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Velocity Amplification in Collision Experiments Involving Superballs

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mentioned phenomena are of course well known, however it seems profitable, from a didactic point of view, to have available a Hamiltonian of the form (12), which is intuitively simple and which encompasses all these phenomena in the same formulation.

It should be stressed that we have been working entirely within a frame in which the Lorentz expression, Eq. (2), is supposed to provide an adequate description of the forces on the charged particle. The Aharonov–Bohm effect,[10] which appears to depend uniquely upon the vector potential **A**, is therefore outside the scope of the present discussion.

[1] H. Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, Mass., 1950).

[2] M. Goeppert-Mayer, Ann. Physik **9**, 273 (1931).

[3] P. I. Richards, Phys. Rev. **73**, 254 (1948).

[4] J. Fiutak, Can. J. Phys. **41**, 12 (1963).

[5] E. A. Power and R. Shail, Proc. Cambridge Phil. Soc. **55**, 87 (1959).

[6] Aa. E. Hansen, Theoret. Chim. Acta **16**, 217 (1970).

[7] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[8] L. I. Schiff, *Quantum Mechanics* (McGraw–Hill, New York, 1968).

[9] H. Eyring, J. Walter, and G. E. Kimball, *Quantum Chemistry* (Wiley, New York, 1960).

[10] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1958).

Velocity Amplification in Collision Experiments Involving Superballs

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If a pen is stuck in a hard rubber ball and dropped from a certain height, the pen may bounce to several times that height. The results of two such experiments, which can easily be duplicated in any undergraduate physics laboratory, are plotted for a range of mass ratios. A simple theoretical discussion which provides a qualitative understanding of the phenomenon is presented. A more complicated formulation which agrees very well with one of the experiments is also presented. The latter involves a simple analog computer program. Finally, an intriguing generalization of the phenomenon is considered.

INTRODUCTION

Shortly after the well-known Superball[1] appeared on the market, one of the authors quite accidentally discovered a surprising effect.[2] The point of a ball point pen is imbedded in the surface of a 3-in. diam Superball, and the pen and ball are dropped from a height of 4 or 5 ft so that the pen remains above the ball and perpendicular to a hard floor below. As the ball strikes the floor, the pen may be ejected so violently that it will strike the ceiling of the average room with considerable force. Furthermore, one can adjust the mass of the pen so that the ball remains completely at rest on the floor after ejecting the pen.

Little attempt was made to obtain a qualitative much less a quantitative understanding of this velocity amplification until the fall of 1968, when the authors came together as students and teacher of the introductory physics course for *non*physics majors at the University of Southern California.

The object of our investigation was to find the final velocities of the pen and the ball as a function of their masses and as a function of the initial velocity just before contact with the floor.

Sufficient qualitative understanding of the effect was obtained early in the project to predict that even greater velocity amplification would be obtained if one or more idler stages consisting of smaller Superballs were introduced between the pen and the bottom ball. Quantitative understanding of this multistage system is more difficult and only a partial analysis of this is given at the end of the paper.

I. APPARATUS AND EXPERIMENTAL RESULTS

An apparatus consisting of a ball and "pen" of variable mass along with a magnetic dropping device (described below) was constructed, and a 12-ft vertical height measuring scale was marked off on the wall next to the apparatus.

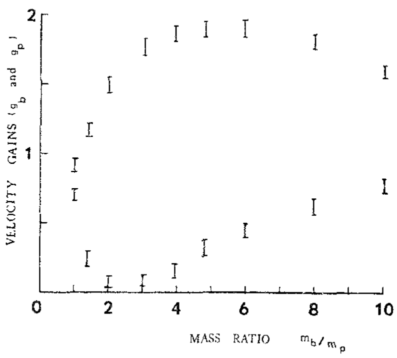

FIG. 1. Velocity gains in first experiment.

When the ball and pen were dropped from initial height h_i, the two objects would achieve a velocity $v_i=(2gh_i)^{1/2}$ just before hitting the floor. After collision with the floor, the pen separated from the ball and rose to a maximum height h_p which was observed by an experimenter standing on a stepladder. Another experimenter observed the final maximum height h_b of the ball.

Just after the collision, the pen is ejected with a velocity v_p much greater than v_i. The gain in velocity is obtained by computing the quotient

$$v_p/v_i=(2gh_p)^{1/2}/(2gh_i)^{1/2}=(h_p/h_i)^{1/2}\equiv g_p. \quad (1)$$

A similar coefficient may be defined for the velocity of the ball immediately after the collision

$$v_b/v_i=(h_b/h_i)^{1/2}\equiv g_b. \quad (2)$$

The two ratios g_p and g_b were plotted against the ratio of mass m_b/m_p. A typical set of results is shown in Fig. 1.

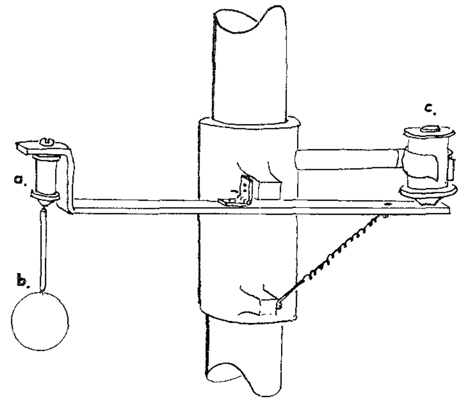

FIG. 2. Magnet (a) is wired to release ball and pen (b) an instant before the other magnet (c) releases supporting arm which swings clear.

The initial velocity v_i could be controlled by adjusting the height of the magnetic dropping device shown in Fig. 2. This device was designed to release the ball and pen before springing out of the path of the ascending pen.

Several different types of pens were tried since a metal pen tip gouged the ball after many trials. One arrangement that works well is shown in Fig. 3. The moment of inertia of the pen shaft increases the directional stability of the projectile.

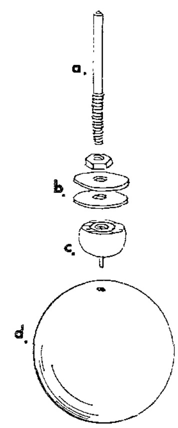

FIG. 3. The pen consisted of a steel-tipped aluminum rod (a) which could hold various weights (b) and screwed into a miniature Superball (c) which was fitted with a threaded sleeve and pin. Large Superball (d) attached more or less weakly to the pin. Another arrangement that can be used is shown in Fig. 14(a).

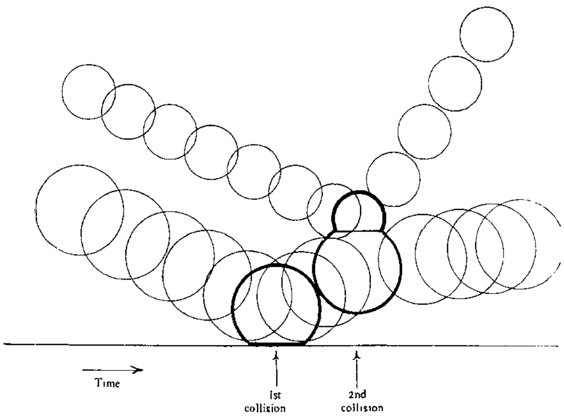

FIG. 4. Exaggerated plot of trajectories of ball and pen shows the assumptions made in the independent-collision model.

II. THEORY: INDEPENDENT COLLISION MODELS

A. Elastic Collisions

In the simplest theory one assumes that the ball collides elastically with the floor and returns to strike the pen in a second collision that is elastic and independent of the first. This process is indicated in Fig. 4.

By applying energy and momentum conservation laws to the second collision (Fig. 4), one obtains formulas for the velocity gains g_p and g_b, which are plotted in Fig. 5.

The dotted line indicates the actual final velocity the ball would have after an inevitable *third* independent elastic collision with the floor. Naturally, these curves are poor approximations

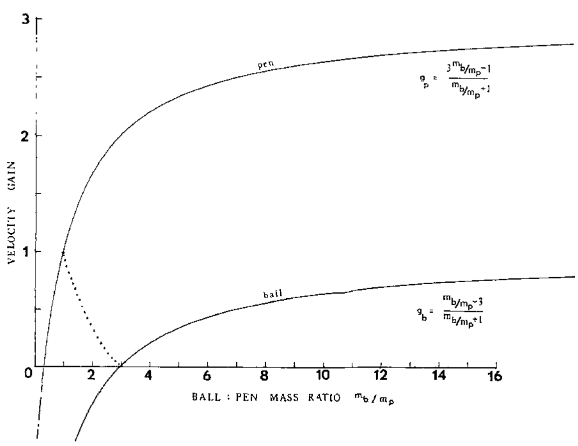

FIG. 5. Elastic independent-collision gain formulas.

for large $m_p > m_b$ since then the ball suffers four or more collisions.[3] They are also poor approximations for small $m_p < \frac{1}{6}m_b$, but nevertheless predict zero final velocity for the ball at $m_p = \frac{1}{3}m_b$, which agrees fairly closely with the experiment. Notice that the maximum gain g_p is 3.0, which is the upper limit for any two-body collision model.

B. Inelastic Collisions

Initially the ball and pen are weakly stuck together and a certain amount of "binding energy" will be lost when they separate. Additional energy is lost in every collision since the ball is not perfectly elastic. In fact, a good Superball will bounce back to about 90% of its original height when dropped by itself onto a hard surface.

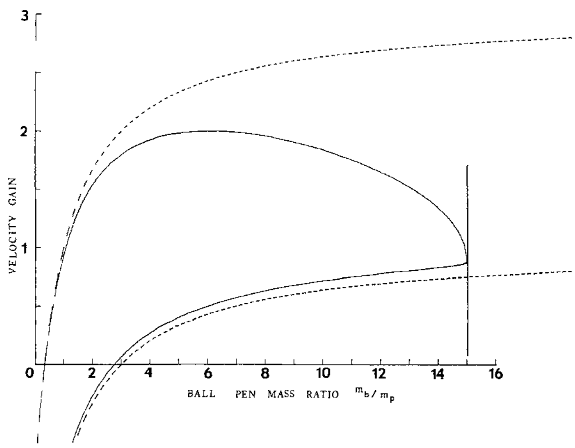

FIG. 6. Gain curves with binding quotient $\alpha = \frac{1}{4}$ and bounce coefficient $\beta = 1$. Threshold occurs at $M_b/M_p = 15$. Dotted curves are gains with $\alpha = 0$ and $\beta = 1$.

Let us assume that energy $\Delta E = \alpha(m_b V_I^2/2)$ is lost in the second independent collision. In the first collision let us assume that the ball returns from the floor with velocity βV_I $(0 < \beta \leq 1)$ just before striking the pen. We assume that α and β are practically constant in a given experiment since V_I and m_b are held constant while m_p is varied. Again one must solve an equation for energy conservation (3) with one for momentum conservation (4):

$$\tfrac{1}{2}m_b(\beta V_I)^2 + \tfrac{1}{2}m_p V_I^2 = \tfrac{1}{2}m_p v_p^2 + \tfrac{1}{2}m_b V_b^2 + \alpha(m_b V_I^2/2), \quad (3)$$

$$m_b \beta V_I - m_p V_I = m_b V_b + m_p v_p. \quad (4)$$

These yield g_p and g_b as functions of mass ratio m_b/m_p,

$$g_b=\frac{V_b}{V_I}=\frac{(\beta m_b/m_p)-[(\beta+1)^2-\alpha(m_b/m_p+1)]^{1/2}-1}{m_b/m_p+1}, \tag{5}$$

$$g_p=\frac{V_p}{V_I}=\frac{\dfrac{\beta m_b}{m_p}+\dfrac{m_b}{m_p}\left[(\beta+1)^2-\alpha\left(\dfrac{m_b}{m_p}+1\right)\right]^{1/2}-1}{[(m_b/m_p)+1]}, \tag{6}$$

which are plotted in Fig. 6 for values $\alpha=\frac{1}{4}$ and $\beta=1$. The dashed lines are the plots in Fig. 5 in which α was zero.

The velocity gain peaks at 2.0 when $m_b/m_p=6$ and falls with decreasing pen mass until $v_p=v_b$ when $m_b/m_p=15$. At this point the pen is so small it cannot free itself from the ball because of the binding energy. Of course, if α is decreased by increasing V_I or lubricating the connecting pins to decrease the binding energy, this "threshold" will move to the right. In Fig. 7 are superimposed plots for $\alpha=0$, $\alpha=\frac{1}{4}$, and $\alpha=\frac{1}{2}$. The experimental results for one experiment (Fig. 1) are plotted over these.

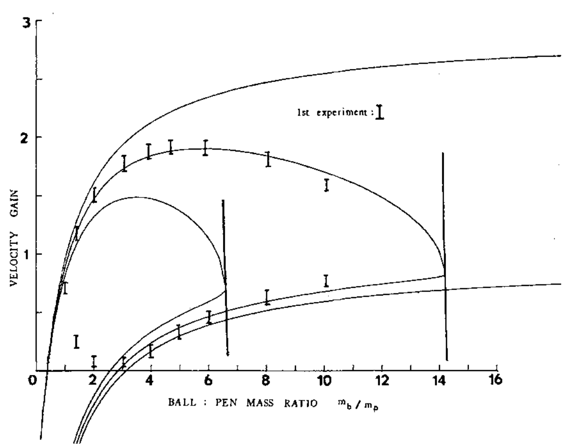

FIG. 7. Gain curves for $\beta=0.95$ and $\alpha=0$, $\frac{1}{4}$, and $\frac{1}{2}$. The $\alpha=\frac{1}{4}$ curves come reasonably close to the results of the first experiment.

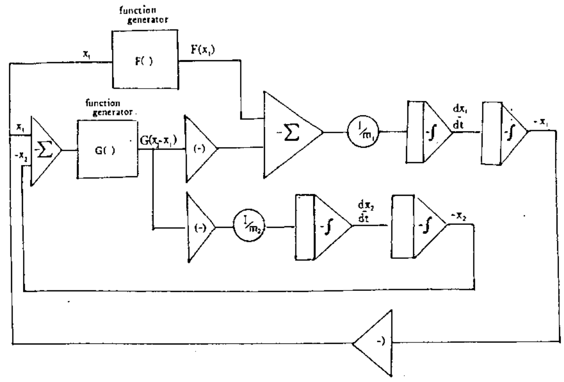

FIG. 8. Analog computer program. In the linear case, the function generators $F(x)$ and $G(x)$ are replaced by coefficient potentiometers.

However, later experiments in which the binding energy was kept to a minimum did not seem to agree with formulas (5) and (6) for any α or β. Furthermore, previous experimentation had shown that the shape and material of the pen point affects the results a great deal.

Therefore, we sought to obtain curves resembling those in Fig. 6 by constructing a more detailed model of the collision process assuming purely elastic forces.

III. ELASTIC CONTINUOUS-FORCE MODELS ANALOG-COMPUTER RESULTS

In this section we discuss solutions of the simultaneous differential Eq. (7) for the positions $x_1(t)$ and $x_2(t)$ of the ball and pen, respectively.

The initial conditions are $\dot{x}_1(0)=\dot{x}_2(0)=-V_I$ and $x_1(0)=x_2(0)=\epsilon$. Forces of gravity are not studied since they are small compared to the forces of collision. Finally, the terminal velocities $\dot{x}_1(\infty)$ and $\dot{x}_2(\infty)$ are plotted one above the other as functions of the mass ratio m_1/m_2, so they may be compared with Figs. 5 and 6.

$$m_1\ddot{x}_1=F(x_1)-G(x_2-x_1),$$
$$m_2\ddot{x}_2=G(x_2-x_1); \tag{7}$$

in Eq. (7) the function $F(x)$ represents the force of the floor on the Superball as a function of the distance x that the floor enters the Superball. The function $G(x)$ is the repulsive force between the ball and pen as a function of the distance x that the pen moves toward the center of the ball.

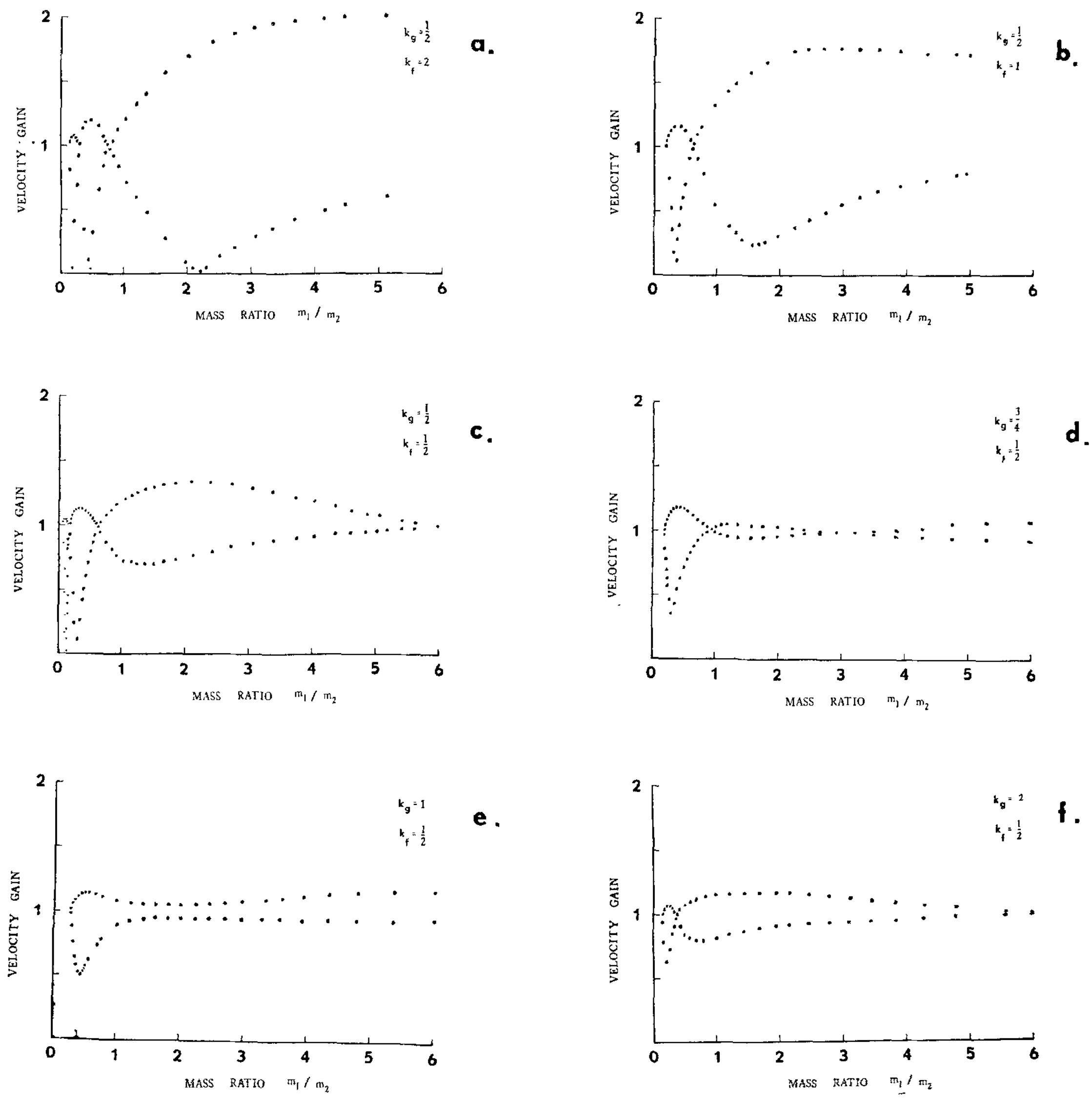

FIG. 9. (a)–(f) Analog computer produced gain curves for various linear forces.

A. Quasi Linear Model

If the force functions $F(x)$ and $G(x)$ were linear, i.e., $F(x)=k_f x$ and $G(x)=k_g x$, one could solve Eq. (7) analytically. These functions are not linear for the Superball and pen experiments, but they could be for linear air-trough experiments involving two sliding masses with bumper springs. In any case, we thought it worthwhile to solve the problem assuming linear repulsive forces $F(x)$ and $G(x)$.

However, these functions $F(x)$ and $G(x)$ would be zero for positive x if there is no attractive-force mechanism present. Because of this complication we chose to solve the problem on an analog computer.[4] The computer program shown in Fig. 8 is immediately applicable to the Superball–pen experiment once one obtains $F(x)$ and $G(x)$ for these two objects.[5]

In each of the Figs. 9(a)–9(f) the final velocity of the pen and ball are plotted against mass ratio m_1/m_2 as in Figs. 1 and 5–7. m_1 is held constant at 60 g while m_2 varies. Six different force combinations are shown.

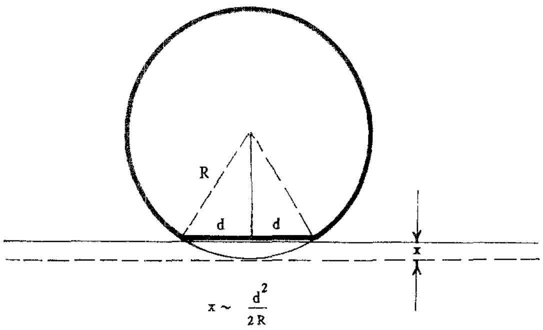

FIG. 10. Sagittal formula.

If $F(x)$ and $G(x)$ were linear for all x, then the form of the velocity gain curves would change only for a change of ratio $F(x)/G(x)$. The above curves, however, would be affected slightly by changes in absolute value of F and G.

To compare these results with those of Fig. 1, we imagine that the Superball and pen pictured in Fig. 3 are masses connected by a spring. To the soft skin of the balls we assign a spring constant k and let $F=k$ and $G=\frac{1}{2}k$. The corresponding gain vs mass ratio curves are shown in Fig. 9(b).

Apparently, the quasilinear model is not good for the Superball since the top curve in Fig. 9(b) is too low, and the bottom curve approaches zero for m_1/m_2 less than 2. However the quasilinear results become more and more like the independent-collision model results as the ratio F/G becomes large. In particular it is interesting to note the behavior of the final velocity curves when $m_1/m_2<1$. In this case the ball underneath must

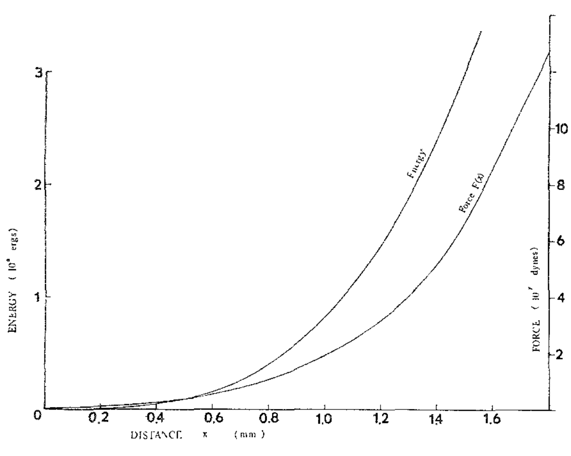

FIG. 11. Adiabatic force $F(x)$ and energy curves for Superball.

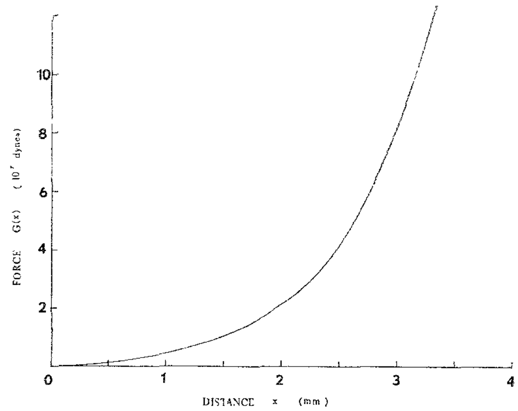

FIG. 12. Adiabatic force function $G(x)$.

oscillate up and down several times to change the momentum of the massive pen above.

B. The Nonlinear Model

The force function $F(x)$ was obtained by dropping a Superball of known mass onto a painted flat metal surface from varying heights. The radius of the paint spot on the skin of the ball was then recorded. In this way the potential energy could be plotted as a function of spot size. Using the sagittal formula, one can easily relate the spot size to the small value x of the depression of the ball, as shown in Fig. 10.

With the potential energy plotted against x (Fig. 11) one could approximate $F(x)$ by plotting the average derivative of the potential energy.

This was also done for the smaller Superball which formed the tip of the pen. In this way the force function $G(x)$ could be computed. (Fig. 12)

Functions $F(x)$ and $G(x)$ were then placed on the function generators of the analog computer.

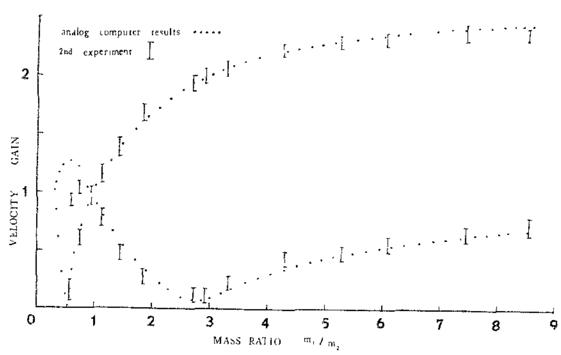

FIG. 13. Comparison between analog computer gain curves and second experiment.

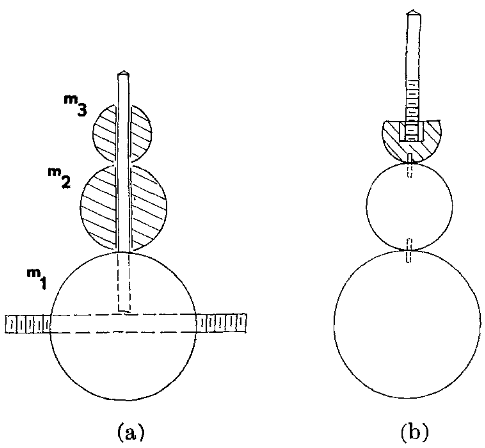

Fig. 14. Two designs for a multiple stage tower of balls. (a) Large number of balls can slide on a shaft. (b) Balls connected by small pins stand to lose appreciable amounts of binding energy.

One of the computer results [which are accurate solutions of Eq. (7) to within 2%] are displayed in Fig. 13. In this the simulated initial velocity was 300 cm/sec, and the results are compared with an experiment which was conducted with the coupling between the two balls made as small as possible. The experimental results remarkably close to the computer predictions. Unfortunately, this delicate arrangement could not withstand the forces that occur when $m_2 > m_1$ and the smaller ball split, making further readings impossible.

These large forces could be read directly from the analog computer while it was operating, as could the position functions $x_1(t)$ and $x_2(t)$. The turning point of the ball was between 1 and 2 mm and it experienced a force of between 20 and 150 kg. Forces of this order were felt by the upper mass m_2 when it is as large as m_1.

The balls are in contact for at least 0.7 msec. We estimate that a shock wave moving with the velocity of sound would take no more than 1.5×10^{-4} sec to travel the diameter of the Superball. This is greater than the collision time by a sufficient margin to make our spring mass model a good approximation. But one could probably not make this assumption for masses composed of steel. It would be very interesting to perform experiments with magnetized steel ball bearings.

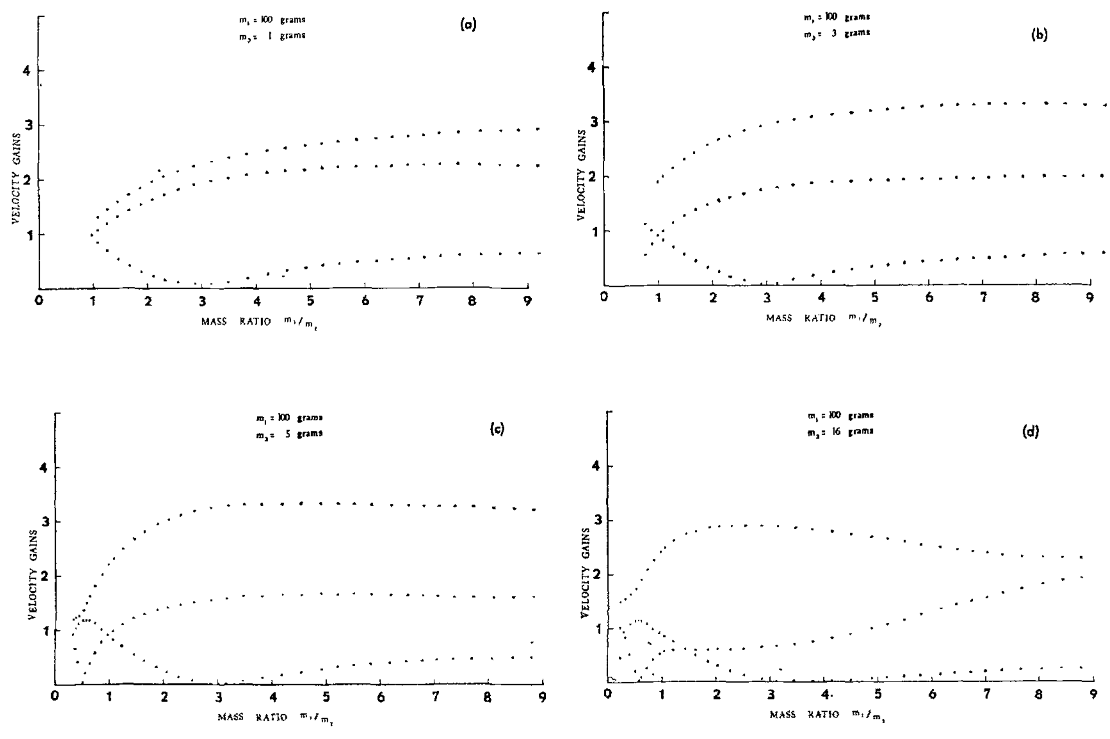

Fig. 15. (a)–(d) Analog computer output for velocity gains of three-ball system.

IV. MULTIPLE-STAGE COLLISIONS

According to the independent-collision model, a gain in velocity of 3.0 is the upper limit for a pen and ball system. We leave it to the reader to show that the same theory predicts a maximum possible gain of 2^n-1 for an n-stage tower of balls.

Furthermore, if the ratio of the masses of nearest neighbors is

$$m_j/(m_{j+1}) = (j+2)/j,$$

the independent-collision model predicts the final velocity of all the balls will be zero except for the top one, which will speed away at n times the initial velocity.

To test those predictions we constructed two arrangements of three balls pictured in Fig. 14.

In the first arrangement [Fig. 14(a)] a $\frac{1}{8}$ in. rod served as a guide upon which the smaller Superballs, which were fitted with metal sleeves, could slide easily.

With these arrangements, velocity gains for the smallest ball over 3.0 have been observed, but the independent-collision model does not give good predictions of the behavior of either system.

We used the analog computer to solve the three-ball problem, and one set of results for an initial velocity of 200 cm/sec is shown in Fig. 15. Note that a peak gain of 3.3 was obtained for the smallest ball when $m_1=100$ g, $m_2=23$ g, and $m_3=5$ g. Subsequently we found that the computer predicted a gain of 4.05 for the smallest ball when the initial velocity was 400 cm/sec and $m_1=100$ g, $m_2=17$ g, and $m_3=3$ g.

We were not able to verify these predictions since our laboratory room was only 12 ft high, and we were not able to move the apparatus elsewhere.

However, these results correspond well with Hart and Herrmann's[6] mechanical analog of an acoustical horn. In their formulation they claim that if masses $m_1, m_2, m_3, \cdots$ are hung together in a horizontal line, as in the famous Newton momentum conservation apparatus, then the greatest energy transfer occurs when the following relations hold.

$$m_2=(m_1m_3)^{1/2}, \quad m_3=(m_2m_4)^{1/2}, \quad m_4=(m_3m_5)^{1/2}\cdots.$$

The formula $m_2=(m_1m_3)^{1/2}$ gives the high point of each of the outputs in Fig. 15.

Clearly these formulas can greatly simplify the search for the optimum launch that involves three or more masses.

ACKNOWLEDGMENT

We would like to thank John C. Fakan, John E. Heighway, and John H. Marburger for help during the initial and final stages of this project.

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[1] Trade name of product by Whammo Manufacturing Co., San Gabriel, Calif.

[2] A similar effect was discovered independently by W. R. Mellen, Amer. J. Phys. **36**, 845 (1968).

[3] In Fig. 9 one can see the form these curves would take if these additional collisions are considered.

[4] G. A. Korn and T. M. Korn, *Electronic Analog Computers* (McGraw–Hill, New York, 1960).

[5] Difficulties with scaling and overload can easily be overcome as shown in Ref. 4. Also, simpler programs can be constructed. However, the one shown can be immediately made to accommodate additional stages of balls as in Sec. IV. Linear frictional forces could easily be simulated, too.

[6] J. B. Hart and R. B. Herrmann, Amer. J. Phys. **36**, 46 (1968).