Group Theory in Quantum Mechanics

Lecture 10 (2.12.15)

Applications of U(2) and R(3) representations

(Quantum Theory for Computer Age - Ch. 10A-B of Unit 3)

(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5 and Ch. 7)

Review: Fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ and Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ representations of U(2) and R(3)

- Euler $\mathbf{R}(\alpha\beta\gamma)$ derived from Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ and vice versa
- Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed (and "real-world" applications)

U(2) density operator approach to symmetry dynamics

- Bloch equation for density operator
- Quick U(2) way to find eigen-solutions for 2-by-2 **H**

The ABC's of U(2) dynamics-Archetypes

- Asymmetric-Diagonal A-Type motion
- Bilateral-Balanced B-Type motion
- Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

- AB-Type motion and Wigner's Avoided-Symmetry-Crossings
- ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

- Conventional amp-phase ellipse coordinates
- Euler Angle $(\alpha\beta\gamma)$ ellipse coordinates

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

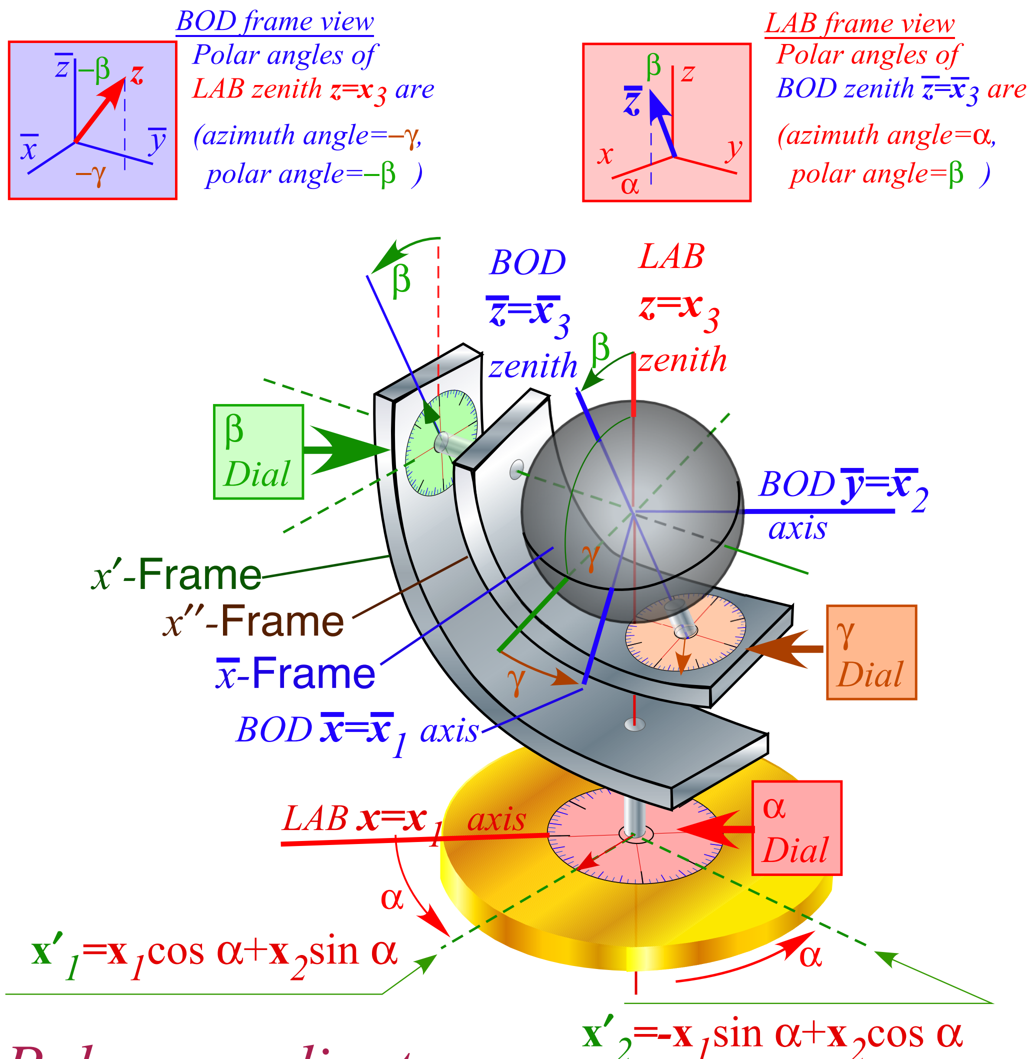
Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Here spin-rotor **S***-polar coordinates are Euler angles*

From Lecture 7 page 86



Polar coordinates for unit Spin vector $\hat{\mathbf{S}}$

$$\hat{S}_X = \cos\alpha \ \sin\beta$$
$$\hat{S}_Y = \sin\alpha \ \sin\beta$$
$$\hat{S}_Z = \cos\beta$$

Spin State

$$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$$

Euler angles

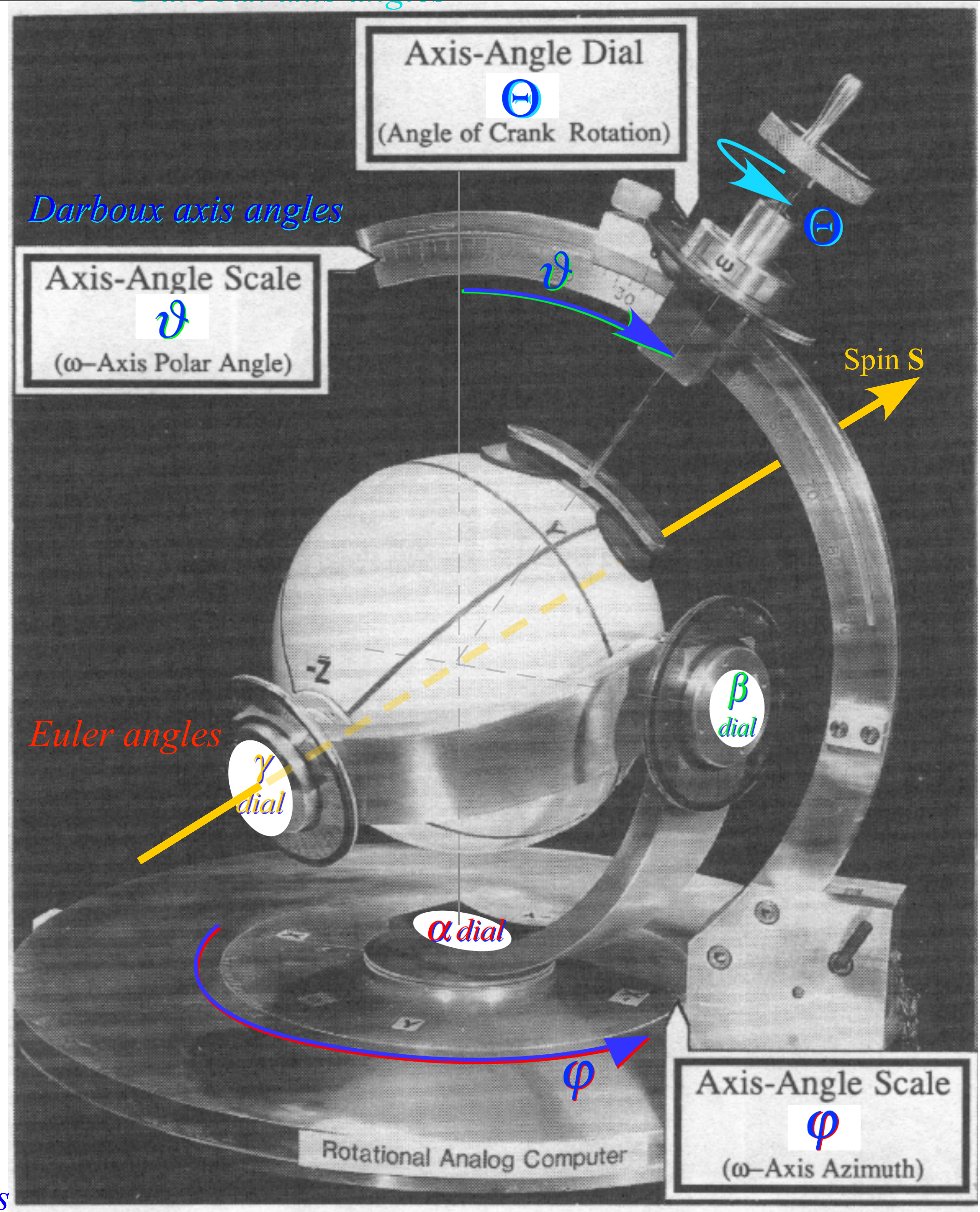
Polar coordinates for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

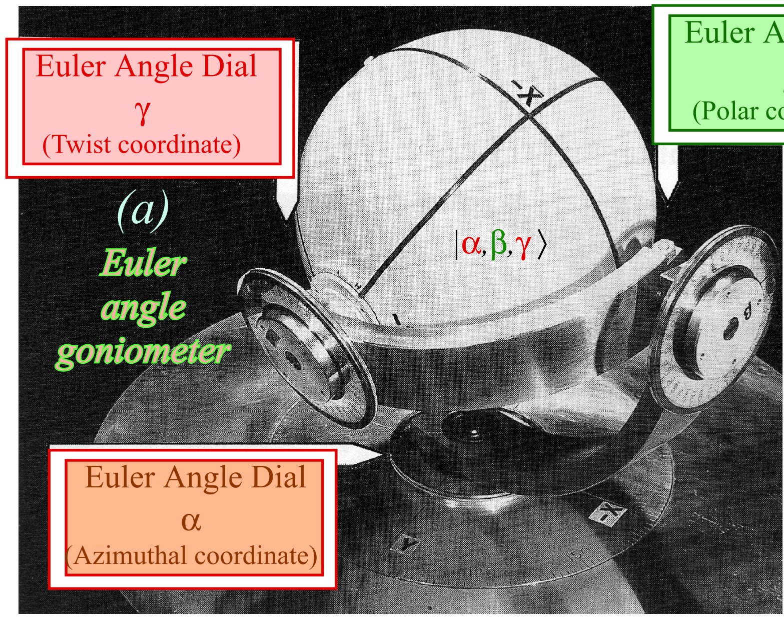
Operator

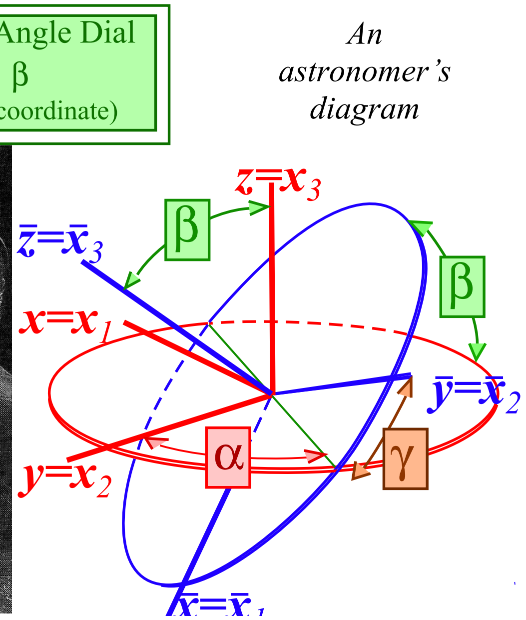
$$|[\varphi\vartheta\Theta]\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\uparrow\rangle$$

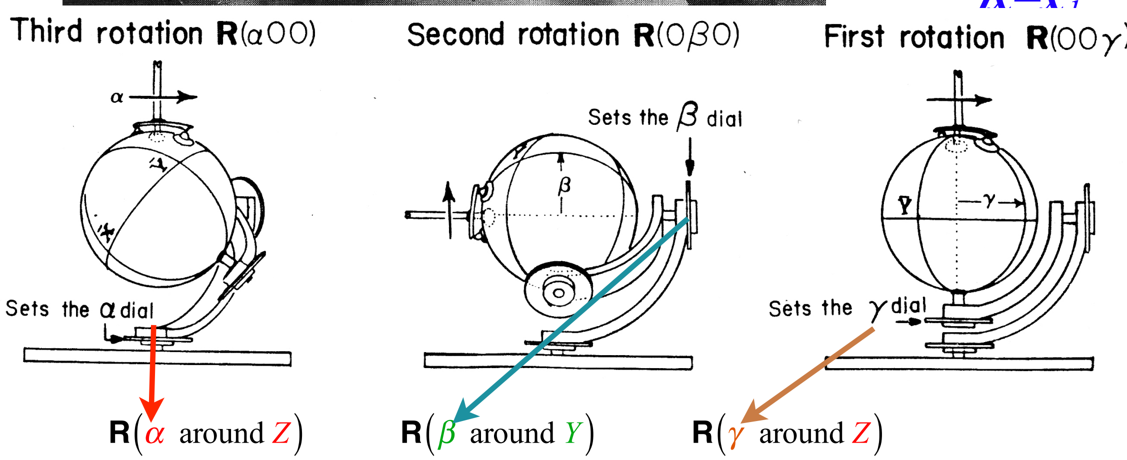
Darboux axis angles



Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

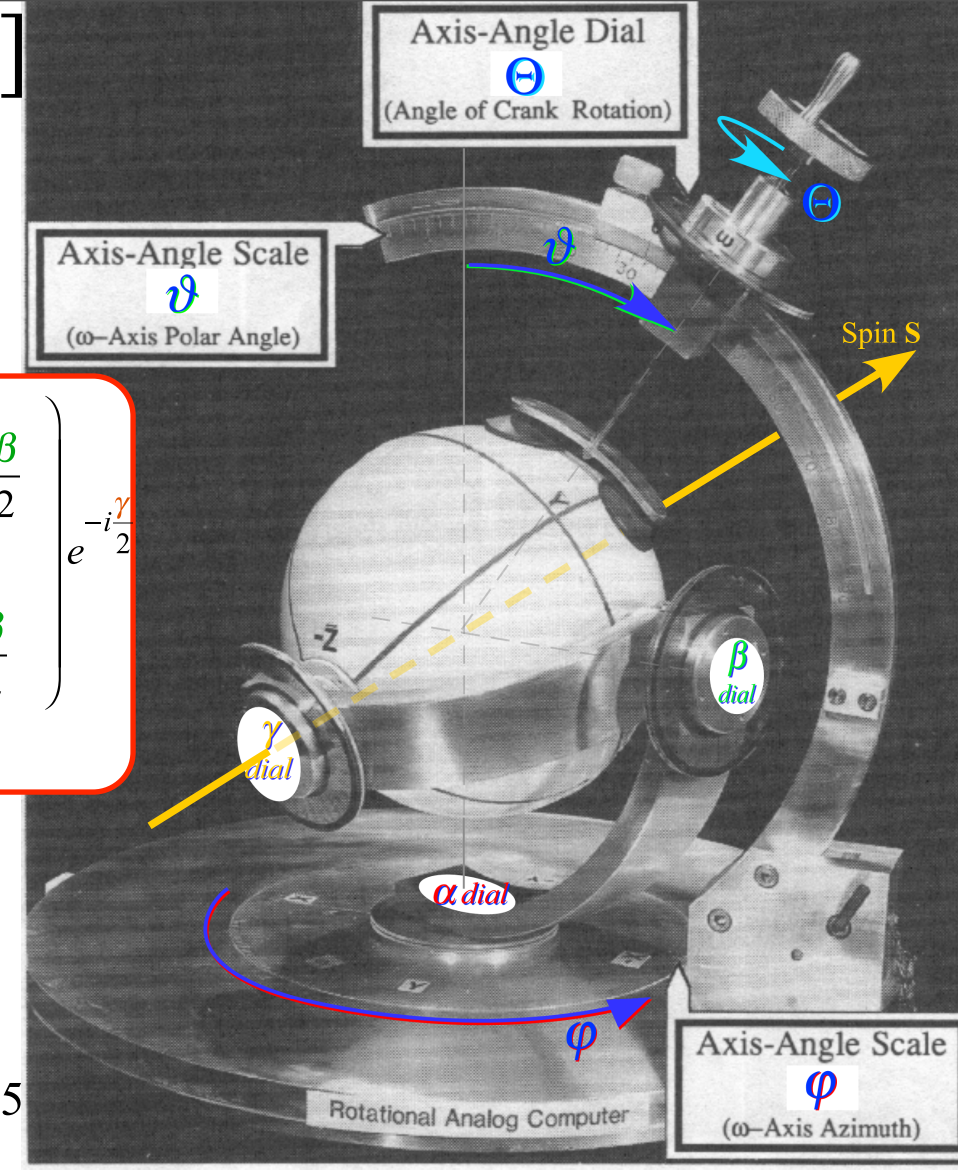






From Lecture 7
page 80 to 89

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Lecture 8
page 21 to 25

$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X - i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X + i\hat{\Theta}_Y) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\Theta}_X\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\Theta}_Y\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\Theta}_Z\sin\frac{\Theta}{2}$$

$\hat{\Theta}_X$: $\cos\varphi\ \sin\vartheta$ $\hat{\Theta}_Y$: $\sin\varphi\ \sin\vartheta$ $\hat{\Theta}_Z$: $\cos\vartheta$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$

$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler* **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

U(2) density operator approach to symmetry dynamics

- *Bloch equation for density operator*

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Example: *Euler angles* $(\alpha=50°\ \beta=60°\ \gamma=70°)$

$$\varphi = (50° - 70° + 180°)/2 = 80°$$

$$\vartheta = \tan^{-1}[\tan 60°/2/\sin(50°+\gamma)/2] = 33.7°$$

$$\Theta = 2\cos^{-1}[\cos 60°/2\cos(50°+\gamma)/2] = 128.7°$$

Reverse check: $(\alpha\beta\gamma)$ in terms of $[\varphi\vartheta\Theta]$

$$\alpha = 80° - 90° + \tan^{-1}(\tan(128.7°/2)\cos 33.7°) = 50.007°$$

$$\beta = 2\sin^{-1}(\sin 128.7°/2\ \sin 33.7°) = 60.022°$$

$$\gamma = \pi/2 - 128.7° + \tan^{-1}(\tan(128.7°/2) = 70.007°$$

Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

➡ *Euler* **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed (and "real-world" applications)*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

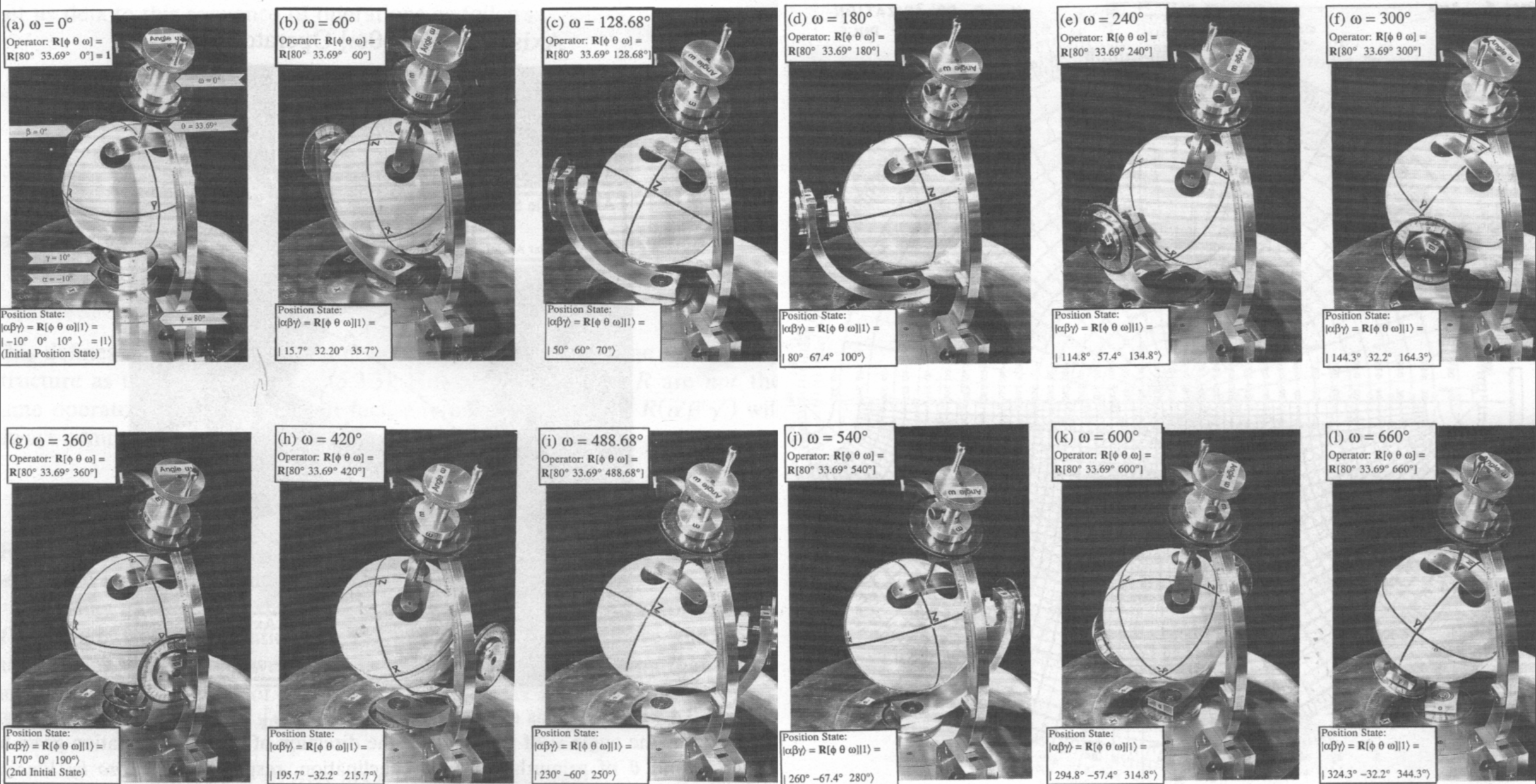
Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

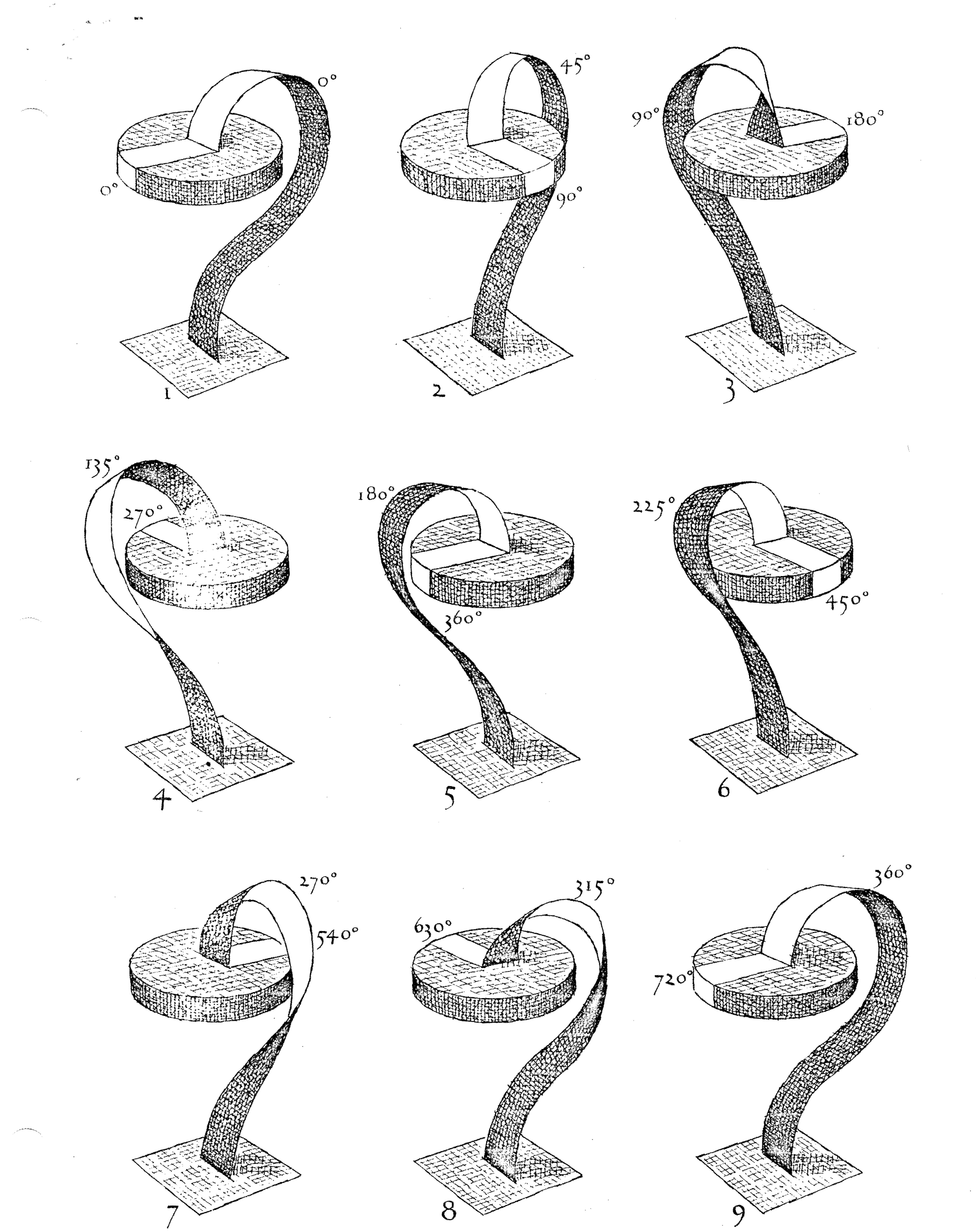
Euler **R**$(\alpha\beta\gamma)$ *rotation* Θ=0−4π-*sequence* [φϑ] *fixed*

Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°



Θ=360° Θ=420° Θ=488.7° Θ=540° Θ=600° Θ=660°

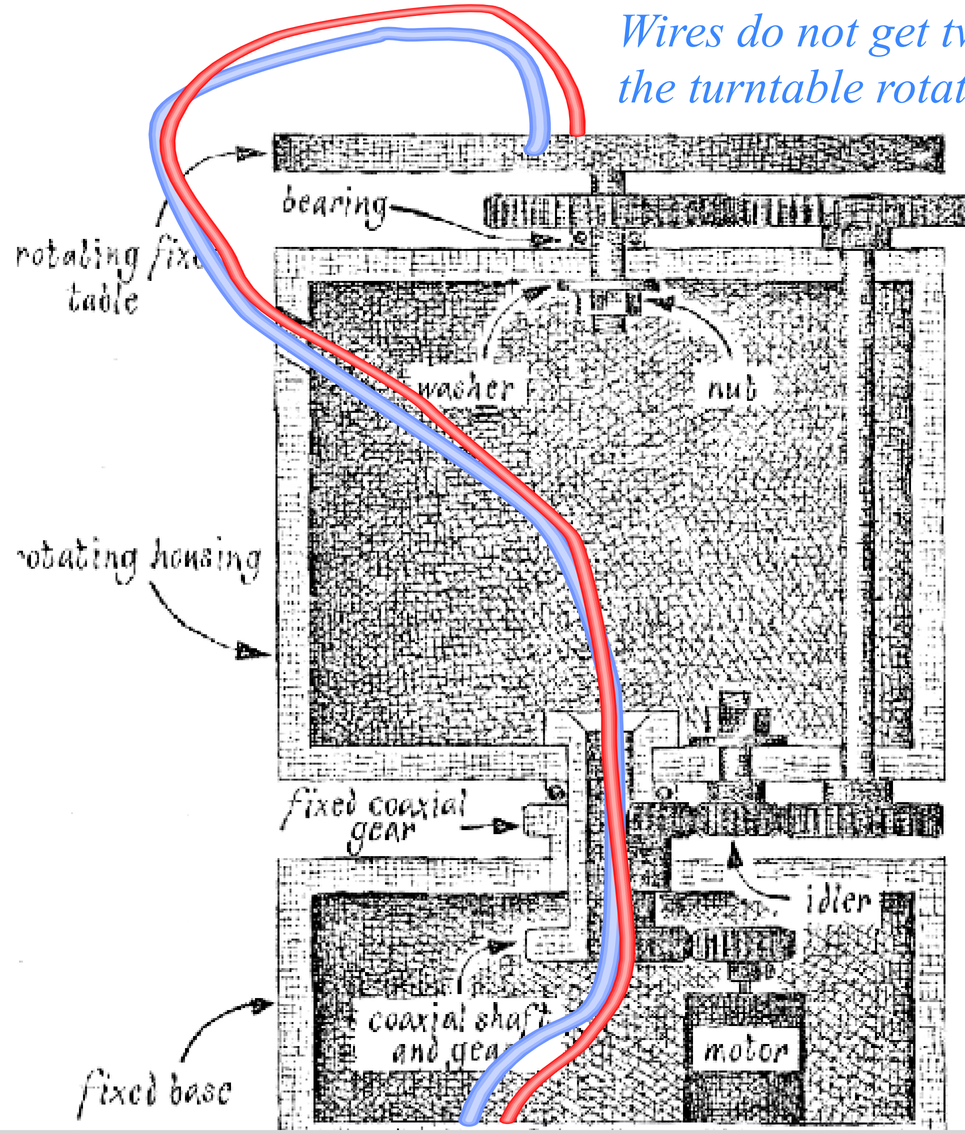
Some "real-world" applications of the U(2)-R(3) spinor-vector topology

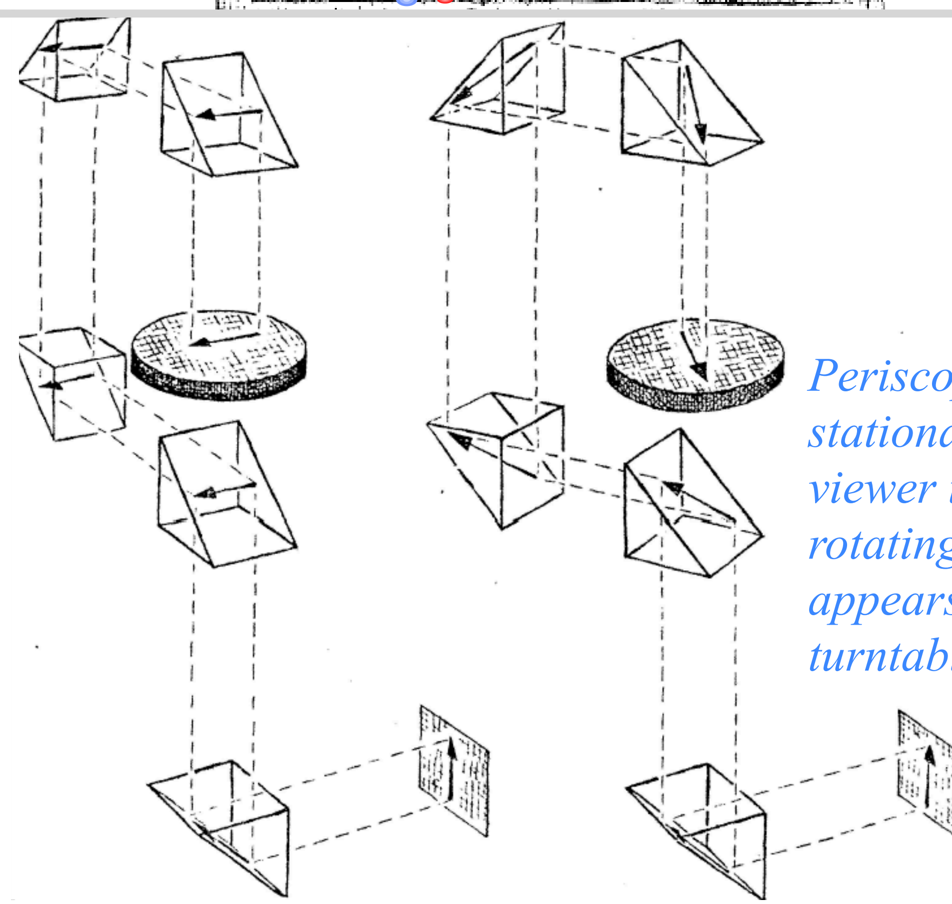


Sequential models of D. A. Adams' antitwister mechanism

From Scientific American December 1975-p.120-125

Wires do not get twisted up as the turntable rotates





Periscope allows stationary outside viewer to see into a rotating frame that appears fixed as the turntable rotates

Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* Θ=0−4π*-sequence* [φϑ] *fixed*

R(3)-U(2) slide rule for converting **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and* ***norm N*** *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N} \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N} \begin{pmatrix} e^{-i\alpha/2} \cos\beta/2 \\ e^{i\alpha/2} \sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N \underbrace{\left(p_1^2 + x_1^2 + p_2^2 + x_2^2 \right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$:

$$\frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2 \right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N \left(p_1^2 + x_1^2 - p_2^2 - x_2^2 \right)$$

scaled by $\frac{1}{2}$:

$$S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2 \right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2} \right) = \frac{N}{2}\cos\beta$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and* ***norm** N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1{}^2 + x_1{}^2 + p_2{}^2 + x_2{}^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\quad \frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1{}^2 + x_1{}^2 - p_2{}^2 - x_2{}^2\right)$$

scaled by $\frac{1}{2}$: $\quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2 + p_1p_2\right)$$

scaled by $\frac{1}{2}$: $\quad S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\,\sin\beta$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1^2 + x_1^2 + p_2^2 + x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right)$$

scaled by $\frac{1}{2}$: $S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2 + p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2 - x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}\otimes\begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$

$U(2)$ density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1{}^2+x_1{}^2+p_2{}^2+x_2{}^2\right)}_{4D\text{-}norm=1}$$
scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1{}^2+x_1{}^2-p_2{}^2-x_2{}^2\right)$$
scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$
scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$
scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

$\rho_{11}=\Psi_1^*\Psi_1=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2=\frac{1}{2}N-S_Z$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}$$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

...2-by-2 *density operator* ρ

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1^2 + x_1^2 + p_2^2 + x_2^2\right)}_{4D\text{-}norm=1}$$
scaled by $\frac{1}{2}$: $\quad \frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right)$$
scaled by $\frac{1}{2}$: $\quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2 + p_1p_2\right)$$
scaled by $\frac{1}{2}$: $\quad S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\,\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2 - x_2p_1\right)$$
scaled by $\frac{1}{2}$: $\quad S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\,\sin\beta$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| =$

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}\otimes\begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$$

$\rho_{11} = \Psi_1^*\Psi_1 = \frac{1}{2}N + S_Z$	$\rho_{12} = \Psi_2^*\Psi_1 = S_X - iS_Y$,
$\rho_{21} = \Psi_1^*\Psi_2 = S_X + iS_Y$	$\rho_{22} = \Psi_2^*\Psi_2 = \frac{1}{2}N - S_Z$

Norm: $N = \Psi1^*\Psi1 + \Psi2^*\Psi2$

$$\rho = \begin{pmatrix} \frac{1}{2}N + S_Z & S_X - iS_Y \\ S_X + iS_Y & \frac{1}{2}N - S_Z \end{pmatrix} = \frac{1}{2}N\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + S_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + S_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + S_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

...so state *density operator* ρ has **σ**-expansion

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ***-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2+x_1^2+p_2^2+x_2^2\right)$$

4D-norm=1

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$: $S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2-x_2p_1\right)$$

scaled by $\frac{1}{2}$: $S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}\otimes\begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$

$\rho_{11}=\Psi_1^*\Psi_1 = \frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1 = S_X-iS_Y$,
$\rho_{21}=\Psi_1^*\Psi_2 = S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2 = \frac{1}{2}N-S_Z$

Norm: $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$

$$= \begin{pmatrix} \frac{1}{2}N+S_Z & S_X-iS_Y \\ S_X+iS_Y & \frac{1}{2}N-S_Z \end{pmatrix} = \frac{1}{2}N\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + S_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + S_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + S_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\rho = \frac{1}{2}N\,\mathbf{1} + S_X\,\boldsymbol{\sigma}_X + S_Y\,\boldsymbol{\sigma}_Y + S_Z\,\boldsymbol{\sigma}_Z = \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has **σ**-expansion

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$ *scaled by* $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right) = \frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$ *scaled by* $\frac{1}{2}$: $S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$ *scaled by* $\frac{1}{2}$: $S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2-x_2p_1\right)$$ *scaled by* $\frac{1}{2}$: $S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}\otimes\begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$

$\rho_{11} = \Psi_1^*\Psi_1 = \frac{1}{2}N + S_Z$	$\rho_{12} = \Psi_2^*\Psi_1 = S_X - iS_Y$,
$\rho_{21} = \Psi_1^*\Psi_2 = S_X + iS_Y$	$\rho_{22} = \Psi_2^*\Psi_2 = \frac{1}{2}N - S_Z$

Norm: $N = \Psi_1^*\Psi_1 + \Psi_2^*\Psi_2$

$$= \begin{pmatrix} \frac{1}{2}N+S_Z & S_X-iS_Y \\ S_X+iS_Y & \frac{1}{2}N-S_Z \end{pmatrix} = \frac{1}{2}N\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + S_X\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_X} + S_Y\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_Y} + S_Z\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_Z}$$

$$\rho = \frac{1}{2}N\,\mathbf{1} + S_X\,\sigma_X + S_Y\,\sigma_Y + S_Z\,\sigma_Z = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma$$

...so state *density operator* ρ has **σ**-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_A} + B\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_B} + C\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_C}$$

$$\mathbf{H} = \omega_0\,\sigma_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C = \omega_0\sigma_0 + \frac{\vec{\Omega}}{2}\bullet\sigma$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ**-*operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$
scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$
scaled by $\frac{1}{2}$: $S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)$$
scaled by $\frac{1}{2}$: $S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$
scaled by $\frac{1}{2}$: $S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$

$\rho_{11}=\Psi_1^*\Psi_1=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2=\frac{1}{2}N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\begin{pmatrix}0&1\\1&0\end{pmatrix}+S_Y\begin{pmatrix}0&-i\\i&0\end{pmatrix}+S_Z\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\rho=\frac{1}{2}N\,\mathbf{1}+S_X\,\sigma_X+S_Y\,\sigma_Y+S_Z\,\sigma_Z=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has **σ**-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$\mathbf{H}=\omega_0\sigma_0+\frac{\Omega_A}{2}\sigma_A+\frac{\Omega_B}{2}\sigma_B+\frac{\Omega_C}{2}\sigma_C=\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H}=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\mathbf{S}$$

$$\rho=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H}=\Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler* **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*
- *R(3)-U(2) slide rule for converting* **R**$(\alpha\beta\gamma) \leftrightarrow$ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

U(2) density operator approach to symmetry dynamics

- ➡ *Bloch equation for density operator*
- *Quick U(2) way to find eigen-solutions for 2-by-2* **H**

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

U(2) density operator approach to symmetry dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Bloch equation for density operator

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\sigma$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \qquad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\mathbf{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\Omega}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\Omega}\bullet\boldsymbol{\sigma}\right)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\mathbf{\Omega}$ *point in different directions*.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar|\dot{\Psi}\rangle = \mathbf{H}|\Psi\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\langle\dot{\Psi}| = \langle\Psi|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar|\dot{\Psi}\rangle\langle\Psi| + i\hbar|\Psi\rangle\langle\dot{\Psi}| = \mathbf{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = [\mathbf{H},\rho]$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta\sigma_\alpha\sigma_\beta = A_\alpha B_\beta(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

This cancels   *This remains*

Given ρ and $\mathbf{H}$ in terms *spin* S-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\Omega}\bullet\boldsymbol{\sigma})(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma})$$

$$-\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathrm{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathrm{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma})(\vec{\Omega}\bullet\boldsymbol{\sigma})$$

Last terms don't cancel if the *spin* S and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}(\vec{\Omega}\bullet\boldsymbol{\sigma})(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma}) - \frac{\hbar}{2}(\vec{\mathrm{S}}\bullet\boldsymbol{\sigma})(\vec{\Omega}\bullet\boldsymbol{\sigma})$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \qquad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = \left[\mathbf{H},\rho\right]$$

$$(\mathbf{A}\bullet\boldsymbol{\sigma})(\mathbf{B}\bullet\boldsymbol{\sigma}) = A_\alpha B_\beta \sigma_\alpha\sigma_\beta = A_\alpha B_\beta\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)$$

$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta\sigma_\gamma$$

$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

This cancels *This remains*

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})$$

$$\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\boldsymbol{\Omega}$ *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}) - \frac{\hbar}{2}(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma})(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma})$$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}(\vec{\boldsymbol{\Omega}}\times\vec{\mathbf{S}})\bullet\boldsymbol{\sigma} - \frac{i\hbar}{2}(\vec{\mathbf{S}}\times\vec{\boldsymbol{\Omega}})\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{S}\bullet\sigma$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar|\dot{\Psi}\rangle = \mathbf{H}|\Psi\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\langle\dot{\Psi}| = \langle\Psi|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar|\dot{\Psi}\rangle\langle\Psi| + i\hbar|\Psi\rangle\langle\dot{\Psi}| = \mathbf{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = [\mathbf{H},\rho]$$

$$(\mathbf{A}\bullet\sigma)(\mathbf{B}\bullet\sigma) = A_\alpha B_\beta \sigma_\alpha \sigma_\beta = A_\alpha B_\beta(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta \sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\sigma$$

This cancels | *This remains*

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\mathbf{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right)\left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{S}\bullet\sigma + \frac{\hbar}{2}(\vec{\Omega}\bullet\sigma)(\vec{S}\bullet\sigma)$$

$$\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{S}\bullet\sigma + \frac{\hbar}{2}(\vec{S}\bullet\sigma)(\vec{\Omega}\bullet\sigma)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\mathbf{\Omega}$ *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}(\vec{\Omega}\bullet\sigma)(\vec{S}\bullet\sigma) - \frac{\hbar}{2}(\vec{S}\bullet\sigma)(\vec{\Omega}\bullet\sigma)$$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}(\vec{\Omega}\times\vec{S})\bullet\sigma - \frac{i\hbar}{2}(\vec{S}\times\vec{\Omega})\bullet\sigma$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right) = i\hbar\dot{\vec{S}}\bullet\sigma = i\hbar(\vec{\Omega}\times\mathbf{S})\bullet\sigma$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{S}\bullet\sigma$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\sigma$$

Ket equation (time forward) and "daggered" bra-equation (time reversed).

$$i\hbar|\dot{\Psi}\rangle = \mathbf{H}|\Psi\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\langle\dot{\Psi}| = \langle\Psi|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$. $\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar|\dot{\Psi}\rangle\langle\Psi| + i\hbar|\Psi\rangle\langle\dot{\Psi}| = \mathbf{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\mathbf{H}$$

The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \mathbf{H}\rho - \rho\mathbf{H} = [\mathbf{H},\rho]$$

$$(\mathbf{A}\bullet\sigma)(\mathbf{B}\bullet\sigma) = A_\alpha B_\beta \sigma_\alpha \sigma_\beta = A_\alpha B_\beta(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma)$$
$$= A_\alpha B_\alpha + i\varepsilon_{\alpha\beta\gamma}A_\alpha B_\beta \sigma_\gamma$$
$$= \mathbf{A}\bullet\mathbf{B} + i(\mathbf{A}\times\mathbf{B})\cdot\sigma$$

Given ρ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\mathbf{\Omega}$-vector:

$$\mathbf{H}\rho = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right)\left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{S}\bullet\sigma + \frac{\hbar}{2}(\vec{\Omega}\bullet\sigma)(\vec{S}\bullet\sigma)$$

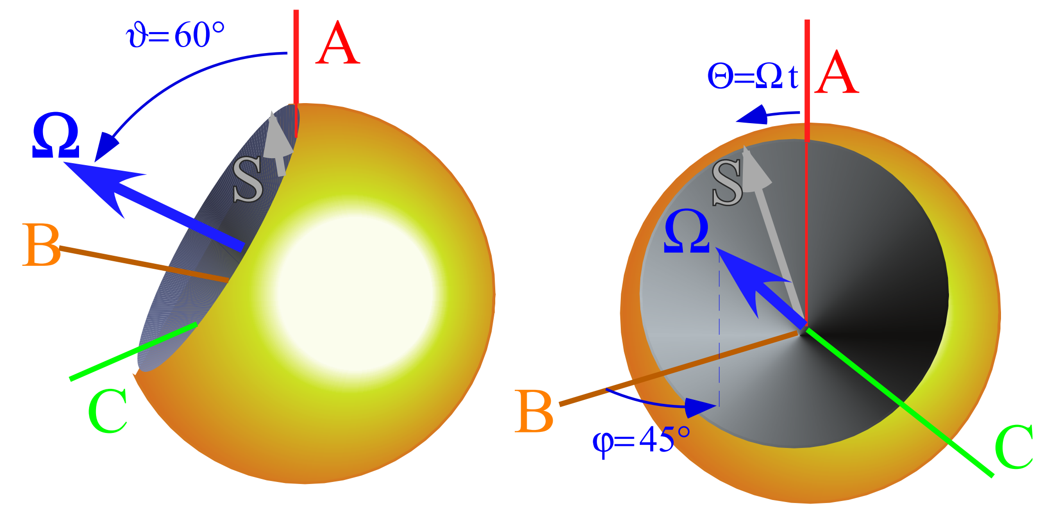
$$\rho\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\Omega}\bullet\sigma\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\Omega}\bullet\sigma + \hbar\Omega_0\vec{S}\bullet\sigma + \frac{\hbar}{2}(\vec{S}\bullet\sigma)(\vec{\Omega}\bullet\sigma)$$

Last terms don't cancel if the *spin* $\mathbf{S}$ and *crank* $\mathbf{\Omega}$ *point in different directions*.

$$\mathbf{H}\rho - \rho\mathbf{H} = \frac{\hbar}{2}(\vec{\Omega}\bullet\sigma)(\vec{S}\bullet\sigma) - \frac{\hbar}{2}(\vec{S}\bullet\sigma)(\vec{\Omega}\bullet\sigma)$$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = \frac{i\hbar}{2}(\vec{\Omega}\times\vec{S})\bullet\sigma - \frac{i\hbar}{2}(\vec{S}\times\vec{\Omega})\bullet\sigma$$

$$i\hbar\frac{\partial}{\partial t}\left(\frac{N}{2}\mathbf{1} + \vec{S}\bullet\sigma\right) = i\hbar\dot{\vec{S}}\bullet\sigma = i\hbar(\vec{\Omega}\times\mathbf{S})\bullet\sigma$$



Factoring out $\bullet\sigma$ gives a classical/quantum *gyro-precession equation*.

$$\frac{\partial\vec{S}}{\partial t} = \dot{\vec{S}} = \vec{\Omega}\times\vec{S}$$

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler* **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*
- *R(3)-U(2) slide rule for converting* **R**$(\alpha\beta\gamma) \leftrightarrow$ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

U(2) density operator approach to symmetry dynamics

- *Bloch equation for density operator*
- *Quick U(2) way to find eigen-solutions for 2-by-2* **H**

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \Omega_B \quad \mathbf{S}_B \quad + \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \boldsymbol{\Omega}_A \quad \mathbf{S}_A \quad + \quad \boldsymbol{\Omega}_B \quad \mathbf{S}_B \quad + \quad \boldsymbol{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\boldsymbol{\Omega}} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\boldsymbol{\Omega}_A{}^2 + \boldsymbol{\Omega}_B{}^2 + \boldsymbol{\Omega}_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \mathbf{\Omega}_A \quad \mathbf{S}_A \quad + \quad \mathbf{\Omega}_B \quad \mathbf{S}_B \quad + \quad \mathbf{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi \sin\vartheta, \quad \Omega_C = \Omega\sin\varphi \sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\mathbf{\Omega}_A{}^2 + \mathbf{\Omega}_B{}^2 + \mathbf{\Omega}_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

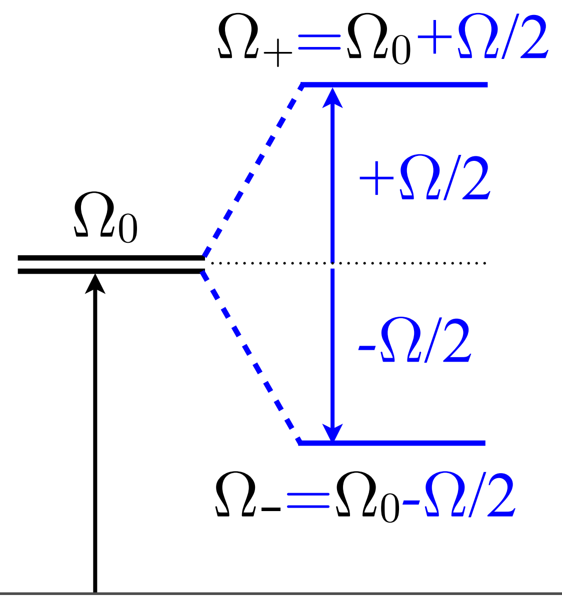
$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \mathbf{\Omega}_A \quad \mathbf{S}_A \quad + \quad \mathbf{\Omega}_B \quad \mathbf{S}_B \quad + \quad \mathbf{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\mathbf{\Omega}_A{}^2 + \mathbf{\Omega}_B{}^2 + \mathbf{\Omega}_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\boldsymbol{\Omega}=\boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

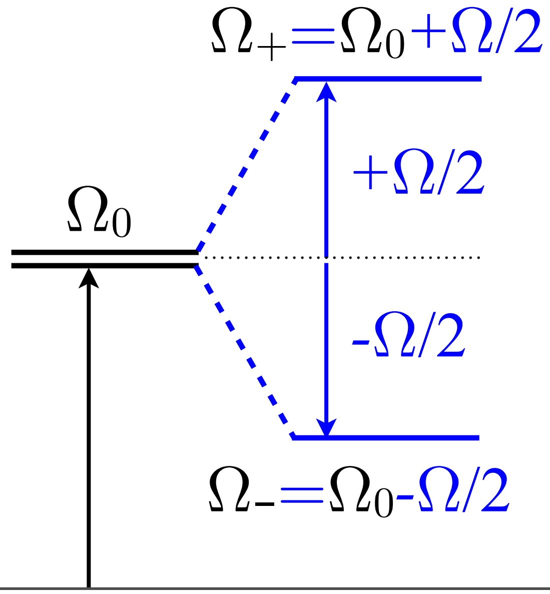
$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \boldsymbol{\Omega}_A \quad \mathbf{S}_A \quad + \quad \boldsymbol{\Omega}_B \quad \mathbf{S}_B \quad + \quad \boldsymbol{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0\mathbf{1} + \vec{\boldsymbol{\Omega}}\bullet\mathbf{S}$$

Step 2. Convert Cartesian to polar form: $(\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\,\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\,\sin\vartheta)$

where: $\Omega_0=\dfrac{A+D}{2}$ *and:* $\Omega=\sqrt{\boldsymbol{\Omega}_A{}^2+\boldsymbol{\Omega}_B{}^2+\boldsymbol{\Omega}_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}$

Eigenvalues: $\Omega_\pm=\Omega_0\pm\Omega/2 = \dfrac{A+D\pm\sqrt{(A-D)^2+4B^2+4C^2}}{2}$

and: $\vartheta=\cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi=\cos^{-1}(\Omega_B/\Omega\sin\vartheta)=\cos^{-1}[\Omega_B/\sqrt{\boldsymbol{\Omega}_B{}^2+\boldsymbol{\Omega}_C{}^2}\ ]$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\mathbf{\Omega}=\mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \mathbf{\Omega}_A \mathbf{S}_A + \mathbf{\Omega}_B \mathbf{S}_B + \mathbf{\Omega}_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

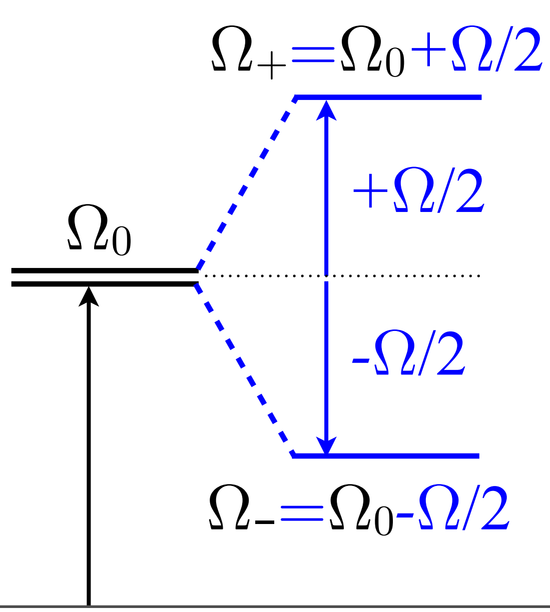
Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$
$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\ ]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2 + 4B^2 + 4C^2}\ ]$, $\quad\varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\ ]$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\mathbf{\Omega}=\mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0=\frac{A+D}{2}$ *and:* $\Omega=\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}$

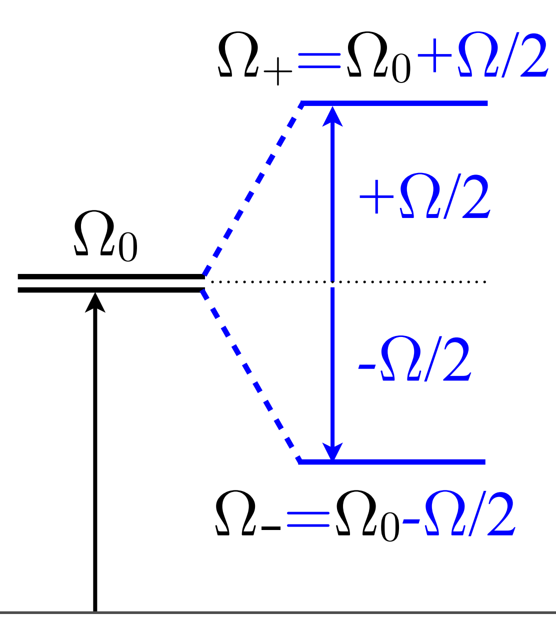
Eigenvalues: $\Omega_\pm=\Omega_0\pm\Omega/2 = \frac{A+D\pm\sqrt{(A-D)^2+4B^2+4C^2}}{2}$

and: $\vartheta=\cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi=\cos^{-1}(\Omega_B/\Omega\sin\vartheta)=\cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2+\Omega_C{}^2}\ ]$

or: $\vartheta=\cos^{-1}[(A-D)/\sqrt{(A-D)^2+4B^2+4C^2}\ ]$, $\quad\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}\ ]$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

$$|\uparrow_{\alpha\beta\gamma}\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)|\uparrow_{000}\rangle$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega}=\mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta,\quad \Omega_B = \Omega\cos\varphi\sin\vartheta,\quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2 = \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$

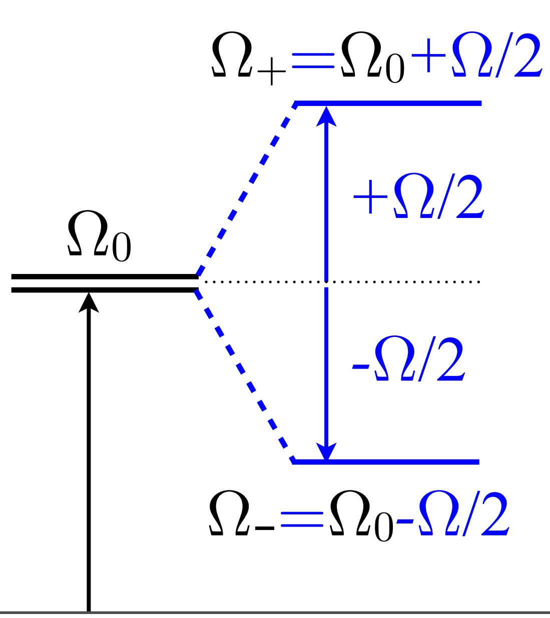
and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\ ]$

or: $\vartheta = \cos^{-1}[(A-D)/\sqrt{(A-D)^2 + 4B^2 + 4C^2}\ ]$, $\quad\varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\ ]$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

with the Darboux axis polar angles (azimuth φ *, polar* ϑ *) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \Omega_B \quad \mathbf{S}_B \quad + \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2 = \dfrac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\;]$

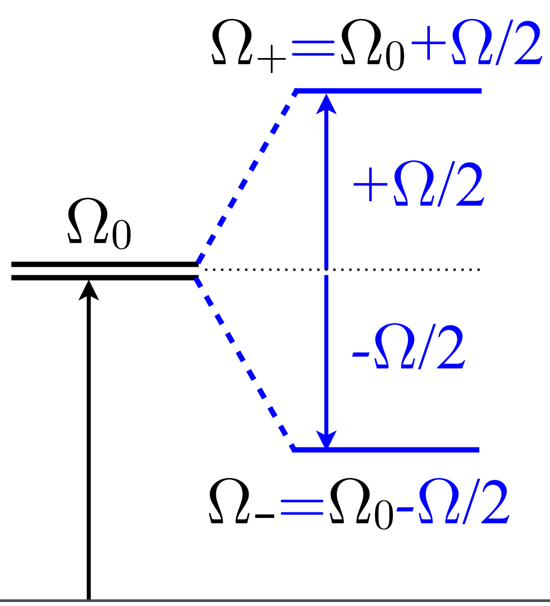
or: $\vartheta = \cos^{-1}[(A-D)/\sqrt{(A-D)^2 + 4B^2 + 4C^2}\;]$, $\quad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\;]$

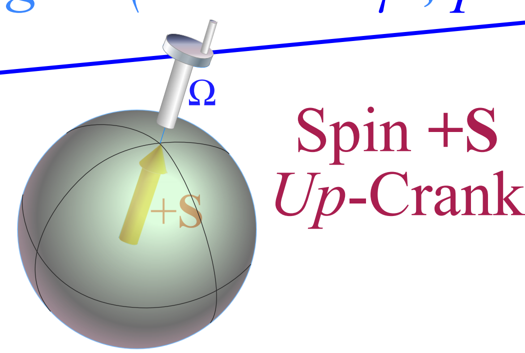
Step 3.To find eigenvectors replace Euler angles (azimuth α, *polar* β *) of Euler-state*

with the Darboux axis polar angles (azimuth φ *, polar* ϑ *) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$





Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A,\Omega_B,\Omega_C)$ *of crank vector* $\mathbf{\Omega}=\mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \Omega_A \mathbf{S}_A + \Omega_B \mathbf{S}_B + \Omega_C \mathbf{S}_C = \Omega_0\mathbf{1} + \vec{\Omega}\bullet\mathbf{S}$$

Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta$, $\Omega_B=\Omega\cos\varphi\sin\vartheta$, $\Omega_C=\Omega\sin\varphi\sin\vartheta$)

where: $\Omega_0 = \frac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2} = \sqrt{(A-D)^2+4B^2+4C^2}$

Eigenvalues: $\Omega_\pm=\Omega_0\pm\Omega/2 = \frac{A+D\pm\sqrt{(A-D)^2+4B^2+4C^2}}{2}$

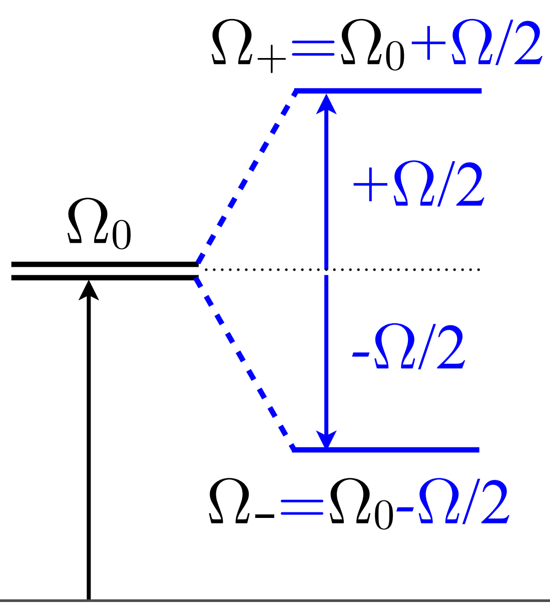
and: $\vartheta=\cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi=\cos^{-1}(\Omega_B/\Omega\sin\vartheta)=\cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2+\Omega_C{}^2}\,]$

or: $\vartheta=\cos^{-1}[(A-D)/\sqrt{(A-D)^2+4B^2+4C^2}\,]$, $\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}\,]$

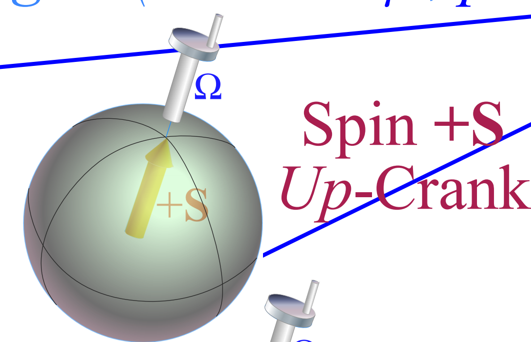
Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta\pm\pi$*) of* **H***-matrix*

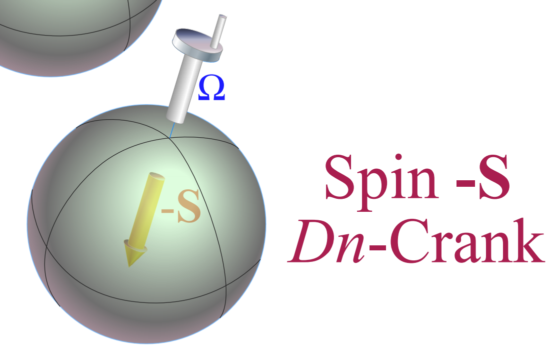
$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$



$$\left|\Omega_-\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta\pm\pi}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta\pm\pi}{2} \end{pmatrix}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \mathbf{\Omega}_A \quad \mathbf{S}_A \quad + \quad \mathbf{\Omega}_B \quad \mathbf{S}_B \quad + \quad \mathbf{\Omega}_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\mathbf{\Omega}} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: $(\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\sin\vartheta)$

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\mathbf{\Omega}_A^{\,2} + \mathbf{\Omega}_B^{\,2} + \mathbf{\Omega}_C^{\,2}} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$
$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

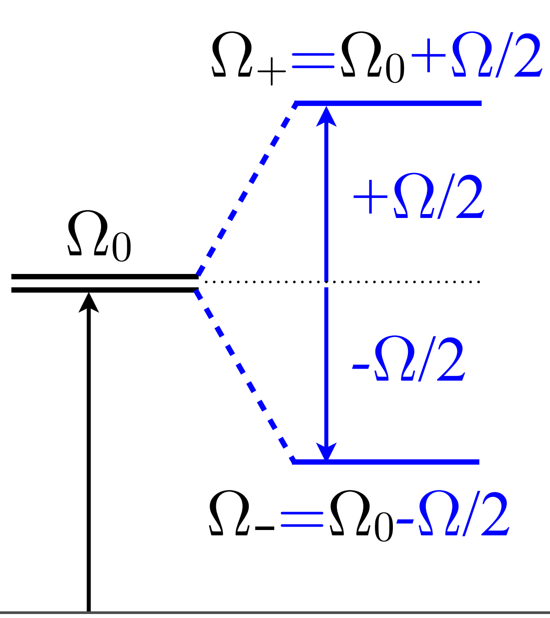
and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\mathbf{\Omega}_B^{\,2} + \mathbf{\Omega}_C^{\,2}}\ ]$

or: $\vartheta = \cos^{-1}[(A-D)/\sqrt{(A-D)^2 + 4B^2 + 4C^2}\ ]$, $\quad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\ ]$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

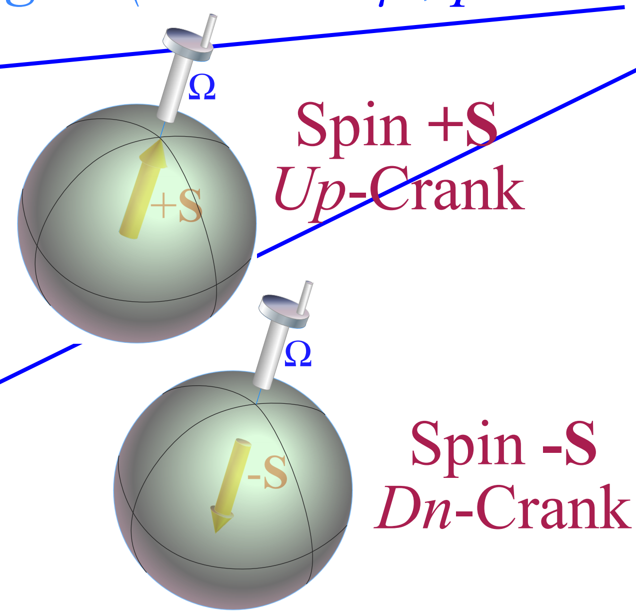
$$|\uparrow_{\alpha\beta\gamma}\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)|\uparrow_{000}\rangle$$

with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta \pm \pi$*) of* **H***-matrix*



$$|\Omega_+\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$

$$|\Omega_-\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta\pm\pi}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta\pm\pi}{2} \end{pmatrix}$$



More reliable computation:

$$\varphi = atan2(C, B)$$

$[\tan^{-1}(C/B)$ *is unreliable*$]$

$$\vartheta = atan2(2\sqrt{B^2 + C^2}, A-D)$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

$$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$$

Step 2.Convert Cartesian to polar form: ($\Omega_A = \Omega\cos\vartheta$, $\Omega_B = \Omega\cos\varphi\sin\vartheta$, $\Omega_C = \Omega\sin\varphi\sin\vartheta$)

$$\Omega_0 = \frac{A+D}{2} = 10$$

and: $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2} = \sqrt{(4)^2 + 4\sqrt{6}^2 + 4\sqrt{6}^2} = \sqrt{16 + 24 + 24} = \sqrt{64} = 8$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

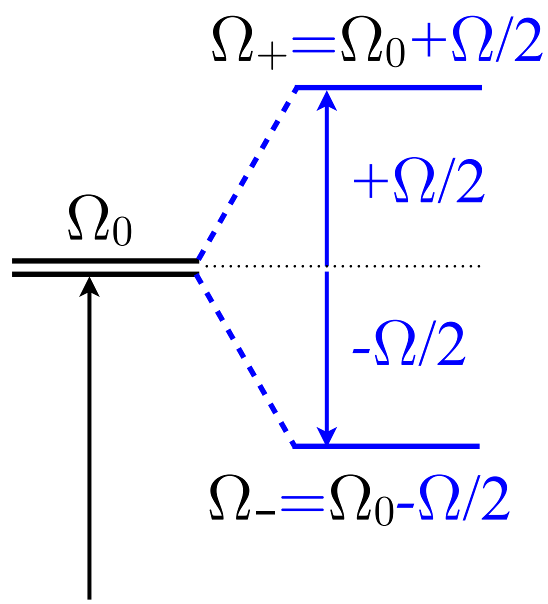
$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

$$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$$

Step 2.Convert Cartesian to polar form: ($\Omega_A = \Omega\cos\vartheta$, $\Omega_B = \Omega\cos\varphi\sin\vartheta$, $\Omega_C = \Omega\sin\varphi\sin\vartheta$)

$$\Omega_0 = \frac{A+D}{2} = 10$$

and: $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2} = \sqrt{(4)^2 + 4\sqrt{6}^2 + 4\sqrt{6}^2} = \sqrt{16+24+24} = \sqrt{64} = 8$



$\underline{|eigenvalue-1}$

$$\omega_\uparrow = 10 + \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2}$$
$$= 10 + 4 = 14$$

$\underline{|eigenvalue-2}$

$$\omega_\downarrow = 10 - \sqrt{\left(\frac{12-8}{2}\right)^2 + \left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2}$$
$$= 10 - 4 = 6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$

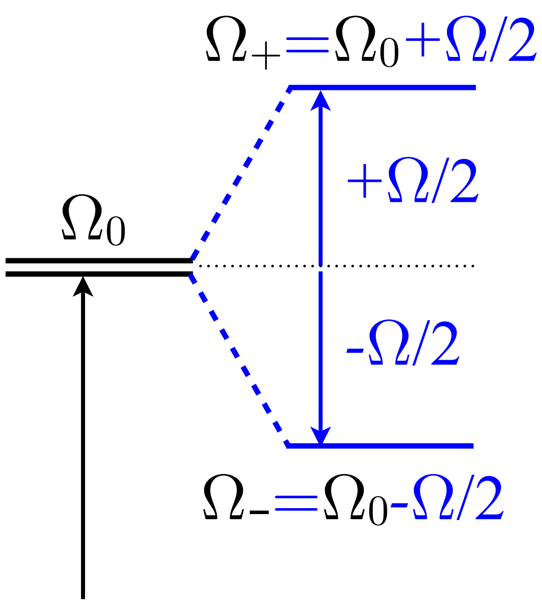
Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta$)

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega=\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$

or: $\vartheta=\cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2+4B^2+4C^2}]=\cos^{-1}[(4)/8]=\pi/3,$

$\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}]=\cos^{-1}[\sqrt{6}/\sqrt{12}]=\pi/4$



$|eigenvalue-1$

$$\omega_\uparrow=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}=10+4=14$$

$|eigenvalue-2$

$$\omega_\downarrow=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}=10-4=6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$

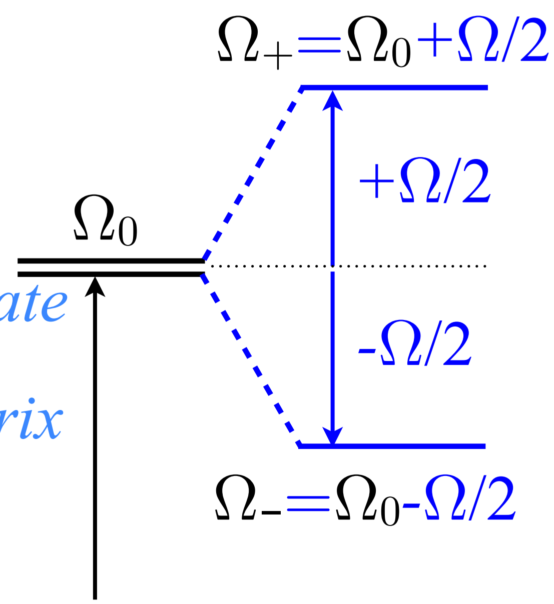
Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta$)

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega\ =\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$

or: $\vartheta=\cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2+4B^2+4C^2}\ ]=\cos^{-1}[(4)/8]=\pi/3,$

$\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}\ ]=\cos^{-1}[\sqrt{6}/\sqrt{12}]=\pi/4$



Step 3.To find eigenvectors replace Euler angles (azimuth α, polar β) of Euler-state

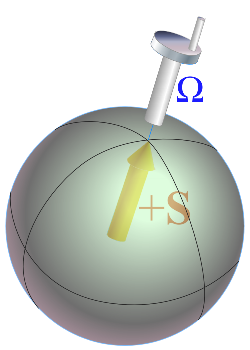
with the Darboux axis polar angles (azimuth φ , polar ϑ or ϑ±π) of **H***-matrix*

$\lfloor eigenvalue-1$

$$\omega_\uparrow=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10+4=14$$

$\lfloor eigenvector-1$

$$\left|\uparrow\right\rangle=\begin{pmatrix} e^{-i\frac{\pi}{8}}\cos\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\sin\frac{\pi}{6} \end{pmatrix}=\begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$

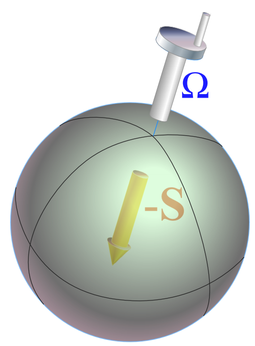


$\lfloor eigenvalue-2$

$$\omega_\downarrow=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10-4=6$$

$\lfloor eigenvector-2$

$$\left|\downarrow\right\rangle=\begin{pmatrix} -e^{-i\frac{\pi}{8}}\sin\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\cos\frac{\pi}{6} \end{pmatrix}=\begin{pmatrix} -e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \\ 1 \end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$



Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* $\Theta = 0 - 4\pi$*-sequence* [φϑ] *fixed*

R(3)-U(2) slide rule for converting **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

→ *Asymmetric-Diagonal A-Type motion*

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

The ABC's of U(2) dynamics

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

$$\text{Crank}: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad \text{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

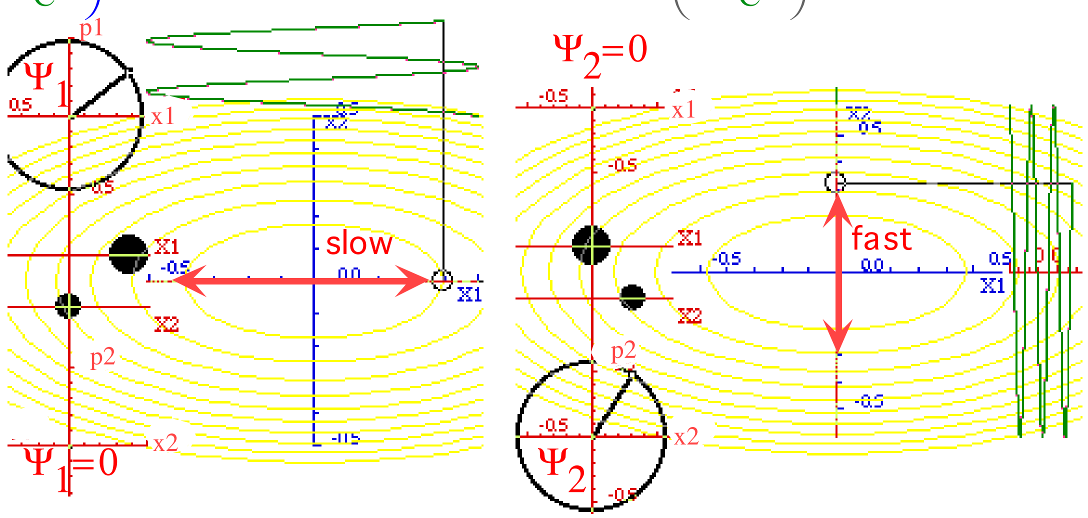
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$



The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

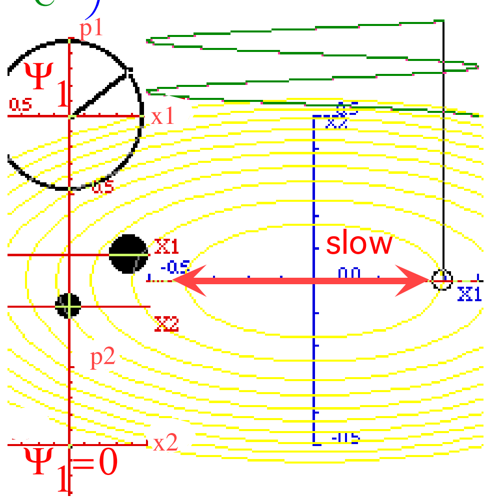
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

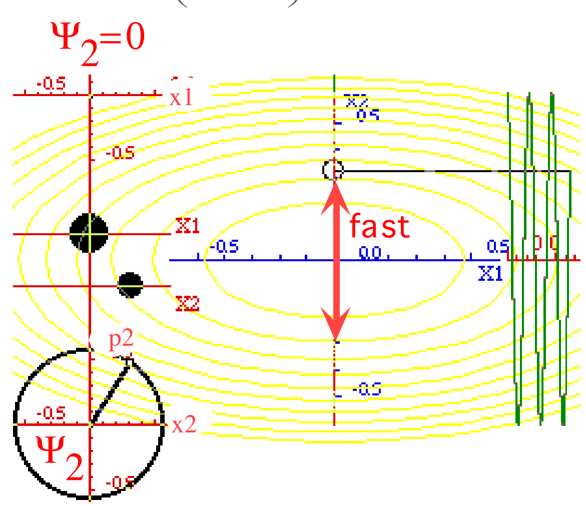
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

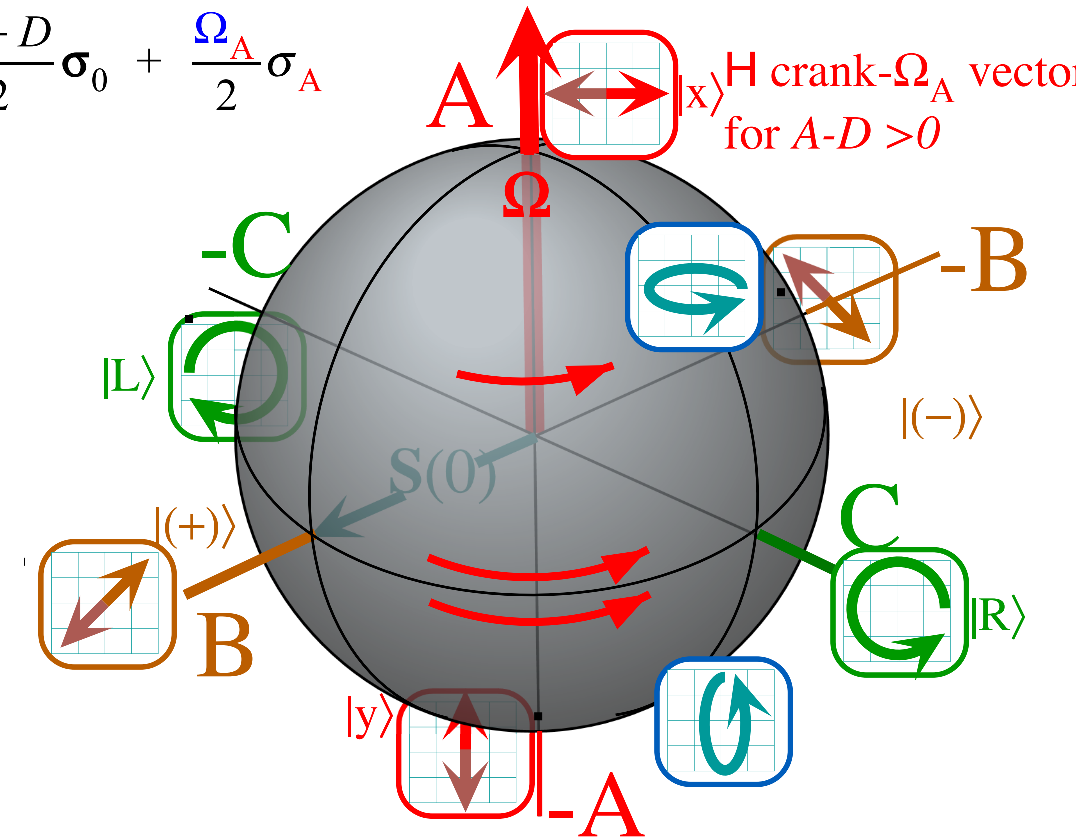
Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$







The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

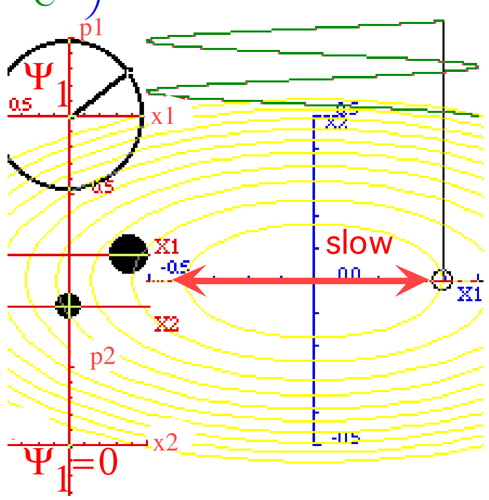
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

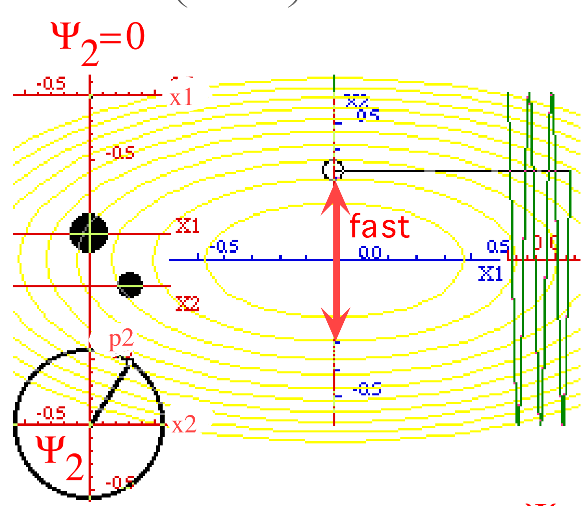
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

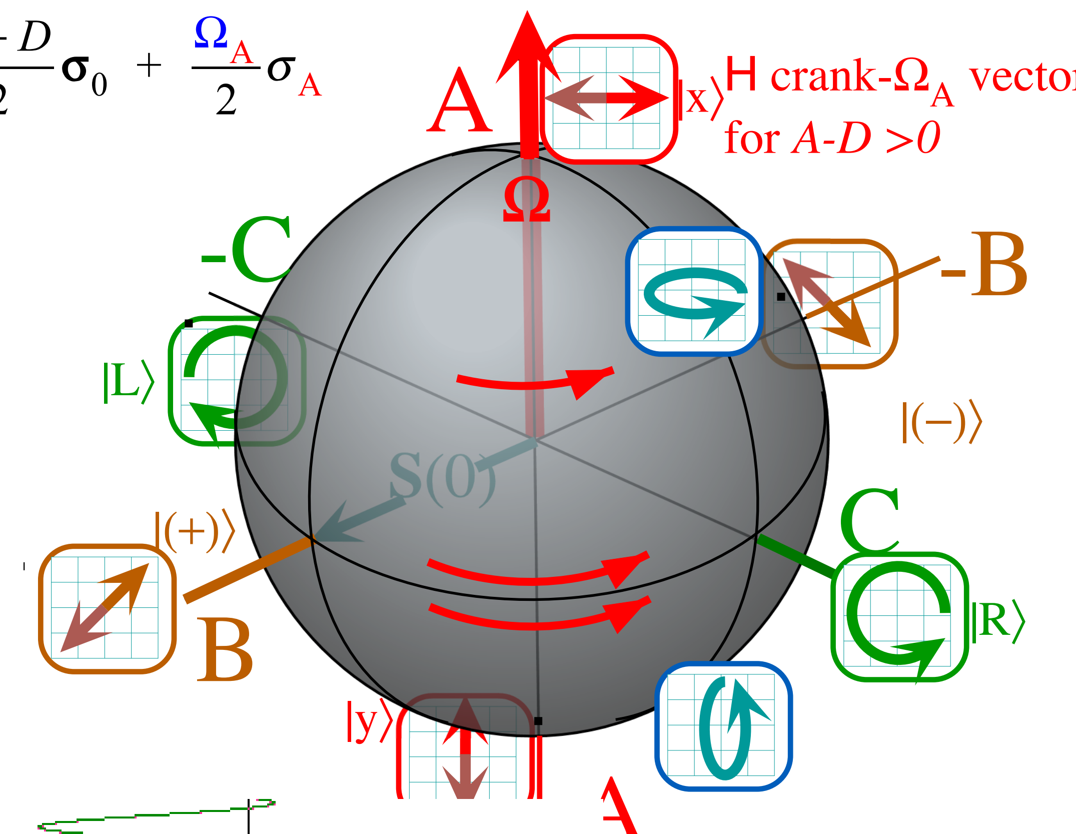
Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

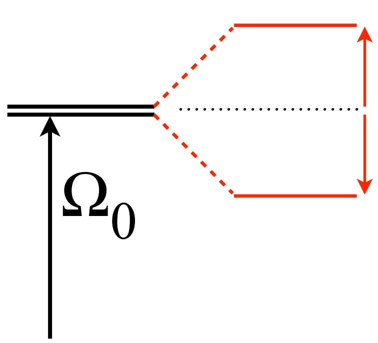
$$\textit{Crank}:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad \textit{Eigen-Spin}:\ \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$



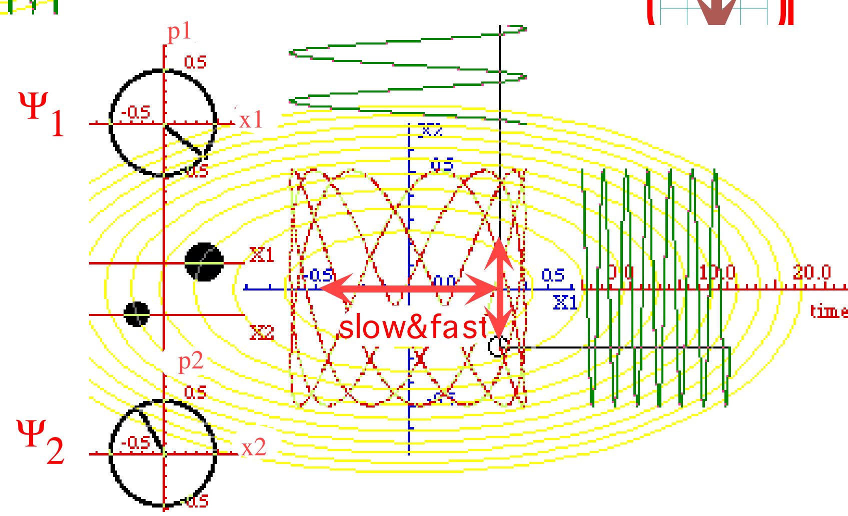




Beat dynamics:



Total beat frequency Ω_A



Review: Fundamental Euler **R***(αβγ) and Darboux* **R***[φϑΘ] representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R***[φϑΘ] and vice versa*

Euler **R***(αβγ) rotation Θ=0−4π-sequence [φϑ] fixed*

R(3)-U(2) slide rule for converting **R***(αβγ) ↔* **R***[φϑΘ] and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

→ *Bilateral-Balanced B-Type motion*

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B$$

The *ABC*'s of *U*(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

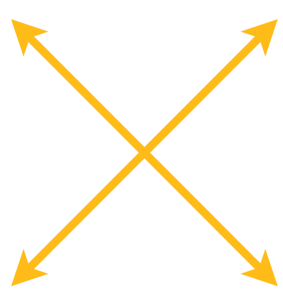
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

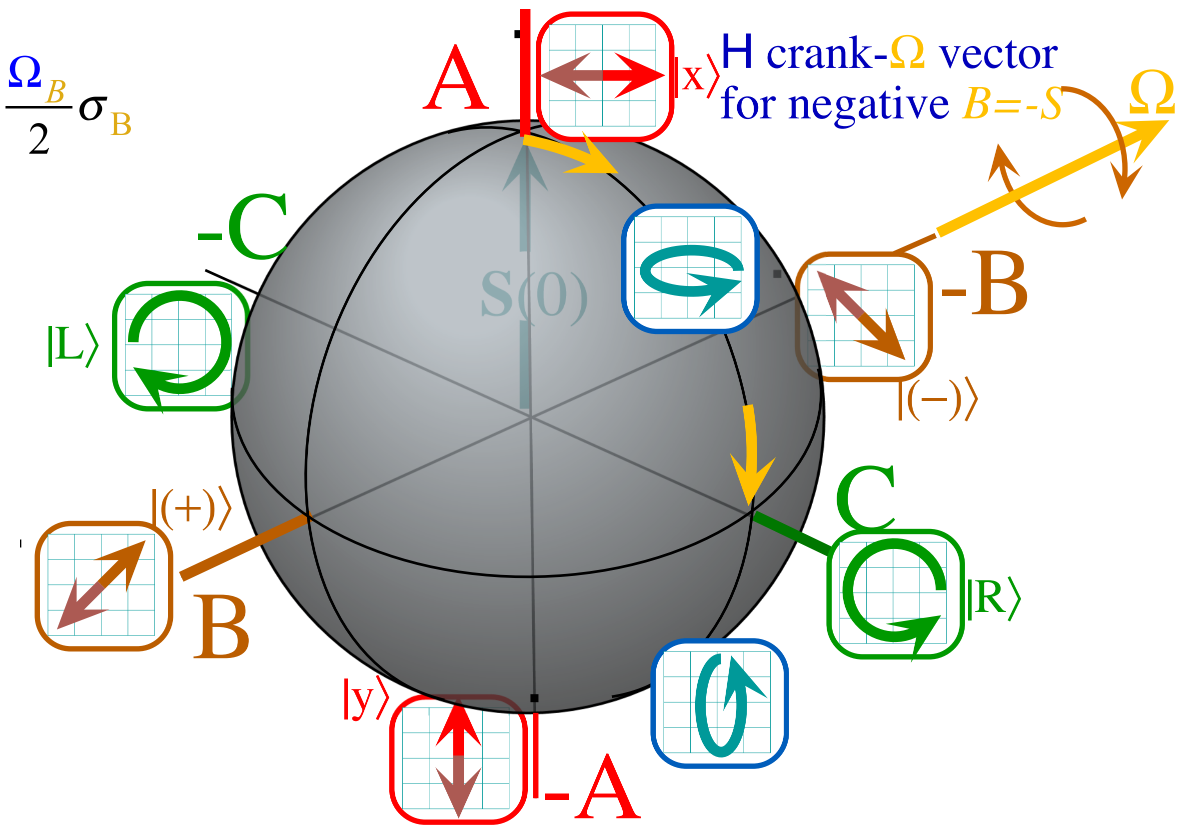
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced *B*-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$$





The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

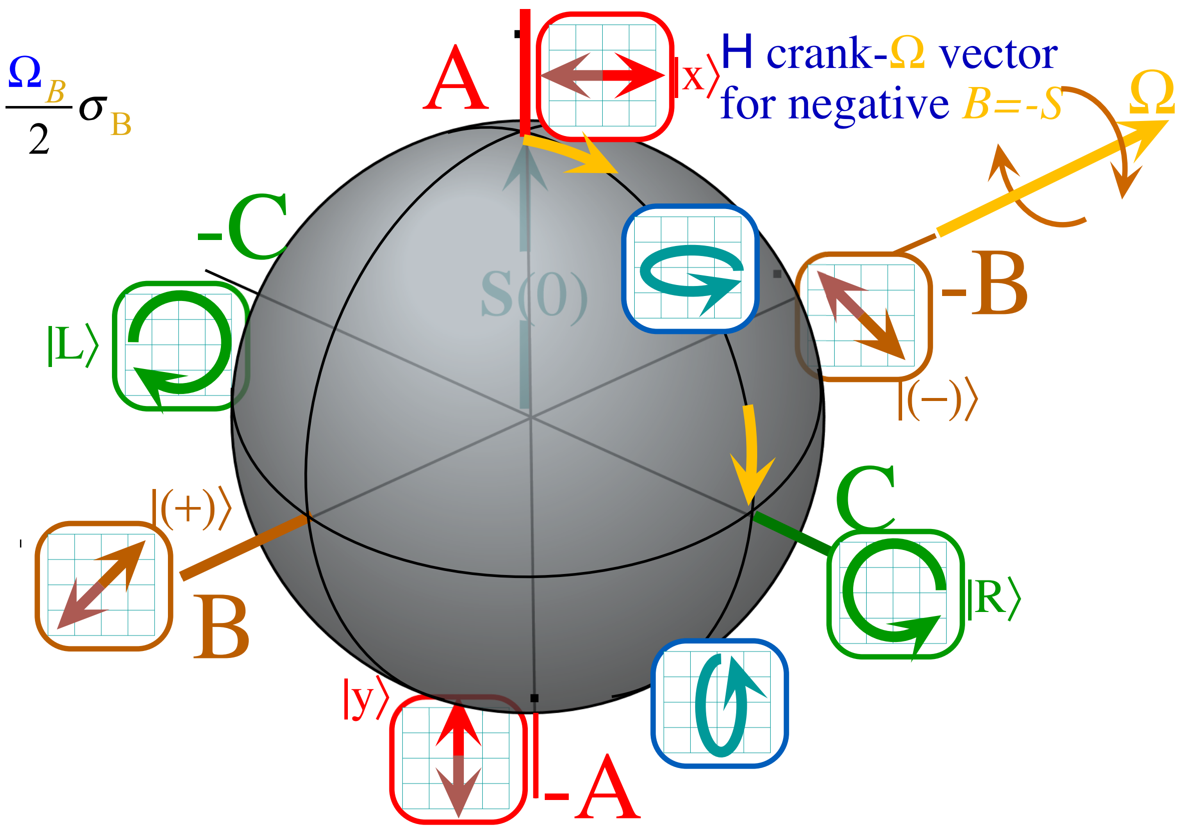
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

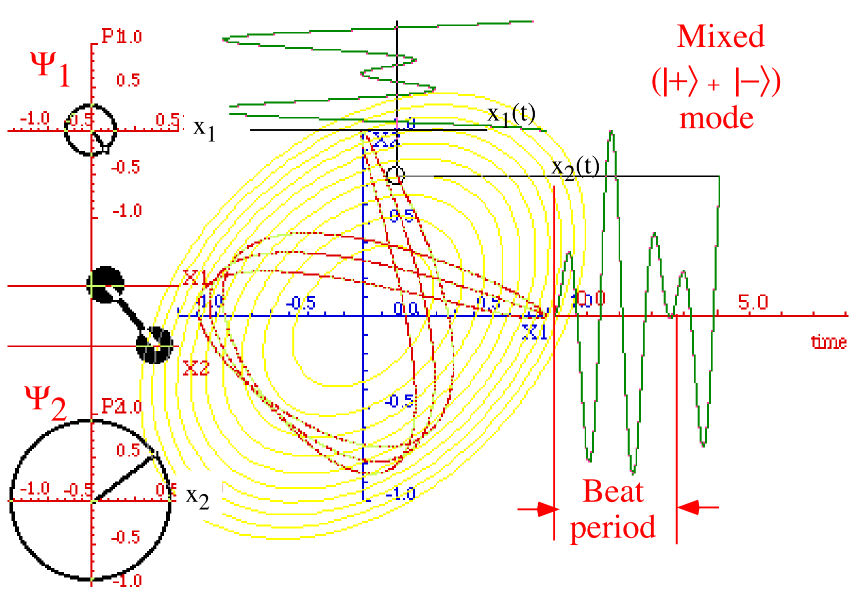
Bilateral-Balanced B-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B$$

$$Crank: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$$



Beat dynamics:



The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

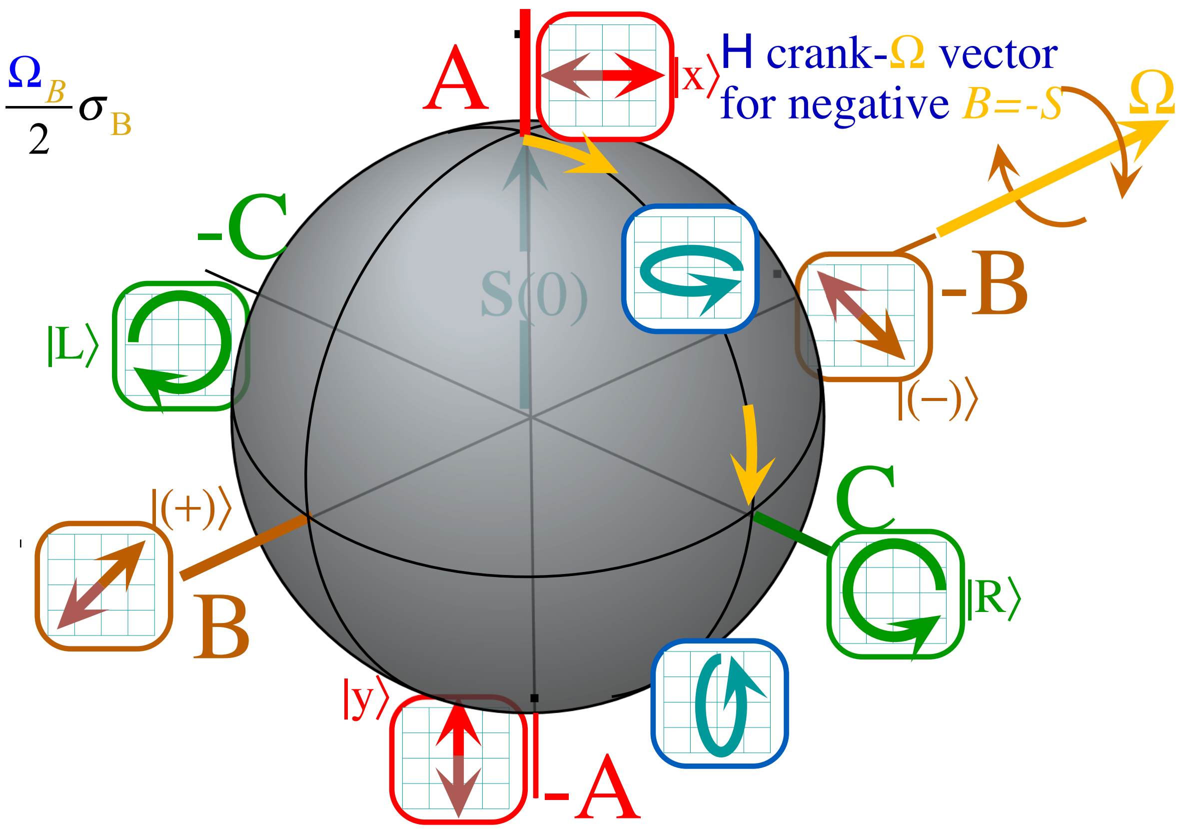
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

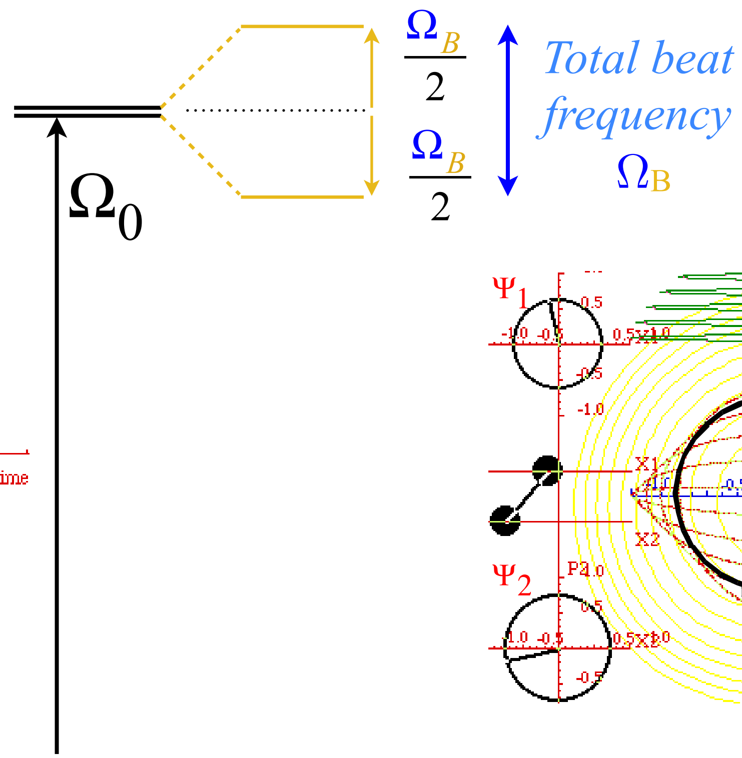
Bilateral-Balanced B-Type motion

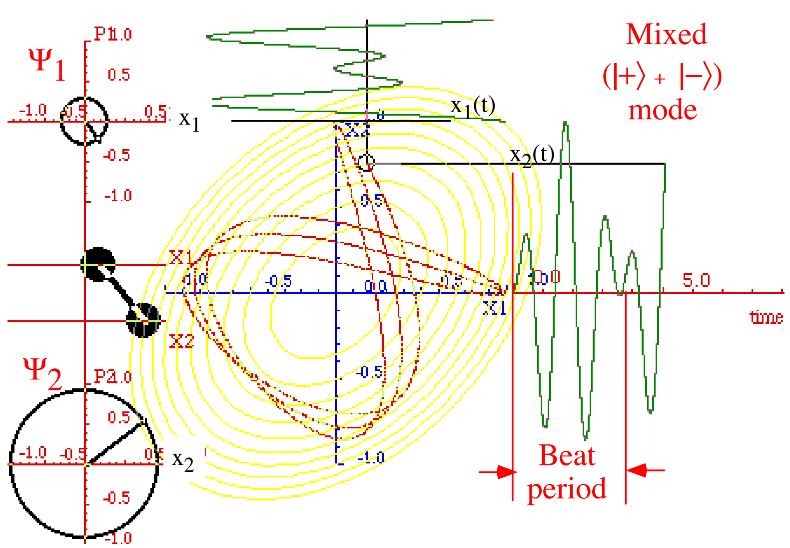
$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B$$

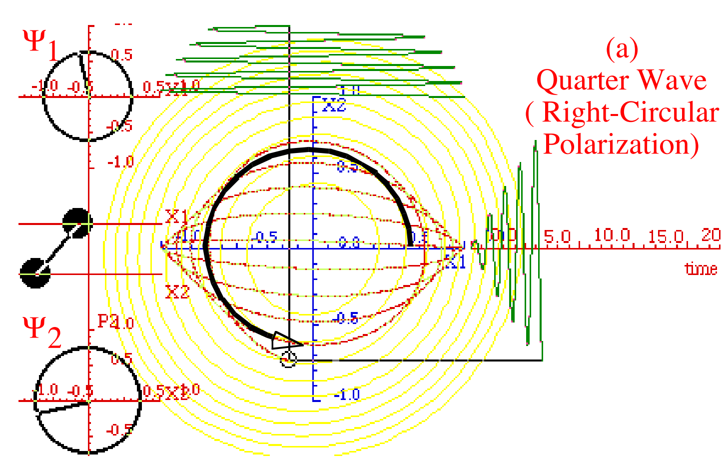
$$Crank: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$$

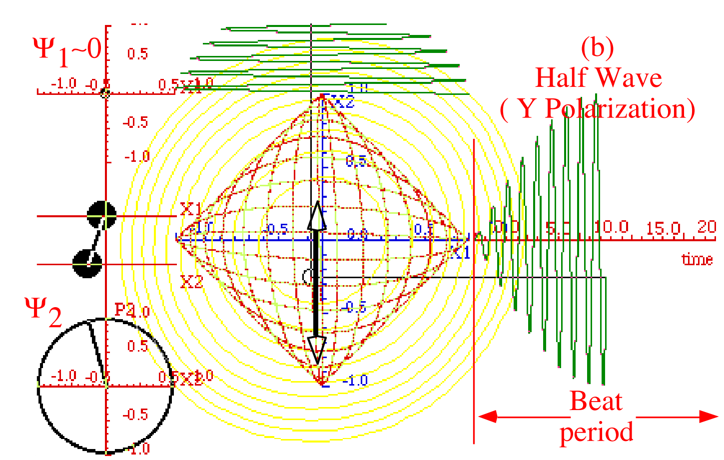


Beat dynamics:









Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting **R**$(\alpha\beta\gamma) \leftrightarrow$ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

→ *Circular-Coriolis... C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

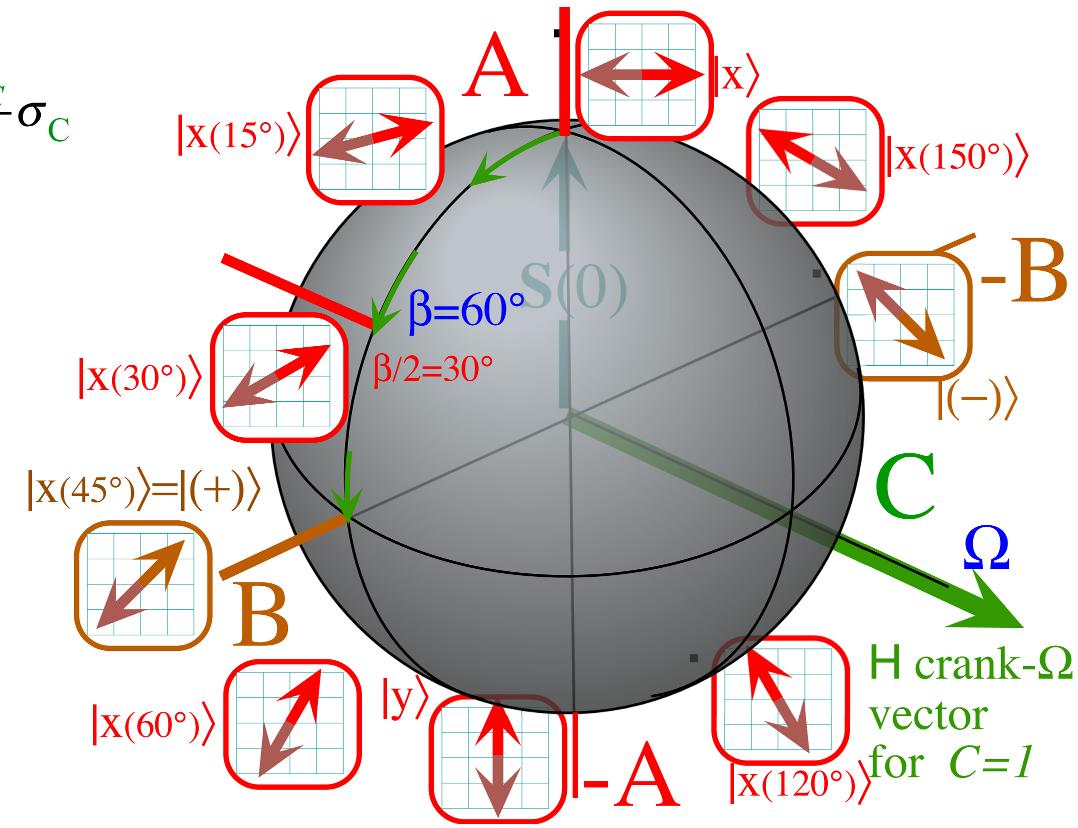
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C$$

$$\textit{Crank}: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad \textit{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$



The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

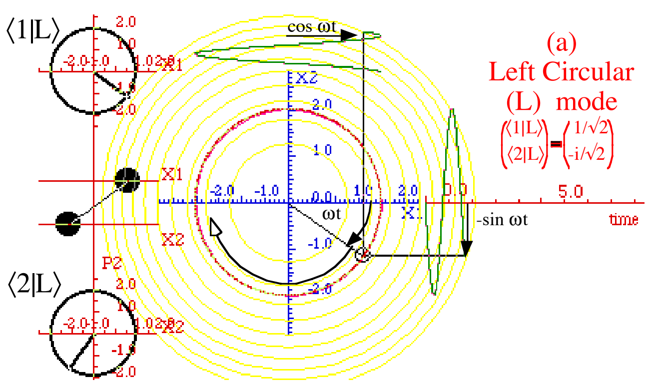
$$= \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

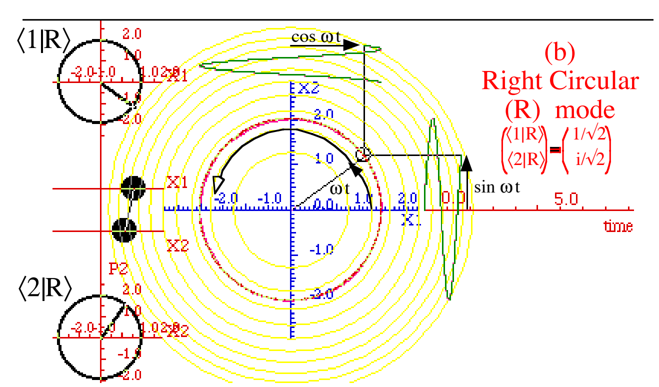
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

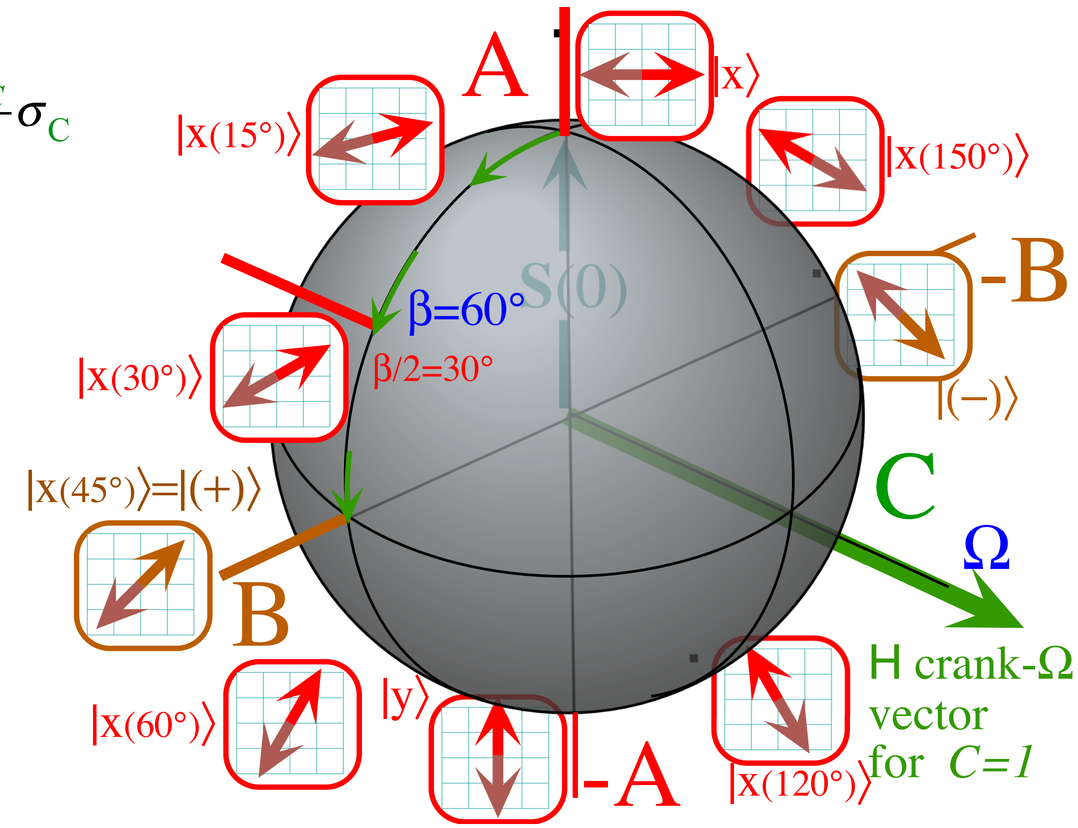
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad Eigen-Spin:\ \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$







The ABC's of U(2) dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

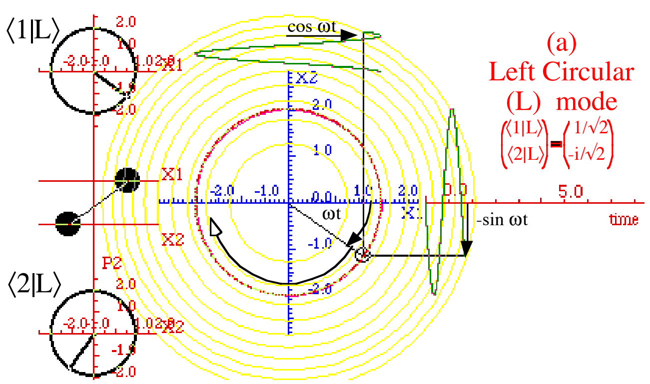
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

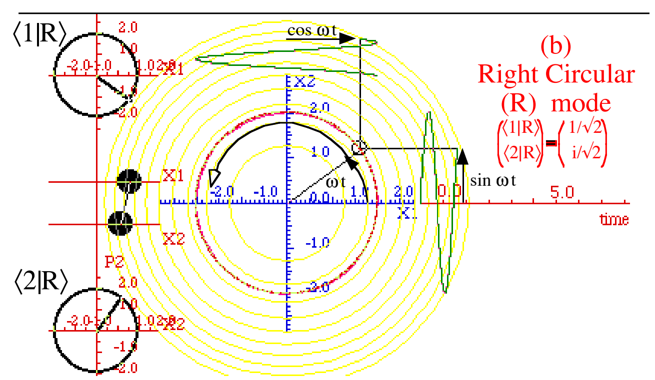
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

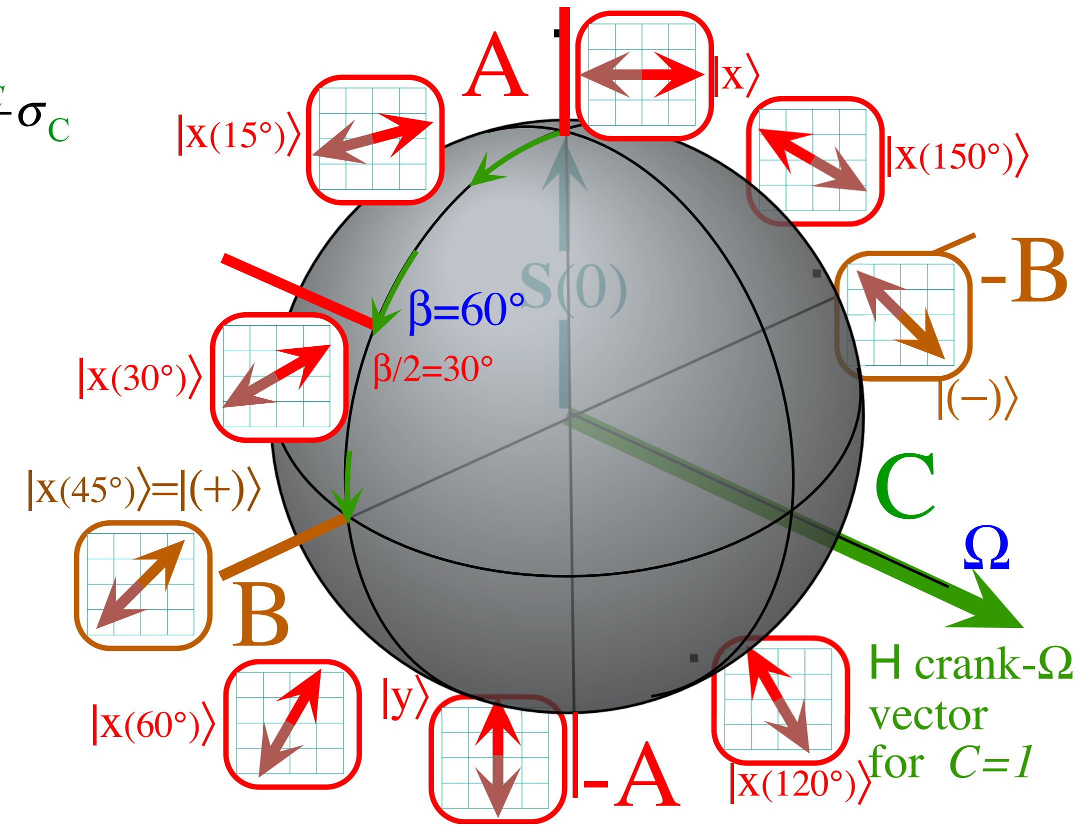
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

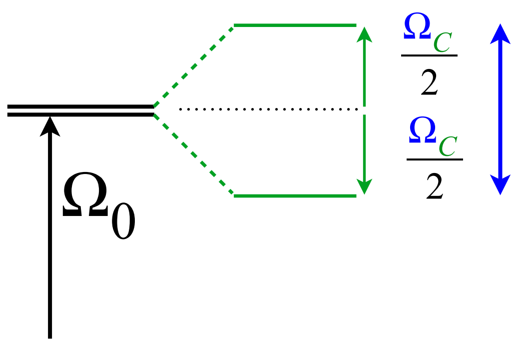
$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$



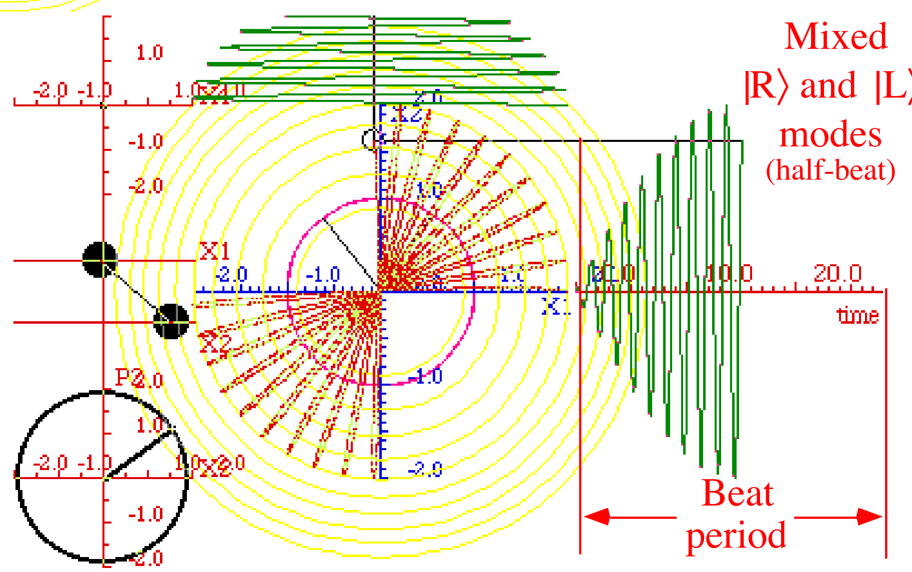




Beat dynamics:



Total beat frequency Ω_C



Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation Θ=0−4π-sequence [φϑ] fixed*

R(3)-U(2) slide rule for converting **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

→ *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

The ABC's of U(2) dynamics-Mixed modes

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

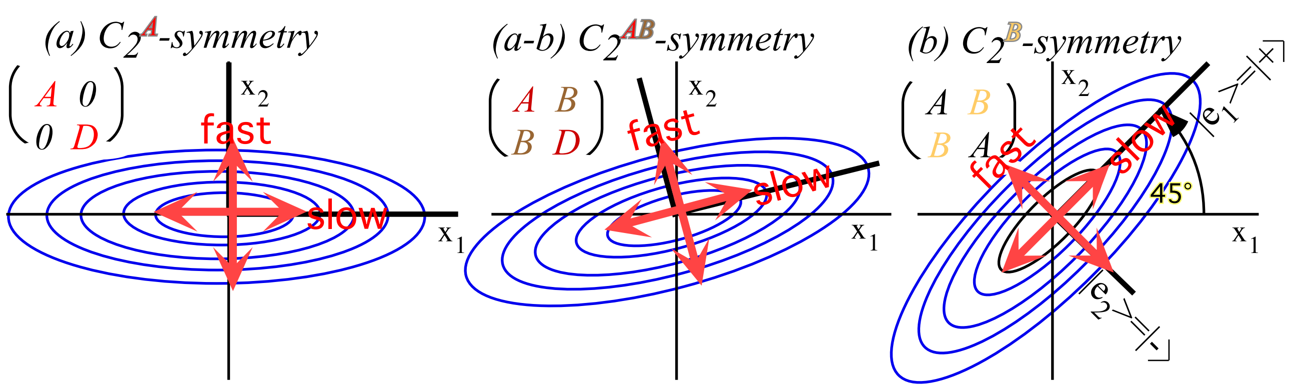
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

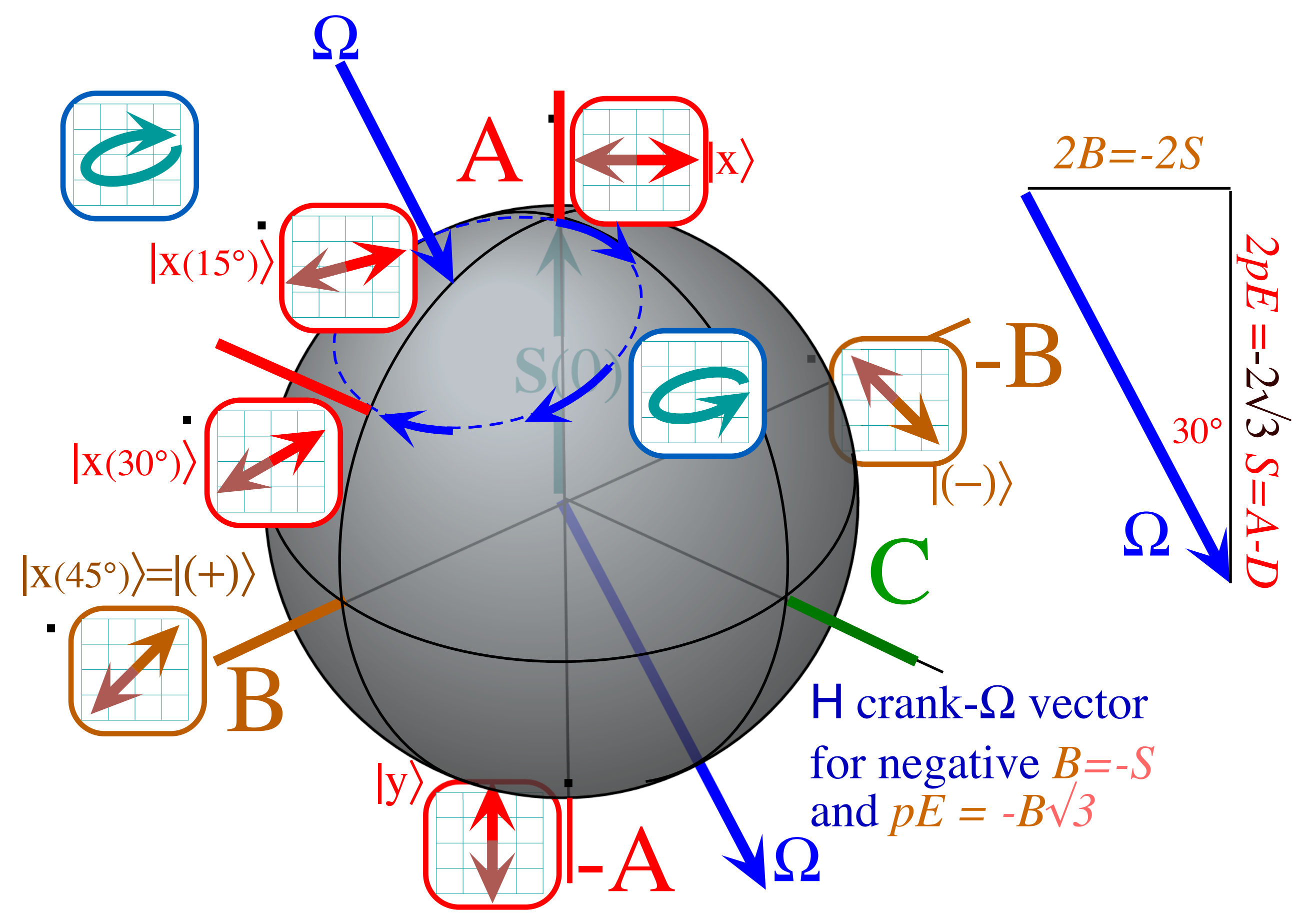
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Tilted-plane polarization AB-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

$$\textit{Crank}:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix} \qquad \textit{Eigen-Spin}: \vec{\mathbf{S}} = \pm S\cdot\hat{\Omega}$$





Beat dynamics:

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

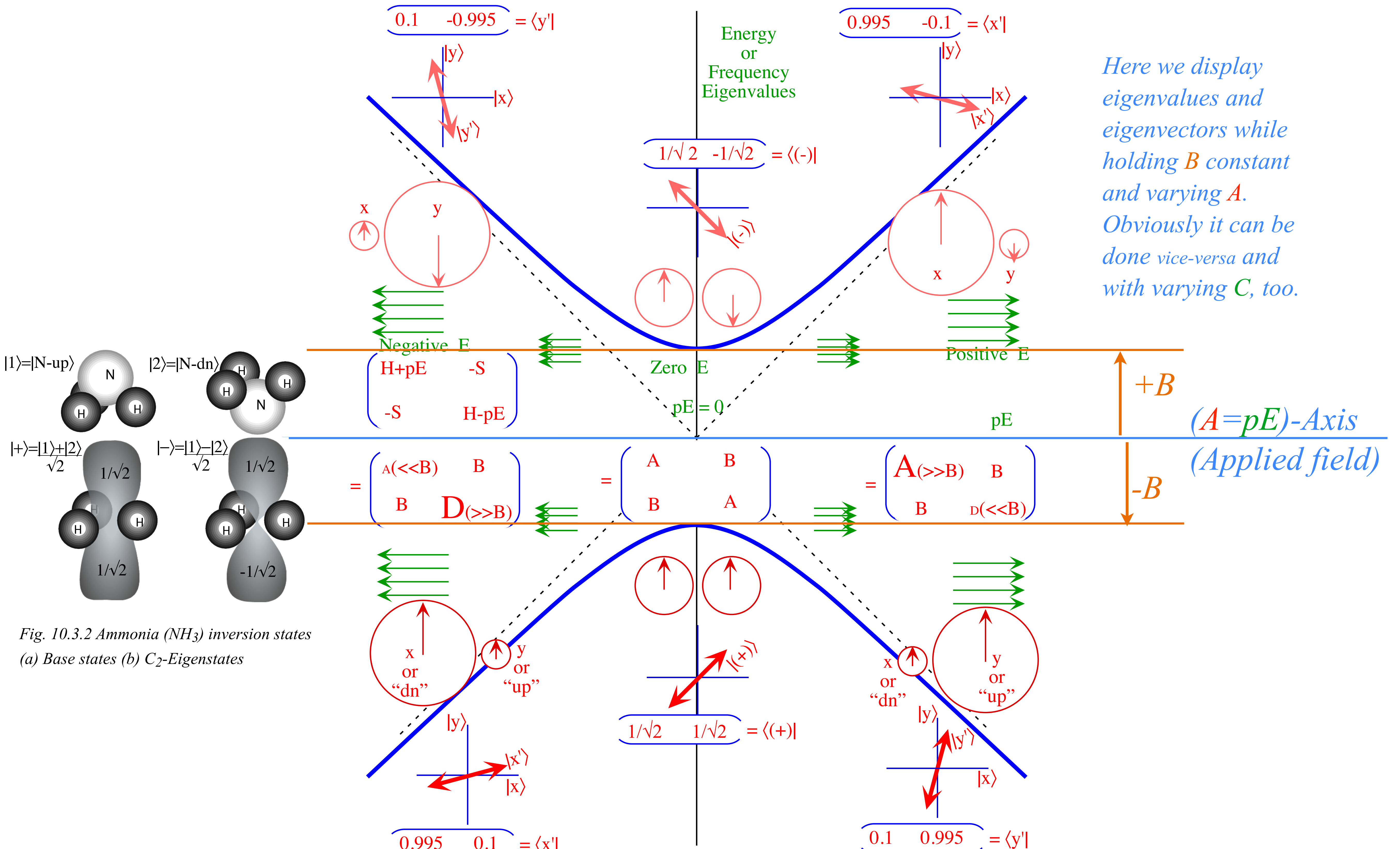


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \quad \text{Secular equation: } \varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2) \quad \text{gives } \textit{hyperbolic} \text{ energy levels: } \varepsilon = \pm\sqrt{A^2 + B^2}$$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

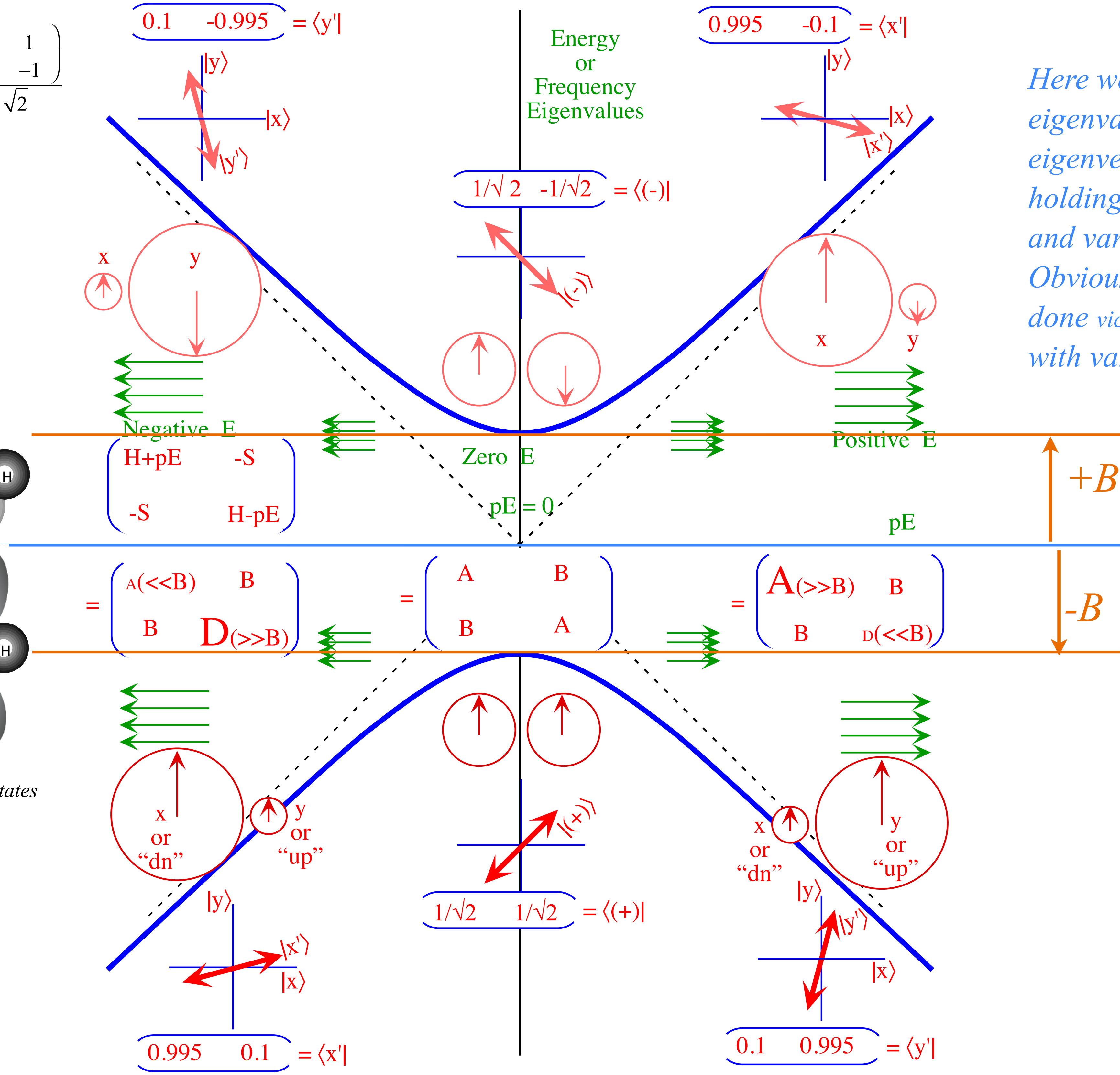


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

(A=pE)-Axis
(Applied field)

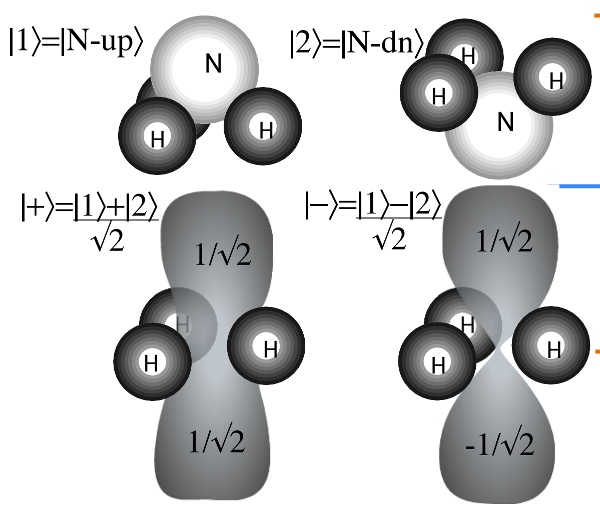


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\quad$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

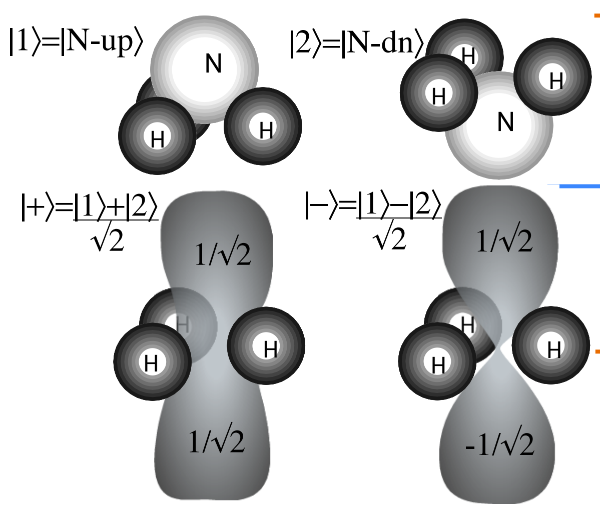


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

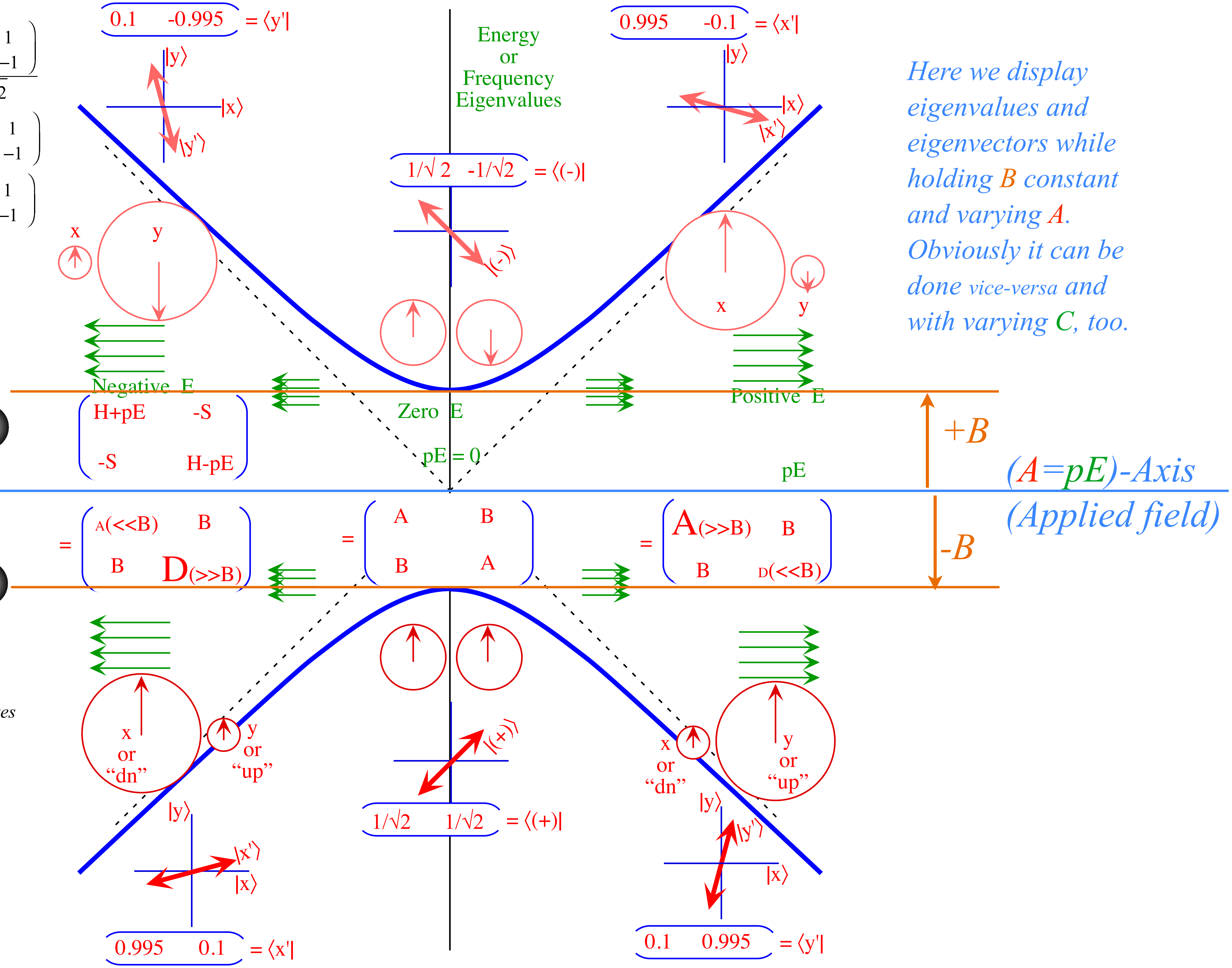


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$$ Secular equation: $\varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

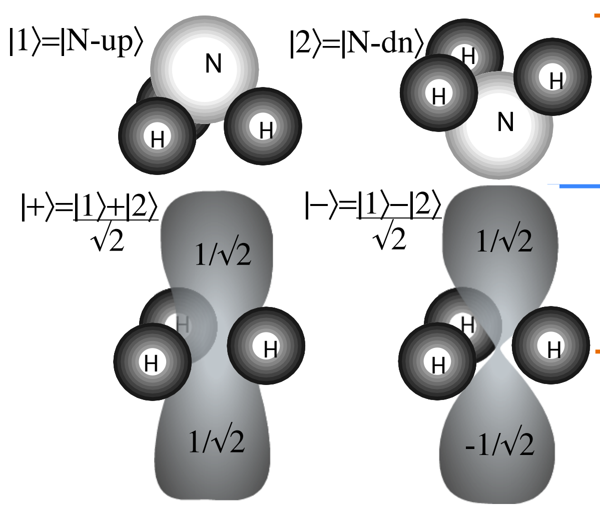


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

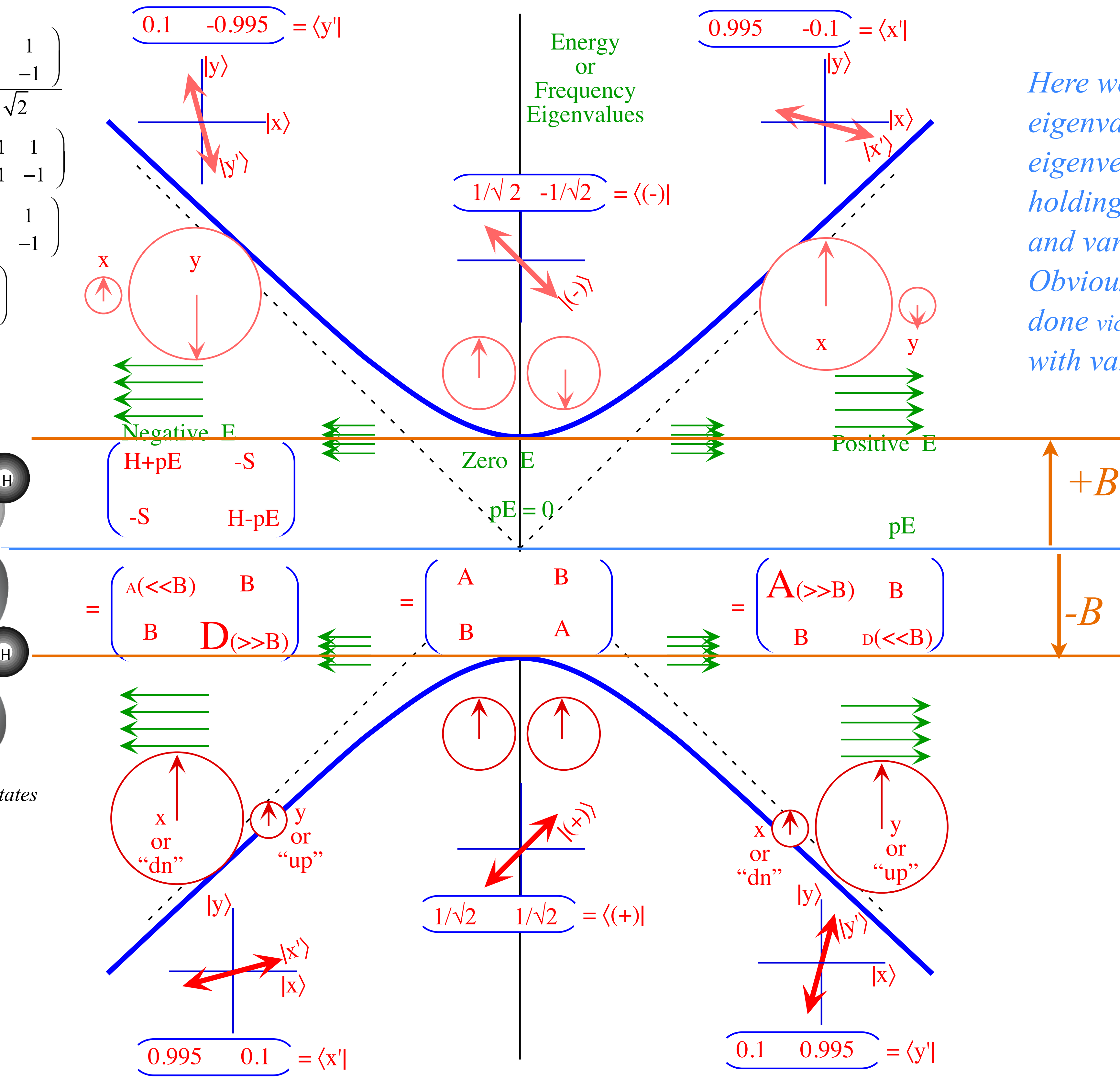


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

$\mathbf{H}(B-basis)$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

$$= \begin{pmatrix} +B & A \\ A & -B \end{pmatrix}$$

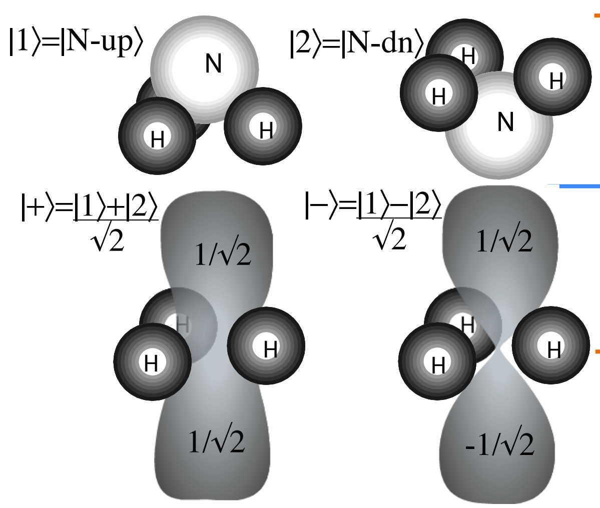
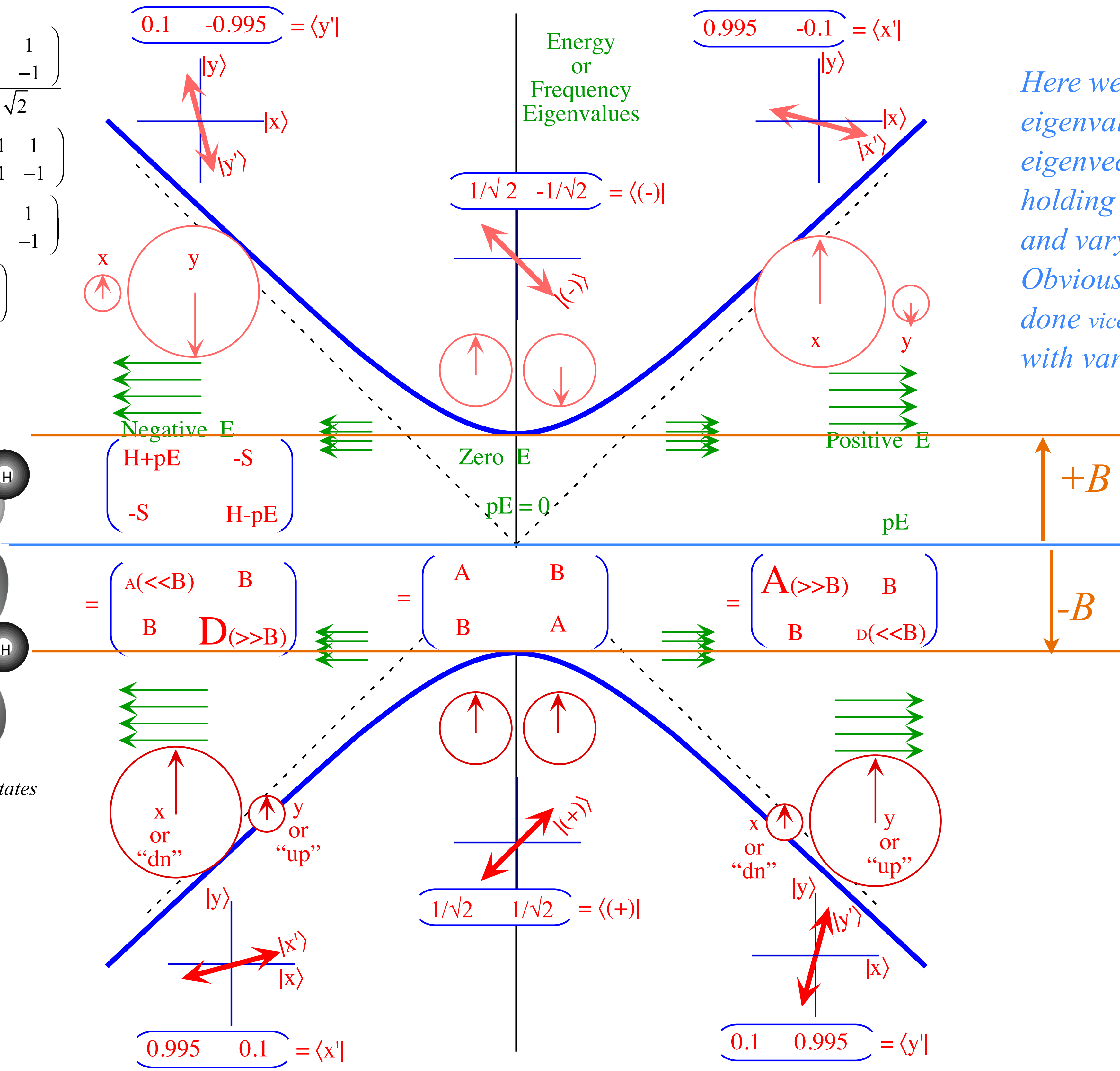


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



(A=pE)-Axis (Applied field)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

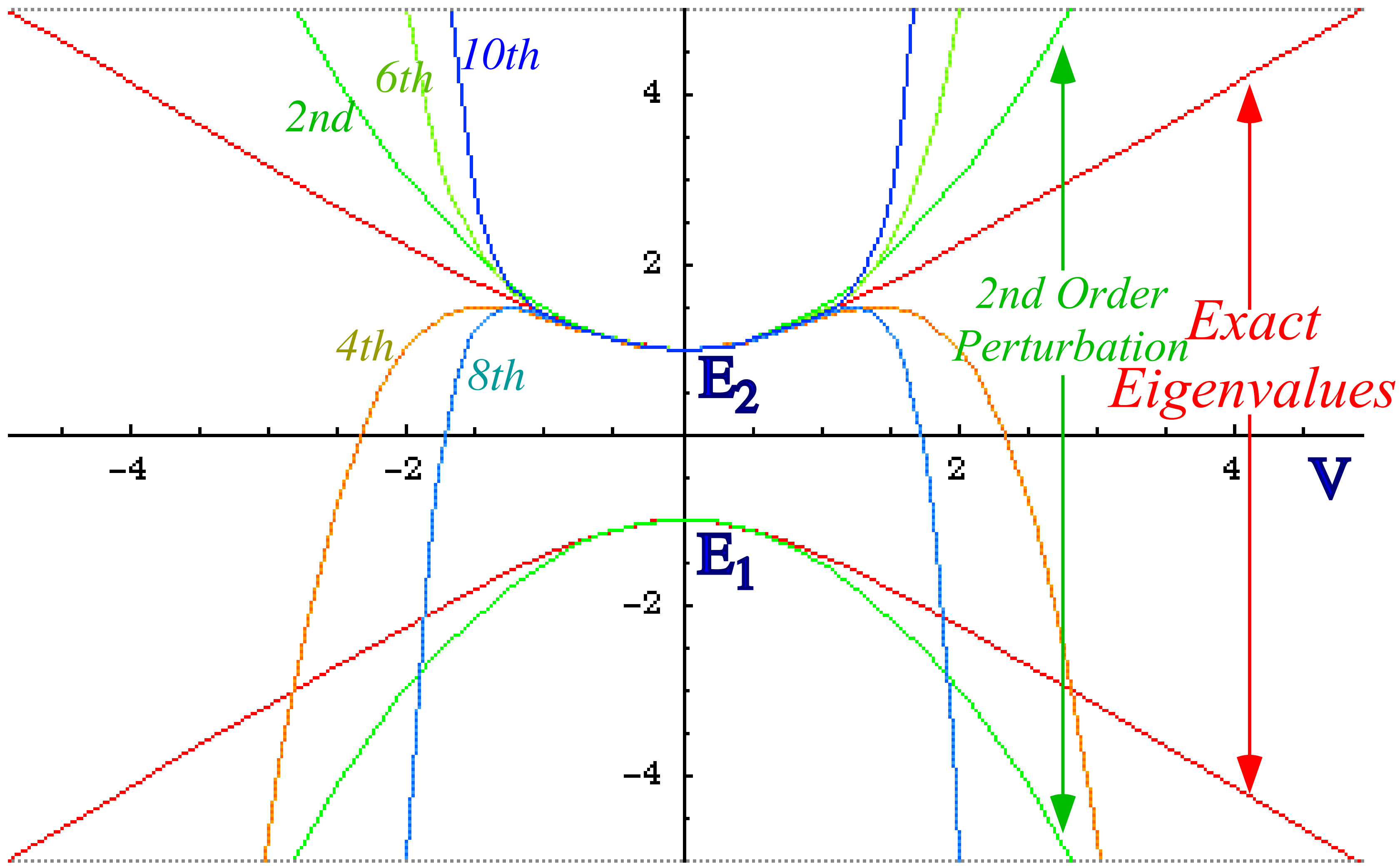
Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$$

2nd order perturbation terms

$$\lambda_1 = E_1 + \frac{V^2}{E_1 - E_2} ,$$

$$\lambda_2 = E_2 + \frac{V^2}{E_2 - E_1} .$$



$$\lambda^2 - (Trace\mathbf{H})\lambda + \det|\mathbf{H}| = 0 = \lambda^2 - (E_1 + E_2)\lambda + (E_1E_2 - V^2)$$

$$\lambda_{1,2} = \frac{E_1 + E_2 \pm \sqrt{(E_1+E_2)^2 - 4E_1E_2 + 4V^2}}{2} = \frac{E_1 + E_2 \pm \sqrt{(E_1-E_2)^2 + 4V^2}}{2} ,$$

Fig. 3.2.2 Comparison of exact vs. 2nd-order thru 10th-order perturbation approximations

$$E_2 = \frac{\Delta}{2} + \frac{V^2}{\Delta} - \frac{V^4}{\Delta^3} + \frac{V^6}{\Delta^5} - \frac{V^8}{\Delta^7} + \frac{V^{10}}{\Delta^9} \cdots \text{, where: } \Delta = |E_1 - E_2|$$

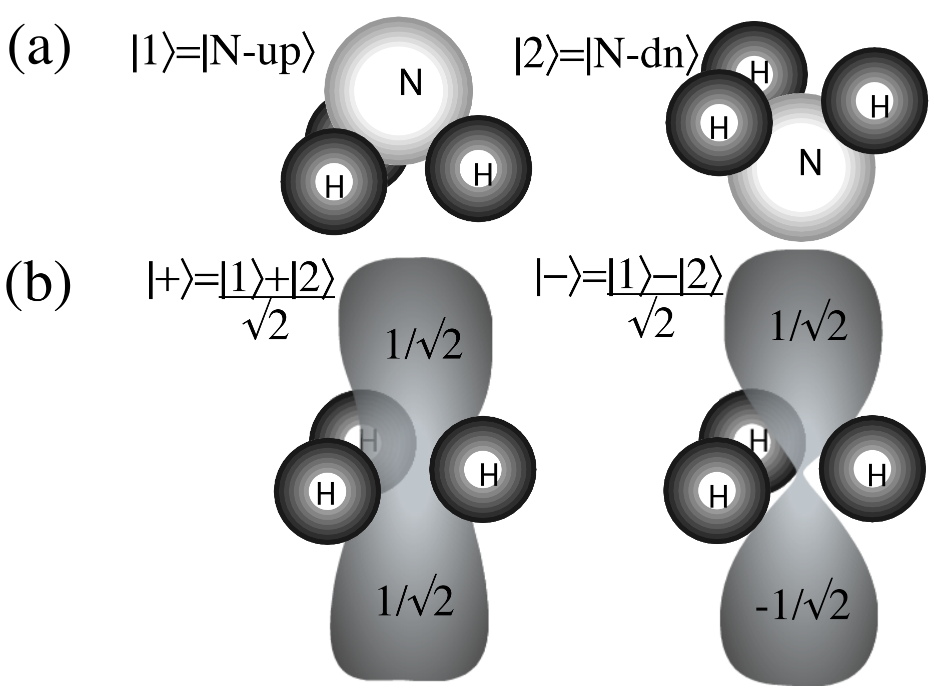
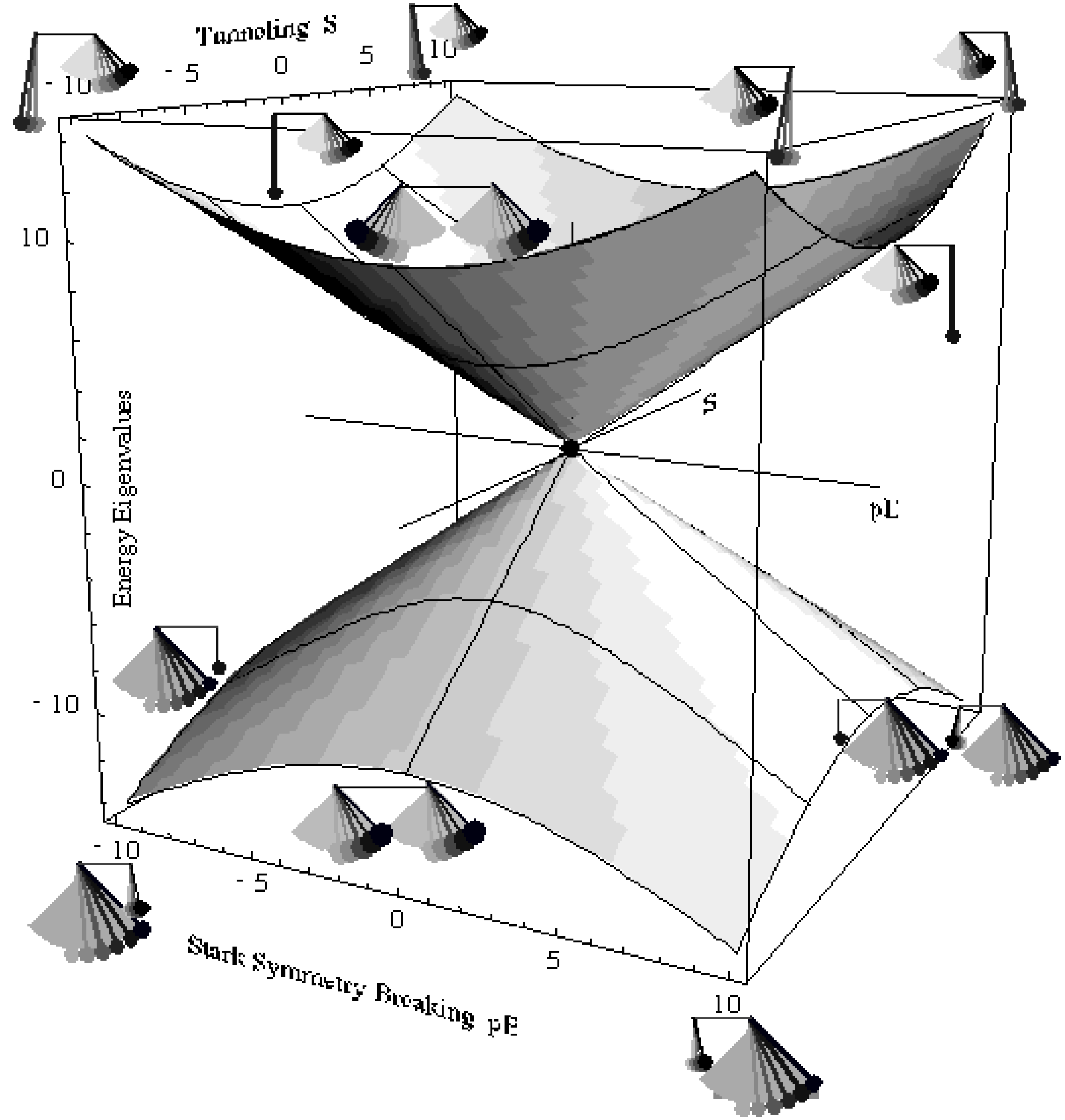


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

Review: Fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler **R***(αβγ) derived from Darboux* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation Θ=0−4π-sequence [φϑ] fixed*

R(3)-U(2) slide rule for converting **R***(αβγ)* ↔ **R**[φϑΘ] *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

→ *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

ABC-Type elliptical polarized motion

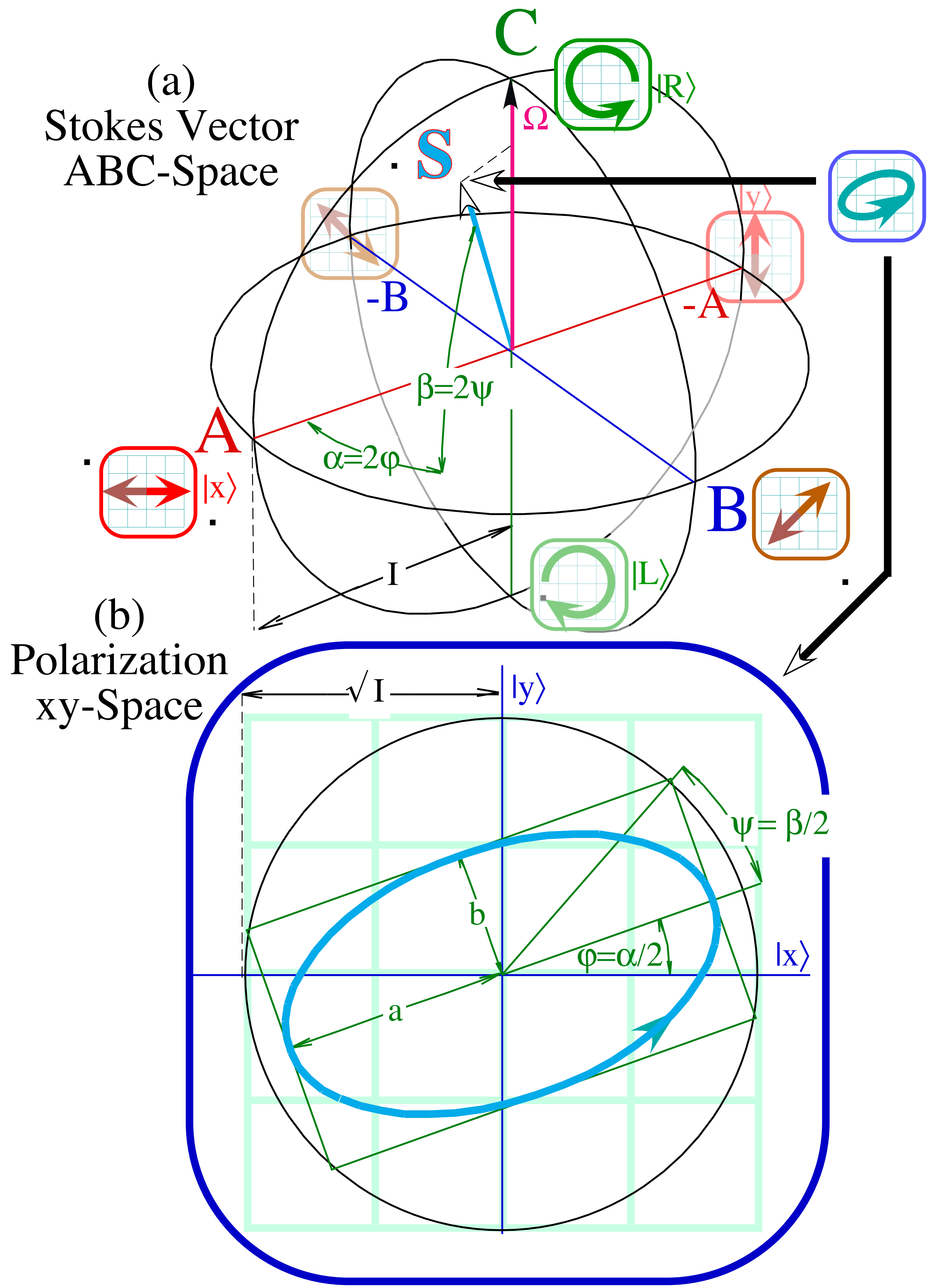


Fig. 10.B.3

Euler-like coordinates for
(a) R(3) spin vector
(b) U(2) polarization ellipse

ABC-Type elliptical polarized motion

(from Principles of Symmetry, Dynamics, and Spectroscopy)

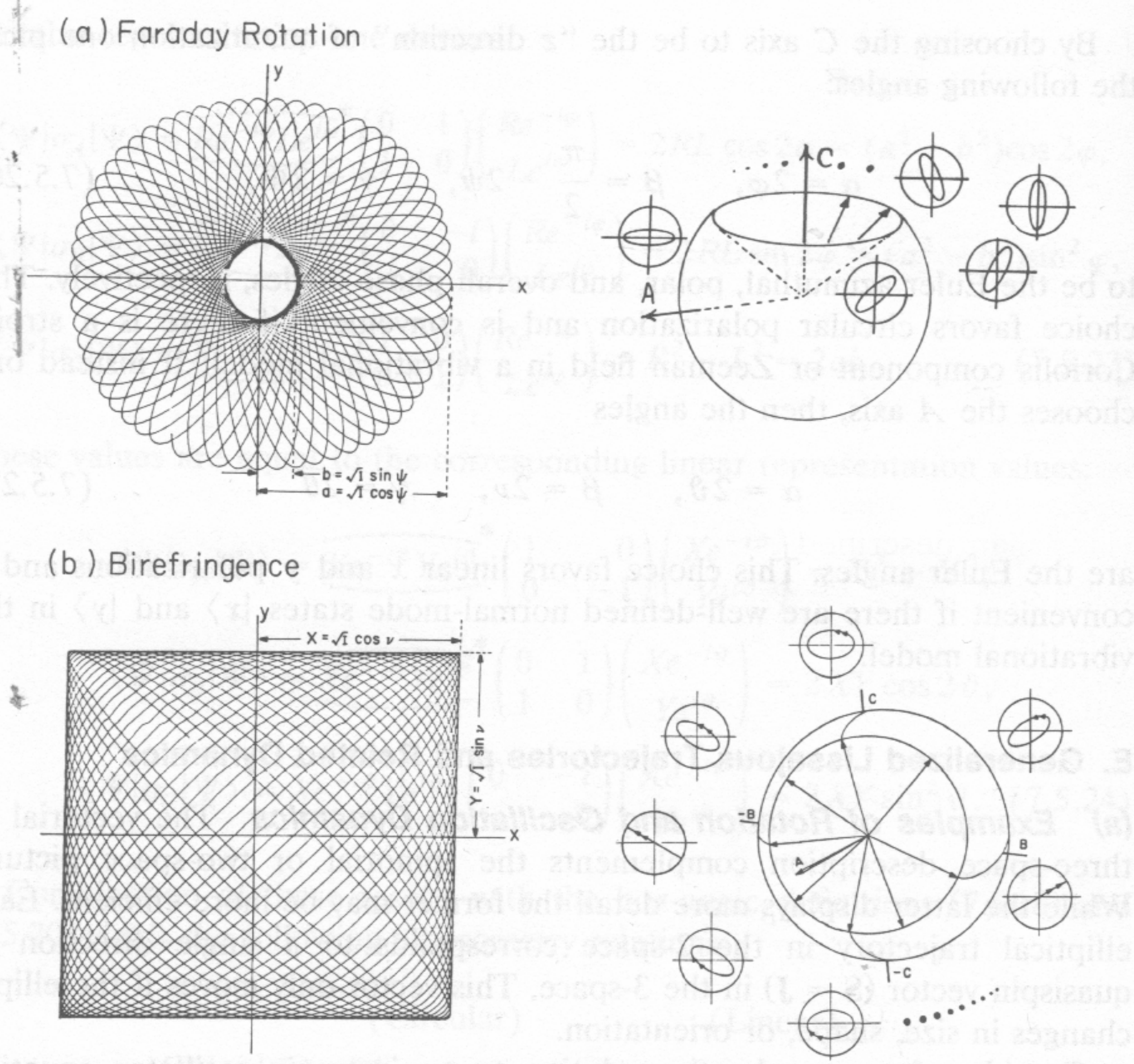
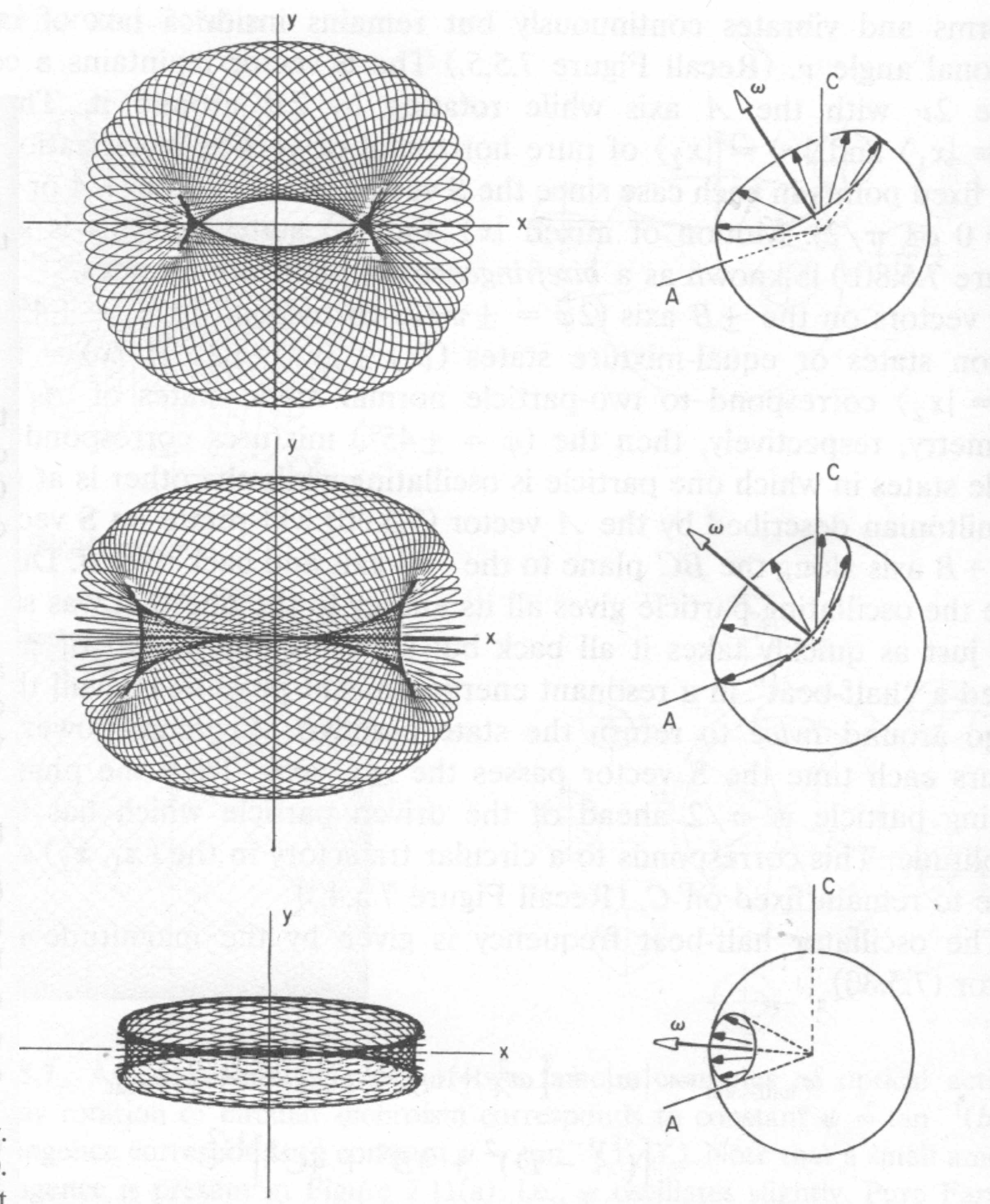
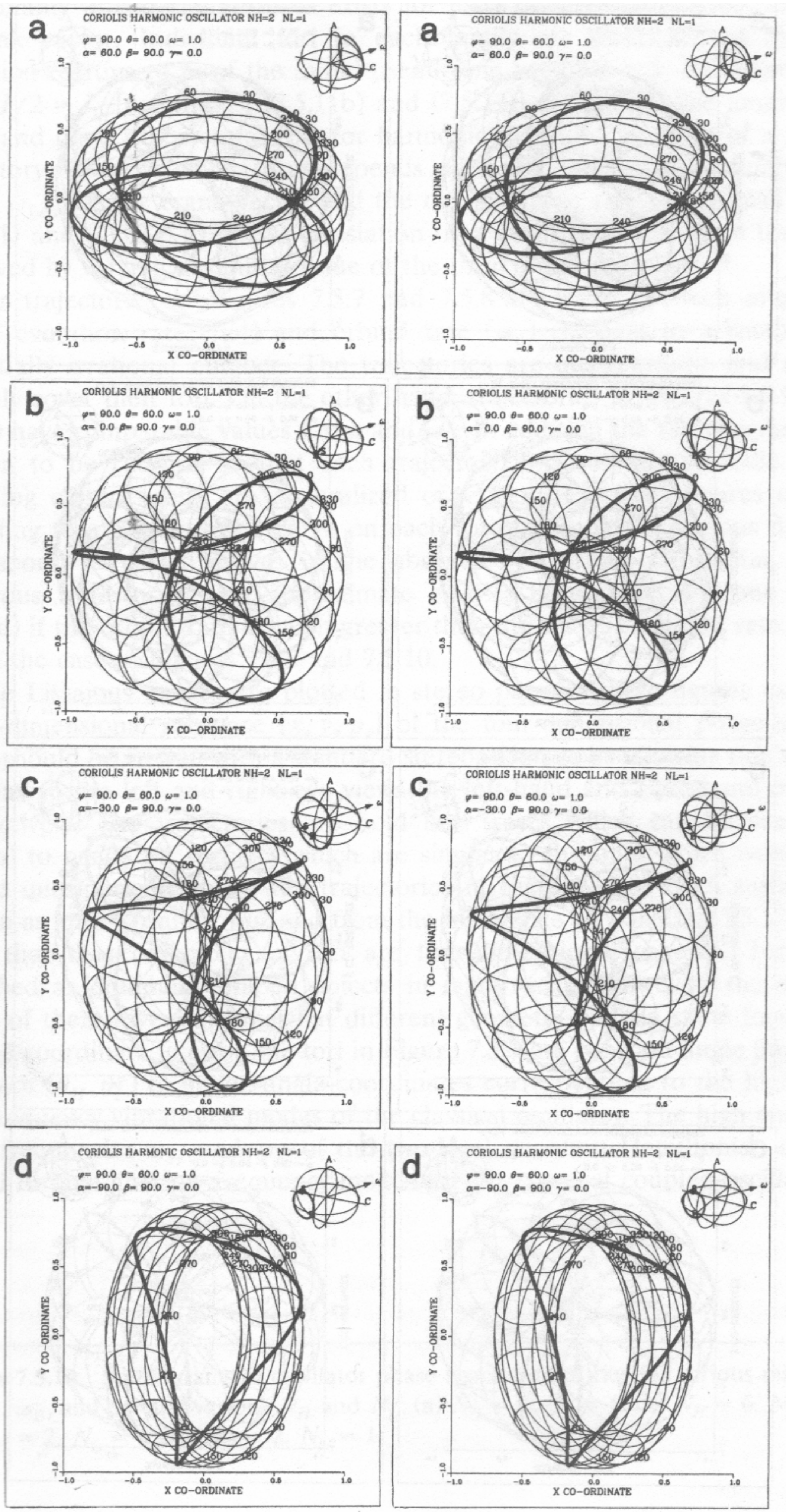


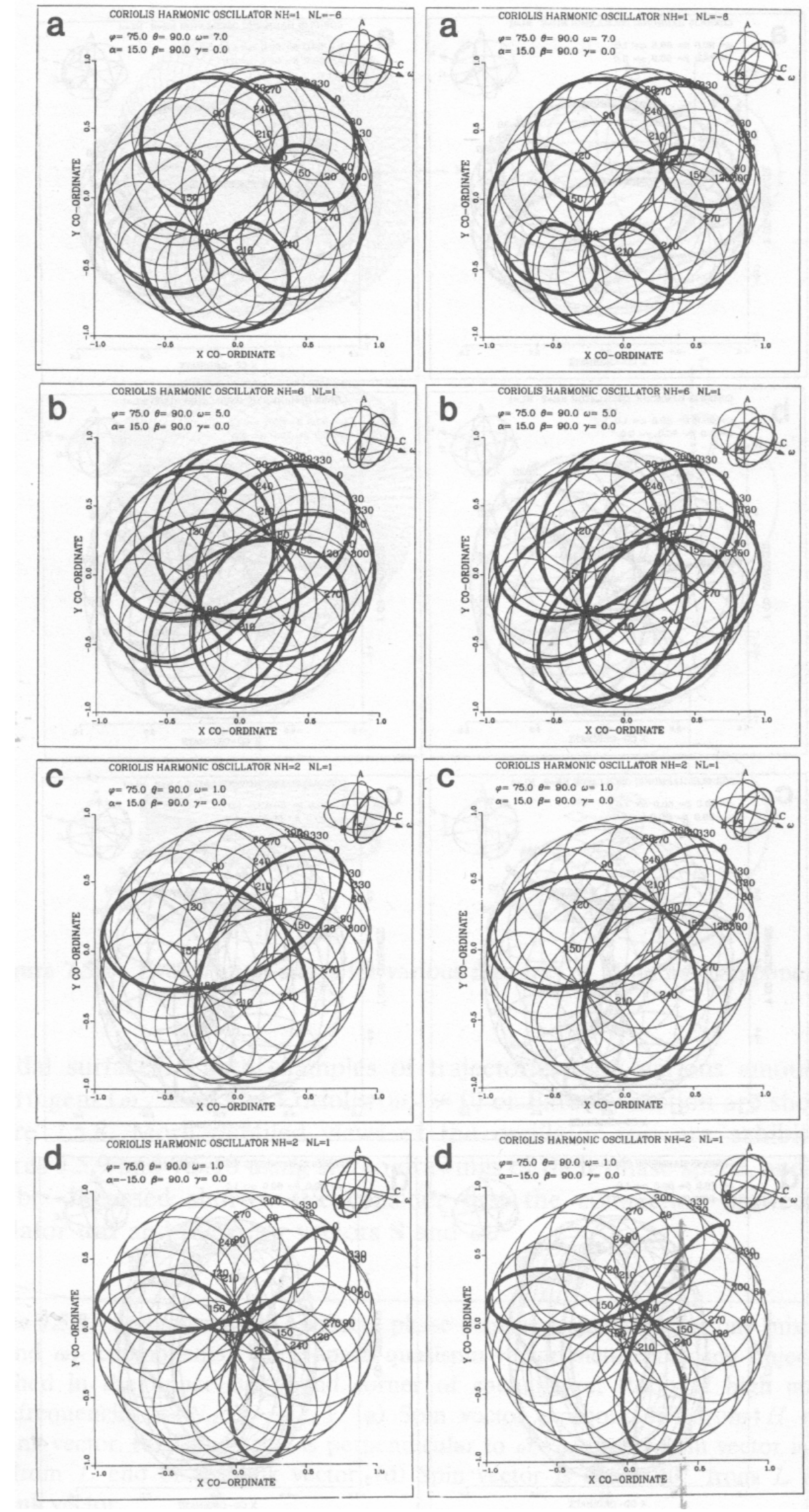
Figure 7.5.7 Analog computer plots of two famous examples of optical activity. (a) Faraday rotation or circular dichroism corresponds to constant $\psi = \tan^{-1}(b/a)$. (b) Birefringence corresponds to constant $\nu = \tan^{-1}(Y/X)$. Note that a small amount of birefringence is present in Figure 7.11(a); i.e., ψ oscillates slightly. Pure Faraday rotation is difficult to achieve on an analog computer.



7.5.8 Evolution of states for various mixtures of A and C components.



ABC-Type elliptical polarized dynamics



Review: Fundamental Euler **R**(αβγ) and Darboux **R**[φϑΘ] representations of U(2) and R(3)

Euler **R**(αβγ) derived from Darboux **R**[φϑΘ] and vice versa
Euler **R**(αβγ) rotation Θ=0−4π-sequence [φϑ] fixed
R(3)-U(2) slide rule for converting **R**(αβγ) ↔ **R**[φϑΘ] and Sundial

U(2) density operator approach to symmetry dynamics
Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes
Asymmetric-Diagonal A-Type motion
Bilateral-Balanced B-Type motion
Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes
AB-Type motion and Wigner's Avoided-Symmetry-Crossings
ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates
→ Conventional amp-phase ellipse coordinates
Euler Angle (αβγ) ellipse coordinates

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates and related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes

A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Amp-phase parameters $(A_1, A_2, \omega t, \rho_1)$

$$\begin{pmatrix} A_1 e^{-i(\omega t + \rho_1)} \\ A_2 e^{-i(\omega t - \rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + i p_1 \\ x_2 + i p_2 \end{pmatrix}$$

$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

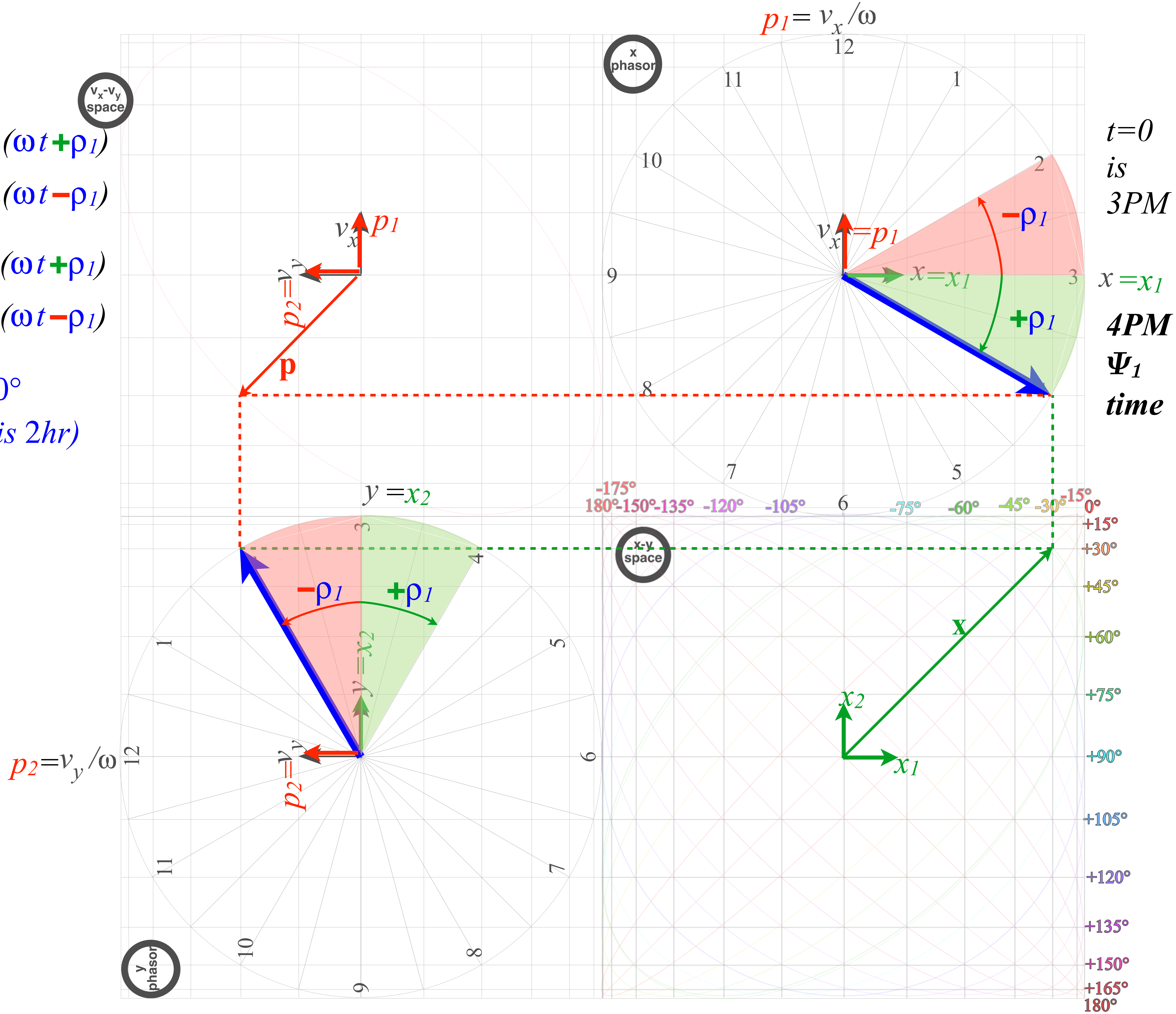
$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)

2PM
Ψ_2
time



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

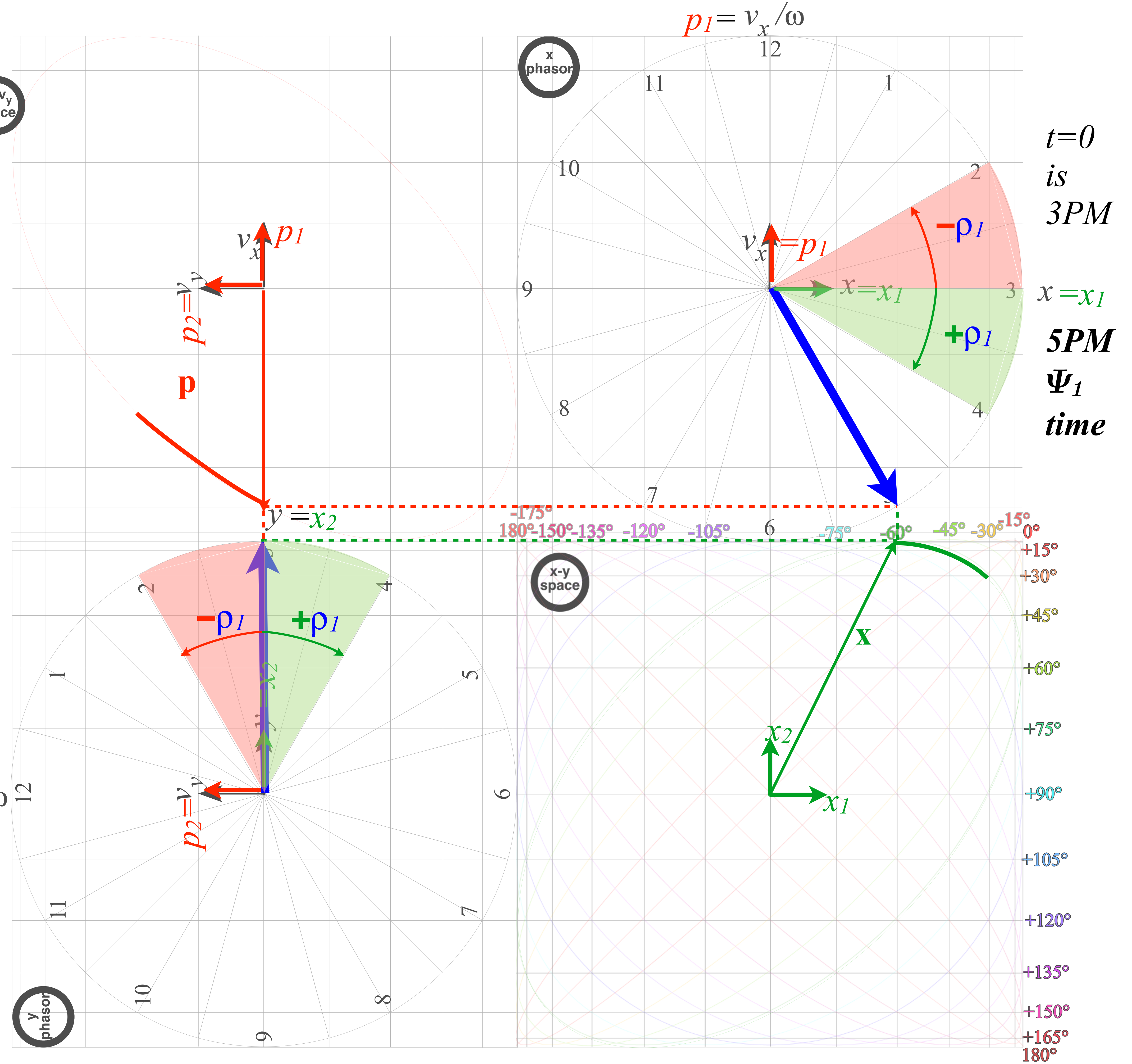
$x_2 = A_2 \cos(\omega t - \rho_1)$

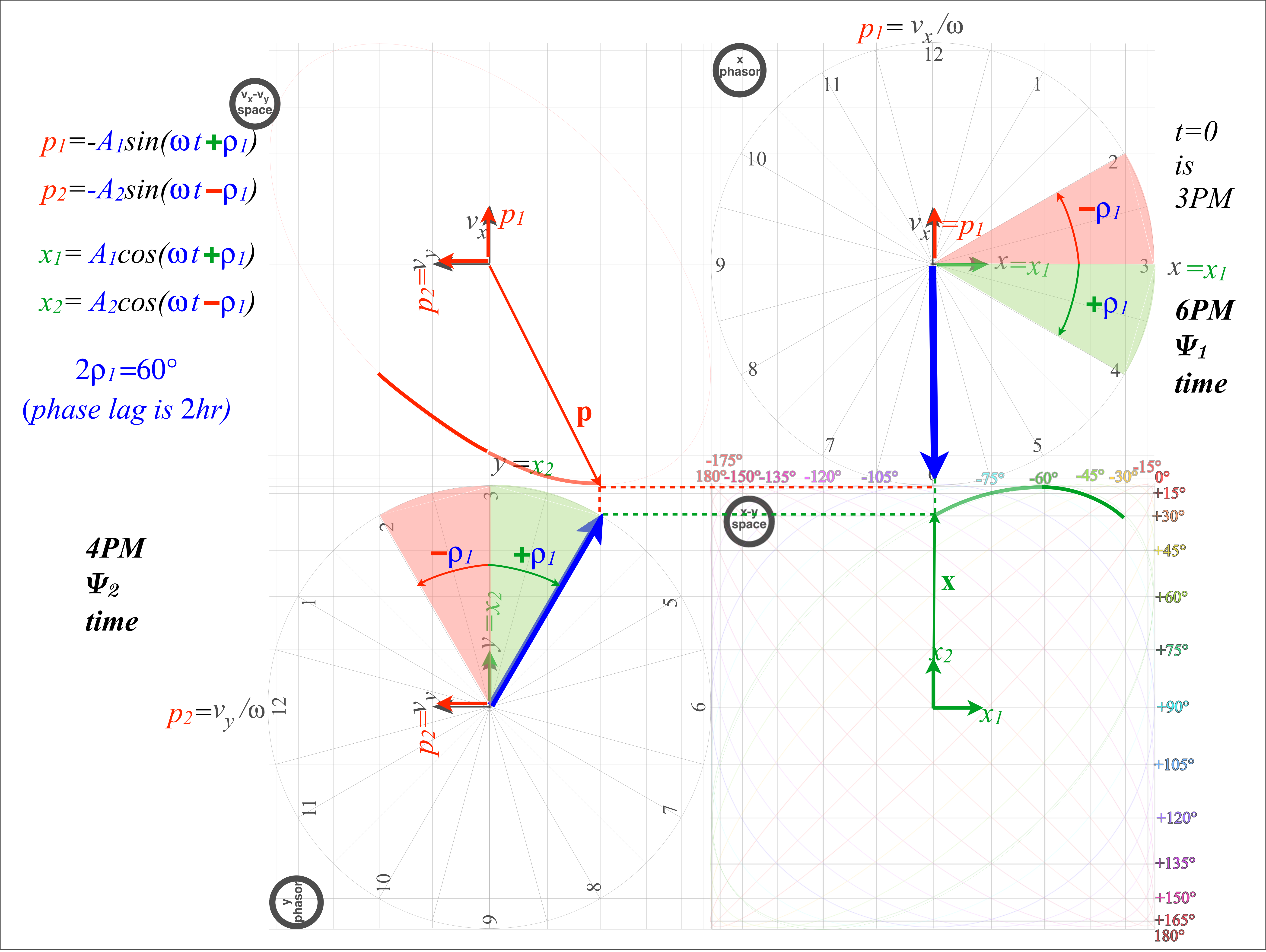
$2\rho_1 = 60°$

(phase lag is 2hr)

3PM
Ψ_2
time

$p_2 = v_y/\omega$




p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0 is 3PM
6PM Ψ1 time
4PM Ψ2 time
x =x1
y=x2
p
x
x1
x2
−ρ1
+ρ1
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°

$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

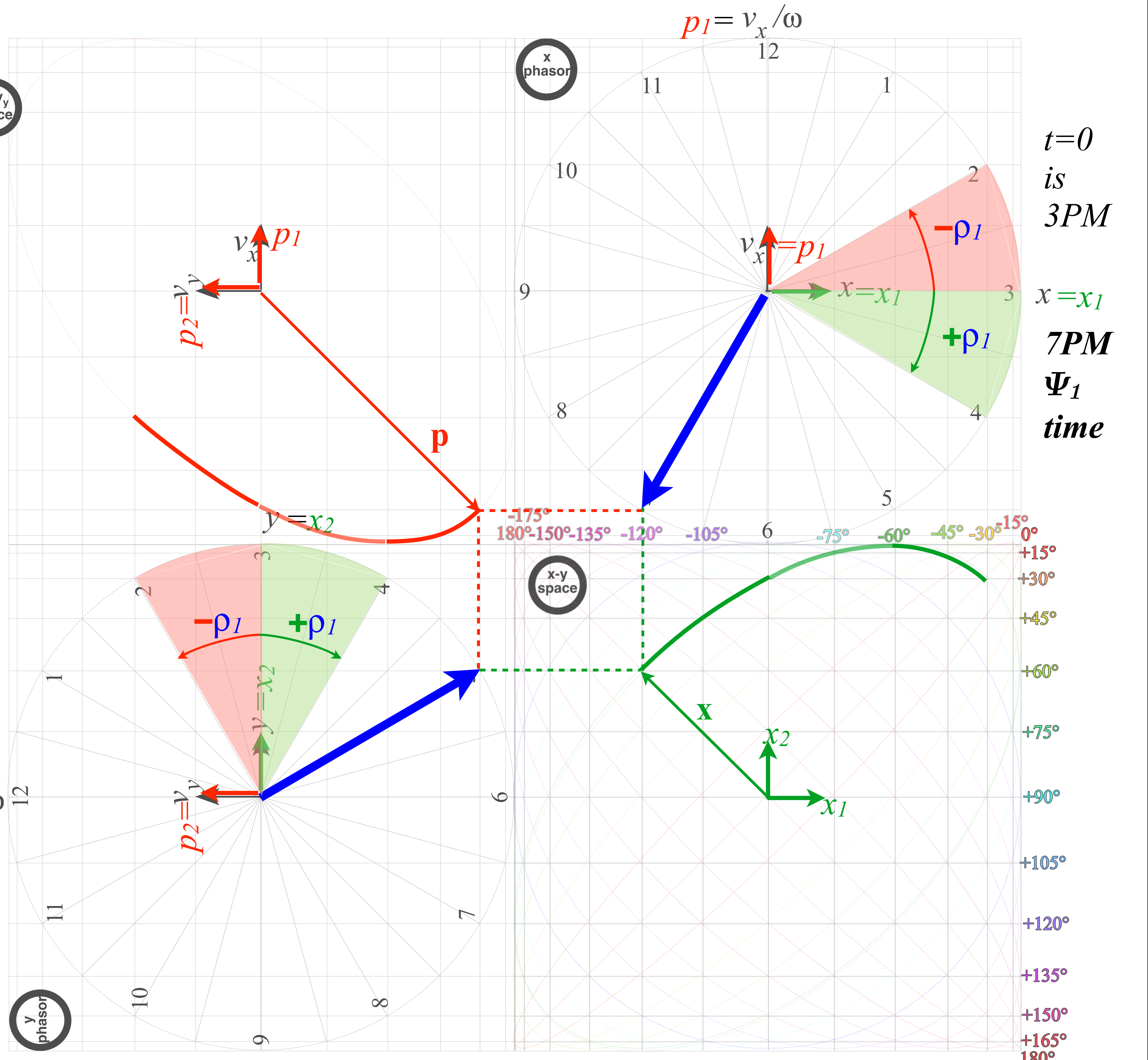
$2\rho_1 = 60°$

(phase lag is 2hr)

5PM

Ψ_2

time



$p_1 = -A_1 \sin(\omega t + \rho_1)$

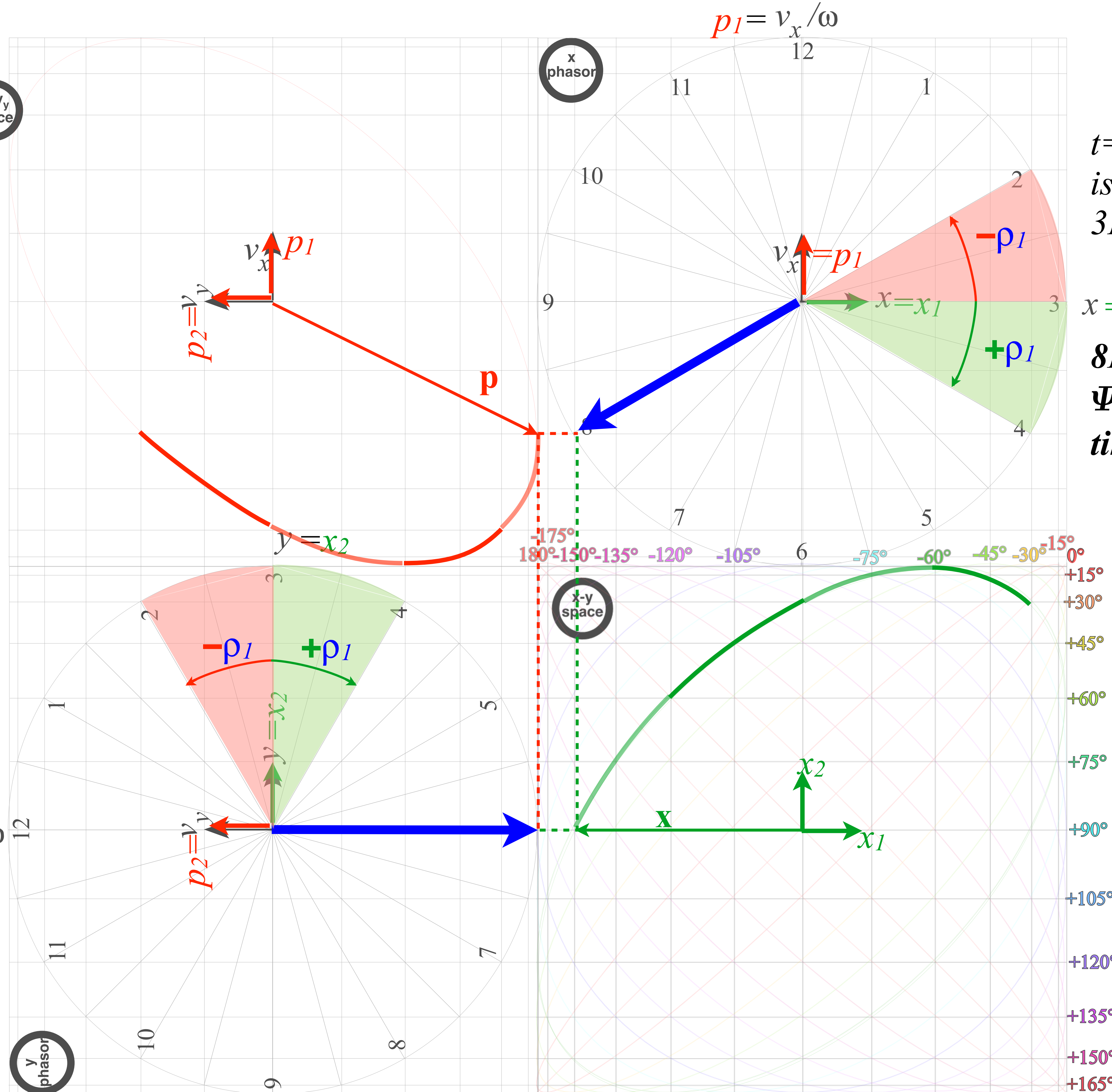
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)



t=0 is 3PM

$x = x_1$

8PM Ψ_1 time

6PM Ψ_2 time

$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

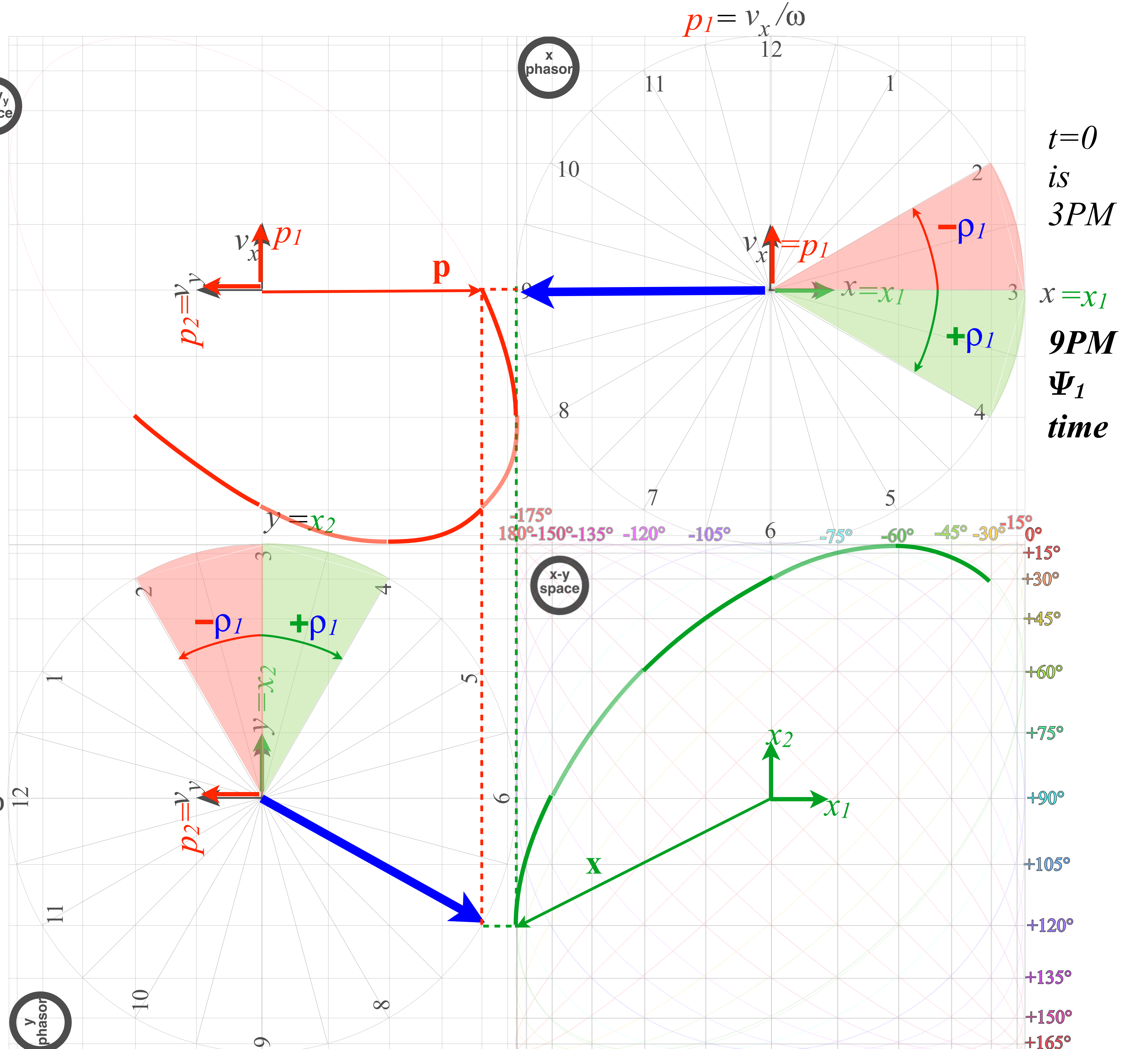
(phase lag is 2hr)

7PM

Ψ_2

time

$p_2 = v_y/\omega$



$p_1 = -A_1 \sin(\omega t + \rho_1)$

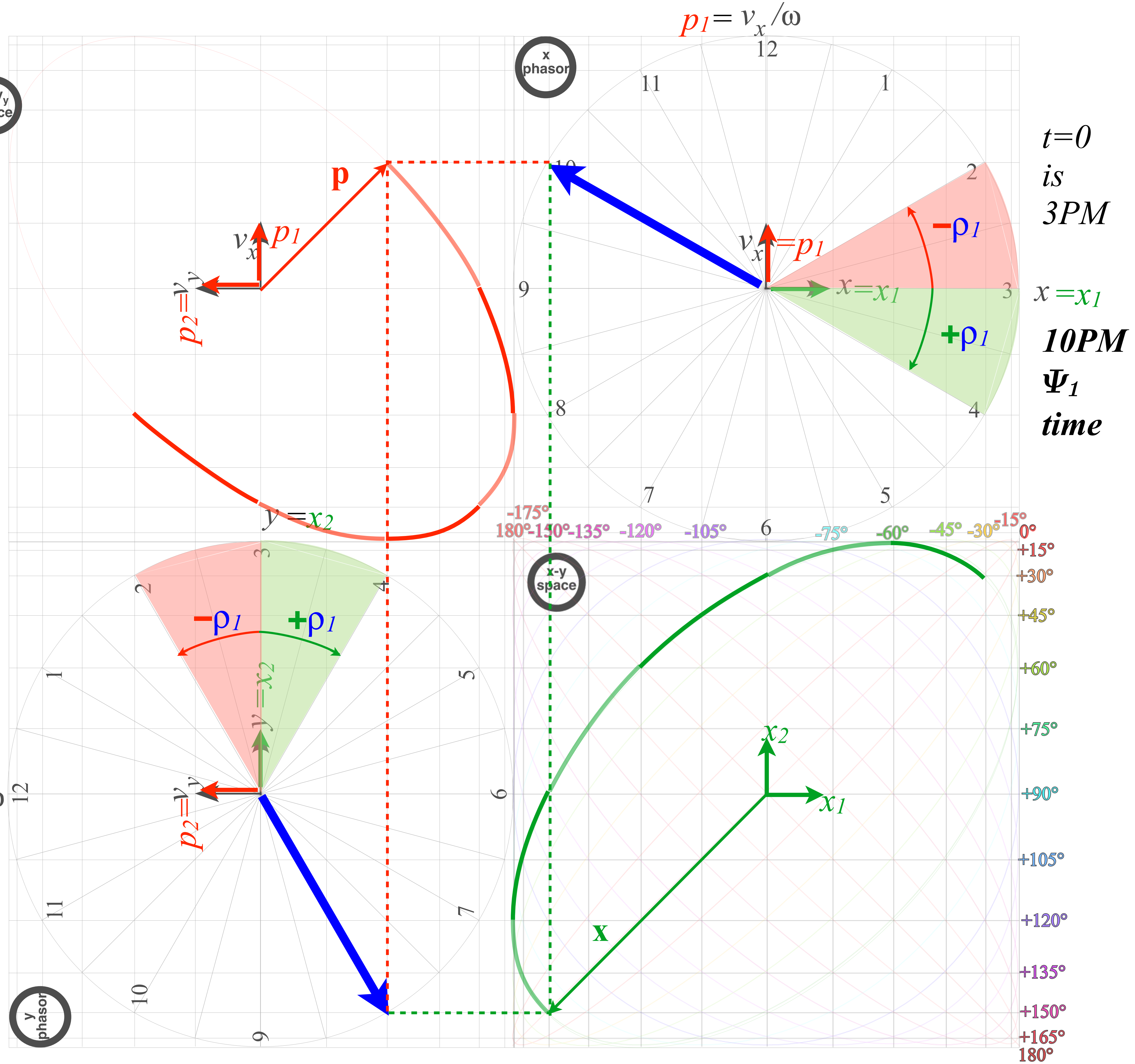
$p_2 = -A_2 \sin(\omega t - \rho_1)$

$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

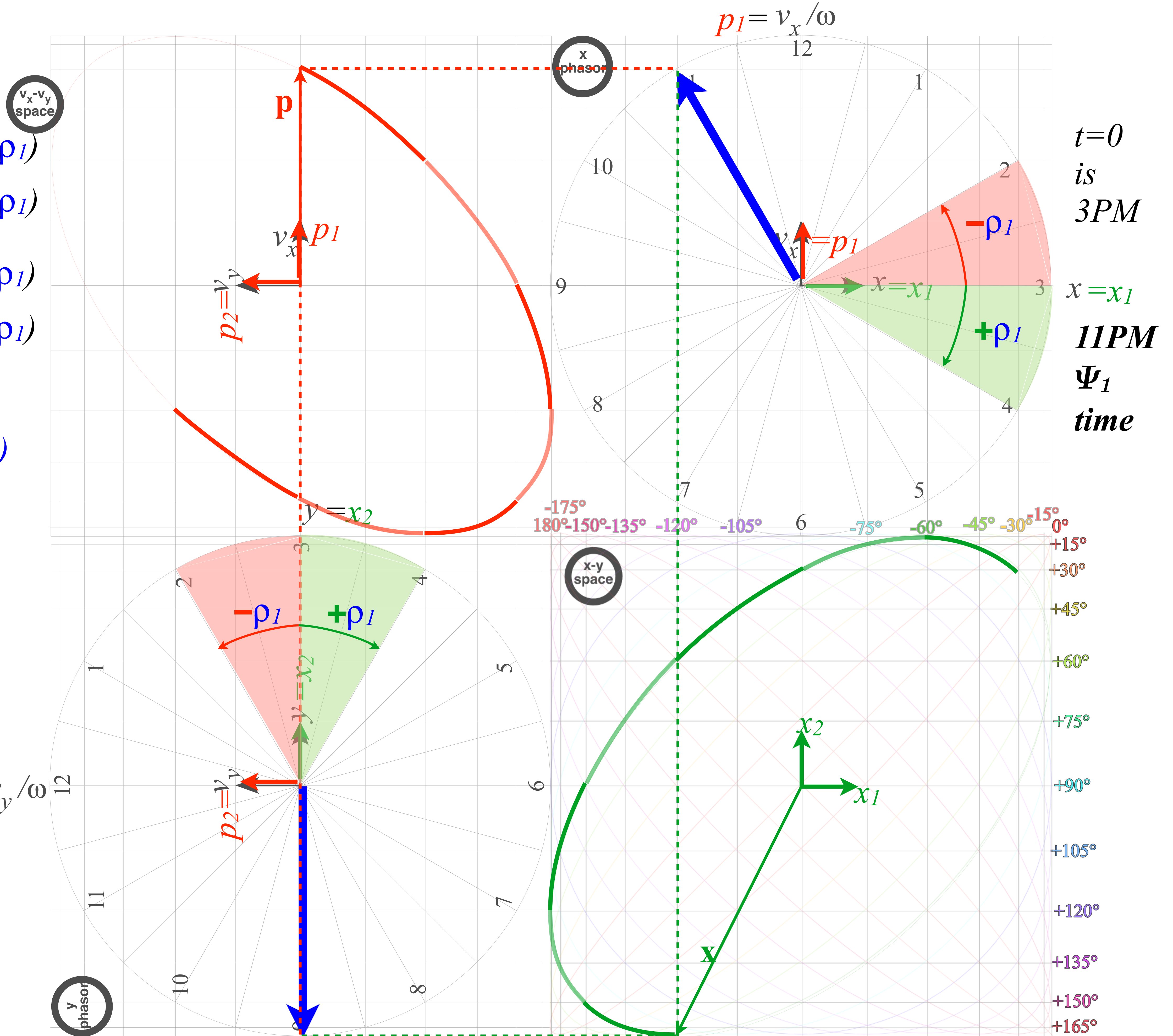
$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)

9PM
Ψ_2
time



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

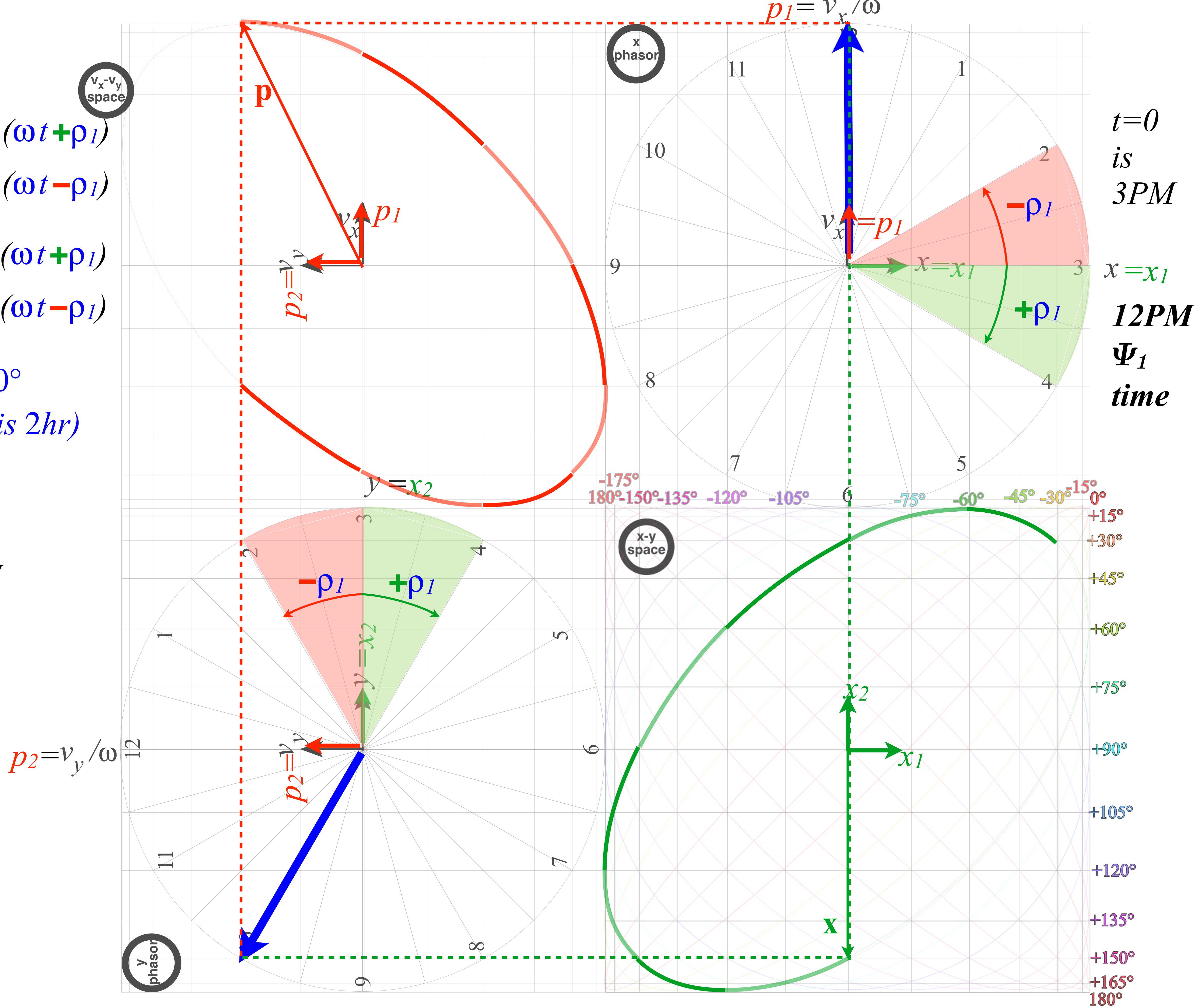
$x_1 = A_1 \cos(\omega t + \rho_1)$

$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)

10PM
Ψ_2
time



$p_1 = -A_1 \sin(\omega t + \rho_1)$

$p_2 = -A_2 \sin(\omega t - \rho_1)$

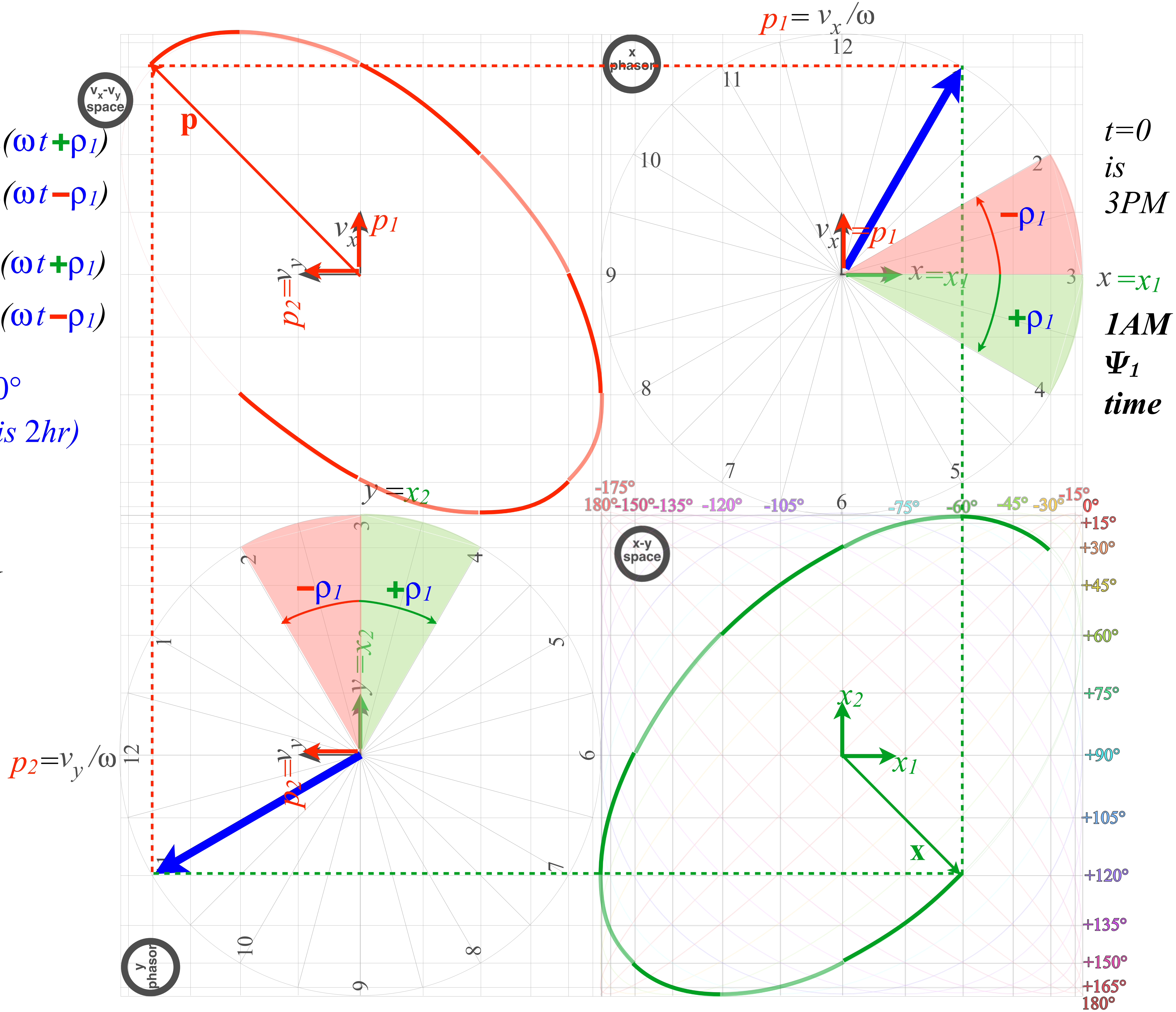
$x_1 = A_1 \cos(\omega t + \rho_1)$

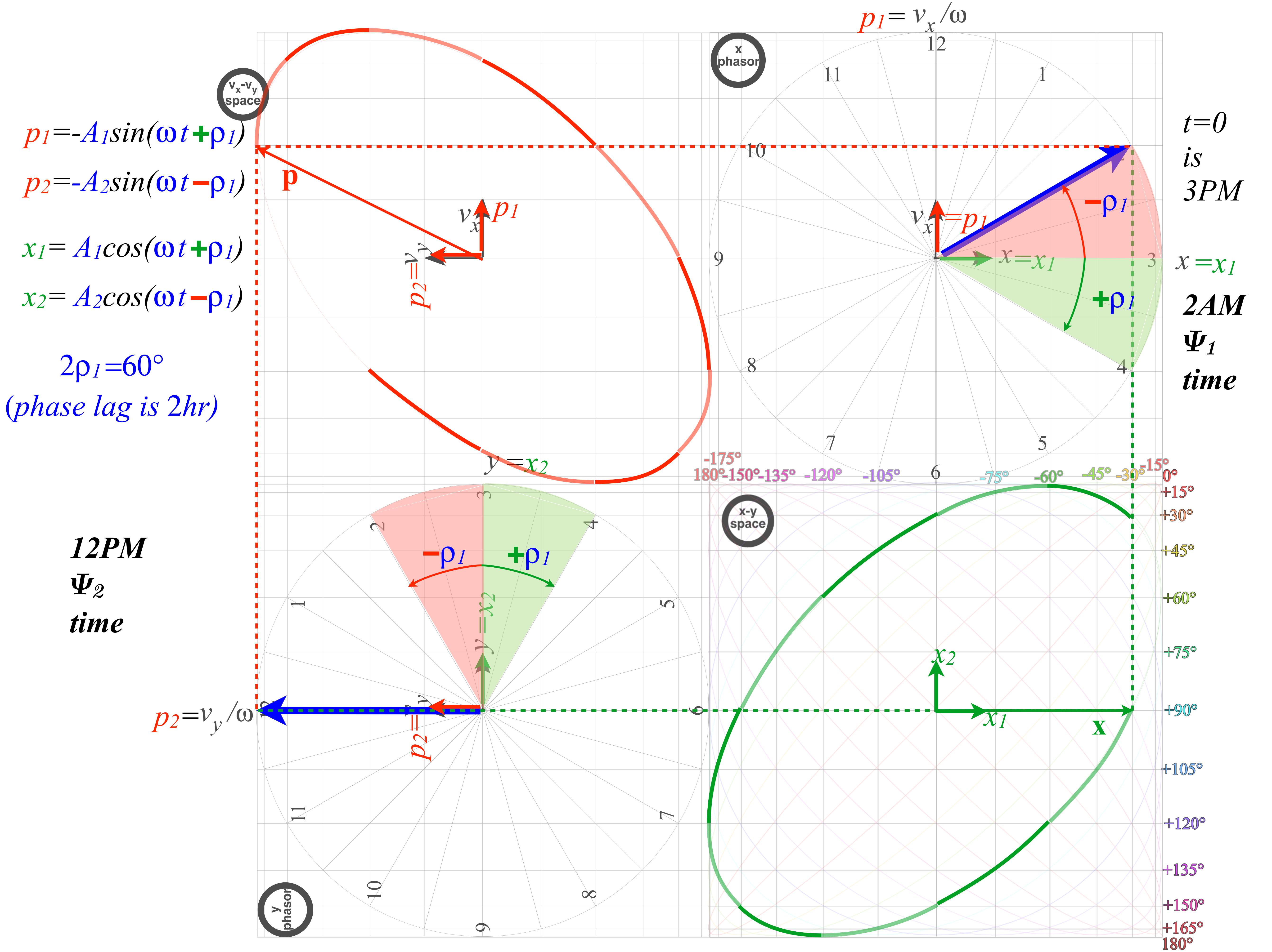
$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

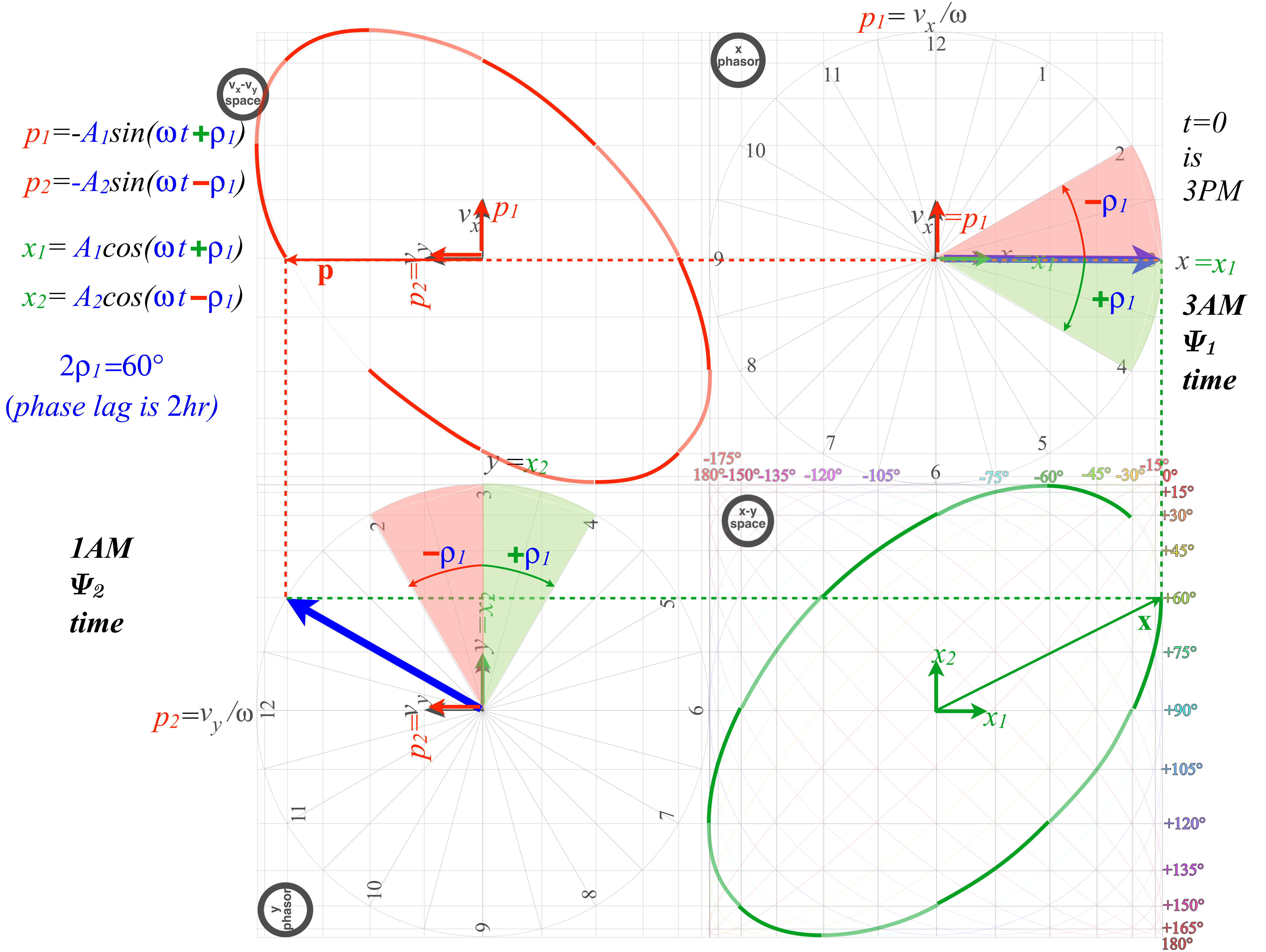
(phase lag is 2hr)

11PM
Ψ_2
time





p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0 is 3PM
2AM Ψ1 time
12PM Ψ2 time
−ρ1
+ρ1
x =x1
y =x2
p
x



p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0 is 3PM
3AM Ψ1 time
1AM Ψ2 time
x =x1
y =x2
−ρ1
+ρ1
p
x
x1
x2

$p_1 = -A_1 \sin(\omega t + \rho_1)$

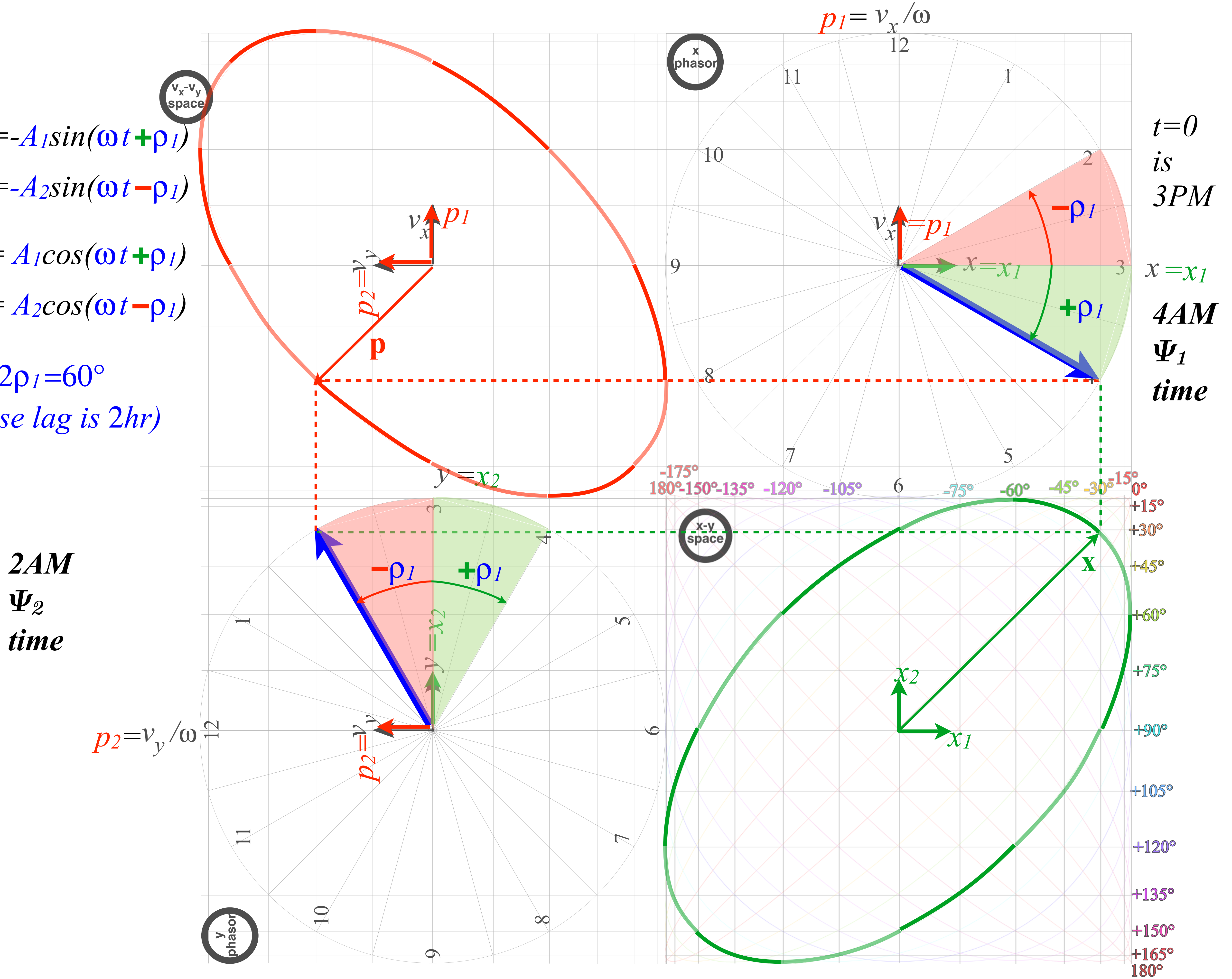
$p_2 = -A_2 \sin(\omega t - \rho_1)$

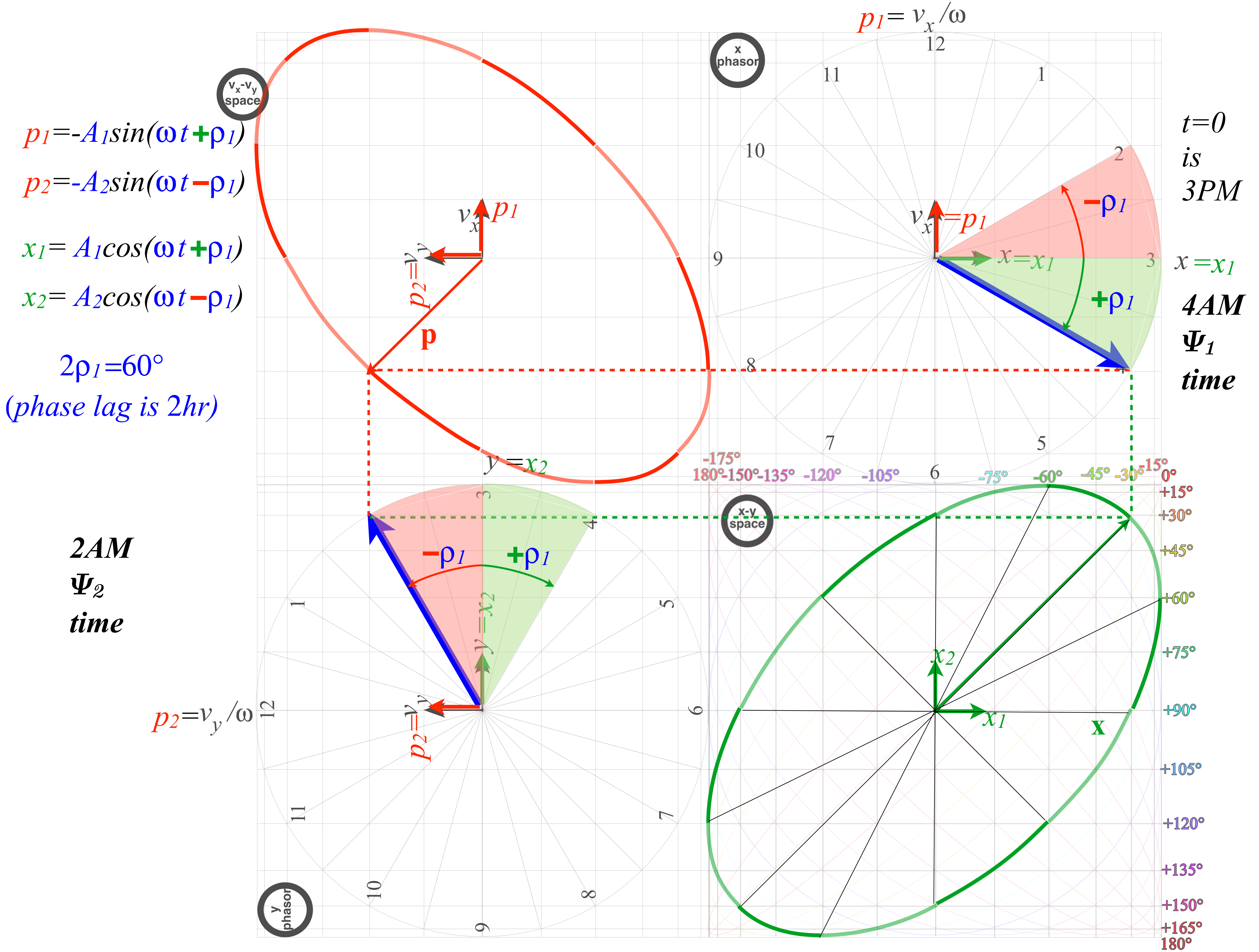
$x_1 = A_1 \cos(\omega t + \rho_1)$

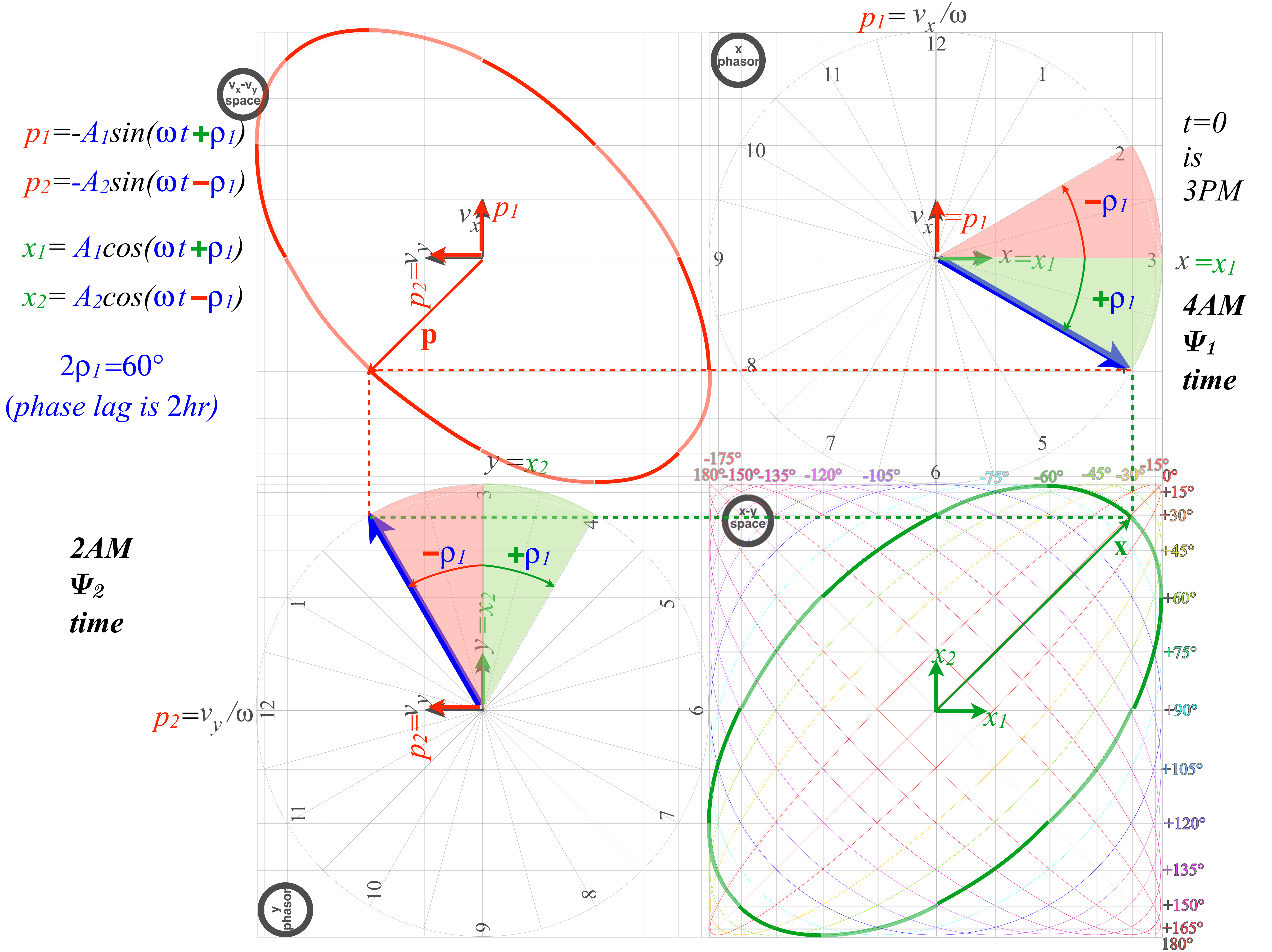
$x_2 = A_2 \cos(\omega t - \rho_1)$

$2\rho_1 = 60°$

(phase lag is 2hr)




p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0 is 3PM
4AM Ψ1 time
2AM Ψ2 time
x =x1
y =x2
p
x



p_1=-A_1sin(ωt+ρ1)
p_2=-A_2sin(ωt−ρ1)
x_1= A_1cos(ωt+ρ1)
x_2= A_2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2hr)
vx-vy space
x phasor
y phasor
x-y space
p_1= v_x/ω
p_2=v_y/ω
t=0 is 3PM
4AM Ψ1 time
2AM Ψ2 time

Review: Fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler **R**$(\alpha\beta\gamma)$ *derived from Darboux* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed*

R(3)-U(2) slide rule for converting **R**$(\alpha\beta\gamma)$ $\leftrightarrow$ **R**$[\varphi\vartheta\Theta]$ *and Sundial*

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates

➡ *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

$2D$ elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + i p_1 \\ x_2 + i p_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + i p_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1\cos(\omega t+\rho_1)$$
$$-p_1 = A_1\sin(\omega t+\rho_1)$$
$$x_2 = A_2\cos(\omega t-\rho_1)$$
$$-p_2 = A_2\sin(\omega t-\rho_1)$$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$
$$x_2 = A_2 \cos(\omega t - \rho_1)$$
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Let: $A_1 = A\cos\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles ($\alpha\beta\gamma$) and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$
$$x_2 = A_2 \cos(\omega t - \rho_1)$$
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2 \cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2 \sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2 \cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2 \sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2 \cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2 \sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2 \cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2 \sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t+\rho_1)$$
$$-p_1 = A_1 \sin(\omega t+\rho_1)$$
$$x_2 = A_2 \cos(\omega t-\rho_1)$$
$$-p_2 = A_2 \sin(\omega t-\rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Let: $\omega t+\rho_1 = (\gamma+\alpha)/2$

$\omega t-\rho_1 = (\gamma-\alpha)/2$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$
$$x_2 = A_2 \cos(\omega t - \rho_1)$$
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

Let: $\omega t + \rho_1 = (\gamma+\alpha)/2$

$\omega t - \rho_1 = (\gamma-\alpha)/2$

$$\tan\beta/2 = A_2/A_1 \qquad A^2 = A_1^2 + A_2^2$$

$$\alpha = 2\rho_1 \qquad \gamma = 2\omega\cdot t$$

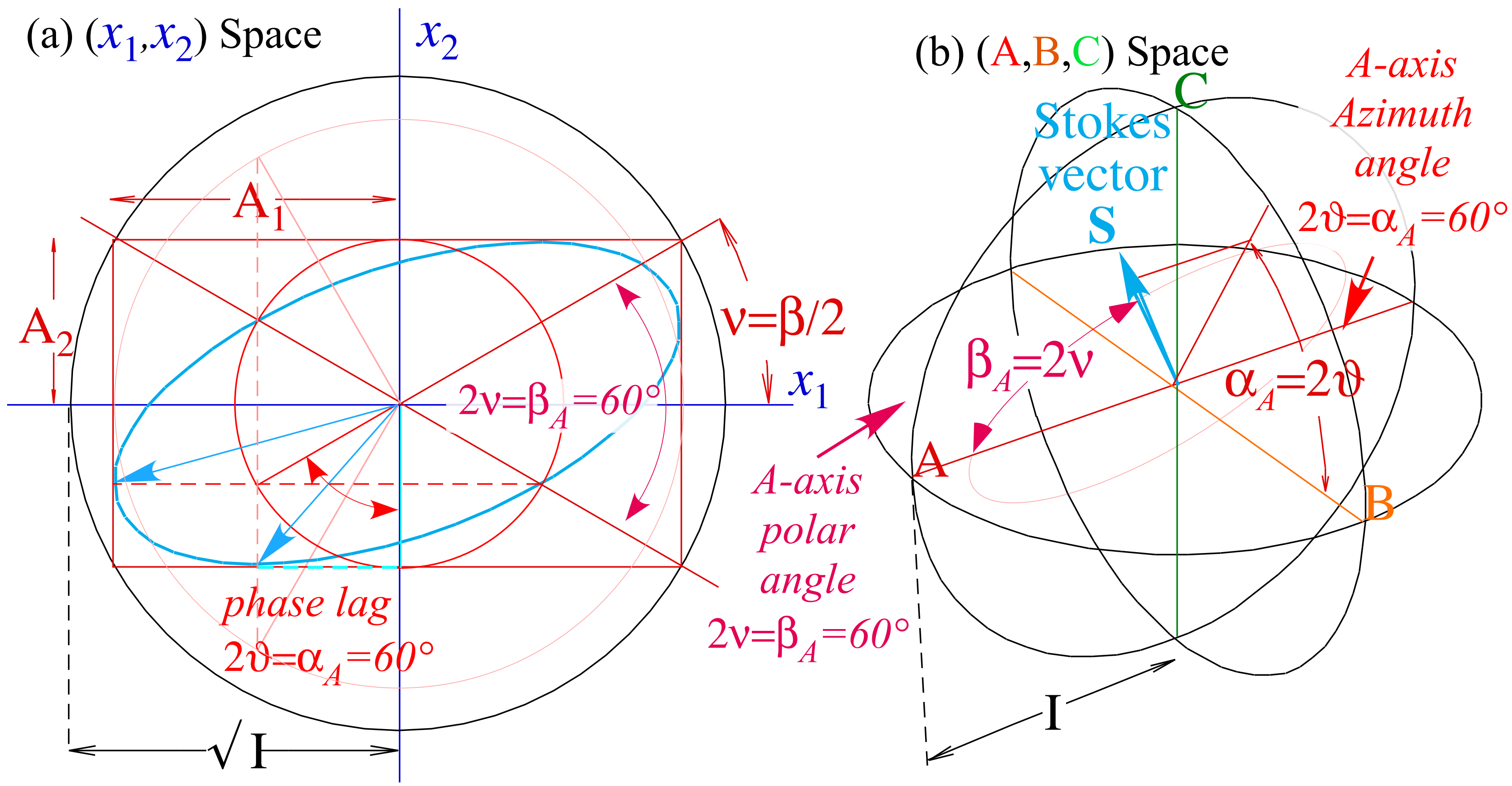
Euler parameters (α,β,γ, A) in terms of *amp-phase parameters* $(A_1, A_2, \omega t, \rho_1)$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

The A-view in $\{x_1,x_2\}$-basis

Angles $\alpha_A = \rho_1 - \rho_2 = 2\rho_1$, $\beta_A = 2\tan^{-1} A_2/A_1$, $\gamma_A = 2\omega \cdot t$
define ellipses with intensity $I = A^2 = A_1^2 + A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



A or *Z*-axis Euler angles

$\alpha = \alpha_A = \rho_1 - \rho_2 = 2\rho_1 = 60°$

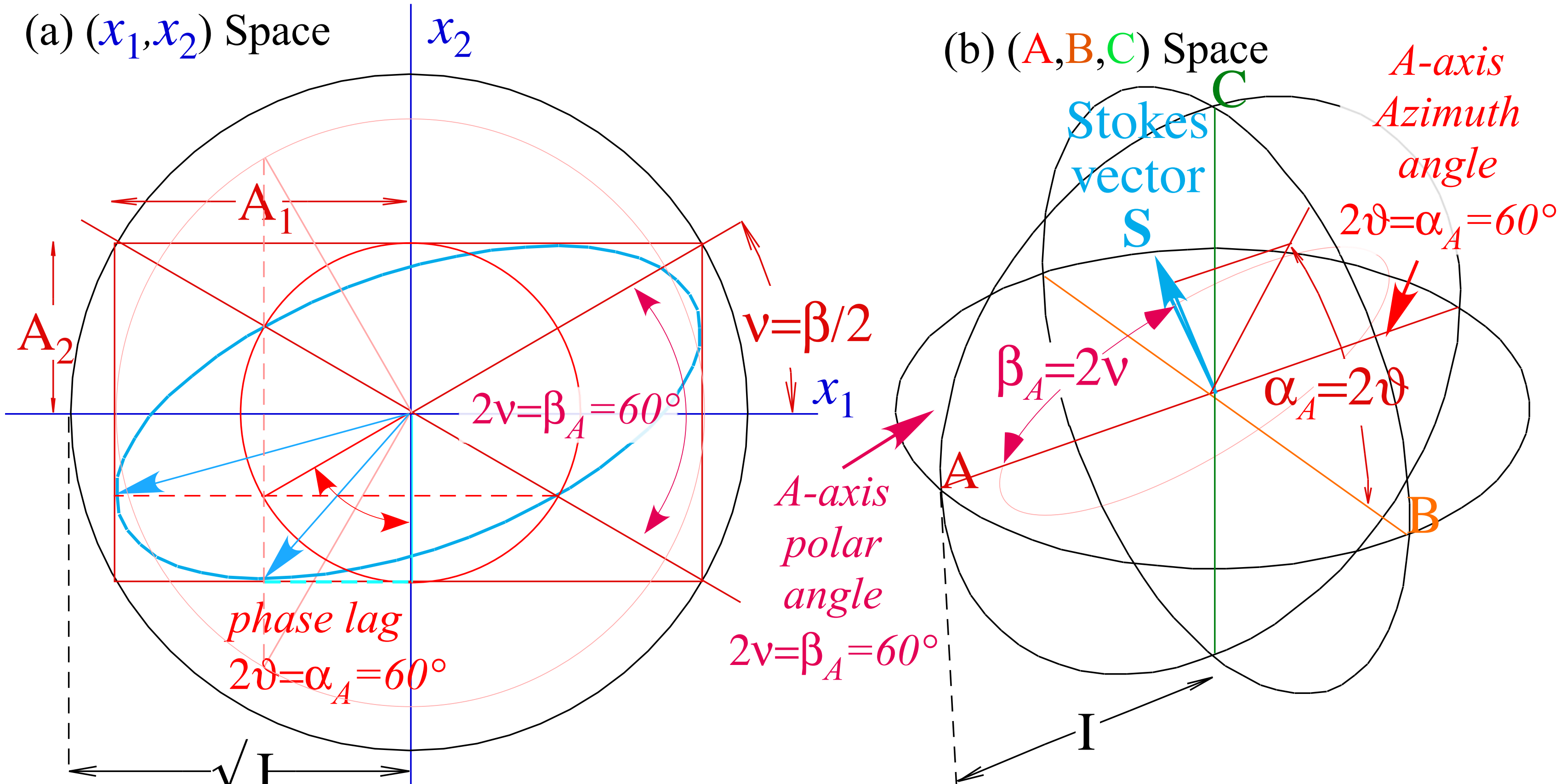
$\beta = \beta_A = 2\tan^{-1} A_2/A_1 = 60°$

$\gamma_A = 2\omega \cdot t$

The A*-view in* {x1,x2}*-basis*

Angles $\alpha_A = \rho_1 - \rho_2 = 2\rho_1$, $\beta_A = 2\tan^{-1} A_2/A_1$, $\gamma_A = 2\omega\cdot t$
define ellipses with intensity $I = A^2 = A_1^2 + A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$



A or *Z*-axis Euler angles

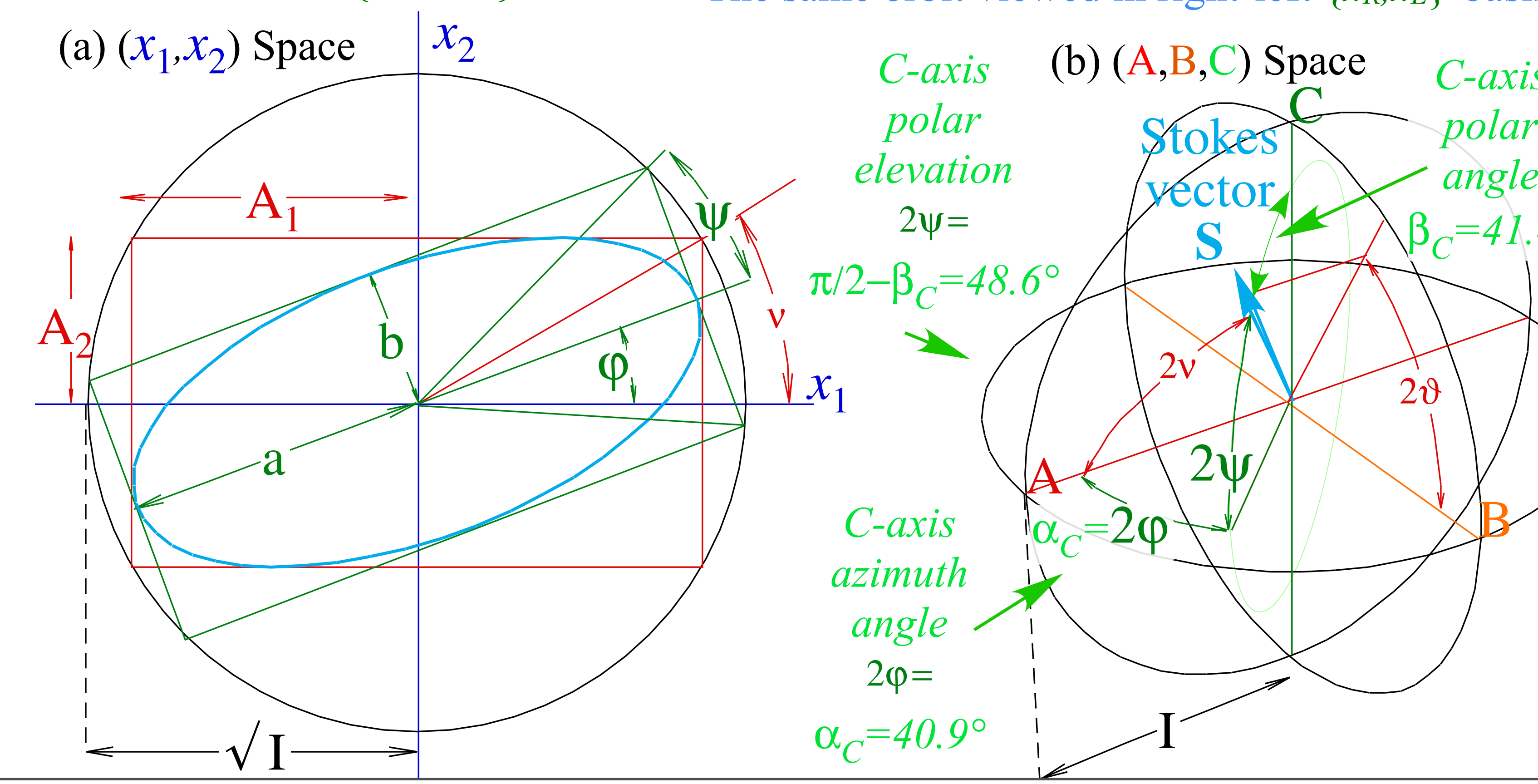
$\alpha = \alpha_A = \rho_1 - \rho_2 = 2\rho_1 = 60°$

$\beta = \beta_A = 2\tan^{-1} A_2/A_1 = 60°$

$\gamma_A = 2\omega\cdot t$

The C*-view in* {xR,xL}*-basis*

The same orbit viewed in right-left $\{x_R,x_L\}$-basis of circular polarization with angles $(\alpha_C,\beta_C,\gamma_C)$.



$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$

Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

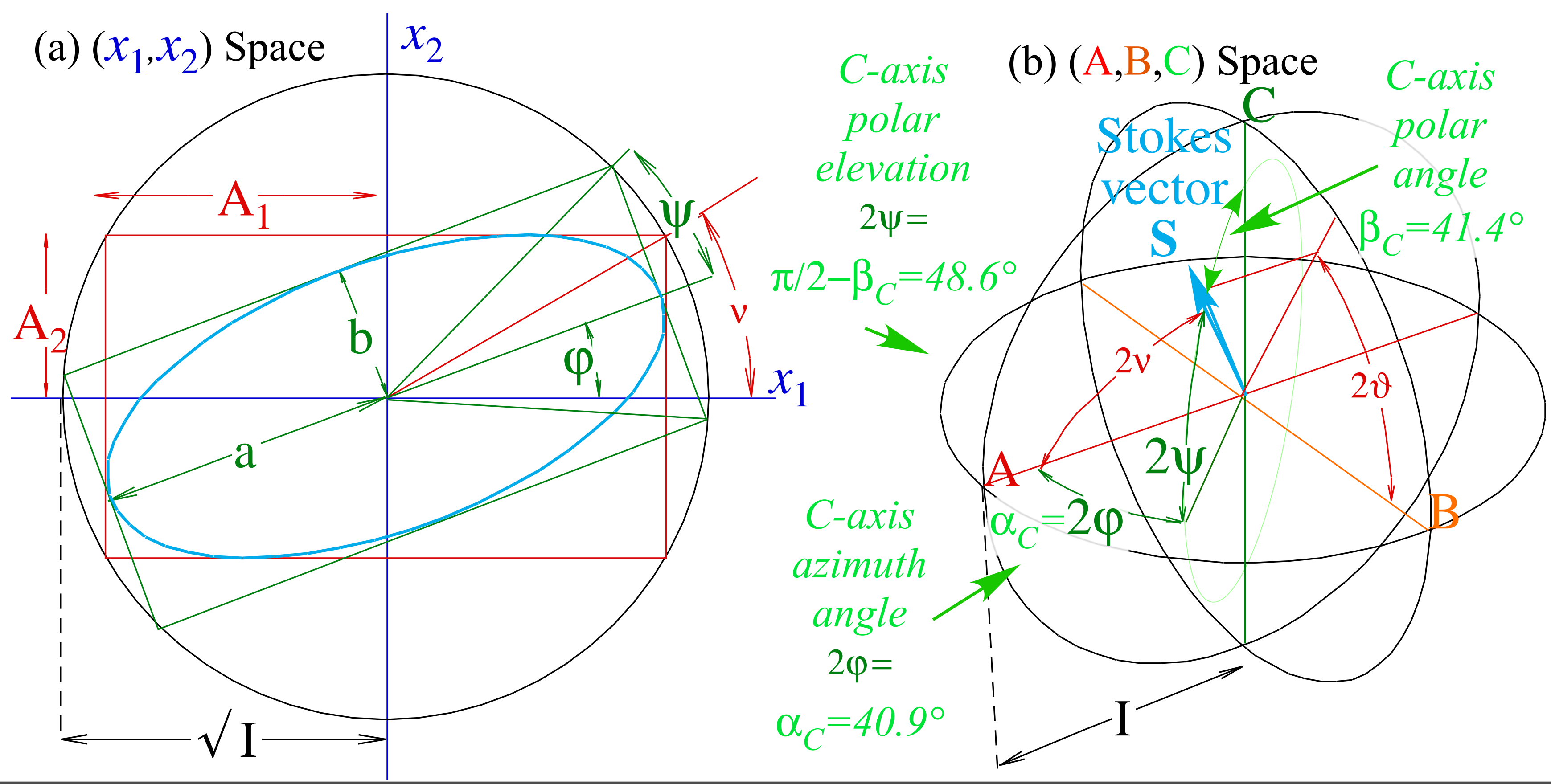
$$\text{Asymmetry } S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B\sin\beta_B = \frac{I}{2}\cos\alpha_C\sin\beta_C$$

$$\text{Balance } S_B = \frac{I}{2}\cos\alpha_A\sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C\sin\beta_C$$

$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A\sin\beta_A = \frac{I}{2}\cos\alpha_B\sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R,x_L\}$-basis

The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.



Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

Asymmetry $S_A = \frac{I}{2}\cos\beta_A \qquad\qquad = \frac{I}{2}\sin\alpha_B \sin\beta_B = \frac{I}{2}\cos\alpha_C \sin\beta_C$

Balance $S_B = \frac{I}{2}\cos\alpha_A \sin\beta_A = \frac{I}{2}\cos\beta_B \qquad\qquad = \frac{I}{2}\sin\alpha_C \sin\beta_C$

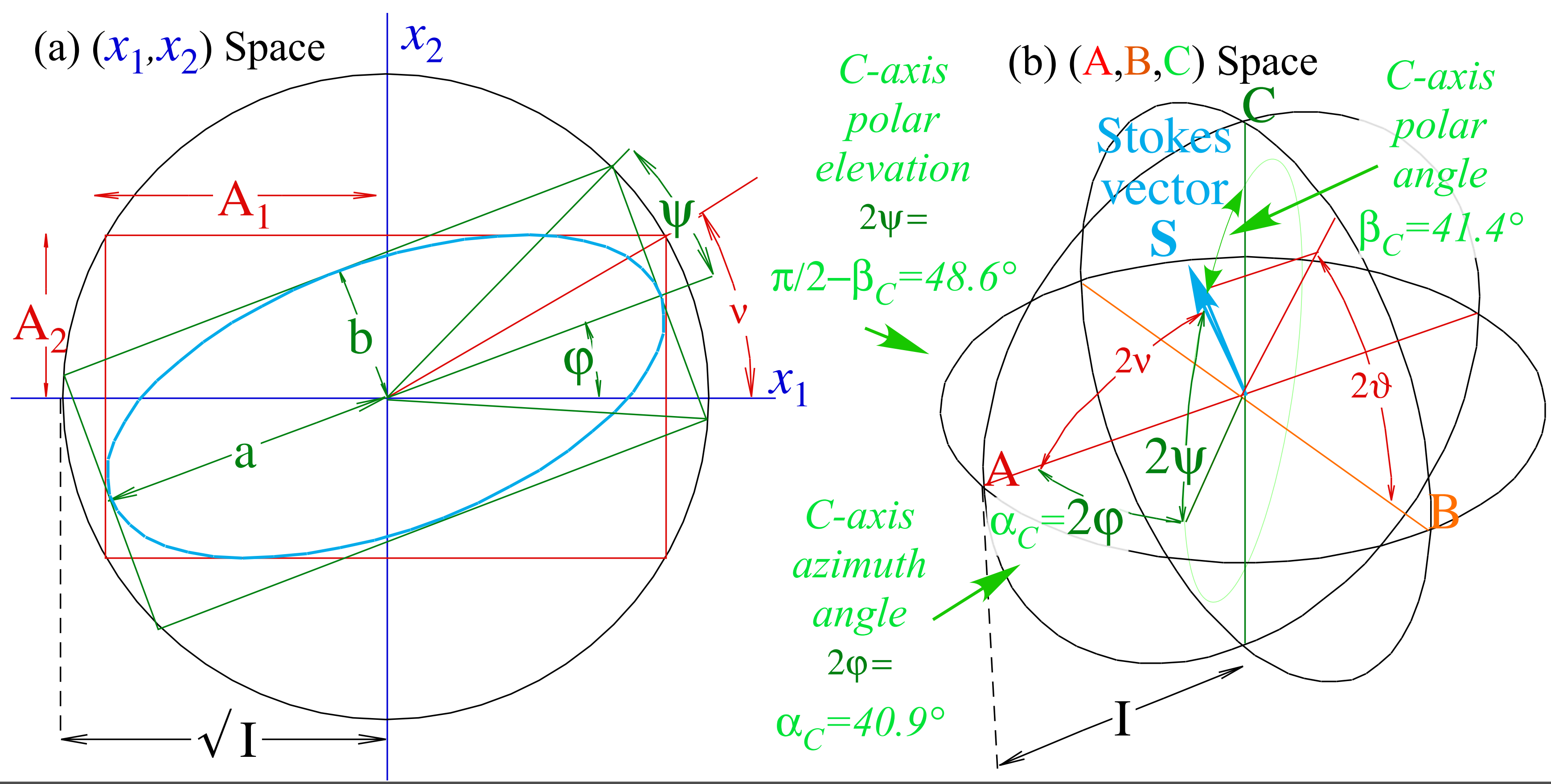
Chirality $S_C = \frac{I}{2}\sin\alpha_A \sin\beta_A = \frac{I}{2}\cos\alpha_B \sin\beta_B = \frac{I}{2}\cos\beta_C$

The C*-view in* $\{x_R, x_L\}$*-basis*

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

Angles (α_C, β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:} \quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4^\circ$$



Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

$$\text{Asymmetry } S_A = \frac{I}{2}\cos\beta_A = \frac{I}{2}\sin\alpha_B\sin\beta_B = \frac{I}{2}\cos\alpha_C\sin\beta_C$$

$$\text{Balance } S_B = \frac{I}{2}\cos\alpha_A\sin\beta_A = \frac{I}{2}\cos\beta_B = \frac{I}{2}\sin\alpha_C\sin\beta_C$$

$$\text{Chirality } S_C = \frac{I}{2}\sin\alpha_A\sin\beta_A = \frac{I}{2}\cos\alpha_B\sin\beta_B = \frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R, x_L\}$-basis

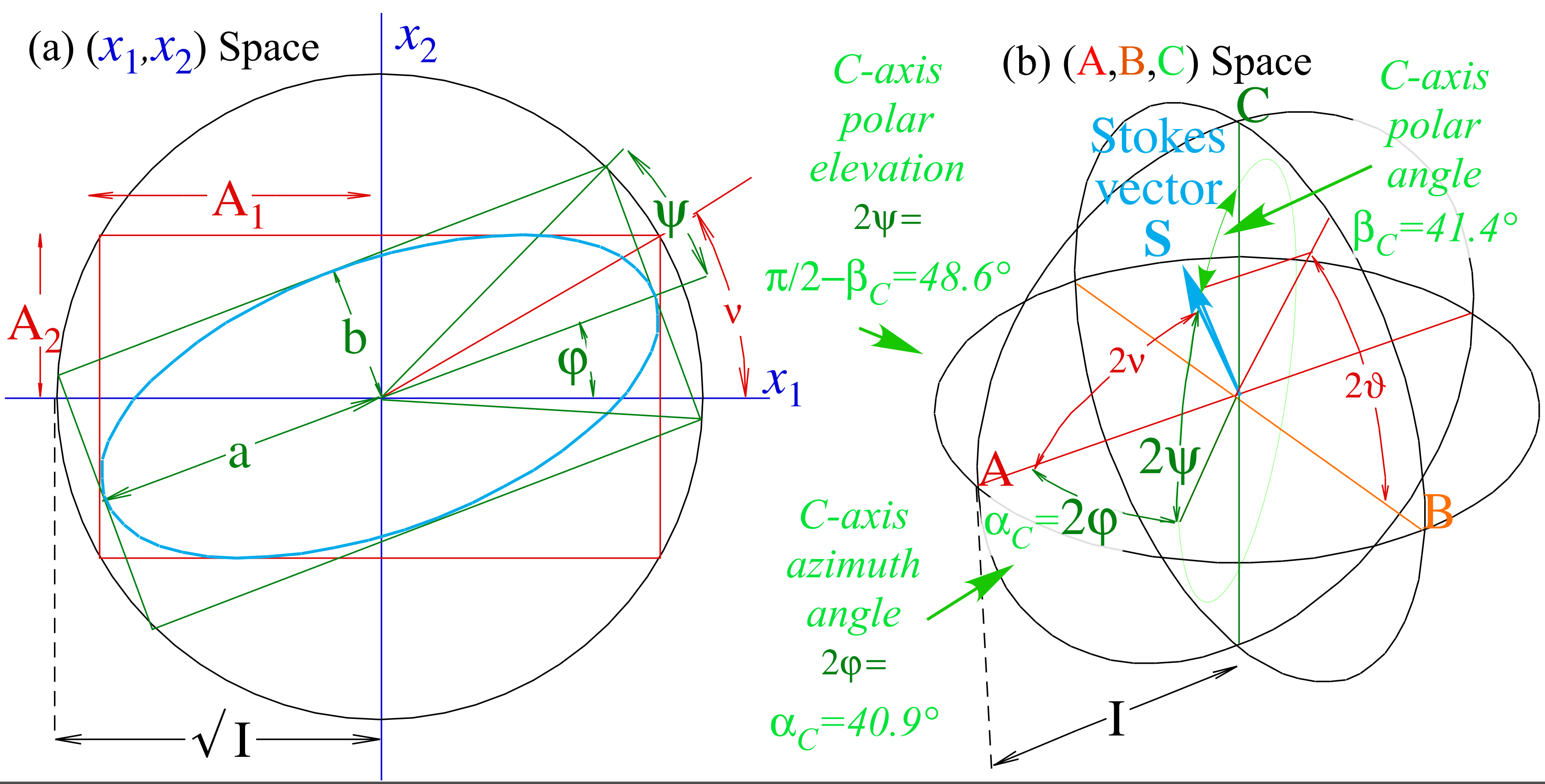
The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

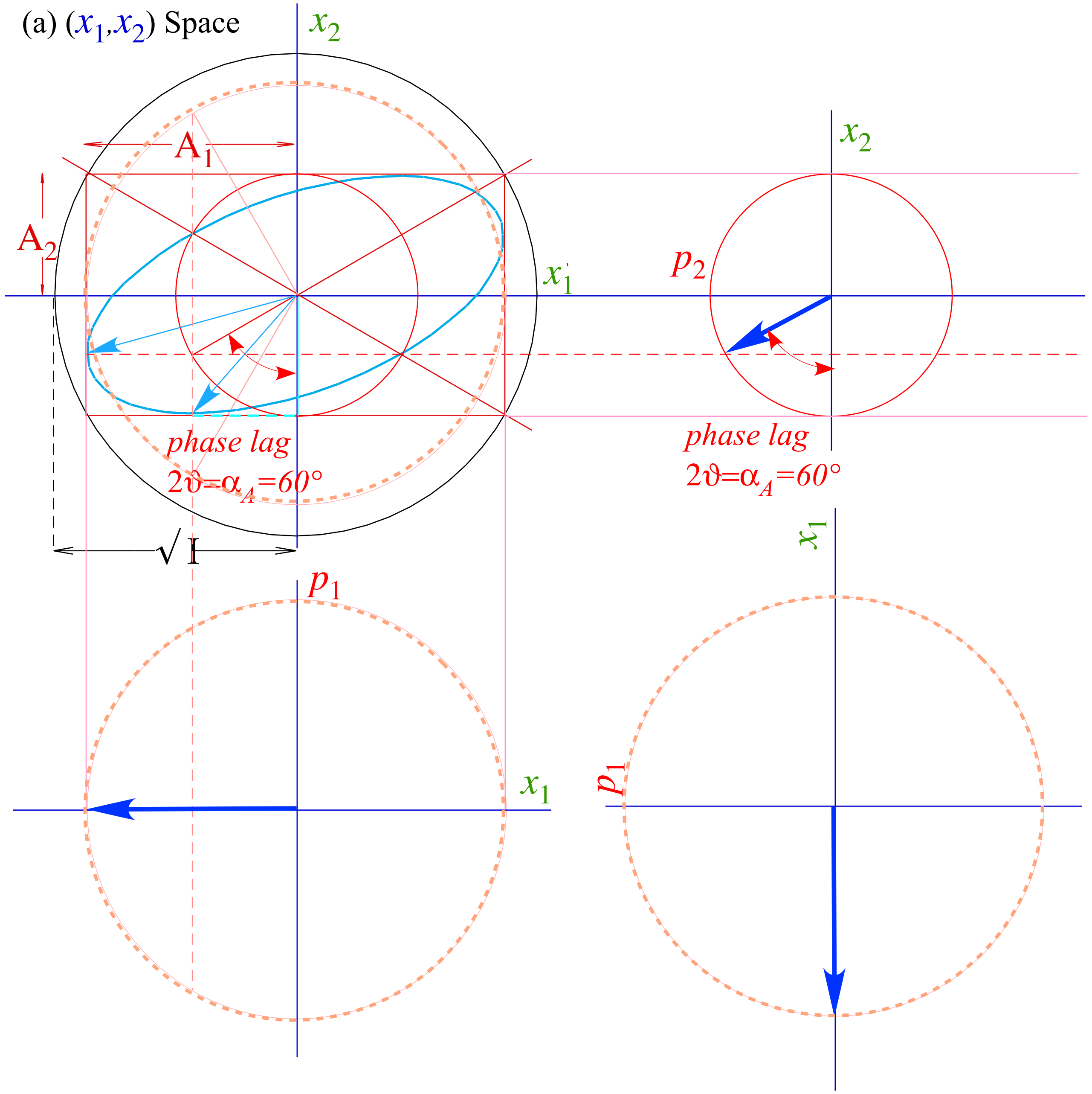
Angles (α_C, β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A\sin\beta_A = \cos\beta_C \qquad \text{or: } \beta_C = \cos^{-1}(\sin\alpha_A\sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4^\circ$$

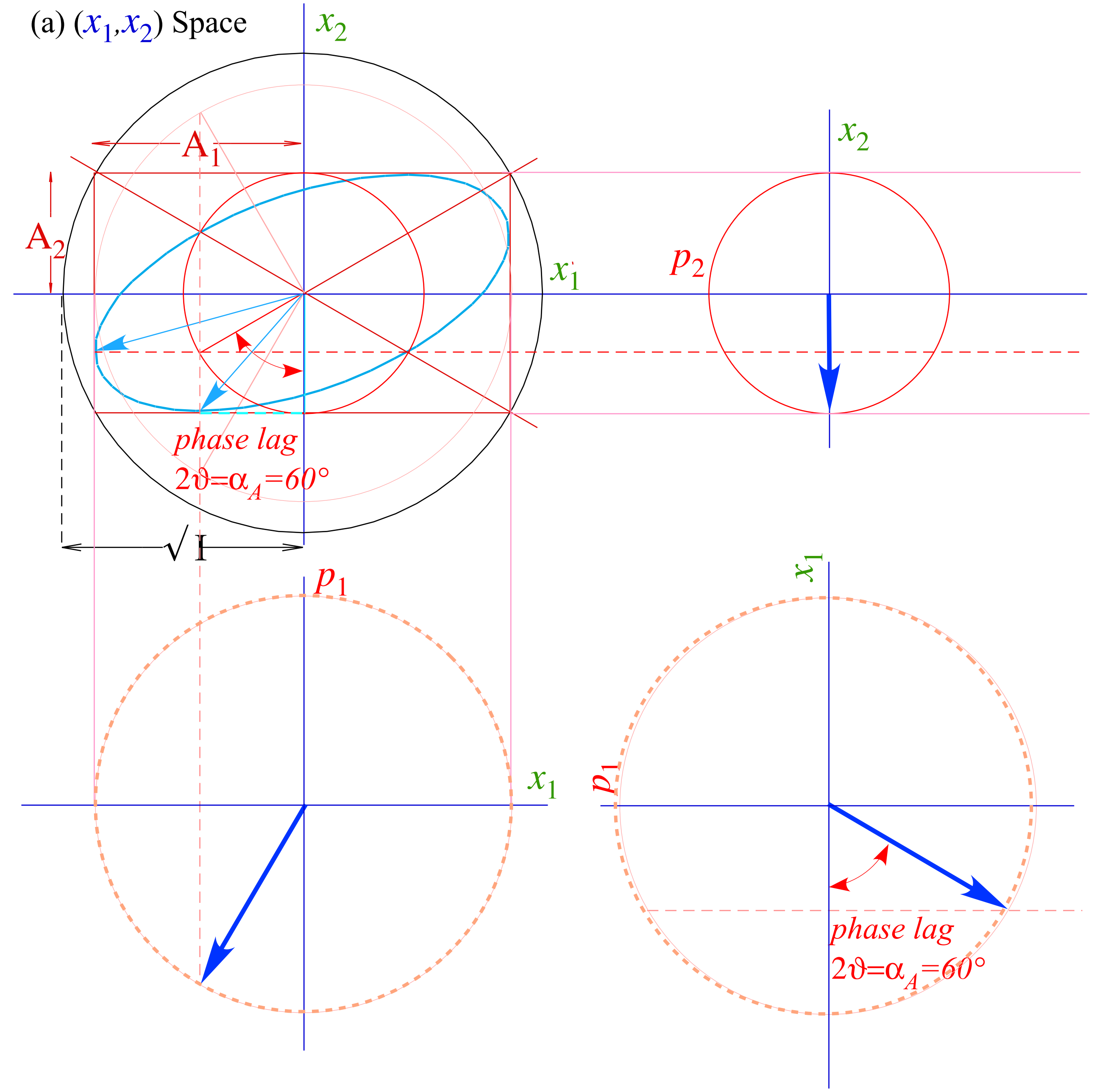
C-axis azimuth angle α_C relates to A-axis angles α_A and β_A . See $\alpha_C = 2\varphi$ below.

$$\frac{\cos\alpha_A\sin\beta_A}{\cos\beta_A} = \tan\alpha_C \qquad \text{or: } \alpha_C = ATN2(\cos\alpha_A\sin\beta_A / \cos\beta_A) = ATN2\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2} / \frac{1}{2}\right) = 40.9^\circ$$

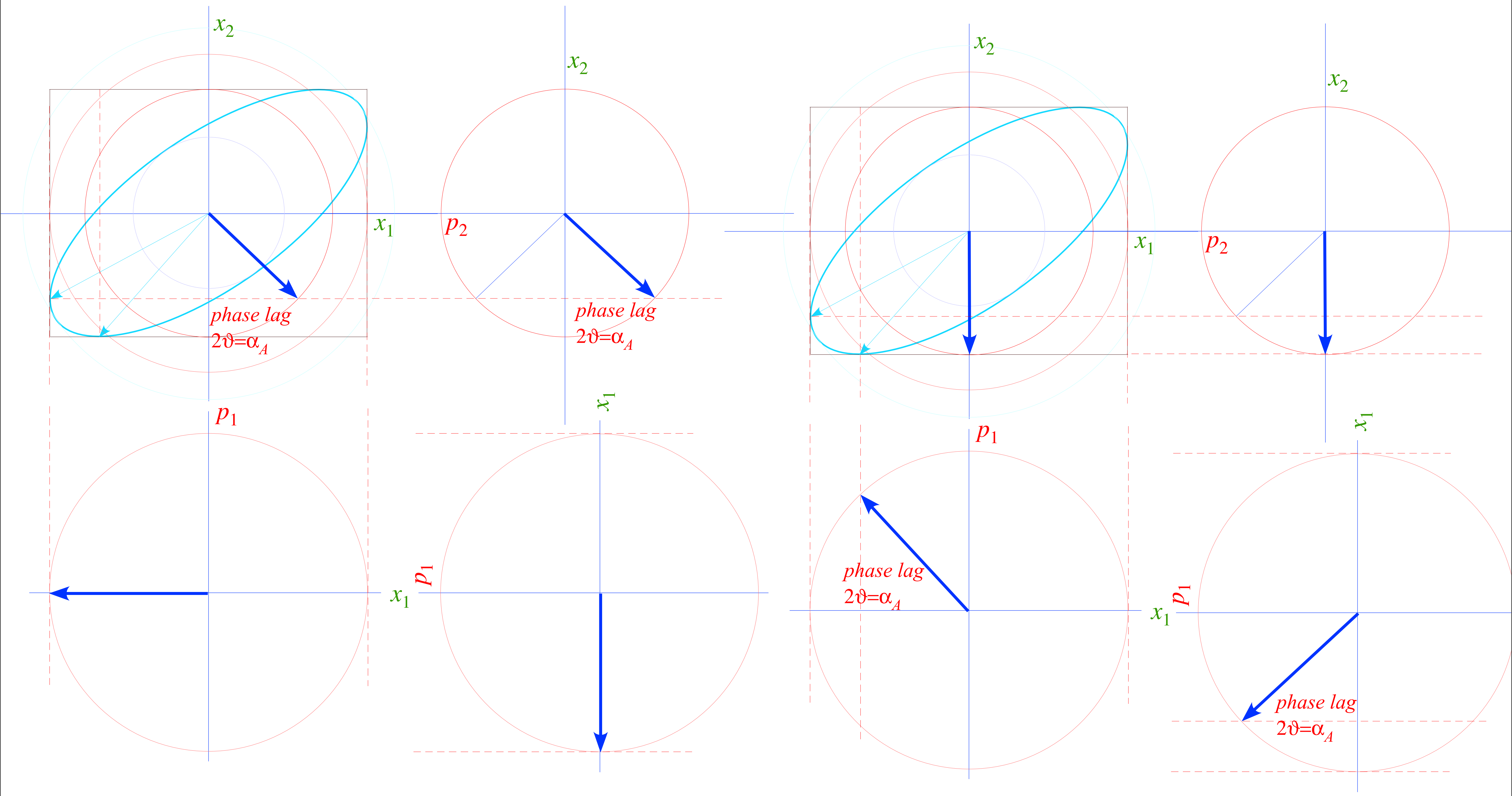




(a) (x_1,x_2) Space
x_2
A_1
A_2
x_1
p_2
phase lag
$2\vartheta=\alpha_A=60°$
phase lag
$2\vartheta=\alpha_A=60°$
$\sqrt{I}$
x_1
p_1
x_1
p_1



(a) (x_1,x_2) Space
x_2
A_1
A_2
x_1
p_2
phase lag
$2\vartheta=\alpha_A=60°$
$\sqrt{I}$
p_1
x_1
x_1
p_1
phase lag
$2\vartheta=\alpha_A=60°$

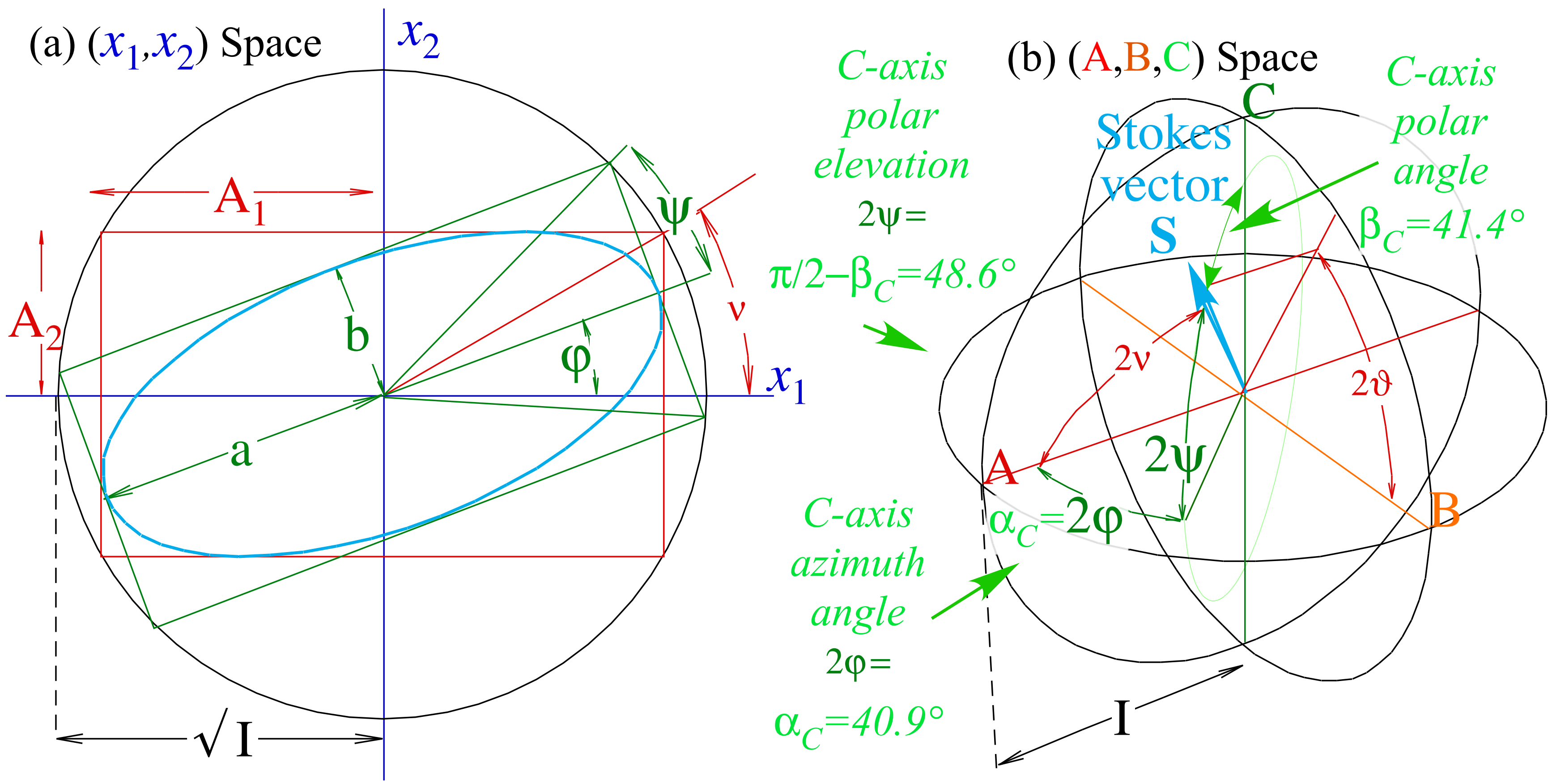


x2
x1
p2
x2
phase lag
2ϑ=αA
phase lag
2ϑ=αA
p1
x1
x1
p1
x2
x1
p2
x2
p1
x1
phase lag
2ϑ=αA
x1
p1
phase lag
2ϑ=αA

The C*-view in* {x_R,x_L}*-basis*

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$



A 90° *B* –rotation $\mathbf{R}(\pi/4)\left|x_1\right\rangle = \left|x_R\right\rangle$ of axis *A* into *C* gets $(\alpha_C, \beta_C, \gamma_C)$ from $(\alpha_A, \beta_A, \gamma_A)$ all at once.

$$\begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} Ae^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ Ae^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\frac{\gamma_A}{2}} = \begin{pmatrix} Ae^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ Ae^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$

Polarization ellipse and spinor state dynamics

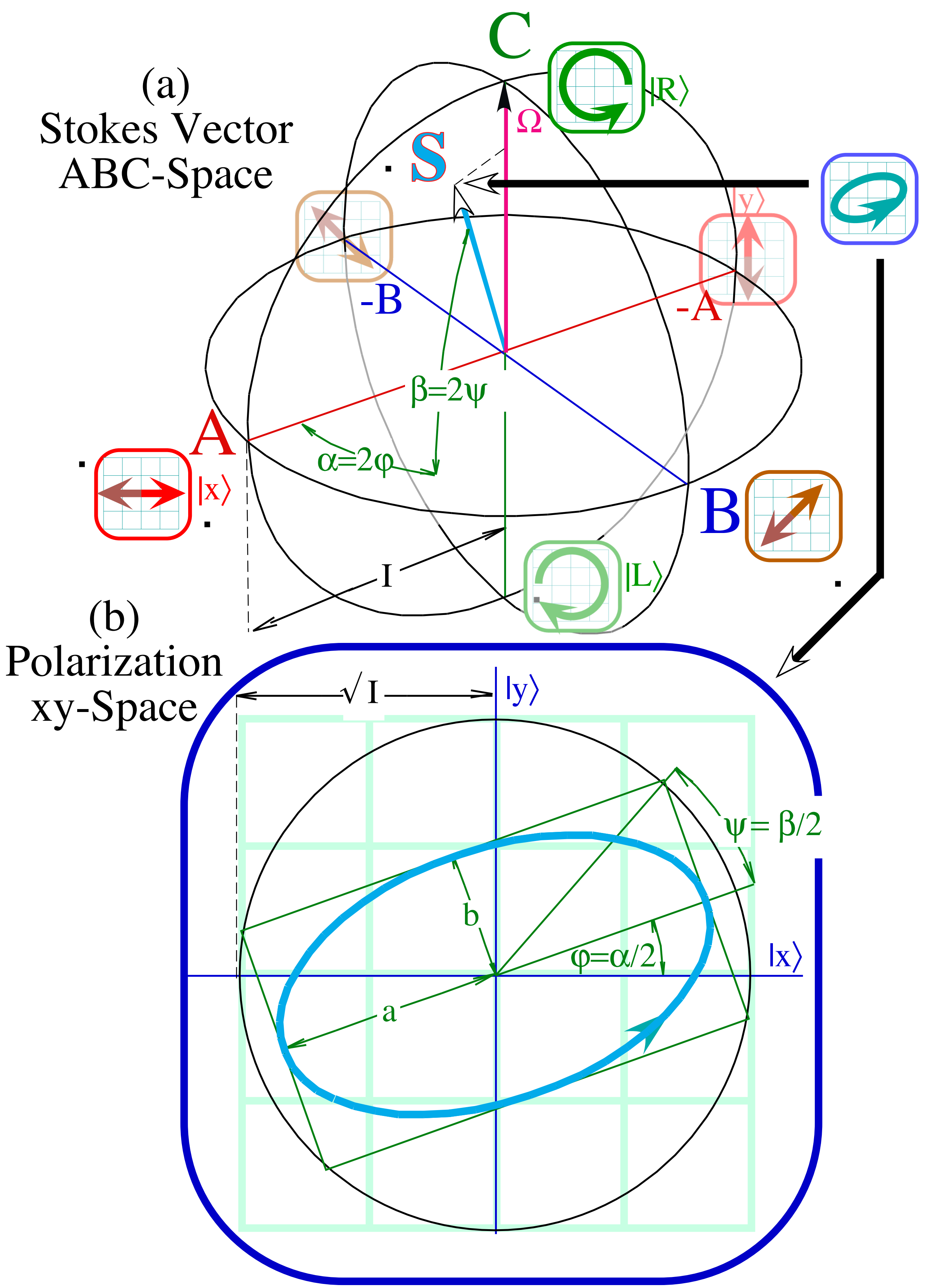


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

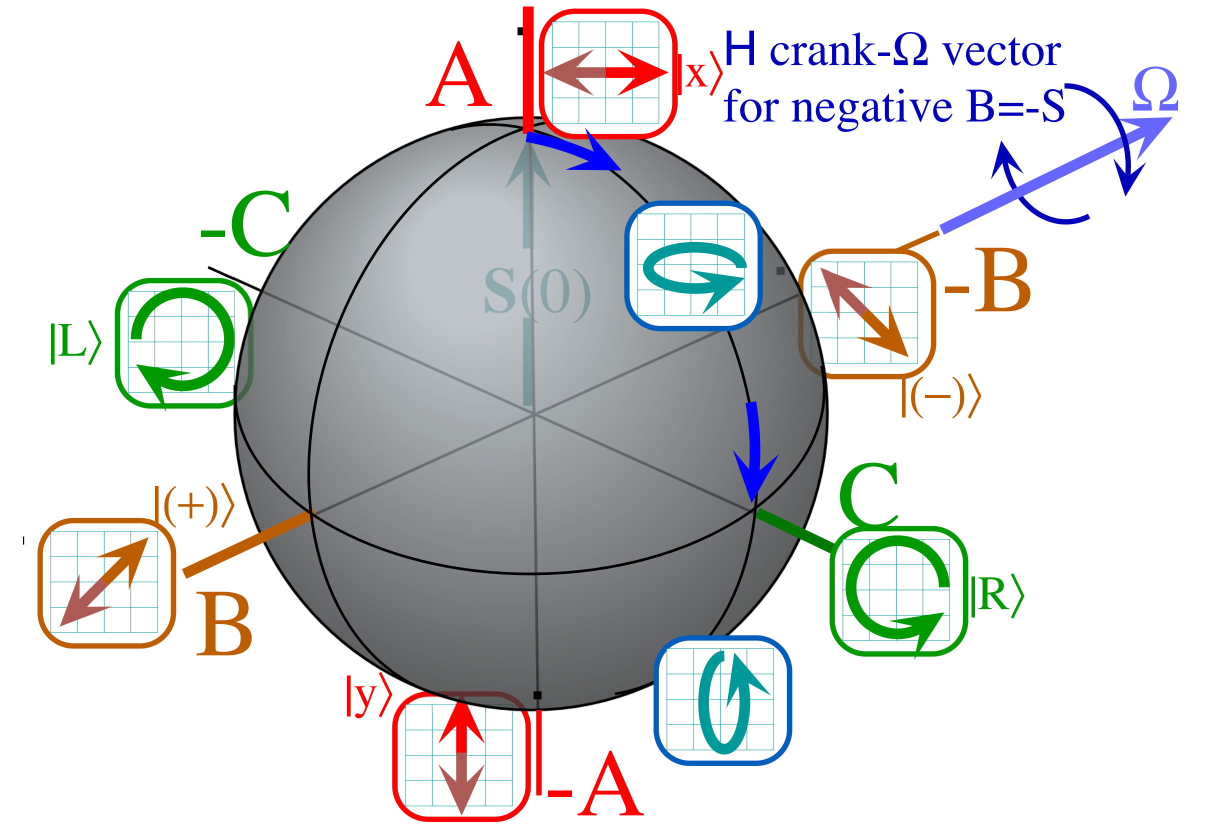
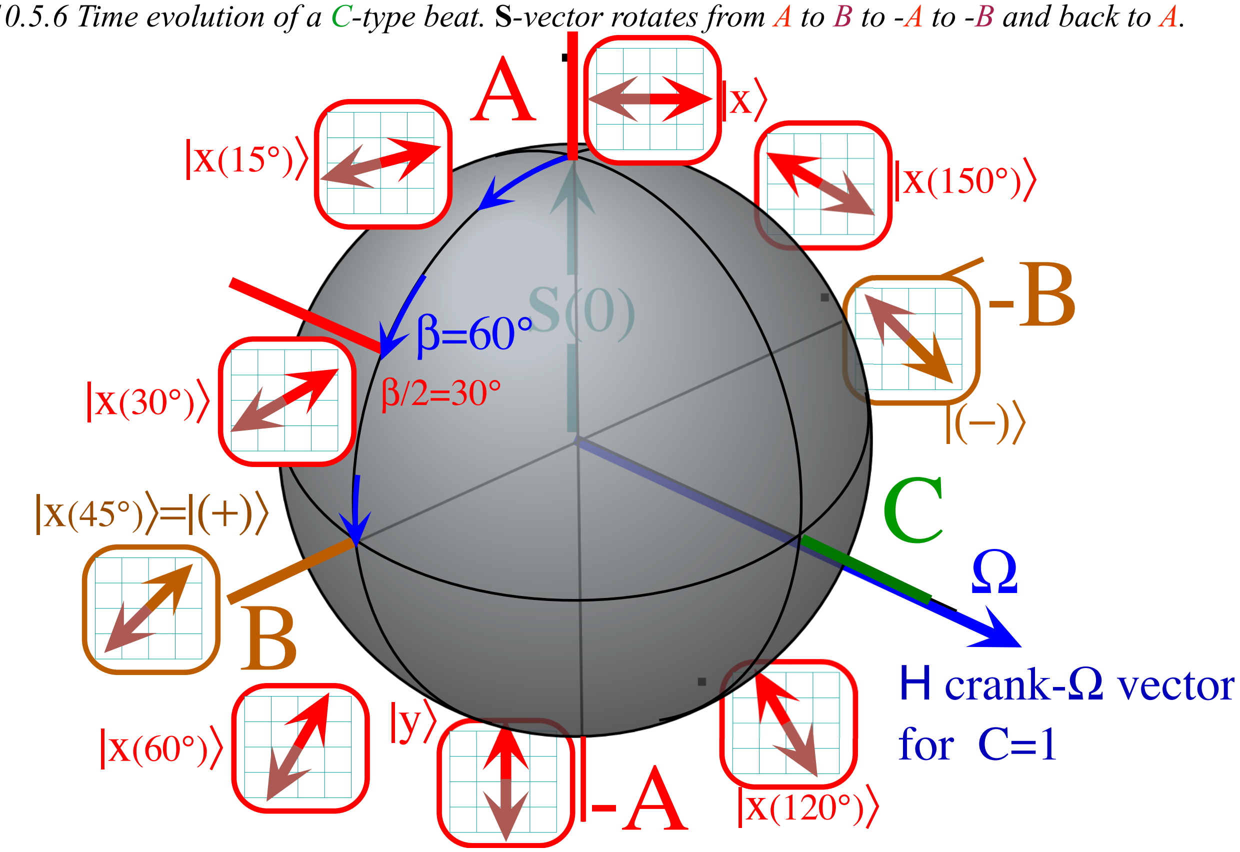


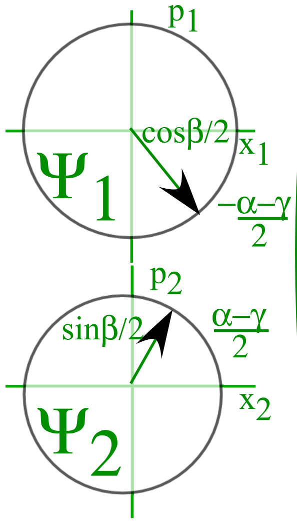
Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Fig. 10.5.6 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*

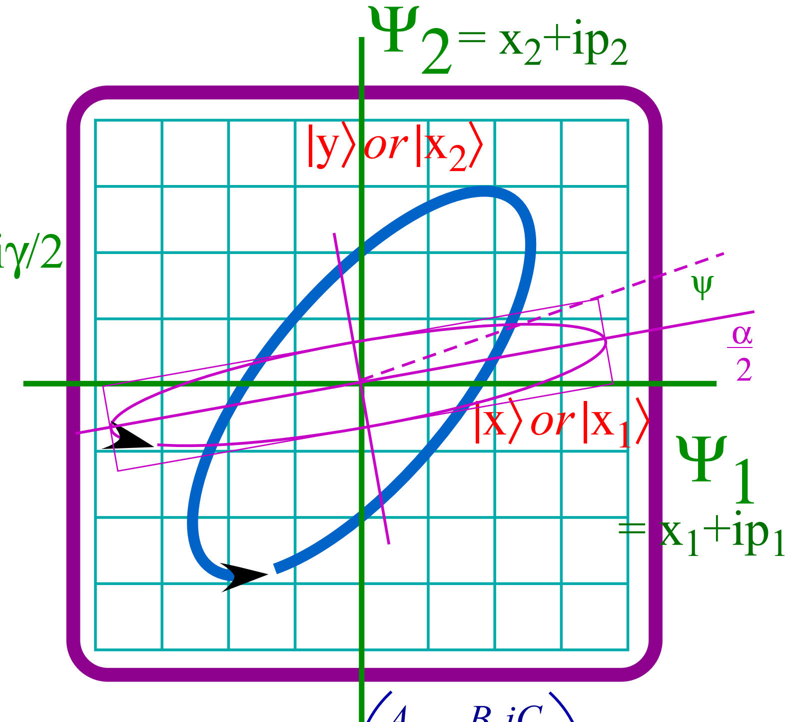


U(2) World : Complex 2D Spinors

2-State ket $|\Psi\rangle =$



$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \begin{pmatrix}\sqrt{N}e^{-i\alpha/2}\cos\beta/2\\ \sqrt{N}e^{i\alpha/2}\sin\beta/2\end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S

$$\begin{pmatrix}s_B\\ s_C\\ s_A\end{pmatrix} = \begin{pmatrix}N\sin\beta\cos\alpha\\ N\sin\beta\sin\alpha\\ N\cos\beta\end{pmatrix}\frac{1}{2}$$

(for α=15° β=45°)

H-Operator $\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}$

Angular velocity

$$\mathbf{\Omega} = \begin{pmatrix}\Omega_B\\ \Omega_C\\ \Omega_A\end{pmatrix} = \begin{pmatrix}2B\\ 2C\\ A-D\end{pmatrix} = \begin{pmatrix}\Omega\sin\vartheta\cos\varphi\\ \Omega\sin\vartheta\sin\varphi\\ \Omega\cos\vartheta\end{pmatrix}$$