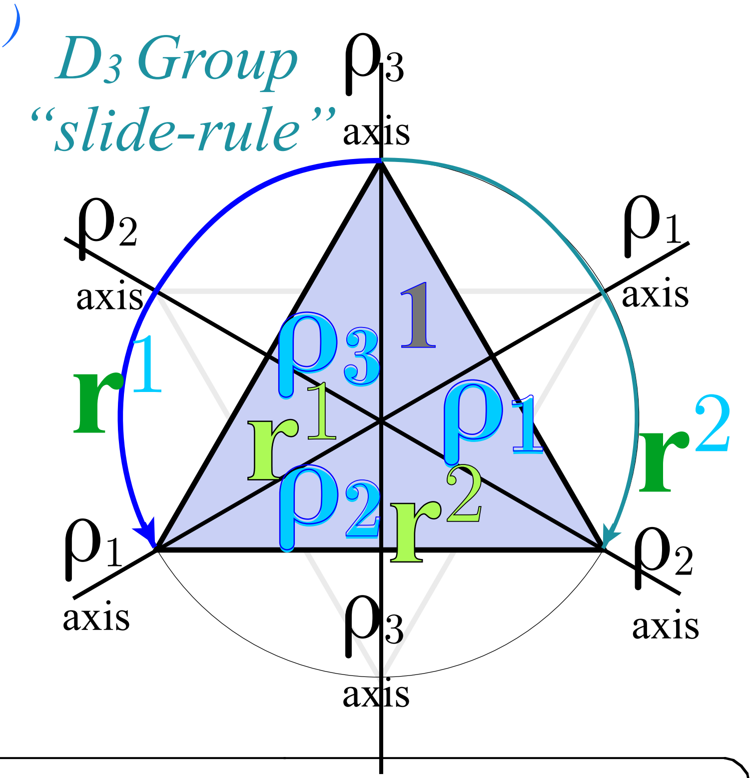
Group Theory in Quantum Mechanics

Lecture 14 (3.10.15)

Smallest non-Abelian group D_3 (and isomorphic $C_{3v} \sim D_3$)

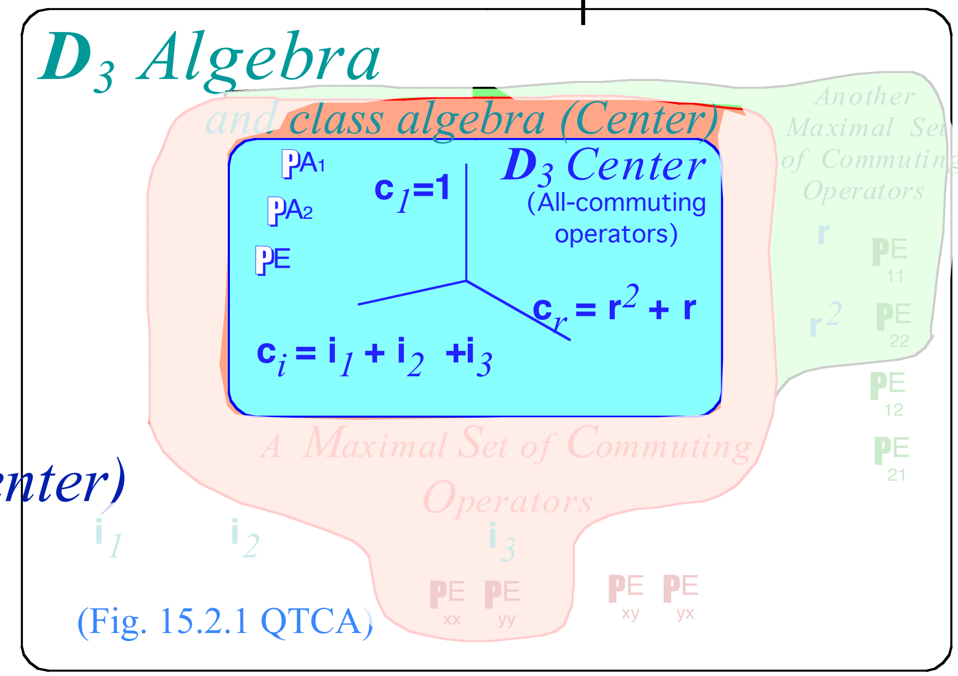
(Int.J.Mol.Sci, 14, 714(2013) p.755-774 , QTCA Unit 5 Ch. 15)
(PSDS - Ch. 3)



3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}
- *D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)*
- *Deriving $D_3 \sim C_{3v}$ products:*
  - *By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$*
  - *By nomograms based on U(2) Hamilton-turns*
- *Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes*

Non-commutative symmetry expansion and Global-Local solution
- *Global vs Local symmetry and Mock-Mach principle*
- *Global vs Local matrix duality for D_3*
  - *Global vs Local symmetry expansion of D_3 Hamiltonian*

1st-Stage spectral decomposition of global/local D_3 Hamiltonian
- *Group theory of equivalence transformations and classes*
  - *Lagrange theorems*
- *All-commuting operators and D_3-invariant class algebra (center)*
- *All-commuting projectors and D_3-invariant characters*
- *Group invariant numbers: Centrum, Rank, and Order*



(Fig. 15.2.1 QTCA)

Lecture 14 ends p.93

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Group invariant numbers: Centrum, Rank, and Order

Fig. 3.1.1 PSDS

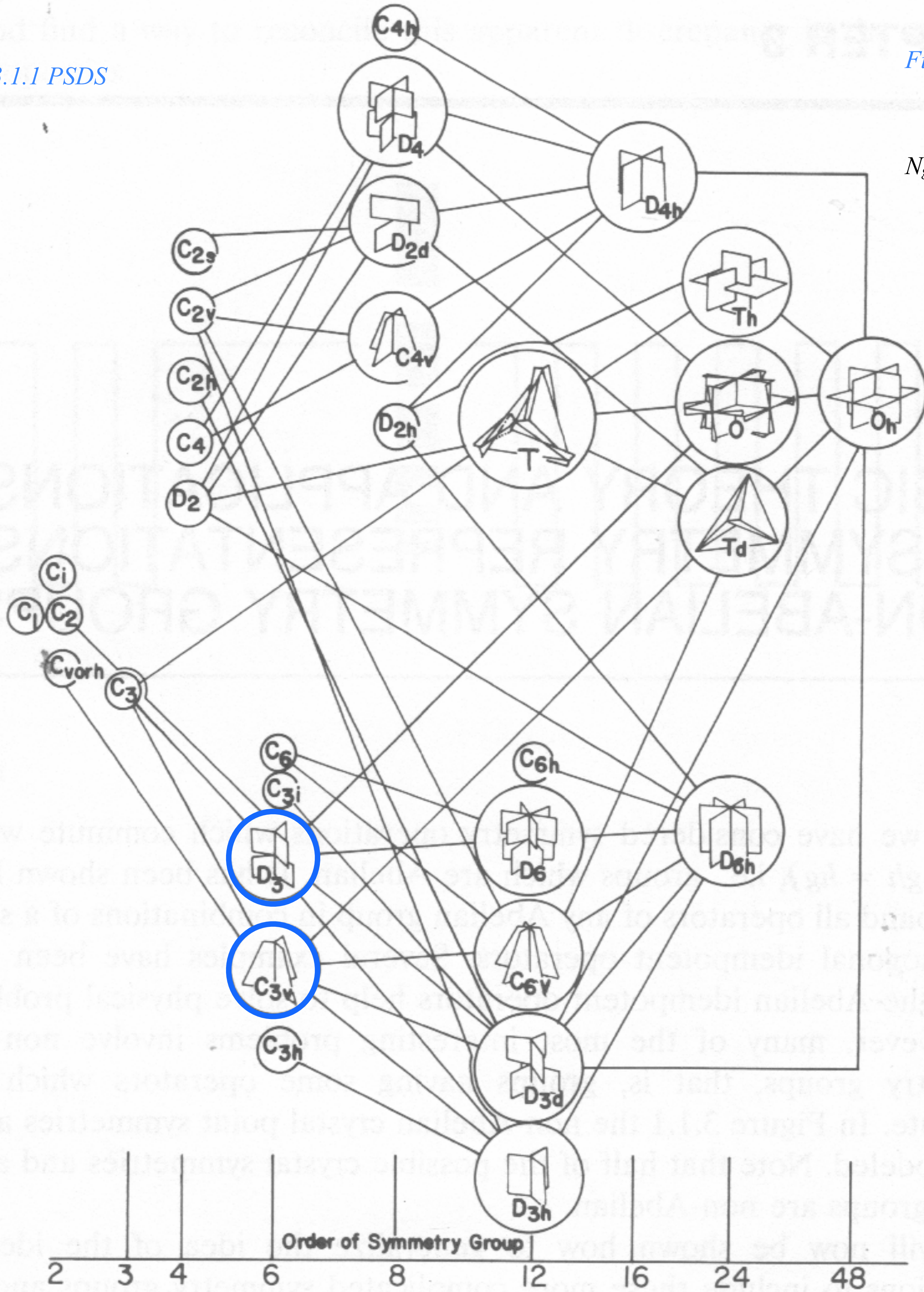
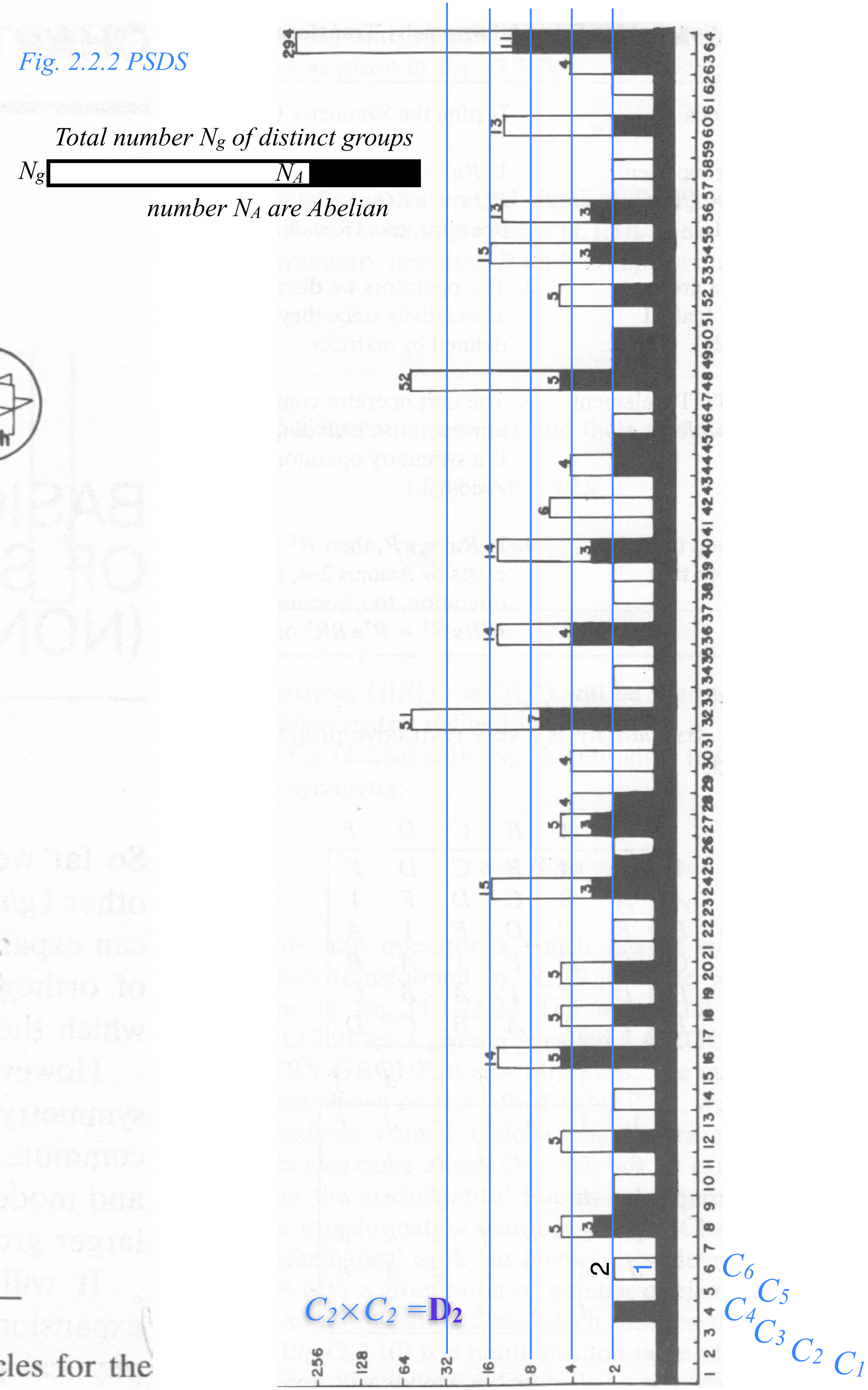
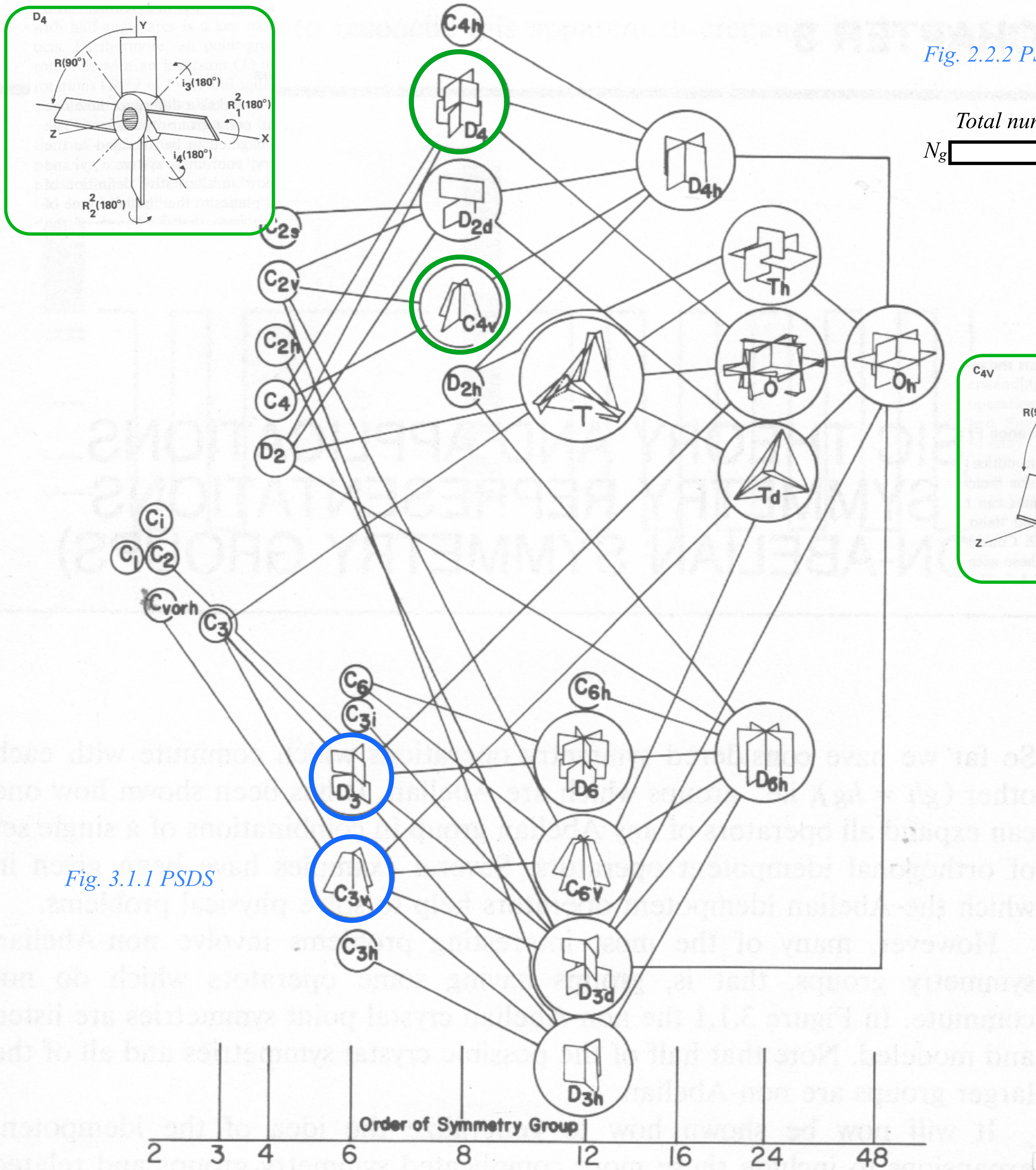


Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

Fig. 2.2.2 PSDS





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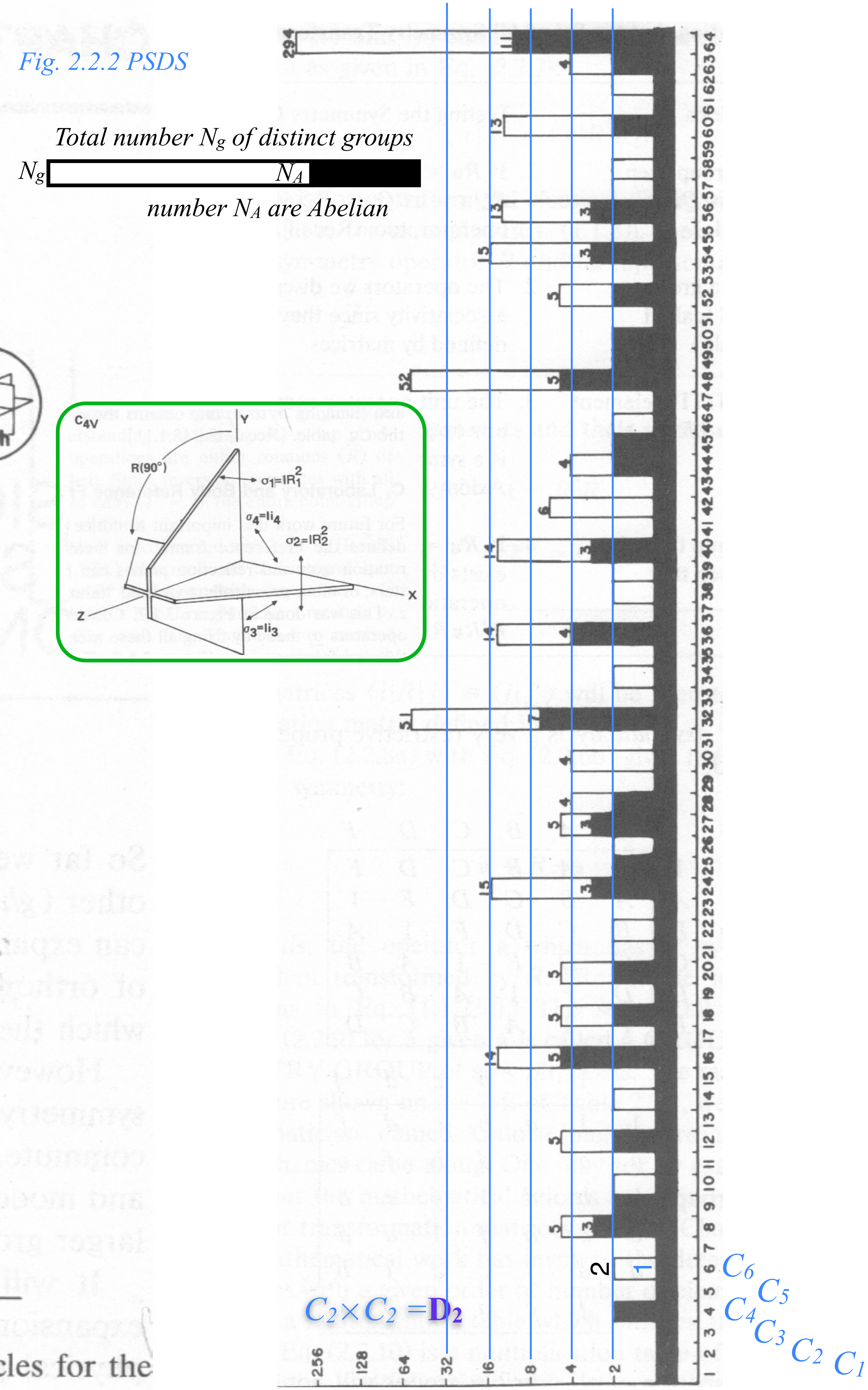

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Fig. 3.1.3 PSDS

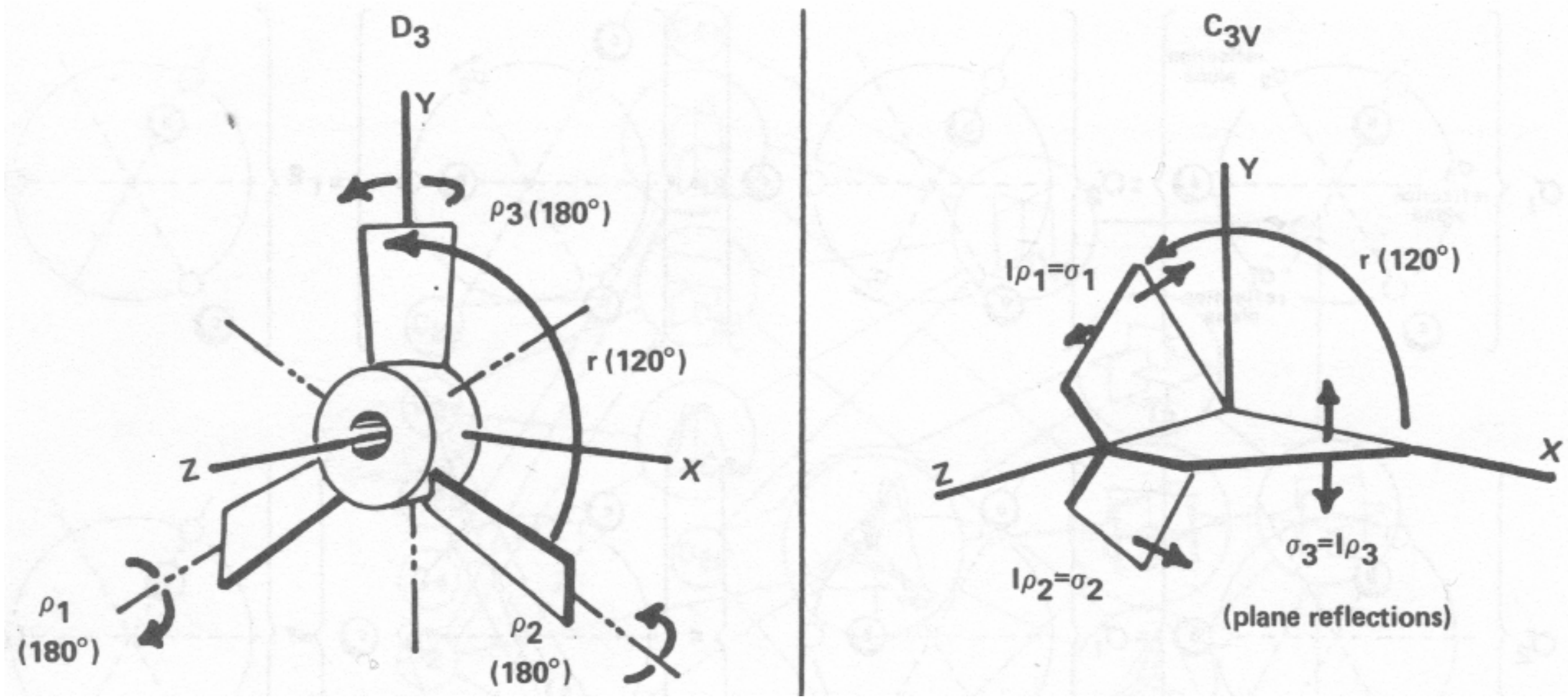
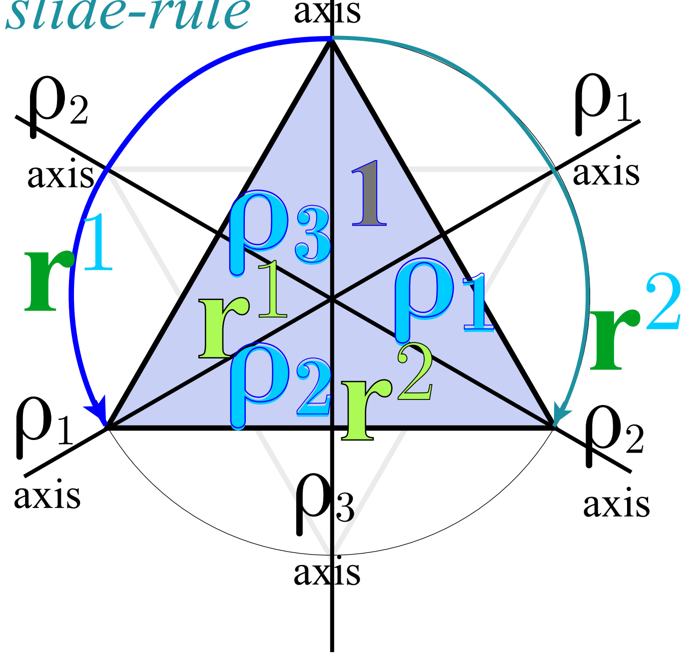
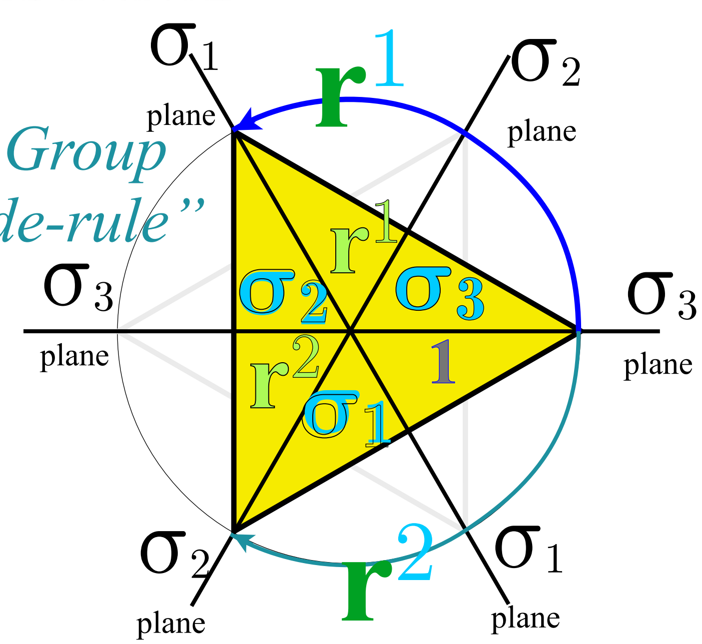


Figure 3.1.3 Pictorial comparison of D_3 and C_{3v} symmetry. A propeller having D_3 symmetry is shown next to a three-plane paddle having C_{3v} symmetry. The group operations are labeled by arrows, which indicate the effect they have. For example, ρ_3 is a 180° rotation around the y axis, while $I\rho_3 = \sigma_3$ is a reflection through the xz plane. (Here axes are fixed and the objects rotate.)

D_3 Group "slide-rule"



C_{3v} Group "slide-rule"



**isomorphic means mathematically the same abstract group even if physically different action.*

*Showing that D_3 and C_{3v} are **isomorphic*** ($D_3 \sim C_{3v}$ share product table)*

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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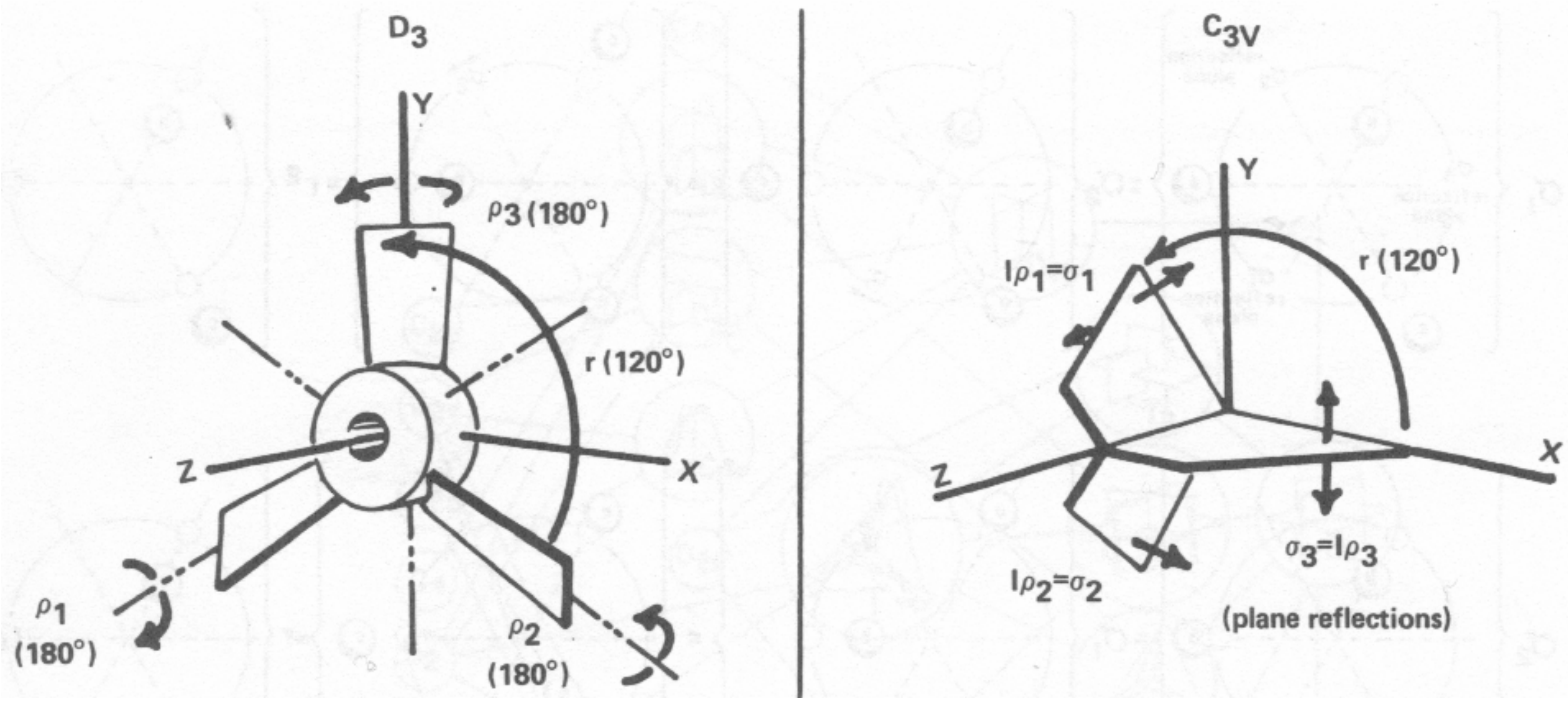

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$$180°\,D_3\text{-Y-axis-rotation: } \rho_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \quad \textit{maps to}: \text{ XZ-mirror-plane reflection: } \sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$$

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3-Dihedral-axes group D_3 *vs.* *3-Vertical-mirror-plane group C_{3v}*

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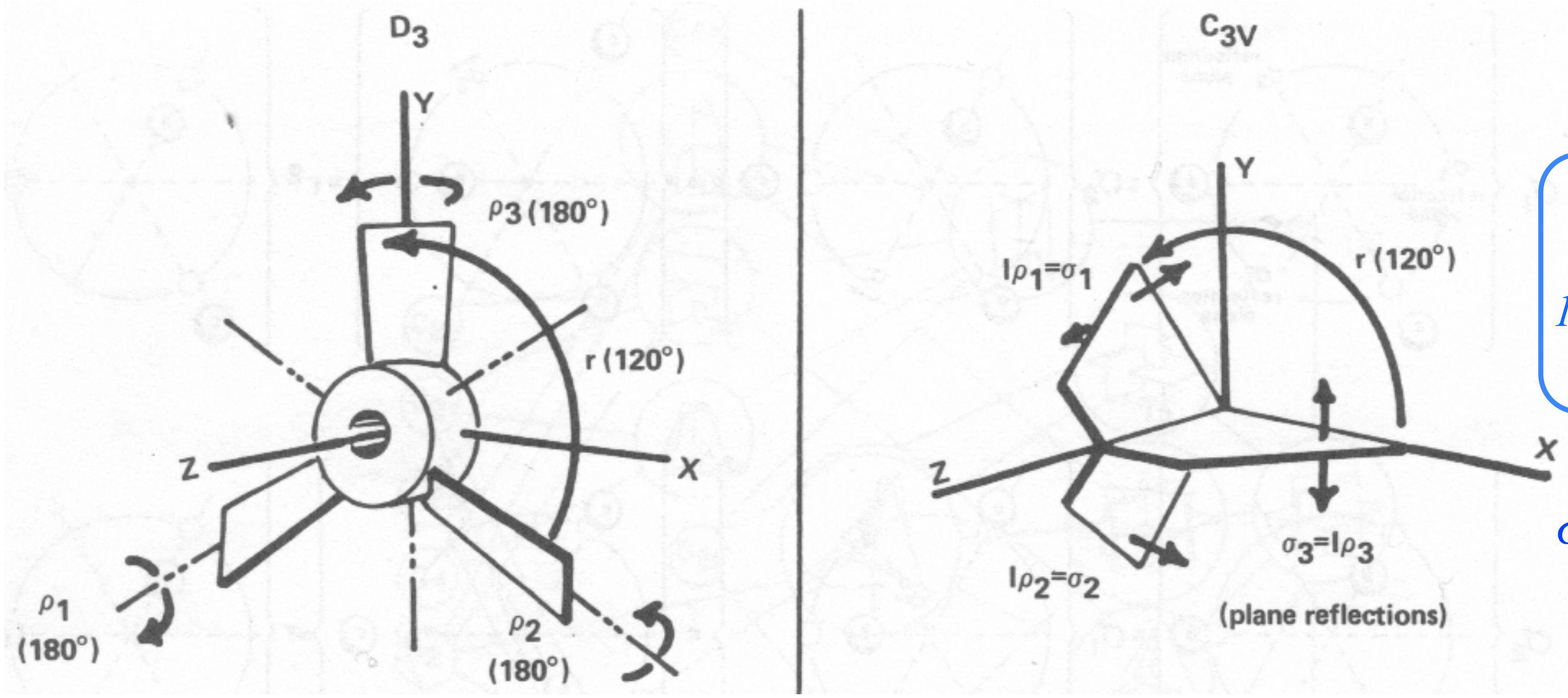

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$\boldsymbol{\sigma} = \mathbf{R}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{R}$

$$\sigma_3 = \begin{pmatrix} +1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & +1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & +1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}\begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix}$$

$$= \rho_3 \cdot \mathbf{I} = \mathbf{I} \cdot \rho_3$$

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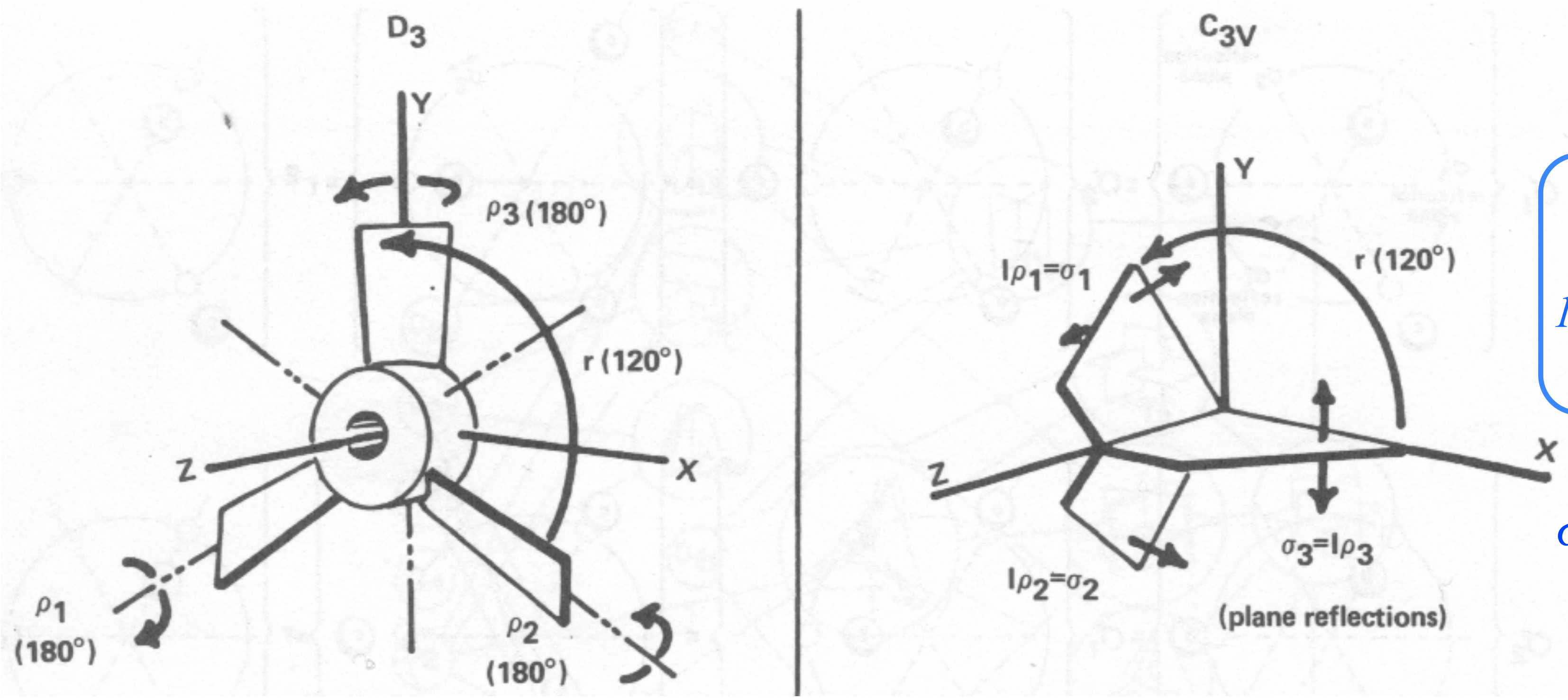

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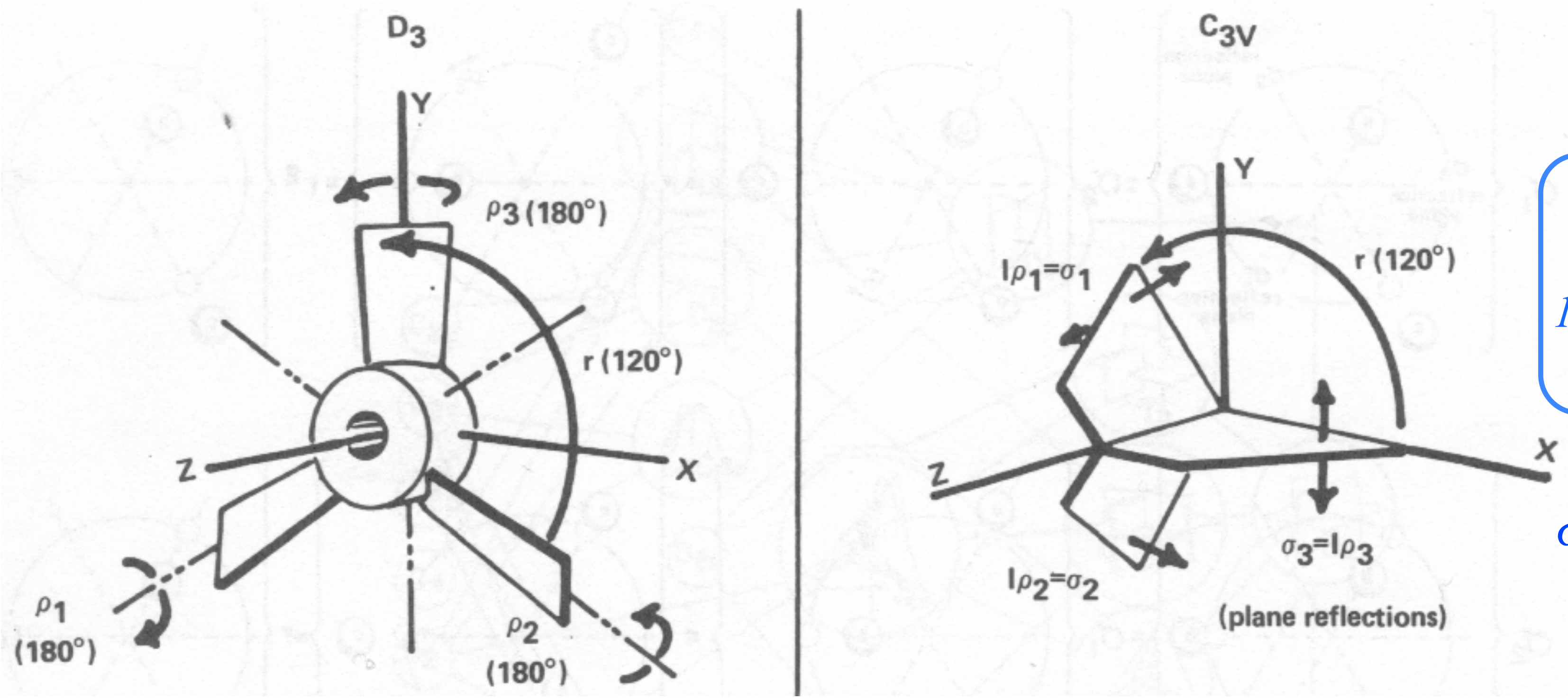

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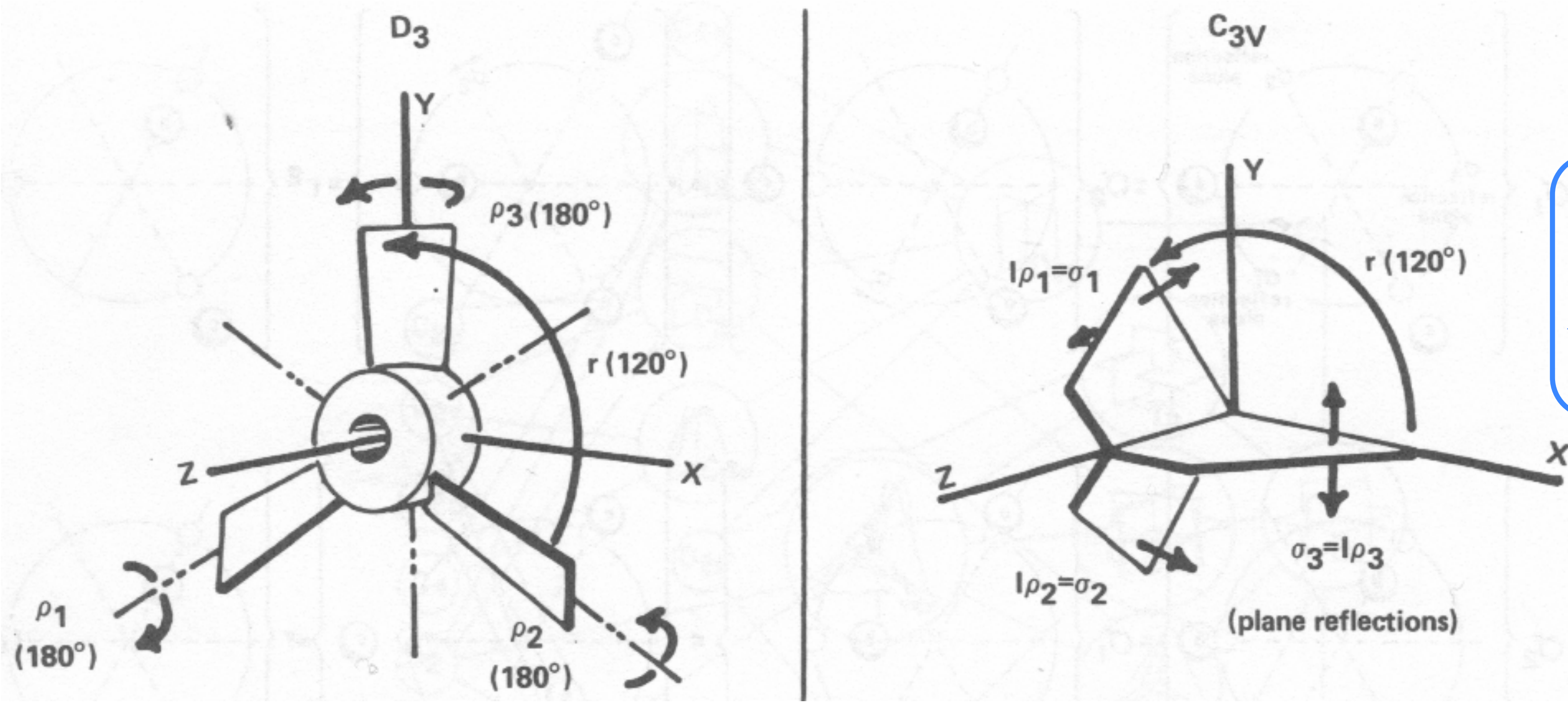

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$$180°D_3\text{-}\rho_1\text{-axis-rotation: } \rho_1 \quad \textit{maps to}: \perp\rho_1\text{-mirror-plane reflection: } \sigma_1 = \rho_1\cdot\mathbf{I} = \mathbf{I}\rho_1$$

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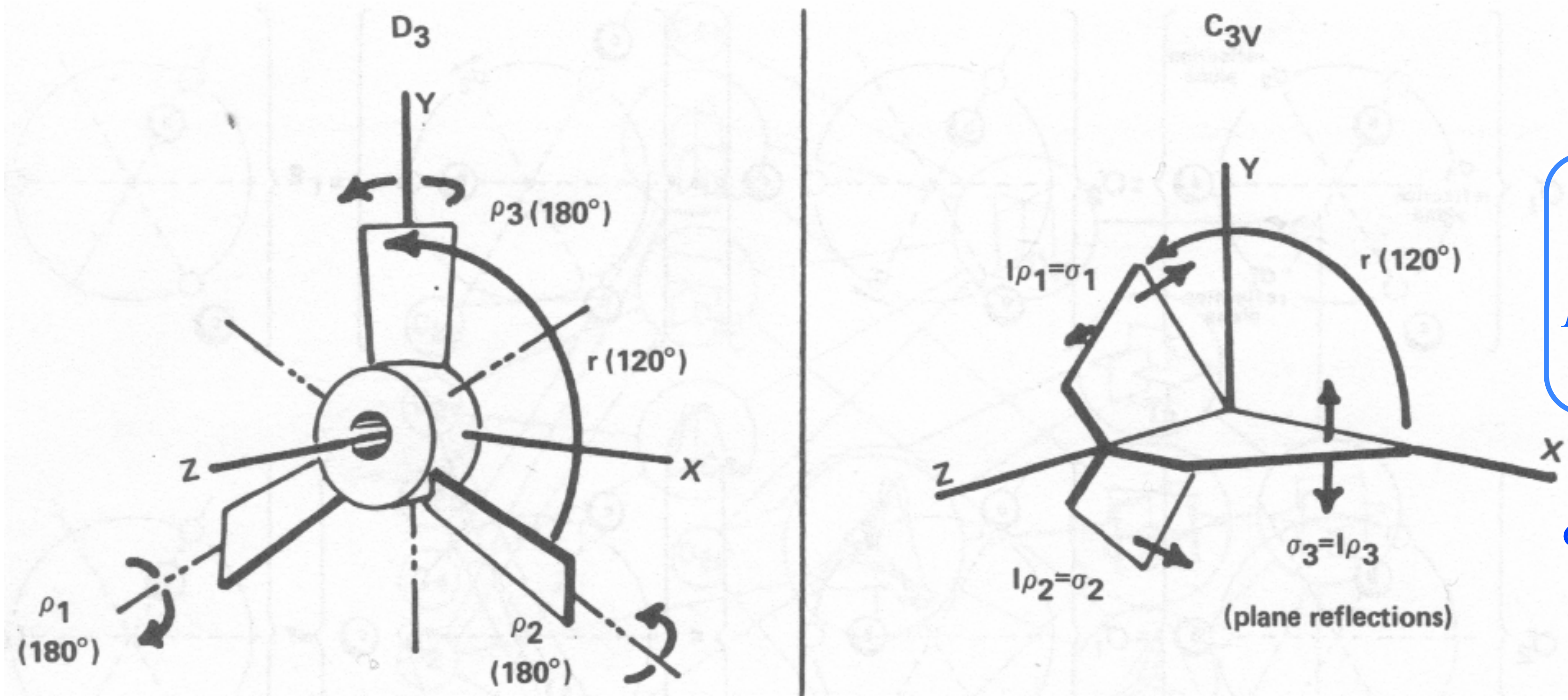

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$180°D_3$-ρ_1-axis-rotation: ρ_1 *maps to* : $\perp\rho_1$-mirror-plane reflection: $\sigma_1 = \rho_1\cdot\mathbf{I} = \mathbf{I}\cdot\rho_1$

D_3-product: $\rho_1\rho_2$ *maps to* : C_{3v}-product: $\sigma_1\sigma_2 = \rho_1\mathbf{I}\mathbf{I}\rho_2 = \rho_1\rho_2$

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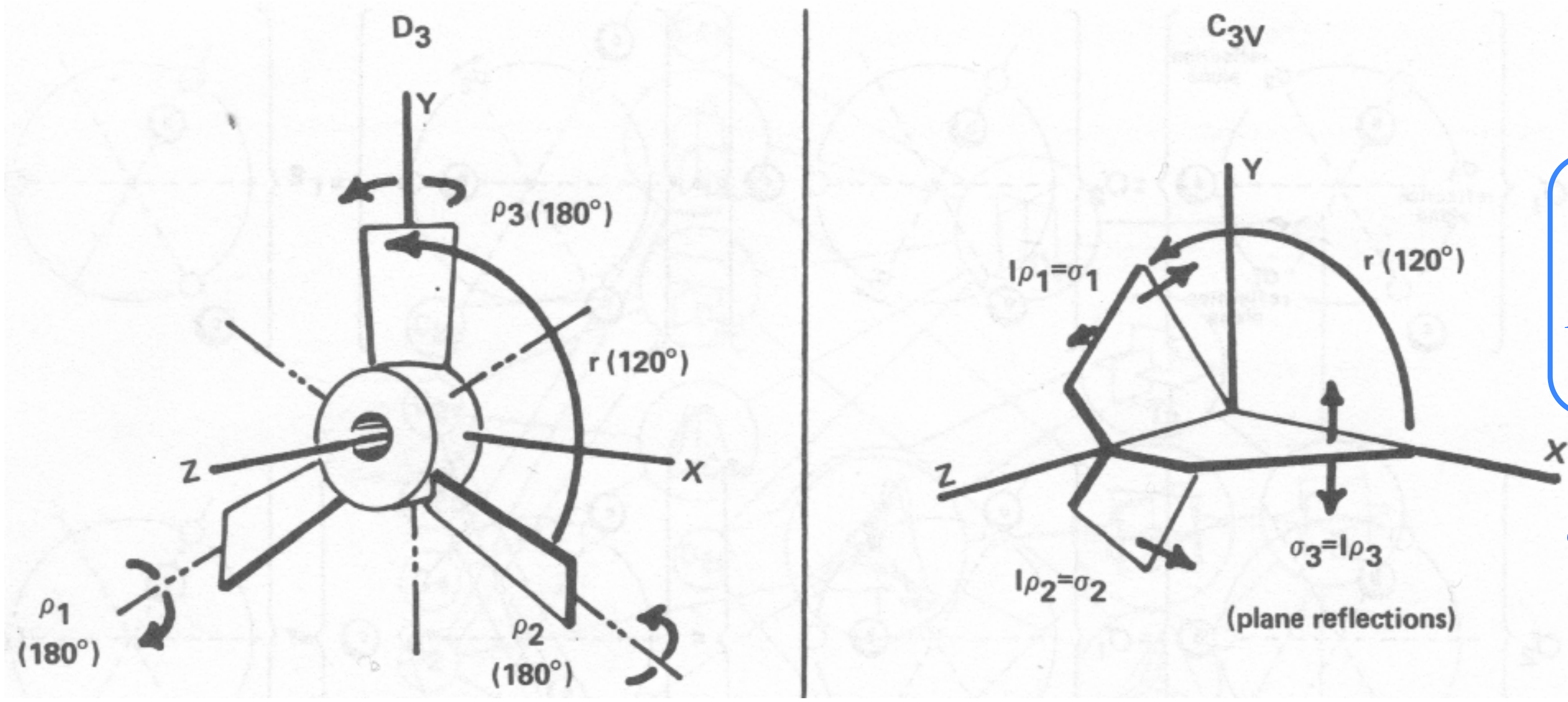

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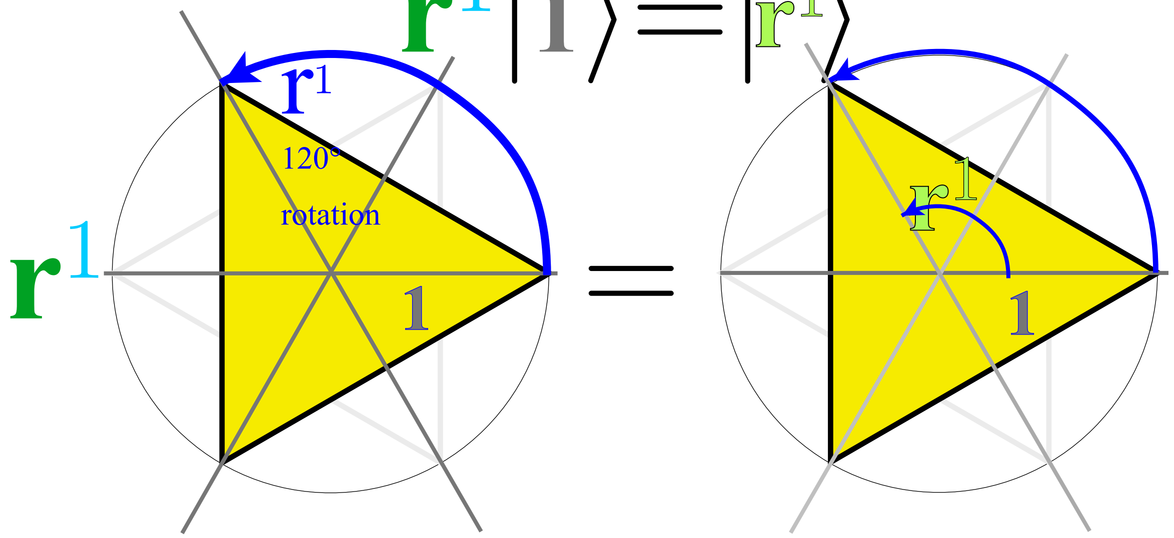

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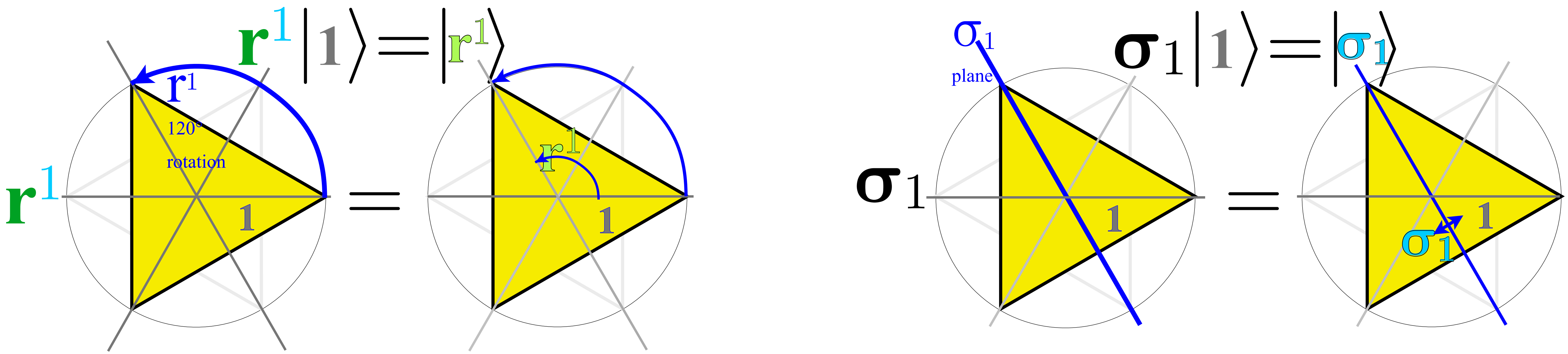

1

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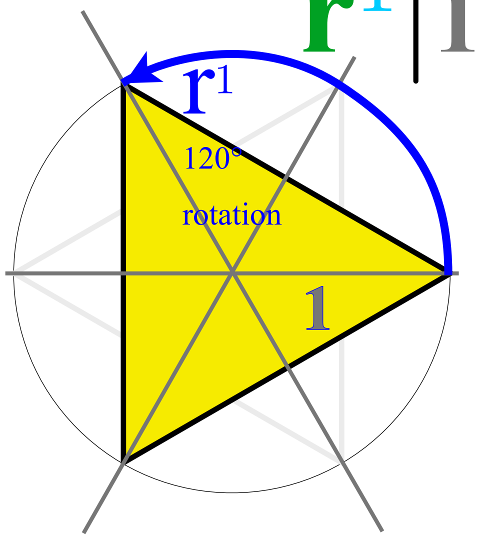
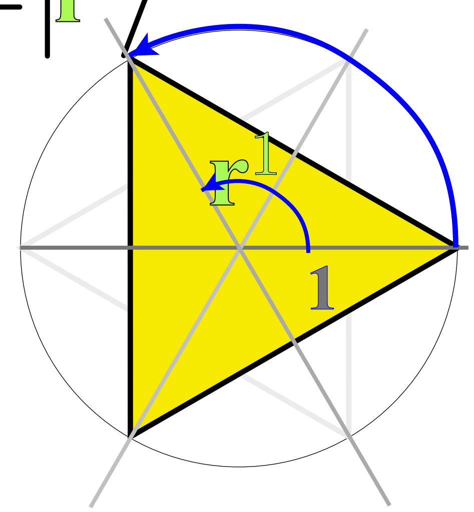
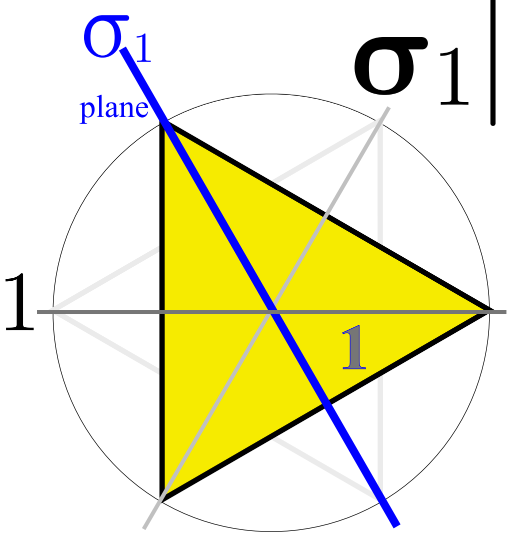
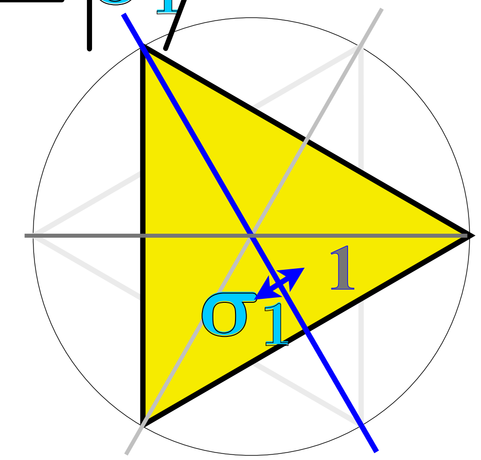
$\mathbf{r}^1$ 

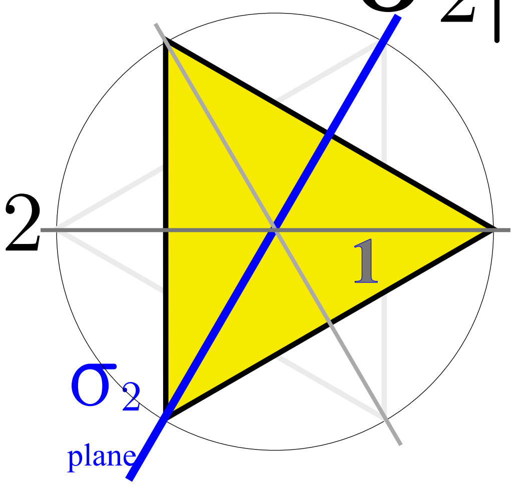
$=$ 

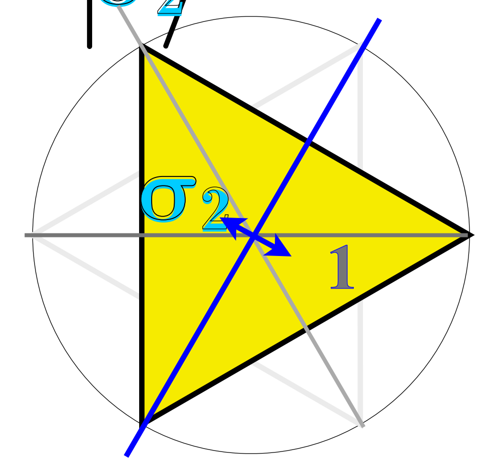
$$\boldsymbol{\sigma}_1|1\rangle = |\boldsymbol{\sigma}_1\rangle$$

$\boldsymbol{\sigma}_1$ 

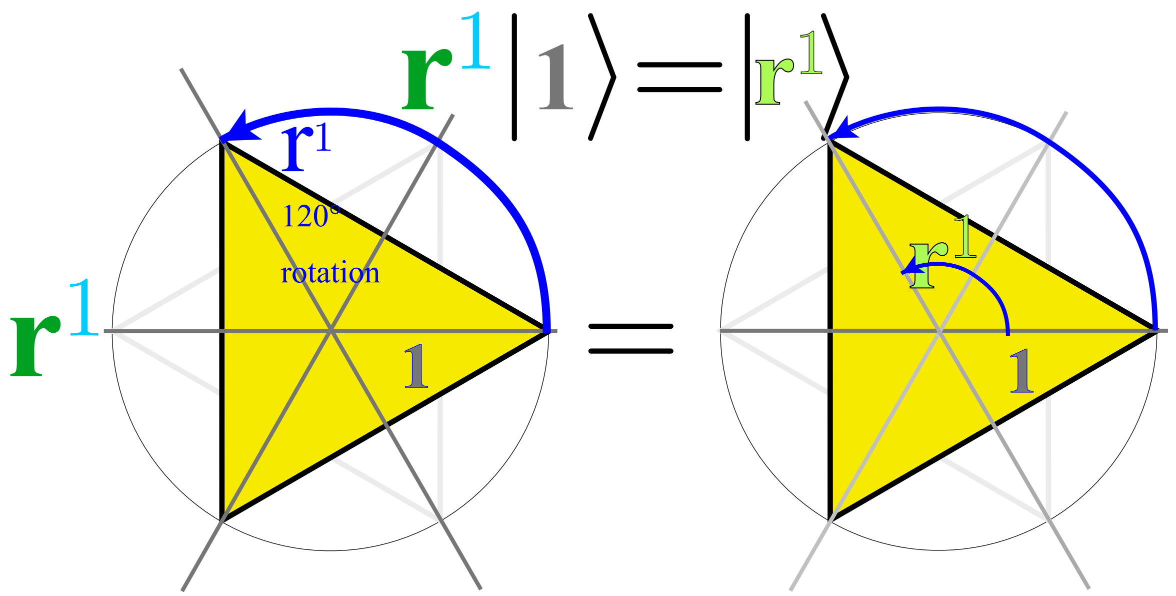
$=$ 

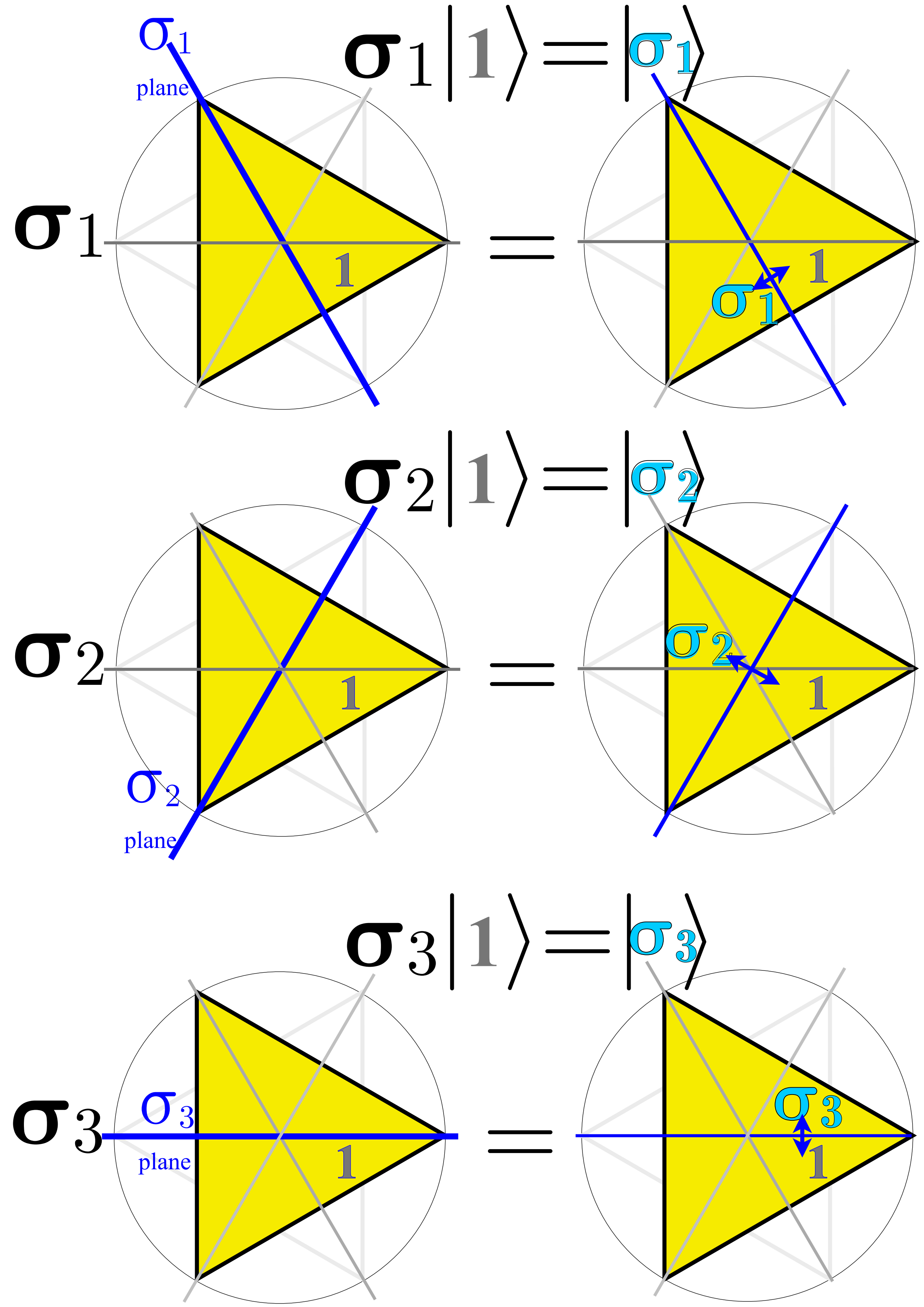
$$\boldsymbol{\sigma}_2|1\rangle = |\boldsymbol{\sigma}_2\rangle$$

$\boldsymbol{\sigma}_2$ 

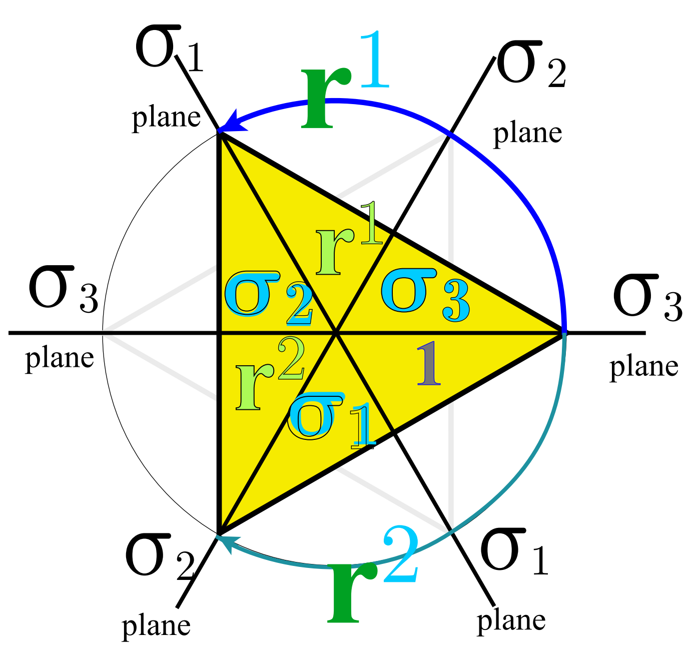
$=$ 

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





Building
C_{3v} Group
"slide-rule"



3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

Group theory of equivalence transformations and classes

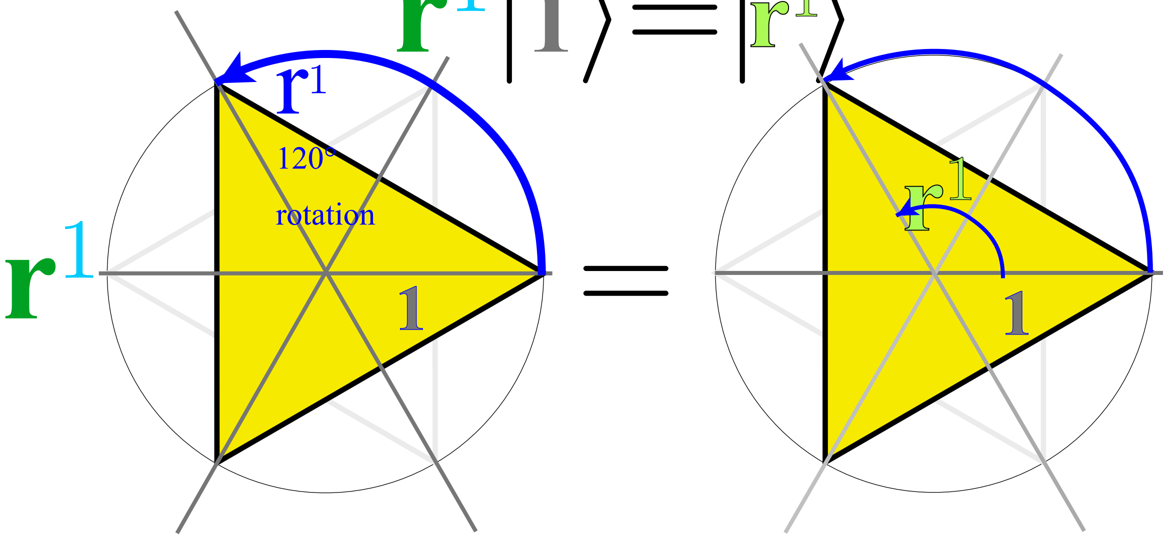
Lagrange theorems

All-commuting operators and D_3-invariant class algebra

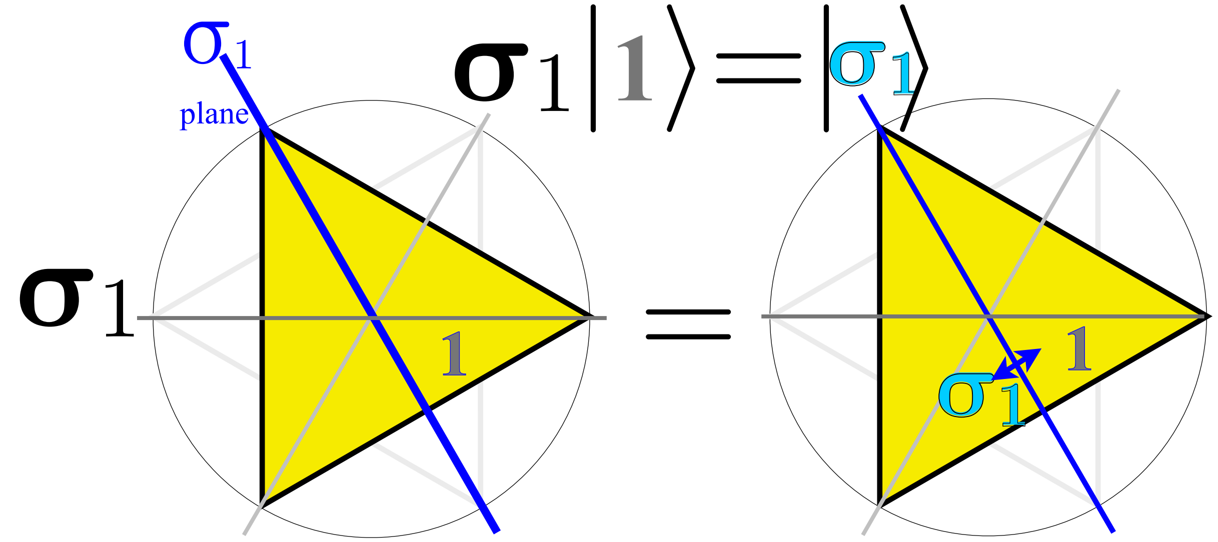
All-commuting projectors and D_3-invariant characters

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

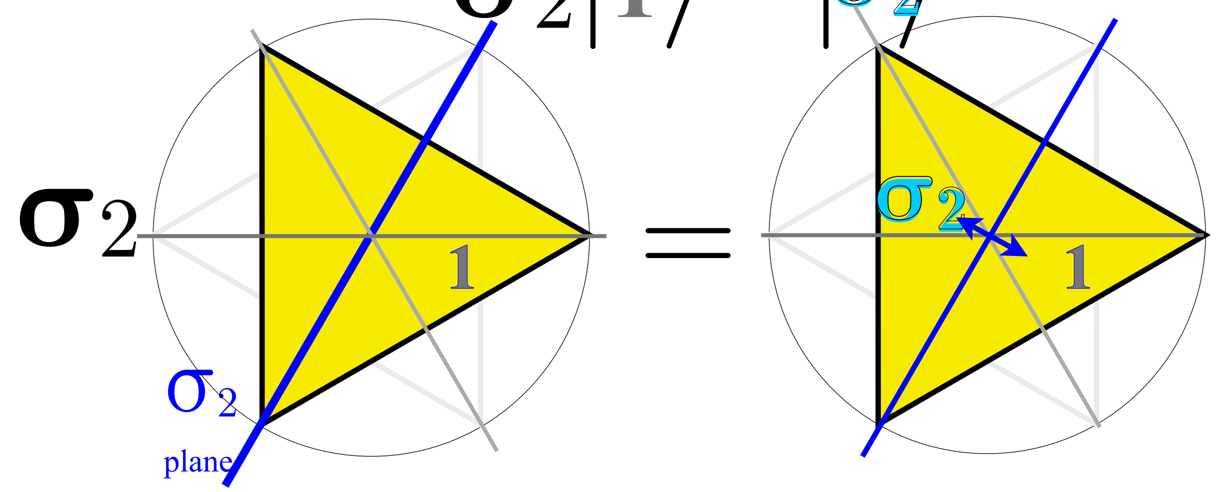
$$\mathbf{r}^1|1\rangle = |\mathbf{r}^1\rangle$$



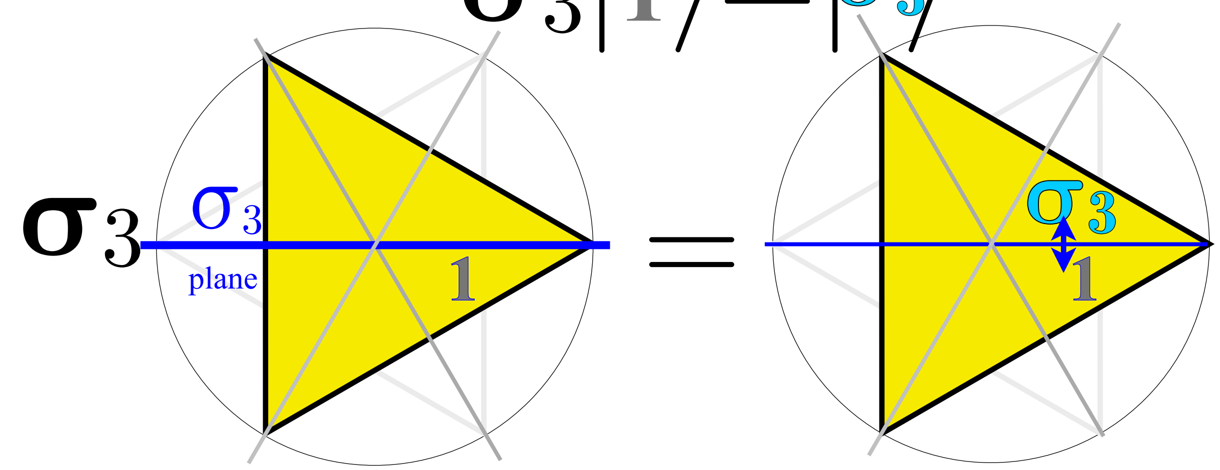
$$\boldsymbol{\sigma}_1|1\rangle = |\boldsymbol{\sigma}_1\rangle$$



$$\boldsymbol{\sigma}_2|1\rangle = |\boldsymbol{\sigma}_2\rangle$$

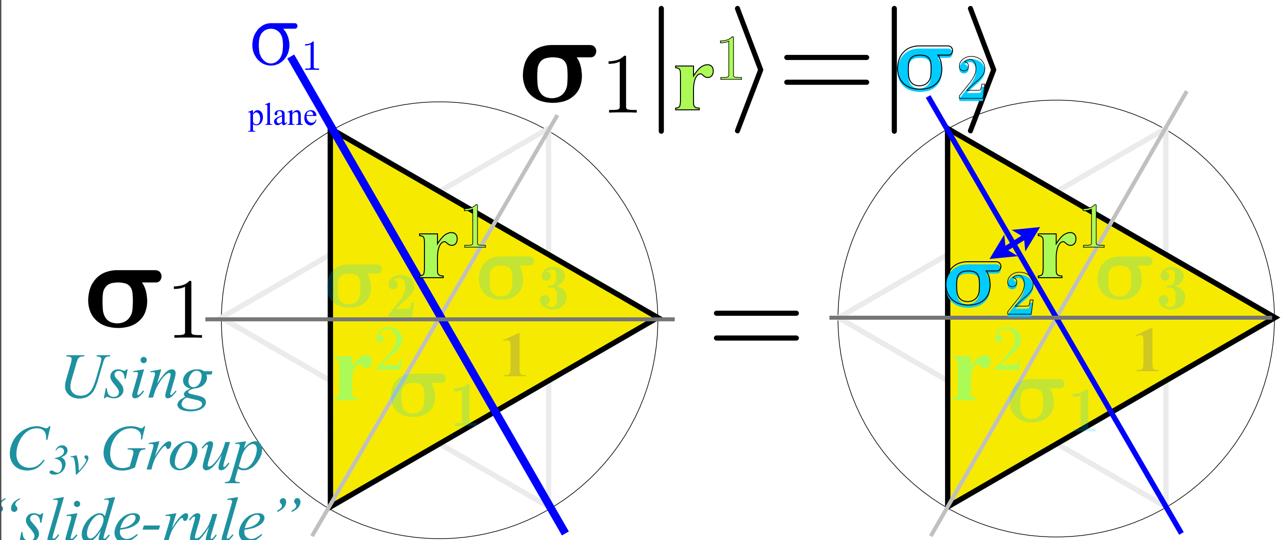


$$\boldsymbol{\sigma}_3|1\rangle = |\boldsymbol{\sigma}_3\rangle$$



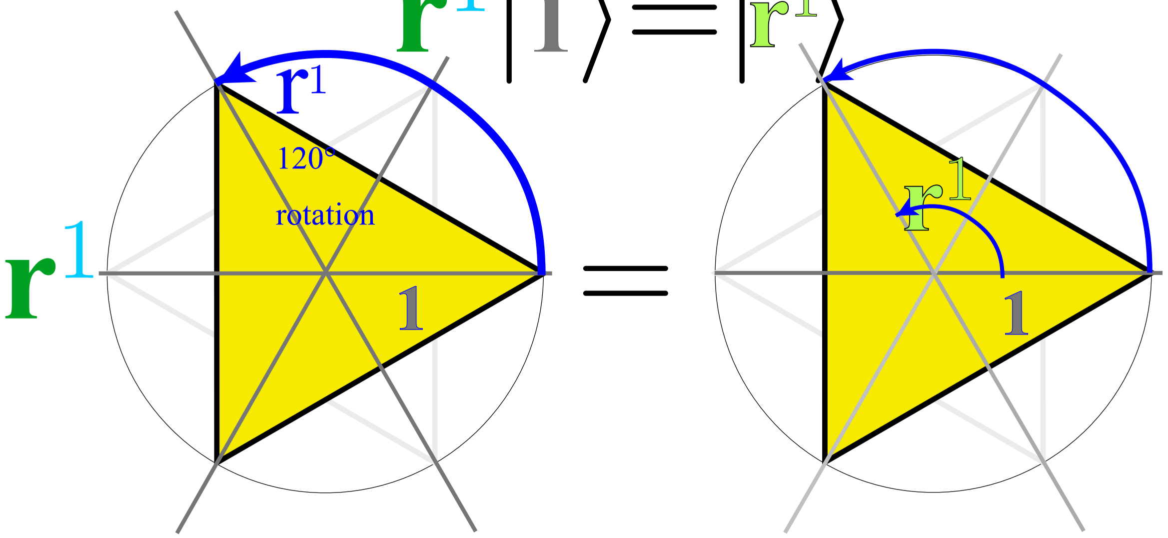
Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$

$$\boldsymbol{\sigma}_1|\mathbf{r}^1\rangle = |\boldsymbol{\sigma}_2\rangle$$

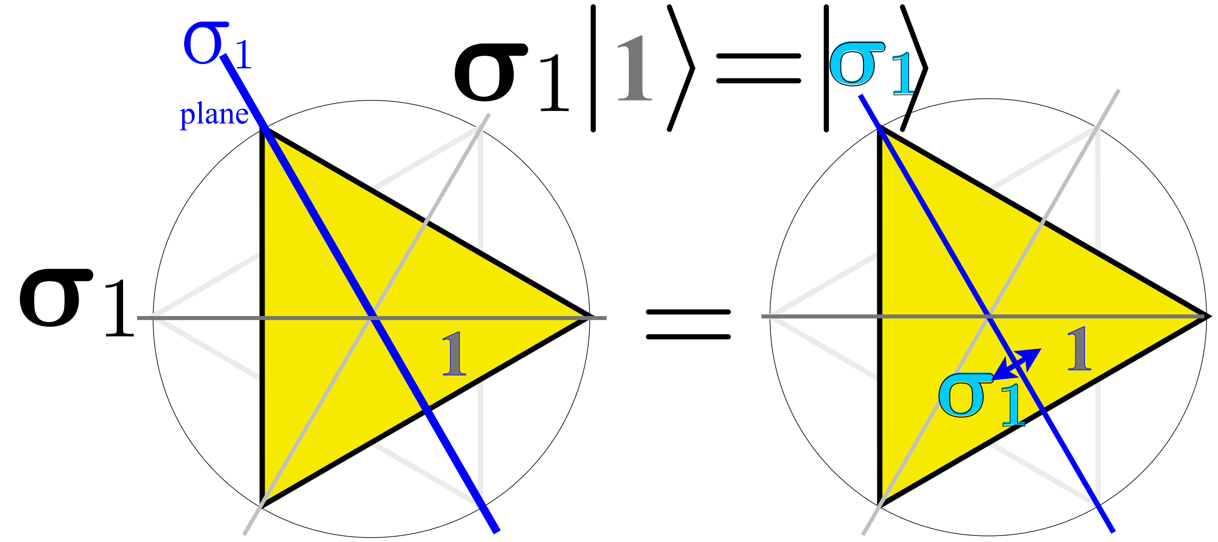


Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

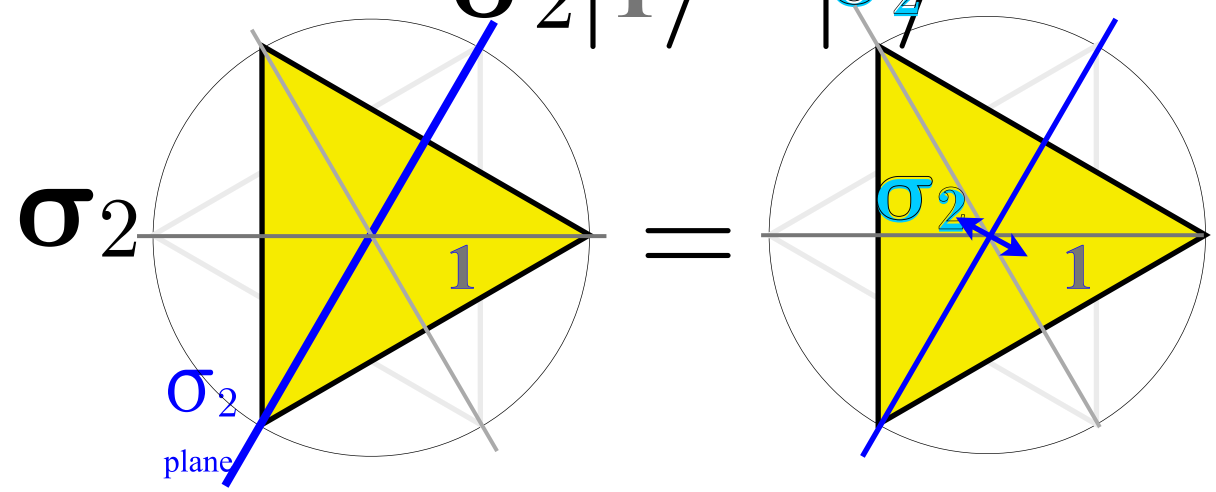
$$\mathbf{r}^1|1\rangle = |\mathbf{r}^1\rangle$$



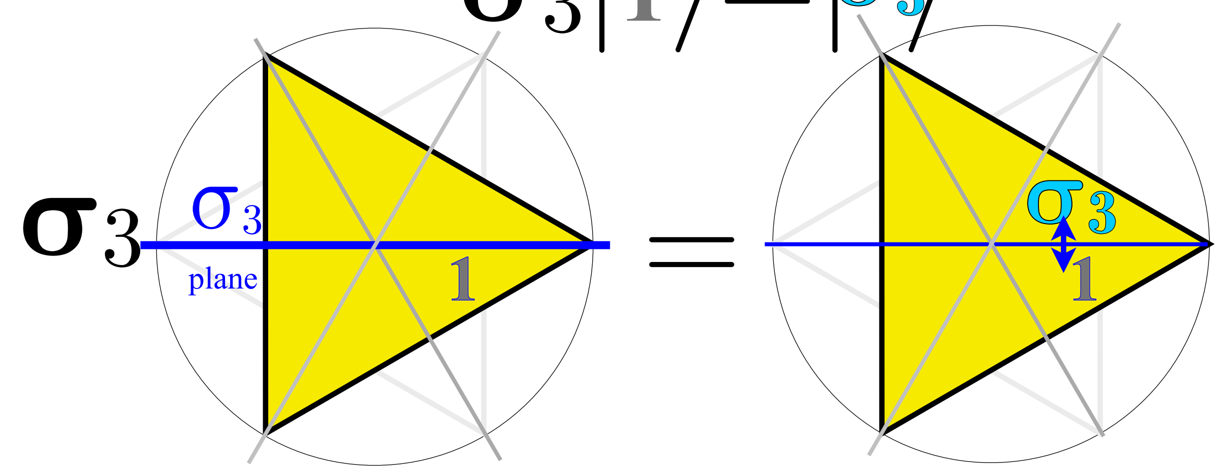
$$\boldsymbol{\sigma}_1|1\rangle = |\boldsymbol{\sigma}_1\rangle$$



$$\boldsymbol{\sigma}_2|1\rangle = |\boldsymbol{\sigma}_2\rangle$$

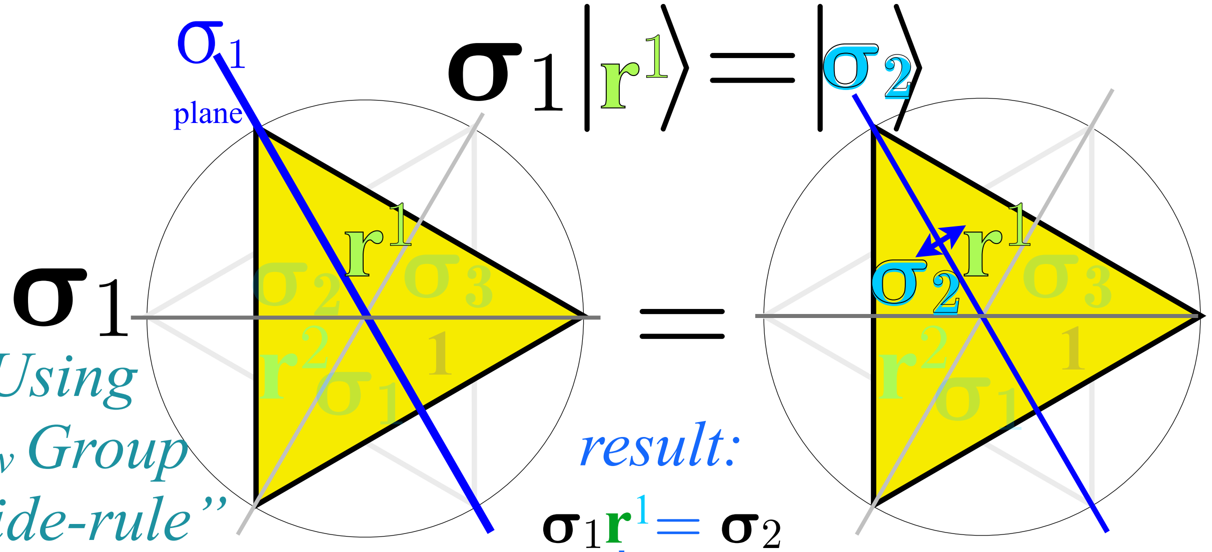


$$\boldsymbol{\sigma}_3|1\rangle = |\boldsymbol{\sigma}_3\rangle$$



Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$

$$\boldsymbol{\sigma}_1|\mathbf{r}^1\rangle = |\boldsymbol{\sigma}_2\rangle$$



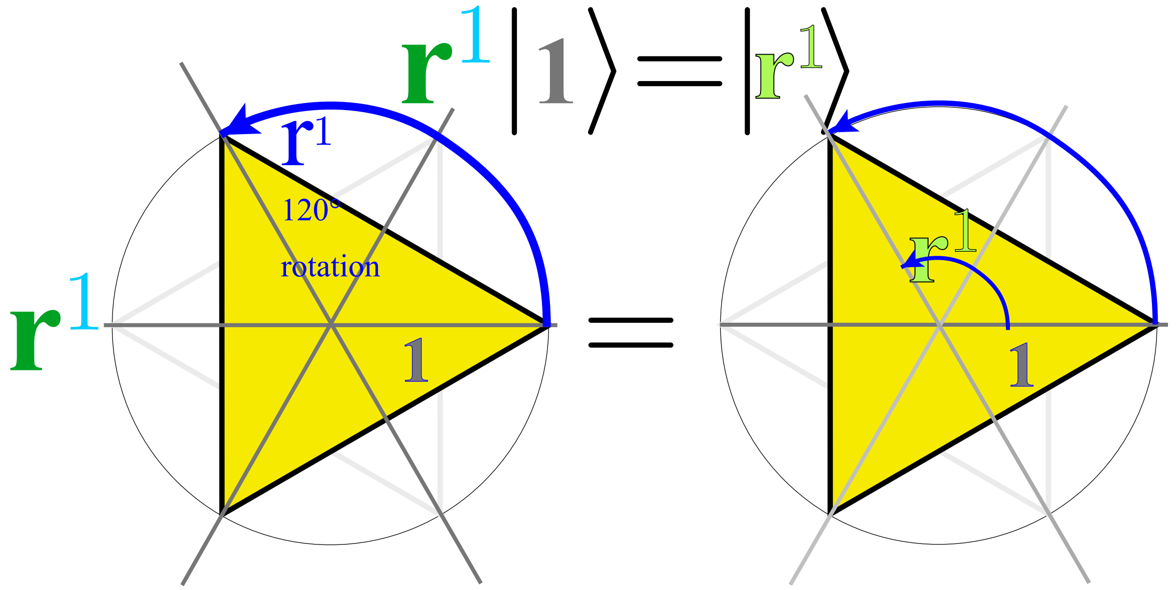
Using C_{3v} Group "slide-rule"

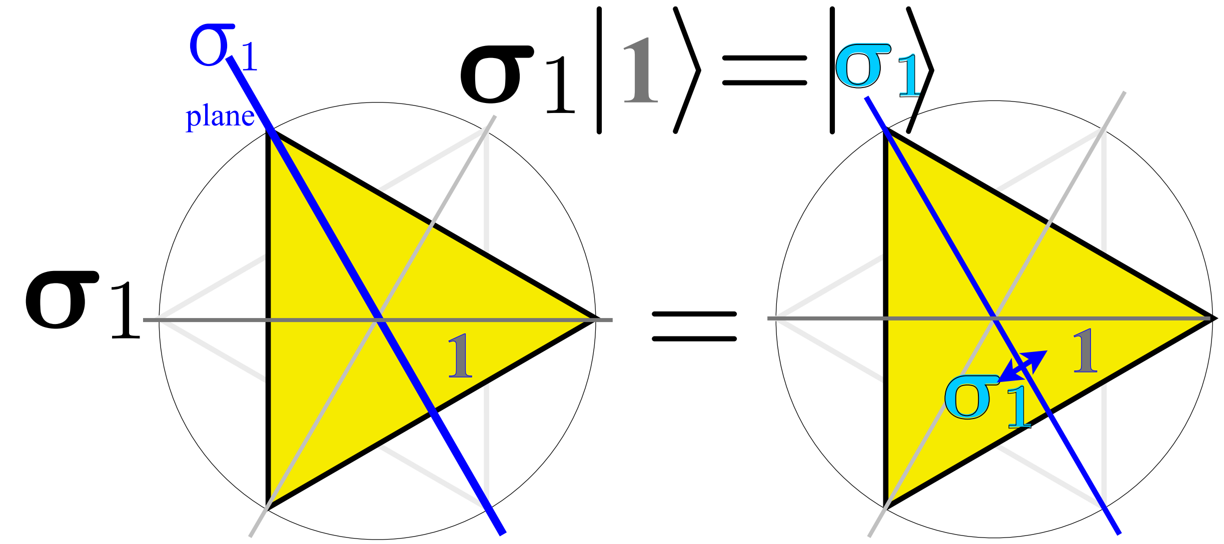
result:
$\boldsymbol{\sigma}_1\mathbf{r}^1 = \boldsymbol{\sigma}_2$

Factor $\mathbf{r}^1$ *on right acts first*

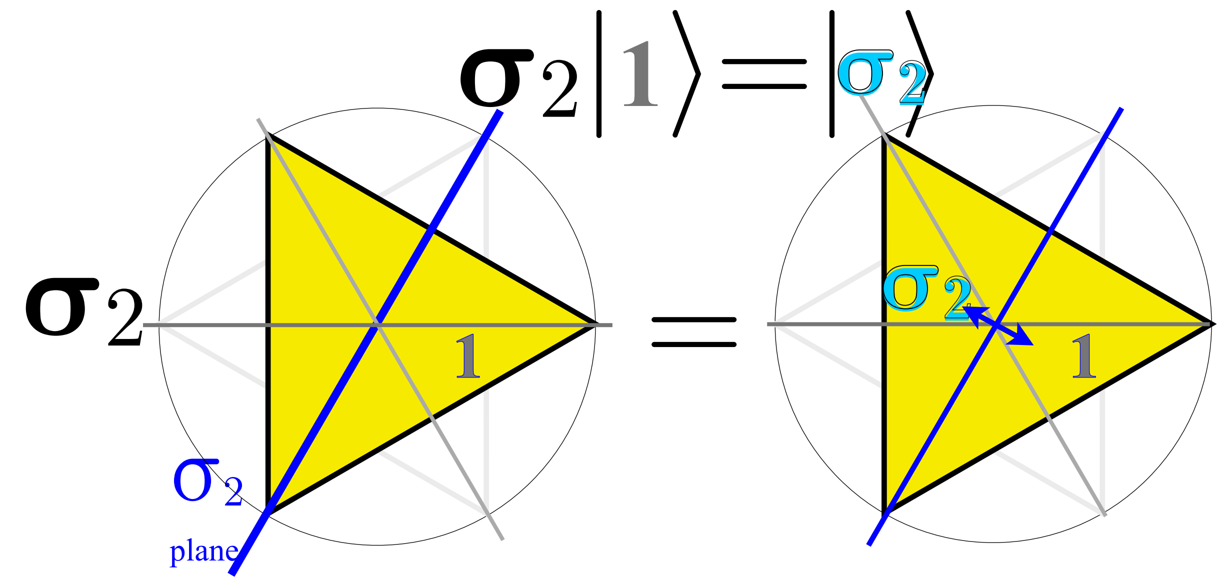
<u>left</u> is <u>last</u>
(like Hebrew)

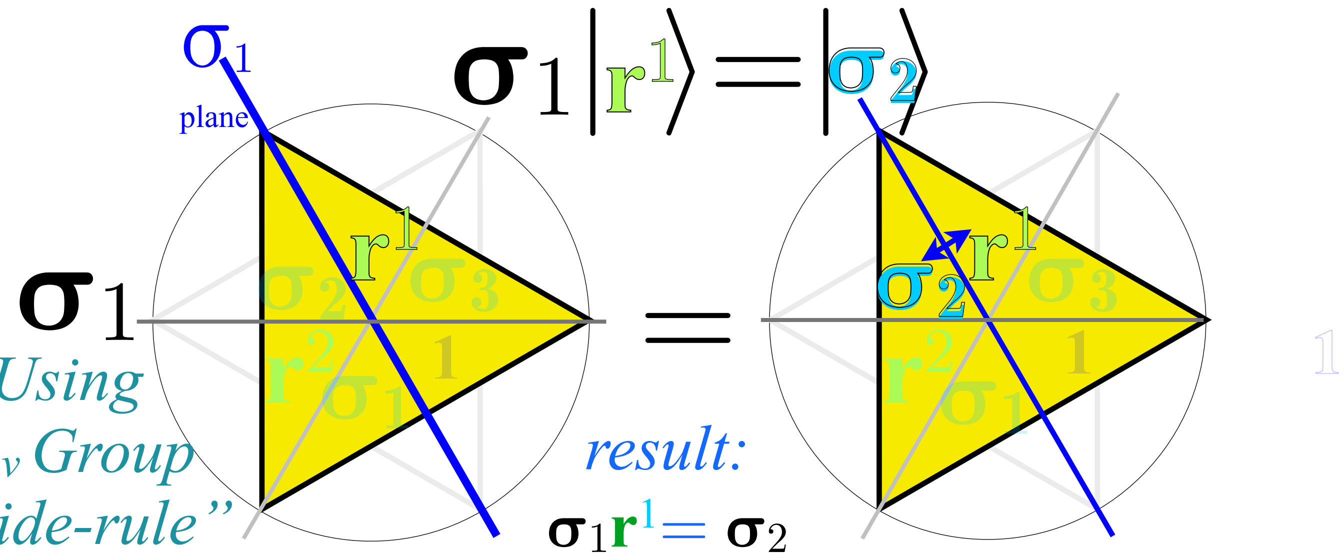
Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





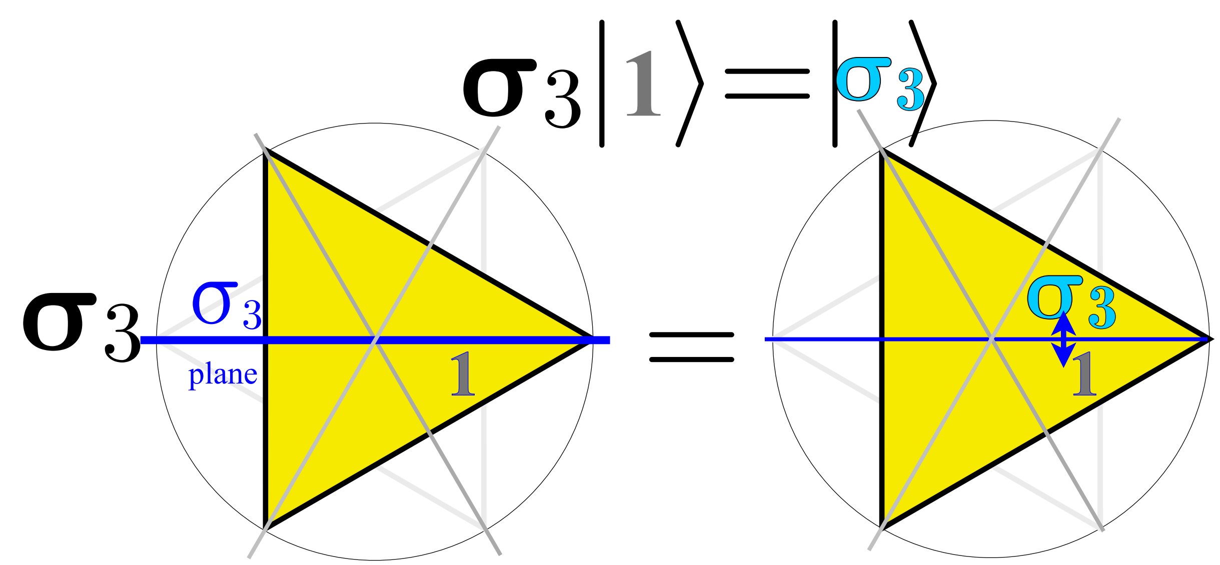
Example: Find C_{3v} product $\boldsymbol{\sigma}_1\mathbf{r}^1|1\rangle = \boldsymbol{\sigma}_1|\mathbf{r}^1\rangle$





Using C_{3v} Group "slide-rule"

result: $\boldsymbol{\sigma}_1\mathbf{r}^1 = \boldsymbol{\sigma}_2$



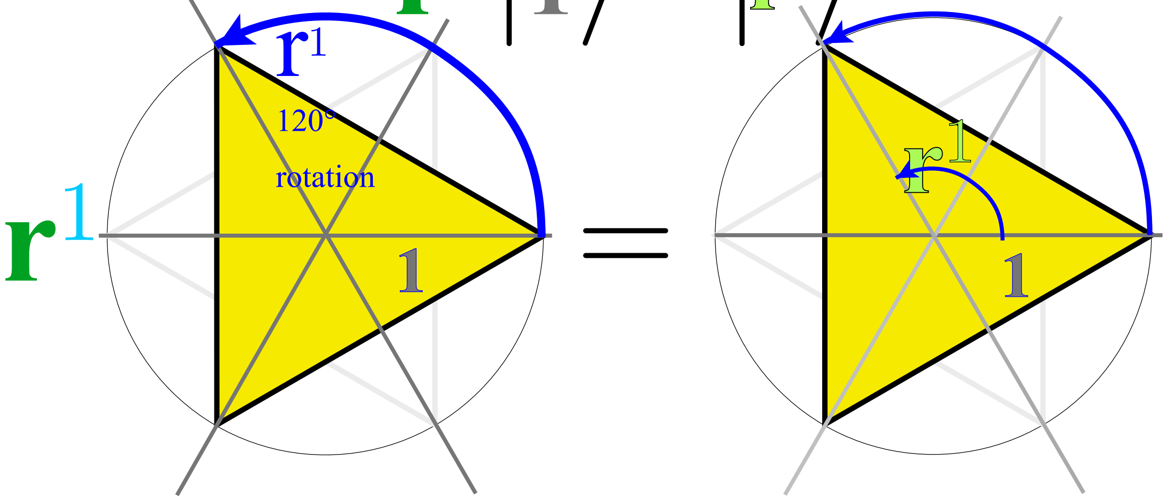
Other $\boldsymbol{\sigma}_1$ results from graph:

$\boldsymbol{\sigma}_1\{\mathbf{1},\ \mathbf{r}^1,\ \mathbf{r}^2,\ \boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \boldsymbol{\sigma}_3\}$

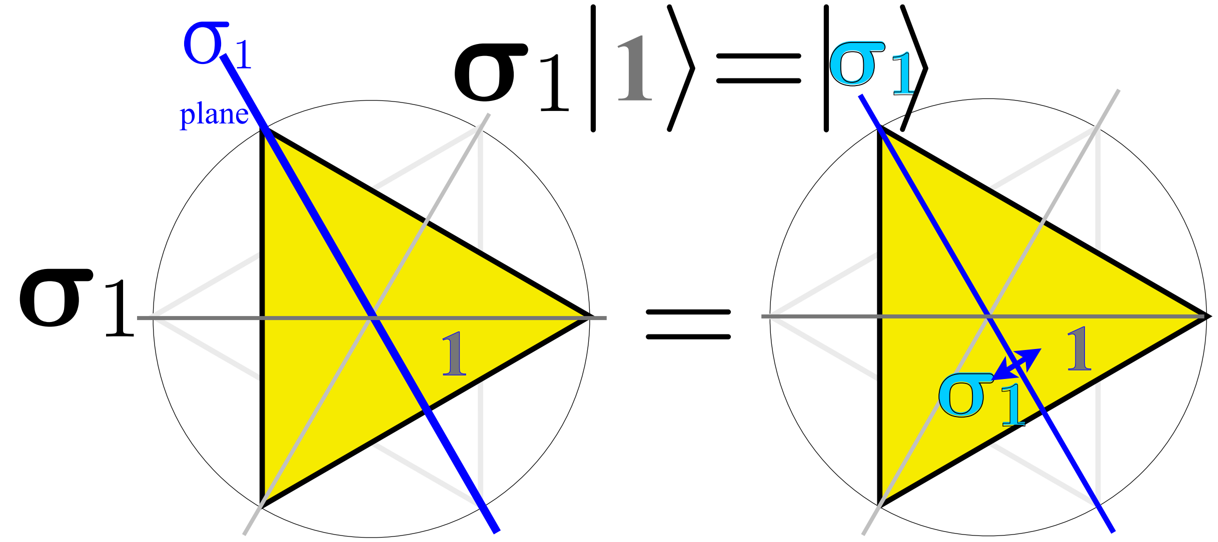
$= \{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \mathbf{1},\ \mathbf{r}^1,\ \mathbf{r}^2\}$

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

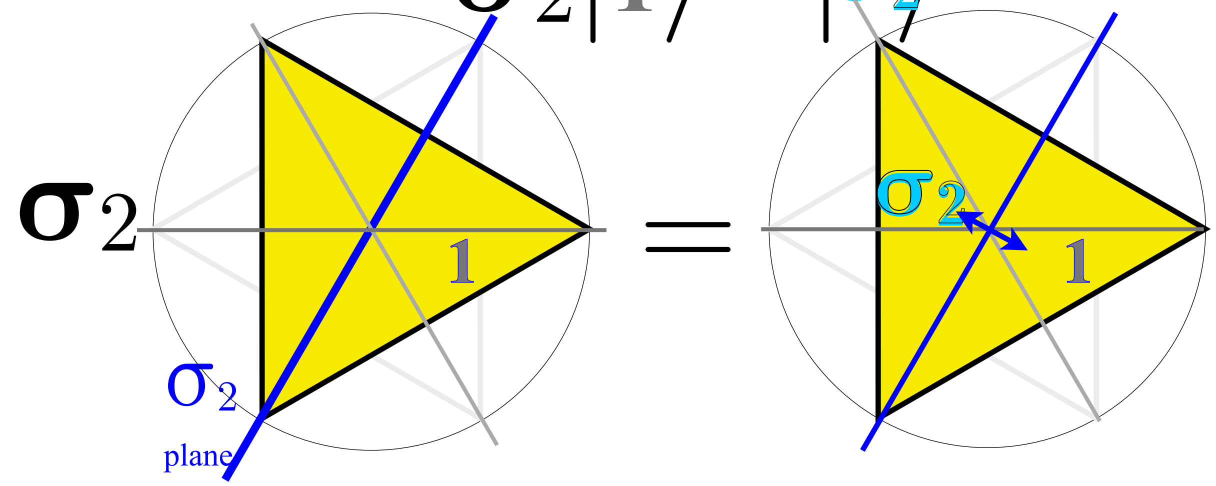
$$\mathbf{r}^1|1\rangle = |\mathbf{r}^1\rangle$$



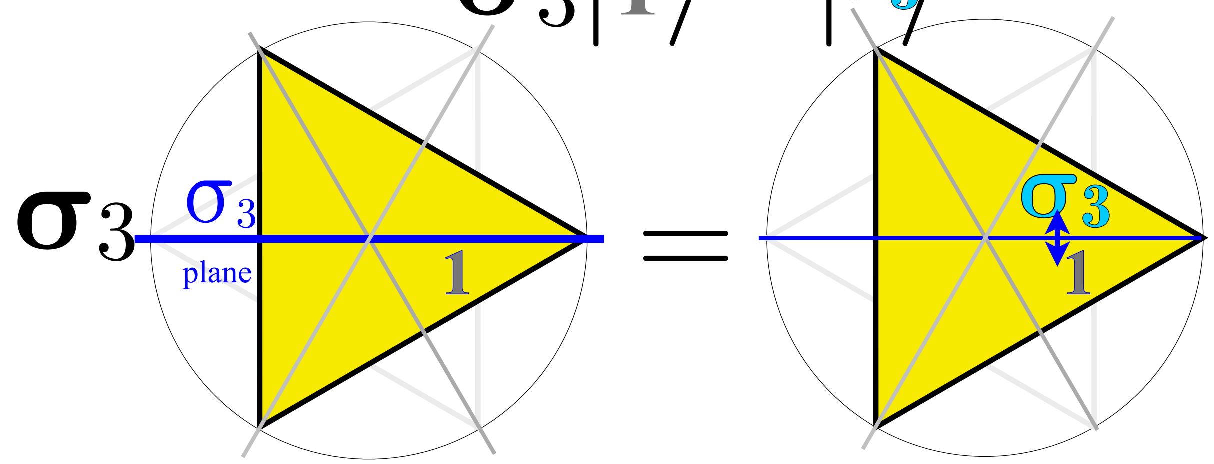
$$\sigma_1|1\rangle = |\sigma_1\rangle$$



$$\sigma_2|1\rangle = |\sigma_2\rangle$$

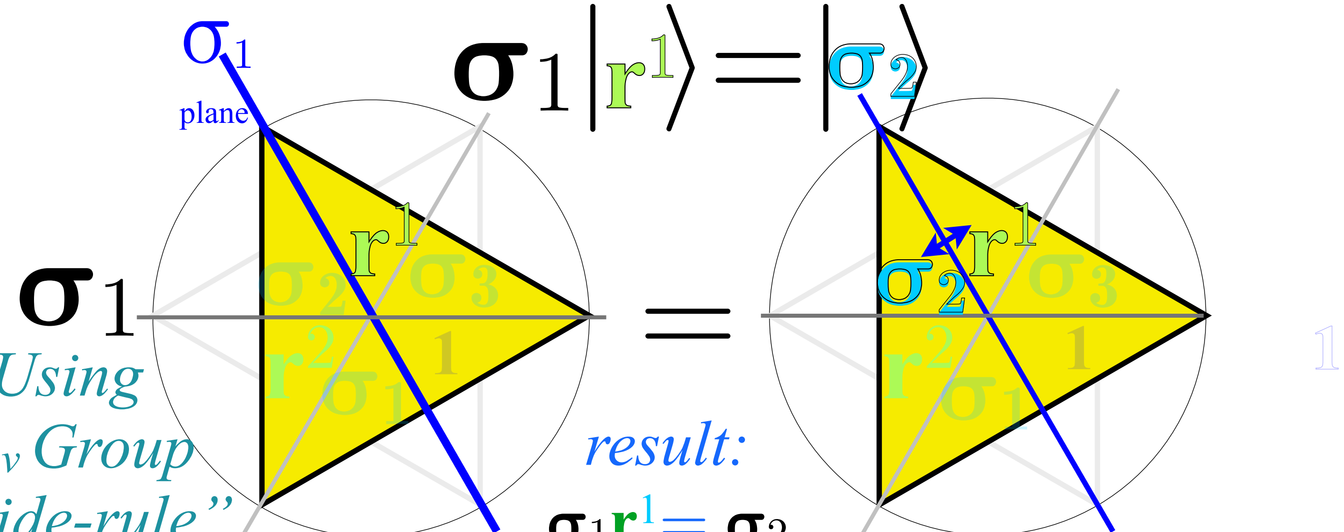


$$\sigma_3|1\rangle = |\sigma_3\rangle$$



Example: Find C_{3v} product $\sigma_1\mathbf{r}^1|1\rangle = \sigma_1|\mathbf{r}^1\rangle$

$$\sigma_1|\mathbf{r}^1\rangle = |\sigma_2\rangle$$



Using C_{3v} Group "slide-rule"

result: $\sigma_1\mathbf{r}^1 = \sigma_2$

Other σ_1 results from graph:

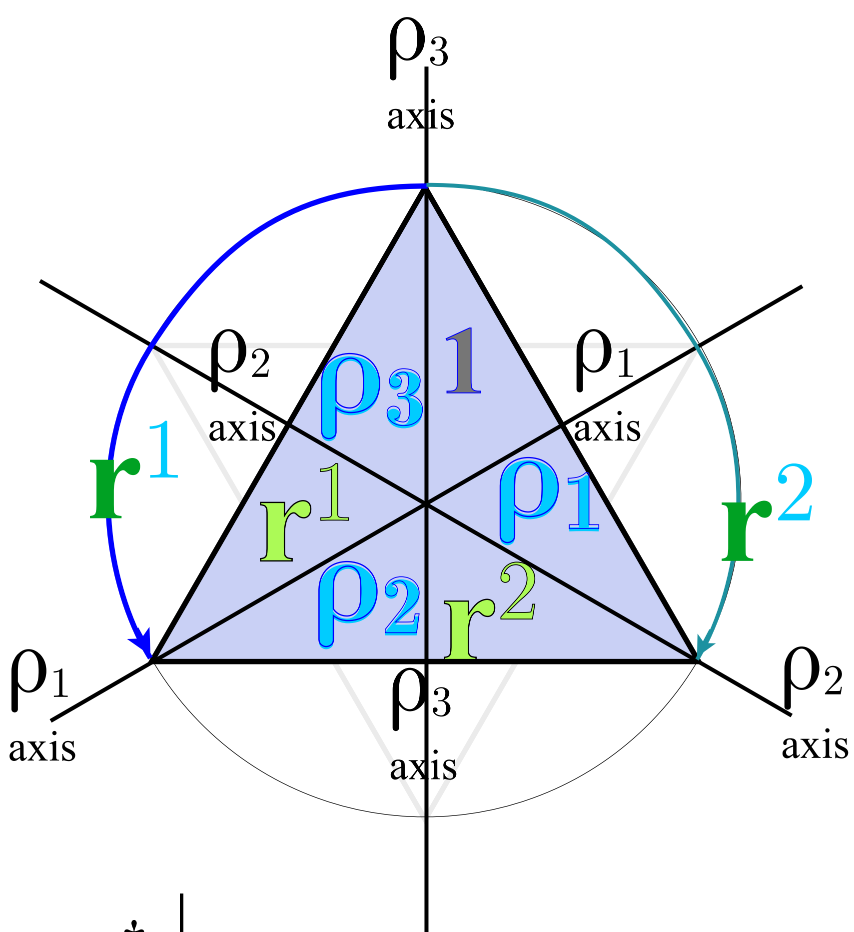
$\sigma_1\{\mathbf{1}, \mathbf{r}^1, \mathbf{r}^2, \sigma_1, \sigma_2, \sigma_3\}$

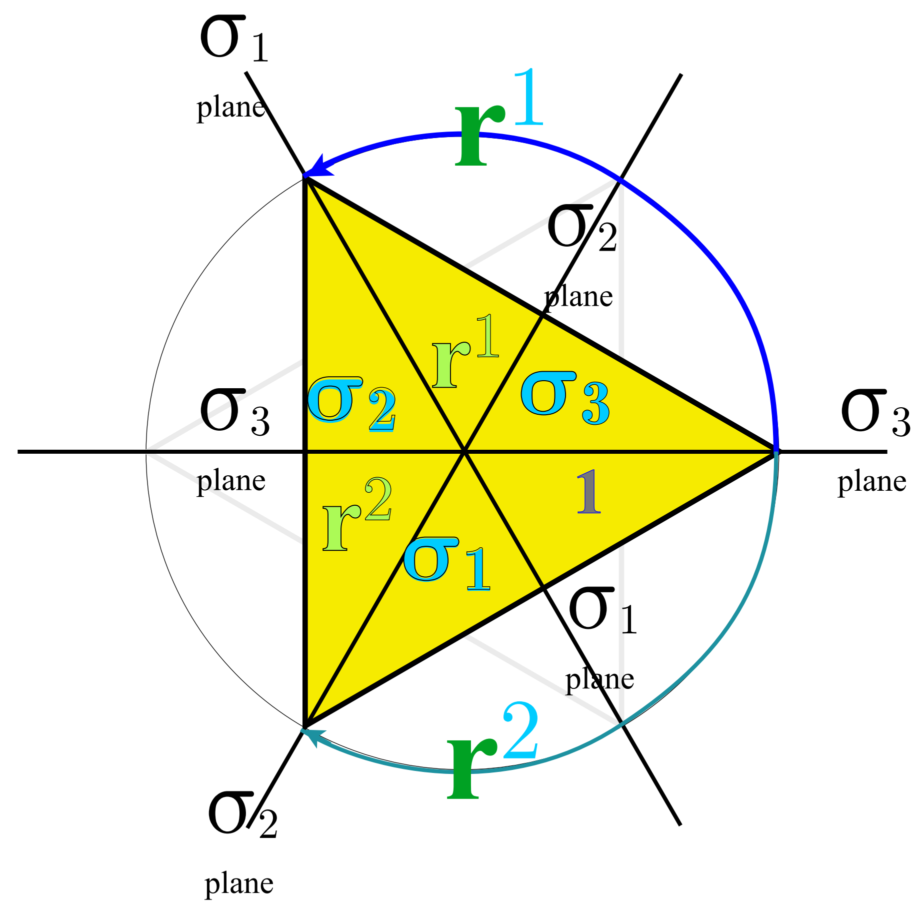
$= \{\sigma_1, \sigma_2, \sigma_3, \mathbf{1}, \mathbf{r}^1, \mathbf{r}^2\}$

....whole C_{3v} group table:

C_{3v} $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_3	σ_1	σ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_2	σ_3	σ_1
σ_1	σ_1	σ_3	σ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
σ_2	σ_2	σ_1	σ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
σ_3	σ_3	σ_2	σ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Deriving $D_3 \sim C_{3v}$ products - By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$





D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

D_3 and C_{3v} clearly are isomorphic $D_3 \sim C_{3v}$ share group table

...except for notation $\rho_k \leftrightarrow \boldsymbol{\sigma}_k$

C_{3v} $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	σ_1	σ_2	σ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	σ_3	σ_1	σ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	σ_2	σ_3	σ_1
σ_1	σ_1	σ_3	σ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
σ_2	σ_2	σ_1	σ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
σ_3	σ_3	σ_2	σ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

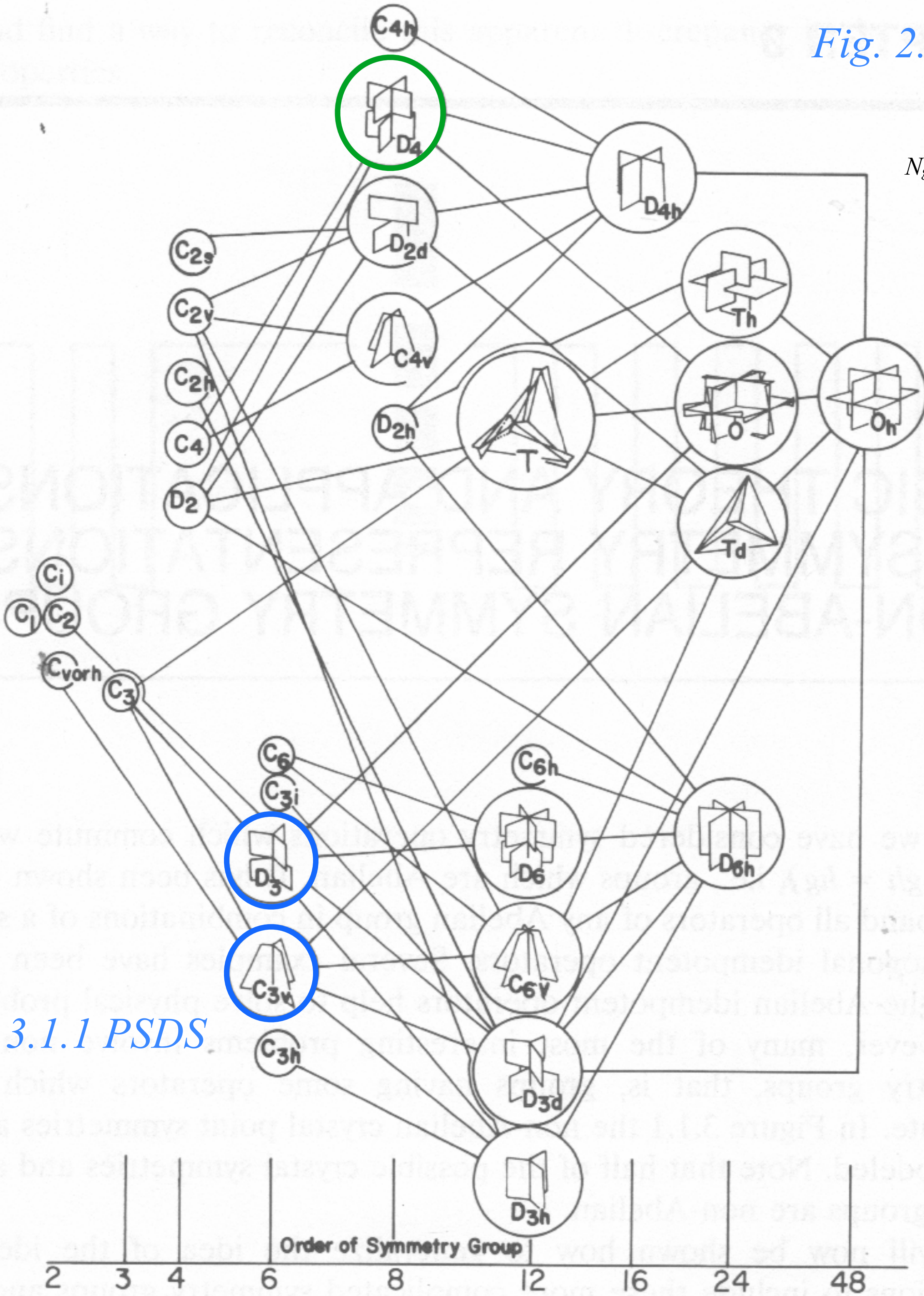
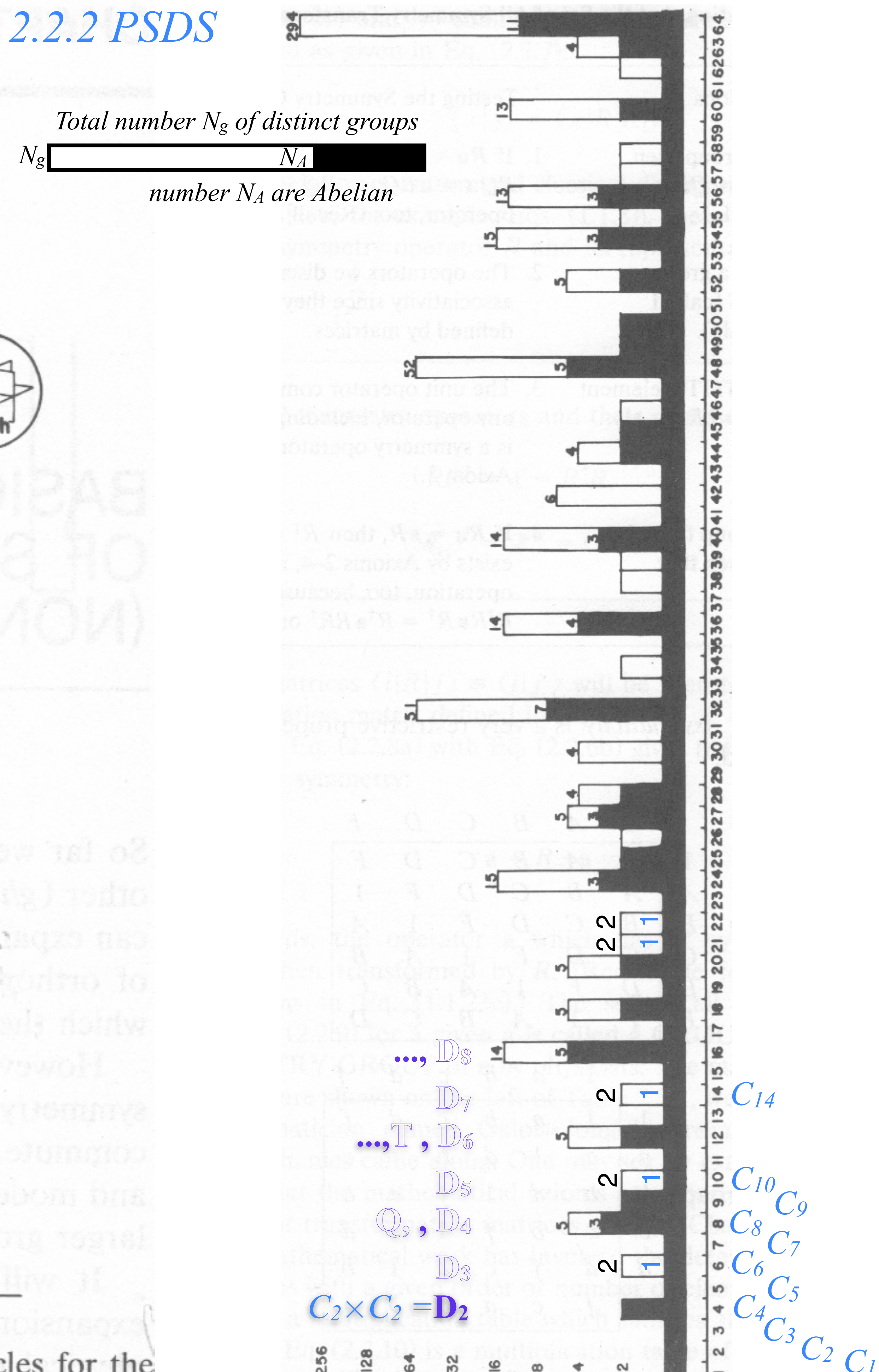


Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)



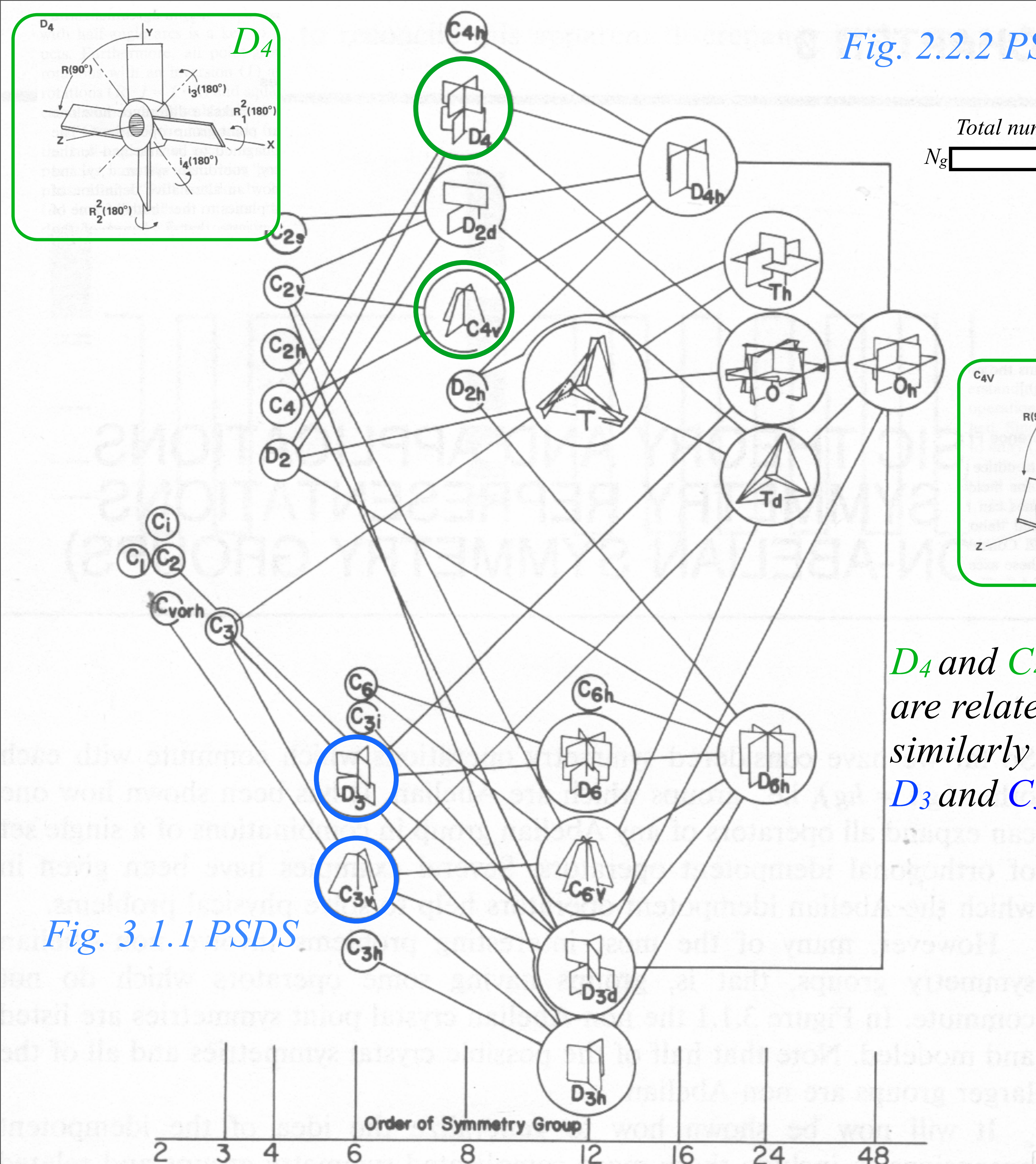


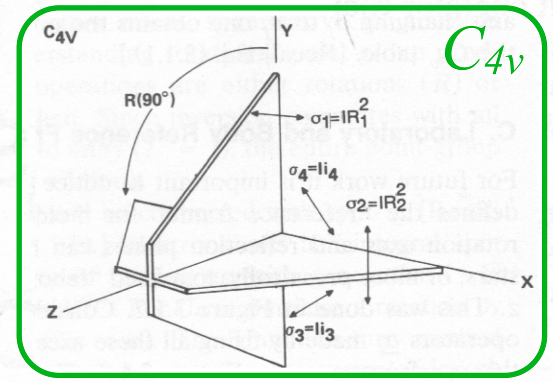
Figure 3.1.1 Crystal point symmetry groups. Models are sketched in circles for the 16 non-Abelian groups. (See also Figure 2.11.1.)

Fig. 2.2.2 PSDS

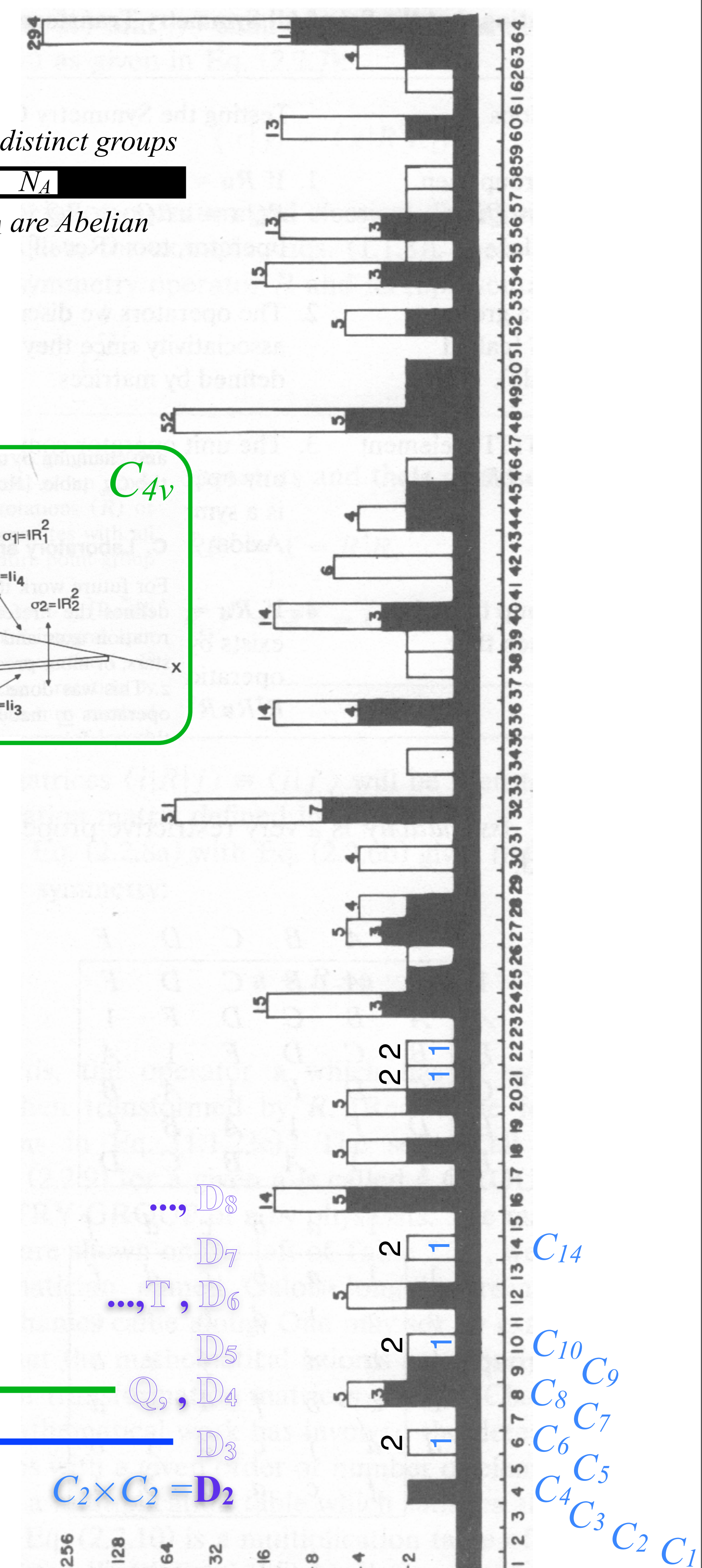
Total number N_g of distinct groups

N_g N_A

number N_A are Abelian



D_4 and C_{4v} are related similarly to D_3 and C_{3v}



3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

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Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

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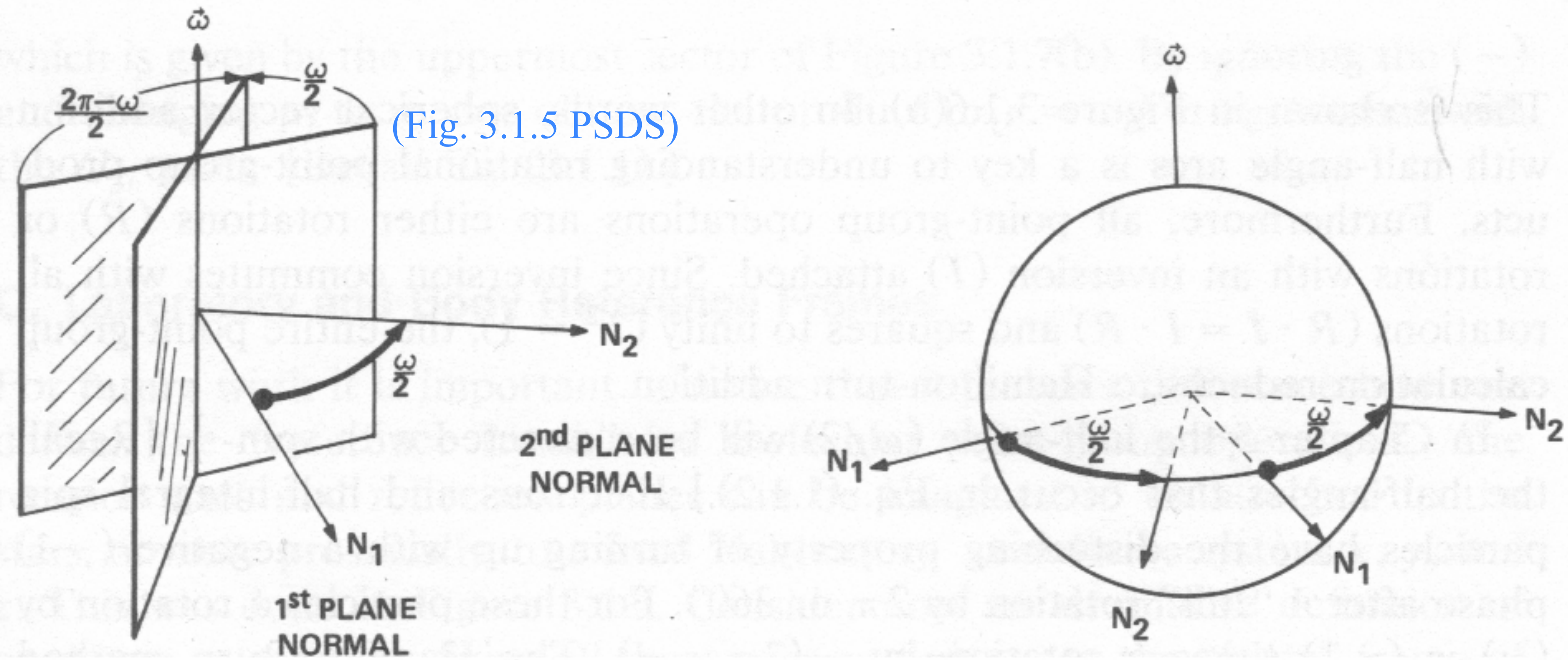
Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

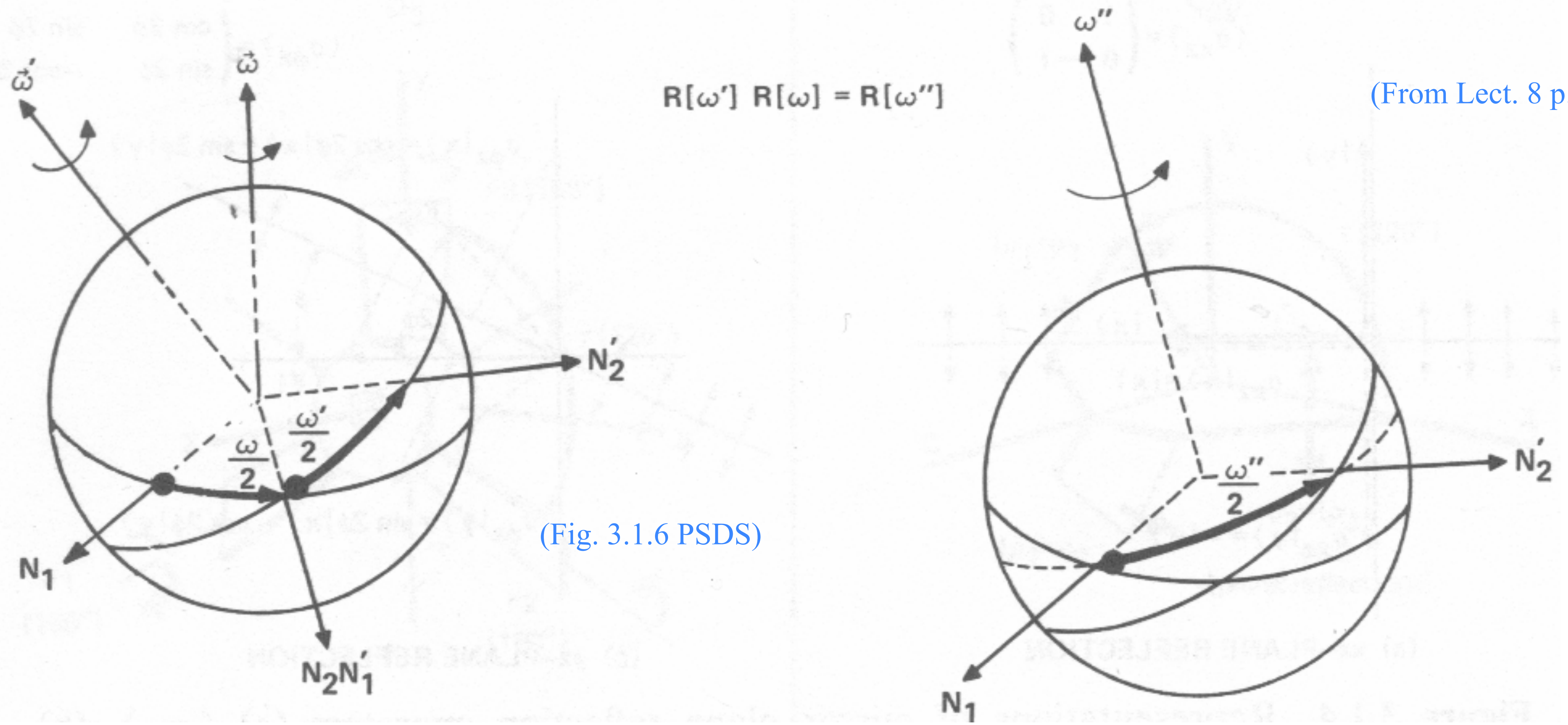
Group invariant numbers: Centrum, Rank, and Order



(Fig. 3.1.5 PSDS)

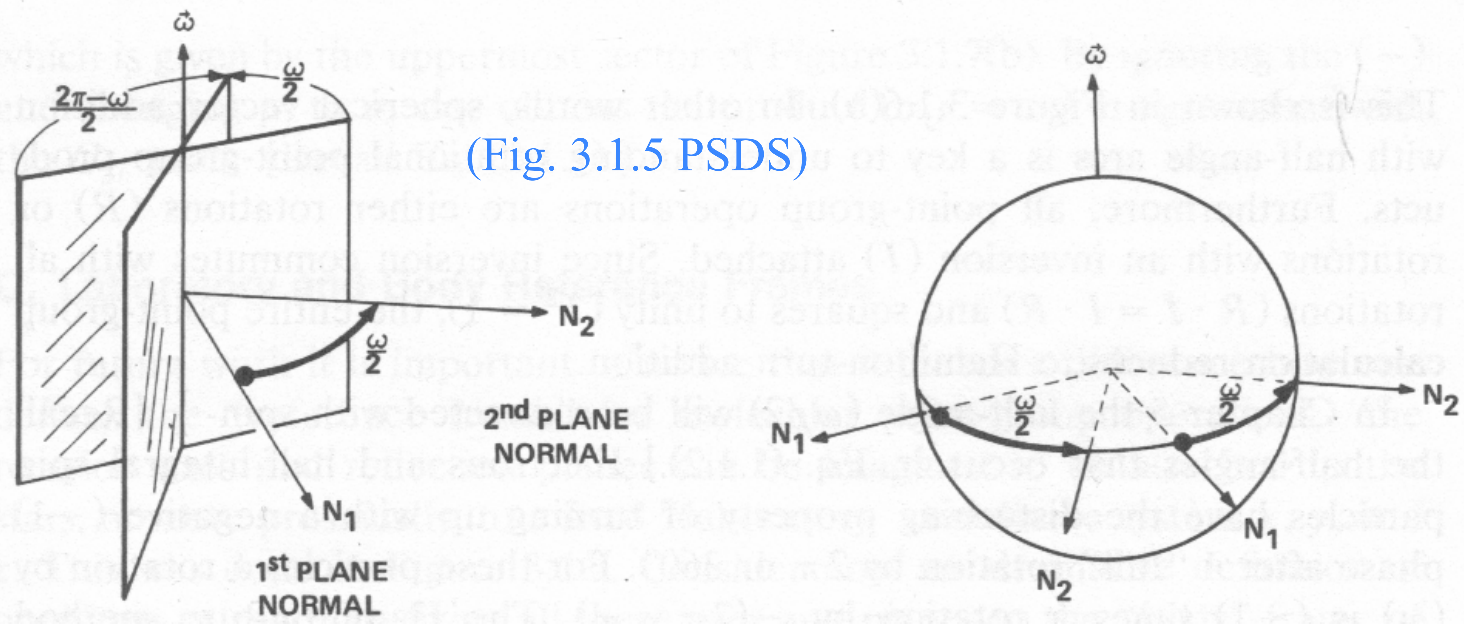
$R[\omega'] \; R[\omega] = R[\omega'']$

(From Lect. 8 p. 65-78)

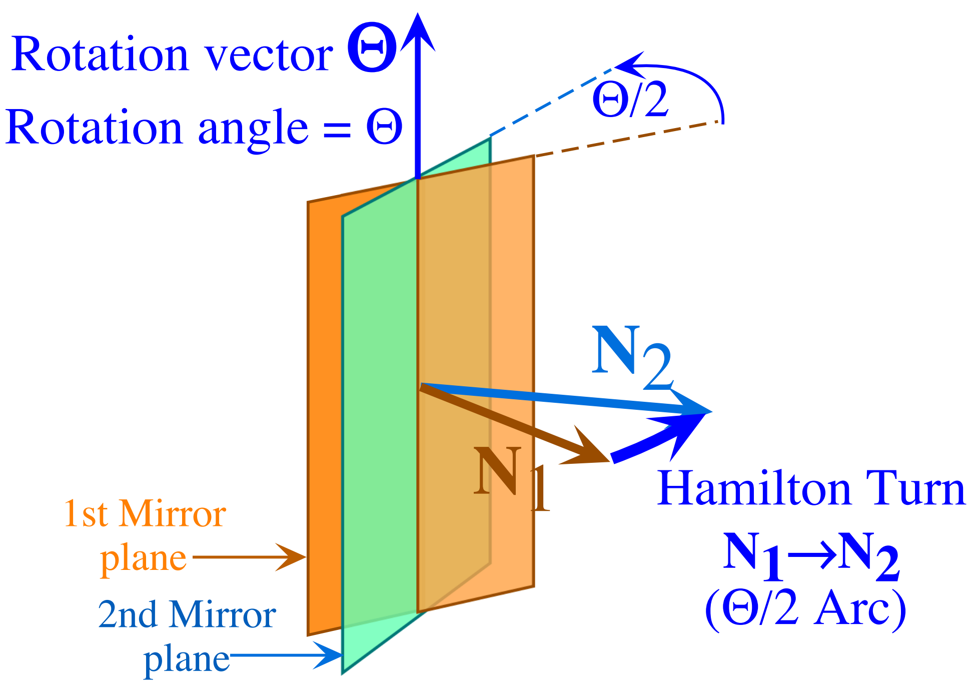


(Fig. 3.1.6 PSDS)

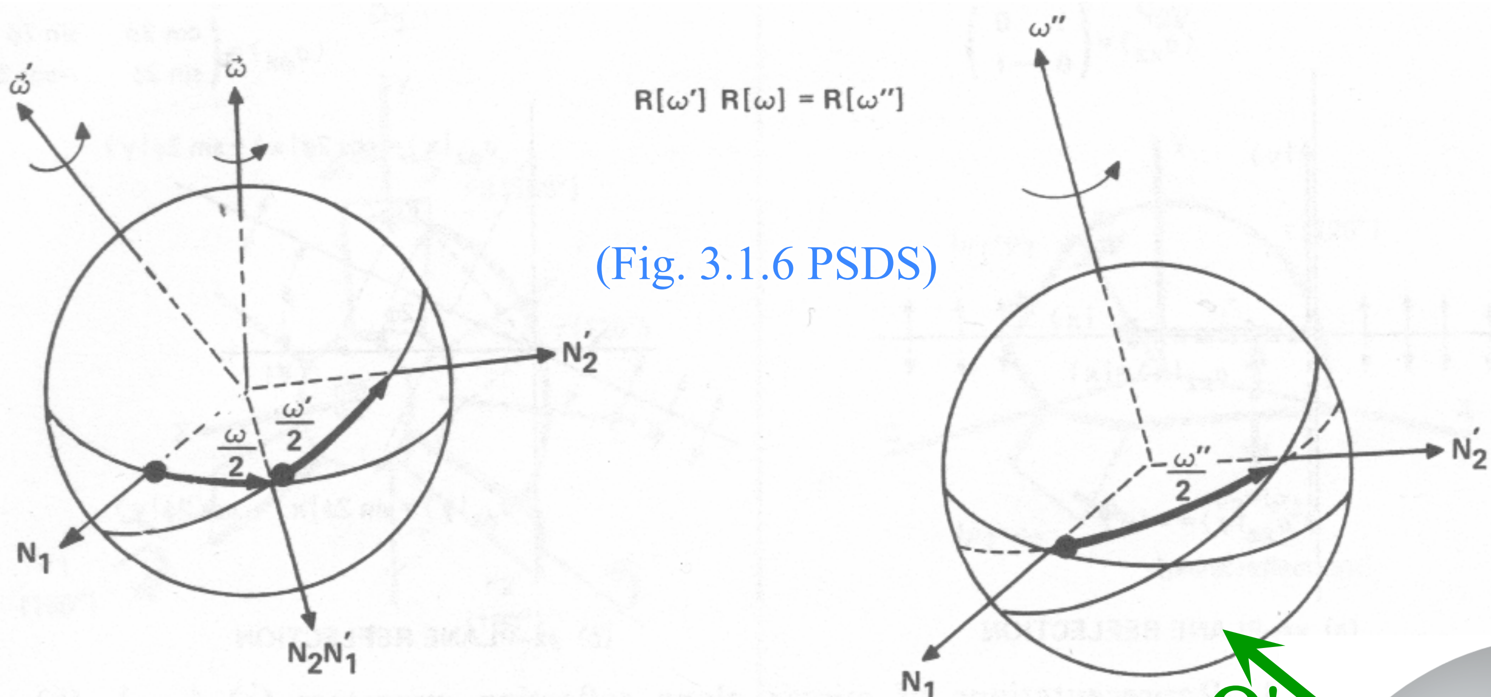
Deriving $D_3 \sim C_{3v}$ products by nomograms based on U(2) Hamilton-turns



(Fig. 3.1.5 PSDS)

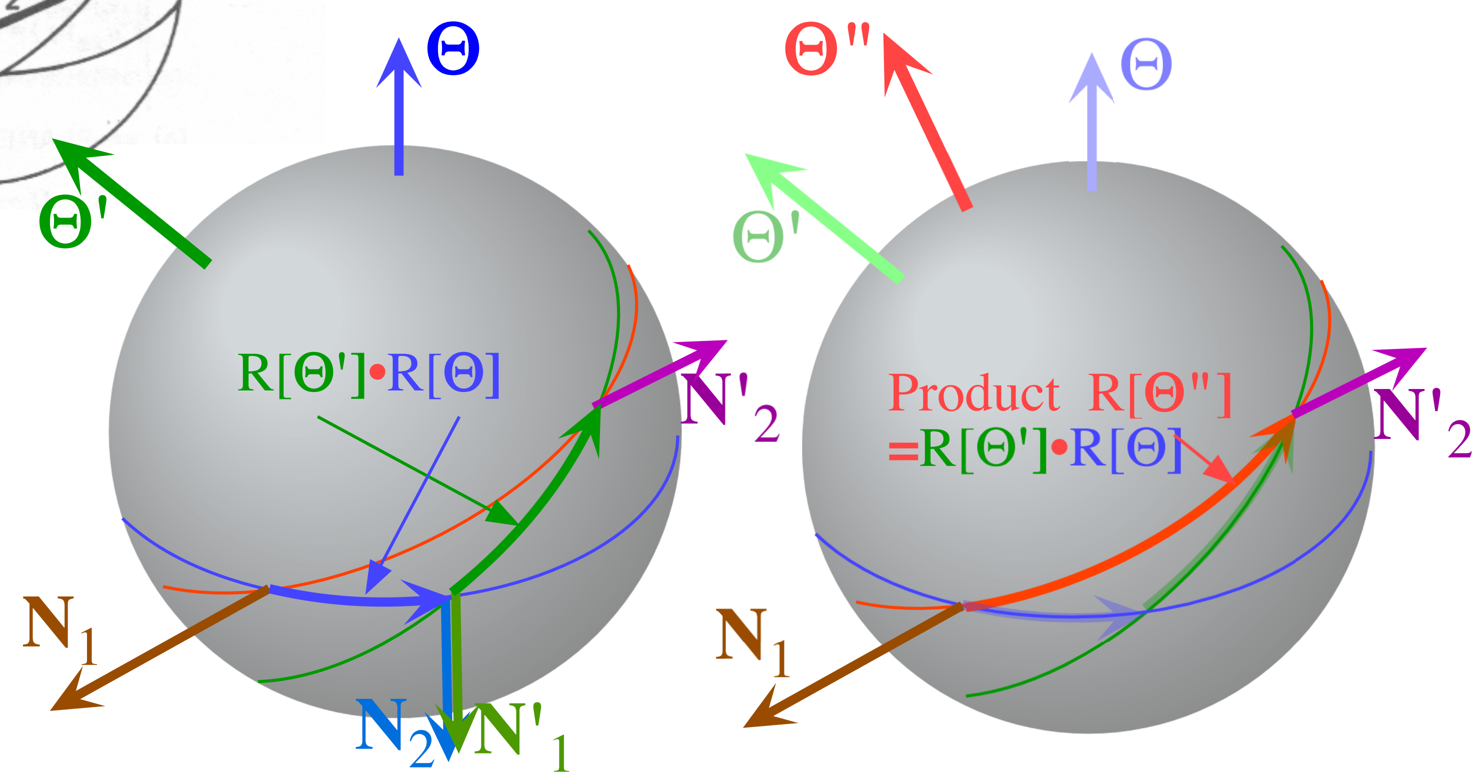


(Fig. 10.A.7 QTCA)

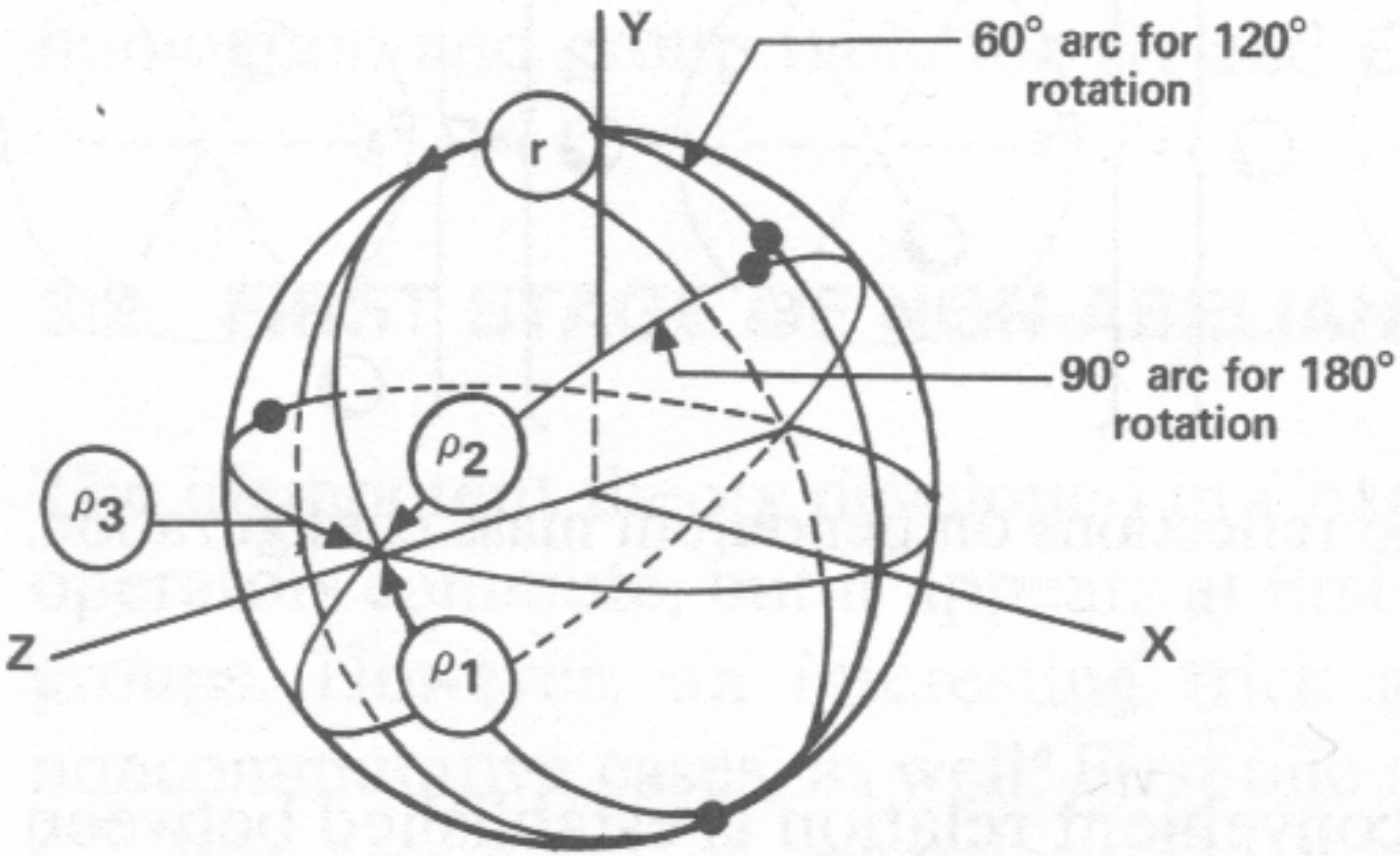


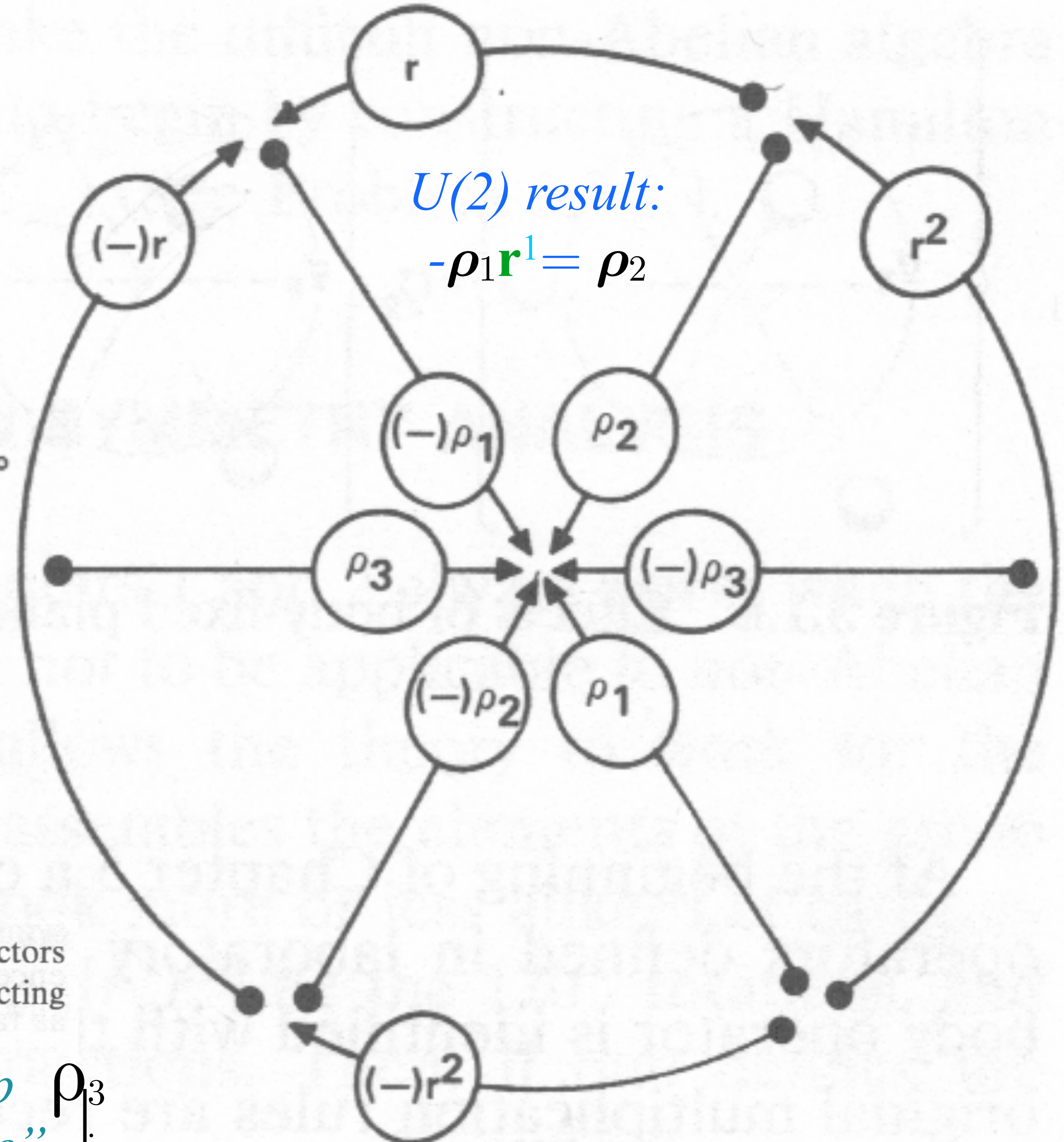
(Fig. 3.1.6 PSDS)

(From Lect. 8 p. 65-78)



(Fig. 10.A.8 QTCA)





U(2) result:

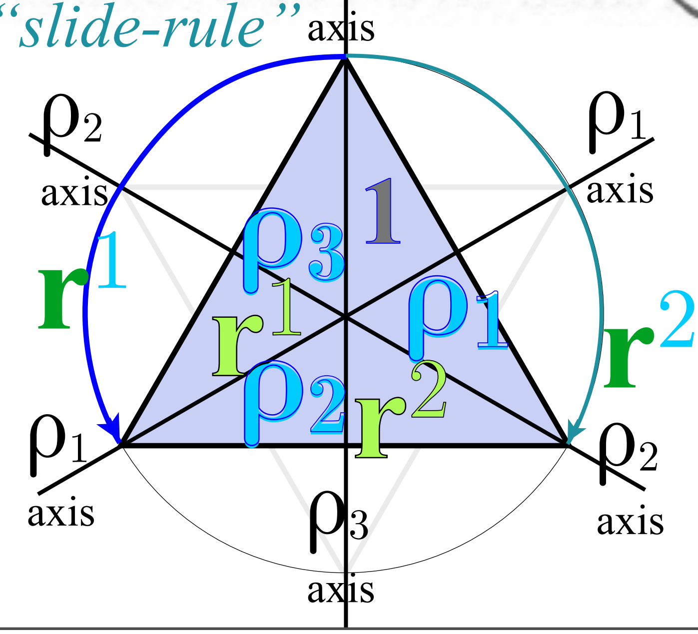
$-\boldsymbol{\rho}_1\mathbf{r}^1 = \boldsymbol{\rho}_2$

Figure 3.1.7 Geometrical definition of symmetry group D_3. (a) Hamilton arc vectors are drawn for rotations r, i_1, and i_3. (b) Group nomogram is obtained by projecting (a) onto the xy plane.

Note $\boldsymbol{h}^2=\mathbf{r}^1$ and $\boldsymbol{h}^4=\mathbf{r}^2$ for D_6 notation

1	h^2	h^4	ρ_1	ρ_2	ρ_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1

D_3 Group "slide-rule"



R(3) result:

$\boldsymbol{\rho}_1\mathbf{r}^1 = \boldsymbol{\rho}_2$

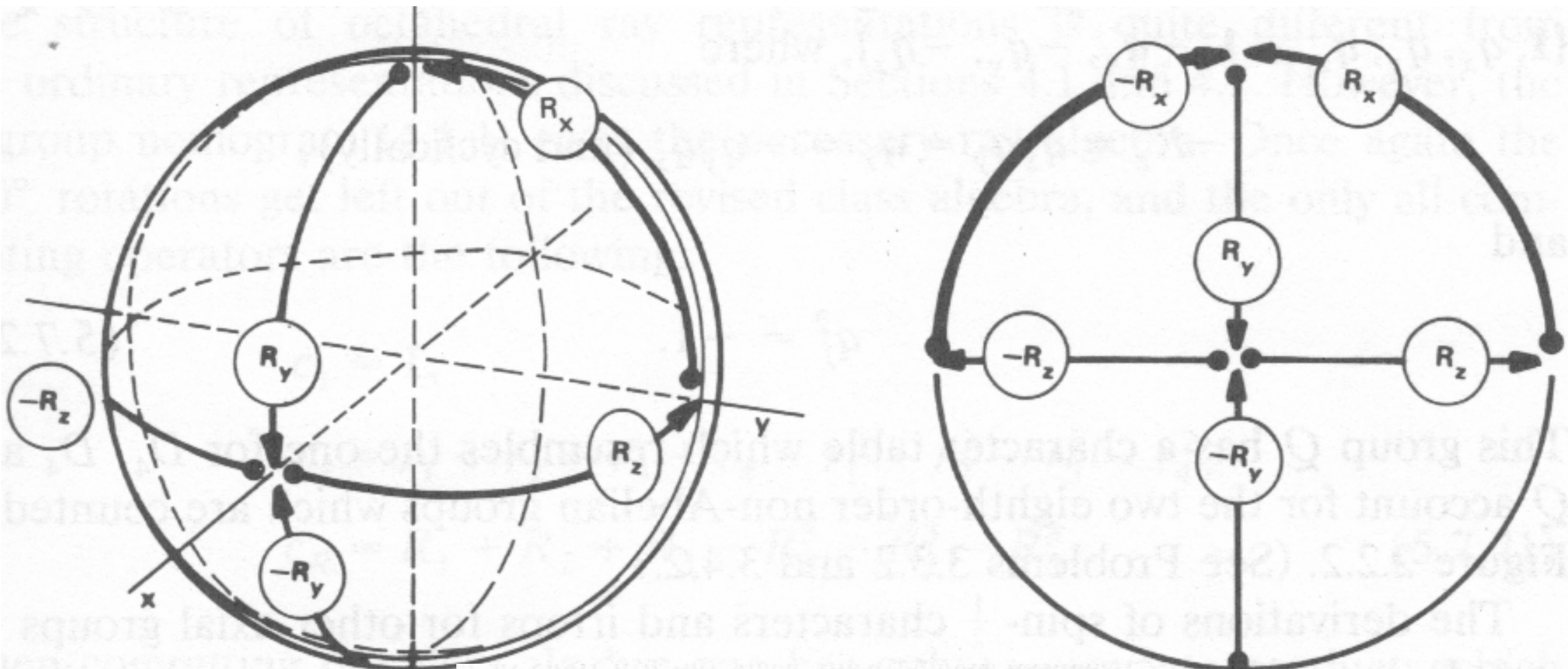


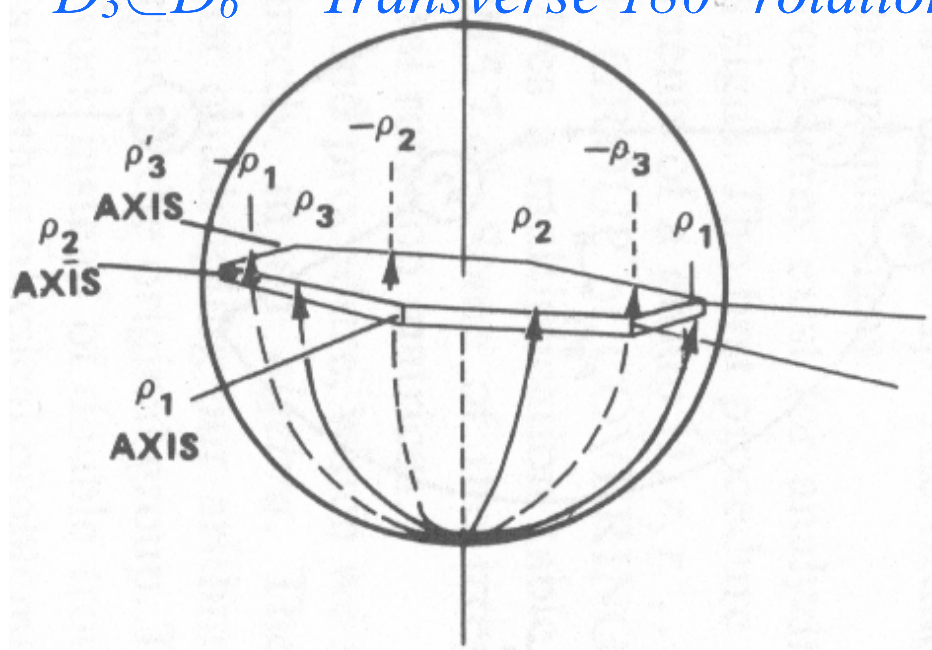
Figure 5.7.4 Hamilton arcs and vector nomogram for D_2

1	R_x	R_y	R_z
R_x	$-\mathbf{1}$	R_z	$-R_y$
R_y	$-R_z$	$-\mathbf{1}$	R_x
R_z	R_y	$-R_x$	$-\mathbf{1}$

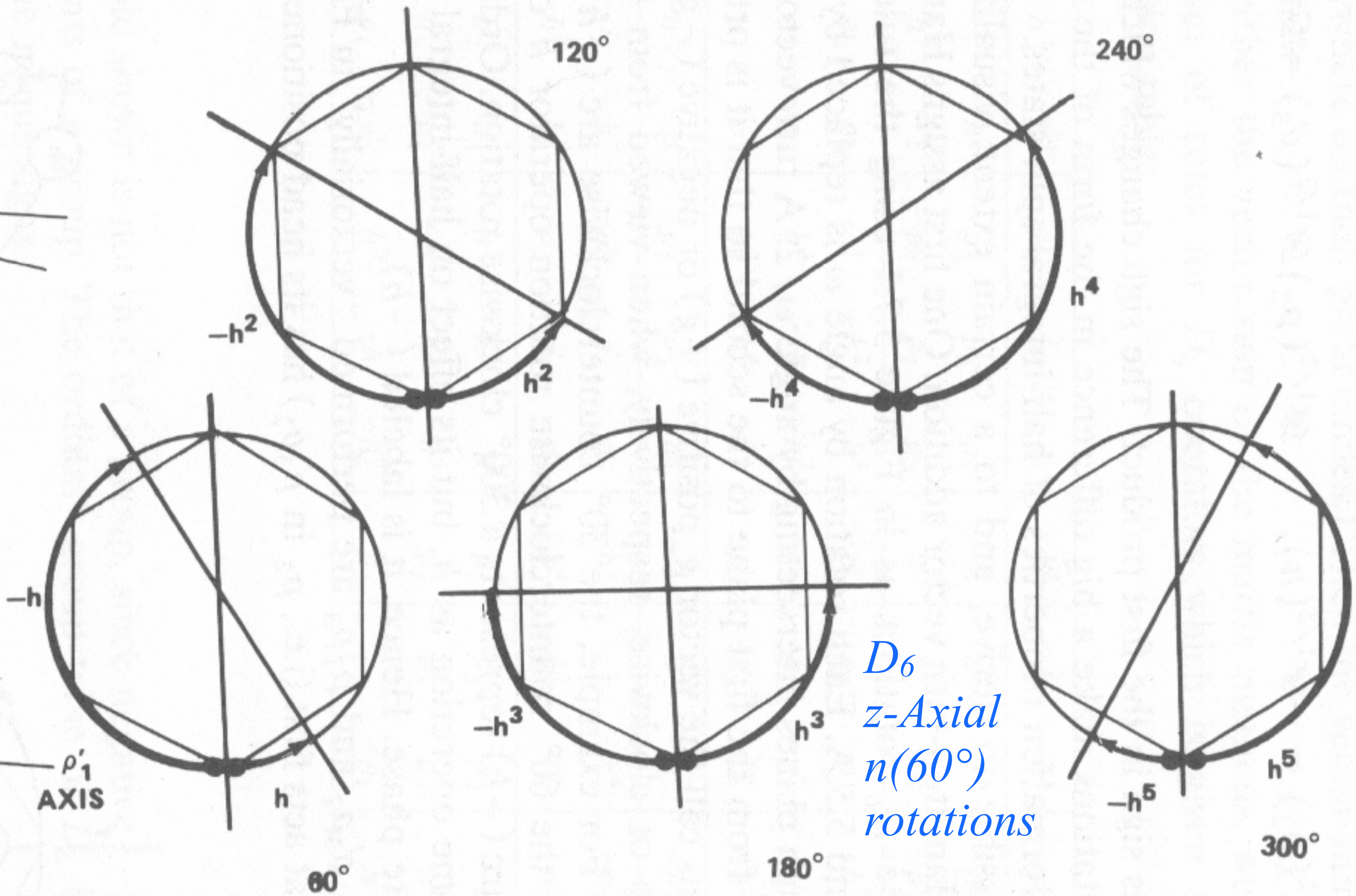
$-i\sigma_B$ $\quad$ $-i\sigma_C$ $\quad$ $-i\sigma_A$

$$\mathscr{D}^E(R_x) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathscr{D}^E(R_y) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathscr{D}^E(R_z) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

$D_3 \subset D_6$ *Transverse 180° rotations*

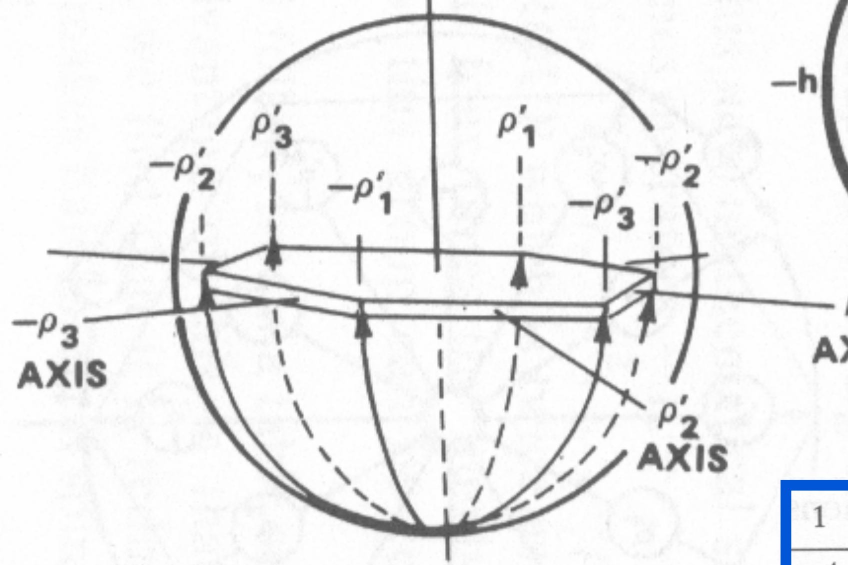


$D_3 \subset D_6$ *z-Axial 120° rotations*



D_6 z-Axial n(60°) rotations

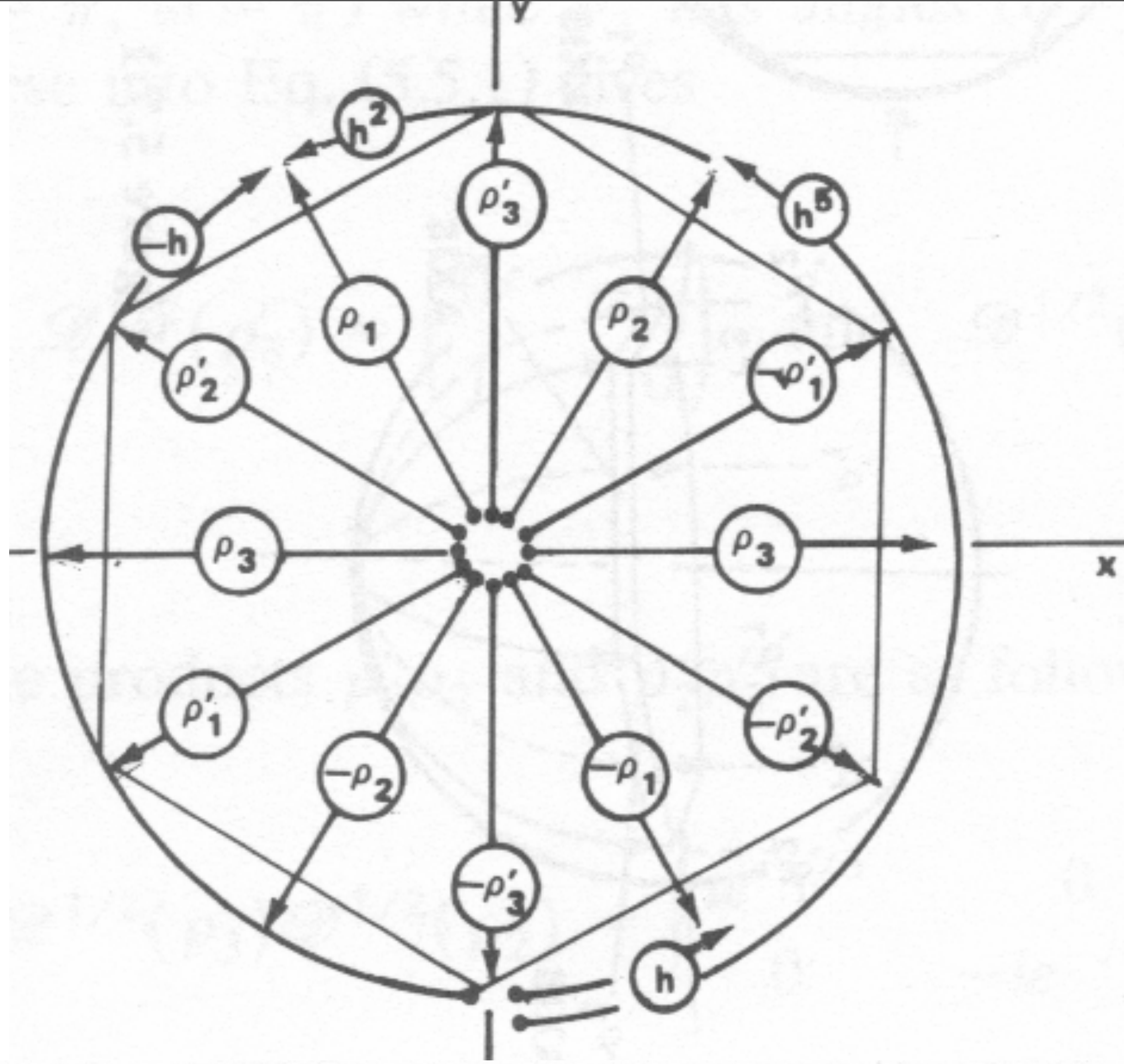
D_6 Transverse 180° rotations



1	h^2	h^4	ρ_1	ρ_2	ρ_3	h^3	h	h^5	ρ'_1	ρ'_2	ρ'_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1	$-h$	h^5	$-h^3$	$-\rho'_2$	$-\rho'_3$	ρ'_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2	h^5	h^3	$-h$	$-\rho'_3$	ρ'_1	ρ'_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$	$-\rho'_1$	ρ'_3	$-\rho'_2$	h^3	h^5	$-h$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$	$-\rho'_2$	$-\rho'_1$	$-\rho'_3$	h	h^3	h^5
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1	$-\rho'_3$	$-\rho'_2$	ρ'_1	$-h^5$	h	h^3
h^3	h^5	$-h$	ρ'_1	ρ'_2	ρ'_3	-1	h^4	$-h^2$	$-\rho_1$	$-\rho_2$	$-\rho_3$
h^5	$-h$	$-h^3$	$-\rho'_3$	ρ'_1	ρ'_2	$-h^2$	-1	$-h^4$	ρ_3	$-\rho_1$	$-\rho_2$
h	h^3	h^5	ρ'_2	ρ'_3	$-\rho'_1$	h^4	h^2	-1	$-\rho_2$	$-\rho_3$	ρ_1
ρ'_1	ρ'_2	ρ'_3	$-h^3$	$-h^5$	h	ρ_1	$-\rho_3$	ρ_2	-1	$-h^2$	$-h^4$
ρ'_2	ρ'_3	$-\rho'_1$	$-h$	$-h^3$	$-h^5$	ρ_2	ρ_1	ρ_3	h^4	-1	$-h^2$
ρ'_3	$-\rho'_1$	$-\rho'_2$	h^5	$-h$	$-h^3$	ρ_3	ρ_2	$-\rho_1$	h^2	h^4	-1

Note $\mathbf{h}^2=\mathbf{r}^1$ *and* $\mathbf{h}^4=\mathbf{r}^2$ *for* $D_6 \supset D_3$ *notation*

Later we show:
$D_6 = D_3 \times C_2$



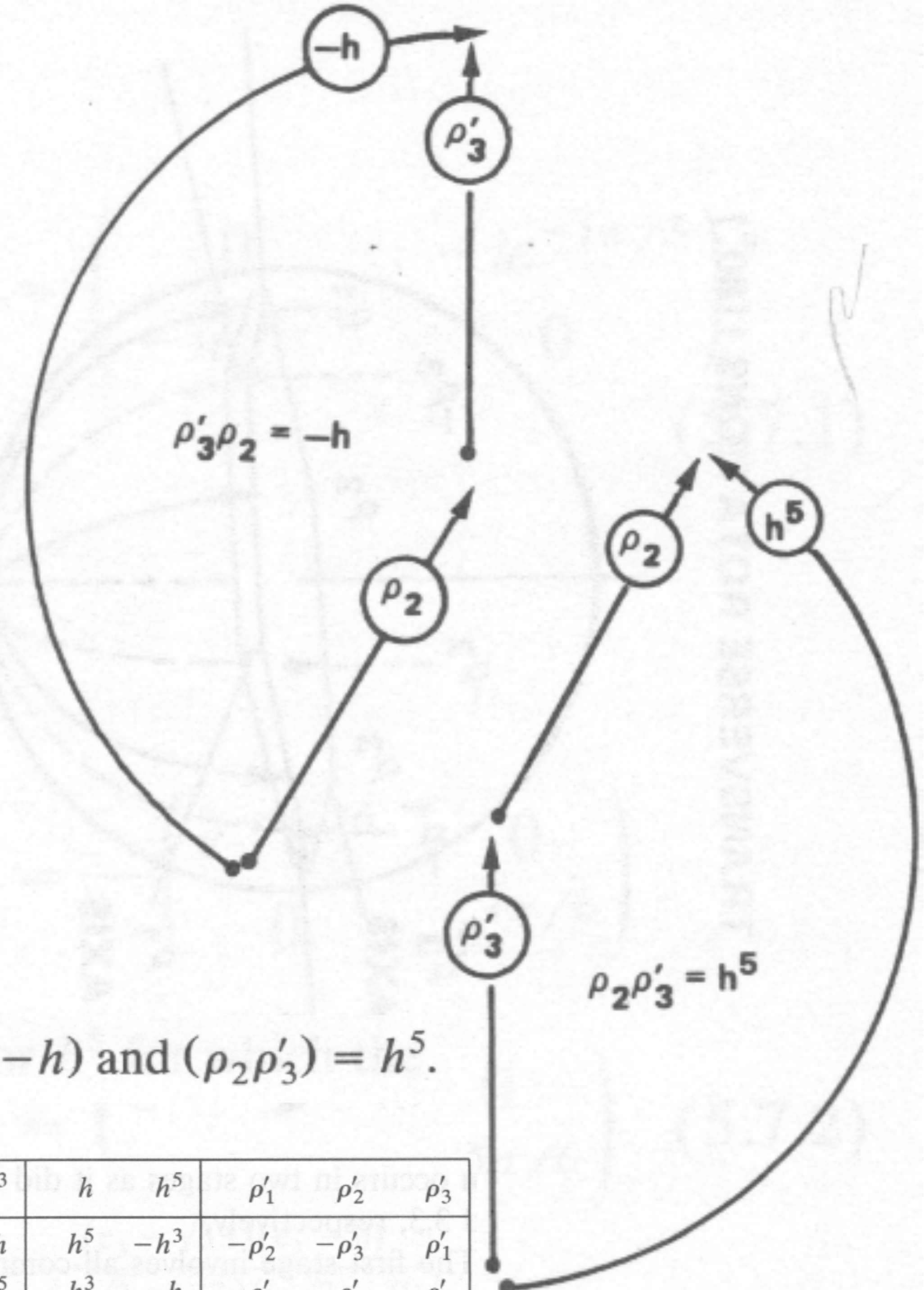


Figure 5.7.2 D_6 nomogram and example of products $(\rho'_3\rho_2 = -h)$ and $(\rho_2\rho'_3) = h^5$.

Note $\boldsymbol{h}^2=\mathbf{r}^1$ *and* $\boldsymbol{h}^4=\mathbf{r}^2$
for $D_6 \supset D_3$ *notation*

Later we show:
$D_6 = D_3 \times C_2$

1	h^2	h^4	ρ_1	ρ_2	ρ_3	h^3	h	h^5	ρ'_1	ρ'_2	ρ'_3
h^4	-1	$-h^2$	$-\rho_2$	$-\rho_3$	ρ_1	$-h$	h^5	$-h^3$	$-\rho'_2$	$-\rho'_3$	ρ'_1
h^2	h^4	-1	$-\rho_3$	ρ_1	ρ_2	h^5	h^3	$-h$	$-\rho'_3$	ρ'_1	ρ'_2
ρ_1	ρ_2	ρ_3	-1	$-h^2$	$-h^4$	$-\rho'_1$	ρ'_3	$-\rho'_2$	h^3	h^5	$-h$
ρ_2	ρ_3	$-\rho_1$	h^4	-1	$-h^2$	$-\rho'_2$	$-\rho'_1$	$-\rho'_3$	h	h^3	h^5
ρ_3	$-\rho_1$	$-\rho_2$	h^2	h^4	-1	$-\rho'_3$	$-\rho'_2$	ρ'_1	$-h^5$	h	h^3
h^3	h^5	$-h$	ρ'_1	ρ'_2	ρ'_3	-1	h^4	$-h^2$	$-\rho_1$	$-\rho_2$	$-\rho_3$
h^5	$-h$	$-h^3$	$-\rho'_3$	ρ'_1	ρ'_2	$-h^2$	-1	$-h^4$	ρ_3	$-\rho_1$	$-\rho_2$
h	h^3	h^5	ρ'_2	ρ'_3	$-\rho'_1$	h^4	h^2	-1	$-\rho_2$	$-\rho_3$	ρ_1
ρ'_1	ρ'_2	ρ'_3	$-h^3$	$-h^5$	h	ρ_1	$-\rho_3$	ρ_2	-1	$-h^2$	$-h^4$
ρ'_2	ρ'_3	$-\rho'_1$	$-h$	$-h^3$	$-h^5$	ρ_2	ρ_1	ρ_3	h^4	-1	$-h^2$
ρ'_3	$-\rho'_1$	$-\rho'_2$	h^5	$-h$	$-h^3$	ρ_3	ρ_2	$-\rho_1$	h^2	h^4	-1

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

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Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

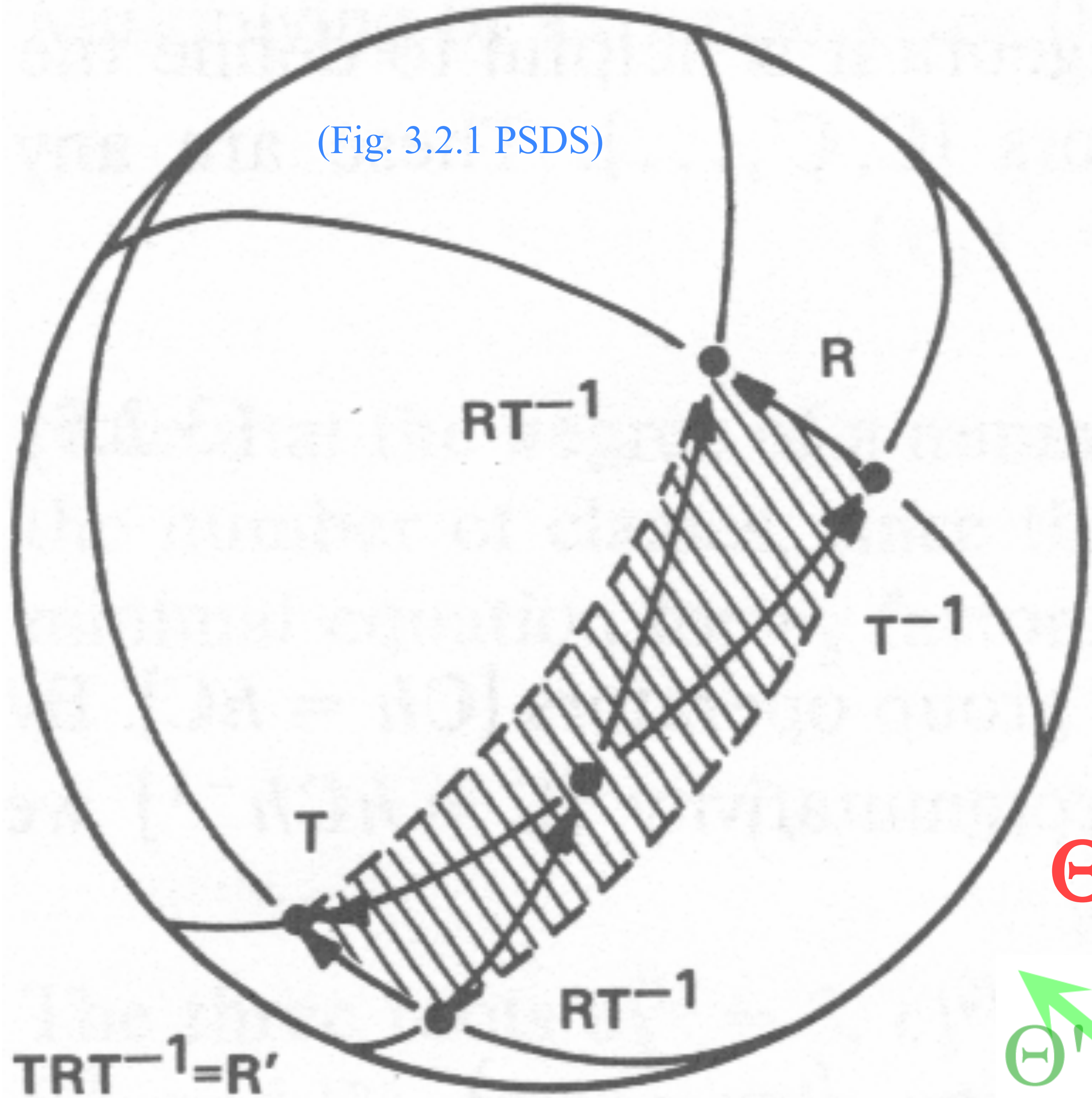
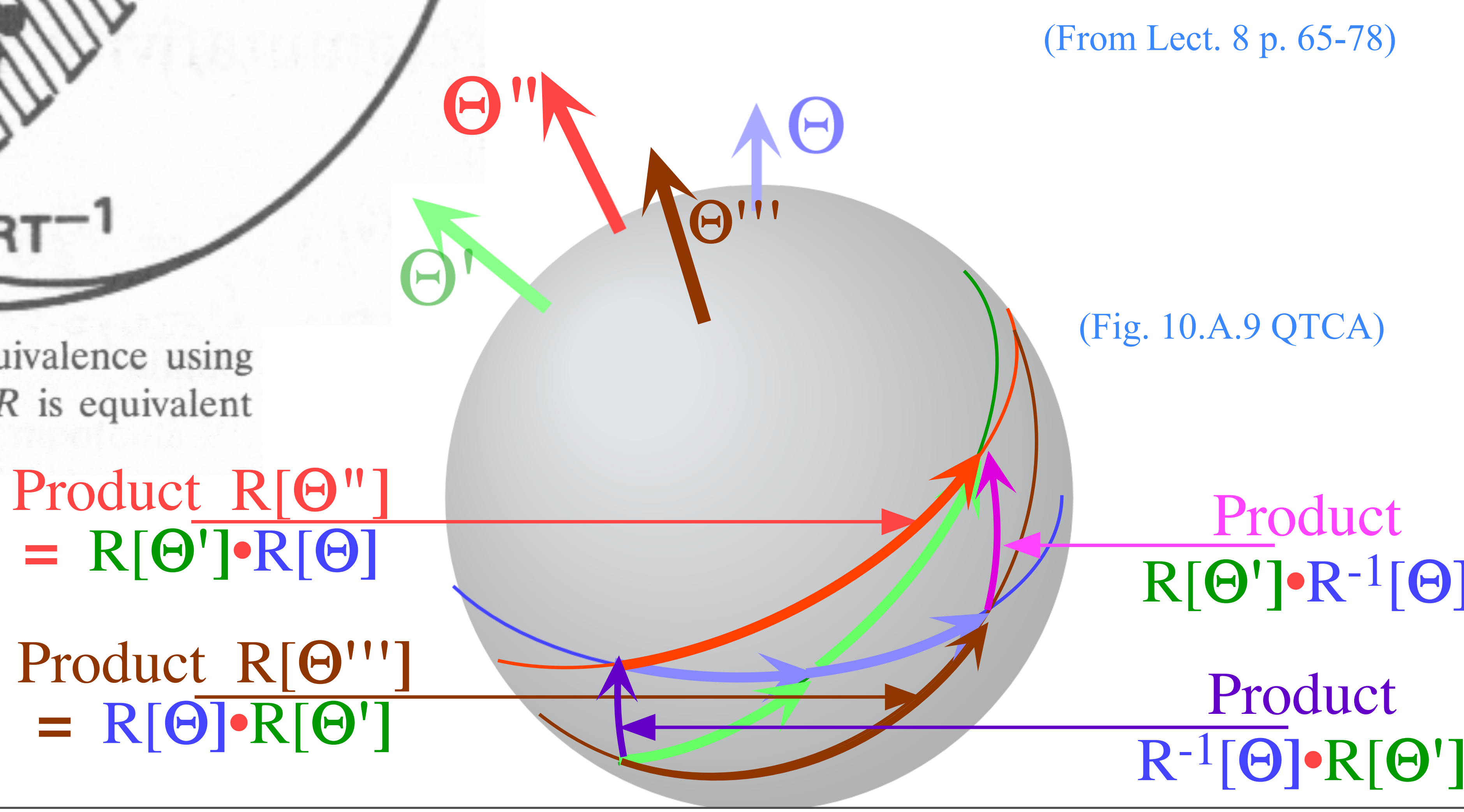
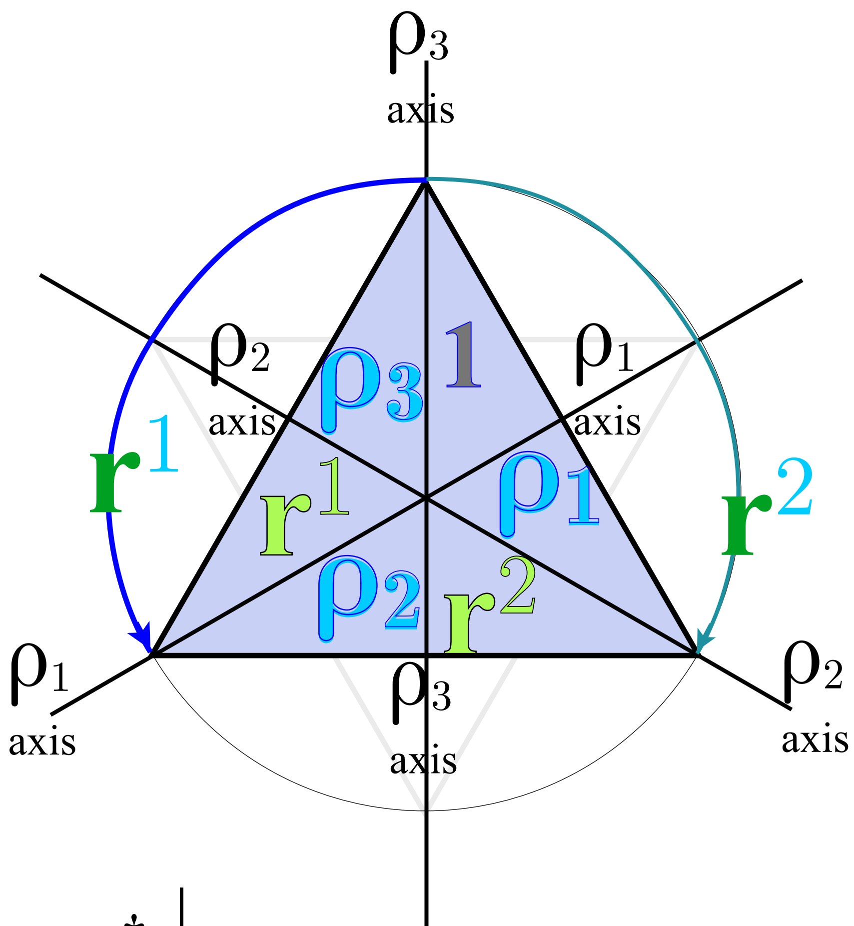


Figure 3.2.1 Showing class equivalence using Hamilton's vectors. Operation R is equivalent to $R' = TRT^{-1}$.



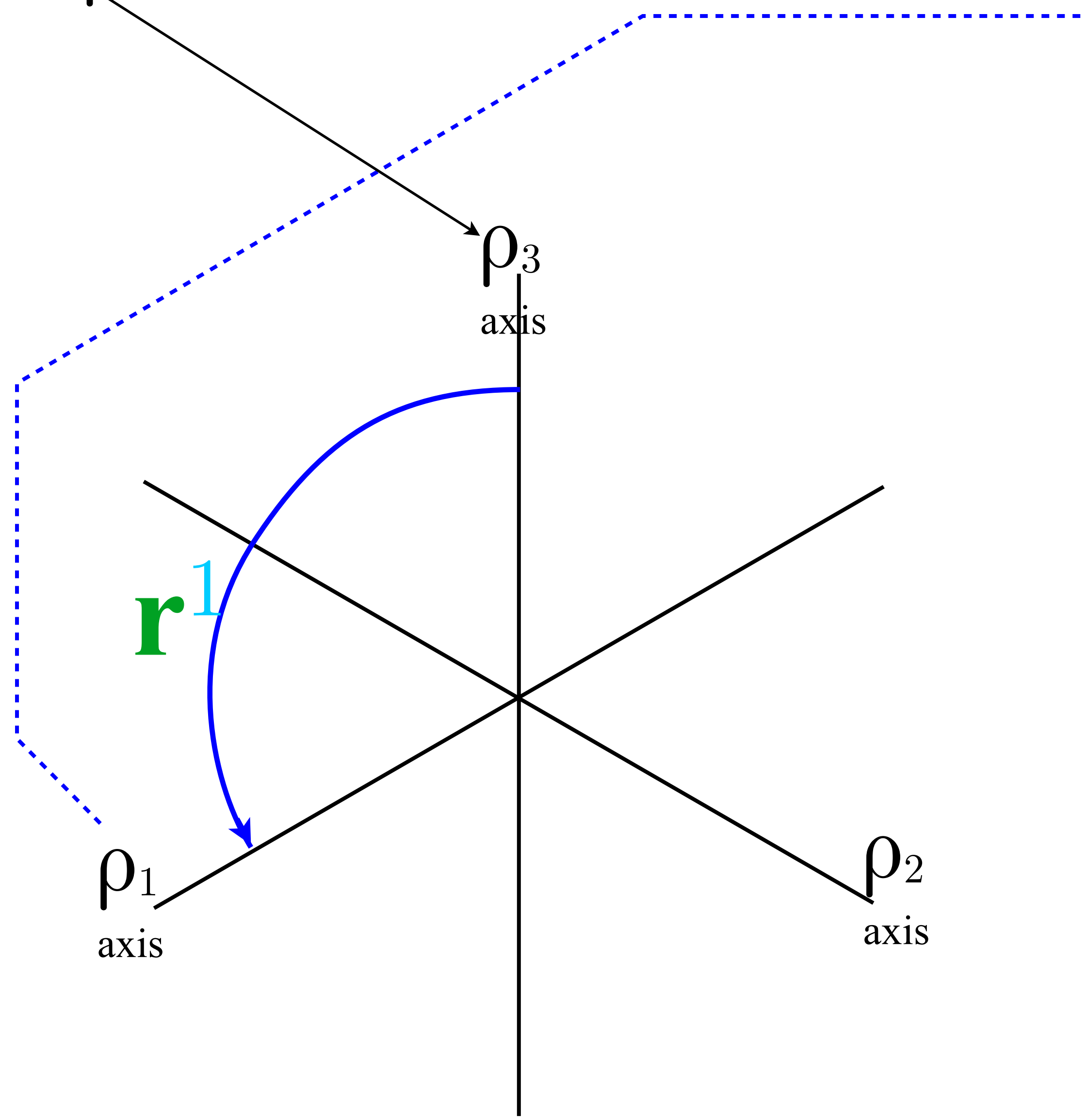
Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



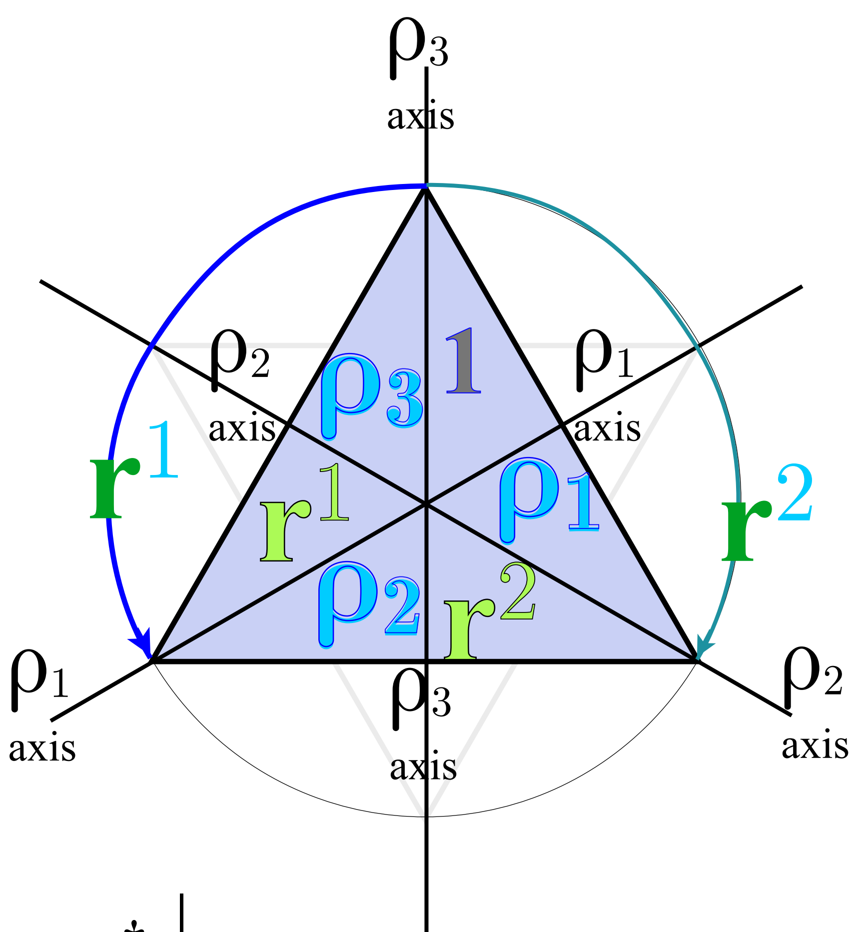
D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Transforming D_3 operators using D_3 operators

Example 1: Rotating ρ_3 axis crank using $\mathbf{r}^1$ puts it down onto ρ_1



Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Transforming D_3 operators using D_3 operators

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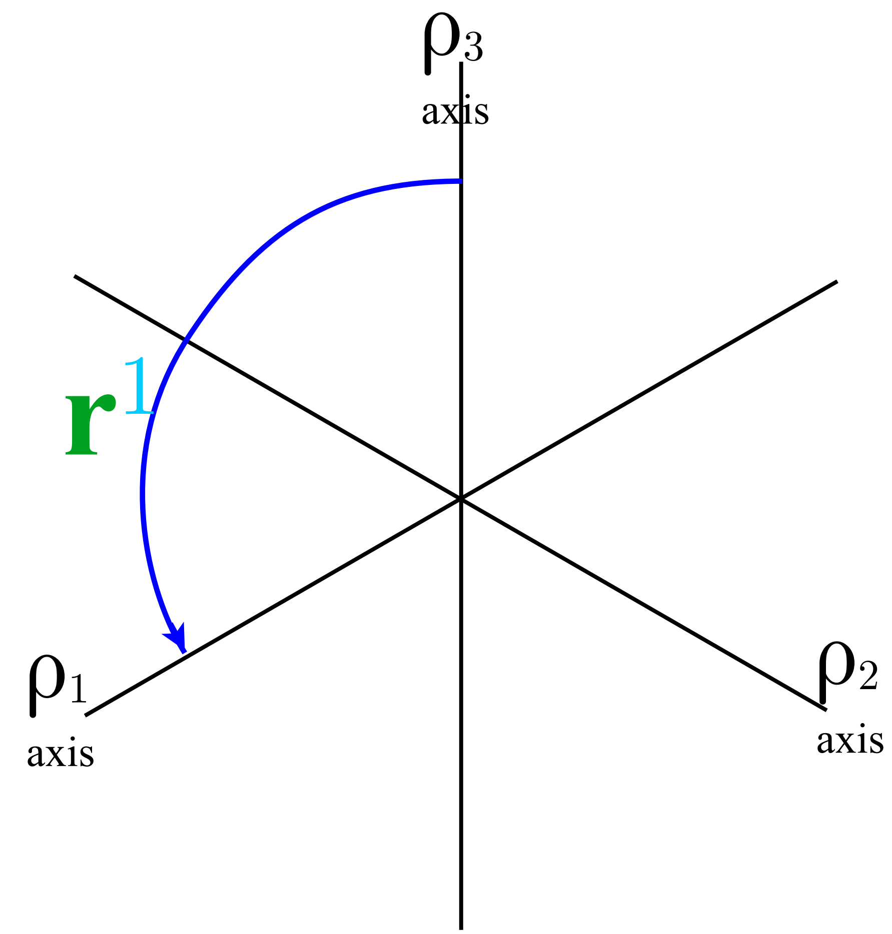
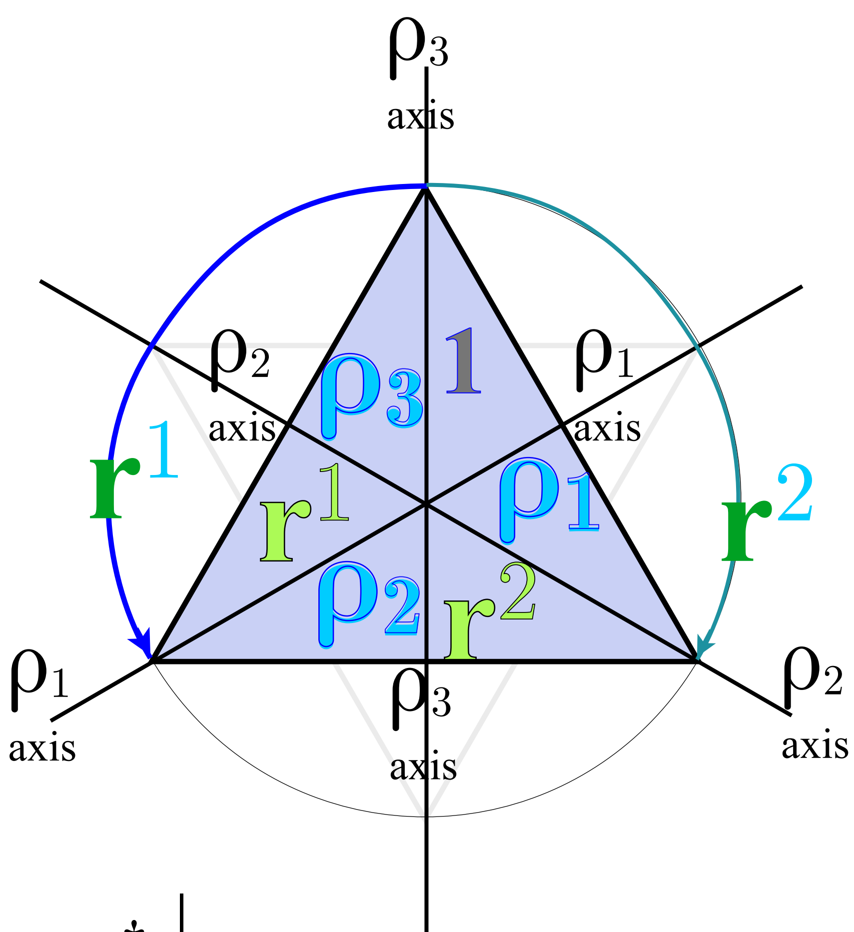
Seems to imply: $\mathbf{r}^1\rho_3(\mathbf{r}^1)^{-1}=\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_1$



Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



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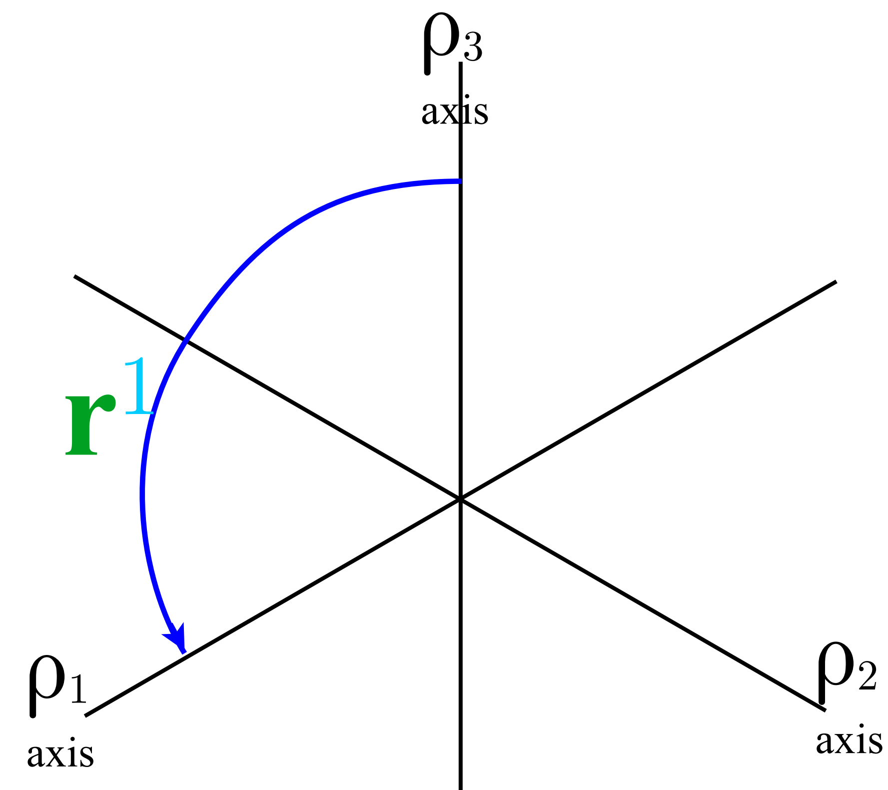
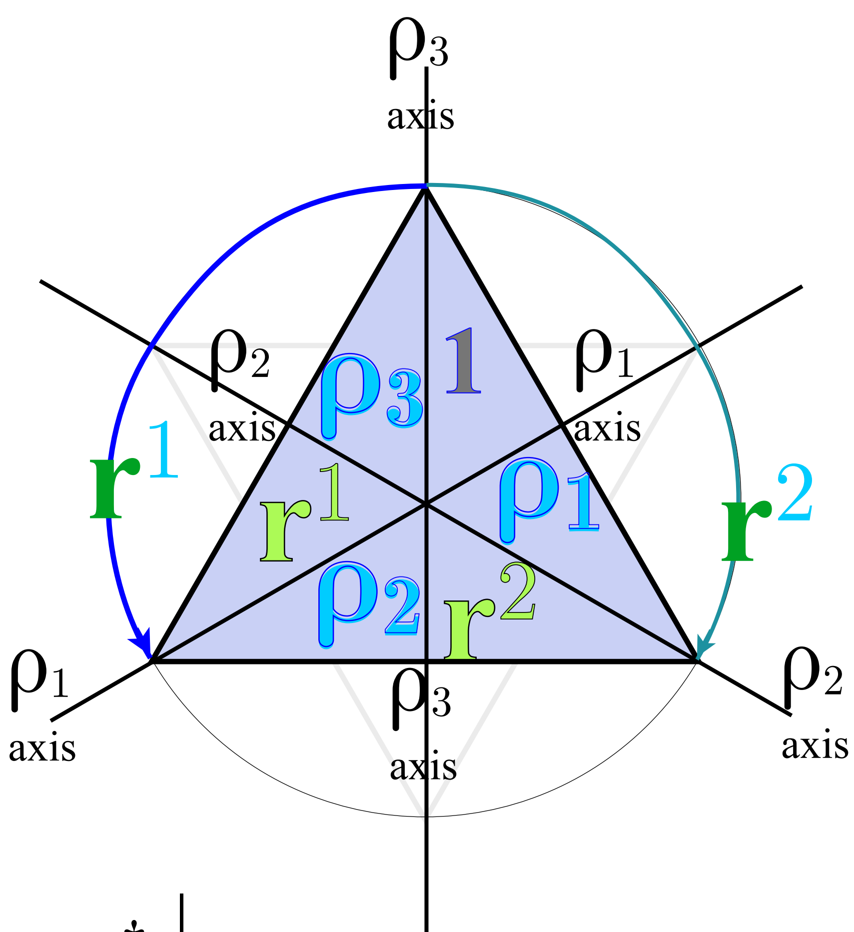


D_3 $gg^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Need to check that with table:

$$\mathbf{r}^1\rho_3\mathbf{r}^2=\rho_2\mathbf{r}^2$$

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



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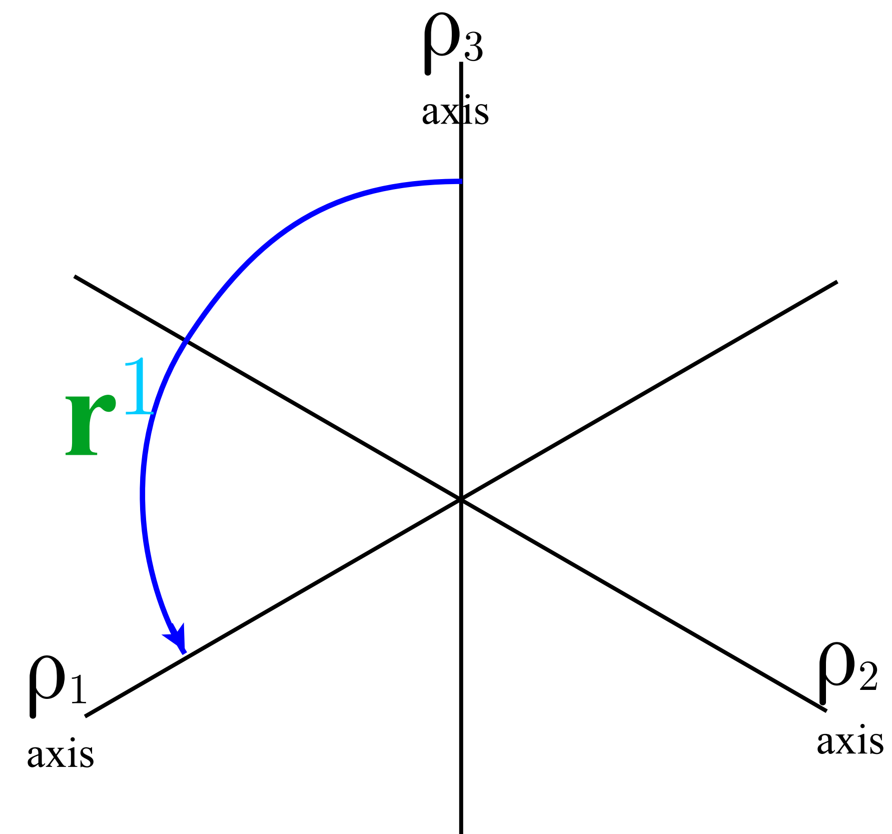


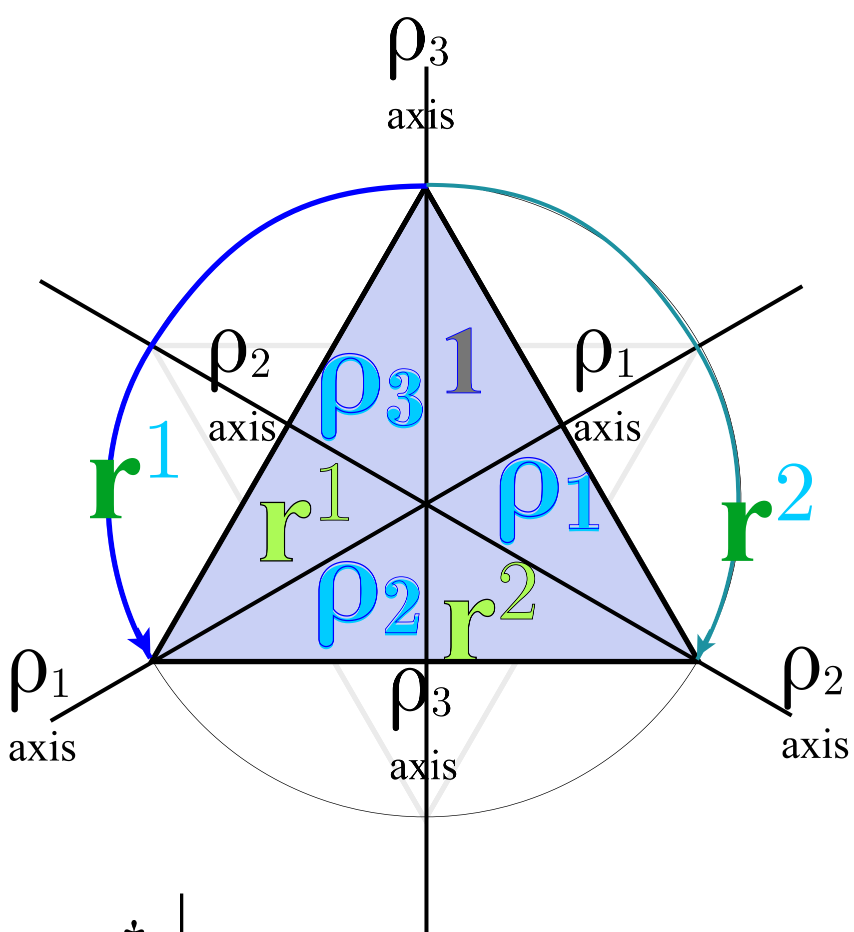
D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

Need to check that with table:

$\mathbf{r}^1\rho_3\mathbf{r}^2 = \rho_2\mathbf{r}^2 = \rho_1$

Checks out!

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes



D_3 $\mathbf{gg}^\dagger$ *form*	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}^1$	ρ_1	ρ_2	ρ_3
$\mathbf{r}^1$	$\mathbf{r}^1$	$\mathbf{1}$	$\mathbf{r}^2$	ρ_3	ρ_1	ρ_2
$\mathbf{r}^2$	$\mathbf{r}^2$	$\mathbf{r}^1$	$\mathbf{1}$	ρ_2	ρ_3	ρ_1
ρ_1	ρ_1	ρ_3	ρ_2	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
ρ_2	ρ_2	ρ_1	ρ_3	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
ρ_3	ρ_3	ρ_2	ρ_1	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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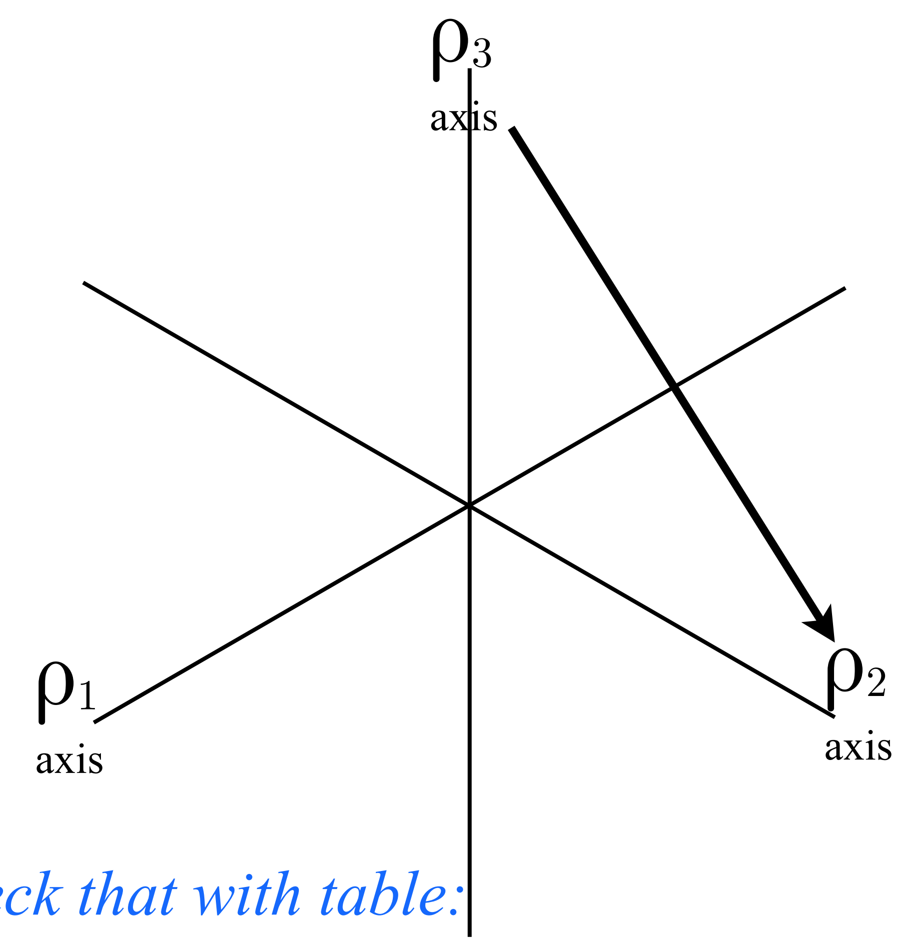
Example 2: Rotating ρ_3 axis crank using ρ_1 puts it down onto ρ_2

Seems to imply: $\rho_1\rho_3(\rho_1)^{-1} = \rho_1\rho_3\,\rho_1 = \rho_2$



Need to check that with table:

$\rho_1\rho_3\,\rho_1 = \mathbf{r}^2\rho_1 = \rho_2$ *Checks out!*

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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What has been done so far:

Abelian (Commutative) C_2, C_3, ..., C_6 ...

H diagonalized by r^p *symmetry operators that COMMUTE with H* $(r^p H = H r^p)$,

and with each other $(r^p r^q = r^{p+q} = r^q r^p)$.

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What we need to learn now:

Non-Abelian (do not commute) D_3, O_h,...

While all H symmetry operations COMMUTE with H $(\mathbf{U} H = H \mathbf{U})$

most do not with each other $(\mathbf{U}\,\mathbf{V} \neq \mathbf{V}\mathbf{U})$.

What has been done so far:

Abelian (Commutative) $C_2, C_2, \ldots, C_6 \ldots$

H diagonalized by r^p *symmetry operators that COMMUTE*

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While all H symmetry operations COMMUTE

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most do not with each other $(\mathbf{U}\,\mathbf{V} \neq \mathbf{V}\mathbf{U})$.

Q: *So how do we write* H *in terms of non-commutative* $\mathbf{U}$ *?*

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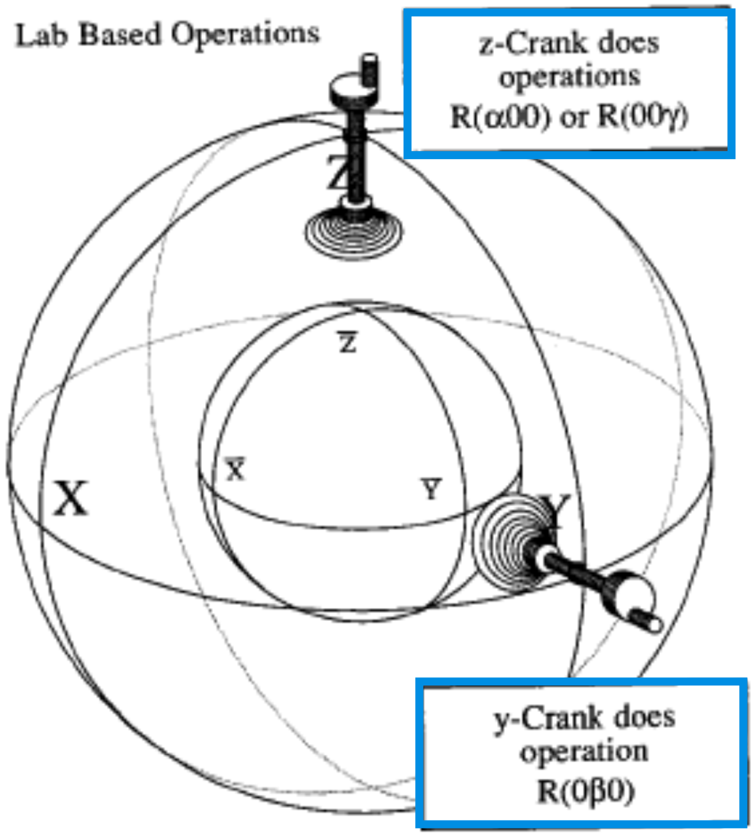
"Give me a place to stand...
and I will move the Earth"
Archimedes 287-212 B.C.E

Ideas of duality/relativity go *way* back (...VanVleck, Casimir..., Mach, Newton, Archimedes...)

Lab-fixed (Extrinsic-Global)**R**



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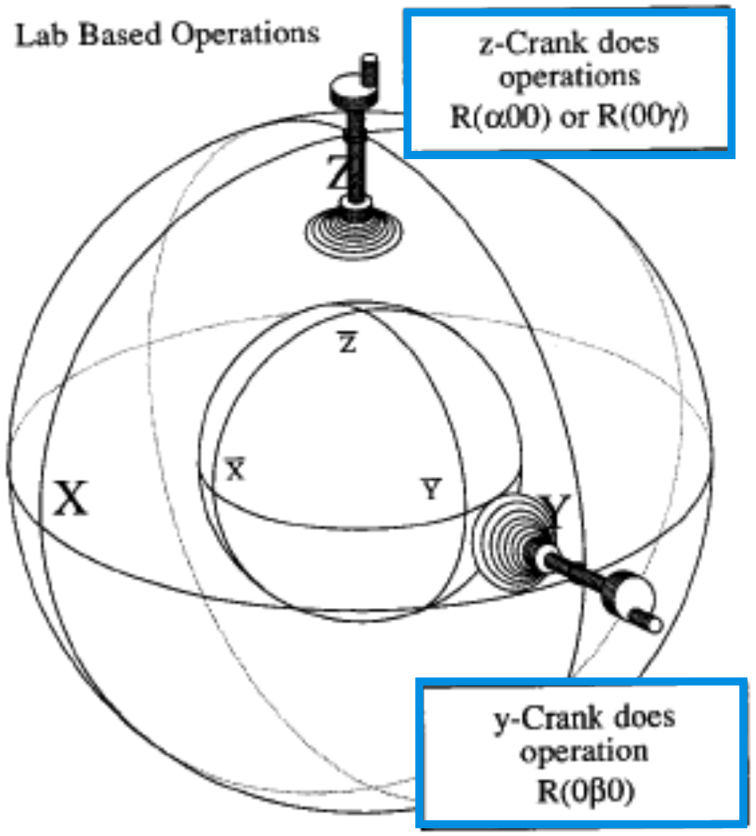

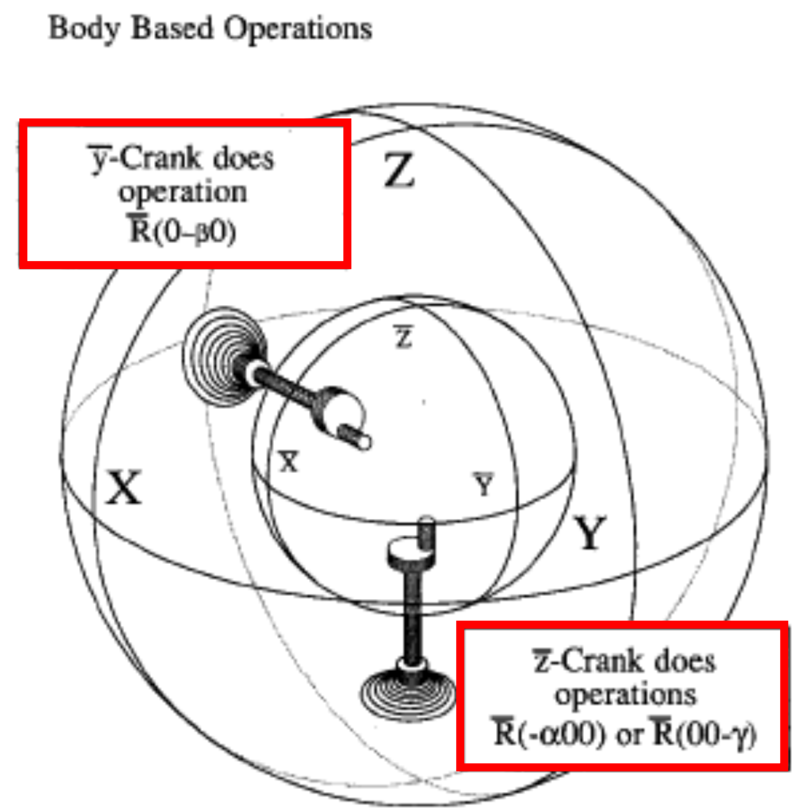

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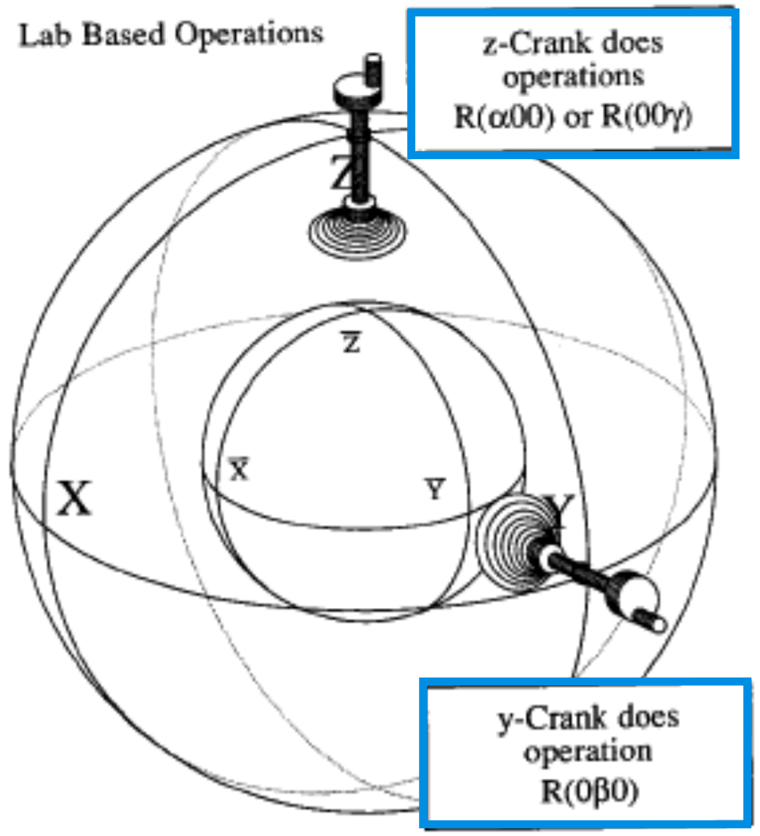
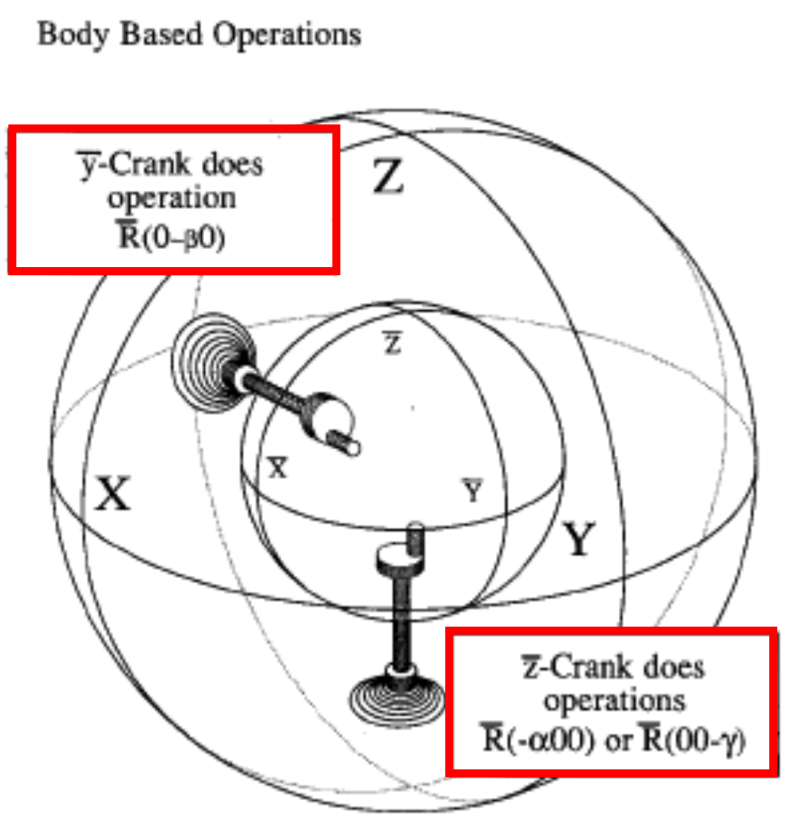


R *commutes*
with *all* $\bar{\mathbf{R}}$
(because they're independent or "unentangled")



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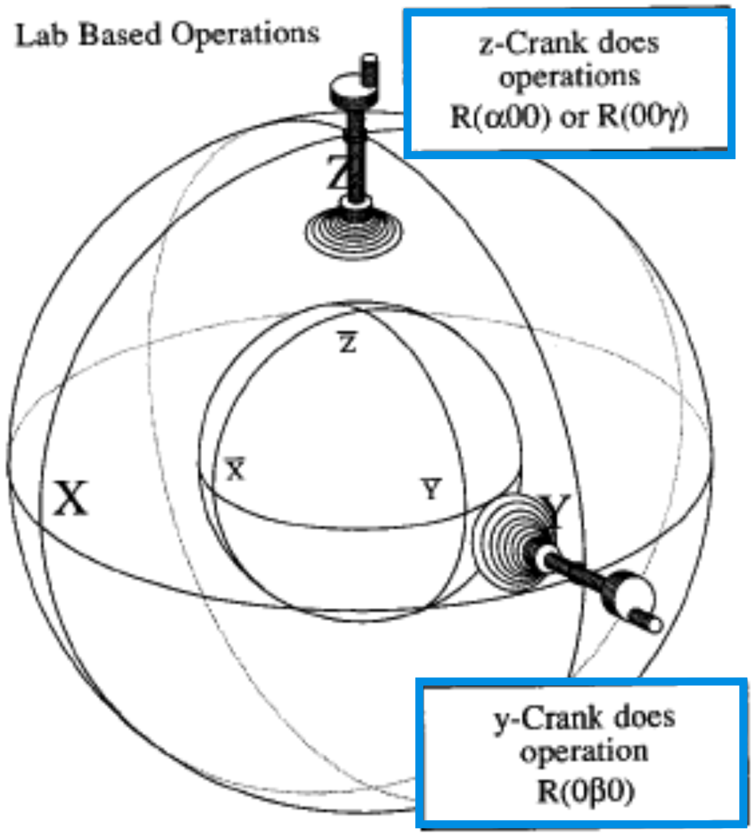


$\mathbf{R}$ *commutes*

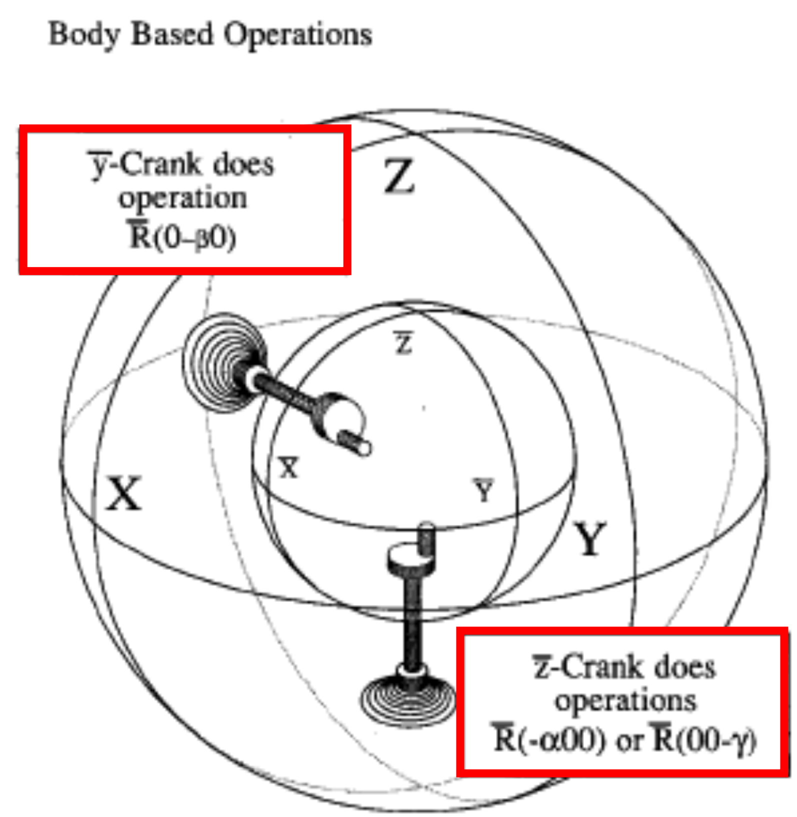
with *all* $\bar{\mathbf{R}}$

(because they're independent or "unentangled")

Mock-Mach relativity principle

$$\mathbf{R}|1\rangle=\bar{\mathbf{R}}^{-1}|1\rangle$$

...for *one* state $|1\rangle$ *only!*



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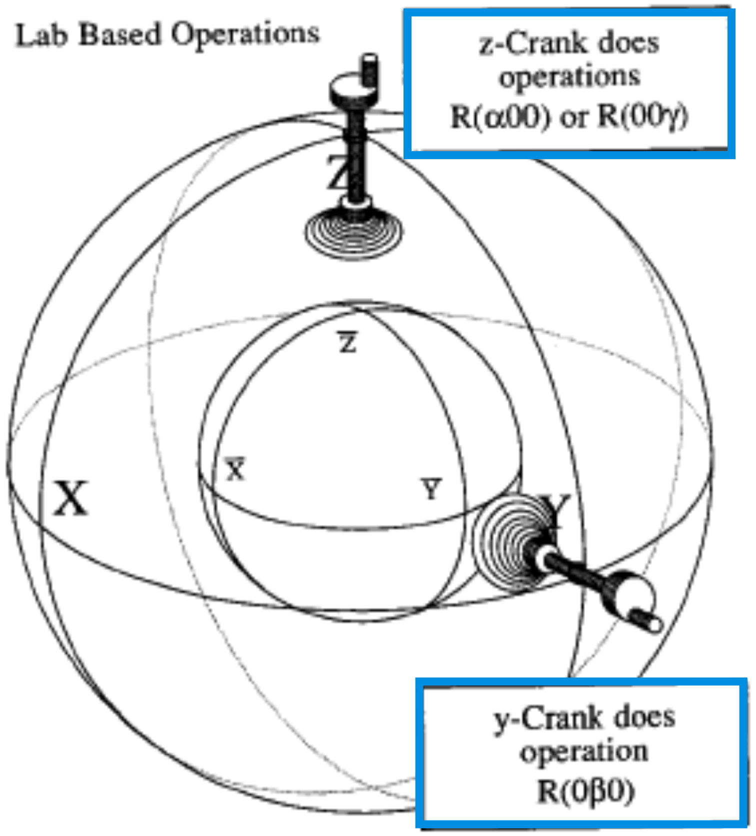
Ideas of duality/relativity go *way* back (...VanVleck, Casimir..., Mach, Newton, Archimedes...)

Lab-fixed (Extrinsic-Global)**R** *vs.* Body-fixed (Intrinsic-Local)$\bar{\mathbf{R}}$

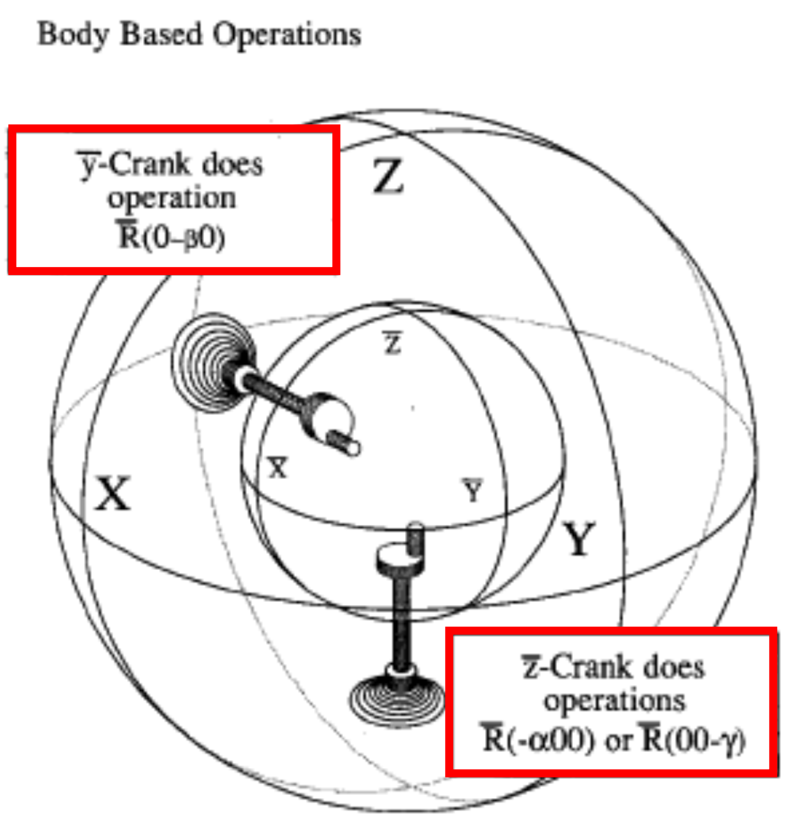


R *commutes*
with *all* $\bar{\mathbf{R}}$
(because they're independent or "unentangled")

Mock-Mach relativity principle

$\mathbf{R}|1\rangle=\bar{\mathbf{R}}^{-1}|1\rangle$

...for *one* state |1⟩ *only!*



...But *how* do you actually *make* the **R** and $\bar{\mathbf{R}}$ operations?

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

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Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

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Group theory of equivalence transformations and classes

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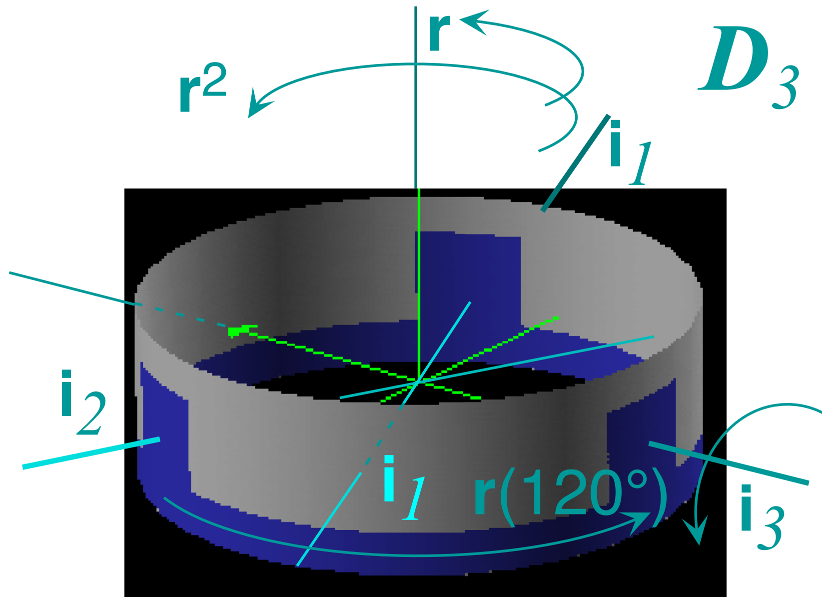
All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

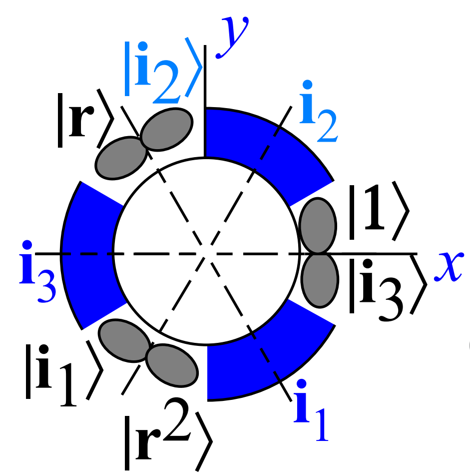
Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$

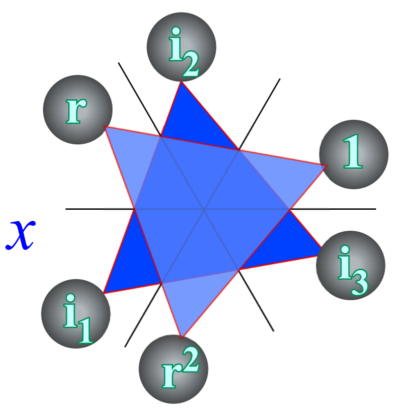


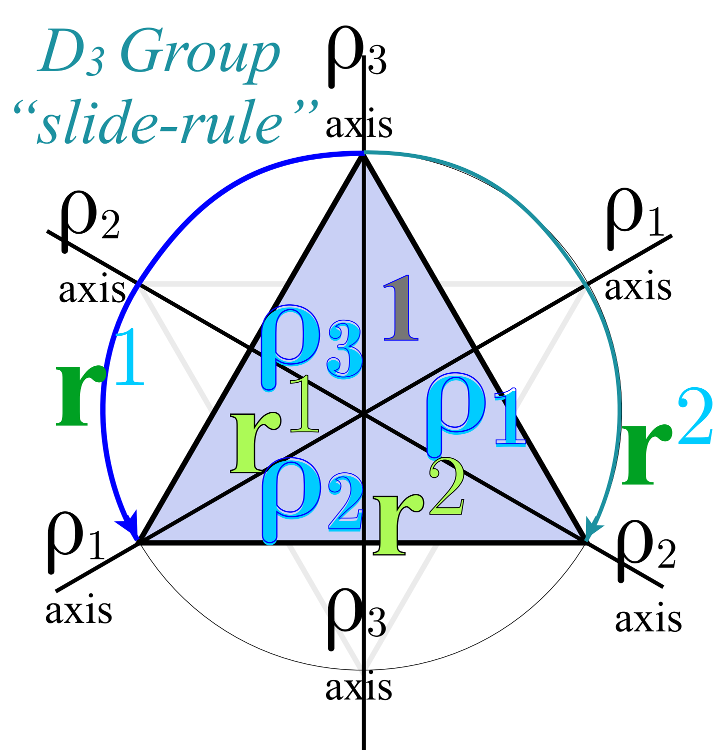
D_3

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

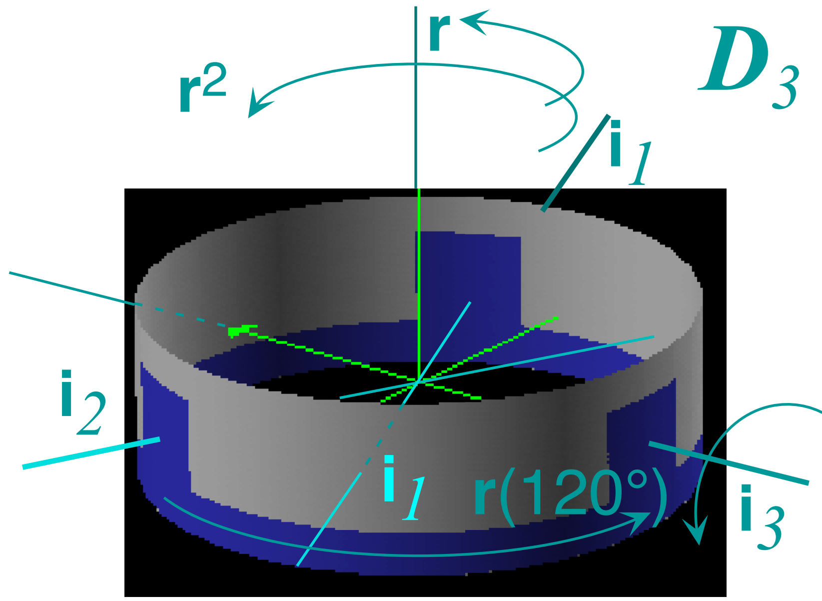
D_3-defined local-wave bases







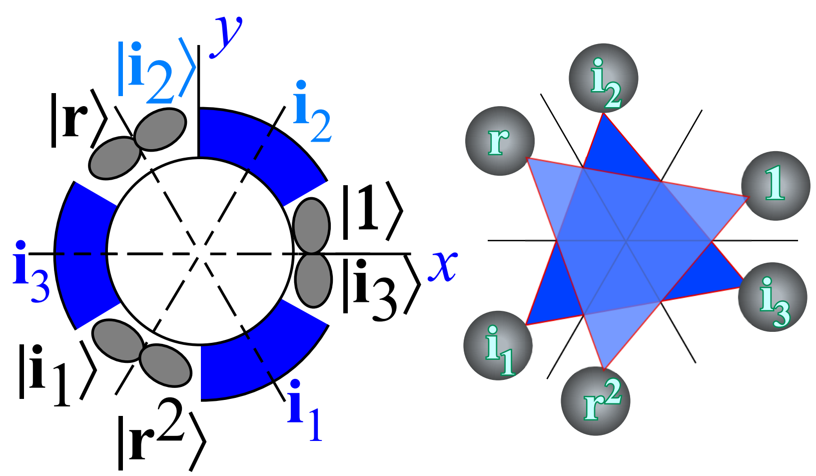
Example of GLOBAL vs LOCAL symmetry algebra for $D_3 \sim C_{3v}$

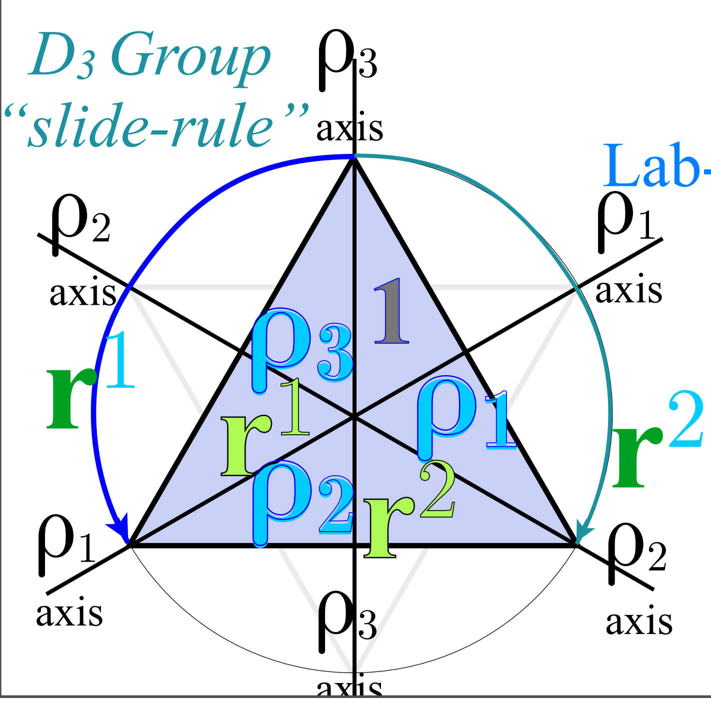


D_3

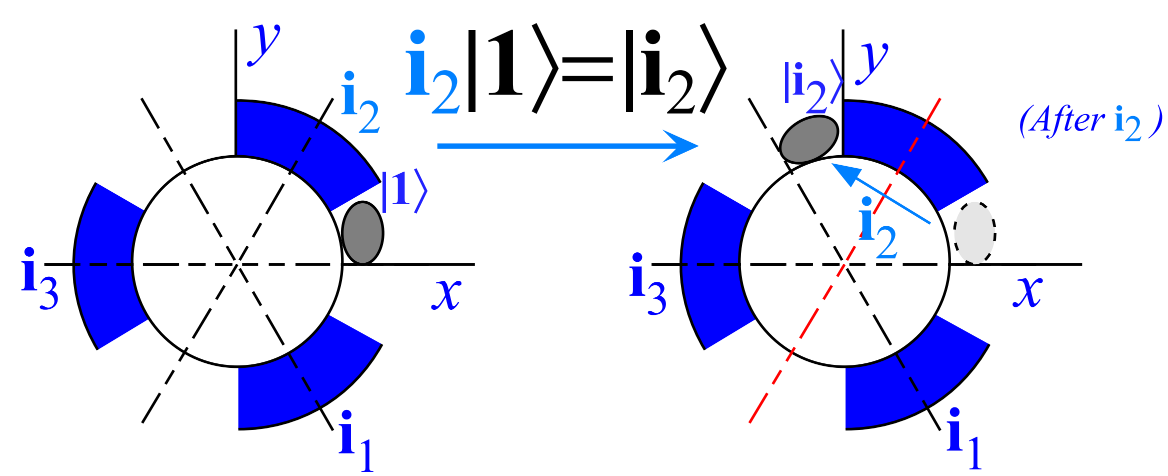
1	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	**1**

D_3-defined local-wave bases





Lab-fixed (Extrinsic-Global) operations and rotation axes



Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

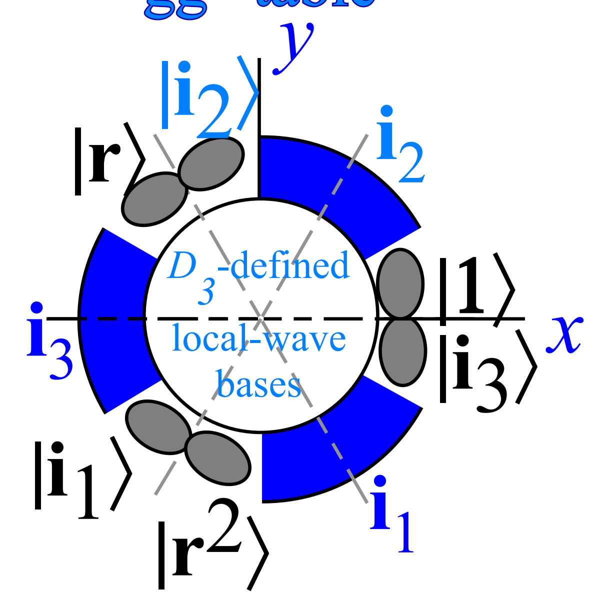
To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

D_3 global $gg^\dagger$-table



Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

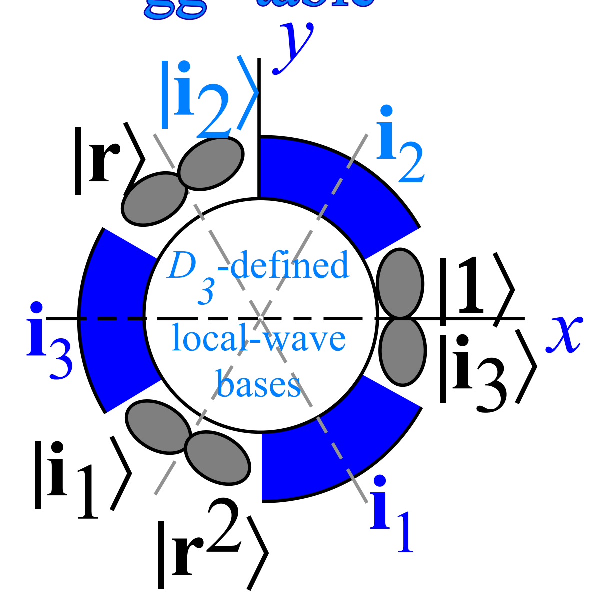
To represent *external* {..**T**,**U**,**V**,... }switch **g** ⇄ **g**† on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	r^2	r	i$_1$	i$_2$	i$_3$
r	1	r^2	i$_3$	i$_1$	i$_2$
r^2	r	1	i$_2$	i$_3$	i$_1$
i$_1$	i$_3$	i$_2$	1	r	r^2
i$_2$	i$_1$	i$_3$	r^2	1	r
i$_3$	i$_2$	i$_1$	r	r^2	1

D_3 global
gg†-table



D_3 local
g†g-table

To represent *internal* {..$\overline{\mathbf{T}}$,$\overline{\mathbf{U}}$,$\overline{\mathbf{V}}$,... } switch **g** ⇄ **g**† on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\quad R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\quad R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	r	r^2	i$_1$	i$_2$	i$_3$
r^2	1	r	i$_2$	i$_3$	i$_1$
r	r^2	1	i$_3$	i$_1$	i$_2$
i$_1$	i$_2$	i$_3$	1	r	r^2
i$_2$	i$_3$	i$_2$	r^2	1	r
i$_3$	i$_1$	i$_2$	r	r^2	1

Example of RELATIVITY-DUALITY for D_3~C_{3v}

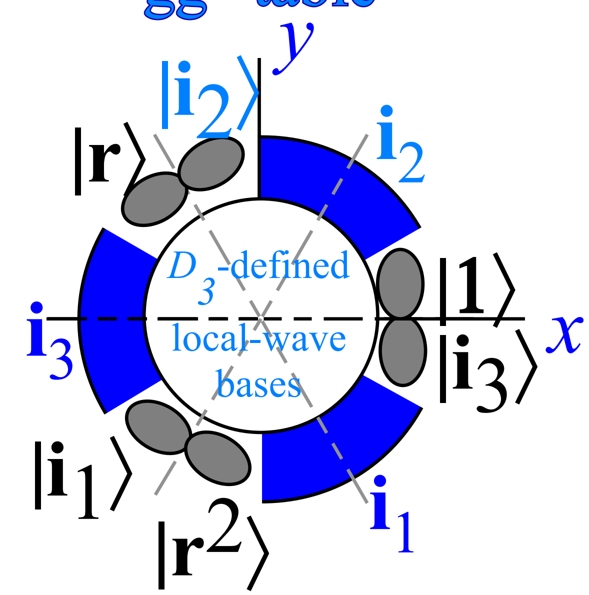
To represent *external* {..**T**,**U**,**V**,... }switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

D_3 global $\mathbf{g}\mathbf{g}^\dagger$-table



D_3 local $\mathbf{g}^\dagger\mathbf{g}$-table

RESULT:
Any $R(\mathbf{T})$
commute (Even if **T** and **U** do not...)
with any $R(\bar{\mathbf{U}})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix},\ R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix},\ R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot\\1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

1	**r**	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	**1**	**r**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
r	$\mathbf{r}^2$	**1**	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	**r**	$\mathbf{r}^2$	**1**

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

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All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

D_3 global $\mathbf{g}\mathbf{g}^\dagger$-table

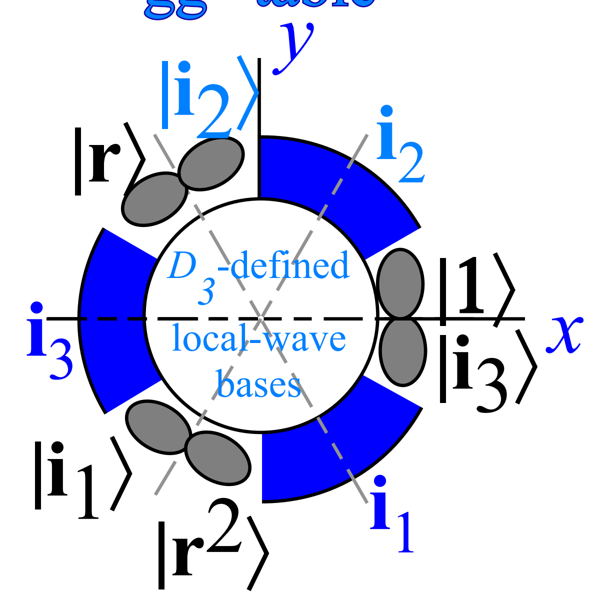
RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...) with any $R(\mathbf{U})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

$$\mathbb{H} = H\mathbf{1}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*



D_3 local $\mathbf{g}^\dagger\mathbf{g}$-table

To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$ switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}^2) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

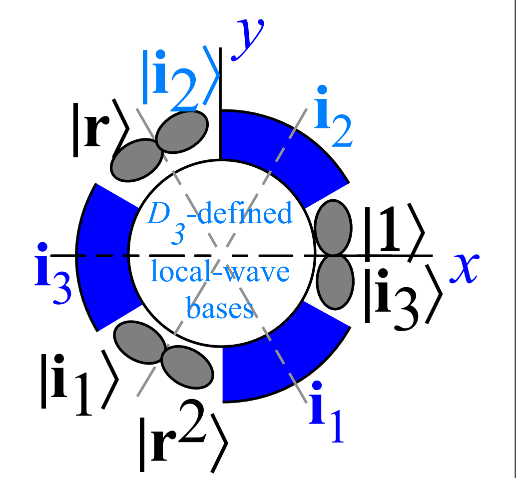
1	**r**	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	**1**	**r**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
r	$\mathbf{r}^2$	**1**	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	**r**	$\mathbf{r}^2$	**1**

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* $\{..\mathbf{T},\mathbf{U},\mathbf{V},...\}$ switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on top of group table

$$R^G(\mathbf{1}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \quad R^G(\mathbf{r}) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\mathbf{i}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\mathbf{i}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$



Local $\mathbb{H}$ *matrix parametrized by* $\bar{\mathbf{g}}$*'s*

RESULT:
Any $R(\mathbf{T})$ *commute* (Even if $\mathbf{T}$ and $\mathbf{U}$ do not...) with any $R(\bar{\mathbf{U}})$...

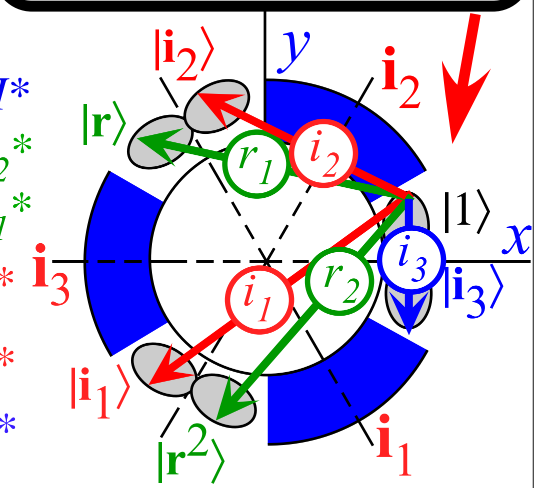
...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

So an $\mathbb{H}$*-matrix having* ***Global*** *symmetry* D_3

$$\mathbb{H} = H\bar{\mathbf{1}}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

is made from ***Local*** *symmetry matrices*

$H = \langle 1|\mathbb{H}|1\rangle = H^*$
$r_1 = \langle \mathbf{r}|\mathbb{H}|1\rangle = r_2^*$
$r_2 = \langle \mathbf{r}^2|\mathbb{H}|1\rangle = r_1^*$
$i_1 = \langle \mathbf{i}_1|\mathbb{H}|1\rangle = i_1^*$
$i_2 = \langle \mathbf{i}_2|\mathbb{H}|1\rangle = i_2^*$
$i_3 = \langle \mathbf{i}_3|\mathbb{H}|1\rangle = i_3^*$



To represent *internal* $\{..\bar{\mathbf{T}},\bar{\mathbf{U}},\bar{\mathbf{V}},...\}$ switch $\mathbf{g} \leftrightarrow \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{r}}^2) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1) = \begin{pmatrix} \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_2) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot \end{pmatrix}, \quad R^G(\bar{\mathbf{i}}_3) = \begin{pmatrix} \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot \end{pmatrix}$$

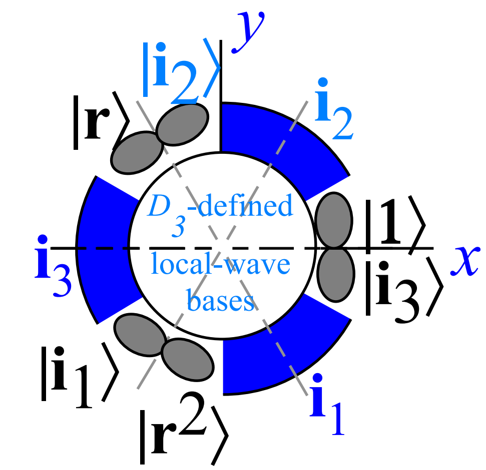
local D_3 defined Hamiltonian matrix

$\mathbb{H} =$

	$\vert\mathbf{1})$	$\vert\mathbf{r})$	$\vert\mathbf{r}^2)$	$\vert\mathbf{i}_1)$	$\vert\mathbf{i}_2)$	$\vert\mathbf{i}_3)$
$(\mathbf{1}\vert$	H	r_1	r_2	i_1	i_2	i_3
$(\mathbf{r}\vert$	r_2	H	r_1	i_2	i_3	i_1
$(\mathbf{r}^2\vert$	r_1	r_2	H	i_3	i_1	i_2
$(\mathbf{i}_1\vert$	i_1	i_2	i_3	H	r_1	r_2
$(\mathbf{i}_2\vert$	i_2	i_3	i_2	r_2	H	r_1
$(\mathbf{i}_3\vert$	i_3	i_1	i_2	r_1	r_2	H

Example of RELATIVITY-DUALITY for $D_3 \sim C_{3v}$

To represent *external* {..**T**,**U**,**V**,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on top of group table



$$R^G(\mathbf{1})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}, R^G(\mathbf{r})=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix}, R^G(\mathbf{r}^2)=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\mathbf{i}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix}, R^G(\mathbf{i}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}, R^G(\mathbf{i}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

Local $\mathbb{H}$ matrix parametrized by $\bar{\mathbf{g}}$'s

RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...)
with any $R(\bar{\mathbf{U}})$...

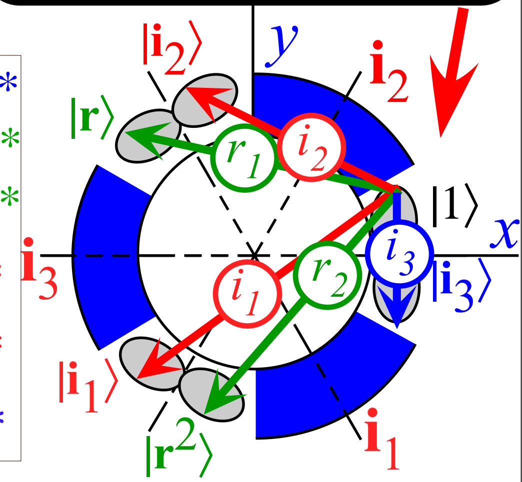
...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\bar{\mathbf{T}}\cdot\bar{\mathbf{U}}=\bar{\mathbf{V}}$.

*So an $\mathbb{H}$-matrix having **Global** symmetry D_3*

$$\mathbb{H} = H\bar{\mathbf{1}}^0 + r_1\bar{\mathbf{r}}^1 + r_2\bar{\mathbf{r}}^2 + i_1\bar{\mathbf{i}}_1 + i_2\bar{\mathbf{i}}_2 + i_3\bar{\mathbf{i}}_3$$

*is made from **Local** symmetry matrices*

$H = \langle 1|\mathbb{H}|1\rangle = H^*$
$r_1 = \langle \mathbf{r}|\mathbb{H}|1\rangle = r_2^*$
$r_2 = \langle \mathbf{r}^2|\mathbb{H}|1\rangle = r_1^*$
$i_1 = \langle \mathbf{i}_1|\mathbb{H}|1\rangle = i_1^*$
$i_2 = \langle \mathbf{i}_2|\mathbb{H}|1\rangle = i_2^*$
$i_3 = \langle \mathbf{i}_3|\mathbb{H}|1\rangle = i_3^*$



All the global g commute with general local $\mathbb{H}$ matrix.

local D_3 defined Hamiltonian matrix

To represent *internal* {..$\bar{\mathbf{T}}$,$\bar{\mathbf{U}}$,$\bar{\mathbf{V}}$,... } switch $\mathbf{g} \leftrightarrows \mathbf{g}^\dagger$ on side of group table

$$R^G(\bar{\mathbf{1}})=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}, R^G(\bar{\mathbf{r}})=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\end{pmatrix}, R^G(\bar{\mathbf{r}}^2)=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\end{pmatrix},$$

$$R^G(\bar{\mathbf{i}}_1)=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}, R^G(\bar{\mathbf{i}}_2)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\end{pmatrix}, R^G(\bar{\mathbf{i}}_3)=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

$\mathbb{H} =$

	$\vert 1\rangle$	$\vert \mathbf{r}\rangle$	$\vert \mathbf{r}^2\rangle$	$\vert \mathbf{i}_1\rangle$	$\vert \mathbf{i}_2\rangle$	$\vert \mathbf{i}_3\rangle$
$\langle 1\vert$	H	r_1	r_2	i_1	i_2	i_3
$\langle \mathbf{r}\vert$	r_2	H	r_1	i_2	i_3	i_1
$\langle \mathbf{r}^2\vert$	r_1	r_2	H	i_3	i_1	i_2
$\langle \mathbf{i}_1\vert$	i_1	i_2	i_3	H	r_1	r_2
$\langle \mathbf{i}_2\vert$	i_2	i_3	i_1	r_2	H	r_1
$\langle \mathbf{i}_3\vert$	i_3	i_1	i_2	r_1	r_2	H

Example of RELATIVITY-DUALITY for D

To represent *external* {..**T**,**U**,**V**,.... }...

$R^G(\mathbf{1}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}$, $R^G(\mathbf{r}) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}$, $R^G(\mathbf{r}^2) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix}$, $R^G(\mathbf{i}_1) =$

RESULT:
Any $R(\mathbf{T})$ *commute* (Even if **T** and **U** do not...)
with any $R(\overline{\mathbf{U}})$...

...and $\mathbf{T}\cdot\mathbf{U}=\mathbf{V}$ if & only if $\overline{\mathbf{T}}\cdot\overline{\mathbf{U}}=\overline{\mathbf{V}}$.

To represent *internal* {..$\overline{\mathbf{T}}$,$\overline{\mathbf{U}}$,$\overline{\mathbf{V}}$,.... }....

$R^G(\overline{\mathbf{1}}) = \begin{pmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1 \end{pmatrix}$, $R^G(\overline{\mathbf{r}}) = \begin{pmatrix} \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot \end{pmatrix}$, $R^G(\overline{\mathbf{r}}^2) = \begin{pmatrix} \cdot&\cdot&1&\cdot&\cdot&\cdot\\ 1&\cdot&\cdot&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\cdot&1&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot&1&\cdot \end{pmatrix}$, $R^G(\overline{\mathbf{i}}_1)$

$H = \langle 1|\mathbb{H}|1\rangle = H^*$

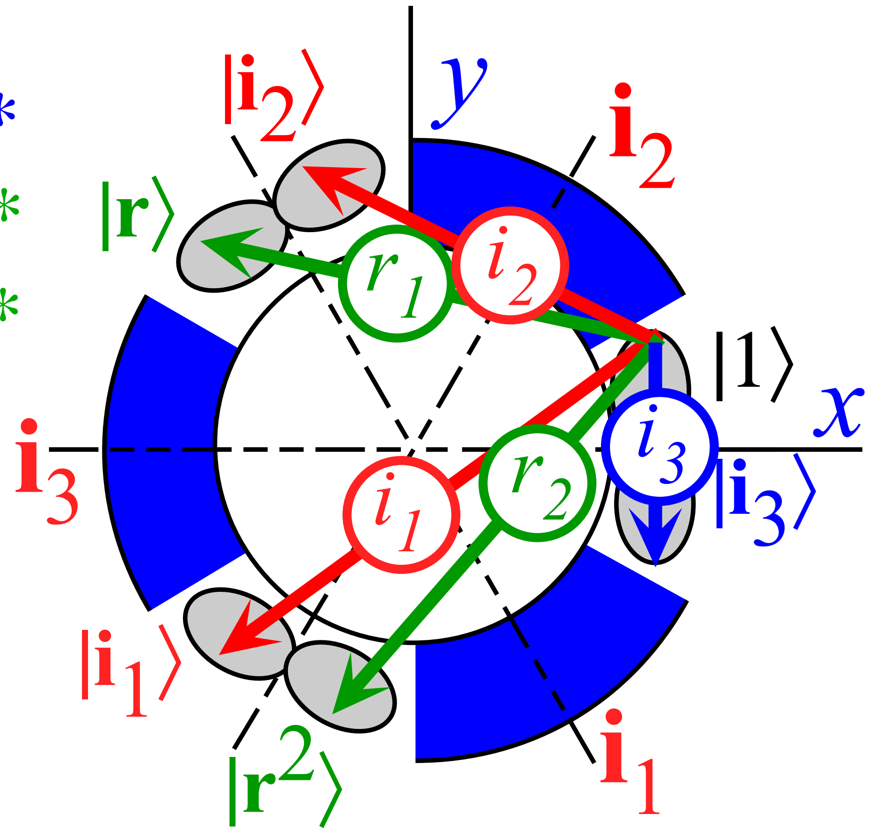
$r_1 = \langle \mathbf{r}|\mathbb{H}|1\rangle = r_2^*$

$r_2 = \langle \mathbf{r}^2|\mathbb{H}|1\rangle = r_1^*$

$i_1 = \langle \mathrm{i}_1|\mathbb{H}|1\rangle = i_1^*$

$i_2 = \langle \mathrm{i}_2|\mathbb{H}|1\rangle = i_2^*$

$i_3 = \langle \mathrm{i}_3|\mathbb{H}|1\rangle = i_3^*$



local-D_3-defined Hamiltonian matrix

So an $\mathbb{H}$*-matrix having **Global** symmetry* D_3

$$\mathbb{H} = H\overline{\mathbf{1}}^0 + r_1\overline{\mathbf{r}}^1 + r_2\overline{\mathbf{r}}^2 + i_1\overline{\mathbf{i}}_1 + i_2\overline{\mathbf{i}}_2 + i_3\overline{\mathbf{i}}_3$$

*is made from **Local** symmetry matrices*

$\mathbb{H} =$

	$\vert\mathbf{1})$	$\vert\mathbf{r})$	$\vert\mathbf{r}^2)$	$\vert\mathbf{i}_1)$	$\vert\mathbf{i}_2)$	$\vert\mathbf{i}_3)$
$(\mathbf{1}\vert$	H	r_1	r_2	i_1	i_2	i_3
$(\mathbf{r}\vert$	r_2	H	r_1	i_2	i_3	i_1
$(\mathbf{r}^2\vert$	r_1	r_2	H	i_3	i_1	i_2
$(\mathbf{i}_1\vert$	i_1	i_2	i_3	H	r_1	r_2
$(\mathbf{i}_2\vert$	i_2	i_3	i_2	r_2	H	r_1
$(\mathbf{i}_3\vert$	i_3	i_1	i_2	r_1	r_2	H

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

➡ *Group theory of equivalence transformations and classes* ⬅

Lagrange theorems

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

Review: Spectral resolution of ***D_3 Center*** *(Class algebra)*

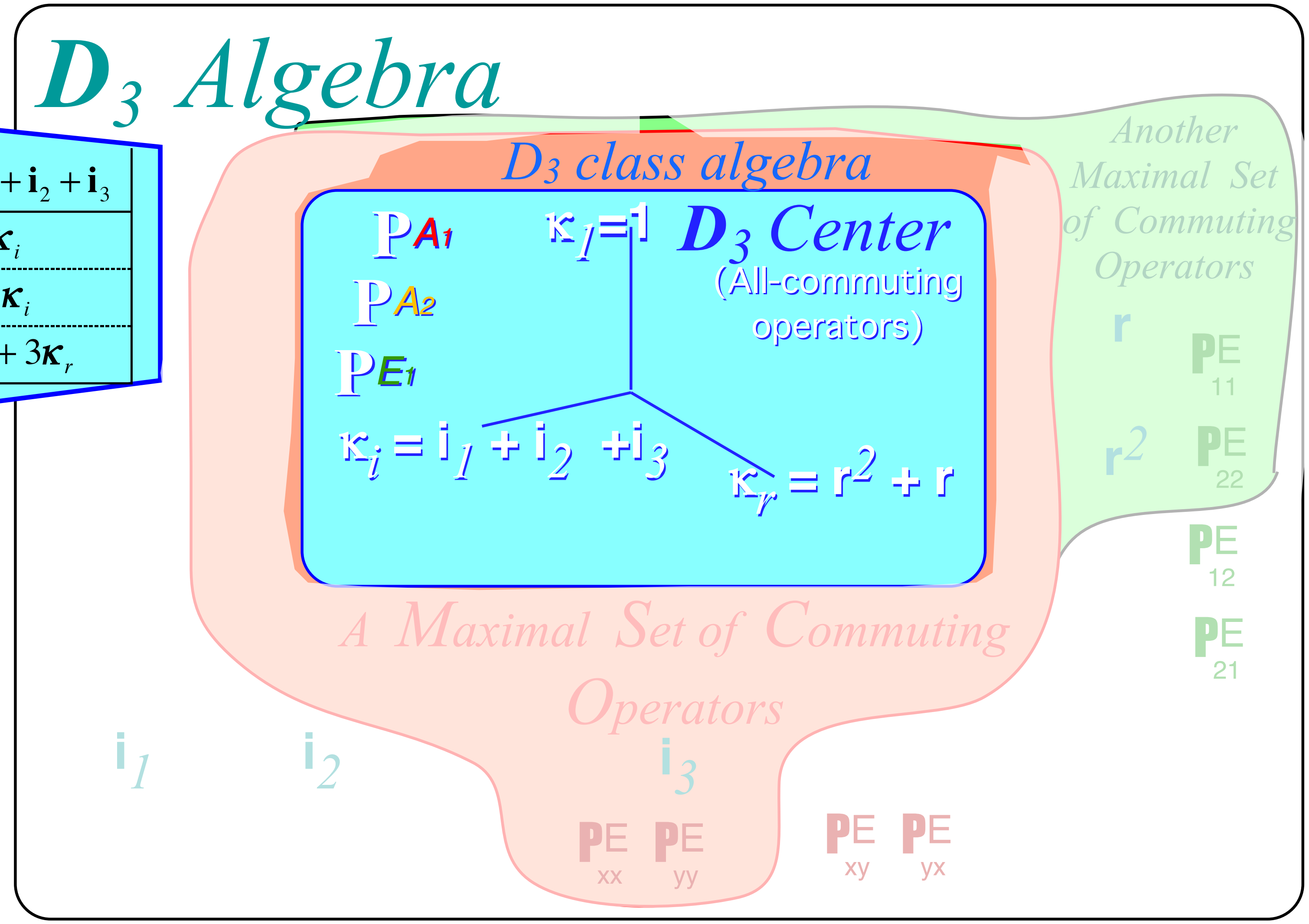
$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$



Review: Spectral resolution of ***D_3 Center*** *(Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

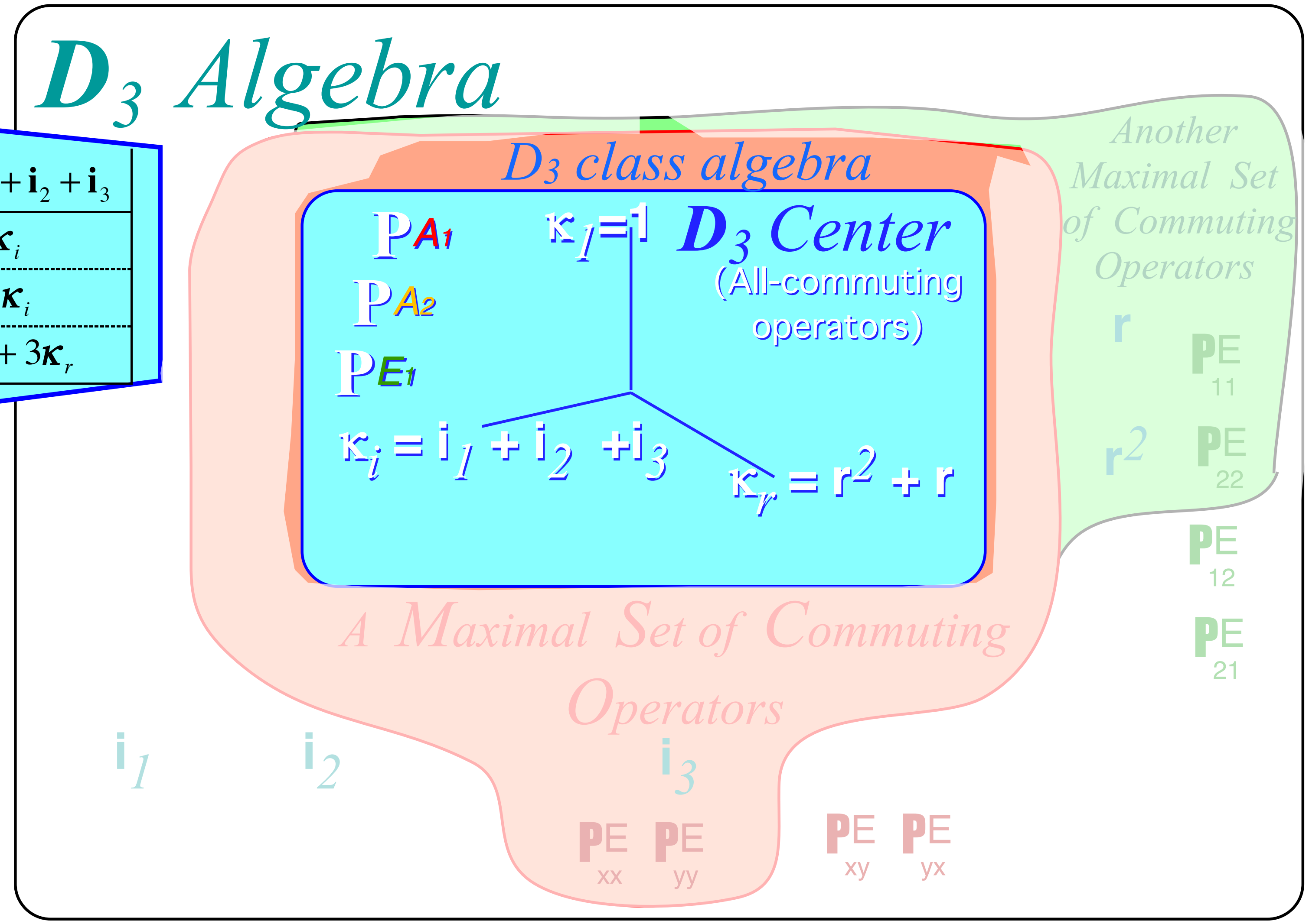
	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3 = 6)$

$^\circ\kappa_k$ = order of class κ_k: $(^\circ\kappa_1 = 1,\ ^\circ\kappa_r = 2,\ ^\circ\kappa_i = 3)$



Review: *Spectral resolution of D_3 Center (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

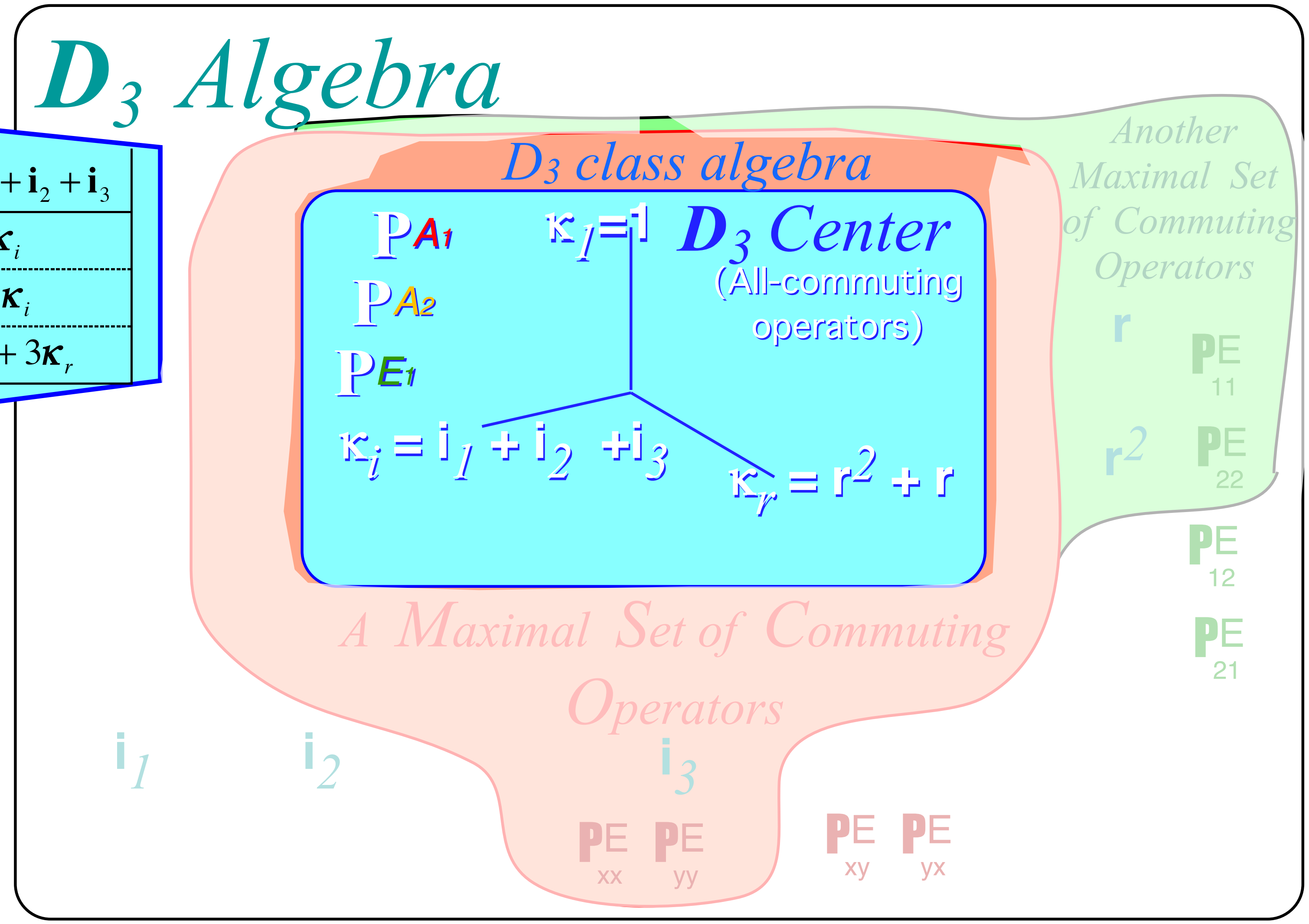
Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t \boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k \mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t \boldsymbol{\kappa}_k \mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j$$



Review: Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r}+\mathbf{r}^2$	$\kappa_i = \mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
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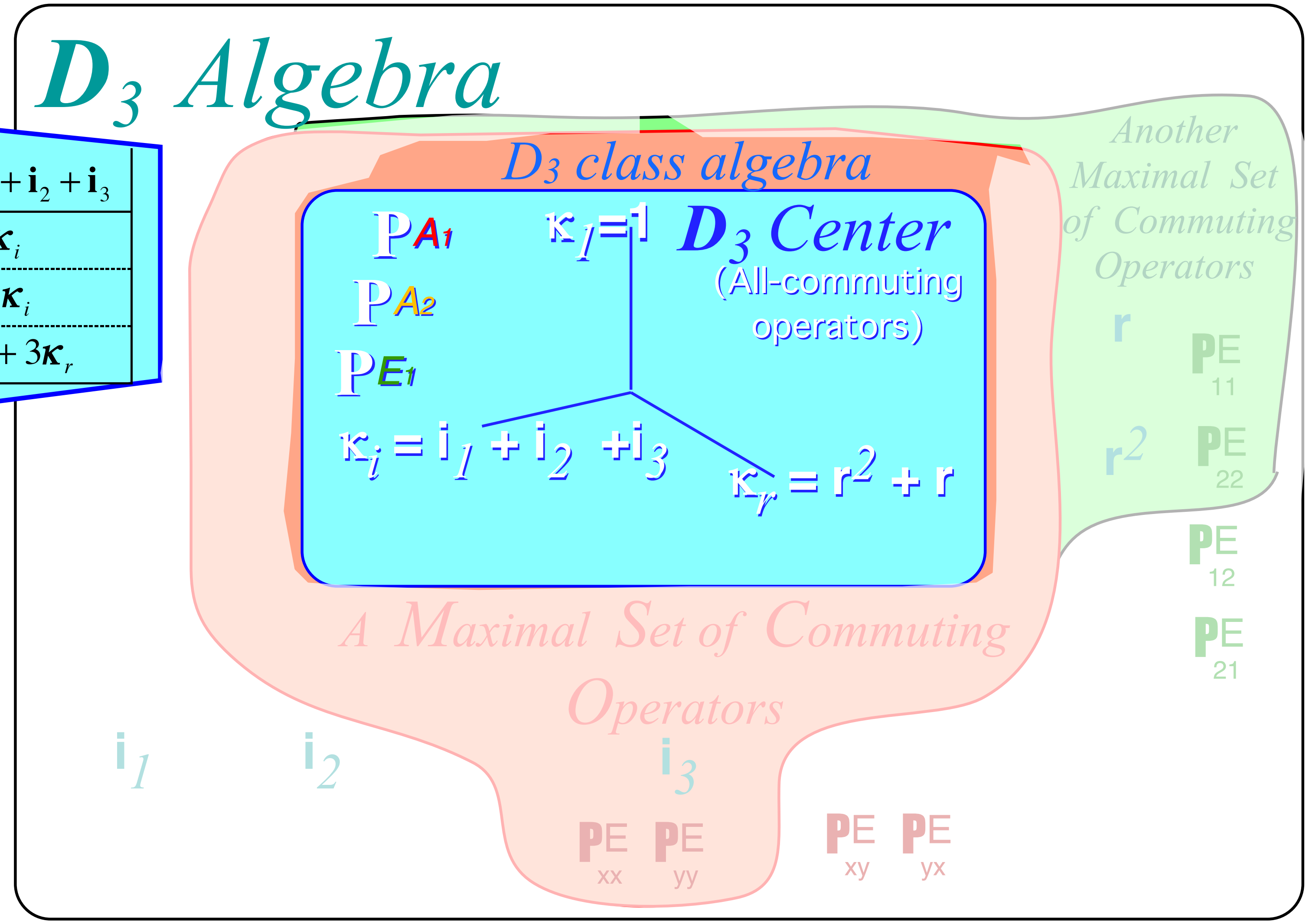
$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j = \frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$



Review: Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
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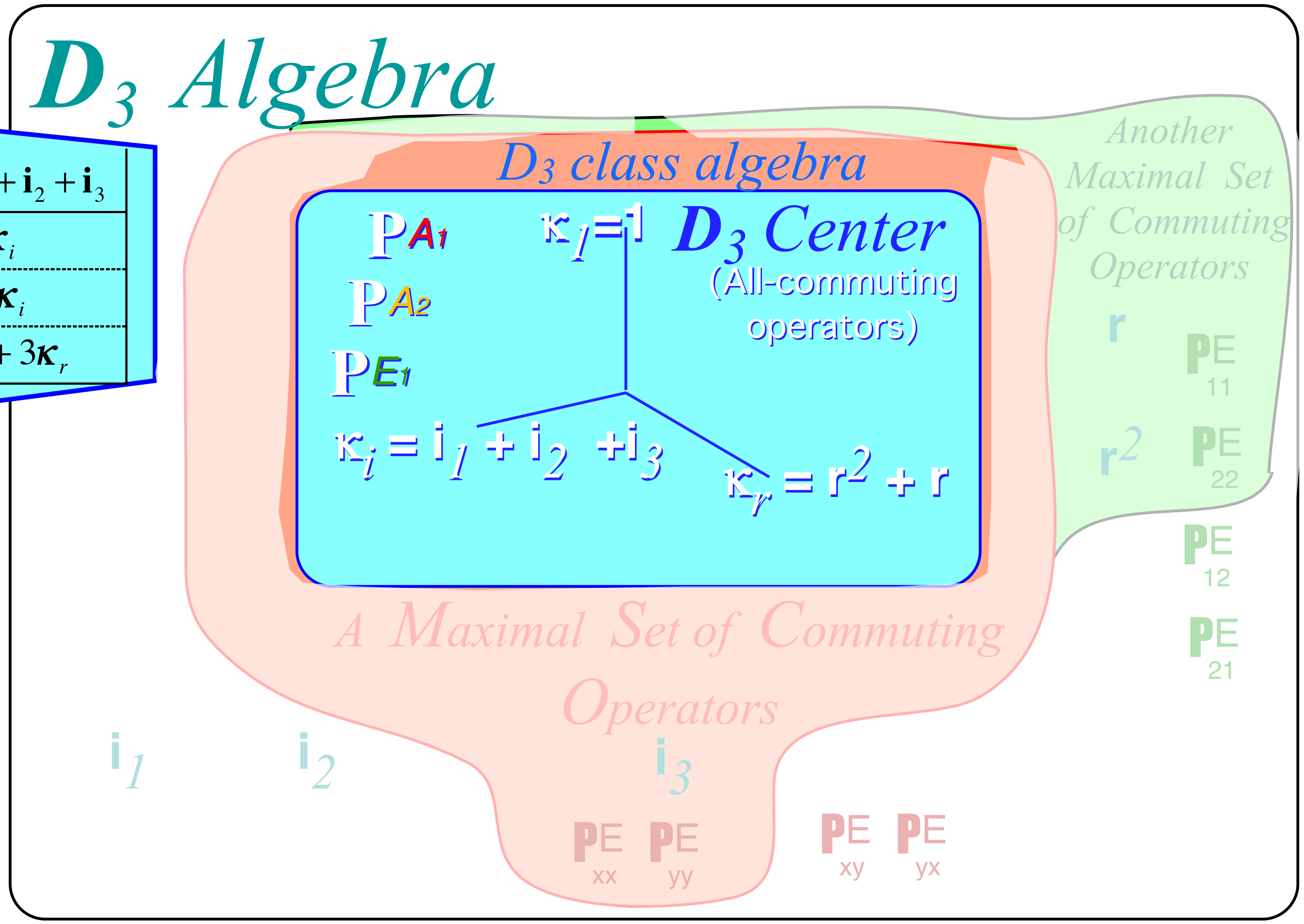
$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j=^\circ\kappa_k}\mathbf{g}_j=\frac{1}{^\circ s_k}\sum_{t=1}^{t=^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1=6,\ ^\circ s_r=3,\ ^\circ s_i=2)$

$^\circ s_k = {^\circ G} / {^\circ\kappa_k}$ *$^\circ s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*



3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle = \mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Stage spectral decomposition of global/local D_3 Hamiltonian

Group theory of equivalence transformations and classes

Lagrange theorems

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order

Review: Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$

Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

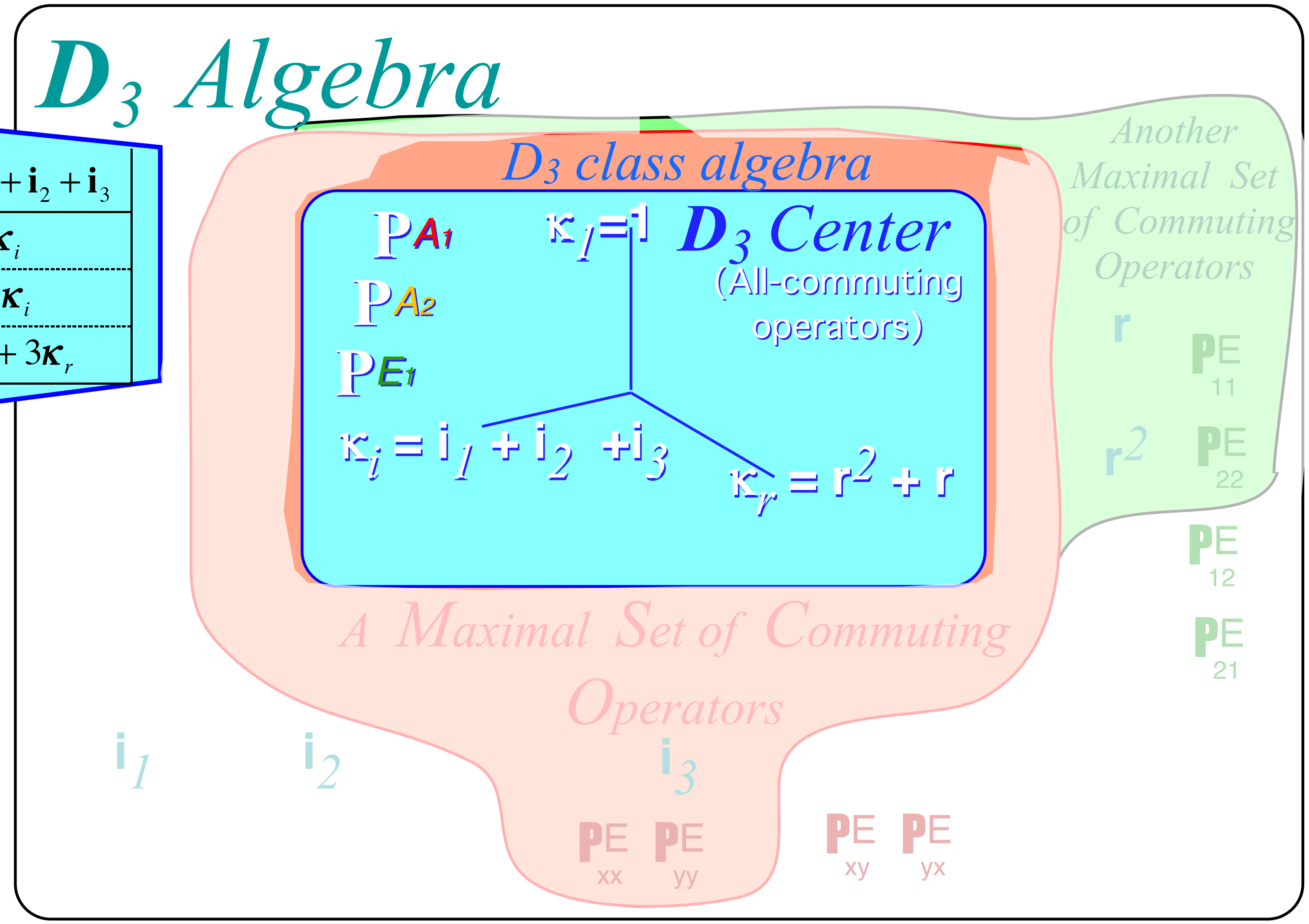
$°G$ = order of group: $(°D_3=6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1=1,\ °\kappa_r=2,\ °\kappa_i=3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j=°\kappa_k}\mathbf{g}_j=\frac{1}{°s_k}\sum_{t=1}^{t=°G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1=6,\ °s_r=3,\ °s_i=2)$

$°s_k=°G/°\kappa_k$ $\quad$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

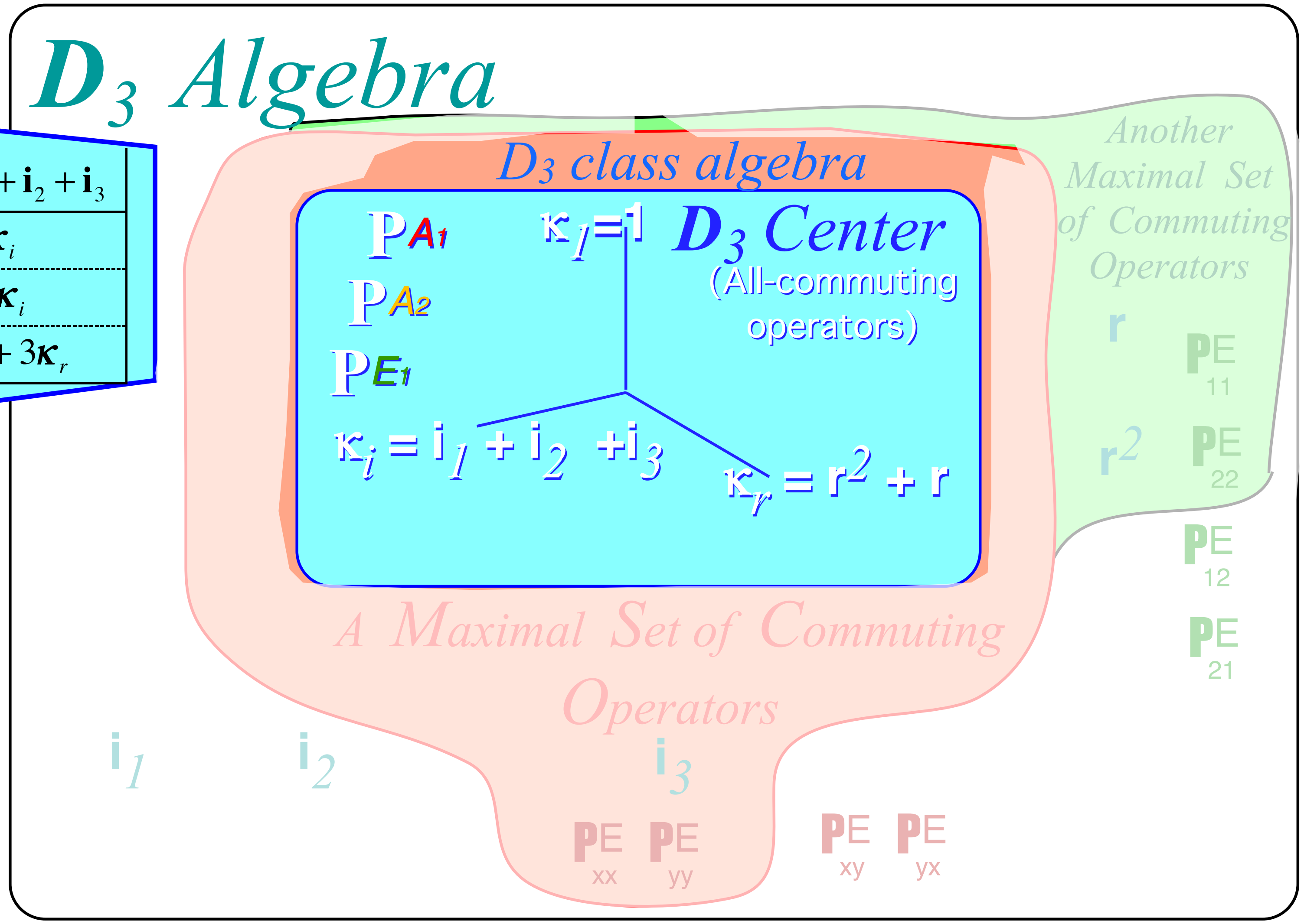


Review: Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1+3\kappa_r$



Class-sum $\boldsymbol{\kappa}_k$ commutes with all $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k = \boldsymbol{\kappa}_k\mathbf{g}_t$

$°G$ = order of group: $(°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $(°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1} = \boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k} = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k}\sum_{t=1}^{t=°G} \mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

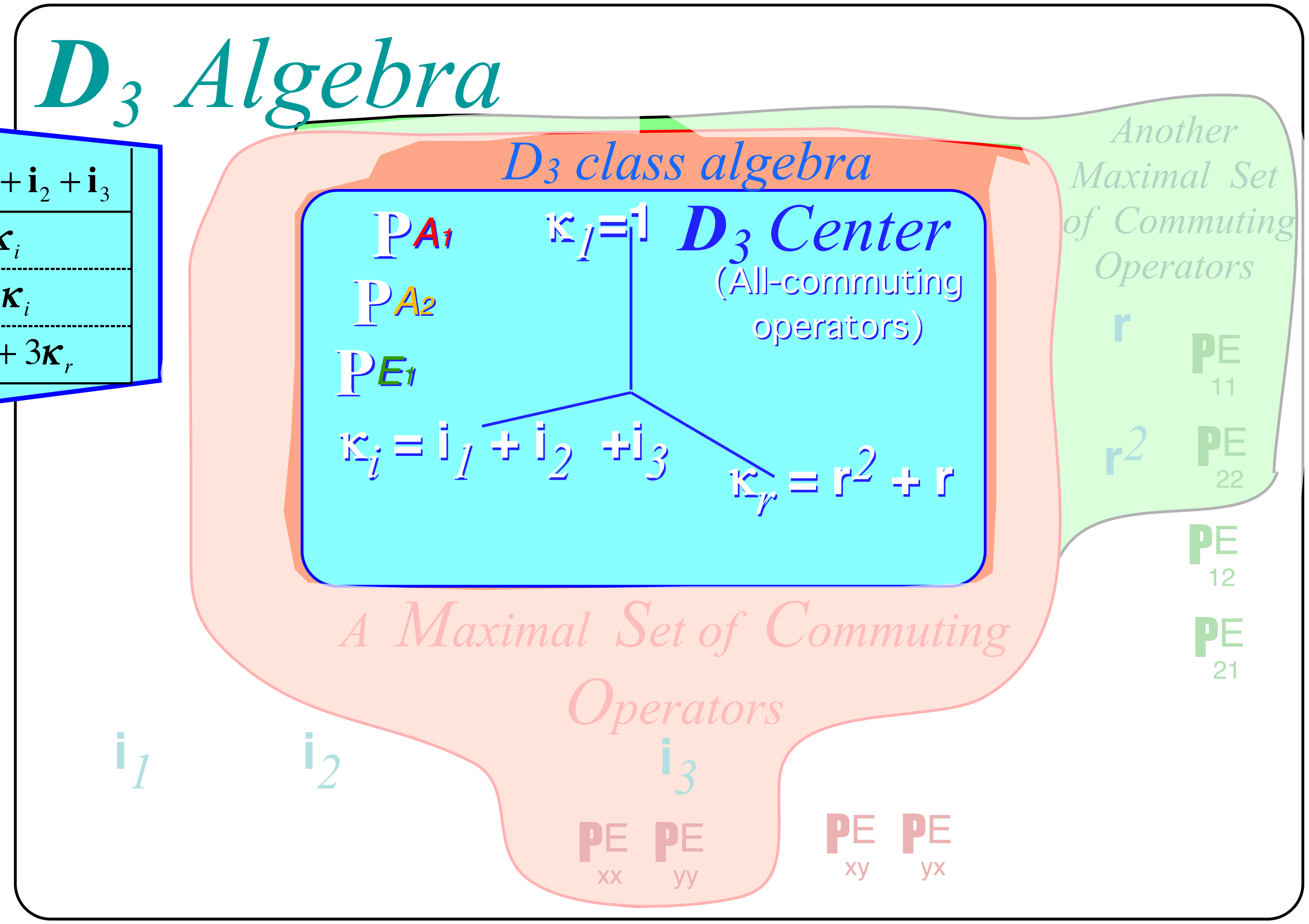
*Review: Spectral resolution of **D₃ Center** (Class algebra)*

1	r^2	r	i_1	i_2	i_3
r	1	r^2	i_3	i_1	i_2
r^2	r	1	i_2	i_3	i_1
i_1	i_3	i_2	1	r	r^2
i_2	i_1	i_3	r^2	1	r
i_3	i_2	i_1	r	r^2	1

$\rightarrow$

	$\kappa_1 = 1$	$\kappa_r = r + r^2$	$\kappa_i = i_1 + i_2 + i_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$

Class-sum κ_k commutes with all g_t



Class-sum κ_k invariance: $\quad g_t \kappa_k = \kappa_k g_t$

$°G$ = order of group: $\quad (°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $\quad (°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$g_t \kappa_k g_t^{-1} = \kappa_k \quad \text{where: } \kappa_k = \sum_{j=1}^{j=°\kappa_k} g_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} g_t g_k g_t^{-1}$$

$°s_k$=order of g_k-self-symmetry: $(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$ $\qquad$ *$°s_k$ is an integer count of D_3 operators g_s that commute with g_k.*

These operators g_s form the g_k-self-symmetry group s_k. Each g_s transforms g_k into itself: $g_s g_k g_s^{-1} = g_k$

If an operator g_t transforms g_k into a different element g'_k of its class: $g_t g_k g_t^{-1} = g'_k$, then so does $g_t g_s$.
that is: $g_t g_s g_k (g_t g_s)^{-1} = g_t g_s g_k g_s^{-1} g_t^{-1} = g_t g_k g_t^{-1} = g'_k$,

⋮

⋮

Review: *Spectral resolution of **D_3 Center** (Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1+\kappa_r$	$2\kappa_i$
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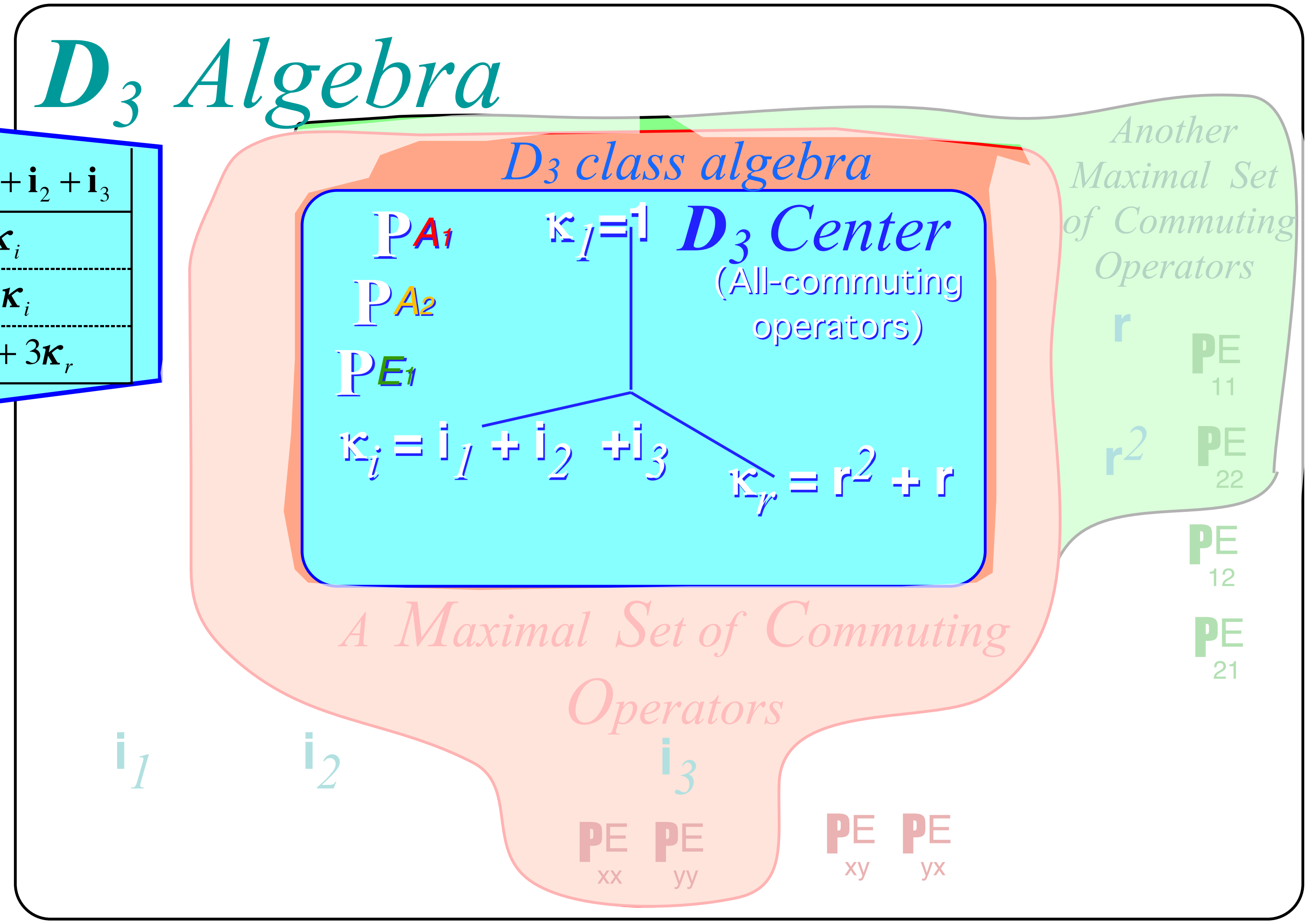

Class-sum κ_k commutes with all $\mathbf{g}_t$

Class-sum κ_k invariance: $\mathbf{g}_t\kappa_k=\kappa_k\mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3=6)$

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$$\mathbf{g}_t\kappa_k\mathbf{g}_t^{-1}=\kappa_k \quad \text{where: } \kappa_k=\sum_{j=1}^{j=^\circ\kappa_k}\mathbf{g}_j=\frac{1}{^\circ s_k}\sum_{t=1}^{t=^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}$$

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These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k . Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}=\mathbf{g}_k$

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that is: $\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k(\mathbf{g}_t\mathbf{g}_s)^{-1}=\mathbf{g}_t\mathbf{g}_s\mathbf{g}_k\mathbf{g}_s^{-1}\mathbf{g}_t^{-1}=\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}=\mathbf{g}'_k$,

Subgroup $s_k=\{\mathbf{g}_0=\mathbf{1},\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$ has $\ell=(^\circ\kappa_k-1)$ ***Left Cosets*** (one coset for each member of class κ_k).

$\mathbf{g}_1 s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\ldots\}$,

$\vdots$

$\vdots$

Review: Spectral resolution of D_3 Center (Class algebra)

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
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$\rightarrow$

	$\kappa_1=\mathbf{1}$	$\kappa_r=\mathbf{r}+\mathbf{r}^2$	$\kappa_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
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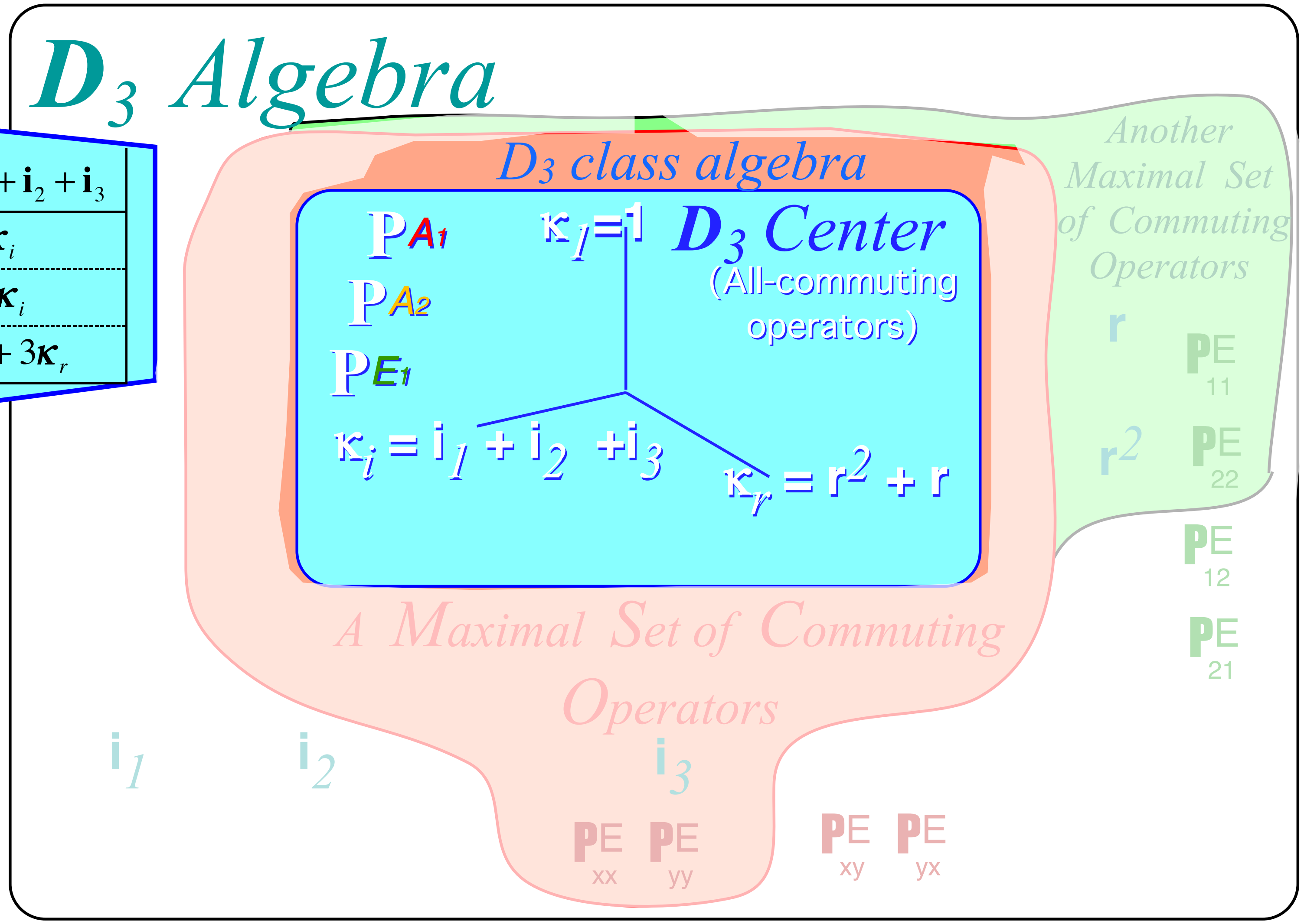

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$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1=6,\ ^\circ s_r=3,\ ^\circ s_i=2)$

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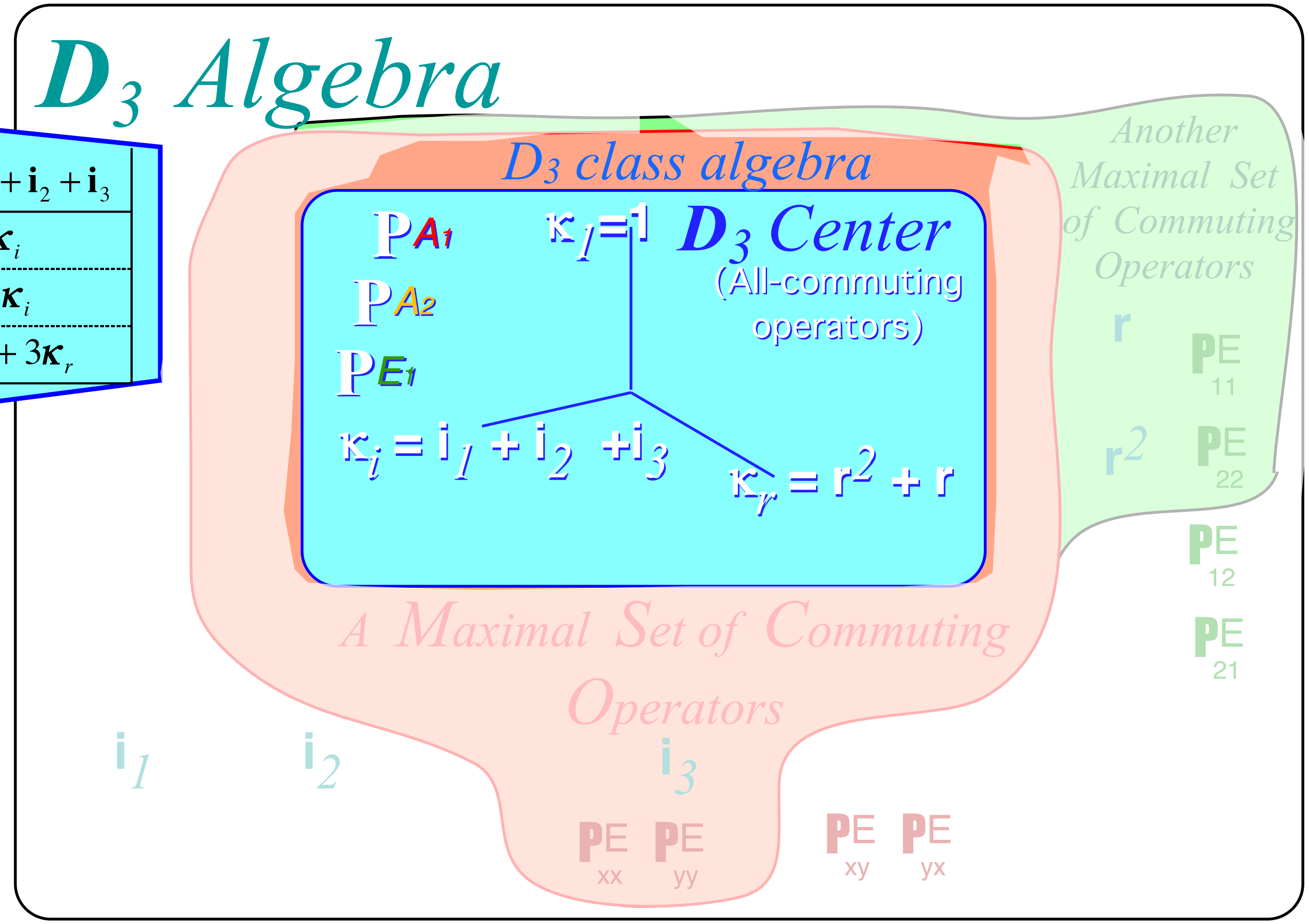
$$\mathbf{g}_1 s_k=\mathbf{g}_1\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\},$$
$$\mathbf{g}_2 s_k=\mathbf{g}_2\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\},\dots$$
$$\vdots$$
$$\mathbf{g}_\ell s_k=\mathbf{g}_\ell\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$$

Review: *Spectral resolution of* ***D_3 Center*** *(Class algebra)*

1	$\mathbf{r}^2$	**r**	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
r	**1**	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	**r**	**1**	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	**1**	**r**	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	**1**	**r**
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	**r**	$\mathbf{r}^2$	**1**

$\rightarrow$

	$\kappa_1 = \mathbf{1}$	$\kappa_r = \mathbf{r} + \mathbf{r}^2$	$\kappa_i = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_1	κ_1	κ_r	κ_i
κ_r	κ_r	$2\kappa_1 + \kappa_r$	$2\kappa_i$
κ_i	κ_i	$2\kappa_i$	$3\kappa_1 + 3\kappa_r$



Class-sum κ_k commutes with all $\mathbf{g}_t$

Class-sum κ_k invariance: $\qquad \mathbf{g}_t \kappa_k = \kappa_k \mathbf{g}_t$

$°G$ = order of group: $\qquad (°D_3 = 6)$

$°\kappa_k$ = order of class κ_k: $\qquad (°\kappa_1 = 1,\ °\kappa_r = 2,\ °\kappa_i = 3)$

$$\mathbf{g}_t \kappa_k \mathbf{g}_t^{-1} = \kappa_k \quad \text{where: } \kappa_k = \sum_{j=1}^{j=°\kappa_k} \mathbf{g}_j = \frac{1}{°s_k} \sum_{t=1}^{t=°G} \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1}$$

$°s_k$=order of $\mathbf{g}_k$-self-symmetry:$(°s_1 = 6,\ °s_r = 3,\ °s_i = 2)$

$°s_k = °G / °\kappa_k$ $\qquad$ *$°s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

These operators $\mathbf{g}_s$ form the $\mathbf{g}_k$-self-symmetry group s_k. Each $\mathbf{g}_s$ transforms $\mathbf{g}_k$ into itself: $\mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} = \mathbf{g}_k$

If an operator $\mathbf{g}_t$ transforms $\mathbf{g}_k$ into a different element $\mathbf{g}'_k$ of its class: $\mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$, then so does $\mathbf{g}_t \mathbf{g}_s$.

that is: $\mathbf{g}_t \mathbf{g}_s \mathbf{g}_k (\mathbf{g}_t \mathbf{g}_s)^{-1} = \mathbf{g}_t \mathbf{g}_s \mathbf{g}_k \mathbf{g}_s^{-1} \mathbf{g}_t^{-1} = \mathbf{g}_t \mathbf{g}_k \mathbf{g}_t^{-1} = \mathbf{g}'_k$,

Subgroup $s_k = \{\mathbf{g}_0 = \mathbf{1},\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$ has $\ell = (°\kappa_k - 1)$ ***Left Cosets*** (one coset for each member of class κ_k).

$\mathbf{g}_1 s_k = \mathbf{g}_1 \{\mathbf{g}_0 = 1,\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$,

$\mathbf{g}_2 s_k = \mathbf{g}_2 \{\mathbf{g}_0 = 1,\ \mathbf{g}_1 = \mathbf{g}_k,\ \mathbf{g}_2, \ldots\}$,...

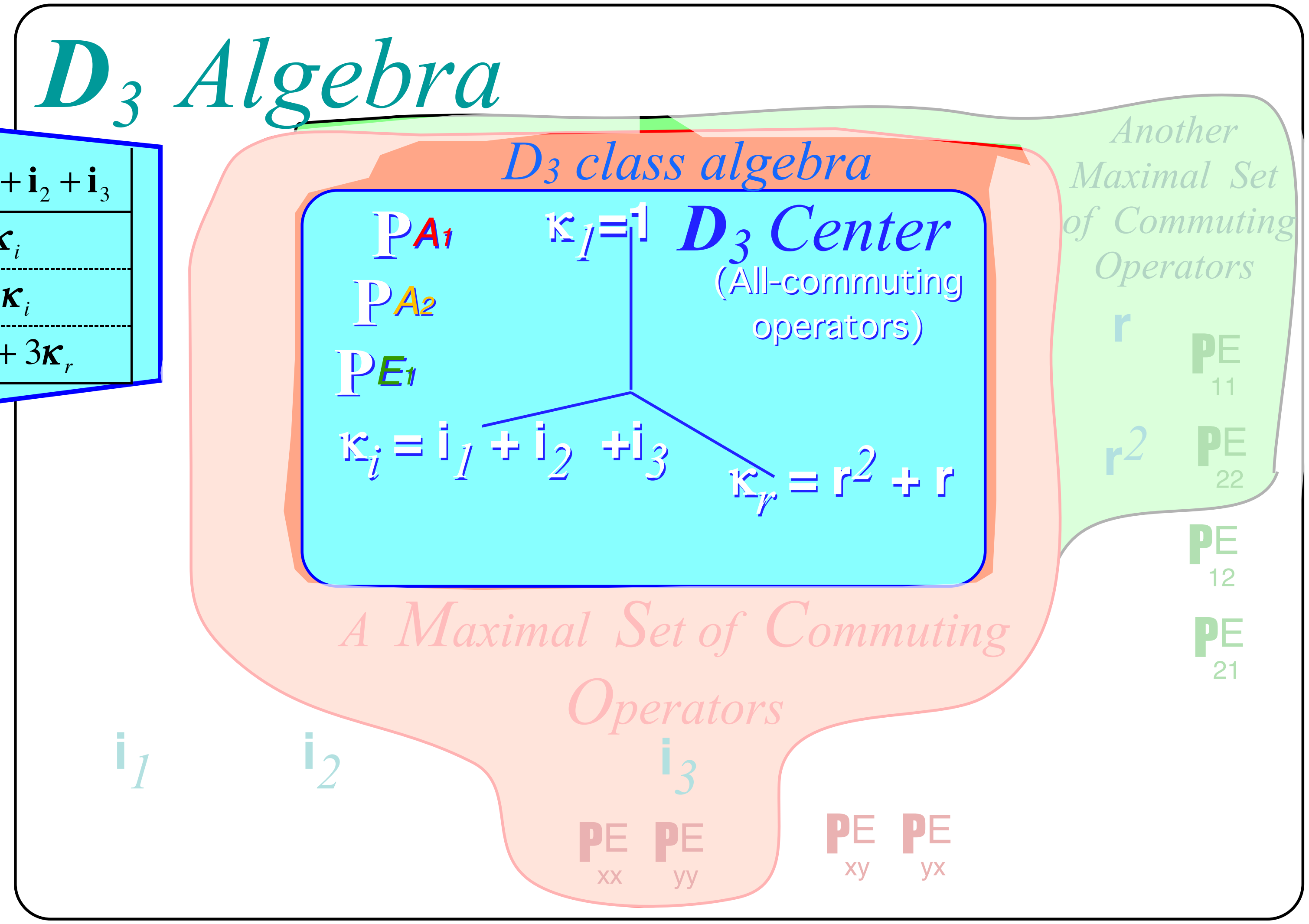
They will divide the group of order $°D_3 = °\kappa_k \cdot °s_k$ evenly into $°\kappa_k$ subsets each of order $°s_k$.

Review: *Spectral resolution of* D_3 ***Center*** *(Class algebra)*

$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}$	$\mathbf{1}$	$\mathbf{r}^2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{r}^2$	$\mathbf{r}$	$\mathbf{1}$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{1}$	$\mathbf{r}$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{i}_3$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}$
$\mathbf{i}_3$	$\mathbf{i}_2$	$\mathbf{i}_1$	$\mathbf{r}$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

	$\boldsymbol{\kappa}_1=\mathbf{1}$	$\boldsymbol{\kappa}_r=\mathbf{r}+\mathbf{r}^2$	$\boldsymbol{\kappa}_i=\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$
$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_1$	$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_r$	$\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_1+\boldsymbol{\kappa}_r$	$2\boldsymbol{\kappa}_i$
$\boldsymbol{\kappa}_i$	$\boldsymbol{\kappa}_i$	$2\boldsymbol{\kappa}_i$	$3\boldsymbol{\kappa}_1+3\boldsymbol{\kappa}_r$



Class-sum $\boldsymbol{\kappa}_k$ *commutes with all* $\mathbf{g}_t$

Class-sum $\boldsymbol{\kappa}_k$ invariance: $\mathbf{g}_t\boldsymbol{\kappa}_k=\boldsymbol{\kappa}_k\mathbf{g}_t$

$^\circ G$ = order of group: $(^\circ D_3=6)$

$^\circ\kappa_k$ = order of class κ_k: $(^\circ\kappa_1=1,\ ^\circ\kappa_r=2,\ ^\circ\kappa_i=3)$

$$\boxed{\mathbf{g}_t\boldsymbol{\kappa}_k\mathbf{g}_t^{-1}=\boldsymbol{\kappa}_k \quad \text{where: } \boldsymbol{\kappa}_\mathbf{k}=\sum_{j=1}^{j={}^\circ\kappa_k}\mathbf{g}_j=\frac{1}{^\circ s_k}\sum_{t=1}^{t={}^\circ G}\mathbf{g}_t\mathbf{g}_k\mathbf{g}_t^{-1}}$$

$^\circ s_k$=order of $\mathbf{g}_k$-self-symmetry: $(^\circ s_1=6,\ ^\circ s_r=3,\ ^\circ s_i=2)$

$\boxed{^\circ s_k={}^\circ G/{}^\circ\kappa_k}$ *$^\circ s_k$ is an integer count of D_3 operators $\mathbf{g}_s$ that commute with $\mathbf{g}_k$.*

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$\mathbf{g}_2 s_k=\mathbf{g}_2\{\mathbf{g}_0=1,\ \mathbf{g}_1=\mathbf{g}_k,\ \mathbf{g}_2,\dots\}$,...

These results are known as ***Lagrange's Coset Theorem(s)***

They will divide the group of order $\boxed{^\circ D_3={}^\circ\kappa_k\cdot{}^\circ s_k}$ evenly into $^\circ\kappa_k$ subsets each of order $^\circ s_k$.

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

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Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

$\boldsymbol{\kappa}_g$'s are *mutually commuting* with respect to themselves and *all-commuting* with respect to the whole group.

$$\mathbf{r}\,\boldsymbol{\kappa}_i\,\mathbf{r}^{-1} = \mathbf{i}_2 + \mathbf{i}_3 + \mathbf{i}_1 = \boldsymbol{\kappa}_i \quad \text{or:} \quad \mathbf{r}\,\boldsymbol{\kappa}_i = \boldsymbol{\kappa}_i\,\mathbf{r}$$

$$\sum_{\mathbf{h}=1}^{\circ G} \mathbf{hgh}^{-1} = \upsilon_g \boldsymbol{\kappa}_g\ , \qquad \text{where: } \upsilon_g = \frac{\circ G}{\circ \kappa_g} = integer$$

$^\circ\kappa_g$ is order of class $\boldsymbol{\kappa}_g$ and must evenly divide group order $^\circ G$.

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$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

$$\boldsymbol{\kappa}_3^2 = 3\cdot\boldsymbol{\kappa}_2 + 3\cdot\mathbf{1}$$

Note also:

$$\boldsymbol{\kappa}_2^2 - \boldsymbol{\kappa}_2 - 2\cdot\mathbf{1} = 0$$

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$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

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Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)}$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$\leftarrow \boldsymbol{\kappa}_3^2 = 3\cdot\boldsymbol{\kappa}_2 + 3\cdot\mathbf{1}$

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

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1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

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$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$0 = (\kappa_2 - 2\cdot\mathbf{1})(\kappa_2 + \mathbf{1})$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad 0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad 0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2} \qquad \kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})}{(+0-3)(+0+3)}$$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot 1 = 0$

$0 = (\kappa_2 - 2\cdot 1)(\kappa_2 + 1)$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad 0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad 0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2} \qquad \kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

Class resolution into sum of eigenvalue $\cdot$ **Projector**

$$\kappa_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$$

$$\kappa_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\kappa_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 - 0\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\mathbf{1})(\kappa_3 + 3\mathbf{1})}{(+0-3)(+0+3)}$$

Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

$0 = (\kappa_2 - 2\cdot\mathbf{1})(\kappa_2 + 1)$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

Note also:

$\kappa_2^2 - \kappa_2 - 2\cdot\mathbf{1} = 0$

$0 = (\kappa_2 - 2\cdot\mathbf{1})(\kappa_2 + \mathbf{1})$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad 0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad 0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2} \qquad \kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

Class resolution into sum of eigenvalue · Projector

$$\kappa_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$$

$$\kappa_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E} \longleftarrow \quad \kappa_r^2 = \kappa_r + 2\cdot\mathbf{1} \Rightarrow (\kappa_r - 2\cdot\mathbf{1})(\kappa_r + \mathbf{1}) = \mathbf{0}$$

$$\kappa_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})}{(+0-3)(+0+3)}$$

Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = (\kappa_1 + \kappa_2 + \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\kappa_1 - \kappa_2 + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$

$\rightarrow$

Each class-sum $\underline{\kappa}_k$ commutes with all of D_3.

$\kappa_1 = \mathbf{1}$	$\kappa_2 = \mathbf{r}^1 + \mathbf{r}^2$	$\kappa_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$
κ_2	$2\kappa_1 + \kappa_2$	$2\kappa_3$
κ_3	$2\kappa_3$	$3\kappa_1 + 3\kappa_2$

Class products give spectral polynomial and all-commuting projectors $\mathbf{P}^{(\alpha)} = \mathbf{P}^{A_1}$, $\mathbf{P}^{A_2}$, and $\mathbf{P}^{E}$

$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\kappa_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad 0 = (\kappa_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad 0 = (\kappa_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\kappa_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \kappa_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2} \qquad \kappa_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

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$$\mathbf{P}^{A_1} = \frac{(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

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$$\mathbf{P}^{A_2} = (\kappa_1 + \kappa_2 - \kappa_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\kappa_1 - \kappa_2 + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

χ_k^α	χ_1^α	χ_2^α	χ_3^α
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	−1
$\alpha = E$	2	−1	0

Spectral analysis of non-commutative "Group-table Hamiltonian"

1st Step: Spectral resolution of D_3-Center (Class algebra of D_3)

$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$
$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{i}_2$
$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{i}_2$	$\mathbf{i}_3$	$\mathbf{i}_1$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
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$$0 = \kappa_3^3 - 9\kappa_3 = (\kappa_3 - 3\cdot\mathbf{1})(\kappa_3 + 3\cdot\mathbf{1})(\kappa_3 - 0\cdot\mathbf{1})$$

$$0 = (\boldsymbol{\kappa}_3 - 3\cdot\mathbf{1})\mathbf{P}^{A_1} \qquad 0 = (\boldsymbol{\kappa}_3 + 3\cdot\mathbf{1})\mathbf{P}^{A_2} \qquad 0 = (\boldsymbol{\kappa}_3 - 0\cdot\mathbf{1})\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_3\mathbf{P}^{A_1} = +3\cdot\mathbf{P}^{A_1} \qquad \boldsymbol{\kappa}_3\mathbf{P}^{A_2} = -3\cdot\mathbf{P}^{A_2} \qquad \boldsymbol{\kappa}_3\mathbf{P}^{E} = +0\cdot\mathbf{P}^{E}$$

Class resolution into sum of eigenvalue $\cdot$ **Projector**

$$\boldsymbol{\kappa}_1 = 1\cdot\mathbf{P}^{A_1} + 1\cdot\mathbf{P}^{A_2} + 1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_r = 2\cdot\mathbf{P}^{A_1} + 2\cdot\mathbf{P}^{A_2} - 1\cdot\mathbf{P}^{E}$$

$$\boldsymbol{\kappa}_i = 3\cdot\mathbf{P}^{A_1} - 3\cdot\mathbf{P}^{A_2} + 0\cdot\mathbf{P}^{E}$$

$$\mathbf{P}^{A_1} = \frac{(\boldsymbol{\kappa}_3 + 3\cdot\mathbf{1})(\boldsymbol{\kappa}_3 - 0\cdot\mathbf{1})}{(+3+3)(+3-0)}$$

$$\mathbf{P}^{A_2} = \frac{(\boldsymbol{\kappa}_3 - 3\cdot\mathbf{1})(\boldsymbol{\kappa}_3 - 0\cdot\mathbf{1})}{(-3-3)(-3-0)}$$

$$\mathbf{P}^{E} = \frac{(\boldsymbol{\kappa}_3 - 3\cdot\mathbf{1})(\boldsymbol{\kappa}_3 + 3\cdot\mathbf{1})}{(+0-3)(+0+3)}$$

Inverse resolution gives D_3 Character Table

$$\mathbf{P}^{A_1} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 + \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3)/6$$

$$\mathbf{P}^{A_2} = (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_3)/6 = (\mathbf{1} + \mathbf{r} + \mathbf{r}^2 - \mathbf{i}_1 - \mathbf{i}_2 - \mathbf{i}_3)/6$$

$$\mathbf{P}^{E} = (2\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2 + 0)/3 = (2\mathbf{1} - \mathbf{r} - \mathbf{r}^2)/3$$

*Irreducible **characters** are traces*
$\chi_\kappa^{(\alpha)} = Tr\, D^{(\alpha)}(\mathbf{r}_\kappa)$
of irreducible representations
$D^{(\alpha)}(\mathbf{r}_\kappa)$

χ_k^α	χ_1^α	χ_2^α	χ_3^α
$\alpha = A_1$	1	1	1
$\alpha = A_2$	1	1	-1
$\alpha = E$	2	-1	0

3-Dihedral-axes group D_3 vs. 3-Vertical-mirror-plane group C_{3v}

D_3 and C_{3v} are isomorphic ($D_3 \sim C_{3v}$ share product table)

Deriving $D_3 \sim C_{3v}$ products:

By group definition $|g\rangle=\mathbf{g}|1\rangle$ of position ket $|g\rangle$

By nomograms based on U(2) Hamilton-turns

Deriving $D_3 \sim C_{3v}$ equivalence transformations and classes

Non-commutative symmetry expansion and Global-Local solution

Global vs Local symmetry and Mock-Mach principle

Global vs Local matrix duality for D_3

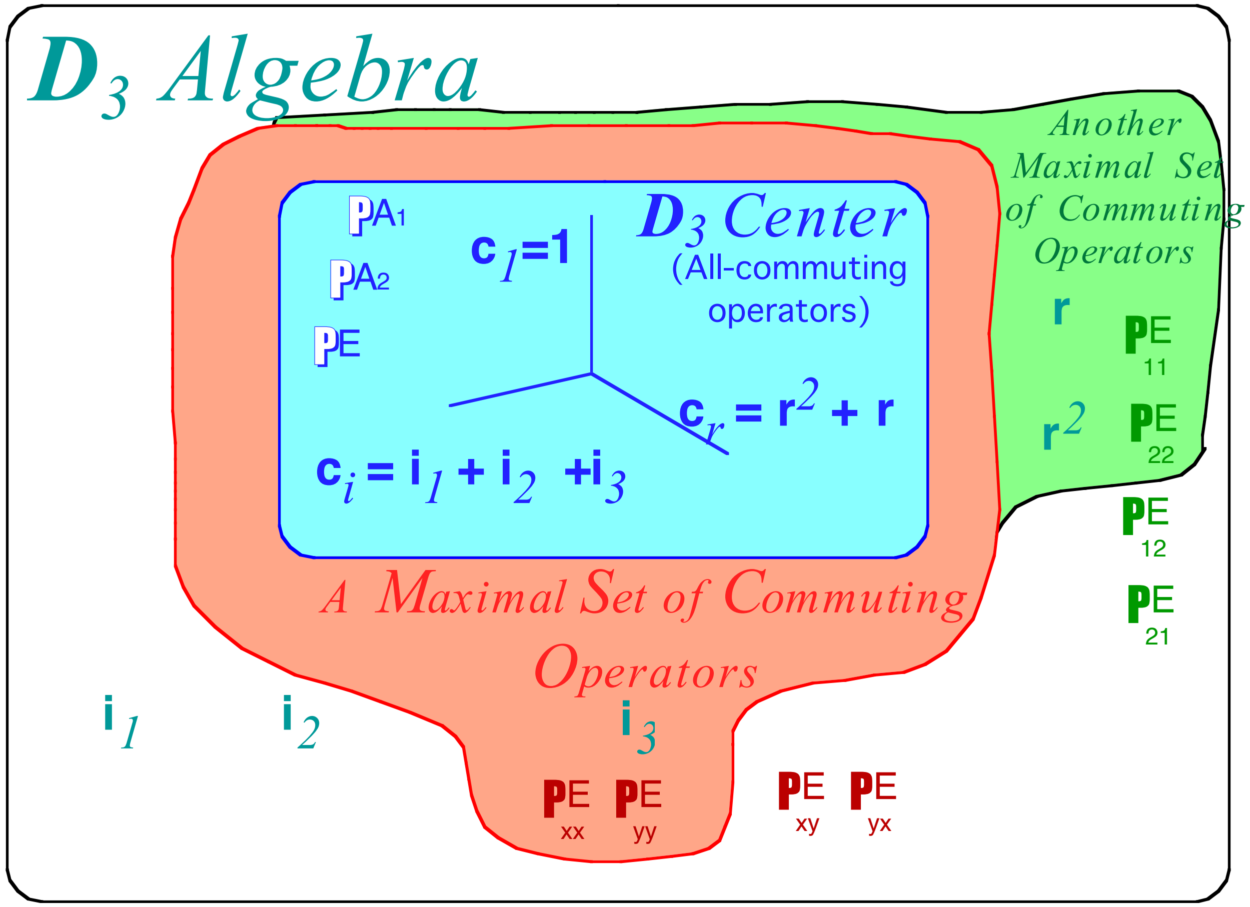
Global vs Local symmetry expansion of D_3 Hamiltonian

1st-Step in spectral analysis of D_3 "group-table" Hamiltonian: Algebra of D_3 Center(Classes)

All-commuting operators and D_3-invariant class algebra

All-commuting projectors and D_3-invariant characters

Group invariant numbers: Centrum, Rank, and Order



(Fig. 15.2.1 QTCA)

D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

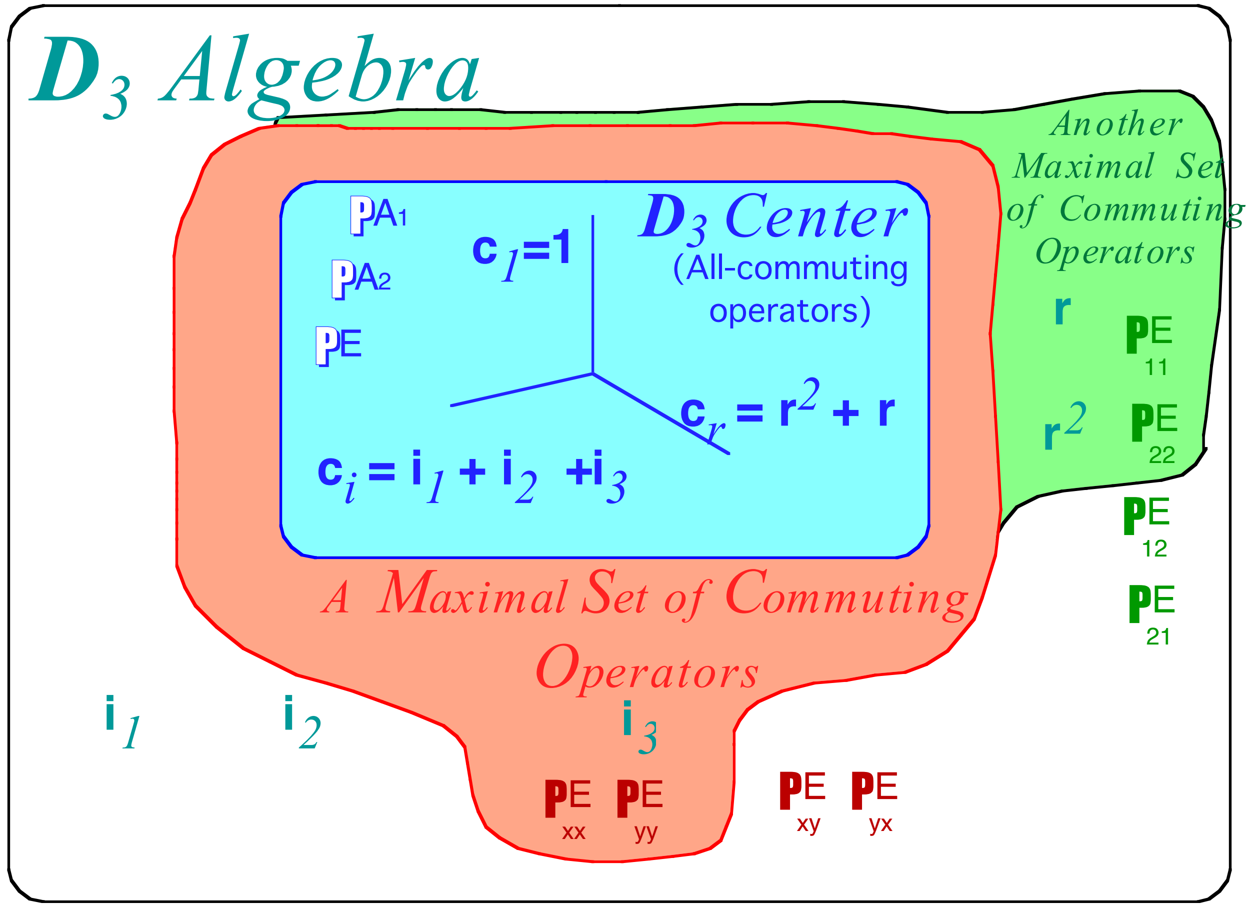
Important invariant numbers or "characters"

$\ell^{\alpha}=$ Irreducible representation (irrep) *dimension* or level *degeneracy*
For symmetry group or algebra G

Centrum: $\kappa(G)=\Sigma_{irrep(\alpha)}(\ell^{\alpha})^0$ =Number of classes, invariants, irrep types, *all-commuting* ops

Rank: $\rho(G)=\Sigma_{irrep(\alpha)}(\ell^{\alpha})^1$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

Order: $^{o}(G)=\Sigma_{irrep(\alpha)}(\ell^{\alpha})^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops



D_3 $\kappa=$	$\mathbf{1}$	$\mathbf{r}^1+\mathbf{r}^2$	$\mathbf{i}_1+\mathbf{i}_2+\mathbf{i}_3$	
$\mathbf{P}^{A_1}=$	1	1	1	/6
$\mathbf{P}^{A_2}=$	1	1	-1	/6
$\mathbf{P}^{E}=$	2	-1	0	/3

Important invariant numbers or "characters"

$\ell^\alpha =$ Irreducible representation (irrep) *dimension* or level *degeneracy*
For symmetry group or algebra G

Centrum: $\kappa(G)=\Sigma_{irrep(\alpha)}(\ell^\alpha)^0$ =Number of classes, invariants, irrep types, *all-commuting* ops

Rank: $\rho(G)=\Sigma_{irrep(\alpha)}(\ell^\alpha)^1$ =Number of irrep idempotents $\mathbf{P}^{(\alpha)}_{n,n}$, *mutually-commuting* ops

Order: $^o(G)=\Sigma_{irrep(\alpha)}(\ell^\alpha)^2$ =*Total* number of irrep projectors $\mathbf{P}^{(\alpha)}_{m,n}$ or symmetry ops

$$\kappa(D_3)=(1)^0+(1)^0+(2)^0=3$$
$$\rho(D_3)=(1)^1+(1)^1+(2)^1=4$$
$$^\circ(D_3)=(1)^2+(1)^2+(2)^2=6$$