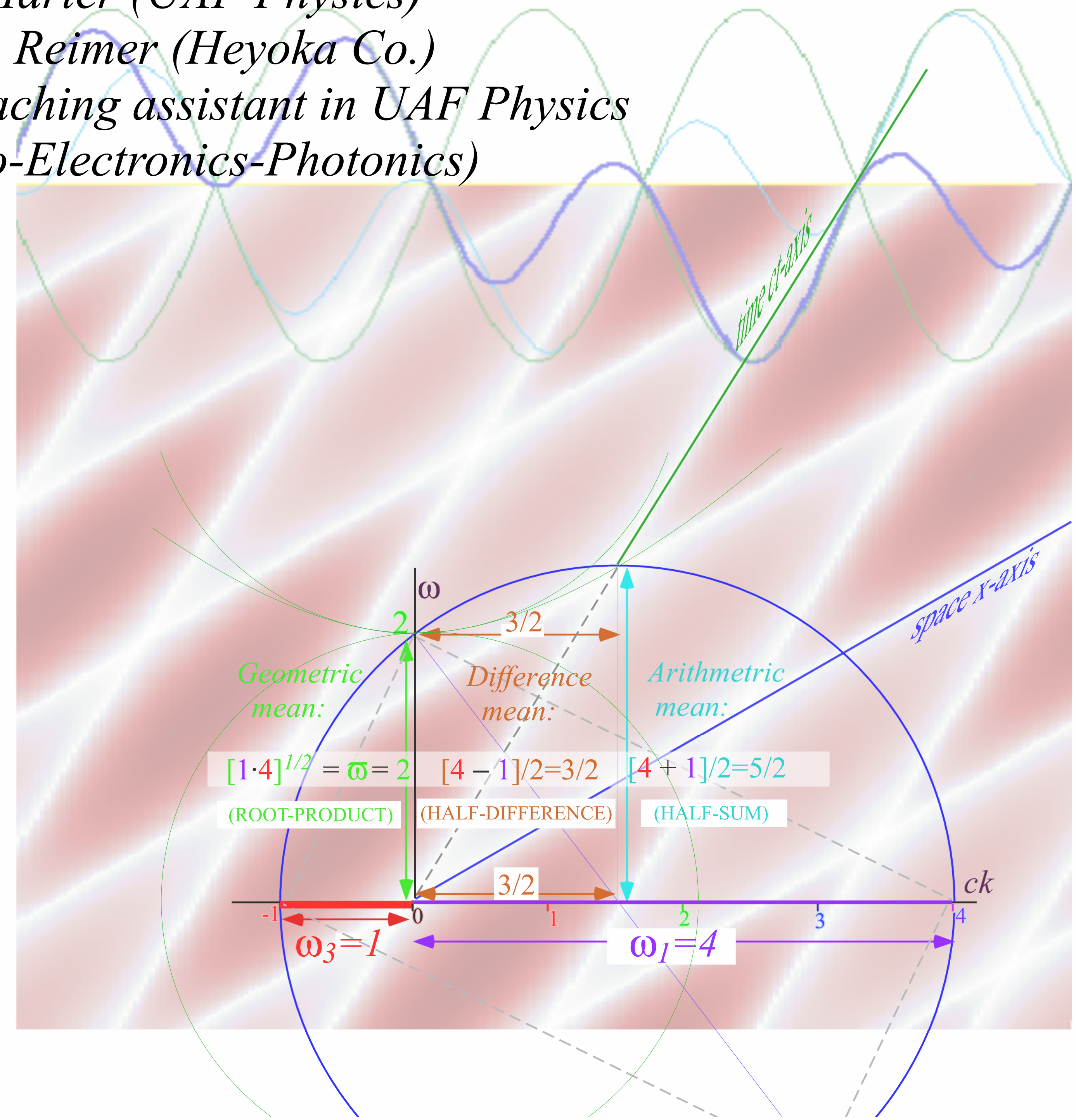
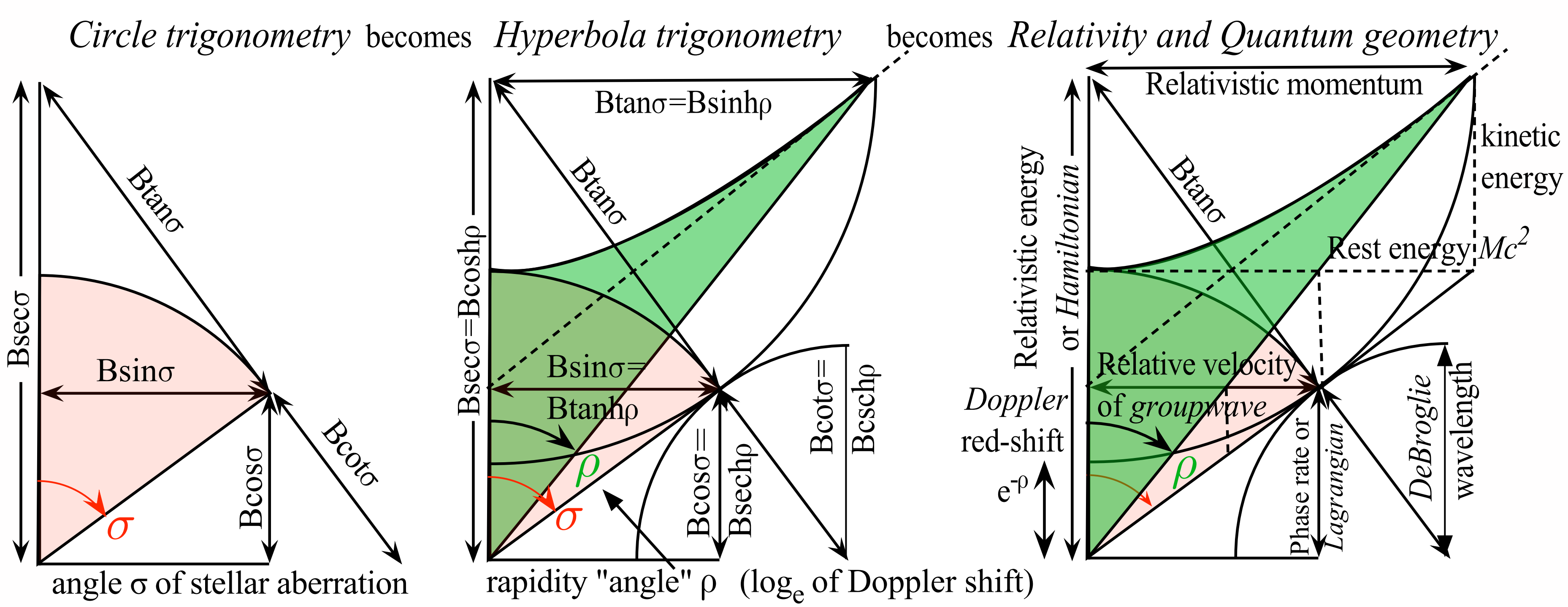
Relawavity: Simple trigonometry leads to understanding of relativity and quantum theory
INBRE 2016

Presentation and production by
W. G. Harter (UAF Physics)
Dr. T.C. Reimer (Heyoka Co.)
Al Calabrese (Teaching assistant in UAF Physics and Micro-Electronics-Photonics)

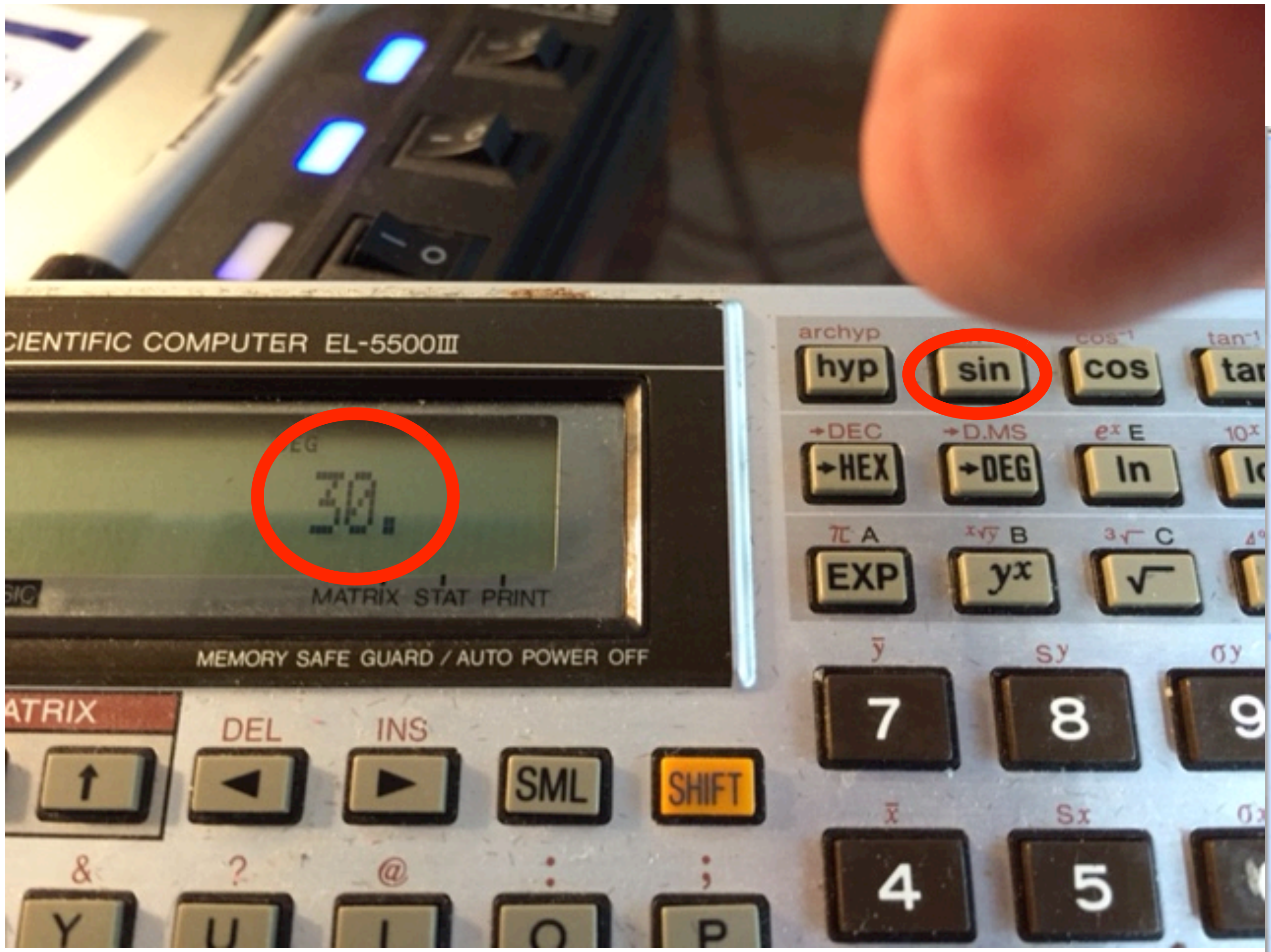


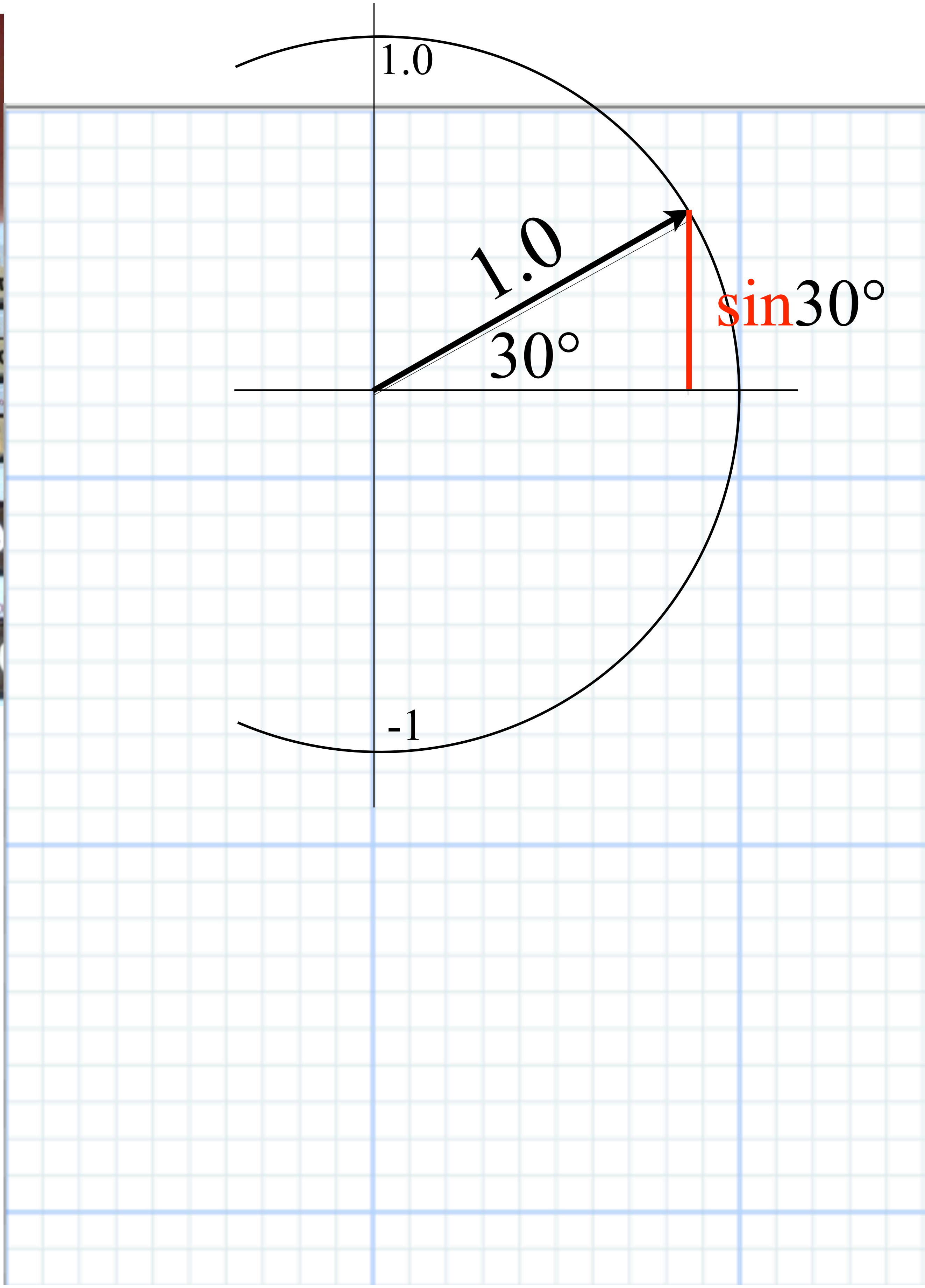
2016 INBRE Workshop - UAF Physics Rm 241 - Saturday Oct. 22 at 10 to 11:15 am.
Relawavity: Simple trigonometry leads to understanding of relativity and quantum theory
Workshop by prof. W. G Harter (UAF Physics), Dr. T.C Reimer (Heyoka Co.), and Al Calabrese (Teaching assistant in UAF Physics and Micro-Electronics-Photonics lab).

Modern science from Astrophysics to Zoology depends increasingly on two pillars of modern physics, special relativity (SR) and and quantum mechanics (QM), that are based on properties of light waves. The bioscience renaissance could not happen without the optics of maser, laser, UV, Xray, and synchrotron effects due to SR and QM theory that is still regarded as esoteric mystery. This workshop seeks to demystify SR and QM theory using high school trigonometry of plane light waves. Using diagrams below of circle trig functions ($\sin\sigma, \cos\sigma, \tan\sigma$) and inverses ($\csc\sigma, \sec\sigma, \cot\sigma$) we show each one is also a hyperbolic function that is key to SR and QM. A circle function like sine is a function $\sin\sigma$ of circular sector area σ that astronomers call a *stellar aberration*. $\sin\sigma$ equals a function $\tanh\rho$ of hyperbola sector area ρ that physicists call *rapidity*. (It happens that equality $\sin\sigma=\tanh\rho$ implies $\tan\sigma=\sinh\rho$, and similarly, $\csc\sigma=\coth\rho$ implies $\cot\sigma=\text{csch}\,\rho$. Finally, $\cos\sigma=\text{sech}\,\rho$ implies $\sec\sigma=\cosh\rho$, a key pair.)
A diagram or "roadmap" of the functions follows from space-time plots of wave interference for a pair of laser plane waves colliding head-on. Similar diagrams arise for inverse or per-time-per-space plots (*i.e.*, frequency υ *vs* wave-number κ). The (x,ct)-plots reveal relativistic space-time mechanics while $(c\kappa,\upsilon)$-plots show quantum energy-momentum effects. Both plots vary with ρ or σ and *Relawavity* web apps provide animation of this.
Rapidity ρ is the natural logarithm of Doppler factor $b=e^\rho$ and key to understanding that a laser wave can be blue shifted two indistinguishable ways: (1) Tune up the laser, or (2) Accelerate the laser toward the observer. Red shifts act similarly. From this can be understood the super-constant nature of light speed *c=299,792,458 meters per second.* Both classical and quantum mechanics follow from this that we call **Evenson's Axiom**: *All Colors go c!* The metrological precision revolution began with Ken Evenson's CW laser speed of light measurements in 1972. What is needed now is similar improvement in our *precision of thinking* about quantum optics.

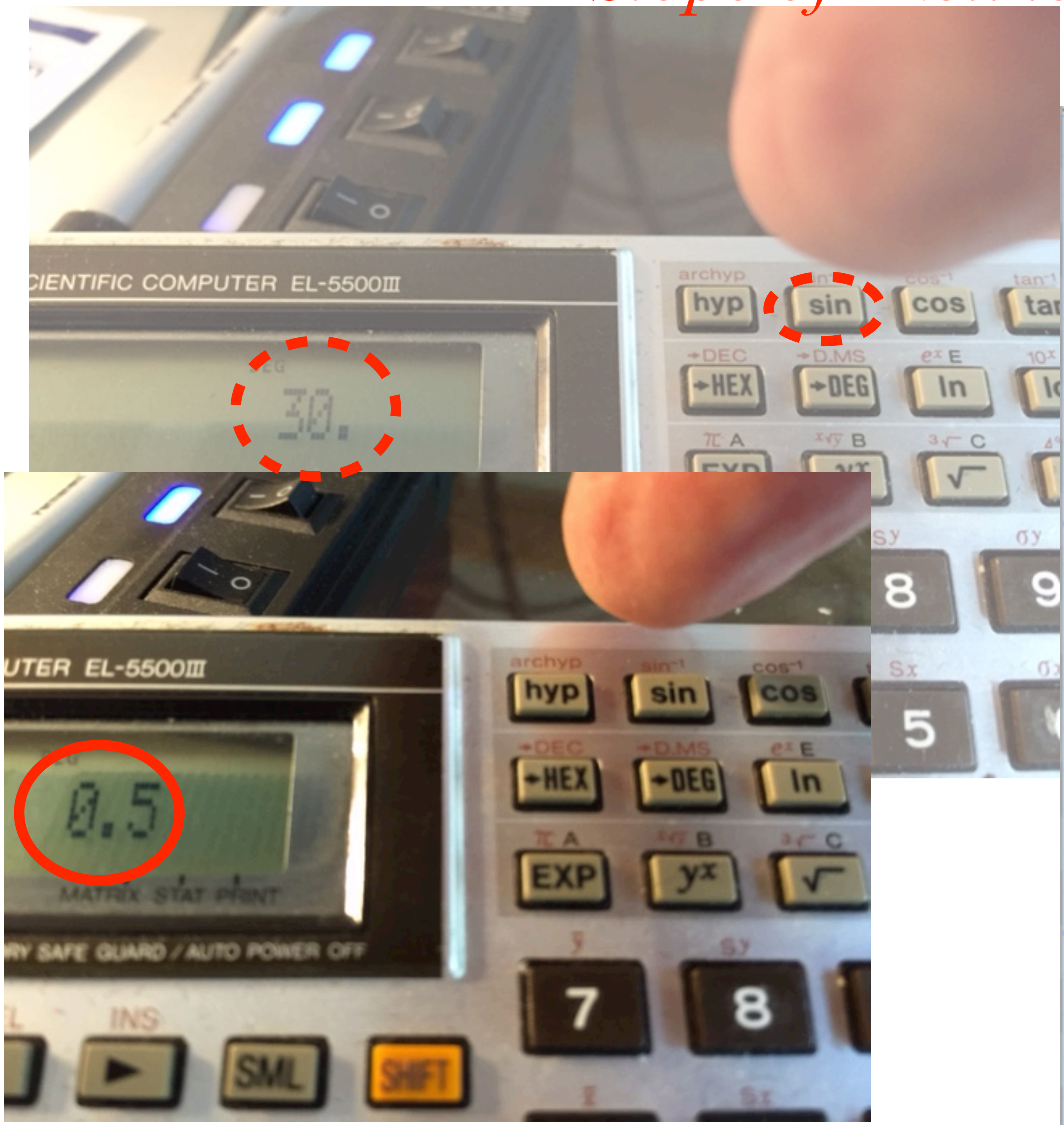


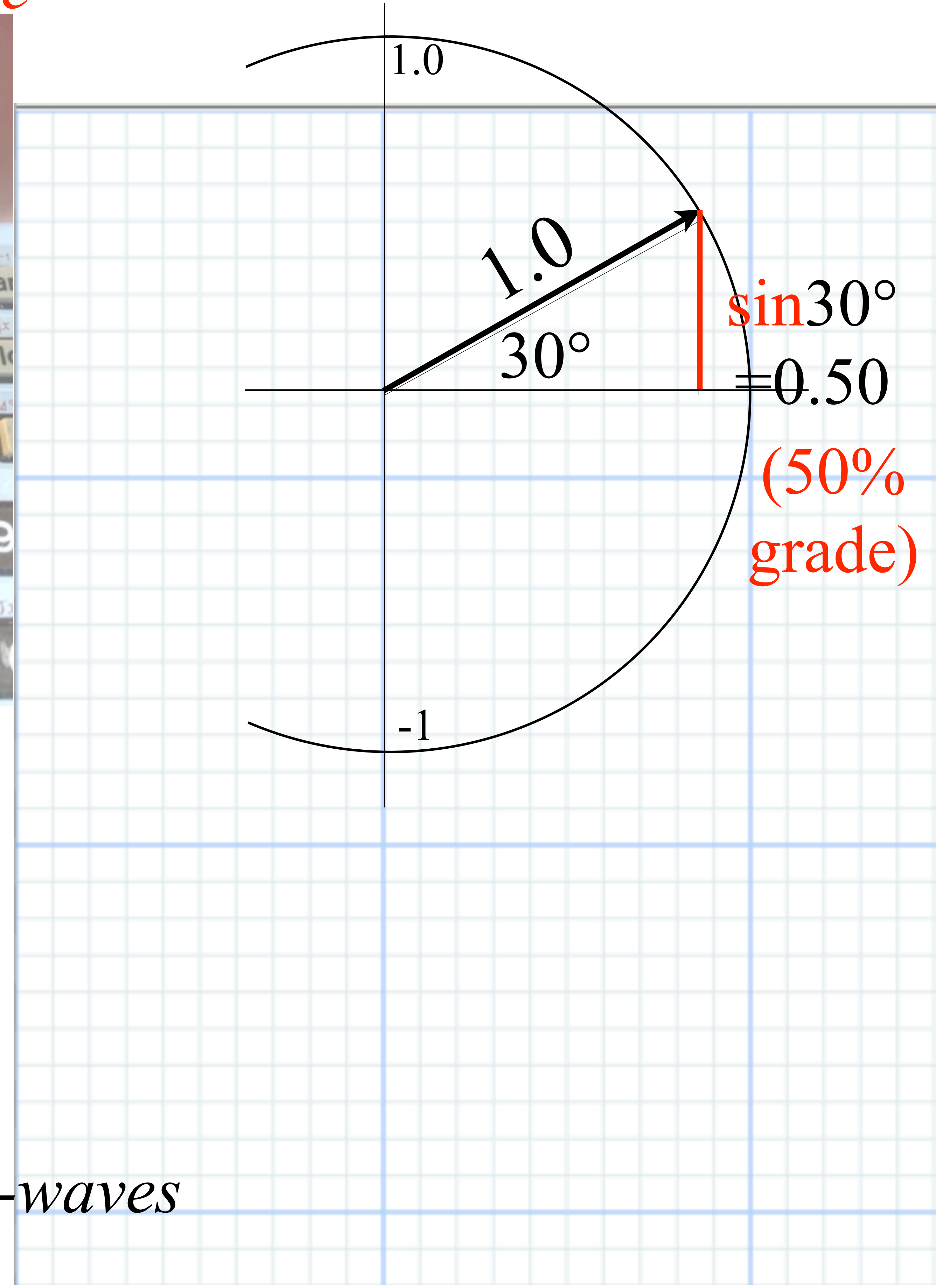
Learning about SIN





Learning about SIN
"Slope of INcline"

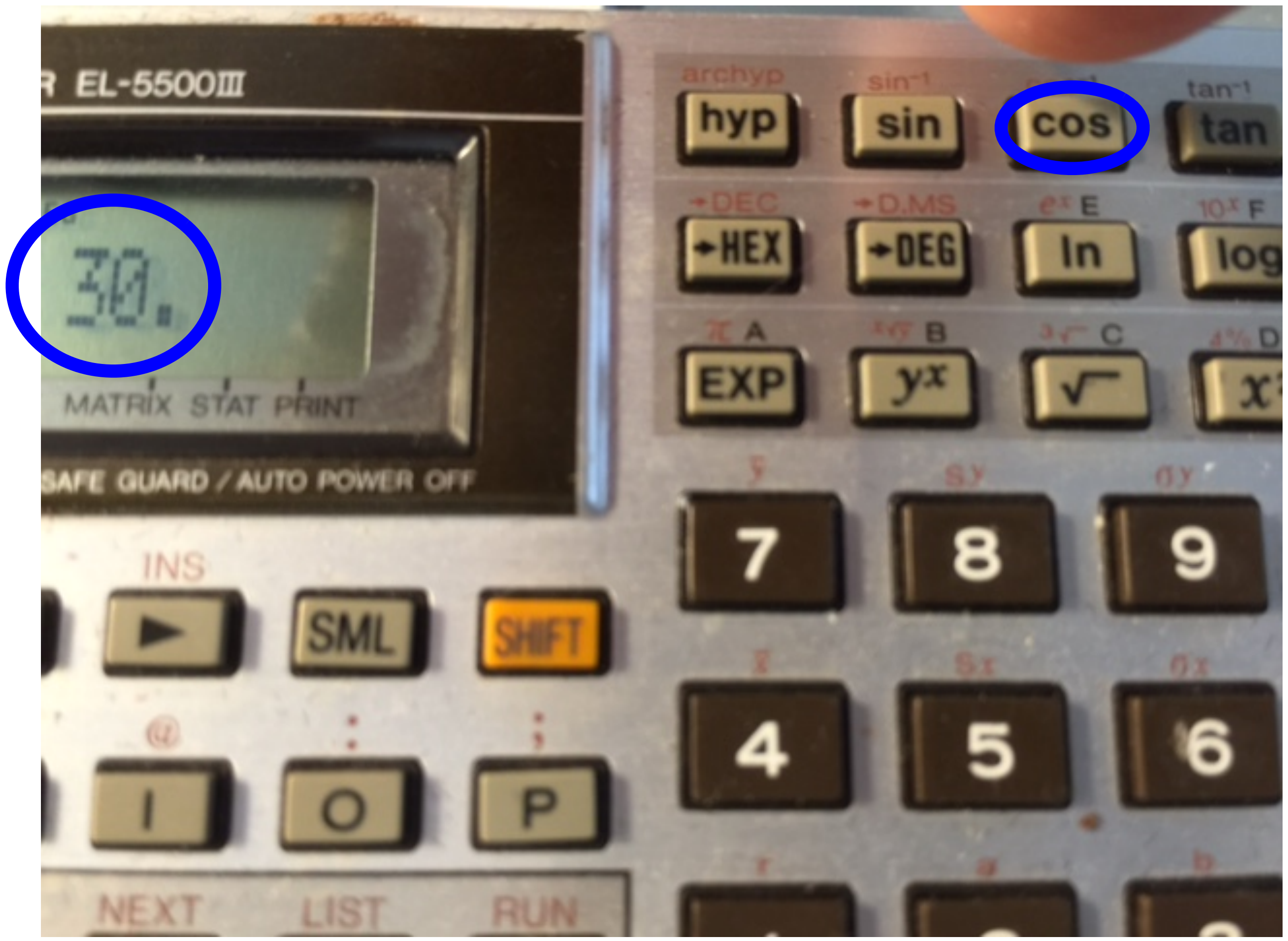


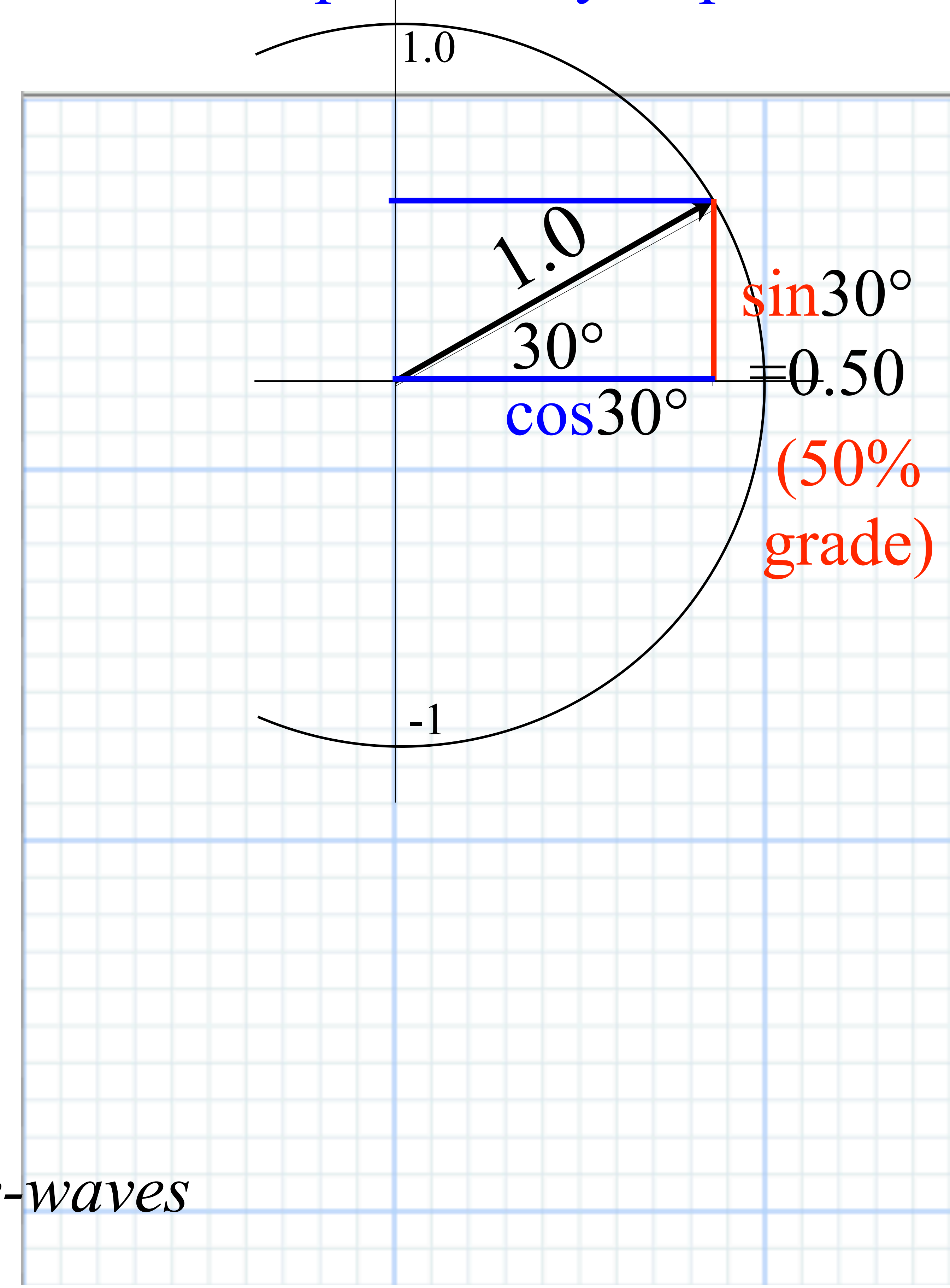


It's mostly about triangles *and sine-waves*

Learning about SIN and the COSin

"*Slope of INcline*" "COmplimentarySlope"





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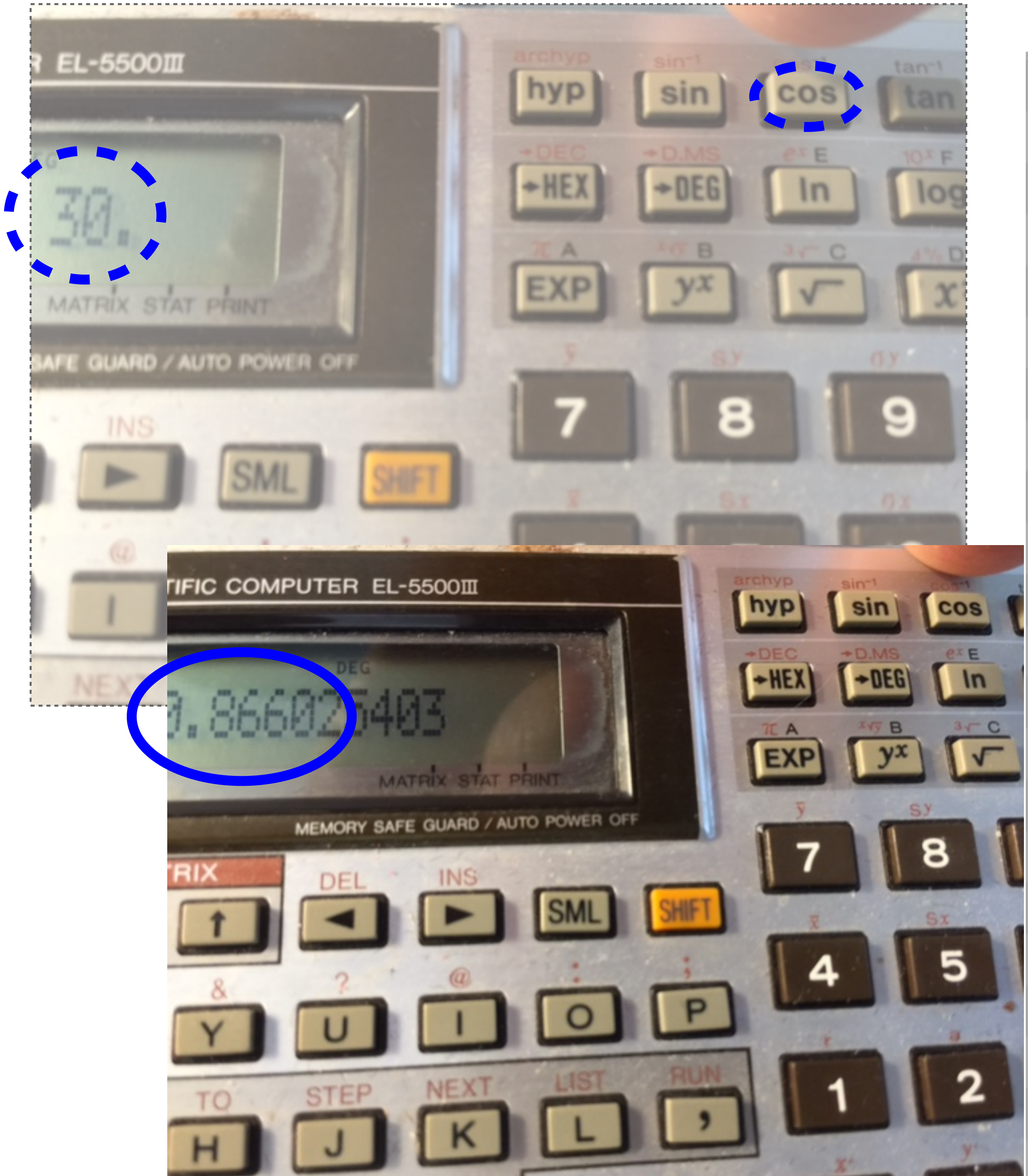

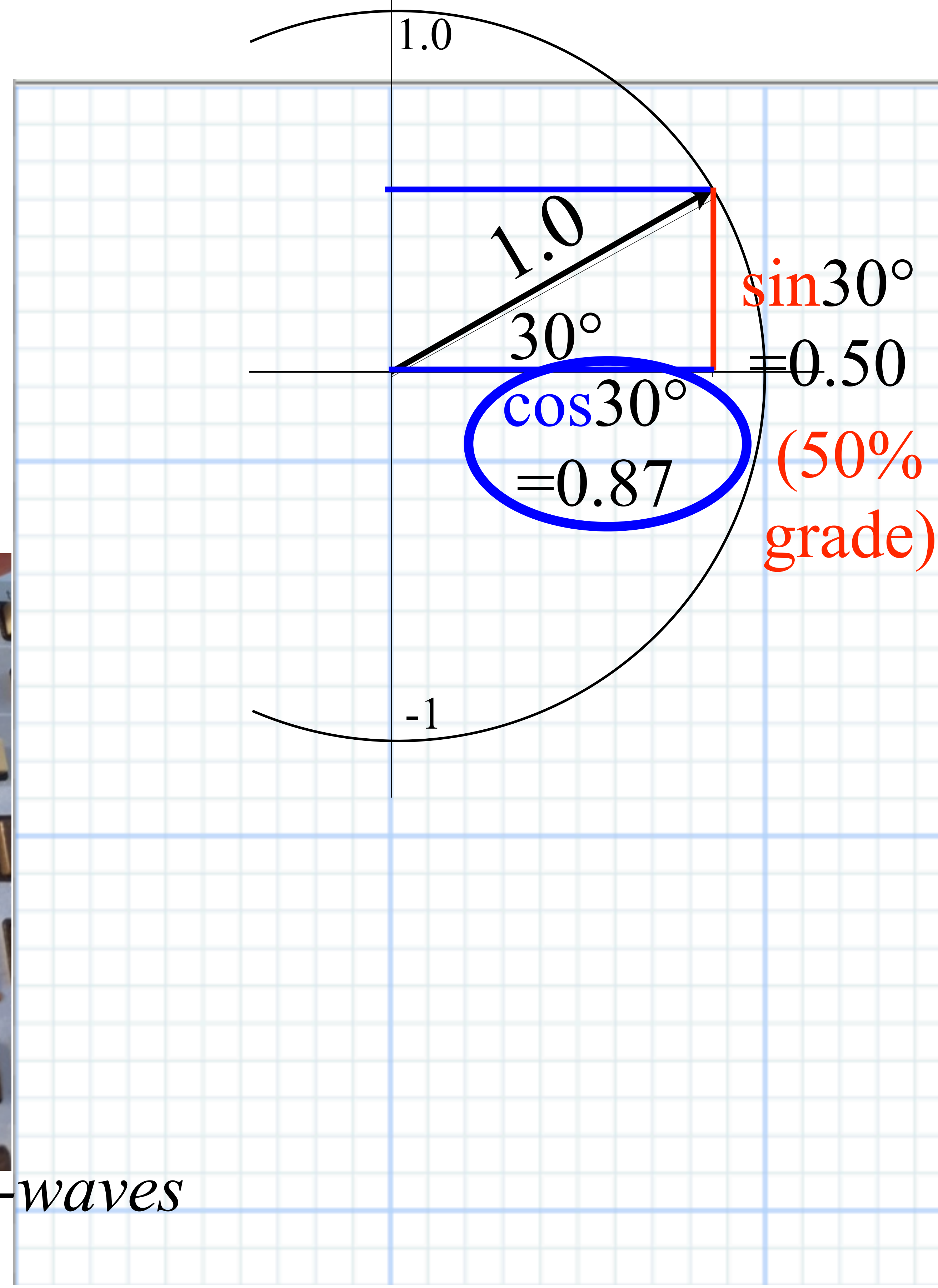

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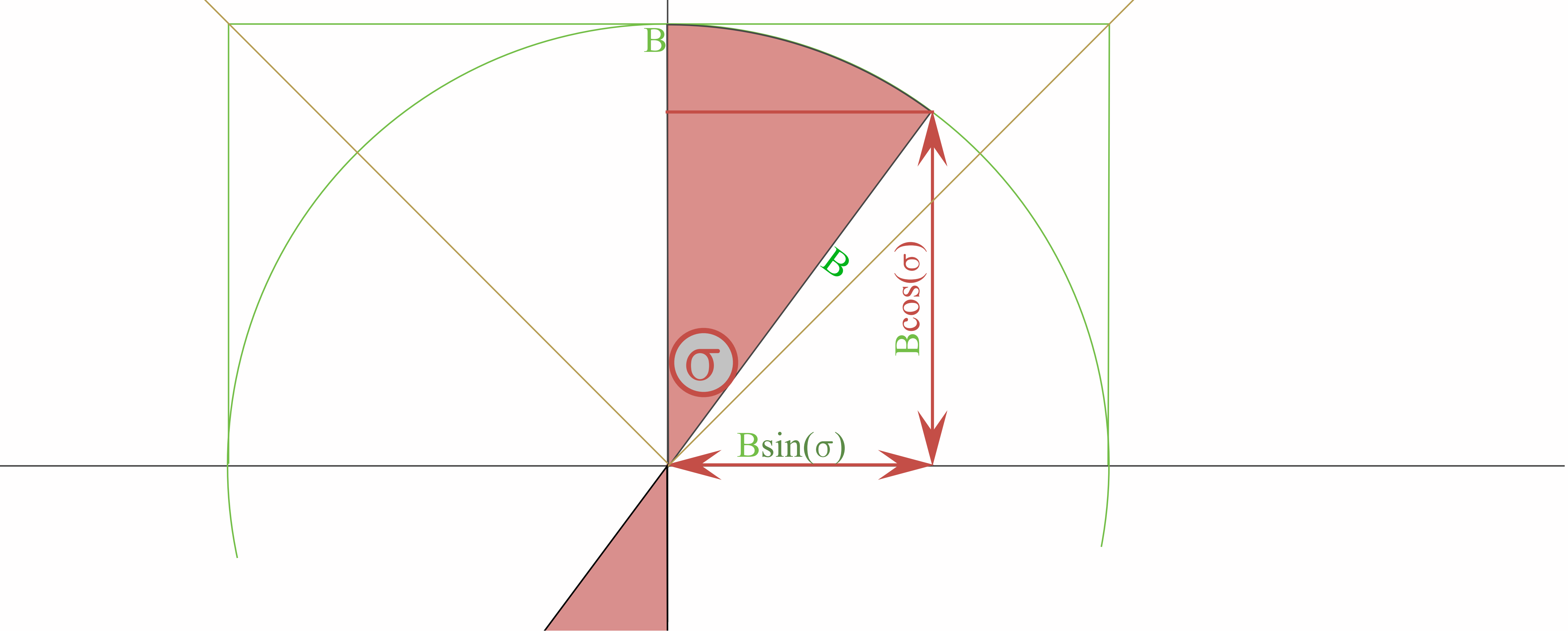
This map has circle sector arc-area $\sigma = 0.6435$

set to angle $\angle\sigma = 36.87° = 0.6435\,radian$

$\sin(\sigma)$=0.6000 = 3/5

$\cos(\sigma)$=0.8000 = 4/5

a small change : we measure <u>angle</u> by sector <u>area</u>



Learning about SIN and the COSin and TANgent

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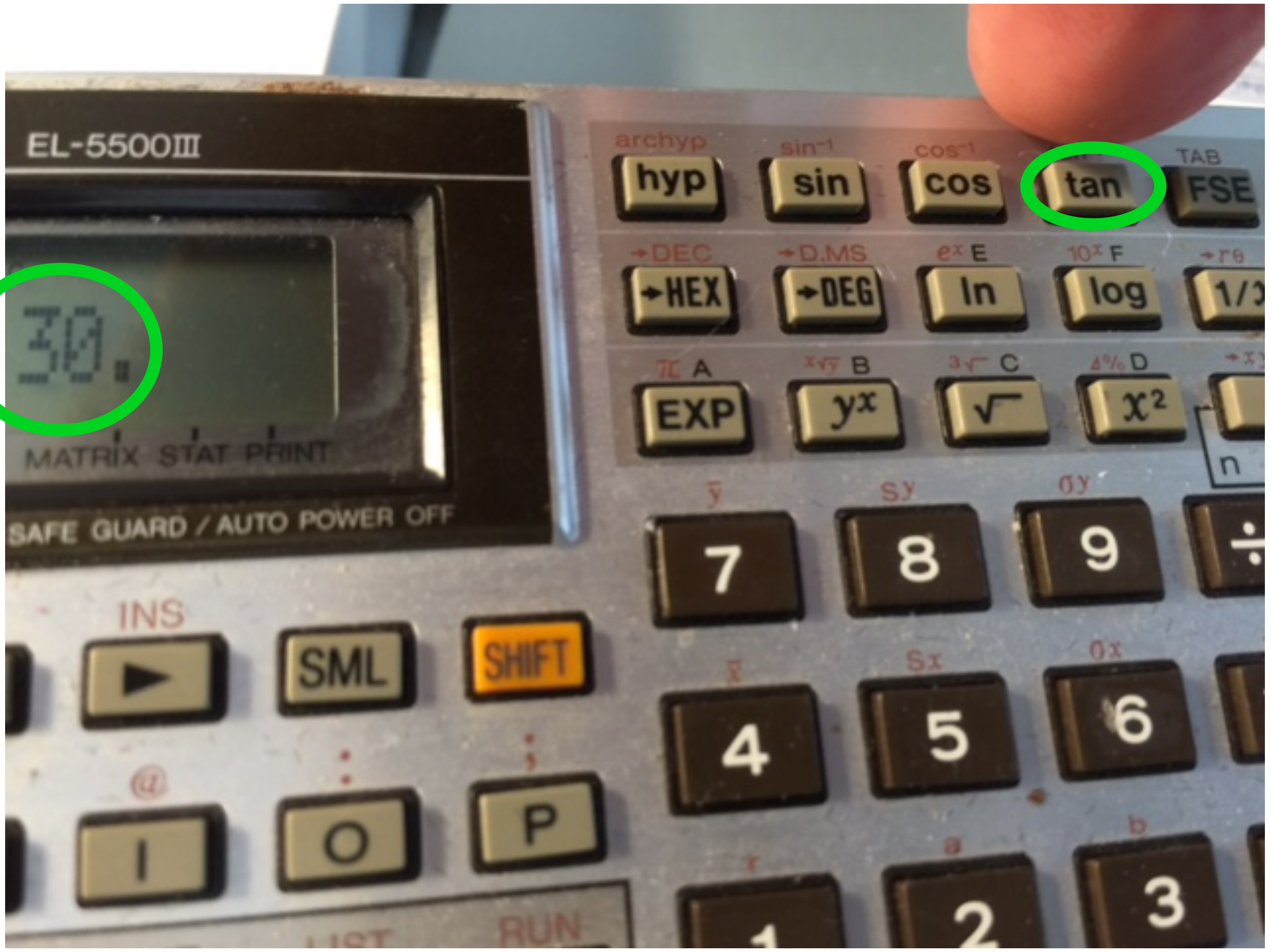

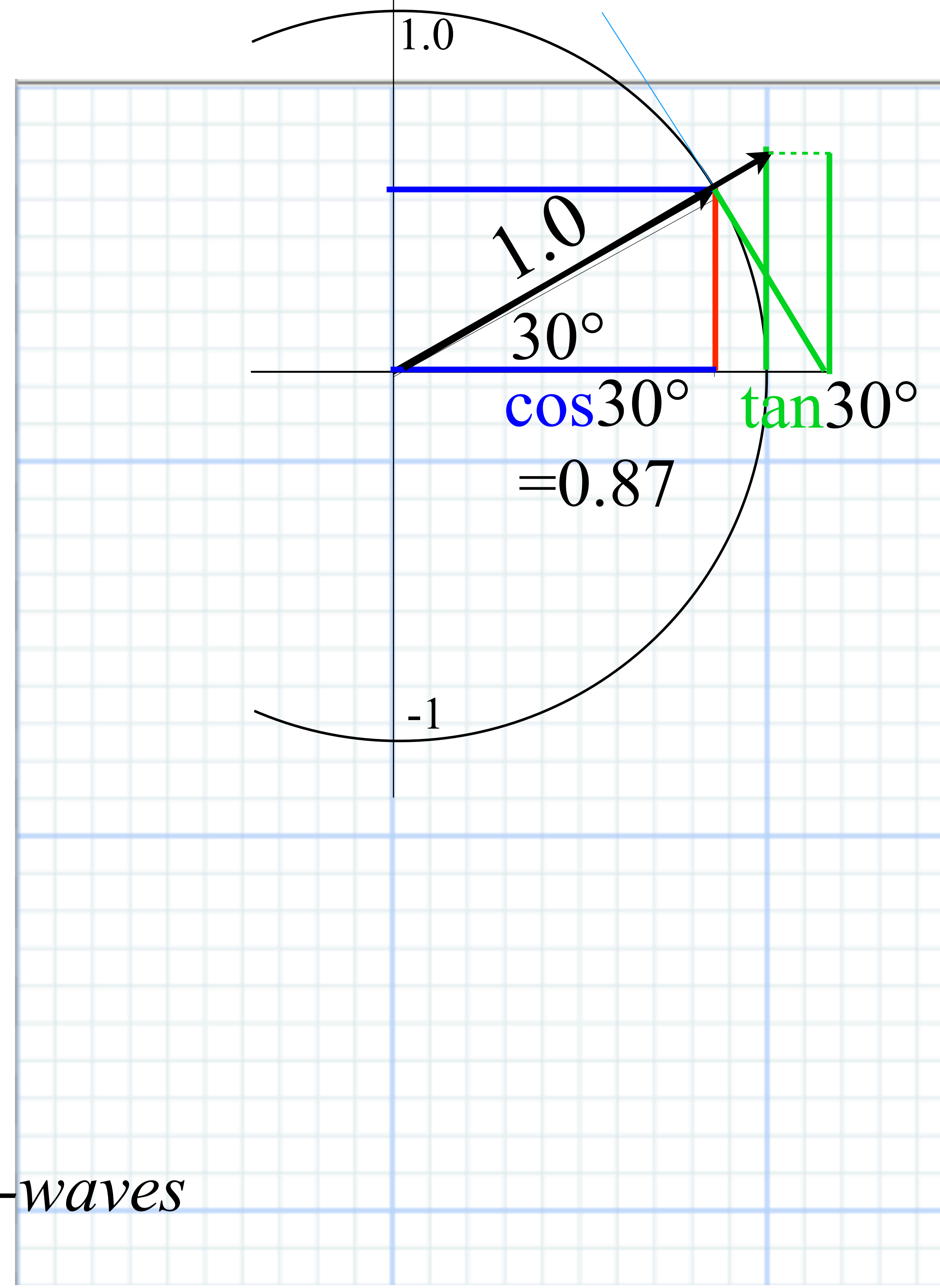

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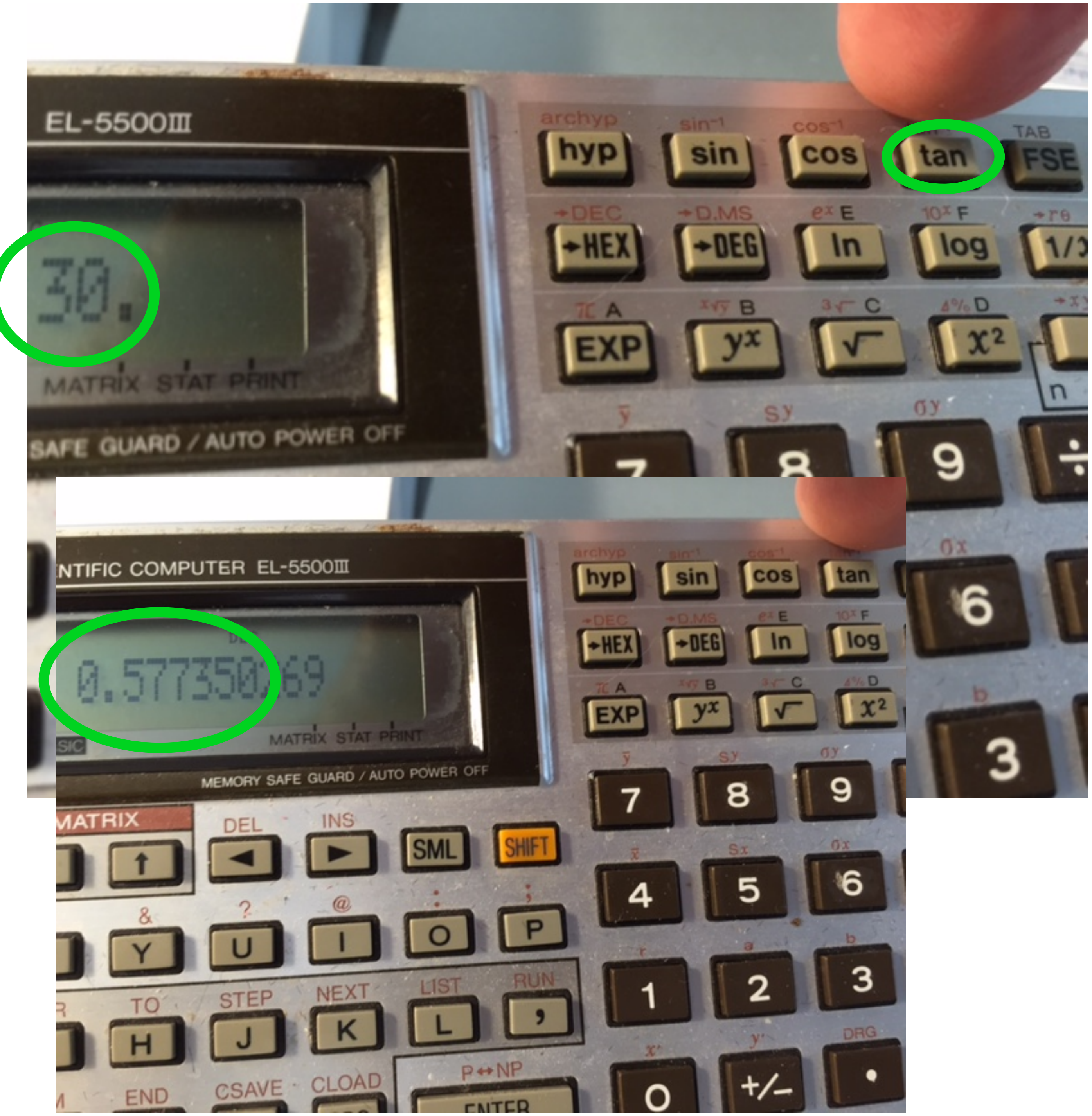

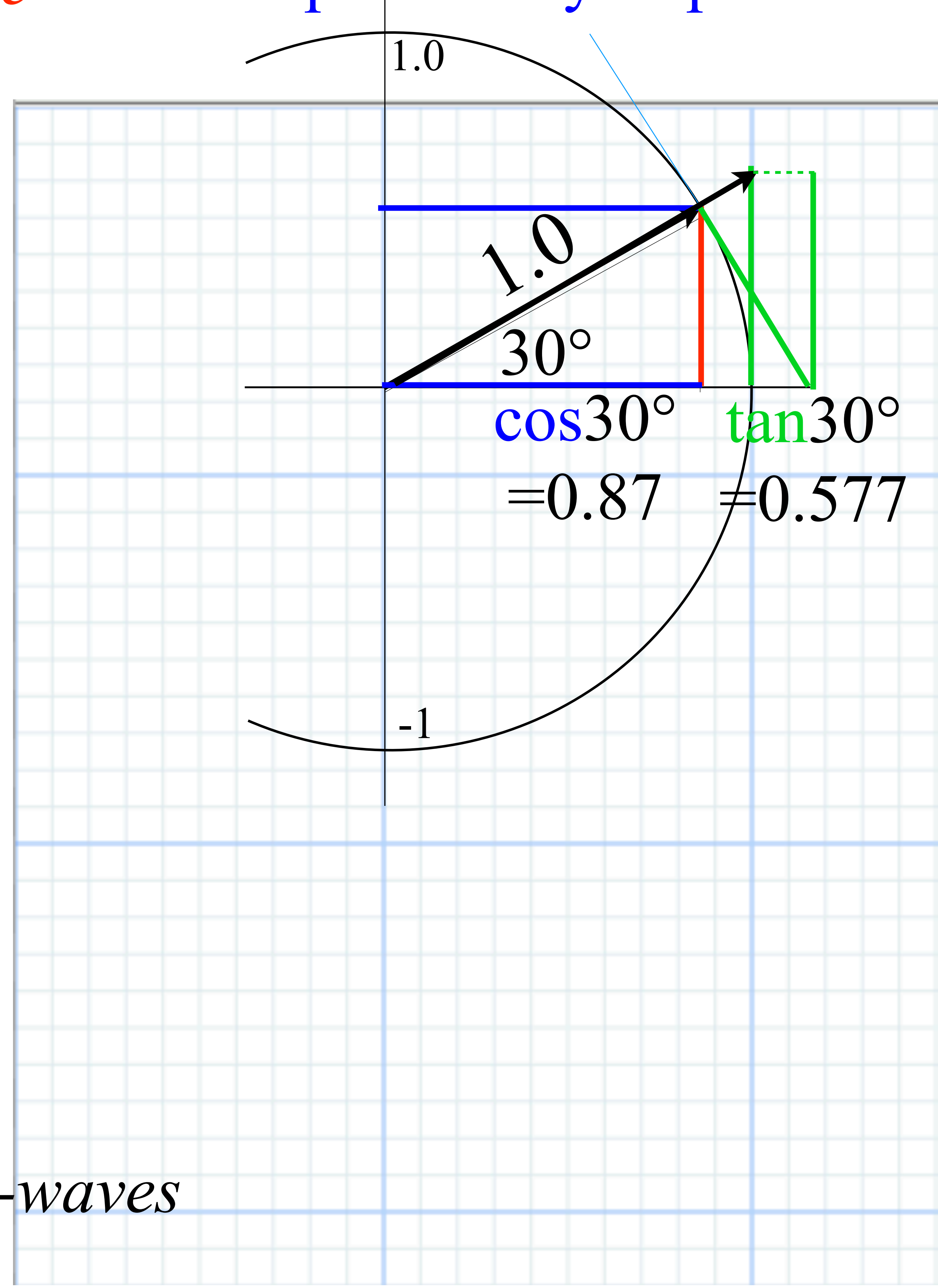

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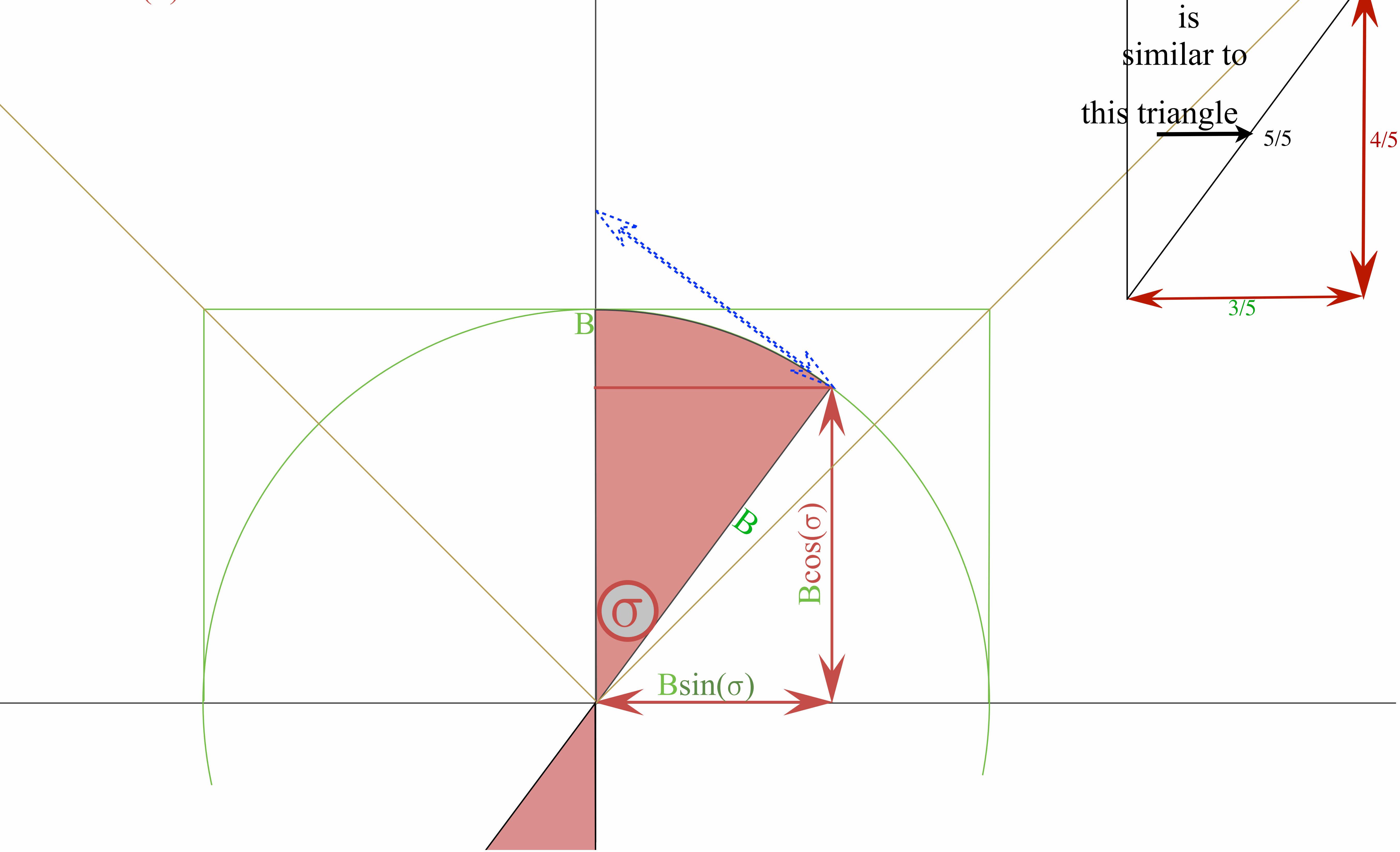

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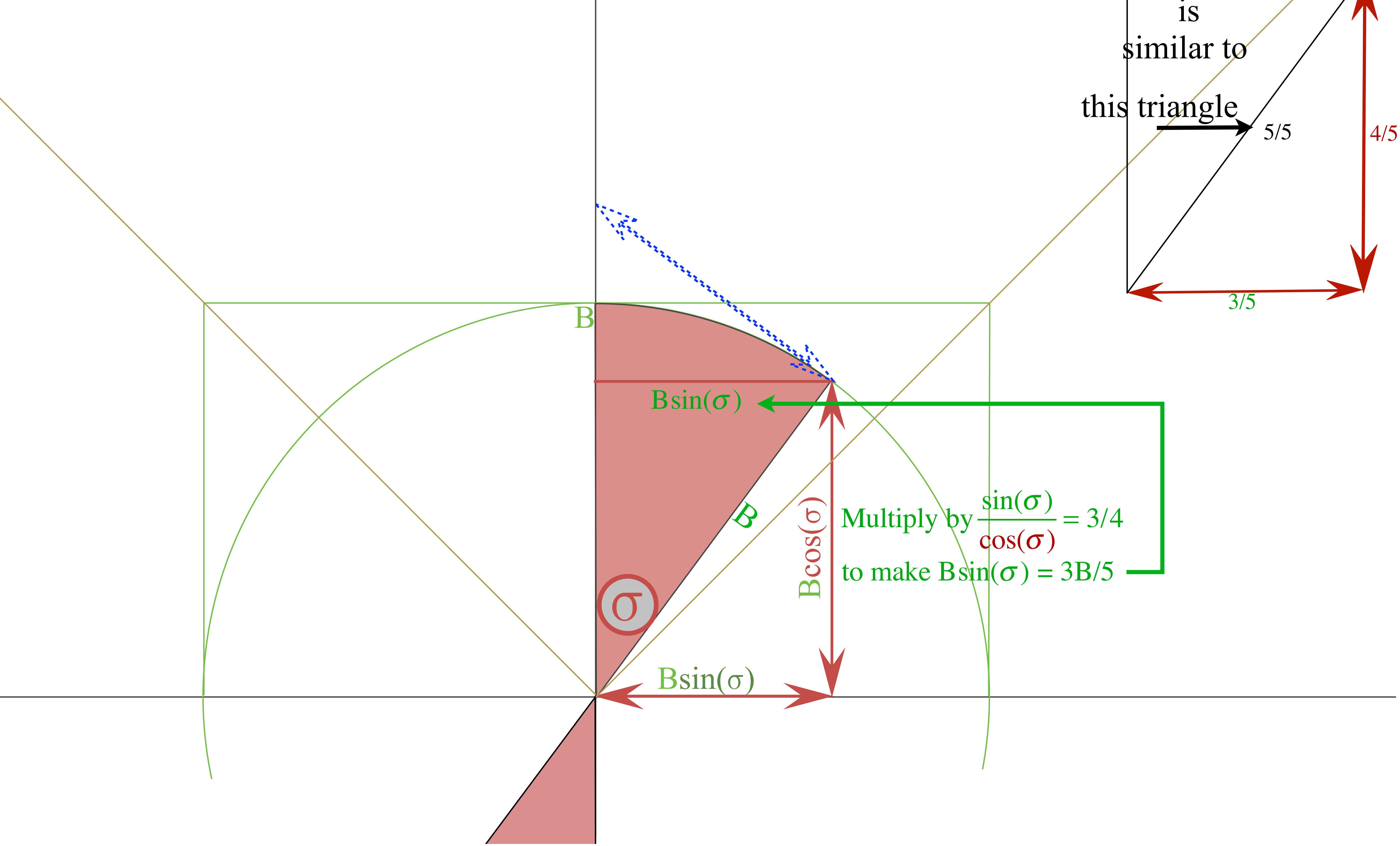

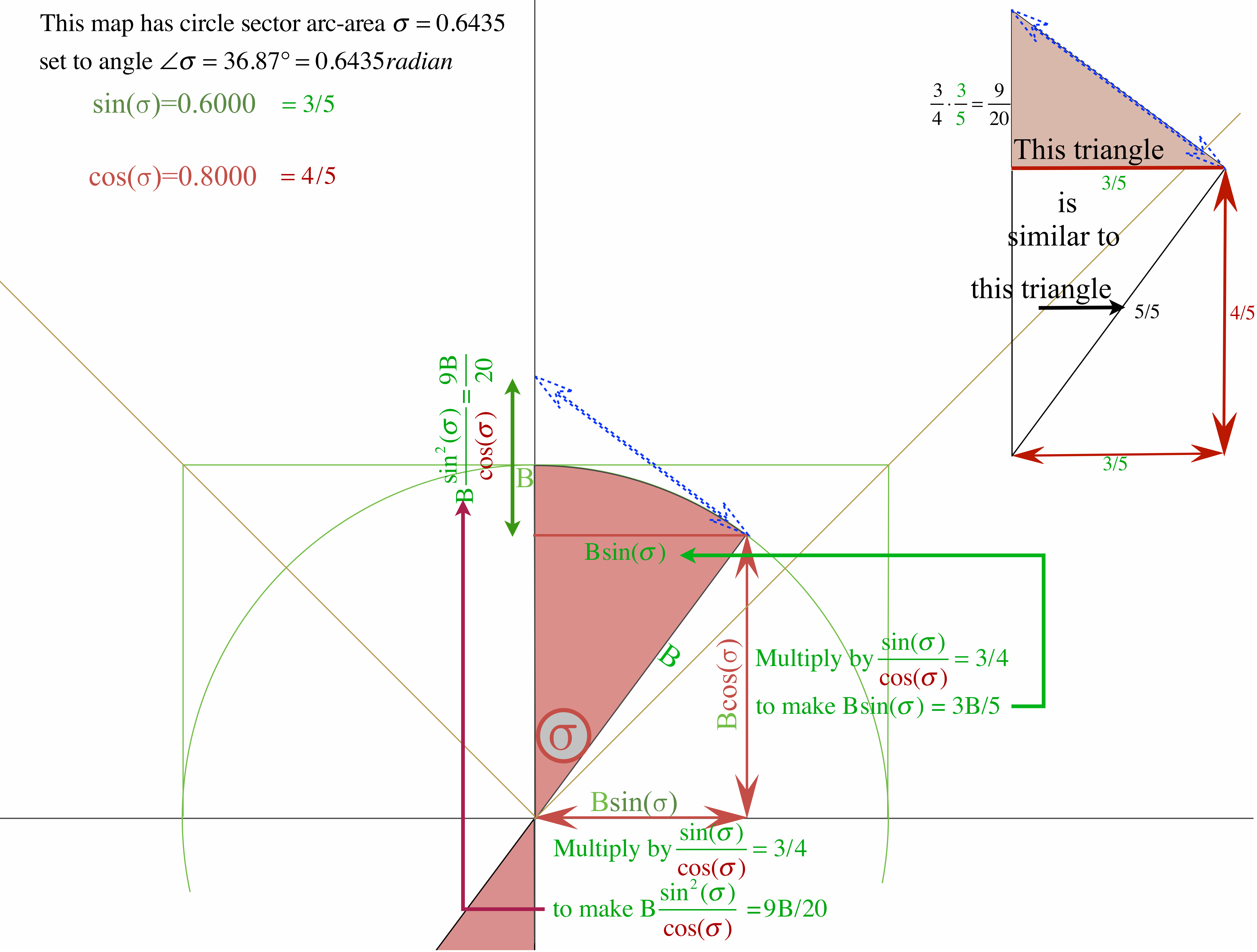
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cos(σ)=0.8000 = 4/5
$\frac{3}{4} \cdot \frac{3}{5} = \frac{9}{20}$
This triangle
3/5
is
similar to
this triangle
5/5
4/5
3/5
$B\frac{\sin^2(\sigma)}{\cos(\sigma)} = \frac{9B}{20}$
B
Bsin(σ)
B
σ
Bcos(σ)
Multiply by $\frac{\sin(\sigma)}{\cos(\sigma)} = 3/4$
to make $B\sin(\sigma) = 3B/5$
Bsin(σ)
Multiply by $\frac{\sin(\sigma)}{\cos(\sigma)} = 3/4$
to make $B\frac{\sin^2(\sigma)}{\cos(\sigma)}$ =9B/20

This map has circle sector arc-area $\sigma = 0.6435$

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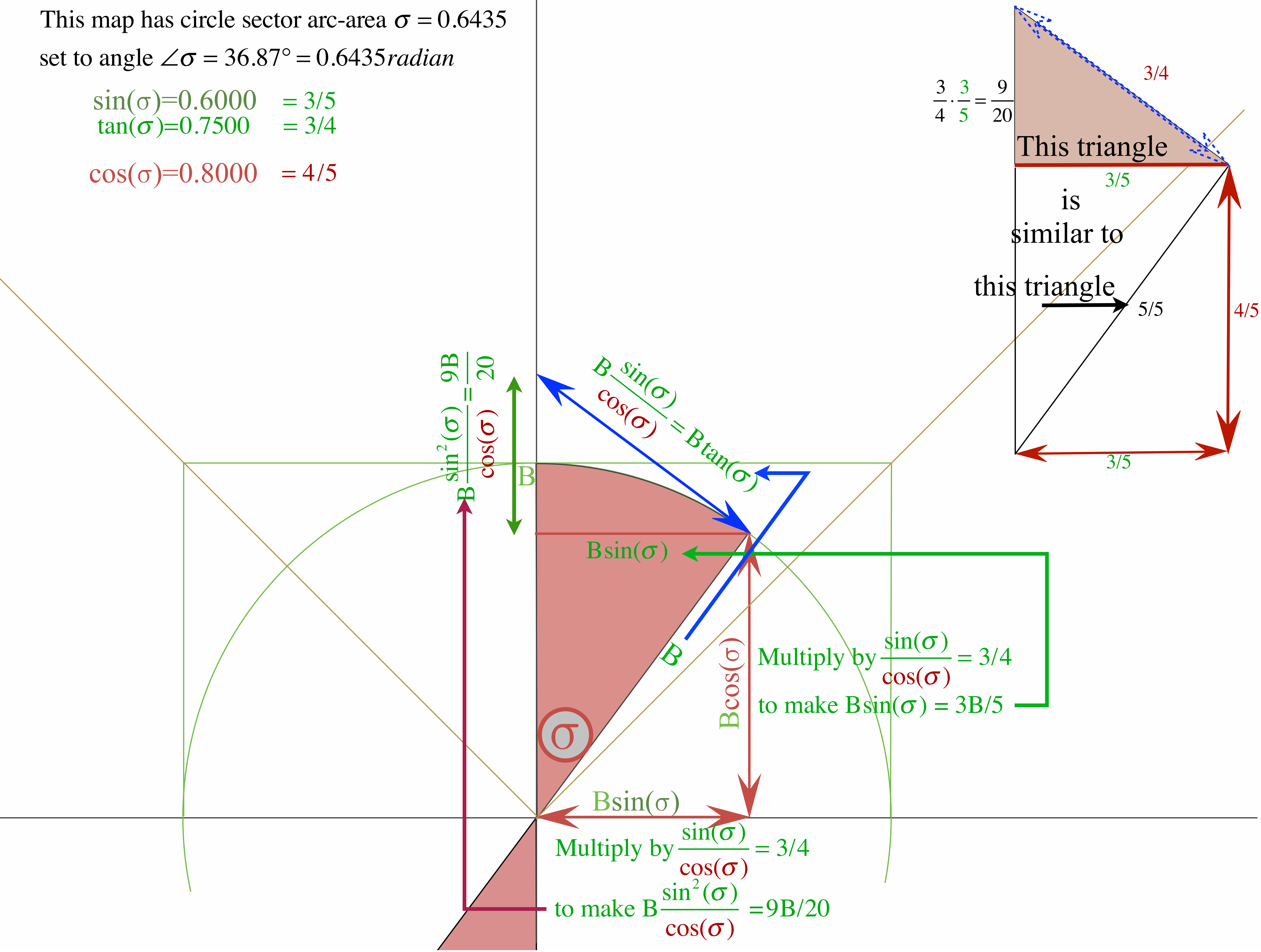

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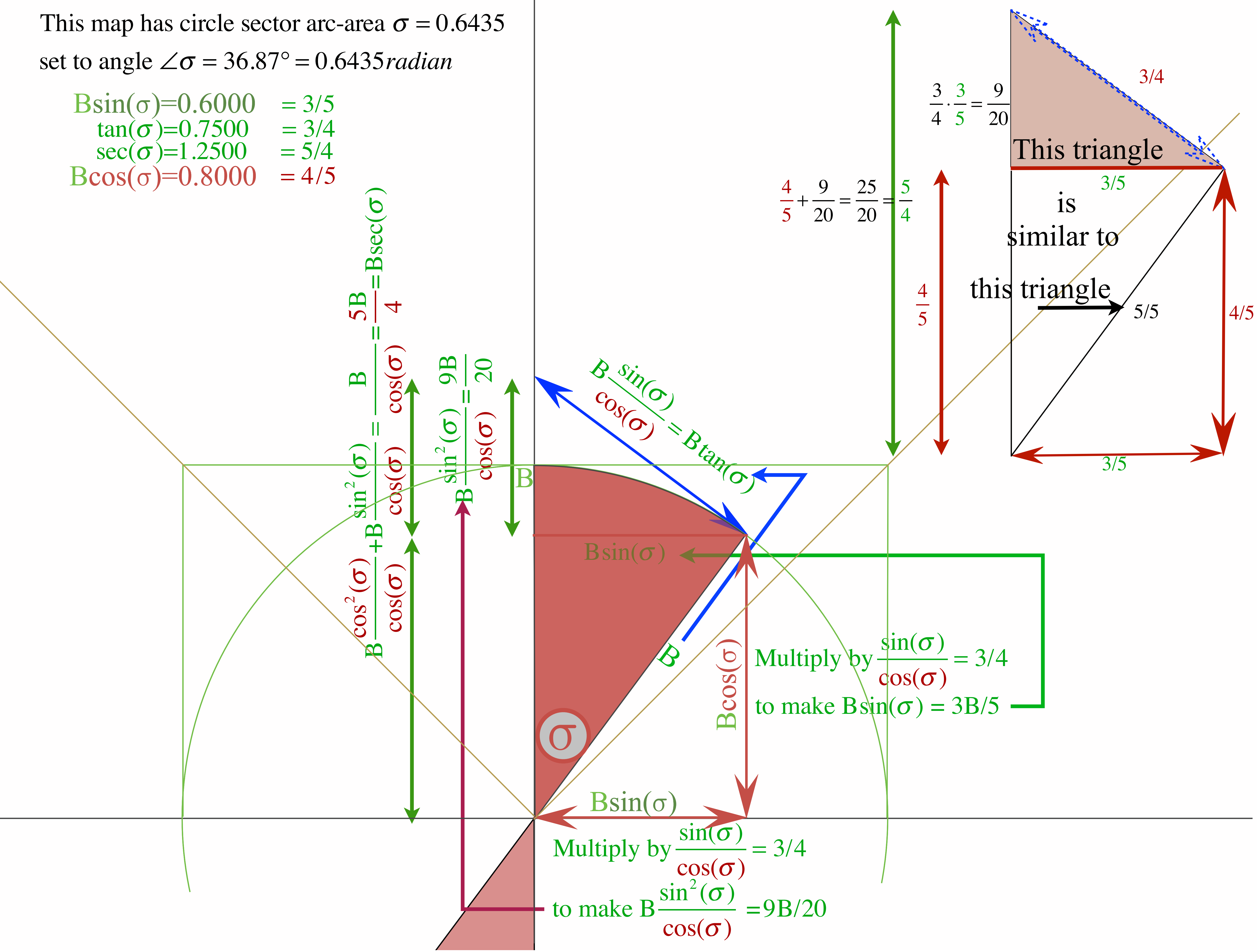

Learning about SIN and the COSin and TANgent and COTangent

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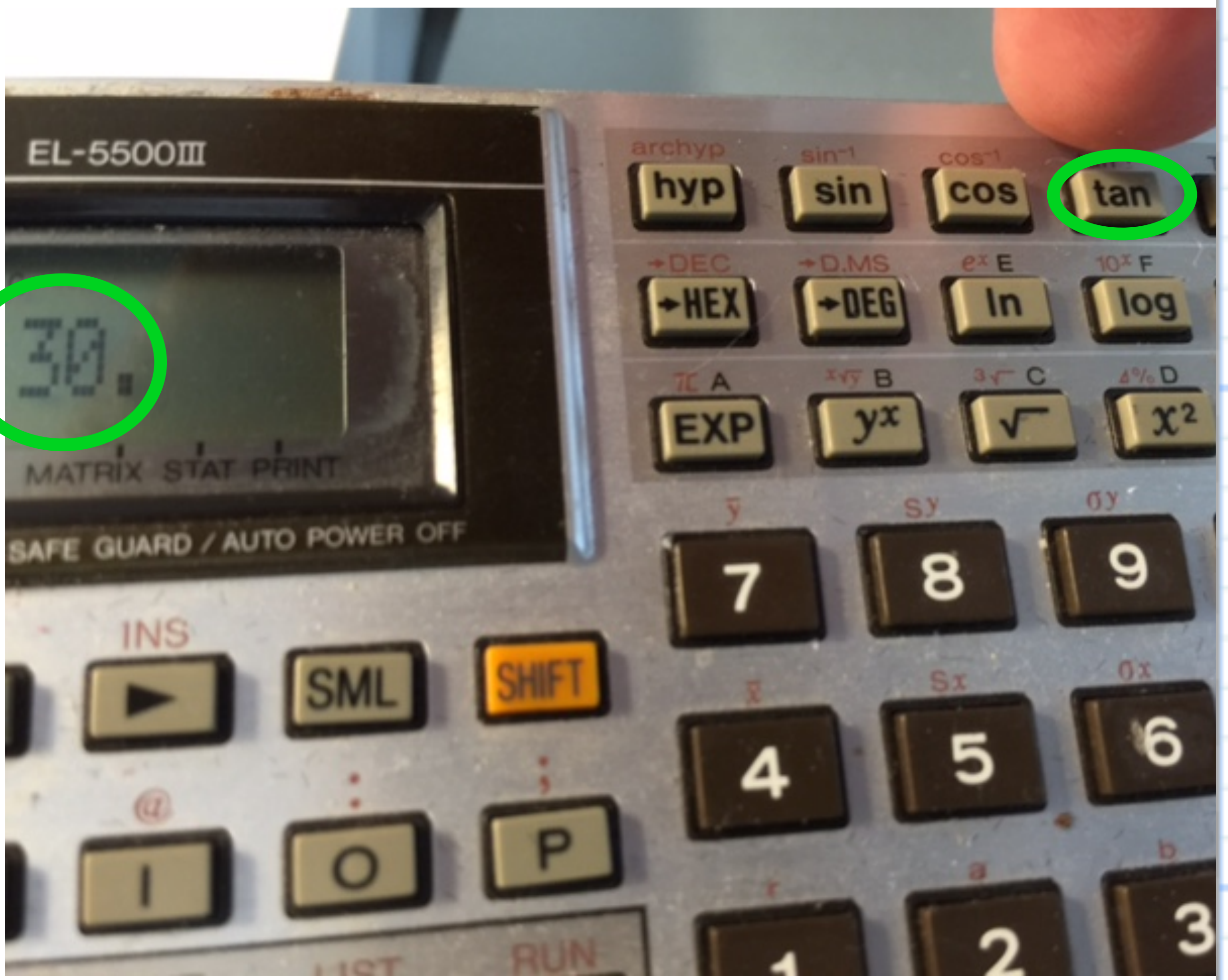

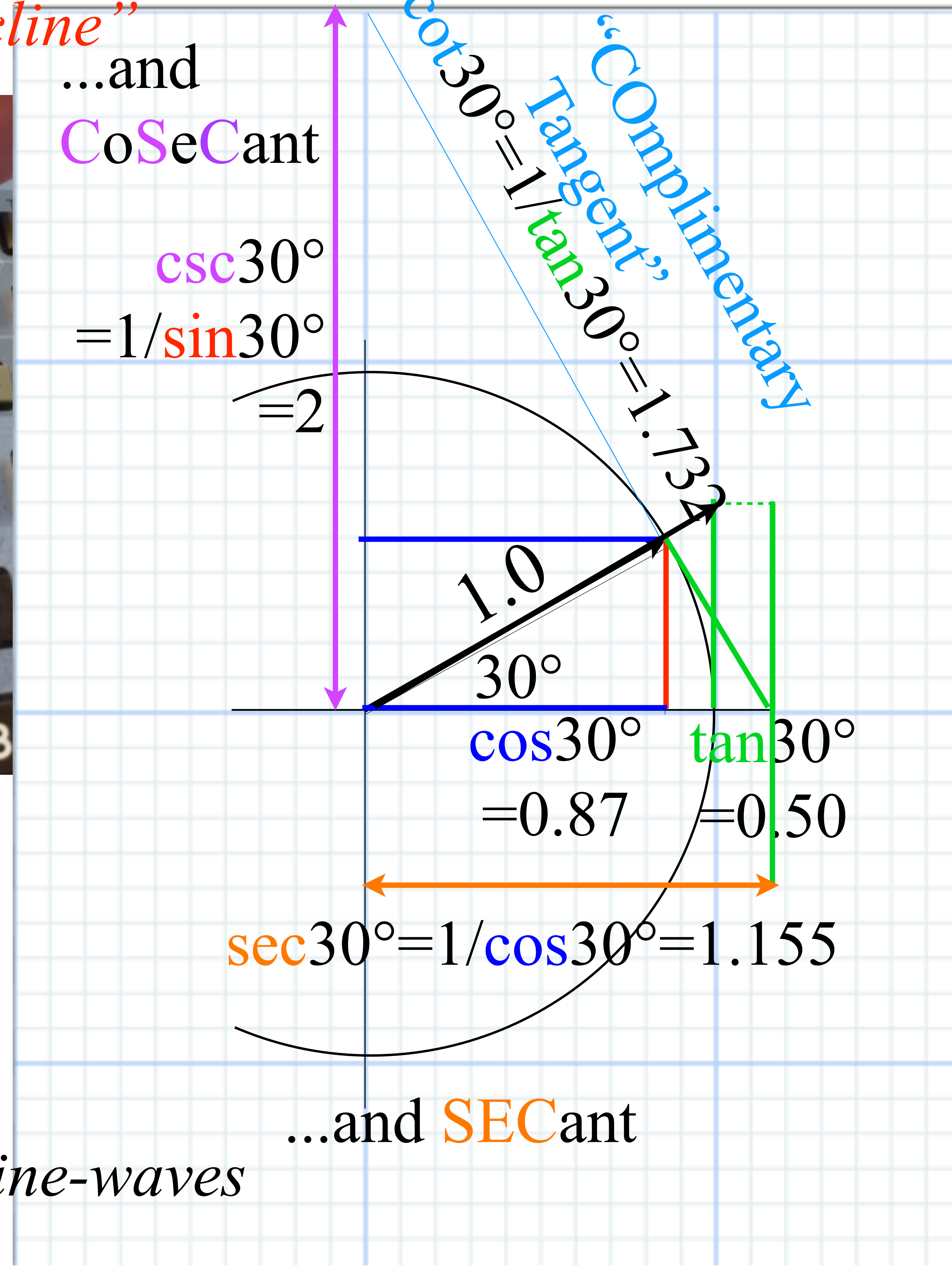

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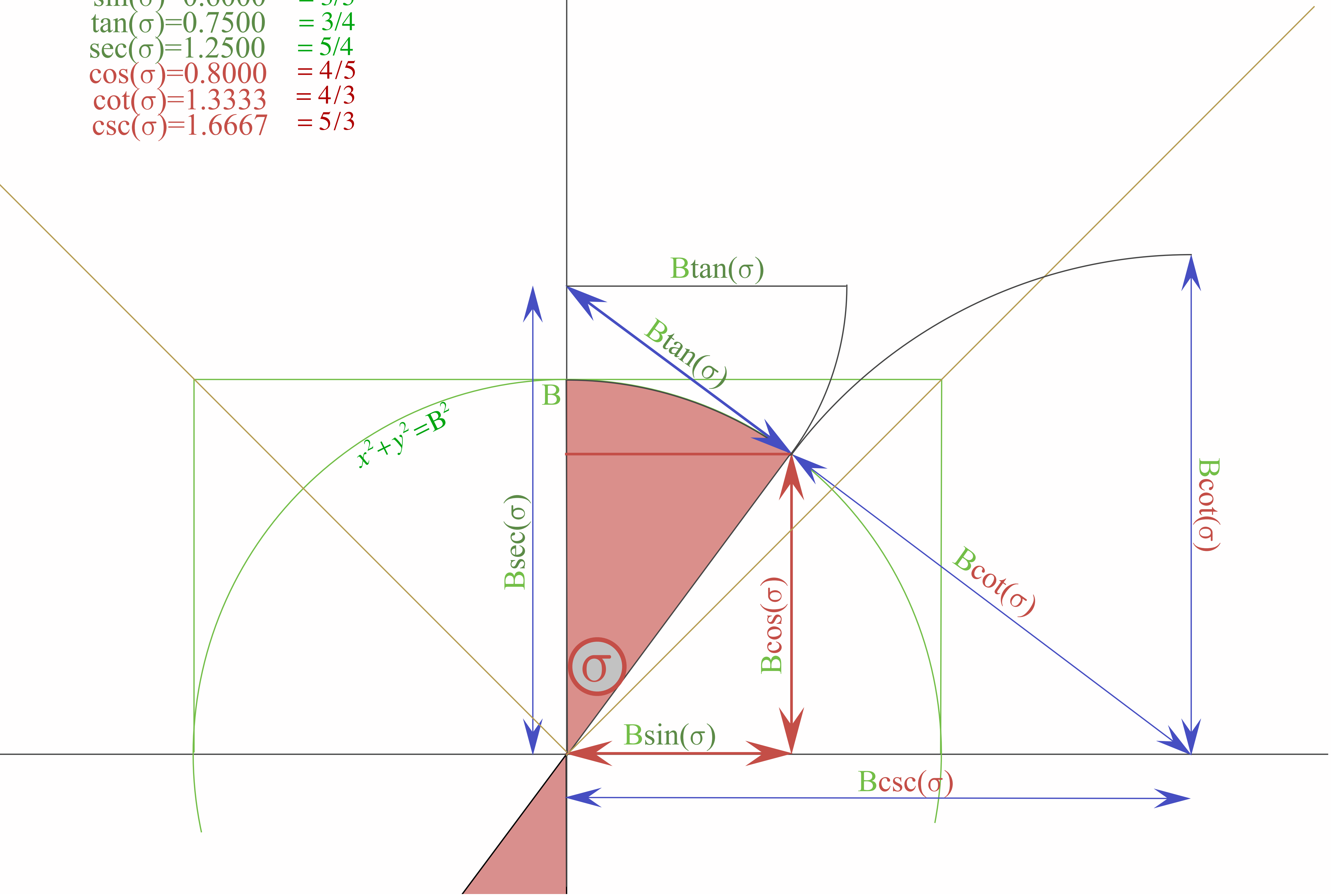

This map has circle sector arc-area $\sigma = 0.6435$

set to angle $\angle\sigma = 36.87° = 0.6435\,radian$

Also it is set to hyperbola sector arc-area $\rho = 0.6931$

angle $\angle\rho = \nu = 30.96°$

sin(σ)=0.6000	=	tanh(ρ)	= 3/5
tan(σ)=0.7500	=	sinh(ρ)	= 3/4
sec(σ)=1.2500	=	cosh(ρ)	= 5/4
cos(σ)=0.8000	=	sech(ρ)	= 4/5
cot(σ)=1.3333	=	csch(ρ)	= 4/3
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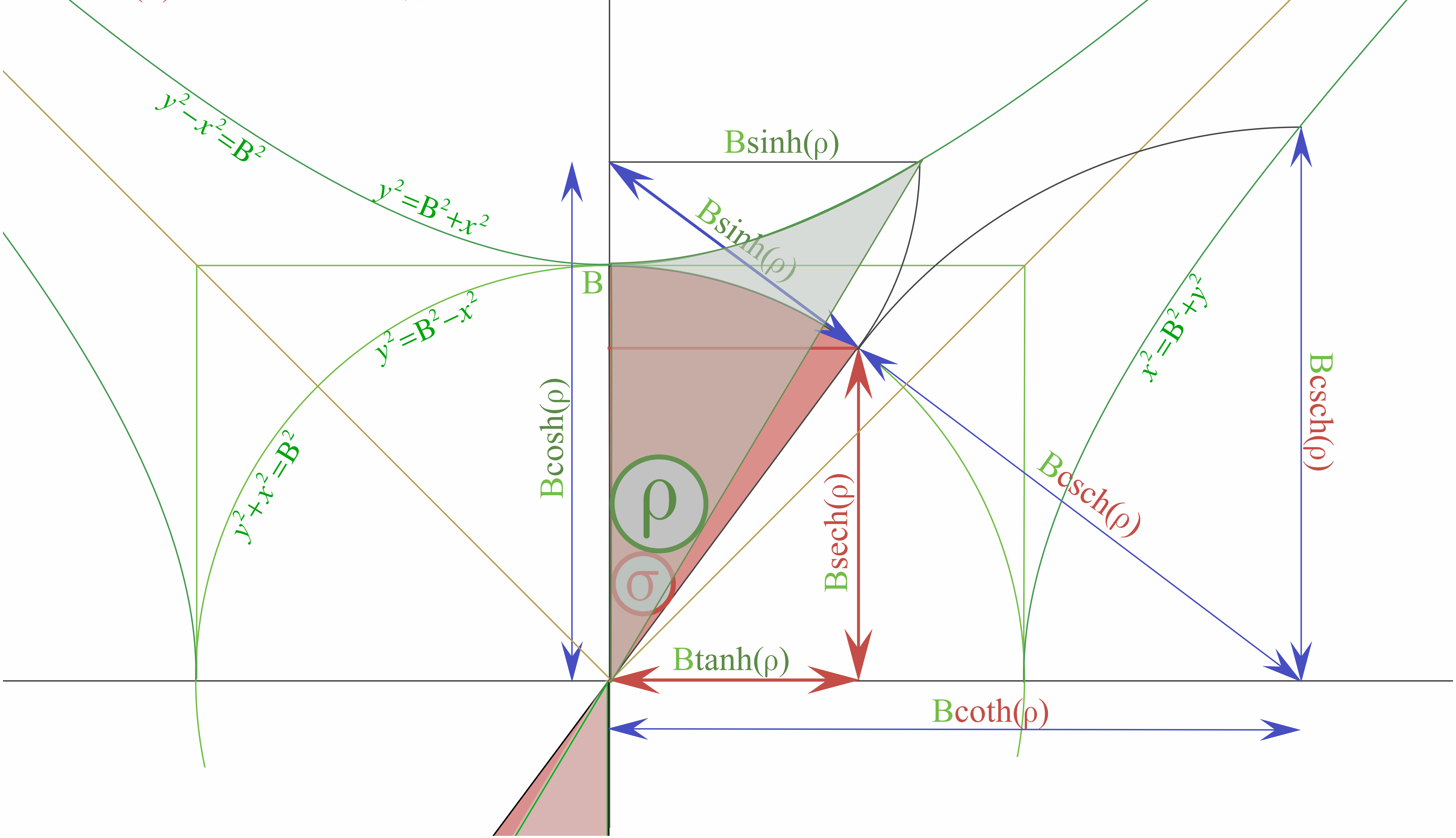

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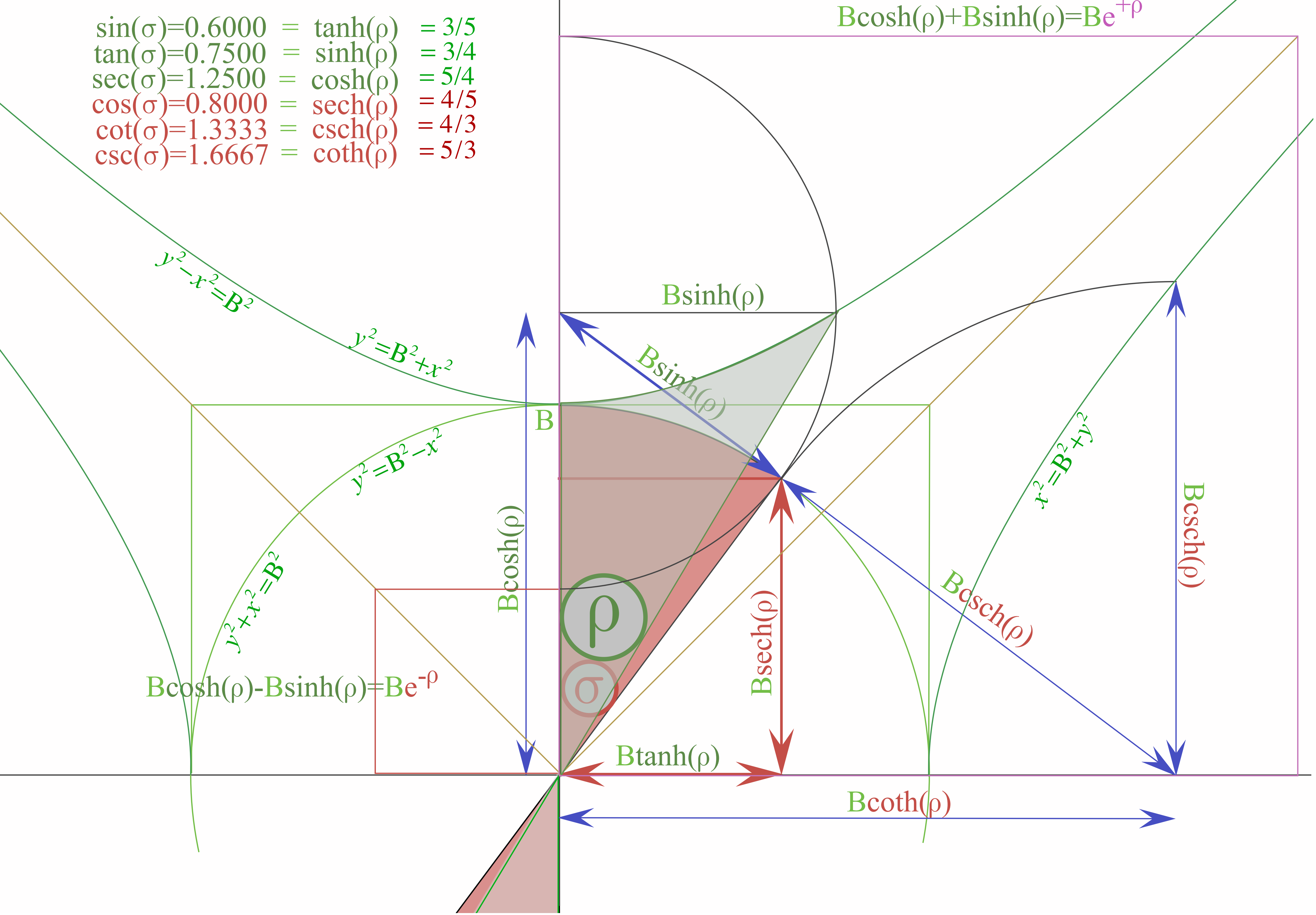

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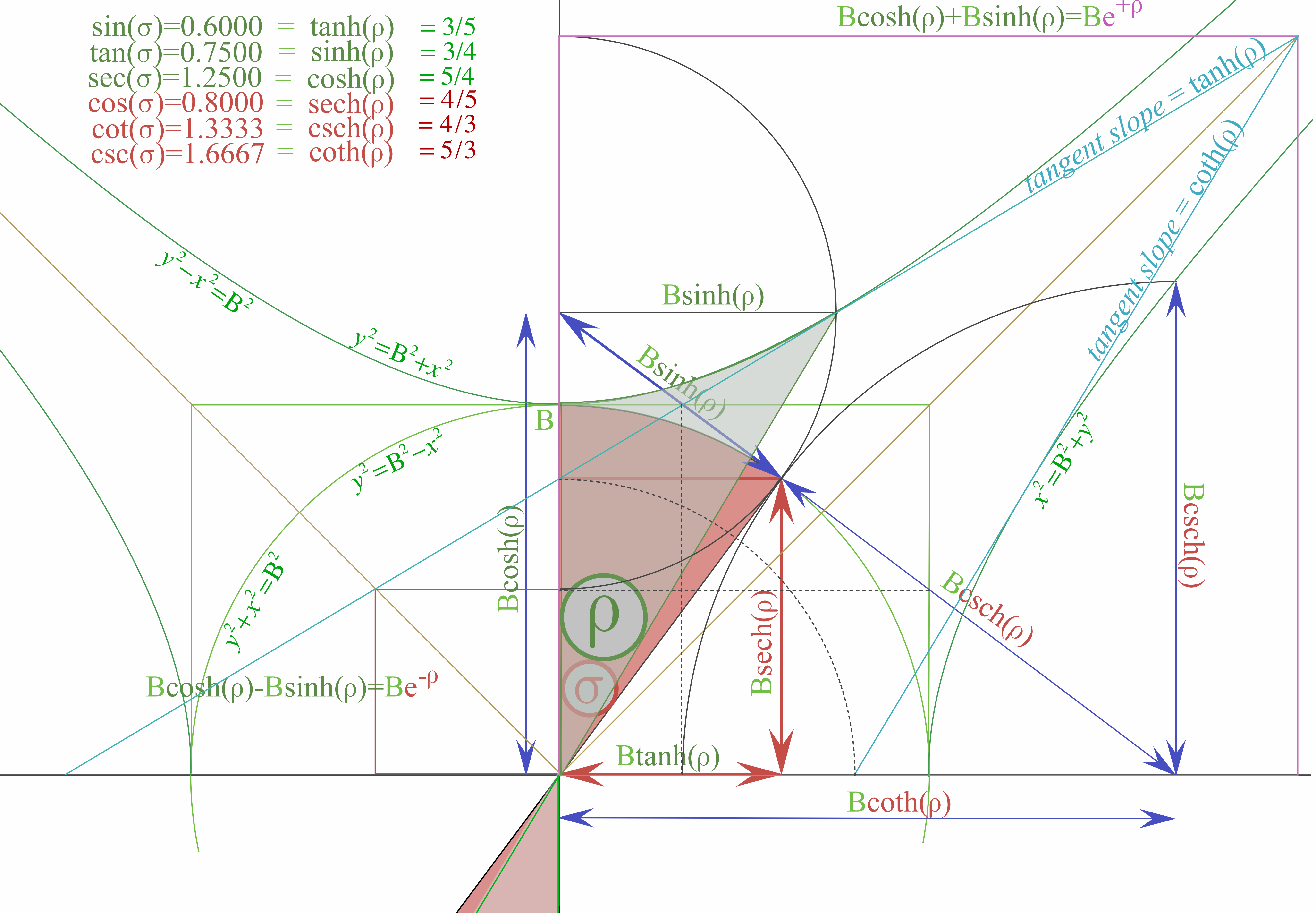

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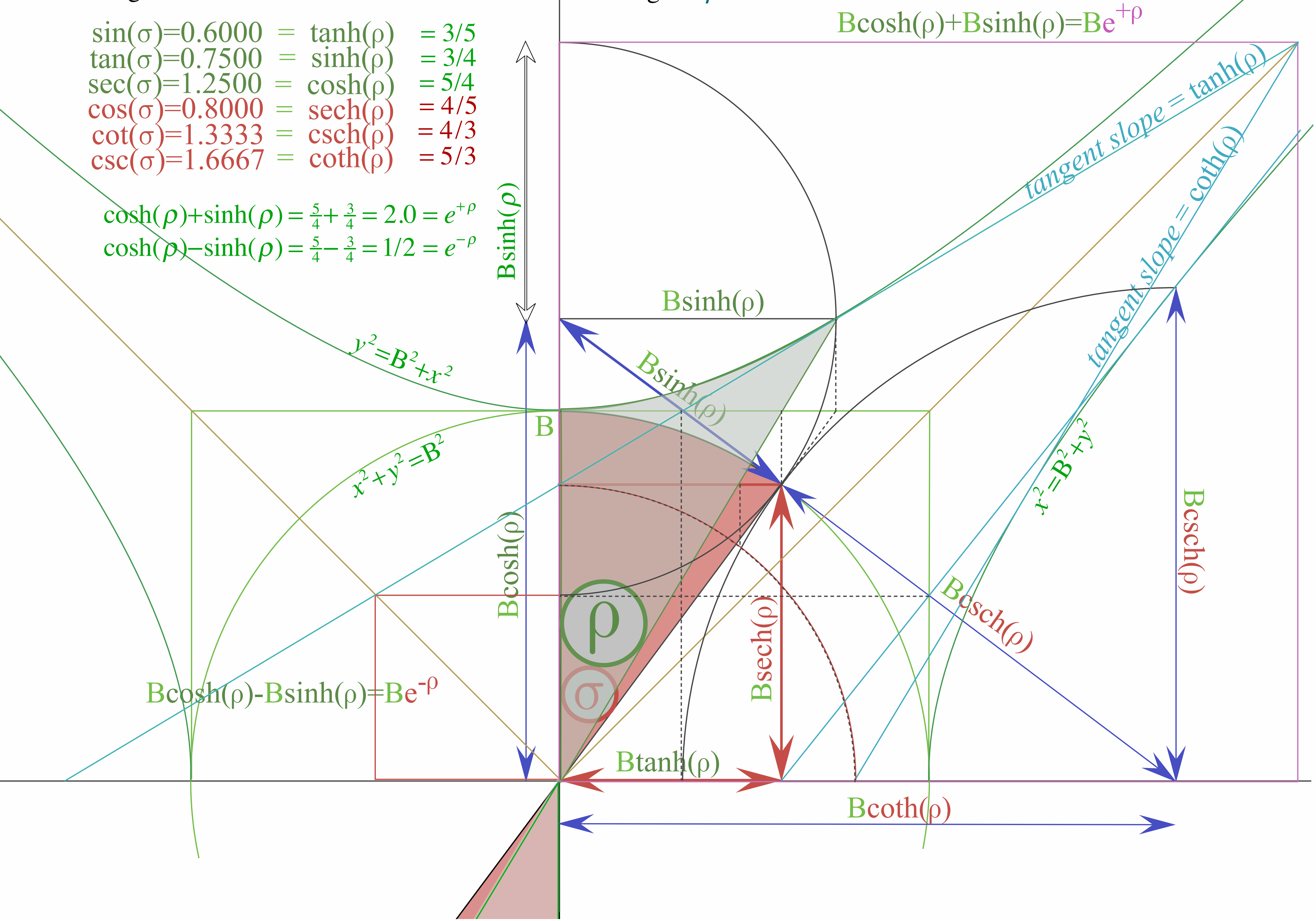

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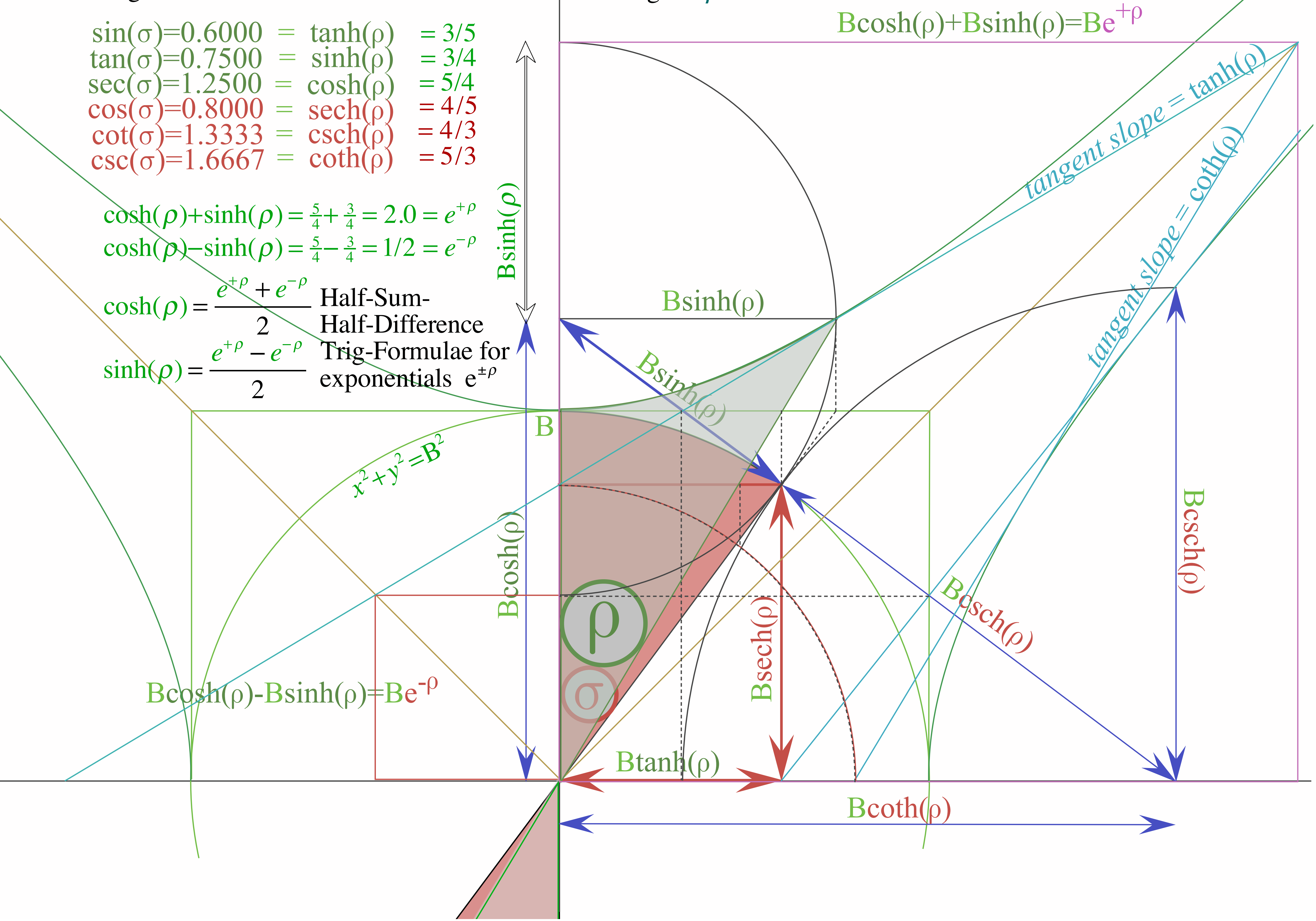

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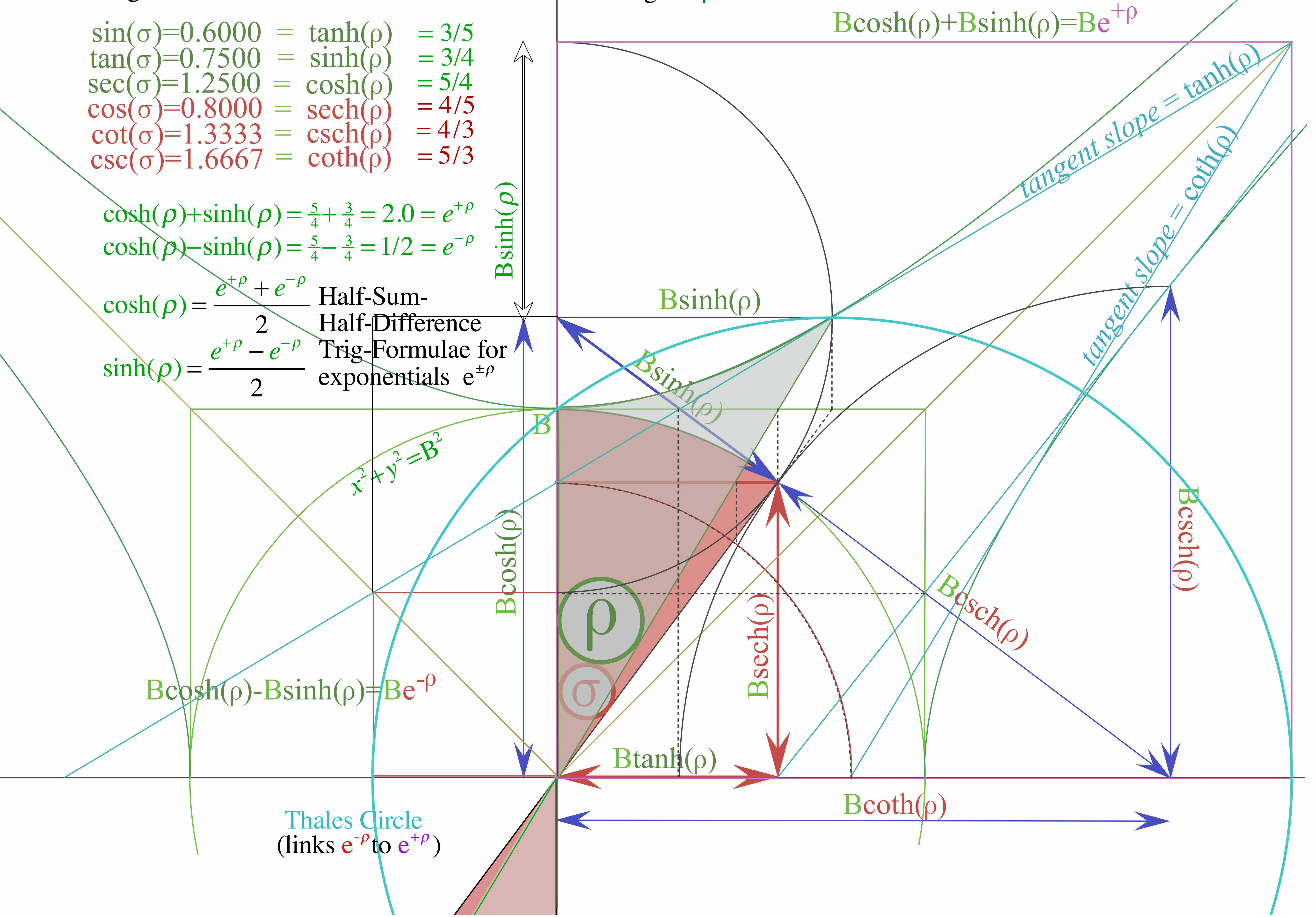
$\csc(\sigma)=1.6667 = \coth(\rho) = 5/3$

$\cosh(\rho)+\sinh(\rho) = \frac{5}{4}+\frac{3}{4} = 2.0 = e^{+\rho}$
$\cosh(\rho)-\sinh(\rho) = \frac{5}{4}-\frac{3}{4} = 1/2 = e^{-\rho}$

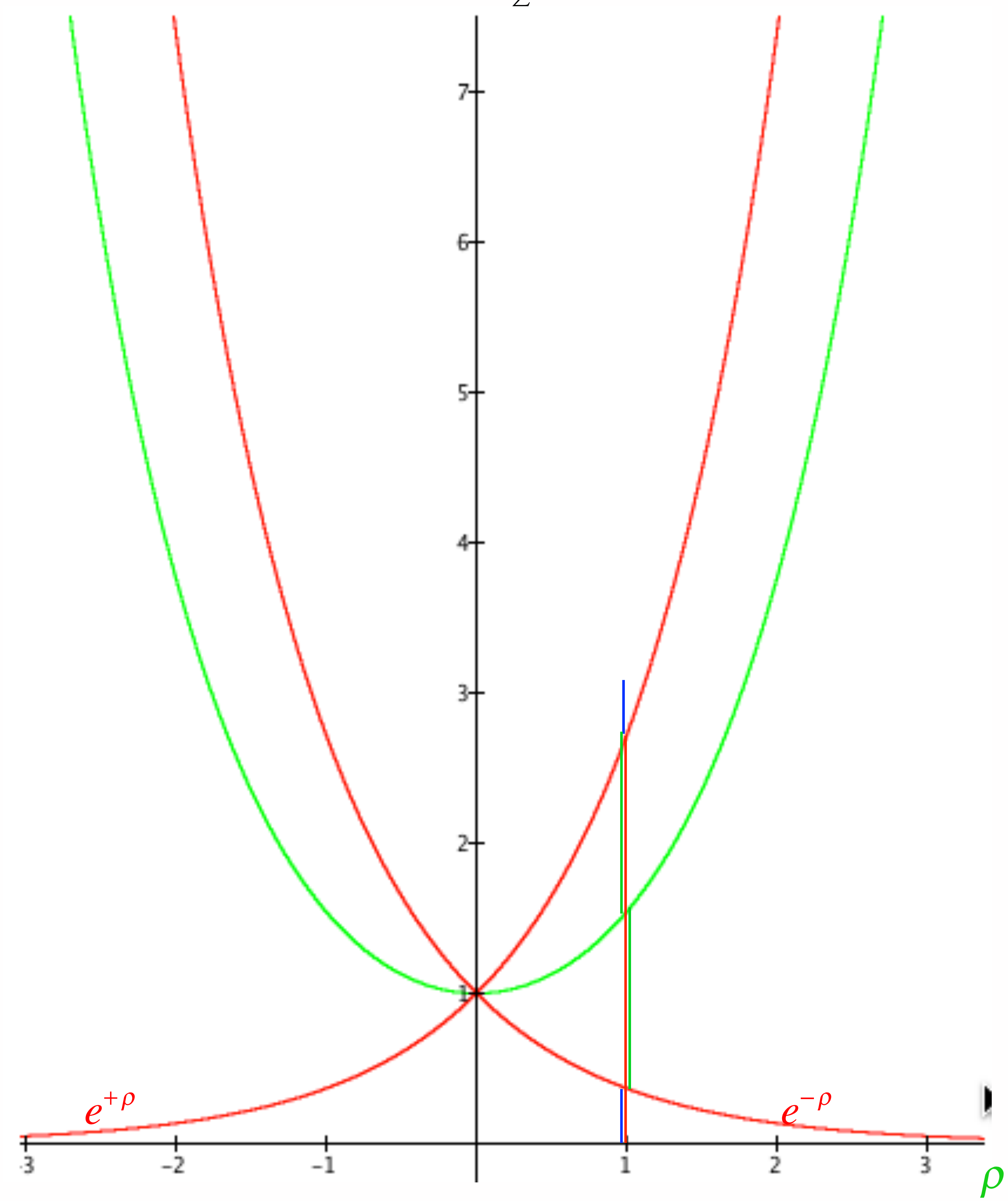
$$\cosh(\rho) = \frac{e^{+\rho}+e^{-\rho}}{2}$$
$$\sinh(\rho) = \frac{e^{+\rho}-e^{-\rho}}{2}$$

Half-Sum-Half-Difference Trig-Formulae for exponentials $e^{\pm\rho}$



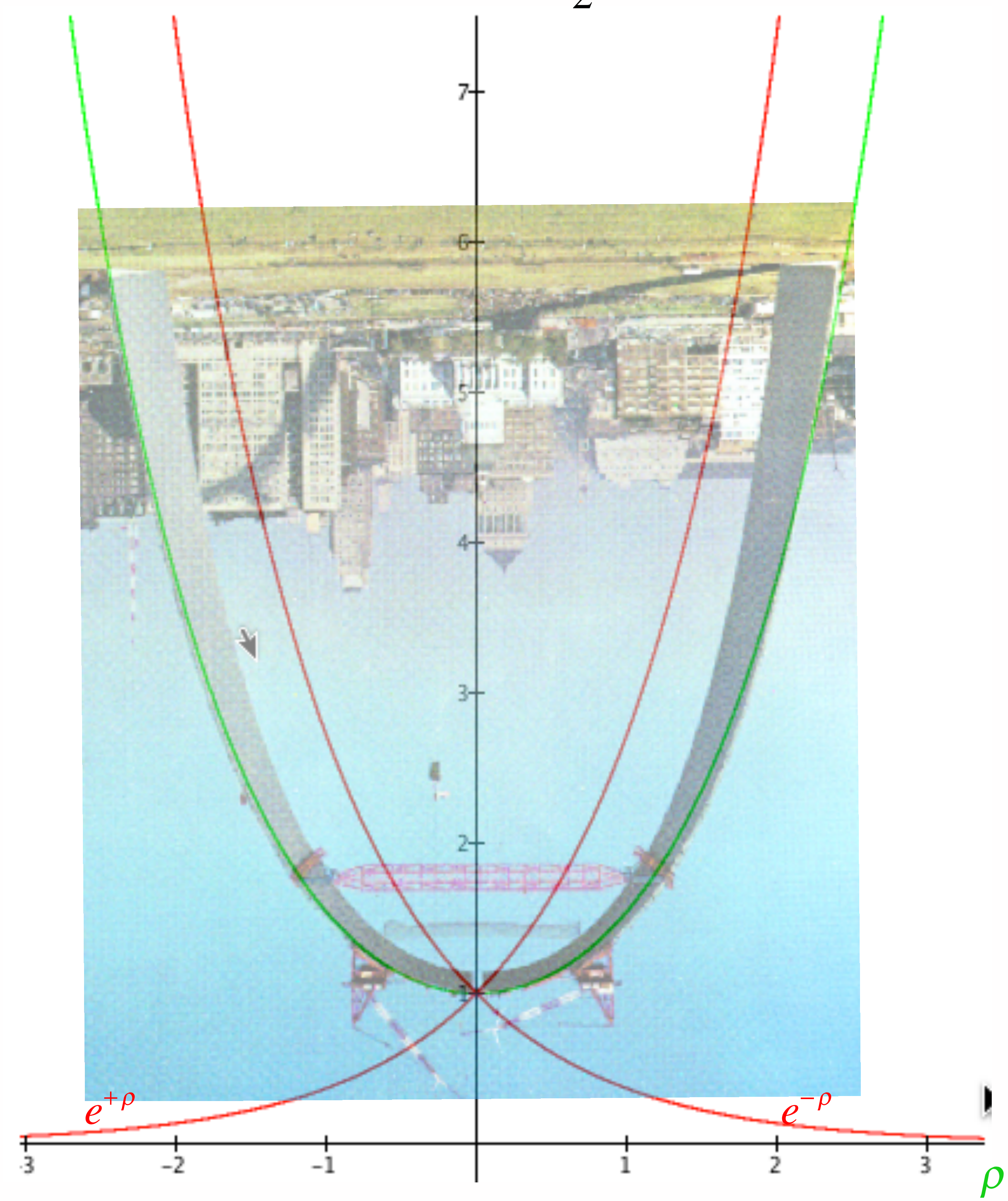
Hyperbolic cosine and the St. Louis Arch (being topped out in 1963)

$$\cosh(\rho) = \frac{e^{+\rho} + e^{-\rho}}{2}$$



Hyperbolic cosine and the St. Louis Arch (being topped out in 1963)

$$y = \cosh(\rho) = \frac{e^{+\rho} + e^{-\rho}}{2}$$



Formulae for Exponentials $e^{\pm\rho}$

$$\cosh(\rho) = \frac{e^{+\rho} + e^{-\rho}}{2}$$

$$\sinh(\rho) = \frac{e^{+\rho} - e^{-\rho}}{2}$$

Half-Sum-
Half-Difference
Trig-Formulae for
exponentials $e^{\pm\rho}$

$$\cosh(\rho)+\sinh(\rho) = e^{+\rho}$$

$$\cosh(\rho)-\sinh(\rho) = e^{-\rho}$$

Formulae for Exponentials $e^{\pm\rho}$ begin with interest-rate formula $e^{rt} = \lim_{n\to\infty}\left(1+\frac{rt}{n}\right)^n$

$$\cosh(\rho) = \frac{e^{+\rho} + e^{-\rho}}{2}$$

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$$\cosh(\rho)+\sinh(\rho) = e^{+\rho}$$

$$\cosh(\rho)-\sinh(\rho) = e^{-\rho}$$

...and its binomial expansion series for exponentials...

$$e^{rt} = 1 + rt + \frac{(rt)^2}{2} + \frac{(rt)^3}{2\cdot 3} + \frac{(rt)^4}{2\cdot 3\cdot 4} + \frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} + \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots$$

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...with rate r sign-flipped on odd powers (*decay*-rate)...

$$e^{-rt} = 1 - rt + \frac{(rt)^2}{2} - \frac{(rt)^3}{2\cdot3} + \frac{(rt)^4}{2\cdot3\cdot4} - \frac{(rt)^5}{2\cdot3\cdot4\cdot5} + \frac{(rt)^6}{2\cdot3\cdot4\cdot5\cdot6} - \ldots$$

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Half-sum has *even* powers of hyper-cosine ...

$$\frac{e^{+rt}+e^{-rt}}{2} = 1 \quad + \frac{(rt)^2}{2} \quad + \frac{(rt)^4}{2\cdot 3\cdot 4} \quad + \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \ldots = \cosh rt$$

Formulae for Exponentials $e^{\pm\rho}$ begin with interest-rate formula $e^{rt} = \lim_{n\to\infty}\left(1+\frac{rt}{n}\right)^n$

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Half-sum has *even* powers of hyper-*cosine* ...

$$\frac{e^{+rt}+e^{-rt}}{2} = 1 + \frac{(rt)^2}{2} + \frac{(rt)^4}{2\cdot 3\cdot 4} + \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \ldots = \cosh rt$$

Half-difference has *odd* powers of hyper-*sine* ...

$$\frac{e^{+rt}-e^{-rt}}{2} = rt + \frac{(rt)^3}{2\cdot 3} + \frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} + \ldots = \sinh rt$$

Formulae for Exponentials $e^{\pm\rho}$ begin with interest-rate formula $e^{rt} = \lim_{n\to\infty}\left(1+\frac{rt}{n}\right)^n$

$$\cosh(\rho) = \frac{e^{+\rho}+e^{-\rho}}{2}$$

$$\sinh(\rho) = \frac{e^{+\rho}-e^{-\rho}}{2}$$

Half-Sum-
Half-Difference
Trig-Formulae for
exponentials $e^{\pm\rho}$

$$\cosh(\rho)+\sinh(\rho) = e^{+\rho}$$

$$\cosh(\rho)-\sinh(\rho) = e^{-\rho}$$

...and its binomial expansion series for exponentials...

$$e^{rt} = 1 + rt + \frac{(rt)^2}{2} + \frac{(rt)^3}{2\cdot 3} + \frac{(rt)^4}{2\cdot 3\cdot 4} + \frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} + \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots$$

...with rate r sign-flipped on odd powers (*decay*-rate)...

$$e^{-rt} = 1 - rt + \frac{(rt)^2}{2} - \frac{(rt)^3}{2\cdot 3} + \frac{(rt)^4}{2\cdot 3\cdot 4} - \frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} + \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \ldots$$

Half-sum has *even* powers of hyper-*cosine* ...

$$\frac{e^{+rt}+e^{-rt}}{2} = 1 + \frac{(rt)^2}{2} + \frac{(rt)^4}{2\cdot 3\cdot 4} + \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \ldots = \cosh rt$$

Half-difference has *odd* powers of hyper-*sine* ...

$$\frac{e^{+rt}-e^{-rt}}{2} = rt + \frac{(rt)^3}{2\cdot 3} + \frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} + \ldots = \sinh rt$$

Complex Exponentials $e^{\pm irt}$ have *imaginary* $i=\sqrt{-1}$ interest-rate

$(r \to ir$ and $\pm rt \to \pm irt)$

$$e^{irt} = 1 + irt + \frac{(irt)^2}{2} + \frac{(irt)^3}{2\cdot 3} + \frac{(irt)^4}{2\cdot 3\cdot 4} + \frac{(irt)^5}{2\cdot 3\cdot 4\cdot 5} + \frac{(irt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots$$

Formulae for Exponentials $e^{\pm\rho}$ begin with interest-rate formula $e^{rt} = \lim_{n\to\infty}\left(1+\frac{rt}{n}\right)^n$

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$$\frac{e^{+rt}+e^{-rt}}{2} = 1 \quad + \frac{(rt)^2}{2} \quad + \frac{(rt)^4}{2\cdot 3\cdot 4} \quad + \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \ldots = \cosh rt$$

Half-difference has *odd* powers of hyper-*sine* ...

$$\frac{e^{+rt}-e^{-rt}}{2} = \quad rt \quad + \frac{(rt)^3}{2\cdot 3} \quad + \frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} + \ldots \quad = \sinh rt$$

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$$e^{irt} = 1 + irt - \frac{(rt)^2}{2} - i\frac{(rt)^3}{2\cdot 3} + \frac{(rt)^4}{2\cdot 3\cdot 4} + i\frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} - \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - i\ldots$$

Note powers of i

$i^0=1, i^1=i, i^2=-1, i^3=-i, i^4=1, i^5=i, i^6=-1, \ldots$

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...and negative-i exponential:

$$e^{-irt} = 1 - irt - \frac{(rt)^2}{2} + i\frac{(rt)^3}{2\cdot3} + \frac{(rt)^4}{2\cdot3\cdot4} - i\frac{(rt)^5}{2\cdot3\cdot4\cdot5} - \frac{(rt)^6}{2\cdot3\cdot4\cdot5\cdot6} + i\ldots$$

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$$e^{irt} = 1+irt-\frac{(rt)^2}{2}-i\frac{(rt)^3}{2\cdot 3}+\frac{(rt)^4}{2\cdot 3\cdot 4}+i\frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5}-\frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6}-i\ldots$$

$$e^{-irt} = 1-irt-\frac{(rt)^2}{2}+i\frac{(rt)^3}{2\cdot 3}+\frac{(rt)^4}{2\cdot 3\cdot 4}-i\frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5}-\frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6}+i\ldots$$

Half-sum has *even* powers of circular-*cosine* ...

$$\frac{e^{+i\,rt}+e^{-i\,rt}}{2} = 1 - \frac{(rt)^2}{2} + \frac{(rt)^4}{2\cdot 3\cdot 4} - \frac{(rt)^6}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \ldots = \cos rt$$

Half-difference has *odd* powers of circular-*sine* ...

$$\frac{e^{+i\,rt}-e^{-i\,rt}}{2} = i\,rt - i\frac{(rt)^3}{2\cdot 3} + i\frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} - \ldots = i\sin rt$$

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Half-Difference
Trig-Formulae for
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$$\cos(\sigma)+i\sin(\sigma) = e^{+i\sigma}$$

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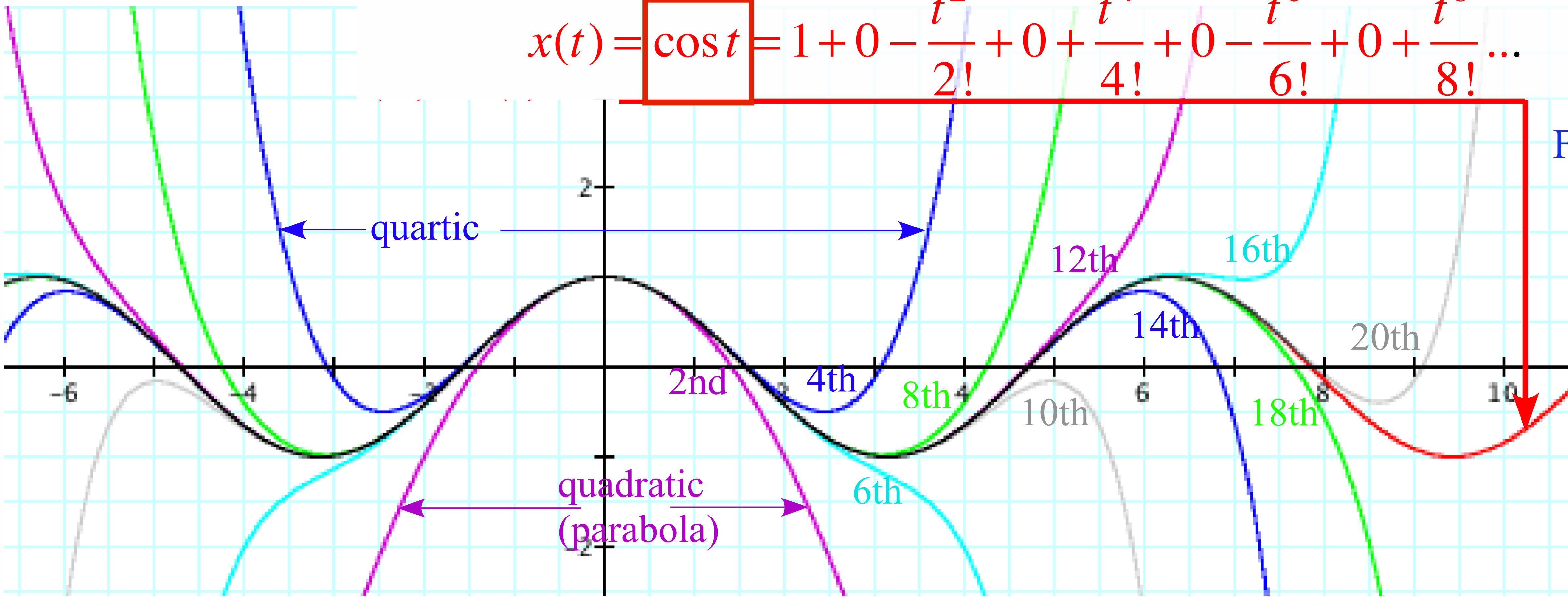
$$\frac{e^{+i\,rt}-e^{-i\,rt}}{2} = i\,rt - i\frac{(rt)^3}{2\cdot 3} + i\frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} - \ldots = i\sin rt$$

$$\frac{e^{+i\,rt}-e^{-i\,rt}}{2i} = rt - \frac{(rt)^3}{2\cdot 3} + \frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} - \ldots = \sin rt$$

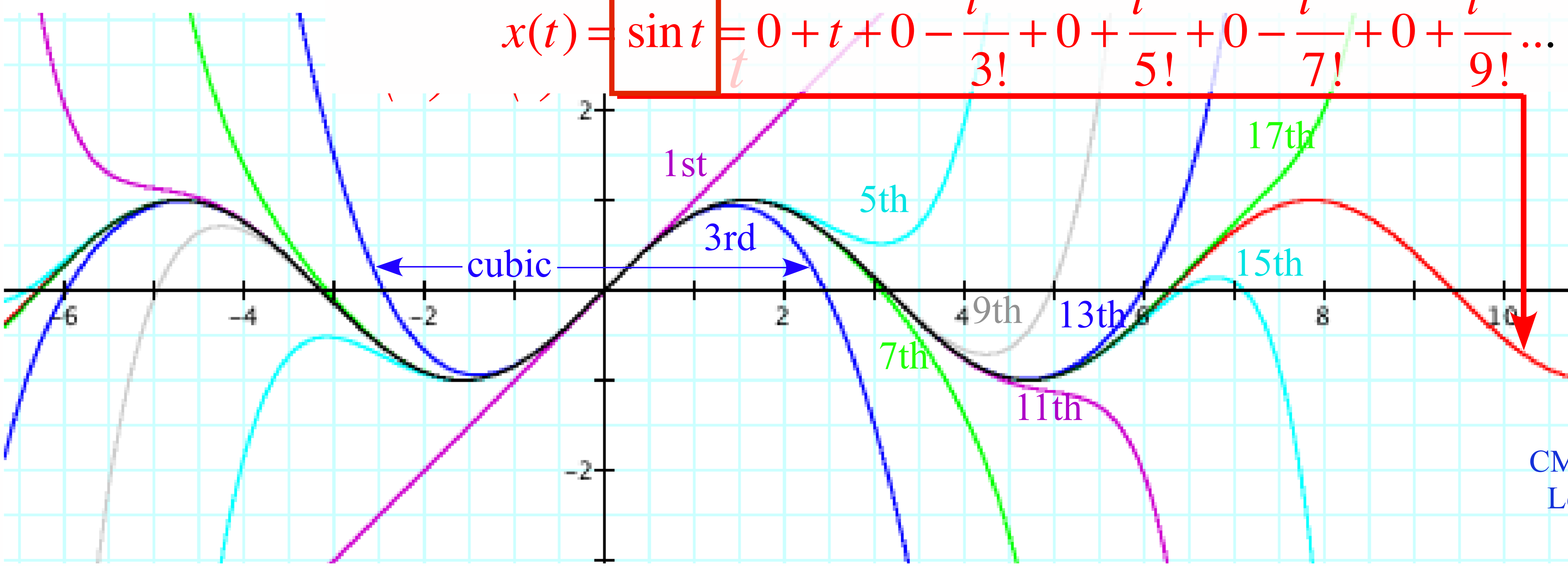
How well do power series make sine curves?

From Classical Mechanics with a BANG!
Unit 1
Fig. 10.3

$$x(t) = \boxed{\cos t} = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 - \frac{t^6}{6!} + 0 + \frac{t^8}{8!} \dots$$



$$x(t) = \boxed{\sin t} = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} + 0 - \frac{t^7}{7!} + 0 + \frac{t^9}{9!} \dots$$



or
CMwBANG!
Lecture 12
p.22

Formulae for Exponentials $e^{\pm\rho}$ begin with interest-rate formula $e^{rt} = \lim_{n\to\infty}\left(1+\frac{rt}{n}\right)^n$

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Half-Sum-
Half-Difference
Trig-Formulae for
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$$\frac{e^{+i\,rt}-e^{-i\,rt}}{2} = i\,rt - i\frac{(rt)^3}{2\cdot 3} + i\frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} - \ldots = i\sin rt$$
$$\frac{e^{+i\,rt}-e^{-i\,rt}}{2i} = rt - \frac{(rt)^3}{2\cdot 3} + \frac{(rt)^5}{2\cdot 3\cdot 4\cdot 5} - \ldots = \sin rt$$

$$\cos(\sigma) = \frac{e^{+i\sigma}+e^{-i\sigma}}{2}$$
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$$\cos(\sigma)+i\sin(\sigma) = e^{+i\sigma}$$
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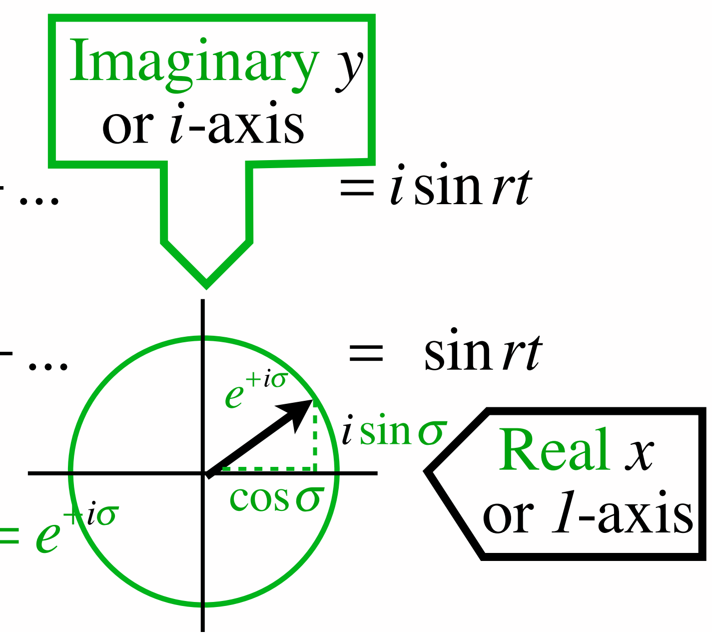


Phasor circle plot of $\cos\sigma + i\sin\sigma = e^{+i\sigma}$

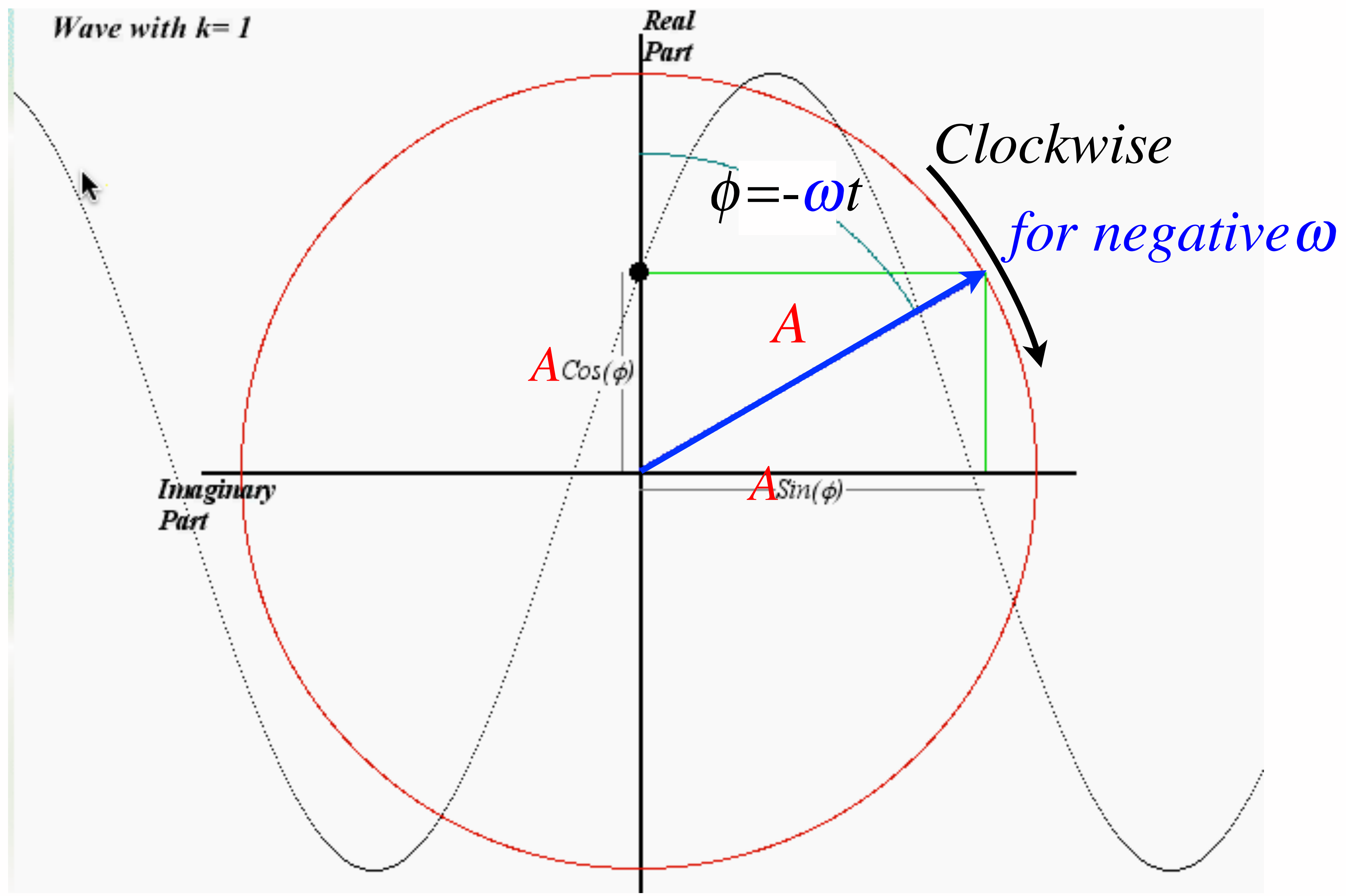
Complex Oscillator Phasors $Ae^{\pm i\phi} = A(\cos\phi \pm i\sin\phi)$

Clockwise angular velocity ω (ϕ=-ωt) *and Amplitude A*

$$Ae^{-i\omega t} = A\cos\omega t - i A\sin\omega t$$

Animated phasor wave k=1

Animated waves k=-4, -1,+2 ,+5



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$Ae^{-i\omega t} = A\cos\omega t - iA\sin\omega t$

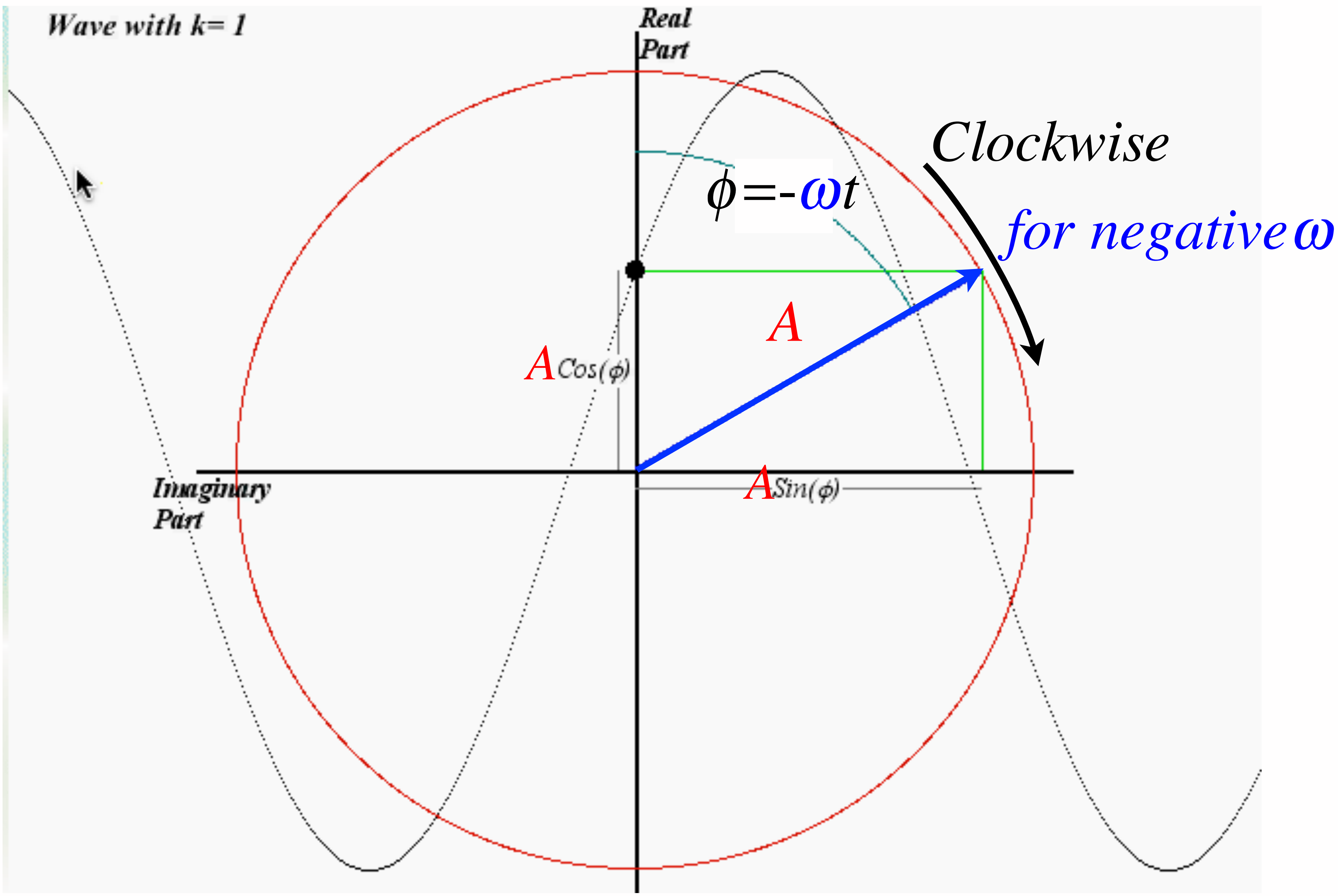
Animated phasor wave k=1 Animated waves k=-4, -1,+2 ,+5

Set: $A = e^{ikx} = \cos kx + i\sin kx$

to make an x-*moving wave*

$e^{ikx}e^{-i\omega t} = e^{ikx}(\cos\omega t - i\sin\omega t)$

$= e^{i(kx-\omega t)}$



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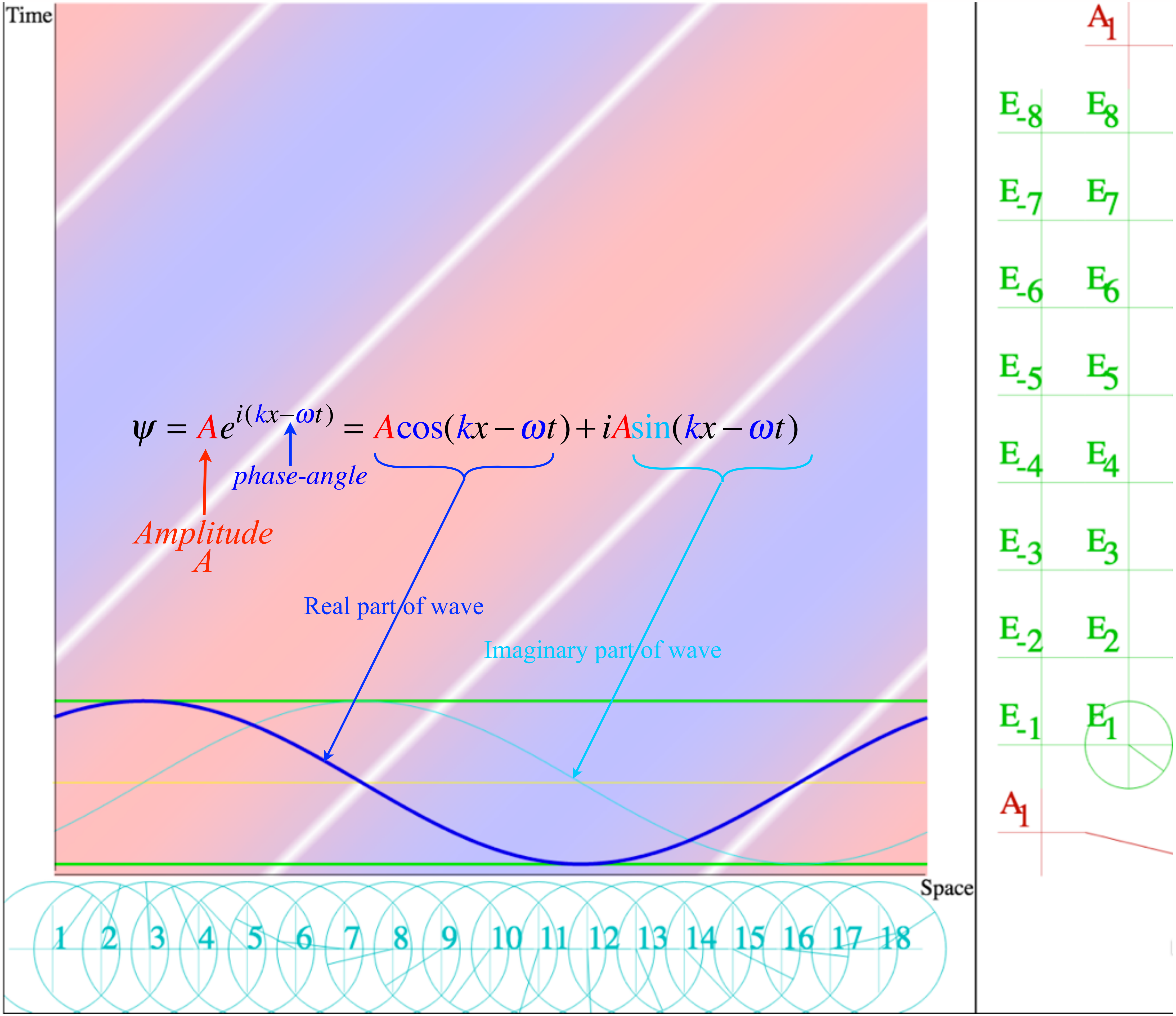
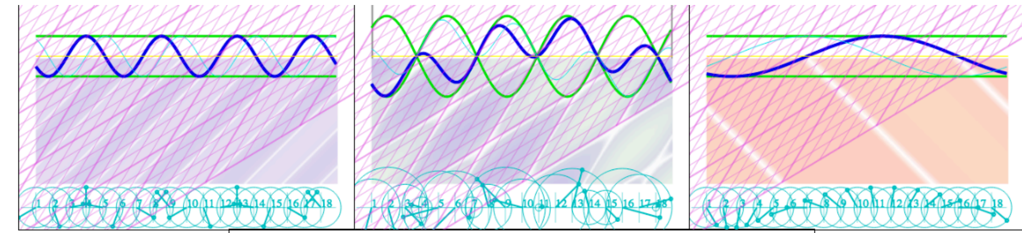
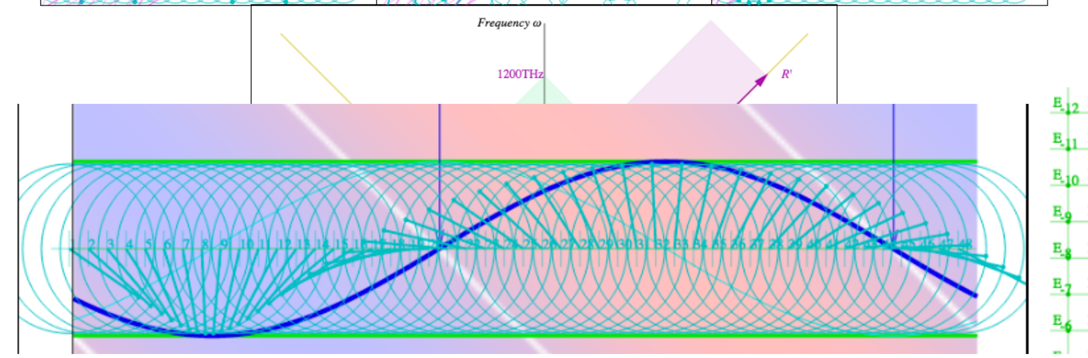
[Animated k=+1 1-CW (continous wave)](#)



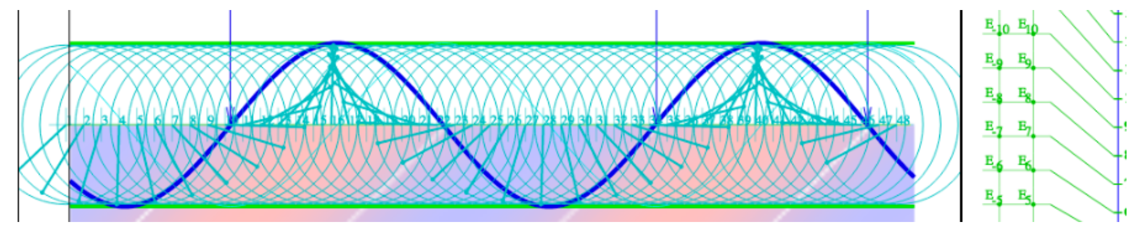
Multi-panel beta=0.6c, with extra coordinate grid. Not quite your black axes, but from other ap
http://www.uark.edu/ua/modphys/markup/BohrItWeb.html?scenario=-30104



http://www.uark.edu/ua/modphys/markup/BohrItWeb.html?scenario=-30022



k=1; Hi Rez, points per well = 48 phasors; w/tracers; x phasors move with now line. URL qualifiers: ?
xPhasorLocationsInd=1& doPhaseZeroTracers=1& doGroupZeroTracers=1
http://www.uark.edu/ua/modphys/markup/BohrItWeb.html?scenario=330002



k=-1 Hi Rez, points per well = 48 phasors; w/tracers; x phasors move with now line.
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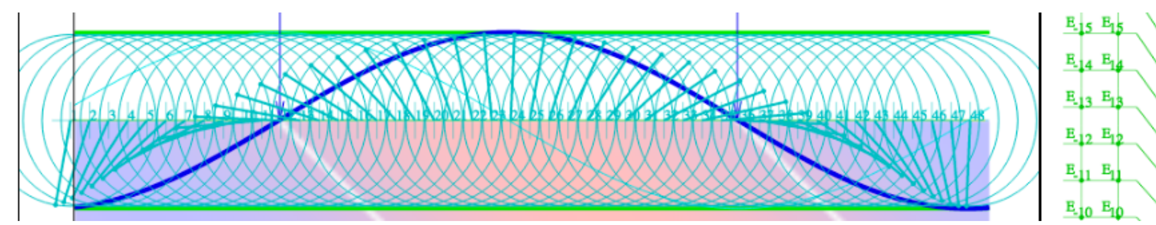


Link to the talk:
http://www.uark.edu/ua/modphys/pdfs/Talk_Pdfs/INBRE_2016.pdf

Link to the RelaWavity Portal:
http://www.uark.edu/ua/modphys/markup/RelaWavityPortal.html

Link to the Harter-Soft Educational Resource Portal:
http://www.uark.edu/ua/modphys/markup/Harter-SoftWebApps.html

1CW Laser-phasor wave function

$$\psi = A\cdot e^{i(kx-\omega t)} = A\cdot\cos(kx-\omega t) + iA\cdot\sin(kx-\omega t)$$

phase-angle

Amplitude A

Dimensionless Light wave-velocity c/c=1

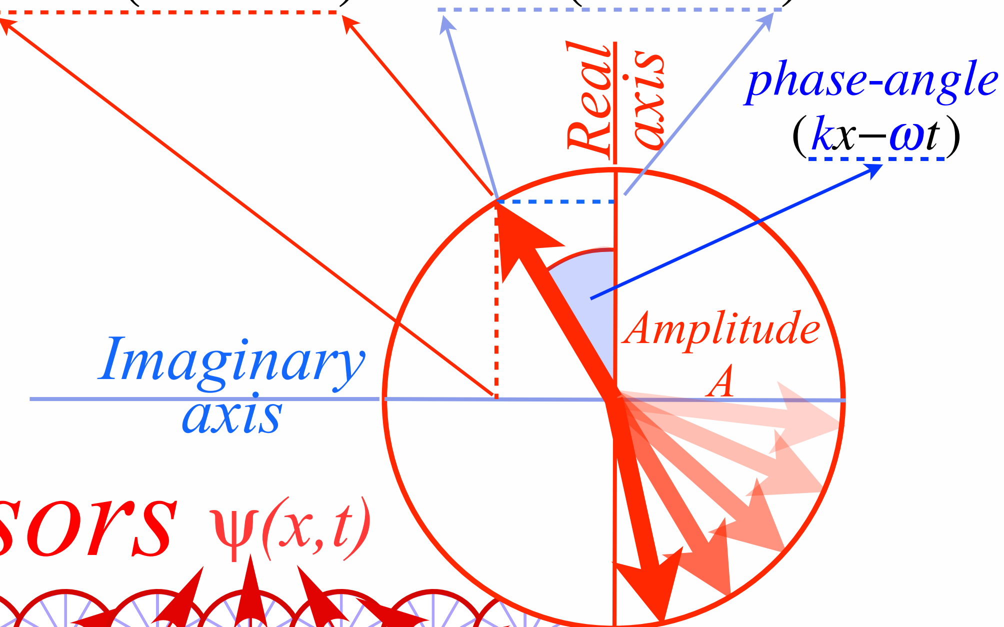
$$\frac{V_{light}}{c} = \frac{\lambda}{c\tau} = \frac{\upsilon}{c\kappa} = 1 = \frac{\omega}{ck} \quad \text{angular units}$$

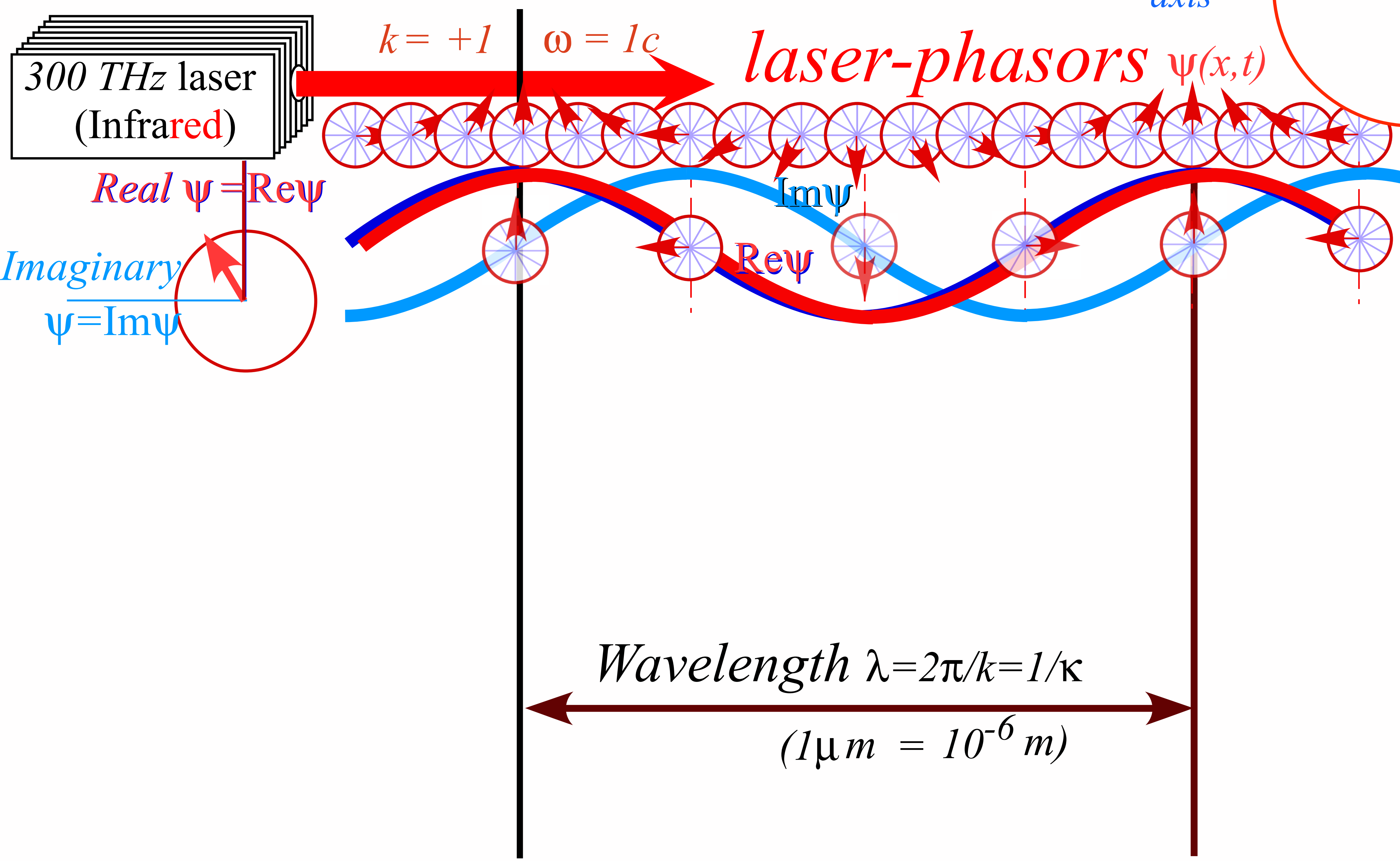
"winks" 'n "kinks"

angular frequency : $\omega = 2\pi\upsilon$

angular wave number : $k = 2\pi\kappa$

k = wave**vector**





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"winks" 'n "kinks"

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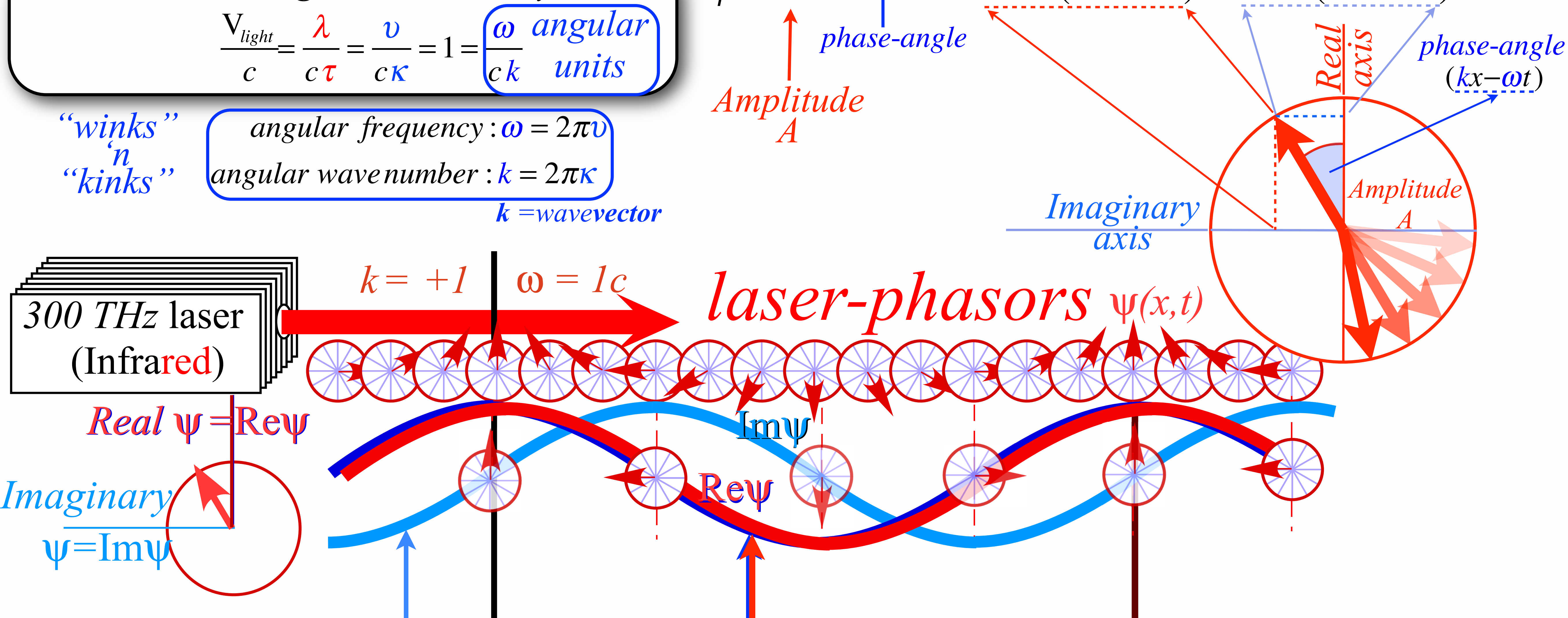

Imagination precedes Reality by exactly One Quarter!

Mantra for US publicly traded corporations

Wavelength $\lambda=2\pi/k=1/\kappa$

$(1\mu m = 10^{-6} m)$

1CW Laser-phasor wave function

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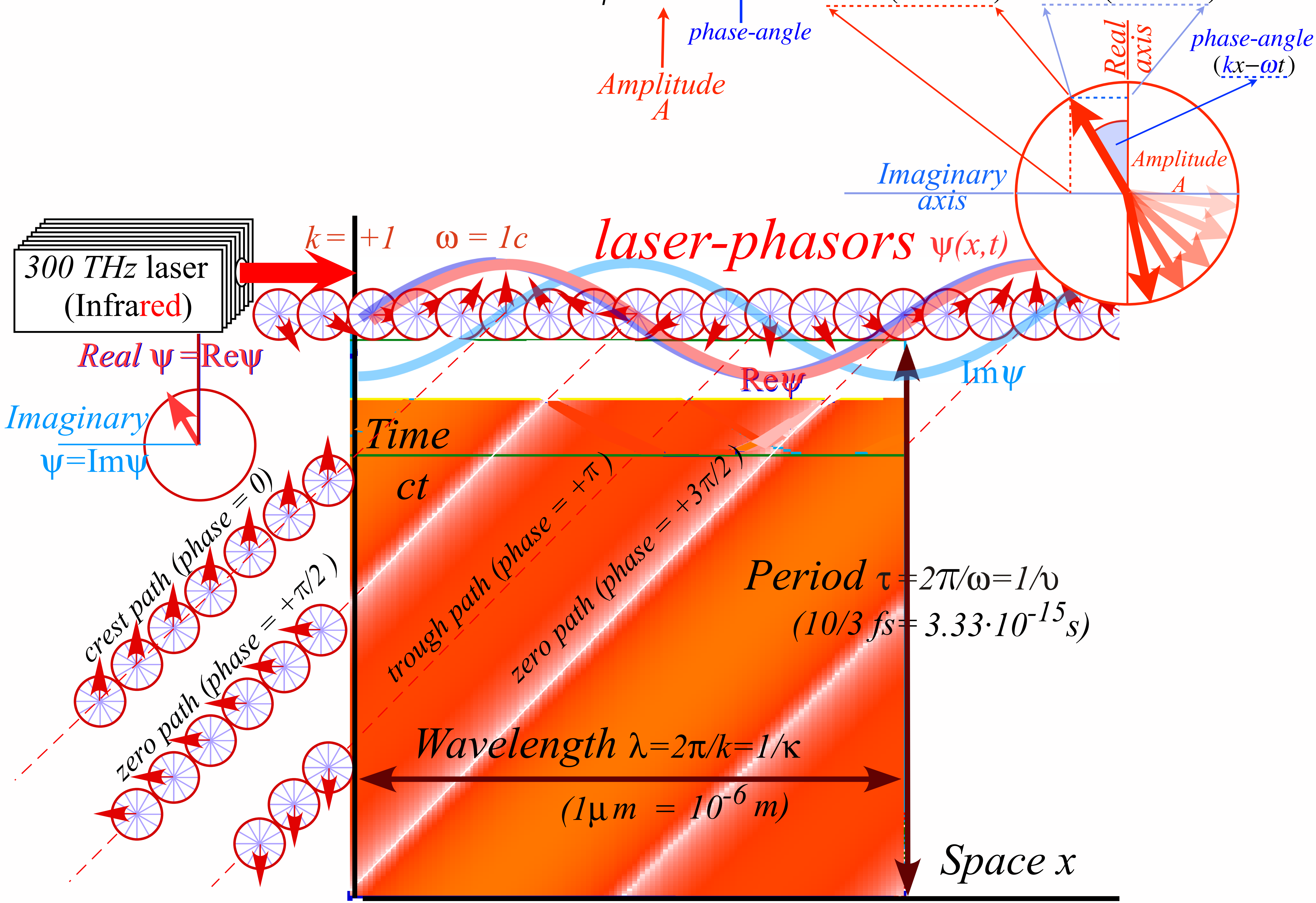
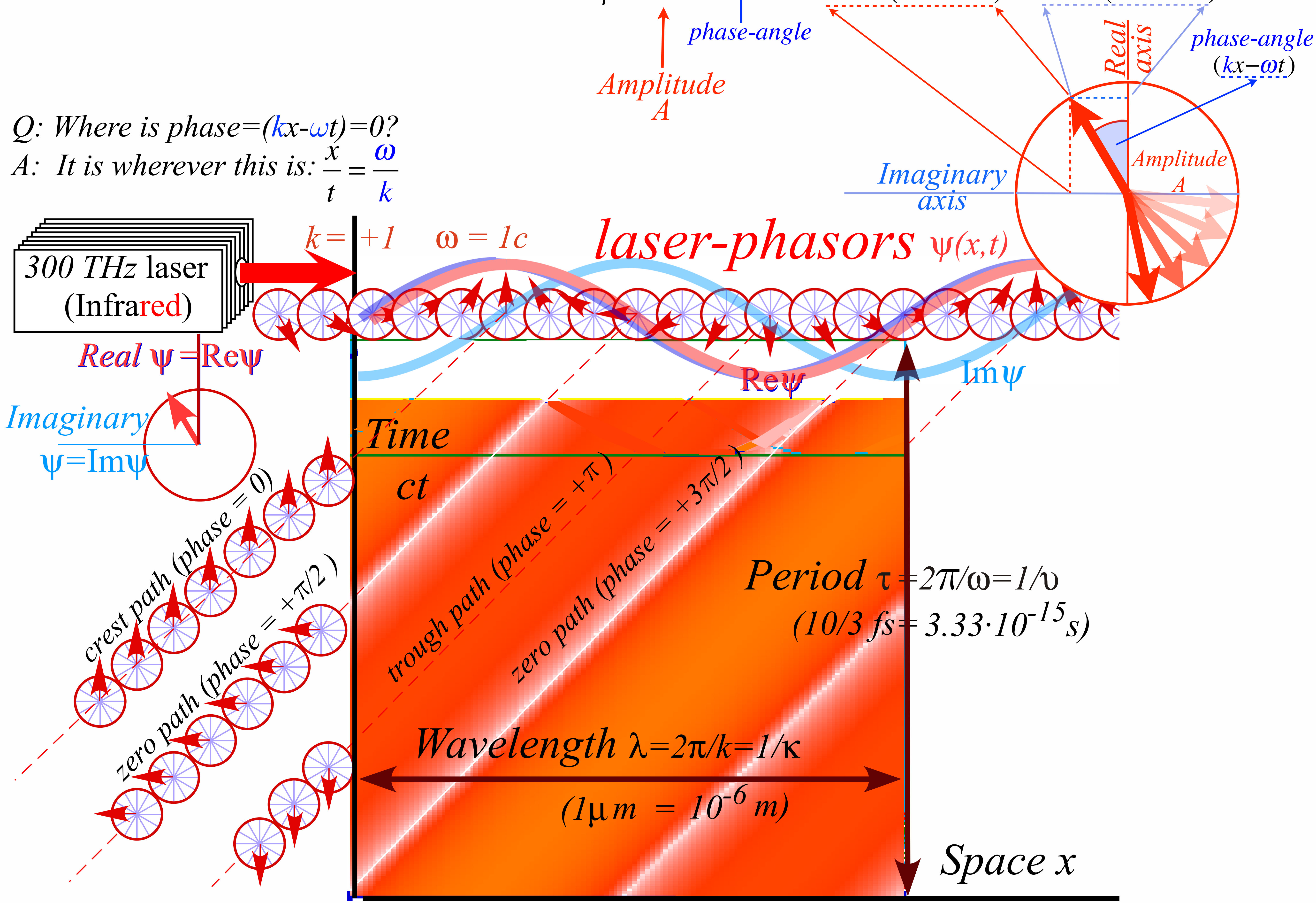


1CW Laser-phasor wave function

$$\psi = A \cdot e^{i(kx-\omega t)} = A \cdot \cos(kx-\omega t) + iA \cdot \sin(kx-\omega t)$$

Q: Where is phase=(kx-ωt)=0?

A: It is wherever this is: $\frac{x}{t} = \frac{\omega}{k}$



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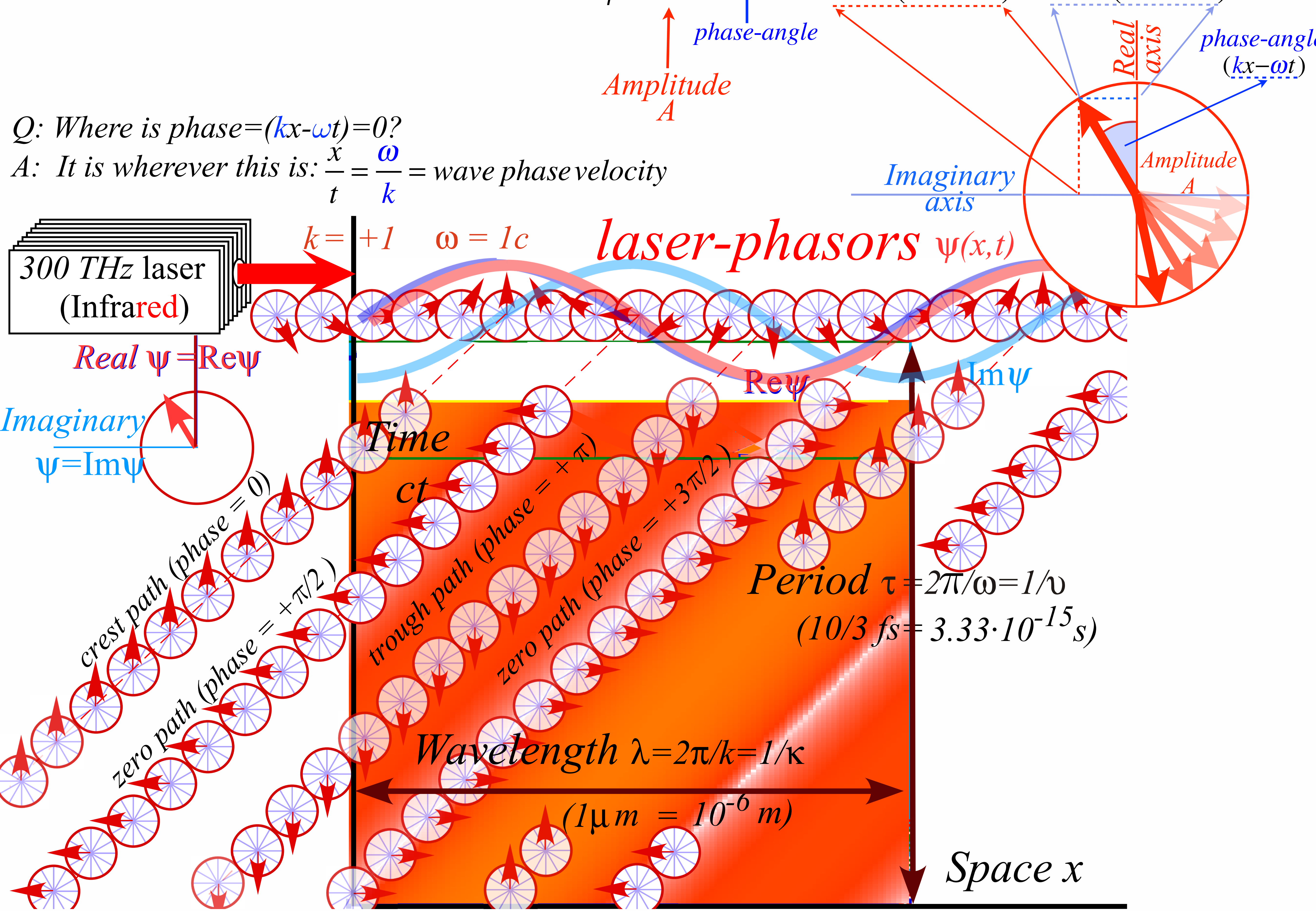
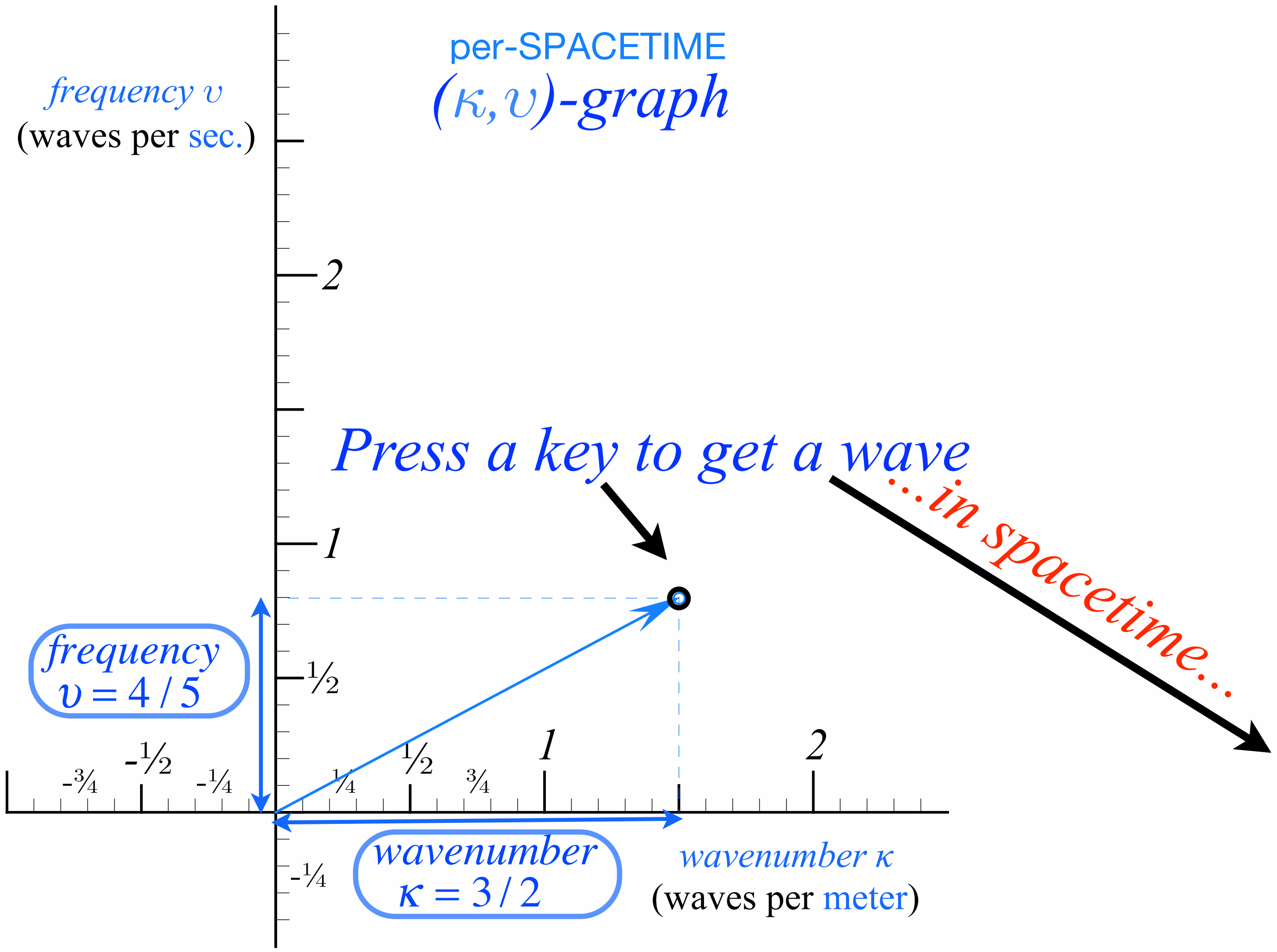
$$\psi = A \cdot e^{i(kx-\omega t)} = A\cdot\cos(kx-\omega t) + iA\cdot\sin(kx-\omega t)$$

Q: Where is phase=(kx-ωt)=0?

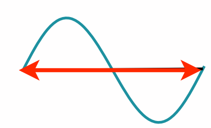
A: It is wherever this is: $\frac{x}{t} = \frac{\omega}{k}$ = *wave phase velocity*



The "Keyboard of the gods" : Introducing per-space-per-time graphs *versus* space-time graphs



SPACETIME
(λ,τ)-graph

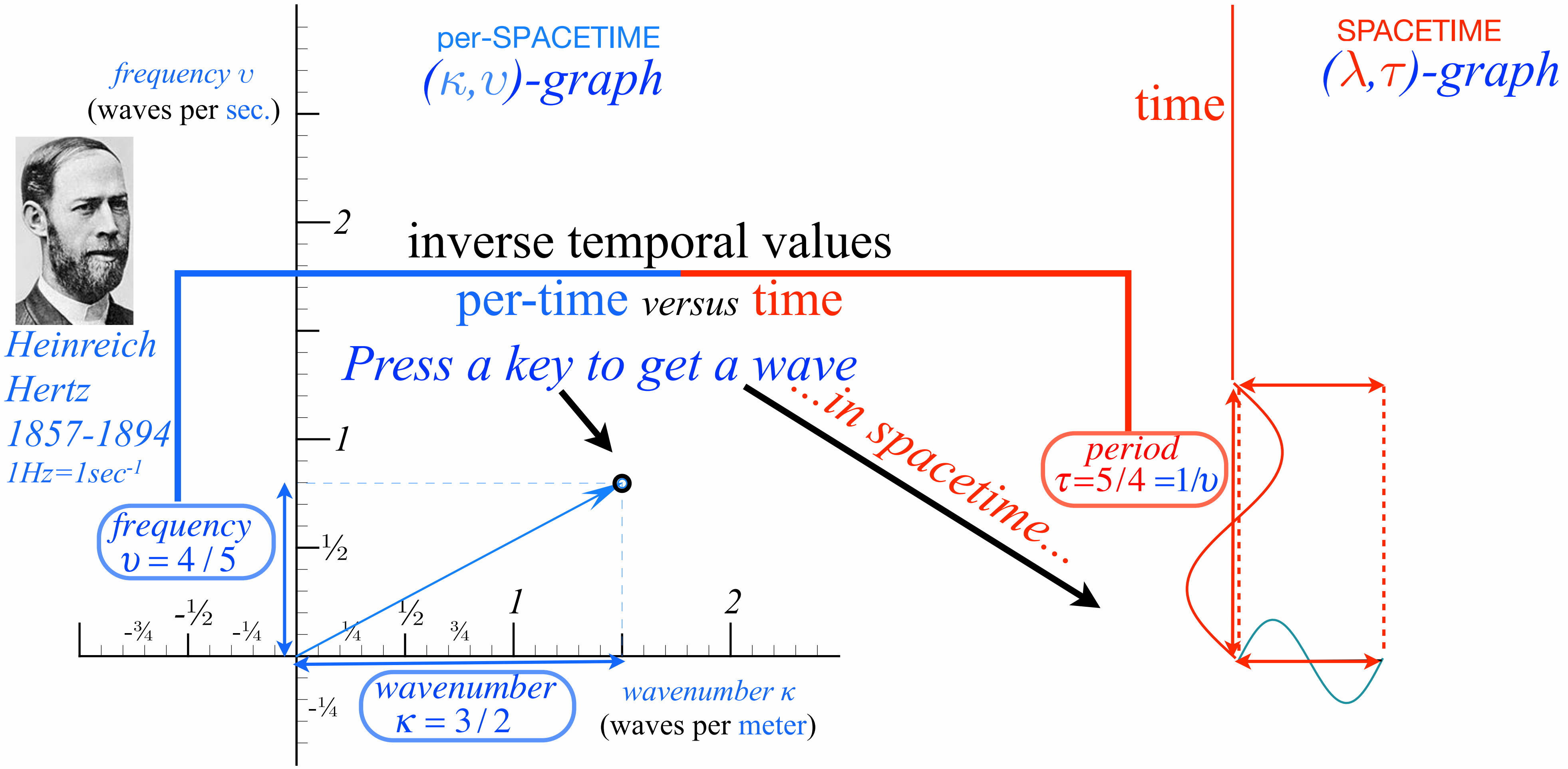


"Keyboard of the gods" is known as "Fourier-space"

Jean-Baptiste
Joseph Fourier
1768-1830

•How to understand waves
and
wave velocity V_{wave}

The "Keyboard of the gods" : Introducing per-space-per-time graphs *versus* space-time graphs



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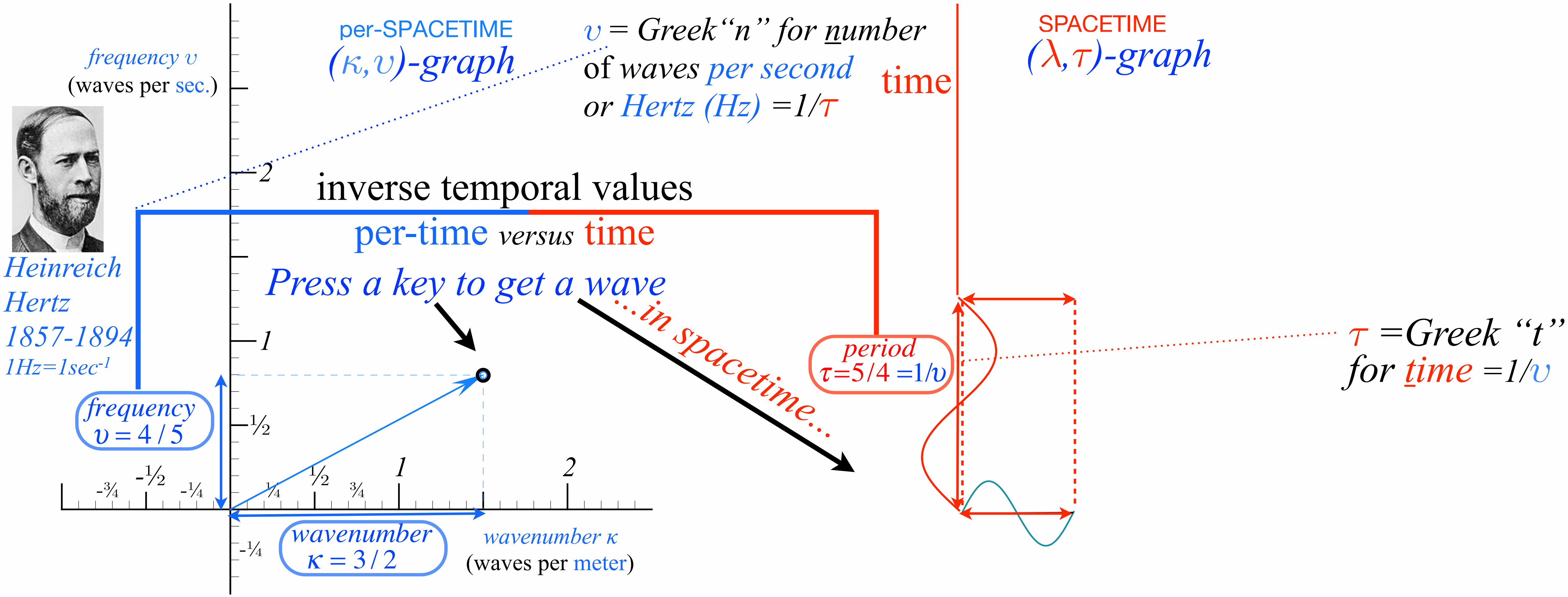

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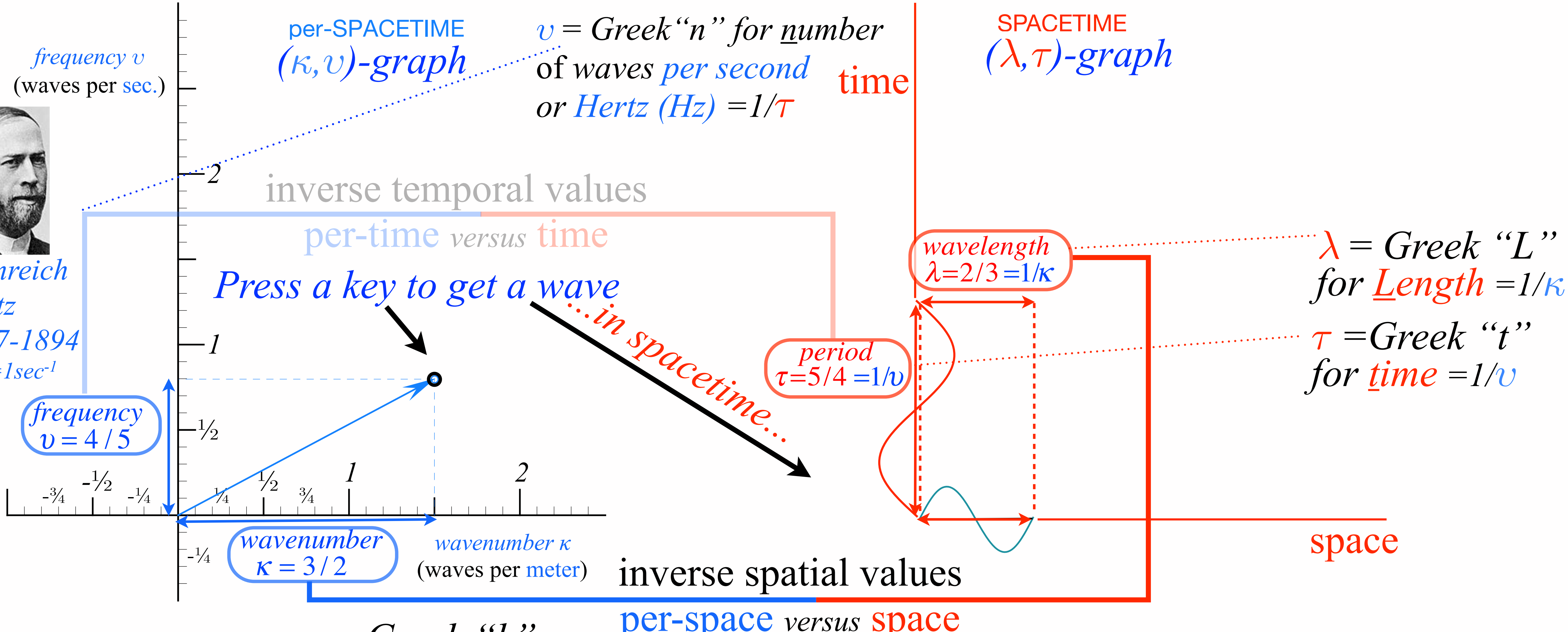

κ =Greek "k"

for Kayser

(or "kinks")=1/λ

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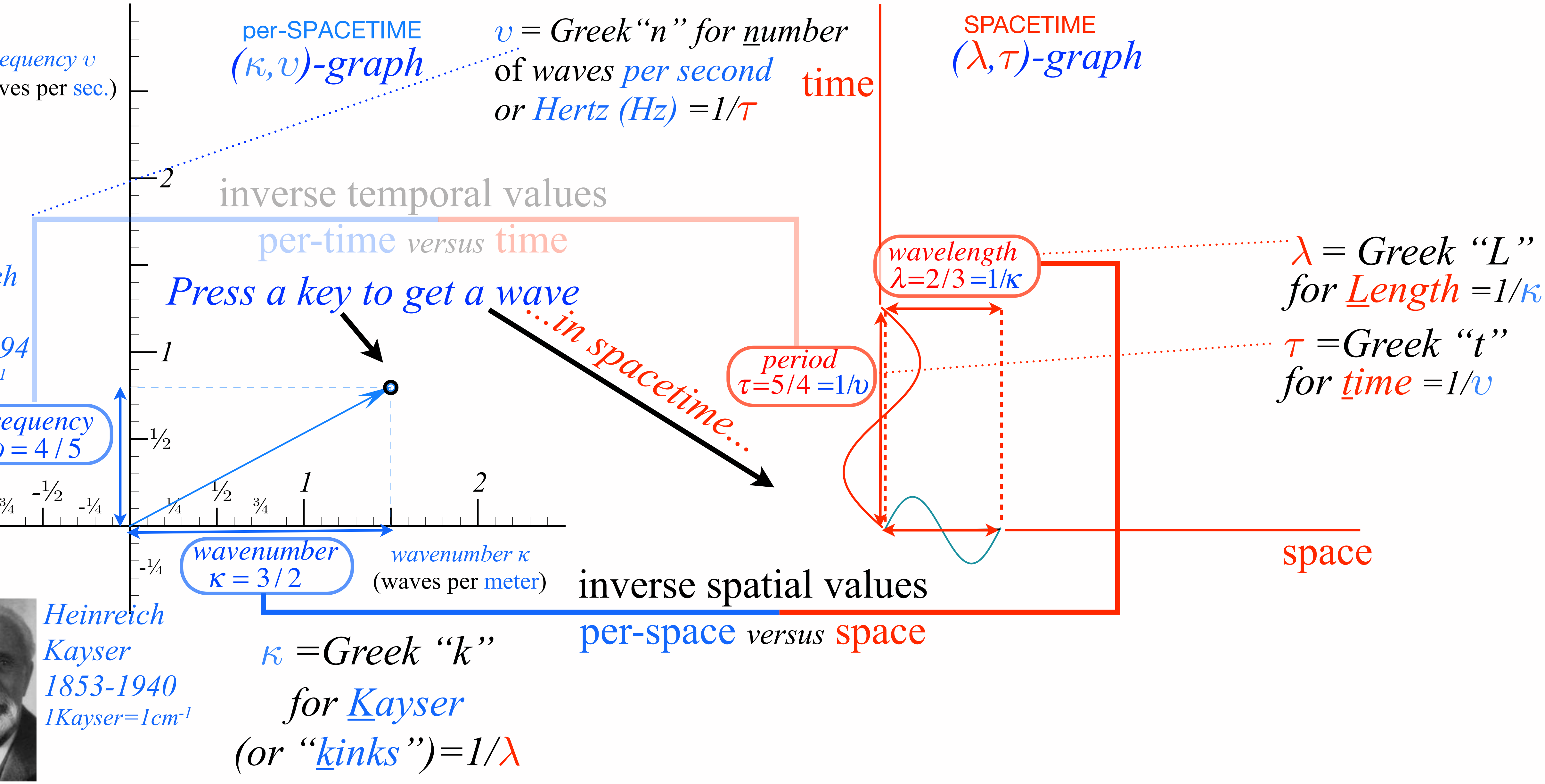
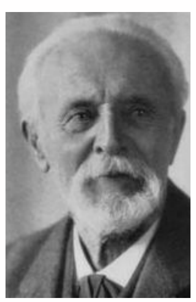


Heinreich Hertz 1857-1894

1Hz=1sec⁻¹



Heinreich Kayser 1853-1940

1Kayser=1cm⁻¹

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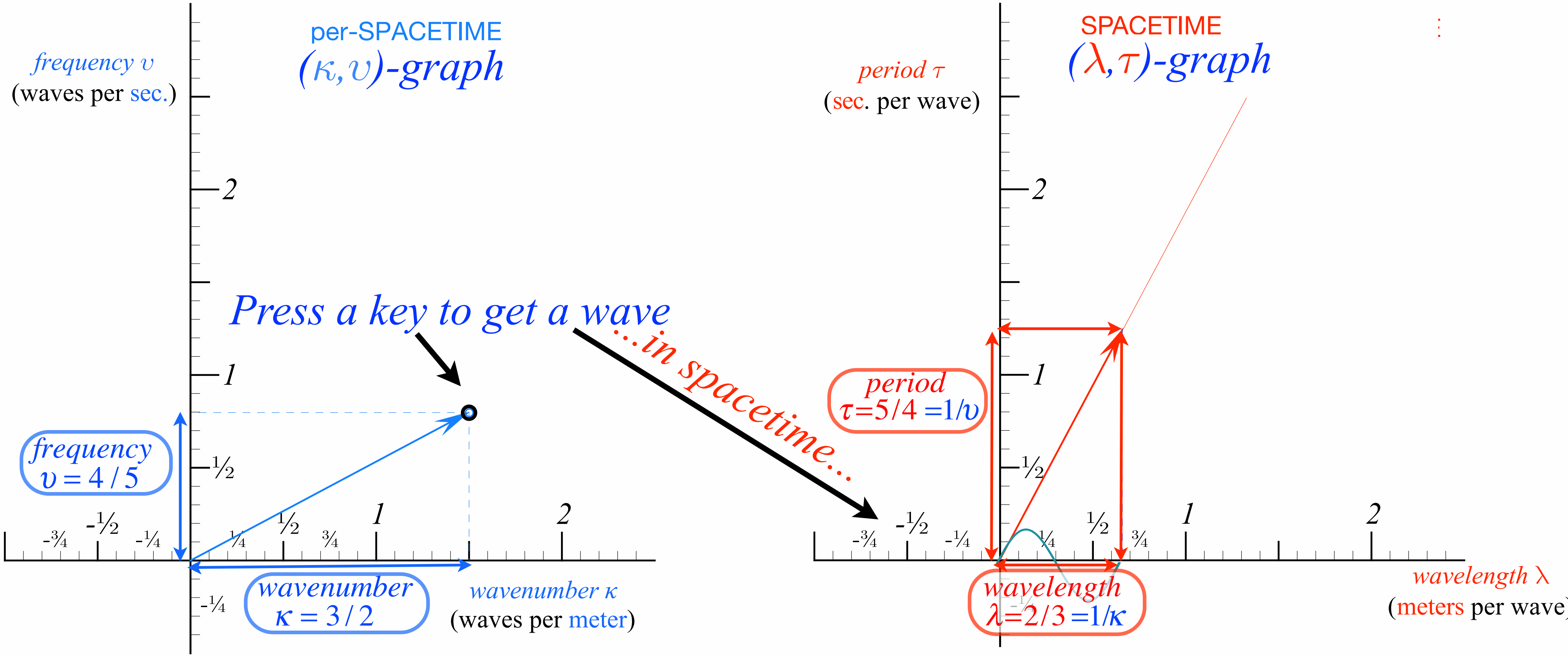

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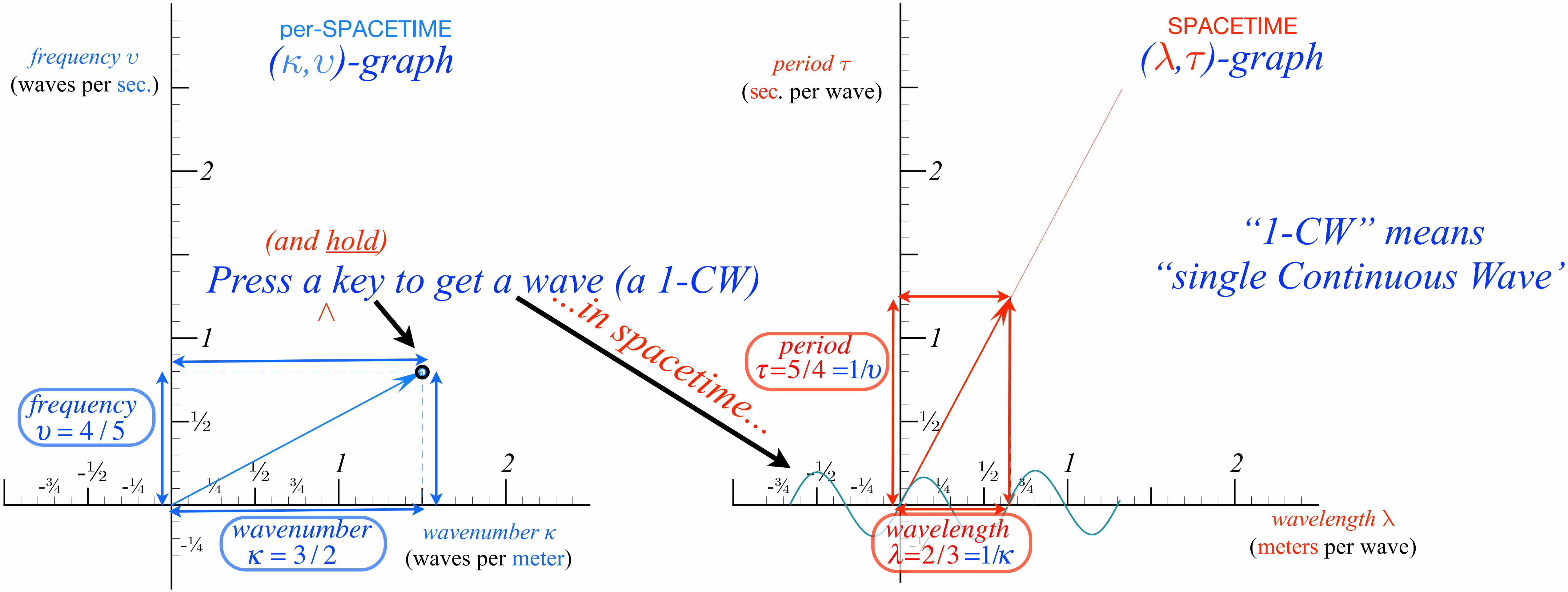

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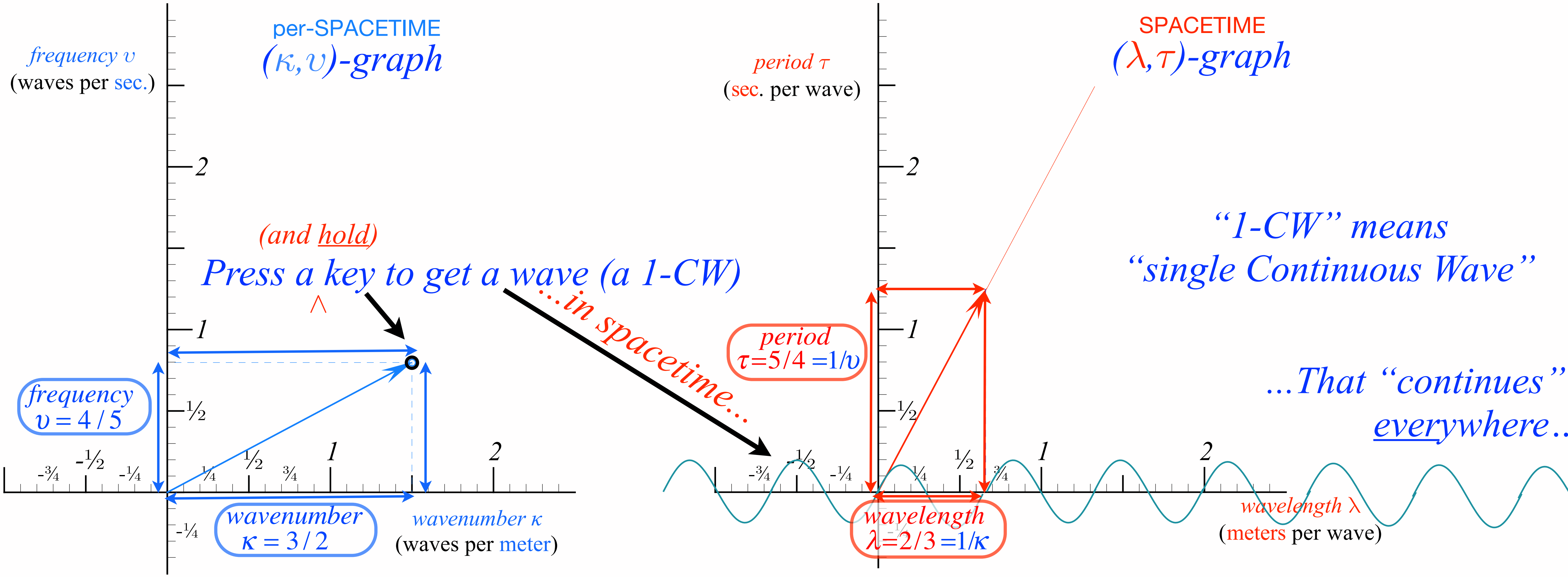

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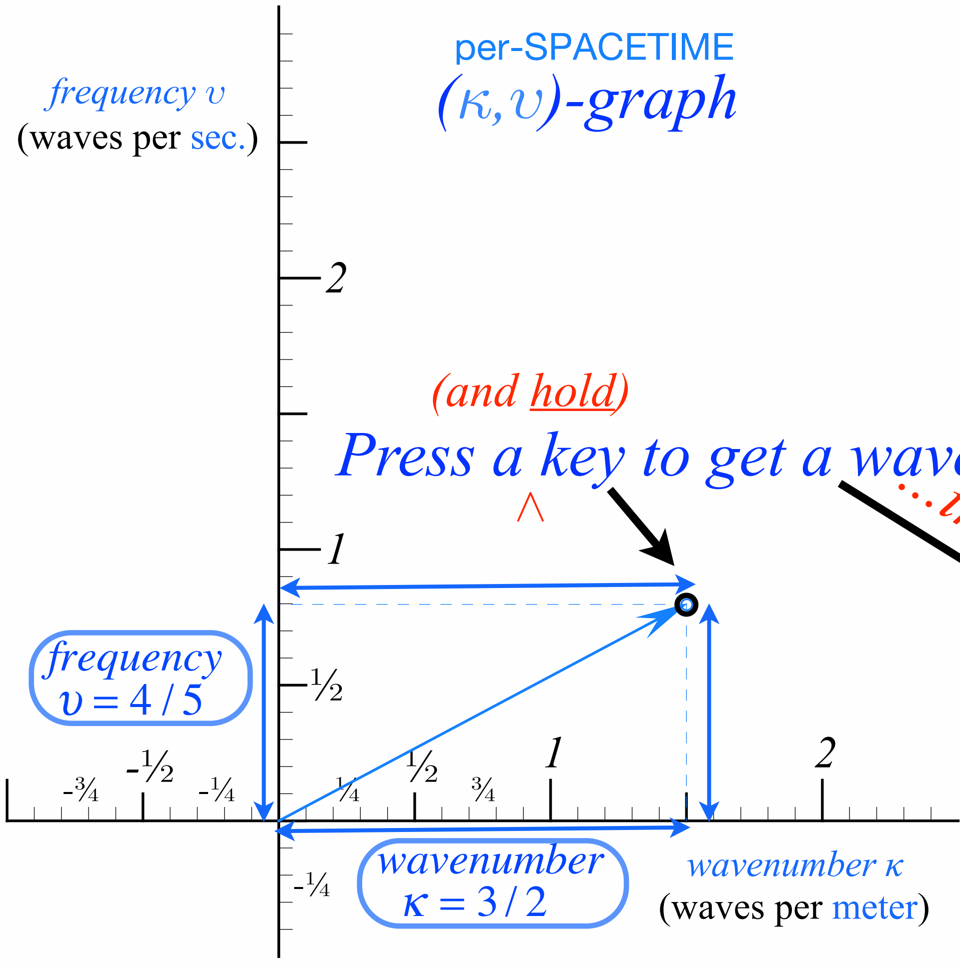
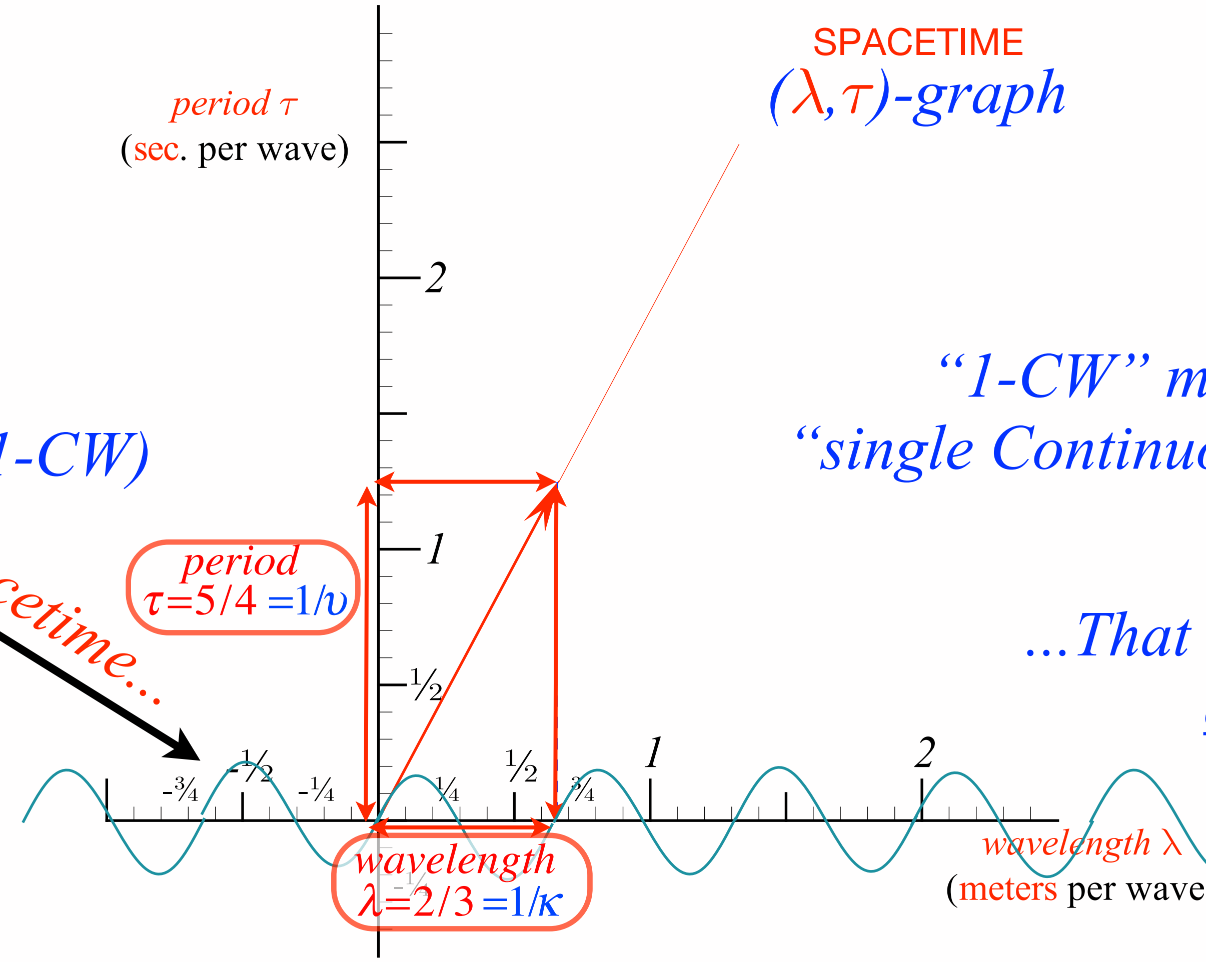


(and hold)

Press a key to get a wave (a 1-CW)

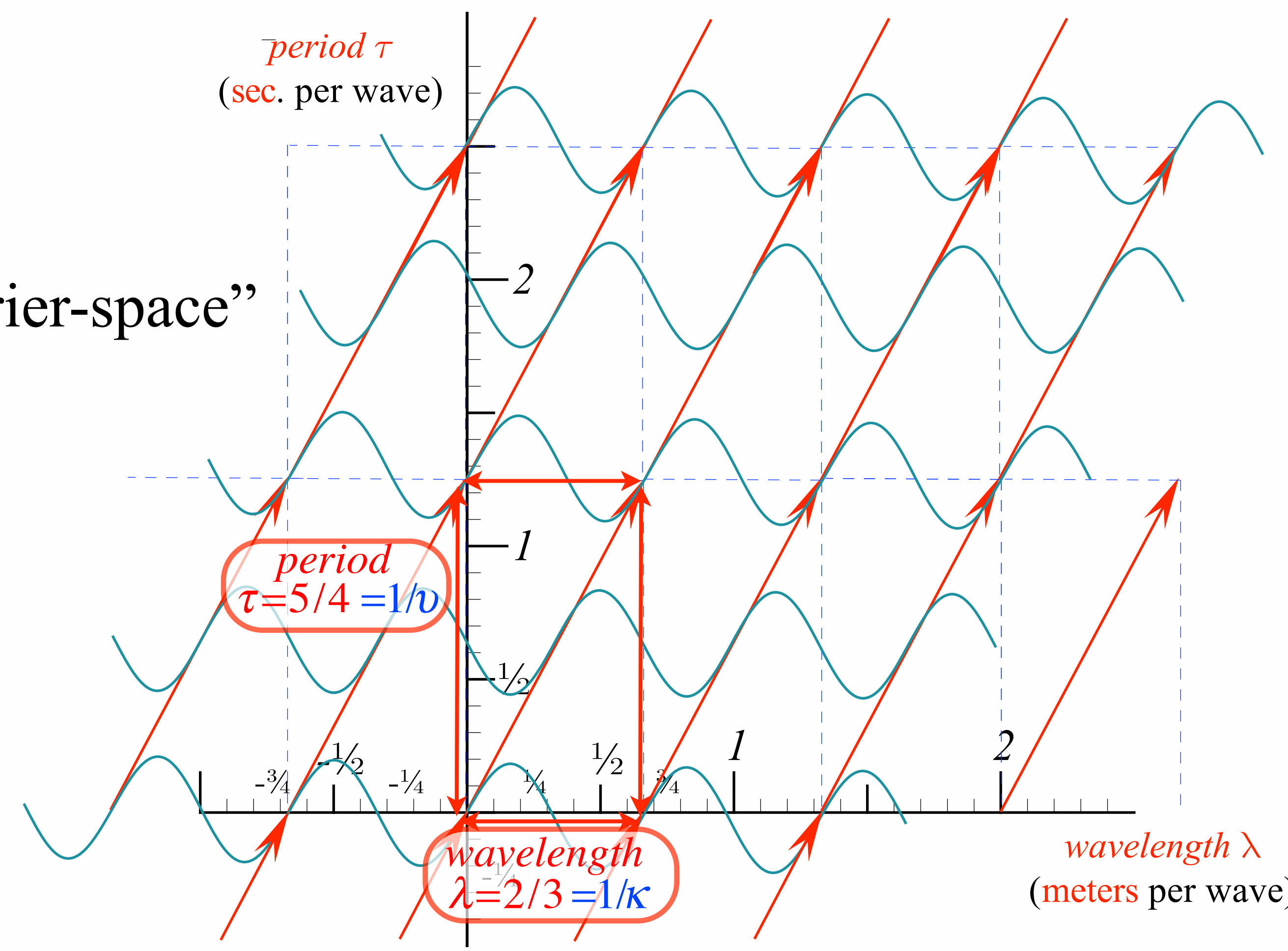
...in spacetime...



"1-CW" means "single Continuous Wave"

...That "continues" everywhere..

...for all time...



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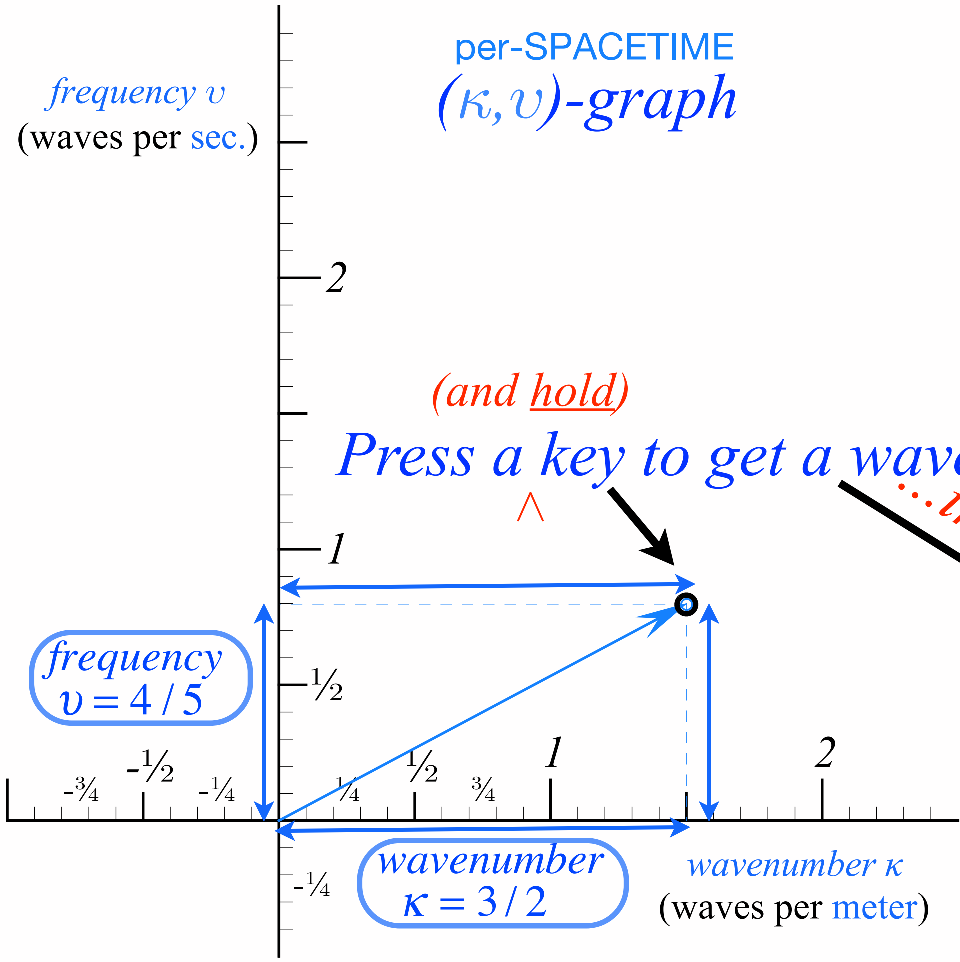

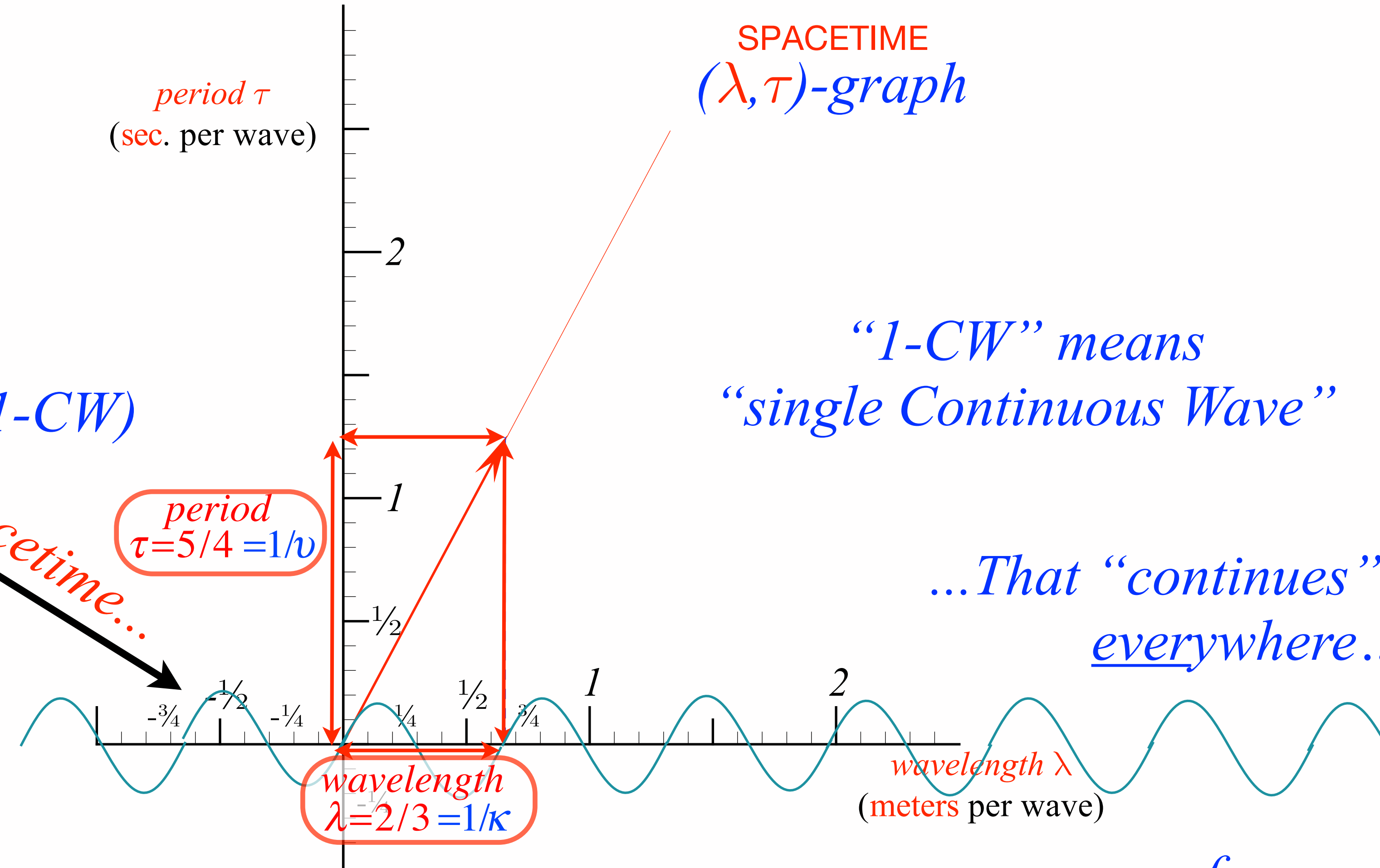

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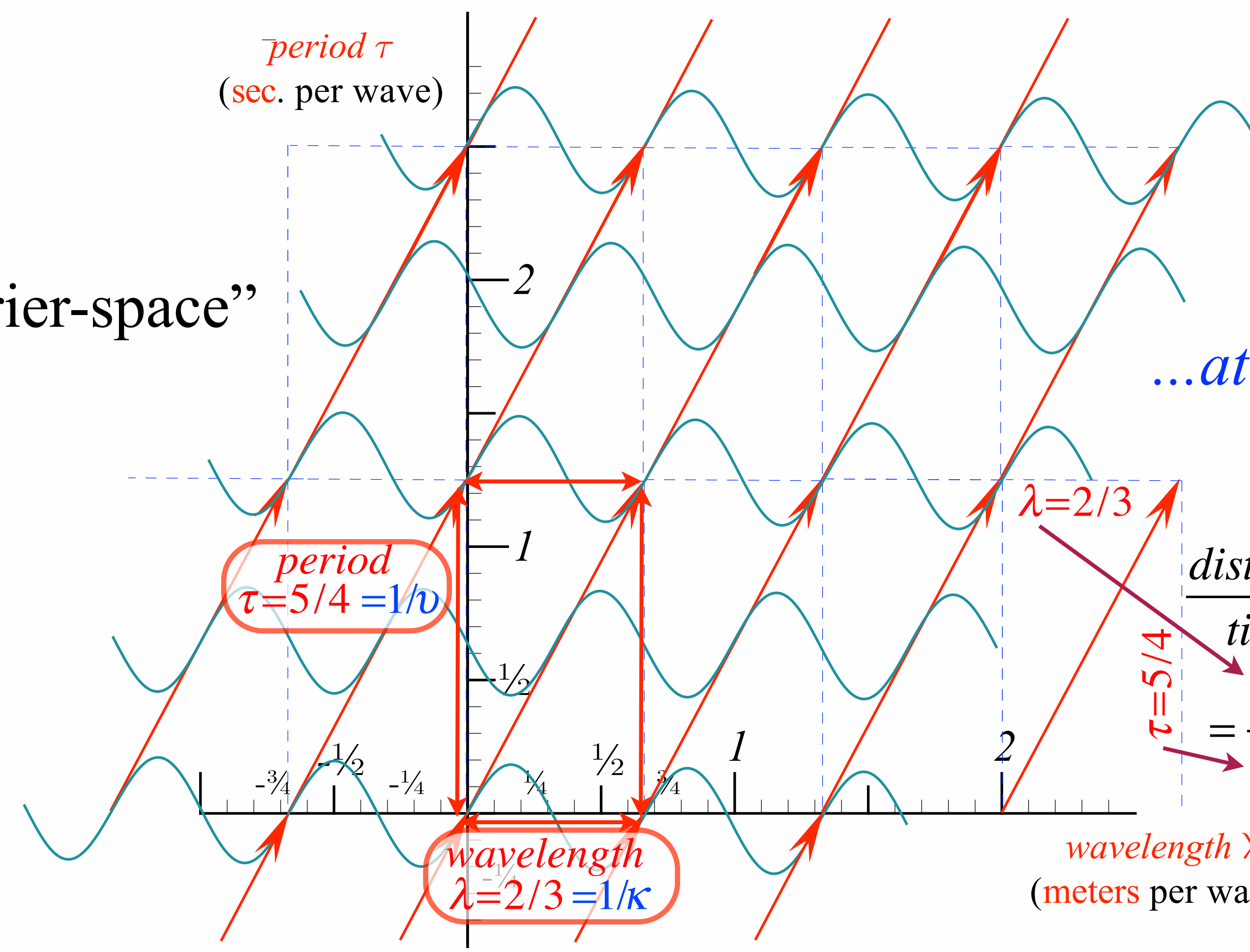

...at a speed of:

$$\frac{distance}{time} = \frac{wavelength}{period} = \frac{\lambda}{\tau}$$

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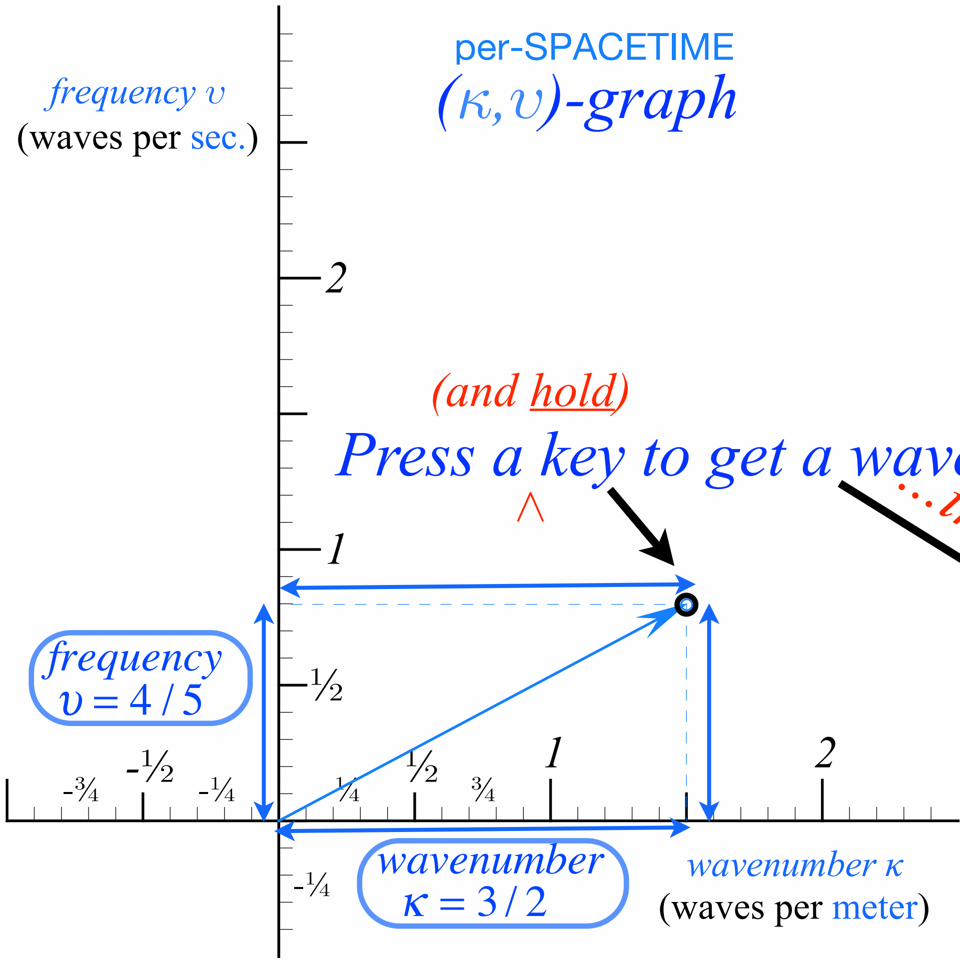
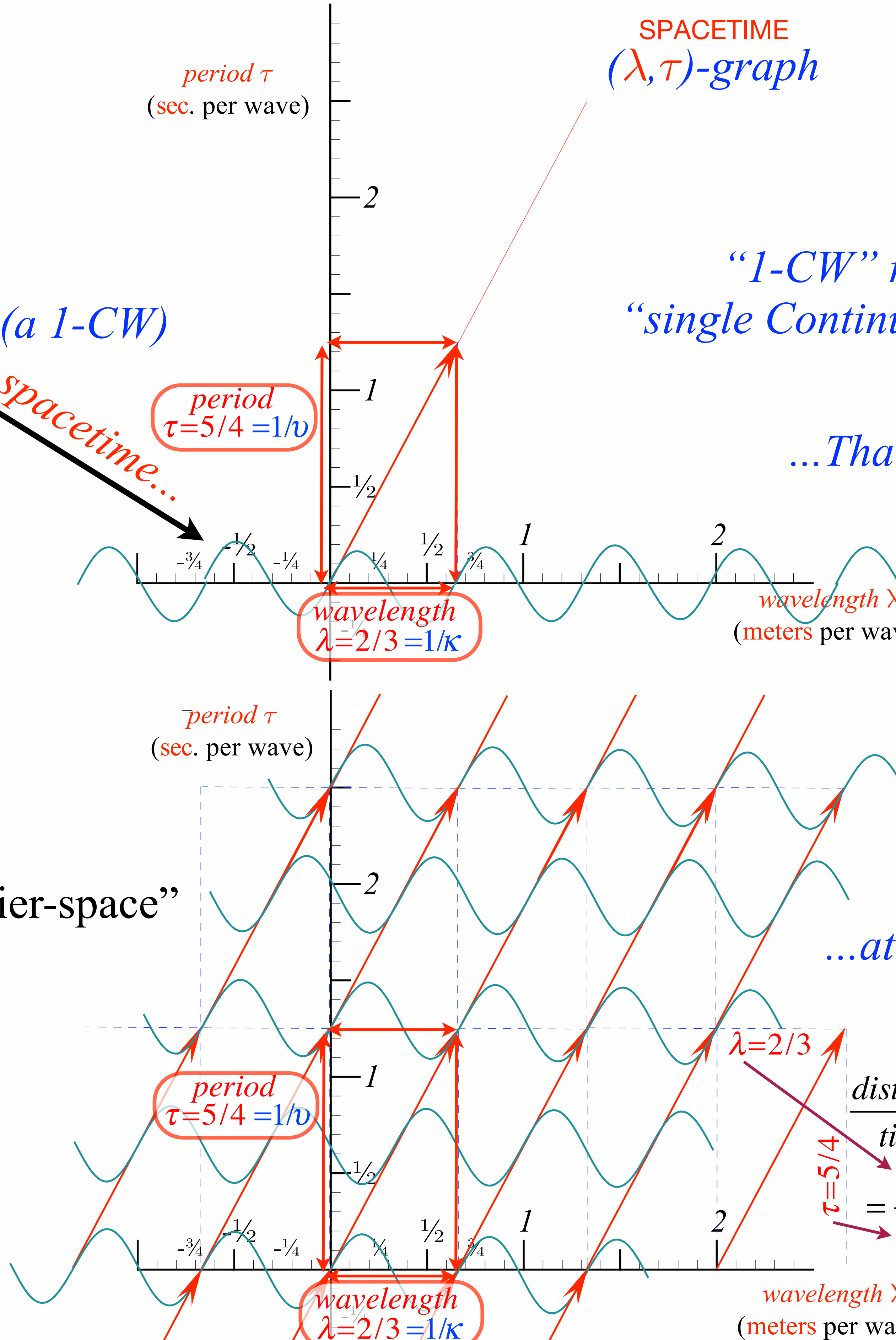


...in spacetime...



"1-CW" means "single Continuous Wave"

...That "continues" everywhere..

...for all time...

...at a speed of:

$$\frac{distance}{time} = \frac{wavelength}{period} = \frac{\lambda}{\tau} = \frac{2/3}{5/4} = \frac{8}{15}\frac{m.}{s.}$$

wave-speed equals slope-to-vertical λ/τ in (λ,τ)-graph

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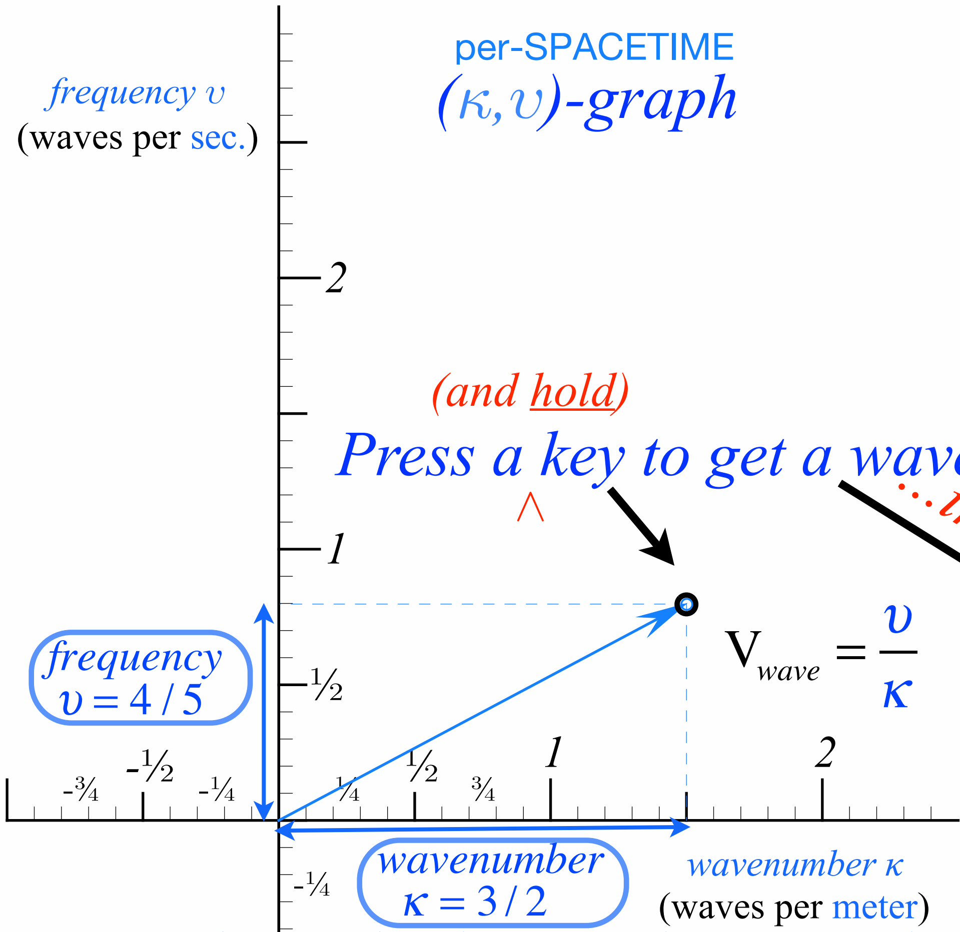
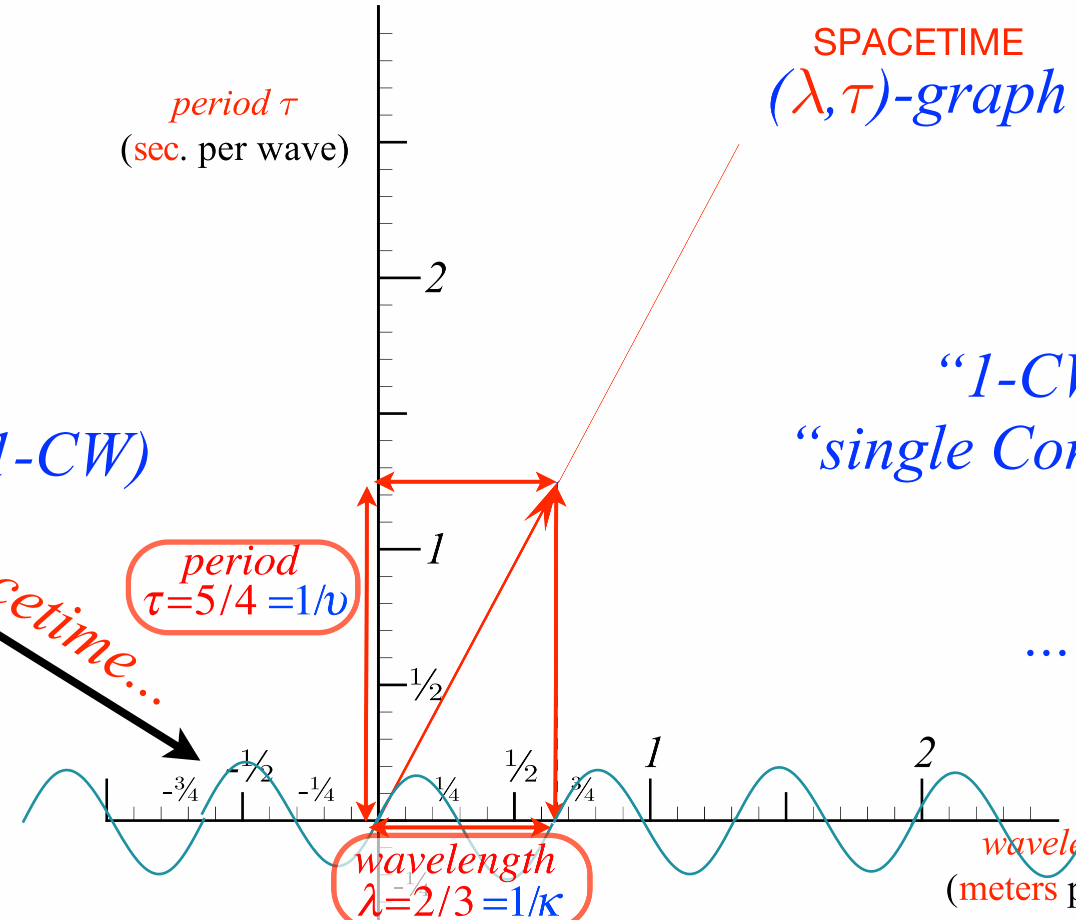


$V_{wave} = \frac{\upsilon}{\kappa}$

...in spacetime...



"1-CW" means
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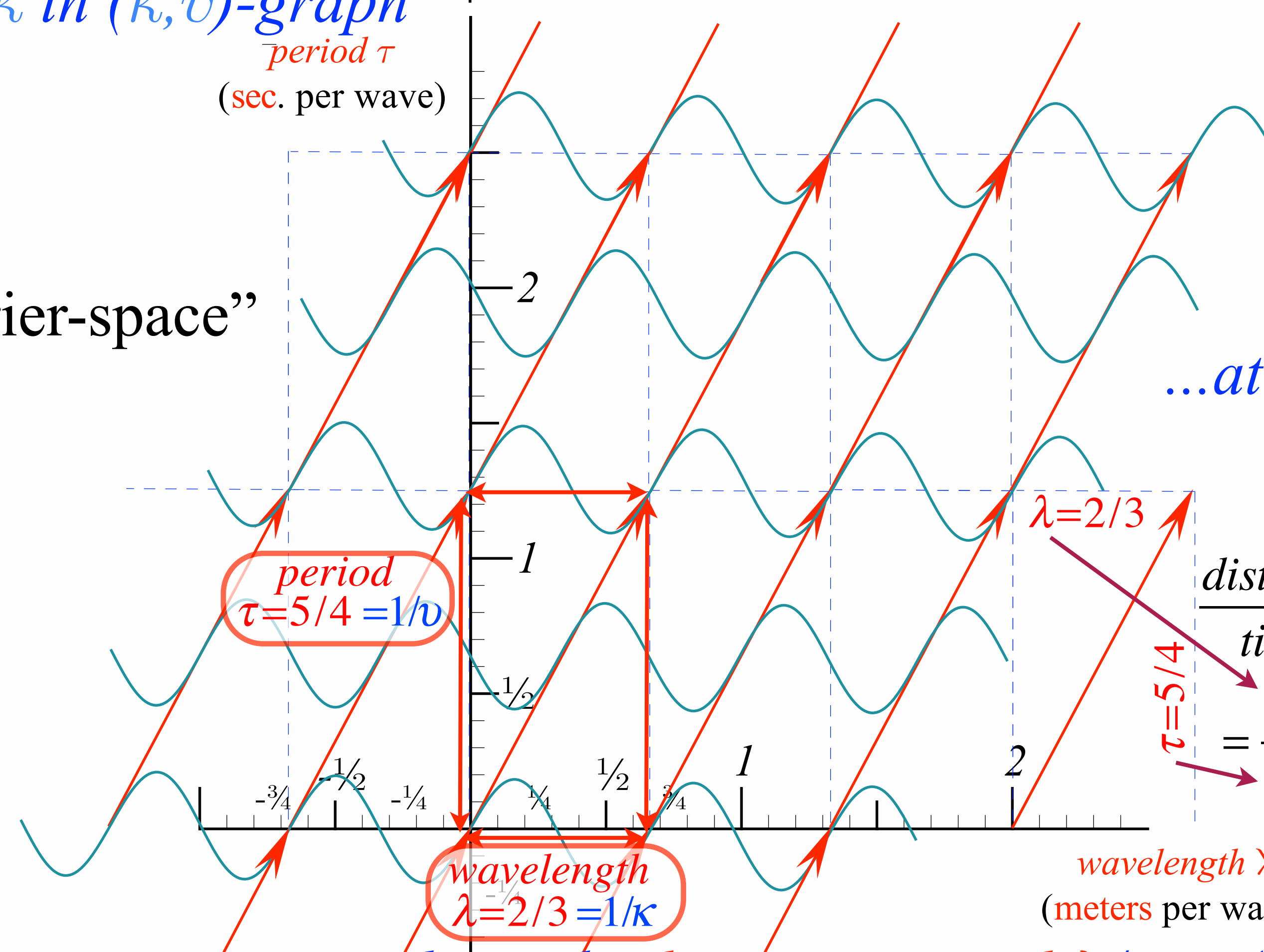
wave-speed equals slope-to-horizontal υ/κ in (κ,υ)-graph



...for all time...

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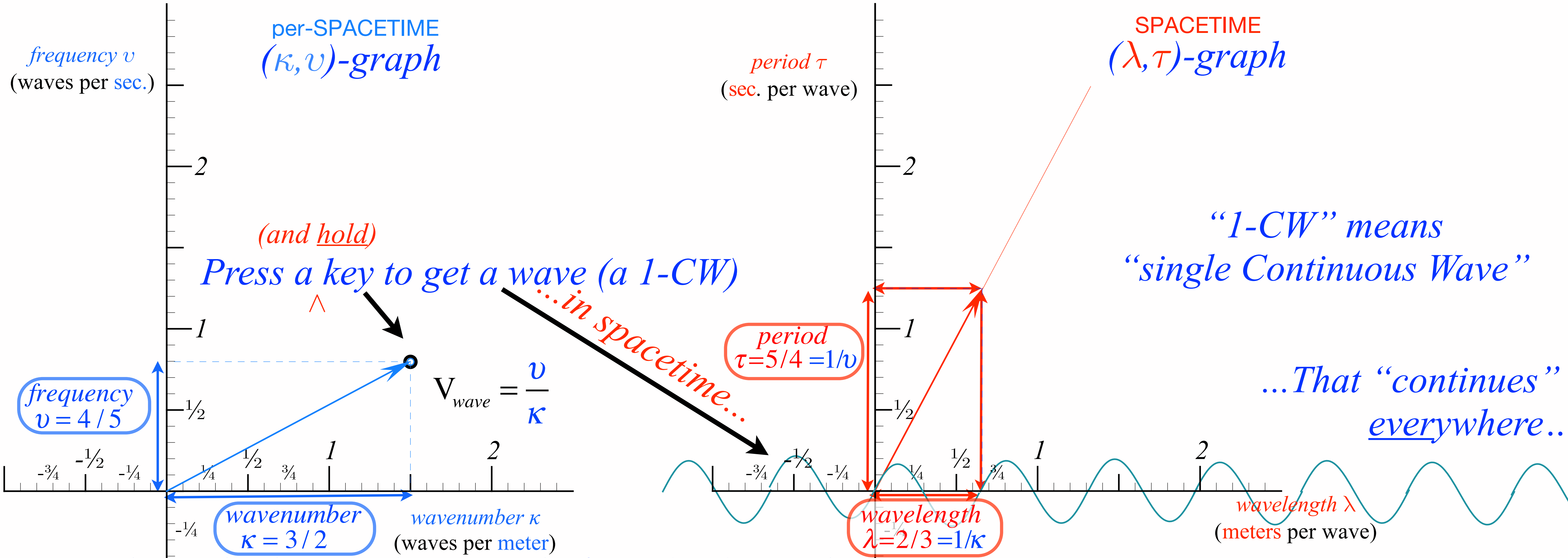
...at a speed of:

$$\frac{distance}{time} = \frac{wavelength}{period} = \frac{\lambda}{\tau} = \frac{2/3}{5/4} = \frac{8}{15}\frac{m.}{s.} = \frac{1/\kappa}{1/\upsilon} = \frac{\upsilon}{\kappa}$$

- *How to understand waves and wave velocity V_{wave}*

wave-speed equals slope-to-vertical λ/τ in (λ,τ)-graph

The "Keyboard of the gods" : Introducing per-space-per-time graphs *versus* space-time graphs



wave-speed equals slope-to-horizontal υ/κ in (κ,υ)-graph

wave-velocity formulas

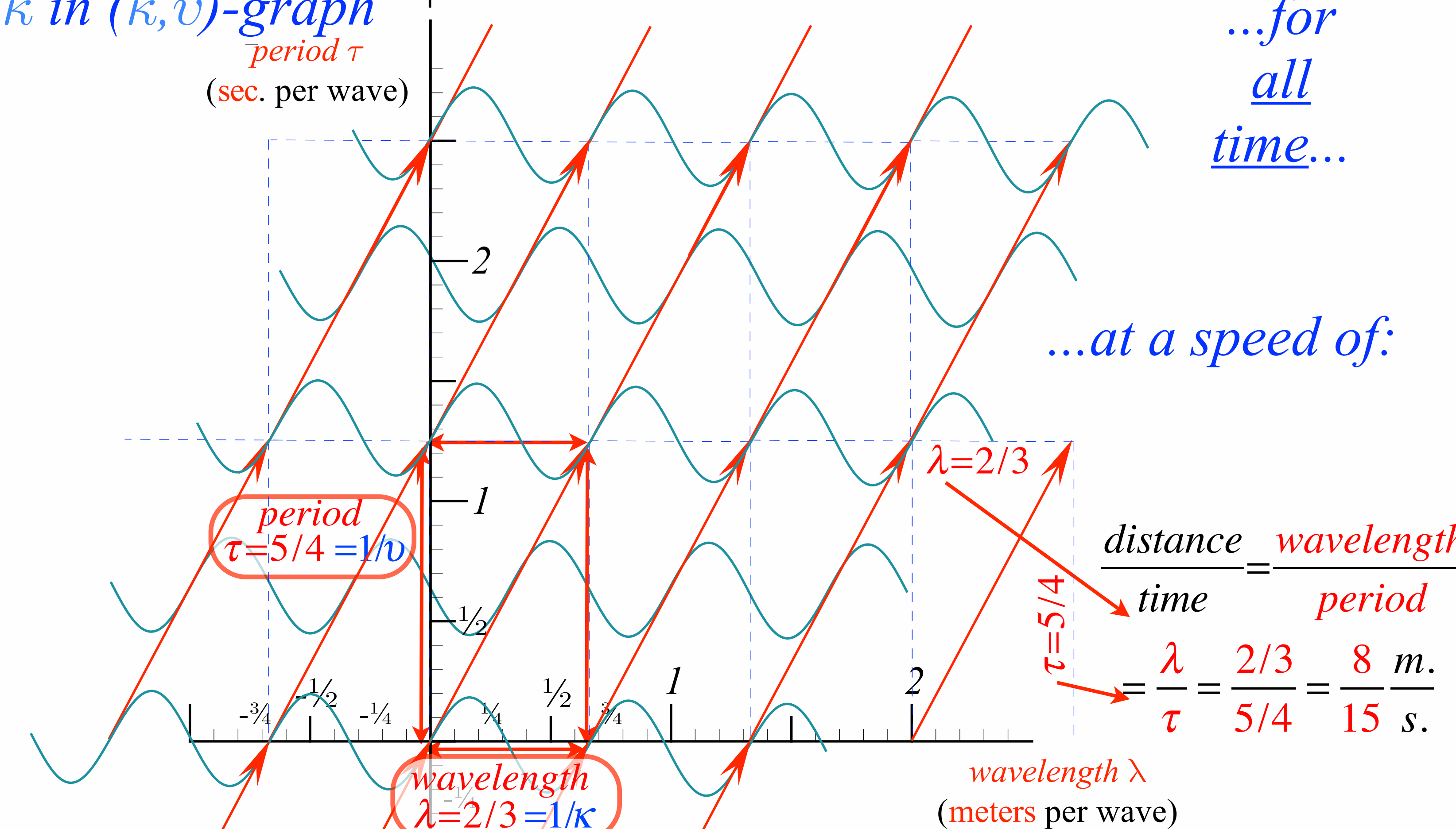
$$\frac{distance}{time} = \frac{wavelength}{period} = \frac{frequency}{wavenumber}$$

$$V_{wave} = \frac{\lambda}{\tau} = \frac{1/\kappa}{1/\upsilon} = \frac{\upsilon}{\kappa} = \frac{1/\tau}{1/\lambda}$$

$$= \frac{2/3}{5/4} = \frac{4/5}{3/2} = \frac{8}{15}\frac{m.}{s.}$$

wave arithmetic is simpler to explain using fractions

- *How to understand waves and "1st quantization"*

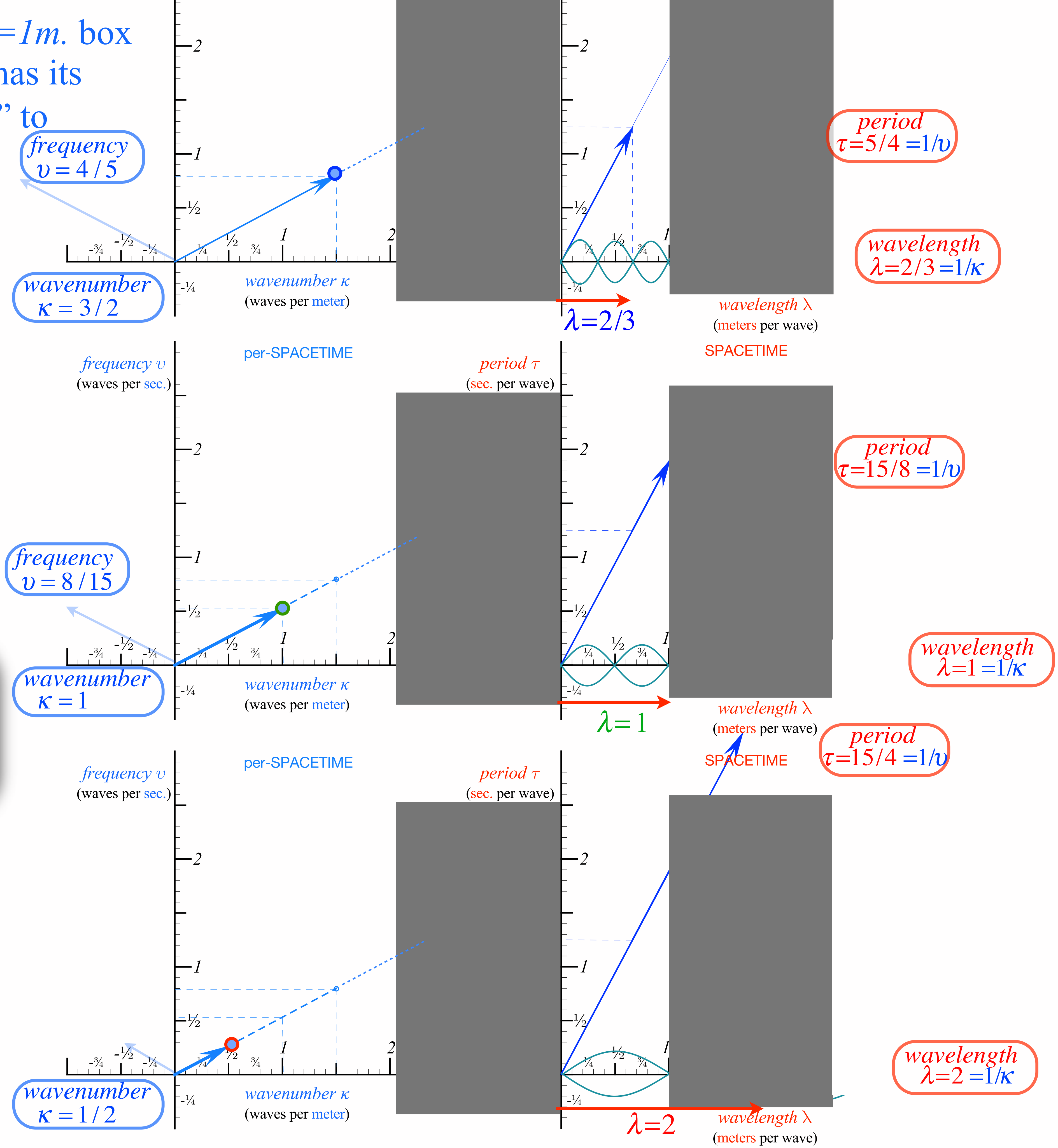


wave-speed equals slope-to-vertical λ/τ in (λ,τ)-graph

If a wave is confined to an *L=1m.* box *the "Keyboard of the gods"* has its wavenumber κ is "quantized" to multiples of *1/2L=1/2*.

$$\kappa = \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, \ldots$$

•*How to understand waves*
and
"1st quantization"
or κ-quantization



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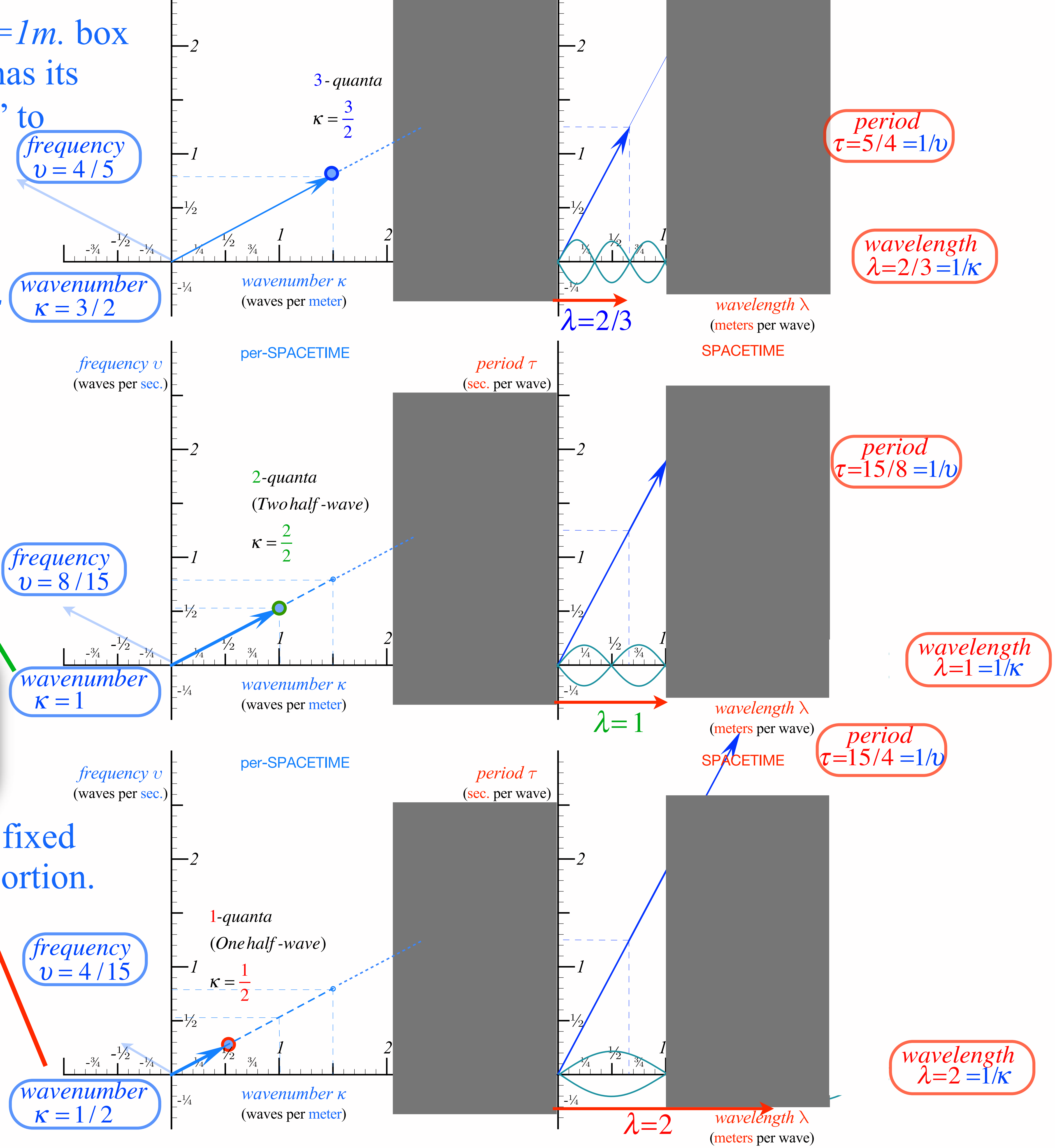
$\kappa = 1/2, 1, 3/2, 2, \ldots$

- *How to understand waves and "1st quantization" or κ-quantization*

If wave velocity $V_{wave}=8/15$ is fixed *frequency* is quantized in proportion.



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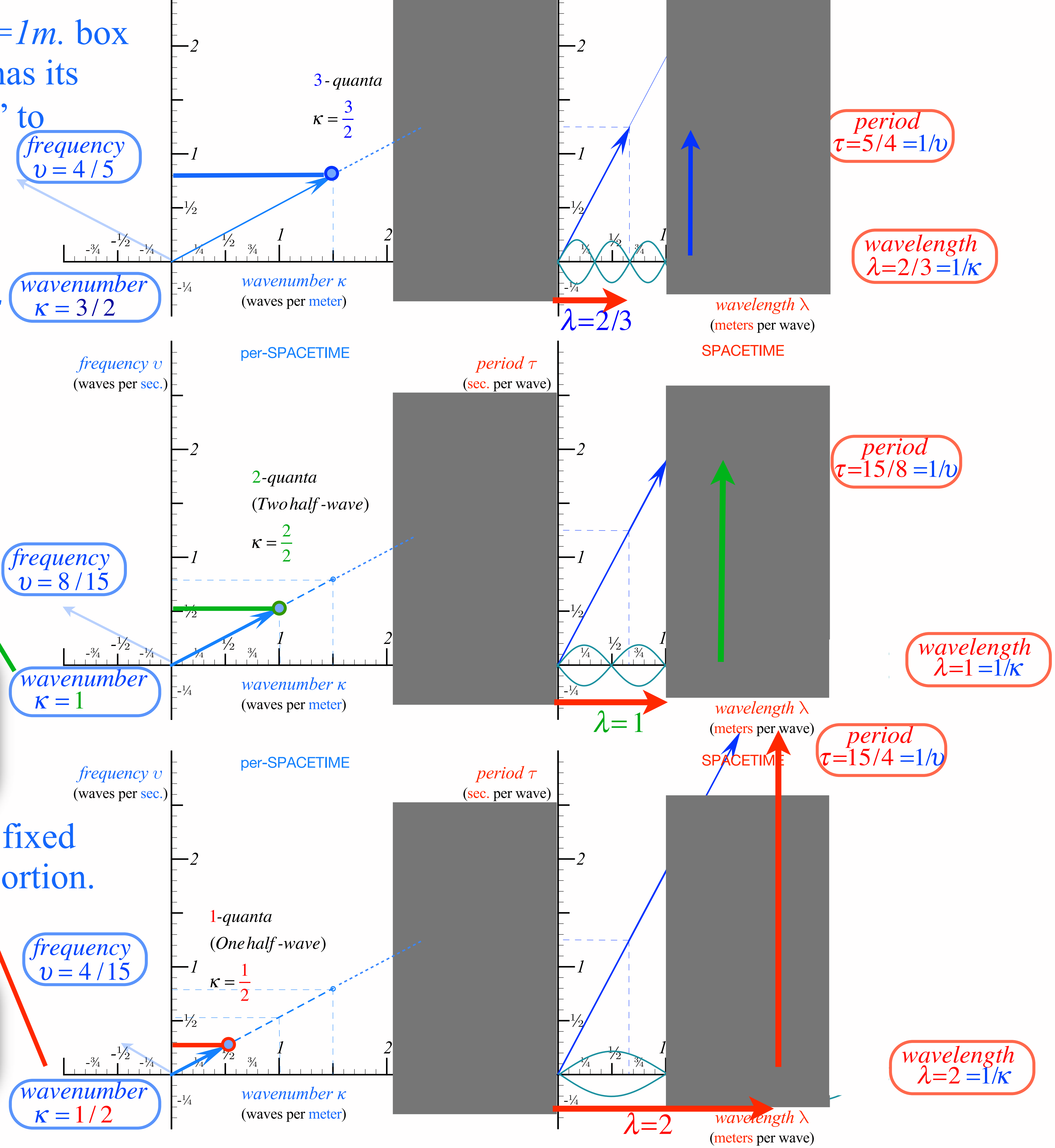
$$\kappa = \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, ...$$

κ = *1/2*, *1*, *3/2*, *2*,...

•*How to understand waves and "1st quantization" or κ-quantization*

If wave velocity V_{wave}=*8/15* is fixed *frequency* is quantized in proportion.

•*Amplitude A-quantization is called "2nd quantization"*



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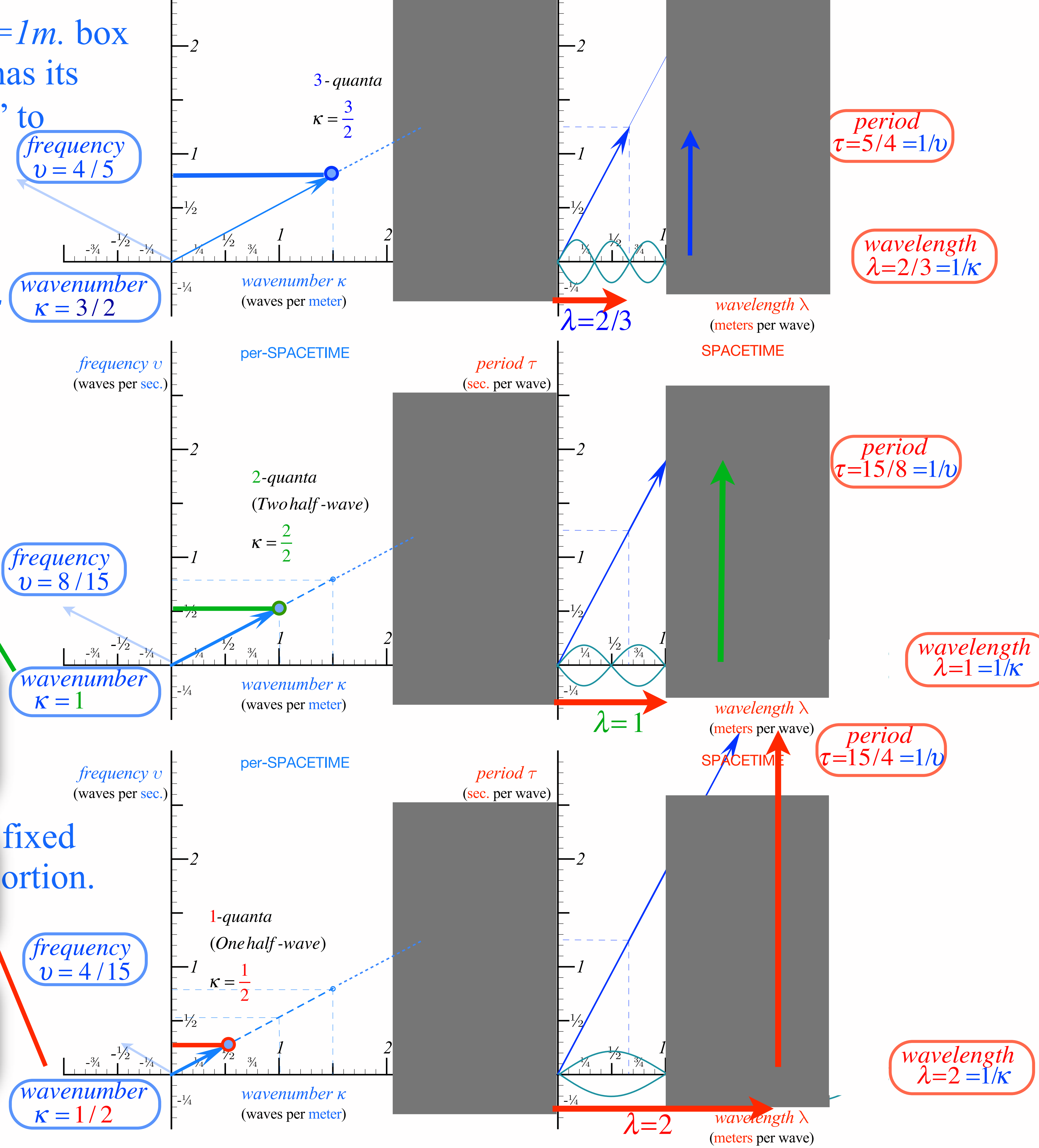
κ = *1/2, 1, 3/2, 2,...*



•How to understand waves and "1st quantization" or κ-quantization

...as QUALITY (color) versus QUANTITY (Number of "photons")

•Amplitude A-quantization is called "2nd quantization"

...s fixed ...portion.

As will be shown:

Light wave-velocity c is VERY fixed

$$V_{light} = c = \frac{\lambda}{\tau} = \frac{1/\kappa}{1/\upsilon} = \frac{\upsilon}{\kappa} = \frac{1/\tau}{1/\lambda} = 299{,}792{,}458 \frac{m.}{s.}$$

As will be shown:

Light wave-velocity c is VERY fixed

$$V_{light} = c = \frac{\upsilon}{\kappa} = \frac{1/\kappa}{1/\upsilon} = \frac{\lambda}{\tau} = \frac{1/\tau}{1/\lambda} = 299{,}792{,}458\,\frac{m.}{s.}$$

Then it's convenient to use:

...or angular variables: $\omega=2\pi\upsilon$

and: $k=2\pi\kappa$

Dimensionless Light wave-velocity c/c=1

$$\frac{V_{light}}{c} = \frac{\upsilon}{c\kappa} = \frac{\lambda}{c\tau} = 1 \quad \textit{instead of:} \quad \frac{\upsilon}{\kappa} = \frac{\lambda}{\tau} = c$$

$$\frac{\upsilon}{\kappa} = \frac{\lambda}{\tau} = c = \frac{\omega}{k}$$

Such graphs use c-units of per-time $\upsilon=c\kappa$ and length $\lambda=c\tau$.

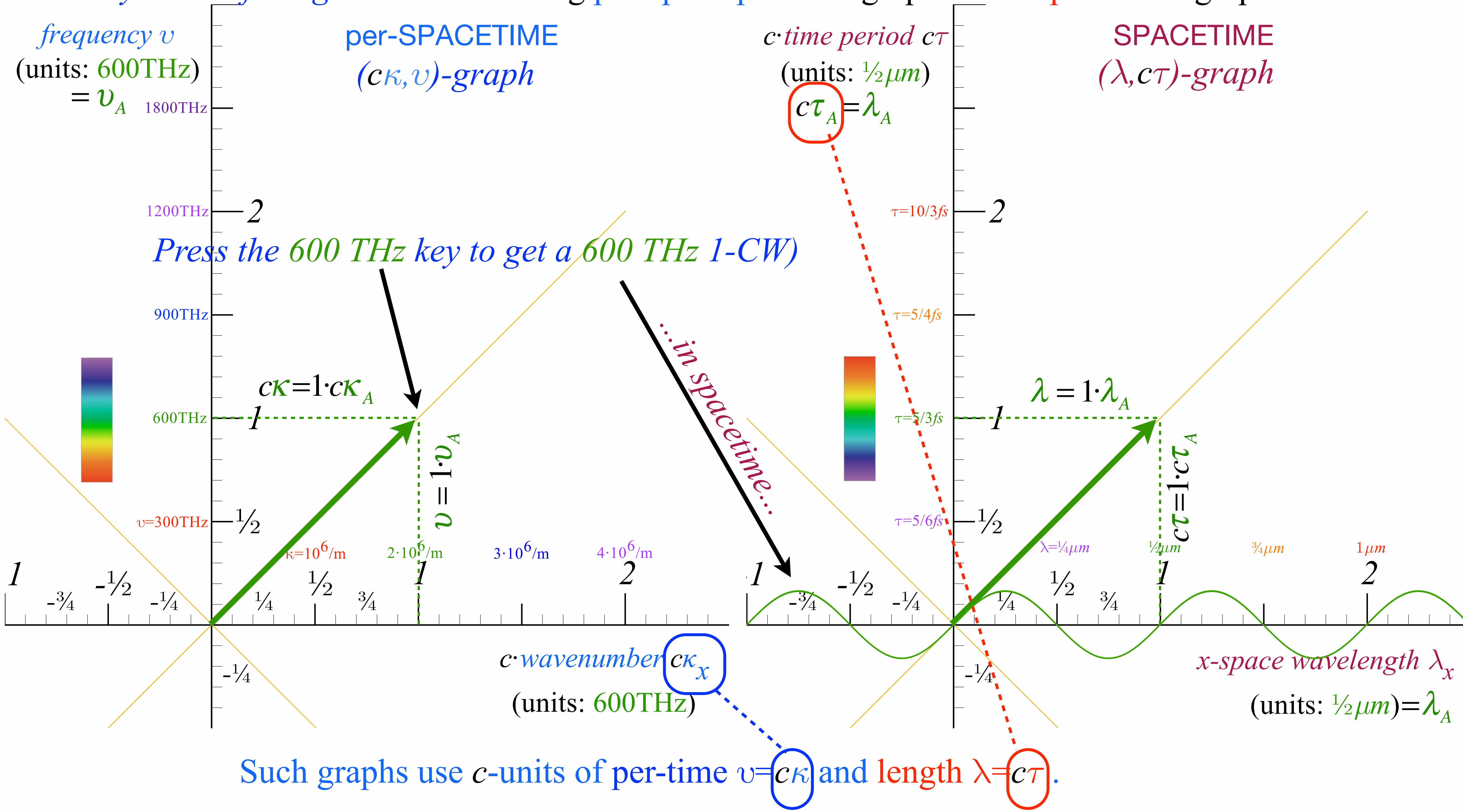
$$\frac{V_{light}}{c} = \frac{\upsilon}{c\kappa} = \frac{1/\kappa}{c/\upsilon} = \frac{\lambda}{c\tau} = \frac{1/\tau}{c/\lambda} = 1$$

$$\frac{\upsilon}{c\kappa} = 1 = \frac{\omega}{ck}$$

$$ck = \omega$$

$$c\kappa = \upsilon$$

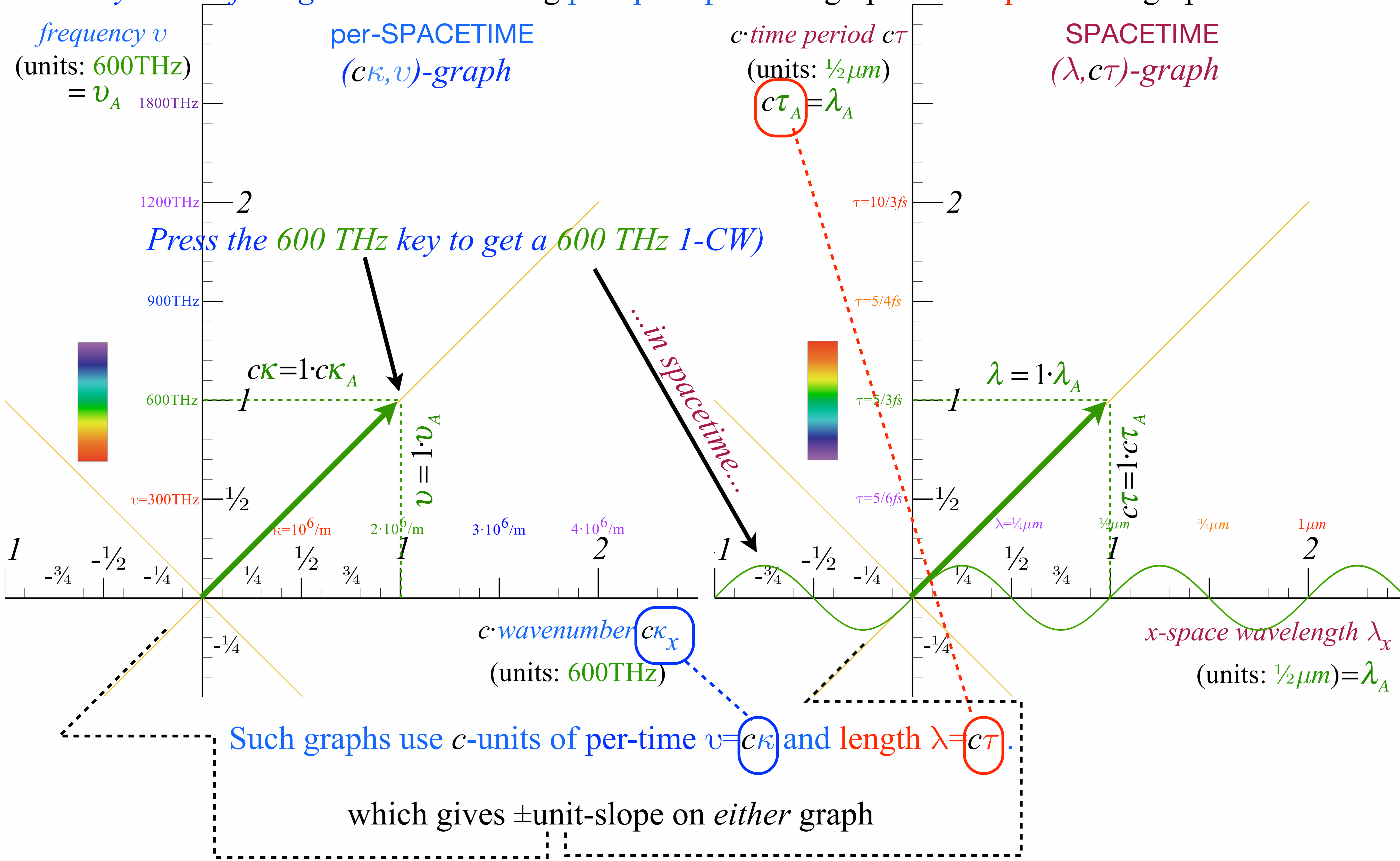
The "Keyboard of the gods" : Introducing per-space-per-time graphs *versus* space-time graphs



Such graphs use *c*-units of per-time υ=cκ and length λ=cτ.

Ways to quantify light waves (600 THz example)

The "Keyboard of the gods" : Introducing per-space-per-time graphs *versus* space-time graphs



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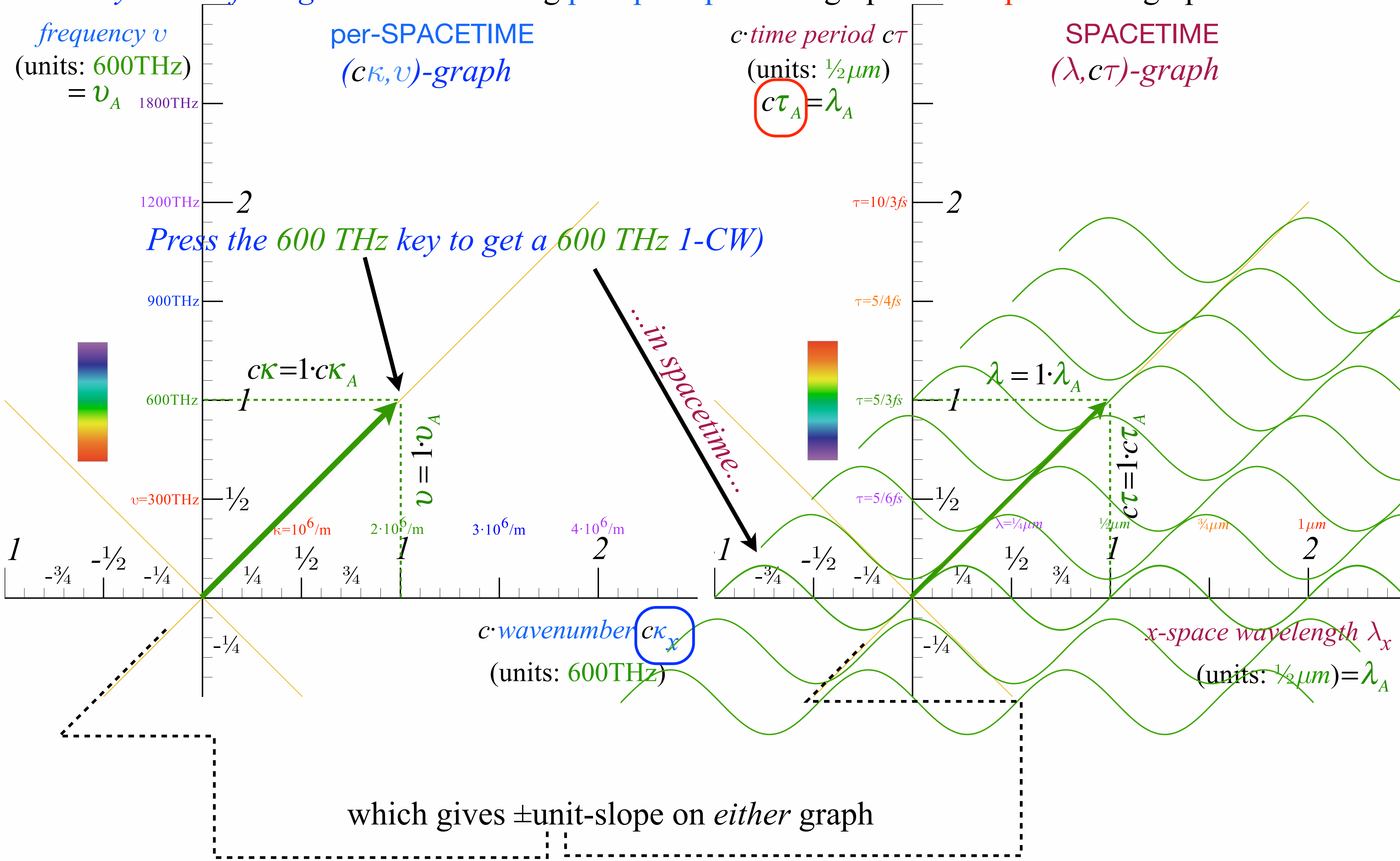
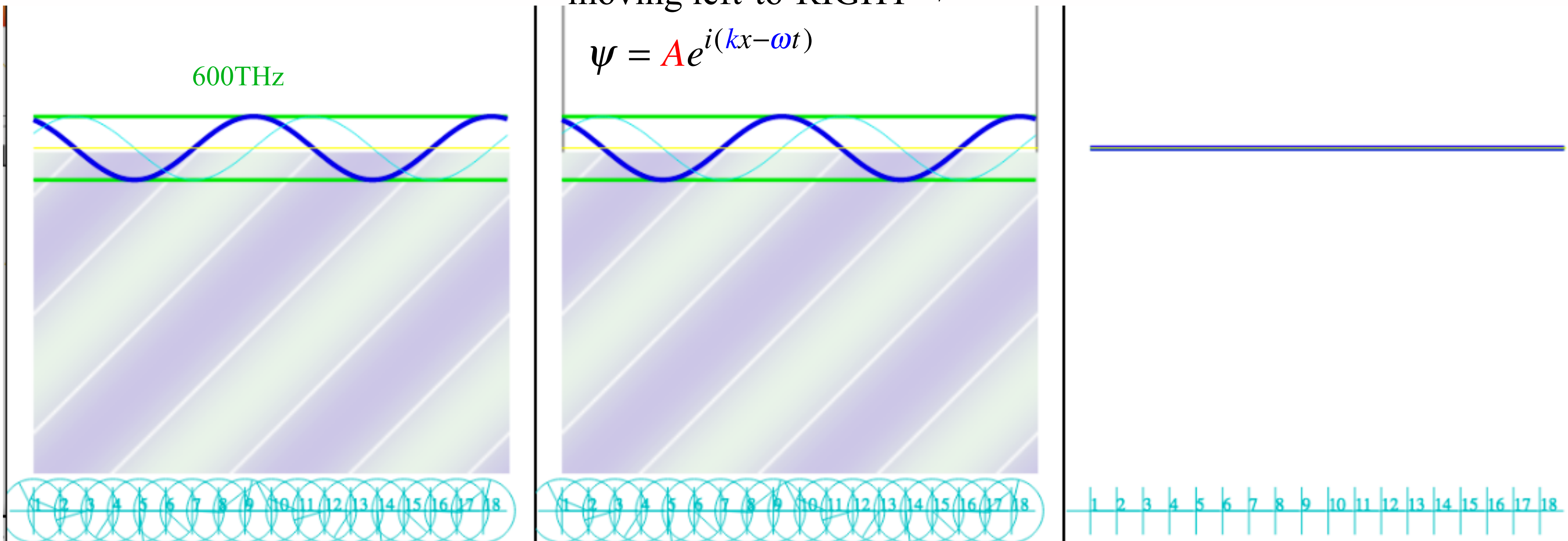
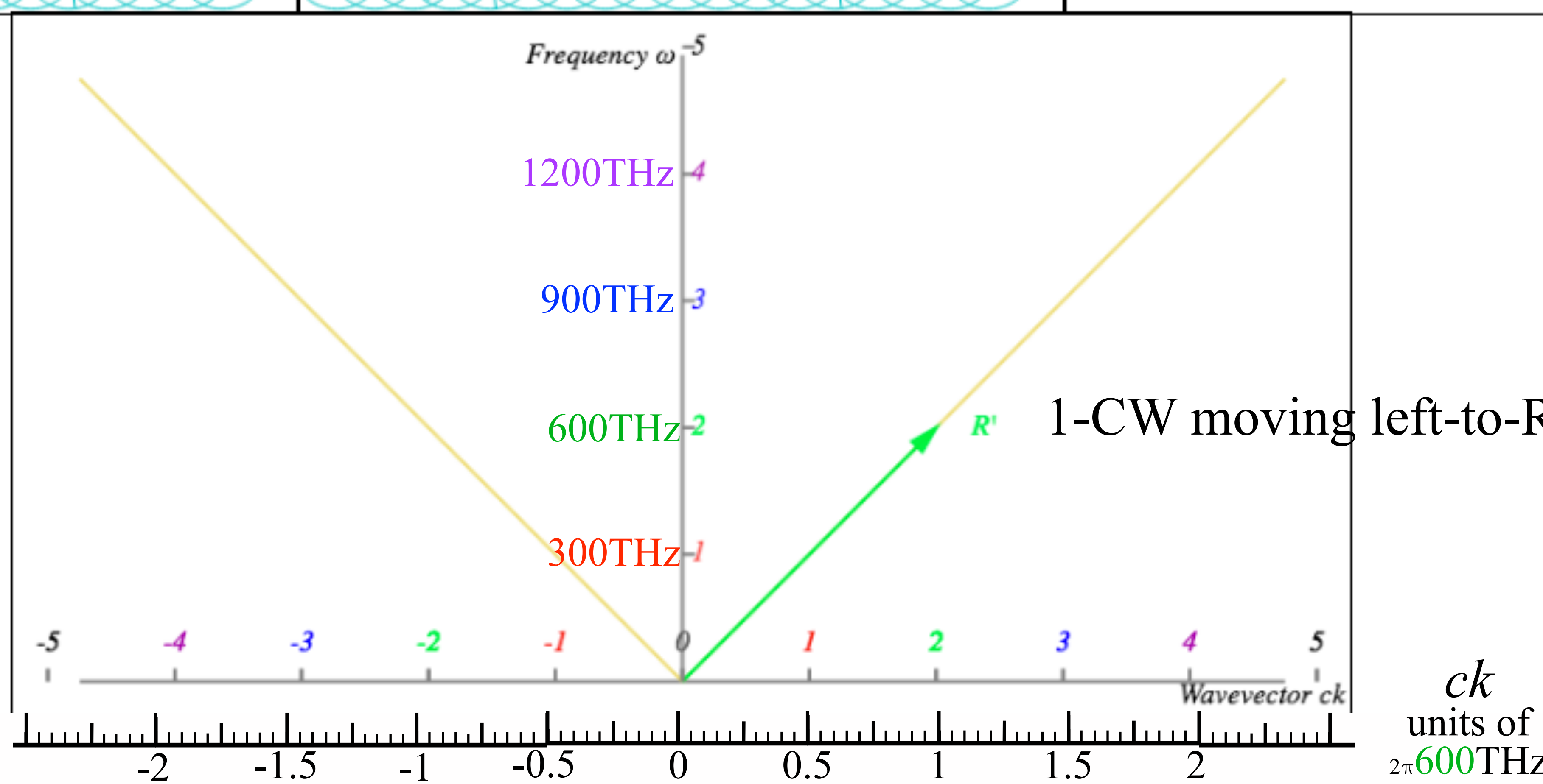


Ways to quantify light waves (600 THz example)

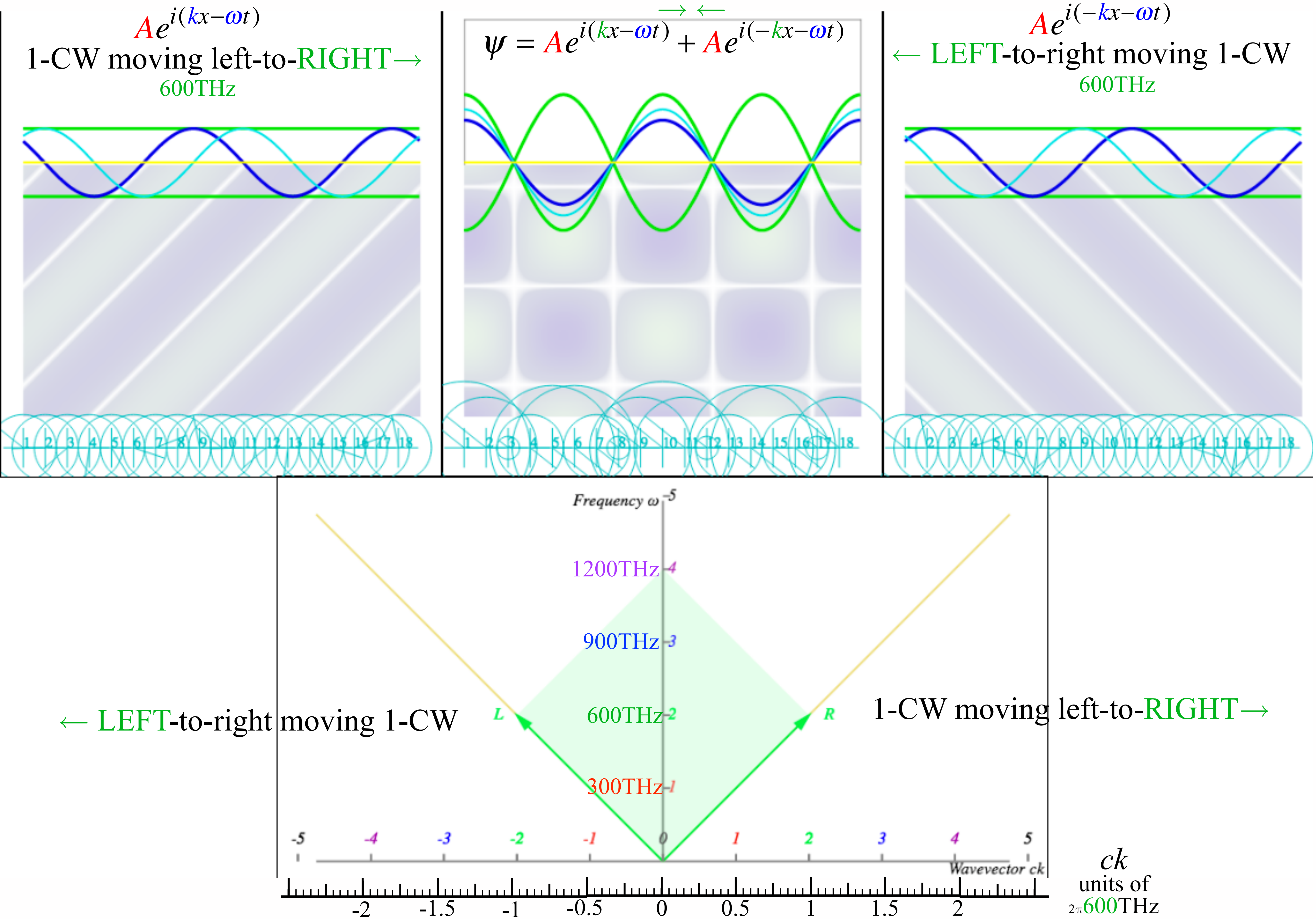
Single continuous wave (1-CW) moving left-to-RIGHT→

$$\psi = Ae^{i(kx-\omega t)}$$

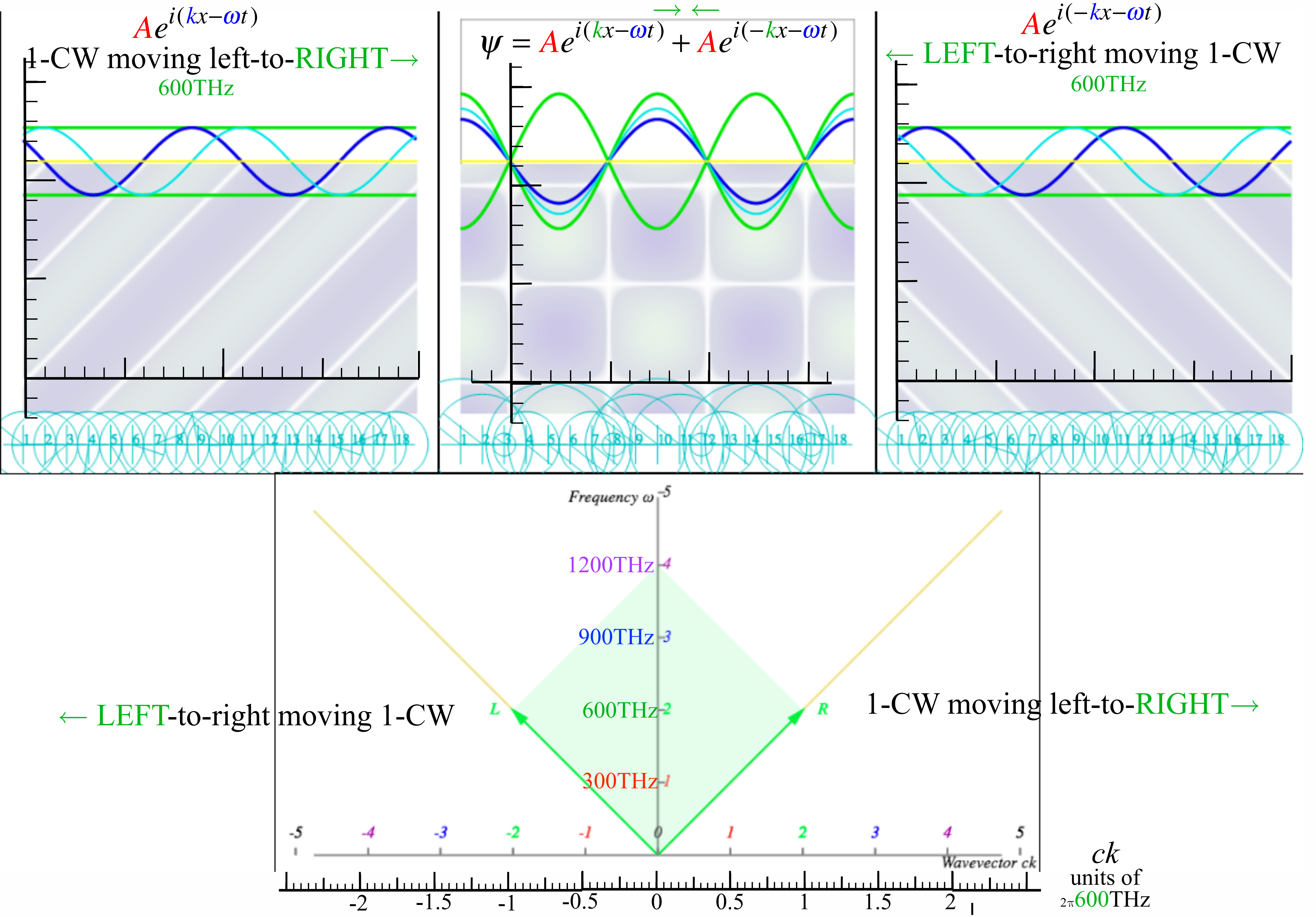




2 colliding waves (2-CW)



2 colliding waves (2-CW)

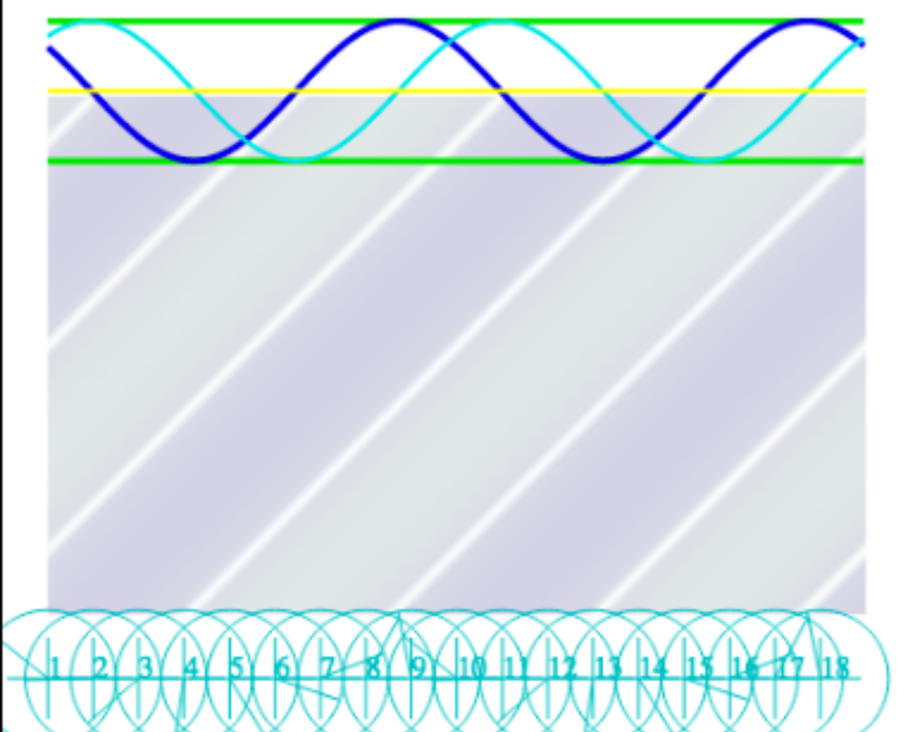


2 colliding waves (2-CW)

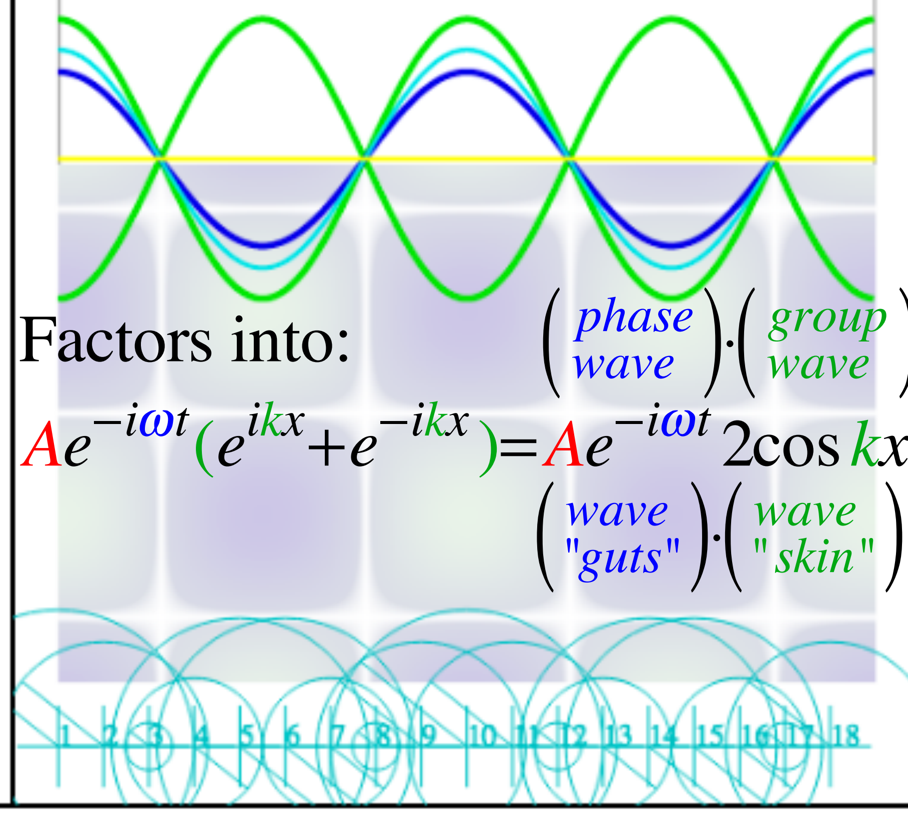
$Ae^{i(kx-\omega t)}$

1-CW moving left-to-RIGHT→

600THz



$$\psi = Ae^{i(kx-\omega t)} + Ae^{i(-kx-\omega t)}$$



Factors into: $\begin{pmatrix} phase \\ wave \end{pmatrix} \cdot \begin{pmatrix} group \\ wave \end{pmatrix}$

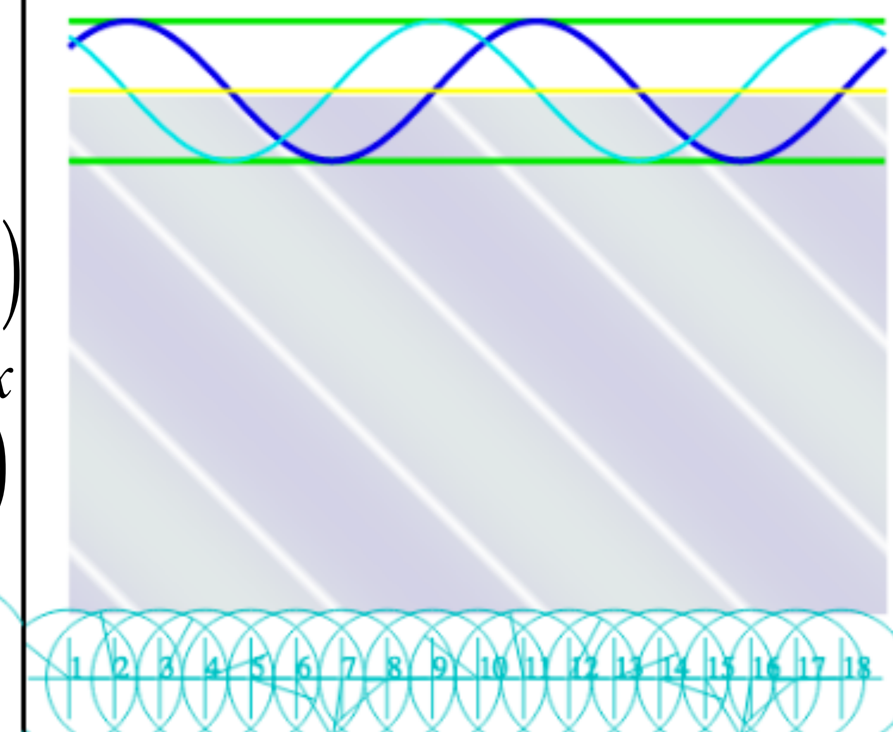
$$Ae^{-i\omega t}(e^{ikx}+e^{-ikx}) = Ae^{-i\omega t}\, 2\cos kx$$

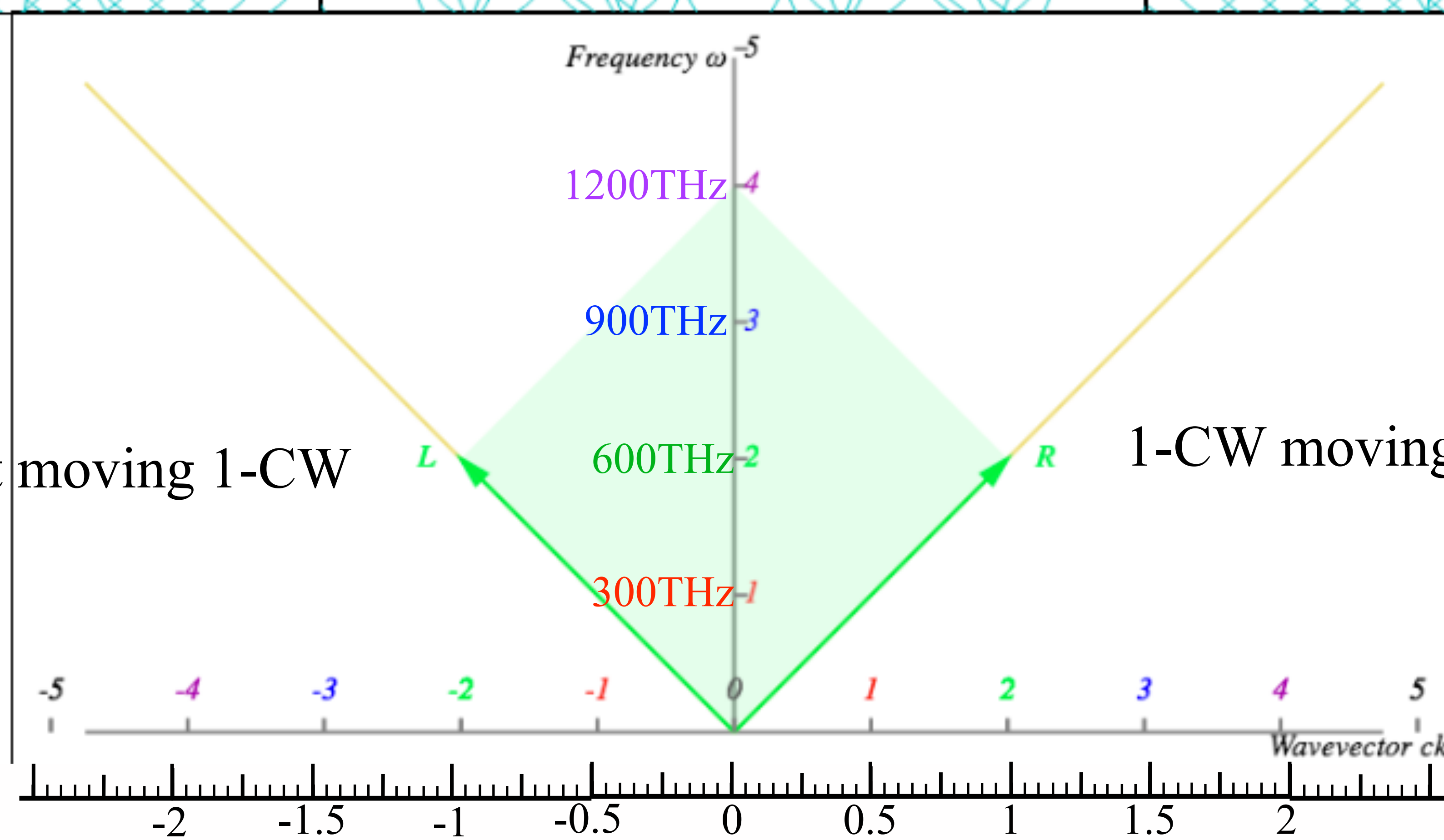
$\begin{pmatrix} wave \\ \text{"guts"} \end{pmatrix} \cdot \begin{pmatrix} wave \\ \text{"skin"} \end{pmatrix}$

$Ae^{i(-kx-\omega t)}$

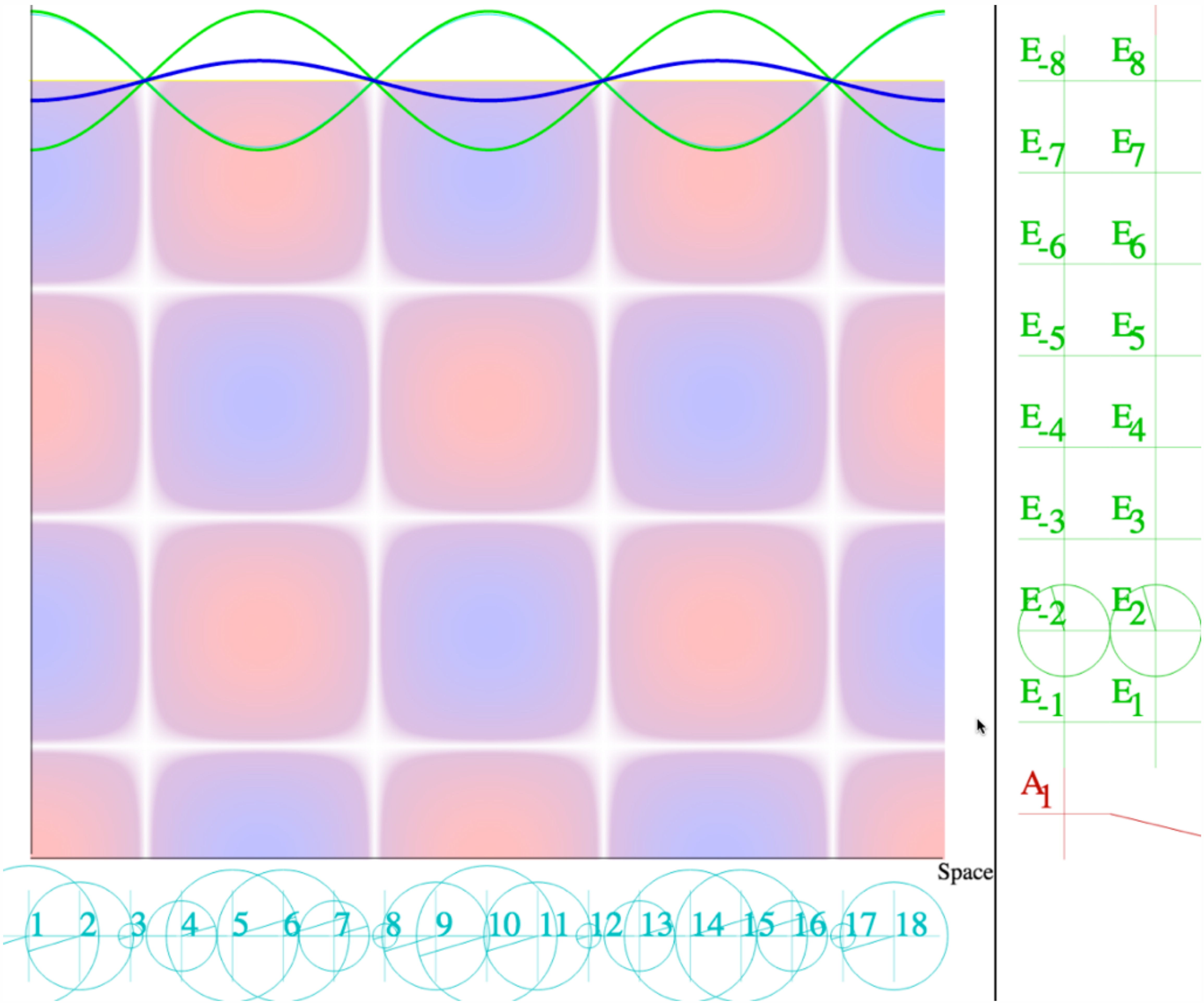
← LEFT-to-right moving 1-CW

600THz





ck units of $_{2\pi}$600THz



E_{-8} E_8
E_{-7} E_7
E_{-6} E_6
E_{-5} E_5
E_{-4} E_4
E_{-3} E_3
E_{-2} E_2
E_{-1} E_1
A_1
Space
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

2 colliding waves (2-CW)

$A e^{i(k_R x - \omega_R t)} = A e^{i(4x - 4ct)}$

1-CW moving left-to-RIGHT→

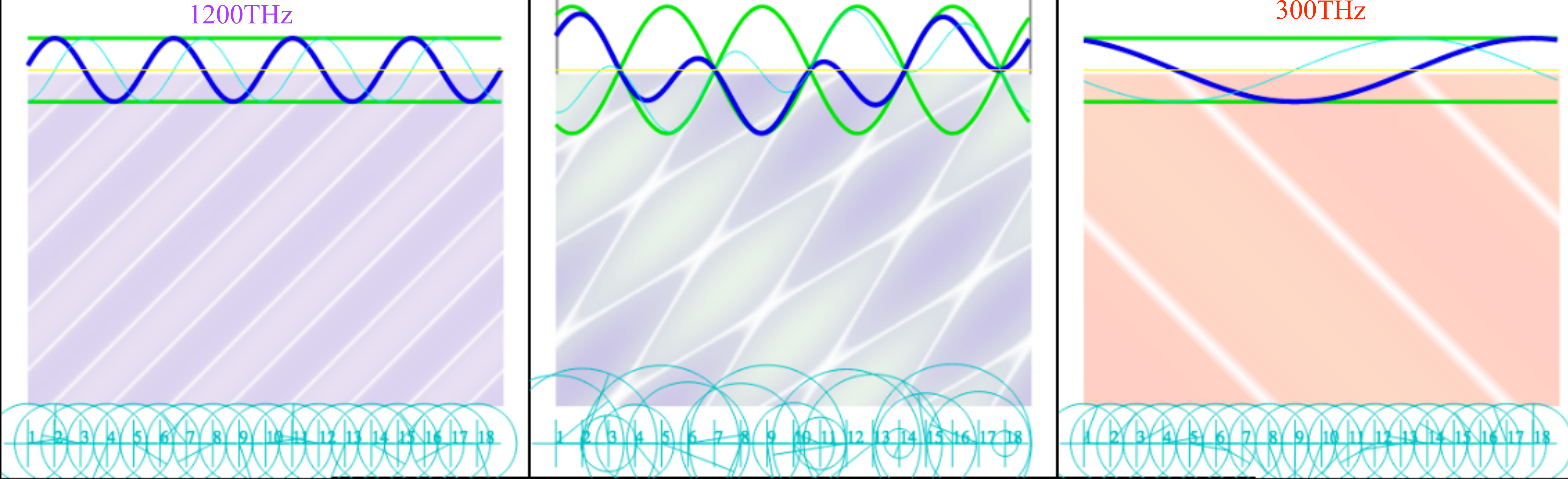
1200THz

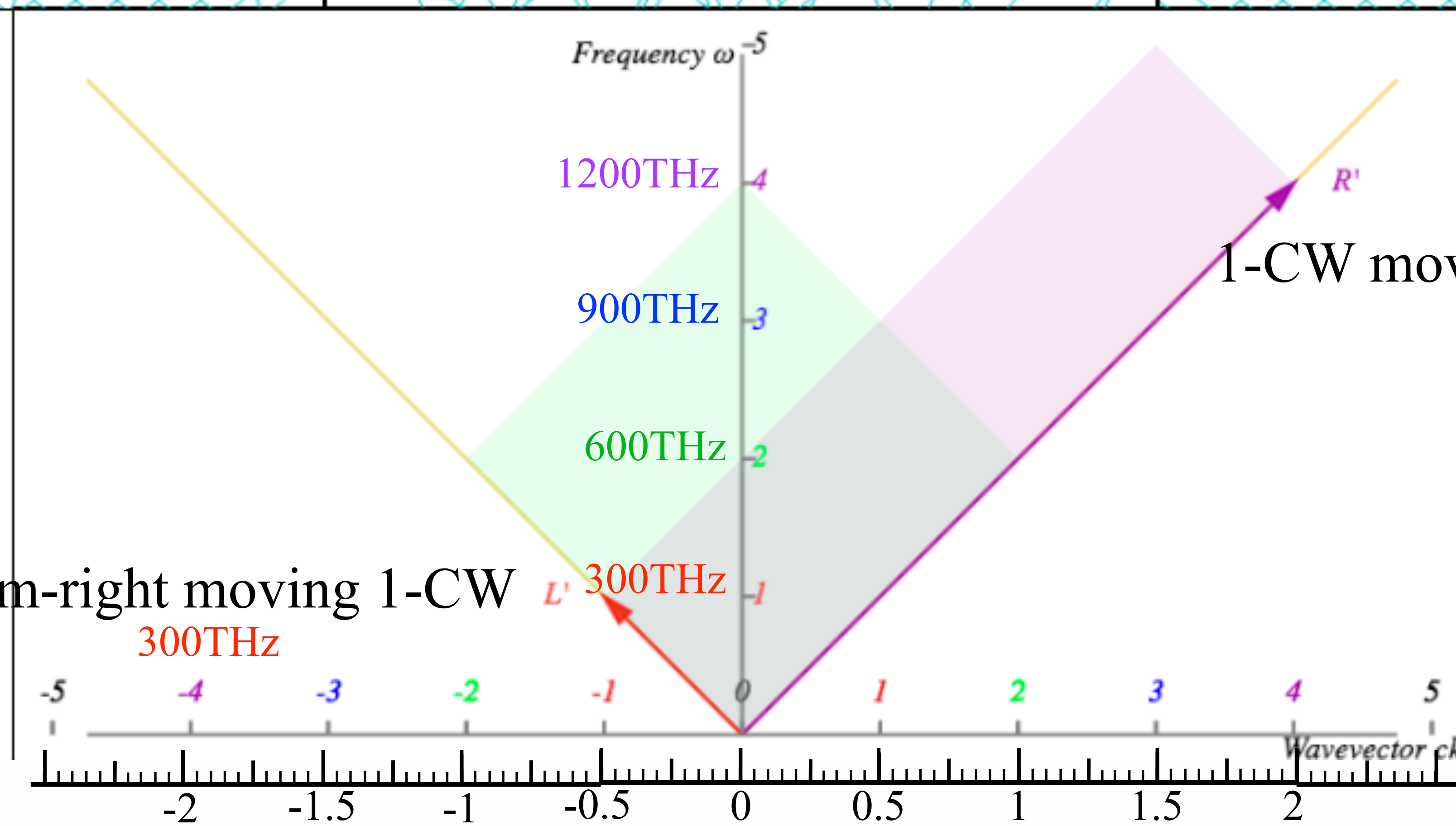
$\psi = A(e^{i(4x-4ct)} + e^{i(-1x-1ct)})$

$A e^{i(k_L x - \omega_L t)} = A e^{i(-1x - 1ct)}$

←LEFT-from-right moving 1-CW

300THz





1-CW moving left-to-RIGHT→

1200THz

← LEFT-from-right moving 1-CW

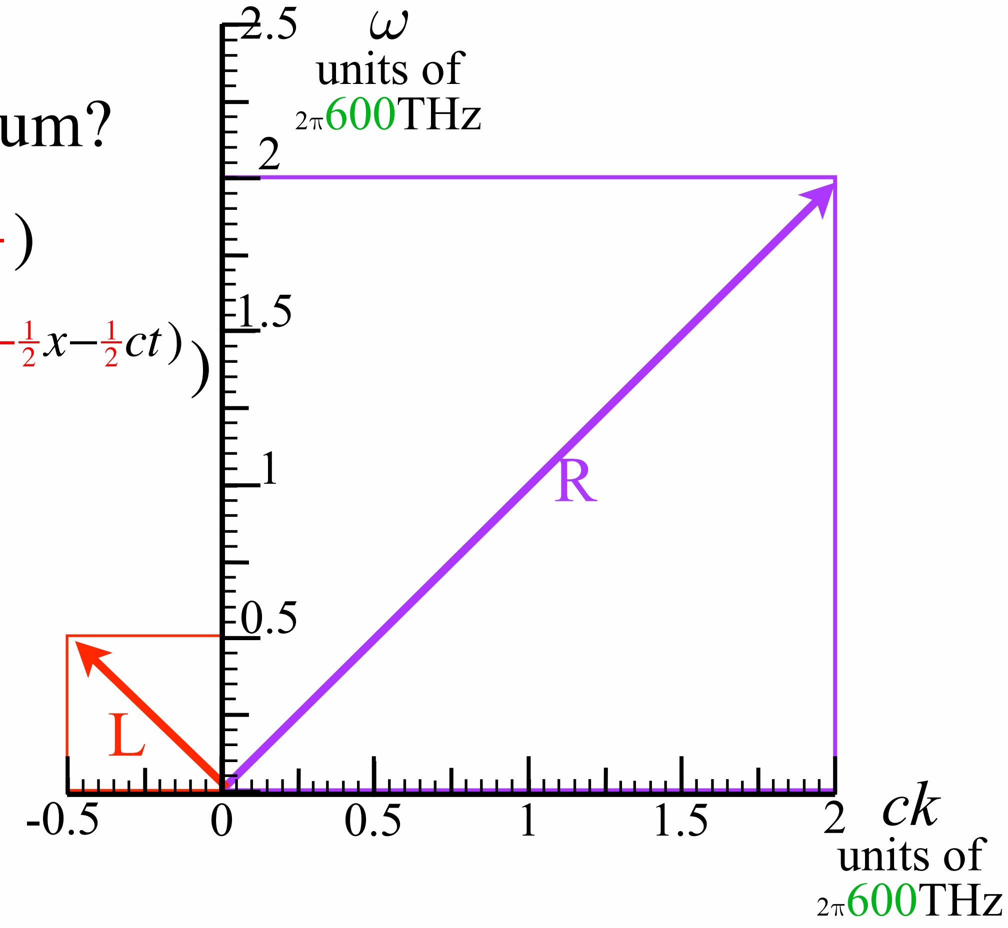
300THz

ck units of $_{2\pi}$600THz

Where are the real-zeros of the colliding-light-wave (2-CW) sum?

RIGHT: $(\omega_R = ck_R = 2c, \quad k_R = 2)$ LEFT: $(\omega_L = -ck_L, \quad k_L = -\frac{1}{2})$

$$\Psi_{\omega_R k_R \omega_L k_L}(x,t) = A\left(e^{i(k_R x - \omega_R t)} + e^{i(k_L x - \omega_L t)}\right) = A\left(e^{i(2x - 2ct)} + e^{i(-\frac{1}{2}x - \frac{1}{2}ct)}\right)$$



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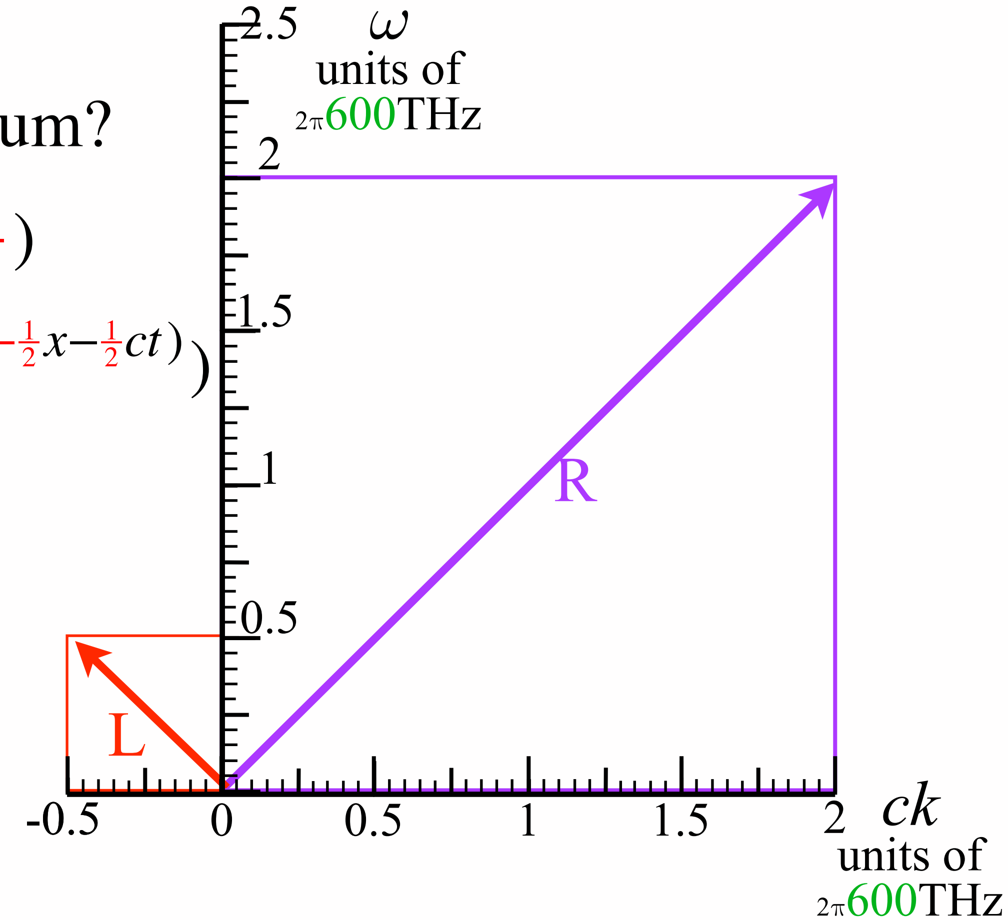
$$\Psi_{\omega_R k_R \omega_L k_L}(x,t) = A\left(e^{i(k_R x - \omega_R t)} + e^{i(k_L x - \omega_L t)}\right) = A\left(e^{i(2x - 2ct)} + e^{i(-\frac{1}{2}x - \frac{1}{2}ct)}\right)$$

To find zeros of a wave sum $e^{iR} + e^{iL}$ we need to factor it



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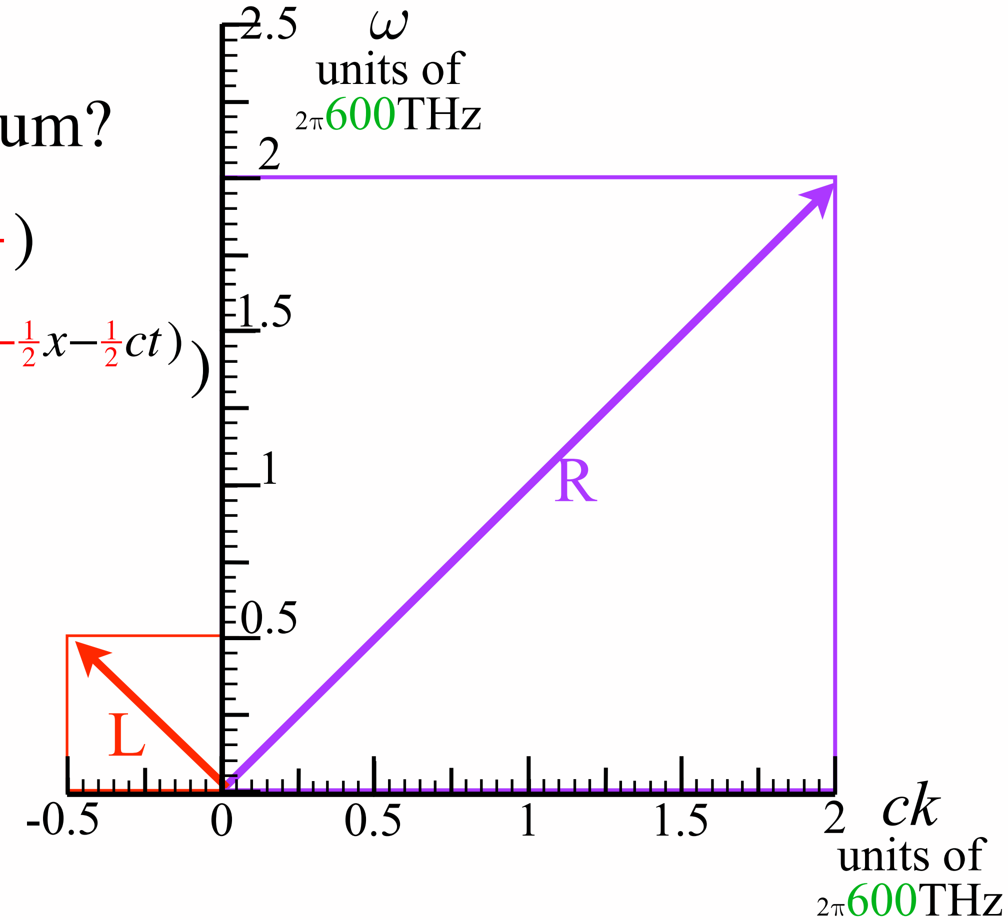

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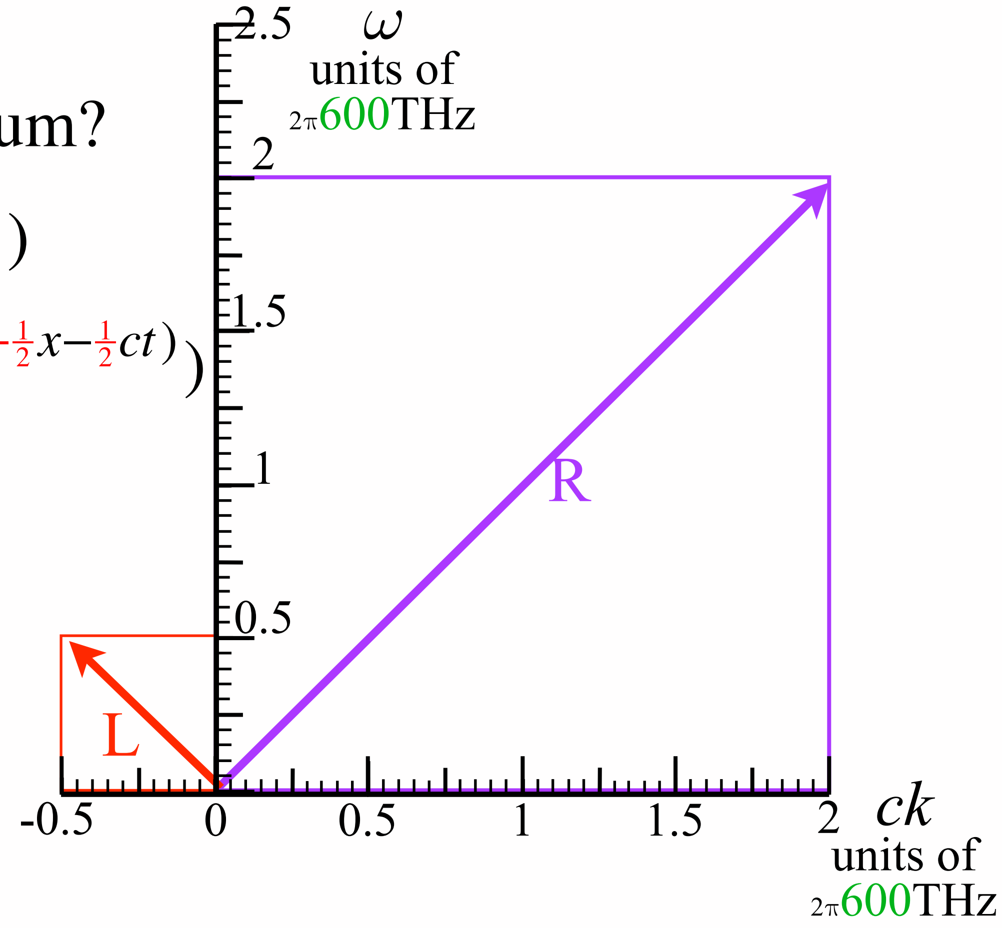

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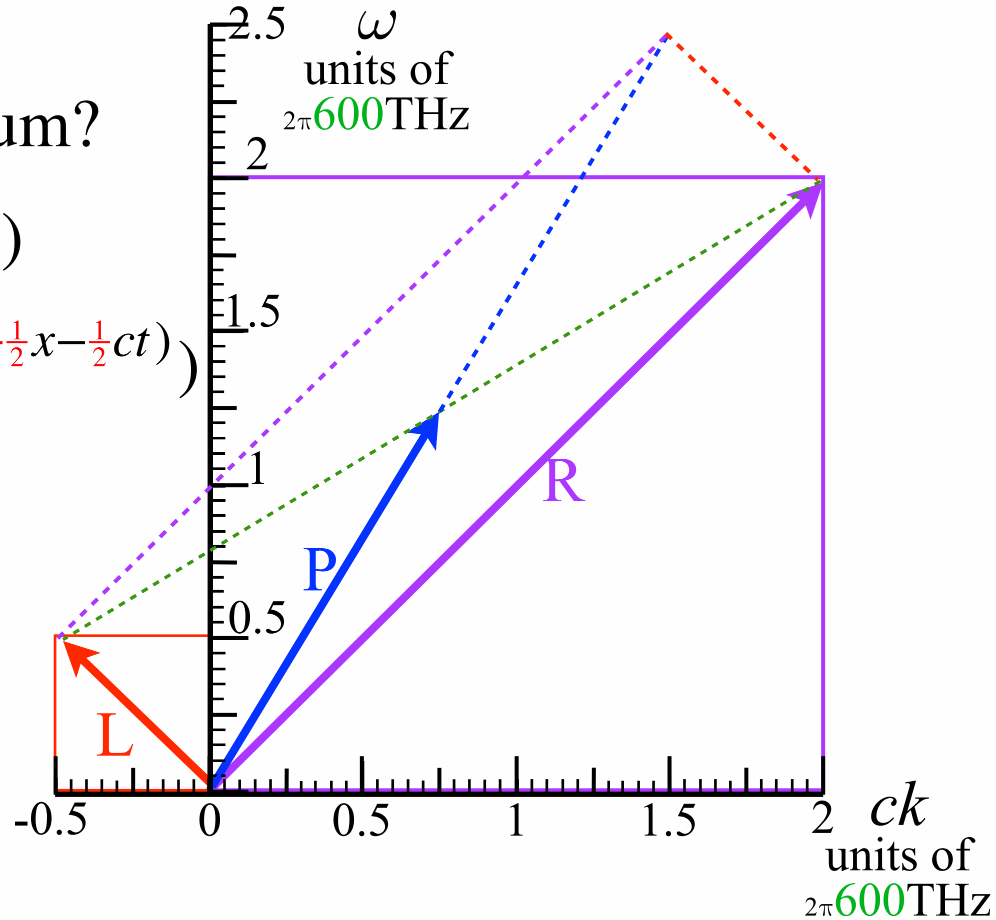

http://www.uark.edu/ua/modphys/markup/RelaWavityWeb.html?plotType=3%7C4&acousticAnalogInd=1

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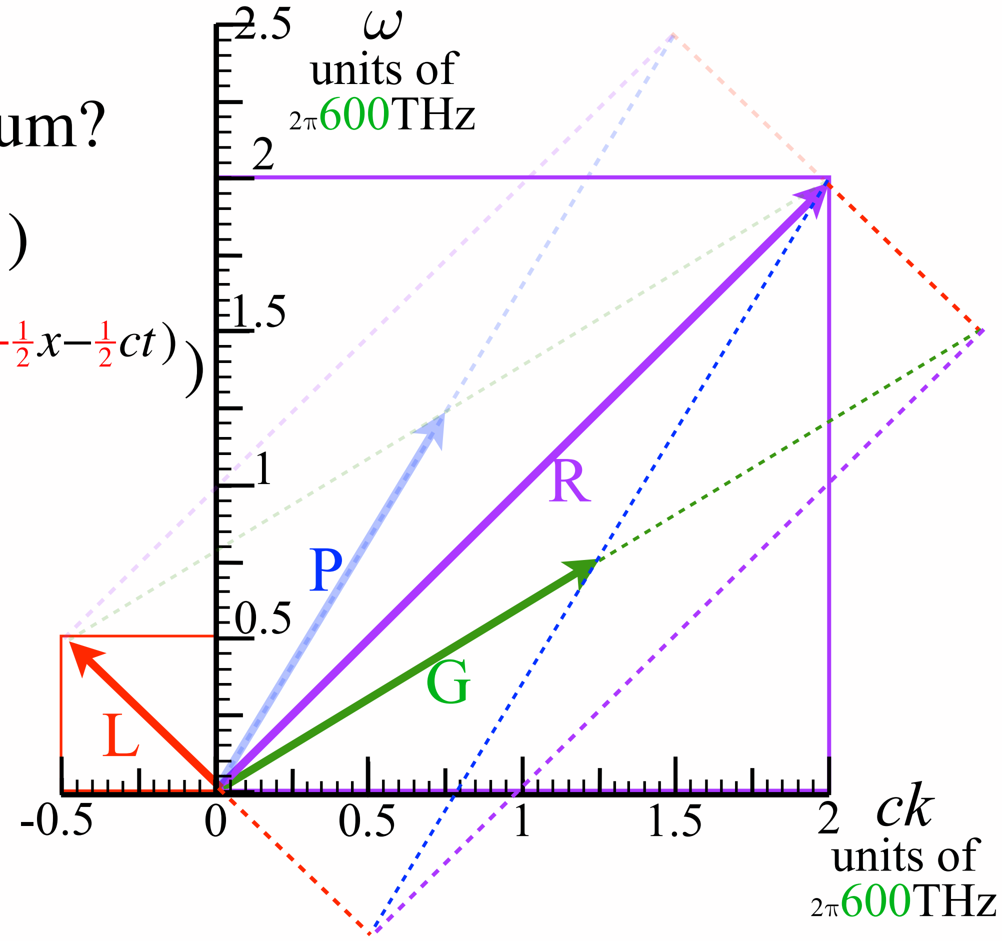

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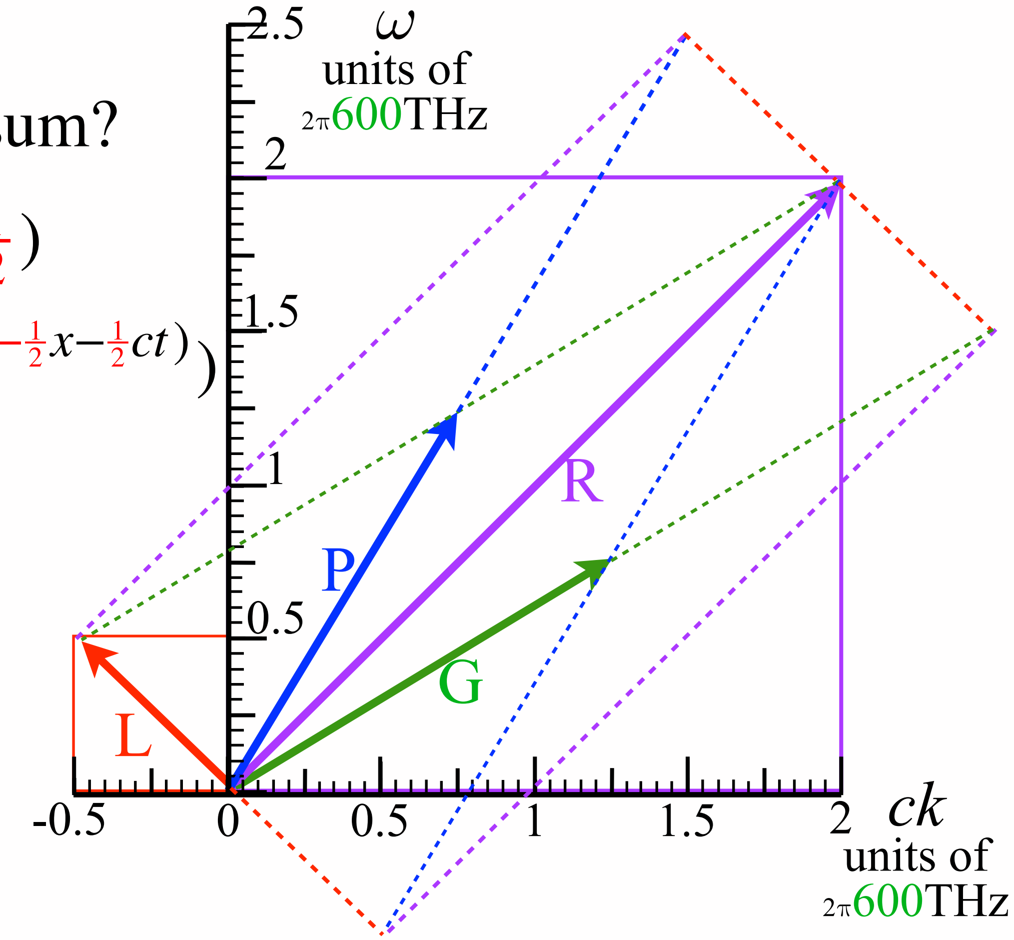

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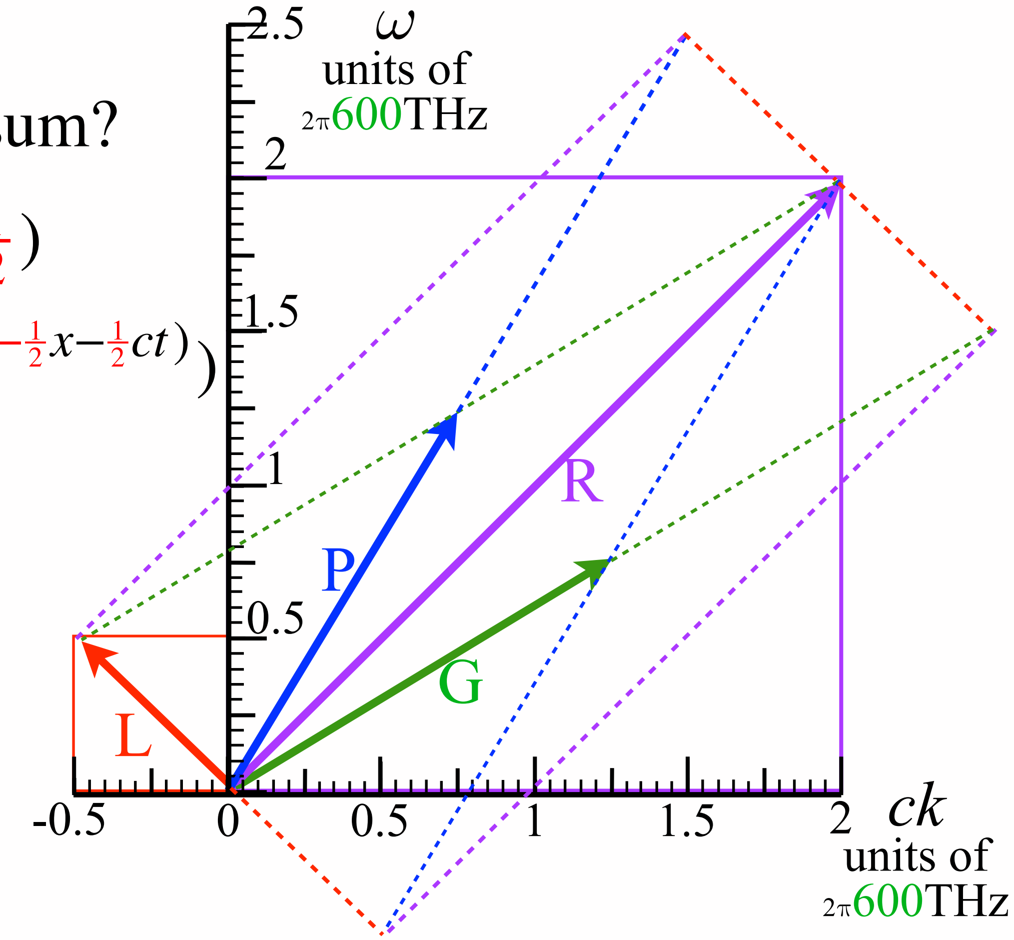

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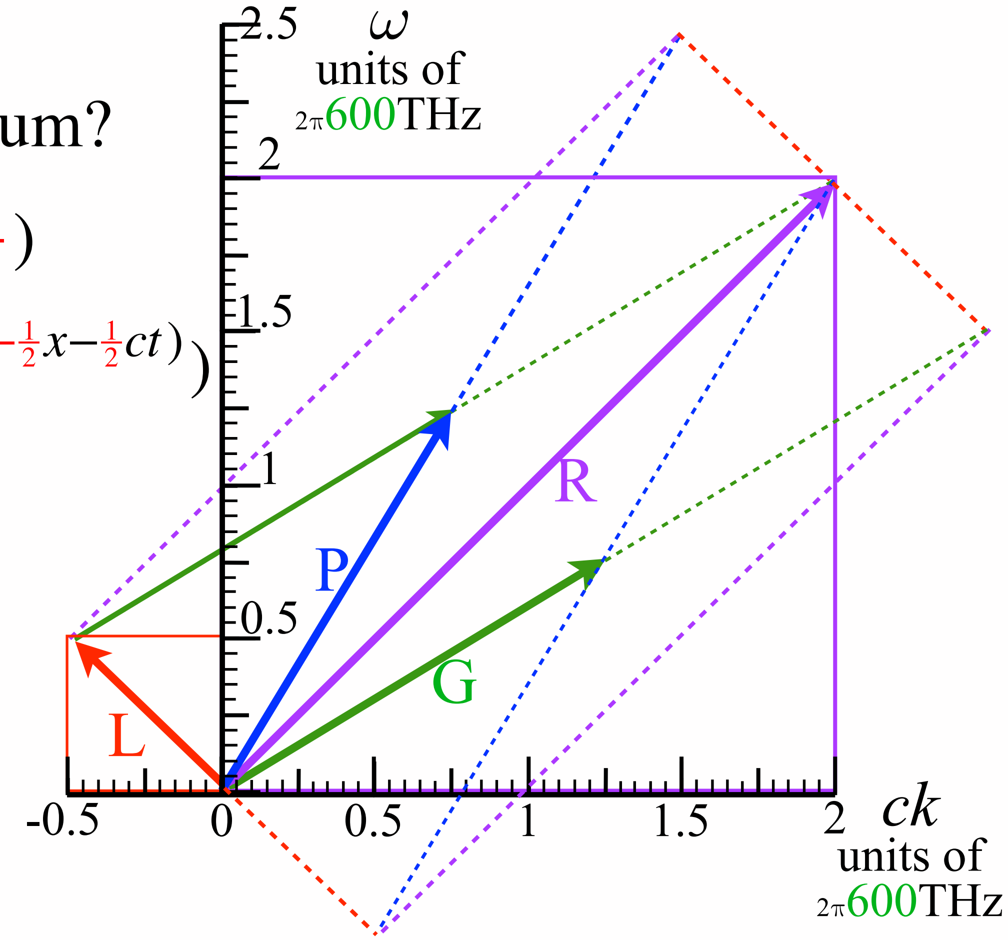

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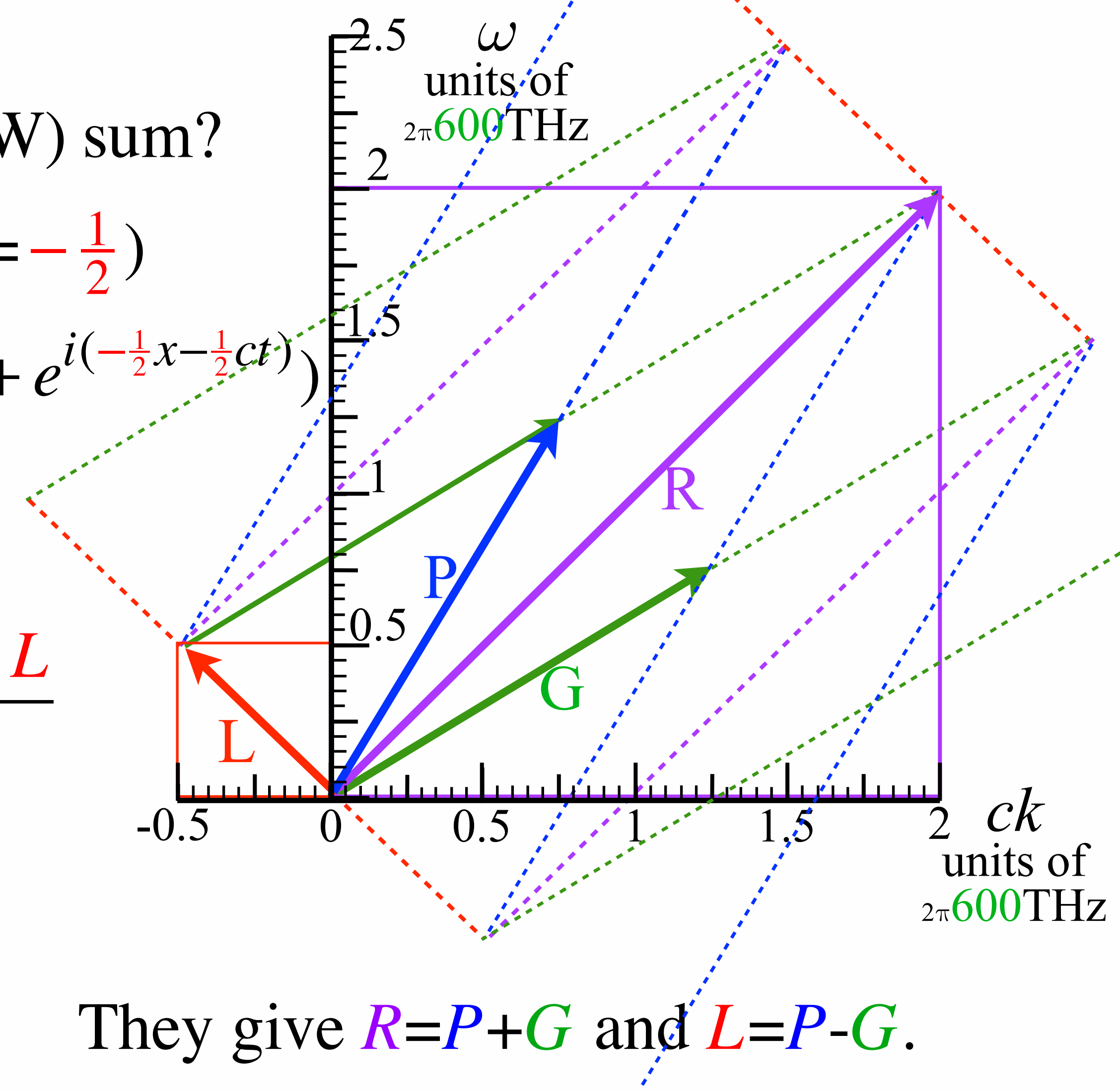

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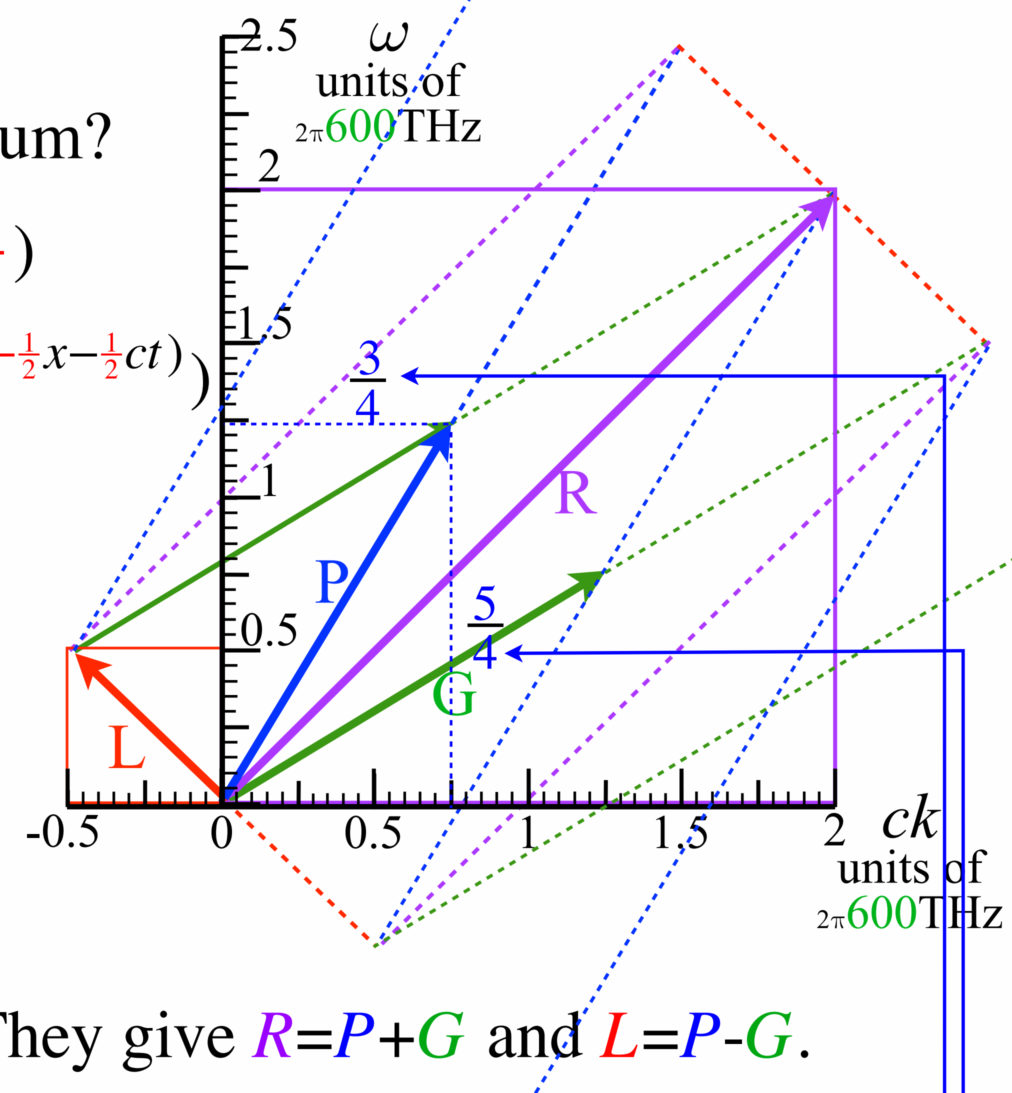

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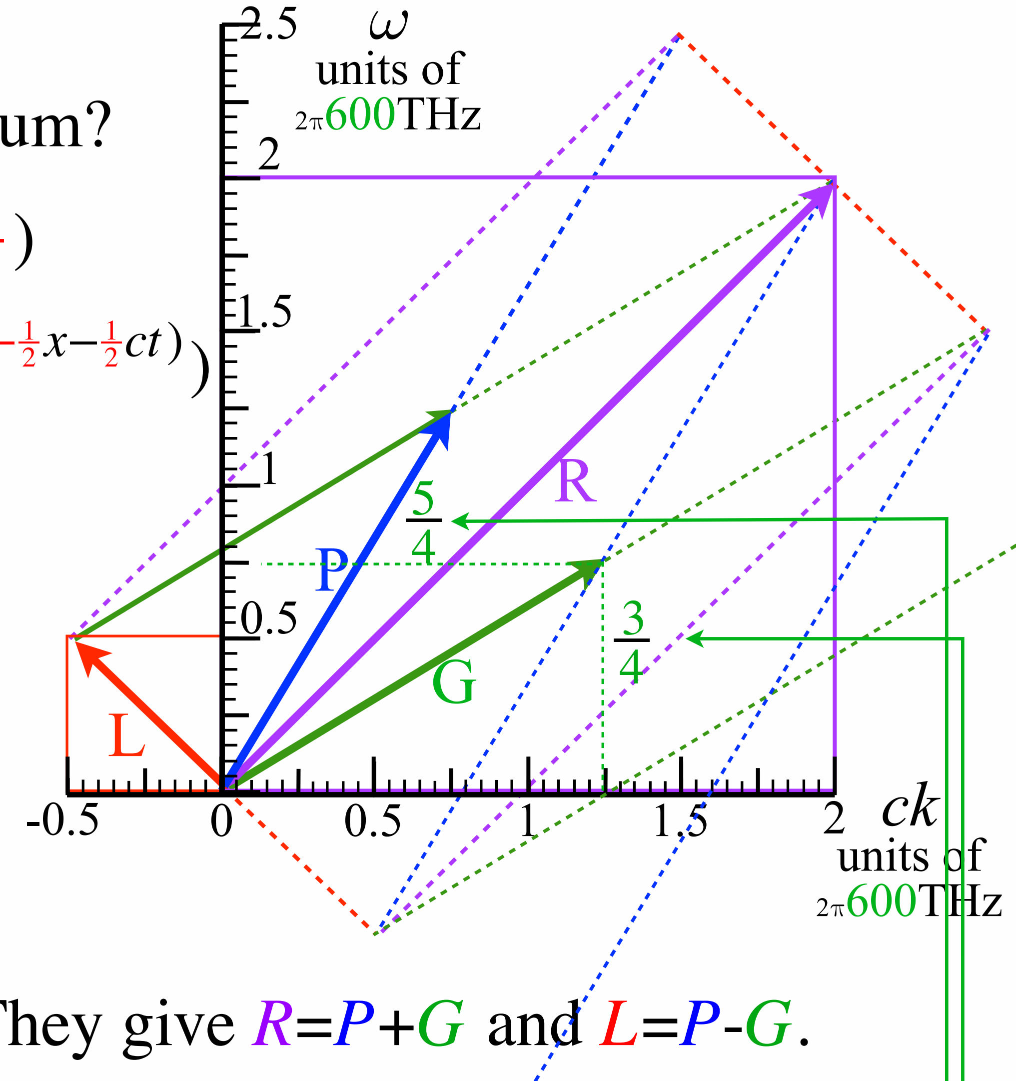

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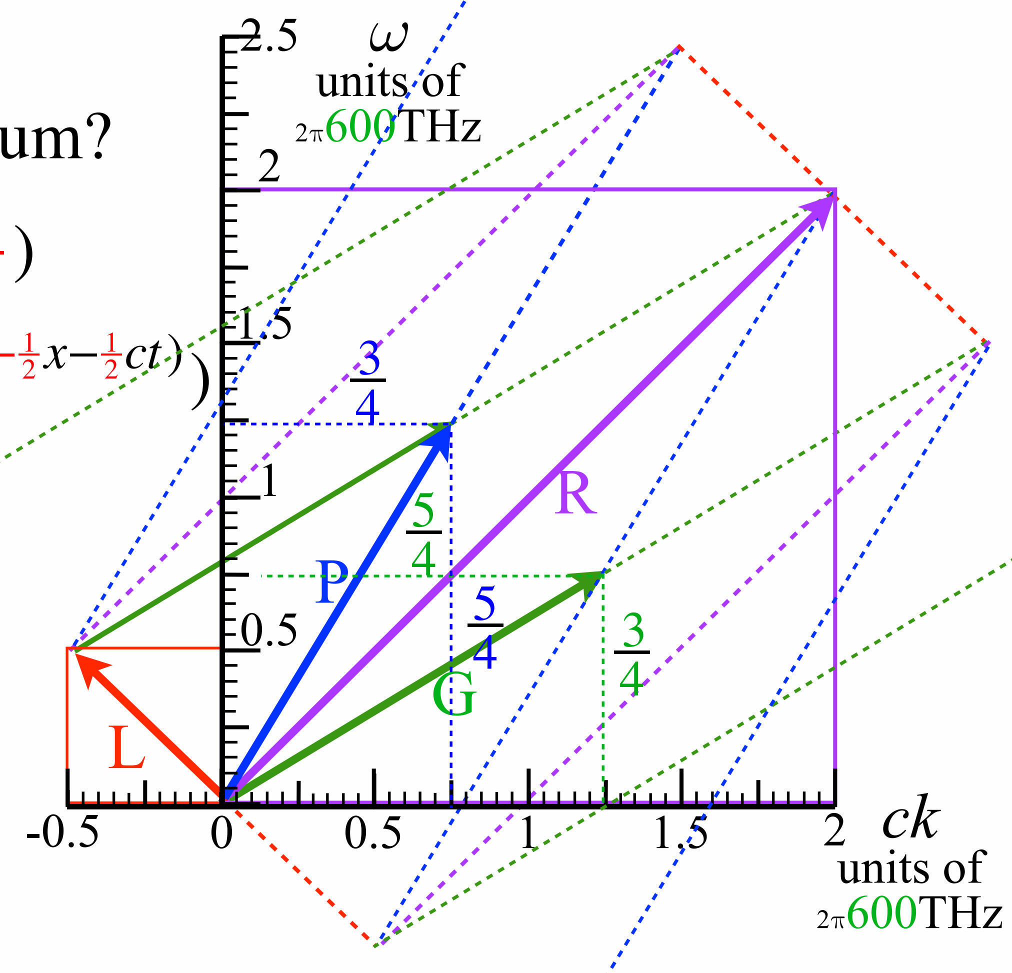

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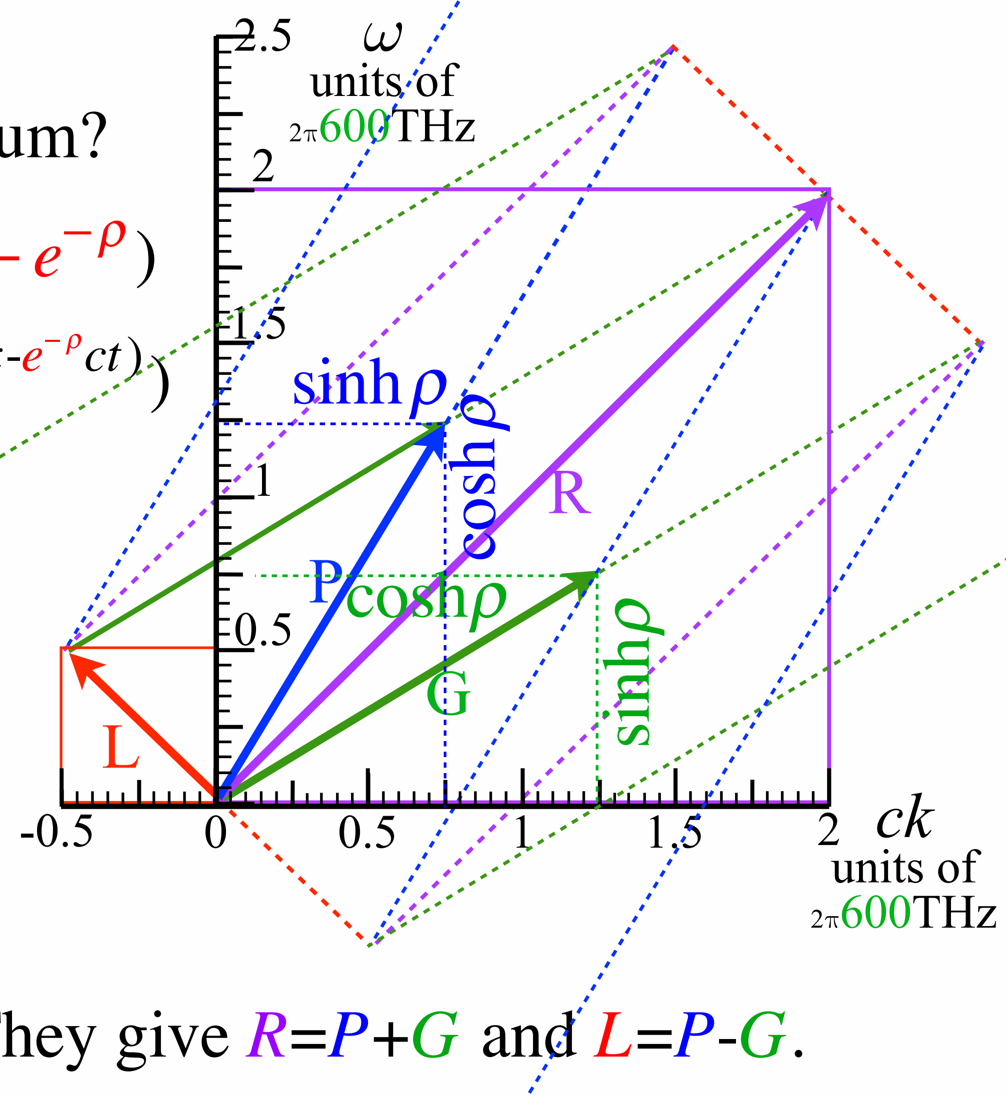

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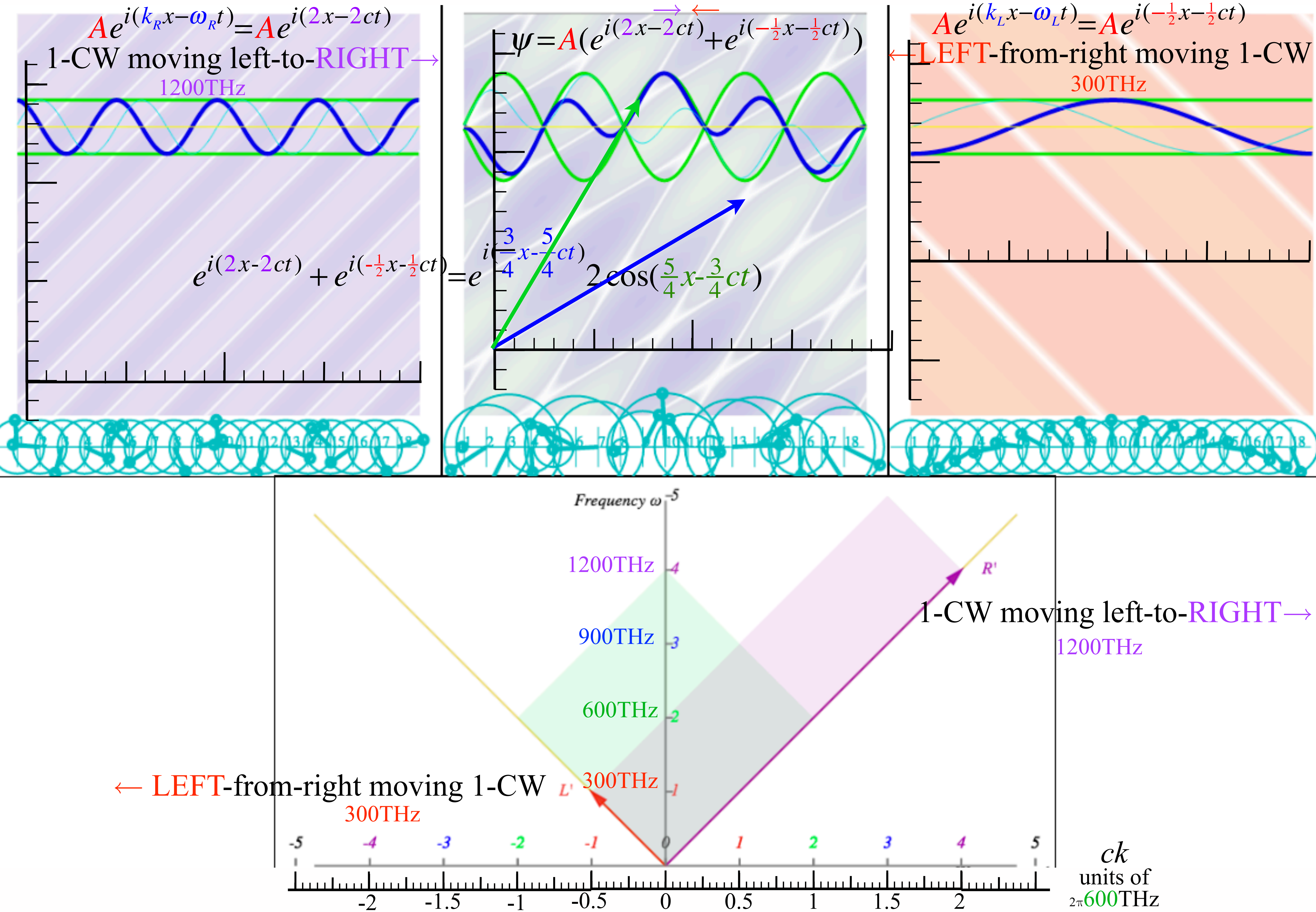
2 colliding waves (2-CW)



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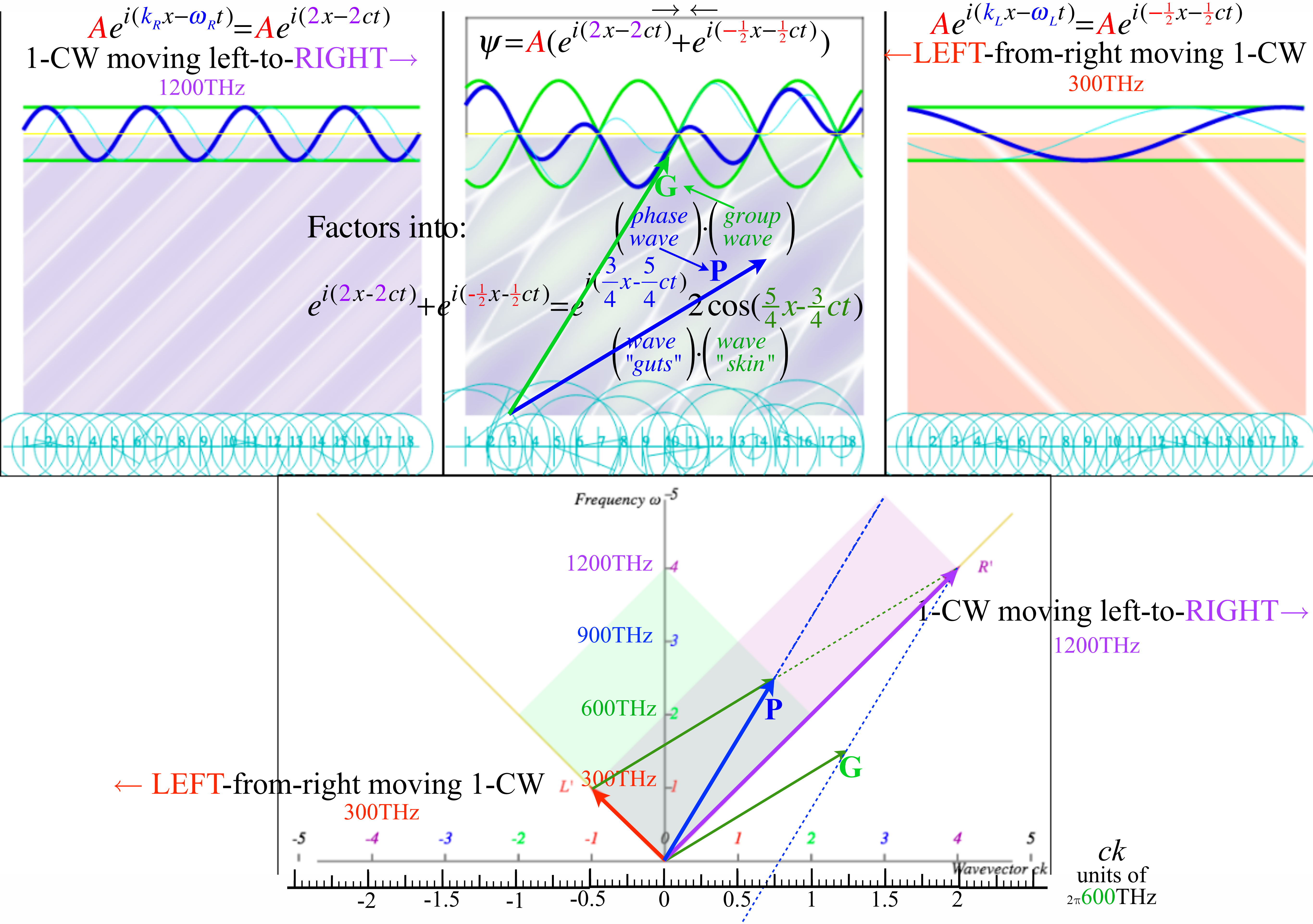

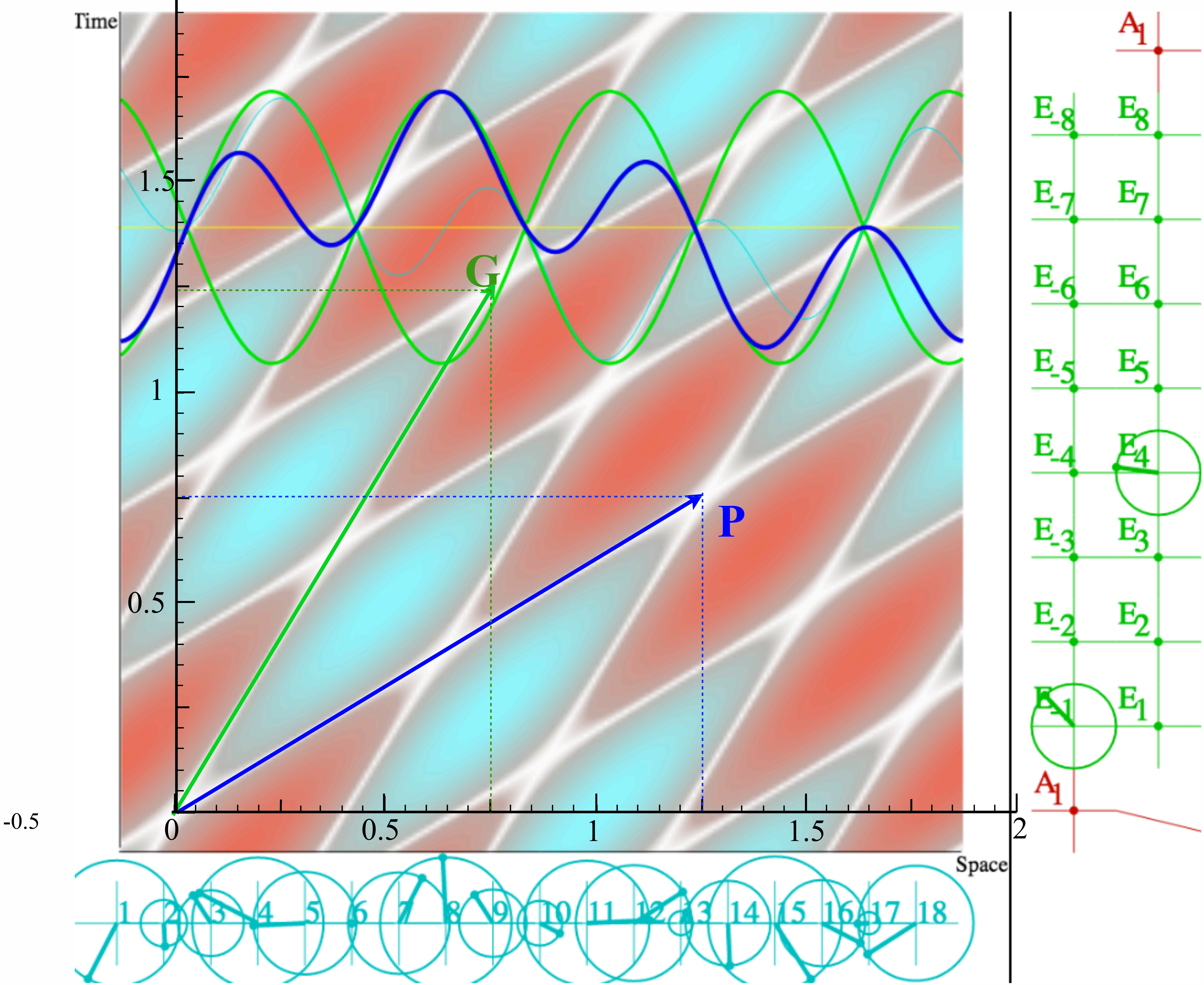

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0.5
-0.5
0
0.5
1
1.5
2
Space
G
P
A1
E-8
E8
E-7
E7
E-6
E6
E-5
E5
E-4
E4
E-3
E3
E-2
E2
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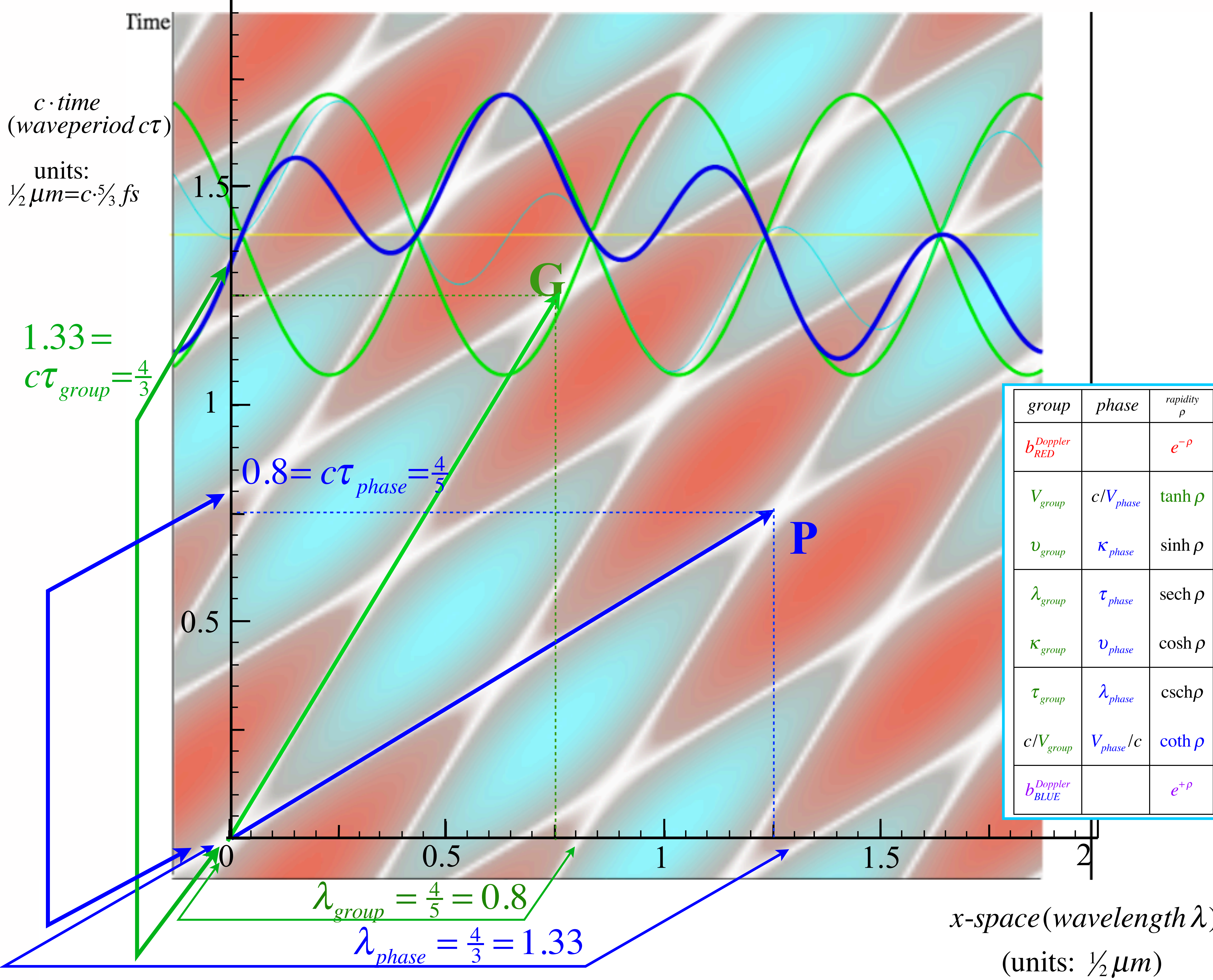
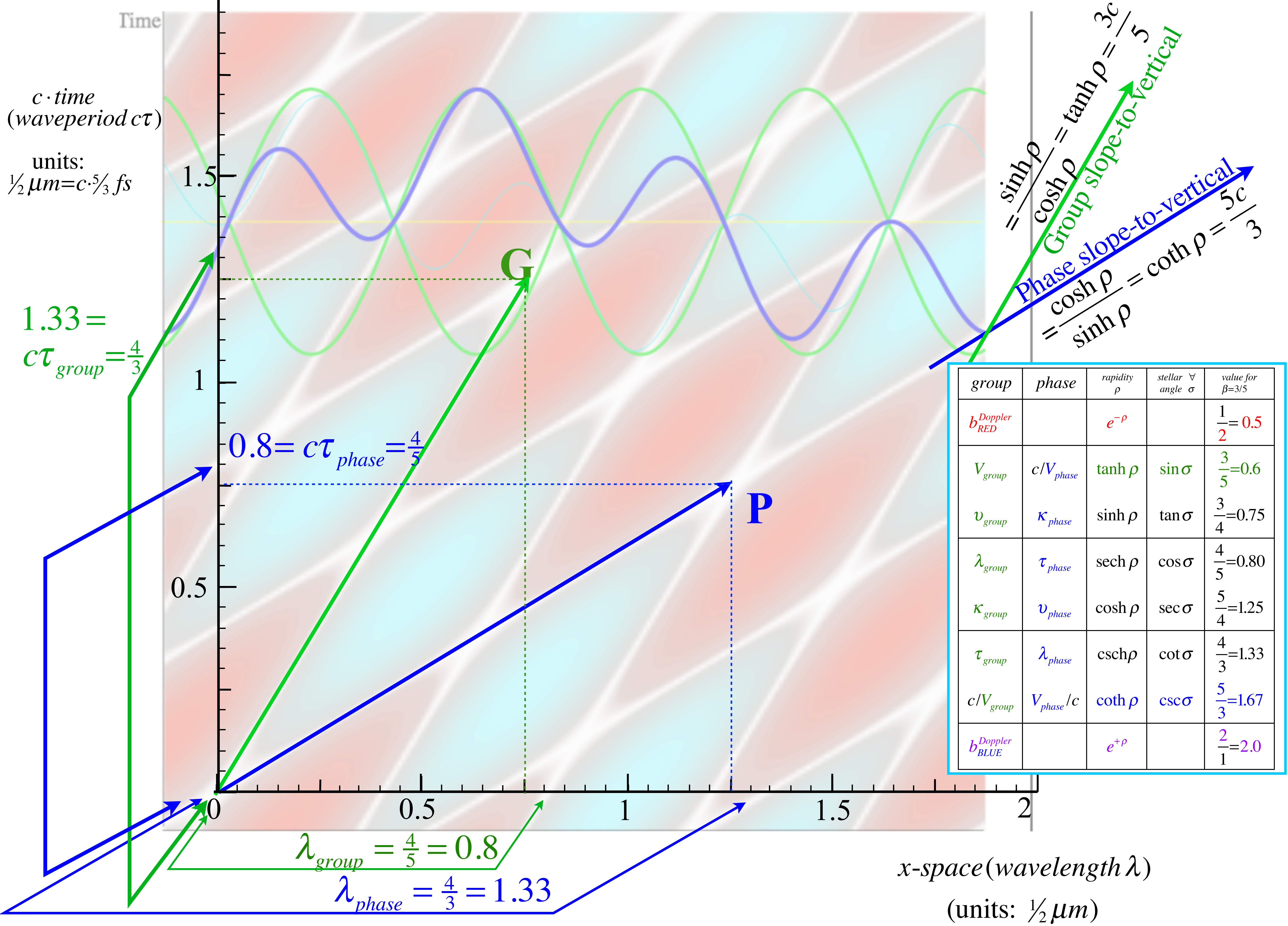


group	phase	rapidity ρ	stellar ∀ angle σ	value for β=3/5
$b_{RED}^{Doppler}$		$e^{-\rho}$		$\frac{1}{2}$= 0.5
V_{group}	c/V_{phase}	$\tanh\rho$	$\sin\sigma$	$\frac{3}{5}$=0.6
υ_{group}	κ_{phase}	$\sinh\rho$	$\tan\sigma$	$\frac{3}{4}$=0.75
λ_{group}	τ_{phase}	$\mathrm{sech}\,\rho$	$\cos\sigma$	$\frac{4}{5}$=0.80
κ_{group}	υ_{phase}	$\cosh\rho$	$\sec\sigma$	$\frac{5}{4}$=1.25
τ_{group}	λ_{phase}	$\mathrm{csch}\rho$	$\cot\sigma$	$\frac{4}{3}$=1.33
c/V_{group}	V_{phase}/c	$\coth\rho$	$\csc\sigma$	$\frac{5}{3}$=1.67
$b_{BLUE}^{Doppler}$		$e^{+\rho}$		$\frac{2}{1}$=2.0



group	phase	rapidity ρ	stellar ∀ angle σ	value for β=3/5
$b_{RED}^{Doppler}$		$e^{-\rho}$		$\frac{1}{2}$= 0.5
V_{group}	c/V_{phase}	$\tanh\rho$	$\sin\sigma$	$\frac{3}{5}$=0.6
υ_{group}	κ_{phase}	$\sinh\rho$	$\tan\sigma$	$\frac{3}{4}$=0.75
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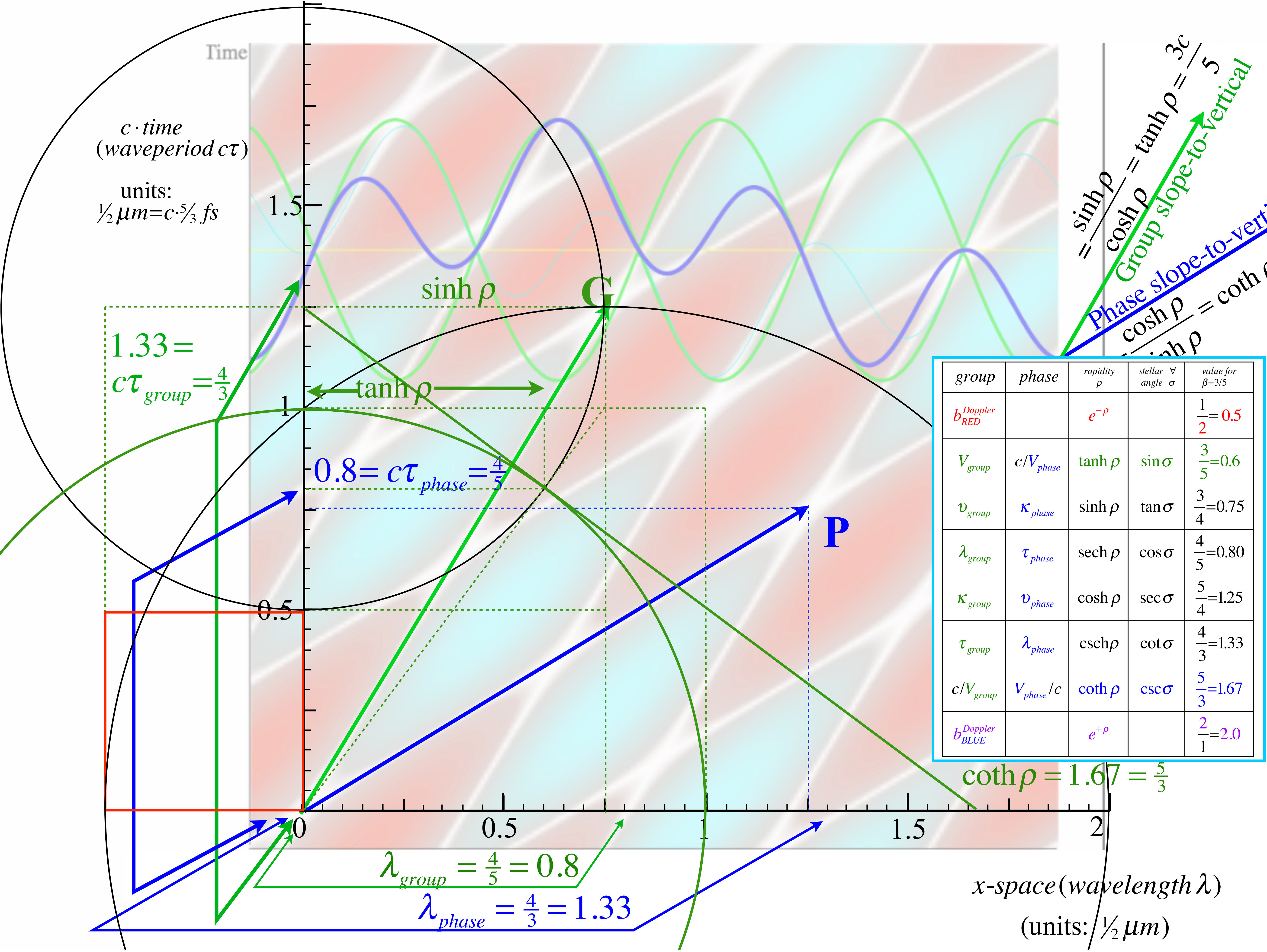

group	phase	rapidity ρ	stellar $\forall$ angle σ	value for $\beta=3/5$
$b^{Doppler}_{RED}$		$e^{-\rho}$		$\frac{1}{2}= 0.5$
V_{group}	c/V_{phase}	$\tanh\rho$	$\sin\sigma$	$\frac{3}{5}=0.6$
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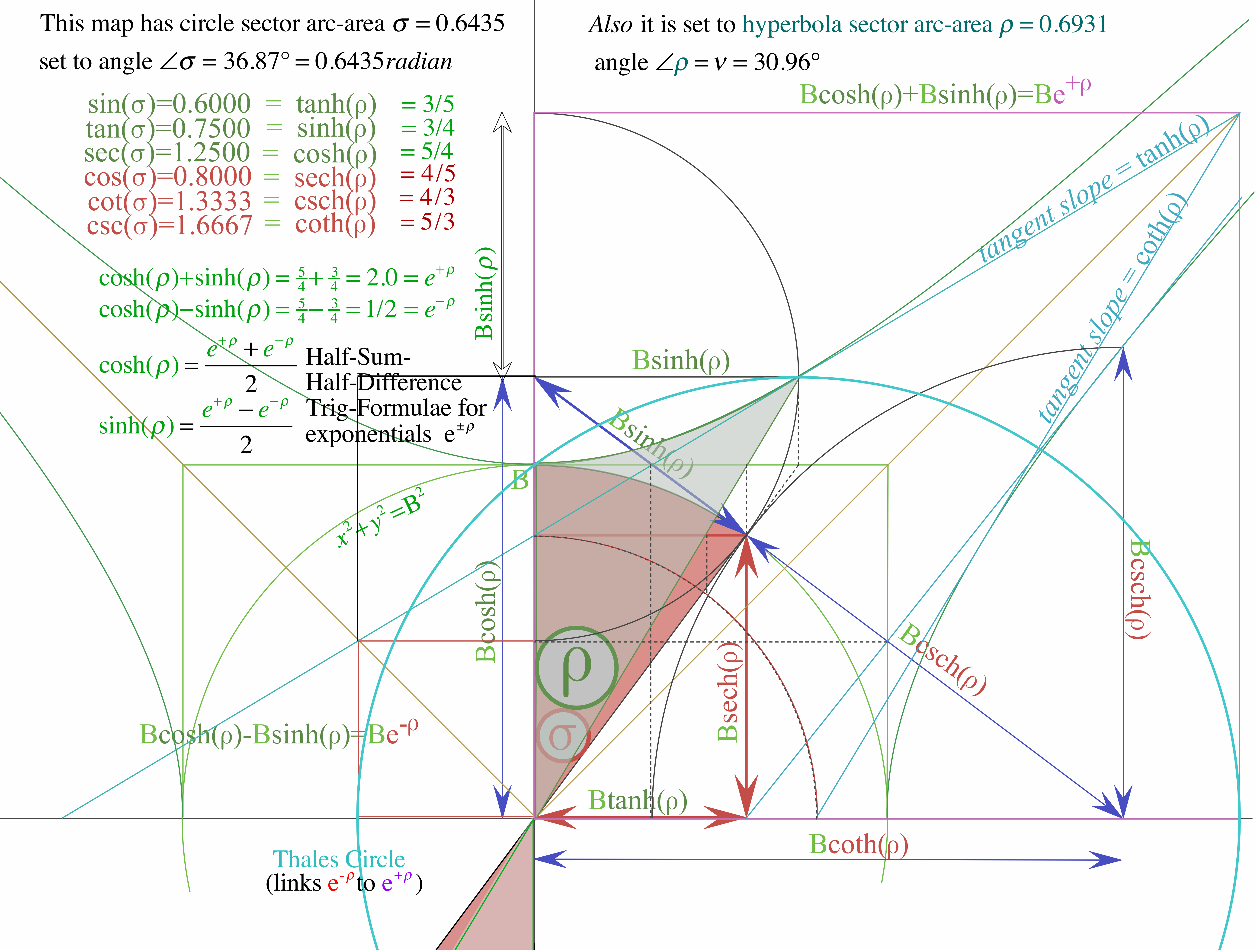
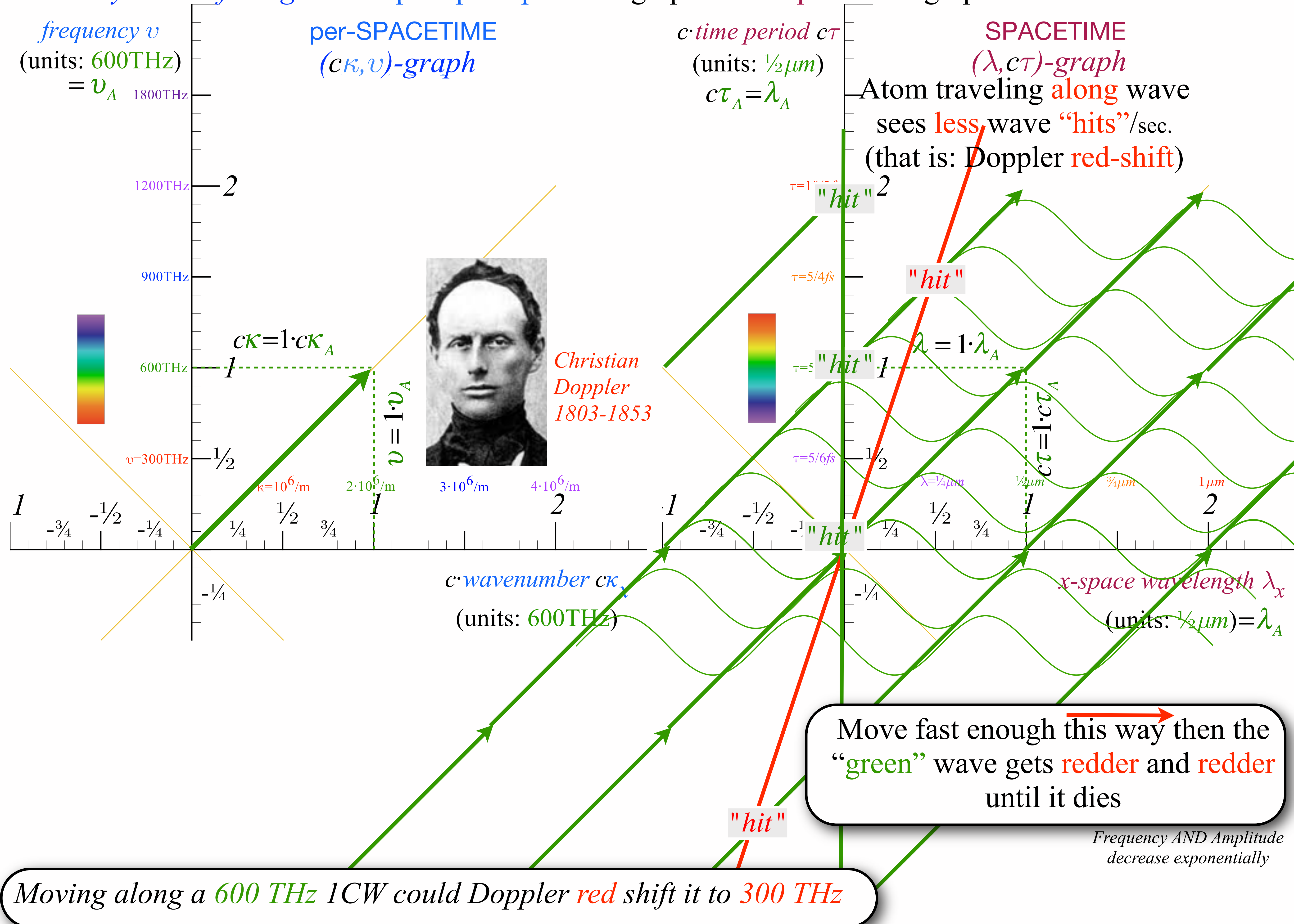
c/V_{group}	V_{phase}/c	$\coth\rho$	$\csc\sigma$	$\frac{5}{3}=1.67$
$b^{Doppler}_{BLUE}$		$e^{+\rho}$		$\frac{2}{1}=2.0$



This map has circle sector arc-area σ = 0.6435
set to angle ∠σ = 36.87° = 0.6435 radian
Also it is set to hyperbola sector arc-area ρ = 0.6931
angle ∠ρ = ν = 30.96°
sin(σ)=0.6000 = tanh(ρ) = 3/5
tan(σ)=0.7500 = sinh(ρ) = 3/4
sec(σ)=1.2500 = cosh(ρ) = 5/4
cos(σ)=0.8000 = sech(ρ) = 4/5
cot(σ)=1.3333 = csch(ρ) = 4/3
csc(σ)=1.6667 = coth(ρ) = 5/3
cosh(ρ)+sinh(ρ) = 5/4 + 3/4 = 2.0 = e^{+ρ}
cosh(ρ)−sinh(ρ) = 5/4 − 3/4 = 1/2 = e^{−ρ}
cosh(ρ) = (e^{+ρ} + e^{−ρ})/2
sinh(ρ) = (e^{+ρ} − e^{−ρ})/2
Half-Sum-
Half-Difference
Trig-Formulae for
exponentials e^{±ρ}
Bcosh(ρ)+Bsinh(ρ)=Be^{+ρ}
Bcosh(ρ)-Bsinh(ρ)=Be^{-ρ}
tangent slope = tanh(ρ)
tangent slope = coth(ρ)
Bsinh(ρ)
Bsinh(ρ)
Bsinh(ρ)
Bcosh(ρ)
B
x²+y²=B²
ρ
σ
Bsech(ρ)
Bcsch(ρ)
Bcsch(ρ)
Btanh(ρ)
Bcoth(ρ)
Thales Circle
(links e^{-ρ} to e^{+ρ})

group	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{c}{V_{group}}$	$b_{BLUE}^{Doppler}$
phase	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{V_{phase}}{c}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\text{sech}\,\rho$	$\cosh\rho$	$\text{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta\equiv\frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
value for $\beta=3/5$	$\frac{1}{2}=0.5$	$\frac{3}{5}=0.6$	$\frac{3}{4}=0.75$	$\frac{4}{5}=0.80$	$\frac{5}{4}=1.25$	$\frac{4}{3}=1.33$	$\frac{5}{3}=1.67$	$\frac{2}{1}=2.0$

The "Keyboard of the gods" or per-space-per-time graphs *versus* space-time graphs



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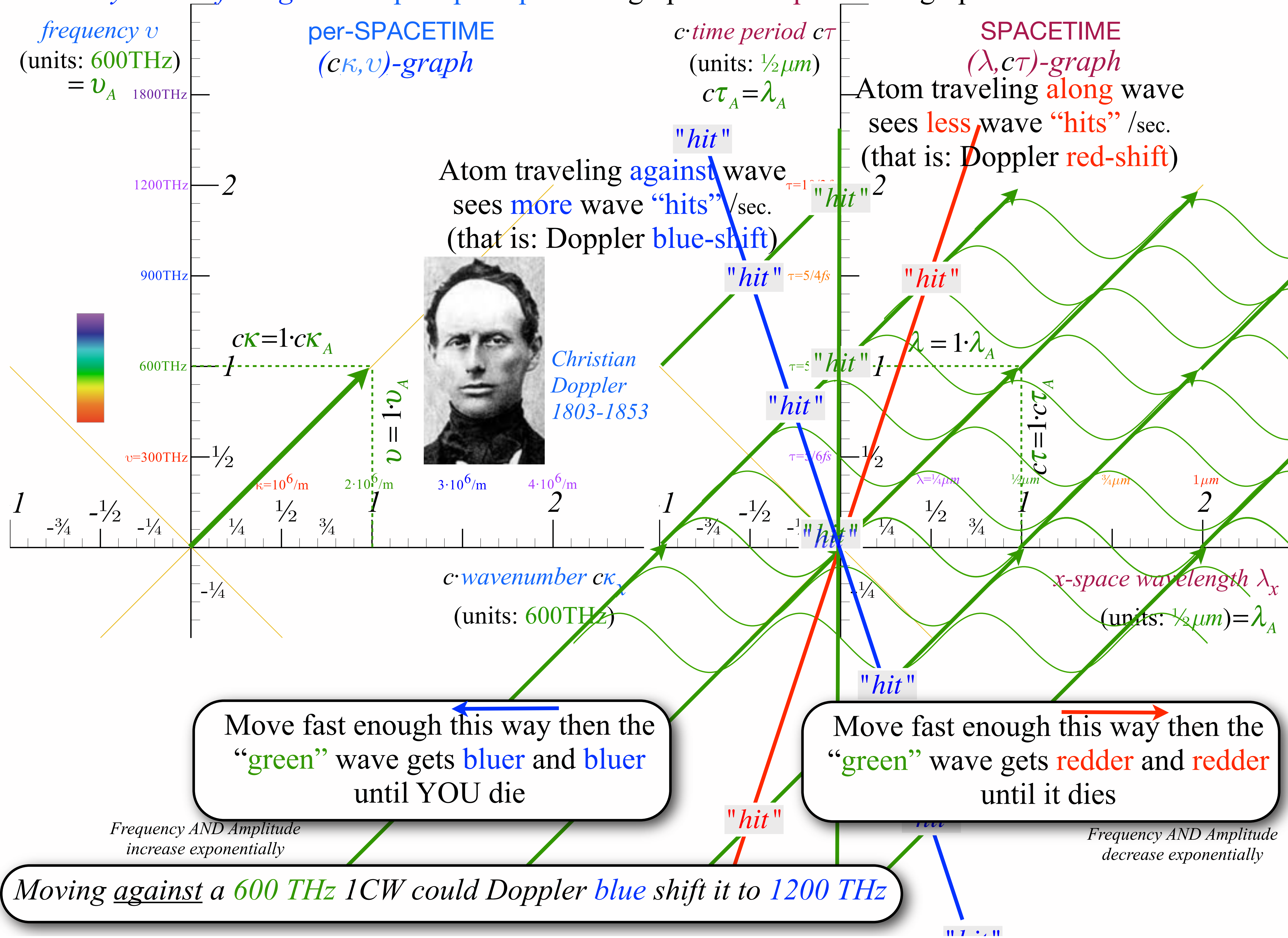
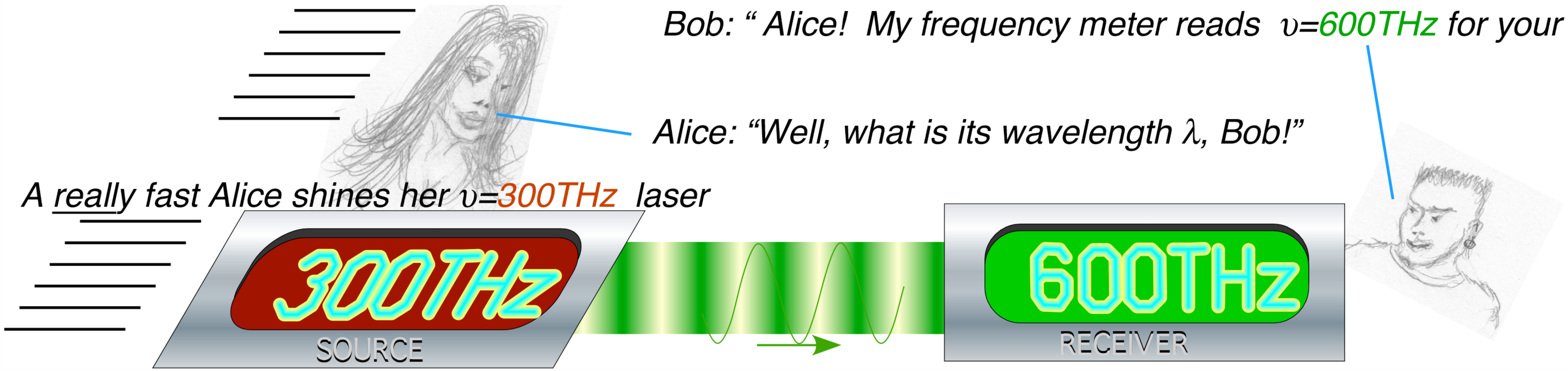


Clarify *Evenson's CW Axiom (All colors go c) by Doppler effects*

Alice tries to fool Bob that she's shining a 600THz laser. (Bob's unaware she's moving *really* fast...)



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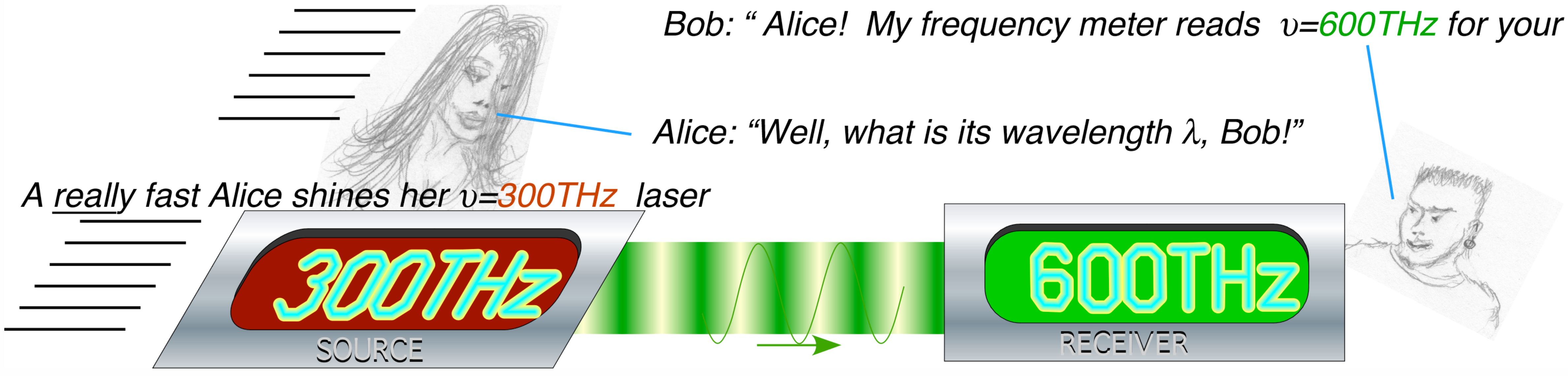

Q1: Can Bob tell it's a *"phony"* 600THz
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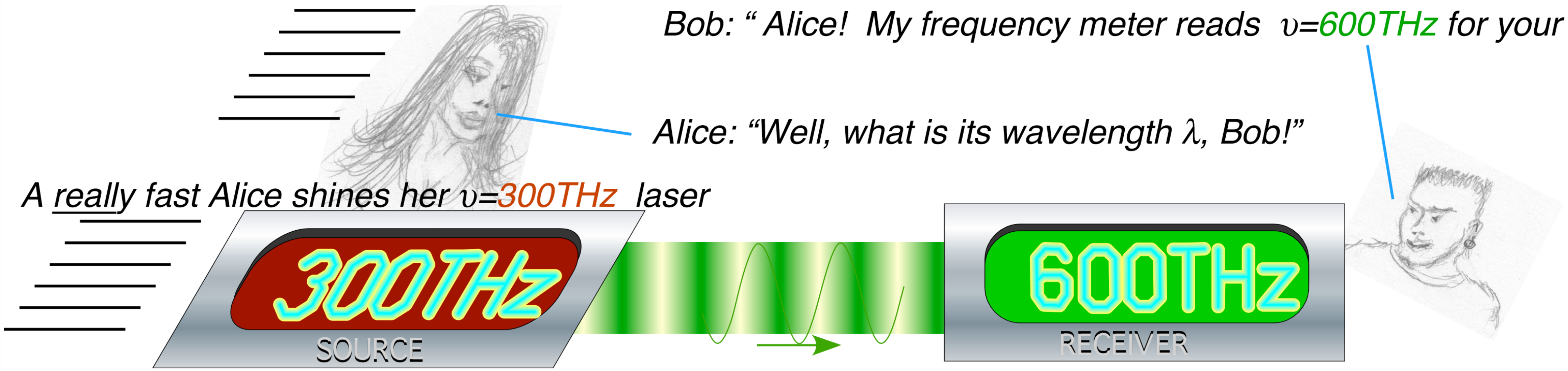

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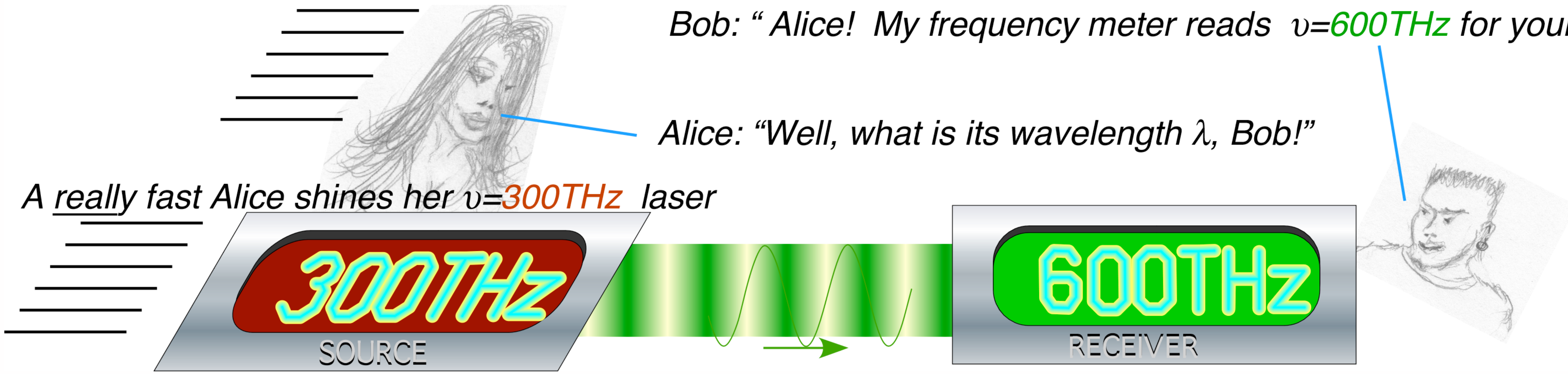

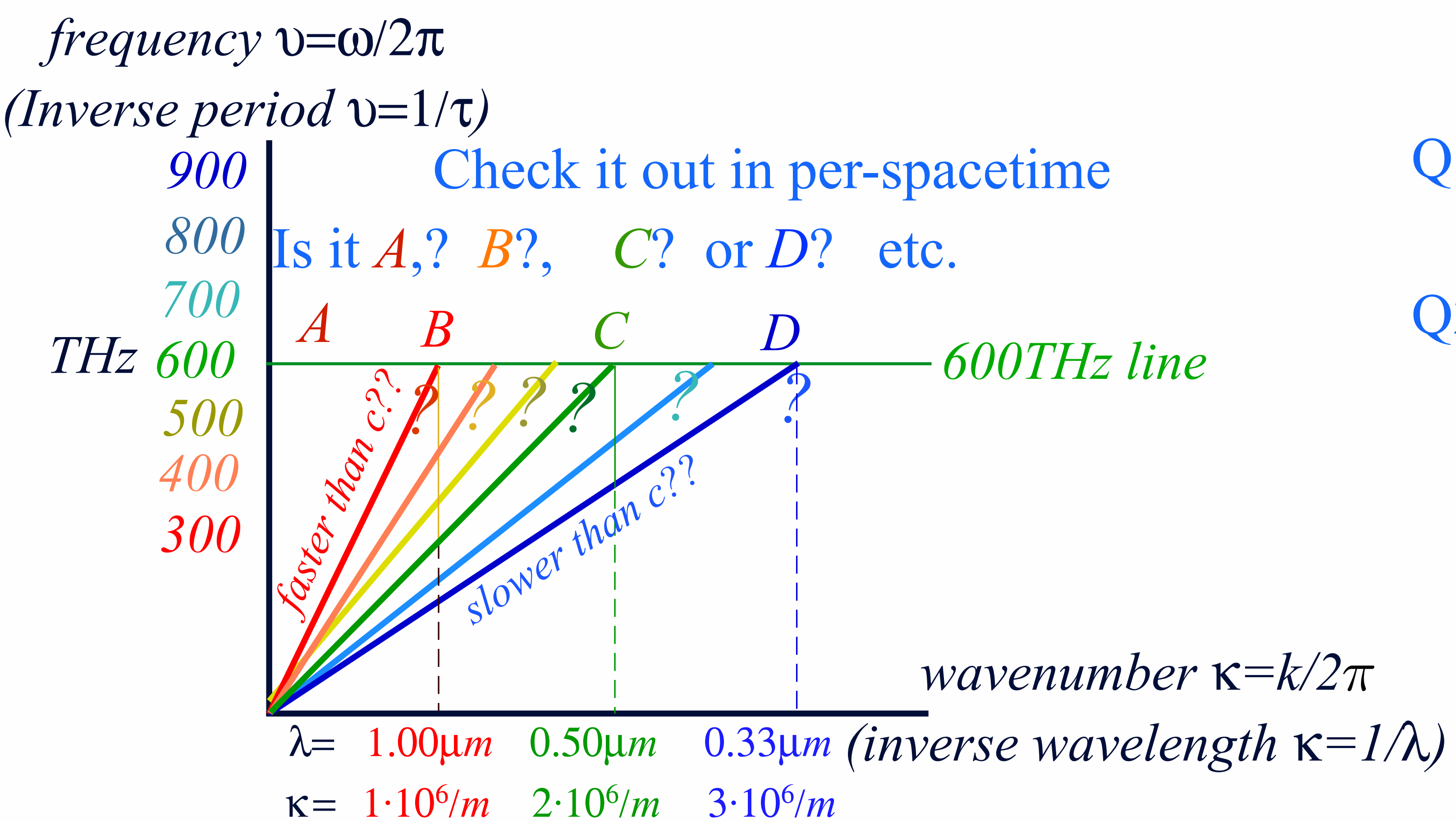

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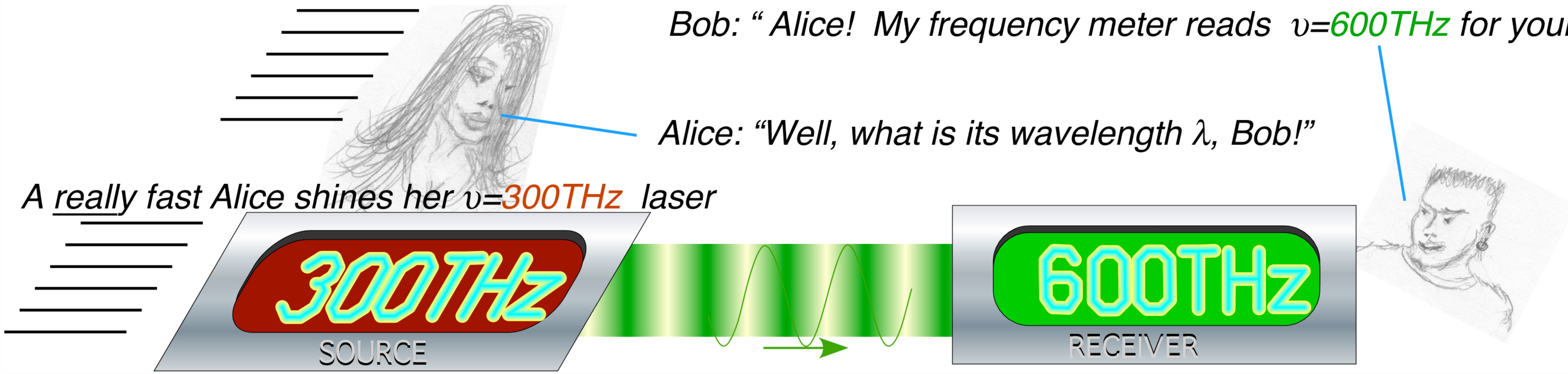

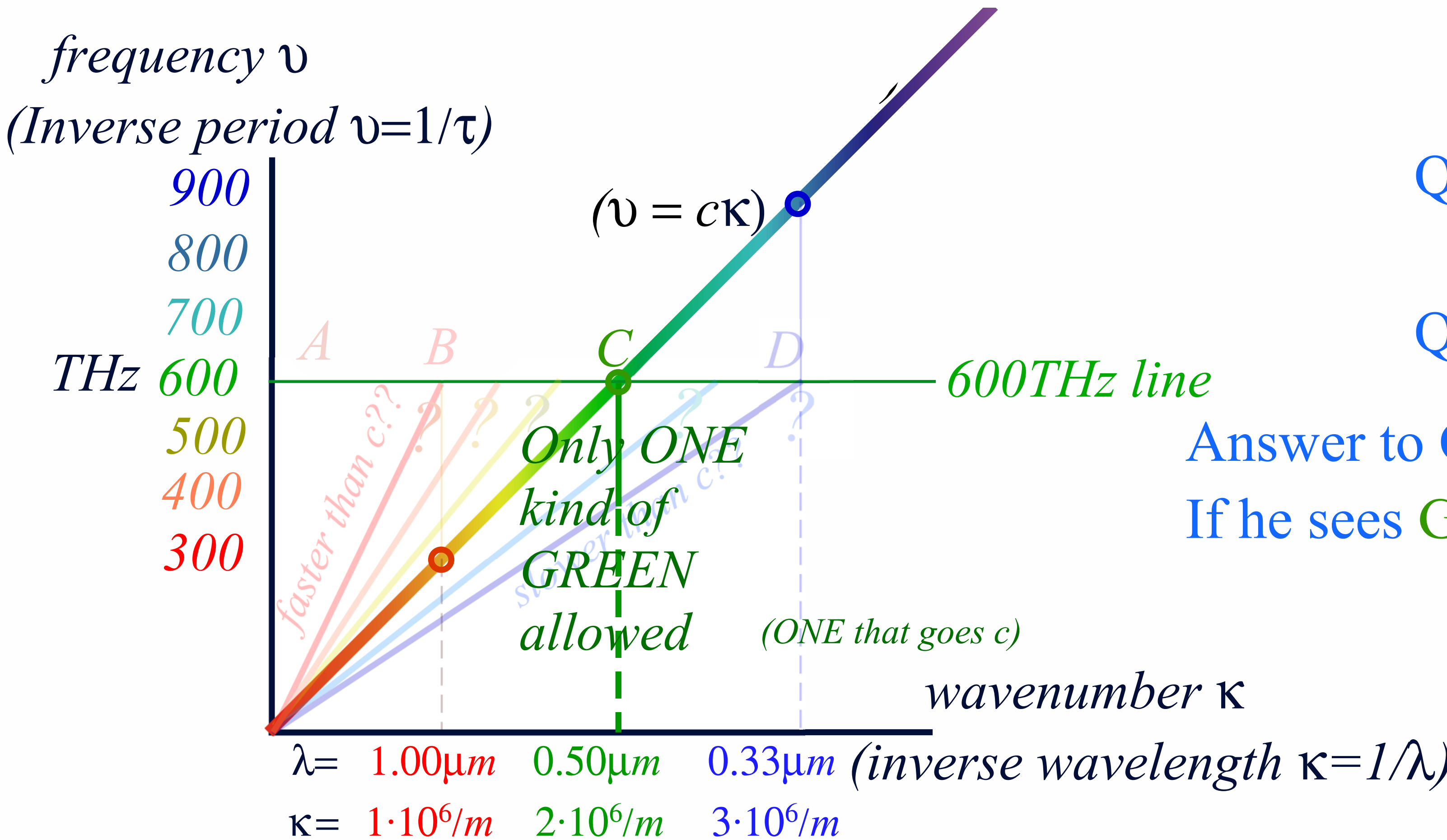

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Answer to Q2 is *C*, the one with *slope* $\upsilon/\kappa=\upsilon\cdot\lambda=c$.

If he sees Green 600THz then he measures λ=0.5μm.

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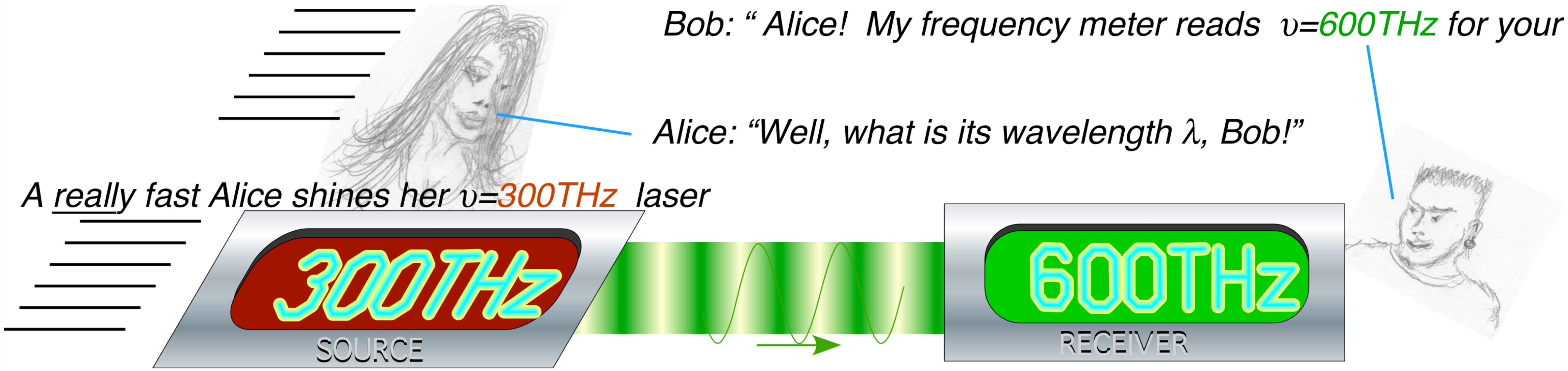

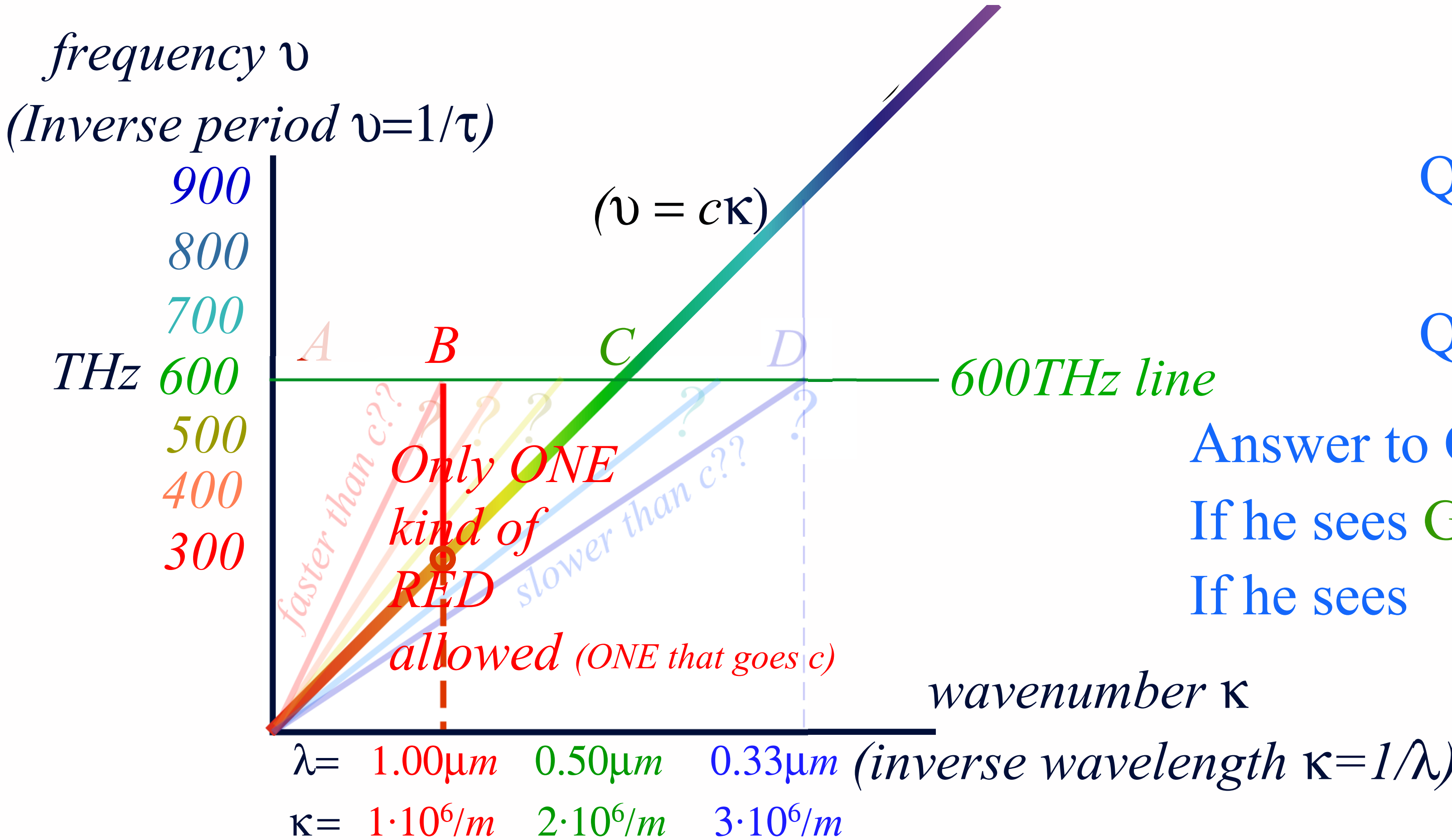

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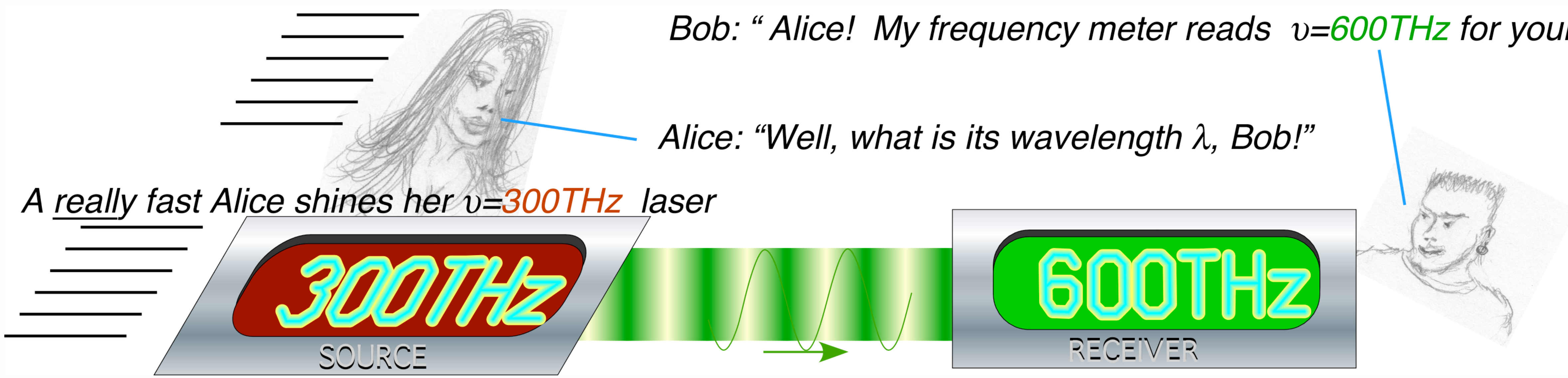

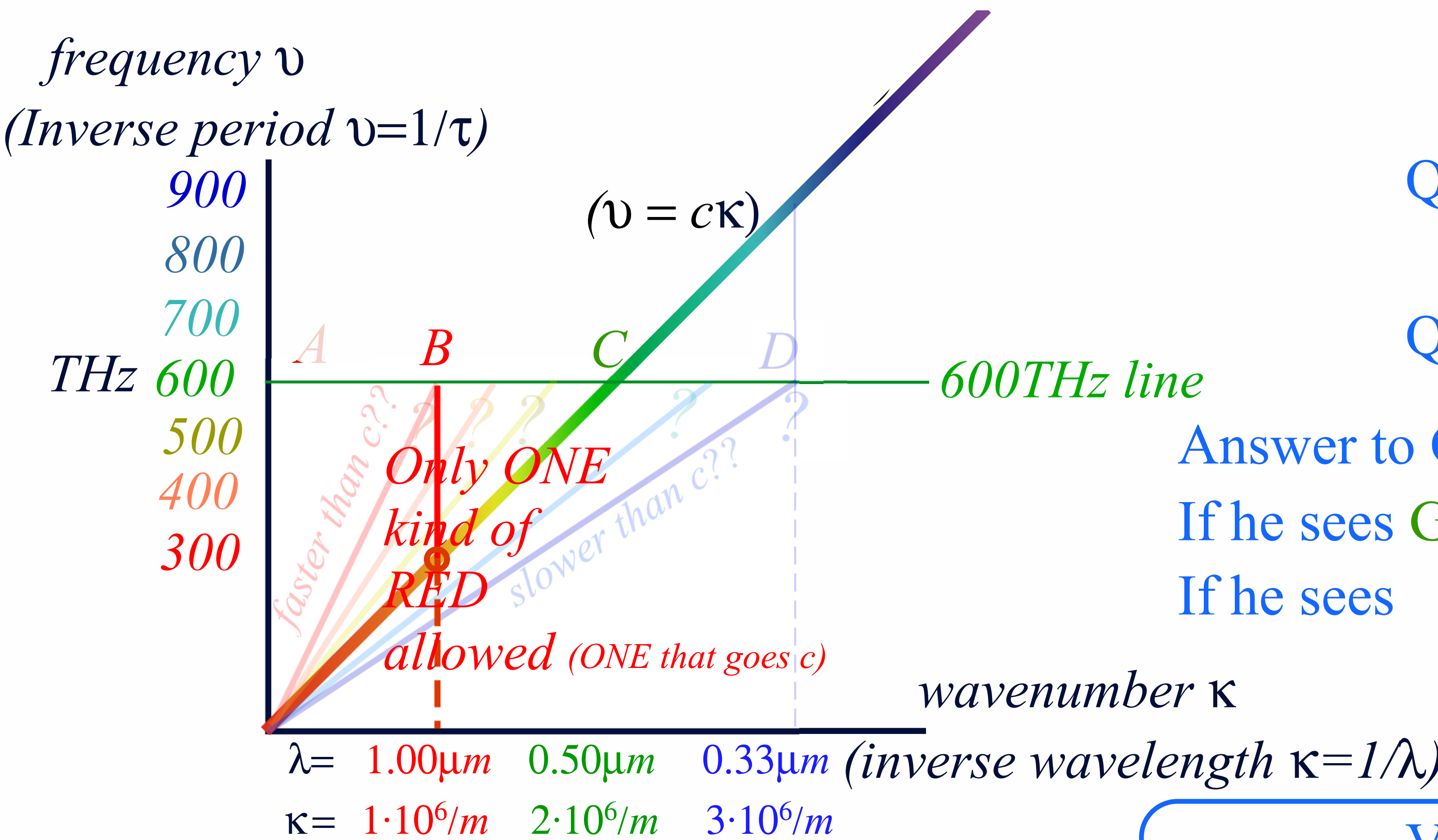

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If he sees Green 600THz then he measures λ=0.5μm.

If he sees Red 300THz then he measures λ=1.0μm.

Answer to Q1 is **NO!**

CW Light carries **no** birth-certificate!

Vacuum only makes one λ for each υ.*

"All colors go $c = \lambda\upsilon = \upsilon/\kappa$ "

Then *Evenson's axiom* holds:

*for each beam and polarization orientation

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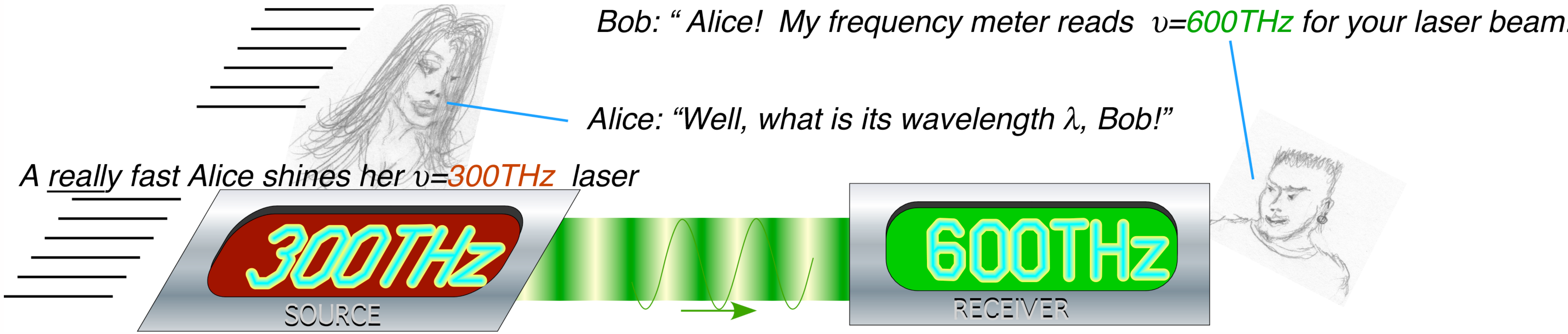
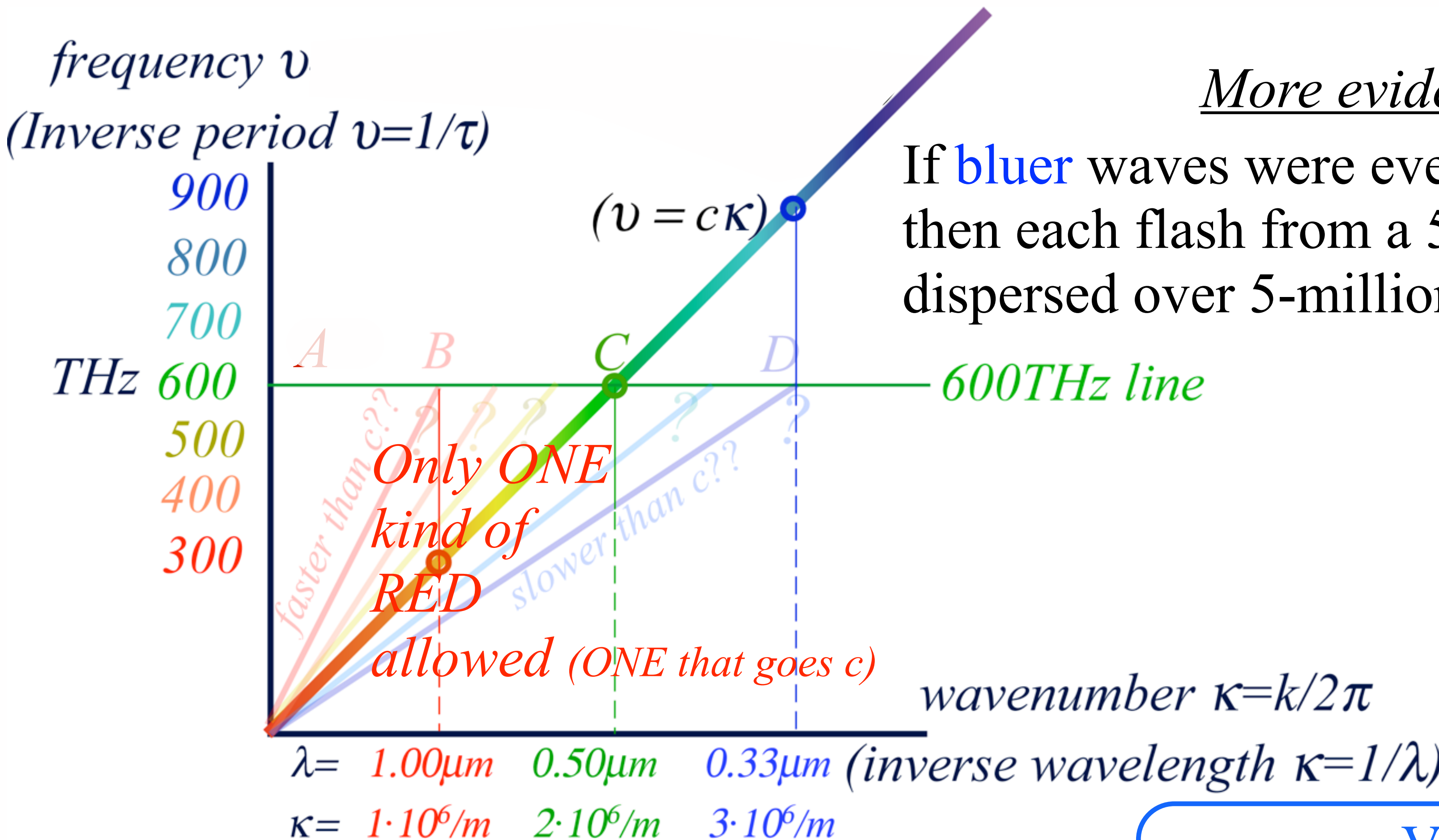




More evidence supporting Evenson's axiom

If bluer waves were even 0.1% faster (or slower) than redder ones then each flash from a 5-billion light-year distant galaxy shows up dispersed over 5-million years. (*Goodbye galactic astronomy!*)

Also could be labeled :

Linear-(non)-dispersion axiom: $\upsilon = c\kappa$

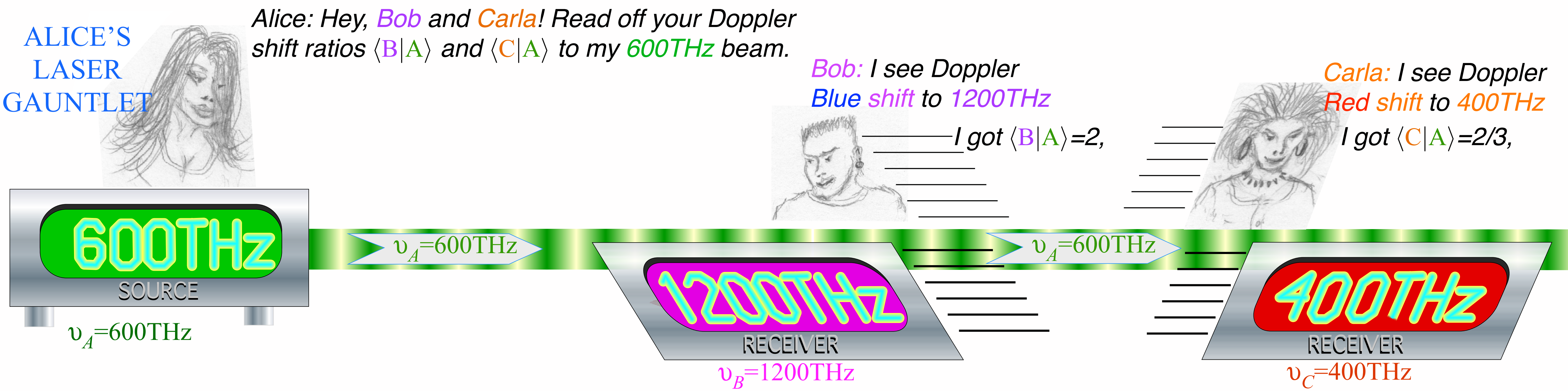
Vacuum only makes one λ for each υ.*

"All colors go $c = \lambda\upsilon = \upsilon/\kappa$ "

Then *Evenson's axiom* holds:

*for each beam and polarization orientation

Easy Doppler-shift and Rapidity calculation



Doppler ratio:

$$\langle R|S\rangle = \frac{\upsilon_{RECEIVER}}{\upsilon_{SOURCE}}$$

Bob-Alice Doppler ratio:

$$\langle B|A\rangle = \frac{\upsilon_B}{\upsilon_A} = \frac{1200}{600} = \frac{2}{1}$$

Carla-Alice Doppler ratio:

$$\langle C|A\rangle = \frac{\upsilon_C}{\upsilon_A} = \frac{400}{600} = \frac{2}{3}$$

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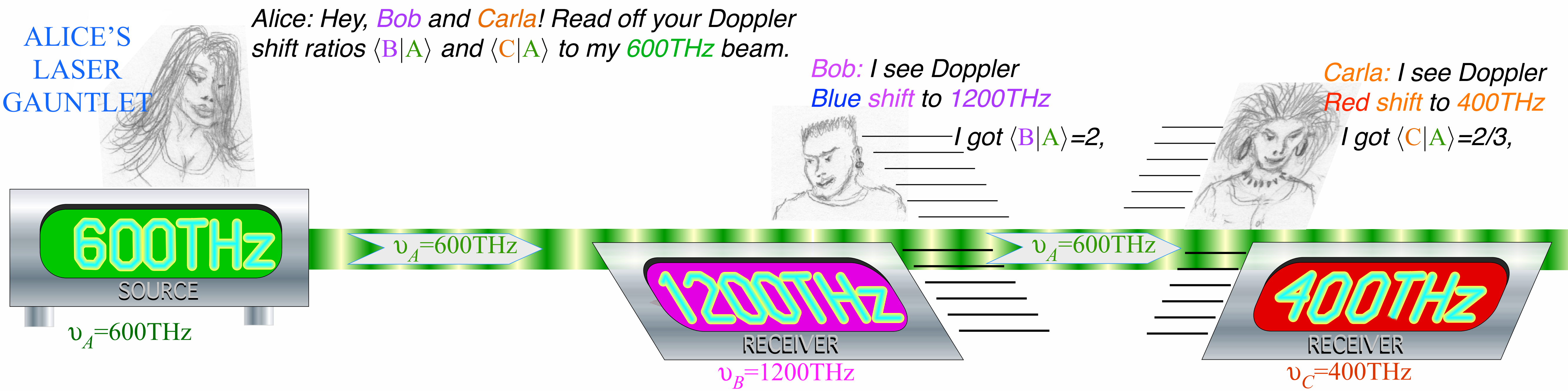

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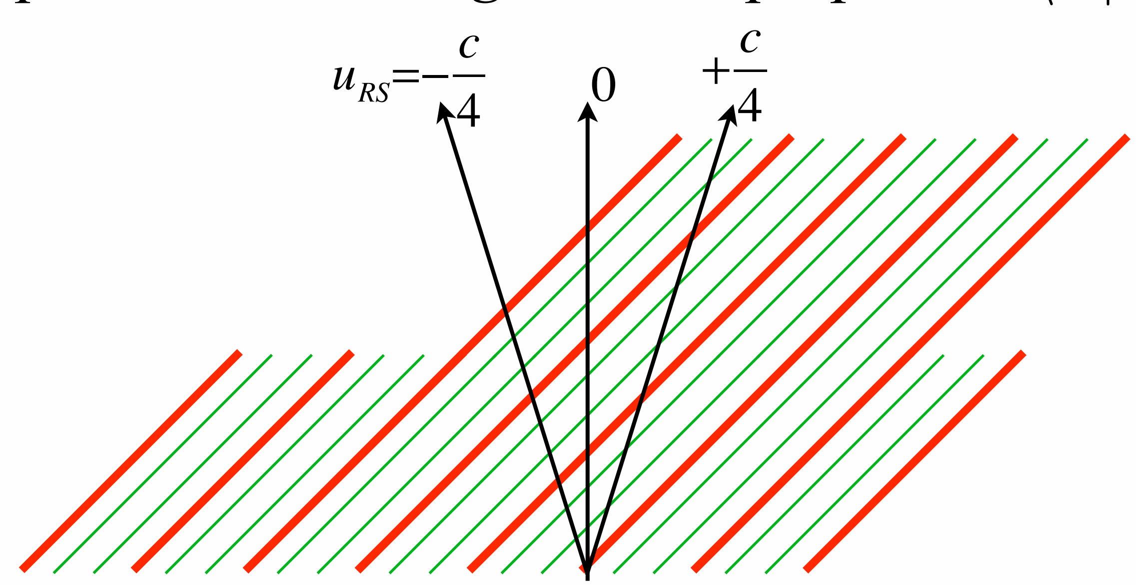
IMPORTANT POINT:

Evenson axiom says Blue, Green, Red, etc. all march in lockstep and so *all* frequencies Doppler shift in same *geometric* proportion ⟨R|S⟩.



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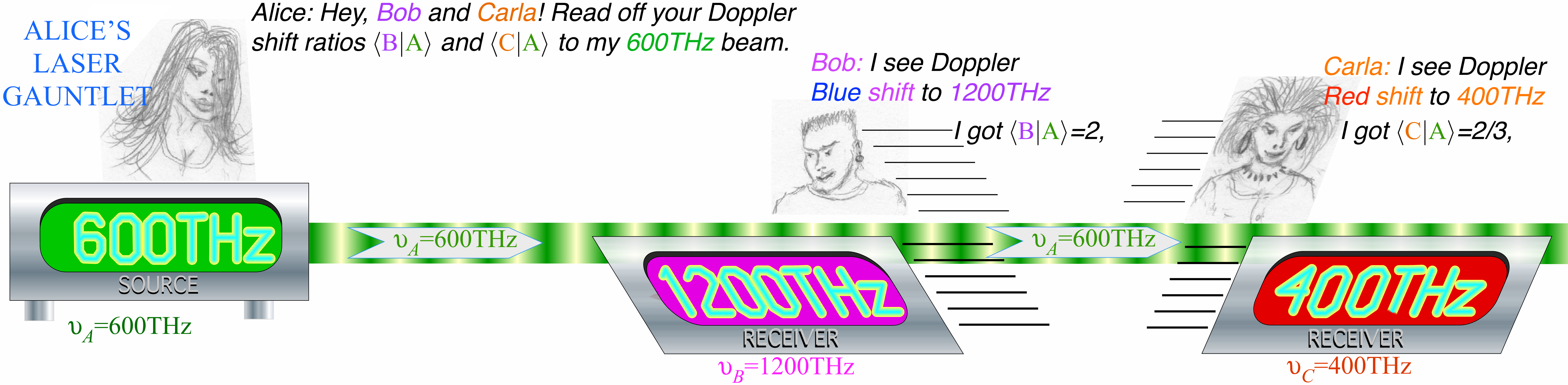

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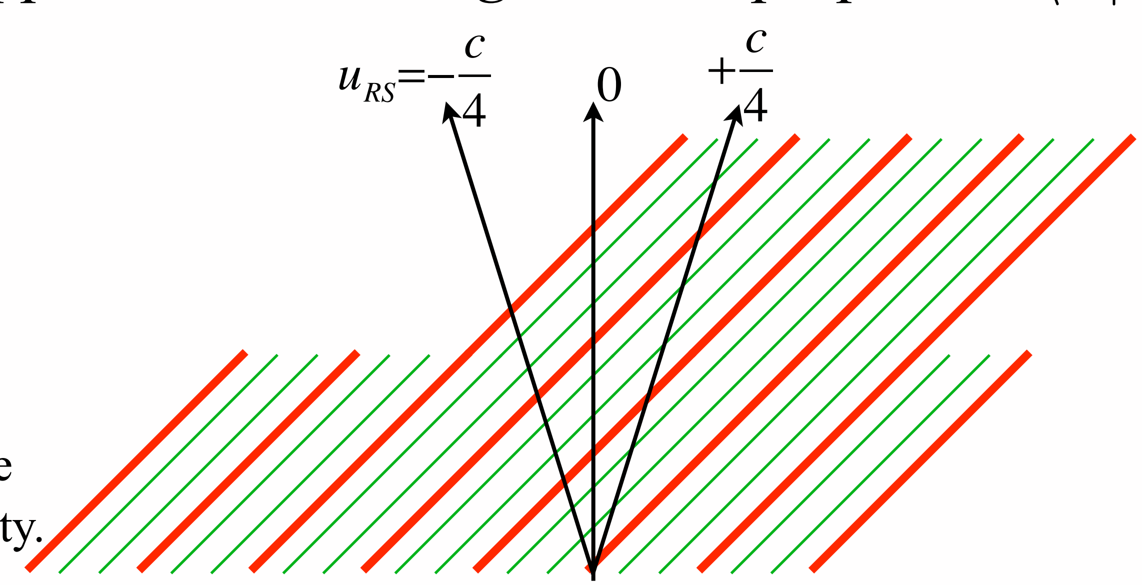
If Alice sends υ_A=600THz Bob sees: $\upsilon_B=\langle B|A\rangle\upsilon_A$=1200THz

If Alice sends υ_A=60 THz Bob sees: $\upsilon_B=\langle B|A\rangle\upsilon_A$=120THz

If Alice sends υ_A=6 Hz Bob sees: $\upsilon_B=\langle B|A\rangle\upsilon_A$=12 Hz

$\langle B|A\rangle$=2 for any frequency Alice and Bob use while they maintain their <u>relative</u> velocity.



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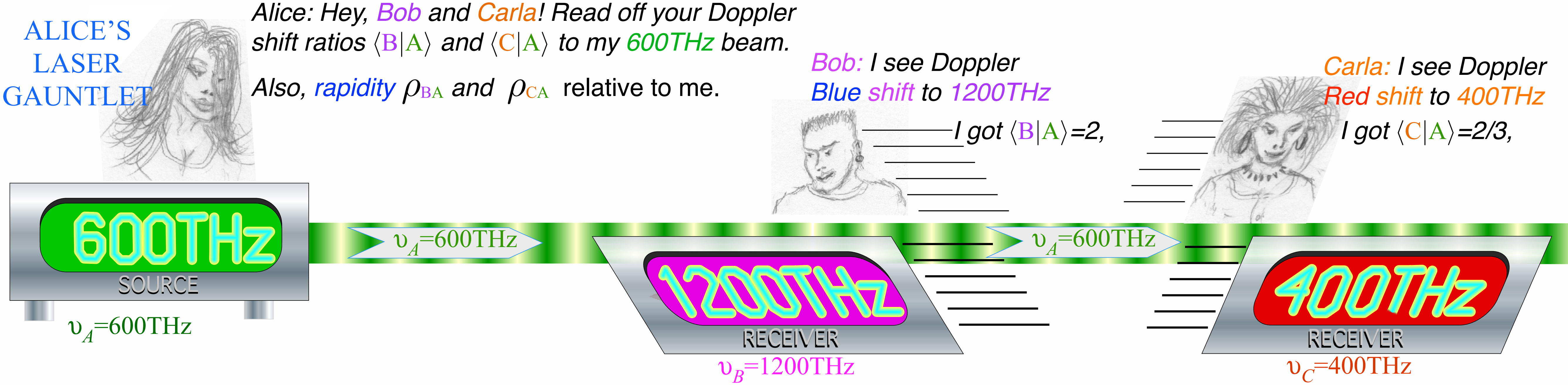

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rapidity:

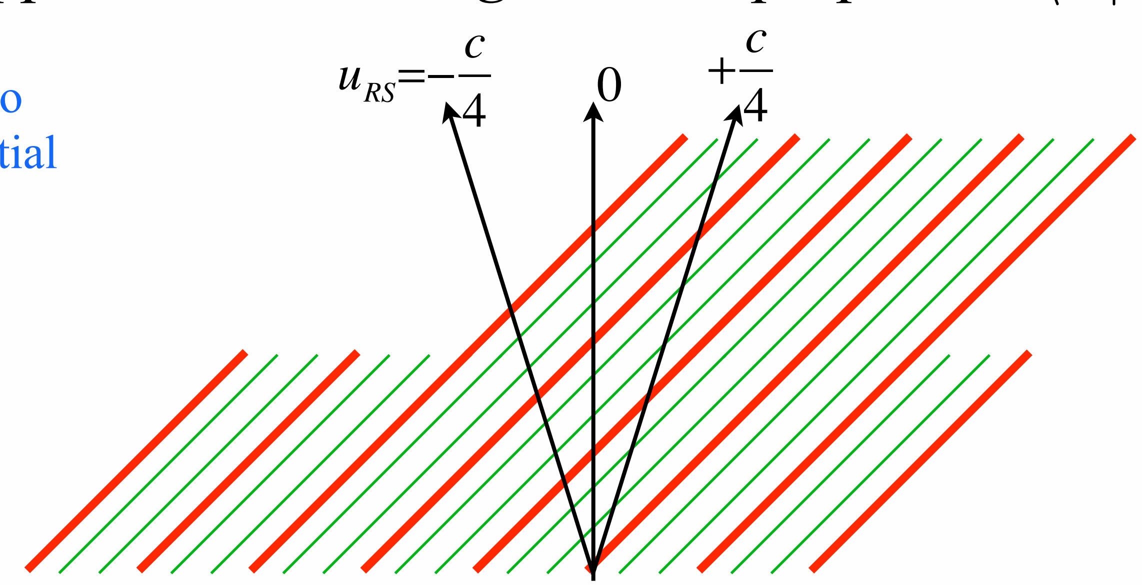
$$\rho_{RS} = \log_e \langle R|S\rangle$$

Definition of Rapidity

IMPORTANT POINTS:

Evenson axiom says Blue, Green, Red, etc. all march in lockstep and so *all* frequencies Doppler shift in same *geometric* proportion $\langle R|S\rangle$.

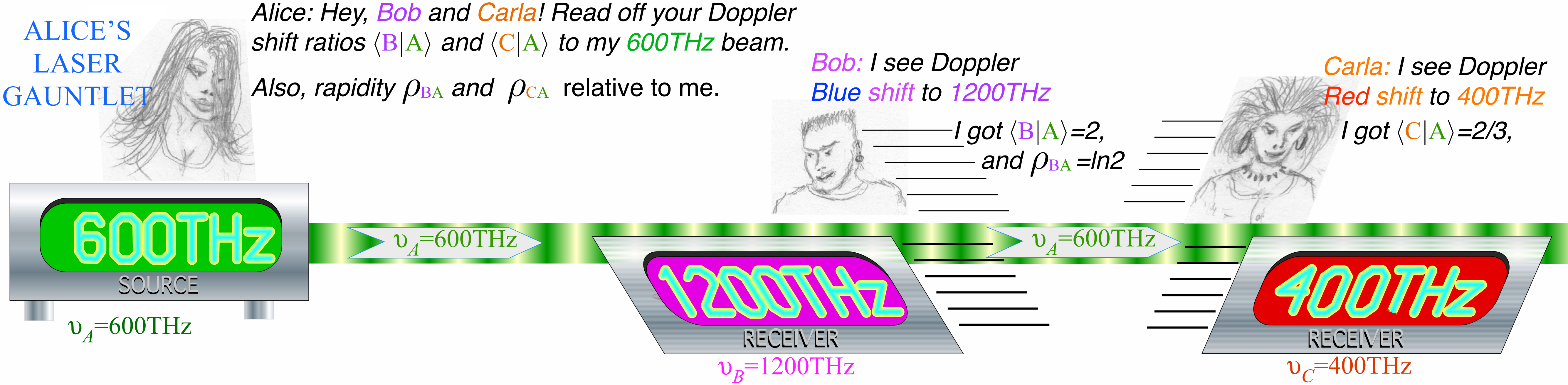
Geometric phenomena tend to involve logarithmic/exponential functionality!



Rapidity is most convenient!
1TeV proton has
u=0.999995598·c (Pain in the A)
or: $\langle R|S\rangle$*=2131.6 (Better)*
or: ρ_{RS}*=7.6646 (Best)*

For low velocity u<<c rapidity ρ_{RS} *approaches u/c*

Easy Doppler-shift and Rapidity calculation



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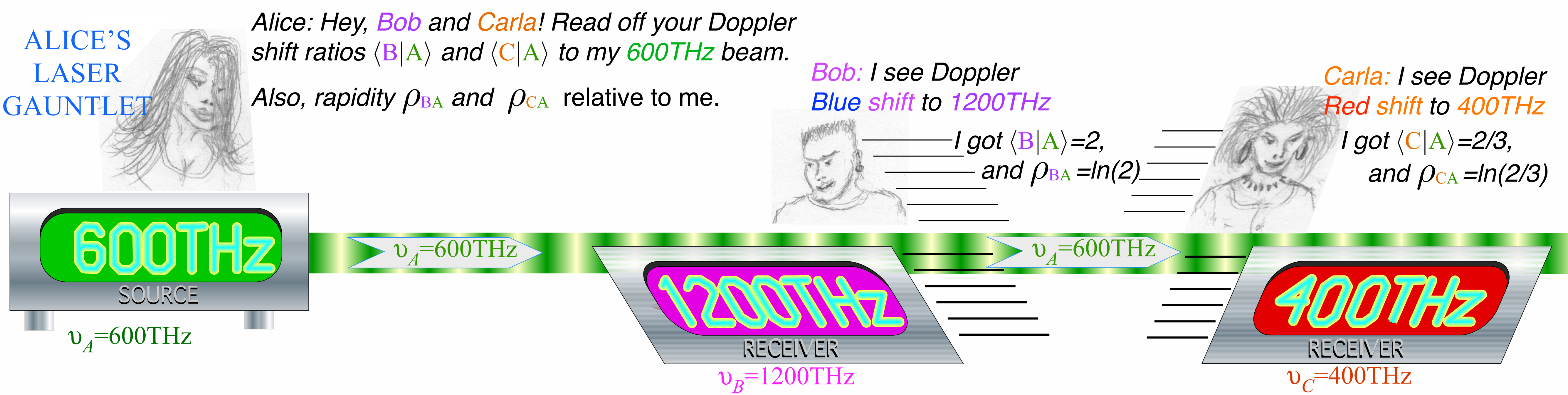

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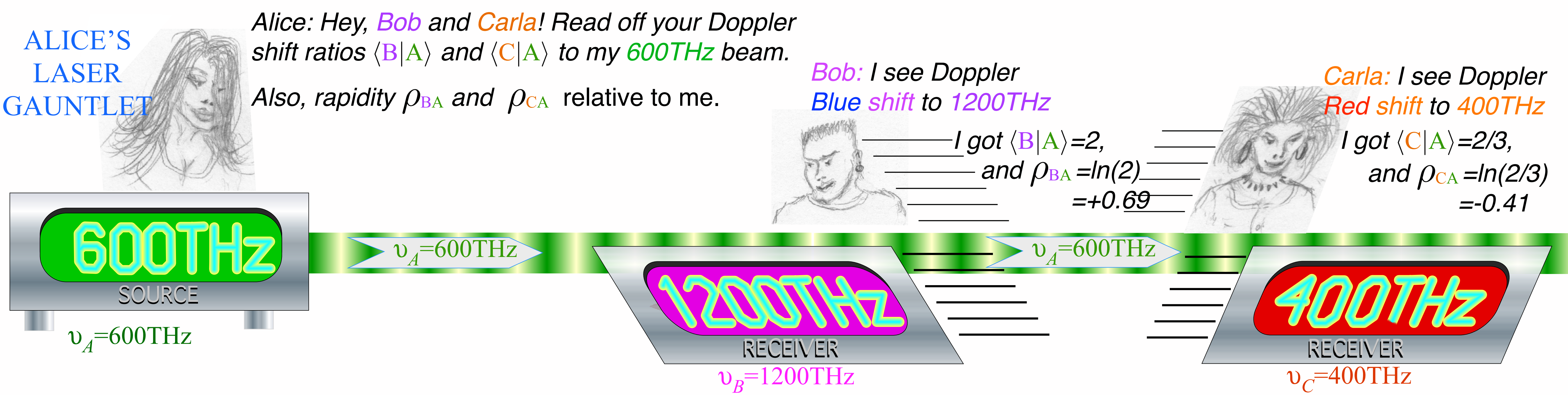
Carla-Alice Doppler ratio:

$$\langle C|A\rangle = \frac{\upsilon_C}{\upsilon_A} = \frac{400}{600} = \frac{2}{3}$$

Carla-Alice rapidity:

$$\rho_{CA} = \log_e \langle C|A\rangle = \log_e \frac{2}{3}$$

Easy Doppler-shift and Rapidity calculation



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$$\rho_{RS} = \log_e \langle R|S\rangle$$

Definition of Rapidity

$$\langle B|A\rangle = \frac{\upsilon_B}{\upsilon_A} = \frac{2}{1}$$

is time-reversal of:

$$\langle A|B\rangle = \frac{\upsilon_A}{\upsilon_B} = \frac{1}{2}$$

Bob-Alice Doppler ratio:

$$\langle B|A\rangle = \frac{\upsilon_B}{\upsilon_A} = \frac{1200}{600} = \frac{2}{1}$$

Bob-Alice rapidity:

$$\rho_{BA} = \log_e \langle B|A\rangle = \log_e \frac{2}{1}$$

$$\rho_{BA} = 0.69 \quad \text{(time-reversed)} \quad (so{:}\ \rho_{AB} = -0.69)$$

Carla-Alice Doppler ratio:

$$\langle C|A\rangle = \frac{\upsilon_C}{\upsilon_A} = \frac{400}{600} = \frac{2}{3}$$

Carla-Alice rapidity:

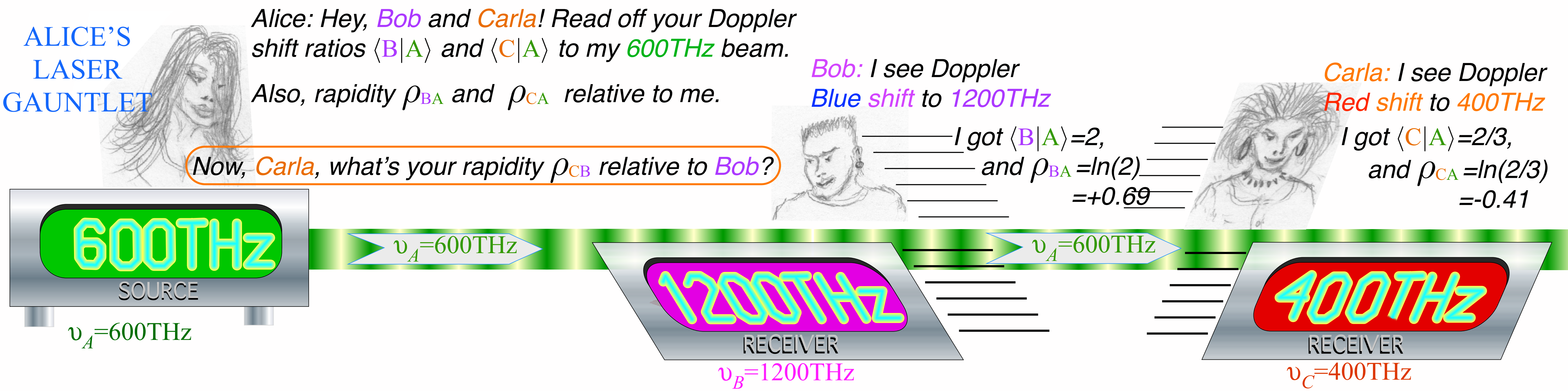
$$\rho_{CA} = \log_e \langle C|A\rangle = \log_e \frac{2}{3}$$

$$\rho_{CA} = -0.41$$

Mnemonic: You can think of rapidity ρ_{BA} as "R" for "Romance"… (+) positive on approach, (-) negative on reproach

Do the stars hate us?

Easy Doppler-shift and Rapidity calculation



Doppler ratio:

$$\langle R|S\rangle = \frac{\upsilon_{RECEIVER}}{\upsilon_{SOURCE}}$$

rapidity:

$$\rho_{RS} = \log_e \langle R|S\rangle$$

Definition of Rapidity

$$\langle B|A\rangle = \frac{\upsilon_B}{\upsilon_A} = \frac{2}{1}$$

is time-reversal of:

$$\langle A|B\rangle = \frac{\upsilon_A}{\upsilon_B} = \frac{1}{2}$$

Bob-Alice Doppler ratio:

$$\langle B|A\rangle = \frac{\upsilon_B}{\upsilon_A} = \frac{1200}{600} = \frac{2}{1}$$

Bob-Alice rapidity:

$$\rho_{BA} = \log_e \langle B|A\rangle = \log_e \frac{2}{1}$$

$$\rho_{BA} = 0.69 \quad \text{(time-reversed)} \ (so{:}\ \rho_{AB} = -0.69)$$

Carla-Alice Doppler ratio:

$$\langle C|A\rangle = \frac{\upsilon_C}{\upsilon_A} = \frac{400}{600} = \frac{2}{3}$$

Carla-Alice rapidity:

$$\rho_{CA} = \log_e \langle C|A\rangle = \log_e \frac{2}{3}$$

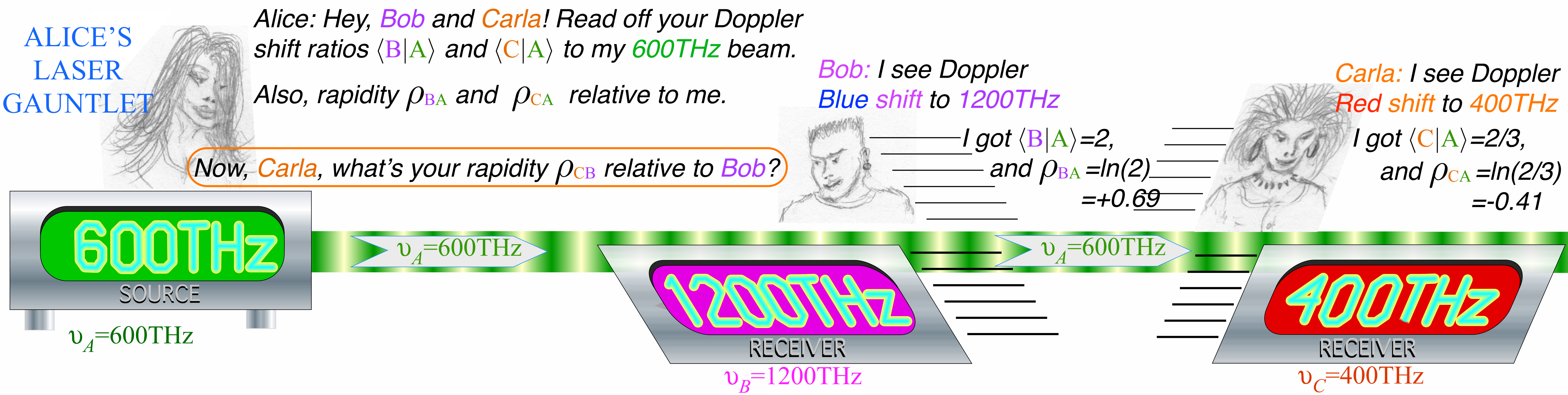
$$\rho_{CA} = -0.41$$

Carla-Bob Doppler ratio:

$$\langle C|B\rangle = \frac{\upsilon_C}{\upsilon_B} = \frac{\upsilon_C}{\upsilon_A}\frac{\upsilon_A}{\upsilon_B} = \langle C|A\rangle\langle A|B\rangle$$

Mnemonic: You can think of rapidity ρ_{BA} as *"R"* for *"Romance"*… (+) positive on approach, (-) negative on reproach

Easy Doppler-shift and Rapidity calculation



Doppler ratio:

$$\langle R|S\rangle = \frac{\upsilon_{RECEIVER}}{\upsilon_{SOURCE}}$$

rapidity:

$$\rho_{RS} = \log_e \langle R|S\rangle$$

so:

$$\langle R|S\rangle = e^{\rho_{RS}}$$

Definition of Rapidity

$$\langle B|A\rangle = \frac{\upsilon_B}{\upsilon_A}$$

is time-reversed

$$\langle A|B\rangle = \frac{\upsilon_A}{\upsilon_B}$$

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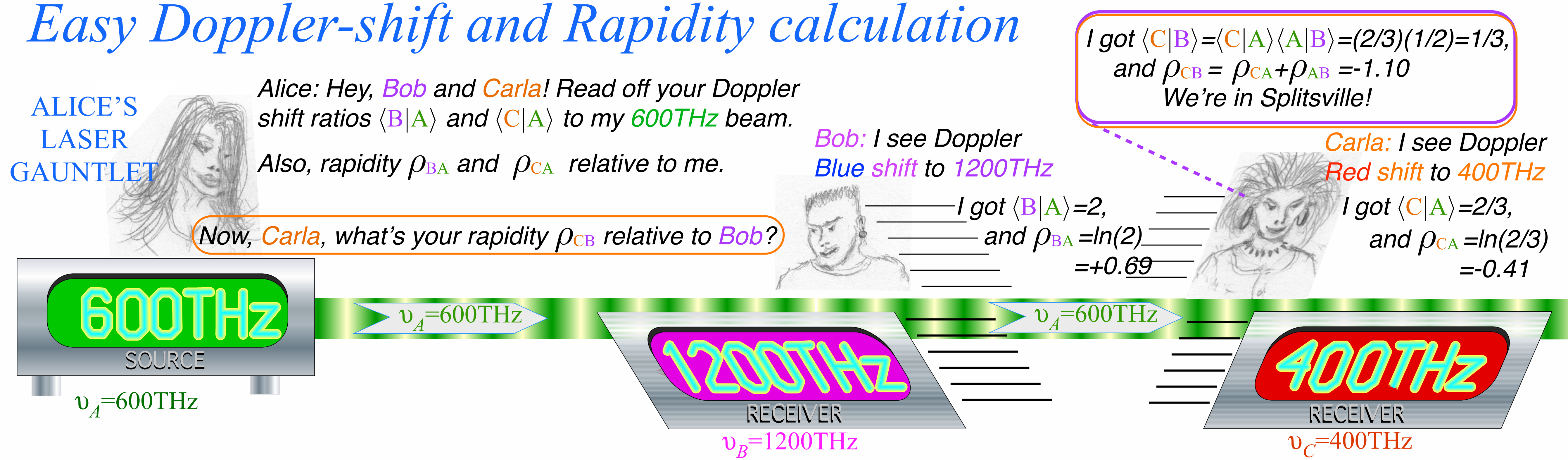
Carla-Bob rapidity:

$$e^{\rho_{CB}} = e^{\rho_{CA}} e^{\rho_{AB}}$$

Easy Doppler-shift and Rapidity calculation



Doppler ratio:

$$\langle R|S\rangle = \frac{\upsilon_{RECEIVER}}{\upsilon_{SOURCE}}$$

rapidity:

$$\rho_{RS} = \log_e \langle R|S\rangle$$

or:

$$\langle R|S\rangle = e^{\rho_{RS}} = e^{-\rho_{SR}}$$

Definition of Rapidity

$$\langle B|A\rangle = \frac{\upsilon_B}{\upsilon_A}$$

is time-reversed

$$\langle A|B\rangle = \frac{\upsilon_A}{\upsilon_B}$$

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Carla-Bob rapidity:

$$e^{\rho_{CB}} = e^{\rho_{CA}} e^{\rho_{AB}} = e^{\rho_{CA}+\rho_{AB}}$$

implies: $\rho_{CB} = \rho_{CA} + \rho_{AB}$

$$= -0.41 - 0.69 = -1.10$$

Easy Doppler-shift and Rapidity calculation

ALICE'S LASER GAUNTLET

Alice: Hey, Bob and Carla! Read off your Doppler shift ratios $\langle B|A\rangle$ and $\langle C|A\rangle$ to my 600THz beam.

Also, rapidity ρ_{BA} and ρ_{CA} relative to me.

Now, Carla, what's your rapidity ρ_{CB} relative to Bob?

Bob: I see Doppler Blue shift to 1200THz

I got $\langle B|A\rangle=2$, and $\rho_{BA}=\ln(2)$ $=+0.69$

Carla: I see Doppler Red shift to 400THz

I got $\langle C|A\rangle=2/3$, and $\rho_{CA}=\ln(2/3)$ $=-0.41$

I got $\langle C|B\rangle=\langle C|A\rangle\langle A|B\rangle=(2/3)(1/2)=1/3$, and $\rho_{CB}=\rho_{CA}+\rho_{AB}=-1.10$ We're in Splitsville!

Doppler ratio:

$$\langle R|S\rangle = \frac{\upsilon_{RECEIVER}}{\upsilon_{SOURCE}}$$

rapidity:

$$\rho_{RS} = \log_e \langle R|S\rangle$$

or:

$$\langle R|S\rangle = e^{\rho_{RS}}$$

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Carla-Bob rapidity:

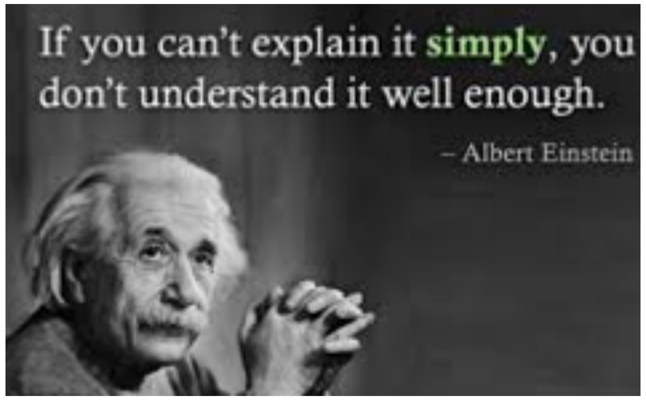
$$e^{\rho_{CB}} = e^{\rho_{CA}} e^{\rho_{AB}} \quad \text{implies:}$$

Galileo's Revenge (part 1)

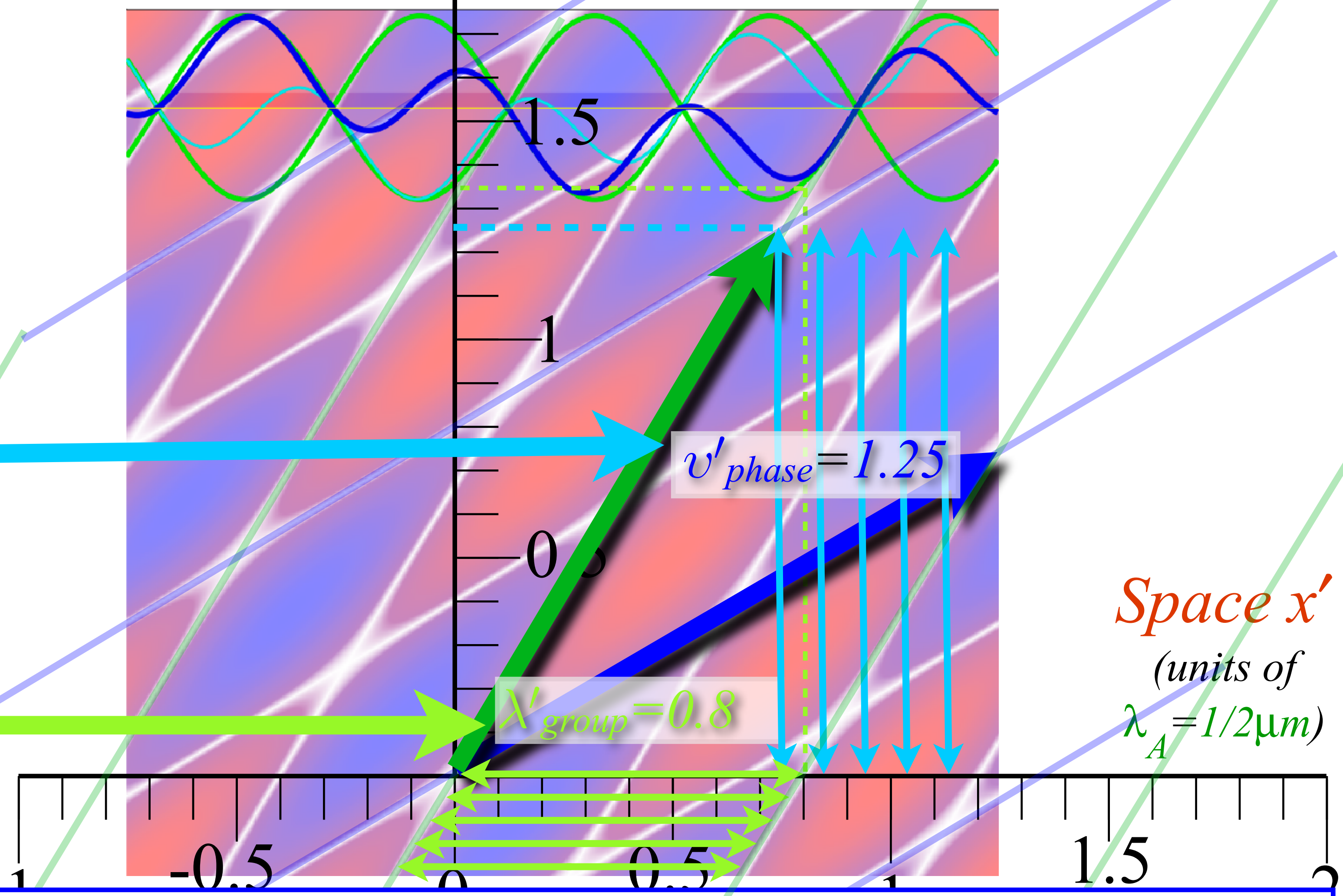
Rapidity adds just like Galilean velocity

$$\rho_{CB} = \rho_{CA} + \rho_{AB}$$
$$= -0.41 - 0.69 = -1.10$$

Two Famous-Name Coefficients



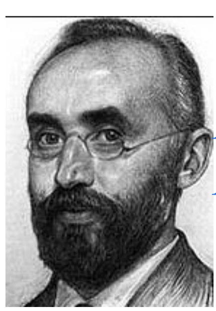
Albert Einstein 1859-1955



Herman Minkowski 1864-1909

This number is called an: ***Einstein time-dilation*** *(dilated by 25% here)*

This number is called a: ***Lorentz length-contraction*** *(contracted by 20% here)*



Hendrik A. Lorentz 1853-1928

Old-Fashioned Notation

phase	$b_{RED}^{Doppler}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
group	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\text{sech}\,\rho$	$\cosh\rho$	$\text{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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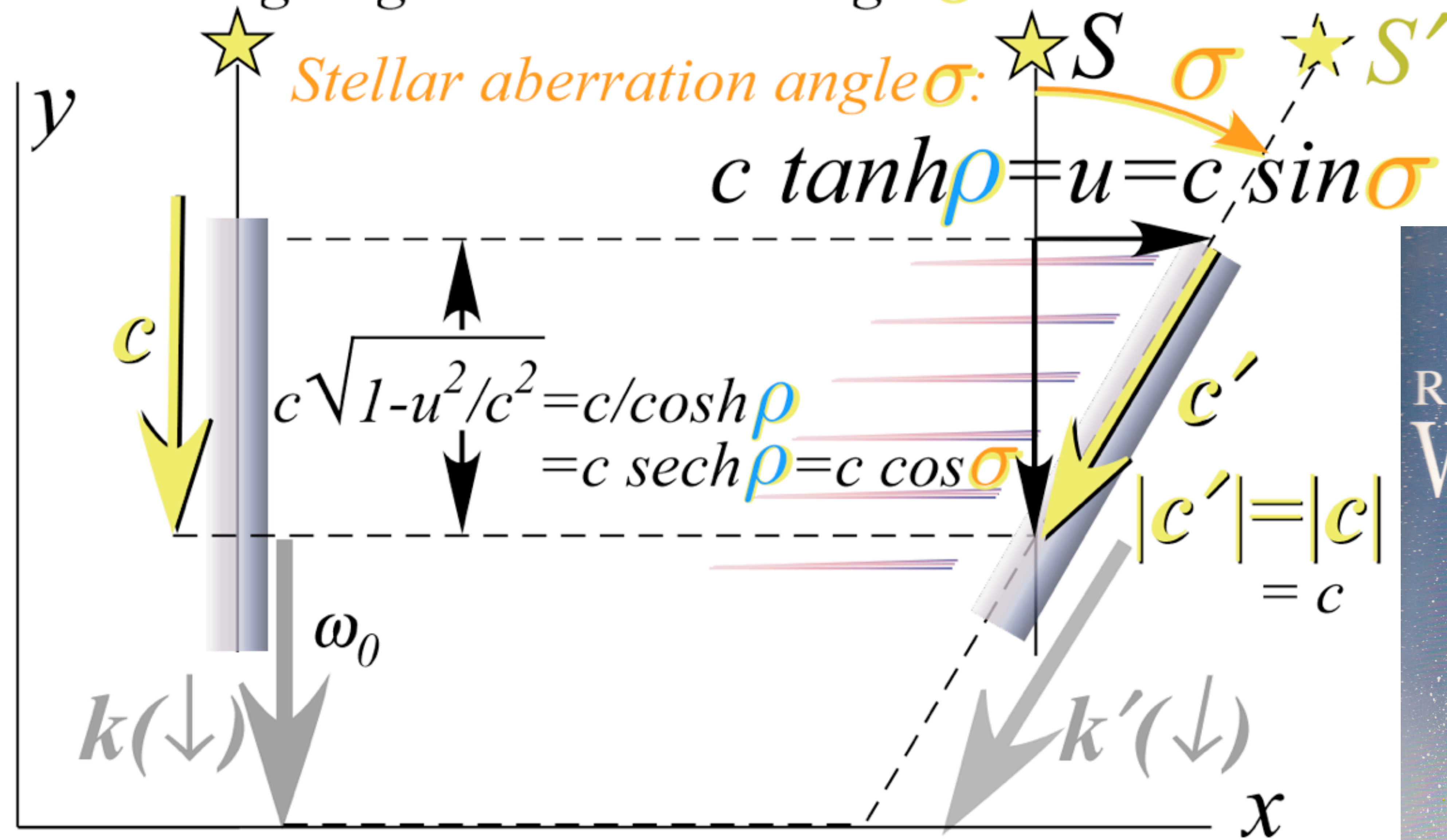
Comparing Longitudinal relativity parameter: Rapidity ρ = $\log_e$(Doppler Shift)

to a Transverse*relativity parameter: Stellar aberration angle σ

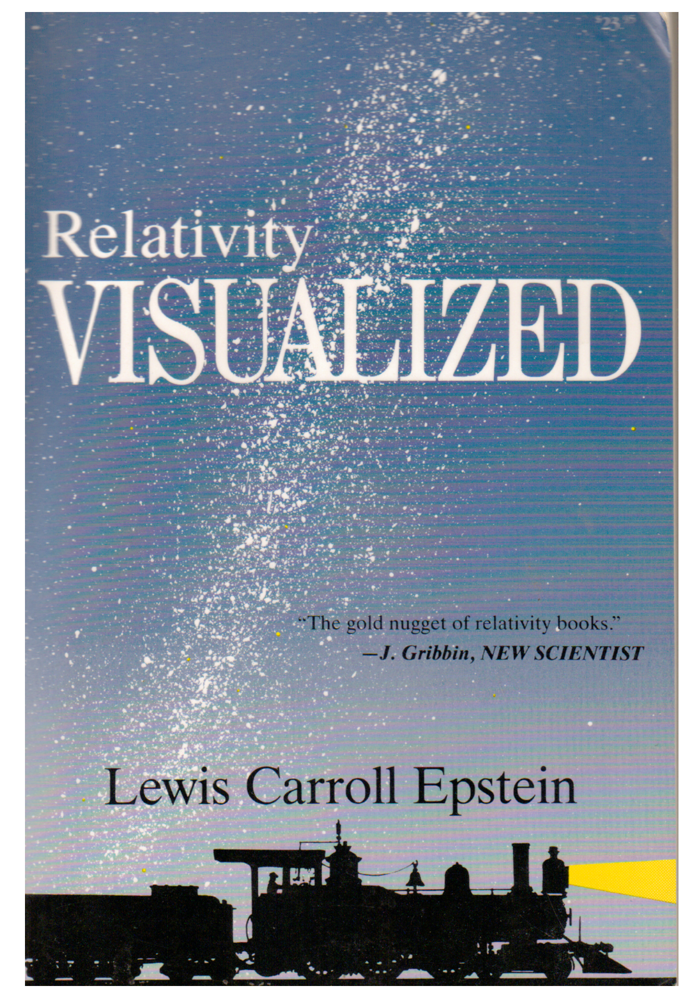
*Lewis Carroll Epstein, ***Relativitätstheorie***, Birkhäuser, (2004) Earlier English version (1985)-

Observer fixed below star sees it directly overhead.

*Observer going **u** sees star at angle σ in **u** direction.*



We used notion σ for stellar-ab-angle, (a "flipped-out" ρ). Epstein not interested in ρ analysis or in relation of σ and ρ.



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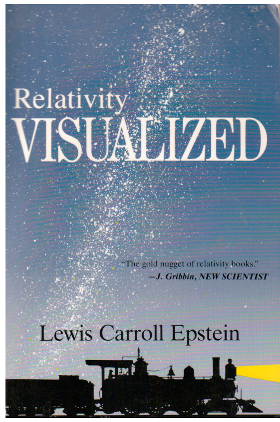
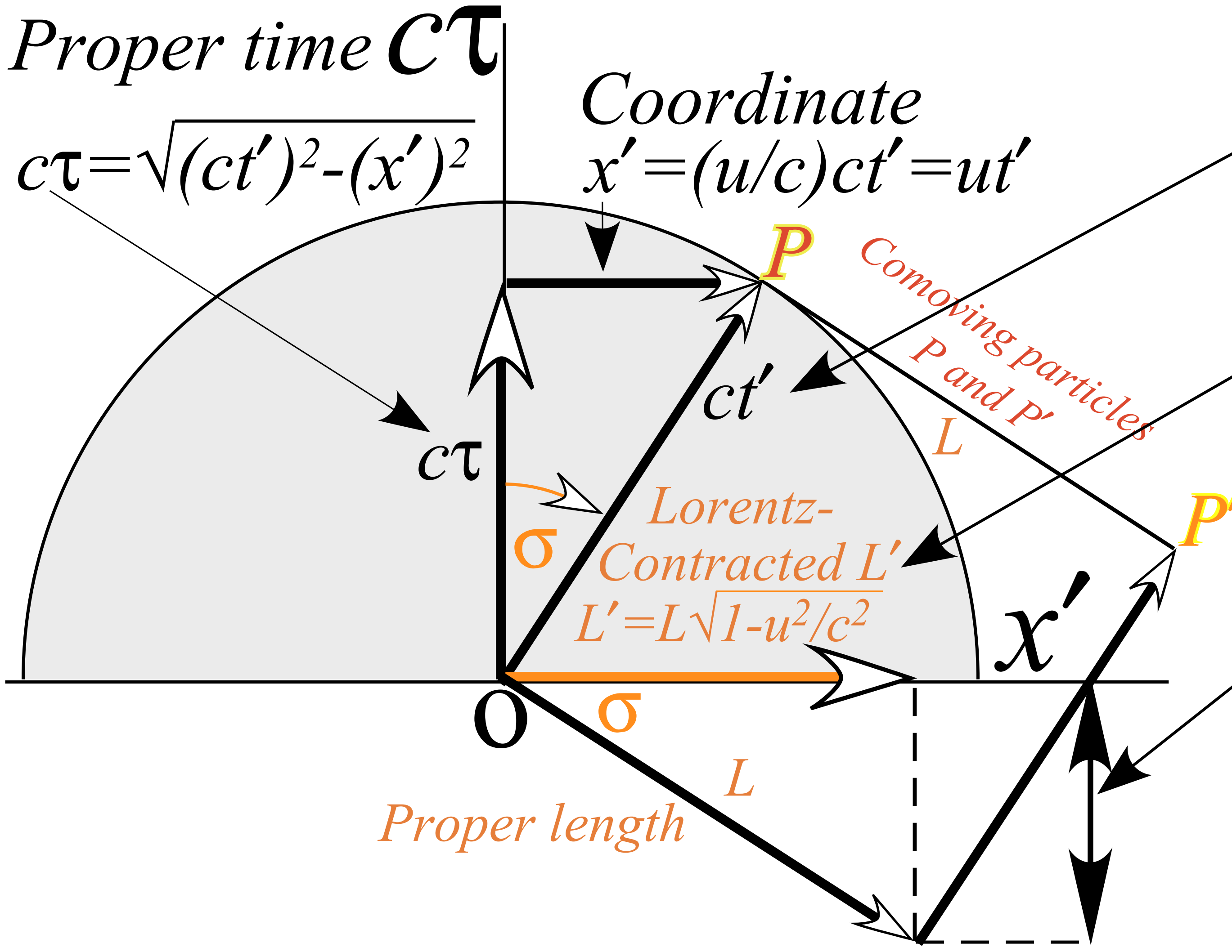


Proper time $c\tau$ vs. coordinate space x - (L. C. Epstein's "Cosmic Speedometer")

Particles P and P' have speed u in (x',ct') and speed c in $(x, c\tau)$



Einstein time dilation:

$$ct'=c\tau\sec\sigma=c\tau\cosh\rho = c\tau/\sqrt{1-u^2/c^2}$$

Lorentz length contraction:

$$L'= L\,\mathrm{sech}\rho = L\cos\sigma = L\cdot\sqrt{1-u^2/c^2}$$

Proper Time asimultaneity:

$$c\,\Delta\tau = L'\sinh\rho = L\cos\sigma\sinh\rho$$
$$= L\cos\sigma\tan\sigma$$
$$= L\sin\sigma \quad = L/\sqrt{c^2/u^2-1} \sim L\,u/c$$

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to a Transverse*relativity parameter: Stellar aberration angle σ

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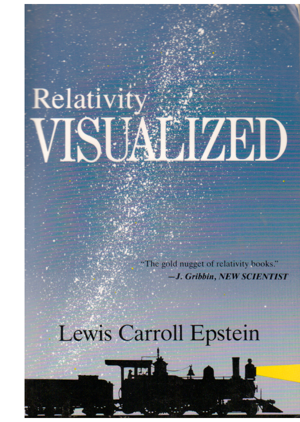

Proper time cτ vs. coordinate space x - (L. C. Epstein's "Cosmic Speedometer")

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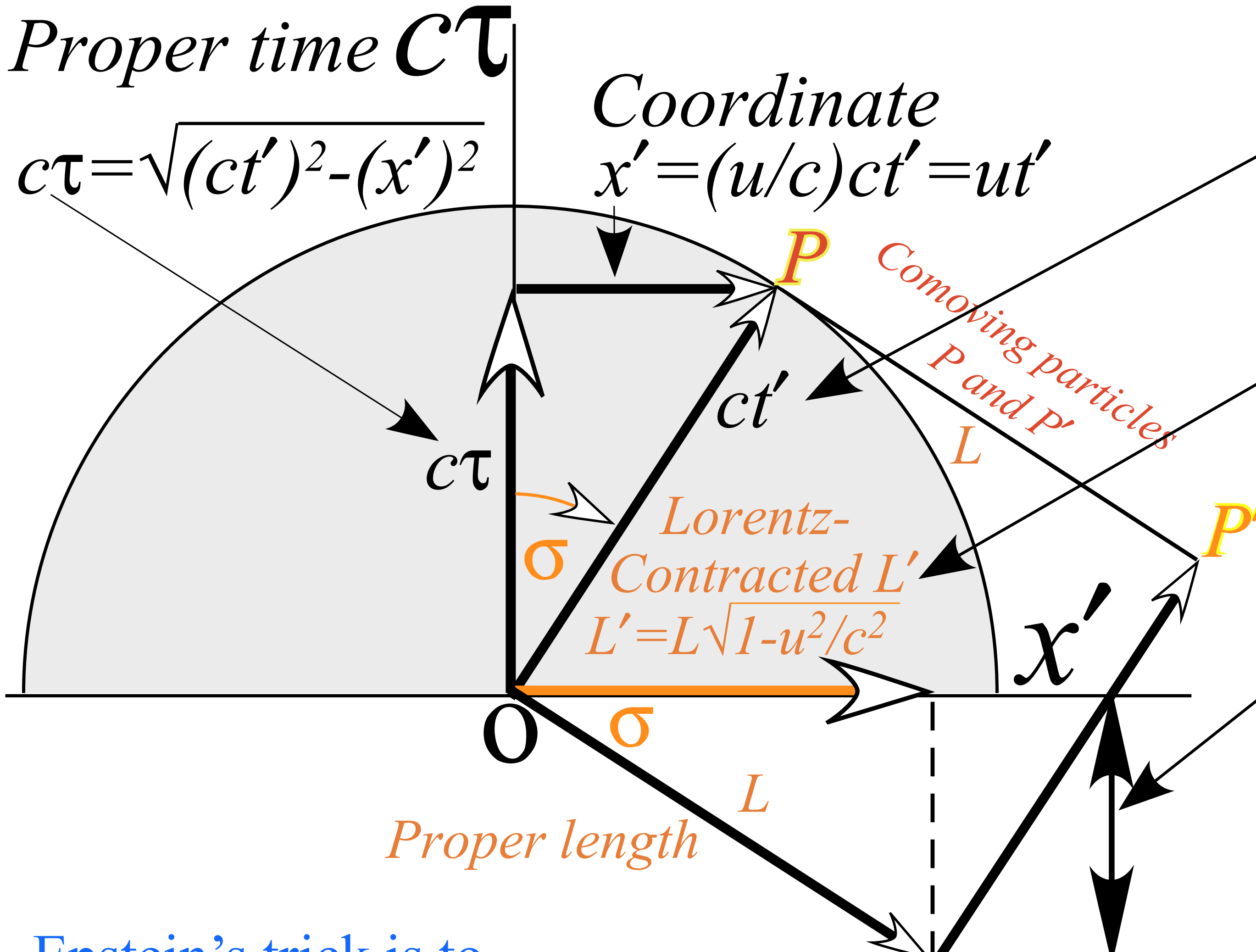

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$$= L\cos\sigma\tan\sigma$$
$$= L\sin\sigma \quad = L/\sqrt{c^2/u^2-1} \sim L\,u/c$$

Epstein's trick is to turn a hyperbolic form $c\tau = \sqrt{(ct')^2 - (x')^2}$ into a circular form:

$$\sqrt{(c\tau)^2 + (x')^2} = (ct')$$

Then everything (and everybody) always goes speed c through $(x', c\tau)$ space!

Using (some) wave parameters to develop relativistic quantum theory

$$\upsilon_{phase} = B\cosh\rho \approx B + \tfrac{1}{2}B\rho^2 \text{ (for } u \ll c)$$

$$c\kappa_{phase} = B\sinh\rho \approx B\rho \quad \text{ (for } u \ll c)$$

At low speeds: $\cosh\rho \approx 1 + \tfrac{1}{2}\rho^2$, $\sinh\rho \approx \rho$

$$B = \upsilon_A$$

$$B = \upsilon_A = c\kappa_A$$

group	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
phase	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\operatorname{sech}\rho$	$\cosh\rho$	$\operatorname{csch}\rho$	$\coth\rho$	$e^{+\rho}$
stellar ∀ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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group	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
phase	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\text{sech}\,\rho$	$\cosh\rho$	$\text{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
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$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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group	$b^{Doppler}_{RED}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b^{Doppler}_{BLUE}$
phase	$\frac{1}{b^{Doppler}_{BLUE}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b^{Doppler}_{RED}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\operatorname{sech}\rho$	$\cosh\rho$	$\operatorname{csch}\rho$	$\coth\rho$	$e^{+\rho}$
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At low speeds:

$$\upsilon_{phase} \approx B + \frac{1}{2}\frac{B}{c^2}u^2 \quad \Leftarrow \text{for } (u \ll c) \Rightarrow \quad \kappa_{phase} \approx \frac{B}{c^2}u$$

time	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
space	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\text{sech}\,\rho$	$\cosh\rho$	$\text{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
stellar angle $\forall\ \sigma$	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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$B = \upsilon_A$

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At low speeds:

$\upsilon_{phase} \approx B + \frac{1}{2}\frac{B}{c^2}u^2 \quad \Leftarrow$ for $(u \ll c) \Rightarrow \quad \kappa_{phase} \approx \frac{B}{c^2}u$

υ_{phase} and κ_{phase} resemble formulae for Newton's kinetic energy and momentum

Resembles: $const. + \frac{1}{2}Mu^2$

Resembles: Mu

group	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
phase	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\mathrm{sech}\,\rho$	$\cosh\rho$	$\mathrm{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
value for β=3/5	$\frac{1}{2}=0.5$	$\frac{3}{5}=0.6$	$\frac{3}{4}=0.75$	$\frac{4}{5}=0.80$	$\frac{5}{4}=1.25$	$\frac{4}{3}=1.33$	$\frac{5}{3}=1.67$	$\frac{2}{1}=2.0$

Using (some) wave parameters to develop relativistic quantum theory

$$\upsilon_{phase} = B\cosh\rho \approx B + \tfrac{1}{2}B\rho^2 \text{ (for } u \ll c)$$

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Rescale υ_{phase} by h so: $M = \frac{hB}{c^2}$

Resembles: $const. + \frac{1}{2}Mu^2$

Resembles: Mu

time	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
space	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\text{sech}\,\rho$	$\cosh\rho$	$\text{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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group	$b^{Doppler}_{RED}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b^{Doppler}_{BLUE}$
phase	$\frac{1}{b^{Doppler}_{BLUE}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b^{Doppler}_{RED}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\mathrm{sech}\,\rho$	$\cosh\rho$	$\mathrm{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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group	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
phase	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\operatorname{sech}\rho$	$\cosh\rho$	$\operatorname{csch}\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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phase	$\frac{1}{b^{Doppler}_{BLUE}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b^{Doppler}_{RED}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\operatorname{sech}\rho$	$\cosh\rho$	$\operatorname{csch}\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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phase	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\text{sech}\,\rho$	$\cosh\rho$	$\text{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ angle σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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group	$b_{RED}^{Doppler}$	$\frac{V_{group}}{c}$	$\frac{\upsilon_{group}}{\upsilon_A}$	$\frac{\lambda_{group}}{\lambda_A}$	$\frac{\kappa_{group}}{\kappa_A}$	$\frac{\tau_{group}}{\tau_A}$	$\frac{V_{phase}}{c}$	$b_{BLUE}^{Doppler}$
phase	$\frac{1}{b_{BLUE}^{Doppler}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b_{RED}^{Doppler}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\operatorname{sech}\rho$	$\cosh\rho$	$\operatorname{csch}\rho$	$\coth\rho$	$e^{+\rho}$
stellar ∀ *angle* σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
value for $\beta=3/5$	$\frac{1}{2}=0.5$	$\frac{3}{5}=0.6$	$\frac{3}{4}=0.75$	$\frac{4}{5}=0.80$	$\frac{5}{4}=1.25$	$\frac{4}{3}=1.33$	$\frac{5}{3}=1.67$	$\frac{2}{1}=2.0$

(old-fashioned notation)

Using (some) wave parameters to develop relativistic quantum theory

$$\upsilon_{phase} = B\cosh\rho \approx B + \tfrac{1}{2}B\rho^2 \text{ (for } u \ll c)$$

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So attach scale factor h to match units.

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phase	$\frac{1}{b^{Doppler}_{BLUE}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b^{Doppler}_{RED}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\mathrm{sech}\,\rho$	$\cosh\rho$	$\mathrm{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
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$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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phase	$\frac{1}{b^{Doppler}_{BLUE}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b^{Doppler}_{RED}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\text{sech}\,\rho$	$\cosh\rho$	$\text{csch}\,\rho$	$\coth\rho$	$e^{+\rho}$
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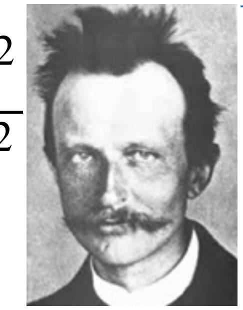

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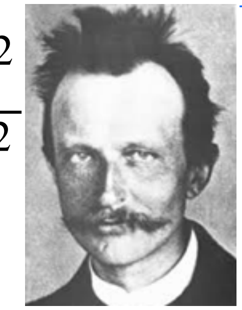

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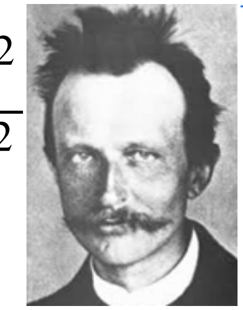

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Resolution and dirty secret: E, N, and υ_{phase} are all frequencies!

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phase	$\frac{1}{b^{Doppler}_{BLUE}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b^{Doppler}_{RED}}$
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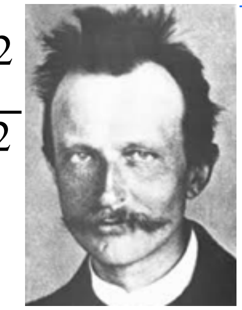

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At low speeds:

$$\upsilon_{phase} \approx B + \frac{1}{2}\frac{B}{c^2}u^2 \quad \Leftarrow \text{for } (u \ll c) \Rightarrow \quad \kappa_{phase} \approx \frac{B}{c^2}u$$

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Rescale υ_{phase} by h so: $M = \frac{hB}{c^2}$ or: $hB = Mc^2$

(The famous Mc^2 shows up here!)

$$h\upsilon_{phase} \approx hB + \frac{1}{2}\frac{hB}{c^2}u^2 \quad \Leftarrow \text{for } (u \ll c) \Rightarrow \quad h\kappa_{phase} \approx \frac{hB}{c^2}u$$

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$$= \text{Total Energy: } E = \frac{Mc^2}{\sqrt{1-u^2/c^2}}$$

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phase	$\frac{1}{b^{Doppler}_{BLUE}}$	$\frac{c}{V_{phase}}$	$\frac{\kappa_{phase}}{\kappa_A}$	$\frac{\tau_{phase}}{\tau_A}$	$\frac{\upsilon_{phase}}{\upsilon_A}$	$\frac{\lambda_{phase}}{\lambda_A}$	$\frac{c}{V_{group}}$	$\frac{1}{b^{Doppler}_{RED}}$
rapidity ρ	$e^{-\rho}$	$\tanh\rho$	$\sinh\rho$	$\operatorname{sech}\rho$	$\cosh\rho$	$\operatorname{csch}\rho$	$\coth\rho$	$e^{+\rho}$
stellar $\forall$ angle σ	$1/e^{+\rho}$	$\sin\sigma$	$\tan\sigma$	$\cos\sigma$	$\sec\sigma$	$\cot\sigma$	$\csc\sigma$	$1/e^{-\rho}$
$\beta \equiv \frac{u}{c}$	$\sqrt{\frac{1-\beta}{1+\beta}}$	$\frac{\beta}{1}$	$\frac{1}{\sqrt{\beta^{-2}-1}}$	$\frac{\sqrt{1-\beta^2}}{1}$	$\frac{1}{\sqrt{1-\beta^2}}$	$\frac{\sqrt{\beta^{-2}-1}}{1}$	$\frac{1}{\beta}$	$\sqrt{\frac{1+\beta}{1-\beta}}$
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Using (some) wave parameters to develop relativistic quantum theory

$\upsilon_{phase} = B\cosh\rho \approx B + \frac{1}{2}B\rho^2$ (for $u \ll c$)

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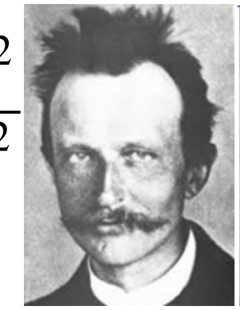

Max Planck 1858-1947

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DeBroglie (1921)

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This motivates the "particle" normalization $\int \Psi^*\Psi\, dV = N$ $\quad \Psi = \sqrt{\dfrac{\varepsilon_0}{h\upsilon}}E$

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