Assignment 4 - **Group Theory and Quantum Physics 5093** - *Due Mar 2, 2017*

Main Reading for problems: In text *QTCA* Unit 3-Ch. 10, 10A-B and Lectures 6-10.

Euler Can Canonize

10.A.1 An 2D-oscillator canonical phase state-(x_1, p_1, x_2, p_2) and a spin-state-$|\alpha,\beta,\gamma\rangle$ are both defined by the Euler angles (α,β,γ) through (10.A.1a-b) as well as by axis angles $[\varphi,\vartheta,\Theta]$ through (10.A.1c). (First, verify all parts of (10.A.1).) If rotation-axis-$\mathbf{\Theta}$ polar angles $[\varphi,\vartheta]$ are fixed while rotation angle $\Theta=\Omega t$ varies uniformly with time, Euler angles (α,β,γ) and phase point (x_1, p_1, x_2, p_2) trace spin and oscillator trajectories, respectively. Verify this for the following cases by discussing plots requested below.

(a) $[\varphi=0,\vartheta=0]$, (b) $[\varphi=0,\vartheta=\pi/2]$, (c) $[\varphi=\pi/2,\vartheta=\pi/2]$ (*option extra credit:* (d) $[\varphi=0,\vartheta=\pi/4]$, (e) $[\varphi=\pi/2,\vartheta=\pi/4]$).

For each case sketch 2D-paths $-p_1$ *vs.* x_1 and x_2 *vs.* x_1 and sketch $\hat{\mathbf{\Theta}}\sin\Theta/2$ in a 3D $(-p_2,x_2,-p_1)$-space which should also have paths for $-p_2$ *vs.* x_2 and x_2 *vs.* $-p_1$ *etc*. Also, indicate the paths followed by the tip of the **S**-spin-vector (10.5.8c) in 3D-spin space(S_X, S_Y, S_Z) and characterize as A-type, B-type, or C-type motion, etc., in each case.

Invariantipodals (Easy)

10.A.2 When an Euler sphere is rotated from origin $|1\rangle$ state $(0=\alpha=\beta=\gamma)$ to some angles (α,β,γ), there are always points on the sphere which end up exactly where they were before the rotation. Verify this and express the polar-coordinates(ϕ,θ) of all such invariant points in terms of (α,β,γ).

Spinor-Vector-Rotor (*Deriving 3x3 matrix* $\mathbf{e}_L\bullet\mathbf{e}_{L'}=\langle\mathbf{R}[\varphi,\vartheta,\Theta]\rangle_{3\times3}$ *is a "rite-of-passage" for group theorists.*)

10.A.3 Prove and develop the result (10.A.15) or *GrpThLect.8 p.47* as described below.

$$\mathbf{R}[\vec{\Theta}]\sigma_L\mathbf{R}[\vec{\Theta}]^\dagger = \left(\cos\frac{\Theta}{2}\,\mathbf{1}-i\sin\frac{\Theta}{2}\,\hat{\mathbf{\Theta}}_K\sigma_K\right)\sigma_L\left(\cos\frac{\Theta}{2}\,\mathbf{1}-i\sin\frac{\Theta}{2}\,\hat{\mathbf{\Theta}}_N\sigma_N\right)^\dagger$$
$$=\sigma_L{}'=\sigma_L\cos\Theta-\varepsilon_{LKM}\hat{\mathbf{\Theta}}_K\sigma_M\sin\Theta+(1-\cos\Theta)\hat{\mathbf{\Theta}}_L\left(\hat{\mathbf{\Theta}}_N\sigma_N\right)$$

(a) Using the σ-product definitions (p. 34-38 of *GrpThLect.8*) and the Levi-Civita tensor identity

$$\varepsilon_{abc}\varepsilon_{dec}=\delta_{ad}\delta_{be}-\delta_{ae}\delta_{bd}\quad\text{(Prove this, too!)}$$

to derive the above result. (In *QTCA* Equation (10.A.15a) yields Eq. (10.A.15b).)

(b) Above result (Eq. (10.A.15)) applies when σ_L are replaced by unit vectors $\mathbf{e}_L$. Sketch resulting vectors Θ and $\mathbf{e}_L$ (before rotation) and $\mathbf{e}_{L'}$ (after rotation) for a rotation of $\mathbf{e}_Z$ by $\Theta=120°$ around an axis with polar angle $\vartheta=54.7° = \arccos(1/\sqrt{3})$ and azimuthal angle $\varphi=45°$. (As is conventional, we measure polar angles off the Z(or A) axis and azimuthal angles from the X(or B) axis counter clockwise in the XY (or BC) plane. Give axis Cartesian coordinates.) Use the above to write down a general 3-by-3 matrix in terms of axis angles $[\varphi,\vartheta,\Theta]$, and test it using angles in (b).

(c) Derive numerical Euler angles (α,β,γ) in degrees for this rotation matrix.

(d) Compare formulas and numerics of 3-by-3 $R(3)$ matrix with 2-by-2 $U(2)$ matrix for the same rotations.

(e) Find 3-by-3 $R(3)$ and 2-by-2 $U(2)$ matrices for rotation $\mathbf{R}_y$ by 90° around Y (or C)-axis.

(f) Do products $\mathbf{R}_y\mathbf{R}[\varphi,\vartheta,\Theta]$ and $\mathbf{R}[\varphi,\vartheta,\Theta]\mathbf{R}_y$ numerically.

You may check products with U(2) product formula (10.A.10) or Lect.8 p.42. Compare results to their Hamilton turns.

Spin erection. Does it phase U(2)? (Extra credit)

10.B.2. The following general problem may certainly become relevant if the mythical quantum computer materializes. It involves erecting an arbitrary state with spin vector **S** to the spin-up Z (or A) position with a particular overall phase Φ. In each case make the description of your solution as simple as possible as though you needed to explain it to engineers.

(a) For a state of 0-phase with spin on the X (or B), describe a single operator that does the above.

(b) For a state of 0-phase with spin at β in the XZ (or AB) plane, describe a single operator that does the above.

(c) Of all possible rotations from β in the XZ plane to spin-up Z, which takes the least energy (that is, least total angle Θ of rotation) regardless of final phase Φ?