Assignment 2 - **Group Theory and Quantum Physics 5093**  Due Th. 2/9/17
Based on: Lectures 3-5 and/or text *QTC A* Unit 1-Ch. 3

**Circle-squish-switched**

1.3.2 The discussion in Sec. 1.6 showed that a unit circle is mapped onto an ellipse \( \langle r | T^{-1} | r \rangle = 1 \) by matrix

\[
T = \begin{pmatrix}
1 & 1/2 \\
1/2 & 1
\end{pmatrix}.
\]

Consider the same mapping by "switched" matrix \( S = \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix} \).

(a) Find eigenvalues of \( S \) and \( S^{-2} \). Spectrally decompose \( S \) and plot its eigenvectors.

(b) Let \( T^{-1} | c \rangle = | r \rangle = | c \rangle | T^{-1} \rangle \) or \( T | c \rangle = | r \rangle = | c \rangle T | \rangle \) so \( \langle r | T^{-1} | r \rangle = \langle c | T | c \rangle = 1 \). Suppose all \( c \)-vectors lie on a curve \( \langle c | T | c \rangle = 1 \).

Discuss curve algebraically and plot this curve and the mapped \( \langle r | T^{-1} | r \rangle = 1 \) curve.

(c) Let \( S^{-1} | r \rangle = | c \rangle = \langle r | S^{-1} \rangle \) or \( S | c \rangle = | r \rangle = \langle c | S \rangle = 1 \). Suppose all \( c \)-vectors lie on a curve \( \langle c | S | c \rangle = 1 \).

Discuss curve algebraically and plot this curve and the mapped \( \langle r | S^{-1} | r \rangle = 1 \) curve.

(d) By conic geometry, derive a map \( M | c \rangle = | r \rangle \) of any real vector \( | c \rangle \) by real-symmetric matrix \( M \).

**Dagger Your Own Ket**

1.3.3 Most quantum matrices have simple relations between eigenvalues \( \epsilon_m \) and their conjugates \( \epsilon_m \dagger \), eigenbras \( \epsilon_m \) and kets \( \epsilon_{m\dagger} \), projectors \( P_m \) and their dagger-conjugates \( (P_m)^\dagger \), and diagonalizing \((d\text{-tran})\) transformations \( T \) and their inverses \( T^{-1} \). Let's see what these relations are for...

(a) ...a Hermitian matrix \( M = H \) such that \( H = H^\dagger \) by spectrally decomposing and diagonalizing a 2x2 reflection matrix

\[
H = \begin{pmatrix}
\cos \varphi & \sin \varphi \\
\sin \varphi & -\cos \varphi
\end{pmatrix}.
\]

(Are its eigenvectors meaningful? Discuss.)

(b) ...a Unitary matrix \( M = U \) such that \( U^{-1} = U^\dagger \) by spectrally decomposing and diagonalizing a general 2x2 rotation matrix

\[
U = \begin{pmatrix}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi
\end{pmatrix}.
\]

(Are its eigenvectors meaningful? Discuss.)

(c) Find all the square-roots of \( H \) and of \( U \). (Test them. There are more than two of each!)

(d) A NORMAL MATRIX \( N \) obeys \( N^\dagger N = NN^\dagger \). Is a Normal matrix always diagonalizable? Prove or disprove.

**Home on Lagrange**

1.3.4 Functional spectral decomposition (3.1.17) is related to Lagrange functional interpolation (3.1.18). Use (3.1.18) to approximate \( \sin x \) given only that \( \sin 0 = 0, \sin \pi/2 = 1 \), and \( \sin \pi = 0 \). Compare your approximation to order-2 Taylor series approximation of \( \sin x \) around \( x = \pi/2 \).

**Bras-ackwards**

1.3.5 See if you can work the spectral decomposition ideas backwards by doing the following "inverse" eigenvalue problems. (Hint: Use ket-bras and \( \otimes \). Normalize first!)

(a) Find a Hermitian 3x3 matrix \( H \) that satisfies.

\[
H \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad H \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad H \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},
\]

Write down and test at least one square root \( \sqrt{H} \). (How many square roots are there?)
Cures for Nilpotency

1.3.6 Can a nilpotent matrix $N$ ($N^m=0$, $N^{m-1}$ not zero, integer $m>1$) be Hermitian $N=N^\dagger$ ever? 
(a) ...for $m=2$? (b) ...for other $m$? (Experiment with 2x2 matrices first.)
Use this and exercise Dagger Your Own Ket to prove Hermitian matrices must be diagonalizable.

Truly Secular

1.3.7 The coefficients $a_k$ of the general $nxn$ secular equation (3.1.5d) and (3.1.5f) of $M$ depend on matrix coefficients $M_{ij}$ and on eigenvalues $\epsilon_m$.
(a) Do they depend on which basis you use to represent $M$? Why or why not?
(b) For a general $4x4$ matrix ($n=4$), compute functions $a_k = a_k(\epsilon_m)$ in an orderly way that clearly shows how they come out for general $n$.
(c) For a general $4x4$ matrix ($n=4$), compute functions $a_k = a_k(M_{ij})$ in an orderly way that clearly shows how they come out for general $n$. Use the $\epsilon$-expansion in Appendix 1.A and (b) above to help express answer in terms of diagonal minor determinants. (NOTE: This is a "crucial" problem whose solutions belongs in your lab "journal" or equivalent.) May do successively $n= 2, 3$, until a pattern emerges.

Adjunct Junk

1.3.8 Given (1.A.5) or $A \cdot A^{ADJ} = 1$ (det$|A|$) with $A = M - \lambda I$ show that $A^{ADJ}$ has $M$ eigenkets $|\lambda\rangle$ if $\lambda$ is an eigenvalue of $M$.
Does $A^{ADJ}$ also harbor $M$'s eigenbras? Use $M=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ as one example and $Q=\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ as another. (Get $P$-ops of $Q$.)

Pair 'em up

1.3.9 An $nxn$ pairing matrix $\Pi$ has 1 for all $n^2$ matrix elements $\Pi_{ij}=1$. It has been used in superconductivity theory and theory of nuclear shell structure.
(a) Use 1(c) above to help derive its eigenvalues and spectral decomposition. (Or, you may develop the theory by doing successively $n= 2, 3$, until the pattern emerges.)
(b) Does the matrix $\Pi+(const.)I$ have the same eigenvectors? eigenvalues? as $\Pi$. How about $(const.)\cdot \Pi$? Explain.

All Together Now

1.3.10 Show how to do a simultaneous spectral decomposition using the projector splitting technique. (a) Spectrally decompose $A=\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, and $B=\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$.
(b) Calculate the "ridiculous function" $B^A$ of these two matrices.

Almost Nilpotent

1.3.11 Matrix $N_{almost}$ is nilpotent for $\epsilon=0$. Find spectral decomposition and eigen-bra-kets. What happens as $\epsilon\rightarrow 0$?
$N_{almost}(\epsilon)=\begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix}$ Which bras or kets survive $\epsilon=0$? Plot for $\epsilon<0$. What ortho-normalization (if any) is left?
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Solutions to Circle-squish-switched

1.3.2 matrix \( T = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \) maps a circle to ellipse \((r|T^{-2}|r)=1\). A "switched" matrix \( S = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \) maps to hyperbola.

\[
T = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix},
\]

\[
S = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{2} \end{pmatrix}.
\]

(a) Find eigenvalues (-1/2,3/2) of S. Spectrally decompose S and plot its eigenvectors.

(b) Let \( T^{-1}|r|=|c\Rightarrow|T^{-1}|r|=|c\Rightarrow|T \) so \( r(T^{-1})|r|=|c(T)|c \) T. c-vectors: \( |c(T)|c \) =1 Same as ellipses on p.24 Ch. 3.

(c) Let \( S^{-1}|r|=|c\Rightarrow|S^{-1}|r|=|c| S |c \Rightarrow|S |c \) S so \( |r| S^{-1}|r|=|c| |S |c \) |c \) S. c-vectors: \( |c |c \) S |c \) =1 Like p.24 Ch. 3 but hyperbolic.

Solutions to Dagger Your Own Ket

1.3.3 Relate \( \epsilon_m \) to \( \epsilon_m^*\) (See below) For NORMAL MATRICES \( N = \overline{N}N \) we have that \( H = (N + \overline{N})/2 \) and \( A = (N + \overline{N})/2i \) are diagonalizable by the same projectors thus proving: \( |e_m| = (|e_m|)^* \) and \( PM = (PM)^* \), and \( T_\phi = T^\dagger \).

(a) .Hermitian \( M = H \Rightarrow \begin{pmatrix} \cos \varphi & \sin \varphi \\ \sin \varphi & -\cos \varphi \end{pmatrix} \): \( H = U^\dagger \). So \( \epsilon_m = \epsilon_m^* \). See Solution 1.2.4 (mirror \( \sigma_0 \)) with \( 2\phi = \varphi \).

(b) .Unitary \( M = U \Rightarrow \begin{pmatrix} \cos \varphi & \sin \varphi \\ \sin \varphi & -\cos \varphi \end{pmatrix} \): \( U = U^\dagger \). So \( \epsilon_m = \epsilon_m^* \). See Solution 1.2.4 (rotation \( R [2\varphi] \)) with \( 2\phi = \varphi \).

(c) See Solution 1.2.4 for square-roots of \( H \) and of \( U \). (Test them. There are four of each!)

Home on Lagrange

1.3.4 Lagrange interpolate \( \sin x \) at \( x = 0, \pi/2, \) and \( \pi = 1 \). Compared to Taylor series of \( \sin x \) around \( x = \pi/2 \).

\[ f(x) = \sin(x - \pi/2) = (x - \pi/2)^2(x - \pi/2) + (x - \pi/2)^3 + \ldots \]  

\[
\text{Lagrange interp'd by: } f(0)P_0 + f(\pi/2)P_{\pi/2} + f(\pi)P_\pi = (-1)P_0 + 0P_{\pi/2} + 1P_{\pi}
\]

\[
P_0 = \frac{(x - \pi/2)(x - \pi/2)}{(0 - \pi/2)(0 - \pi/2)} = \frac{1}{2} (x - \pi/2)(x - \pi/2),
\]

\[
P_\pi = \frac{(0 - x)(0 - x)}{(0 - \pi)(0 - \pi)} = \frac{1}{2} (x - \pi/2)(x - \pi/2) - \frac{1}{2} (x - \pi/2)(x - \pi/2),
\]

\[
P_{\pi/2} = \frac{(0 - x)(0 - x)}{(0 - \pi/2)(0 - \pi/2)} = \frac{1}{2} (x - \pi/2)(x - \pi/2) - \frac{1}{2} (x - \pi/2)(x - \pi/2).
\]

\[
= (-1)P_0 + 0P_{\pi/2} + 1P_{\pi} = \frac{(x - \pi/2)^2(x - \pi/2) + (x - \pi/2)^3 + (x - \pi/2)^4 + \ldots}{2}.
\]

Solutions to Bras-ackwards 1.3.5 Find Hermitian matrix \( H \) given::… Give one square root \( \sqrt{H} \). (How many? \( 2 = 8 \))

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

Solutions to Cures for Nilpotency

1.3.6 Suppose non-zero matrix is nilpotent: \( N = 0 \) but \( N \neq 0 \) and Hermitian \( N = \overline{N} \) implying \( \overline{N} = 0 \). This implies

\[
\sum_{j} N_{i j} N_{j i} = \sum_{j} N_{i j} N_{j i} = \sum_{j} |N_{i j}|^2 = 0
\]

so \( N = 0 \). Nonzero Hermitian matrices can't be nilpotent thus must be diagonalizable.
Solutions to Truly Secular

1.3.7 Coefficients $a_k$ of $\varepsilon^k$ of $n \times n$ SEq of $M$ depend on matrix coefficients $M_{ij}$ and on eigenvalues $\varepsilon_m$.

(a) Do they depend on which basis you use to represent $M$? Eigenvalues are basis independent and hence so is SEq. 
(b) $4 \times 4$ matrix $(n=4)$ functions $a_k = a_k(\varepsilon_m)$ indicate general $n$. 
\[ S(\varepsilon) = (\varepsilon - \varepsilon_1)(\varepsilon - \varepsilon_2)(\varepsilon - \varepsilon_3)(\varepsilon - \varepsilon_4) \]
implies 
\[ S(\varepsilon) = \varepsilon^4 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)\varepsilon^3 + (\varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \varepsilon_1\varepsilon_4 + \varepsilon_2\varepsilon_3 + \varepsilon_2\varepsilon_4 + \varepsilon_3\varepsilon_4)\varepsilon^2 + (\varepsilon_1\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_2\varepsilon_4 + \varepsilon_1\varepsilon_3\varepsilon_4 + \varepsilon_2\varepsilon_3\varepsilon_4 + \varepsilon_2\varepsilon_4\varepsilon_3 + \varepsilon_3\varepsilon_4\varepsilon_2)\varepsilon - (\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4) \]
\[ = (2 - 5\varepsilon_1 - 4\varepsilon_2) \]
for $\lambda = 5$, 
\[ = (2 - 1\varepsilon_1 - 4\varepsilon_2) \]
for $\lambda = 1$. 

Solutions to Adjunct Junk

1.3.8 $A \lambda^{ADJ} = 1\det[M\lambda]$ with $A = M \lambda I$ gives $(M-\lambda I)A^{ADJ} = (\det[M\lambda])I = 0$ so $A^{ADJ}M = MA^{ADJ} = \lambda A^{ADJ}$. So $(M-\lambda I)^{ADJ}$ harbors $M$'s eigensets and bras. For example $(M-\lambda I)^{ADJ}$ below yield non-normalized projection matrices.

\[
\begin{pmatrix}
4 - \lambda & 1 \\
3 & 2 - \lambda
\end{pmatrix}
\begin{pmatrix}
2 - \lambda & -1 \\
-3 & 4 - \lambda
\end{pmatrix}
= \begin{pmatrix}
2 - 5 & -1 \\
-3 & 4 - 5
\end{pmatrix}
\text{for } \lambda = 5,
= \begin{pmatrix}
2 - 1 & -1 \\
-3 & 4 - 1
\end{pmatrix}
\text{for } \lambda = 1.
\]

Solutions to Pair 'em up

1.3.9 An $N \times N$ pairing matrix $\Pi$ has $N^2$ elements $\Pi_{ij} = 1$. One eigenvector $(1,1,1,\ldots,1)$ e-value=$N$. All other e-vals zero. 
(a) Minimal equation is $\Pi^2 = N\Pi$ or $(\Pi - N\Pi)(\Pi - 0) = 0$. 
(b) ...matrix $\Pi + (\text{const.})I$ has same eigenvectors? $\text{YES}$ eigenvalues? $\text{YES}$ +const as $\Pi$. How about $(\text{const.})^{-1}\Pi$? Explain.

Solutions to All Together Now

1.3.10 (a) Spectrally decompose 
\[ A = \begin{pmatrix}
2 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}, \text{ e-val: } \lambda_1 = 1, \lambda_2 = 3, 3 \] and 
\[ B = \begin{pmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{pmatrix}, \text{ e-val: } \lambda_1 = 1, \lambda_2 = 4.4
\]
\[ 1 = \begin{pmatrix}
1 & -1 \\
-1 & 1 \\
0 & 0
\end{pmatrix}, 1 = \begin{pmatrix}
1 & 1 \\
1 & 1 \\
0 & 0
\end{pmatrix}, 1 = \begin{pmatrix}
2 & -1 \\
1 & -1 \\
0 & 0
\end{pmatrix}, 1 = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 2
\end{pmatrix}, 1 = \begin{pmatrix}
1 & -1 \\
1 & -1 \\
0 & 0
\end{pmatrix}, 1 = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 2
\end{pmatrix}
\]

(b) "ridiculous function" $B A = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} + \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
1 & 1 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

(c) "ridiculous function" $A B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix} + \begin{pmatrix}
1 & 1 & -2 \\
1 & 1 & -2 \\
1 & 1 & -2
\end{pmatrix} = \begin{pmatrix}
1 & 1 & -2 \\
1 & 1 & -2 \\
1 & 1 & -2
\end{pmatrix}
\]

Almost Nilpotent

1.3.11 Find spectral decomposition and eigen-bra-kets and what happens as $\varepsilon \to 0$ for 
\[
N_{secut}(\varepsilon) = \begin{pmatrix}
1 & 1 \\
\varepsilon & 1
\end{pmatrix}
\]
Only ket survives.

$\lambda^2 - 2\lambda + 1 - \varepsilon^2 = 0$ gives: $\lambda = 1 \pm \varepsilon$ and: $P_{+\varepsilon} = \frac{1}{2} \begin{pmatrix}
1 & 1/\varepsilon \\
\varepsilon & 1
\end{pmatrix}, P_{-\varepsilon} = \frac{1}{2} \begin{pmatrix}
1 & -1/\varepsilon \\
-\varepsilon & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]