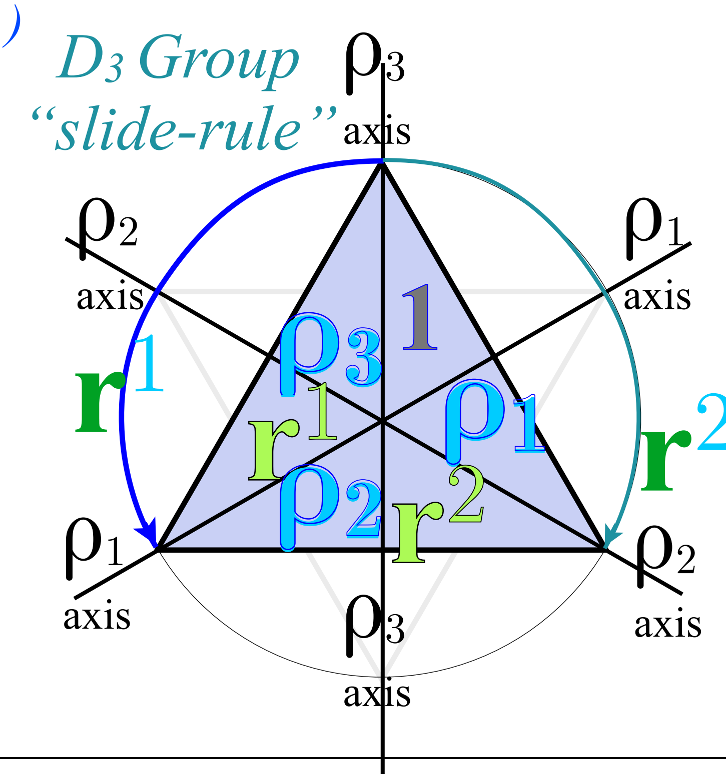
Group Theory in Quantum Mechanics

Lecture 15 *(3.09.17)*

Smallest non-Abelian group D_3 (and isomorphic $C_{3v} \sim D_3$)

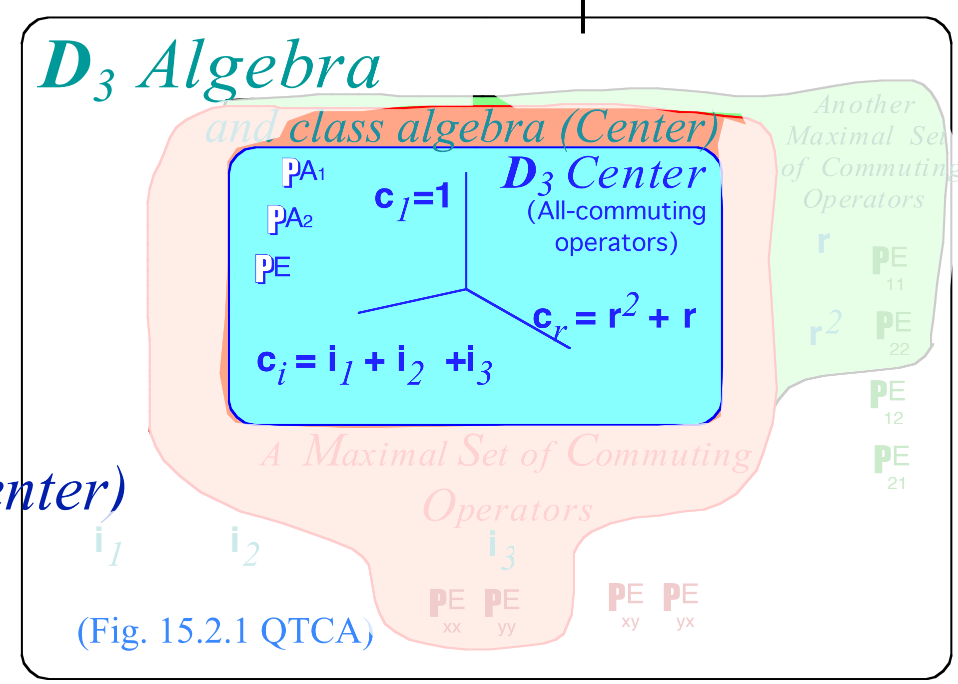
(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)
(PSDS - Ch. 3)



3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}
- *D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*
- *Deriving $D_3 \sim C_{3v}$ products:*
  - *By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$*
  - *By nomograms based on U(2) Hamilton-turns*
- *Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes*

Non-commutative symmetry expansion and Global-Local solution
- *Global vs Local symmetry and Mock-Mach principle*
- *Global vs Local matrix duality for D_3*
  - *Global vs Local symmetry expansion of D_3 Hamiltonian*

1st-Stage spectral decomposition of global/local D_3 Hamiltonian
- *Group theory of equivalence transformations and classes*
  - *Lagrange theorems*
- *All-commuting operators and D_3-invariant class algebra (center)*
- *All-commuting projectors and D_3-invariant characters*
- *Group invariant numbers: Centrum, Rank, and Order*



(Fig. 15.2.1 QTCA)

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

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Lagrange theorems

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Group invariant numbers: Centrum, Rank, and Order

Fig. 3.1.1 PSDS

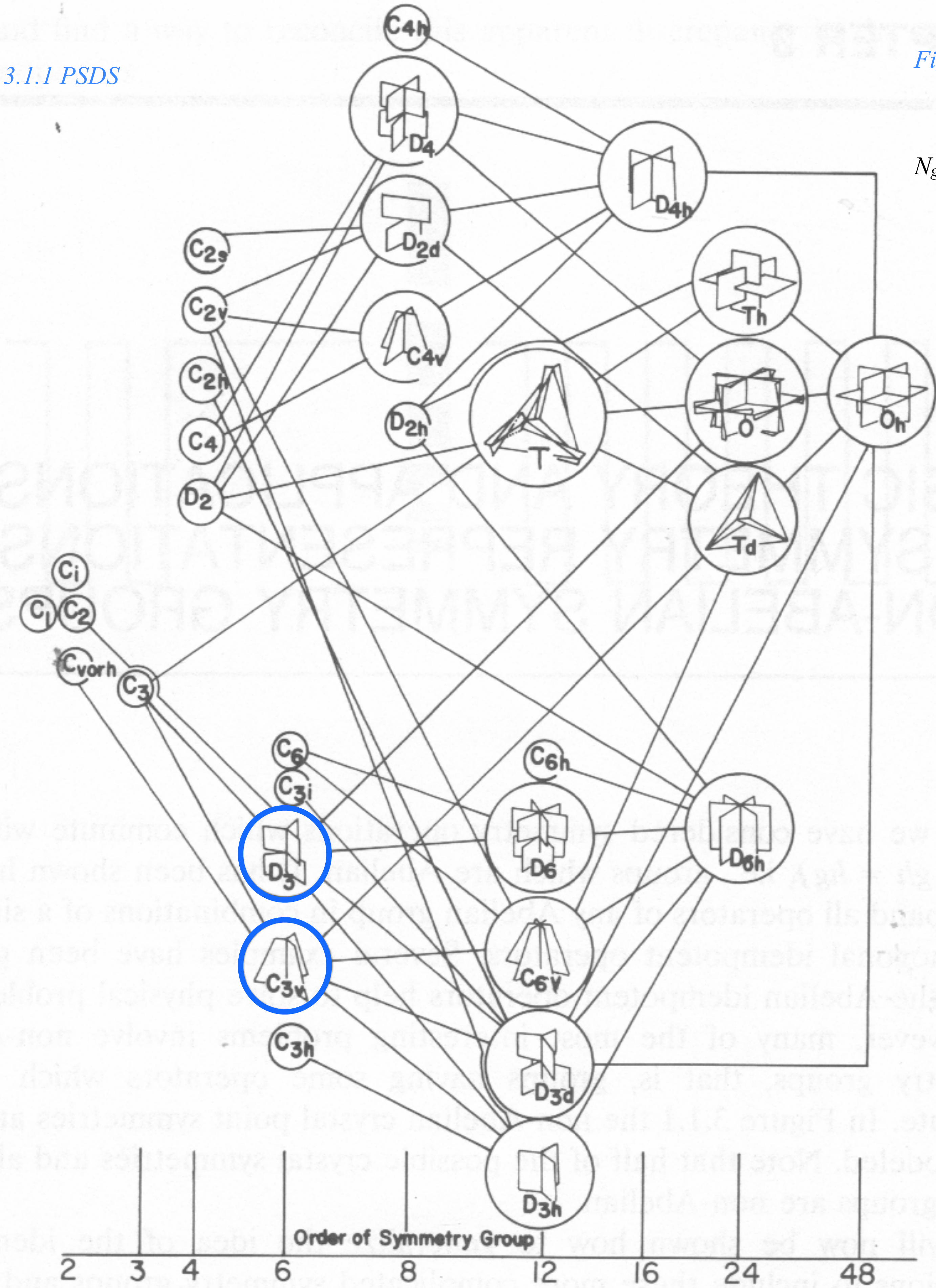
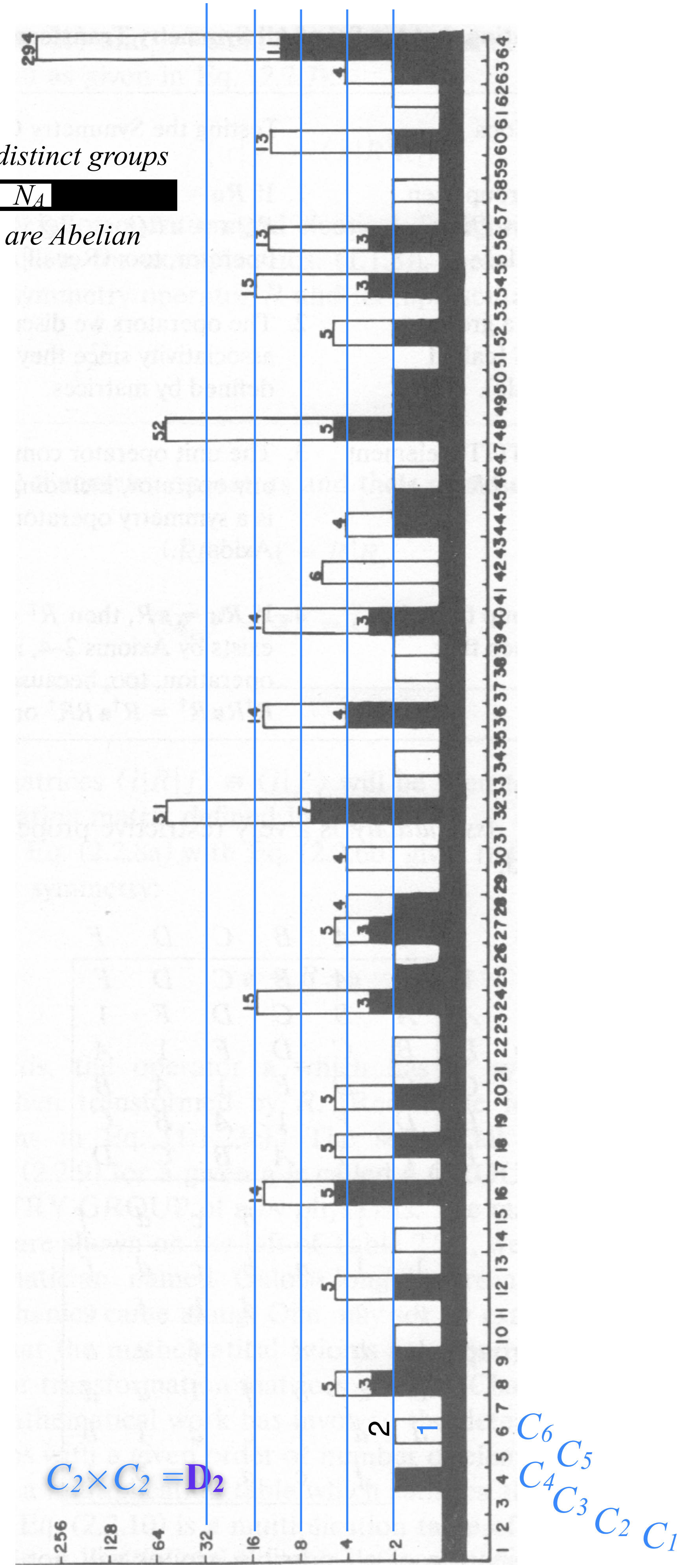
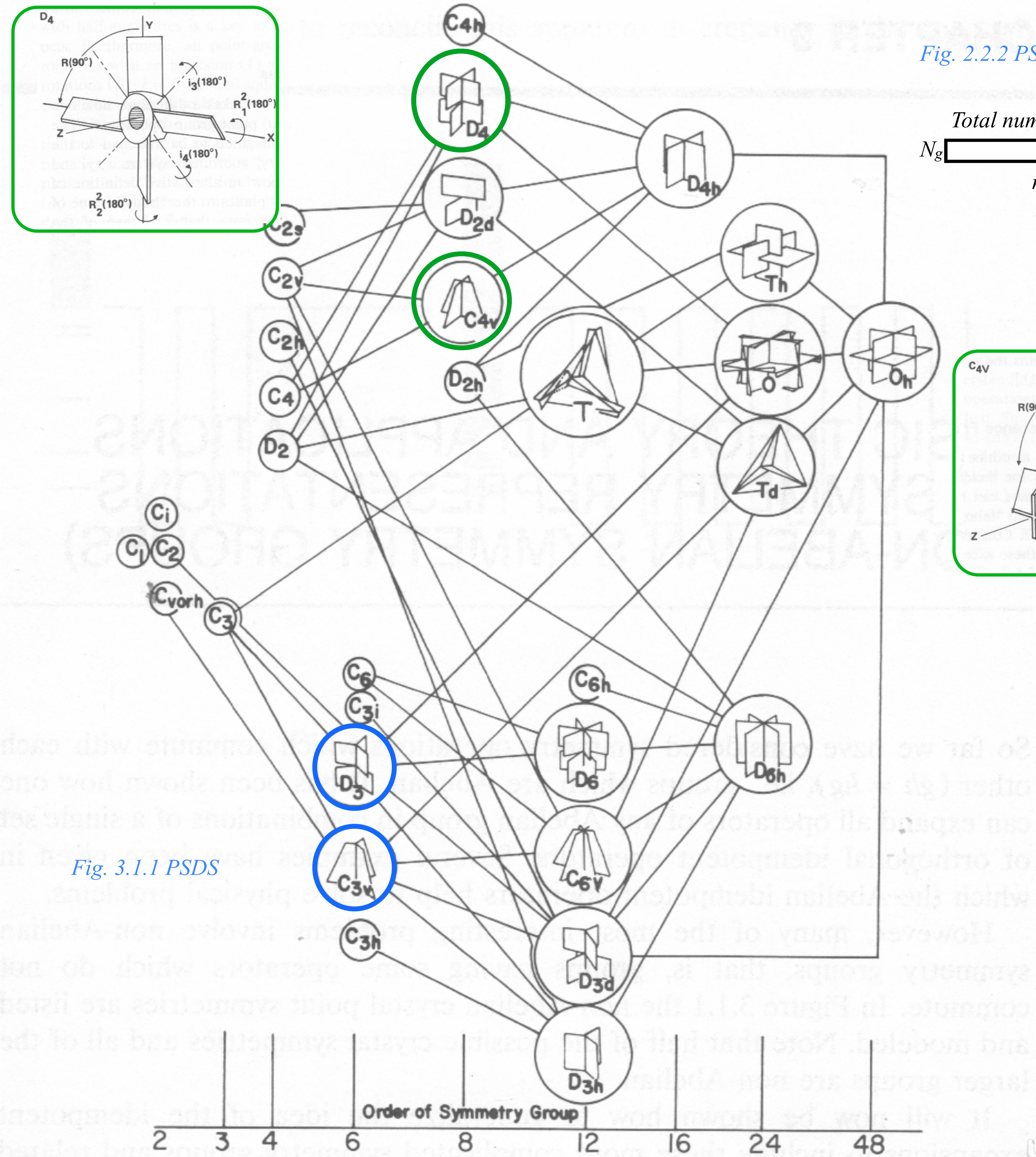


Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

Fig. 2.2.2 PSDS





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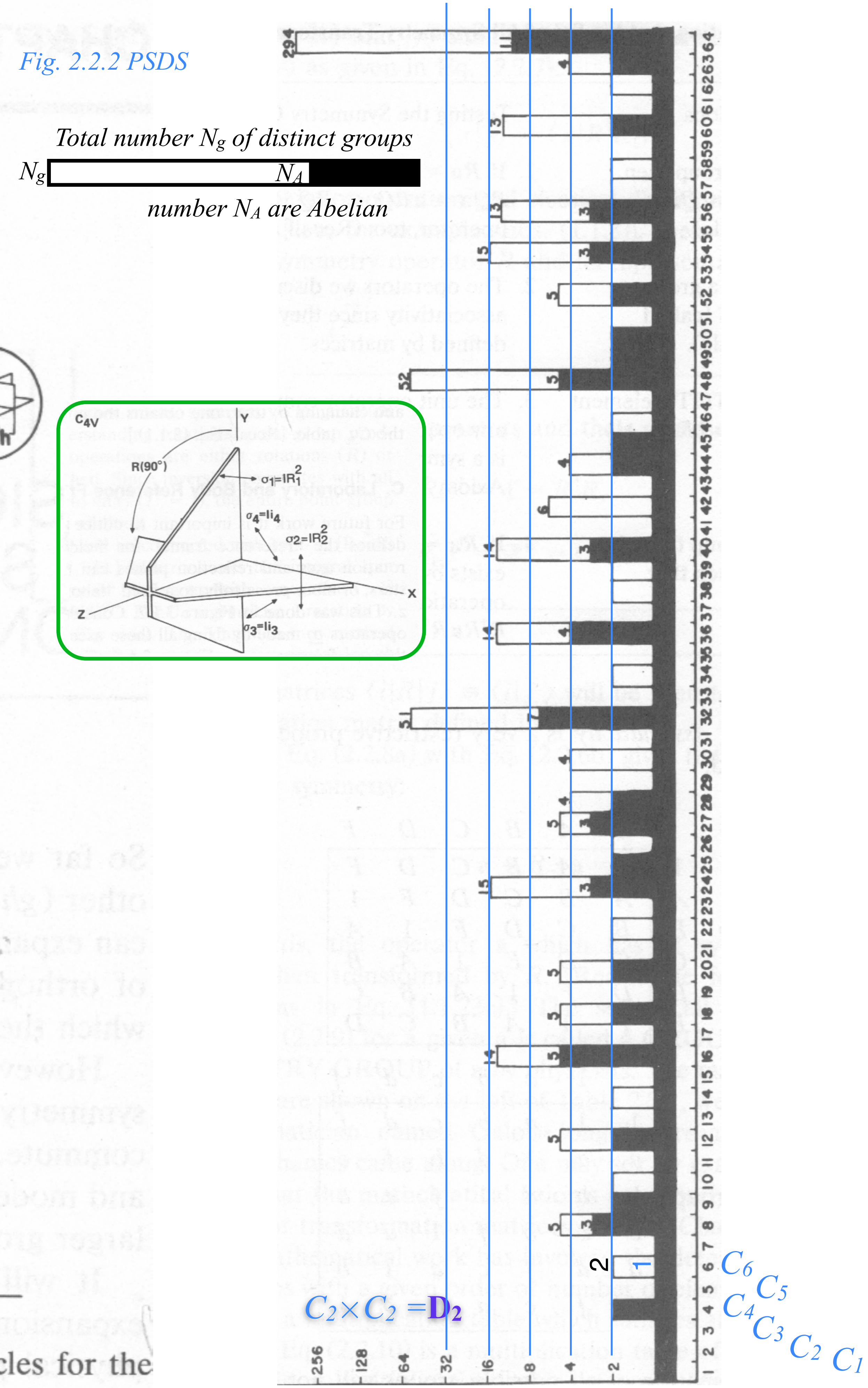

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3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

Fig. 3.1.3 PSDS

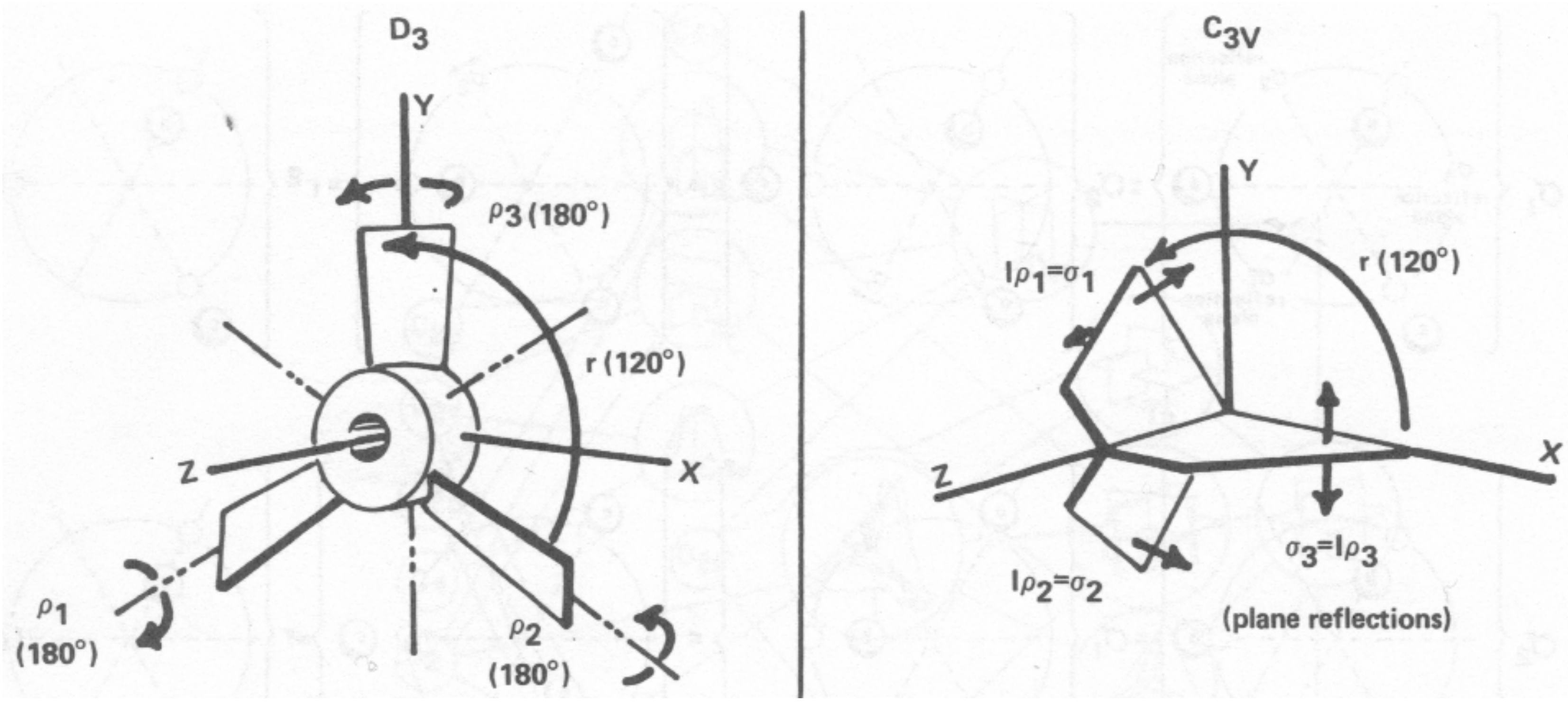
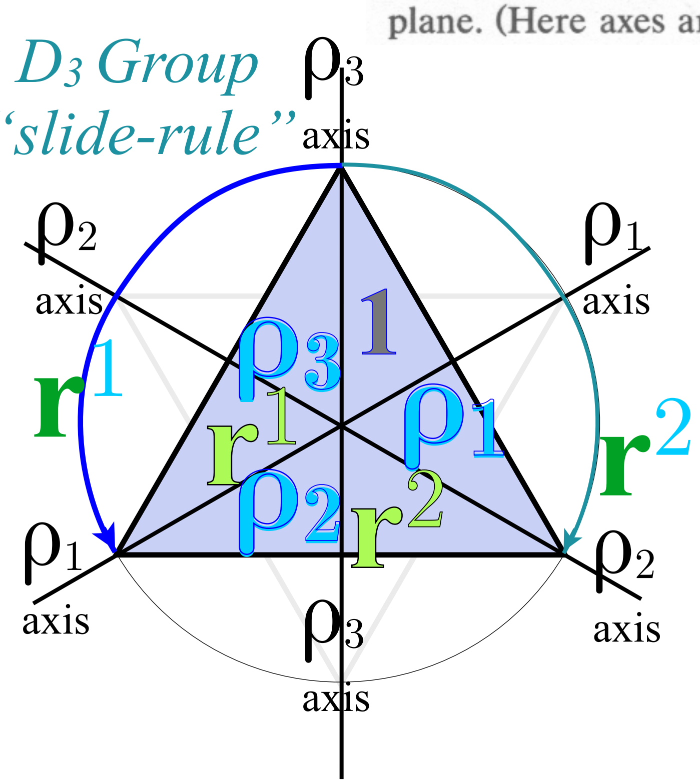
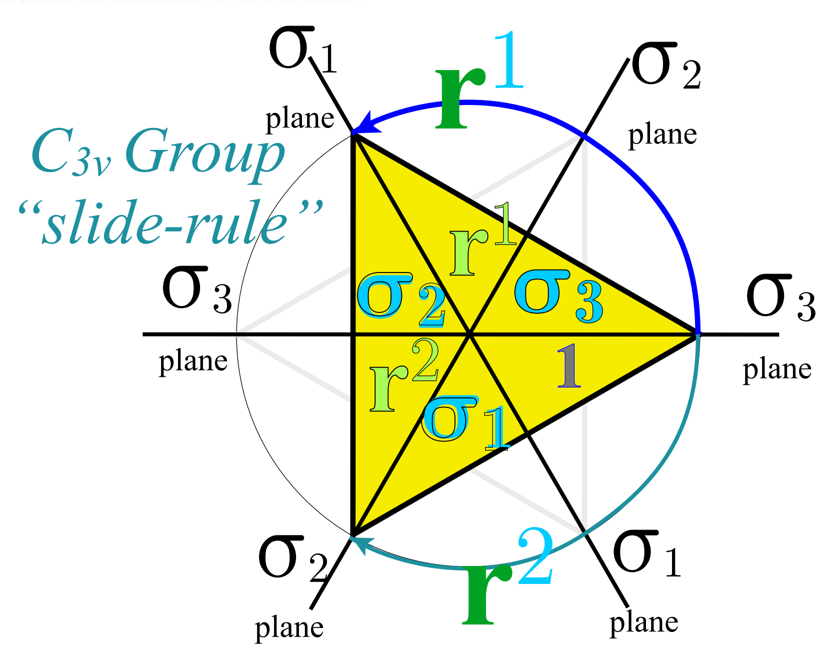


Figure 3.1.3 Pictorial comparison of D_3 and C_{3v} symmetry. A propeller having D_3 symmetry is shown next to a three-plane paddle having C_{3v} symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, ρ_3 is a 180° rotation around the y axis, while $I\rho_3 = \sigma_3$ is a reflection through the xz plane. (Here axes are fixed and the objects rotate.)



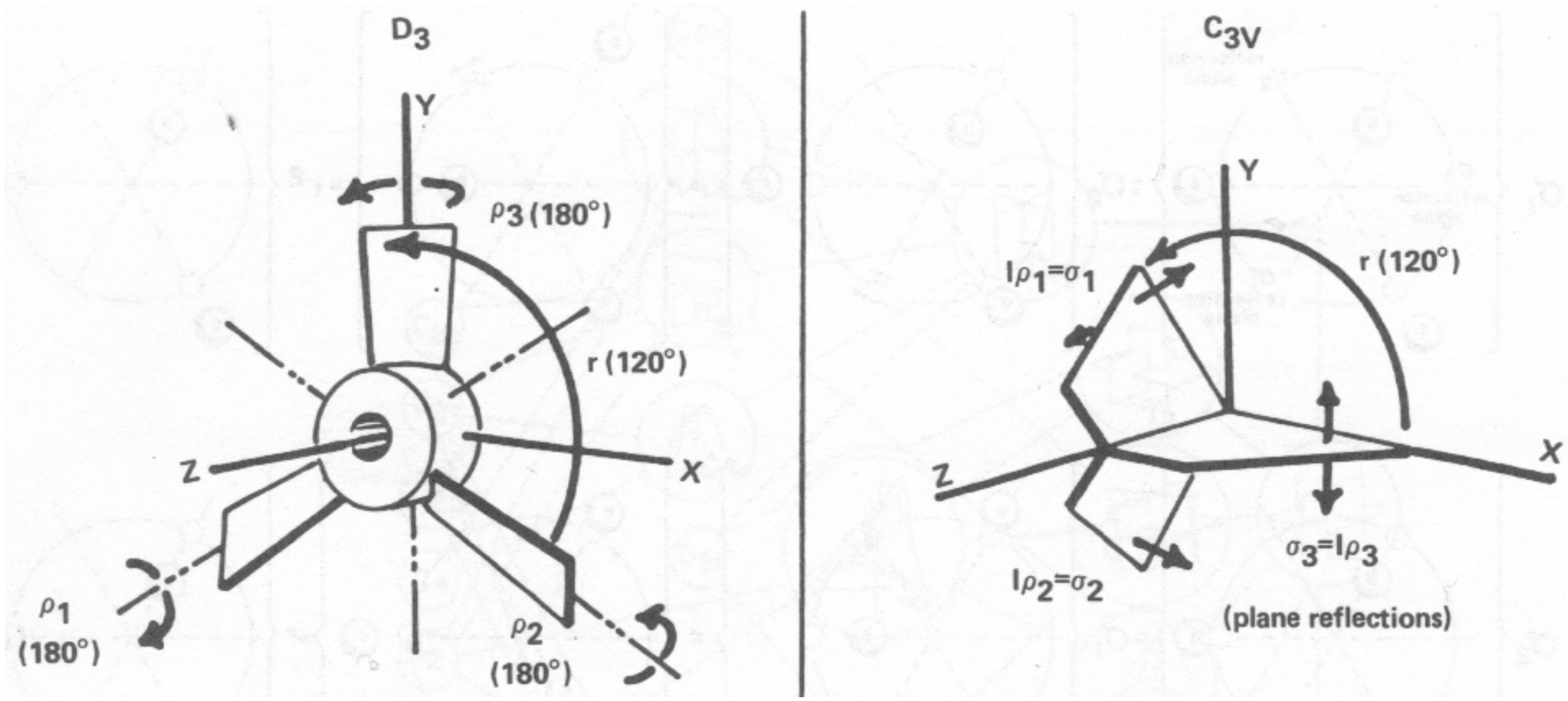


**isomorphic means mathematically the same abstract group even if physically different action.*

Showing that D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*

3-Dihedral-axes group D_3 *vs.* 3-Vertical-mirror-plane group C_{3v}

Fig. 3.1.3 PSDS



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180° D_3-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ *maps to*: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

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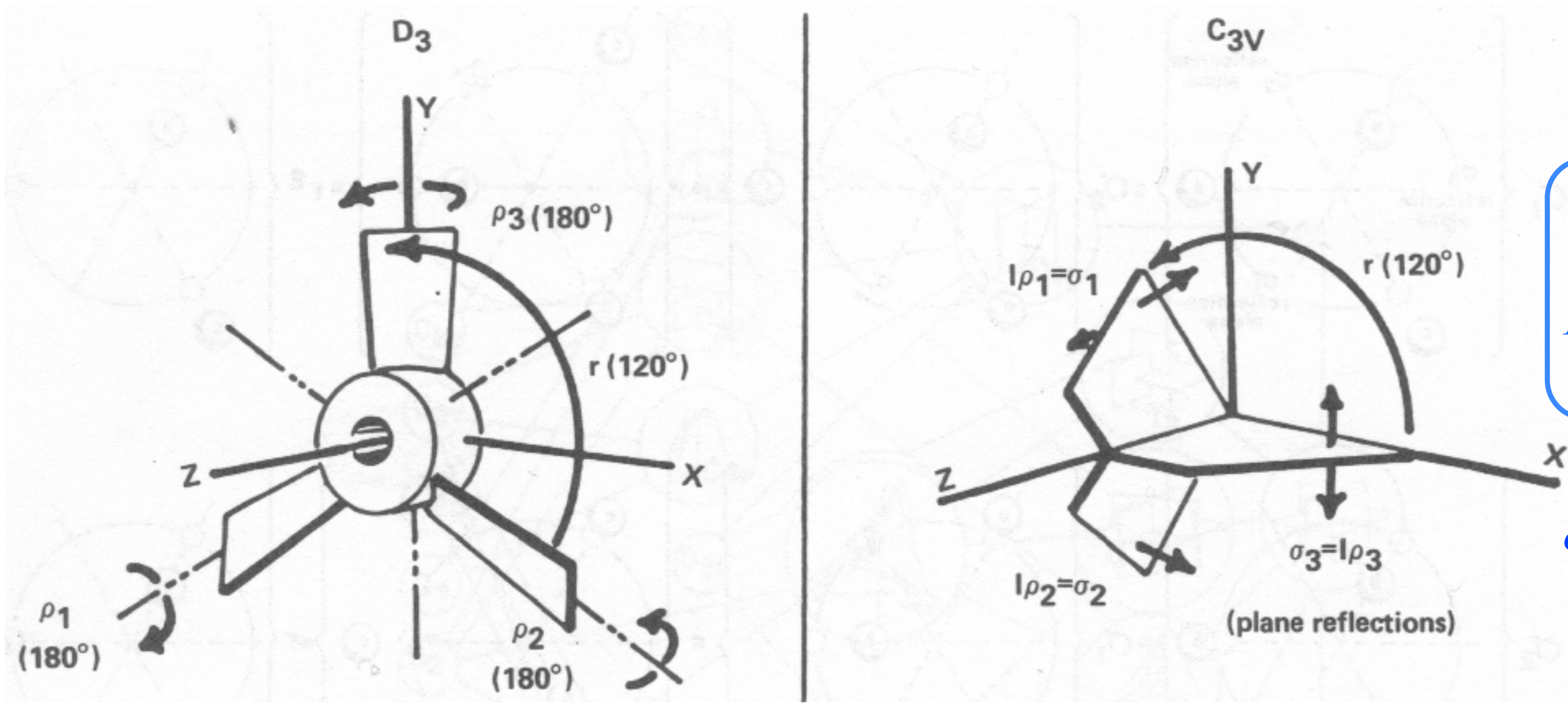

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Mirror-plane-reflection $\boldsymbol{\sigma}$
equals
180° ⊥-axial-rotation-inversion
$\boldsymbol{\sigma} = \mathbf{R}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{R}$

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix} = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = \rho_3\cdot\mathbf{I} = \mathbf{I}\cdot\rho_3$$

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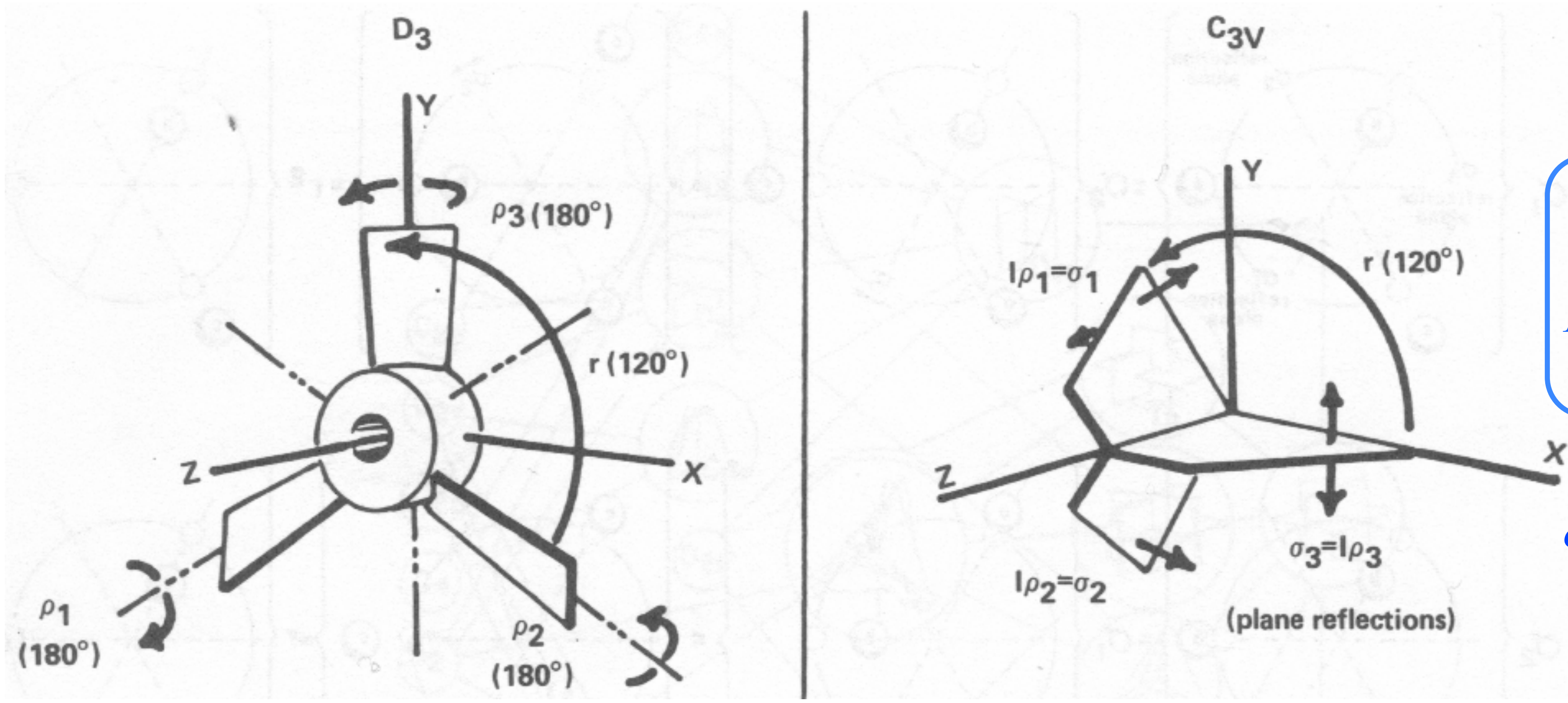

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Inversion
$\mathbf{I} = -\mathbf{1}$
commutes with all $\mathbf{R}$

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3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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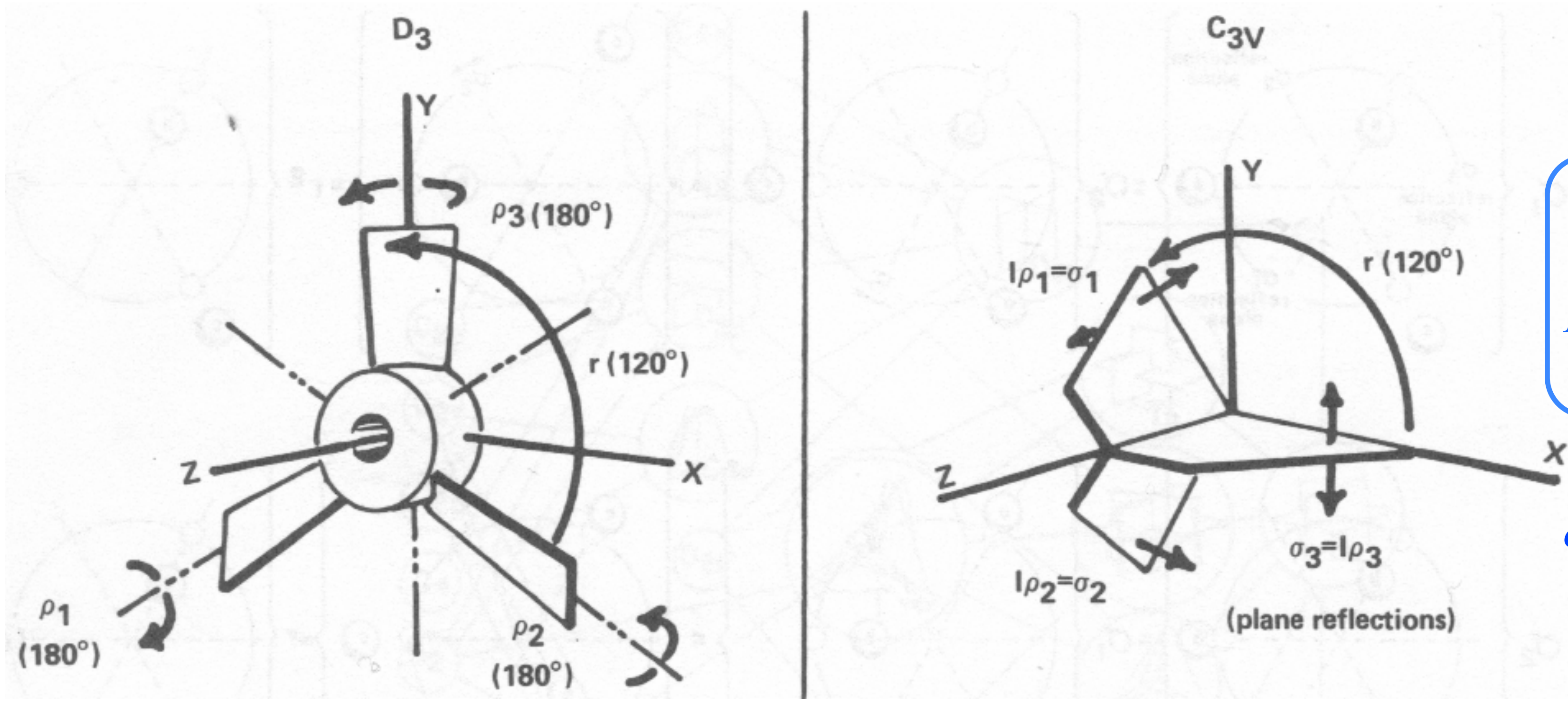

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$\sigma = \mathbf{R}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{R}$

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$$= \rho_3\cdot\mathbf{I} = \mathbf{I}\cdot\rho_3$$

180° D_3-Y-axis-rotation: $\rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$ *maps to*: XZ-mirror-plane reflection: $\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$

180° D_3-ρ_2-axis-rotation: ρ_2 *maps to*: ⊥ρ_2-mirror-plane reflection: $\sigma_2 = \rho_2\cdot\mathbf{I} = \mathbf{I}\cdot\rho_2$

Inversion
$\mathbf{I} = -\mathbf{1}$
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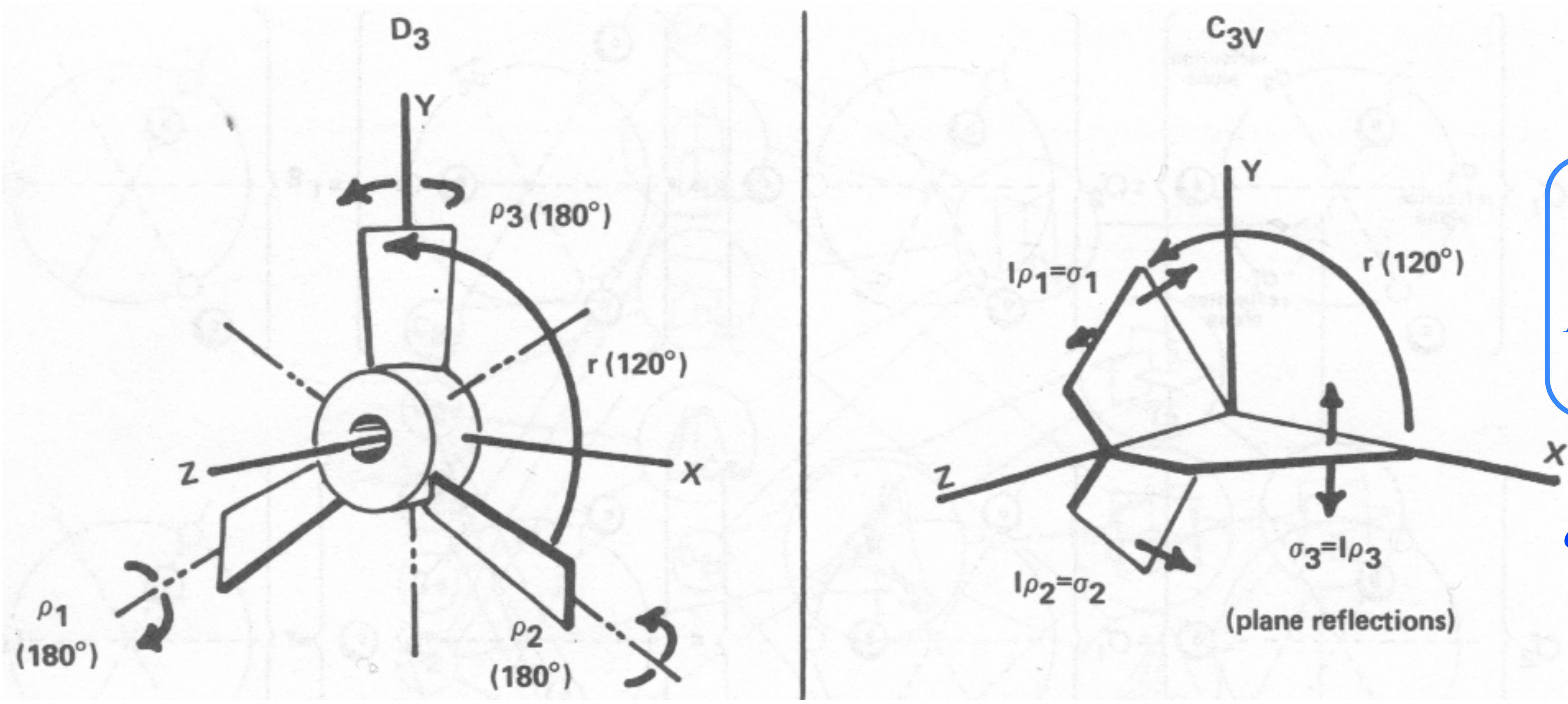

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$\sigma = \mathbf{R}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{R}$

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix} = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = \rho_3\cdot\mathbf{I} = \mathbf{I}\cdot\rho_3$$

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180° D_3-ρ_2-axis-rotation: ρ_2 *maps to*: $\perp\rho_2$-mirror-plane reflection: $\sigma_2 = \rho_2\cdot\mathbf{I} = \mathbf{I}\rho_2$

180° D_3-ρ_1-axis-rotation: ρ_1 *maps to*: $\perp\rho_1$-mirror-plane reflection: $\sigma_1 = \rho_1\cdot\mathbf{I} = \mathbf{I}\rho_1$

Inversion $\mathbf{I} = -\mathbf{1}$ *commutes with all* **R**

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3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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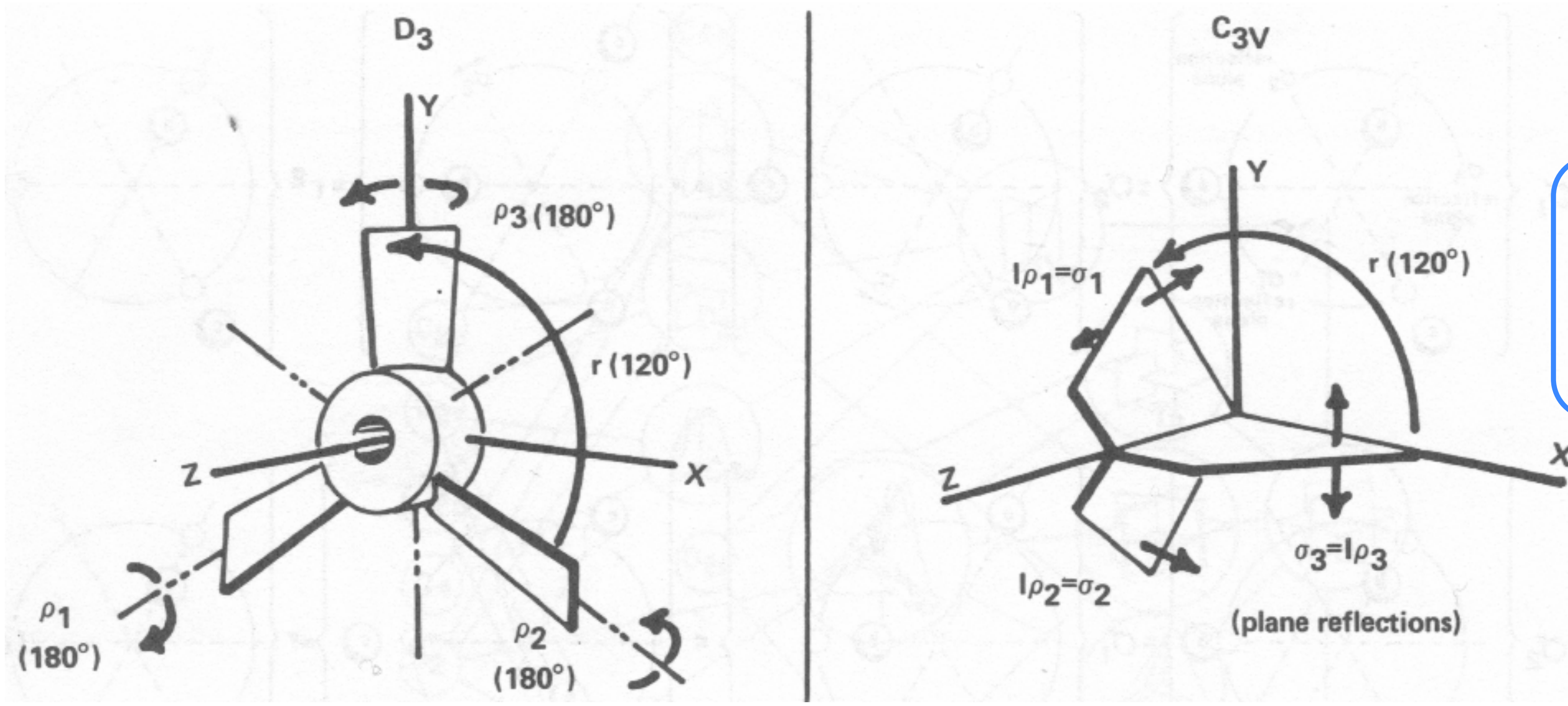

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Mirror-plane-reflection **σ** *equals* *180° ⊥-axial-rotation-inversion*
σ = R·I = I·R

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix} = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} = \rho_3 \cdot \mathbf{I} = \mathbf{I} \cdot \rho_3$$

$$180° D_3\text{-Y-axis-rotation: } \rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \quad \textit{maps to}: \text{ XZ-mirror-plane reflection: } \sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$$

Inversion **I** = **−1** *commutes with all* **R**

$180° D_3$-ρ_2-axis-rotation: ρ_2 *maps to*: ⊥ρ_2-mirror-plane reflection: $\sigma_2 = \rho_2 \cdot \mathbf{I} = \mathbf{I}\rho_2$

$180° D_3$-ρ_1-axis-rotation: ρ_1 *maps to*: ⊥ρ_1-mirror-plane reflection: $\sigma_1 = \rho_1 \cdot \mathbf{I} = \mathbf{I}\rho_1$

D_3-product: $\rho_1\rho_2$ *maps to*: C_{3v}-product: $\sigma_1\sigma_2 = \rho_1 \mathbf{I}\mathbf{I}\rho_2 = \rho_1\rho_2$

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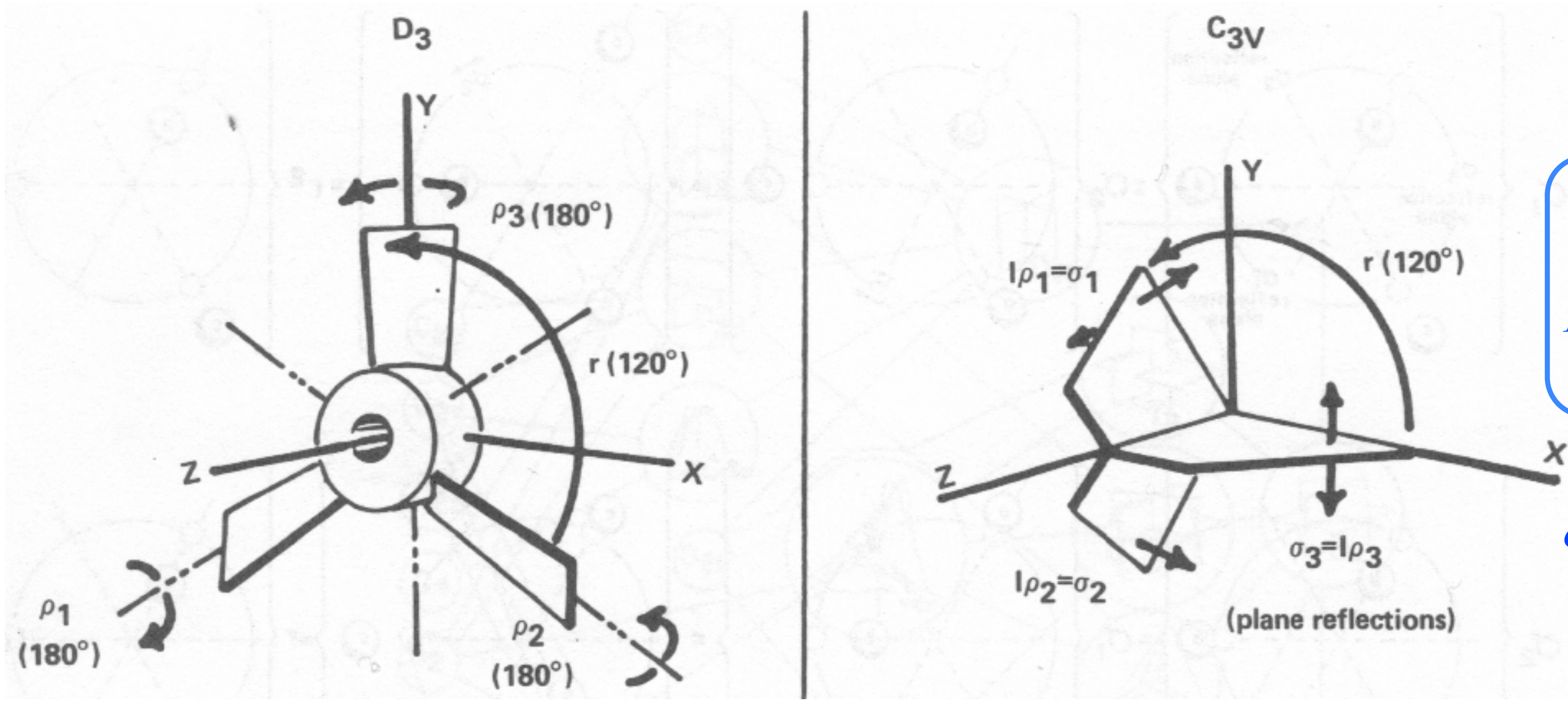

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$180°\ D_3$-ρ_1-axis-rotation: ρ_1 *maps to*: $\perp\rho_1$-mirror-plane reflection: $\sigma_1 = \rho_1\cdot\mathbf{I} = \mathbf{I}\cdot\rho_1$

D_3-product: $\rho_1\rho_2$ *maps to*: C_{3v}-product: $\sigma_1\sigma_2 = \rho_1\mathbf{I}\mathbf{I}\rho_2 = \rho_1\rho_2$

D_3-product: $\rho_1\mathbf{r}^p$ *maps to*: C_{3v}-product: $\sigma_1\mathbf{r}^p = \rho_1\mathbf{I}\mathbf{r}^p = \rho_1\mathbf{r}^p\mathbf{I} = \mathbf{I}\rho_1\mathbf{r}^p$

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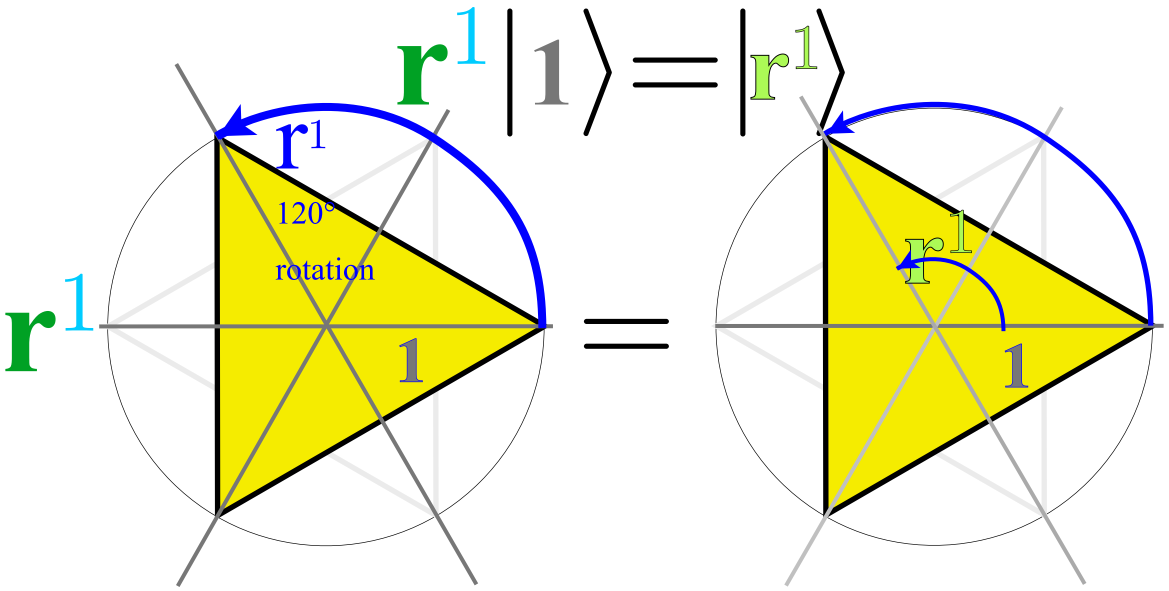
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

$$\mathbf{r}^1 |1\rangle = |\mathbf{r}^1\rangle$$



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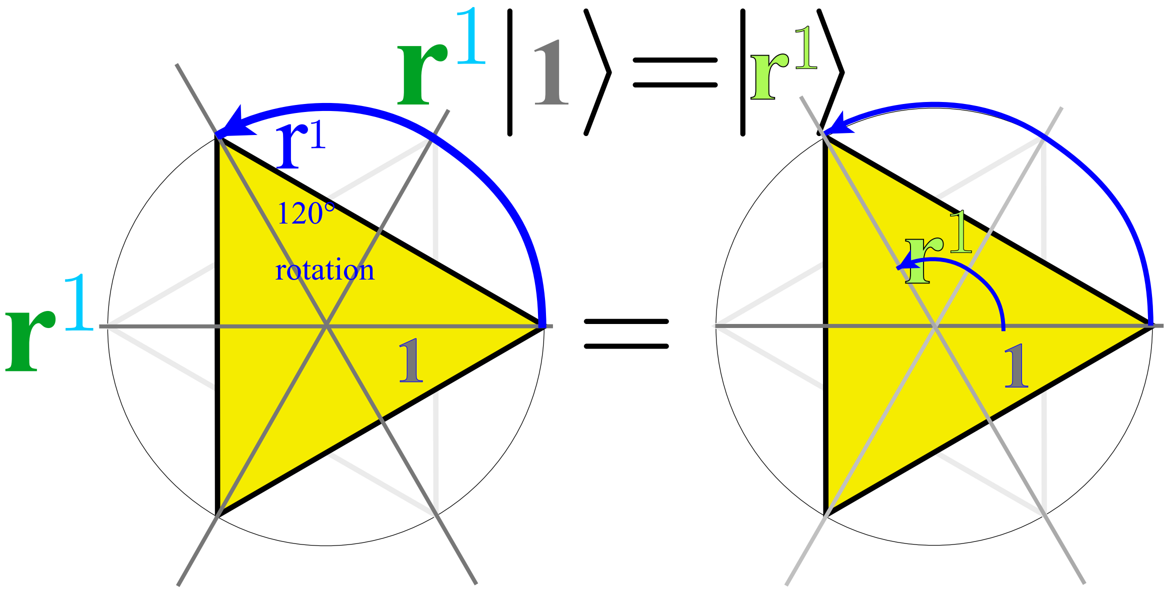

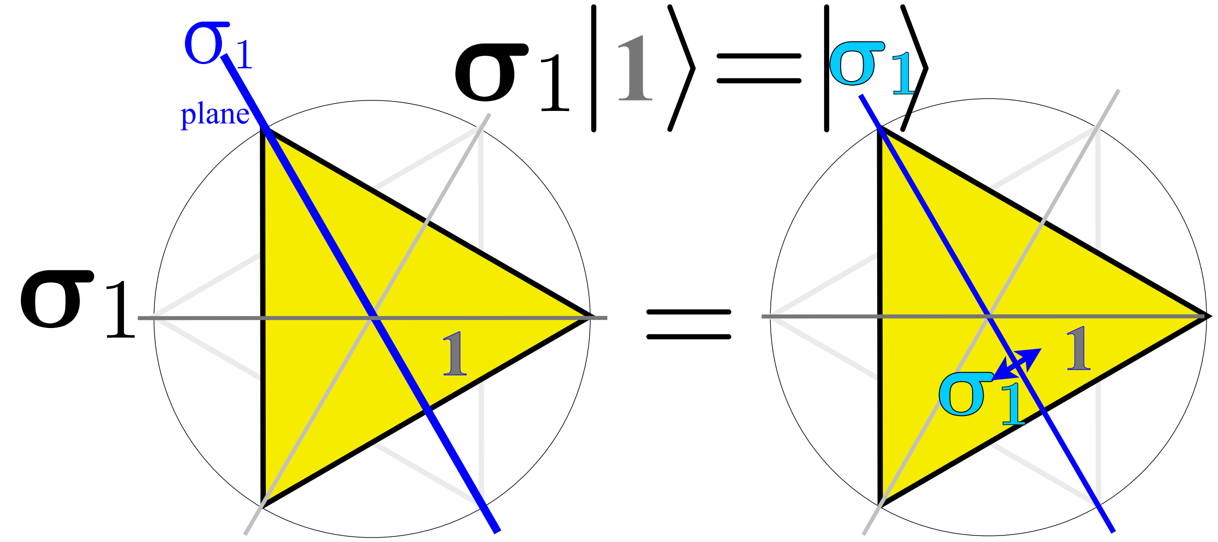

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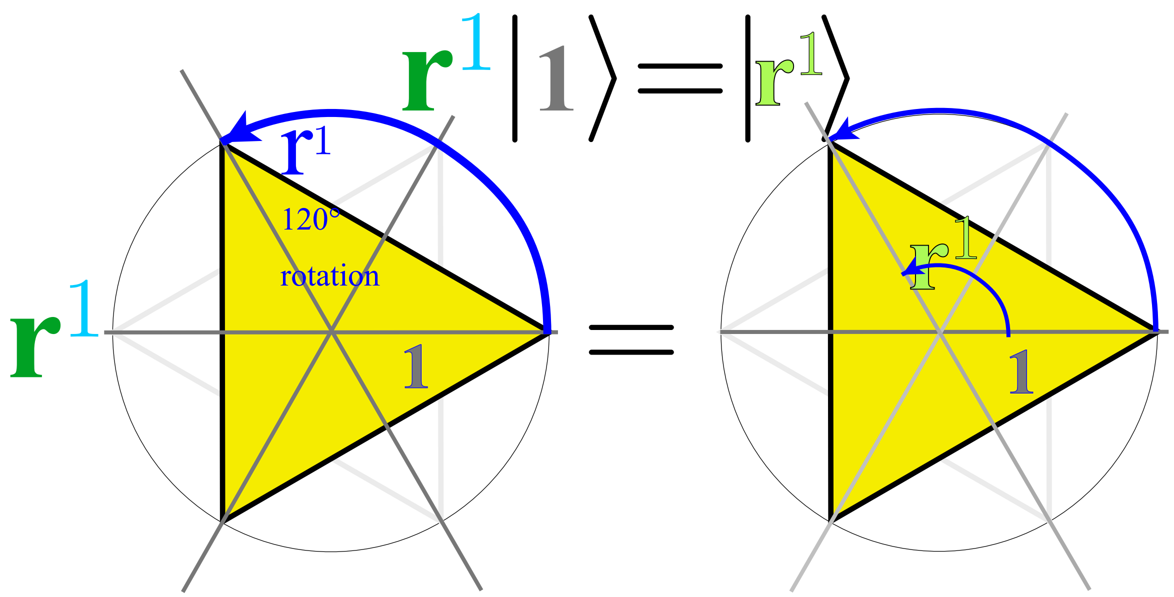
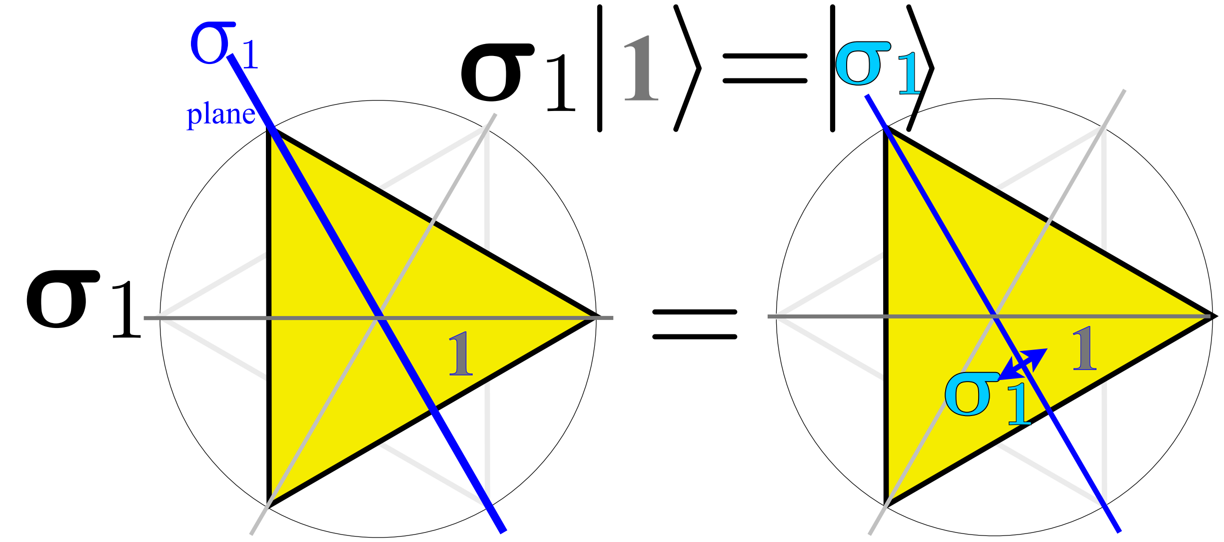
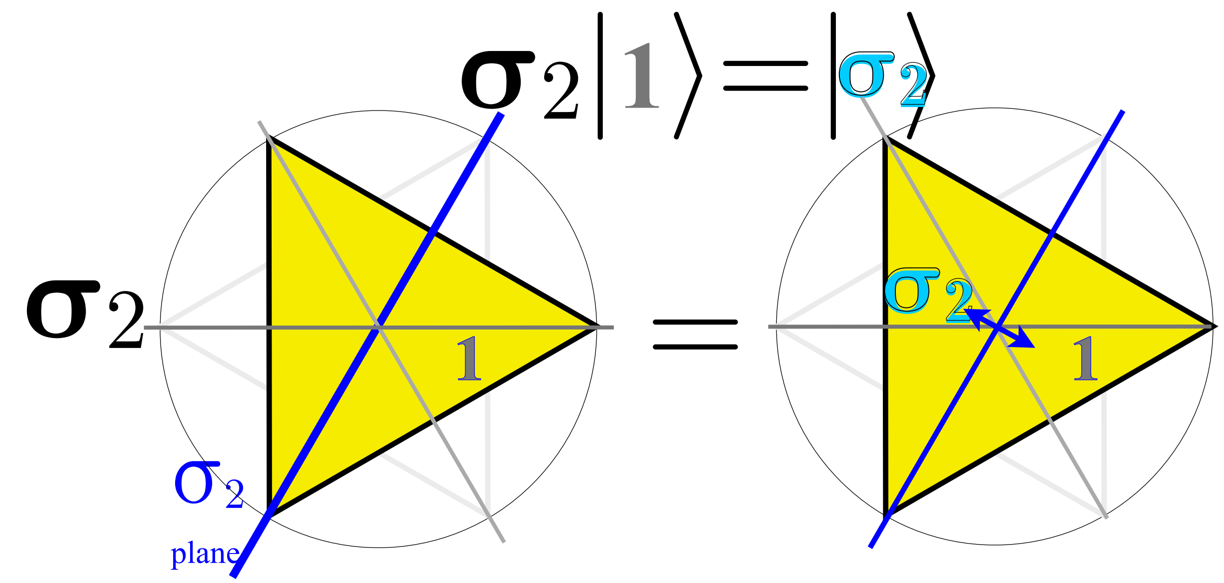
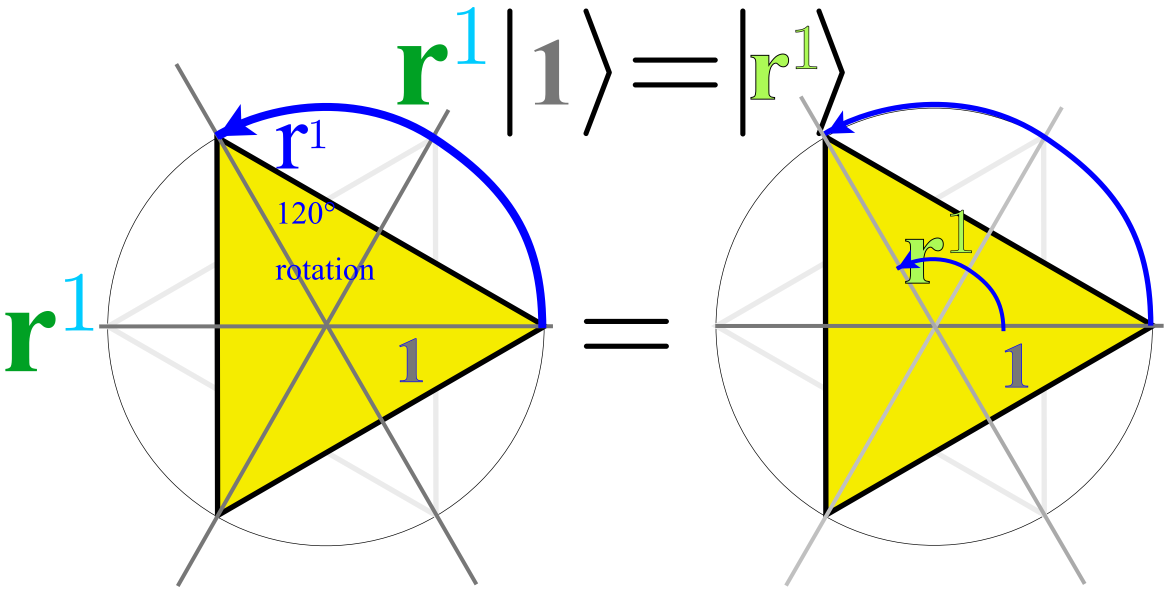
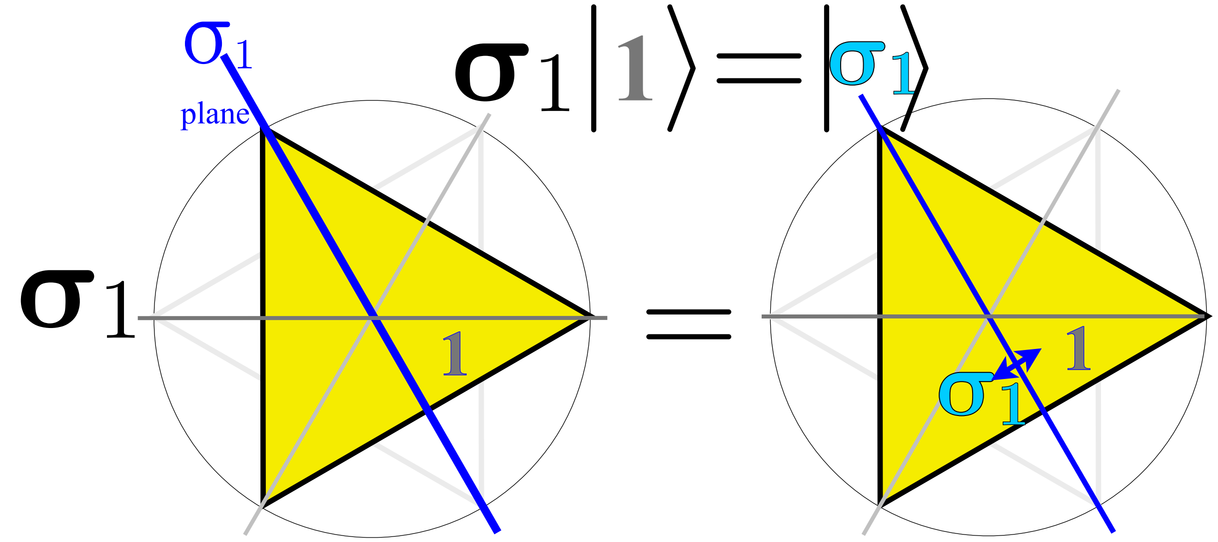


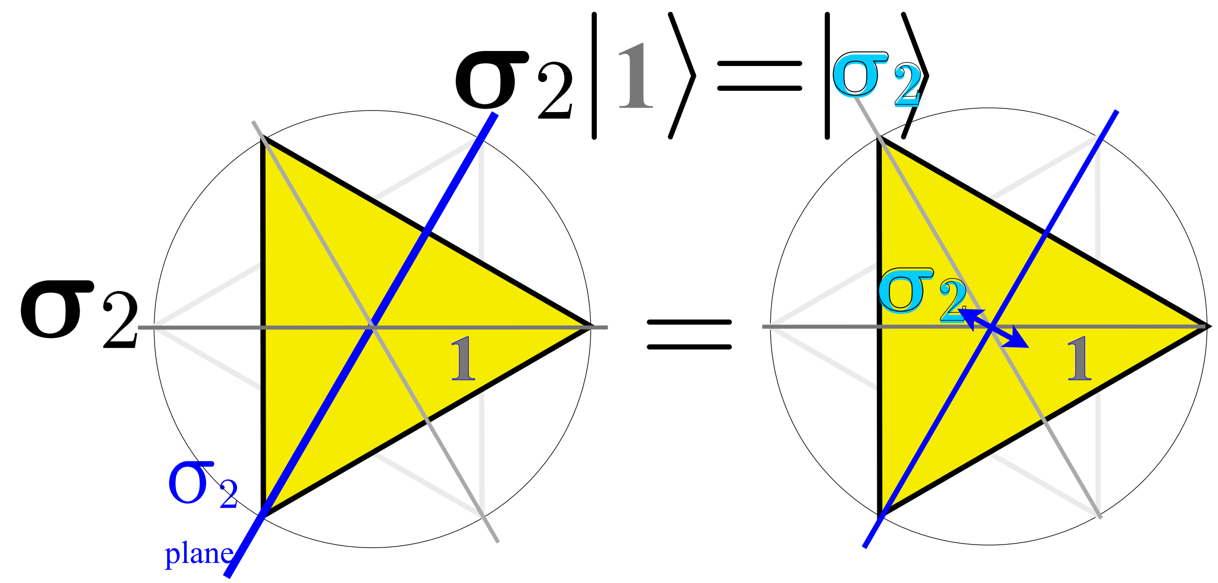




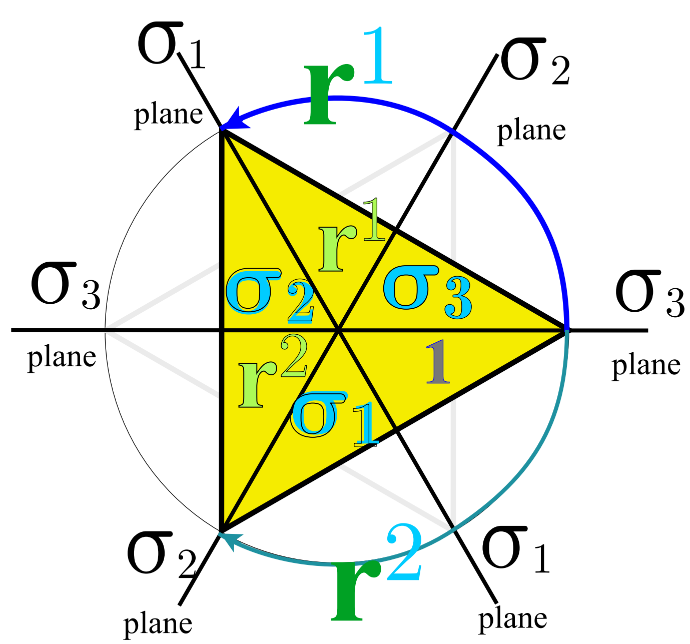
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

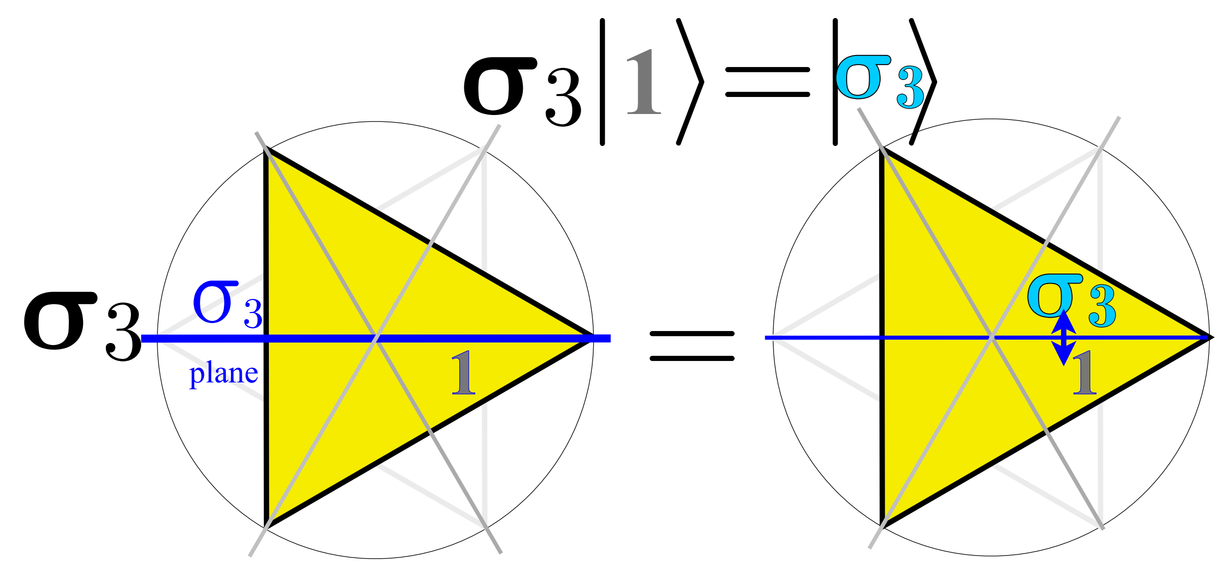






Building
C_{3v} Group
"slide-rule"





3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

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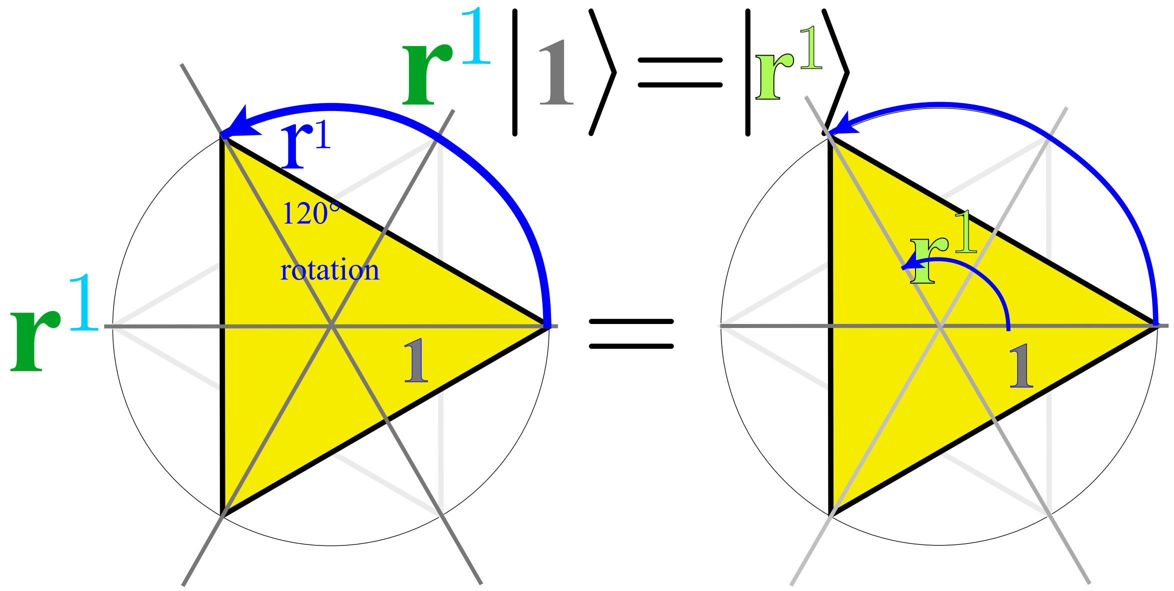
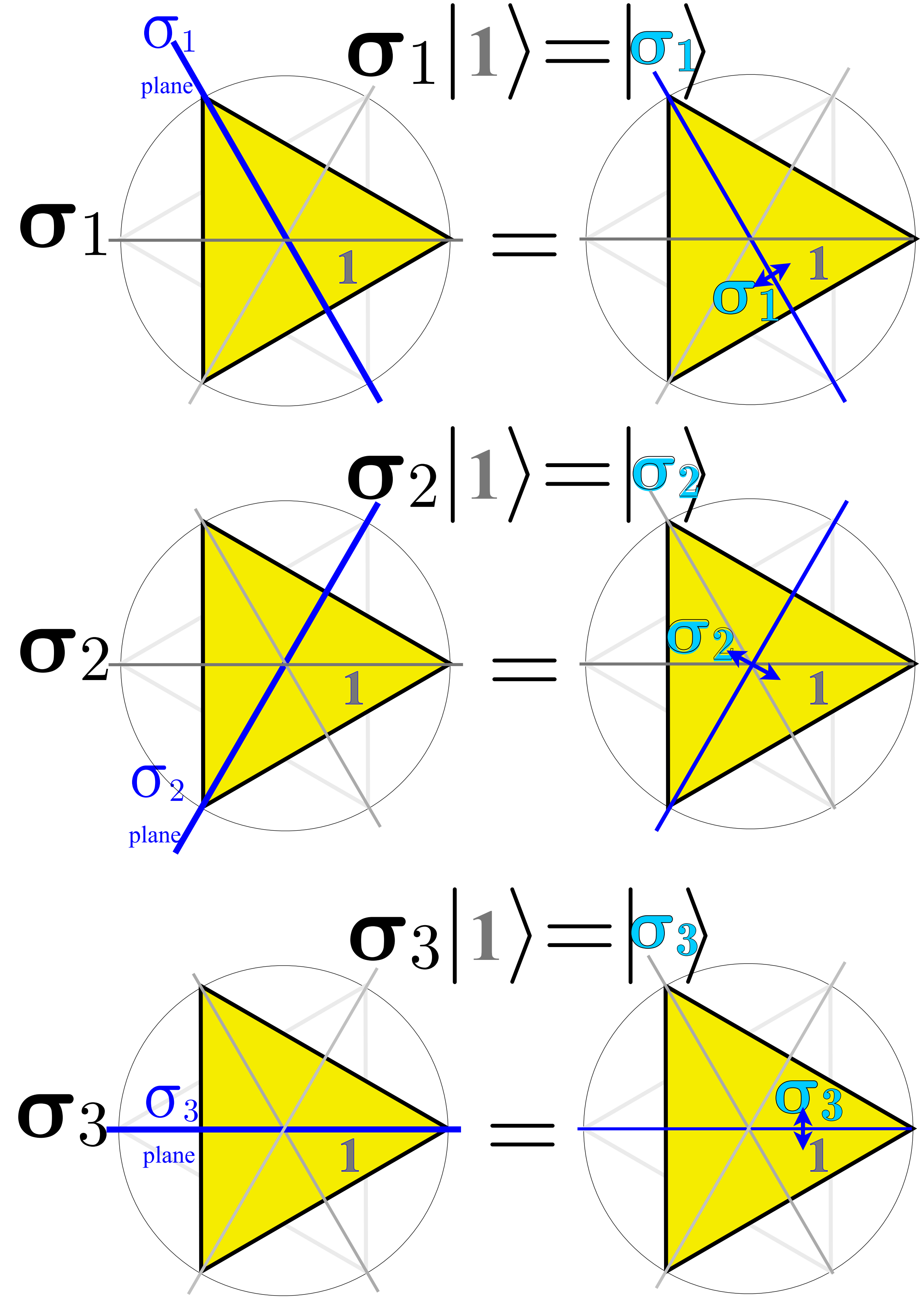
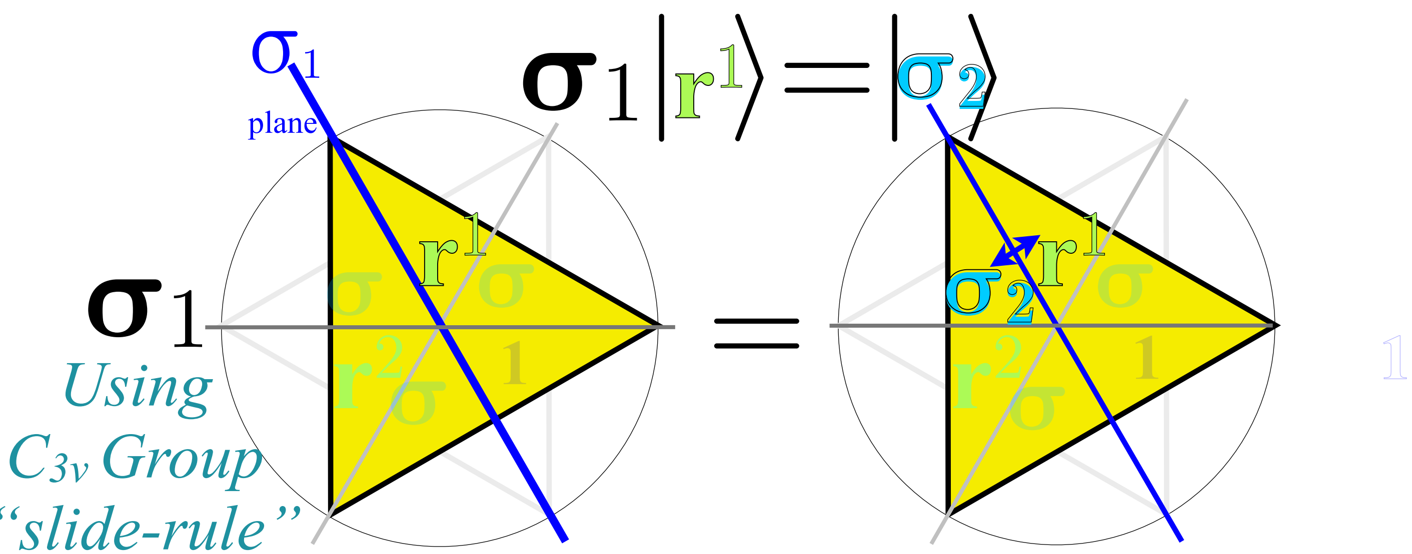
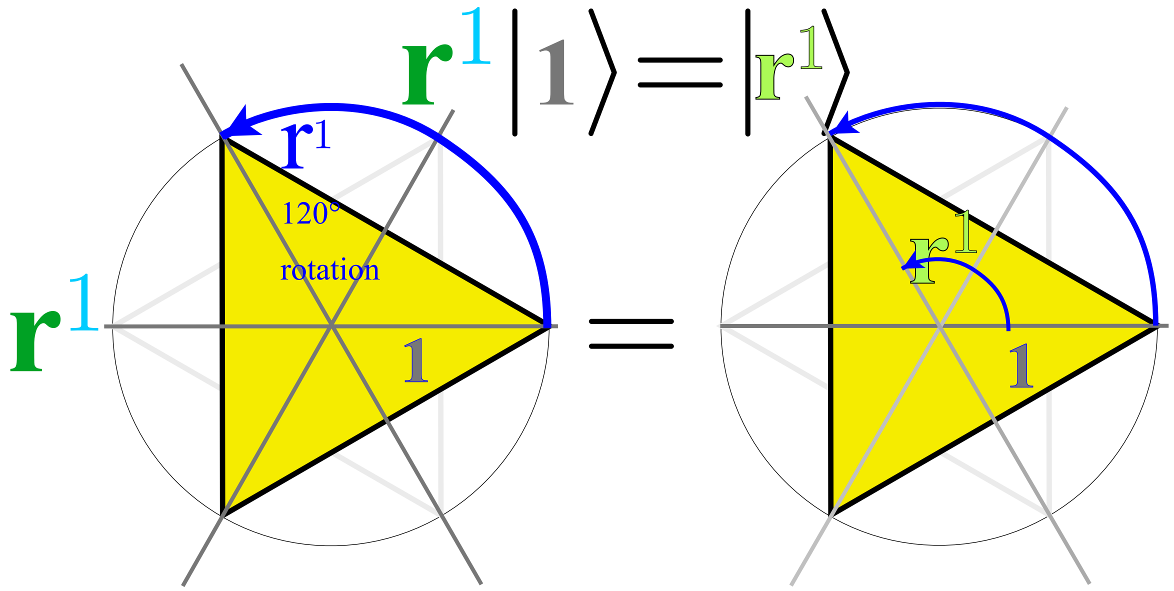


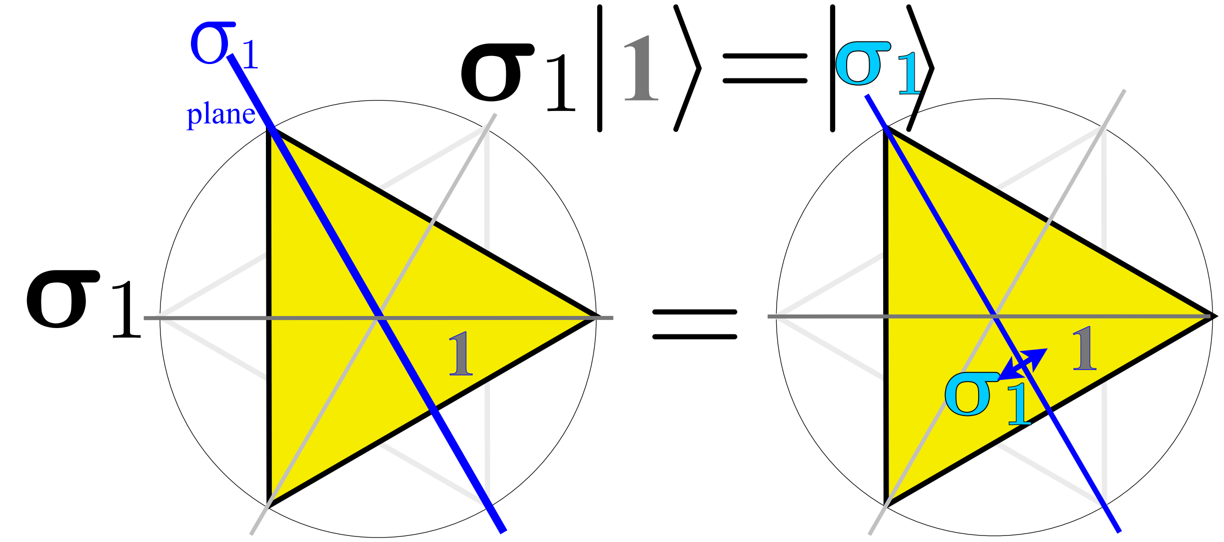


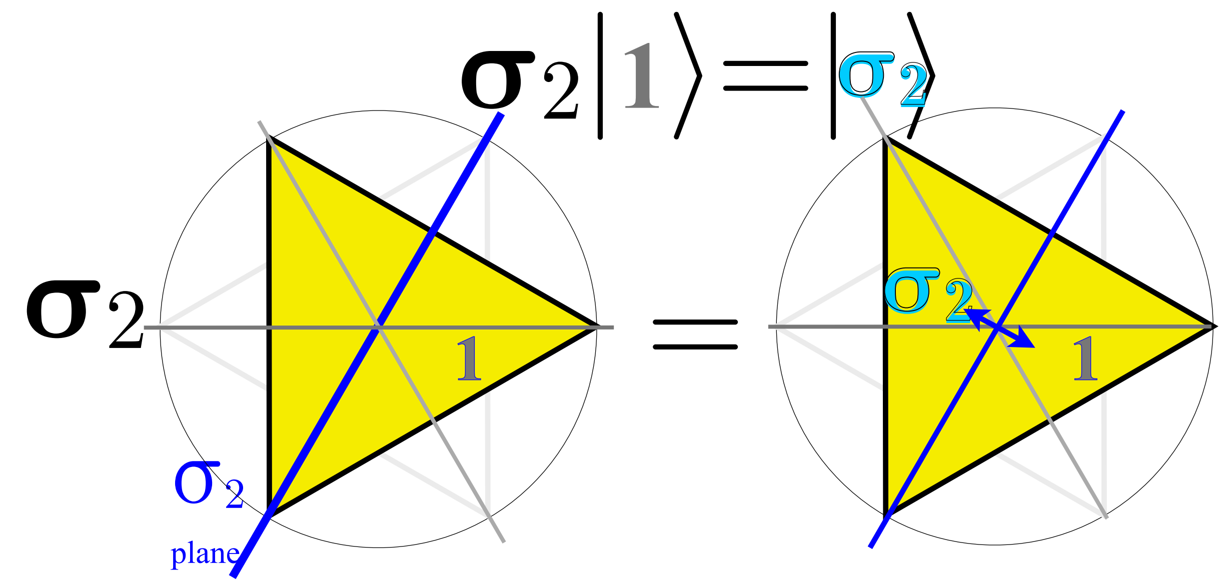
Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$

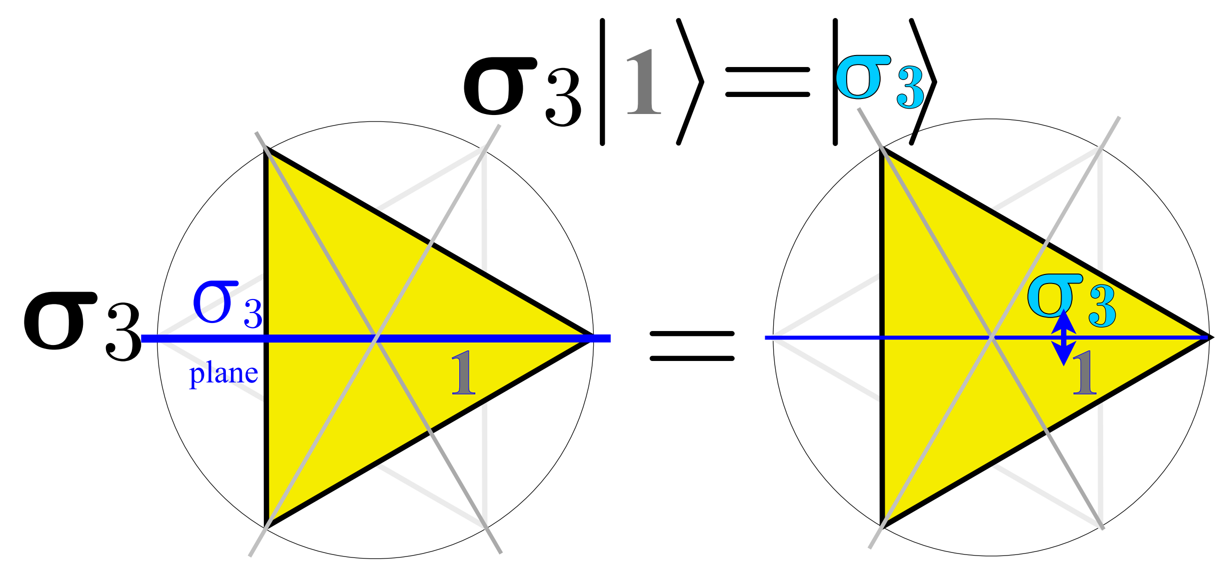


Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

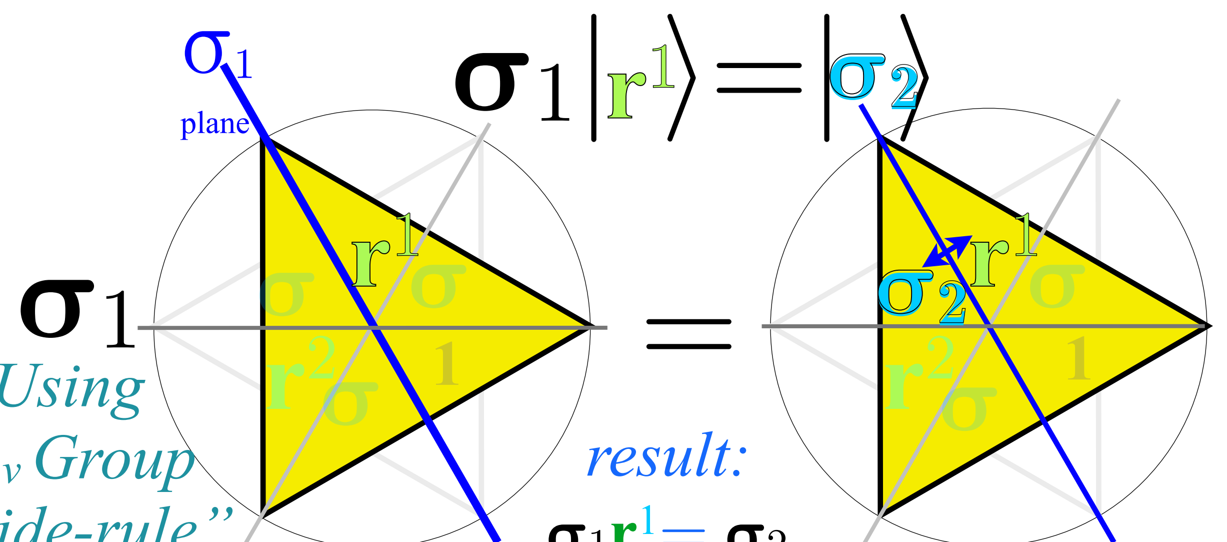








Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$



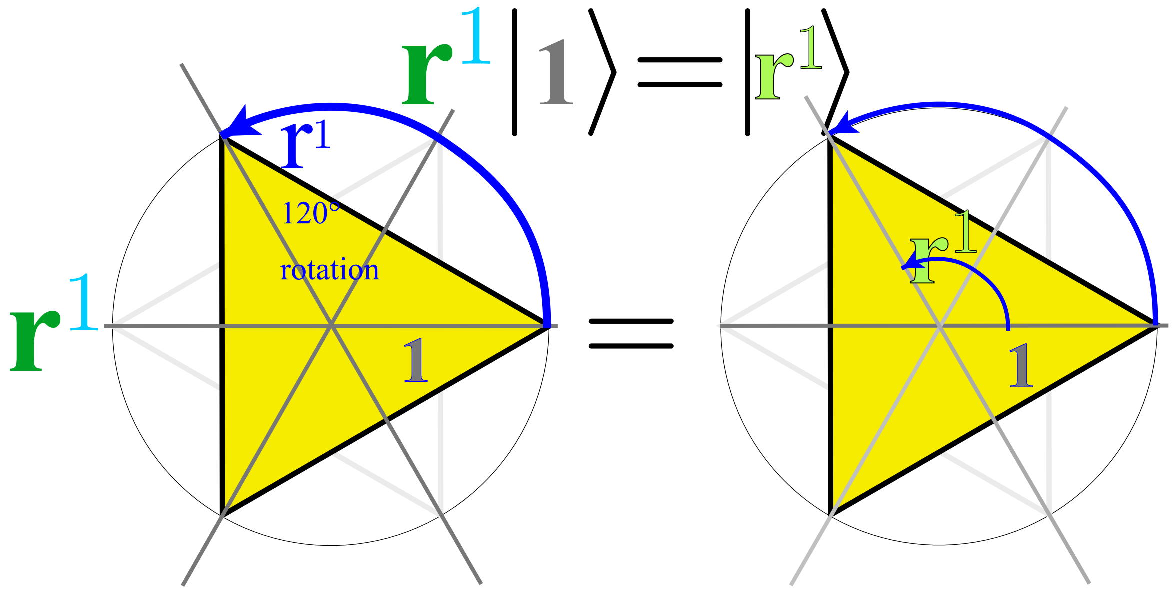
Using C_{3v} Group "slide-rule"

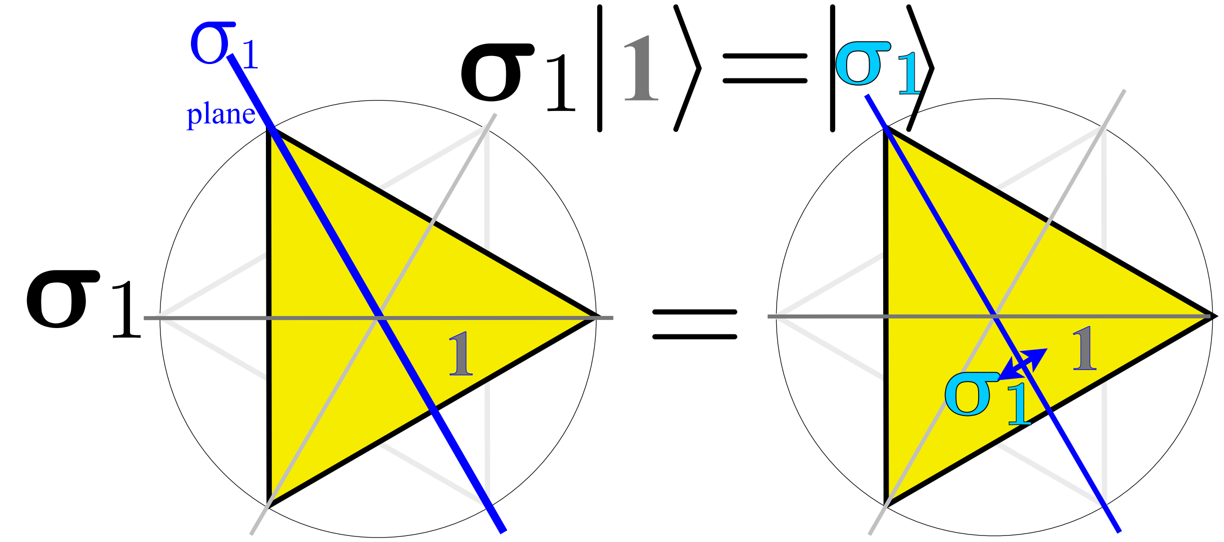
result:
$\boldsymbol{\sigma}_1\mathbf{r}^1 = \boldsymbol{\sigma}_2$

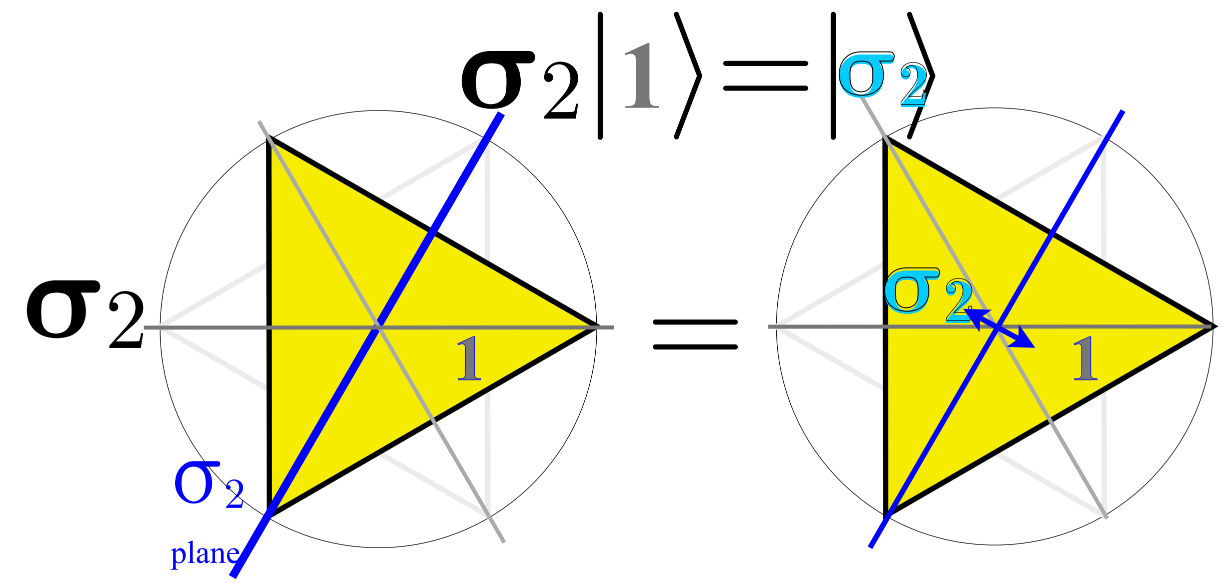
Factor $\mathbf{r}^1$ on right acts first

left is last
(like Hebrew)

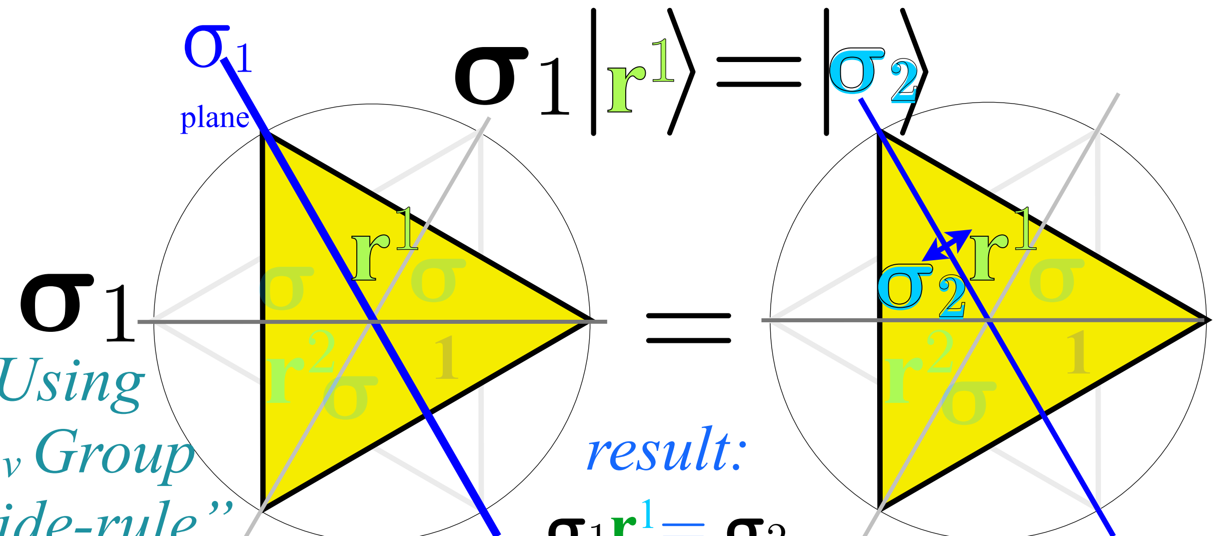
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





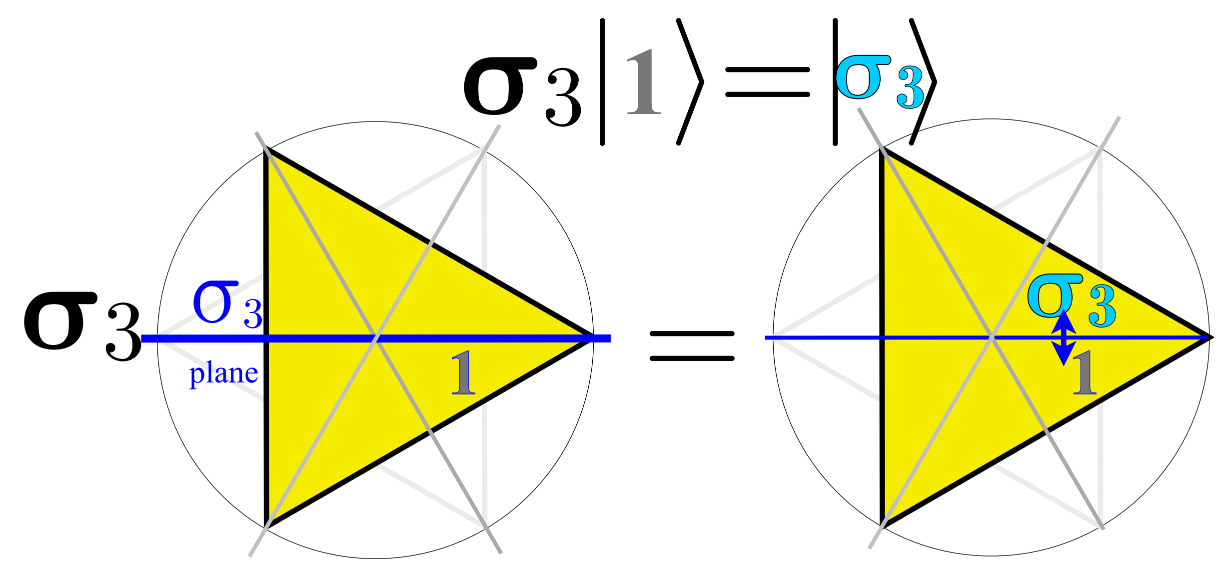


Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$



Using C_{3v} Group "slide-rule"

result: $\boldsymbol{\sigma}_1\mathbf{r}^1 = \boldsymbol{\sigma}_2$

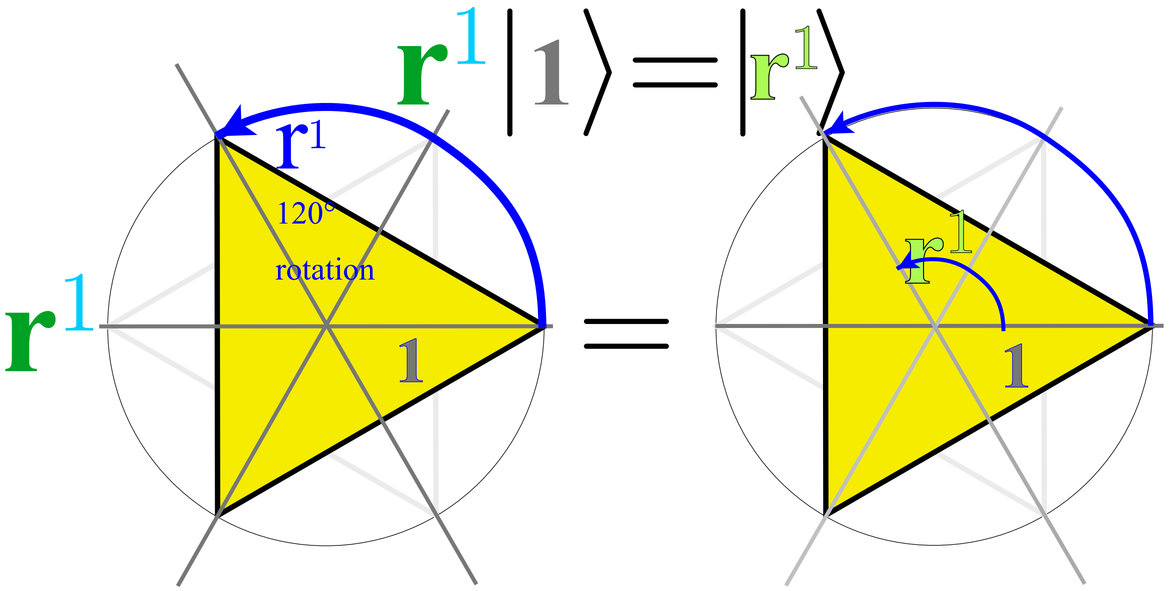


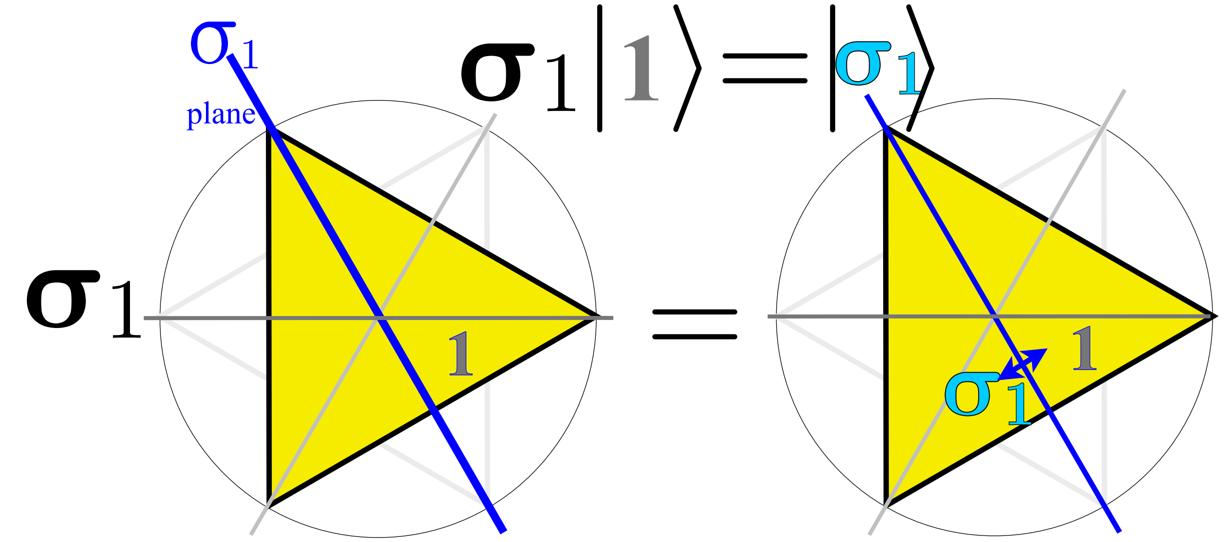
Other $\boldsymbol{\sigma}_1$ results from graph:

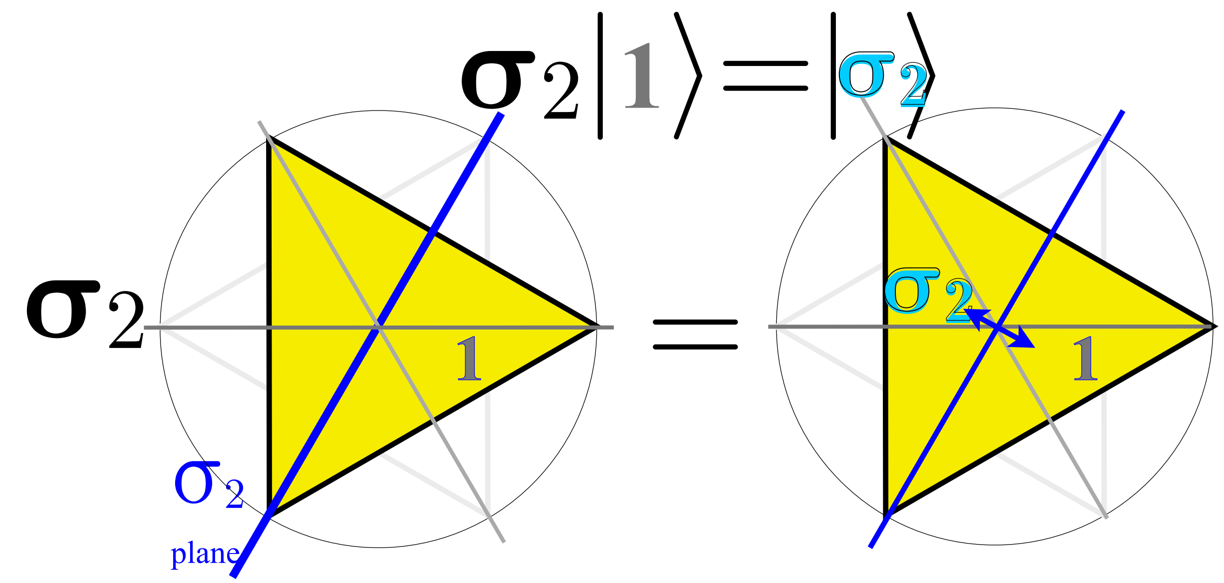
$\boldsymbol{\sigma}_1\{\mathbf{1}, \mathbf{r}^1, \mathbf{r}^2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3\}$

$= \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \mathbf{1}, \mathbf{r}^1, \mathbf{r}^2\}$

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$







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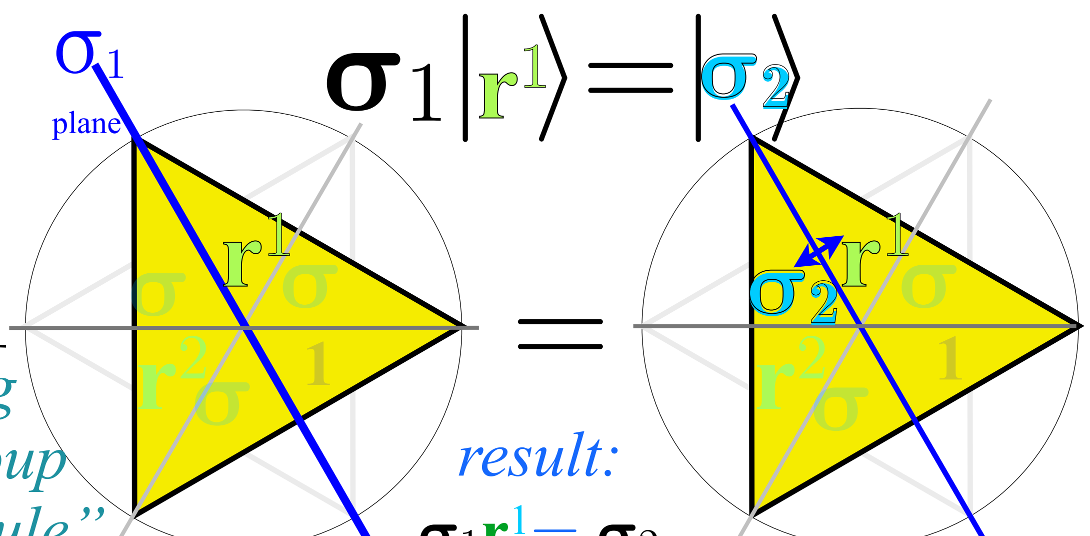

Using C_{3v} Group "slide-rule"

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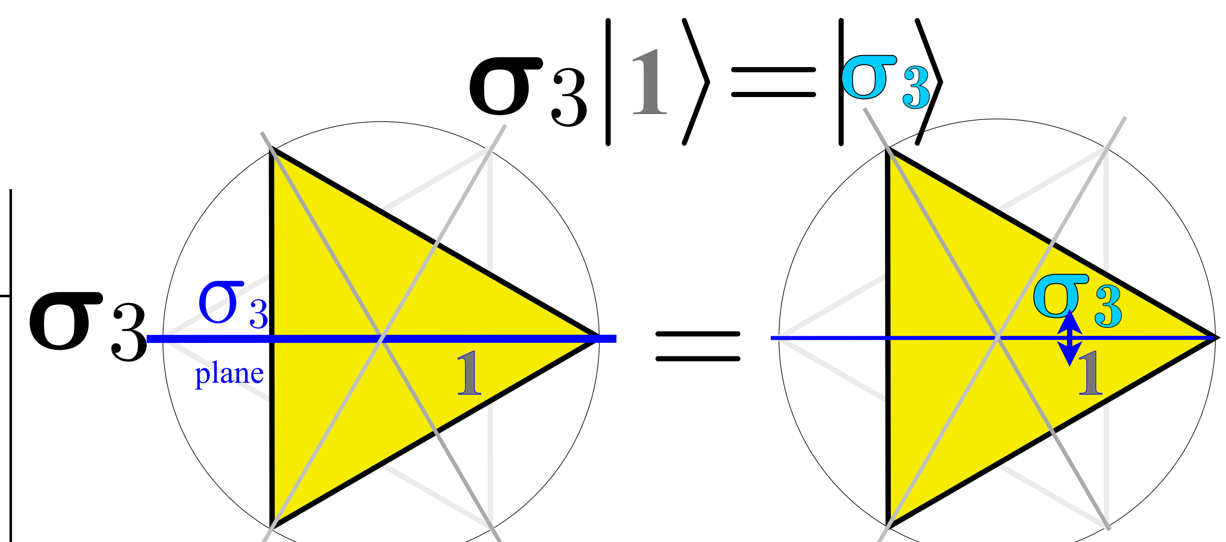

Other $\boldsymbol{\sigma}_1$ results from graph:

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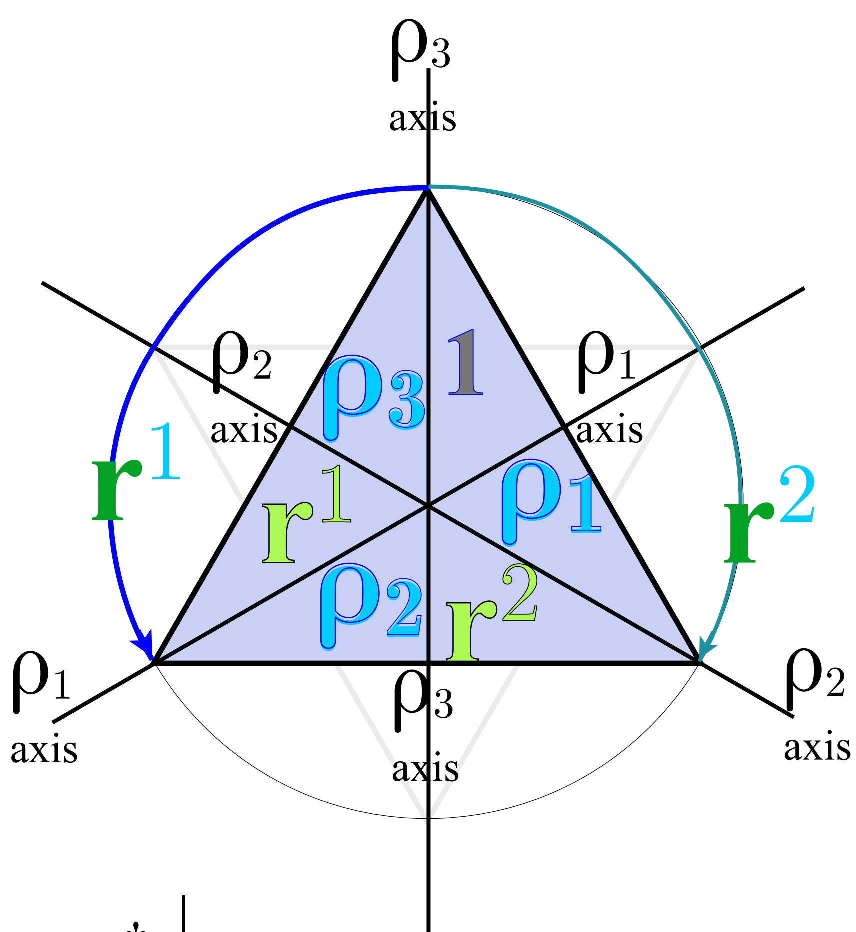
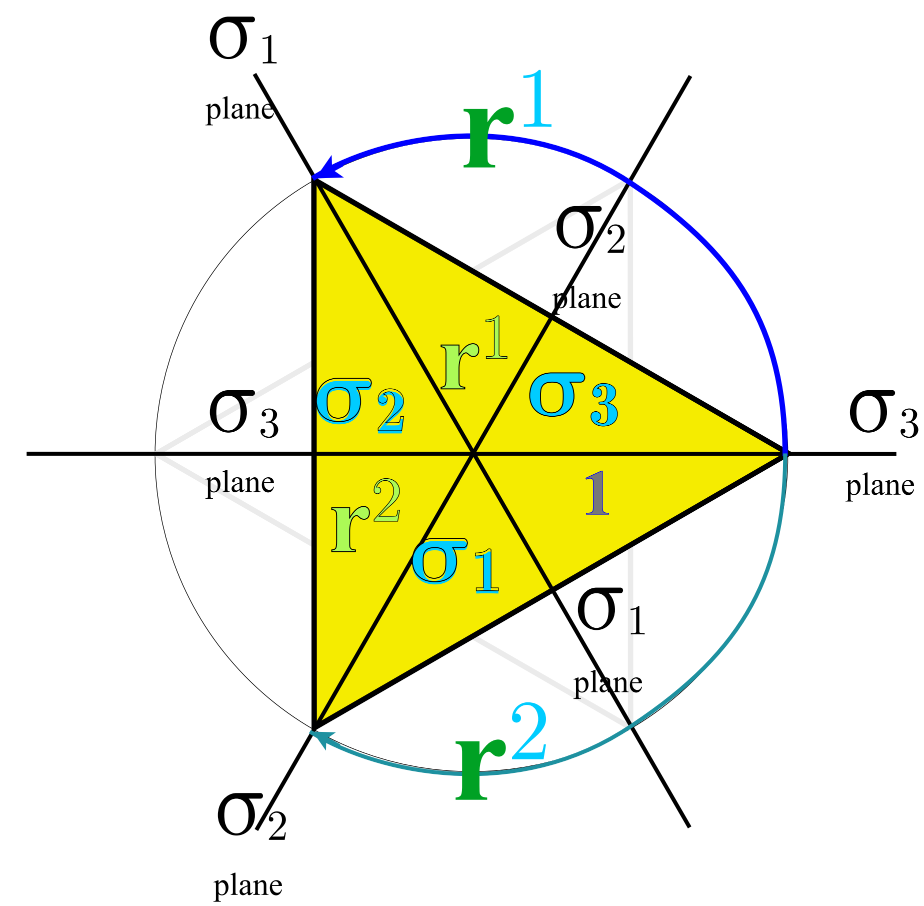
$= \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \mathbf{1}, \mathbf{r}^1, \mathbf{r}^2\}$

....whole C_{3v} group table:

C_{3v} $gg^\dagger$ form	1	r^2	r^1	σ_1	σ_2	σ_3
1	1	r^2	r^1	σ_1	σ_2	σ_3
r^1	r^1	1	r^2	σ_3	σ_1	σ_2
r^2	r^2	r^1	1	σ_2	σ_3	σ_1
σ_1	σ_1	σ_3	σ_2	1	r^1	r^2
σ_2	σ_2	σ_1	σ_3	r^2	1	r^1
σ_3	σ_3	σ_2	σ_1	r^1	r^2	1

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

D_3 and C_{3v} clearly are isomorphic $D_3 \sim C_{3v}$ share group table

...except for notation
$\rho_k \leftrightarrow \sigma_k$

C_{3v} $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_3	σ_1	σ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_2	σ_3	σ_1
σ_1	σ_1	σ_3	σ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
σ_2	σ_2	σ_1	σ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
σ_3	σ_3	σ_2	σ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

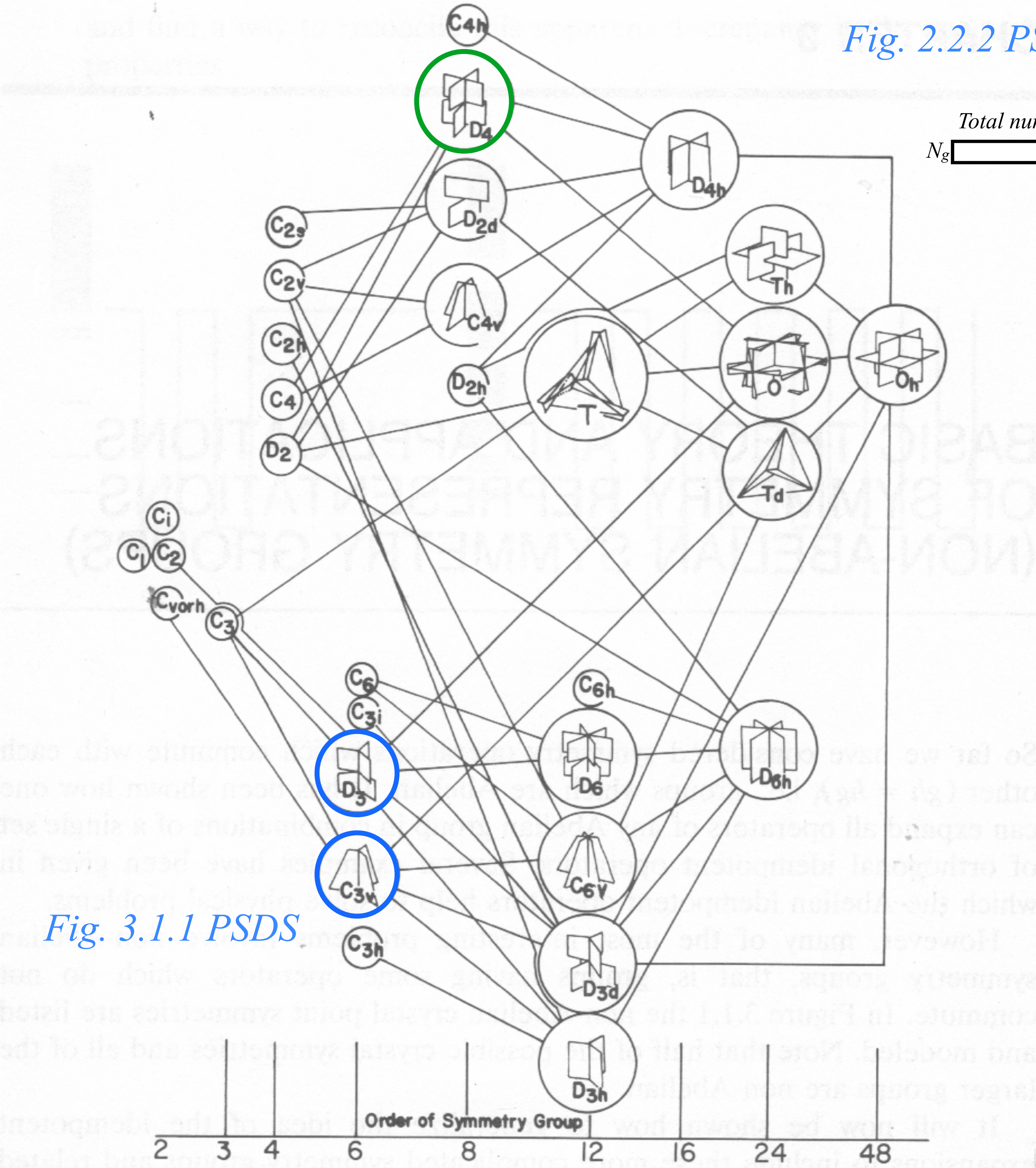
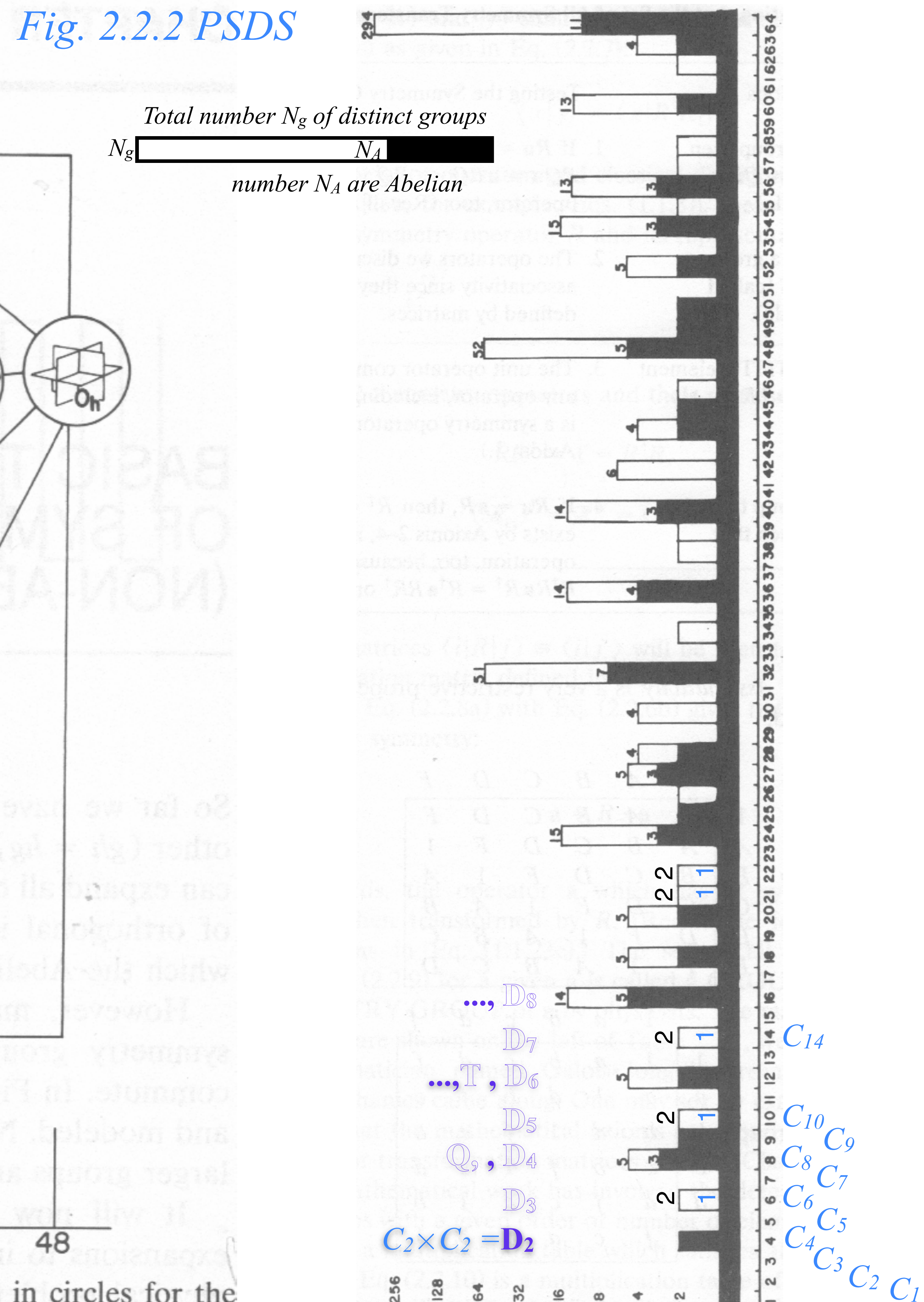


Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)



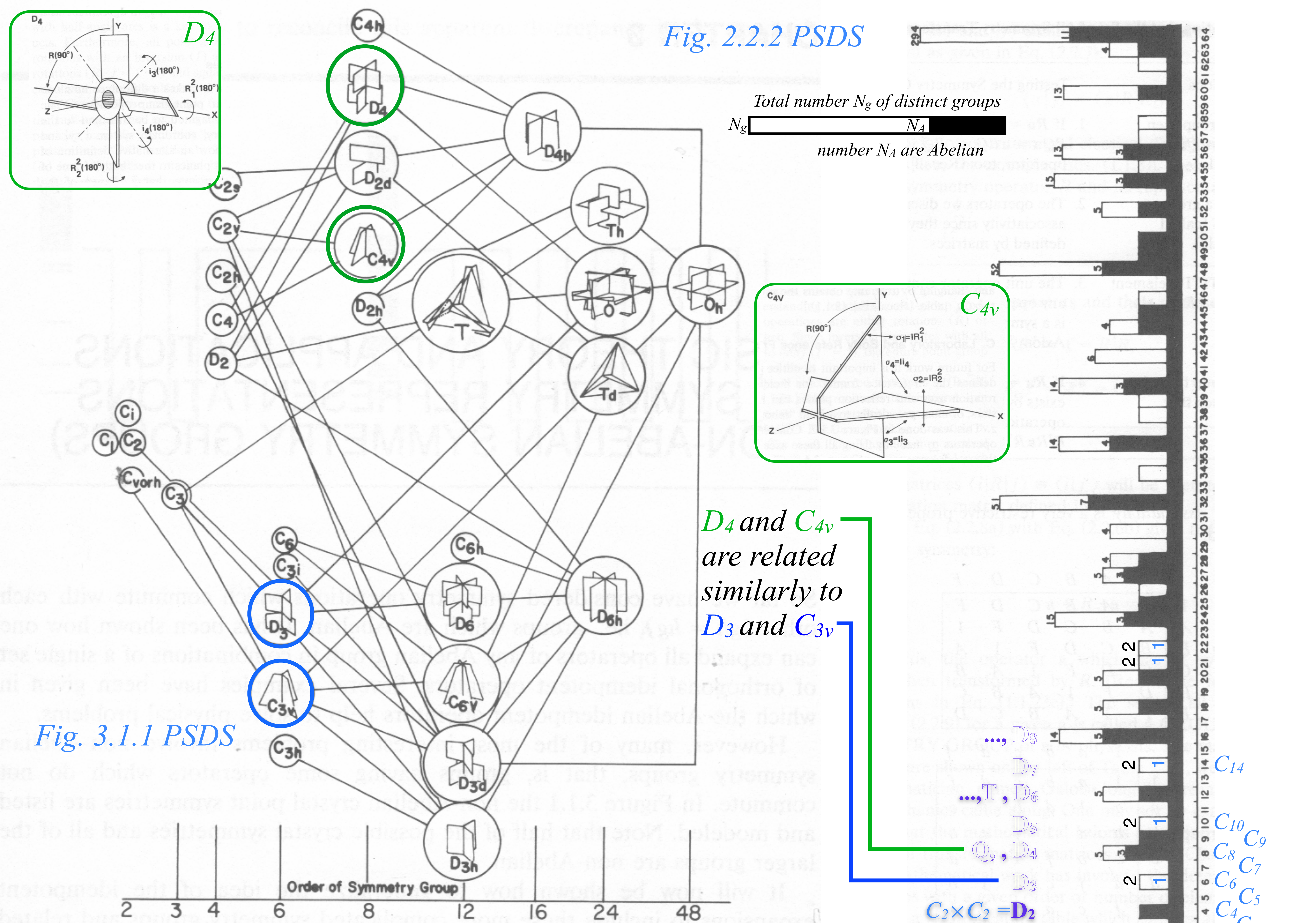


Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

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Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

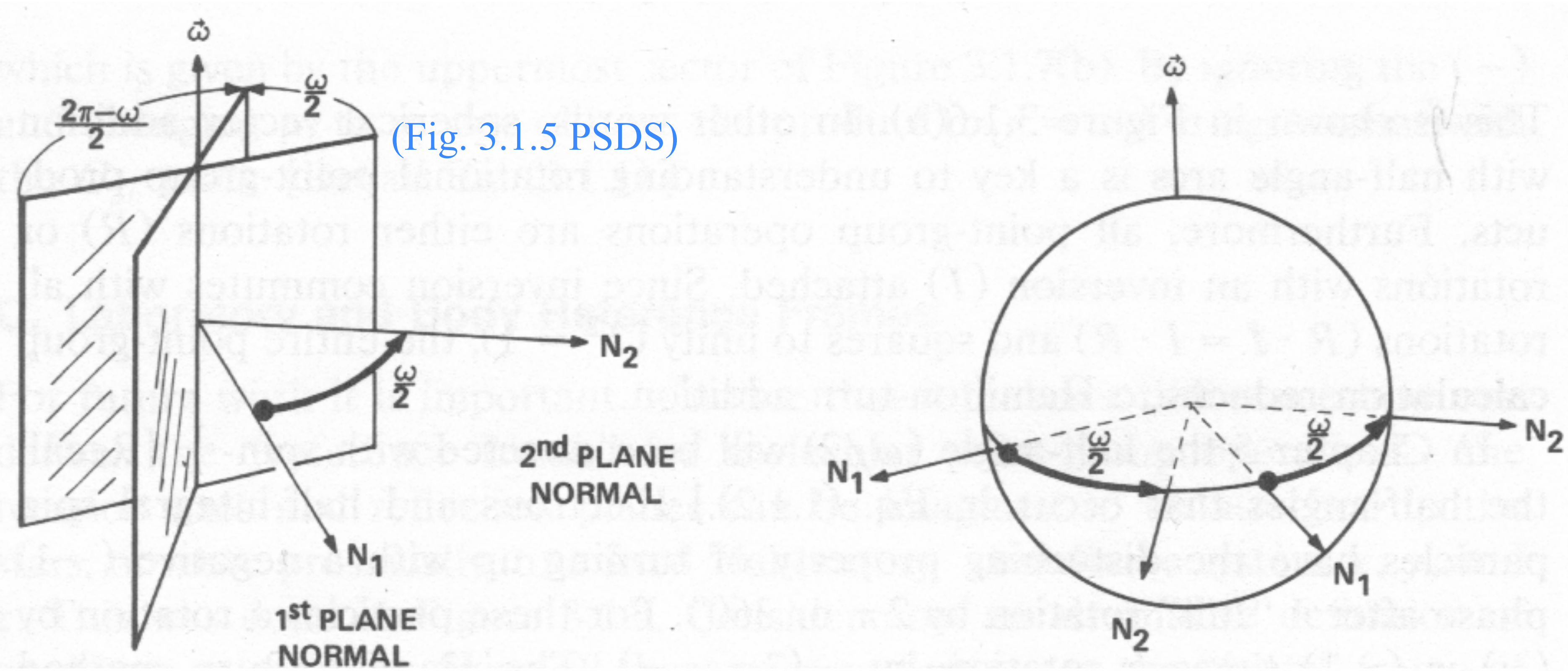
Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting operators and D_3-invariant class algebra

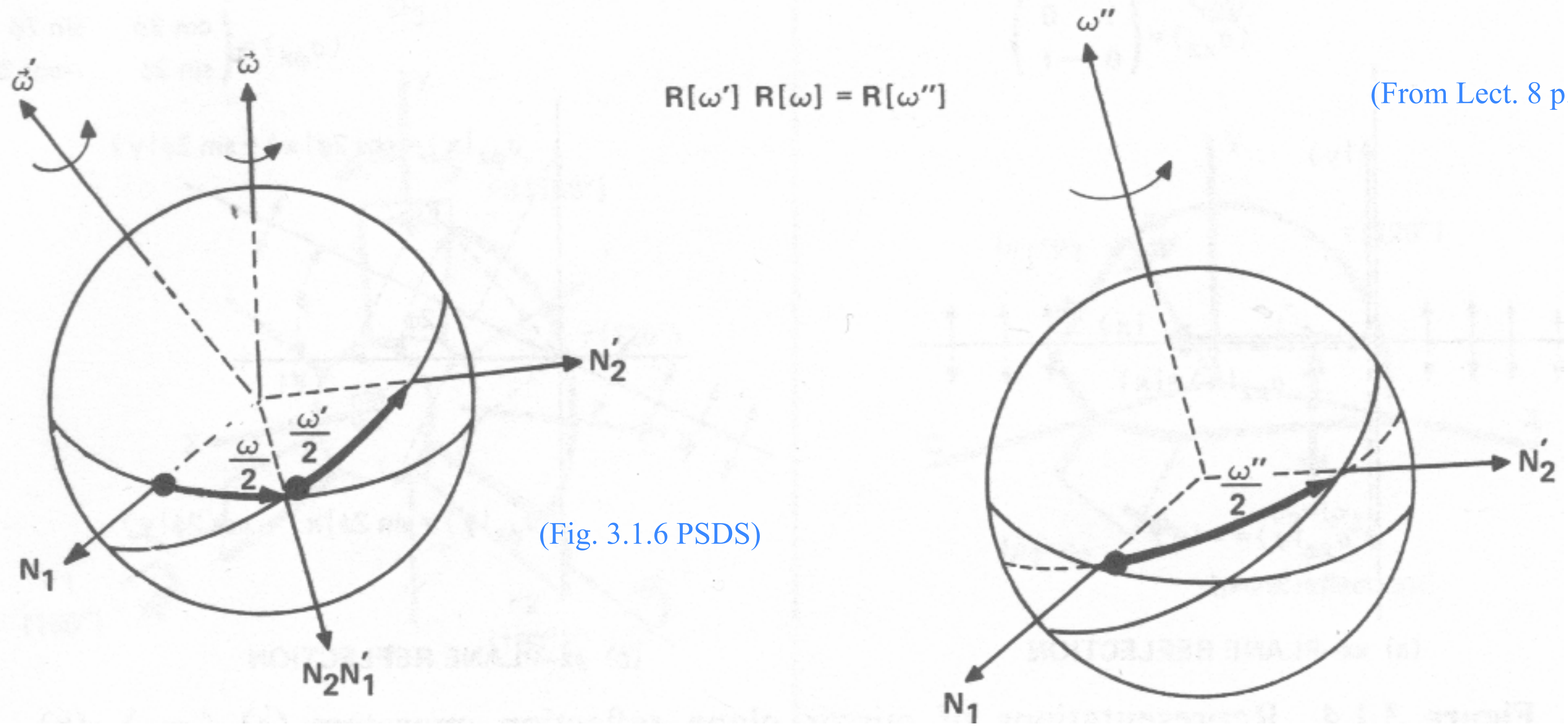
All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order



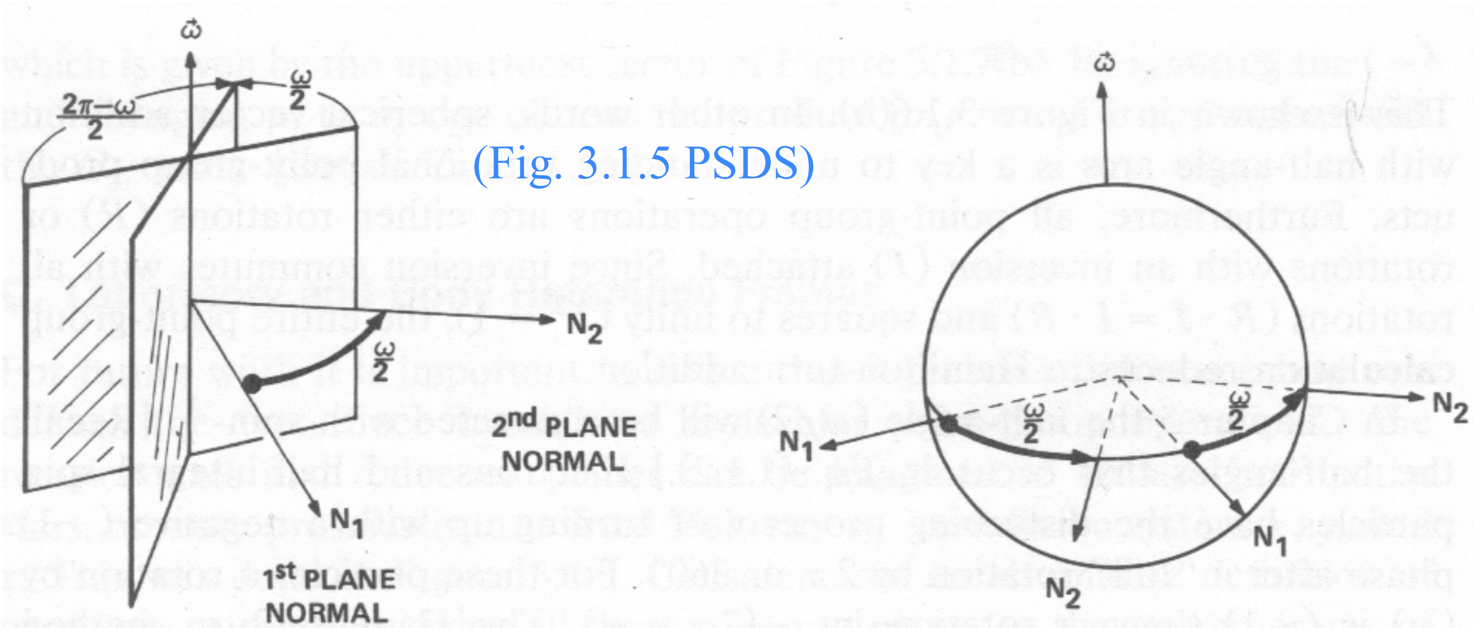
(Fig. 3.1.5 PSDS)

(From Lect. 8 p. 65-78)

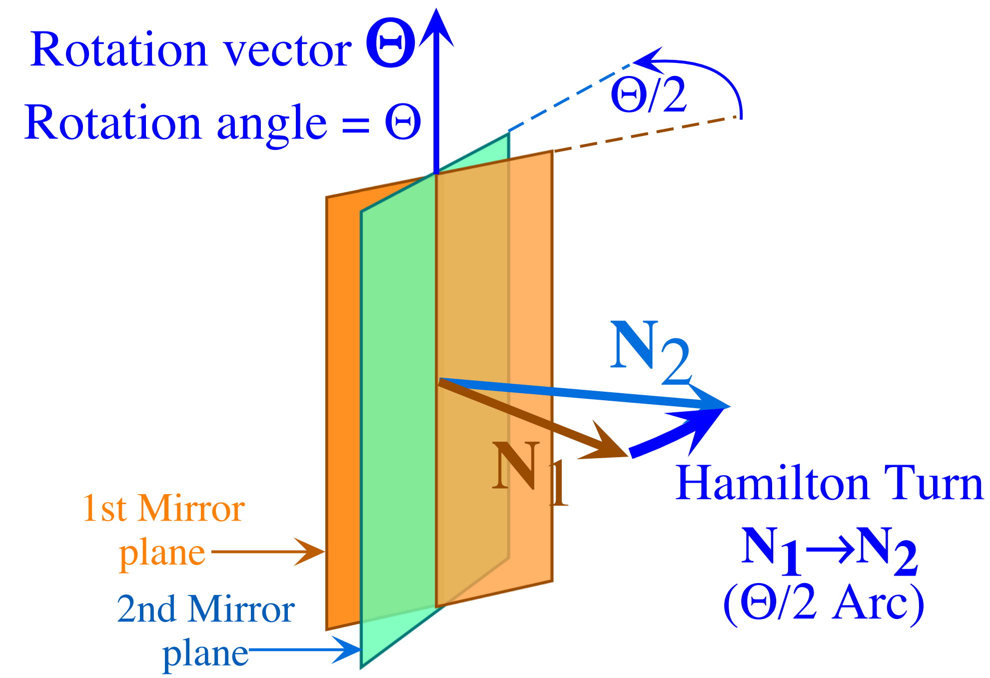


(Fig. 3.1.6 PSDS)

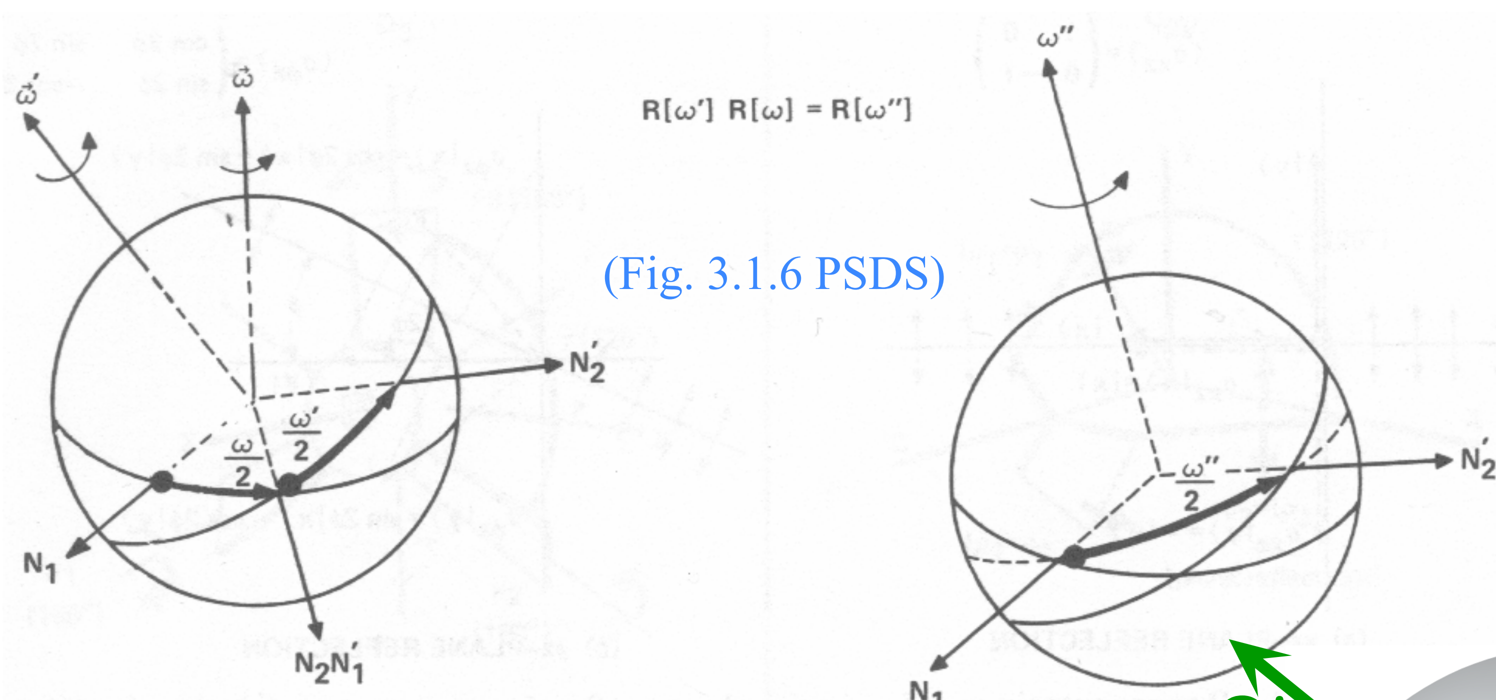
Deriving $D_3 \sim C_{3v}$ products by nomograms based on U(2) Hamilton-turns



(Fig. 3.1.5 PSDS)

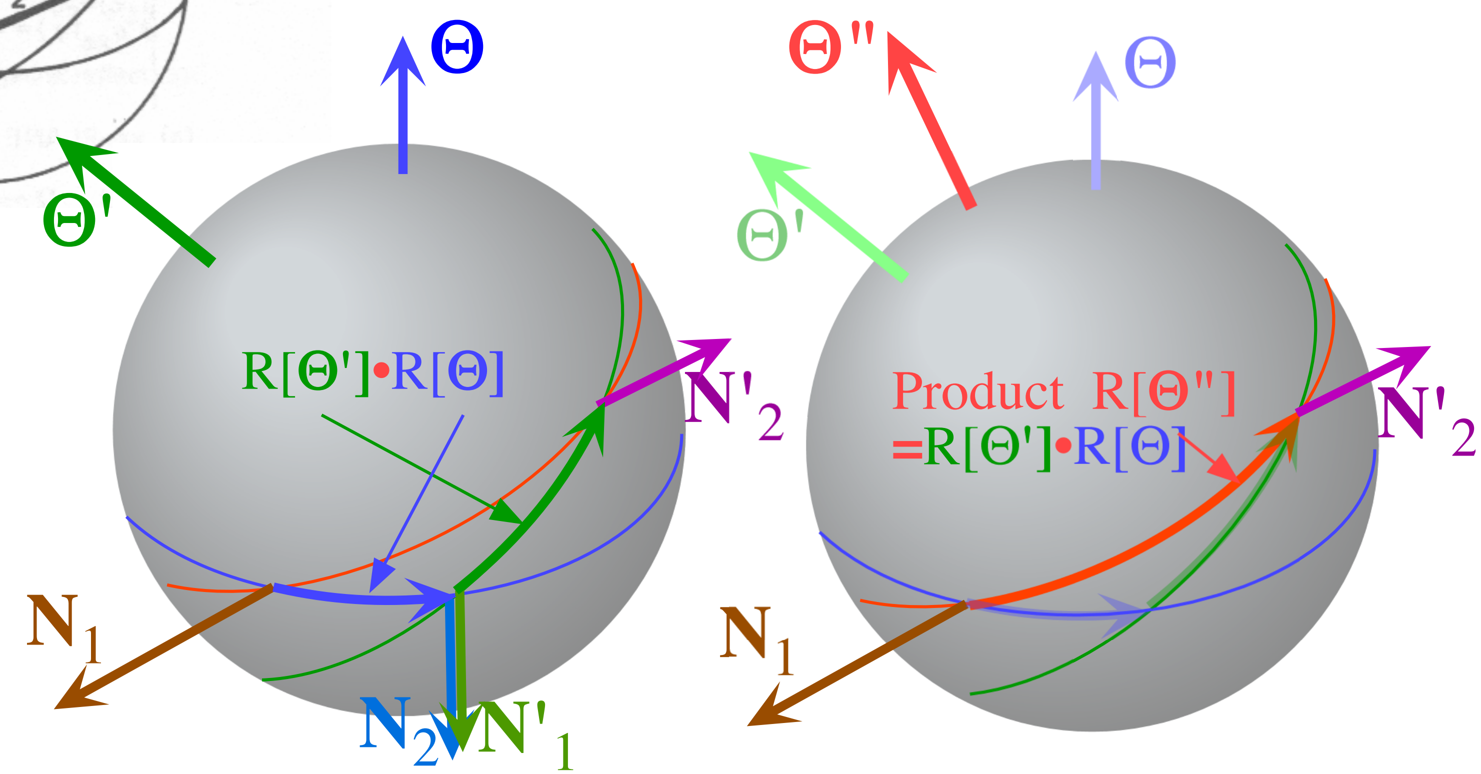


(Fig. 10.A.7 QTCA)



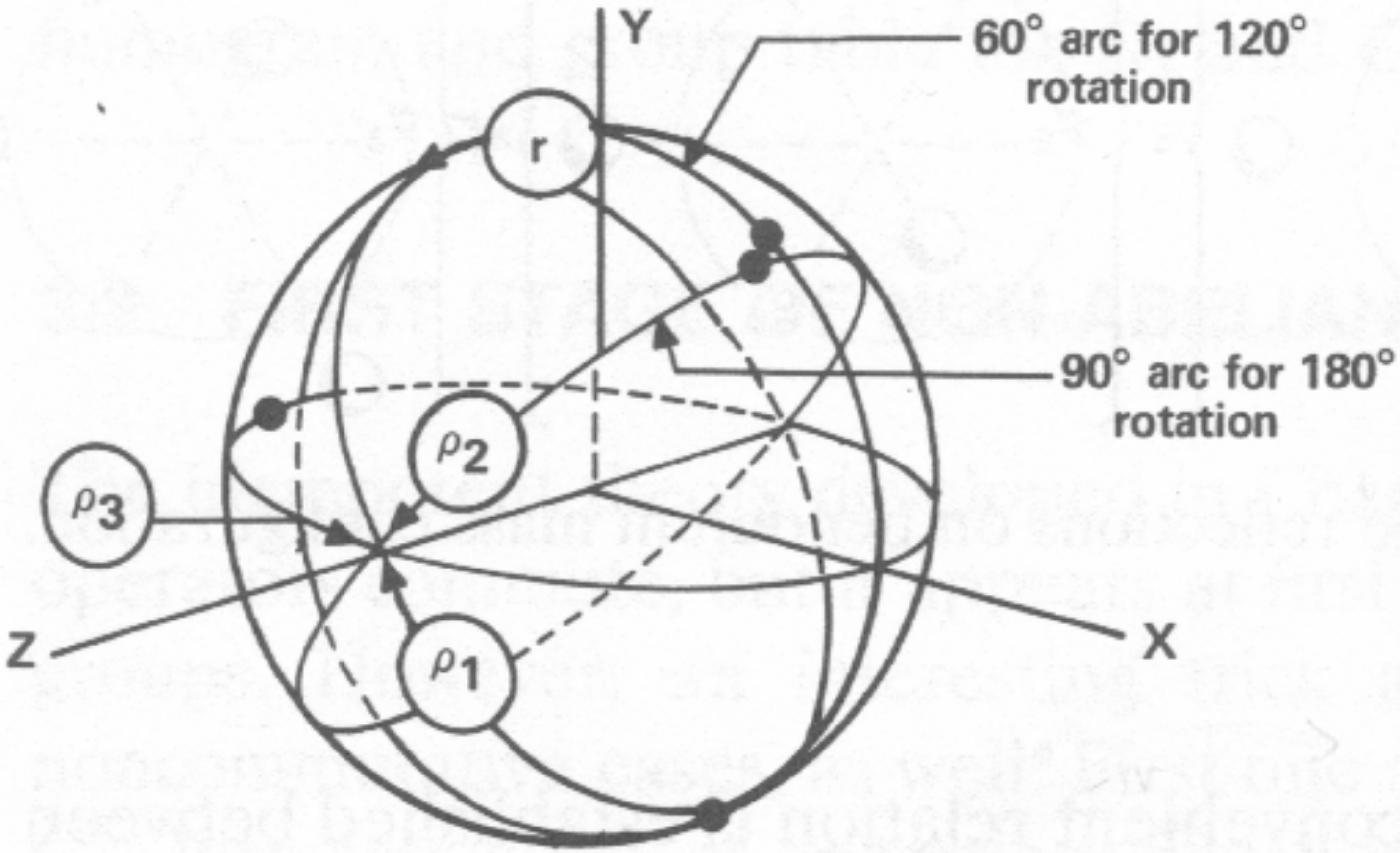
(Fig. 3.1.6 PSDS)

(From Lect. 8 p. 63-78)



(Fig. 10.A.8 QTCA)

Deriving $D_3 \sim C_{3v}$ products by nomograms based on U(2) Hamilton-turns



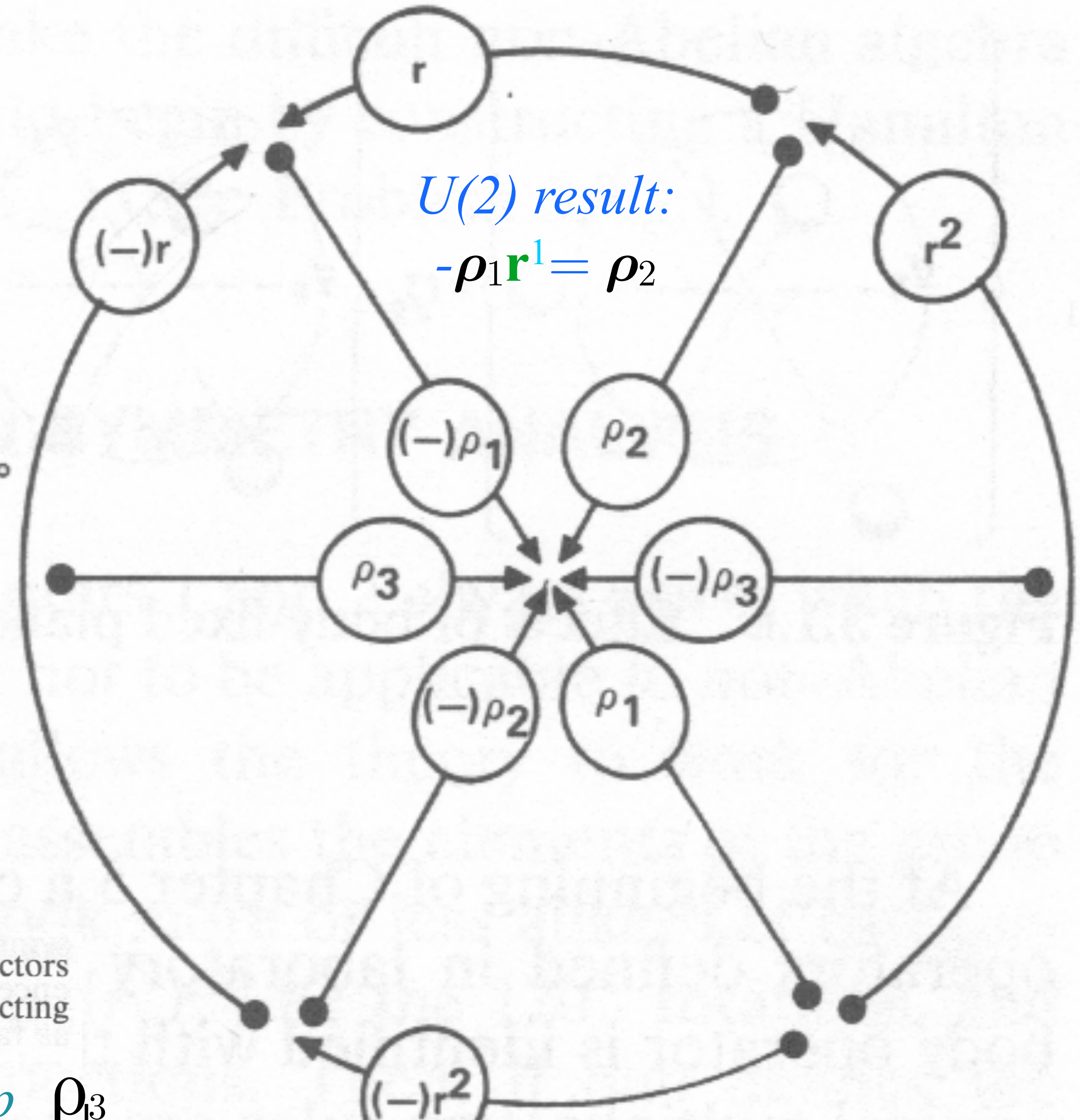
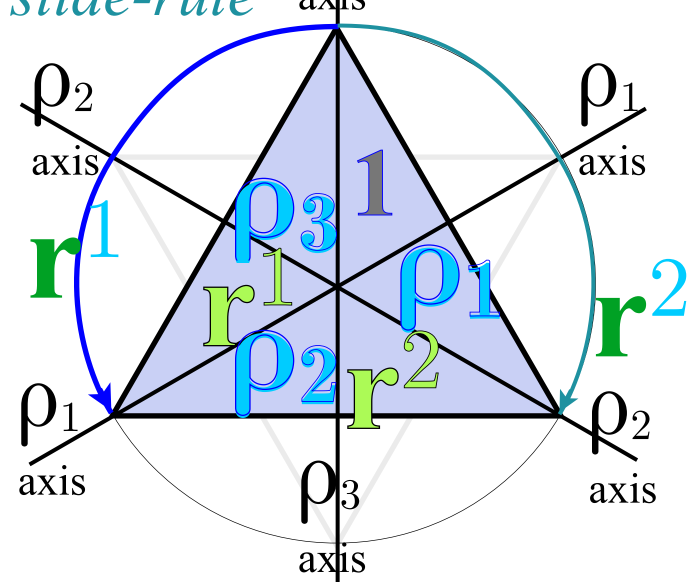


Figure 3.1.7 Geometrical definition of symmetry group D_3. (a) Hamilton arc vectors are drawn for rotations r, i_1, and i_3. (b) Group nomogram is obtained by projecting (a) onto the xy plane.

Note $\boldsymbol{h}^2 = \mathbf{r}^1$ *and* $\boldsymbol{h}^4 = \mathbf{r}^2$ *for* D_6 *notation*

1	h^2	h^4	ρ_1	ρ_2	ρ_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1

D_3 Group "slide-rule"



R(3) result:

$\boldsymbol{\rho}_1 \mathbf{r}^1 = \boldsymbol{\rho}_2$

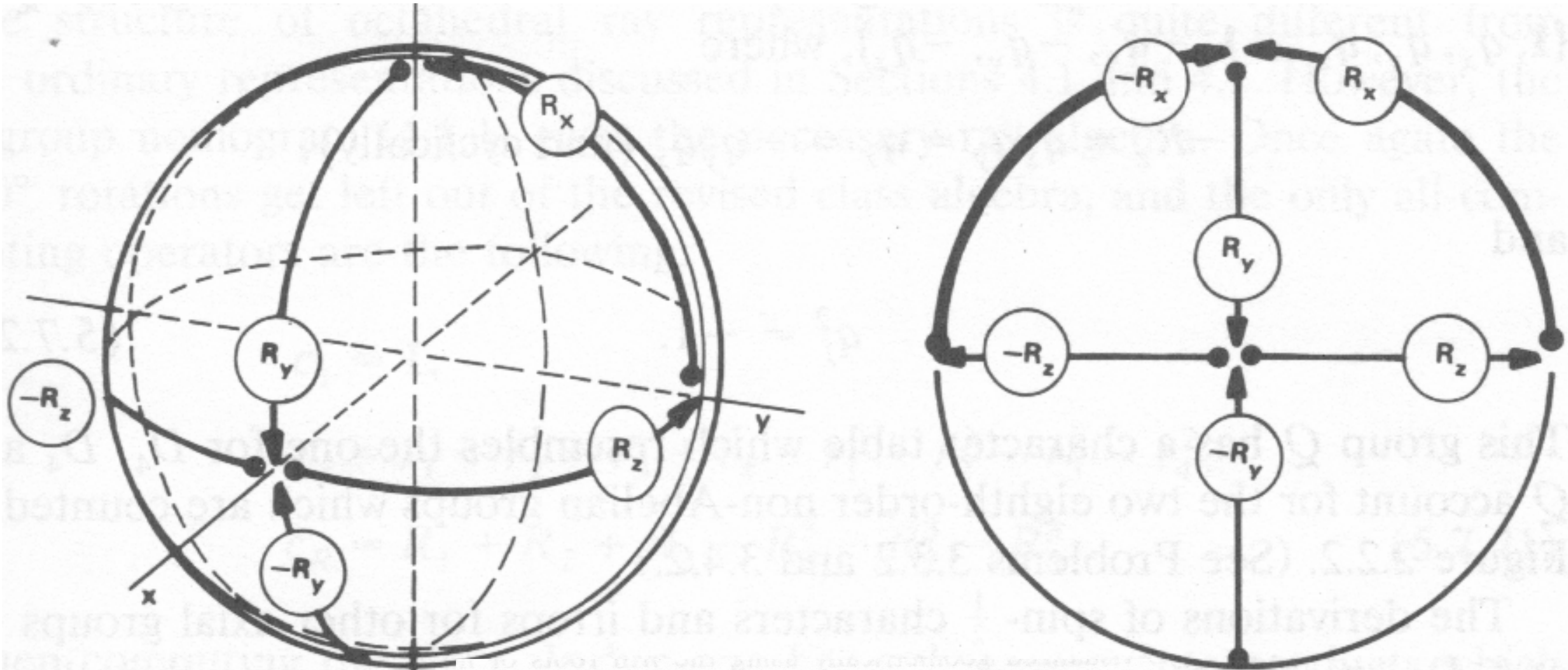


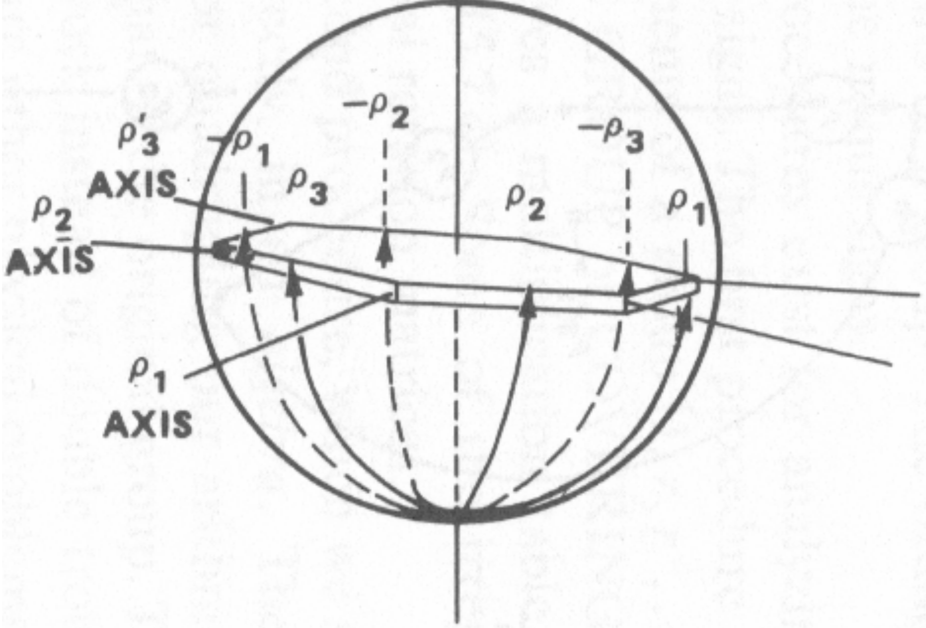
Figure 5.7.4 Hamilton arcs and vector nomogram for D_2

$\mathbf{1}$	R_x	R_y	R_z
R_x	$-\mathbf{1}$	R_z	$-R_y$
R_y	$-R_z$	$-\mathbf{1}$	R_x
R_z	R_y	$-R_x$	$-\mathbf{1}$

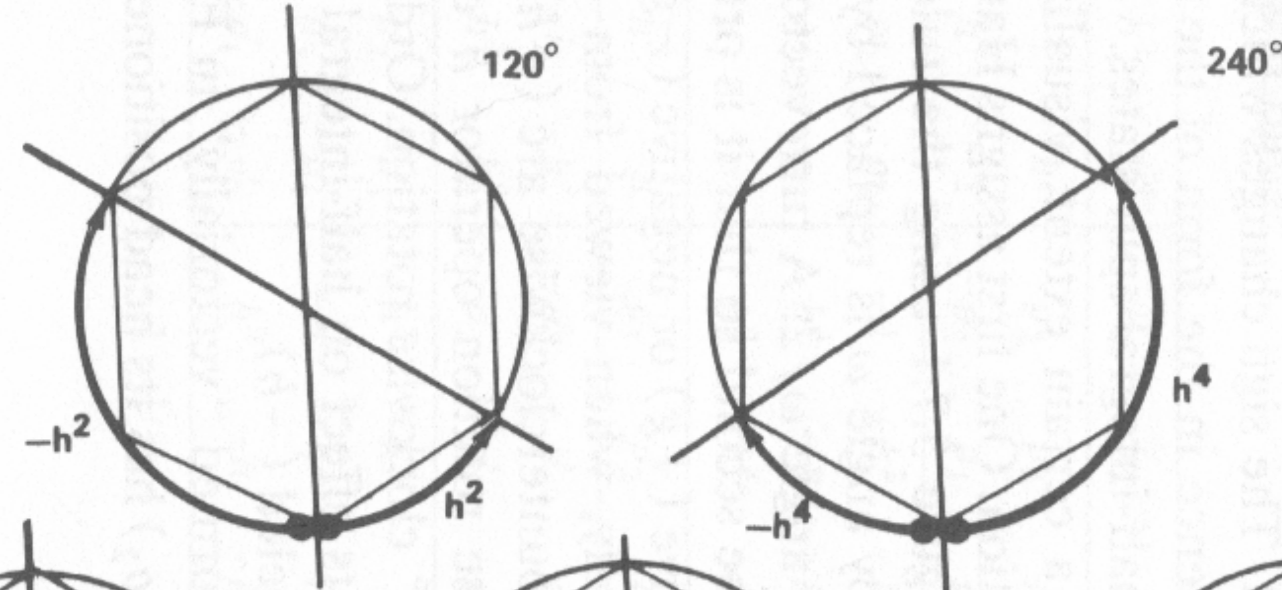
$-i\sigma_B$ $\qquad$ $-i\sigma_C$ $\qquad$ $-i\sigma_A$

$$\mathscr{D}^E(R_x) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathscr{D}^E(R_y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathscr{D}^E(R_z) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

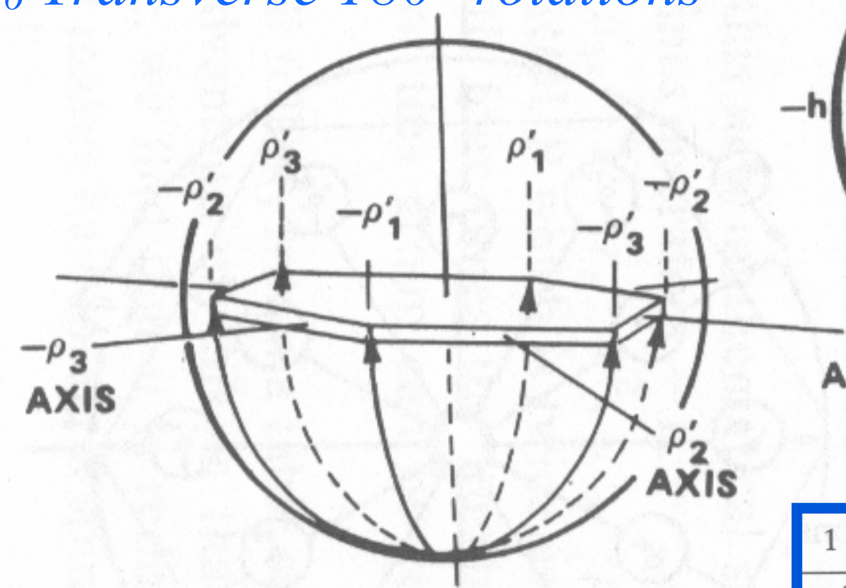
$D_3 \subset D_6$ Transverse 180° rotations

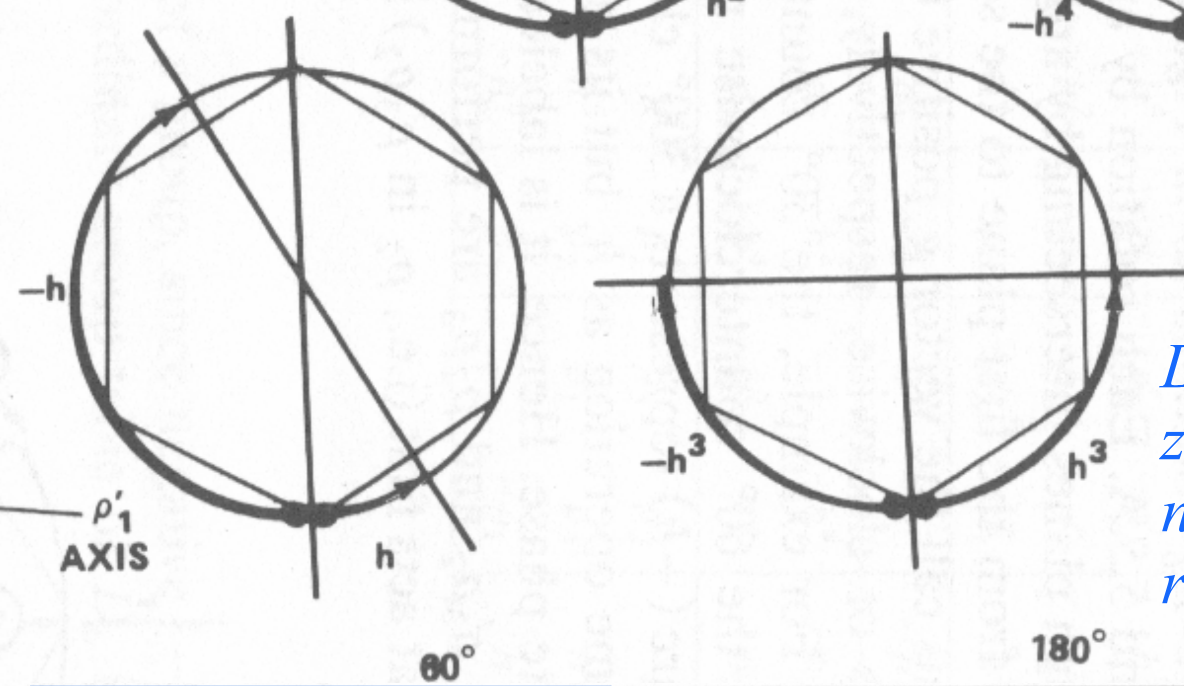


$D_3 \subset D_6$ z-Axial 120° rotations

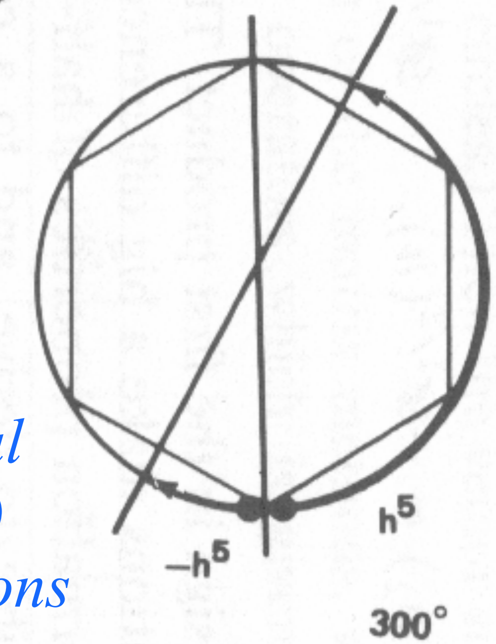


D_6 Transverse 180° rotations





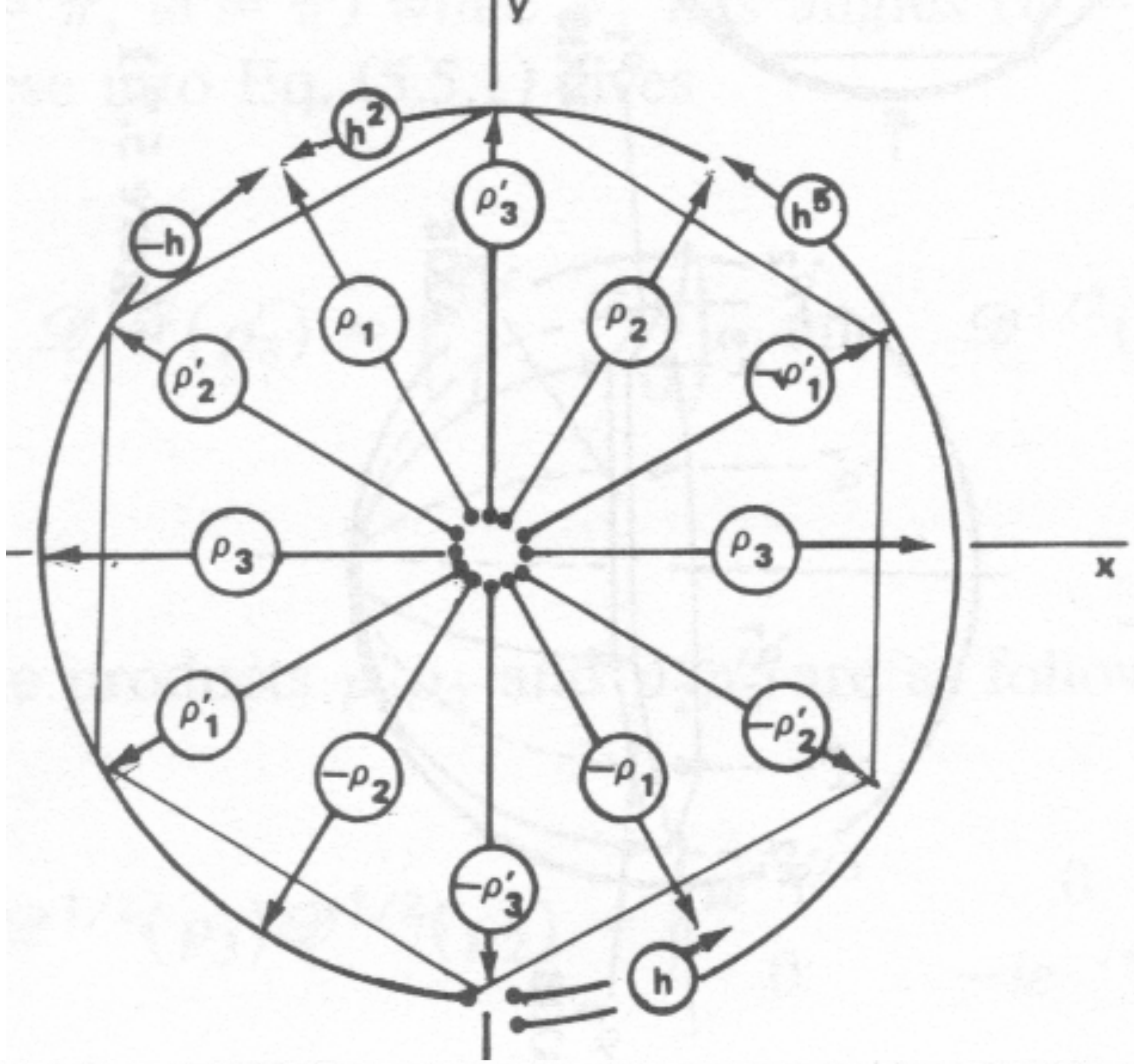
D_6 z-Axial n(60°) rotations



Note $\mathbf{h}^2 = \mathbf{r}^1$ and $\mathbf{h}^4 = \mathbf{r}^2$ for $D_6 \supset D_3$ notation

Later we show: $D_6 = D_3 \times C_2$

1	h^2	h^4	ρ_1	ρ_2	ρ_3	h^3	h	h^5	ρ'_1	ρ'_2	ρ'_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1	$-h$	h^5	$-h^3$	$-\rho'_2$	$-\rho'_3$	ρ'_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2	h^5	h^3	$-h$	$-\rho'_3$	ρ'_1	ρ'_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$	$-\rho'_1$	ρ'_3	$-\rho'_2$	h^3	h^5	$-h$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$	$-\rho'_2$	$-\rho'_1$	$-\rho'_3$	h	h^3	h^5
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1	$-\rho'_3$	$-\rho'_2$	ρ'_1	$-h^5$	h	h^3
h^3	h^5	$-h$	ρ'_1	ρ'_2	ρ'_3	-1	h^4	$-h^2$	$-\rho_1$	$-\rho_2$	$-\rho_3$
h^5	$-h$	$-h^3$	$-\rho'_3$	ρ'_1	ρ'_2	$-h^2$	-1	$-h^4$	ρ_3	$-\rho_1$	$-\rho_2$
h	h^3	h^5	ρ'_2	ρ'_3	$-\rho'_1$	h^4	h^2	-1	$-\rho_2$	$-\rho_3$	ρ_1
ρ'_1	ρ'_2	ρ'_3	$-h^3$	$-h^5$	h	ρ_1	$-\rho_3$	ρ_2	-1	$-h^2$	$-h^4$
ρ'_2	ρ'_3	$-\rho'_1$	$-h$	$-h^3$	$-h^5$	ρ_2	ρ_1	ρ_3	h^4	-1	$-h^2$
ρ'_3	$-\rho'_1$	$-\rho'_2$	h^5	$-h$	$-h^3$	ρ_3	ρ_2	$-\rho_1$	h^2	h^4	-1



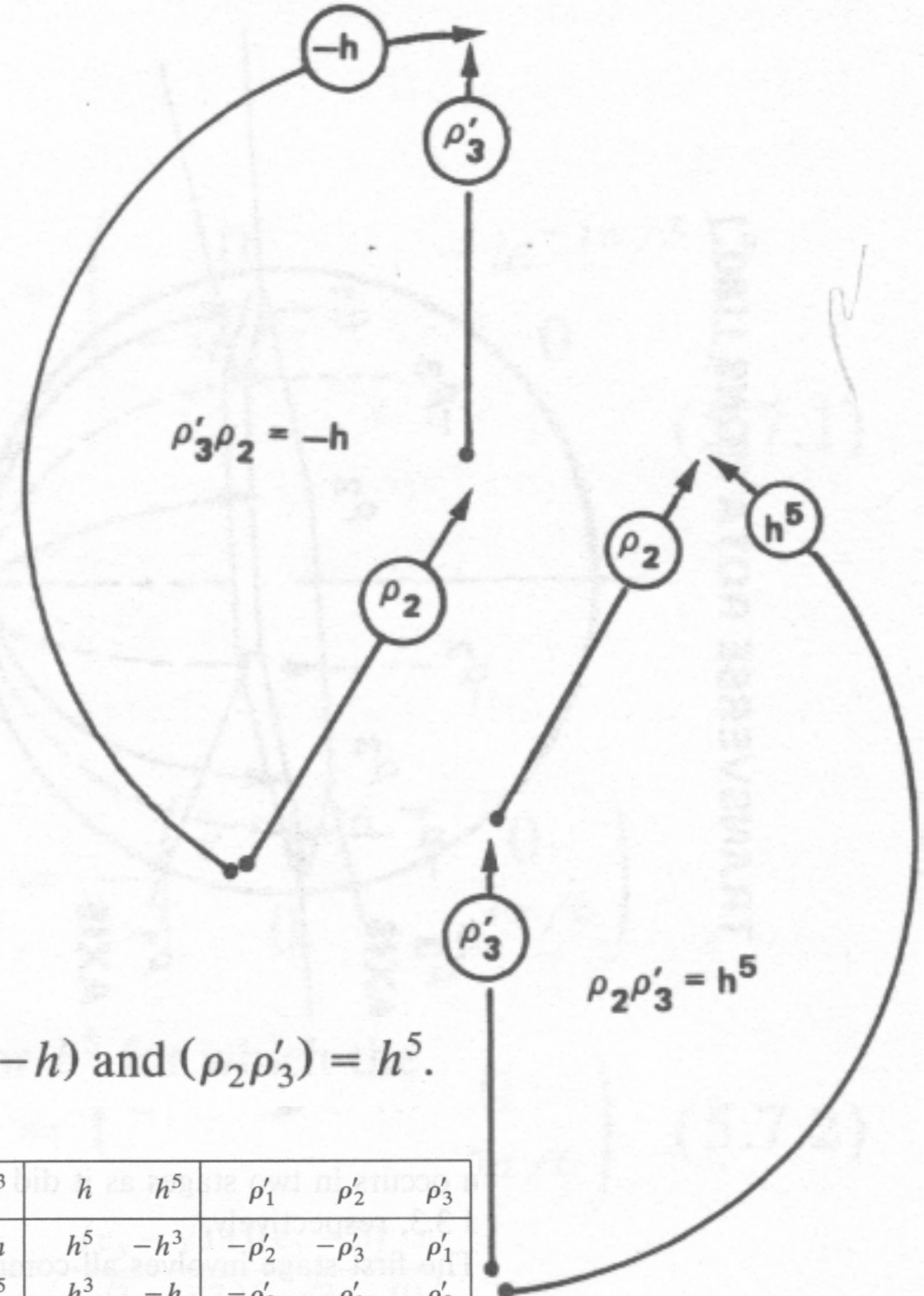


Figure 5.7.2 D_6 nomogram and example of products $(\rho'_3\rho_2 = -h)$ and $(\rho_2\rho'_3) = h^5$.

Note $\boldsymbol{h}^2 = \mathbf{r}^1$ *and* $\boldsymbol{h}^4 = \mathbf{r}^2$ *for* $D_6 \supset D_3$ *notation*

Later we show: $D_6 = D_3 \times C_2$

1	h^2	h^4	ρ_1	ρ_2	ρ_3	h^3	h	h^5	ρ'_1	ρ'_2	ρ'_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1	$-h$	h^5	$-h^3$	$-\rho'_2$	$-\rho'_3$	ρ'_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2	h^5	h^3	$-h$	$-\rho'_3$	ρ'_1	ρ'_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$	$-\rho'_1$	ρ'_3	$-\rho'_2$	h^3	h^5	$-h$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$	$-\rho'_2$	$-\rho'_1$	$-\rho'_3$	h	h^3	h^5
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1	$-\rho'_3$	$-\rho'_2$	ρ'_1	$-h^5$	h	h^3
h^3	h^5	$-h$	ρ'_1	ρ'_2	ρ'_3	-1	h^4	$-h^2$	$-\rho_1$	$-\rho_2$	$-\rho_3$
h^5	$-h$	$-h^3$	$-\rho'_3$	ρ'_1	ρ'_2	$-h^2$	-1	$-h^4$	ρ_3	$-\rho_1$	$-\rho_2$
h	h^3	h^5	ρ'_2	ρ'_3	$-\rho'_1$	h^4	h^2	-1	$-\rho_2$	$-\rho_3$	ρ_1
ρ'_1	ρ'_2	ρ'_3	$-h^3$	$-h^5$	h	ρ_1	$-\rho_3$	ρ_2	-1	$-h^2$	$-h^4$
ρ'_2	ρ'_3	$-\rho'_1$	$-h$	$-h^3$	$-h^5$	ρ_2	ρ_1	ρ_3	h^4	-1	$-h^2$
ρ'_3	$-\rho'_1$	$-\rho'_2$	h^5	$-h$	$-h^3$	ρ_3	ρ_2	$-\rho_1$	h^2	h^4	-1

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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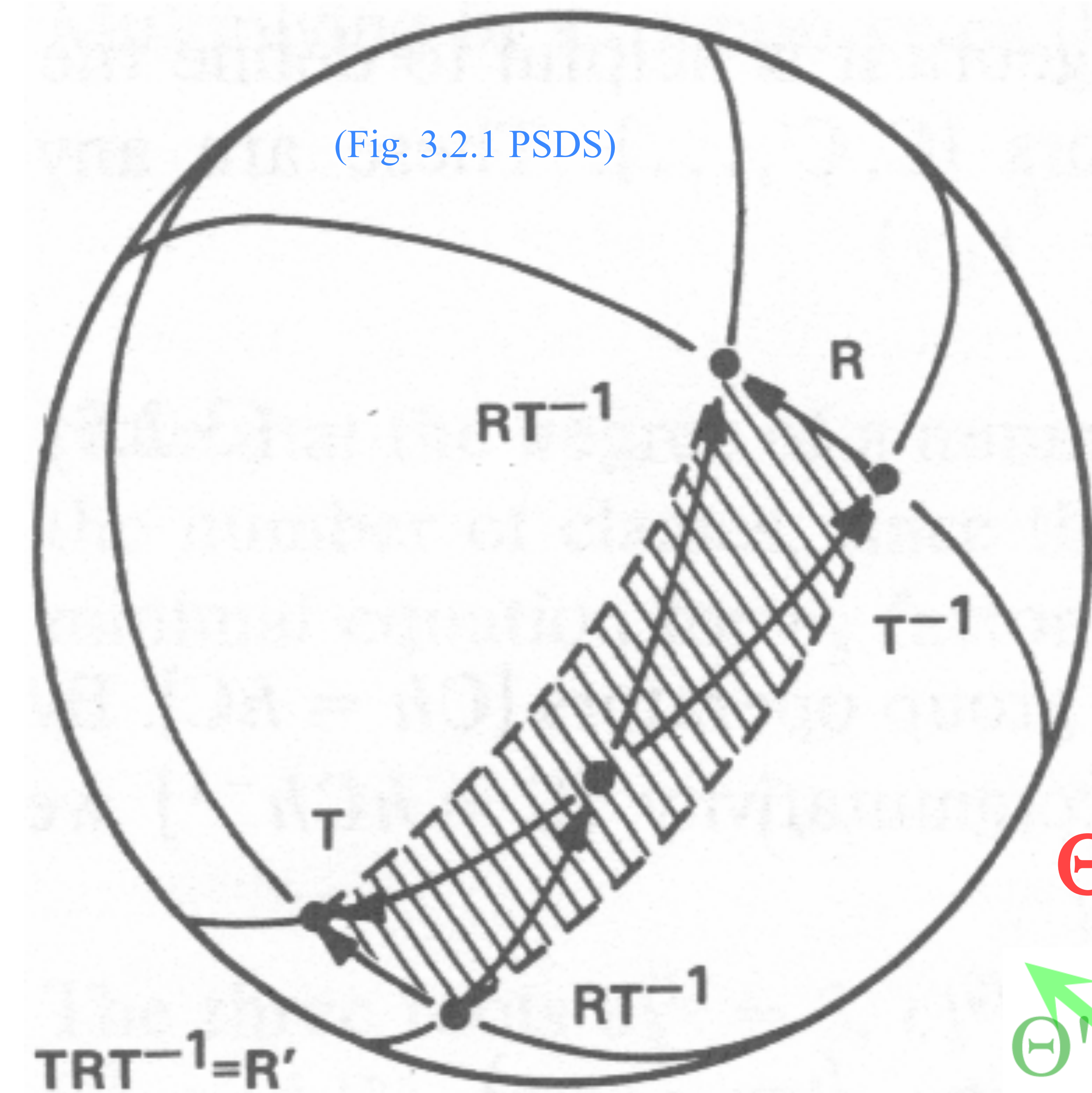
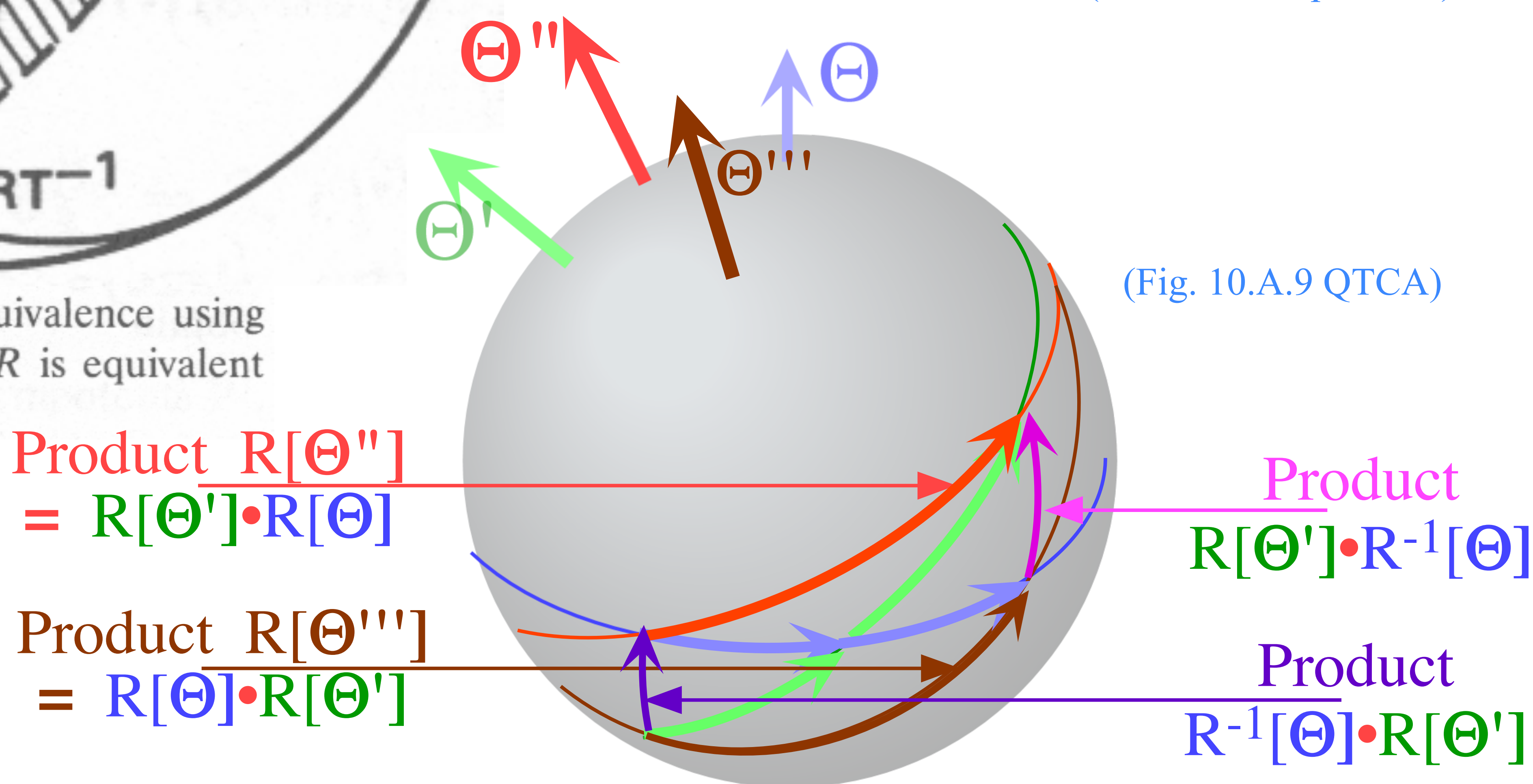
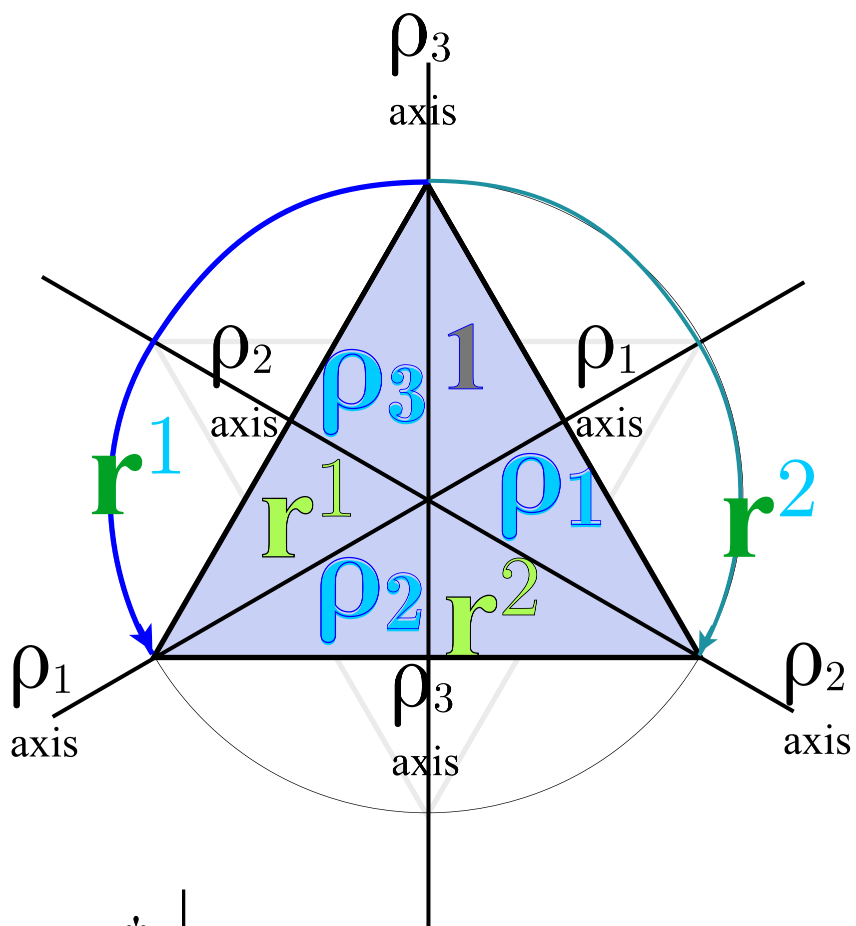


Figure 3.2.1 Showing class equivalence using Hamilton's vectors. Operation R is equivalent to $R' = TRT^{-1}$.

(From Lect. 8 p. 65-78)



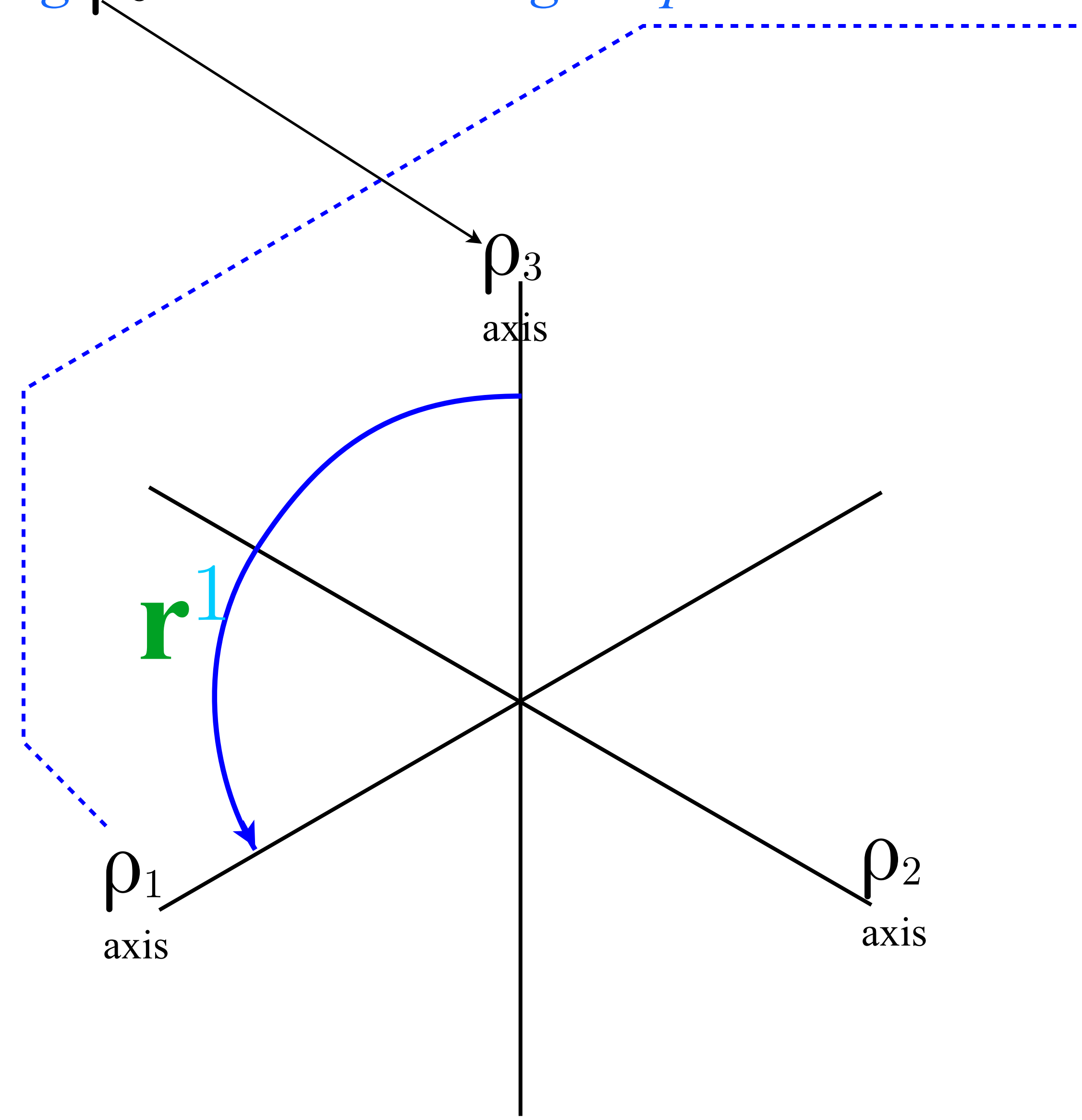
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



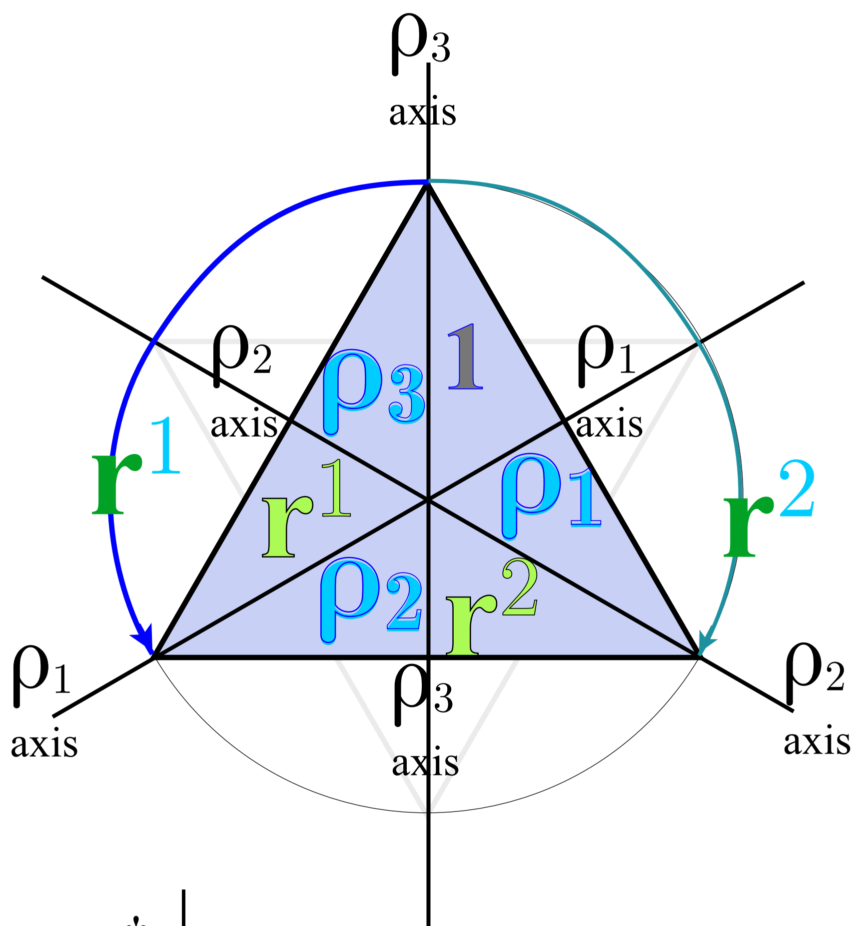
D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Transforming D_3 operators using D_3 operators

Example 1: Rotating ρ_3 axis crank using $\mathbf{r}^1$ puts it down onto ρ_1



Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



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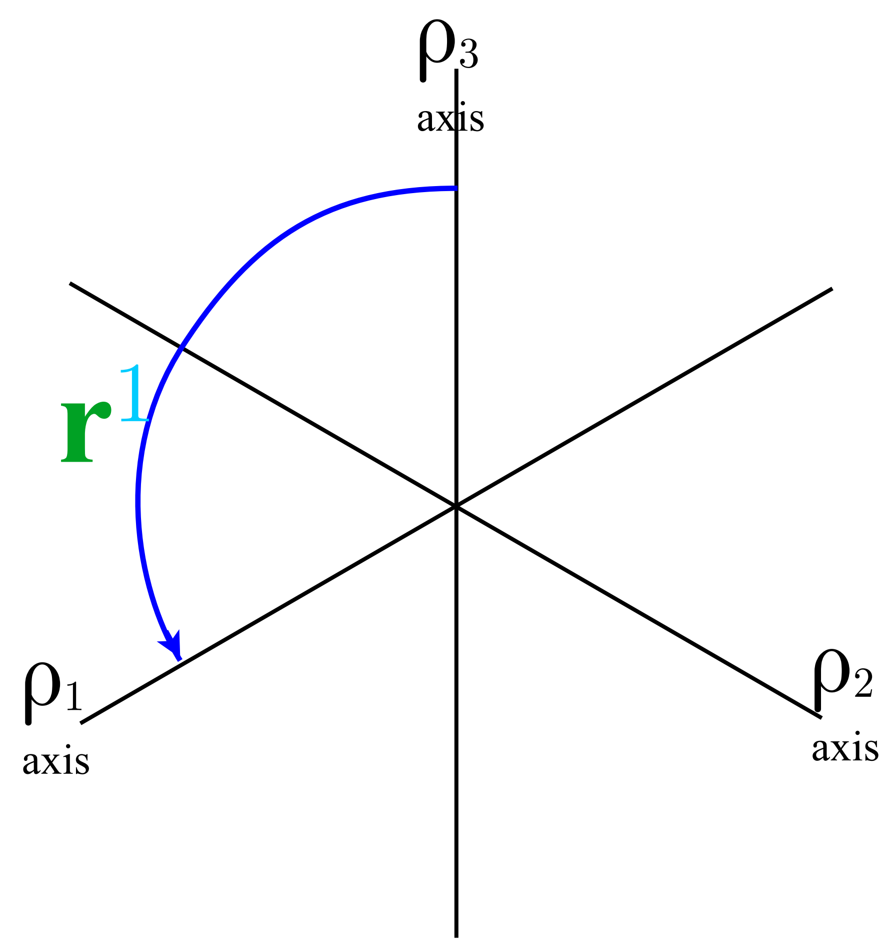
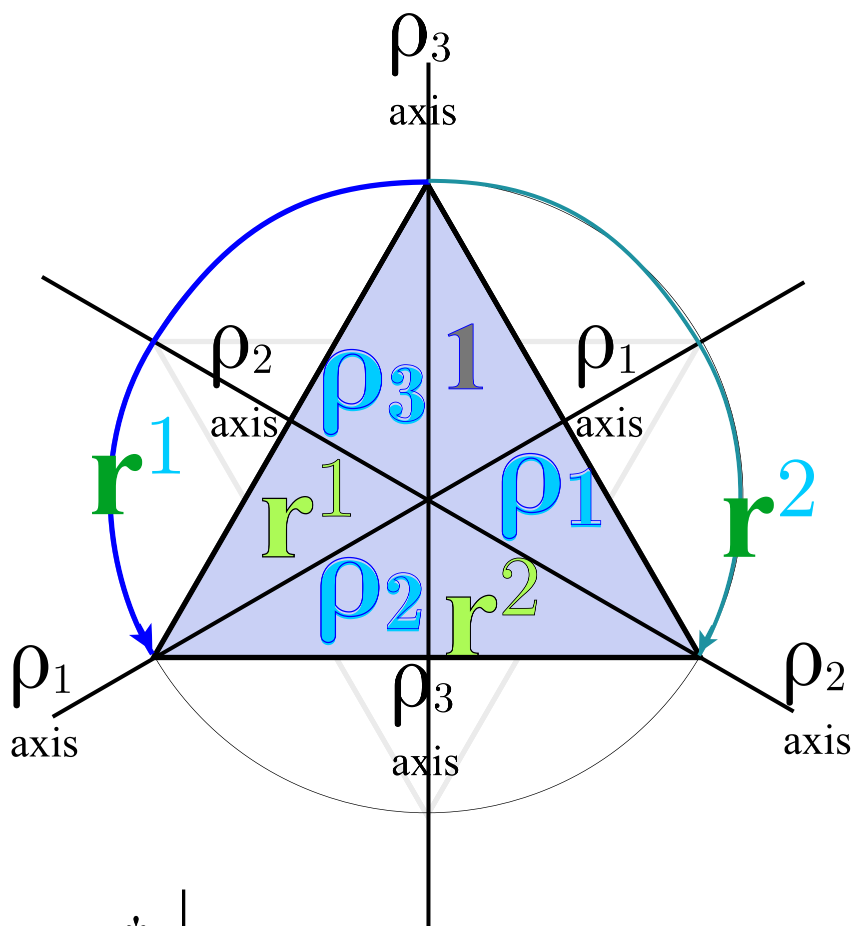


D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



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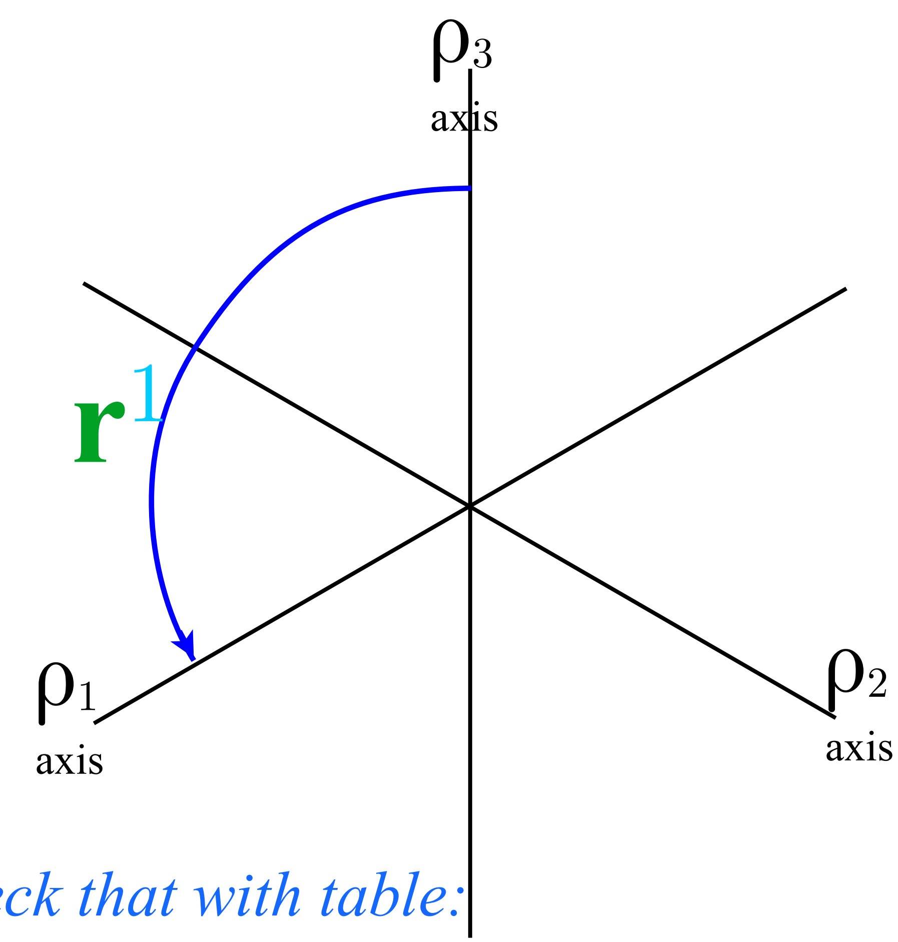
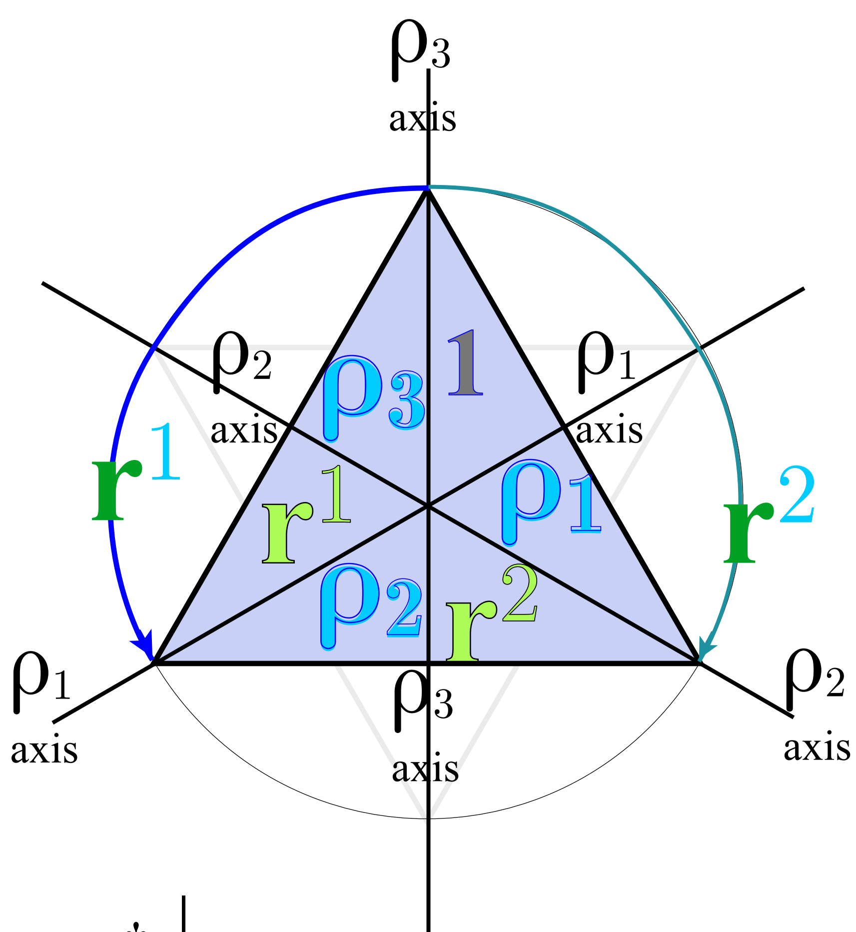


D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Need to check that with table:

$\mathbf{r}^1\rho_3\mathbf{r}^2 = \rho_2\mathbf{r}^2$

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



Transforming D_3 operators using D_3 operators

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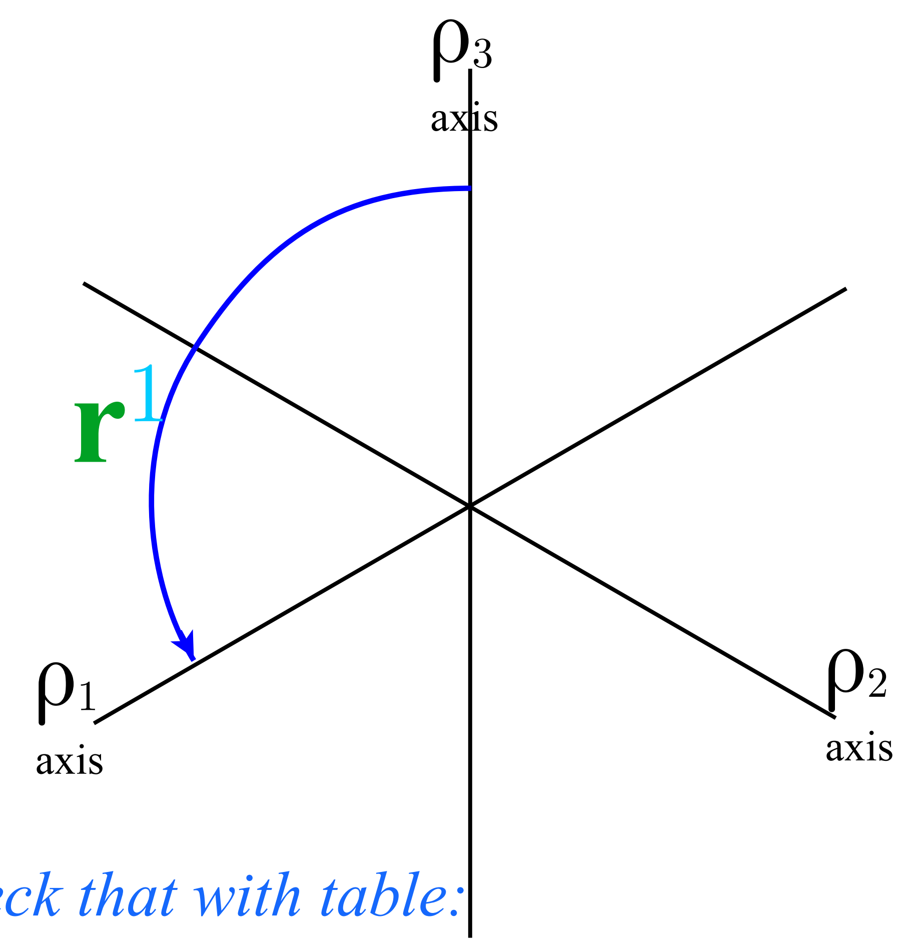
Seems to imply: $\mathbf{r}^1\rho_3(\mathbf{r}^1)^{-1}=\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_1$



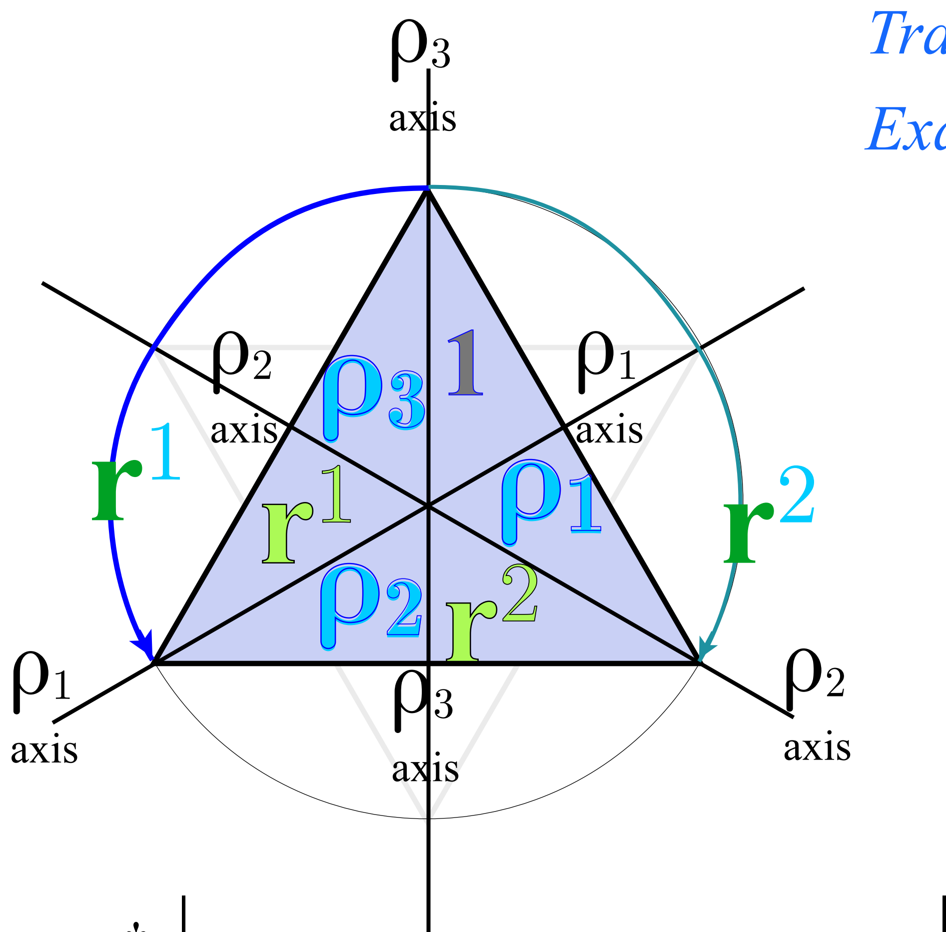
D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Need to check that with table:

$\mathbf{r}^1\rho_3\mathbf{r}^2 = \rho_2\mathbf{r}^2 = \rho_1$

Checks out!

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

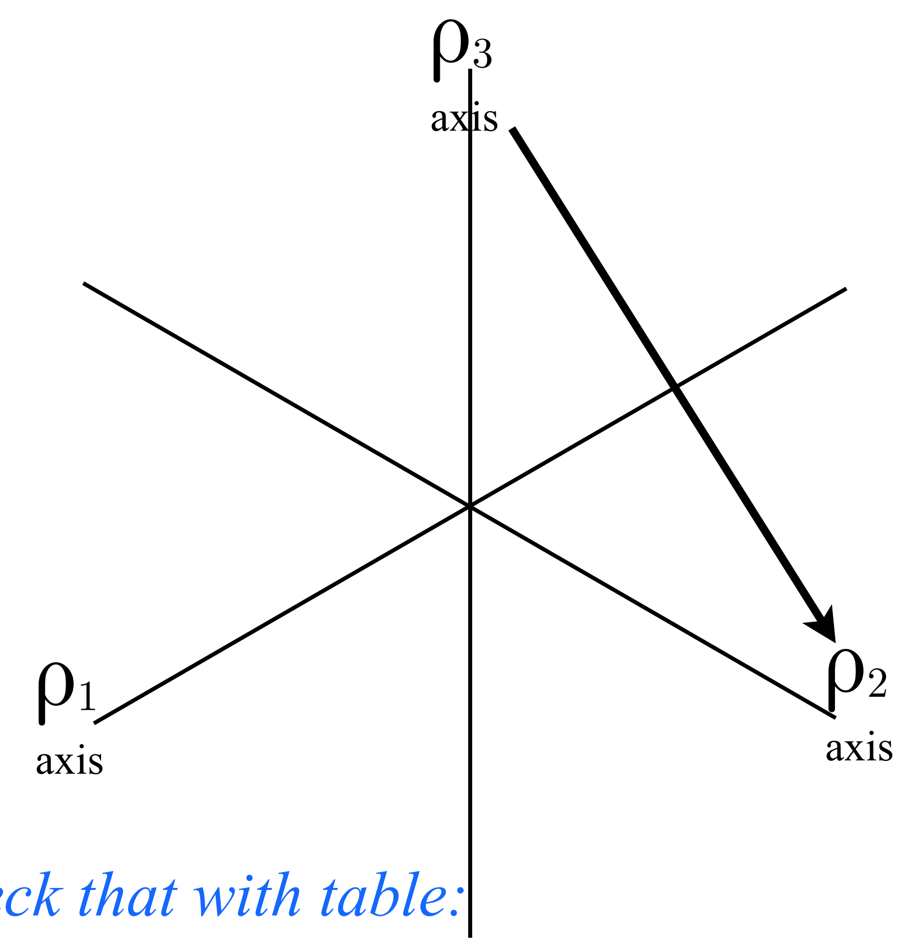


D_3 **gg**† *form*	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
1	**1**	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	**1**	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	**1**	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	**1**	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	**1**	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	**1**

Transforming D_3 operators using D_3 operators

Example 2: Rotating ρ_3 *axis crank using* ρ_1 *puts it down onto* ρ_2

Seems to imply: $\rho_1 \rho_3 (\rho_1)^{-1} = \rho_1 \rho_3 \rho_1 = \rho_2$



Need to check that with table:

$\rho_1 \rho_3 \rho_1 = \mathbf{r}^2 \rho_1 = \rho_2$ *Checks out!*

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What has been done so far:

Abelian (Commutative) C_2, C_3, ..., C_6 ...

H diagonalized by r^p *symmetry operators that COMMUTE with H* ($r^p H = H r^p$),

and with each other ($r^p r^q = r^{p+q} = r^q r^p$).

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What we need to learn now:

Non-Abelian (do not commute) D_3, O_h, ...

While all H symmetry operations COMMUTE with H $(\mathbf{U} H = H \mathbf{U})$

most do not with each other $(\mathbf{U}\mathbf{V} \neq \mathbf{V}\mathbf{U})$.

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most do not with each other $(\mathbf{U}\,\mathbf{V} \neq \mathbf{V}\mathbf{U})$.

Q: *So how do we write H in terms of non-commutative* $\mathbf{U}$ *?*

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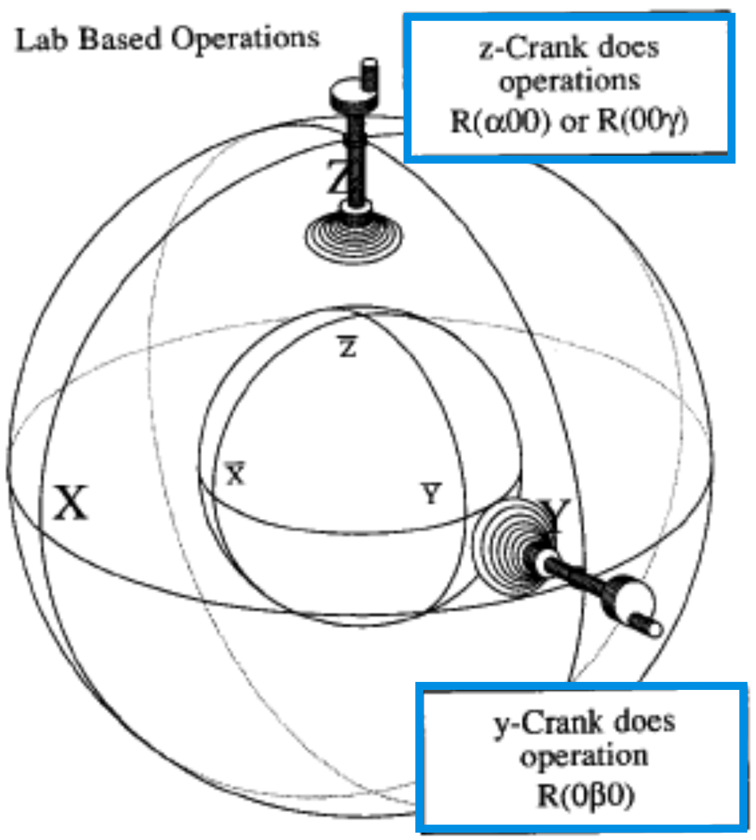
"Give me a place to stand...
and I will move the Earth"
Archimedes 287-212 B.C.E

Ideas of duality/relativity go *way* back (...VanVleck, Casimir..., Mach, Newton, Archimedes...)

Lab-fixed (Extrinsic-Global)**R**



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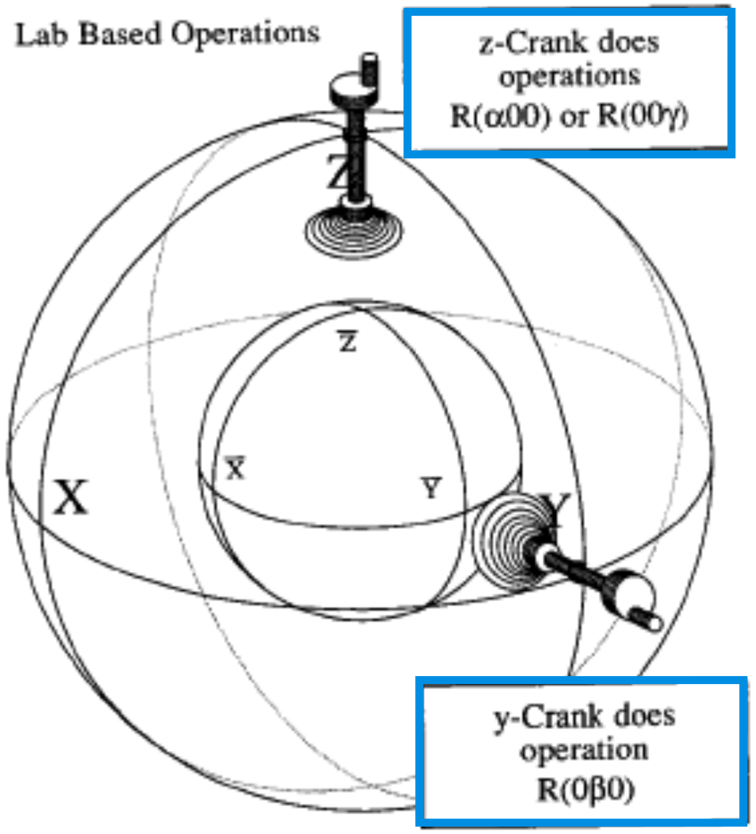

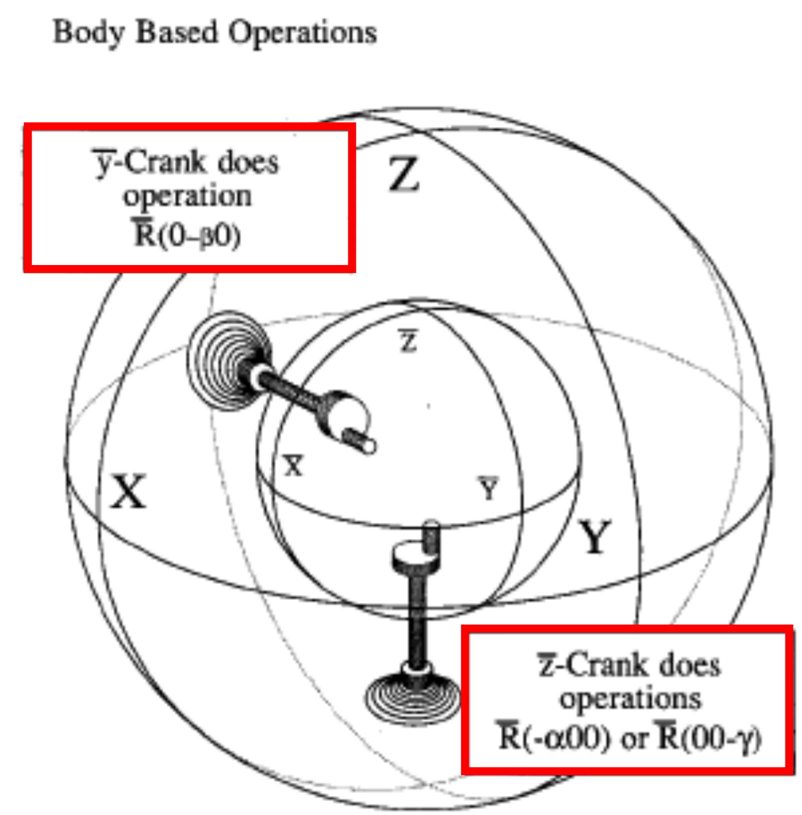

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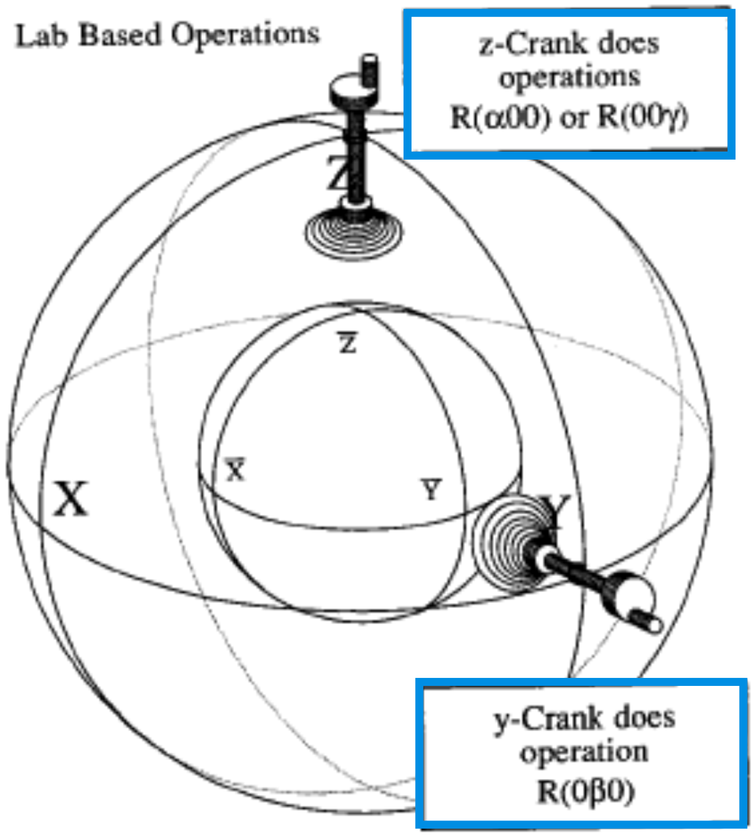
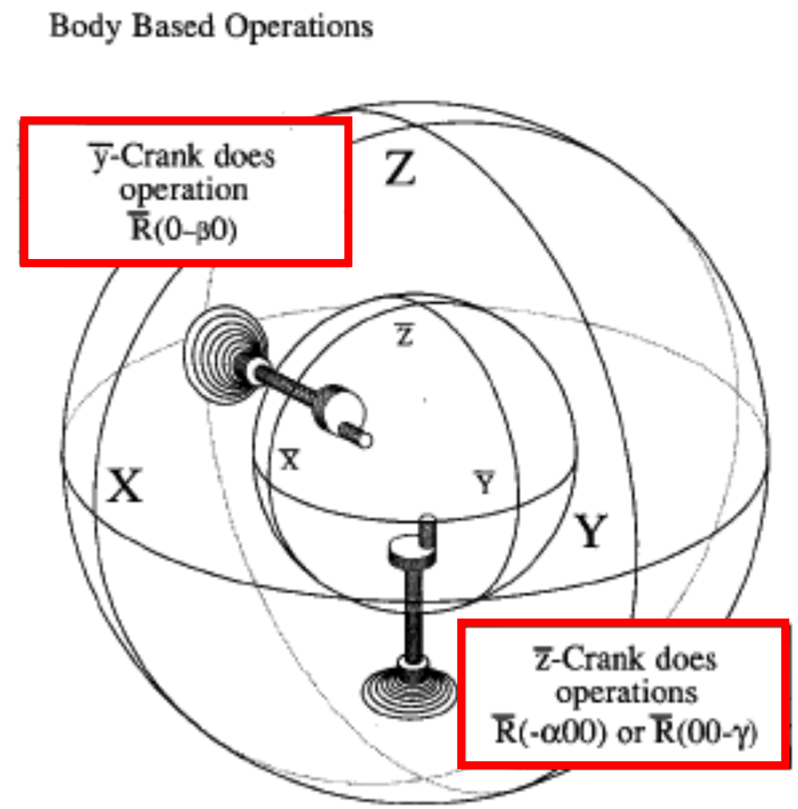


$\mathbf{R}$ *commutes*

with *all* $\bar{\mathbf{R}}$

(because they're independent or "unentangled")



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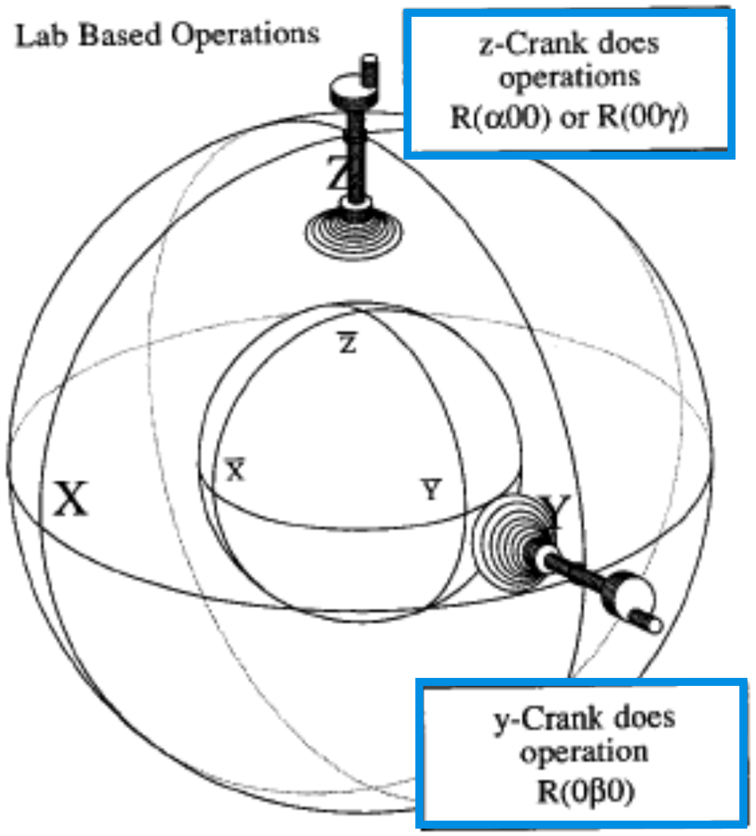

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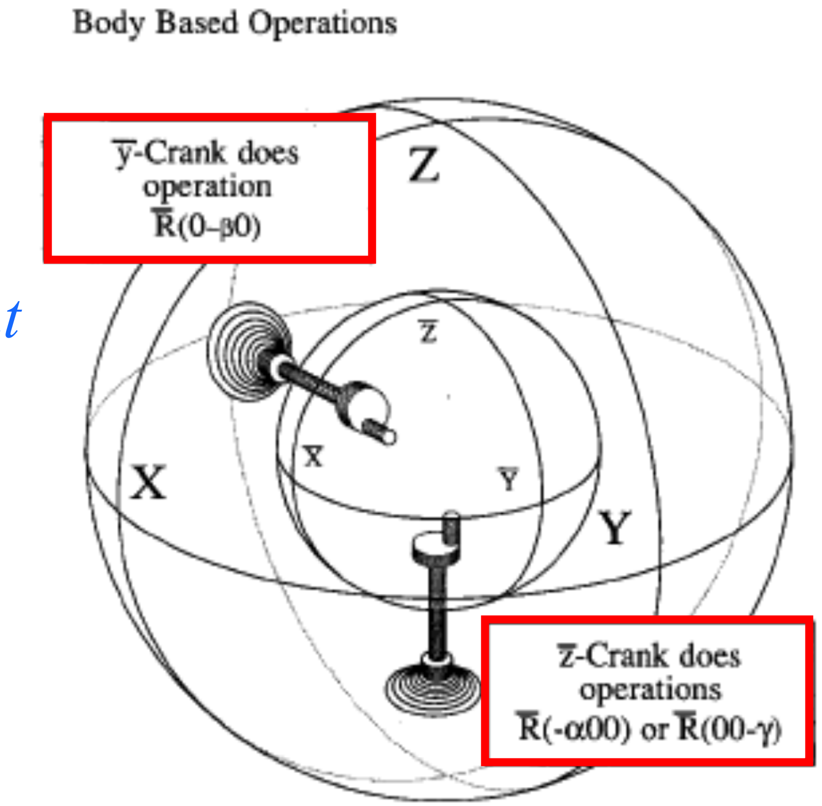
with *all* $\bar{\mathbf{R}}$
(because they're independent or "unentangled")

Mock-Mach relativity principle

$$\mathbf{R}|1\rangle = \bar{\mathbf{R}}^{-1}|1\rangle$$

...for *one* state |1⟩ *only!*



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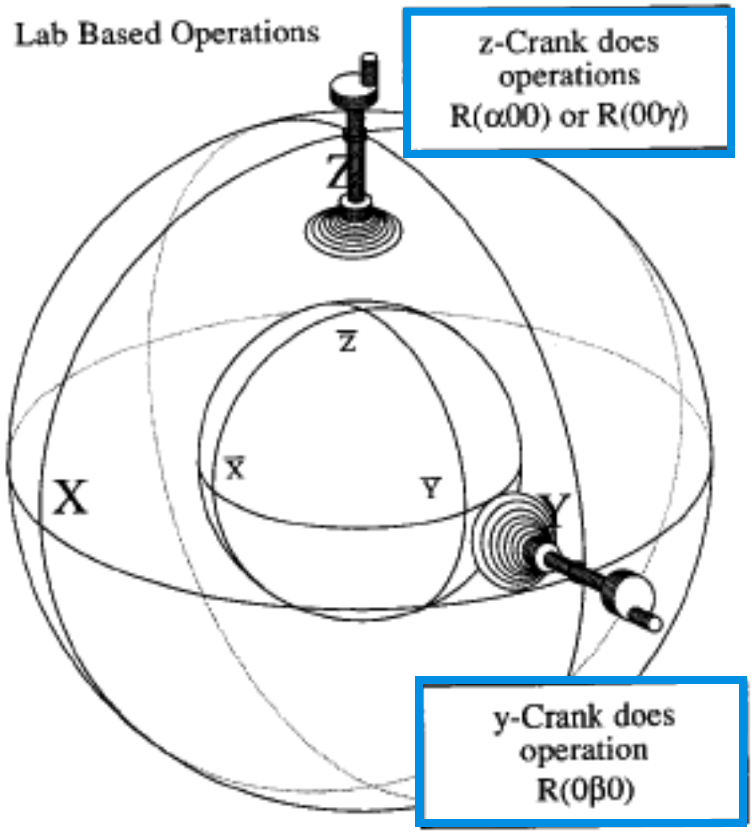

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(because they're independent or "unentangled")

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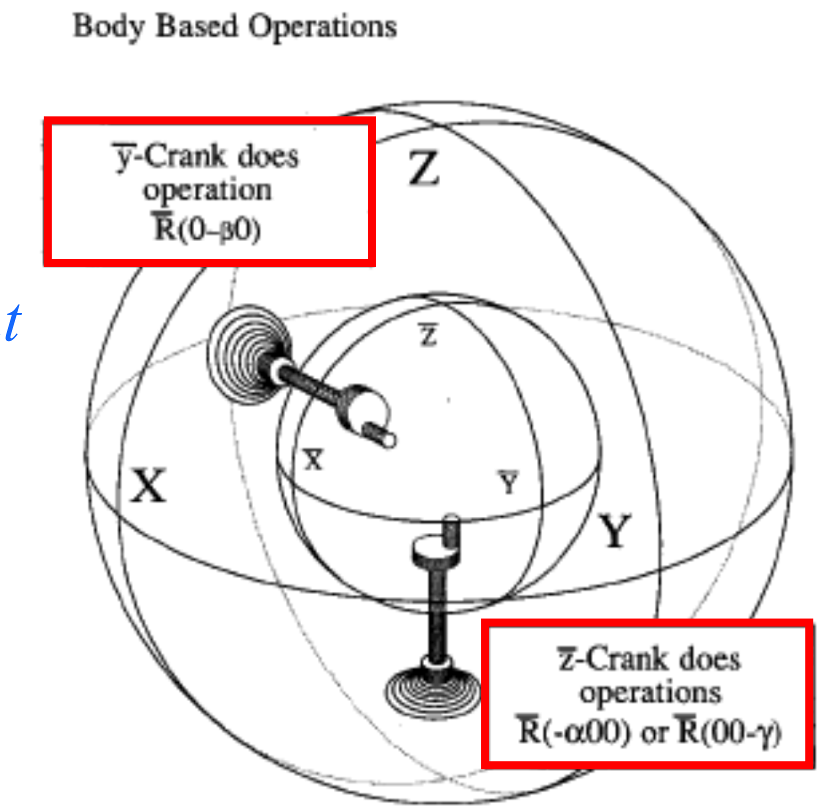

...But *how* do you actually *make* the $\mathbf{R}$ and $\bar{\mathbf{R}}$ operations?

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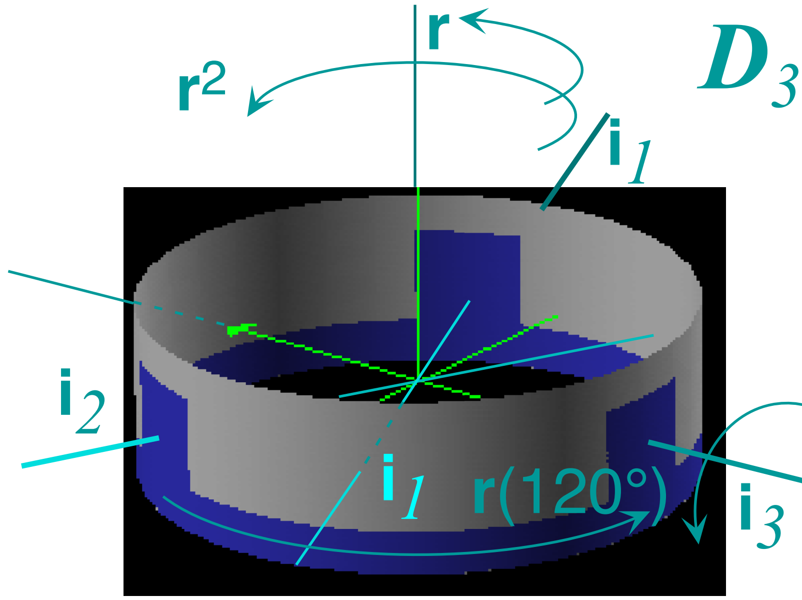
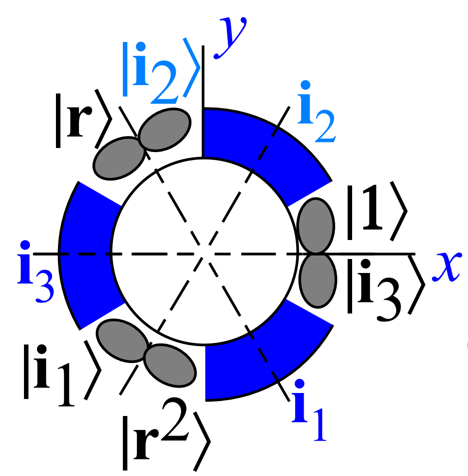
Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$

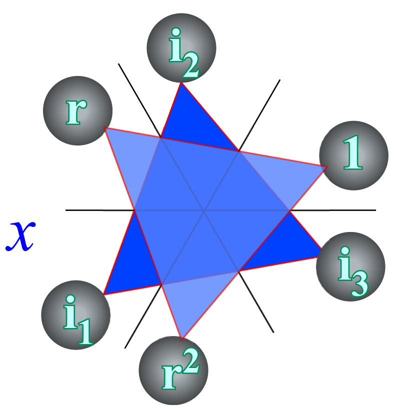


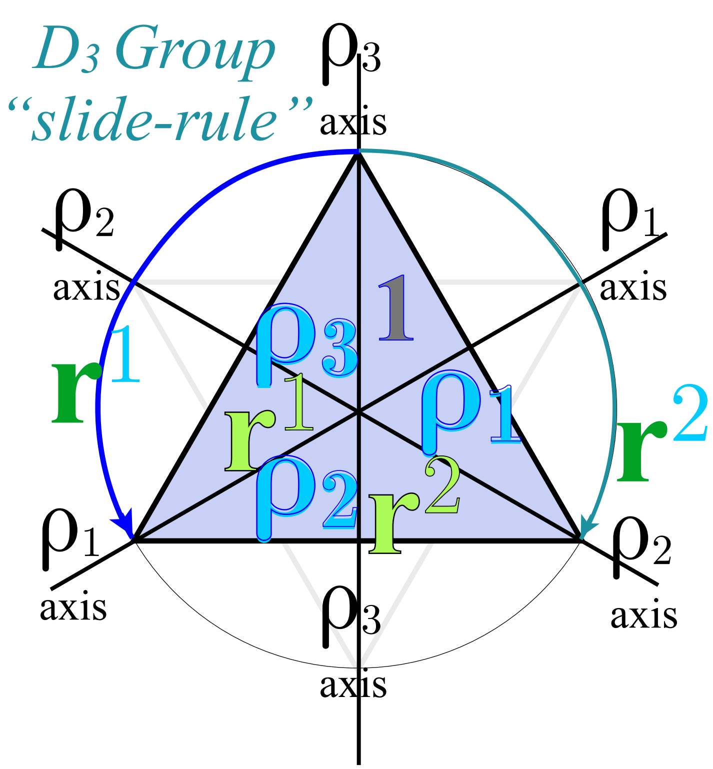
D_3

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

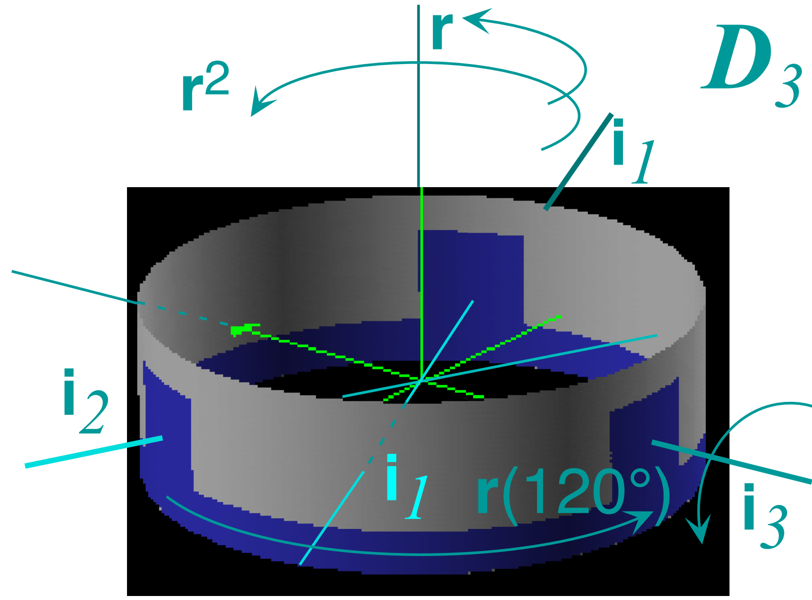
D_3-defined local-wave bases







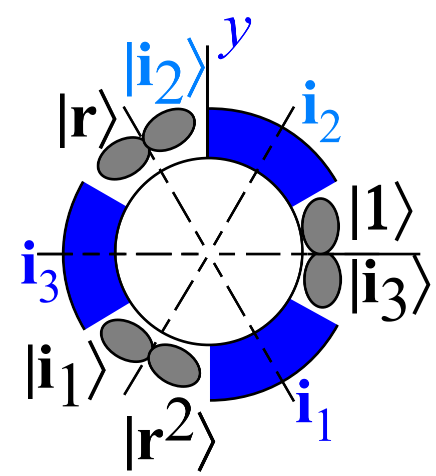
Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$

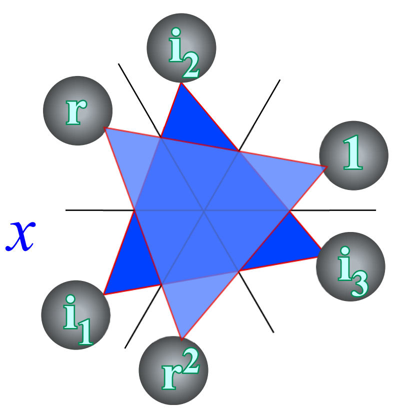


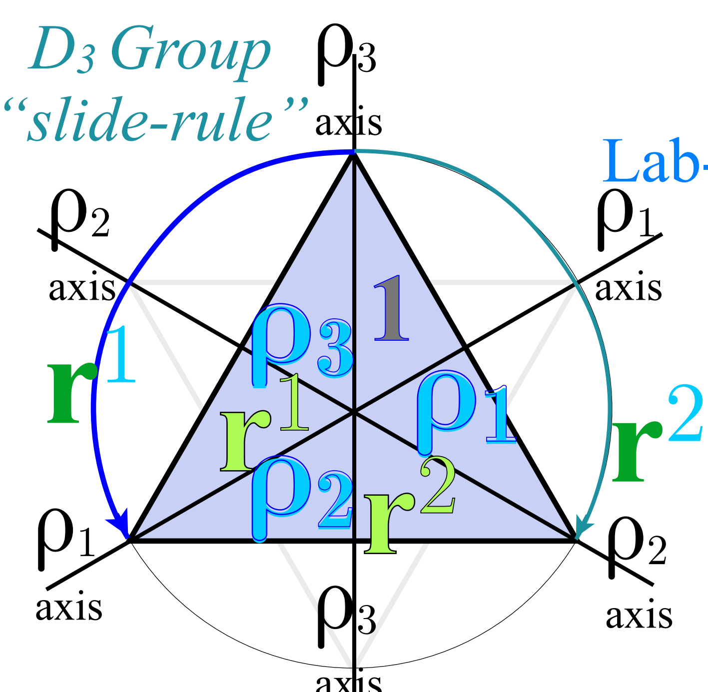
D_3

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

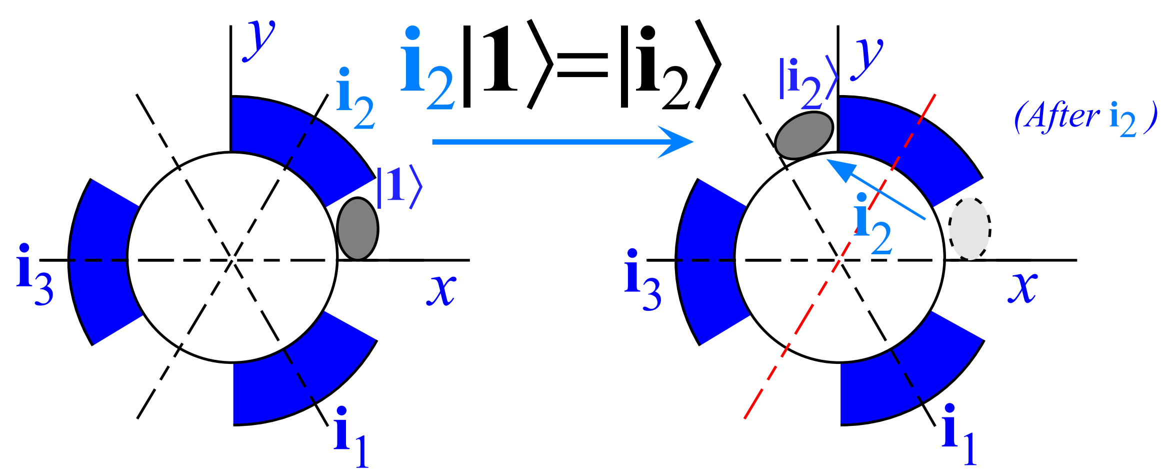
D_3-defined local-wave bases







Lab-fixed (Extrinsic-Global) operations and rotation axes



Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

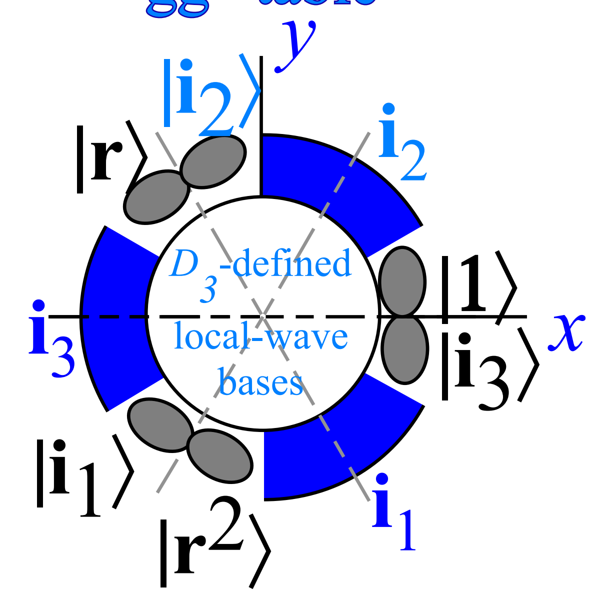
To represent *external* {..**T**,**U**,**V**,... }switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

D_3 global $gg^\dagger$-table



Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

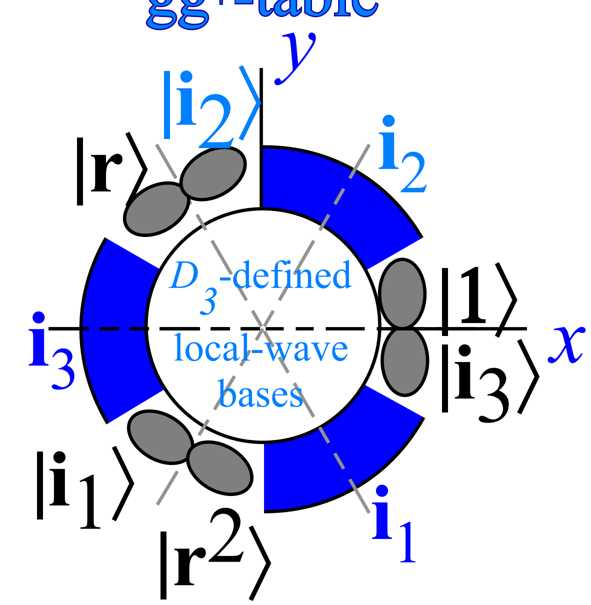
To represent *external* {..**T**,**U**,**V**,... } switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

1	r²	r	i₁	i₂	i₃
r	1	r²	i₃	i₁	i₂
r²	r	1	i₂	i₃	i₁
i₁	i₃	i₂	1	r	r²
i₂	i₁	i₃	r²	1	r
i₃	i₂	i₁	r	r²	1

***D_3* global**
$gg^\dagger$-table



D_3 local
$g^\dagger g$-table

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}}) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}^2) = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

1	r	r²	i₁	i₂	i₃
r²	1	r	i₂	i₃	i₁
r	r²	1	i₃	i₁	i₂
i₁	i₂	i₃	1	r	r²
i₂	i₃	i₂	r²	1	r
i₃	i₁	i₂	r	r²	1

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* {..**T**,**U**,**V**,... }switch **g** ⇄ **g**† on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

RESULT:
Any $R(\mathbf{T})$
commute (Even if **T** and **U** do not...)
with any $R(\bar{\mathbf{U}})$...

...and **T**·**U**=**V** if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

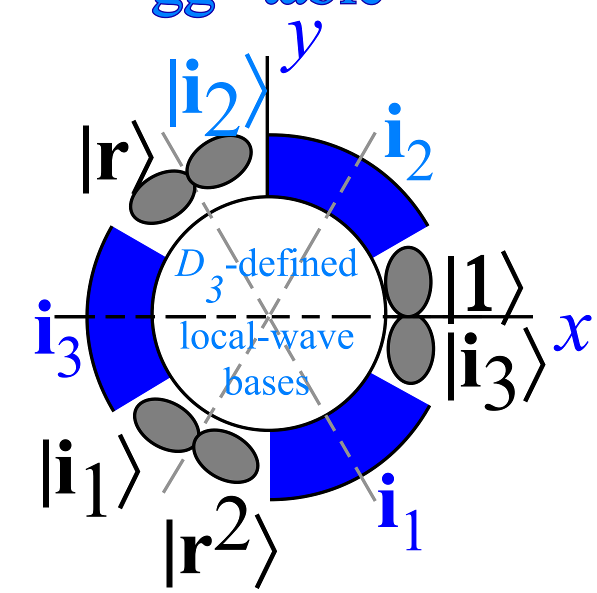
To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch **g** ⇄ **g**† on side of group table

$$R^G(\bar{\mathbf{1}}) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}) = \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}^2) = \begin{pmatrix} \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

D_3 global gg†-table



D_3 local g†g-table

$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* {..**T**,**U**,**V**,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	**r²**	**r**	**i₁**	**i₂**	**i₃**
r	**1**	**r²**	**i₃**	**i₁**	**i₂**
r²	**r**	**1**	**i₂**	**i₃**	**i₁**
i₁	**i₃**	**i₂**	**1**	**r**	**r²**
i₂	**i₁**	**i₃**	**r²**	**1**	**r**
i₃	**i₂**	**i₁**	**r**	**r²**	**1**

D_3 global
$gg^\dagger$-table

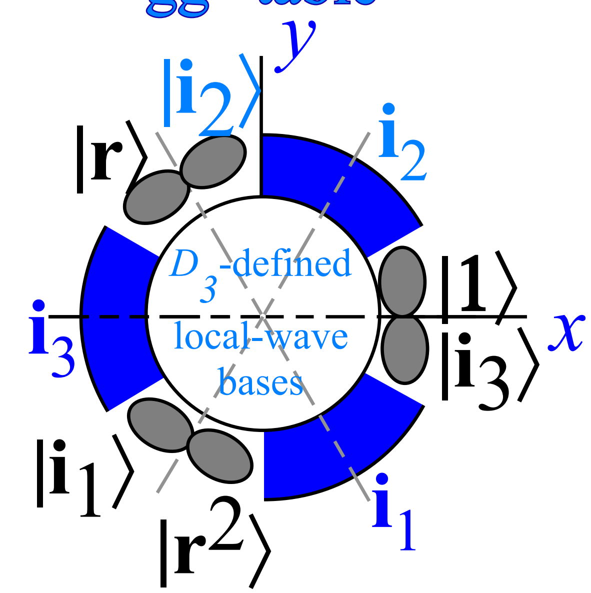
RESULT:
Any *R(**T**)*
commute (Even if **T** and **U** do not...)
with any *R($\bar{\mathbf{U}}$)*...

...and **T**·**U**=**V** if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$-matrix
*having **Global** symmetry D_3*

$$\mathbb{H} = H\bar{\mathbf{1}}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

is made from
***Local** symmetry matrices*



D_3 local
$g^\dagger g$-table

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

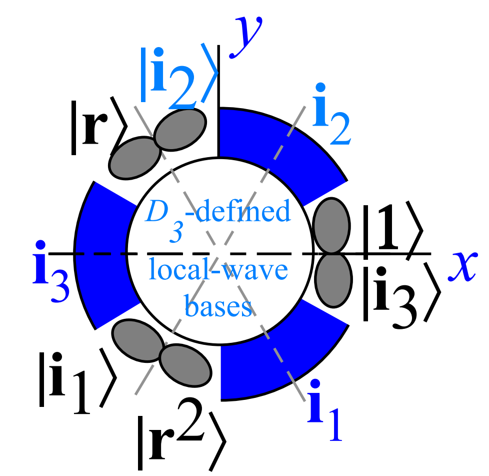
1	**r**	**r²**	**i₁**	**i₂**	**i₃**
r²	**1**	**r**	**i₂**	**i₃**	**i₁**
r	**r²**	**1**	**i₃**	**i₁**	**i₂**
i₁	**i₂**	**i₃**	**1**	**r**	**r²**
i₂	**i₃**	**i₂**	**r²**	**1**	**r**
i₃	**i₁**	**i₂**	**r**	**r²**	**1**

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* {..**T**,**U**,**V**,... }switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$



Local $\mathbb{H}$ *matrix parametrized by* $\bar{\mathbf{g}}$*'s*

RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...)
with any $R(\bar{\mathbf{U}})$...

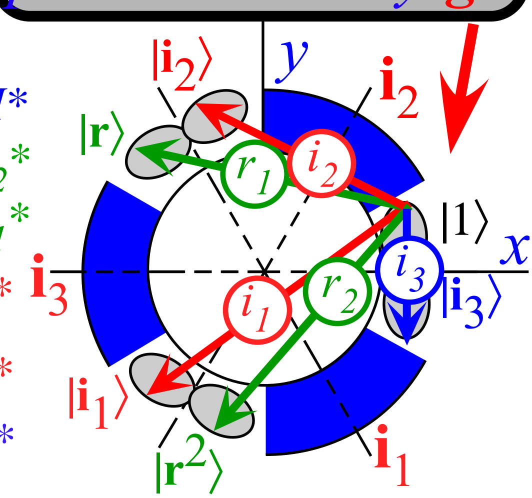
...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

$$\mathbb{H} = H\bar{\mathbf{1}}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*

$H = \langle 1|\mathbb{H}|1\rangle = H^*$
$r_1 = \langle r|\mathbb{H}|1\rangle = r_2^*$
$r_2 = \langle r^2|\mathbb{H}|1\rangle = r_1^*$
$i_1 = \langle i_1|\mathbb{H}|1\rangle = i_1^*$
$i_2 = \langle i_2|\mathbb{H}|1\rangle = i_2^*$
$i_3 = \langle i_3|\mathbb{H}|1\rangle = i_3^*$



To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

local D_3 defined Hamiltonian matrix

$\mathbb{H}=$

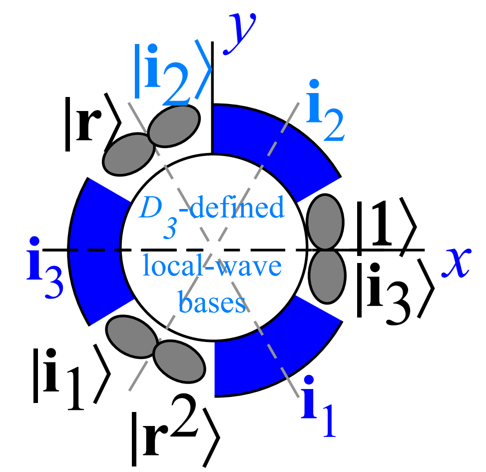
	$\vert 1)$	$\vert r)$	$\vert r^2)$	$\vert i_1)$	$\vert i_2)$	$\vert i_3)$
$(1\vert$	H	r_1	r_2	i_1	i_2	i_3
$(r\vert$	r_2	H	r_1	i_2	i_3	i_1
$(r^2\vert$	r_1	r_2	H	i_3	i_1	i_2
$(i_1\vert$	i_1	i_2	i_3	H	r_1	r_2
$(i_2\vert$	i_2	i_3	i_1	r_2	H	r_1
$(i_3\vert$	i_3	i_1	i_2	r_1	r_2	H

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* {..**T**,**U**,**V**,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\;
R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\;
R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\;
R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\;
R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$



Local $\mathbb{H}$ *matrix parametrized by* $\bar{\mathbf{g}}$*'s*

RESULT:
Any $R(\mathbf{T})$
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with any $R(\bar{\mathbf{U}})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

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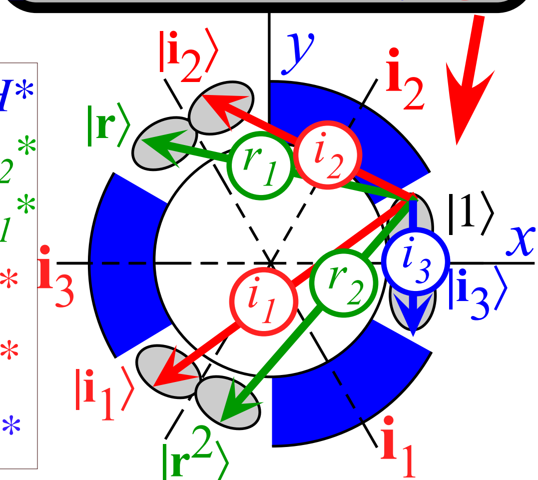
is made from ***Local*** *symmetry matrices*

$H = \langle 1|\mathbb{H}|1\rangle = H^*$
$r_1 = \langle \mathbf{r}|\mathbb{H}|1\rangle = r_2^*$
$r_2 = \langle \mathbf{r}^2|\mathbb{H}|1\rangle = r_1^*$
$i_1 = \langle \mathbf{i}_1|\mathbb{H}|1\rangle = i_1^*$
$i_2 = \langle \mathbf{i}_2|\mathbb{H}|1\rangle = i_2^*$
$i_3 = \langle \mathbf{i}_3|\mathbb{H}|1\rangle = i_3^*$



All the global $\mathbf{g}$ *commute with general local* $\mathbb{H}$ *matrix.*

local D_3 defined Hamiltonian matrix

$\mathbb{H} =$

	$\vert\mathbf{1})$	$\vert\mathbf{r})$	$\vert\mathbf{r}^2)$	$\vert\mathbf{i}_1)$	$\vert\mathbf{i}_2)$	$\vert\mathbf{i}_3)$
$(\mathbf{1}\vert$	H	r_1	r_2	i_1	i_2	i_3
$(\mathbf{r}\vert$	r_2	H	r_1	i_2	i_3	i_1
$(\mathbf{r}^2\vert$	r_1	r_2	H	i_3	i_1	i_2
$(\mathbf{i}_1\vert$	i_1	i_2	i_3	H	r_1	r_2
$(\mathbf{i}_2\vert$	i_2	i_3	i_1	r_2	H	r_1
$(\mathbf{i}_3\vert$	i_3	i_1	i_2	r_1	r_2	H

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\;
R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\;
R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\;
R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\;
R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

Example of RELATIVITY-DUALITY for D

To represent *external* {..**T**,**U**,**V**,... }...

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&1\end{pmatrix}$$

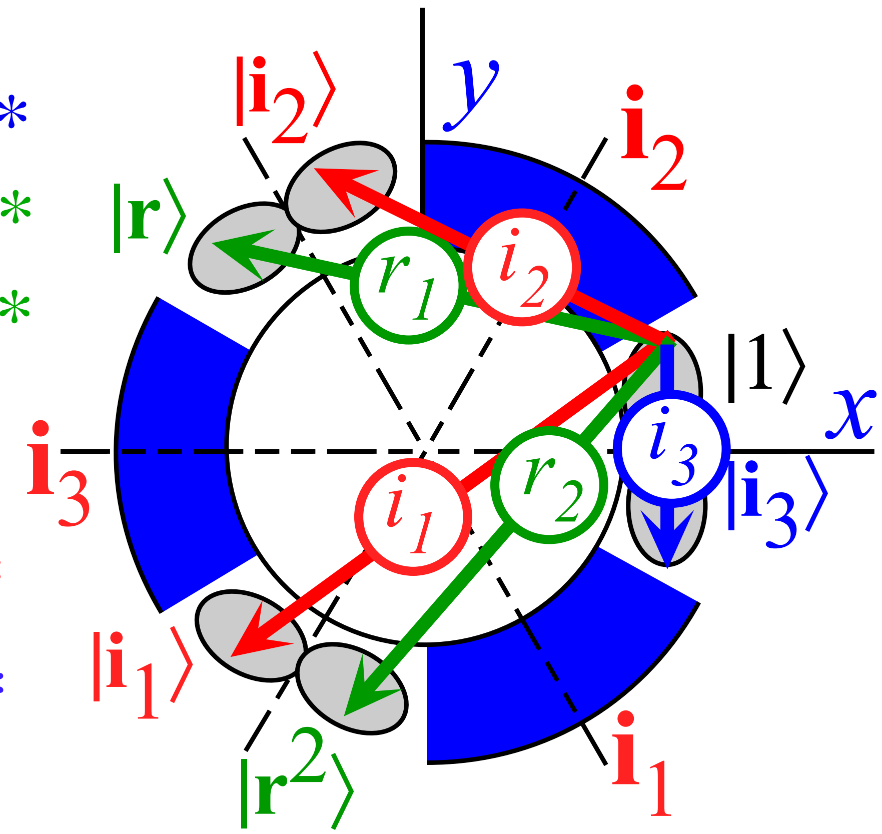
RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...)
with any $R(\bar{\mathbf{U}})$...

...and **T**·**U**=**V** if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... }....

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\1&\cdot&\cdot\\\cdot&\cdot&1\\\cdot&1&\cdot\end{pmatrix}$$

$$H = \langle 1|\mathbb{H}|1\rangle = H^*$$
$$r_1 = \langle \mathbf{r}|\mathbb{H}|1\rangle = r_2^*$$
$$r_2 = \langle \mathbf{r}^2|\mathbb{H}|1\rangle = r_1^*$$
$$i_1 = \langle \mathbf{i}_1|\mathbb{H}|1\rangle = i_1^*$$
$$i_2 = \langle \mathbf{i}_2|\mathbb{H}|1\rangle = i_2^*$$
$$i_3 = \langle \mathbf{i}_3|\mathbb{H}|1\rangle = i_3^*$$



So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

local-D_3-defined Hamiltonian matrix

$$\mathbb{H} = H\mathbf{1}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2 + i_1\mathbf{i}_1 + i_2\mathbf{i}_2 + i_3\mathbf{i}_3$$

is made from ***Local*** *symmetry matrices*

$\mathbb{H} =$

	$\lvert\mathbf{1})$	$\lvert\mathbf{r})$	$\lvert\mathbf{r}^2)$	$\lvert\mathbf{i}_1)$	$\lvert\mathbf{i}_2)$	$\lvert\mathbf{i}_3)$
$(\mathbf{1}\rvert$	H	r_1	r_2	i_1	i_2	i_3
$(\mathbf{r}\rvert$	r_2	H	r_1	i_2	i_3	i_1
$(\mathbf{r}^2\rvert$	r_1	r_2	H	i_3	i_1	i_2
$(\mathbf{i}_1\rvert$	i_1	i_2	i_3	H	r_1	r_2
$(\mathbf{i}_2\rvert$	i_2	i_3	i_2	r_2	H	r_1
$(\mathbf{i}_3\rvert$	i_3	i_1	i_2	r_1	r_2	H

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

Review: Spectral resolution of D_3 Center (Class algebra)

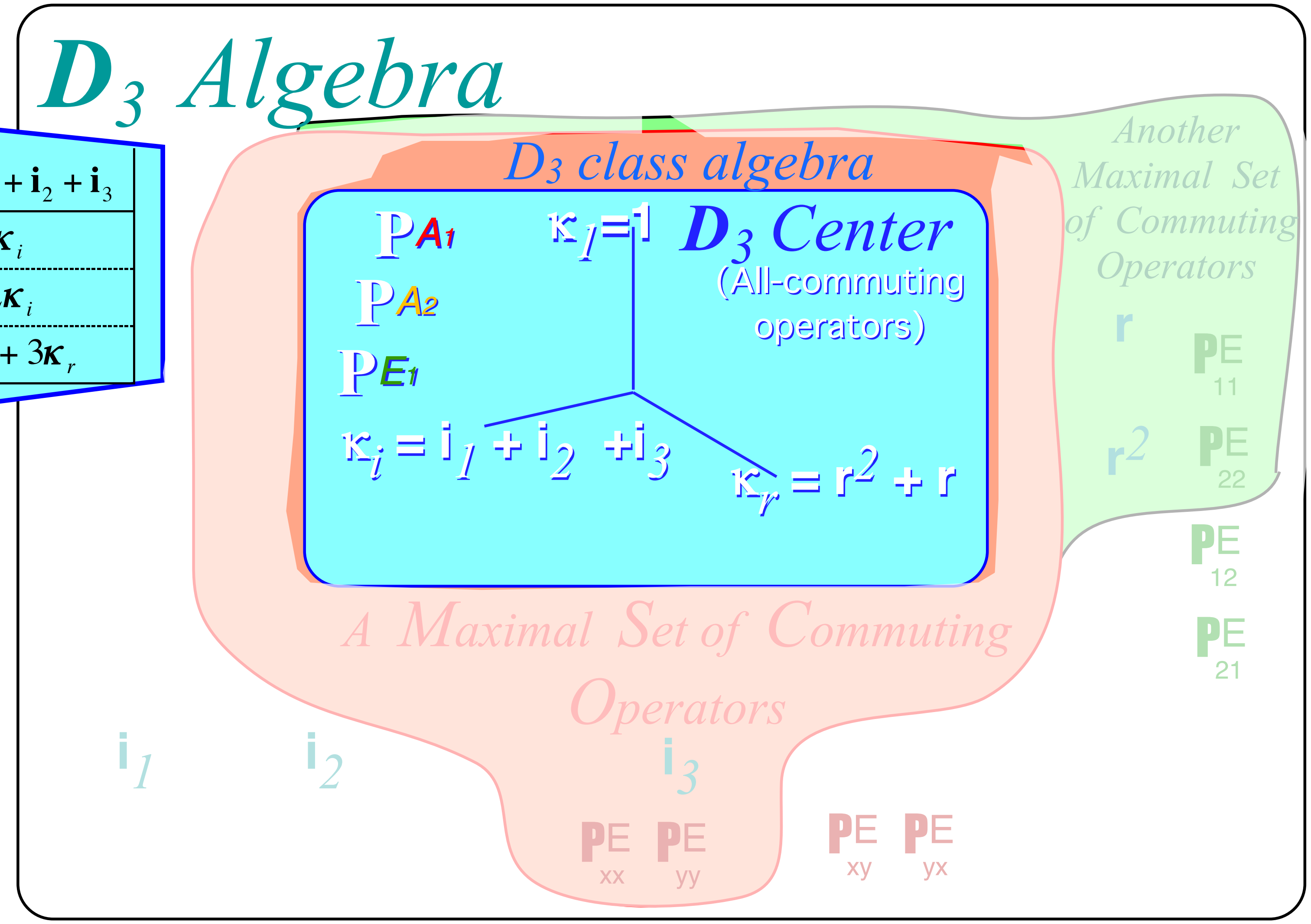
$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum κ_k commutes with all $\mathbf{g}_t$

Class-sum κ_k invariance: $\mathbf{g}_t \kappa_k = \kappa_k \mathbf{g}_t$



Review: Spectral resolution of D_3 Center (Class algebra)

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

→

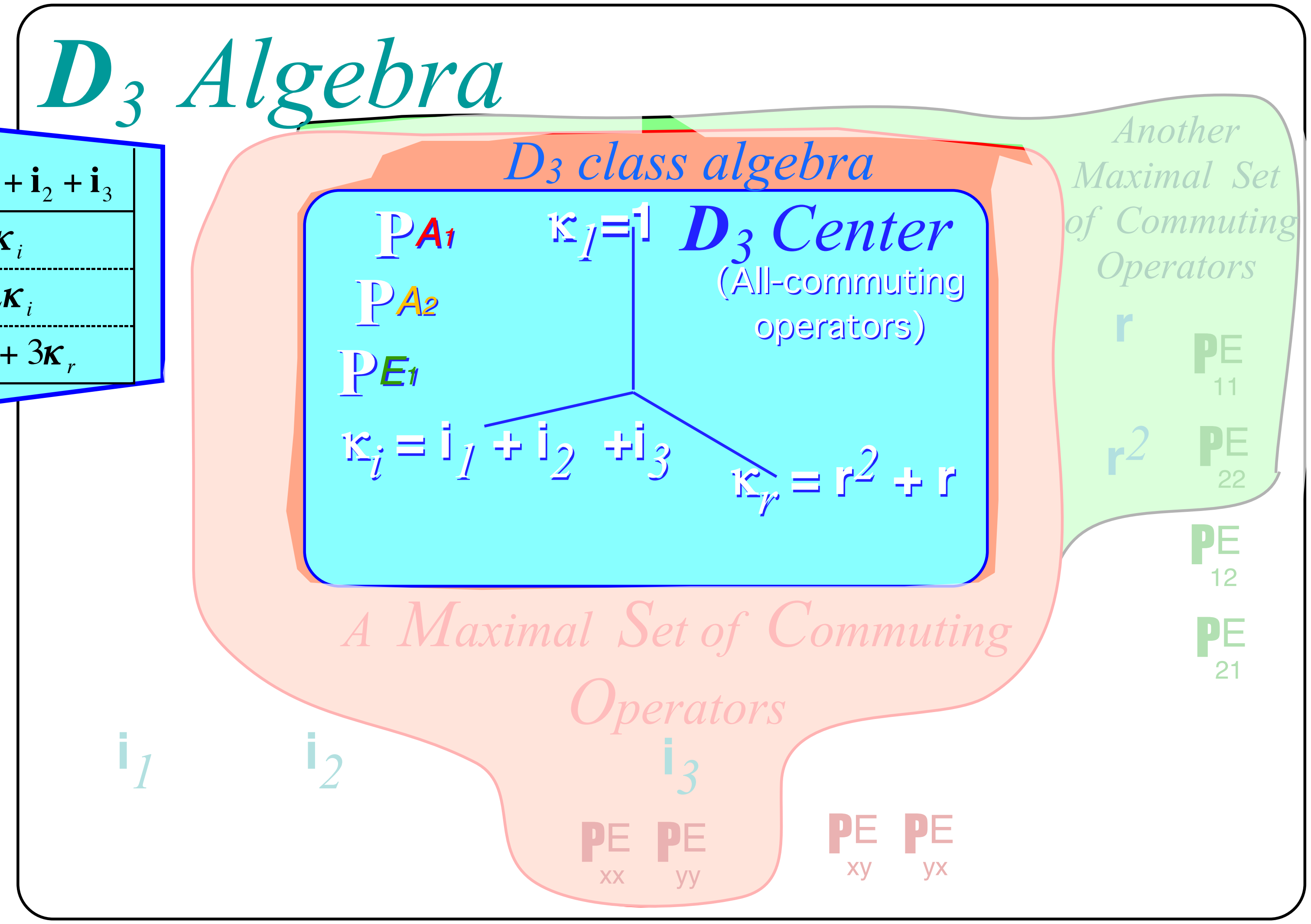
	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$



Review: *Spectral resolution of **D_3 Center** (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

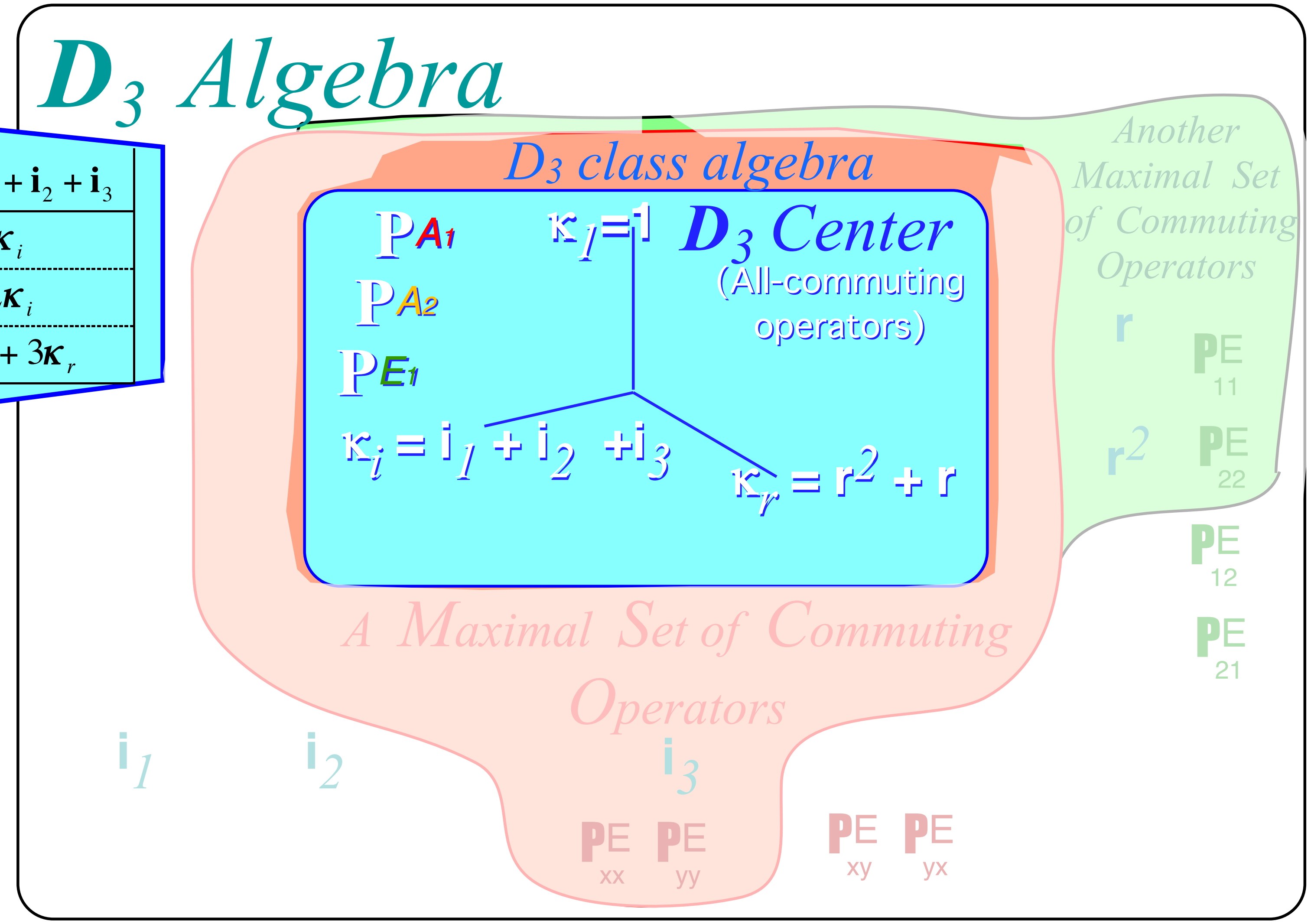
Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k$ where: $\boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j$



Review: Spectral resolution of $\mathbf{D_3}$ Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

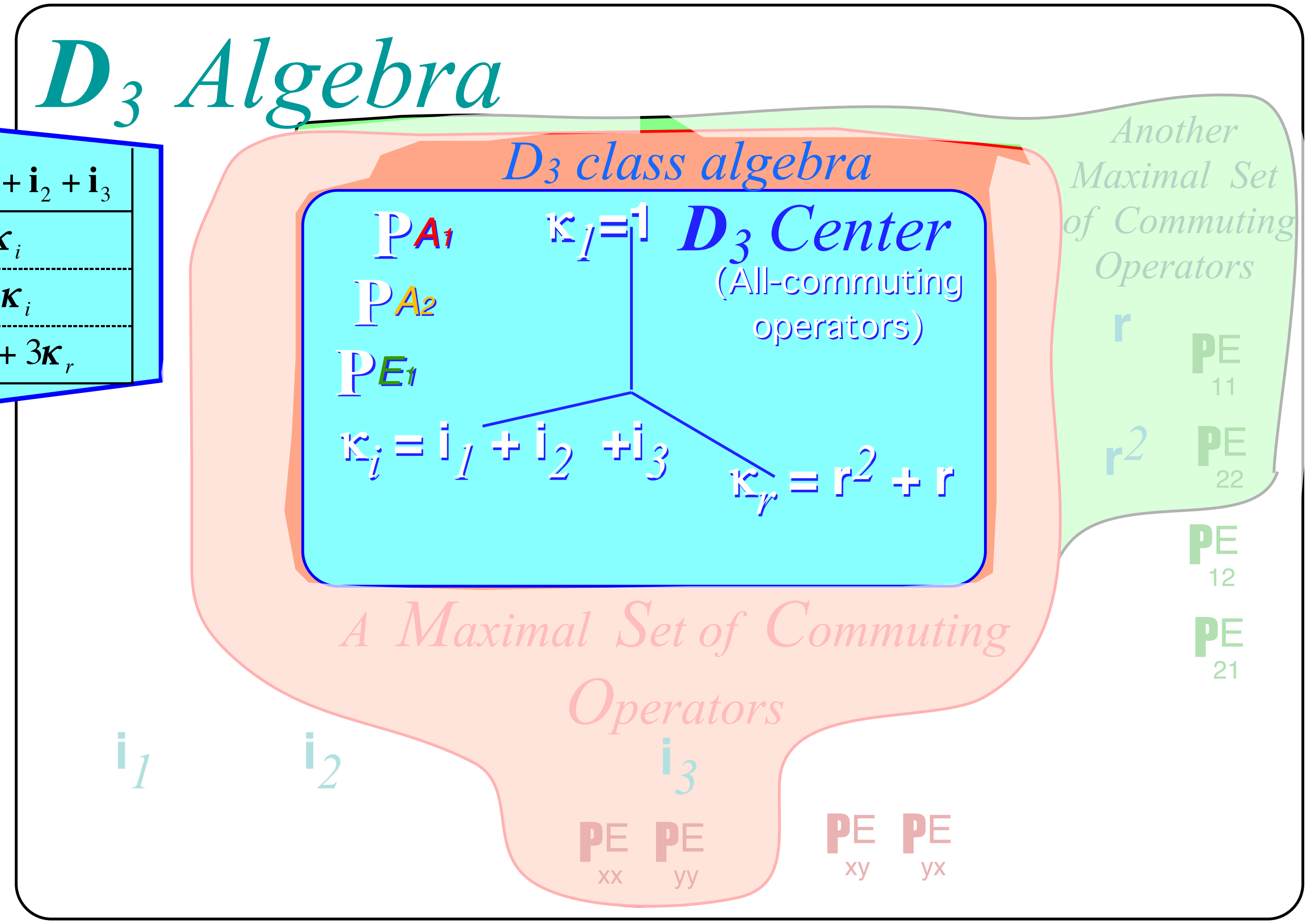
Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3 = 6)$

$^\circ \kappa_k$ = order of class κ_k: $(^\circ\kappa_1 = 1,\ ^\circ\kappa_r = 2,\ ^\circ\kappa_i = 3)$

$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=^\circ\kappa_k} \mathbf{g}_j = \frac{1}{^\circ s_k} \sum_{t=1}^{t=^\circ G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1 = 6,\ ^\circ s_r = 3,\ ^\circ s_i = 2)$



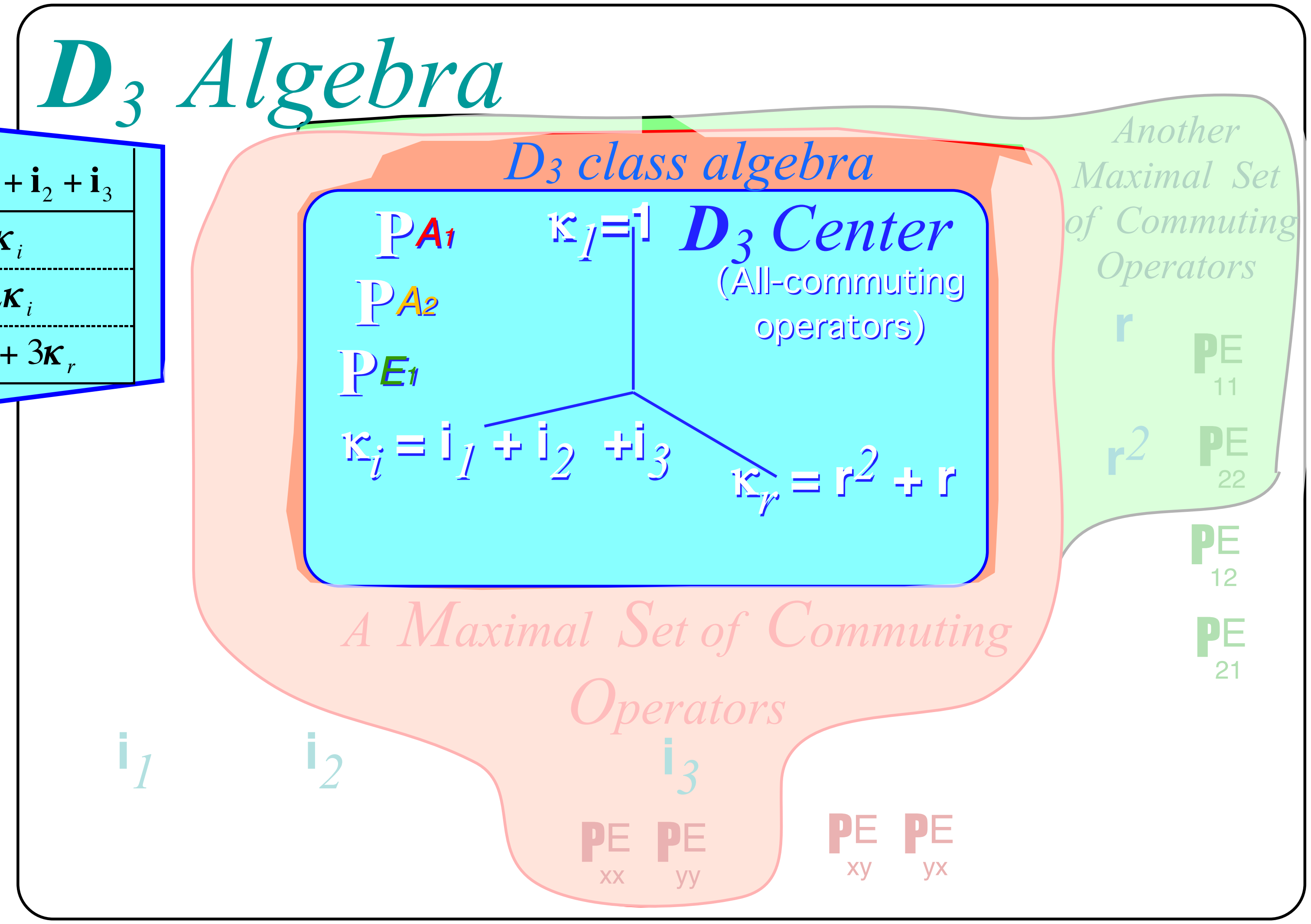
Review: *Spectral resolution of* $\mathbf{D_3}$ ***Center*** *(Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$



Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$°s_k$ = order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

Group theory of equivalence transformations and classes

➡ *Lagrange theorems* ⬅

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

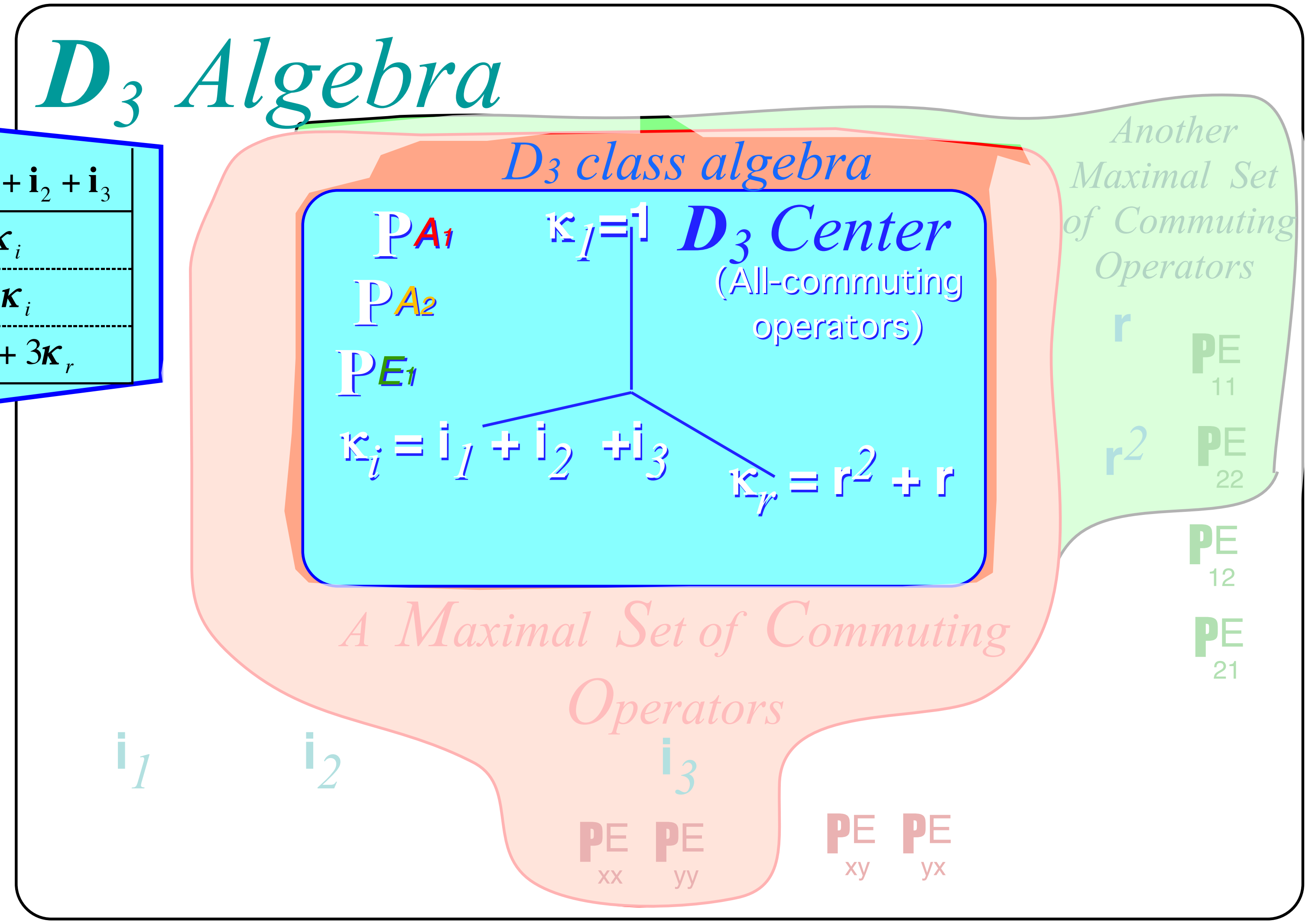
Review: *Spectral resolution of $\mathbf{D_3}$ Center (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$



Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3=6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j=\frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1=6,\ °s_r=3,\ °s_i=2)$

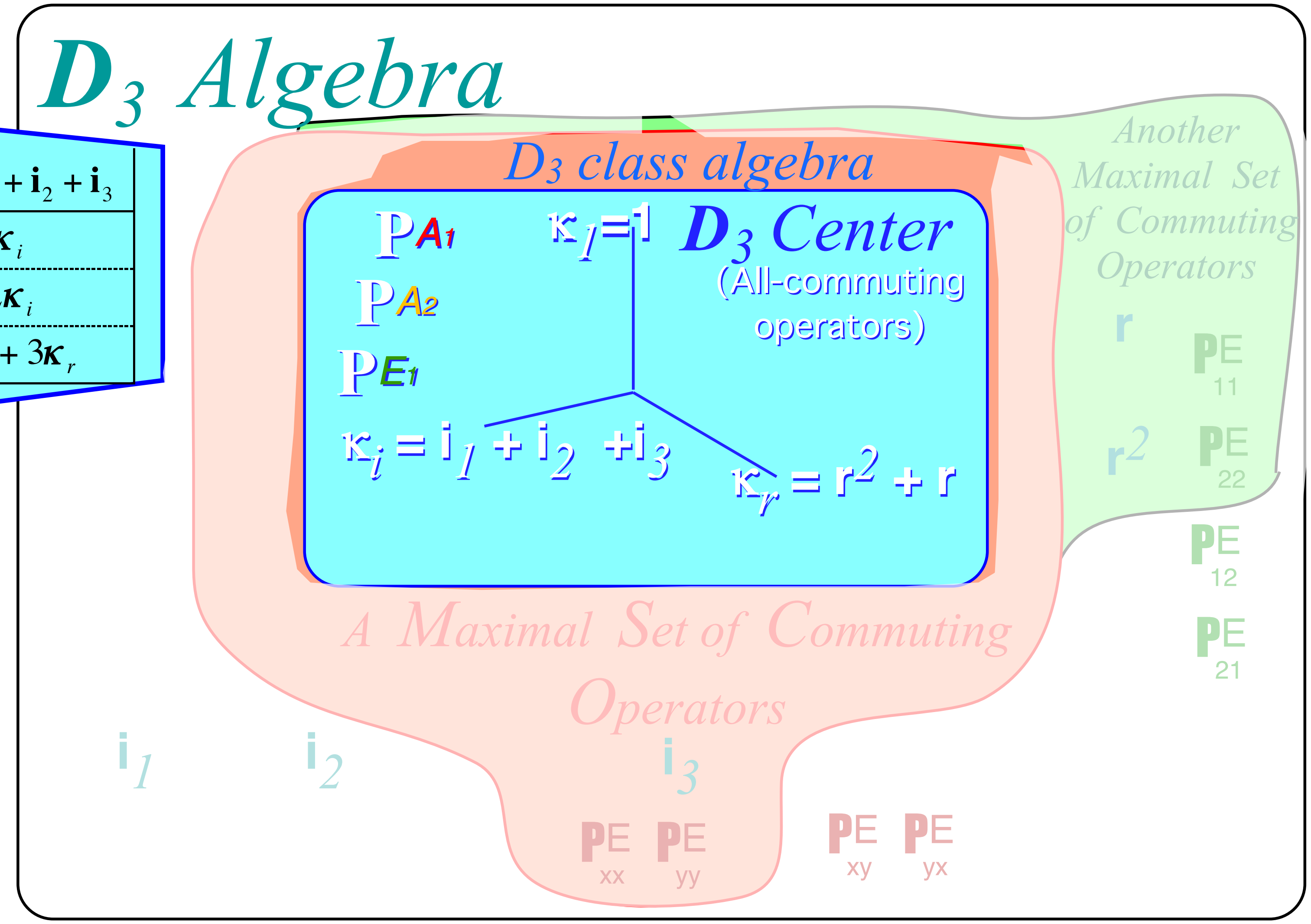
$°s_k=°G/°\kappa_k$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

Review: Spectral resolution of $\mathbf{D_3}$ ***Center*** *(Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3=6)$

$^\circ\kappa_k$ = order of class κ_k: $(^\circ\kappa_1=1,\ ^\circ\kappa_r=2,\ ^\circ\kappa_i=3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j=^\circ\kappa_k}\mathbf{g}_j=\frac{1}{^\circ s_k}\sum_{t=1}^{t=^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1=6,\ ^\circ s_r=3,\ ^\circ s_i=2)$

$^\circ s_k={}^\circ G/{}^\circ\kappa_k$ *$^\circ s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

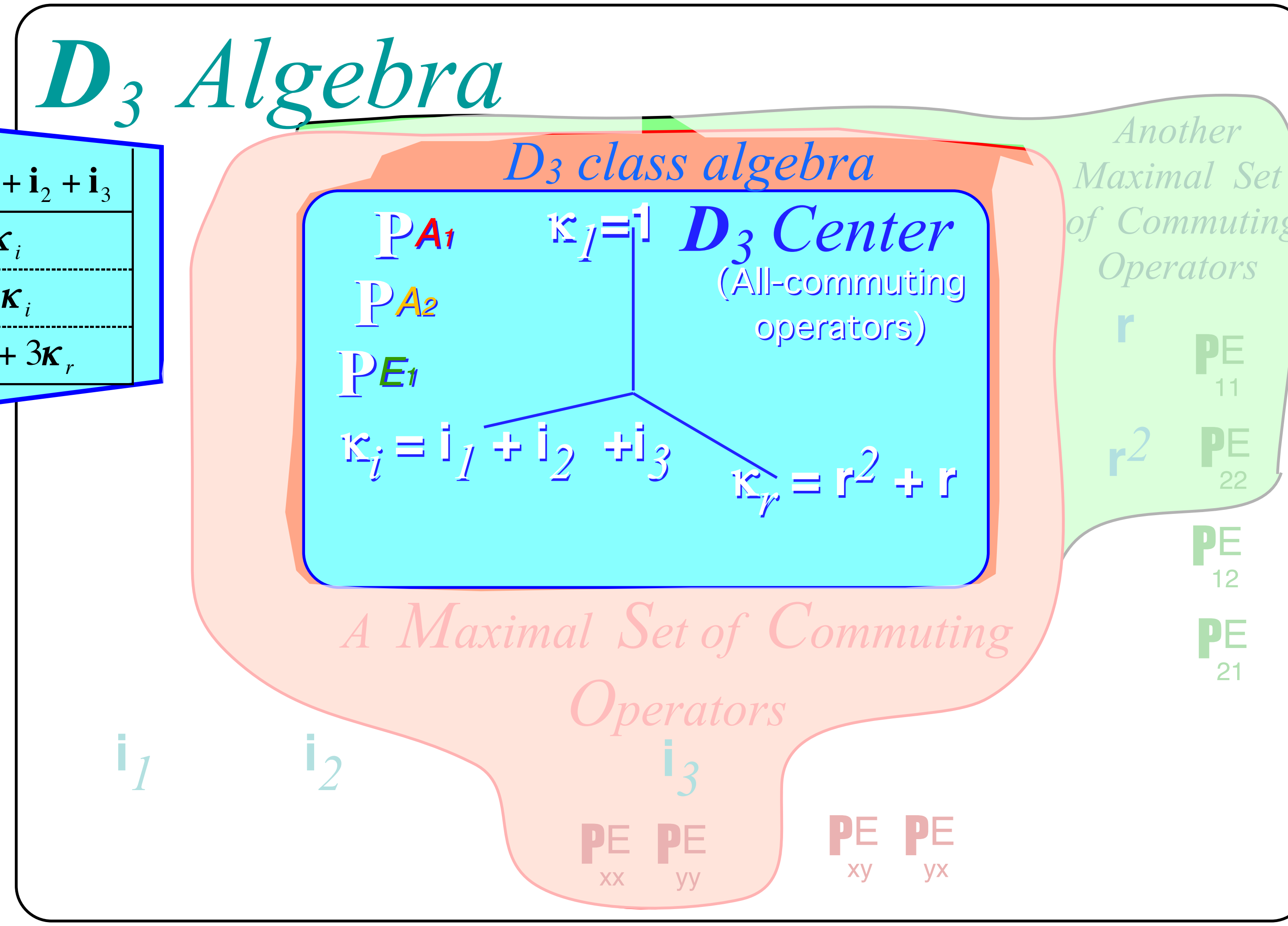
These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

Review: *Spectral resolution of **D_3 Center** (Class algebra)*

1	**r^2**	**r**	**i_1**	**i_2**	**i_3**
r	**1**	**r^2**	**i_3**	**i_1**	**i_2**
r^2	**r**	**1**	**i_2**	**i_3**	**i_1**
i_1	**i_3**	**i_2**	**1**	**r**	**r^2**
i_2	**i_1**	**i_3**	**r^2**	**1**	**r**
i_3	**i_2**	**i_1**	**r**	**r^2**	**1**

→

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G\,/\,°\kappa_k$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

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If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$, then so does $\mathbf{g}_t \mathbf{g}_s$.
that is: $\mathbf{g}_t \mathbf{g}_s \mathbf{g}_k (\mathbf{g}_t \mathbf{g}_s)^{-1} = \mathbf{g}_t \mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} \mathbf{g}_t^{-1} = \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$,

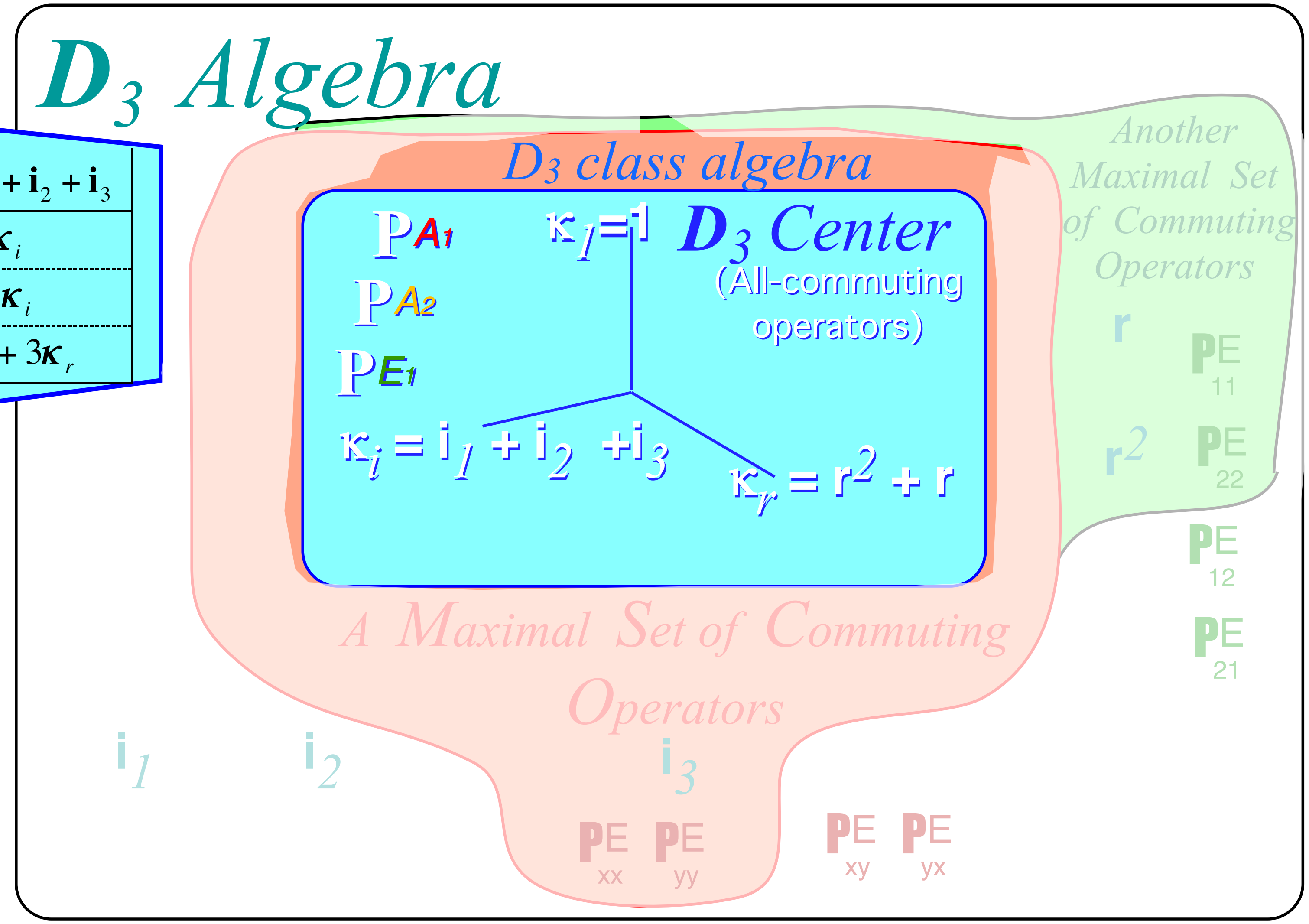
⋮
⋮

Review: *Spectral resolution of* ***D_3 Center*** *(Class algebra)*

1	**r²**	**r**	**i_1**	**i_2**	**i_3**
r	**1**	**r²**	**i_3**	**i_1**	**i_2**
r²	**r**	**1**	**i_2**	**i_3**	**i_1**
i_1	**i_3**	**i_2**	**1**	**r**	**r²**
i_2	**i_1**	**i_3**	**r²**	**1**	**r**
i_3	**i_2**	**i_1**	**r**	**r²**	**1**

→

	$\kappa_1 = 1$	$\kappa_r = r + r^2$	$\kappa_i = i_1 + i_2 + i_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

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These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k . Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} = \mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$, then so does $\mathbf{g}_t \mathbf{g}_s$.

that is:

Subgroup $s_k = \{\mathbf{g}_0 = \mathbf{1},\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$ *has* $\mathbf{l} = (°\kappa_k - 1)$ ***Left Cosets*** (one coset for each member of class κ_k).

$\mathbf{g}_1 s_k = \mathbf{g}_1 \{\mathbf{g}_0 = 1,\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$,

⋮

⋮

Review: Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

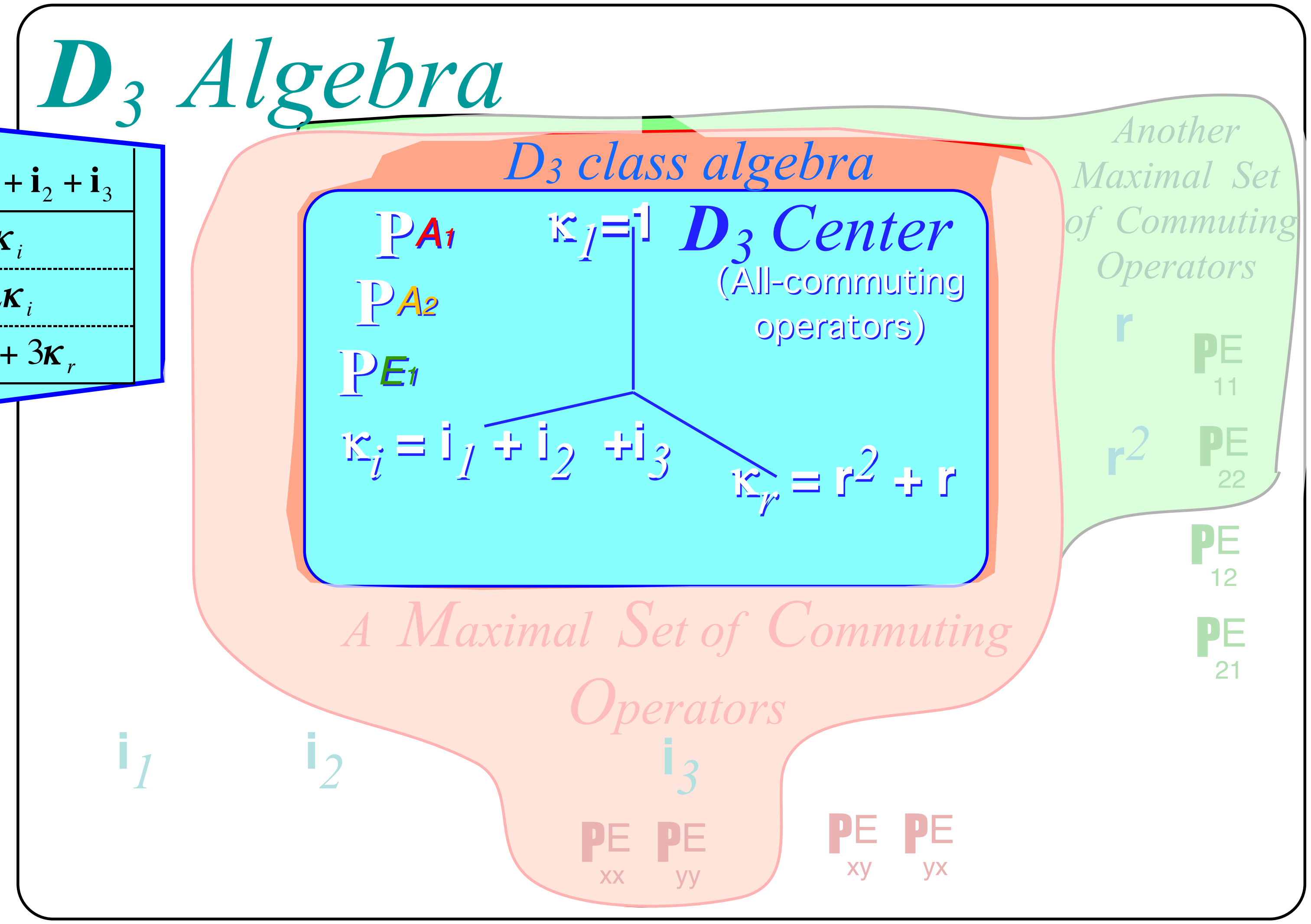
$°G$ = order of group: $(°D_3=6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j=\frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1=6,\ °s_r=3,\ °s_i=2)$

$°s_k=°G/°\kappa_k$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*



These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.

that is:

Subgroup $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$ *has* $\mathbf{l}=(°\kappa_k-1)$ ***Left Cosets*** (one coset for each member of class κ_k).

$\mathbf{g}_1 s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$,

$\mathbf{g}_2 s_k=\mathbf{g}_2\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\},\dots$

$\vdots$

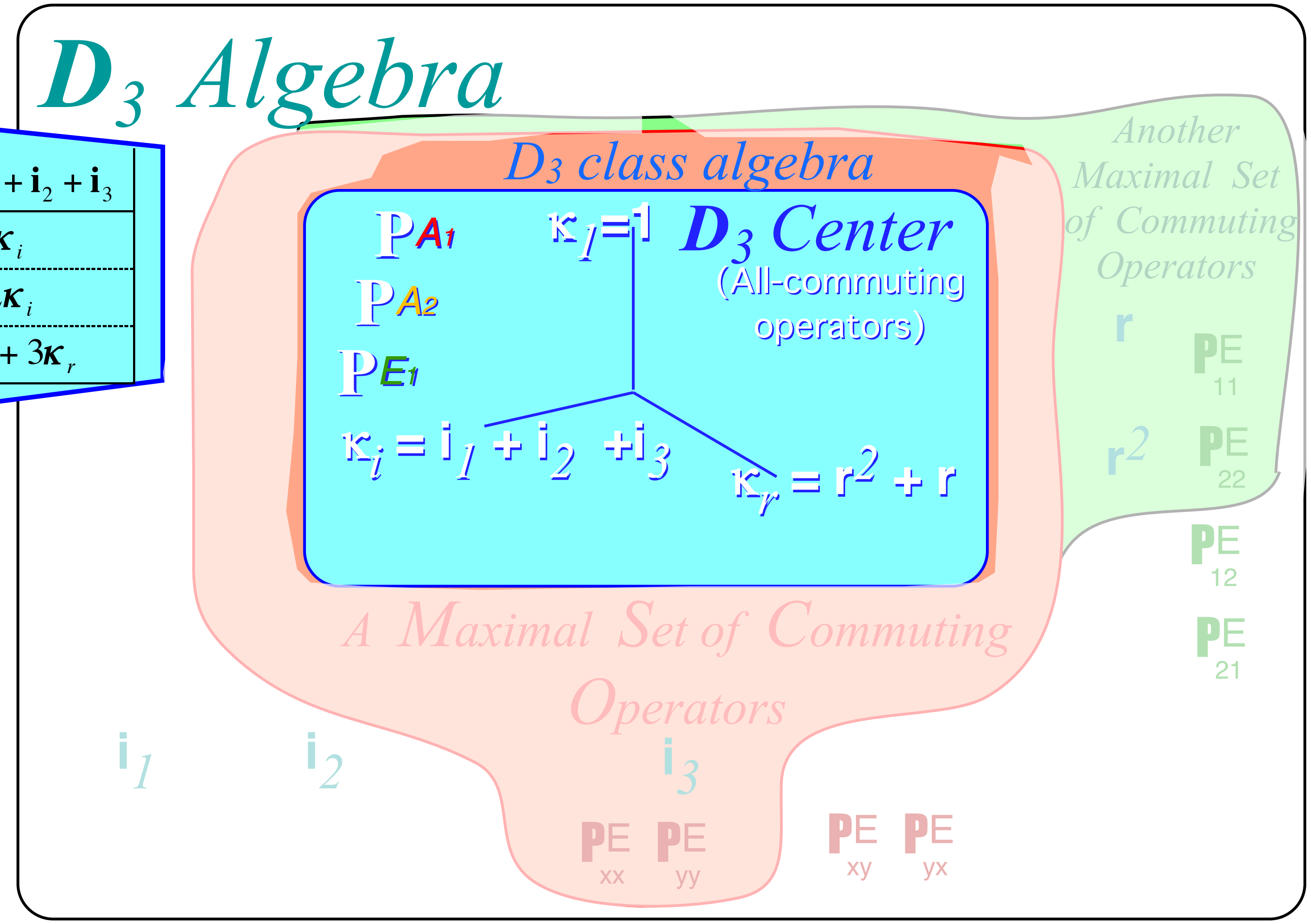
$\mathbf{g}_\mathbf{l} s_k=\mathbf{g}_\mathbf{l}\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$

Review: Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j = \frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$, then so does $\mathbf{g}_t\mathbf{g}_s$.

that is:

Subgroup $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$ has $\mathbf{l}=(°\kappa_k-1)$ ***Left Cosets*** (one coset for each member of class κ_k).

$\mathbf{g}_1\, s_k = \mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$,

$\mathbf{g}_2\, s_k = \mathbf{g}_2\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$,...

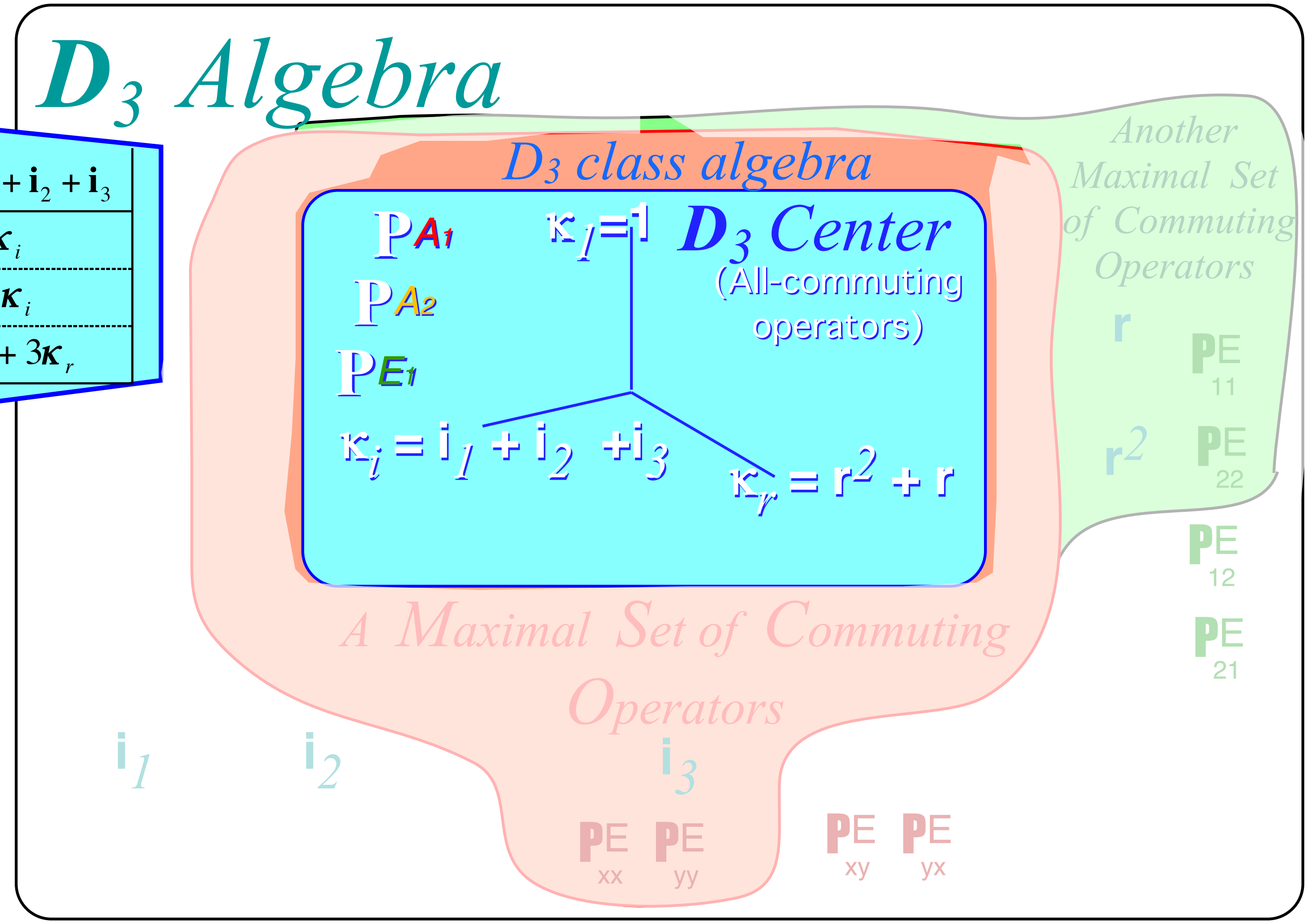
They will divide the group of order $°D_3 = °\kappa_k \cdot °s_k$ evenly into $°\kappa_k$ subsets each of order $°s_k$.

Review: Spectral resolution of D_3 ***Center*** *(Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

→

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$



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$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_k=\sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j=\frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

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$\mathbf{g}_1\, s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$,

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*These results are known as **Lagrange's Coset Theorem(s)***

They will divide the group of order $°D_3=°\kappa_k\cdot°s_k$ evenly into $°\kappa_k$ subsets each of order $°s_k$.

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

$\boldsymbol{\kappa}_g$'s are *mutually commuting* with respect to themselves and *all-commuting* with respect to the whole group.

$$\mathbf{r}\,\boldsymbol{\kappa}_i\,\mathbf{r}^{-1} = \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_1 = \boldsymbol{\kappa}_i \quad \text{or:} \quad \mathbf{r}\,\boldsymbol{\kappa}_i = \boldsymbol{\kappa}_i\,\mathbf{r}$$

$$\sum_{\mathbf{h}=1}^{\circ G} \mathbf{h}\mathbf{g}\mathbf{h}^{-1} = \upsilon_g \boldsymbol{\kappa}_g\ , \qquad \text{where: } \upsilon_g = \frac{{}^\circ G}{{}^\circ\kappa_g} = \textit{integer}$$

${}^\circ\kappa g$ is order of class $\boldsymbol{\kappa}g$ and must evenly divide group order ${}^\circ G$.

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$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

$$\boldsymbol{\kappa}^2_3 = 3\cdot\boldsymbol{\kappa}_2 + 3\cdot\mathbf{1}$$

Note also:

$$\boldsymbol{\kappa}^2_2 - \boldsymbol{\kappa}_2 - 2\cdot\mathbf{1} = 0$$

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$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)}$

$$0 = \kappa_{\mathbf{3}}^3 - 9\kappa_{\mathbf{3}} = (\kappa_{\mathbf{3}} - 3\cdot\mathbf{1})(\kappa_{\mathbf{3}} + 3\cdot\mathbf{1})(\kappa_{\mathbf{3}} - 0\cdot\mathbf{1})$$

$\leftarrow \boldsymbol{\kappa}^2{}_3 = 3\cdot\boldsymbol{\kappa}_2 + 3\cdot\mathbf{1}$

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➡ *All-commuting projectors and D_3-invariant characters* ⬅

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1}$$

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$

$\kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})}{(+0-3)(+0+3)}$$

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

$0 = (\kappa_2 - 2\cdot\mathbf{1})(\kappa_2 + \mathbf{1})$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1}$

$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1}$

$0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2}$

$\kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2}$

$0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$

$\kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$

Class resolution into sum of eigenvalue · **Projector**

$\kappa_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$

$\kappa_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$

$\kappa_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 + 3\mathbf{1})}{(+0-3)(+0+3)}$$

Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

$0 = (\kappa_2 - 2\cdot\mathbf{1})(\kappa_2 + \mathbf{1})$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

Note also:

$\kappa^2_2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

$0 = (\kappa_2 - 2\cdot\mathbf{1})(\kappa_2 + \mathbf{1})$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1}$ $\qquad$ $0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2}$ $\qquad$ $0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$

$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1}$ $\qquad$ $\kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2}$ $\qquad$ $\kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$

Class resolution into sum of eigenvalue · **Projector**

$\kappa_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$

$\kappa_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$ $\longleftarrow$ $\kappa^2_r = \kappa_r + 2\cdot\mathbf{1} \Rightarrow (\kappa_r - 2\cdot\mathbf{1})(\kappa_r + \mathbf{1}) = \mathbf{0}$

$\kappa_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})}{(+0-3)(+0+3)}$$

Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\kappa_1 - \kappa_2 + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1}$$

$$0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2}$$

$$0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E} \qquad \kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

Class resolution into sum of eigenvalue · Projector

$$\kappa_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$$

$$\kappa_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E} \longleftarrow \kappa_r^2 = \kappa_r + 2\cdot\mathbf{1} \Rightarrow (\kappa_r - 2\cdot\mathbf{1})(\kappa_r + \mathbf{1}) = \mathbf{0}$$

$$\kappa_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 + 3\mathbf{1})}{(+0-3)(+0+3)}$$

Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = (\kappa_1 + \kappa_2 - \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\kappa_1 - \kappa_2 + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

χ_k^α	χ_1^α	χ_2^α	χ_3^α
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	−1
$\alpha = E$	2	−1	0

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1=\mathbf{1}$	$\kappa_2=\mathbf{r}^1+\mathbf{r}^2$	$\kappa_3=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_2	$2\kappa_1+\kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1+3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)}=\mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

$$0=\kappa_3^3-9\kappa_3=(\kappa_3-3\cdot\mathbf{1})(\kappa_3+3\cdot\mathbf{1})(\kappa_3-0\cdot\mathbf{1})$$

$$0=(\kappa_3-3\cdot\mathbf{1})\mathbf{P}^{A_1}\qquad 0=(\kappa_3+3\cdot\mathbf{1})\mathbf{P}^{A_2}\qquad 0=(\kappa_3-0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\kappa_3\mathbf{P}^{A_1}=+3\cdot\mathbf{P}^{A_1}\qquad \kappa_3\mathbf{P}^{A_2}=-3\cdot\mathbf{P}^{A_2}\qquad \kappa_3\mathbf{P}^{E}=+0\cdot\mathbf{P}^{E}$$

Class resolution into sum of eigenvalue · Projector

$$\kappa_1=1\cdot\mathbf{P}^{A_1}+1\cdot\mathbf{P}^{A_2}+1\cdot\mathbf{P}^{E}$$

$$\kappa_r=2\cdot\mathbf{P}^{A_1}+2\cdot\mathbf{P}^{A_2}-1\cdot\mathbf{P}^{E}$$

$$\kappa_i=3\cdot\mathbf{P}^{A_1}-3\cdot\mathbf{P}^{A_2}+0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1}=\frac{(\kappa_3+3\mathbf{1})(\kappa_3-0\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2}=\frac{(\kappa_3-3\mathbf{1})(\kappa_3-0\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E}=\frac{(\kappa_3-3\mathbf{1})(\kappa_3+3\mathbf{1})}{(+0-3)(+0+3)}$$

Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1}=(\kappa_1+\kappa_2+\kappa_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2+\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2}=(\kappa_1+\kappa_2-\kappa_3)/6=(\mathbf{1}+\mathbf{r}+\mathbf{r}^2-\mathbf{i}_1-\mathbf{i}_2-\mathbf{i}_3)/6$$

$$\mathbf{P}^{E}=(2\kappa_1-\kappa_2+0)/3=(2\mathbf{1}-\mathbf{r}-\mathbf{r}^2)/3$$

Irreducible ***characters*** *are traces* $\chi_\kappa^{(\alpha)}=Tr\,D^{(\alpha)}(\mathbf{r}_\kappa)$ *of irreducible representations* $D^{(\alpha)}(\mathbf{r}_\kappa)$

χ_k^α	χ_1^α	χ_2^α	χ_3^α
$\alpha=A_1$	1	1	1
$\alpha=A_2$	1	1	−1
$\alpha=E$	2	−1	0

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle=\mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

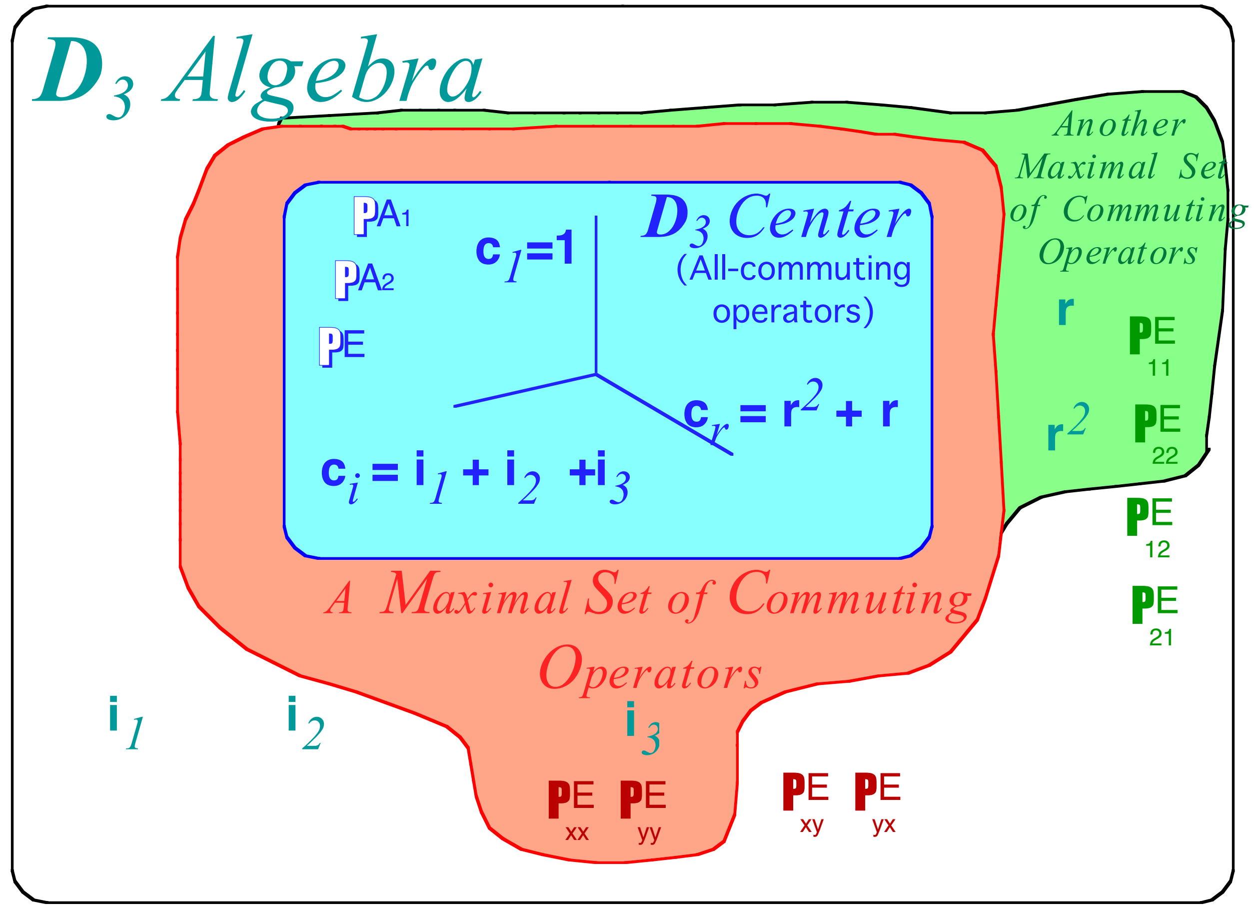
Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Step in spectral analysis of D_3 "group-table" Hamiltonian: Algebra of D_3 Center(Classes)

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

→ *Group invariant numbers: Centrum, Rank, and Order* ←



(Fig. 15.2.1 QTCA)

D_3	$\kappa =$ **1**	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Important invariant numbers or "characters"

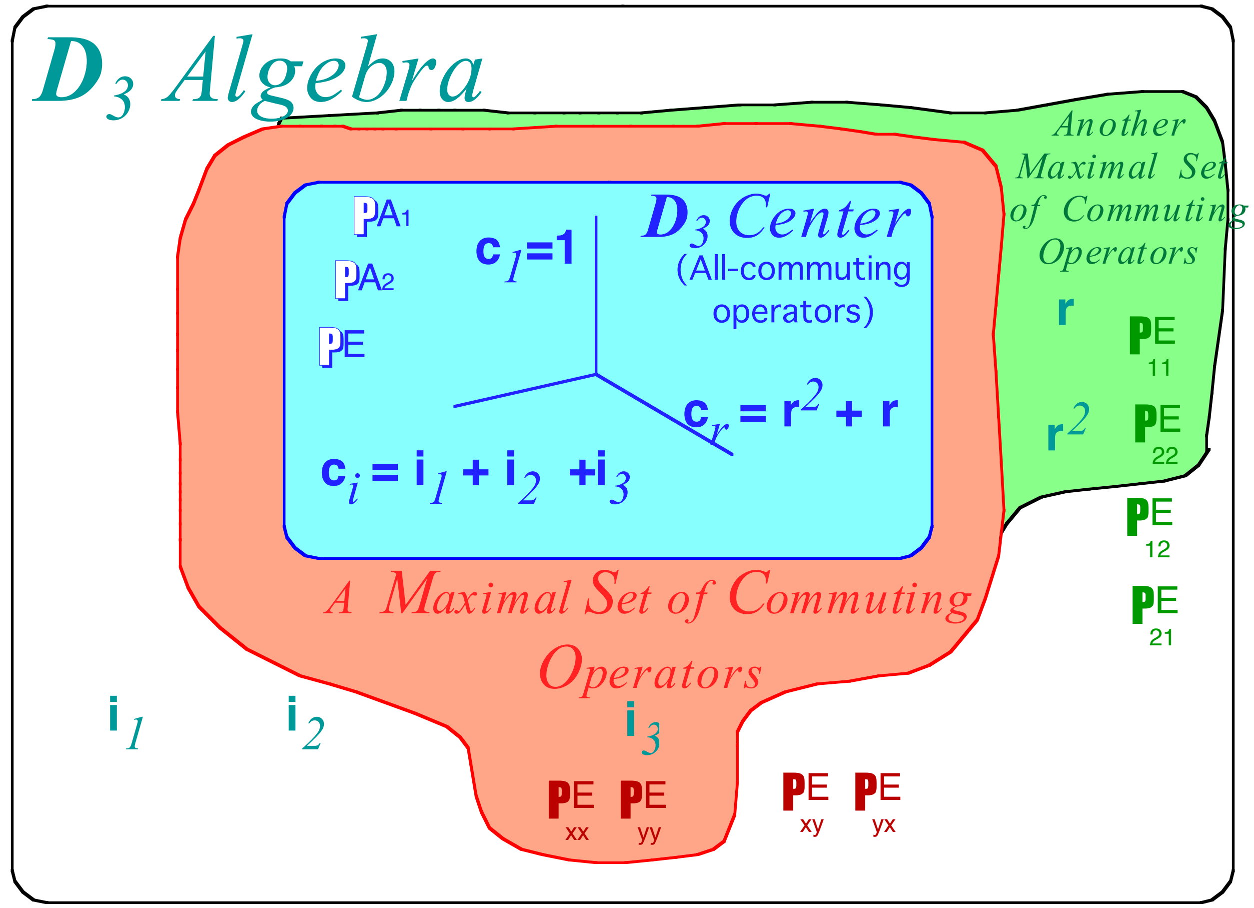
$\ell^\alpha =$ Irreducible representation (irrep) *dimension* or level *degeneracy*

For symmetry group or algebra G

Centrum: $\kappa(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^0$ =Number of classes, invariants, irrep types, *all-commuting* ops

Rank: $\rho(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^1$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

Order: $^o(G)=\Sigma_{irrep(\alpha)} (\ell^\alpha)^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops



D_3 κ =	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Important invariant numbers or "characters"

$\ell^{\alpha}=$ Irreducible representation (irrep) *dimension* or level *degeneracy*

For symmetry group or algebra G

Centrum: $\kappa(G)=\Sigma_{irrep(\alpha)} (\ell^{\alpha})^0$ =Number of classes, invariants, irrep types, *all-commuting* ops

Rank: $\rho(G)=\Sigma_{irrep(\alpha)} (\ell^{\alpha})^1$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

Order: $^{o}(G)=\Sigma_{irrep(\alpha)} (\ell^{\alpha})^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops

$$\kappa(D_3)=(1)^0+(1)^0+(2)^0=3$$
$$\rho(D_3)=(1)^1+(1)^1+(2)^1=4$$
$$^{o}(D_3)=(1)^2+(1)^2+(2)^2=6$$