Spectral resolution to ***irreducible representations*** *(or "**irreps**") foretold by* ***characters*** *or traces*
Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$
Atomic ℓ-level or $2\ell+1$-multiplet splitting
D_3 examples for ℓ=1-6
Group invariant numbers: Centrum, Rank, and Order

2nd-Stage spectral decompositions of global/local D_3
Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$

3rd-stage spectral resolution to ***irreducible representations*** *(ireps) and Hamiltonian eigensolutions*
Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains

Review: 1st-Stage Spectral resolution of ***D_3 Center*** *(All-commuting class projectors)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\boldsymbol{\kappa}_1 = \mathbf{1}$	$\boldsymbol{\kappa}_r = \mathbf{r} + \mathbf{r}^2$	$\boldsymbol{\kappa}_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_i$	$\boldsymbol{\kappa}_i$	$2\boldsymbol{\kappa}_i$	$3\boldsymbol{\kappa}_1 + 3\boldsymbol{\kappa}_r$

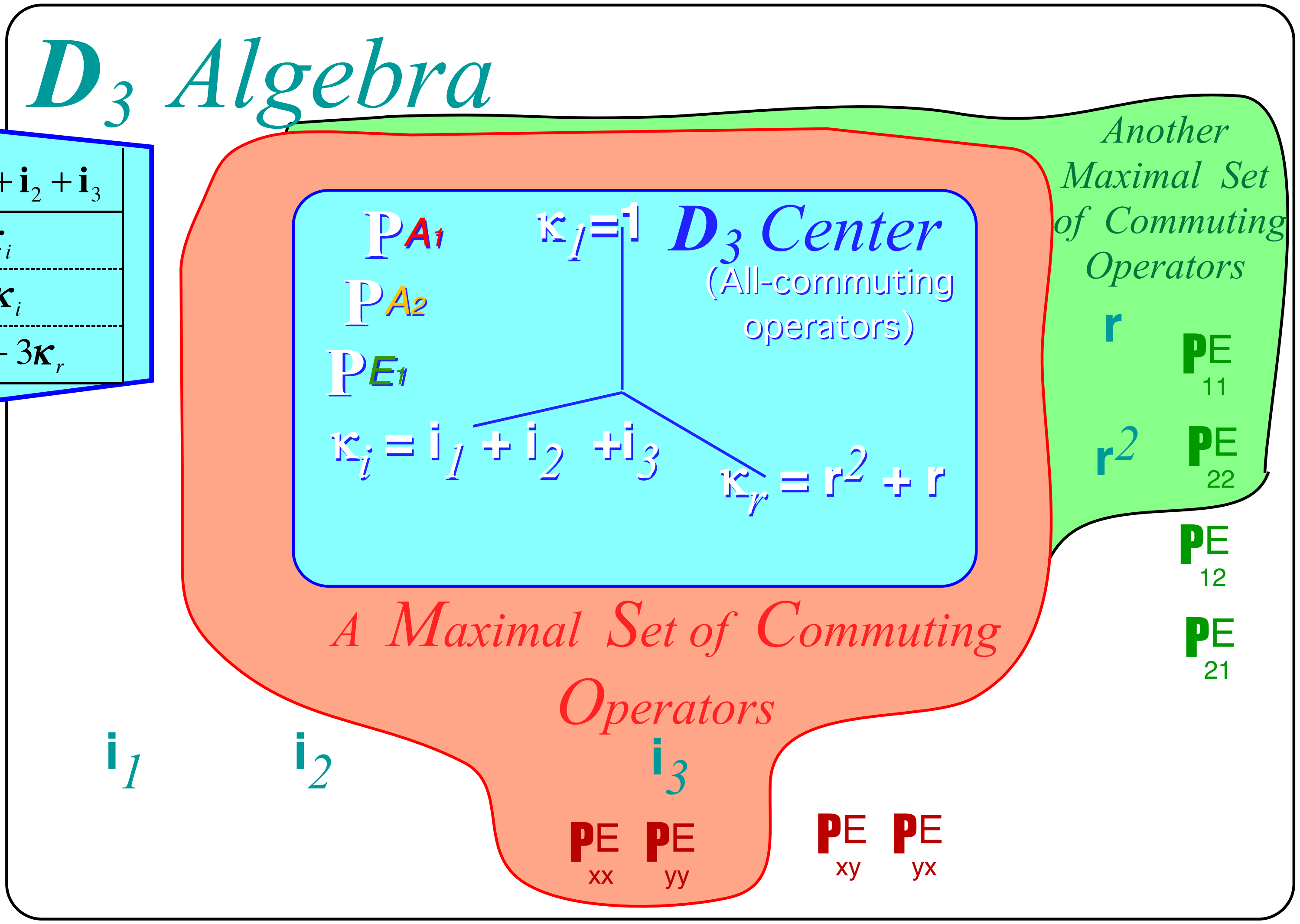
Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G =$ order of group: $(°D_3 = 6)$

$°\kappa_k =$ order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

Class minimal equation



Review: 1^{st}-Stage Spectral resolution of D_3 Center (All-commuting class projectors)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

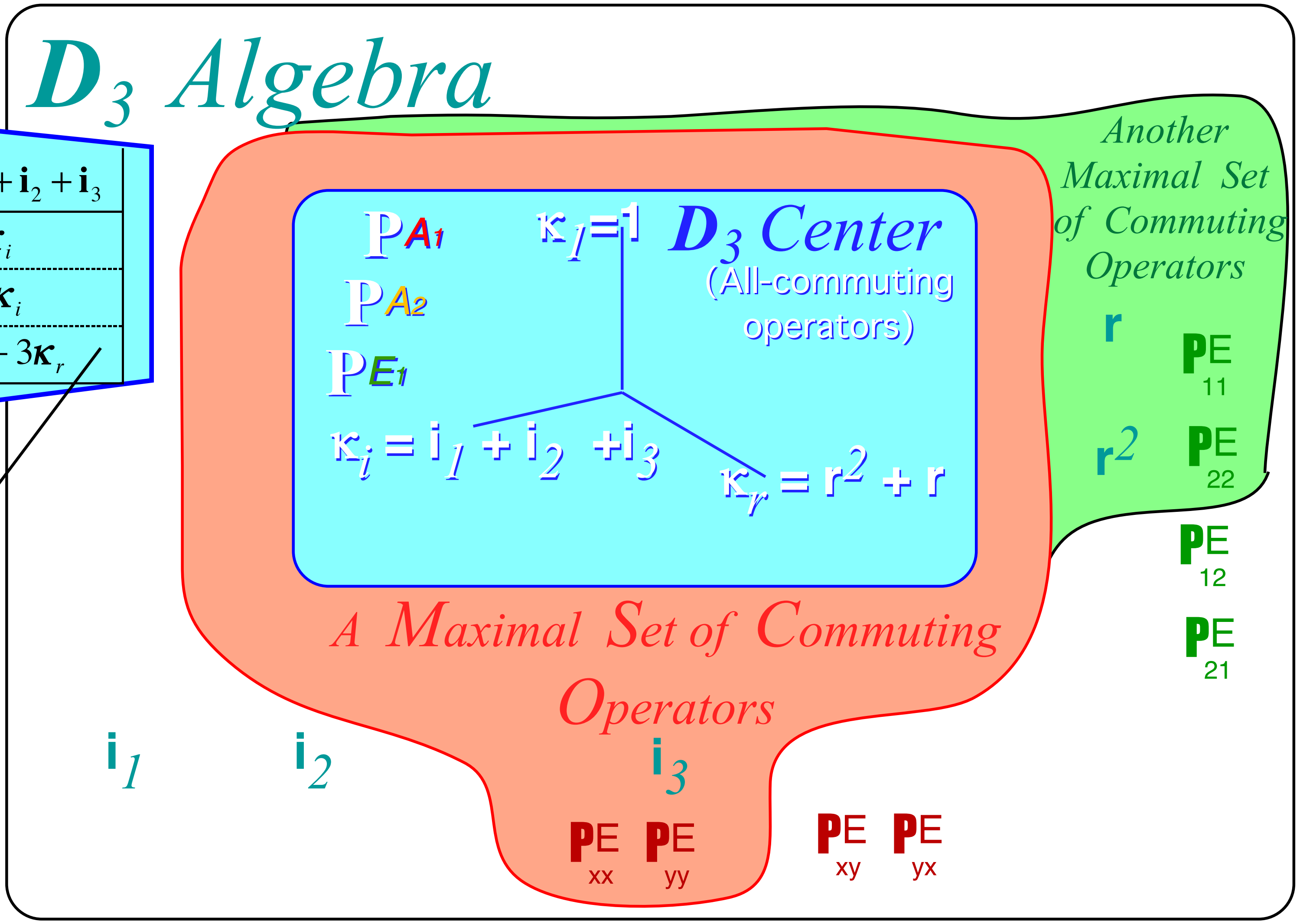
Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^2 = 3 \cdot \boldsymbol{\kappa}_r + 3 \cdot \mathbf{1}$$



*Review: 1st-Stage Spectral resolution of **D3 Center** (All-commuting class projectors)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

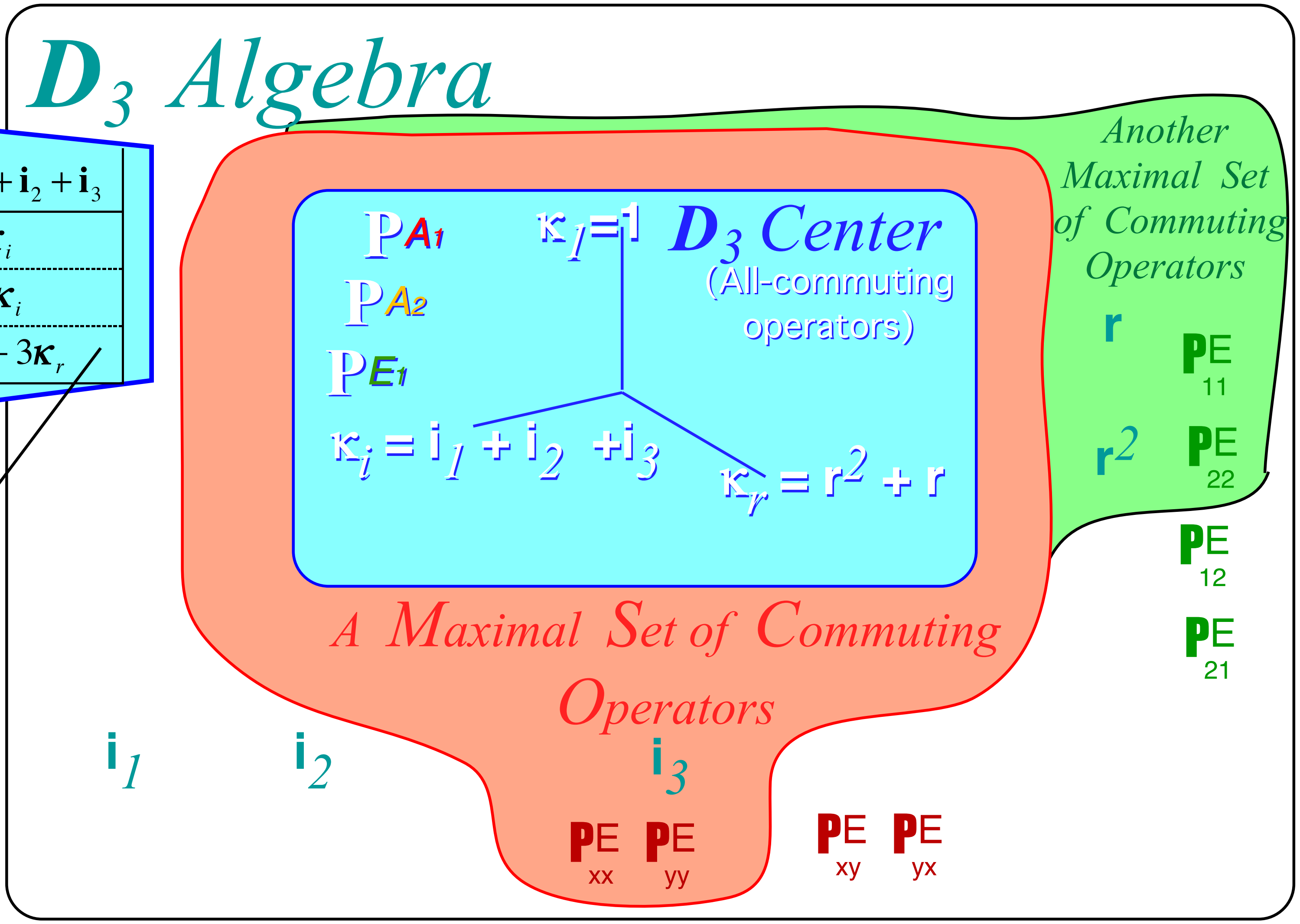
Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G =$ order of group: $(°D_3 = 6)$

$°\kappa_k =$ order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^3 = 3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i + 3\cdot\boldsymbol{\kappa}_i = 9\cdot\boldsymbol{\kappa}_i \leftarrow \boldsymbol{\kappa}_i^2 = 3\cdot\boldsymbol{\kappa}_r + 3\cdot\mathbf{1}$$



*Review: 1st-Stage Spectral resolution of **D3 Center** (All-commuting class projectors)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G =$ order of group: $(°D_3 = 6)$

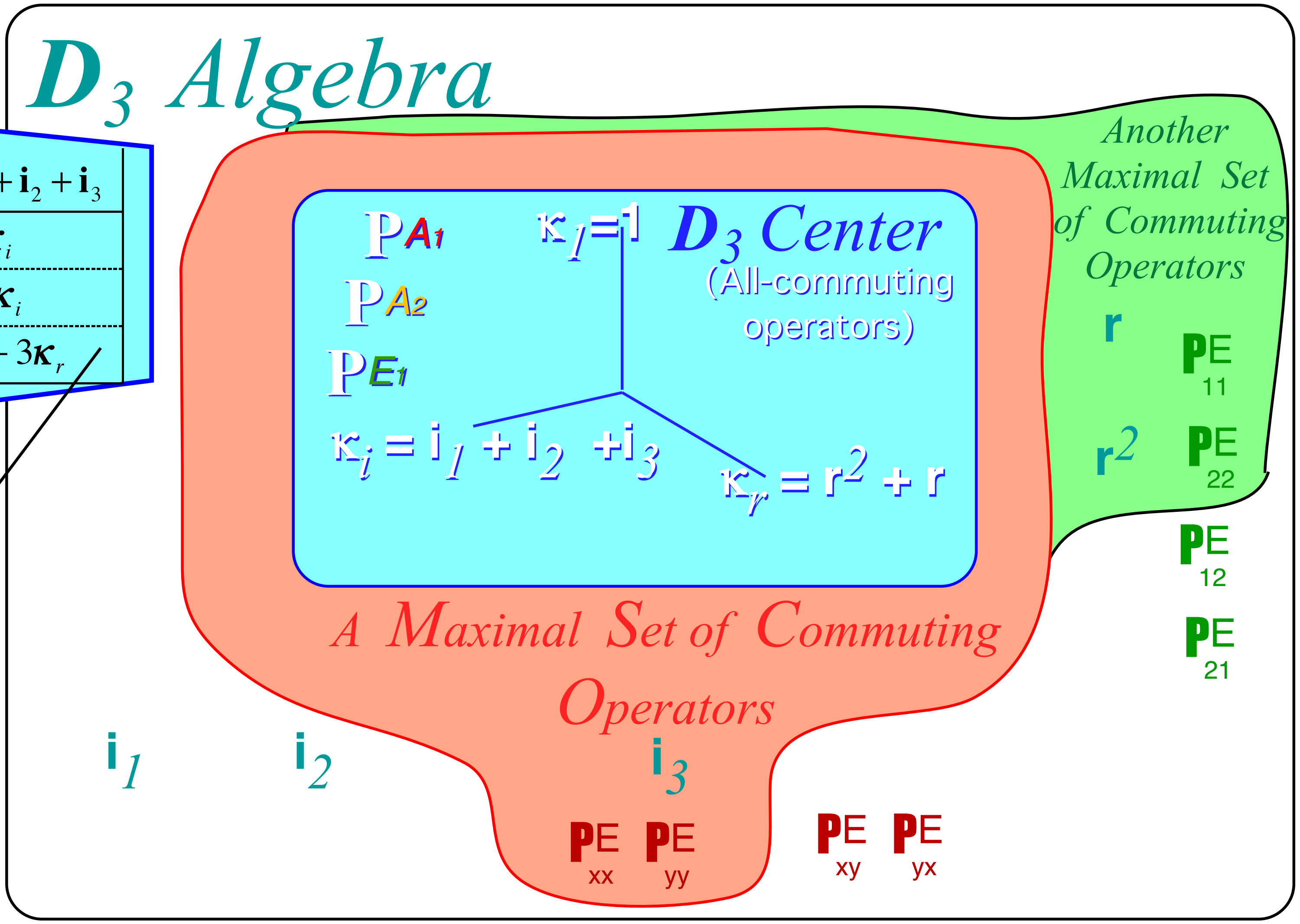
$°\kappa_k =$ order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^2 = 3\cdot\boldsymbol{\kappa}_r + 3\cdot\mathbf{1}$$

$$\boldsymbol{\kappa}_i^3 = 3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i + 3\cdot\boldsymbol{\kappa}_i = 9\cdot\boldsymbol{\kappa}_i$$

$$0 = \boldsymbol{\kappa}_i^3 - 9\cdot\boldsymbol{\kappa}_i = (\boldsymbol{\kappa}_i - 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i + 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i - 0\cdot\mathbf{1})$$



Review: 1^{st}-Stage Spectral resolution of $\mathbf{D_3}$ Center (All-commuting class projectors)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G =$ order of group: $(°D_3 = 6)$

$°\kappa_k =$ order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^2 = 3 \cdot \boldsymbol{\kappa}_r + 3 \cdot \mathbf{1}$$

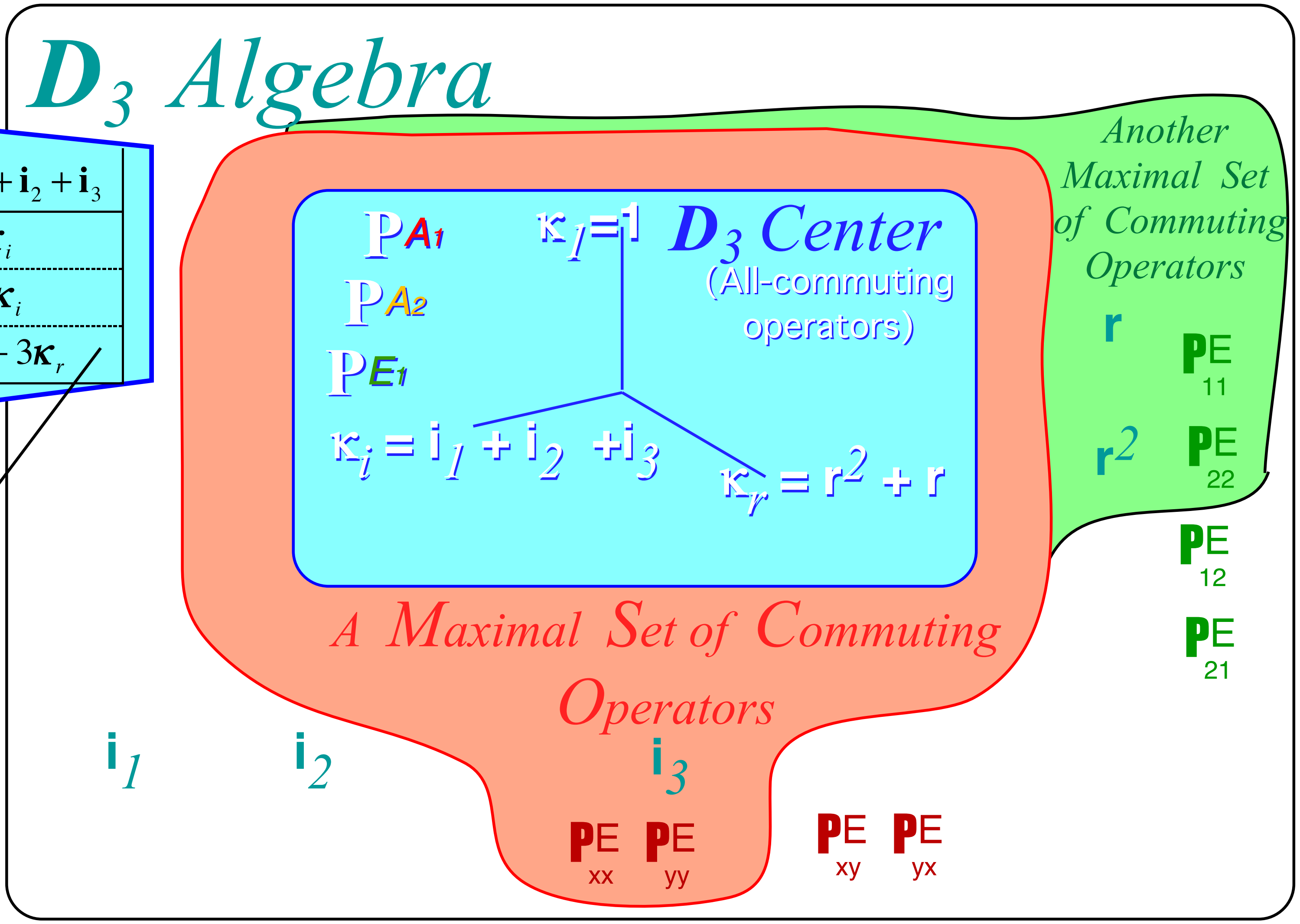
$$\boldsymbol{\kappa}_i^3 = 3 \cdot \boldsymbol{\kappa}_r \boldsymbol{\kappa}_i + 3 \cdot \boldsymbol{\kappa}_i = 9 \cdot \boldsymbol{\kappa}_i$$

$$0 = \boldsymbol{\kappa}_i^3 - 9 \cdot \boldsymbol{\kappa}_i = (\boldsymbol{\kappa}_i - 3 \cdot \mathbf{1})(\boldsymbol{\kappa}_i + 3 \cdot \mathbf{1})(\boldsymbol{\kappa}_i - 0 \cdot \mathbf{1})$$

$$\boldsymbol{\kappa}_1 = 1 \cdot \mathbf{P}^{A_1} + 1 \cdot \mathbf{P}^{A_2} + 1 \cdot \mathbf{P}^{E} = \mathbf{1} \quad \textit{(Completeness)}$$

$$\boldsymbol{\kappa}_r = 2 \cdot \mathbf{P}^{A_1} + 2 \cdot \mathbf{P}^{A_2} - 1 \cdot \mathbf{P}^{E} \longleftarrow \boldsymbol{\kappa}_r^2 = \boldsymbol{\kappa}_r + 2 \cdot \mathbf{1} \Rightarrow (\boldsymbol{\kappa}_r - 2 \cdot \mathbf{1})(\boldsymbol{\kappa}_r + \mathbf{1}) = \mathbf{0}$$

$$\boldsymbol{\kappa}_i = 3 \cdot \mathbf{P}^{A_1} - 3 \cdot \mathbf{P}^{A_2} + 0 \cdot \mathbf{P}^{E}$$



*Review: 1st-Stage Spectral resolution of **D3 Center** (All-commuting class projectors)*

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G =$ order of group: $(°D_3 = 6)$

$°\kappa_k =$ order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^2 = 3 \cdot \boldsymbol{\kappa}_r + 3 \cdot \mathbf{1}$$

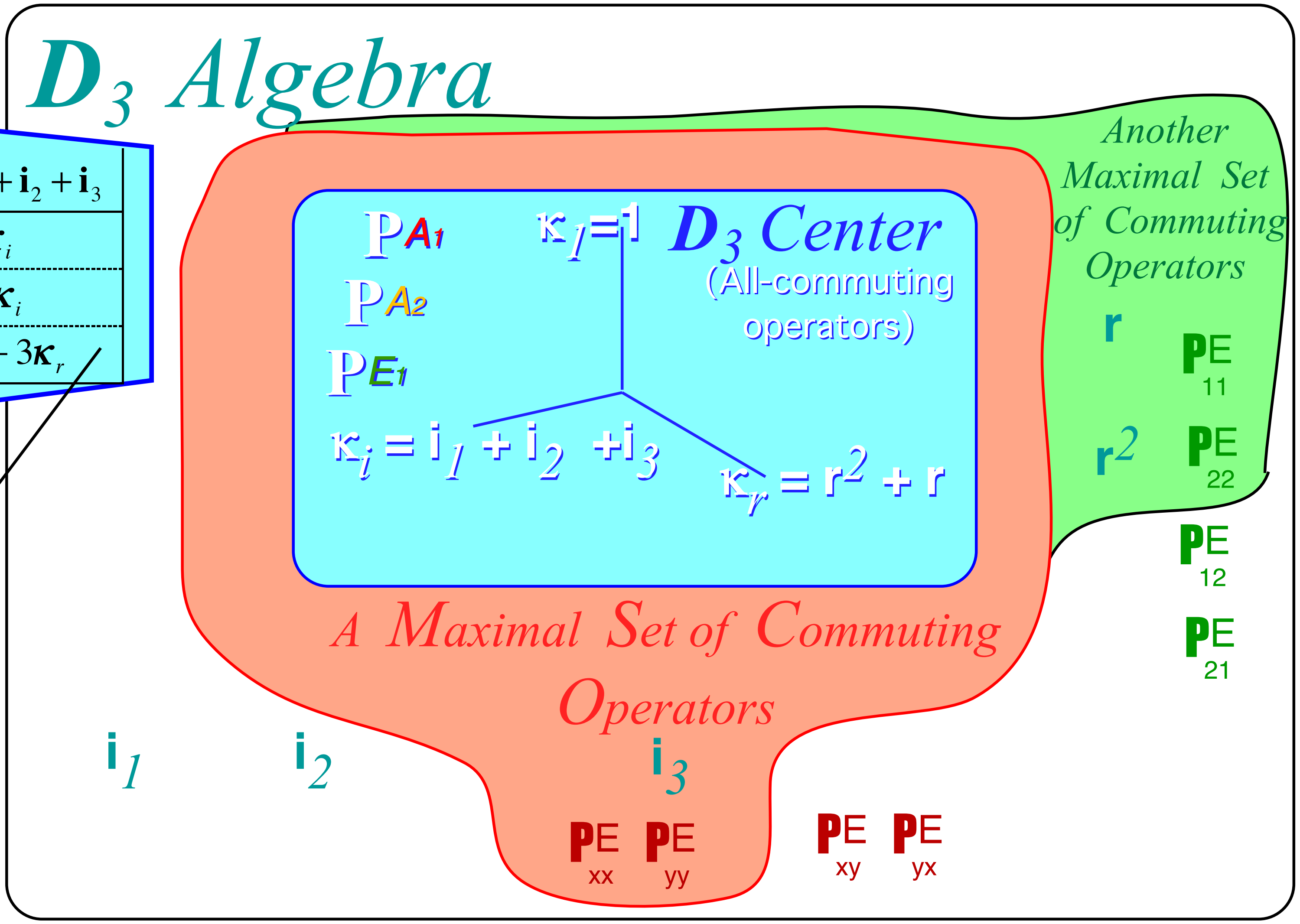
$$\boldsymbol{\kappa}_i^3 = 3 \cdot \boldsymbol{\kappa}_r \boldsymbol{\kappa}_i + 3 \cdot \boldsymbol{\kappa}_i = 9 \cdot \boldsymbol{\kappa}_i$$

$$0 = \boldsymbol{\kappa}_i^3 - 9 \cdot \boldsymbol{\kappa}_i = (\boldsymbol{\kappa}_i - 3 \cdot \mathbf{1})(\boldsymbol{\kappa}_i + 3 \cdot \mathbf{1})(\boldsymbol{\kappa}_i - 0 \cdot \mathbf{1})$$

$$\boldsymbol{\kappa}_1 = 1 \cdot \mathbf{P}^{A_1} + 1 \cdot \mathbf{P}^{A_2} + 1 \cdot \mathbf{P}^{E} = \mathbf{1} \quad \textit{(Completeness)}$$

$$\boldsymbol{\kappa}_r = 2 \cdot \mathbf{P}^{A_1} + 2 \cdot \mathbf{P}^{A_2} - 1 \cdot \mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i = 3 \cdot \mathbf{P}^{A_1} - 3 \cdot \mathbf{P}^{A_2} + 0 \cdot \mathbf{P}^{E}$$



$$\mathbf{P}^{A_1} = \frac{(\boldsymbol{\kappa}_i + 3 \cdot \mathbf{1})(\boldsymbol{\kappa}_i - 0 \cdot \mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\boldsymbol{\kappa}_i - 3 \cdot \mathbf{1})(\boldsymbol{\kappa}_i - 0 \cdot \mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\boldsymbol{\kappa}_i - 3 \cdot \mathbf{1})(\boldsymbol{\kappa}_i + 3 \cdot \mathbf{1})}{(+0-3)(+0+3)}$$

*Review: 1st-Stage Spectral resolution of **D3 Center** (All-commuting class projectors)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3=6)$

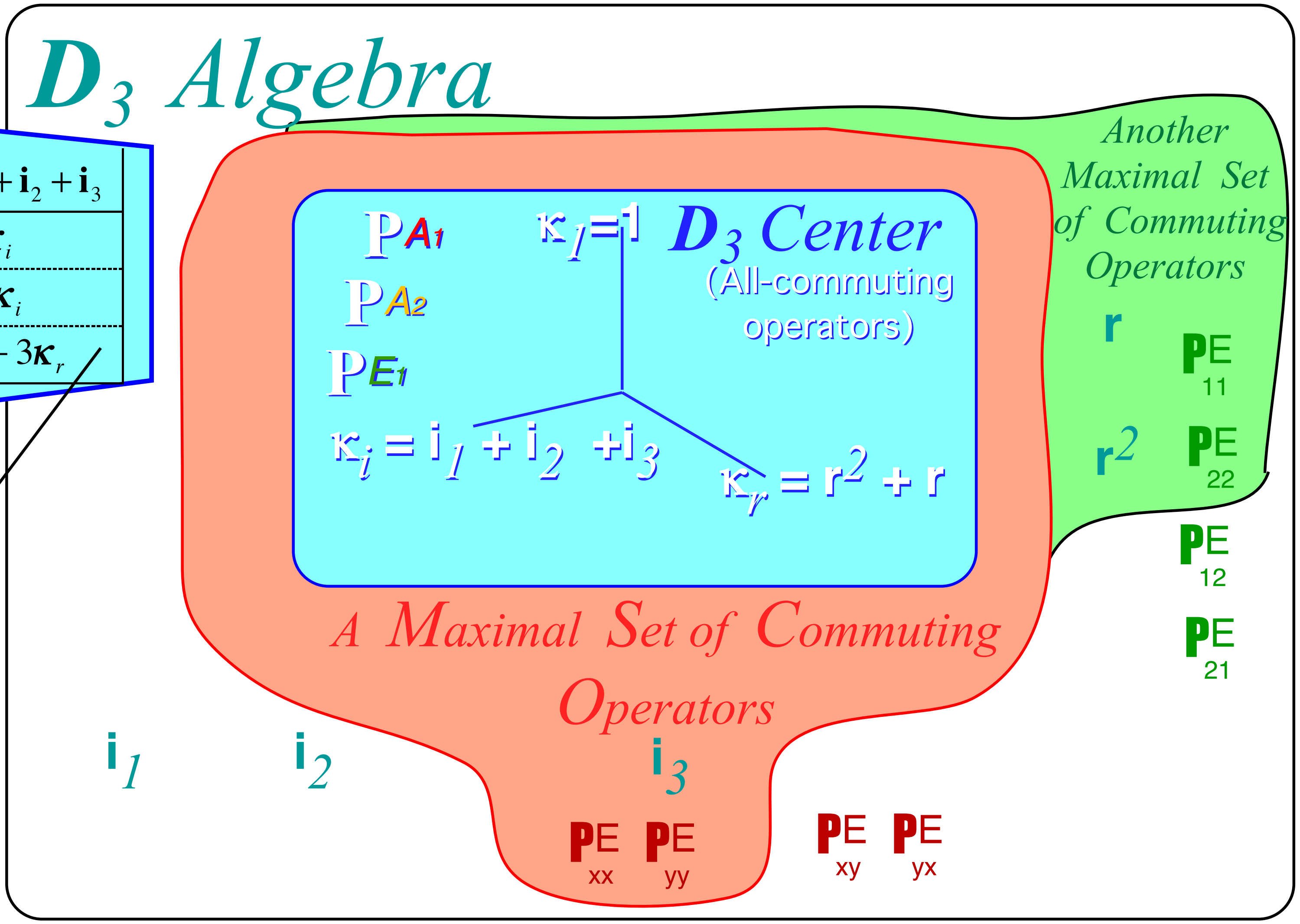
$°\kappa_k$ = order of class κ_k: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^2=3\cdot\boldsymbol{\kappa}_r+3\cdot\mathbf{1}$$

$$\boldsymbol{\kappa}_i^3=3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i+3\cdot\boldsymbol{\kappa}_i=9\cdot\boldsymbol{\kappa}_i$$

$$0=\boldsymbol{\kappa}_i^3-9\cdot\boldsymbol{\kappa}_i=(\boldsymbol{\kappa}_i-3\cdot\mathbf{1})(\boldsymbol{\kappa}_i+3\cdot\mathbf{1})(\boldsymbol{\kappa}_i-0\cdot\mathbf{1})$$



Class ortho-complete projector relations

$$\boldsymbol{\kappa}_1=1\cdot\mathbf{P}^{A_1}+1\cdot\mathbf{P}^{A_2}+1\cdot\mathbf{P}^{E}=\mathbf{1}\quad\textit{(Completeness)}$$

$$\boldsymbol{\kappa}_r=2\cdot\mathbf{P}^{A_1}+2\cdot\mathbf{P}^{A_2}-1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i=3\cdot\mathbf{P}^{A_1}-3\cdot\mathbf{P}^{A_2}+0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r+\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

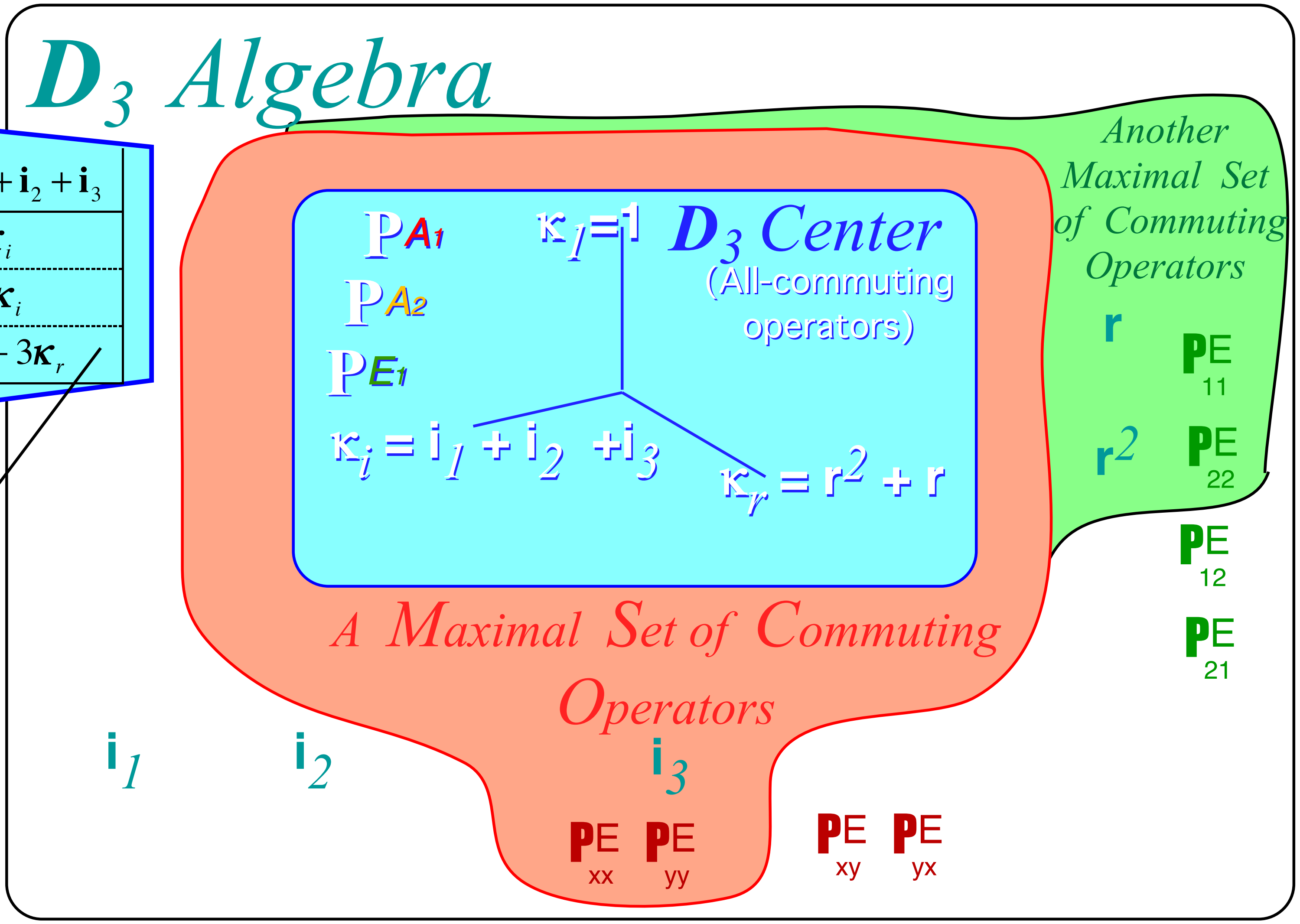
$$\mathbf{P}^{E}=(2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_r+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$$

Review: 1^{st}-Stage Spectral resolution of $\mathbf{D_3}$ Center (All-commuting class projectors)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r}+\mathbf{r}^2$	$\kappa_i = \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^2 = 3\cdot\boldsymbol{\kappa}_r + 3\cdot\mathbf{1}$$

$$\boldsymbol{\kappa}_i^3 = 3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i + 3\cdot\boldsymbol{\kappa}_i = 9\cdot\boldsymbol{\kappa}_i$$

$$0 = \boldsymbol{\kappa}_i^3 - 9\cdot\boldsymbol{\kappa}_i = (\boldsymbol{\kappa}_i - 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i + 3\cdot\mathbf{1})(\boldsymbol{\kappa}_i - 0\cdot\mathbf{1})$$

$$\boldsymbol{\kappa}_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E} = \mathbf{1} \quad \textit{(Completeness)}$$

$$\boldsymbol{\kappa}_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_r + \boldsymbol{\kappa}_i)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_r - \boldsymbol{\kappa}_i)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_r + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

Class ortho-complete projector relations

$$\boldsymbol{\kappa}_k = \sum_{(\alpha)} \frac{°\kappa_k \chi_k^{(\alpha)}}{\ell^{(\alpha)}} \mathbf{P}^{(\alpha)}$$

$$\mathbf{P}^{(\alpha)} = \frac{\ell^{(\alpha)}}{°G} \sum_k \chi_k^{(\alpha)*} \boldsymbol{\kappa}_k = \frac{\ell^{(\alpha)}}{°G} \sum_{g=1}^{°G} \chi_g^{(\alpha)*} \mathbf{g}$$

χ_k^{α}	χ_1^{α}	χ_r^{α}	χ_i^{α}
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	-1
$\alpha = E$	2	-1	0

Review: 1^{st}-Stage Spectral resolution of D_3 Center (All-commuting class projectors)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\boldsymbol{\kappa}_1=\mathbf{1}$	$\boldsymbol{\kappa}_r=\mathbf{r}+\mathbf{r}^2$	$\boldsymbol{\kappa}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_i$	$\boldsymbol{\kappa}_i$	$2\boldsymbol{\kappa}_i$	$3\boldsymbol{\kappa}_1+3\boldsymbol{\kappa}_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3=6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

Class minimal equation

$$\boldsymbol{\kappa}_i^2=3\cdot\boldsymbol{\kappa}_r+3\cdot\mathbf{1}$$

$$\boldsymbol{\kappa}_i^3=3\cdot\boldsymbol{\kappa}_r\boldsymbol{\kappa}_i+3\cdot\boldsymbol{\kappa}_i=9\cdot\boldsymbol{\kappa}_i$$

$$0=\boldsymbol{\kappa}_i^3-9\cdot\boldsymbol{\kappa}_i=(\boldsymbol{\kappa}_i-3\cdot\mathbf{1})(\boldsymbol{\kappa}_i+3\cdot\mathbf{1})(\boldsymbol{\kappa}_i-0\cdot\mathbf{1})$$

$$\boldsymbol{\kappa}_1=1\cdot\mathbf{P}^{A_1}+1\cdot\mathbf{P}^{A_2}+1\cdot\mathbf{P}^{E}=\mathbf{1}\quad\textit{(Completeness)}$$

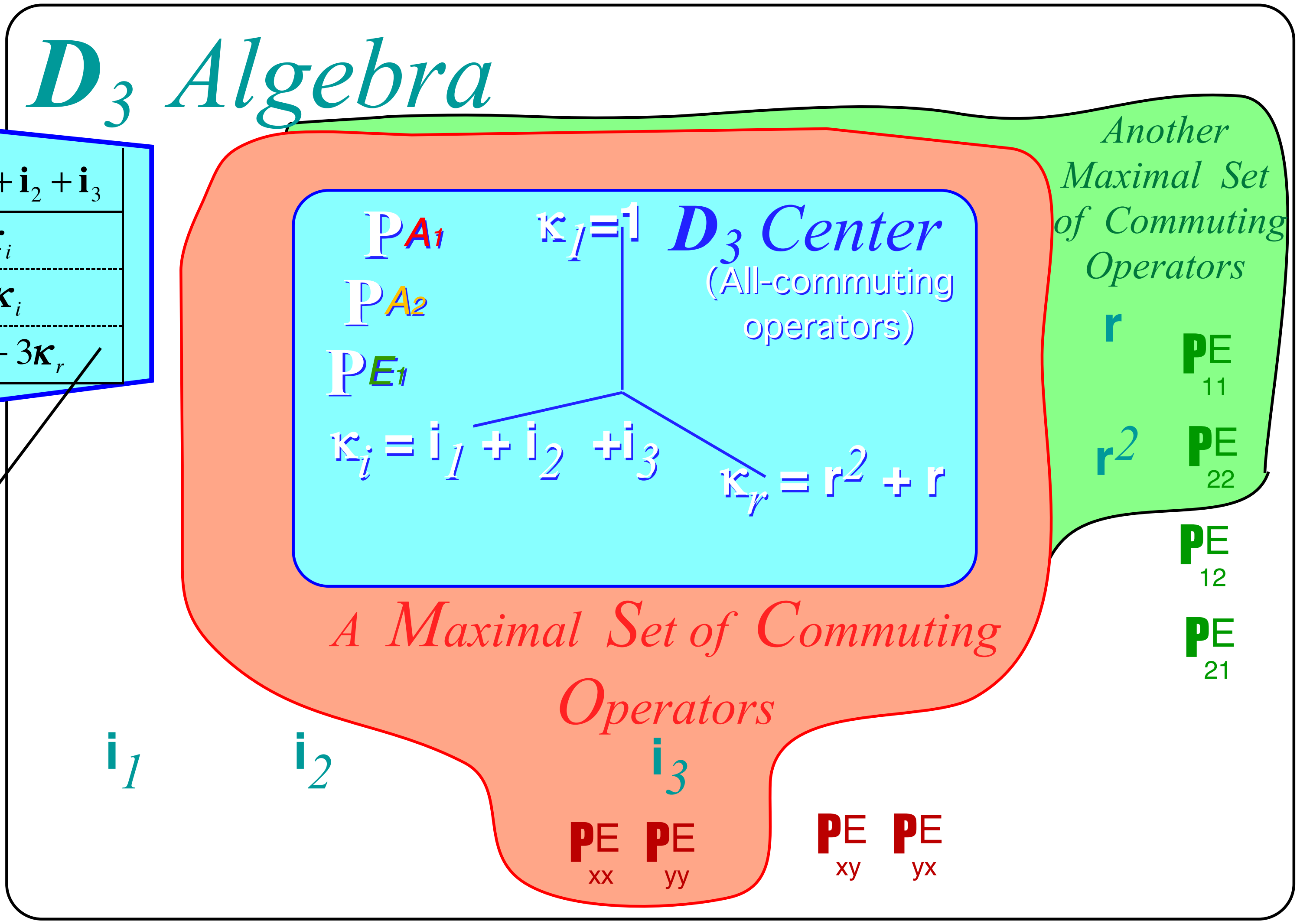
$$\boldsymbol{\kappa}_r=2\cdot\mathbf{P}^{A_1}+2\cdot\mathbf{P}^{A_2}-1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i=3\cdot\mathbf{P}^{A_1}-3\cdot\mathbf{P}^{A_2}+0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r+\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r-\boldsymbol{\kappa}_i)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}=(2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_r+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$$



D_3 Algebra

$\mathbf{P}^{A_1}$

$\mathbf{P}^{A_2}$

$\mathbf{P}^{E_1}$

$\kappa_1=1$

D_3 Center

(All-commuting operators)

$\kappa_i=i_1+i_2+i_3$

$\kappa_r=r^2+r$

A Maximal Set of Commuting Operators

Another Maximal Set of Commuting Operators

$\mathbf{r}$

$\mathbf{r}^2$

$\mathbf{P}^E_{11}$

$\mathbf{P}^E_{22}$

$\mathbf{P}^E_{12}$

$\mathbf{P}^E_{21}$

$\mathbf{i}_1$

$\mathbf{i}_2$

$\mathbf{i}_3$

$\mathbf{P}^E_{xx}$

$\mathbf{P}^E_{yy}$

$\mathbf{P}^E_{xy}$

$\mathbf{P}^E_{yx}$

Class ortho-complete projector relations

$$\boldsymbol{\kappa}_k=\sum_{(\alpha)}\frac{°\kappa_k\chi_k^{(\alpha)}}{\ell^{(\alpha)}}\mathbf{P}^{(\alpha)}$$

$$\mathbf{P}^{(\alpha)}=\frac{\ell^{(\alpha)}}{°G}\sum_k\chi_k^{(\alpha)*}\boldsymbol{\kappa}_k$$

$$=\frac{\ell^{(\alpha)}}{°G}\sum_{g=1}^{°G}\chi_g^{(\alpha)*}\mathbf{g}$$

χ_k^α	χ_1^α	χ_r^α	χ_i^α
$\alpha=A_1$	1	1	1
$\alpha=A_2$	1	1	−1
$\alpha=E$	2	−1	0

Use $\chi_1^{(\alpha)*}=\ell^{(\alpha)}$ *to find* κ_1 *coefficient*

$$\mathbf{P}^{(\alpha)}=\frac{(\ell^{(\alpha)})^2}{°G}\kappa_1+\ldots$$

Review:Spectral resolution of ***D_3 Center*** *(Class algebra)*

- *Group theory of equivalence transformations and classes*
  - *Lagrange theorems*
- *All-commuting class projectors and D_3-invariant character ortho-completeness*

Spectral resolution to ***irreducible representations*** *(or "****irreps****") foretold by* ***characters*** *or traces*

  - *Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*
  - *Atomic ℓ-level or 2ℓ+1-multiplet splitting*
    - *D_3 examples for ℓ=1-6*
- *Group invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3

- *Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*

3rd-stage spectral resolution to ***irreducible representations*** *(ireps) and Hamiltonian eigensolutions*

- *Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

Spectral resolution to ***irreducible representations*** *(or "****irreps****") foretold by* ***characters*** *or* *traces*

$$R^G(\mathbf{1})= \qquad R^G(\mathbf{r})= \qquad R^G(\mathbf{r}^2)= \qquad R^G(\mathbf{i}_1)= \qquad R^G(\mathbf{i}_2)= \qquad R^G(\mathbf{i}_3)=$$

$$\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_1})=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}/6 \qquad TraceR(\mathbf{P}^{A_1})=1$$

*Spectral resolution to **irreducible representations** (or "**irreps**") is foretold by **characters** or traces*

$$R^G(\mathbf{1})= \quad R^G(\mathbf{r})= \quad R^G(\mathbf{r}^2)= \quad R^G(\mathbf{i}_1)= \quad R^G(\mathbf{i}_2)= \quad R^G(\mathbf{i}_3)=$$

$$\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\left(\begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right),
\left(\begin{array}{cccccc}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{array}\right),
\left(\begin{array}{cccccc}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot
\end{array}\right),
\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{array}\right),
\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}\right),
\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_1})=\left(\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)/6 \qquad Trace R(\mathbf{P}^{A_1})=1$$

So: $R(\mathbf{P}^{A_1})$ reduces to:
$$\left(\begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)$$

Spectral resolution to ***irreducible representations*** *(or "****irreps****") is foretold by* ***characters*** *or traces*

$$
\begin{array}{c}
 \\ 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3
\end{array}
\begin{array}{c}
R^G(\mathbf{1}) = \\
\left(\begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right)
\end{array},
\begin{array}{c}
R^G(\mathbf{r}) = \\
\left(\begin{array}{cccccc}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{array}\right)
\end{array},
\begin{array}{c}
R^G(\mathbf{r}^2) = \\
\left(\begin{array}{cccccc}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot
\end{array}\right)
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_1) = \\
\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot
\end{array}\right)
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_2) = \\
\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_3) = \\
\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
\end{array}
$$

$$
\mathbf{P}^{A_1} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_1}) = \left(\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)/6 \qquad TraceR(\mathbf{P}^{A_1}) = 1
$$

So: $R(\mathbf{P}^{A_1})$ reduces to:

$$
\left(\begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

$$
\mathbf{P}^{A_2} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_2}) = \left(\begin{array}{cccccc}
1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & -1 & 1 & 1 & 1
\end{array}\right)/6 \qquad TraceR(\mathbf{P}^{A_2}) = 1
$$

So: $R(\mathbf{P}^{A_2})$ reduces to:

$$
\left(\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right)
$$

*Spectral resolution to **irreducible representations** (or “**irreps**”) is foretold by **characters** or traces*

$$R^G(\mathbf{1})= \qquad R^G(\mathbf{r})= \qquad R^G(\mathbf{r}^2)= \qquad R^G(\mathbf{i}_1)= \qquad R^G(\mathbf{i}_2)= \qquad R^G(\mathbf{i}_3)=$$

$$\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\mathbf{P}^{A_1}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2+\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_1})=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}/6 \qquad Trace R(\mathbf{P}^{A_1})=1$$

So: $R(\mathbf{P}^{A_1})$ reduces to:

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\mathbf{P}^{A_2}=(\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6 \Rightarrow R(\mathbf{P}^{A_2})=\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}/6 \qquad Trace R(\mathbf{P}^{A_2})=1$$

So: $R(\mathbf{P}^{A_2})$ reduces to:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\mathbf{P}^{E}=(2\boldsymbol{\kappa}_1-\boldsymbol{\kappa}_2+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2+0+0+0)/3 \Rightarrow R(\mathbf{P}^{E})=\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}/3 \qquad Trace R(\mathbf{P}^{E})=4$$

So: $R(\mathbf{P}^{E})$ reduces to:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Review:Spectral resolution of ***D_3 Center*** *(Class algebra)*
- *Group theory of equivalence transformations and classes*
    - *Lagrange theorems*
- *All-commuting class projectors and D_3-invariant character ortho-completeness*

Spectral resolution to ***irreducible representations*** *(or "**irreps**") foretold by* ***characters*** *or traces*
- *Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*
- *Atomic ℓ-level or $2\ell+1$-multiplet splitting*
    - *D_3 examples for $\ell=1$-6*
- *Group invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3
- *Splitting class projectors using subgroup chains $D_3\supset C_2$ and $D_3\supset C_3$*

3rd-stage spectral resolution to ***irreducible representations*** *(ireps) and Hamiltonian eigensolutions*
- *Tunneling modes and spectra for $D_3\supset C_2$ and $D_3\supset C_3$ local subgroup chains*

Spectral resolution to ***irreducible representations*** *(or "****irreps****") is foretold by* ***characters*** *or traces*

$$
R^G(\mathbf{1})=\begin{array}{c}1\\r^1\\r^2\\i_1\\i_2\\i_3\end{array}\left(\begin{array}{cccccc}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{array}\right),\quad
R^G(\mathbf{r})=\left(\begin{array}{cccccc}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{array}\right),\quad
R^G(\mathbf{r}^2)=\left(\begin{array}{cccccc}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{array}\right),
$$

$$
R^G(\mathbf{i}_1)=\left(\begin{array}{cccccc}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{array}\right),\quad
R^G(\mathbf{i}_2)=\left(\begin{array}{cccccc}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{array}\right),\quad
R^G(\mathbf{i}_3)=\left(\begin{array}{cccccc}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{array}\right)
$$

$$
R(\mathbf{P}^{A_1})=\left(\begin{array}{cccccc}1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\end{array}\right)/6\quad\Rightarrow\quad Trace R(\mathbf{P}^{A_1})=1\qquad \text{So: } R(\mathbf{P}^{A_1}\mathbf{g})\text{ reduces to:}\left(\begin{array}{cccccc}D^{A_1}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{array}\right)
$$

$$
R(\mathbf{P}^{A_2})=\left(\begin{array}{cccccc}1&1&1&-1&-1&-1\\1&1&1&-1&-1&-1\\1&1&1&-1&-1&-1\\-1&-1&-1&1&1&1\\-1&-1&-1&1&1&1\\-1&-1&-1&1&1&1\end{array}\right)/6\quad\Rightarrow\quad Trace R(\mathbf{P}^{A_2})=1
$$

$$
R(\mathbf{P}^{E})=\left(\begin{array}{cccccc}2&-1&-1&0&0&0\\-1&2&-1&0&0&0\\-1&-1&2&0&0&0\\0&0&0&2&-1&-1\\0&0&0&-1&2&-1\\0&0&0&-1&-1&1\end{array}\right)/3\quad\Rightarrow\quad Trace R(\mathbf{P}^{E})=4
$$

*Spectral resolution to **irreducible representations** (or "**irreps**") is foretold by **characters** or <u>traces</u>*

$$
\begin{array}{c}
R^G(\mathbf{1})= \\
\begin{array}{c|cccccc} 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ r^1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ r^2 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ i_1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ i_2 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ i_3 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}
\end{array},
\begin{array}{c}
R^G(\mathbf{r})= \\
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{r}^2)= \\
\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_1)= \\
\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_2)= \\
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}
\end{array},
\begin{array}{c}
R^G(\mathbf{i}_3)= \\
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}
\end{array}
$$

$$
R(\mathbf{P}^{A_1})=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}/6 \quad \Rightarrow \quad Trace R(\mathbf{P}^{A_1})=1 \qquad \text{So: } R(\mathbf{P}^{A_1}\mathbf{g}) \text{ reduces to:} \begin{pmatrix} D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}
$$

$$
R(\mathbf{P}^{A_2})=\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}/6 \quad \Rightarrow \quad Trace R(\mathbf{P}^{A_2})=1 \qquad \text{So: } R(\mathbf{P}^{A_2}\mathbf{g}) \text{ reduces to:} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}
$$

$$
R(\mathbf{P}^{E})=\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}/3 \quad \Rightarrow \quad Trace R(\mathbf{P}^{E})=4
$$

*Spectral resolution to **irreducible representations** (or "**irreps**") is foretold by **characters** or traces*

$$R^G(\mathbf{1})=\qquad R^G(\mathbf{r})=\qquad R^G(\mathbf{r}^2)=\qquad R^G(\mathbf{i}_1)=\qquad R^G(\mathbf{i}_2)=\qquad R^G(\mathbf{i}_3)=$$

$$\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$R(\mathbf{P}^{A_1})=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}/6 \Rightarrow \quad Trace R(\mathbf{P}^{A_1})=1 \qquad \text{So: } R(\mathbf{P}^{A_1}\mathbf{g}) \text{ reduces to:} \begin{pmatrix} D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

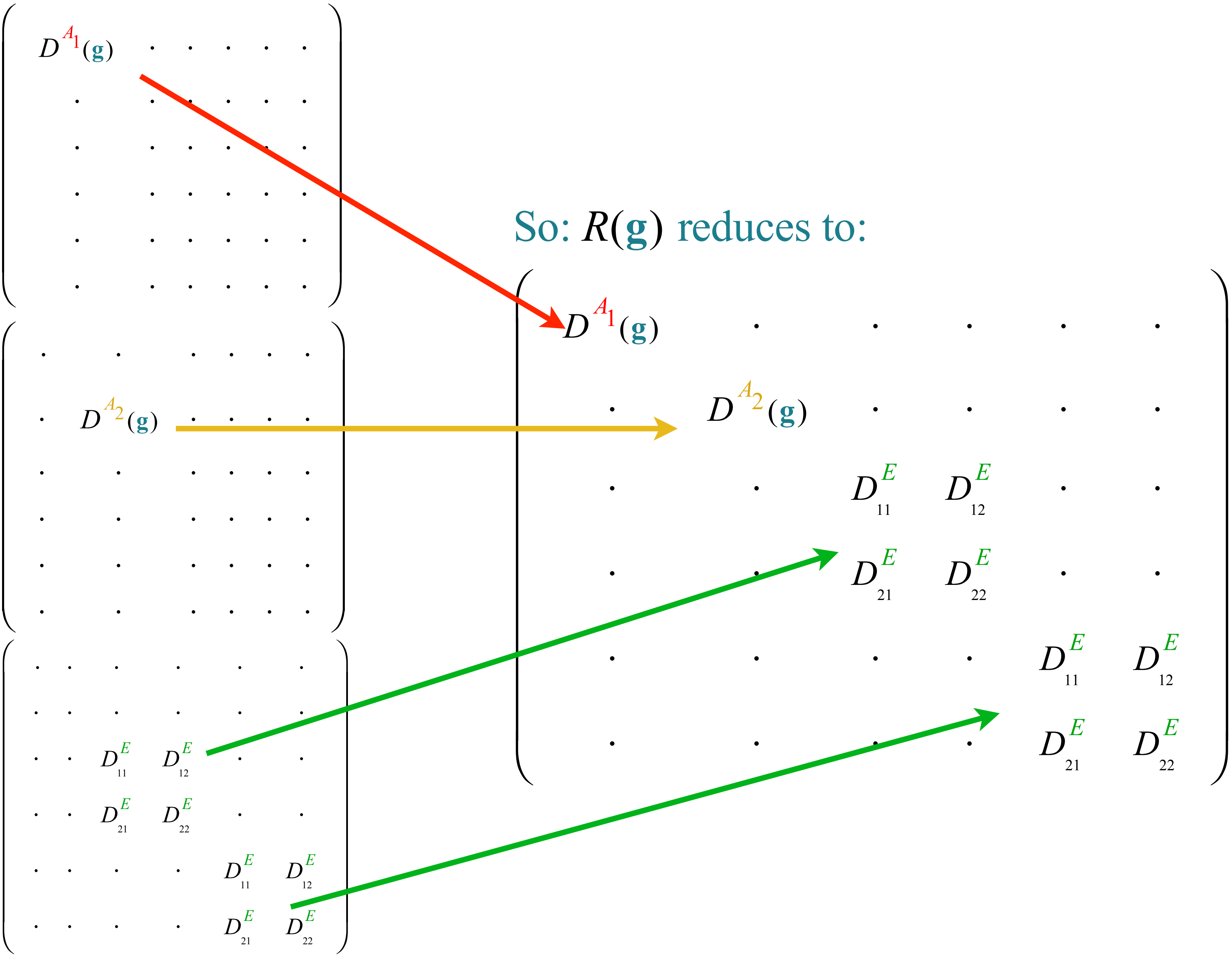
$$R(\mathbf{P}^{A_2})=\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}/6 \Rightarrow \quad Trace R(\mathbf{P}^{A_2})=1 \qquad \text{So: } R(\mathbf{P}^{A_2}\mathbf{g}) \text{ reduces to:} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$R(\mathbf{P}^{E})=\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{pmatrix}/3 \Rightarrow \quad Trace R(\mathbf{P}^{E})=4 \qquad \text{So: } R(\mathbf{P}^{E}\mathbf{g}) \text{ reduces to:} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^E_{11} & D^E_{12} & \cdot & \cdot \\ \cdot & \cdot & D^E_{21} & D^E_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & D^E_{11} & D^E_{12} \\ \cdot & \cdot & \cdot & \cdot & D^E_{21} & D^E_{22} \end{pmatrix}$$

Spectral resolution to ***irreducible representations*** *(or "****irreps****") foretold by* ***characters*** *or* *traces*

$$R^G(\mathbf{1})=\ \ R^G(\mathbf{r})=\ \ R^G(\mathbf{r}^2)=\ \ R^G(\mathbf{i}_1)=\ \ R^G(\mathbf{i}_2)=\ \ R^G(\mathbf{i}_3)=$$

$$\begin{array}{c} 1\\ r^1\\ r^2\\ i_1\\ i_2\\ i_3 \end{array}
\begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix},
\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$



$TraceR(\mathbf{P}^{A_1})=1$ So: $R(\mathbf{P}^{A_1}\mathbf{g})$ reduces to:

$$\begin{pmatrix} D^{A_1}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$TraceR(\mathbf{P}^{A_2})=1$ So: $R(\mathbf{P}^{A_2}\mathbf{g})$ reduces to:

$$\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&D^{A_2}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$TraceR(\mathbf{P}^{E})=4$ So: $R(\mathbf{P}^{E}\mathbf{g})$ reduces to:

$$\begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&D^E_{11}&D^E_{12}&\cdot&\cdot\\ \cdot&\cdot&D^E_{21}&D^E_{22}&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&D^E_{11}&D^E_{12}\\ \cdot&\cdot&\cdot&\cdot&D^E_{21}&D^E_{22} \end{pmatrix}$$

So: $R(\mathbf{g})$ reduces to:

$$\begin{pmatrix} D^{A_1}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&D^{A_2}(\mathbf{g})&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&D^E_{11}&D^E_{12}&\cdot&\cdot\\ \cdot&\cdot&D^E_{21}&D^E_{22}&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&D^E_{11}&D^E_{12}\\ \cdot&\cdot&\cdot&\cdot&D^E_{21}&D^E_{22} \end{pmatrix}$$

*Review: Spectral resolution of **D_3 Center** (Class algebra)*

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting class projectors and D_3-invariant character ortho-completeness

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

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Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains

Spectral resolution to ***irreducible representations*** *(or "****ireps****") foretold by* ***characters*** *or traces*

$$R^G(\mathbf{1})= \qquad R^G(\mathbf{r})= \qquad R^G(\mathbf{r}^2)= \qquad R^G(\mathbf{i}_1)= \qquad R^G(\mathbf{i}_2)= \qquad R^G(\mathbf{i}_3)=$$

$$\begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array}
\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},
\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\{R^G(\mathbf{g})\}$ has lots of empty space and looks like it could be reduced.

But, $\{R^G(\mathbf{g})\}$ cannot be diagonalized all-at-once. (Not all $\mathbf{g}$ commute.)

Nevertheless, $\{R^G(\mathbf{g})\}$ can be *block-diagonalized all-at-once* into ***"ireps"*** A_1, A_2, and E_1

$R(\mathbf{g})$ reduces to:

$$\begin{pmatrix} D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^E_{11} & D^E_{12} & \cdot & \cdot \\ \cdot & \cdot & D^E_{21} & D^E_{22} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & D^E_{11} & D^E_{12} \\ \cdot & \cdot & \cdot & \cdot & D^E_{21} & D^E_{22} \end{pmatrix}$$

*Spectral resolution to **irreducible representations** (or "**ireps**") foretold by **characters** or traces*

$$
R^G(\mathbf{1})= \quad R^G(\mathbf{r})= \quad R^G(\mathbf{r}^2)= \quad R^G(\mathbf{i}_1)= \quad R^G(\mathbf{i}_2)= \quad R^G(\mathbf{i}_3)=
$$

$$
\begin{array}{c}
1\\ r^1\\ r^2\\ i_1\\ i_2\\ i_3
\end{array}
\begin{pmatrix}
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix},
\begin{pmatrix}
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot
\end{pmatrix},
\begin{pmatrix}
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot
\end{pmatrix},
\begin{pmatrix}
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot
\end{pmatrix},
\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot
\end{pmatrix},
\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot\\
1&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}
$$

$\{R^G(\mathbf{g})\}$has lots of empty space and looks like it could be reduced.

But, $\{R^G(\mathbf{g})\}$ cannot be diagonalized all-at-once. (Not all $\mathbf{g}$ commute.)

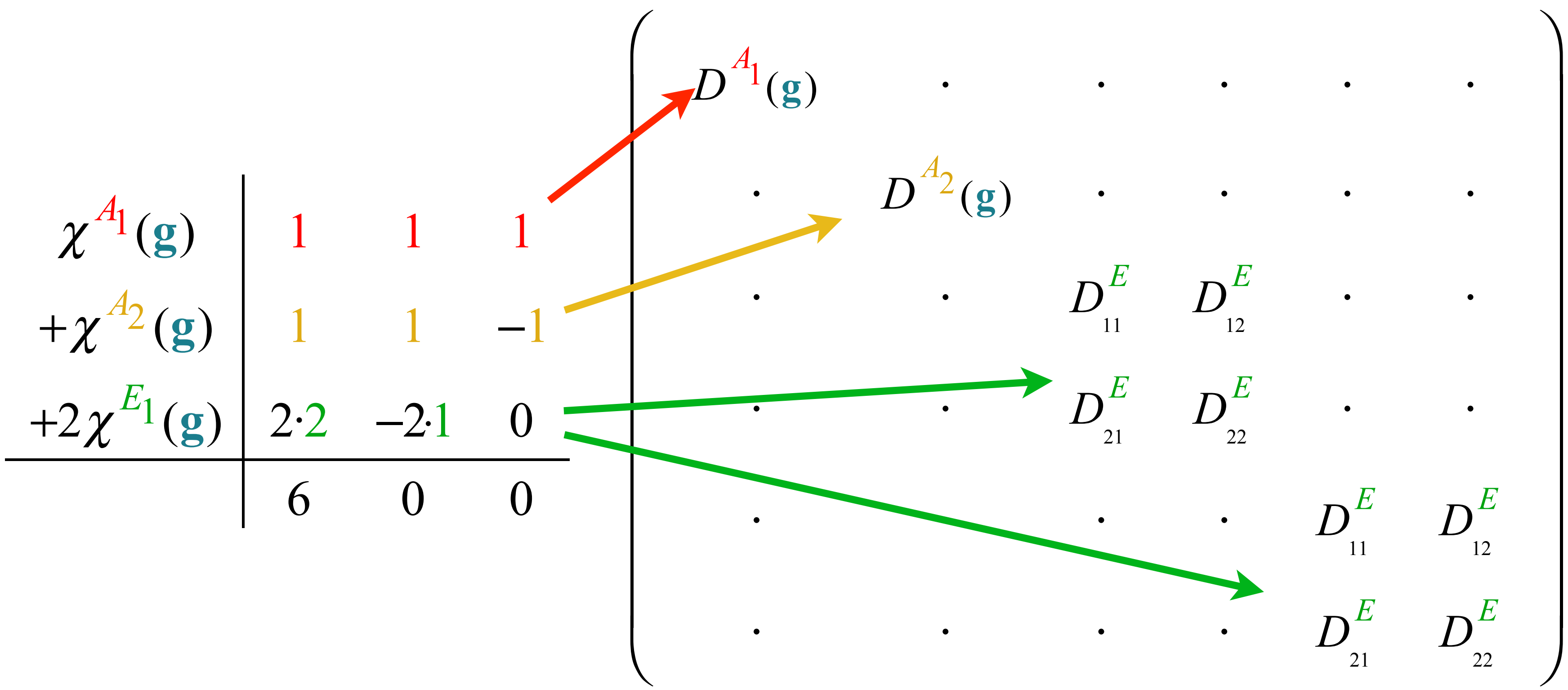
Nevertheless, $\{R^G(\mathbf{g})\}$ can be *block-diagonalized* *all-at-once* into ***"ireps"*** A_1, A_2, and E_1

We relate traces of $\{R^G(\mathbf{g})\}$:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$Trace R^G(\mathbf{g})=$	6	0	0

to D_3 character table:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	-1
$\chi^{E_1}(\mathbf{g})=$	2	-1	0



$R(\mathbf{g})$ reduces to:

$\chi^{A_1}(\mathbf{g})$	1	1	1
$+\chi^{A_2}(\mathbf{g})$	1	1	-1
$+2\chi^{E_1}(\mathbf{g})$	$2\cdot2$	$-2\cdot1$	0
	6	0	0

$$
\begin{pmatrix}
D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & D^E_{11} & D^E_{12} & \cdot & \cdot\\
\cdot & \cdot & D^E_{21} & D^E_{22} & \cdot & \cdot\\
\cdot & & \cdot & \cdot & D^E_{11} & D^E_{12}\\
\cdot & \cdot & \cdot & \cdot & D^E_{21} & D^E_{22}
\end{pmatrix}
$$

Spectral resolution to ***irreducible representations*** *(or "****ireps****") foretold by* ***characters*** *or* *traces*

$$R^G(\mathbf{1})= \begin{array}{c} 1 \\ r^1 \\ r^2 \\ i_1 \\ i_2 \\ i_3 \end{array} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\mathbf{r})= \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2)= \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1)= \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2)= \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3)= \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\{R^G(\mathbf{g})\}$has lots of empty space and looks like it could be reduced.

But, $\{R^G(\mathbf{g})\}$ cannot be diagonalized all-at-once. (Not all $\mathbf{g}$ commute.)

Nevertheless, $\{R^G(\mathbf{g})\}$ can be *block-diagonalized* *all-at-once* into ***"ireps"*** A_1, A_2, and E_1

We relate traces of $\{R^G(\mathbf{g})\}$:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$Trace R^G(\mathbf{g})=$	6	0	0

to D_3 character table:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	-1
$\chi^{E_1}(\mathbf{g})=$	2	-1	0

$\chi^{A_1}(\mathbf{g})$	1	1	1
$+\chi^{A_2}(\mathbf{g})$	1	1	-1
$+2\chi^{E_1}(\mathbf{g})$	$2\cdot 2$	$-2\cdot 1$	0
	6	0	0

$R(\mathbf{g})$ reduces to:

$$\begin{pmatrix} D^{A_1}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & D^{A_2}(\mathbf{g}) & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & D^E_{11} & D^E_{12} & \cdot & \cdot \\ \cdot & \cdot & D^E_{21} & D^E_{22} & \cdot & \cdot \\ \cdot & & \cdot & \cdot & D^E_{11} & D^E_{12} \\ \cdot & \cdot & \cdot & \cdot & D^E_{21} & D^E_{22} \end{pmatrix}$$

So $\{R^G(\mathbf{g})\}$ can be *block-diagonalized* into a ***direct sum***$\oplus$ of ***"ireps"*** $R^G(\mathbf{g})=D^{A_1}(\mathbf{g})\oplus D^{A_2}(\mathbf{g})\oplus 2D^{E_1}(\mathbf{g})$

Review:Spectral resolution of ***D_3 Center*** *(Class algebra)*

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting class projectors and D_3-invariant character ortho-completeness

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Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

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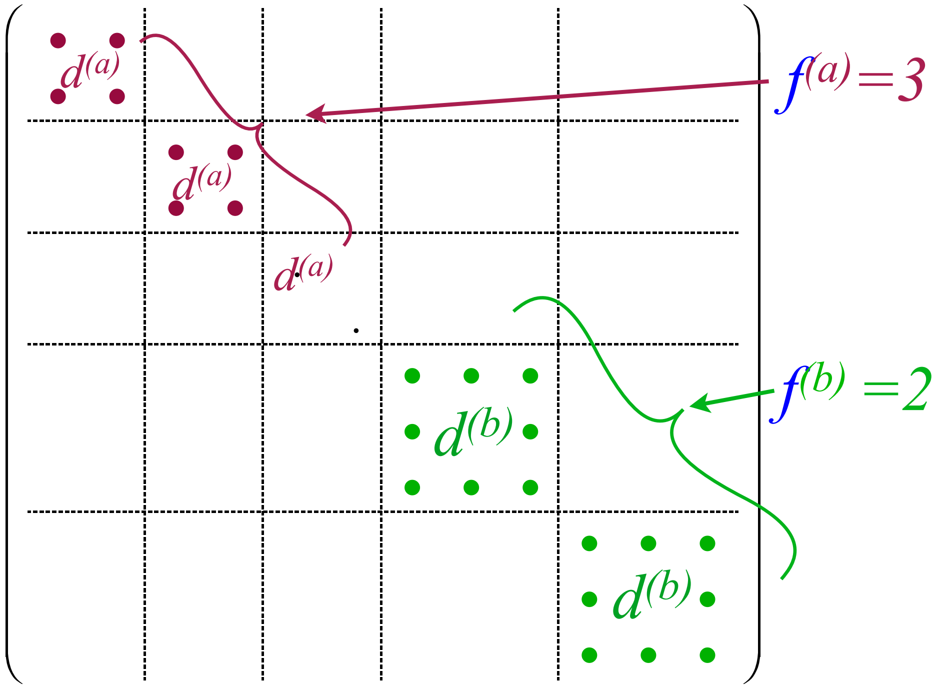
Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

(irep ≡ irreducible representation)

Symmetry reduction of G to $H\subset G$ *involves splitting of G-ireps* $D^{(\alpha)}(G)$ *into smaller H-ireps* $d^{(a)}(H)$

$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H)$ *is reducible to:* $T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$

$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$



The following derives formulae for integral $H\subset G$ *correlation coefficients* $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

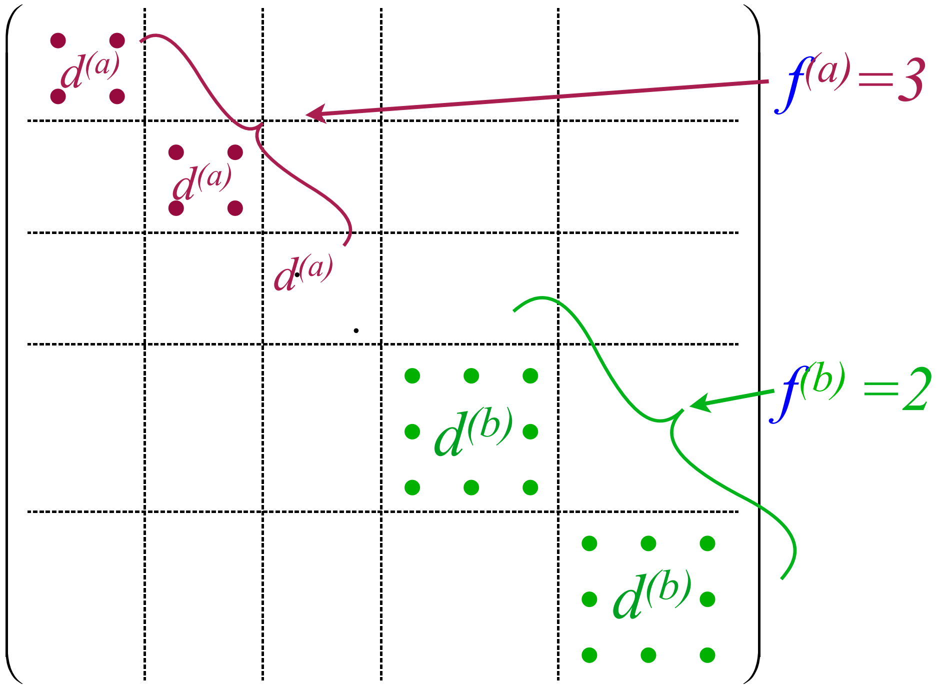
Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

($irep \equiv$ irreducible representation)

Symmetry reduction of G to $H \subset G$ involves splitting of G-ireps $D^{(\alpha)}(G)$ into smaller H-ireps $d^{(a)}(H)$

$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H)$ is reducible to: $T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}{}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$

$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$



The following derives formulae for integral $H \subset G$ correlation coefficients $f^{(b)}(D^{(\alpha)}(G)\downarrow H)$

$$Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = f^{(b)} \cdot \ell^{(b)}$$

Since each $d^{(b)}(\mathbf{P}^{(b)})$ is $\ell^{(b)}$-by-$\ell^{(b)}$ unit matrix

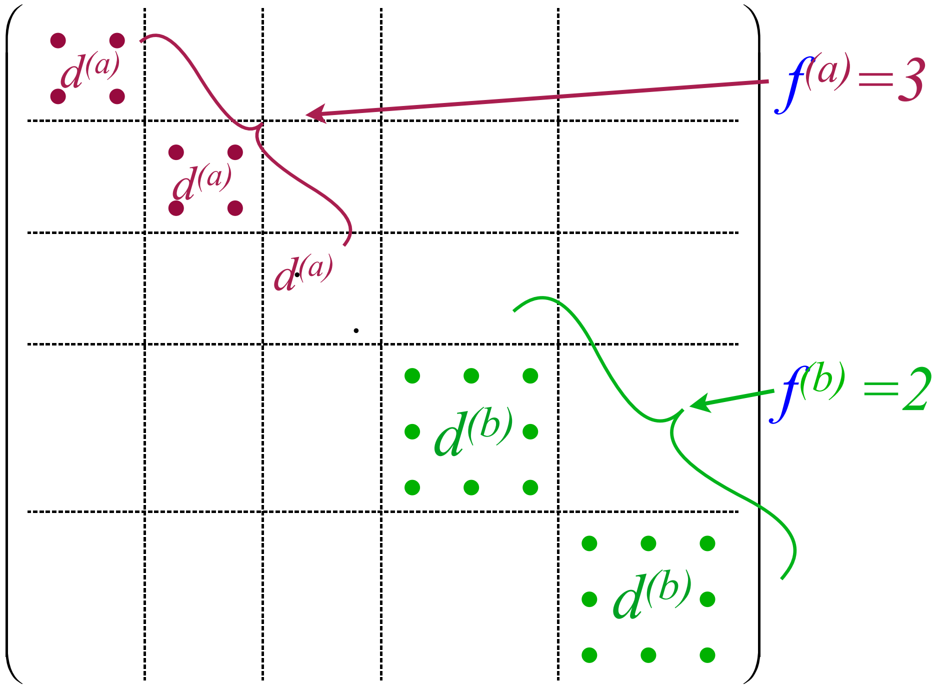
Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

(*irep* ≡ *irreducible representation*)

Symmetry reduction of G to H⊂G involves splitting of G-ireps $D^{(\alpha)}(G)$ into smaller H-ireps $d^{(a)}(H)$

$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H)$ *is reducible to:* $T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$

$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$



The following derives formulae for integral H⊂G correlation coefficients $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

$$Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = f^{(b)} \cdot \ell^{(b)}$$

$$f^{(b)} = \frac{1}{\ell^{(b)}} Trace D^{(\alpha)}(\mathbf{P}^{(b)})$$

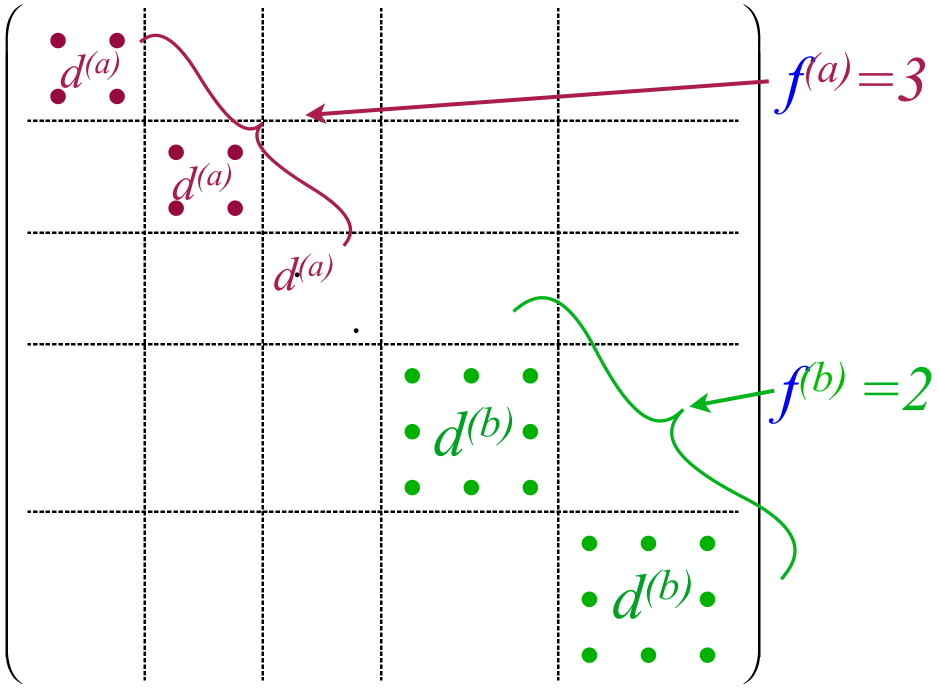
Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

(irep ≡ irreducible representation)

Symmetry reduction of G to $H \subset G$ involves splitting of G-ireps $D^{(\alpha)}(G)$ into smaller H-ireps $d^{(a)}(H)$

$$D^{(\alpha)}(G)\downarrow H \equiv D^{(\alpha)}(H) \text{ is reducible to: } T_{reducer}\, D^{(\alpha)}(H) T^{\dagger}_{reducer} = f^{(a)} d^{(a)}(H) \oplus f^{(b)} d^{(b)}(H) \oplus \ldots$$

$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$



The following derives formulae for integral $H \subset G$ correlation coefficients $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

$$Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = f^{(b)} \cdot \ell^{(b)}$$

Class ortho-complete projector relations (p.24)

$$\mathbf{P}^{(\alpha)} = \frac{\ell^{(\alpha)}}{{}^{\circ}G} \sum_{k \in G} \chi_k^{(\alpha)*} \boldsymbol{\kappa}_k$$

$$\mathbf{P}^{(b)} = \frac{\ell^{(b)}}{{}^{\circ}H} \sum_{k \in H} \chi_k^{(b)*} \boldsymbol{\kappa}_k$$

$$f^{(b)} = \frac{1}{\ell^{(b)}} Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = \frac{1}{\ell^{(b)}} \frac{\ell^{(b)}}{{}^{\circ}H} \sum_{\substack{classes \\ \boldsymbol{\kappa}_k \in H}} \chi_k^{(b)*} Trace D^{(\alpha)}(\boldsymbol{\kappa}_k)$$

Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

(irep ≡ irreducible representation)

Symmetry reduction of G to H⊂G involves splitting of G-ireps $D^{(\alpha)}(G)$ into smaller H-ireps $d^{(a)}(H)$

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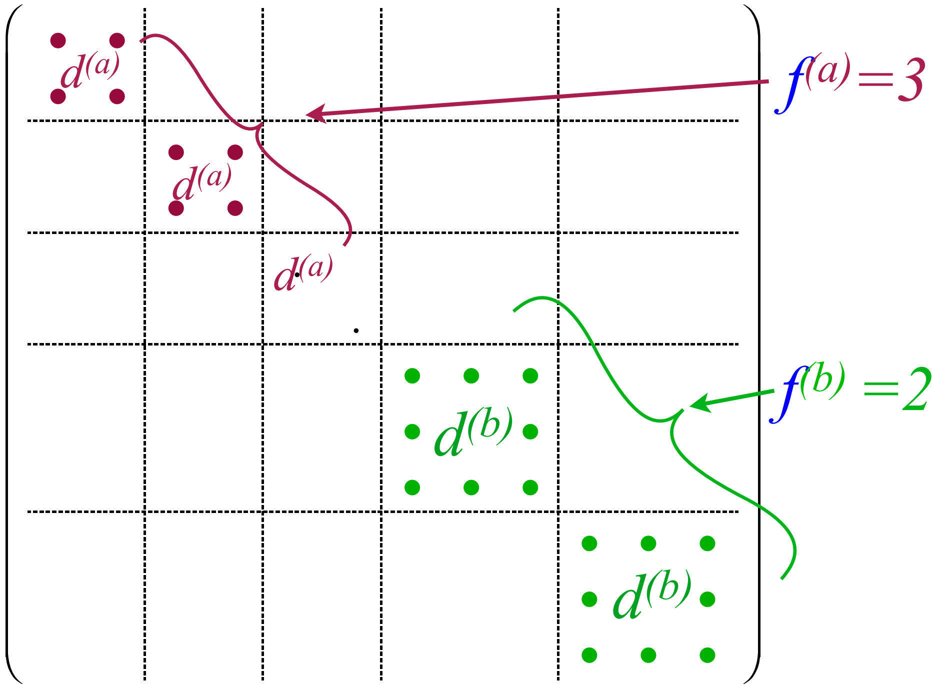
$$T_{reducer} D^{(\alpha)}(h) T^{\dagger}_{reducer} =$$



The following derives formulae for integral H⊂G correlation coefficients $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

$$Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = f^{(b)} \cdot \ell^{(b)}$$

$$f^{(b)} = \frac{1}{\ell^{(b)}} Trace D^{(\alpha)}(\mathbf{P}^{(b)}) = \frac{1}{\ell^{(b)}} \frac{\ell^{(b)}}{{}^{\circ}H} \sum_{\substack{classes\\ \boldsymbol{\kappa}_k \in H}} \chi_k^{(b)*} \underbrace{Trace D^{(\alpha)}(\boldsymbol{\kappa}_k)}_{\chi^{(\alpha)}(\boldsymbol{\kappa}_k) = {}^{\circ}\kappa_k \chi_k^{(\alpha)}}$$

Class ortho-complete projector relations (p.24)

$$\boxed{\begin{aligned} \mathbf{P}^{(\alpha)} &= \frac{\ell^{(\alpha)}}{{}^{\circ}G} \sum_{k\in G} \chi_k^{(\alpha)*} \boldsymbol{\kappa}_k \\ \mathbf{P}^{(b)} &= \frac{\ell^{(b)}}{{}^{\circ}H} \sum_{k\in H} \chi_k^{(b)*} \boldsymbol{\kappa}_k \end{aligned}}$$

$$\boxed{f^{(b)} = \frac{1}{{}^{\circ}H} \sum_{\substack{classes\\ \boldsymbol{\kappa}_k \in H}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\alpha)}}$$

Character relation for frequency $f^{(b)}$ of $d^{(b)}$ of subgroup H in $D^{(\alpha)}\downarrow H$ of G

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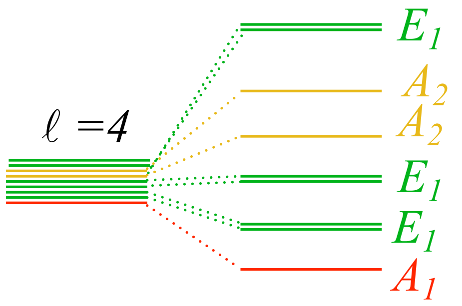
Formula from p.44

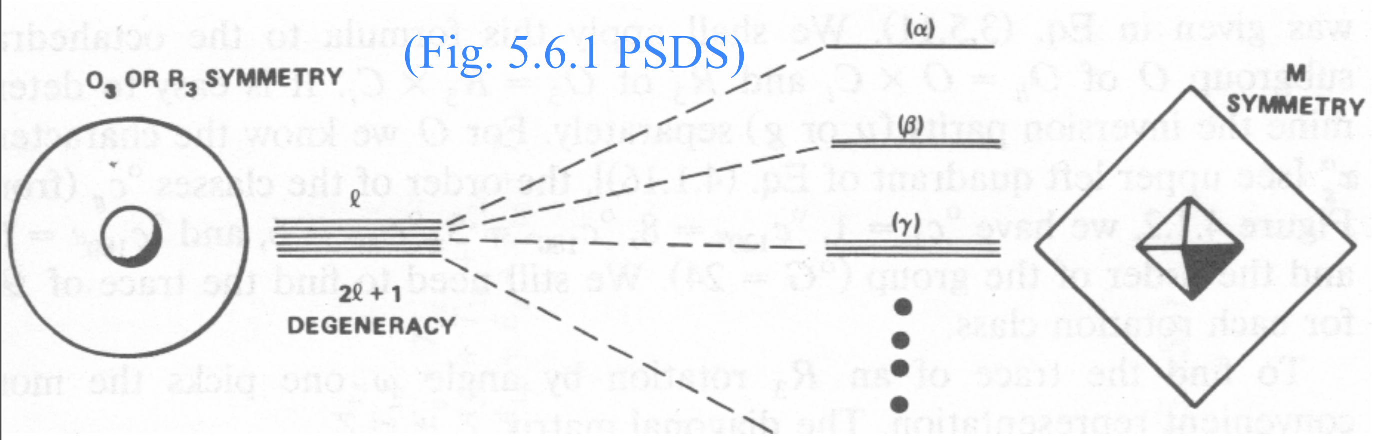
Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



ℓ=0, s-singlet
2ℓ+1=1
ℓ=1, p-triplet
2ℓ+1=3

Crystal-field splitting: O(3)⊃D3 symmetry reduction

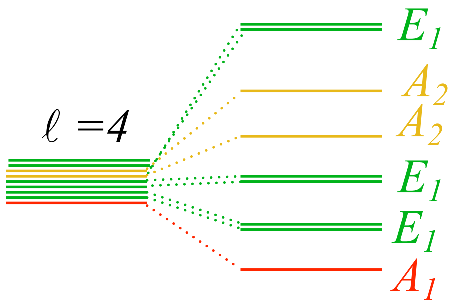

(Fig. 5.6.1 PSDS)

Atomic ℓ-level or 2ℓ+1-multiplet splitting

Formula from p.44

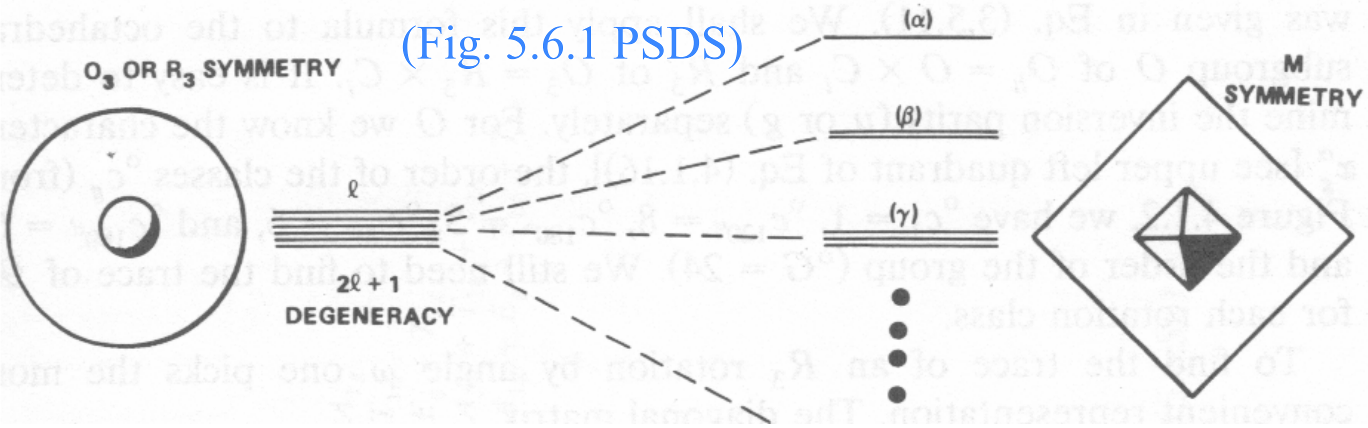
Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



ℓ=0, s-singlet
2ℓ+1=1
ℓ=1, p-triplet
2ℓ+1=3
ℓ=2, d-quintet
2ℓ+1=5

Crystal-field splitting: O(3)⊃D3 symmetry reduction



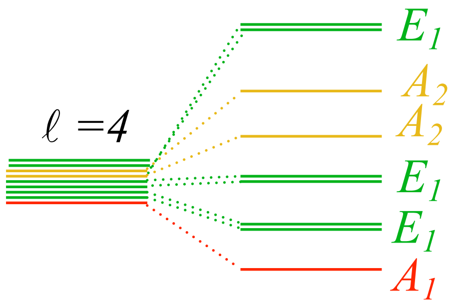
(Fig. 5.6.1 PSDS)

Atomic ℓ-level or 2ℓ+1-multiplet splitting

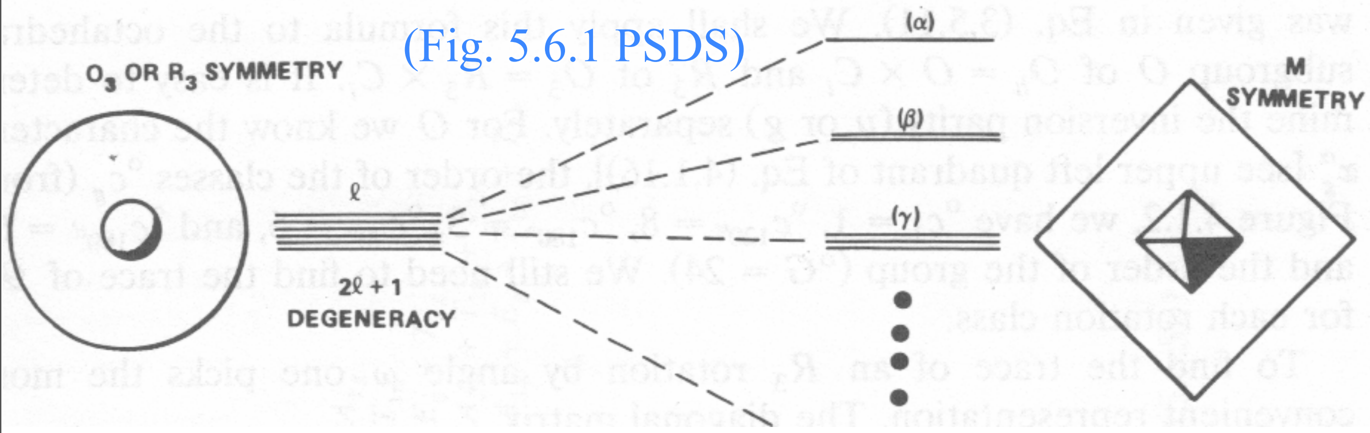
Formula from p.44

Example: (ℓ=4)

$$f^{(b)}=\frac{1}{{}^{\circ}D_3}\sum_{\substack{classes\\ \kappa_k\in D_3}}{}^{\circ}\kappa_k\chi_k^{(b)*}\chi_k^{(\ell)}$$



Crystal-field splitting: O(3)⊃D₃ symmetry reduction



(Fig. 5.6.1 PSDS)

ℓ=0, s-singlet

2ℓ+1=1

ℓ=1, p-triplet

2ℓ+1=3

ℓ=2, d-quintet

2ℓ+1=5

ℓ=3, f-septet

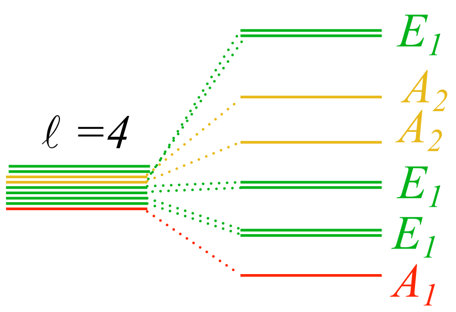
2ℓ+1=7

Atomic ℓ-level or 2ℓ+1-multiplet splitting

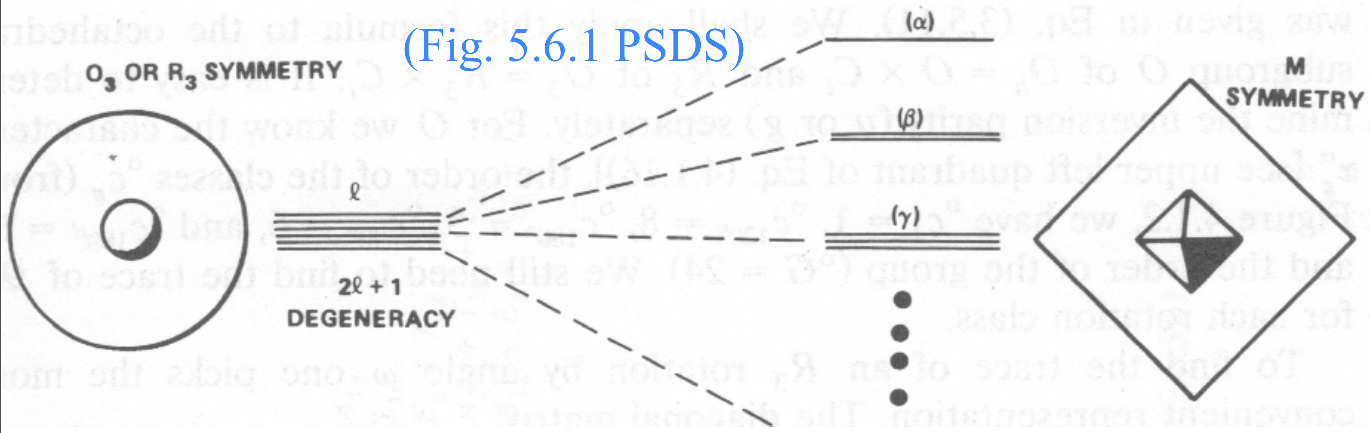
Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{\text{classes} \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



Crystal-field splitting: O(3)⊃D3 symmetry reduction



(Fig. 5.6.1 PSDS)

ℓ=0, s-singlet
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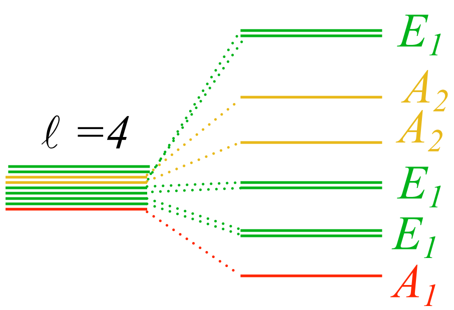
2ℓ+1=5
ℓ=3, f-septet
2ℓ+1=7
ℓ=4, g-nonet
2ℓ+1=9
ℓ=5, h-(11)-let
2ℓ+1=11
...

Atomic ℓ-level or 2ℓ+1-multiplet splitting

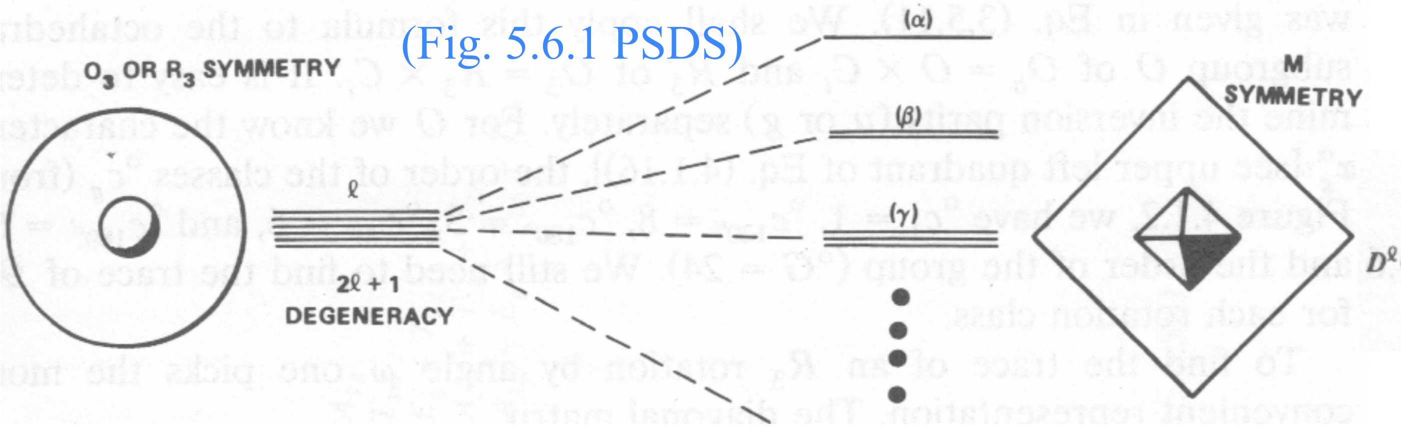
Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{\text{classes}\\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



Crystal-field splitting: O(3)⊃D₃ symmetry reduction and $D^\ell \downarrow D_3$ splitting



(Fig. 5.6.1 PSDS)

$$D^{\ell}(R) = \begin{pmatrix} D^{\ell}_{\ell,\ell}(R) & \cdots\cdots & D_{\ell,-\ell} \\ D_{\ell-1,\ell} & & \\ D_{-\ell,\ell} & \cdots\cdots & D_{-\ell,-\ell} \end{pmatrix} \downarrow M \cong \begin{pmatrix} \boxed{D^{\alpha}(R)} & & \\ & \boxed{D^{\beta}(R)} & \\ & & \boxed{D^{\gamma}(R)} \end{pmatrix}$$

ℓ=0, s-singlet
2ℓ+1=1
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2ℓ+1=5
ℓ=3, f-septet
2ℓ+1=7
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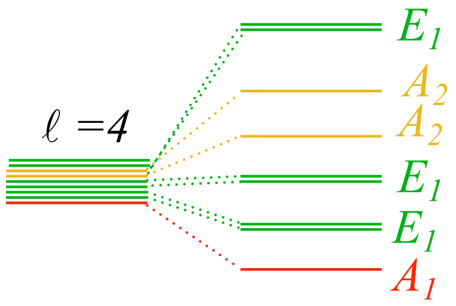
...

Atomic ℓ-level or $2\ell+1$-multiplet splitting

Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{\text{classes}\\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



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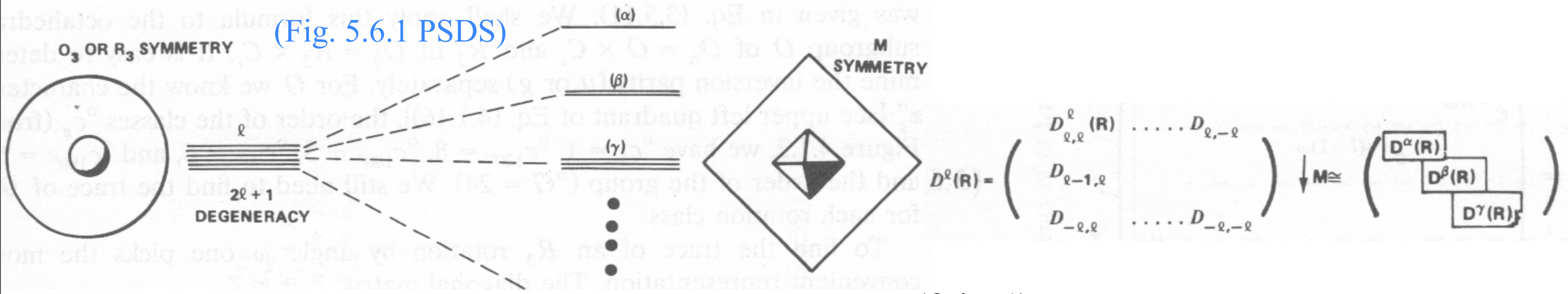
$2\ell+1=1$
ℓ=1, p-triplet
$2\ell+1=3$
ℓ=2, d-quintet
$2\ell+1=5$
ℓ=3, f-septet
$2\ell+1=7$
ℓ=4, g-nonet
$2\ell+1=9$
ℓ=5, h-(11)-let
$2\ell+1=11$
...

Crystal-field splitting: $O(3) \supset D_3$ symmetry reduction and $D^{\ell} \downarrow D_3$ splitting



(Fig. 5.6.1 PSDS)

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$$\chi^{\ell}\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

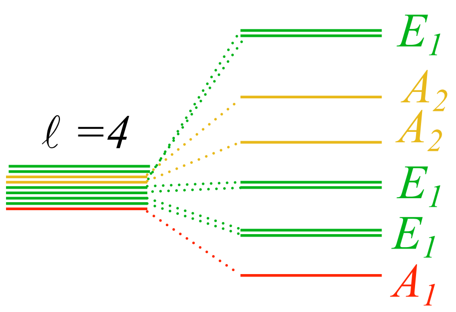
R(3) character
where: $2\ell+1$
is ℓ-orbital dimension

Atomic ℓ-level or 2ℓ+1-multiplet splitting

Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes\\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



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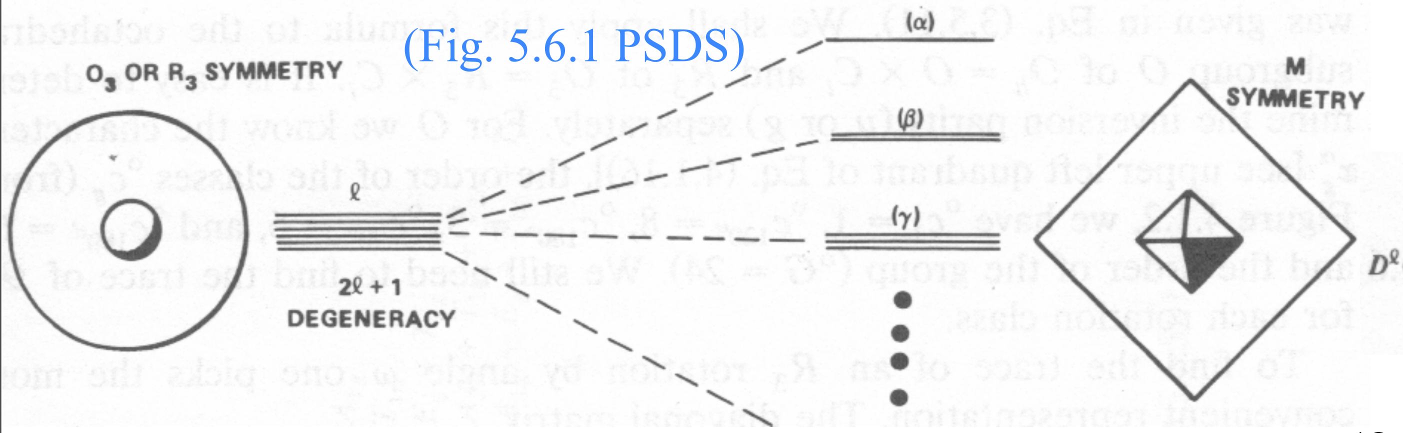
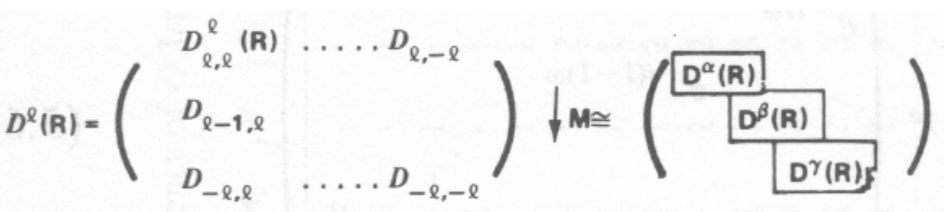
2ℓ+1=7
ℓ=4, g-nonet
2ℓ+1=9
ℓ=5, h-(11)-let
2ℓ+1=11
…

Crystal-field splitting: O(3)⊃D3 symmetry reduction and D^ℓ↓D3 splitting


(Fig. 5.6.1 PSDS)



U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^\ell\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

$$\chi^\ell(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

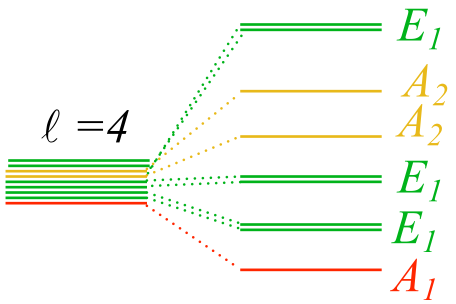
R(3) character
where: 2ℓ+1
is ℓ-orbital dimension

Atomic ℓ-level or 2ℓ+1-multiplet splitting

Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



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2ℓ+1=1
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2ℓ+1=3
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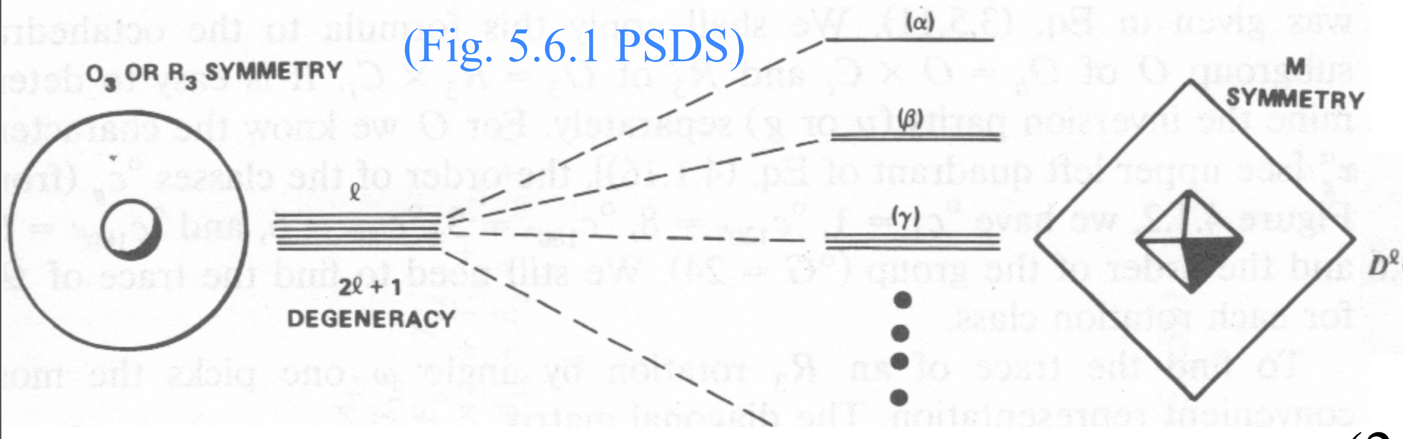
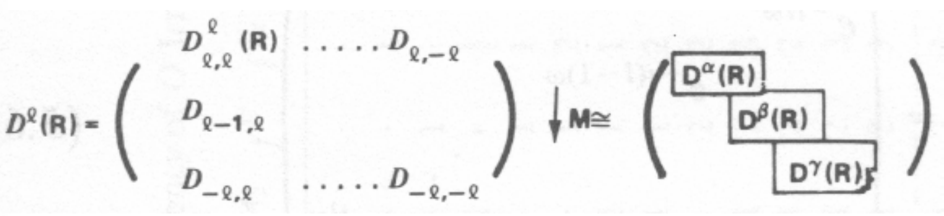
2ℓ+1=5
ℓ=3, f-septet
2ℓ+1=7
ℓ=4, g-nonet
2ℓ+1=9
ℓ=5, h-(11)-let
2ℓ+1=11
…

Crystal-field splitting: O(3)⊃D₃ symmetry reduction and $D^\ell \downarrow D_3$ splitting

(Fig. 5.6.1 PSDS)





U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
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6	13	1	1
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$$\chi^\ell\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

R(3) character
where: 2ℓ+1
is ℓ-orbital dimension

$$\chi^\ell(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and *D₃* character table from p. 24:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

*Review: Spectral resolution of **D_3 Center** (Class algebra)*
- *Group theory of equivalence transformations and classes*
  - *Lagrange theorems*
- *All-commuting class projectors and D_3-invariant character ortho-completeness*

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*
- *Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*
- *Atomic ℓ-level or 2ℓ+1-multiplet splitting*
  - *D_3 examples for ℓ=1-6*
- *Group invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3
- *Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*

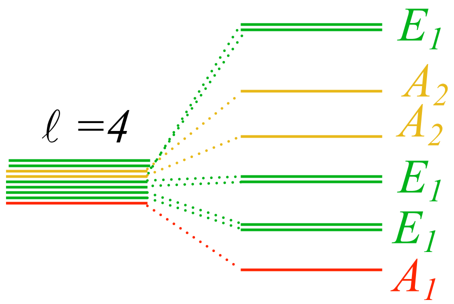
*3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions*
- *Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

Atomic ℓ-level or 2ℓ+1-multiplet splitting

Formula from p.44

Example: (ℓ=4)

$$f^{(b)}=\frac{1}{{}^\circ D_3}\sum_{\substack{classes\\ \kappa_k\in D_3}}{}^\circ\kappa_k\,\chi_k^{(b)*}\,\chi_k^{(\ell)}$$



ℓ=0, s-singlet
2ℓ+1=1
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2ℓ+1=7
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ℓ=5, h-(11)-let
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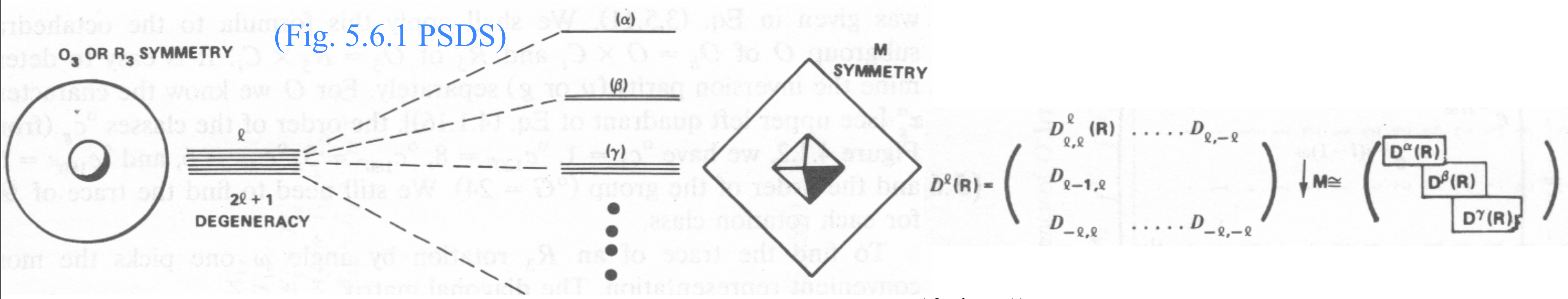
…

Crystal-field splitting: O(3)⊃D3 symmetry reduction and $D^\ell\downarrow D_3$ splitting


(Fig. 5.6.1 PSDS)

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
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3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^\ell\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

$$\chi^\ell(\Theta)=\frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

R(3) character
where: 2ℓ+1
is ℓ-orbital dimension

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$

...and *D*3 character table from p. 24:

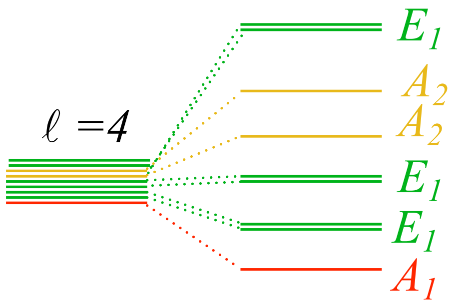
$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

Atomic ℓ-level or 2ℓ+1-multiplet splitting

Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



ℓ=0, s-singlet
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ℓ=4, g-nonet
2ℓ+1=9
ℓ=5, h-(11)-let
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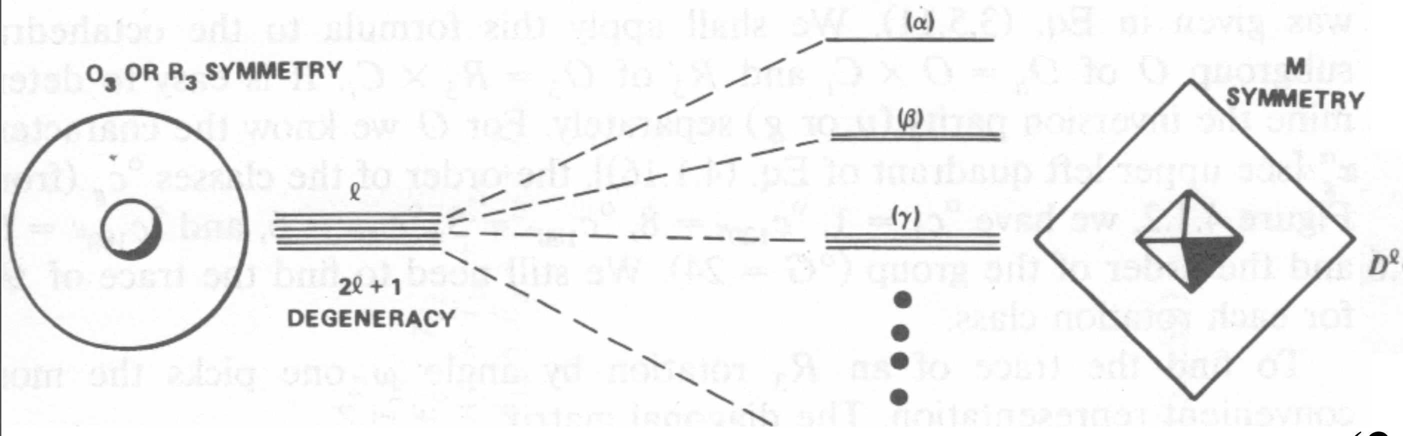
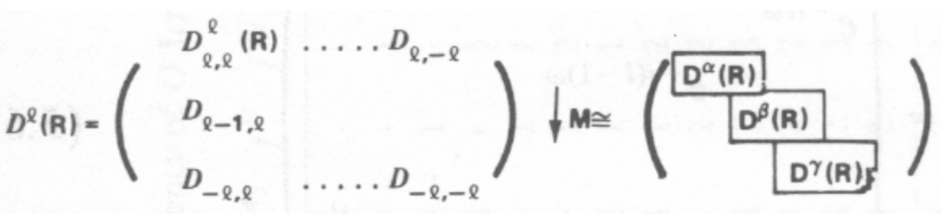




U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^{\ell}(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^{\ell}\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

$$\chi^{\ell}(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

R(3) character
where: 2ℓ+1
is ℓ-orbital dimension

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	.	.	$1A_1$
1	.	1	1	$0A_1 \oplus A_2 \oplus E_1$

...and D_3 character table from p. 24:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

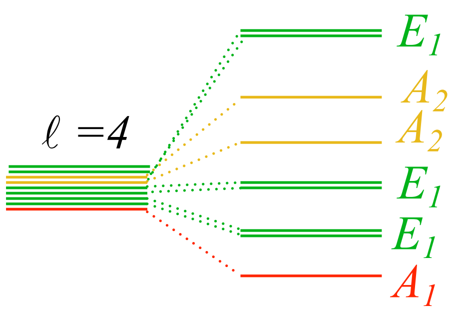
	3	0	−1
$0\chi^{A_1}(\mathbf{g})=$	0	0	0
$1\chi^{A_2}(\mathbf{g})=$	1	1	−1
$1\chi^{E_1}(\mathbf{g})=$	2	−1	0

Atomic ℓ-level or 2ℓ+1-multiplet splitting

Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



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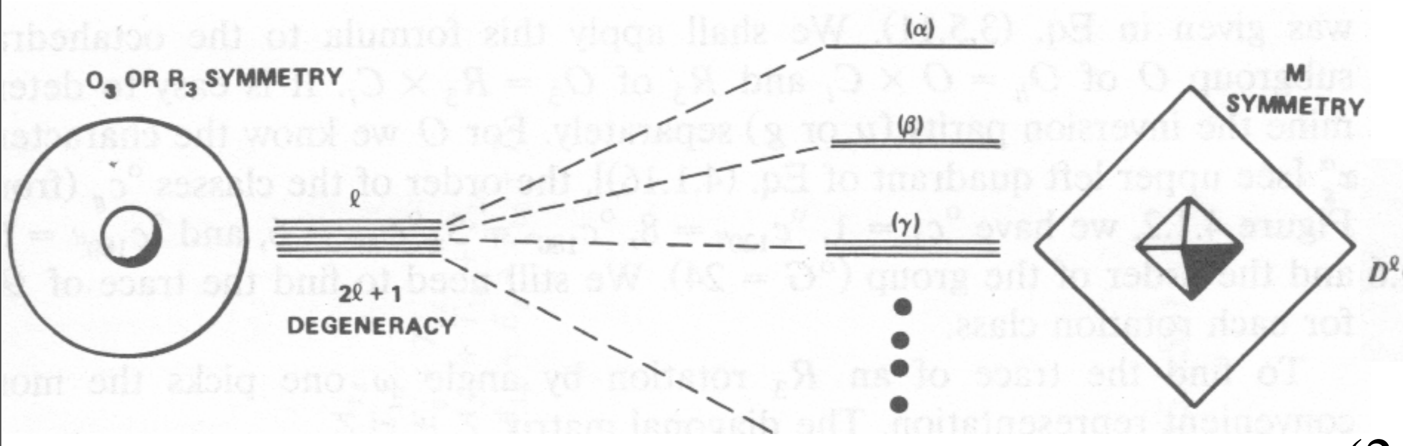

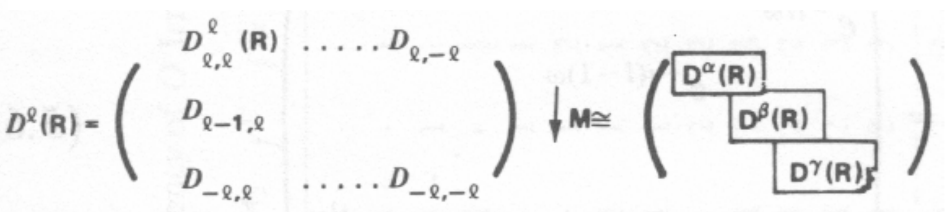

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^{\ell}(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
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5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^{\ell}\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

$$\chi^{\ell}(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

R(3) character
where: 2ℓ+1
is ℓ-orbital dimension

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1 \oplus A_2 \oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$

...and D_3 character table from p. 24:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
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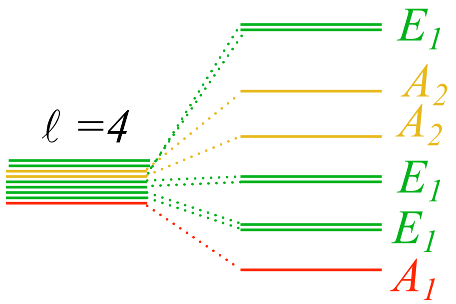
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

Atomic ℓ-level or 2ℓ+1-multiplet splitting

Formula from p.44

Example: (ℓ=4)

$$f^{(b)}=\frac{1}{{}^{\circ}D_3}\sum_{\substack{\text{classes}\\ \kappa_k\in D_3}}{}^{\circ}\kappa_k\chi_k^{(b)*}\chi_k^{(\ell)}$$



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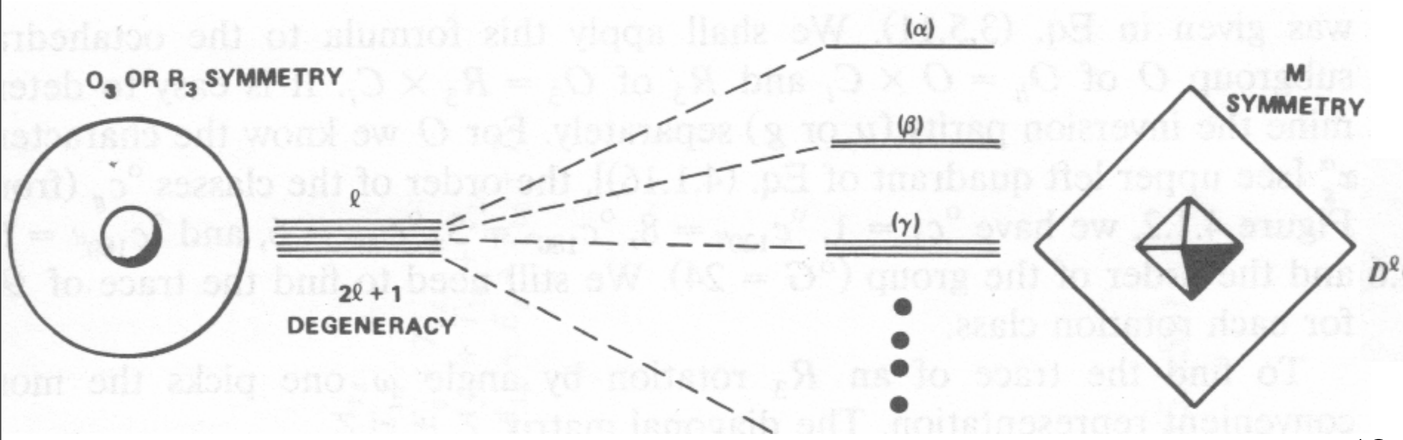
ℓ=2, d-quintet
2ℓ+1=5
ℓ=3, f-septet
2ℓ+1=7
ℓ=4, g-nonet
2ℓ+1=9
ℓ=5, h-(11)-let
2ℓ+1=11
...



$$D^{\ell}(R)=\begin{pmatrix}D^{\ell}_{\ell,\ell}(R) & \ldots\ldots D_{\ell,-\ell}\\ D_{\ell-1,\ell} & \\ D_{-\ell,\ell} & \ldots\ldots D_{-\ell,-\ell}\end{pmatrix}\Big\downarrow M\cong\begin{pmatrix}\boxed{D^{\alpha}(R)} & & \\ & \boxed{D^{\beta}(R)} & \\ & & \boxed{D^{\gamma}(R)}\end{pmatrix}$$

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^{\ell}(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^{\ell}\left(\frac{2\pi}{n}\right)=\frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

R(3) character
where: 2ℓ+1
is ℓ-orbital dimension

$$\chi^{\ell}(\Theta)=\frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1\quad\oplus 2E_1$

...and *D*₃ character table from p. 24:

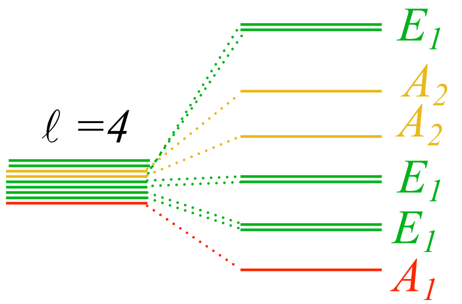
$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

	5	−1	1
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$0\chi^{A_2}(\mathbf{g})=$	0	0	0
$2\chi^{E_1}(\mathbf{g})=$	4	−2	0

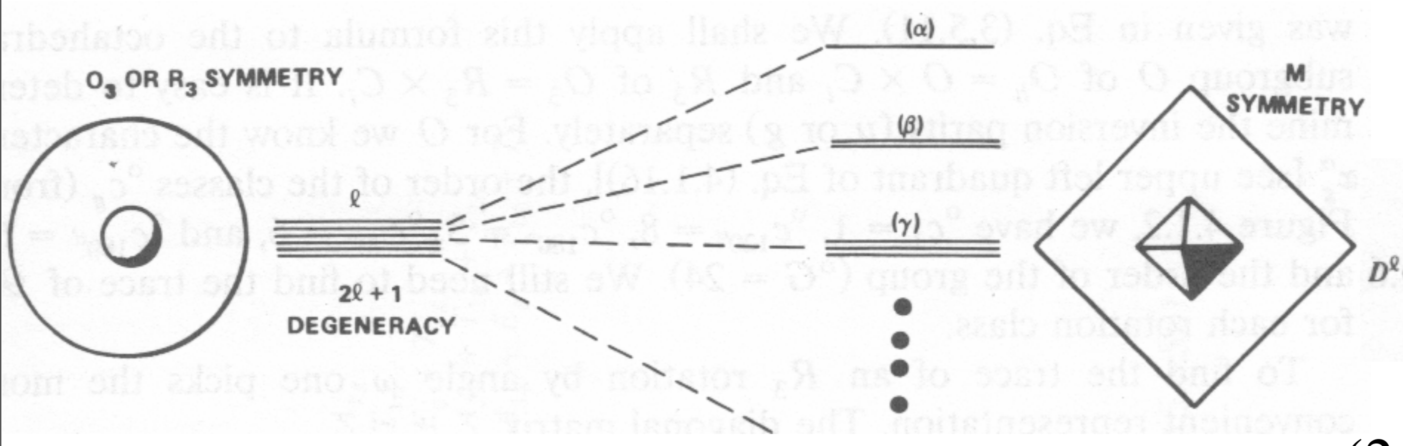
Atomic ℓ-level or 2ℓ+1-multiplet splitting
Formula from p.44

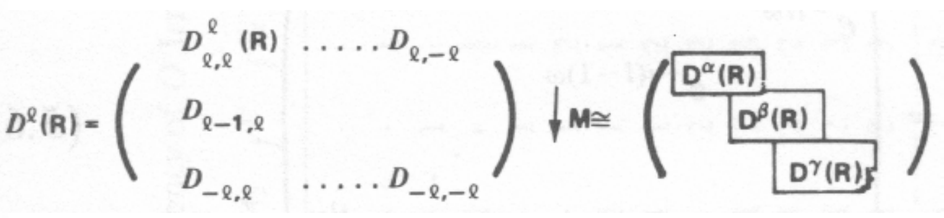
Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^\circ D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^\circ\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



ℓ=0, s-singlet
2ℓ+1=1
ℓ=1, p-triplet
2ℓ+1=3
ℓ=2, d-quintet
2ℓ+1=5
ℓ=3, f-septet
2ℓ+1=7
ℓ=4, g-nonet
2ℓ+1=9
ℓ=5, h-(11)-let
2ℓ+1=11
...





U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^\ell\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

R(3) character
where: 2ℓ+1
is ℓ-orbital dimension

$$\chi^\ell(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and D_3 character table from p. 24:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

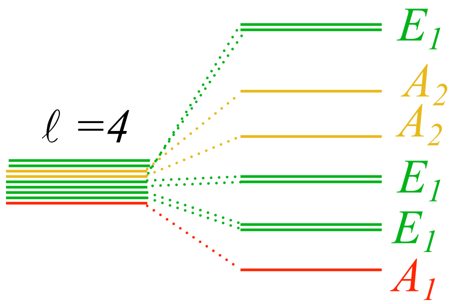
$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$
3	1	2	2	$1A_1\oplus 2A_2\oplus 2E_1$

Atomic ℓ-level or 2ℓ+1-multiplet splitting

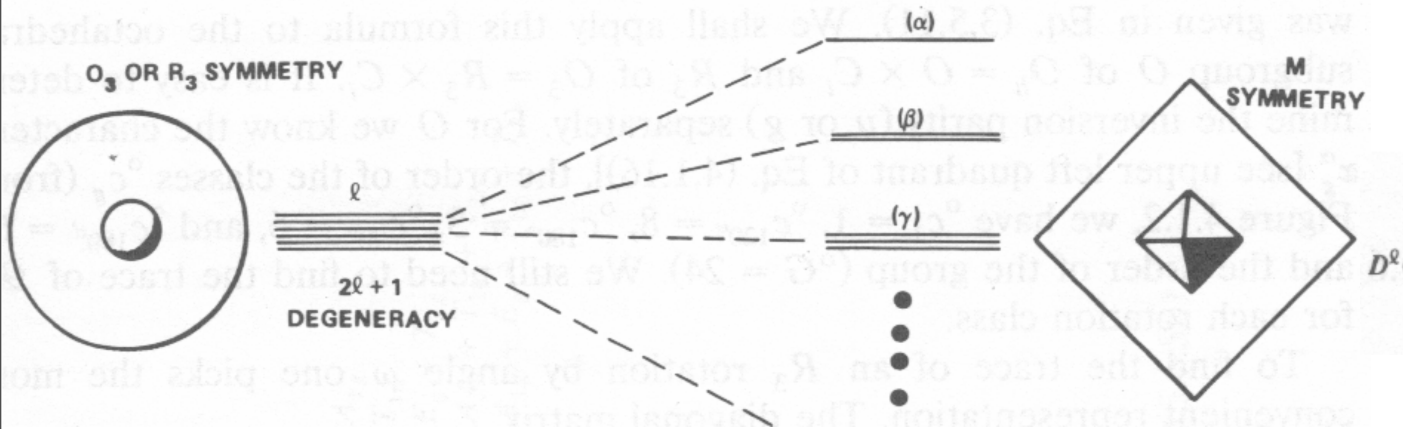
Formula from p.44

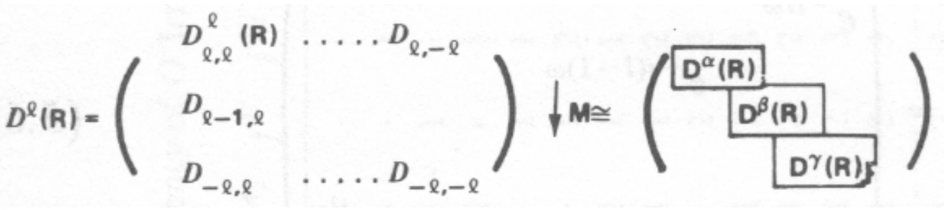
Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



ℓ=0, s-singlet
2ℓ+1=1
ℓ=1, p-triplet
2ℓ+1=3
ℓ=2, d-quintet
2ℓ+1=5
ℓ=3, f-septet
2ℓ+1=7
ℓ=4, g-nonet
2ℓ+1=9
ℓ=5, h-(11)-let
2ℓ+1=11
…





U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$$\chi^{\ell}\left(\frac{2\pi}{n}\right) = \frac{\sin\frac{(2\ell+1)\pi}{n}}{\sin\frac{\pi}{n}}$$

$$\chi^{\ell}(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

R(3) character
where: 2ℓ+1
is ℓ-orbital dimension

$\chi^{\ell}(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

...and *D₃* character table from p. 24:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

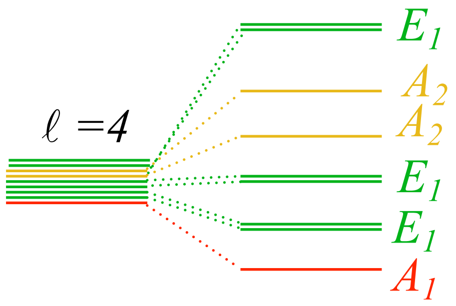
$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$
3	1	2	2	$1A_1\oplus 2A_2\oplus 2E_1$

	7	1	−1
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$2\chi^{A_2}(\mathbf{g})=$	2	2	−2
$2\chi^{E_1}(\mathbf{g})=$	4	−2	0

Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^\circ D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^\circ\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



$$f^{(E_1)} = \frac{1}{{}^\circ D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^\circ\kappa_k \chi_k^{(E_1)*} \chi_k^{(\ell=4)} = \frac{1}{{}^\circ D_3}\left({}^\circ\kappa_{0^\circ} \chi_{0^\circ}^{(E_1)*} \chi_{0^\circ}^{(\ell=4)} + {}^\circ\kappa_{120^\circ} \chi_{120^\circ}^{(E_1)*} \chi_{120^\circ}^{(\ell=4)} + {}^\circ\kappa_{180^\circ} \chi_{180^\circ}^{(E_1)*} \chi_{180^\circ}^{(\ell=4)} \right)$$

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^\ell(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and *D*₃ character table from p. 24:

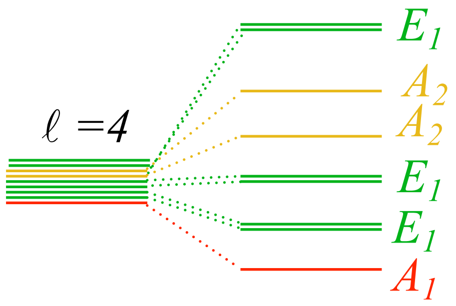
$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1 \oplus A_2 \oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$
3	1	2	2	$1A_1 \oplus 2A_2 \oplus 2E_1$
4	1	2	3	

Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



$$f^{(E_1)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(E_1)*} \chi_k^{(\ell=4)} = \frac{1}{{}^{\circ}D_3}\left({}^{\circ}\kappa_{0^\circ} \chi_{0^\circ}^{(E_1)*} \chi_{0^\circ}^{(\ell=4)} + {}^{\circ}\kappa_{120^\circ} \chi_{120^\circ}^{(E_1)*} \chi_{120^\circ}^{(\ell=4)} + {}^{\circ}\kappa_{180^\circ} \chi_{180^\circ}^{(E_1)*} \chi_{180^\circ}^{(\ell=4)} \right)$$

$$= \frac{1}{6}\left(1 \cdot 2^* \cdot 9 \quad + \quad 2 \cdot -1^* \cdot 0 \quad + \quad 3 \cdot 0^* \cdot 1 \quad \right)$$

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^{\ell}(\Theta)$	$\Theta = 0$	$\frac{2\pi}{3}$	π
$\ell = 0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^{\ell}(\Theta) = \frac{\sin(\ell + \frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and D_3 character table from p. 24:

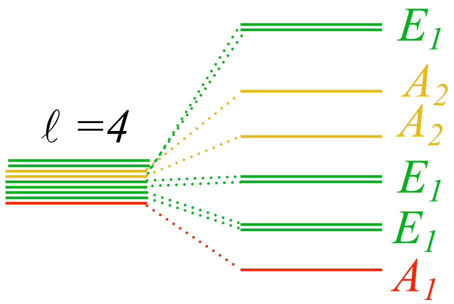
$(\mathbf{g}) =$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1, \mathbf{r}^2\}$	$\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g}) =$	1	1	1
$\chi^{A_2}(\mathbf{g}) =$	1	1	−1
$\chi^{E_1}(\mathbf{g}) =$	2	−1	0

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell = 0$	1	·	·	$1A_1$
1	·	1	1	$0A_1 \oplus A_2 \oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$
3	1	2	2	$1A_1 \oplus 2A_2 \oplus 2E_1$
4	1	2	3	

Formula from p.44

Example: (ℓ=4)

$$f^{(b)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(b)*} \chi_k^{(\ell)}$$



$$f^{(E_1)} = \frac{1}{{}^{\circ}D_3} \sum_{\substack{classes \\ \kappa_k \in D_3}} {}^{\circ}\kappa_k \chi_k^{(E_1)*} \chi_k^{(\ell=4)} = \frac{1}{{}^{\circ}D_3}\left({}^{\circ}\kappa_{0^\circ} \chi_{0^\circ}^{(E_1)*} \chi_{0^\circ}^{(\ell=4)} + {}^{\circ}\kappa_{120^\circ} \chi_{120^\circ}^{(E_1)*} \chi_{120^\circ}^{(\ell=4)} + {}^{\circ}\kappa_{180^\circ} \chi_{180^\circ}^{(E_1)*} \chi_{180^\circ}^{(\ell=4)} \right)$$

$$= \frac{1}{6}\left(1 \cdot 2^* \cdot 9 \quad + \quad 2 \cdot -1^* \cdot 0 \quad + \quad 3 \cdot 0^* \cdot 1 \right)$$

$$f^{(E_1)} = 3$$

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^\ell(\Theta) = \frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and *D*₃ character table from p. 24:

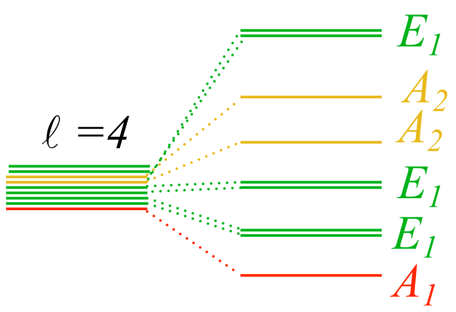
$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1 \oplus A_2 \oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$
3	1	2	2	$1A_1 \oplus 2A_2 \oplus 2E_1$
4	2	1	3	$\oplus 3E_1$

Formula from p.44

Example: (ℓ=4)

$$f^{(b)}=\frac{1}{{}^{\circ}D_3}\sum_{\substack{classes\\ \kappa_k\in D_3}}{}^{\circ}\kappa_k\chi_k^{(b)*}\chi_k^{(\ell)}$$



$$f^{(E_1)}=\frac{1}{{}^{\circ}D_3}\sum_{\substack{classes\\ \kappa_k\in D_3}}{}^{\circ}\kappa_k\chi_k^{(E_1)*}\chi_k^{(\ell=4)}=\frac{1}{{}^{\circ}D_3}\left({}^{\circ}\kappa_{0^\circ}\chi_{0^\circ}^{(E_1)*}\chi_{0^\circ}^{(\ell=4)}+{}^{\circ}\kappa_{120^\circ}\chi_{120^\circ}^{(E_1)*}\chi_{120^\circ}^{(\ell=4)}+{}^{\circ}\kappa_{180^\circ}\chi_{180^\circ}^{(E_1)*}\chi_{180^\circ}^{(\ell=4)}\right)$$

$$=\frac{1}{6}\left(\ 1\cdot 2^*\cdot 9\quad+\quad 2\cdot -1^*\cdot 0\quad+\quad 3\cdot 0^*\cdot 1\ \right)$$

$$f^{(E_1)}=3$$

$$f^{(A_2)}=\frac{1}{6}\left(\ 1\cdot 1^*\cdot 9\quad+\quad 2\cdot 1^*\cdot 0\quad+\quad 3\cdot -1^*\cdot 1\right)=1$$

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
$\boxed{4}$	$\boxed{9}$	$\boxed{0}$	$\boxed{1}$
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^\ell(\Theta)=\frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and *D*₃ character table from p. 24:

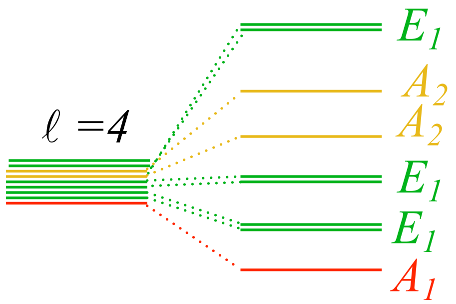
$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1\quad\oplus 2E_1$
3	1	2	2	$1A_1\oplus 2A_2\oplus 2E_1$
4	2	1	3	$\oplus 1A_2\oplus 3E_1$

Formula from p.44

Example: (ℓ=4)

$$f^{(b)}=\frac{1}{{}^{\circ}D_3}\sum_{\substack{classes\\ \kappa_k\in D_3}}{}^{\circ}\kappa_k\chi_k^{(b)*}\chi_k^{(\ell)}$$



$$f^{(E_1)}=\frac{1}{{}^{\circ}D_3}\sum_{\substack{classes\\ \kappa_k\in D_3}}{}^{\circ}\kappa_k\chi_k^{(E_1)*}\chi_k^{(\ell=4)}=\frac{1}{{}^{\circ}D_3}\left({}^{\circ}\kappa_{0^\circ}\chi_{0^\circ}^{(E_1)*}\chi_{0^\circ}^{(\ell=4)}+{}^{\circ}\kappa_{120^\circ}\chi_{120^\circ}^{(E_1)*}\chi_{120^\circ}^{(\ell=4)}+{}^{\circ}\kappa_{180^\circ}\chi_{180^\circ}^{(E_1)*}\chi_{180^\circ}^{(\ell=4)}\right)$$

$$=\frac{1}{6}\left(\ 1\cdot 2^{*}\cdot 9\quad+\quad 2\cdot -1^{*}\cdot 0\quad+\quad 3\cdot 0^{*}\cdot 1\ \right)$$

$$f^{(E_1)}=3$$

$$f^{(A_2)}=\frac{1}{6}\left(\ 1\cdot 1^{*}\cdot 9\quad+\quad 2\cdot 1^{*}\cdot 0\quad+\quad 3\cdot -1^{*}\cdot 1\right)=1$$

$$f^{(A_1)}=\frac{1}{6}\left(\ 1\cdot 1^{*}\cdot 9\quad+\quad 2\cdot 1^{*}\cdot 0\quad+\quad 3\cdot\ 1^{*}\cdot 1\right)=2$$

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^{\ell}(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^{\ell}(\Theta)=\frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and *D*3 character table from p. 24:

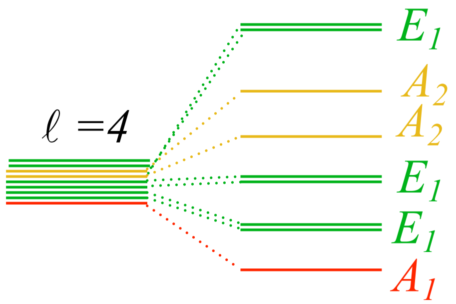
$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1\quad\oplus 2E_1$
3	1	2	2	$1A_1\oplus 2A_2\oplus 2E_1$
4	2	1	3	$2A_1\oplus 1A_2\oplus 3E_1$

Formula from p.44

Example: $(\ell=4)$

$$f^{(b)}=\frac{1}{{}^\circ D_3}\sum_{\substack{classes\\ \kappa_k\in D_3}} {}^\circ\kappa_k \chi_k^{(b)*}\chi_k^{(\ell)}$$



$$f^{(E_1)}=\frac{1}{{}^\circ D_3}\sum_{\substack{classes\\ \kappa_k\in D_3}} {}^\circ\kappa_k \chi_k^{(E_1)*}\chi_k^{(\ell=4)}=\frac{1}{{}^\circ D_3}\left({}^\circ\kappa_{0^\circ}\chi_{0^\circ}^{(E_1)*}\chi_{0^\circ}^{(\ell=4)}+{}^\circ\kappa_{120^\circ}\chi_{120^\circ}^{(E_1)*}\chi_{120^\circ}^{(\ell=4)}+{}^\circ\kappa_{180^\circ}\chi_{180^\circ}^{(E_1)*}\chi_{180^\circ}^{(\ell=4)}\right)$$

$$=\frac{1}{6}\left(\ 1\cdot 2^*\cdot 9\quad+\quad 2\cdot -1^*\cdot 0\quad+\quad 3\cdot 0^*\cdot 1\ \right)$$

$$f^{(E_1)}=3$$

$$f^{(A_2)}=\frac{1}{6}\left(\ 1\cdot 1^*\cdot 9\quad+\quad 2\cdot 1^*\cdot 0\quad+\quad 3\cdot -1^*\cdot 1\right)=1$$

$$f^{(A_1)}=\frac{1}{6}\left(\ 1\cdot 1^*\cdot 9\quad+\quad 2\cdot 1^*\cdot 0\quad+\quad 3\cdot\ 1^*\cdot 1\right)=2$$

U(2) characters
from Lecture 12.6 p.134:
(or end of this lecture)

$\chi^\ell(\Theta)$	$\Theta=0$	$\frac{2\pi}{3}$	π
$\ell=0$	1	1	1
1	3	0	−1
2	5	−1	1
3	7	1	−1
4	9	0	1
5	11	−1	−1
6	13	1	1
7	15	0	−1

$$\chi^\ell(\Theta)=\frac{\sin(\ell+\frac{1}{2})\Theta}{\sin\frac{\Theta}{2}}$$

...and D_3 character table from p. 24:

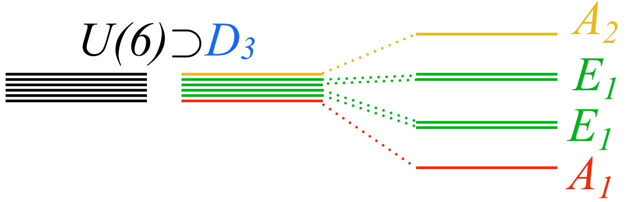
$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	−1
$\chi^{E_1}(\mathbf{g})=$	2	−1	0

$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1\quad\oplus 2E_1$
3	1	2	2	$1A_1\oplus 2A_2\oplus 2E_1$
4	2	1	3	$2A_1\oplus 1A_2\oplus 3E_1$
5	1	2	4	$1A_1\oplus 2A_2\oplus 4E_1$
6	3	2	4	$3A_1\oplus 2A_2\oplus 4E_1$
7	2	3	5	$2A_1\oplus 3A_2\oplus 5E_1$

$Note:\ \ \ell=6\ \big|\ 13\quad 1\quad 1\ \big|= A_1\ \big|\ 1\quad 1\quad 1\ \big|\ \oplus 2R^G\ \big|\ 12\quad 0\quad 0\ \big| = A_1\oplus 2[A_1\oplus A_2\oplus 2E_1]$

($\ell=6$ is 1st re-cycling point)

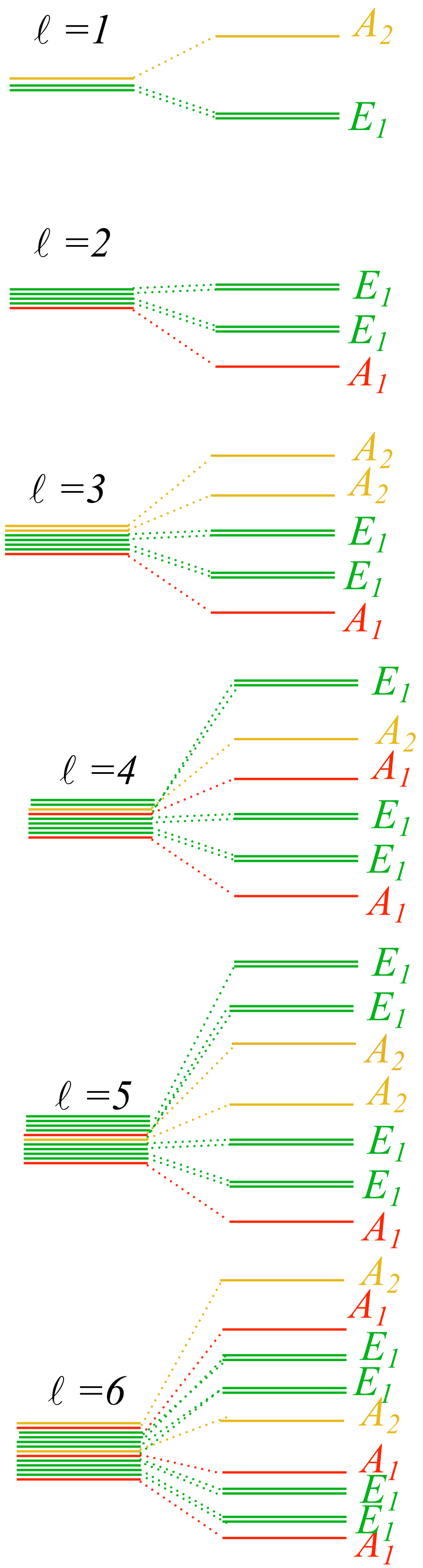
Spectral splitting in symmetry breaking foretold by character analysis (on p. 38)



$$R^G(U(6))\downarrow D_3 = D^{A_1}(\mathbf{g})\oplus D^{A_2}(\mathbf{g})\oplus 2D^{E_1}(\mathbf{g})$$

Crystal-field splitting: $O(3)\supset D_3$ symmetry reduction and $D^{\ell}\downarrow D_3$ splitting

$R(3)\supset D_3$



$f^{(\alpha)}(\ell)$	f^{A_1}	f^{A_2}	f^{E_1}	
$\ell=0$	1	·	·	$1A_1$
1	·	1	1	$0A_1\oplus A_2\oplus E_1$
2	1	·	2	$1A_1 \quad \oplus 2E_1$
3	1	2	2	$1A_1\oplus 2A_2\oplus 2E_1$
4	2	1	3	$2A_1\oplus 1A_2\oplus 3E_1$
5	1	2	4	$1A_1\oplus 2A_2\oplus 4E_1$
6	3	2	4	$3A_1\oplus 2A_2\oplus 4E_1$
7	2	3	5	$2A_1\oplus 3A_2\oplus 5E_1$

D_3 character table:

$(\mathbf{g})=$	$\{\mathbf{1}\}$	$\{\mathbf{r}^1,\mathbf{r}^2\}$	$\{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3\}$
$\chi^{A_1}(\mathbf{g})=$	1	1	1
$\chi^{A_2}(\mathbf{g})=$	1	1	-1
$\chi^{E_1}(\mathbf{g})=$	2	-1	0

*Review: Spectral resolution of **D_3 Center** (Class algebra)*

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting class projectors and D_3-invariant character ortho-completeness

*Spectral resolution to **irreducible representations** (or "**irreps**") foretold by **characters** or traces*

Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

Atomic ℓ-level or $2\ell+1$-multiplet splitting

D_3 examples for $\ell=1$-6

Group invariant numbers: Centrum, Rank, and Order

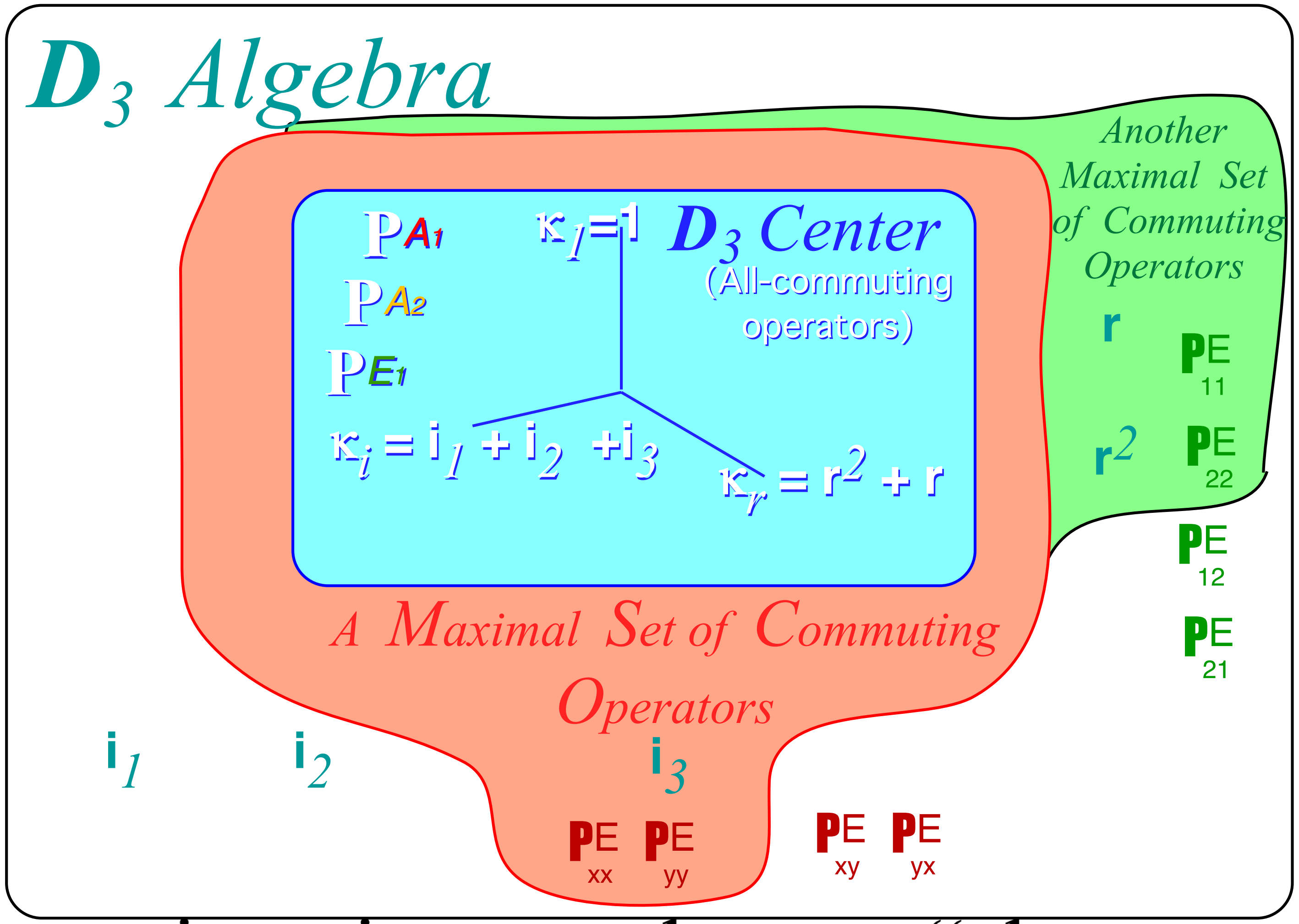
2nd-Stage spectral decompositions of global/local D_3

Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$

*3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions*

Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains

D_3 Algebra



P^{A_1}

$\kappa_1=1$

D_3 Center

(All-commuting operators)

P^{A_2}

P^{E_1}

$\kappa_i = i_1 + i_2 + i_3$

$\kappa_r = r^2 + r$

Another Maximal Set of Commuting Operators

$\mathbf{r}$ $\mathbf{P}^E_{11}$

$\mathbf{r}^2$ $\mathbf{P}^E_{22}$

$\mathbf{P}^E_{12}$

$\mathbf{P}^E_{21}$

A Maximal Set of Commuting Operators

$\mathbf{i}_1$ $\mathbf{i}_2$ $\mathbf{i}_3$

$\mathbf{P}^E_{xx}$ $\mathbf{P}^E_{yy}$ $\mathbf{P}^E_{xy}$ $\mathbf{P}^E_{yx}$

D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Important invariant numbers or "characters"

$\ell^\alpha=$ Irreducible representation (irrep) *dimension* or level *degeneracy*

For symmetry group or algebra G

Centrum: $\kappa(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^0$ =Number of classes, invariants, irrep types, *all-commuting* ops

Rank: $\rho(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^1$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

Order: $^\circ(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops

$$\kappa(D_3)=(1)^0+(1)^0+(2)^0=3$$

$$\rho(D_3)=(1)^1+(1)^1+(2)^1=4$$

$$^\circ(D_3)=(1)^2+(1)^2+(2)^2=6$$

*Review: Spectral resolution of **D_3 Center** (Class algebra)*

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting class projectors and D_3-invariant character ortho-completeness

Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

Group invariant numbers: Centrum, Rank, and Order

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Splitting classes

*3rd-stage spectral resolution to **irreducible representations** (ireps) and Hamiltonian eigensolutions*

Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains

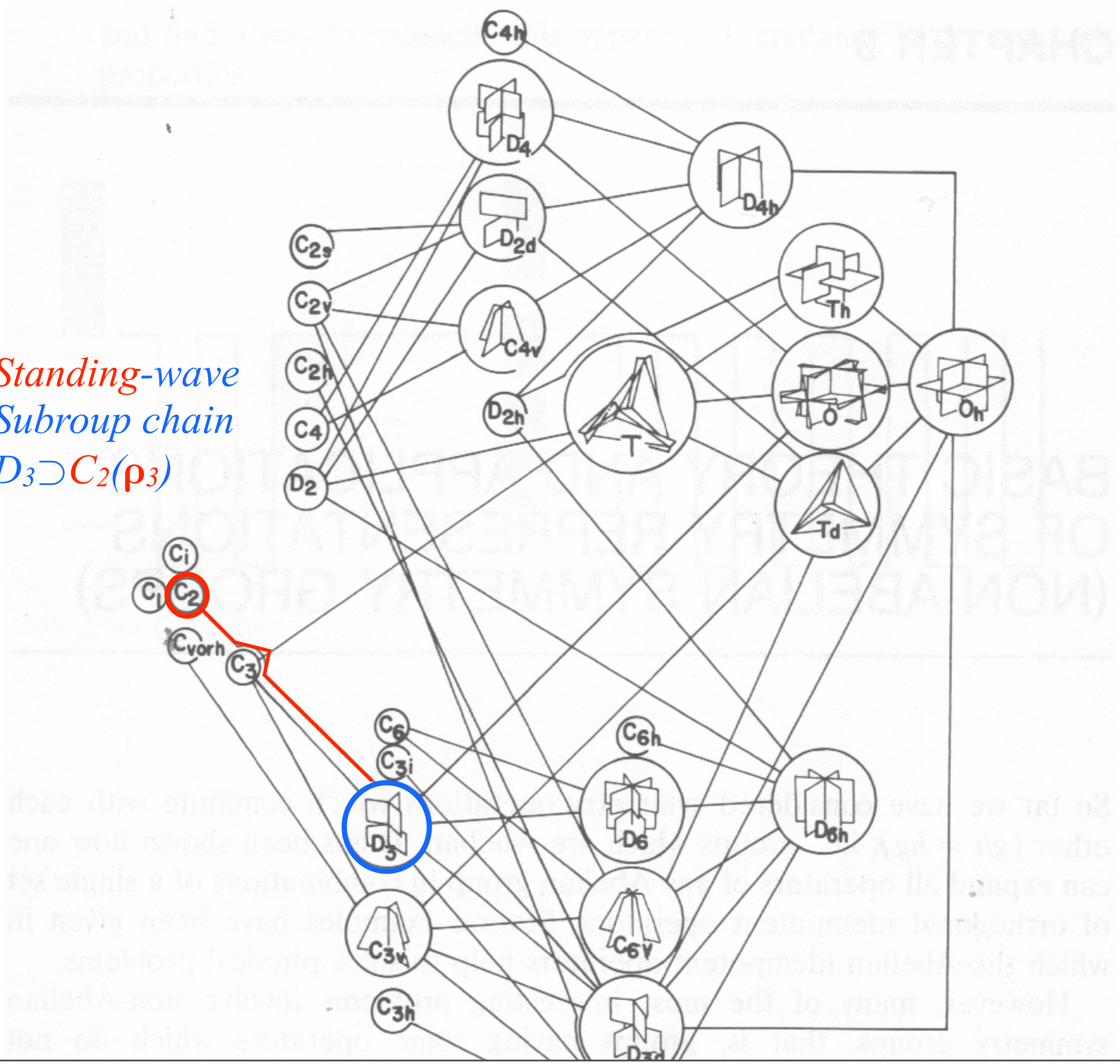
Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

*Correlate D_3 characters with its subgoup(s) C_2(**i**)*

Standing-wave
Subroup chain
$D_3 \supset C_2(\rho_3)$



Spectral reduction of non-commutative “Group-table Hamiltonian”

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) C_2(**i**)

D_3 κ =	**1**	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 κ =	**1**	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

Spectral reduction of non-commutative “Group-table Hamiltonian”

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$

D_3 $\kappa =$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa =$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Spectral reduction of non-commutative “Group-table Hamiltonian”

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$

$$\begin{array}{c|ccc|c} D_3 \quad \kappa = & \mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 & \\ \hline \mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\ \mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\ \mathbf{P}^{E}= & 2 & -1 & 0 & /3 \end{array}$$

$$\begin{array}{c|cc|c} C_2 \quad \kappa = & \mathbf{1} & \mathbf{i}_3 & \\ \hline p^{0_2}= & 1 & 1 & /2 \\ p^{1_2}= & 1 & -1 & /2 \end{array}$$

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)

$$\begin{array}{c|cc} D_3 \supset C_2 & 0_2 & 1_2 \\ \hline n^{A_1}= & 1 & \cdot \\ n^{A_2}= & \cdot & 1 \\ n^{E}= & 1 & 1 \end{array}$$

Rank $\rho(D_3)$=4 implies
there will be exactly 4
“C_2-friendly” irep projectors

$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)} (p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

*Correlate D_3 characters with its subgoup(s) C_2(**i**)*

D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 $\kappa=$	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Rank $\rho(D_3)$=4 implies there will be exactly 4 "C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$$1 = p^{0_2} + p^{1_2}$$

	0_2	1_2
$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_2 0_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_2 1_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_2 0_2}$	$\mathbf{P}^{E}_{1_2 1_2}$

Spectral reduction of non-commutative “Group-table Hamiltonian”

D_3 *Example*

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$

D_3 κ=	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2 κ=	$\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Rank $\rho(D_3)$=4 implies there will be exactly 4 “C_2-friendly” irep projectors

$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2}) = \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$$1 = p^{0_2} + p^{1_2}$$

$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_2 0_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_2 1_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_2 0_2}$	$\mathbf{P}^{E}_{1_2 1_2}$

↓

$$\mathbf{P}^{A_1} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2 = (2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2 = (2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

Review:Spectral resolution of ***D_3 Center*** *(Class algebra)*
- *Group theory of equivalence transformations and classes*
  - *Lagrange theorems*
- *All-commuting class projectors and D_3-invariant character ortho-completeness*
  - *Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*
- *Group invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3
- *Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*
  - *Splitting classes*

3rd-stage spectral resolution to ***irreducible representations*** *(ireps) and Hamiltonian eigensolutions*
- *Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

2nd-Stage

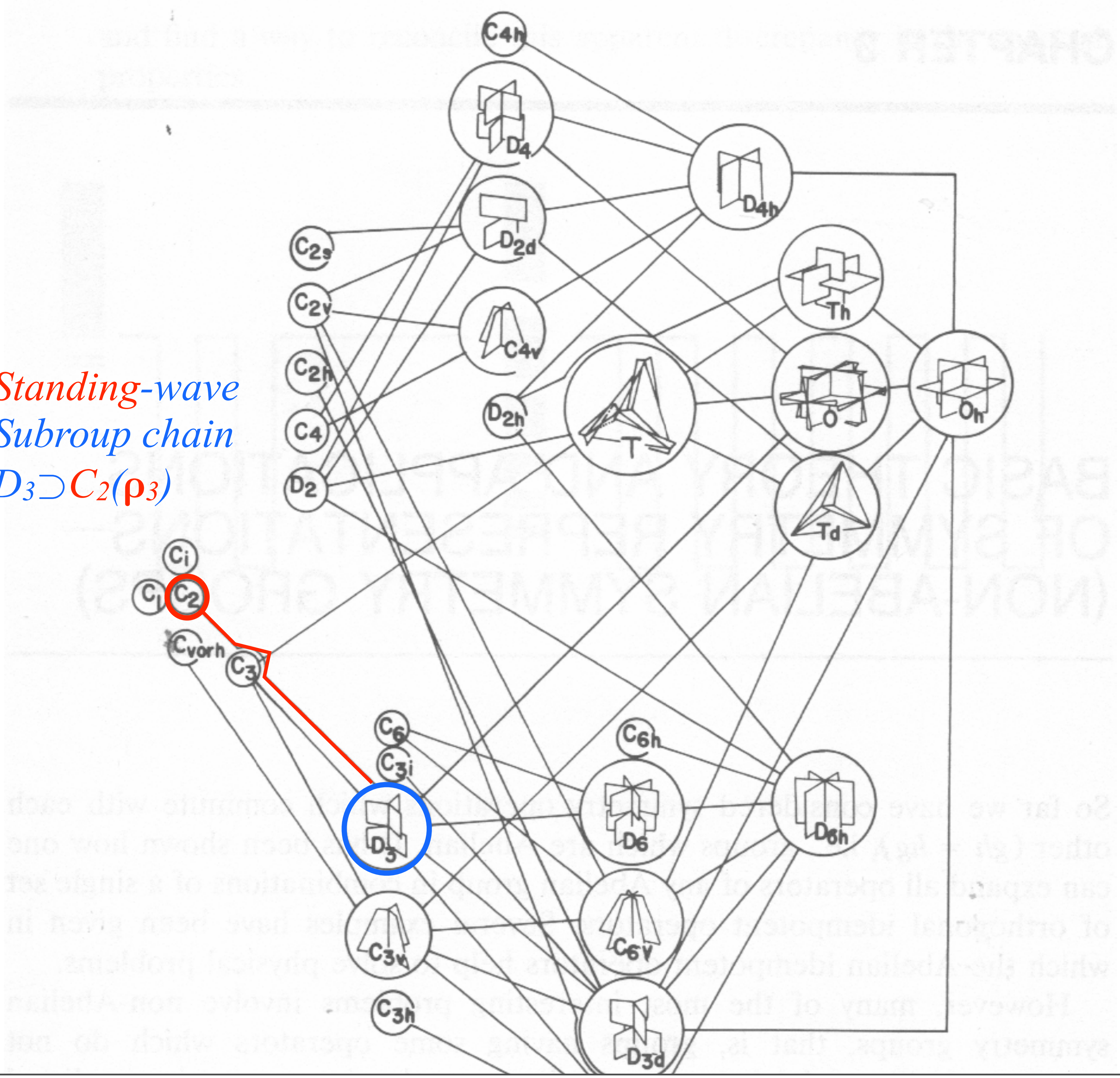
Spectral reduction of non-commutative “Group-table Hamiltonian”

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

*Correlate D_3 characters with its subgoup(s) C_2(**i**)*

Standing-wave Subroup chain $D_3 \supset C_2(\rho_3)$



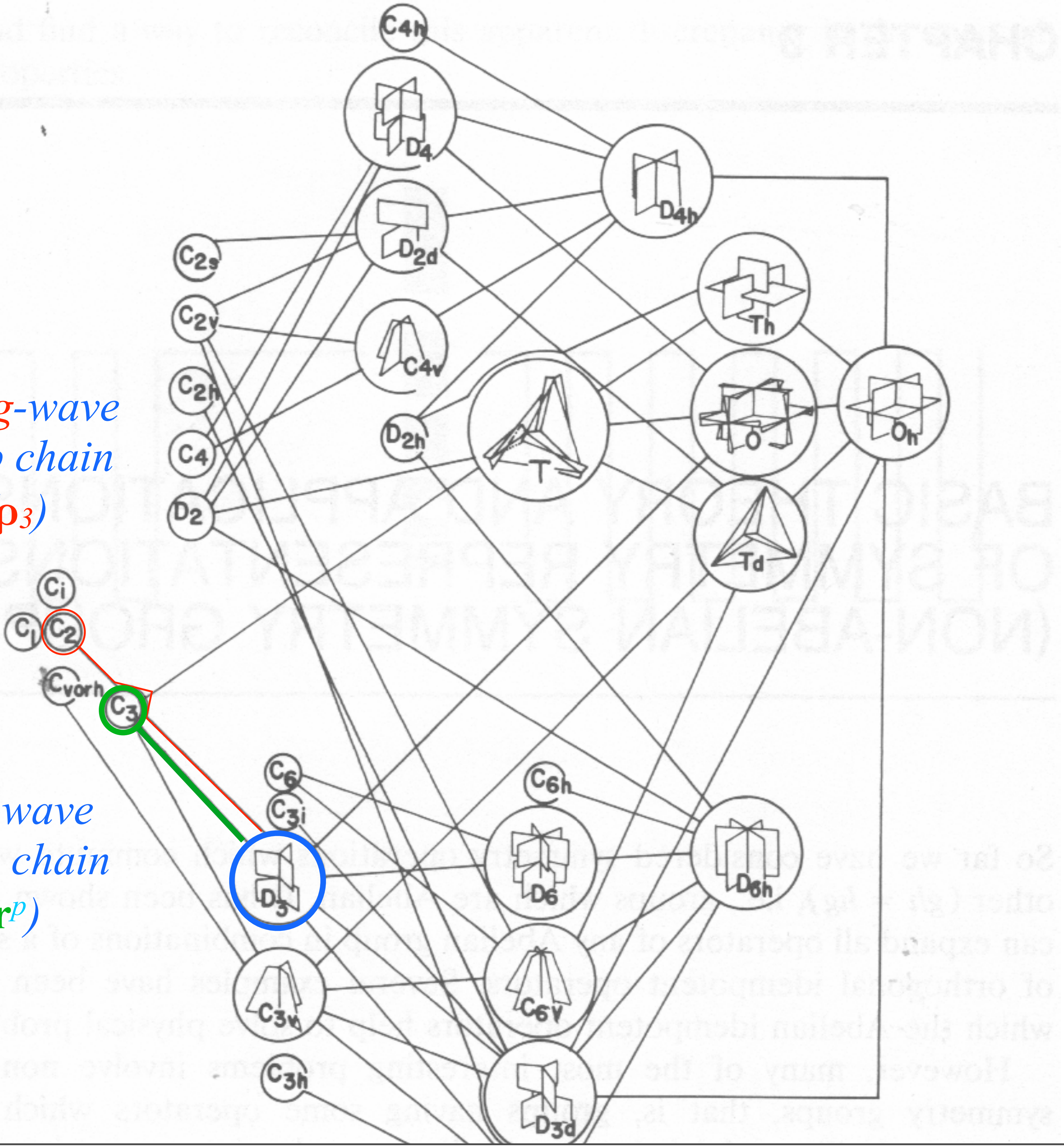
2nd-Stage

Spectral reduction of non-commutative “Group-table Hamiltonian”

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ (C_2 and C_3 don’t commute)



Standing-wave
Subroup chain
$D_3 \supset C_2(\rho_3)$

Moving-wave
Subroup chain
$D_3 \supset C_3(\mathbf{r}^p)$

2nd-Stage

Spectral reduction of non-commutative “Group-table Hamiltonian”

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ (C_2 and C_3 don’t commute)

$$\begin{array}{c|ccc|c} D_3\ \kappa= & \mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 & \\ \hline \mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\ \mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\ \mathbf{P}^{E}= & 2 & -1 & 0 & /3 \end{array}$$

$$\begin{array}{c|cc|c} C_2\ \kappa= & \mathbf{1} & \mathbf{i}_3 & \\ \hline p^{0_2}= & 1 & 1 & /2 \\ p^{1_2}= & 1 & -1 & /2 \end{array}$$

Let:

$\varepsilon = e^{-2\pi i/3}$

$$\begin{array}{c|ccc|c} C_3\ \kappa= & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \\ \hline p^{0_3}= & 1 & 1 & 1 & /3 \\ p^{1_3}= & 1 & \varepsilon & \varepsilon^* & /3 \\ p^{2_3}= & 2 & \varepsilon^* & \varepsilon & /3 \end{array}$$

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)

$$\begin{array}{c|cc} D_3 \supset C_2 & 0_2 & 1_2 \\ \hline n^{A_1}= & 1 & \cdot \\ n^{A_2}= & \cdot & 1 \\ n^{E}= & 1 & 1 \end{array}$$

Rank $\rho(D_3)$=4 implies

there will be exactly 4

“C_2-friendly” irep projectors

$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$

$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$$1 = p^{0_2} + p^{1_2}$$

$$\begin{array}{c|cc} \mathbf{P}^{A_1}= & \mathbf{P}^{A_1}_{0_2 0_2} & \cdot \\ \mathbf{P}^{A_2}= & \cdot & \mathbf{P}^{A_2}_{1_2 1_2} \\ \mathbf{P}^{E}= & \mathbf{P}^{E}_{0_2 0_2} & \mathbf{P}^{E}_{1_2 1_2} \end{array}$$

$$\downarrow$$

$$\mathbf{P}^{A_1} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2 = (\ \mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3\ )/6$$

$$\mathbf{P}^{A_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2 = (\ \mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3\ )/6$$

2nd-Stage

Spectral reduction of non-commutative “Group-table Hamiltonian”

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) C_2(i) or ELSE C_3(r) ***(C_2 and C_3 don’t commute)***

$$\begin{array}{c|ccc|c} D_3\ \kappa= & \boxed{\mathbf{1}} & \boxed{\mathbf{r}^1+\mathbf{r}^2} & \boxed{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3} & \\ \mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\ \mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\ \mathbf{P}^{E}= & 2 & -1 & 0 & /3 \end{array}$$

$$\begin{array}{c|cc|c} C_2\ \kappa= & \mathbf{1} & \mathbf{i}_3 & \\ p^{0_2}= & 1 & 1 & /2 \\ p^{1_2}= & 1 & -1 & /2 \end{array}$$

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $\mathbf{1} = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)

$$\begin{array}{c|cc} D_3\supset C_2 & 0_2 & 1_2 \\ n^{A_1}= & 1 & \cdot \\ n^{A_2}= & \cdot & 1 \\ n^{E}= & 1 & 1 \end{array}$$

Rank $\rho(D_3)$=4 implies

there will be exactly 4

“C_2-friendly” irep projectors

$$\mathbf{P}^{(\alpha)}\mathbf{1} = \mathbf{P}^{(\alpha)}(p^{0_2}+p^{1_2})$$

$$= \mathbf{P}^{(\alpha)}_{0_20_2} + \mathbf{P}^{(\alpha)}_{1_21_2}$$

$$\mathbf{1} = p^{0_2} + p^{1_2}$$

$$\begin{array}{c|cc|} \mathbf{P}^{A_1}= & \mathbf{P}^{A_1}_{0_20_2} & \cdot \\ \mathbf{P}^{A_2}= & \cdot & \mathbf{P}^{A_2}_{1_21_2} \\ \mathbf{P}^{E}= & \mathbf{P}^{E}_{0_20_2} & \mathbf{P}^{E}_{1_21_2} \end{array}$$

$$\mathbf{P}^{A_1} = \mathbf{P}^{A_1}p^{0_2} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{A_2} = \mathbf{P}^{A_2}p^{1_2} = \mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{0_20_2} = \mathbf{P}^{E}p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2 = (2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{1_21_2} = \mathbf{P}^{E}p^{1_2} = \mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2 = (2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3\ )/6$$

Let:

$\varepsilon = e^{-2\pi i/3}$

$$\begin{array}{c|ccc|c} C_3\ \kappa= & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \\ p^{0_3}= & 1 & 1 & 1 & /3 \\ p^{1_3}= & 1 & \varepsilon & \varepsilon^* & /3 \\ p^{2_3}= & 1 & \varepsilon^* & \varepsilon & /3 \end{array}$$

Same for *Correlation table:*

$$\begin{array}{c|ccc} D_3\supset C_3 & 0_3 & 1_3 & 2_3 \\ n^{A_1}= & 1 & \cdot & \cdot \\ n^{A_2}= & 1 & \cdot & \cdot \\ n^{E}= & \cdot & 1 & 1 \end{array}$$

2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ (C_2 and C_3 don't commute)

D_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2	$\kappa=\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

*$D_3\supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1=p^{0_2}+p^{1_2}$ will make **IRREDUCIBLE** $\mathbf{P}^{(\alpha)}_{n,n}$)*

$D_3\supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Rank $\rho(D_3)$=4 implies

there will be exactly 4

"C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_2}+p^{1_2})$$

$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_21_2}$$

$$1=p^{0_2}+p^{1_2}$$

$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_20_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_21_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_20_2}$	$\mathbf{P}^{E}_{1_21_2}$

$$\downarrow$$

$$\mathbf{P}^{A_1}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\ 1+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{A_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1\text{-}\mathbf{i}_3)/2=(\ 1+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\ \mathbf{i}_2-\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{0_20_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\ \mathbf{i}_2+2\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{1_21_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1\text{-}\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\ \mathbf{i}_2-2\mathbf{i}_3\ )/6$$

Let:

$\varepsilon=e^{-2\pi i/3}$

C_3	$\kappa=\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	
$p^{0_3}=$	1	1	1	/3
$p^{1_3}=$	1	ε	ε^*	/3
$p^{2_3}=$	1	ε^*	ε	/3

Same for *Correlation table:*

$D_3\supset C_3$	0_3	1_3	2_3
$n^{A_1}=$	1	·	·
$n^{A_2}=$	1	·	·
$n^{E}=$	·	1	1

Rank $\rho(D_3)$=4 implies

there will be exactly 4

"C_3-friendly" irreducible projectors

$$\mathbf{P}^{(\alpha)}1=\mathbf{P}^{(\alpha)}(p^{0_3}+p^{1_3}+p^{2_3})$$

$$=\mathbf{P}^{(\alpha)}_{0_20_2}+\mathbf{P}^{(\alpha)}_{1_31_3}+\mathbf{P}^{(\alpha)}_{2_32_3}$$

2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) $C_2(\mathbf{i})$ or ELSE $C_3(\mathbf{r})$ (C_2 and C_3 don't commute)

$$\begin{array}{c|ccc|c} D_3 \ \kappa = & \mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 & \\ \hline \mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\ \mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\ \mathbf{P}^{E}= & 2 & -1 & 0 & /3 \end{array}$$

$$\begin{array}{c|cc|c} C_2 \ \kappa = & \mathbf{1} & \mathbf{i}_3 & \\ \hline p^{0_2}= & 1 & 1 & /2 \\ p^{1_2}= & 1 & -1 & /2 \end{array}$$

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)

$$\begin{array}{c|cc} D_3 \supset C_2 & 0_2 & 1_2 \\ \hline n^{A_1}= & 1 & \cdot \\ n^{A_2}= & \cdot & 1 \\ n^{E}= & 1 & 1 \end{array}$$

Rank $\rho(D_3)=4$ implies
there will be exactly 4
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$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_2 1_2}$$

$$1 = p^{0_2} + p^{1_2}$$

$$\begin{array}{c|cc} \mathbf{P}^{A_1}= & \mathbf{P}^{A_1}_{0_2 0_2} & \cdot \\ \mathbf{P}^{A_2}= & \cdot & \mathbf{P}^{A_2}_{1_2 1_2} \\ \mathbf{P}^{E}= & \mathbf{P}^{E}_{0_2 0_2} & \mathbf{P}^{E}_{1_2 1_2} \end{array}$$

↓

$$\mathbf{P}^{A_1} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{A_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2 = (2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2 = (2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3\ )/6$$

Let:
$\varepsilon = e^{-2\pi i/3}$

$$\begin{array}{c|ccc|c} C_3 \ \kappa = & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \\ \hline p^{0_3}= & 1 & 1 & 1 & /3 \\ p^{1_3}= & 1 & \varepsilon & \varepsilon^* & /3 \\ p^{2_3}= & 1 & \varepsilon^* & \varepsilon & /3 \end{array}$$

Same for Correlation table:

$$\begin{array}{c|ccc} D_3 \supset C_3 & 0_3 & 1_3 & 2_3 \\ \hline n^{A_1}= & 1 & \cdot & \cdot \\ n^{A_2}= & 1 & \cdot & \cdot \\ n^{E}= & \cdot & 1 & 1 \end{array}$$

Rank $\rho(D_3)=4$ implies
there will be exactly 4
"C_3-friendly" irreducible projectors

$$\mathbf{P}^{(\alpha)} 1 = \mathbf{P}^{(\alpha)}(p^{0_3} + p^{1_3} + p^{2_3})$$
$$= \mathbf{P}^{(\alpha)}_{0_2 0_2} + \mathbf{P}^{(\alpha)}_{1_3 1_3} + \mathbf{P}^{(\alpha)}_{2_3 2_3}$$

$$1 = p^{0_3} + p^{1_3} + p^{2_3}$$

$$\begin{array}{c|ccc} \mathbf{P}^{A_1}= & \mathbf{P}^{A_1}_{0_3 0_3} & \cdot & \cdot \\ \mathbf{P}^{A_2}= & \mathbf{P}^{A_2}_{0_3 0_3} & \cdot & \cdot \\ \mathbf{P}^{E}= & \cdot & \mathbf{P}^{E}_{1_3 1_3} & \mathbf{P}^{E}_{2_3 2_3} \end{array}$$

2nd-Stage

Spectral reduction of non-commutative "Group-table Hamiltonian"

D_3 Example

2nd Step: Spectral resolution of Class Projector(s) of D_3

Correlate D_3 characters with its subgoup(s) C_2($\mathbf{i}$) or ELSE C_3($\mathbf{r}$) *(C_2 and C_3 don't commute)*

D_3	κ = $\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

C_2	κ = $\mathbf{1}$	$\mathbf{i}_3$	
$p^{0_2}=$	1	1	/2
$p^{1_2}=$	1	-1	/2

Let:
$\varepsilon=e^{-2\pi i/3}$

C_3	κ = $\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	
$p^{0_3}=$	1	1	1	/3
$p^{1_3}=$	1	ε	ε*	/3
$p^{2_3}=$	1	ε*	ε	/3

$D_3 \supset C_2$ Correlation table shows which products of class projector $\mathbf{P}^{(\alpha)}$ with C_2-unit $1 = p^{0_2} + p^{1_2}$ will make IRREDUCIBLE $\mathbf{P}^{(\alpha)}_{n,n}$)

$D_3 \supset C_2$	0_2	1_2
$n^{A_1}=$	1	·
$n^{A_2}=$	·	1
$n^{E}=$	1	1

Same for *Correlation table:*

$D_3 \supset C_3$	0_3	1_3	2_3
$n^{A_1}=$	1	·	·
$n^{A_2}=$	1	·	·
$n^{E}=$	·	1	1

Rank ρ(D_3)=4 implies there will be exactly 4 "C_2-friendly" irep projectors

$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_2} + p^{1_2})$$
$$= \mathbf{P}^{(\alpha)}_{0_20_2} + \mathbf{P}^{(\alpha)}_{1_21_2}$$

$$1 = p^{0_2} + p^{1_2}$$

$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_20_2}$	·
$\mathbf{P}^{A_2}=$	·	$\mathbf{P}^{A_2}_{1_21_2}$
$\mathbf{P}^{E}=$	$\mathbf{P}^{E}_{0_20_2}$	$\mathbf{P}^{E}_{1_21_2}$

Rank ρ(D_3)=4 implies there will be exactly 4 "C_3-friendly" irreducible projectors

$$\mathbf{P}^{(\alpha)}1 = \mathbf{P}^{(\alpha)}(p^{0_3} + p^{1_3} + p^{2_3})$$
$$= \mathbf{P}^{(\alpha)}_{0_20_2} + \mathbf{P}^{(\alpha)}_{1_31_3} + \mathbf{P}^{(\alpha)}_{2_32_3}$$

$$1 = p^{0_3} + p^{1_3} + p^{2_3}$$

$\mathbf{P}^{A_1}=$	$\mathbf{P}^{A_1}_{0_30_3}$	·	·
$\mathbf{P}^{A_2}=$	$\mathbf{P}^{A_2}_{0_30_3}$	·	·
$\mathbf{P}^{E}=$	·	$\mathbf{P}^{E}_{1_31_3}$	$\mathbf{P}^{E}_{2_32_3}$

$$\mathbf{P}^{A_1} = \mathbf{P}^{A_1}p^{0_2} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{A_2} = \mathbf{P}^{A_2}p^{1_2} = \mathbf{P}^{A_2}(\mathbf{1}\text{-}\mathbf{i}_3)/2 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\ \mathbf{i}_2-\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{0_20_2} = \mathbf{P}^{E}p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2 = (2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\ \mathbf{i}_2+2\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{1_21_2} = \mathbf{P}^{E}p^{1_2} = \mathbf{P}^{E}(\mathbf{1}\text{-}\mathbf{i}_3)/2 = (2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\ \mathbf{i}_2-2\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{A_1}_{0_30_3} = \mathbf{P}^{A_1}p^{0_3} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)/3 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{A_2}_{0_30_3} = \mathbf{P}^{A_2}p^{0_3} = \mathbf{P}^{A_2}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)/3 = (\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\ \mathbf{i}_2-\mathbf{i}_3\ )/6$$

$$\mathbf{P}^{E}_{1_31_3} = \mathbf{P}^{E}p^{1_3} = \mathbf{P}^{E}(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3 = (\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2\ )/3$$

$$\mathbf{P}^{E}_{2_32_3} = \mathbf{P}^{E}p^{2_3} = \mathbf{P}^{E}(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3 = (\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2\ )/3$$

Review:Spectral resolution of ***D_3 Center*** *(Class algebra)*

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting class projectors and D_3-invariant character ortho-completeness

Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$

Atomic ℓ-level or $2\ell+1$-multiplet splitting

D_3 examples for ℓ=1-6

Group invariant numbers: Centrum, Rank, and Order

2nd-Stage spectral decompositions of global/local D_3

Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$

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3rd-stage spectral resolution to ***irreducible representations*** *(ireps) and Hamiltonian eigensolutions*

Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains

2nd-Stage

2nd Step: (contd.) While some class projectors $\mathbf{P}^{(\alpha)}$ split in two, so ALSO DO some classes κ_k

$$\begin{array}{c|ccc|l} D_3 & \kappa=\boxed{\mathbf{1}} & \boxed{\mathbf{r}^1+\mathbf{r}^2} & \boxed{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3} & \\ \hline \mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\ \mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\ \mathbf{P}^{E}= & 2 & -1 & 0 & /3 \end{array}$$

Rank $\rho(D_3)=4$

idempotents

$\mathbf{P}^{(\alpha)}$

4 different

idempotent

$\mathbf{P}^{(\alpha)}_{n,n}$

Centrum $\kappa(D_3)=3$

idempotents

$\mathbf{P}^{(\alpha)}$

$$\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}+\boxed{\boxed{\mathbf{i}_1+\mathbf{i}_2}+\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}-\boxed{\boxed{\mathbf{i}_1-\mathbf{i}_2}-\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}-\boxed{\boxed{\mathbf{i}_1-\mathbf{i}_2}+2\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}+\boxed{\boxed{\mathbf{i}_1+\mathbf{i}_2}-2\boxed{\mathbf{i}_3}})/6$$

$\mathbf{P}^E$ splits into $\mathbf{P}^E=\mathbf{P}^E_{0_2 0_2}+\mathbf{P}^E_{1_2 1_2}$,

class $\kappa_\mathbf{i}$ splits into $\kappa_{\mathbf{i}_{12}}$ and $\kappa_{\mathbf{i}_3}$

$$\mathbf{P}^{A_1}_{0_3 0_3}=\mathbf{P}^{A_1}p^{0_3}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)/3=(\boxed{\mathbf{1}}+\boxed{\boxed{\mathbf{r}^1}+\boxed{\mathbf{r}^2}}+\boxed{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3})/6$$

$$\mathbf{P}^{A_2}_{0_3 0_3}=\mathbf{P}^{A_2}p^{0_3}=\mathbf{P}^{A_2}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)/3=(\boxed{\mathbf{1}}+\boxed{\boxed{\mathbf{r}^1}+\boxed{\mathbf{r}^2}}-\boxed{\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3})/6$$

$$\mathbf{P}^{E}_{1_3 1_3}=\mathbf{P}^{E}p^{1_3}=\mathbf{P}^{E}(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3=(\boxed{\mathbf{1}}+\boxed{\boxed{\varepsilon\mathbf{r}^1}+\boxed{\varepsilon^*\mathbf{r}^2}})/3$$

$$\mathbf{P}^{E}_{2_3 2_3}=\mathbf{P}^{E}p^{2_3}=\mathbf{P}^{E}(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3=(\boxed{\mathbf{1}}+\boxed{\boxed{\varepsilon^*\mathbf{r}^1}+\boxed{\varepsilon\mathbf{r}^2}})/3$$

$\varepsilon=e^{-2\pi i/3}$

$\mathbf{P}^E$ splits into $\mathbf{P}^E=\mathbf{P}^E_{1_3 1_3}+\mathbf{P}^E_{2_3 2_3}$,

class $\kappa_\mathbf{r}$ splits into $\kappa_{\mathbf{r}^1}$ and $\kappa_{\mathbf{r}^2}$

2nd Step: (contd.) While some class projectors $\mathbf{P}^{(\alpha)}$ split in two, so ALSO DO some classes κ_k

	D_3 $\kappa =\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}$

4 different
idempotent
$\mathbf{P}^{(\alpha)}_{n,n}$

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

$$\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

$\mathbf{P}^E$ *splits into* $\mathbf{P}^E=\mathbf{P}^E_{0_2 0_2}+\mathbf{P}^E_{1_2 1_2}$,

class $\kappa_{\mathbf{i}}$ *splits into* $\kappa_{\mathbf{i}_{12}}$ *and* $\kappa_{\mathbf{i}_3}$

$$\mathbf{P}^{A_1}_{0_3 0_3}=\mathbf{P}^{A_1}p^{0_3}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{0_3 0_3}=\mathbf{P}^{A_2}p^{0_3}=\mathbf{P}^{A_2}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)/3=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{1_3 1_3}=\mathbf{P}^{E}p^{1_3}=\mathbf{P}^{E}(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon\mathbf{r}^1+\varepsilon^*\mathbf{r}^2)/3$$

$$\mathbf{P}^{E}_{2_3 2_3}=\mathbf{P}^{E}p^{2_3}=\mathbf{P}^{E}(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3=(\mathbf{1}+\varepsilon^*\mathbf{r}^1+\varepsilon\mathbf{r}^2)/3$$

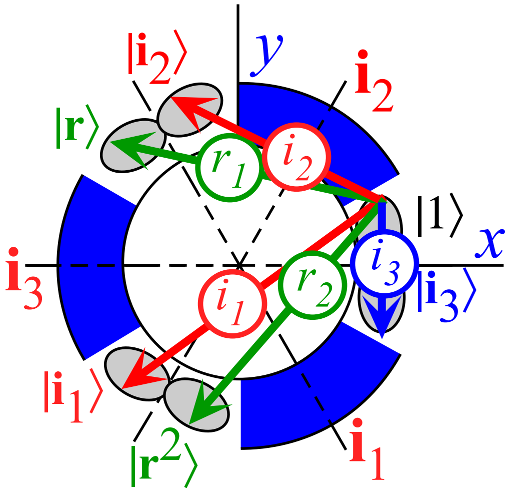
$\varepsilon=e^{-2\pi i/3}$

$\mathbf{P}^E$ *splits into* $\mathbf{P}^E=\mathbf{P}^E_{1_3 1_3}+\mathbf{P}^E_{2_3 2_3}$,

class $\kappa_{\mathbf{r}}$ *splits into* $\kappa_{\mathbf{r}^1}$ *and* $\kappa_{\mathbf{r}^2}$

$r=r_2$	$i=i_2$
must	must
equal	equal
r_1	i_1

For Local
$D_3\supset C_2(\mathbf{i}_3)$
symmetry
i_3 *is free parameter*



Rank $\rho(D_3)$= 4
parameters in
either case

$i=i_1=i_2=i_3$

For Local
$D_3\supset C_3(\mathbf{r}^p)$
symmetry
r_1 *and* r_2 *are free*

Review:Spectral resolution of ***D_3 Center*** *(Class algebra)*

- *Group theory of equivalence transformations and classes*
  - *Lagrange theorems*
- *All-commuting class projectors and D_3-invariant character ortho-completeness*
  - *Subgroup splitting and correlation frequency formula: $f^{(a)}(D^{(\alpha)}(G)\downarrow H)$*
- *Group invariant numbers: Centrum, Rank, and Order*

2nd-Stage spectral decompositions of global/local D_3

- *Splitting class projectors using subgroup chains $D_3 \supset C_2$ and $D_3 \supset C_3$*
  - *Splitting classes*

3rd-stage spectral resolution to ***irreducible representations*** *(ireps) and Hamiltonian eigensolutions*

- *Tunneling modes and spectra for $D_3 \supset C_2$ and $D_3 \supset C_3$ local subgroup chains*

Centrum $\kappa(D_3)=3$

idempotents

$\mathbf{P}^{(\alpha)}$

$$
\begin{array}{r|ccc|l}
D_3\ \kappa = & \boxed{\mathbf{1}} & \boxed{\mathbf{r}^1+\mathbf{r}^2} & \boxed{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3} & \\
\hline
\mathbf{P}^{A_1} = & 1 & 1 & 1 & /6 \\
\mathbf{P}^{A_2} = & 1 & 1 & -1 & /6 \\
\mathbf{P}^{E} = & 2 & -1 & 0 & /3
\end{array}
$$

Rank $\rho(D_3)=4$

idempotents

$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x} = \mathbf{P}^{A_1}_{0_2 0_2} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2 = (\boxed{\mathbf{1}} + \boxed{\mathbf{r}^1 + \mathbf{r}^2} + \boxed{\boxed{\mathbf{i}_1 + \mathbf{i}_2} + \boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{A_2}_{y,y} = \mathbf{P}^{A_2}_{1_2 1_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2 = (\boxed{\mathbf{1}} + \boxed{\mathbf{r}^1 + \mathbf{r}^2} - \boxed{\boxed{\mathbf{i}_1 - \mathbf{i}_2} - \boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{E}_{x,x} = \mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2 = (2\boxed{\mathbf{1}} - \boxed{\mathbf{r}^1 - \mathbf{r}^2} - \boxed{\boxed{\mathbf{i}_1 - \mathbf{i}_2} + 2\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{E}_{y,y} = \mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2 = (2\boxed{\mathbf{1}} - \boxed{\mathbf{r}^1 - \mathbf{r}^2} + \boxed{\boxed{\mathbf{i}_1 + \mathbf{i}_2} - 2\boxed{\mathbf{i}_3}})/6$$

3rd and Final Step:
Spectral resolution of ALL 6 of D3 :

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

$$\begin{array}{c|ccc|c}
D_3 \ \kappa = & \mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 & \\
\hline
\mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\
\mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\
\mathbf{P}^{E}= & 2 & -1 & 0 & /3
\end{array}$$

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x} = \mathbf{P}^{A_1}_{0_2 0_2} = \mathbf{P}^{A_1} p^{0_2} = \mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2 = (\ \mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y} = \mathbf{P}^{A_2}_{1_2 1_2} = \mathbf{P}^{A_2} p^{1_2} = \mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2 = (\ \mathbf{1} + \mathbf{r}^1 + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x} = \mathbf{P}^{E}_{0_2 0_2} = \mathbf{P}^{E} p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 + 2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y} = \mathbf{P}^{E}_{1_2 1_2} = \mathbf{P}^{E} p^{1_2} = \mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2 = (2\mathbf{1} - \mathbf{r}^1 - \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3)/6$$

3rd and Final Step:

Spectral resolution of ALL 6 of D3 :

The old ‘**g**-equals-**1**-times-**g**-times-**1**’ Trick

$$\mathbf{g} = \mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1} = (\mathbf{P}^{A_1}_{x,x} + \mathbf{P}^{A_2}_{y,y} + \mathbf{P}^{E}_{x,x} + \mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x} + \mathbf{P}^{A_2}_{y,y} + \mathbf{P}^{E}_{x,x} + \mathbf{P}^{E}_{y,y})$$

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

$$\begin{array}{c|ccc|l} D_3 \ \kappa= & \mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 & \\ \hline \mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\ \mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\ \mathbf{P}^{E}= & 2 & -1 & 0 & /3 \end{array}$$

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}= \mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}+\boxed{\mathbf{i}_1+\mathbf{i}_2}+\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{A_2}_{y,y}= \mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}-\boxed{\mathbf{i}_1-\mathbf{i}_2}-\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{E}_{x,x}= \mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}-\boxed{\mathbf{i}_1-\mathbf{i}_2}+2\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{E}_{y,y}= \mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}+\boxed{\mathbf{i}_1+\mathbf{i}_2}-2\boxed{\mathbf{i}_3})/6$$

3rd and Final Step:
Spectral resolution of ALL 6 of D3 :

The old ‘g-equals-1-times-g-times-1’ Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\begin{aligned}\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}&+\quad 0 \quad+\quad 0\quad+\quad 0\\ +\ 0\quad+\ \mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}&+\quad 0\quad+\quad 0\\ +\ 0\quad+\quad 0\quad&+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}\\ +\ 0\quad+\quad 0\quad&+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}\end{aligned}$$

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

$$\begin{array}{c|ccc|c} D_3 & \kappa=\boxed{\mathbf{1}} & \boxed{\mathbf{r}^1+\mathbf{r}^2} & \boxed{\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3} & \\ \mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\ \mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\ \mathbf{P}^{E}= & 2 & -1 & 0 & /3 \end{array}$$

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}+\boxed{\mathbf{i}_1+\mathbf{i}_2}+\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}-\boxed{\mathbf{i}_1-\mathbf{i}_2}-\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}-\boxed{\mathbf{i}_1-\mathbf{i}_2}+2\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}+\boxed{\mathbf{i}_1+\mathbf{i}_2}-2\boxed{\mathbf{i}_3})/6$$

3rd and Final Step:
Spectral resolution of ALL 6 of D3 :

The old ‘g-equals-1-times-g-times-1’ Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\begin{aligned}\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}&+\quad 0 \quad+\quad 0\quad+\quad 0\\ +\;0\;+\;\mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}&+\quad 0\quad+\quad 0\\ +\;0\;+\;0\;&+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}\\ +\;0\;+\;0\;&+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}\end{aligned}$$

where:

$$\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}$$

$$\mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}=D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}$$

$$\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}\qquad \mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$

$$\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}\qquad \mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

Need to Define
6 Irreducible
Projectors $\mathbf{P}^{(\alpha)}_{m,n}$
Order $°(D_3)=6$

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

$$\begin{array}{c|ccc|c} D_3\ \kappa= & \mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 & \\ \hline \mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\ \mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\ \mathbf{P}^{E}= & 2 & -1 & 0 & /3 \end{array}$$

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}= \mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(1+\mathbf{i}_3)/2=(\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}= \mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(1-\mathbf{i}_3)/2=(\ \mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}= \mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(1+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}= \mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(1-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

3rd and Final Step:
Spectral resolution of ALL 6 of D3 :

The old ‘g-equals-1-times-g-times-1’ Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+\quad 0\quad+\quad 0\quad+\quad 0$$

$$+\ 0\quad+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+\quad 0\quad+\quad 0$$

$$+\ 0\quad+\quad 0\quad+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$

$$+\ 0\quad+\quad 0\quad+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

where:

$$\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}$$

$$\mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}=D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}$$

$$\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}$$

$$\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$

$$\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}$$

$$\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

Need to Define
6 Irreducible
Projectors $\mathbf{P}^{(\alpha)}_{m,n}$
Order $°(D_3)=6$

Centrum $\kappa(D_3)=3$

idempotents

$\mathbf{P}^{(\alpha)}$

$$\begin{array}{r|ccc|l}
D_3 \quad \kappa = & \mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 & \\
\hline
\mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\
\mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\
\mathbf{P}^{E}= & 2 & -1 & 0 & /3
\end{array}$$

Rank $\rho(D_3)=4$

idempotents

$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}= \mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}_{y,y}= \mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{x,x}= \mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2+2\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}_{y,y}= \mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\mathbf{1}-\mathbf{r}^1-\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2-2\mathbf{i}_3)/6$$

3rd and Final Step:

Spectral resolution of ALL 6 of D3 :

The old ‘g-equals-1-times-g-times-1’ Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$

$$+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

where:

$$\mathbf{P}^{A_1}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{A_1}_{x,x}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}$$

$$\mathbf{P}^{A_2}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{A_2}_{y,y}=D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}$$

$$\mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x} \qquad \mathbf{P}^{E}_{x,x}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}$$

$$\mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{x,x}=D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x} \qquad \mathbf{P}^{E}_{y,y}\cdot\mathbf{g}\cdot\mathbf{P}^{E}_{y,y}=D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}$$

Need to Define

6 Irreducible

Projectors $\mathbf{P}^{(\alpha)}_{m,n}$

Order $°(D_3)=6$

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

$$\begin{array}{l|ccc|}
D_3\ \kappa= & \mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 \\
\hline
\mathbf{P}^{A_1}= & 1 & 1 & 1 \ /6 \\
\mathbf{P}^{A_2}= & 1 & 1 & -1 \ /6 \\
\mathbf{P}^{E}= & 2 & -1 & 0 \ /3 \\
\hline
\end{array}$$

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}=\mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}+\boxed{\mathbf{i}_1+\mathbf{i}_2}+\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{A_2}_{y,y}=\mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}-\boxed{\mathbf{i}_1-\mathbf{i}_2}-\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{E}_{x,x}=\mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}-\boxed{\mathbf{i}_1-\mathbf{i}_2}+2\boxed{\mathbf{i}_3})/6$$

$$\mathbf{P}^{E}_{y,y}=\mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}+\boxed{\mathbf{i}_1+\mathbf{i}_2}-2\boxed{\mathbf{i}_3})/6$$

3rd and Final Step:
Spectral resolution of ALL 6 of D3 :

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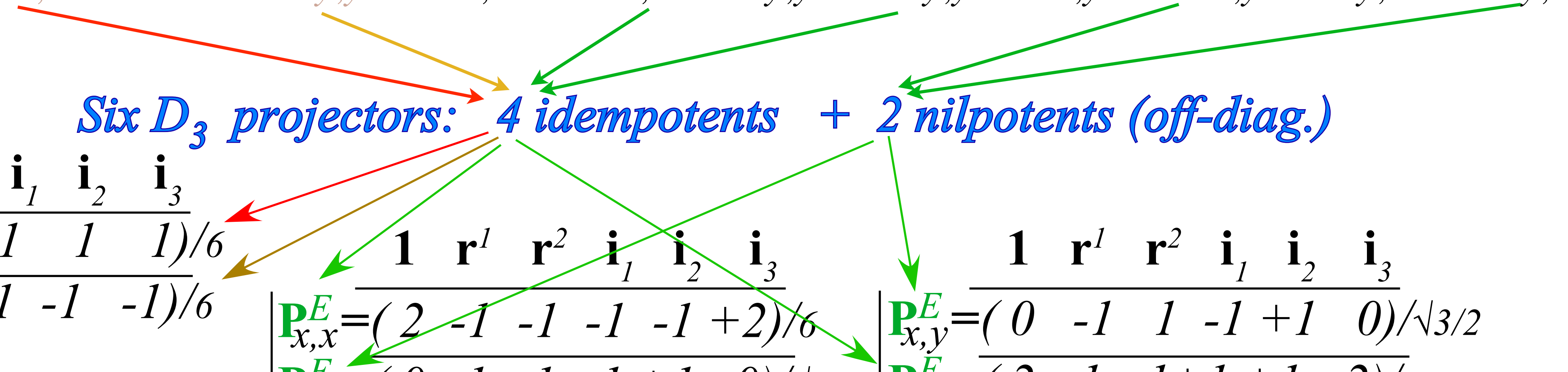
$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}$$



Six D_3 *projectors: 4 idempotents + 2 nilpotents (off-diag.)*

$$\begin{array}{l|cccccc}
 & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\
\hline
\mathbf{P}^{A_1}_{x,x}= & (1 & 1 & 1 & 1 & 1 & 1)/6 \\
\hline
\mathbf{P}^{A_2}_{y,y}= & (1 & 1 & 1 & -1 & -1 & -1)/6
\end{array}$$

$$\begin{array}{l|cccccc}
 & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\
\hline
\mathbf{P}^{E}_{x,x}= & (2 & -1 & -1 & -1 & -1 & +2)/6 \\
\hline
\mathbf{P}^{E}_{y,x}= & (0 & 1 & -1 & -1 & +1 & 0)/\sqrt{3/2} \\
\hline
\end{array}$$

$$\begin{array}{l|cccccc}
 & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\
\hline
\mathbf{P}^{E}_{x,y}= & (0 & -1 & 1 & -1 & +1 & 0)/\sqrt{3/2} \\
\hline
\mathbf{P}^{E}_{y,y}= & (2 & -1 & -1 & +1 & +1 & -2)/6 \\
\hline
\end{array}$$

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

D_3 $\kappa =$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}= \mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2}=\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}+\boxed{\boxed{\mathbf{i}_1+\mathbf{i}_2}+\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{A_2}_{y,y}= \mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2}=\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}-\boxed{\boxed{\mathbf{i}_1-\mathbf{i}_2}-\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{E}_{x,x}= \mathbf{P}^{E}_{0_2 0_2}=\mathbf{P}^{E}p^{0_2}=\mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}-\boxed{\boxed{\mathbf{i}_1-\mathbf{i}_2}+2\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{E}_{y,y}= \mathbf{P}^{E}_{1_2 1_2}=\mathbf{P}^{E}p^{1_2}=\mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}+\boxed{\boxed{\mathbf{i}_1+\mathbf{i}_2}-2\boxed{\mathbf{i}_3}})/6$$

3rd and Final Step:

Spectral resolution of ALL 6 of D3 :

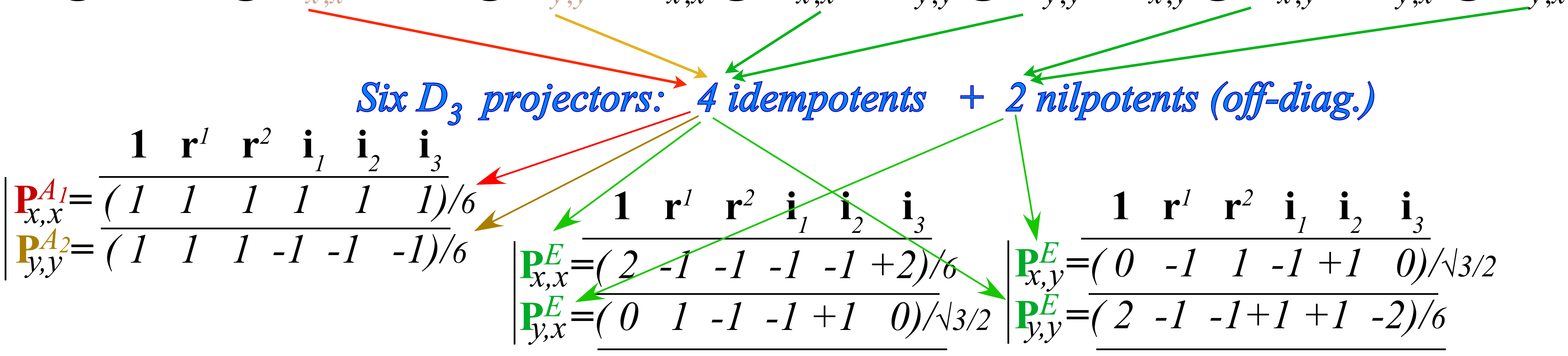
$$\mathbf{g}=\Sigma_m\Sigma_e\Sigma_b D^{(m)}_{eb}(\mathbf{g})\,\mathbf{P}^{(m)}_{eb}$$

$$\mathbf{P}^{(m)}_{eb}= {}_{(norm)}\Sigma_{\mathbf{g}} D^{(m)*}_{eb}(\mathbf{g})\,\mathbf{g}$$

The old ‘g-equals-1-times-g-times-1’ Trick

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_2}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}$$



Six D_3 projectors: 4 idempotents + 2 nilpotents (off-diag.)

$$\begin{array}{l|cccccc|l} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 & \\ \hline \mathbf{P}^{A_1}_{x,x}= & (1 & 1 & 1 & 1 & 1 & 1)/6 & \\ \mathbf{P}^{A_2}_{y,y}= & (1 & 1 & 1 & -1 & -1 & -1)/6 & \end{array}$$

$$\begin{array}{l|cccccc|l} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 & \\ \hline \mathbf{P}^{E}_{x,x}= & (2 & -1 & -1 & -1 & -1 & +2)/6 & \\ \mathbf{P}^{E}_{y,x}= & (0 & 1 & -1 & -1 & +1 & 0)/\sqrt{3/2} & \end{array}$$

$$\begin{array}{l|cccccc|l} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 & \\ \hline \mathbf{P}^{E}_{x,y}= & (0 & -1 & 1 & -1 & +1 & 0)/\sqrt{3/2} & \\ \mathbf{P}^{E}_{y,y}= & (2 & -1 & -1 & +1 & +1 & -2)/6 & \end{array}$$

Centrum $\kappa(D_3)=3$
idempotents
$\mathbf{P}^{(\alpha)}$

$$
\begin{array}{l|ccc|l}
D_3 & \kappa=\mathbf{1} & \mathbf{r}^1+\mathbf{r}^2 & \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3 & \\
\hline
\mathbf{P}^{A_1}= & 1 & 1 & 1 & /6 \\
\mathbf{P}^{A_2}= & 1 & 1 & -1 & /6 \\
\mathbf{P}^{E}= & 2 & -1 & 0 & /3
\end{array}
$$

Rank $\rho(D_3)=4$
idempotents
$\mathbf{P}^{(\alpha)}_{n,n}$

$$\mathbf{P}^{A_1}_{x,x}= \mathbf{P}^{A_1}_{0_2 0_2}=\mathbf{P}^{A_1}p^{0_2} =\mathbf{P}^{A_1}(\mathbf{1}+\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}+\boxed{\boxed{\mathbf{i}_1+\mathbf{i}_2}+\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{A_2}_{y,y}= \mathbf{P}^{A_2}_{1_2 1_2}=\mathbf{P}^{A_2}p^{1_2} =\mathbf{P}^{A_2}(\mathbf{1}-\mathbf{i}_3)/2=(\boxed{\mathbf{1}}+\boxed{\mathbf{r}^1+\mathbf{r}^2}-\boxed{\boxed{\mathbf{i}_1-\mathbf{i}_2}-\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{E}_{x,x}= \mathbf{P}^{E}_{0_2 0_2}= \mathbf{P}^{E}p^{0_2} = \mathbf{P}^{E}(\mathbf{1}+\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}-\boxed{\boxed{\mathbf{i}_1-\mathbf{i}_2}+2\boxed{\mathbf{i}_3}})/6$$

$$\mathbf{P}^{E}_{y,y}= \mathbf{P}^{E}_{1_2 1_2}= \mathbf{P}^{E}p^{1_2} = \mathbf{P}^{E}(\mathbf{1}-\mathbf{i}_3)/2=(2\boxed{\mathbf{1}}-\boxed{\mathbf{r}^1-\mathbf{r}^2}+\boxed{\boxed{\mathbf{i}_1+\mathbf{i}_2}-2\boxed{\mathbf{i}_3}})/6$$

3rd and Final Step:

Spectral resolution of ALL 6 of D3 :

The old ‘g-equals-1-times-g-times-1’ Trick

$$\mathbf{g}=\Sigma_m\Sigma_e\Sigma_b D^{(m)}_{eb}(\mathbf{g})\,\mathbf{P}^{(m)}_{eb}$$

$$\mathbf{P}^{(m)}_{eb}= {}_{(norm)}\Sigma_{\mathbf{g}} D^{(m)*}_{eb}(\mathbf{g})\mathbf{g}$$

$$\mathbf{g}=\mathbf{1}\cdot\mathbf{g}\cdot\mathbf{1}=(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_1}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})\cdot\mathbf{g}\cdot(\mathbf{P}^{A_1}_{x,x}+\mathbf{P}^{A_1}_{y,y}+\mathbf{P}^{E}_{x,x}+\mathbf{P}^{E}_{y,y})$$

$$\mathbf{g}=D^{A_1}(\mathbf{g})\mathbf{P}^{A_1}_{x,x}+D^{A_2}(\mathbf{g})\mathbf{P}^{A_2}_{y,y}+D^{E}_{x,x}(\mathbf{g})\mathbf{P}^{E}_{x,x}+D^{E}_{y,y}(\mathbf{g})\mathbf{P}^{E}_{y,y}+D^{E}_{x,y}(\mathbf{g})\mathbf{P}^{E}_{x,y}+D^{E}_{y,x}(\mathbf{g})\mathbf{P}^{E}_{y,x}$$

Six D_3 projectors: 4 idempotents + 2 nilpotents (off-diag.)

$$
\begin{array}{l|cccccc|l}
 & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 & \\
\hline
\mathbf{P}^{A_1}_{x,x}= & (1 & 1 & 1 & 1 & 1 & 1) & /6 \\
\mathbf{P}^{A_2}_{y,y}= & (1 & 1 & 1 & -1 & -1 & -1) & /6
\end{array}
$$

$$
\begin{array}{l|cccccc|l}
 & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 & \\
\hline
\mathbf{P}^{E}_{x,x}= & (2 & -1 & -1 & -1 & -1 & +2) & /6 \\
\mathbf{P}^{E}_{y,x}= & (0 & 1 & -1 & -1 & +1 & 0) & /\sqrt{3/2}
\end{array}
$$

$$
\begin{array}{l|cccccc|l}
 & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 & \\
\hline
\mathbf{P}^{E}_{x,y}= & (0 & -1 & 1 & -1 & +1 & 0) & /\sqrt{3/2} \\
\mathbf{P}^{E}_{y,y}= & (2 & -1 & -1 & +1 & +1 & -2) & /6
\end{array}
$$

where D_3 *irreducible representations* are: $D^{A_1}(\mathbf{g})=+1$, $D^{A_2}(\mathbf{g})=\pm 1$,

$$D^E(1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},D^E(\mathbf{r})=\begin{pmatrix}-\frac{1}{2}&-\sqrt{\frac{3}{4}}\\\sqrt{\frac{3}{4}}&-\frac{1}{2}\end{pmatrix},D^E(\mathbf{r}^2)=\begin{pmatrix}-\frac{1}{2}&\sqrt{\frac{3}{4}}\\-\sqrt{\frac{3}{4}}&-\frac{1}{2}\end{pmatrix},D^E(\mathbf{i}_1)=\begin{pmatrix}-\frac{1}{2}&-\sqrt{\frac{3}{4}}\\-\sqrt{\frac{3}{4}}&\frac{1}{2}\end{pmatrix},D^E(\mathbf{i}_2)=\begin{pmatrix}-\frac{1}{2}&\sqrt{\frac{3}{4}}\\\sqrt{\frac{3}{4}}&\frac{1}{2}\end{pmatrix},D^E(\mathbf{i}_3)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$D_3 > C_2\,\mathbf{i}_3$ projector states

$$\left|{}^{(m)}_{eb}\right\rangle = \mathbf{P}^{(m)}_{eb}\left|1\right\rangle$$

Global (LAB) symmetry

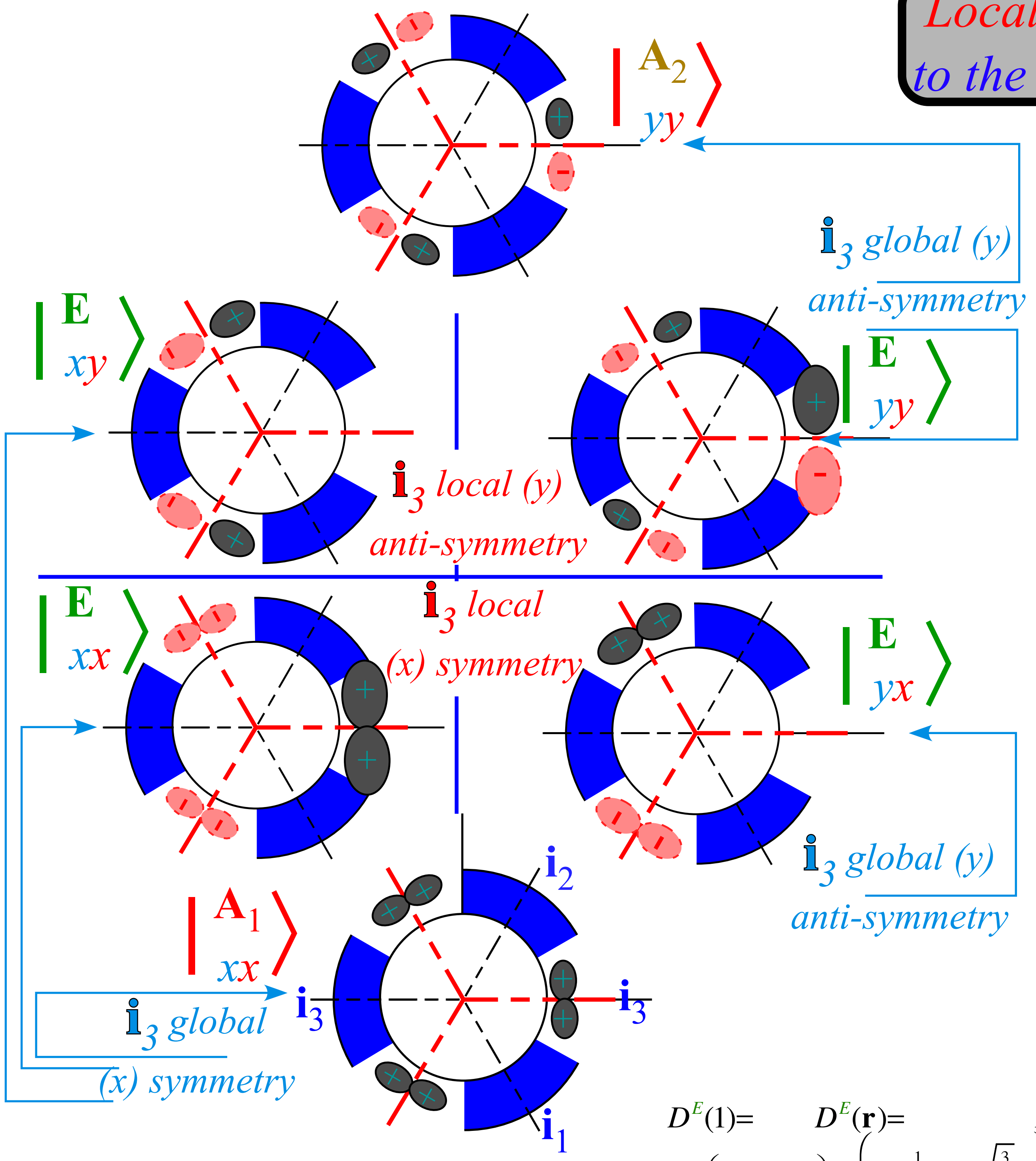
$$\mathbf{i}_3\left|{}^{(m)}_{eb}\right\rangle = \mathbf{i}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle$$
$$=(-1)^e\left|{}^{(m)}\right\rangle$$

Local (BOD) symmetry

$$\bar{\mathbf{i}}_3\left|{}^{(m)}_{eb}\right\rangle = \bar{\mathbf{i}}_3\mathbf{P}^{(m)}_{eb}\left|1\right\rangle = \mathbf{P}^{(m)}_{eb}\bar{\mathbf{i}}_3\left|1\right\rangle$$
$$= \mathbf{P}^{(m)}_{eb}\mathbf{i}_3^{\dagger}\left|1\right\rangle = (-1)^b\left|{}^{(m)}\right\rangle$$

Local $\bar{\mathbf{g}}$ commute through to the "inside" to be a $\mathbf{g}^{\dagger}$

Here the "Mock-Mach" is being applied!



$$\begin{array}{rcccccc} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \end{array}$$

$$\mathbf{P}^{A_2}_{y,y} = (\,1\quad 1\quad 1\quad -1\quad -1\quad -1)/6$$

$$\mathbf{P}^{E}_{x,y} = (\,0\quad -1\quad 1\quad -1\quad +1\quad 0)/\sqrt{3/2}$$
$$\mathbf{P}^{E}_{y,y} = (\,2\quad -1\quad -1\quad +1\quad +1\quad -2)/6$$

$$\mathbf{P}^{E}_{x,x} = (\,2\quad -1\quad -1\quad -1\quad -1\quad +2)/6$$
$$\mathbf{P}^{E}_{y,x} = (\,0\quad 1\quad -1\quad -1\quad +1\quad 0)/\sqrt{3/2}$$

$$\mathbf{P}^{A_1}_{x,x} = (\,1\quad 1\quad 1\quad 1\quad 1\quad 1)/6$$

$$D^{A_1}(\mathbf{g}) = +1,\; D^{A_2}(\mathbf{r}^p) = +1,\; D^{A_2}(\mathbf{i}_q) = -1$$

$$D^E(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D^E(\mathbf{r}) = \begin{pmatrix} -\frac{1}{2} & -\sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix} \quad D^E(\mathbf{r}^2) = \begin{pmatrix} -\frac{1}{2} & \sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & -\frac{1}{2} \end{pmatrix}$$

$$D^E(\mathbf{i}_1) = \begin{pmatrix} -\frac{1}{2} & -\sqrt{\frac{3}{4}} \\ -\sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix} \quad D^E(\mathbf{i}_2) = \begin{pmatrix} -\frac{1}{2} & \sqrt{\frac{3}{4}} \\ \sqrt{\frac{3}{4}} & \frac{1}{2} \end{pmatrix} \quad D^E(\mathbf{i}_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left|\begin{smallmatrix}(m)\\ eb\end{smallmatrix}\right\rangle = \mathbf{P}^{(m)}_{eb}|\mathbf{1}\rangle$$

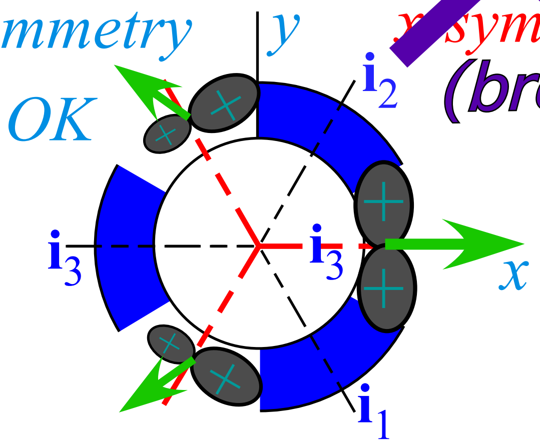
external LAB *internal BOD*

symmety label-e *symmety label-b*

GLOBAL *LOCAL*

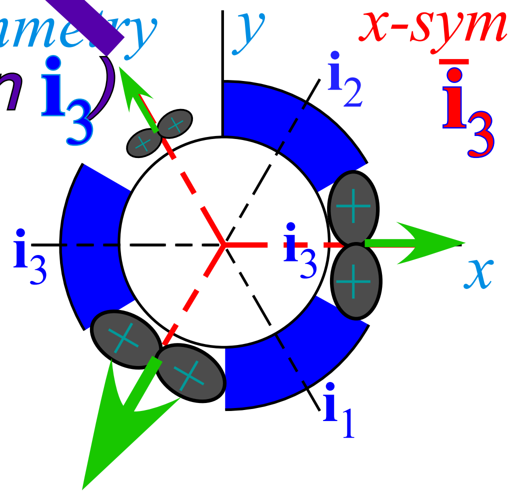
GLOBAL
$(i_3)=0_2$
x-symmetry
$\mathbf{i}_3$ *OK*

~~*LOCAL*~~
~~$(i_3)=0_2$~~
~~*x-symmetry*~~
(broken $\bar{\mathbf{i}}_3$)



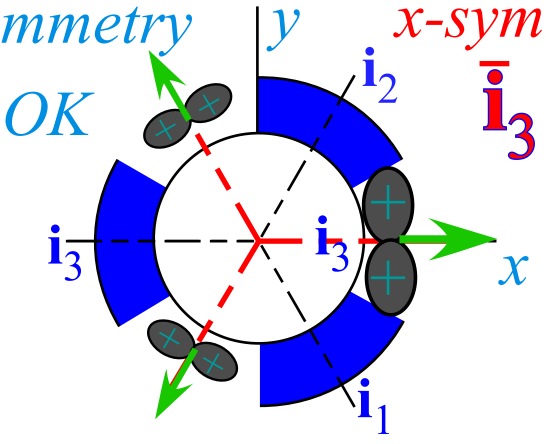
~~*GLOBAL*~~
~~$(i_3)=0_2$~~
~~*x-symmetry*~~
(broken $\mathbf{i}_3$)

LOCAL
$(i_3)=0_2$
x-symmetry
$\bar{\mathbf{i}}_3$ *OK*



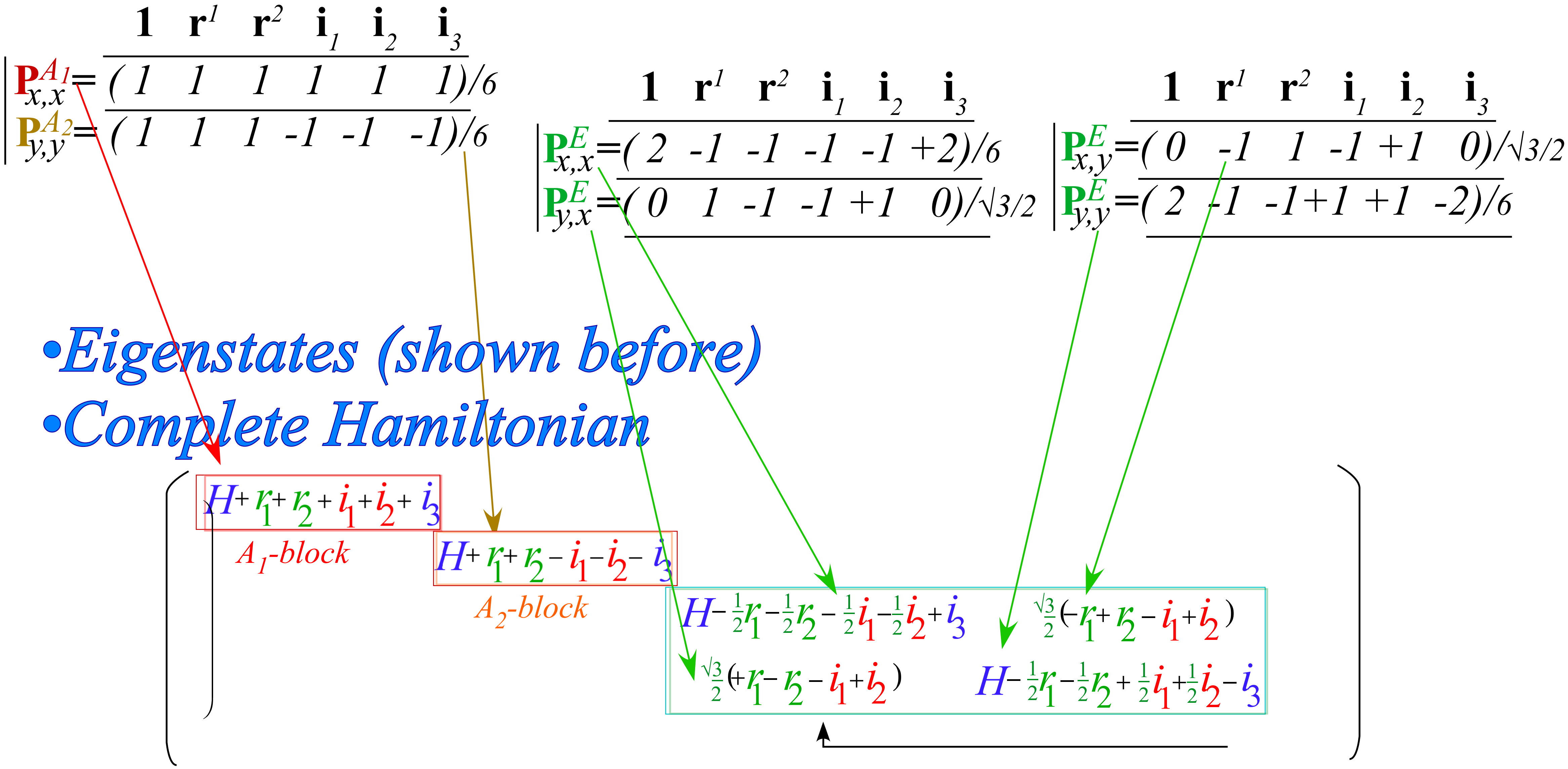
GLOBAL
$(i_3)=0_2$
x-symmetry
$\mathbf{i}_3$ *OK*

LOCAL
$(i_3)=0_2$
x-symmetry
$\bar{\mathbf{i}}_3$ *OK*



$$\mathbf{P}^{(\mu)}_{mn}=\frac{\ell^{(\mu)}}{{}^{\circ}G}\Sigma_{\mathbf{g}} D^{(\mu)*}_{mn}(\mathbf{g})\mathbf{g}$$

Spectral Efficiency: Same $D(a)_{mn}$ projectors give a lot!



$$\begin{array}{l|cccccc} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \hline \mathbf{P}^{A_1}_{x,x}= & (1 & 1 & 1 & 1 & 1 & 1)/6 \\ \mathbf{P}^{A_2}_{y,y}= & (1 & 1 & 1 & -1 & -1 & -1)/6 \end{array}$$

$$\begin{array}{l|cccccc} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \hline \mathbf{P}^{E}_{x,x}= & (2 & -1 & -1 & -1 & -1 & +2)/6 \\ \mathbf{P}^{E}_{y,x}= & (0 & 1 & -1 & -1 & +1 & 0)/\sqrt{3/2} \end{array}$$

$$\begin{array}{l|cccccc} & \mathbf{1} & \mathbf{r}^1 & \mathbf{r}^2 & \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ \hline \mathbf{P}^{E}_{x,y}= & (0 & -1 & 1 & -1 & +1 & 0)/\sqrt{3/2} \\ \mathbf{P}^{E}_{y,y}= & (2 & -1 & -1{+}1 & +1 & -2)/6 & \end{array}$$

- Eigenstates (shown before)
- Complete Hamiltonian

$$\left(\begin{array}{cccc} \boxed{H+r_1+r_2+i_1+i_2+i_3} & & & \\ \text{A}_1\text{-block} & \boxed{H+r_1+r_2-i_1-i_2-i_3} & & \\ & \text{A}_2\text{-block} & H-\frac{1}{2}r_1-\frac{1}{2}r_2-\frac{1}{2}i_1-\frac{1}{2}i_2+i_3 & \frac{\sqrt{3}}{2}(-r_1+r_2-i_1+i_2) \\ & & \frac{\sqrt{3}}{2}(+r_1-r_2-i_1+i_2) & H-\frac{1}{2}r_1-\frac{1}{2}r_2+\frac{1}{2}i_1+\frac{1}{2}i_2-i_3 \end{array}\right)$$

- Local symmetery eigenvalue formulae *(L.S.=> off-diagonal zero.)*

$r_1=r_2=r_1{}^*=r,\qquad i_1=i_2=i_1{}^*=i$

gives:

A_1-level: $H+2r+2i+i_3$

A_1-level: $H+2r-2i-i_3$

E_x-level: $H-r-i+i_3$

E_y-level: $H-r+i-i_3$

$D_3 > C_2\ \mathbf{i}_3$ projector states

$$\left|{}^{(m)}_{eb}\right\rangle = \mathbf{P}^{(m)}_{eb}|1\rangle$$

Global (LAB) symmetry

$$\mathbf{i}_3\left|{}^{(m)}_{eb}\right\rangle = \mathbf{i}_3\mathbf{P}^{(m)}_{eb}|1\rangle$$
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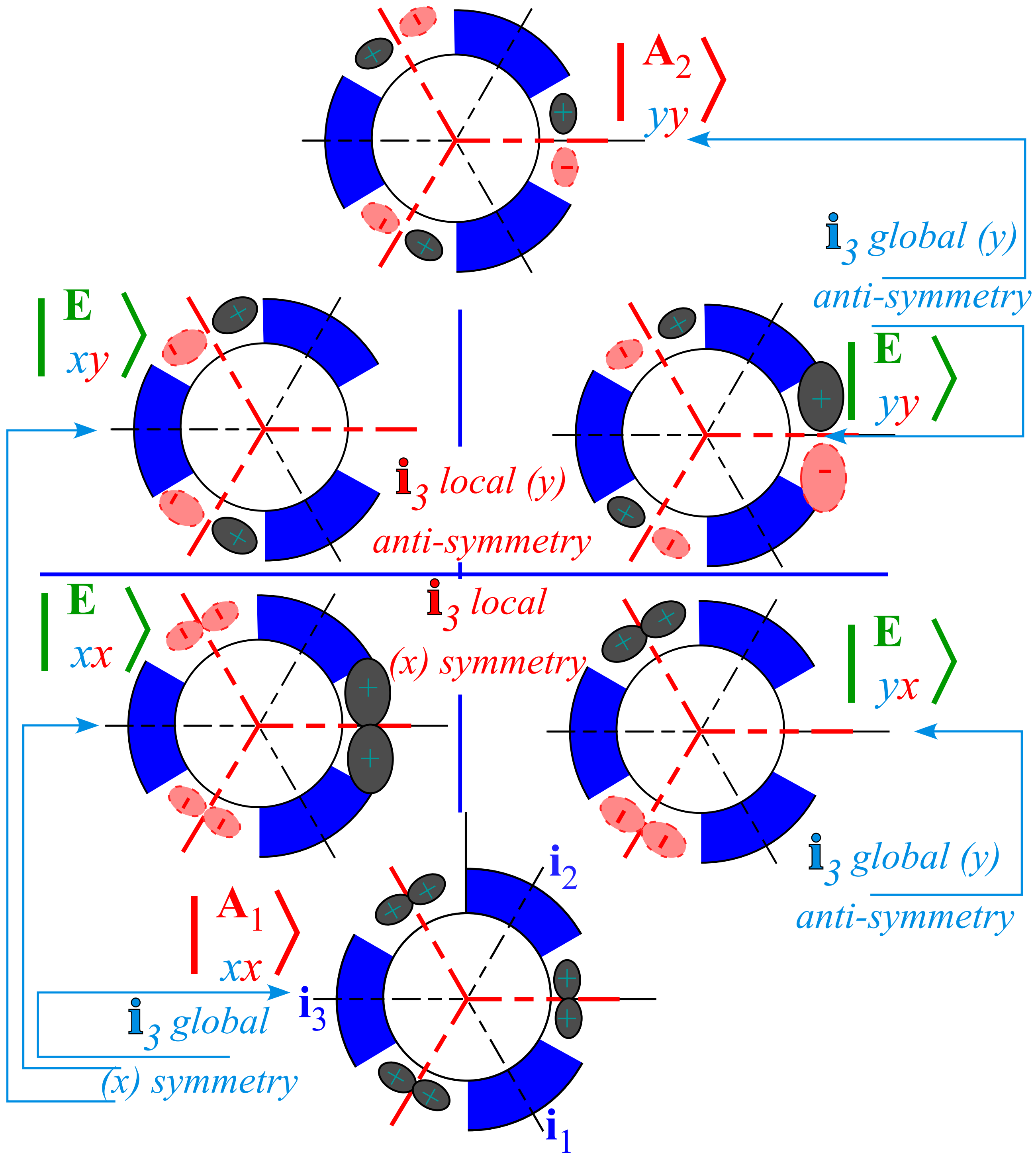
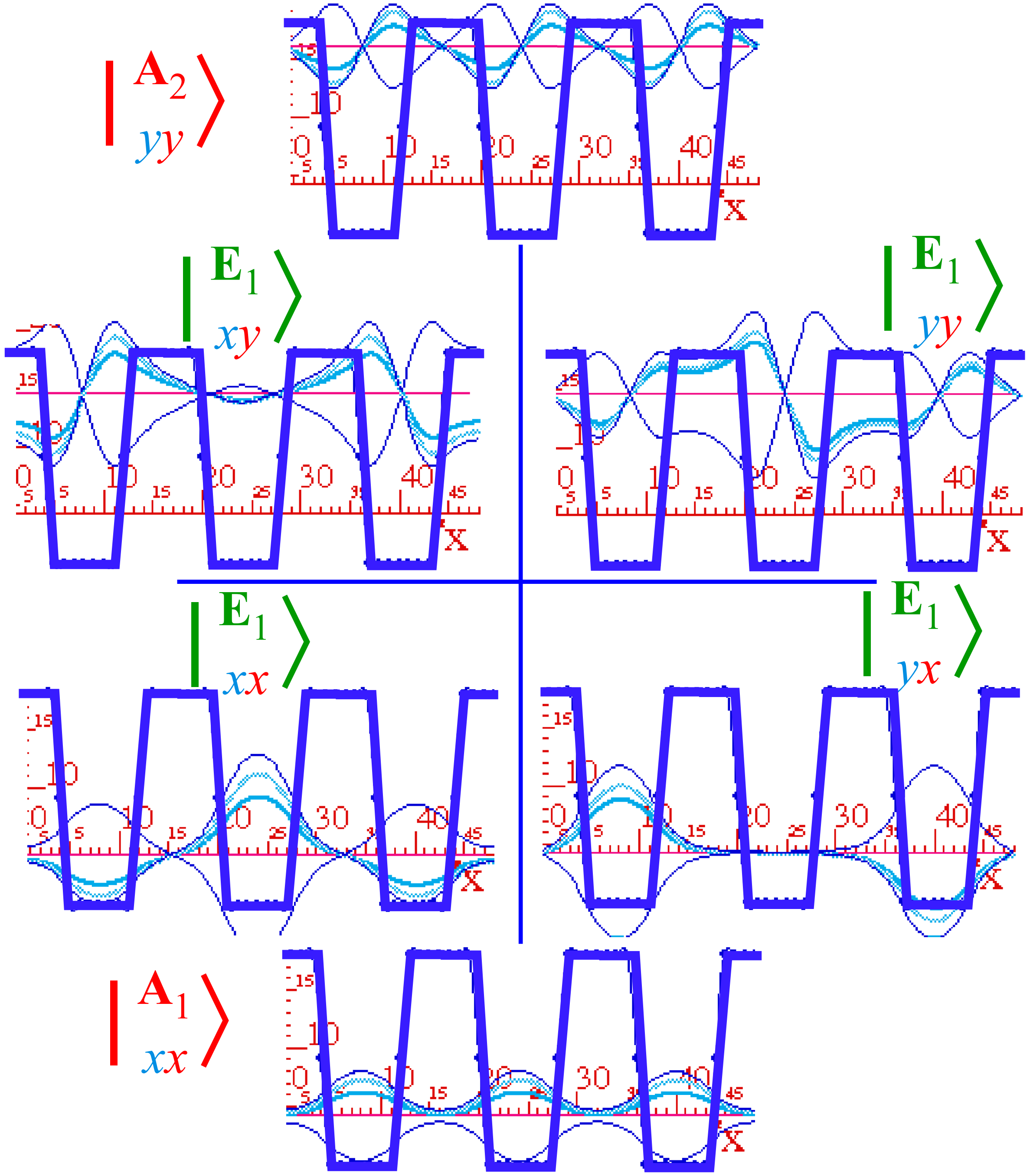
Local (BOD) symmetry

$$\bar{\mathbf{i}}_3\left|{}^{(m)}_{eb}\right\rangle = \bar{\mathbf{i}}_3\mathbf{P}^{(m)}_{eb}|1\rangle = \mathbf{P}^{(m)}_{eb}\bar{\mathbf{i}}_3|1\rangle$$
$$= \mathbf{P}^{(m)}_{eb}\mathbf{i}_3^{\dagger}|1\rangle = (-1)^b\left|{}^{(m)}\right\rangle$$



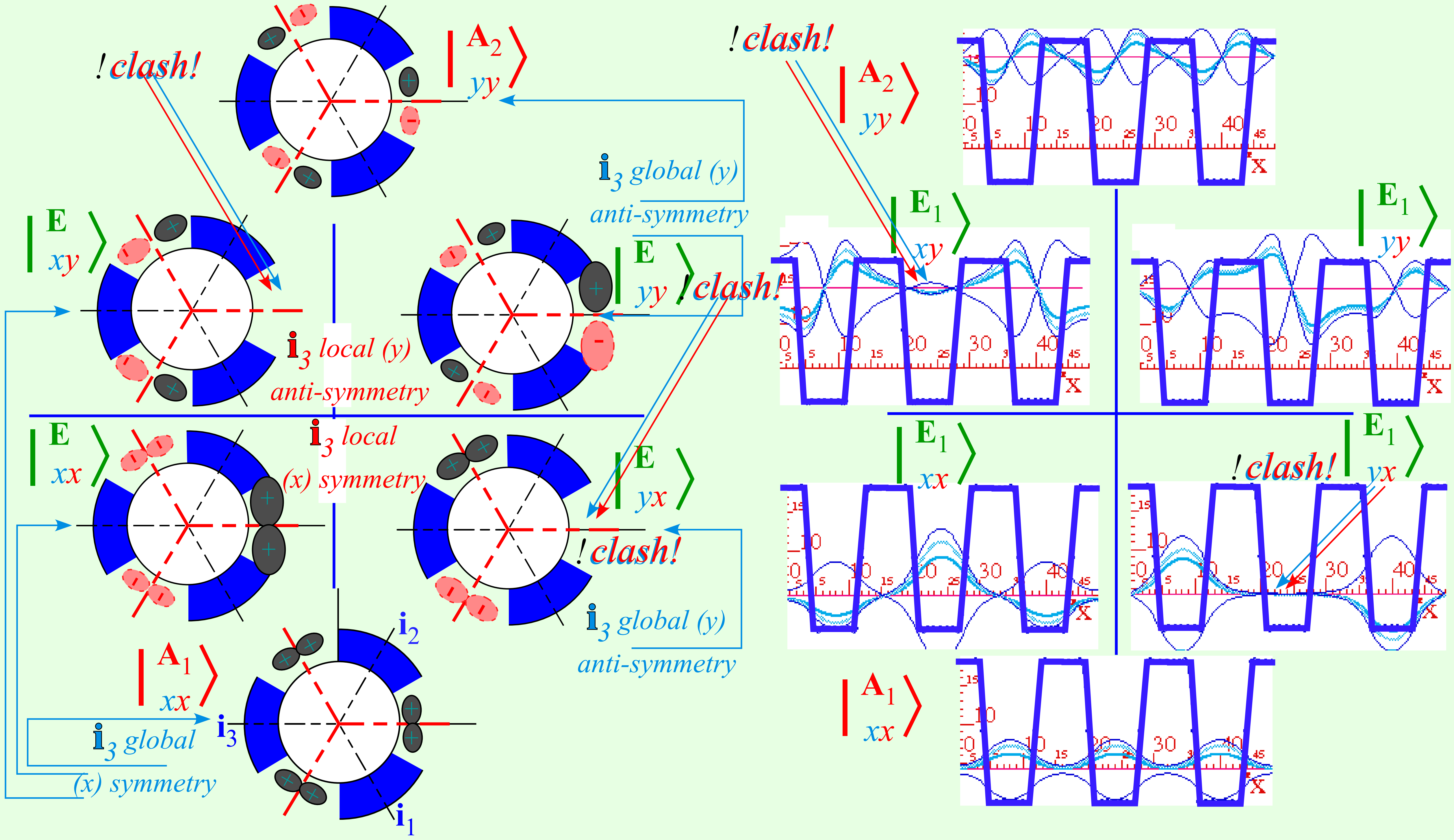


$$D^{E}(1)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad D^{E}(\mathbf{r})=\begin{pmatrix}-\frac{1}{2}&-\sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{4}}&-\frac{1}{2}\end{pmatrix}\quad D^{E}(\mathbf{r}^2)=\begin{pmatrix}-\frac{1}{2}&\sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}}&-\frac{1}{2}\end{pmatrix}\quad D^{E}(\mathbf{i}_1)=\begin{pmatrix}-\frac{1}{2}&-\sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}}&\frac{1}{2}\end{pmatrix}\quad D^{E}(\mathbf{i}_2)=\begin{pmatrix}-\frac{1}{2}&\sqrt{\frac{3}{4}}\\ \sqrt{\frac{3}{4}}&\frac{1}{2}\end{pmatrix}\quad D^{E}(\mathbf{i}_3)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

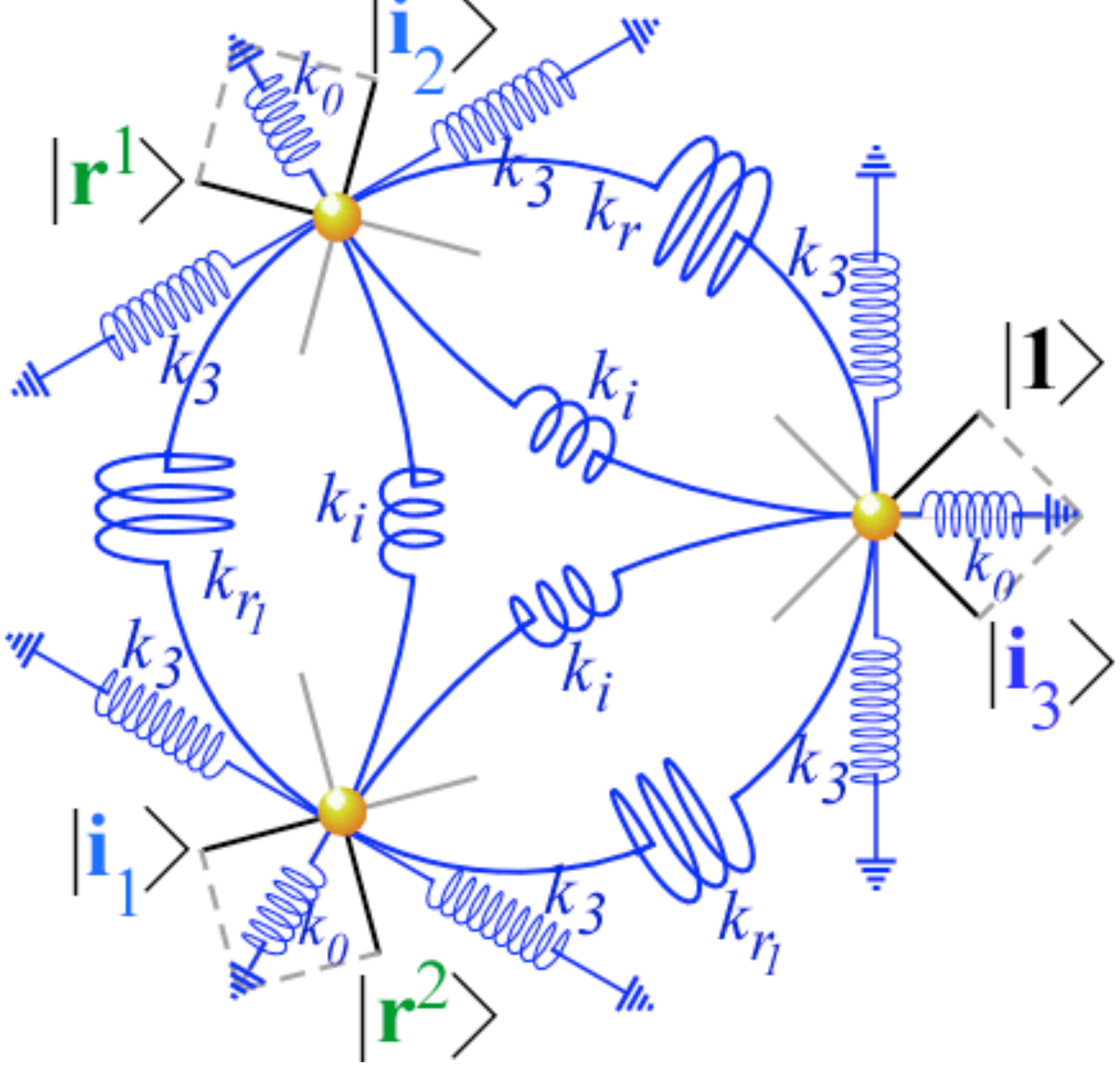
$$D^{A_1}(\mathbf{g})=+1,\ D^{A_2}(\mathbf{r}^p)=+1,\ D^{A_2}(\mathbf{i}_q)=-1$$

When there is no there, there...

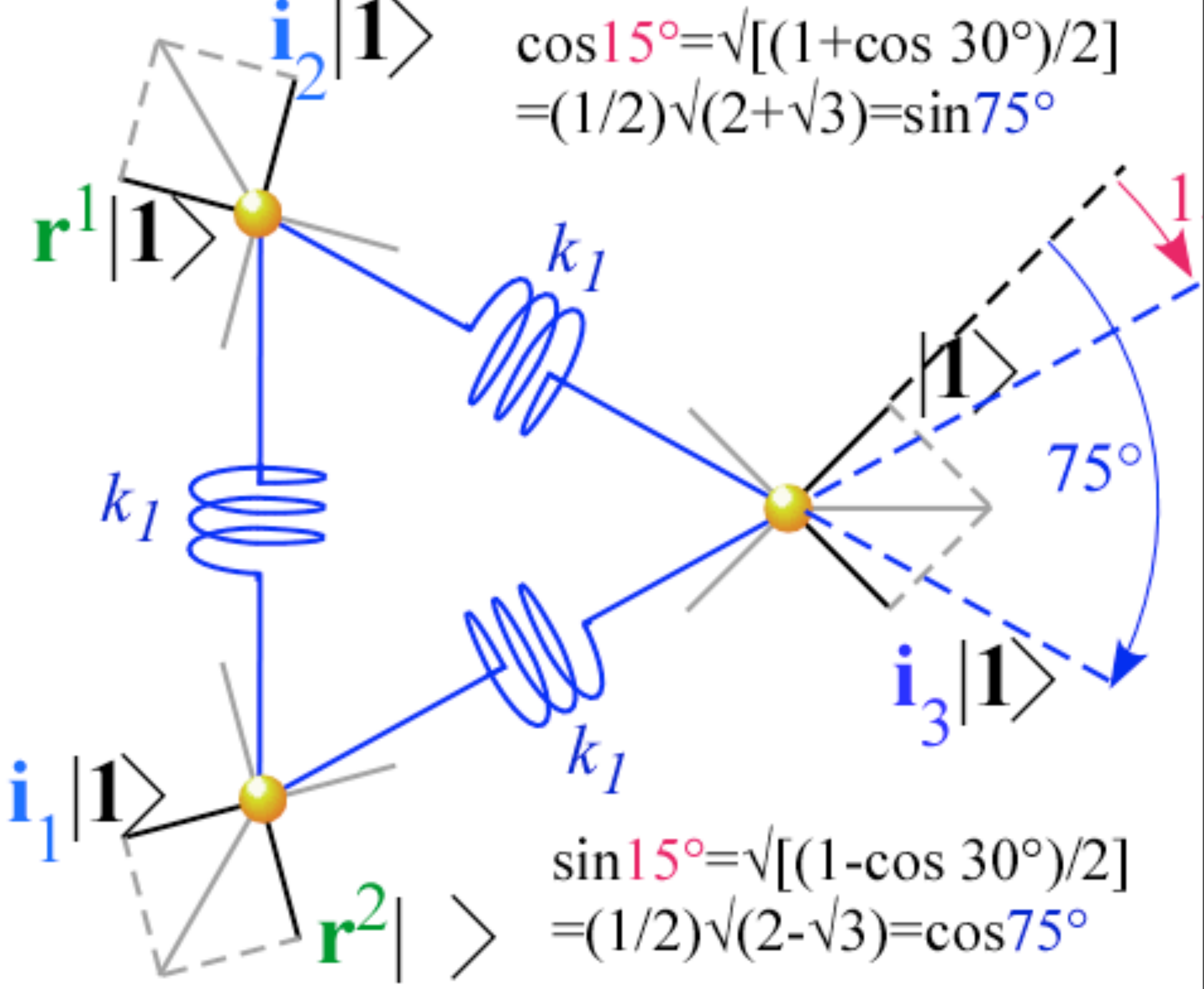
Nobody Home
where LOCAL
and GLOBAL



(a) Local $D_3 \supset C_2(i_3)$ model

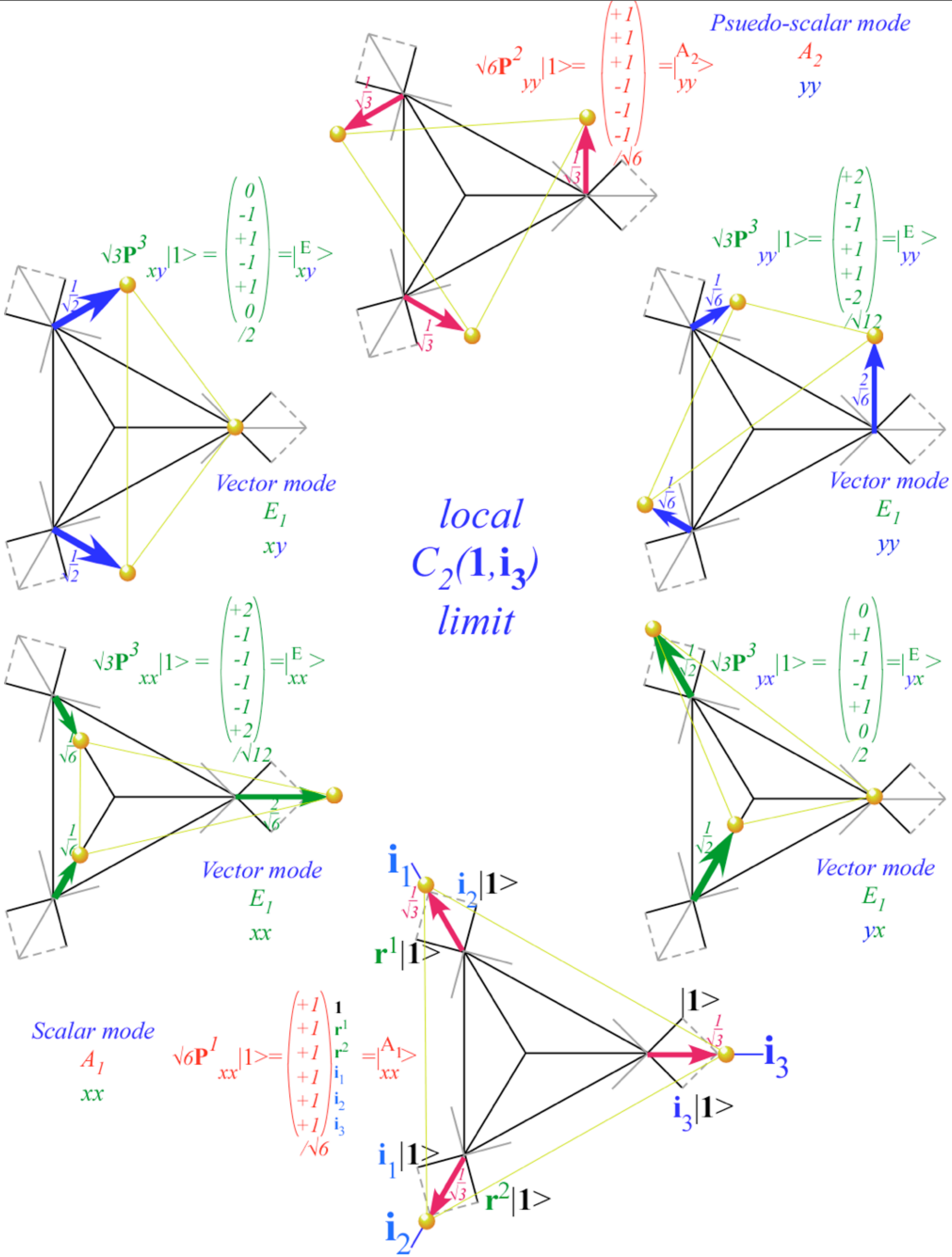


(b) Mixed local symmetry D_3 model

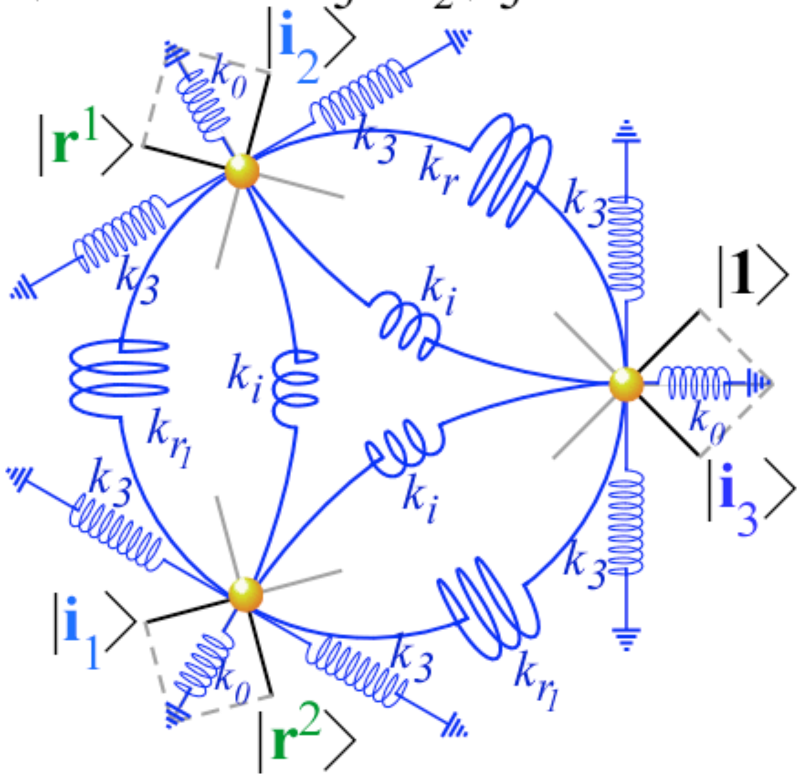


$\cos 15° = \sqrt{[(1+\cos 30°)/2]}$
$= (1/2)\sqrt{(2+\sqrt{3})} = \sin 75°$

$\sin 15° = \sqrt{[(1-\cos 30°)/2]}$
$= (1/2)\sqrt{(2-\sqrt{3})} = \cos 75°$



(a) Local $D_3 \supset C_2(i_3)$ model



Polygonal geometry of U(2)⊃C$_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=trace D^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2} \end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=trace D^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

Excerpts from Lecture 12.6 page 126-136

Polygonal geometry of U(2)⊃C_N character spectral function

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(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \qquad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \qquad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

Excerpts from Lecture 12.6 page 126-136

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(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta} \end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\,\Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}}=2\cos\frac{\Theta}{2}$$

(spinor-j=1/2)

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0}=1$$

(scalar-j=0)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots+e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta}=1+2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots+e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

Excerpts from Lecture 12.6 page 126-136

Polygonal geometry of $U(2)\supset C_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

is an ($\ell^j=2j+1$)-term sum of $e^{-im\Theta}$ over allowed m-quanta $m=\{-j, -j+1, \ldots, j-1, j\}$.

$$\chi^{1/2}(\Theta)=traceD^{1/2}(\Theta)=trace\begin{pmatrix} e^{-i\theta/2} & \cdot \\ \cdot & e^{+i\theta/2}\end{pmatrix}$$

(spinor-j=1/2)

$$\chi^{1}(\Theta)=traceD^{1}(\Theta)=trace\begin{pmatrix} e^{-i\theta} & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & e^{-i\theta}\end{pmatrix}$$

(vector-j=1)

$\chi^j(\Theta)$ involves a sum of $2\cos(m\ \Theta/2)$ for $m\geq 0$ up to $m=j$.

$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}}=2\cos\frac{\Theta}{2}$$

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$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta}=1+2\cos\Theta$$

(vector-j=1)

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots\quad+e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

$\chi^j(\Theta)$ is a geometric series with ratio $e^{i\Theta}$ between each successive term.

$$\chi^{j}(\Theta)=TraceD^{(j)}(\Theta)=e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}+e^{+i\Theta j}$$

$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

Subtracting gives:

$$\chi^{j}(\Theta)(1-e^{-i\Theta})=-e^{-i\Theta(j+1)}\qquad\qquad+\qquad\qquad e^{+i\Theta j}$$

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Polygonal geometry of U(2)⊃C_N character spectral function

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(spinor-j=1/2)

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$$\chi^{1/2}(\Theta)=e^{-i\frac{\Theta}{2}}+e^{i\frac{\Theta}{2}} \qquad =2\cos\frac{\Theta}{2}$$

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$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

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Polygonal geometry of $U(2)\supset C_N$ character spectral function

Trace-character $\chi^j(\Theta)$ of U(2) rotation by C_n angle $\Theta=2\pi/n$

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(spinor-j=1/2)

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(spinor-j=1/2)

$$\chi^{3/2}(\Theta)=e^{-i\frac{3\Theta}{2}}+\ldots \qquad +e^{i\frac{3\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}$$

$$\chi^{5/2}(\Theta)=e^{-i\frac{5\Theta}{2}}+\ldots \qquad +e^{i\frac{5\Theta}{2}}=2\cos\frac{\Theta}{2}+2\cos\frac{3\Theta}{2}+2\cos\frac{5\Theta}{2}$$

$$\chi^{0}(\Theta)=e^{-i\Theta\cdot 0} \qquad =1$$

(scalar-j=0)

$$\chi^{1}(\Theta)=e^{-i\Theta}+1+e^{i\Theta} \quad =1+2\cos\Theta$$

(vector-j=1)

$$\chi^{2}(\Theta)=e^{-i2\Theta}+\ldots e^{i2\Theta}=1+2\cos\Theta+2\cos 2\Theta$$

(tensor-j=2)

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$$\chi^{j}(\Theta)e^{-i\Theta}=e^{-i\Theta(j+1)}+e^{-i\Theta j}+e^{-i\Theta(j-1)}+e^{-i\Theta(j-2)}+\ldots+e^{+i\Theta(j-2)}+e^{+i\Theta(j-1)}$$

Subtracting/dividing gives $\chi^j(\Theta)$ formula.

$$\chi^{j}(\Theta)=\frac{e^{+i\Theta j}-e^{-i\Theta(j+1)}}{1-e^{-i\Theta}}=\frac{e^{+i\Theta(j+\frac{1}{2})}-e^{-i\Theta(j+\frac{1}{2})}}{e^{+i\frac{\Theta}{2}}-e^{-i\frac{\Theta}{2}}}=\frac{\sin\Theta(j+\frac{1}{2})}{\sin\frac{\Theta}{2}}$$

For C_n angle $\Theta=2\pi/n$ this χ^j has a lot of geometric significance.

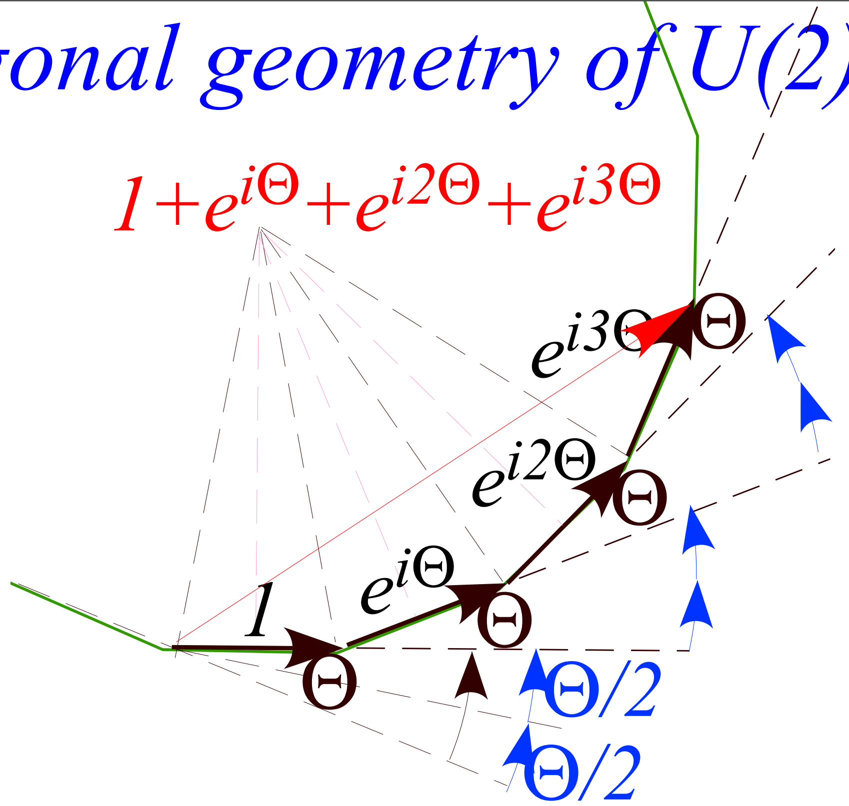
$$\boxed{\chi^{j}(\frac{2\pi}{n})=\frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi\ell^{j}}{n}}{\sin\frac{\pi}{n}}}$$

Character Spectral Function
where: $\ell^j=2j+1$
is U(2) irrep dimension

Polygonal geometry of $U(2)\supset C_N$ character spectral function

$$\chi^j(\frac{2\pi}{n}) = \frac{\sin\frac{\pi}{n}(2j+1)}{\sin\frac{\pi}{n}} = \frac{\sin\frac{\pi\ell^J}{n}}{\sin\frac{\pi}{n}}$$

Character Spectral Function
where: ℓ^j=2j+1
is U(2) irrep dimension



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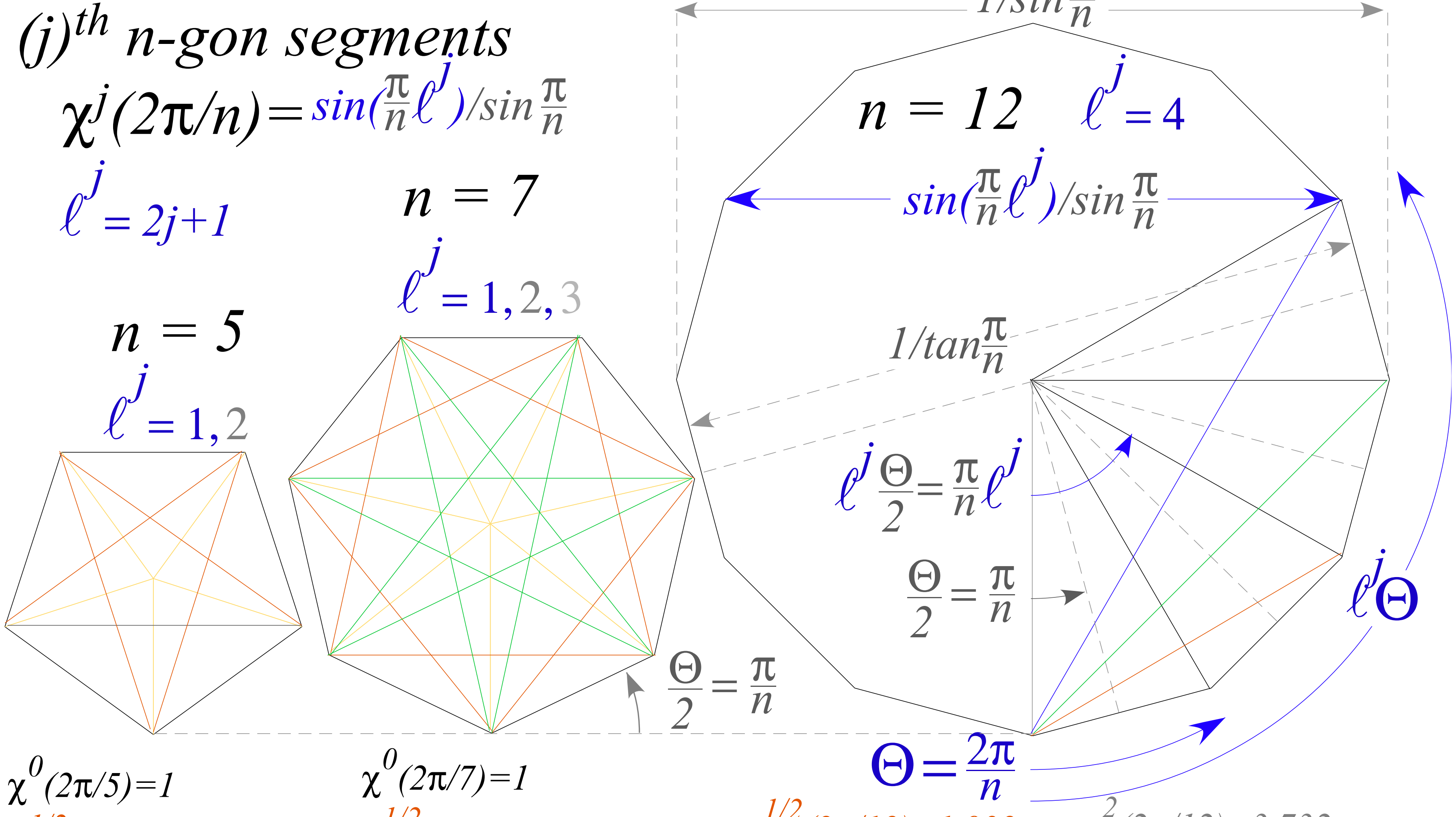

$\chi^0(2\pi/5)=1$
$\chi^{1/2}(2\pi/5)=1.618...$
$=(1+\sqrt{5})/2=$

$\chi^0(2\pi/7)=1$
$\chi^{1/2}(2\pi/7)=1.802...$
$\chi^1(2\pi/7)=2.247...$
$\chi^{3/2}(2\pi/7)=2.247...$

$\chi^{1/2}(2\pi/12)=1.932...$
$\chi^1(2\pi/12)=2.732...$
$\chi^{3/2}(2\pi/12)=3.346...$

$\chi^2(2\pi/12)=3.732...$
$\chi^{5/2}(2\pi/12)=3.864...$
$\chi^3(2\pi/12)=3.732...$