Lecture 17
Tue. 10.23.2012

GCC Lagrange and Hamilton Equations for Trebuchet
or
"How do we ignore all those constraint forces?"
(Ch. 1-5 of Unit 2 and Unit 3)

Review of Lagrangian equation derivation (Elementary trebuchet) (Mostly Unit 2.)
- *Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}*
- *Force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and γ_{mn} tensor*

Equations of motion and force analysis (Mostly Unit 2.)
- *Forces: total, genuine, potential, and/or fictitious*
- *Lagrange equation forms*
- *Riemann equation forms*
- *2nd-guessing Riemann? (More like Unit 3.)*

Chapter 1. The Trebuchet: A dream problem for Galileo?

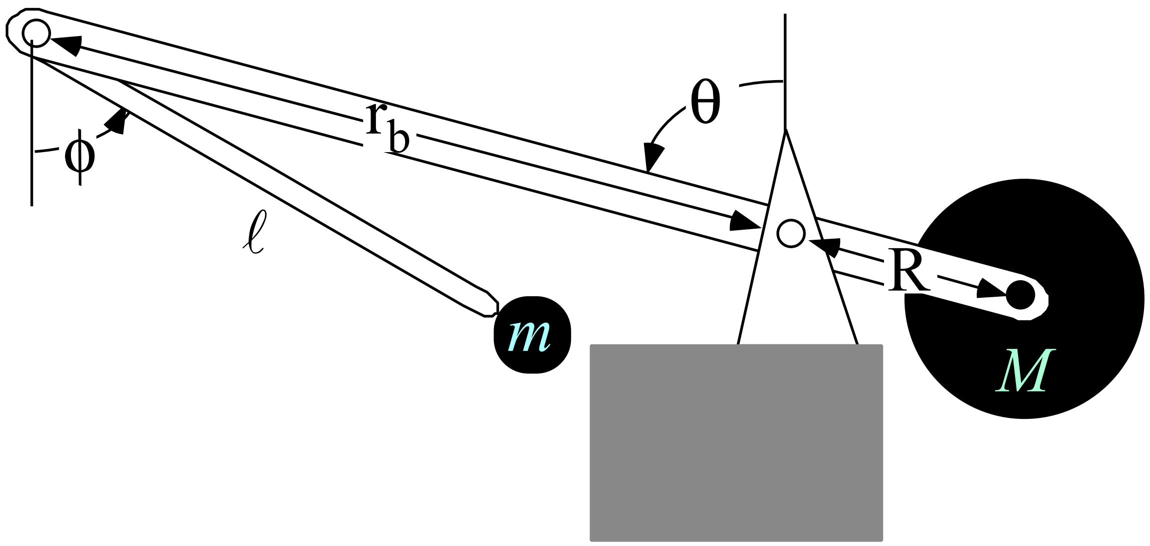
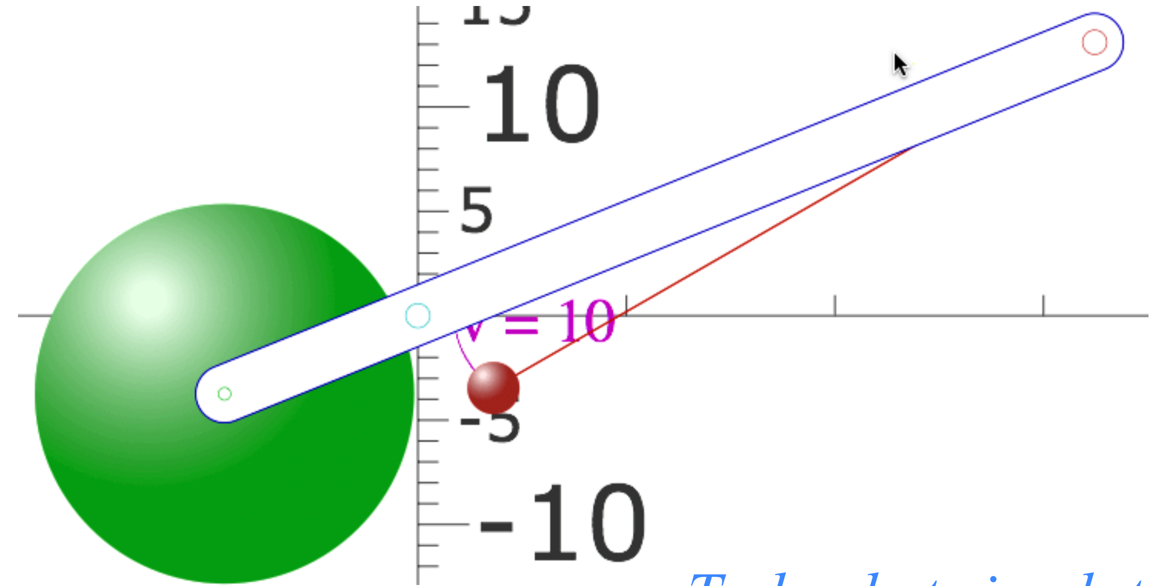
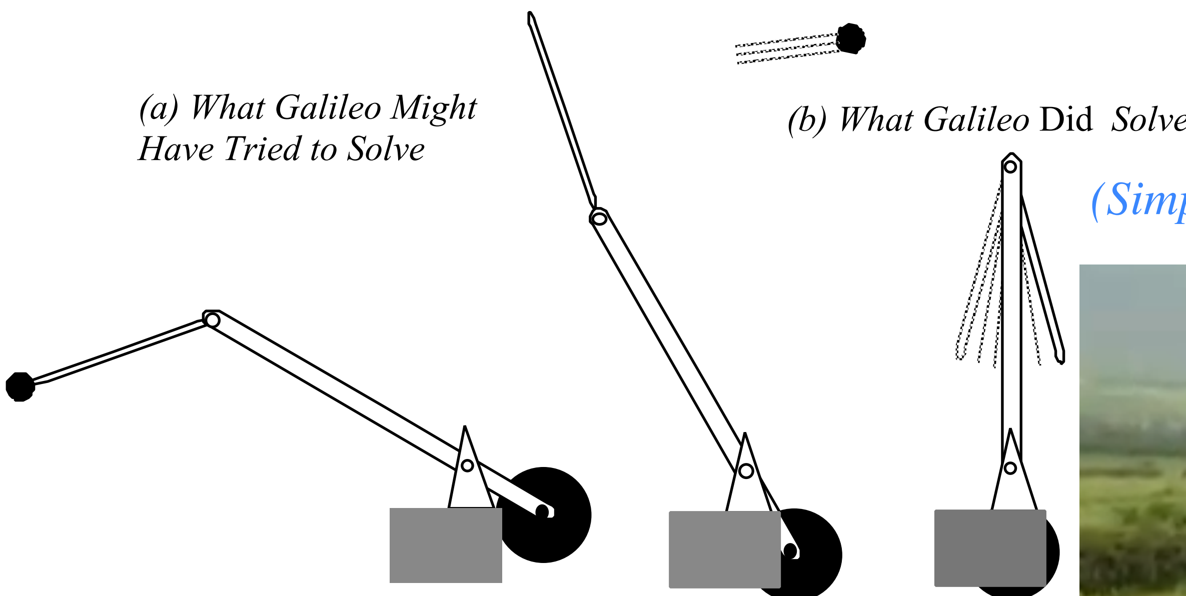


Fig. 2.1.1 An elementary ground-fixed trebuchet



Trebuchet simulator

http://www.uark.edu/rso/modphys/testing/markup/TrebuchetWeb.html



(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed) problem

Review of Lagrangian equation derivation (Elementary trebuchet)

→ *Coordinate geometry, kinetic energy, and dynamic metric tensor* γ_{mn}

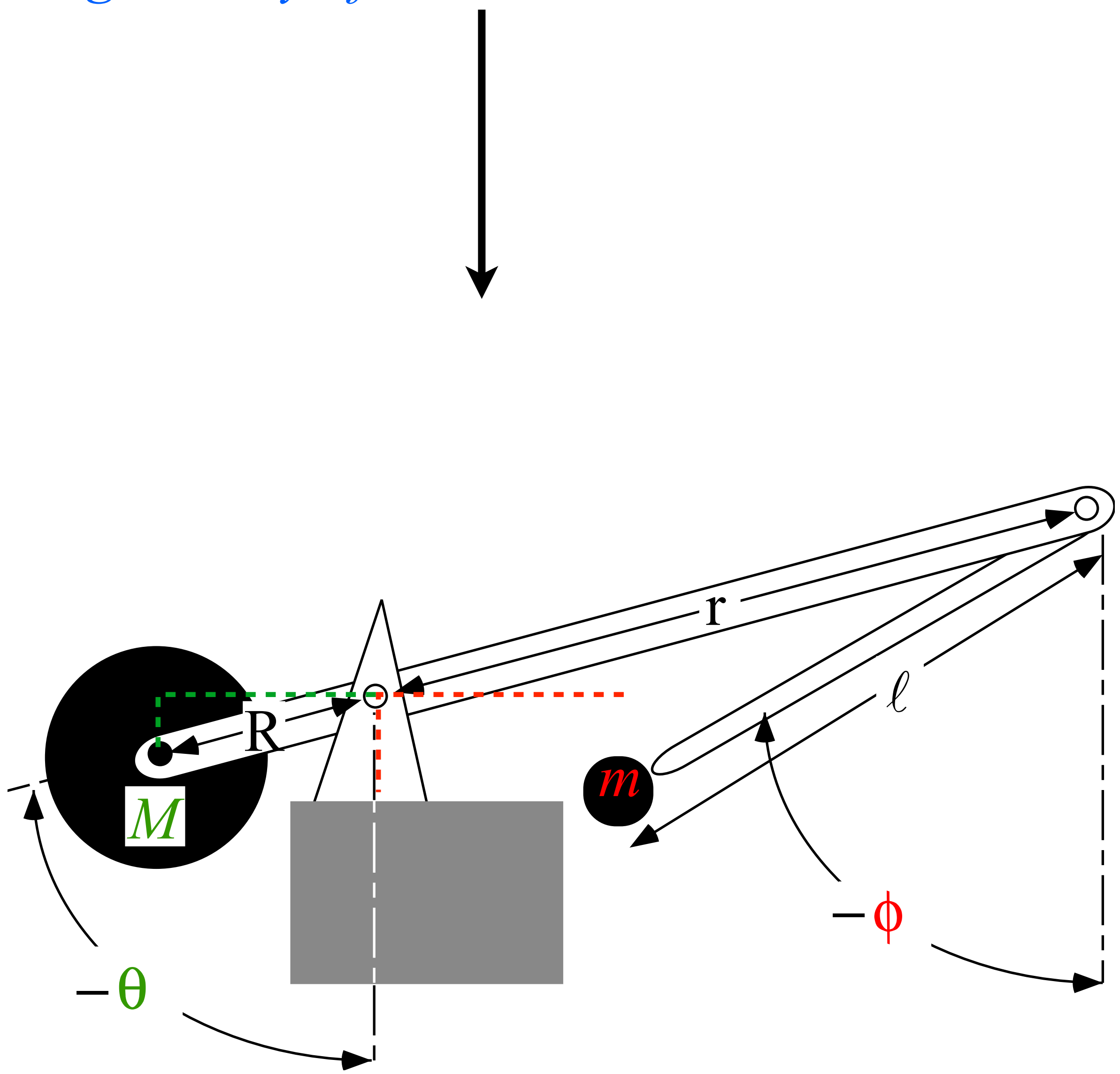
Basic force, work, and acceleration

Lagrangian force equation

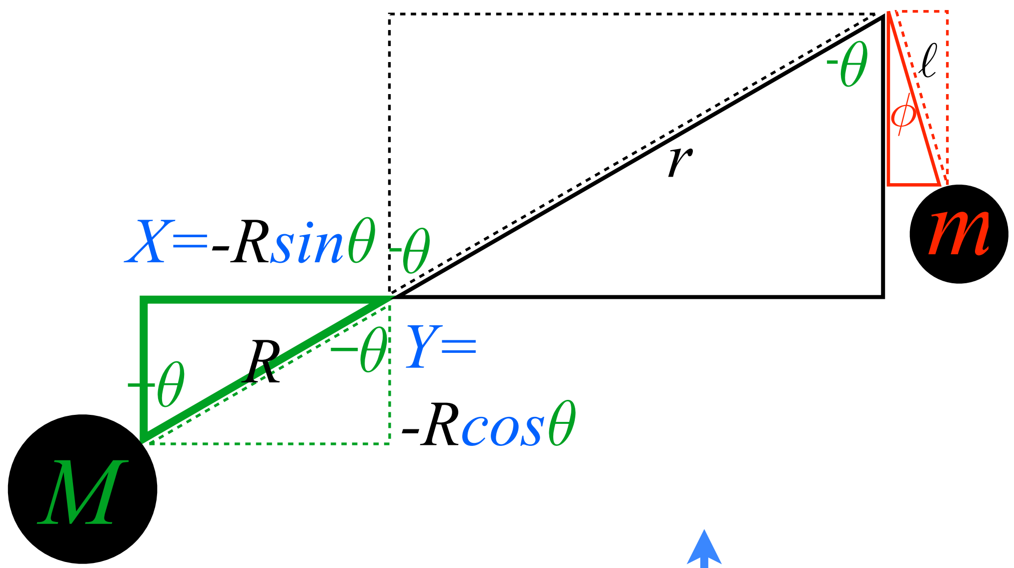
Canonical momentum and γ_{mn} *tensor*

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

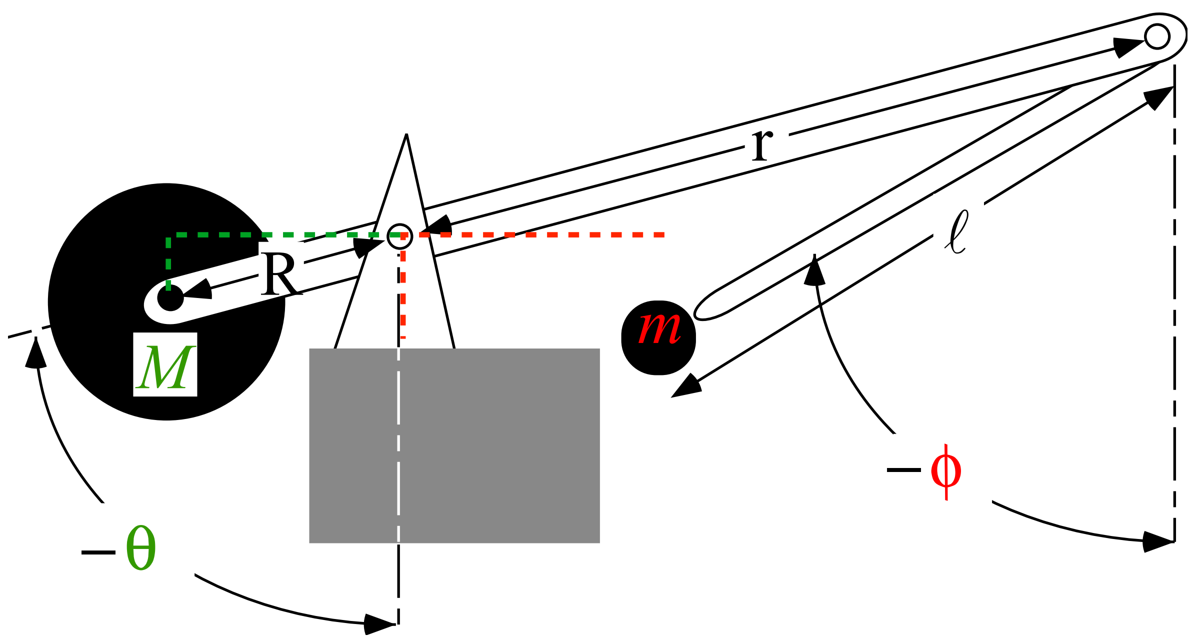
geometry of trebuchet



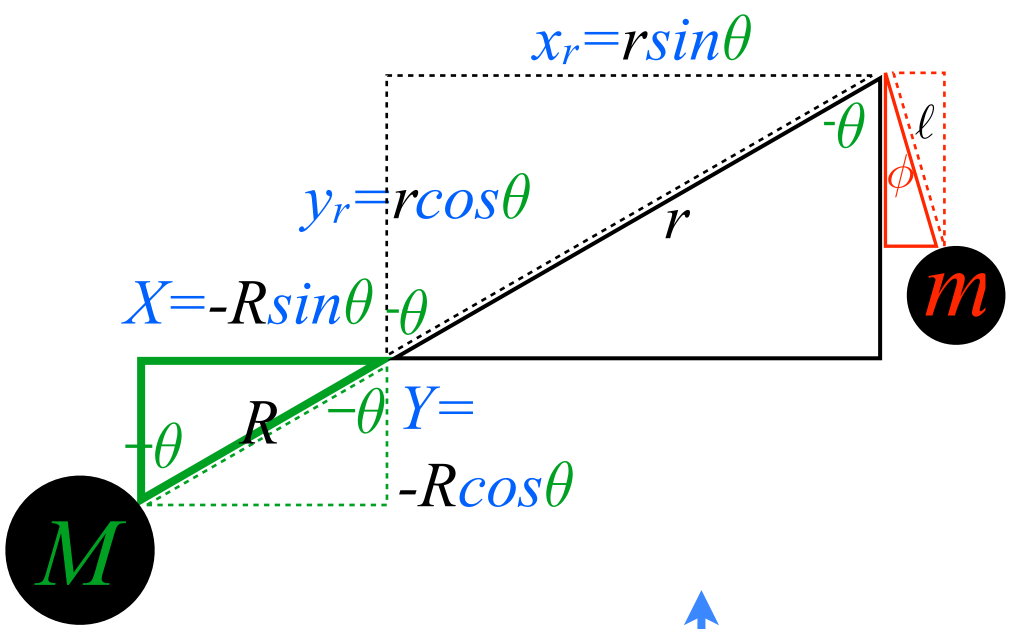
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



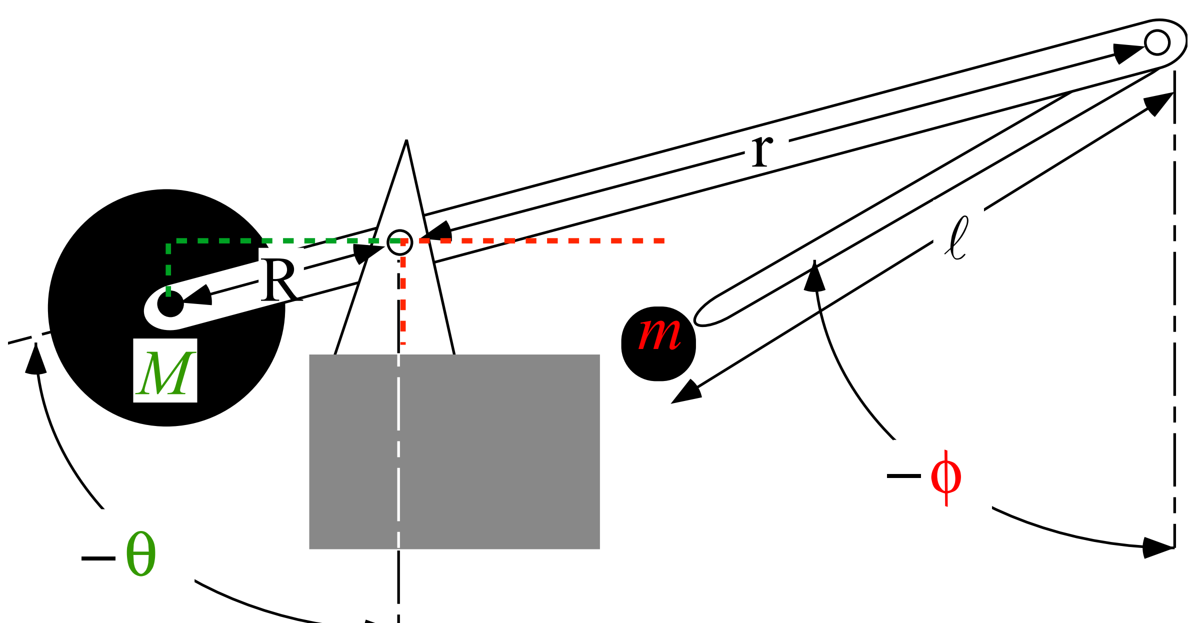
geometry of trebuchet simplified somewhat...



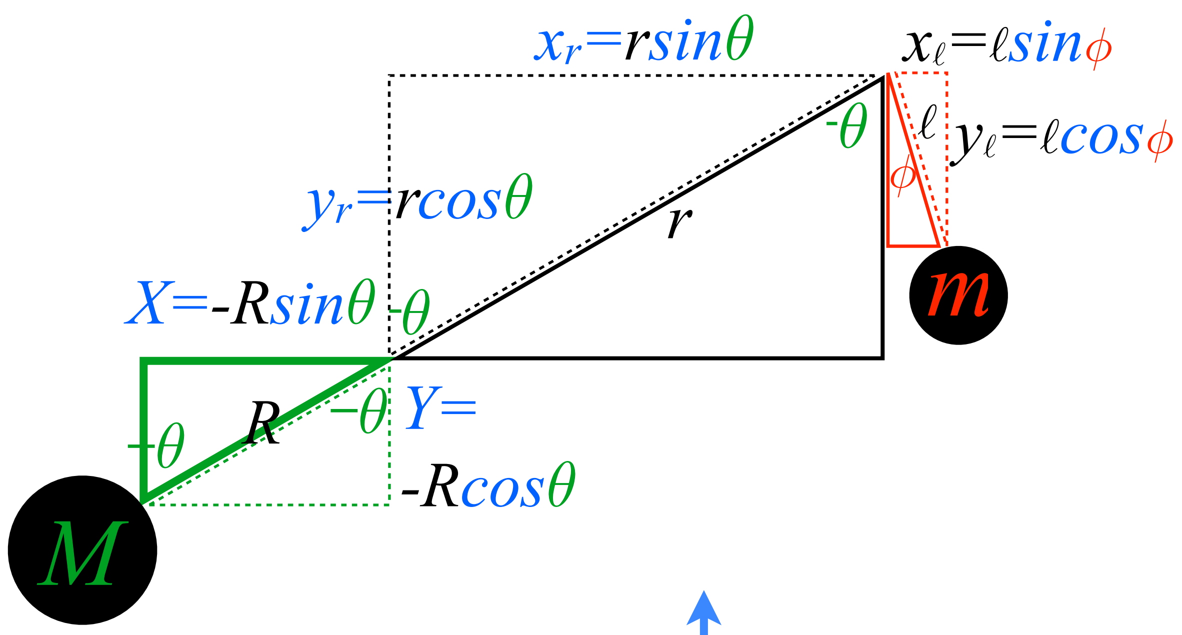
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



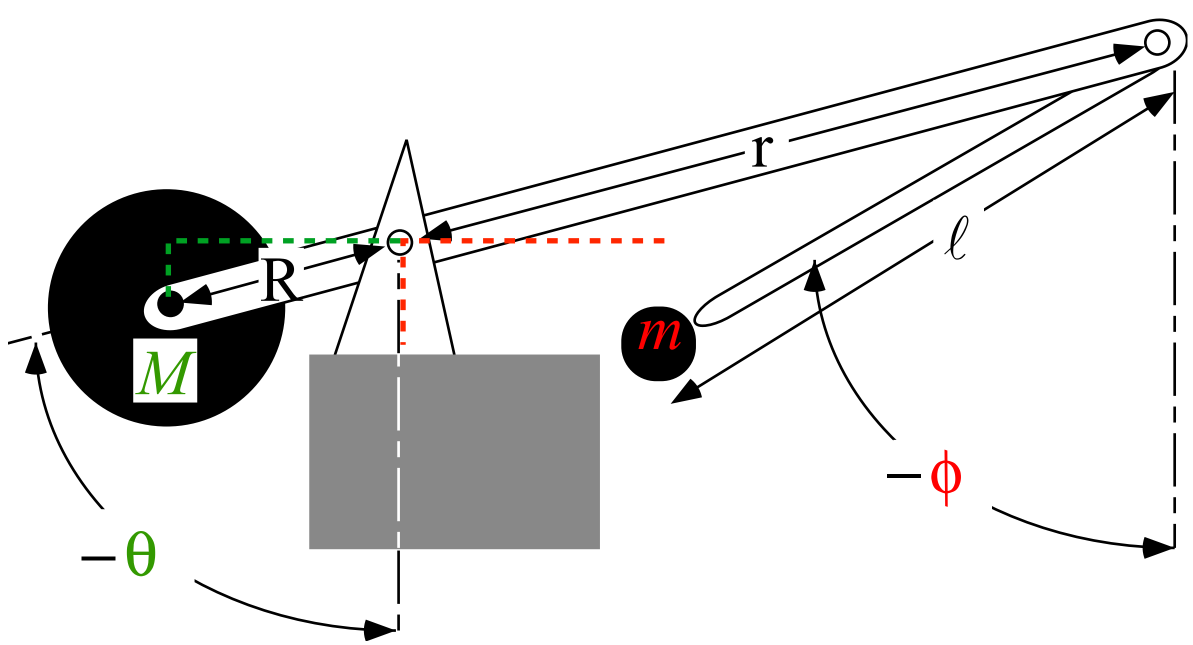
geometry of trebuchet simplified somewhat...



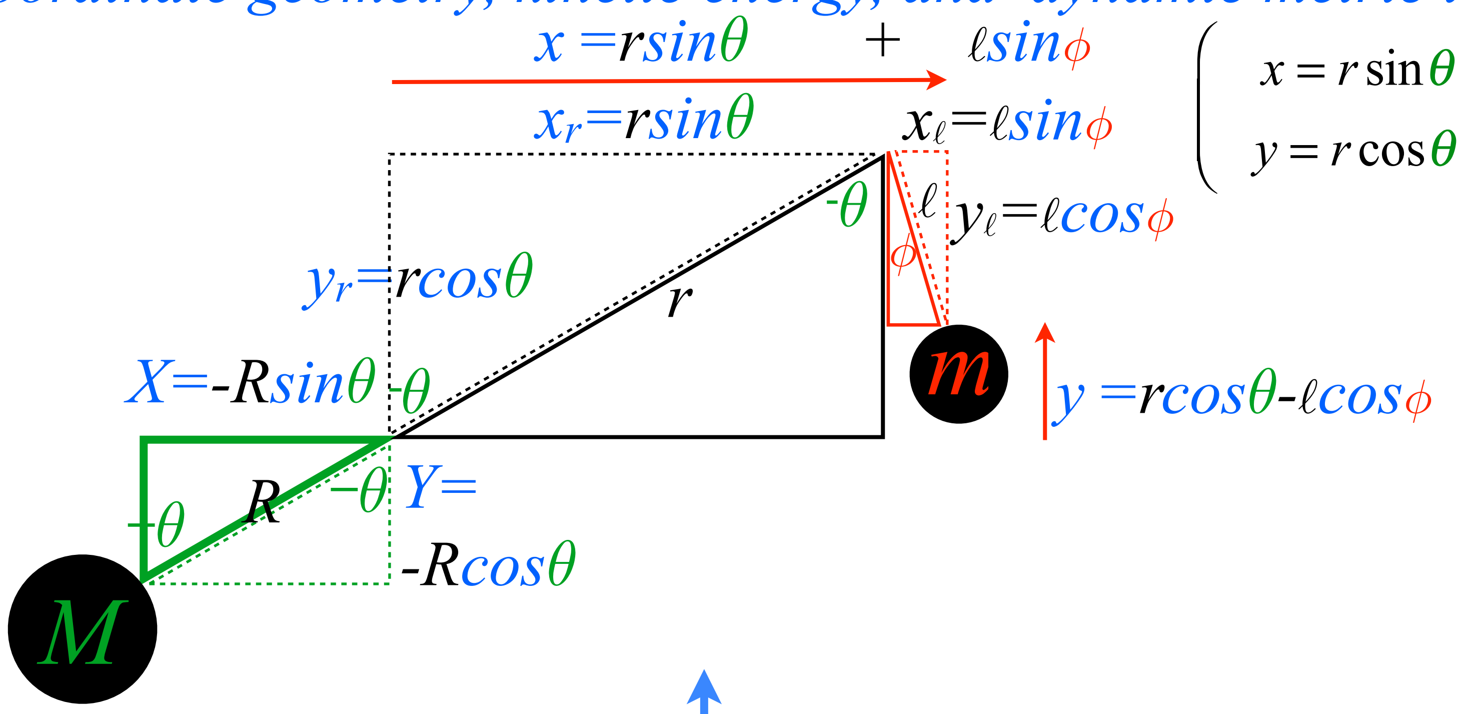
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



geometry of trebuchet simplified somewhat...

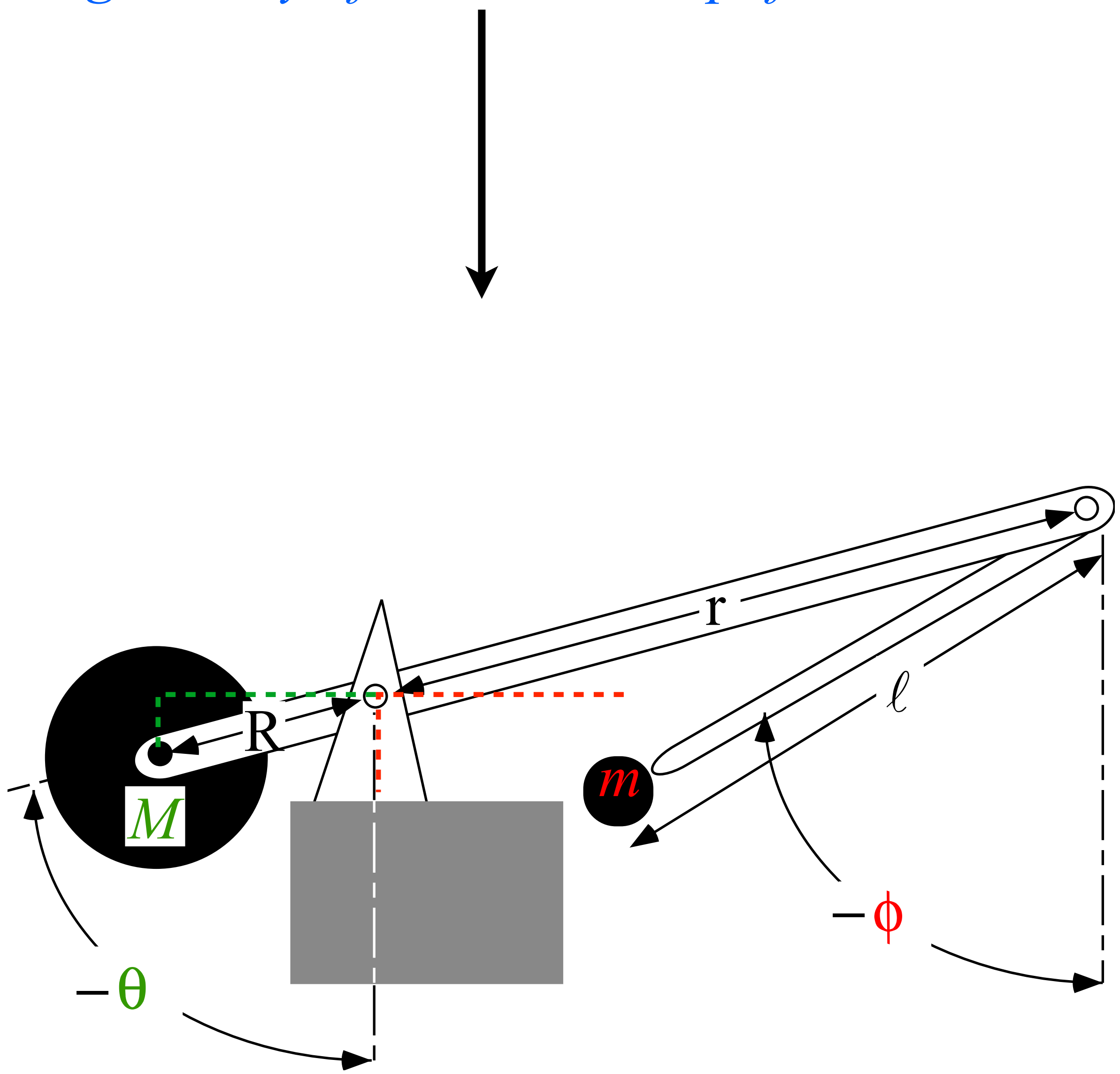


Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

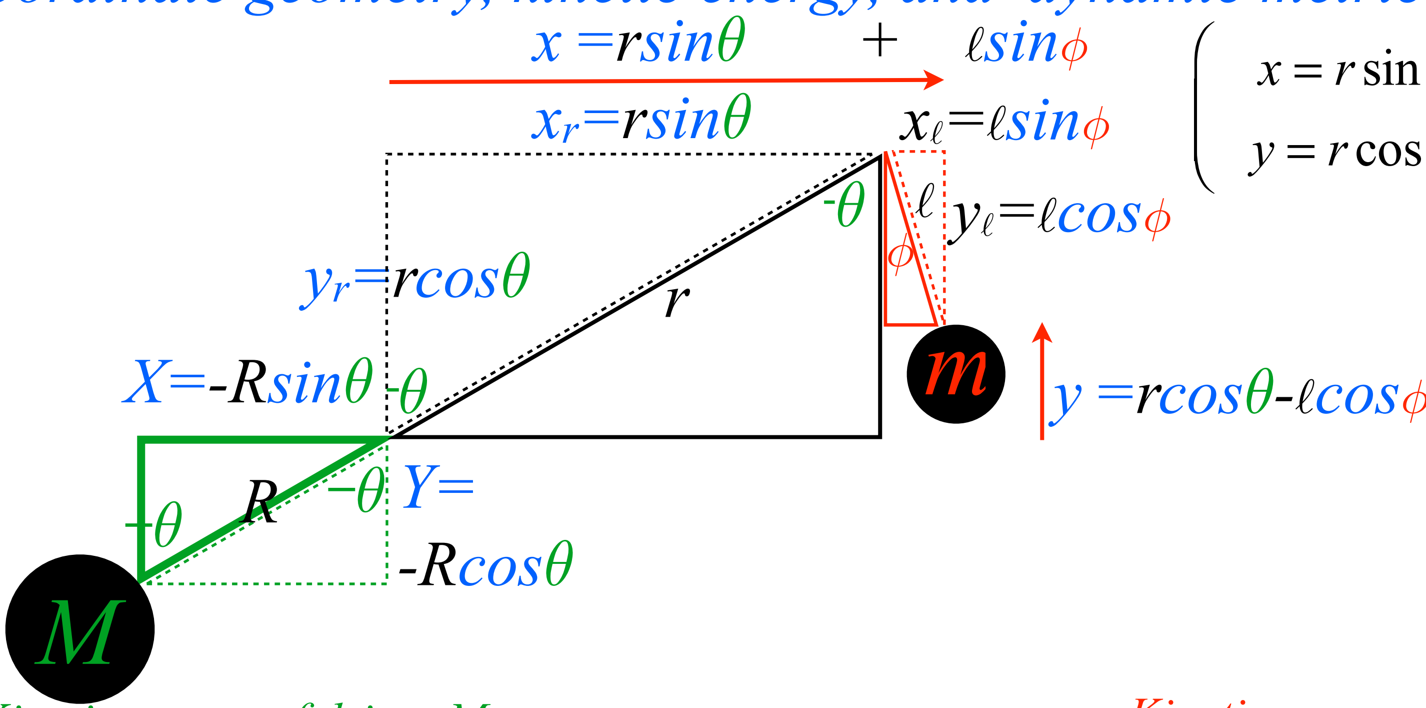


$$\begin{pmatrix} x = r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

geometry of trebuchet simplified somewhat...



Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

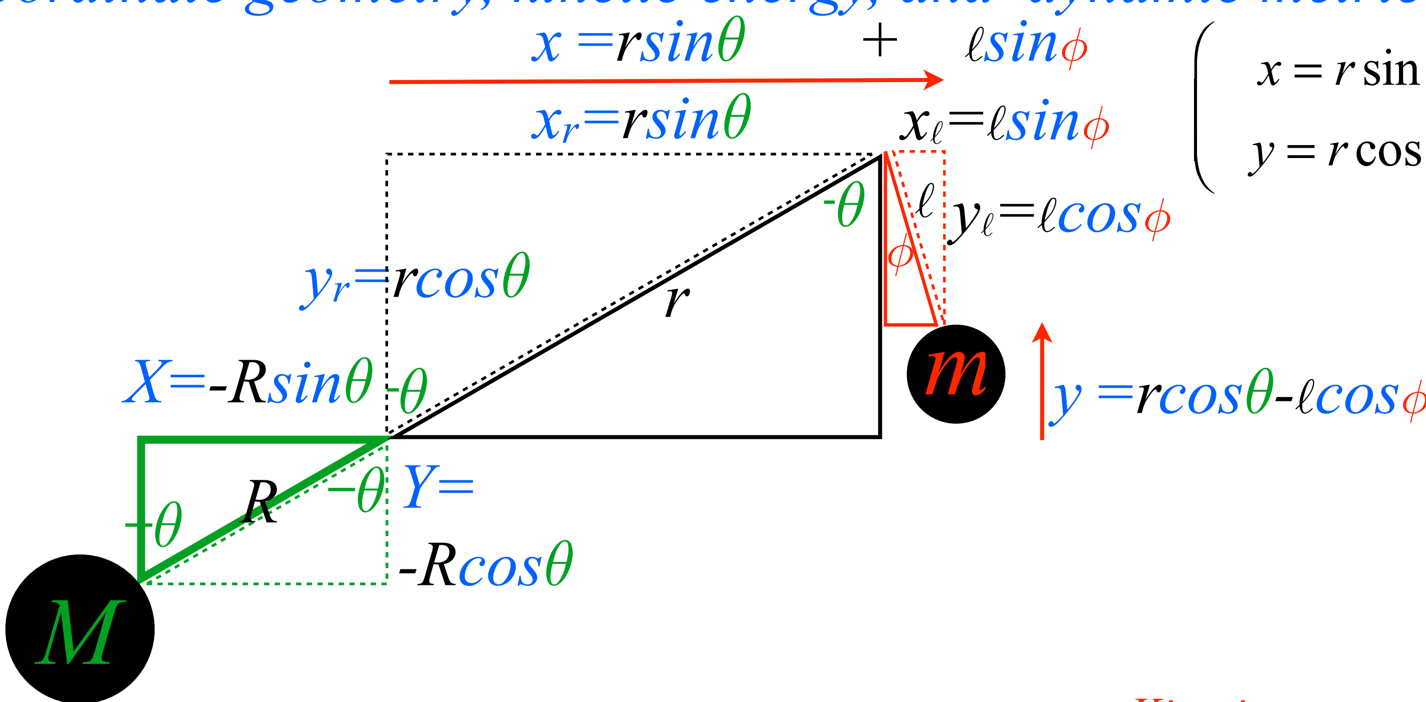
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kinetic energy of driver M

$$T(M) = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} M\dot{Y}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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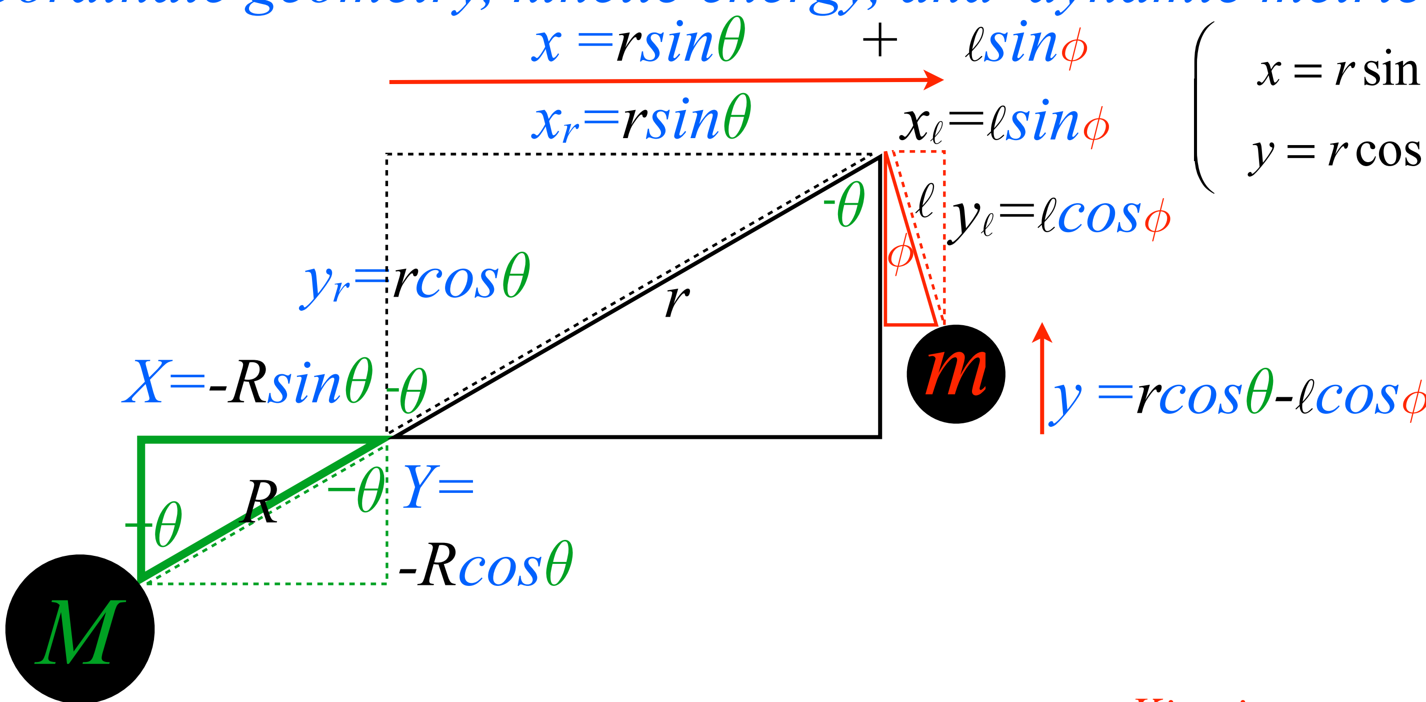
$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kinetic energy of driver M

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2 \right]$

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

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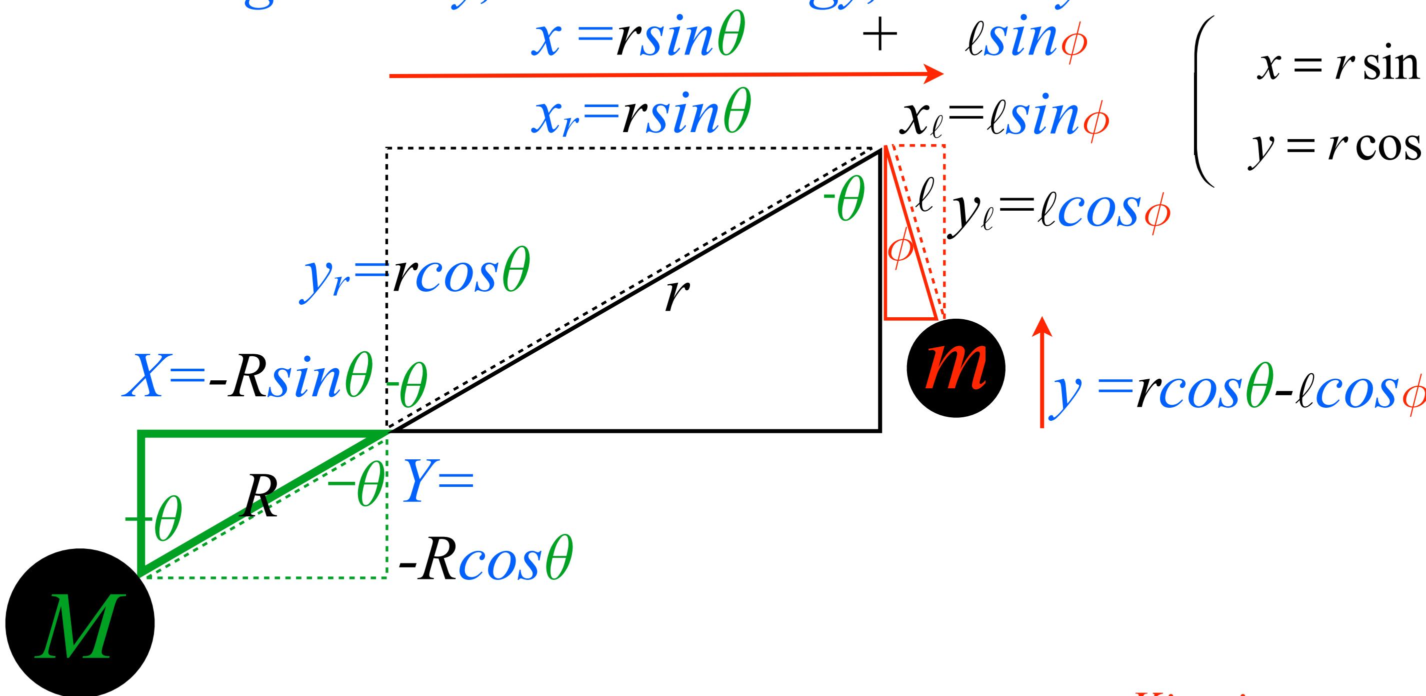
$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right]$$

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



$$\begin{pmatrix} x = r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Kinetic energy of driver M

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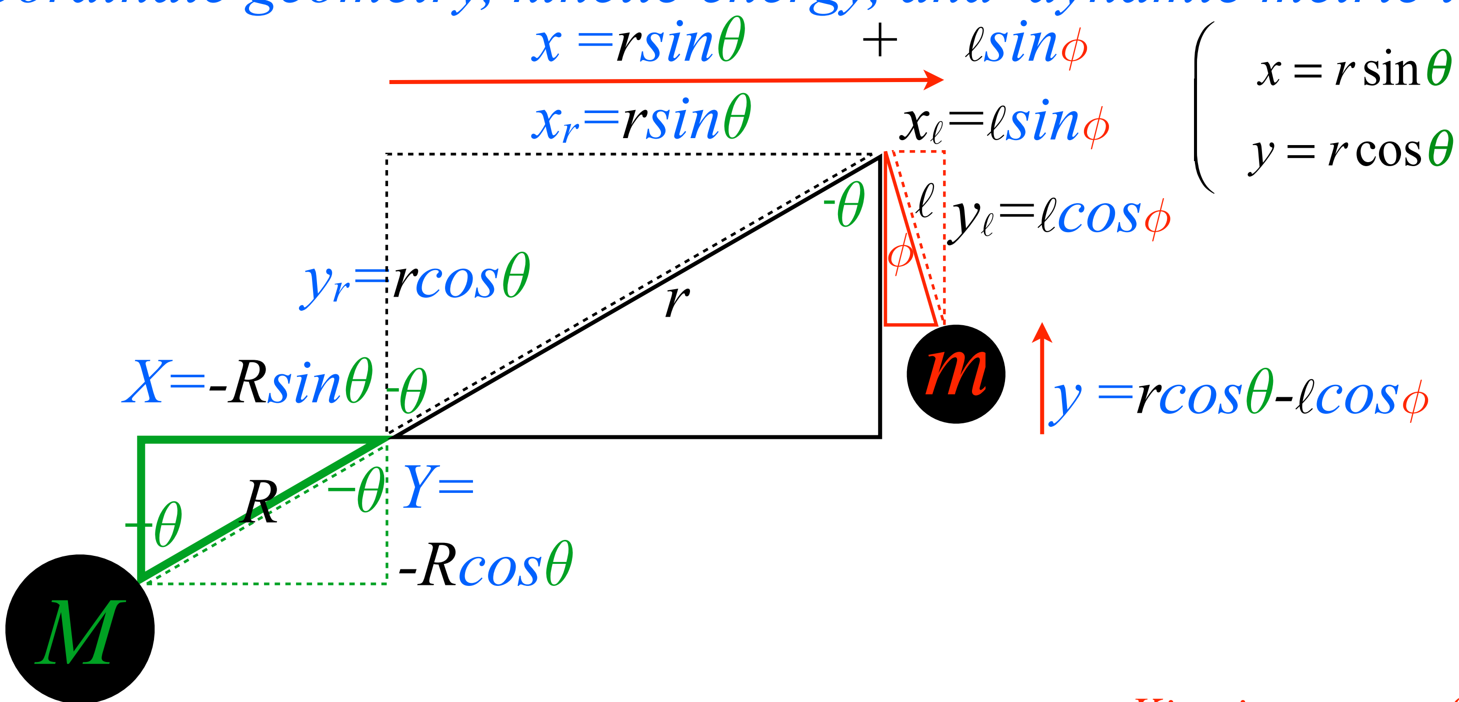
Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}



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$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right]$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor γ_{mn} in raw Cartesian X,Y and x,y

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Dynamic metric tensor γ_{mn} in GCC θ and ϕ

Review of Lagrangian equation derivation (Elementary trebuchet)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

→ *Basic force, work, and acceleration*

Lagrangian force equation

Canonical momentum and γ_{mn} tensor

Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

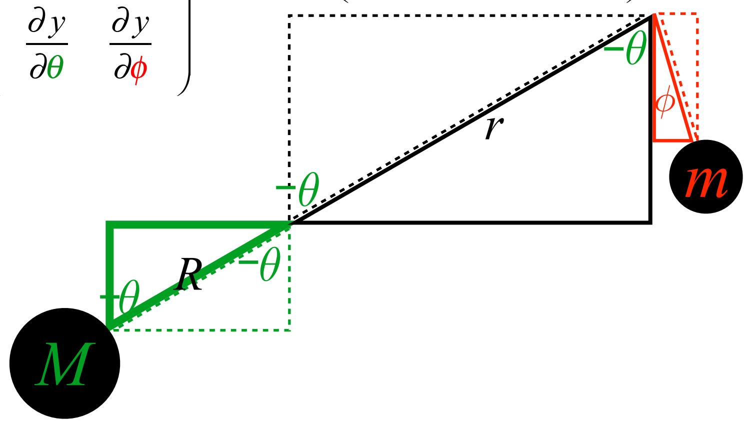
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Write work-sums in columns:

$$\begin{aligned} dW = \; & F_X\,dX && = M\ddot{X}\,dX \\ & + F_Y\,dY && + M\ddot{Y}\,dY \\ & + F_x\,dx && + m\ddot{x}\,dx \\ & + F_y\,dy && + m\ddot{y}\,dy \end{aligned}$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

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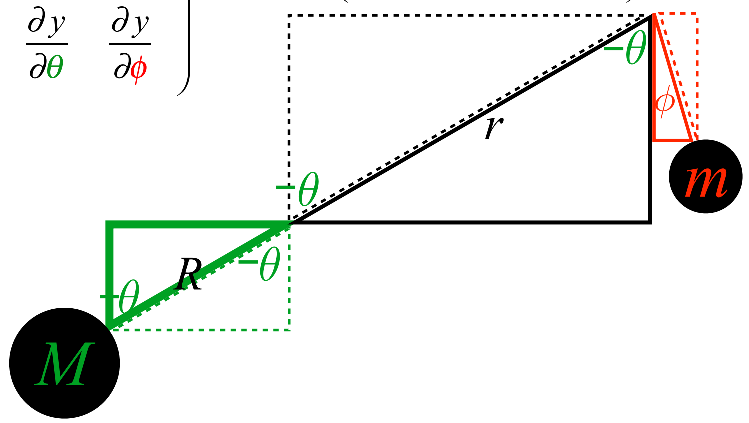
Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = \; & F_X\, dX && = M\ddot{X}\, dX && = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi && = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi \\
& + F_Y\, dY && + M\ddot{Y}\, dY && + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi && + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi \\
& + F_x\, dx && + m\ddot{x}\, dx && + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi && + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi \\
& + F_y\, dy && + m\ddot{y}\, dy && + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi && + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi
\end{aligned}$$



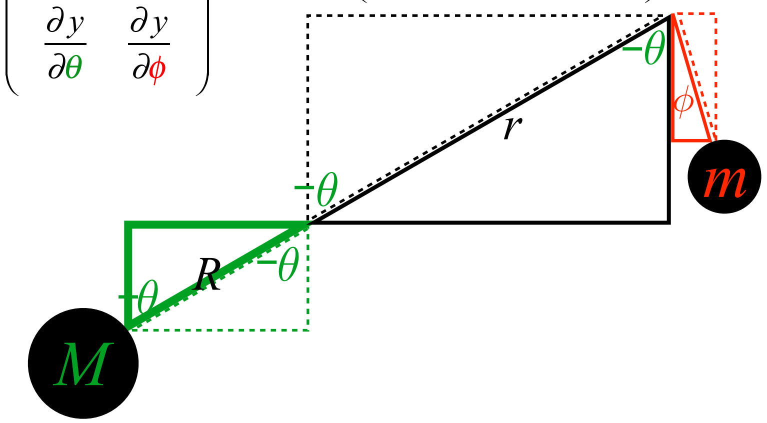
Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\ dX = M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\ dY \quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\ dx \quad +m\ddot{x}\ dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\ dy \quad +m\ddot{y}\ dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

Assuming variables θ and ϕ are independent...

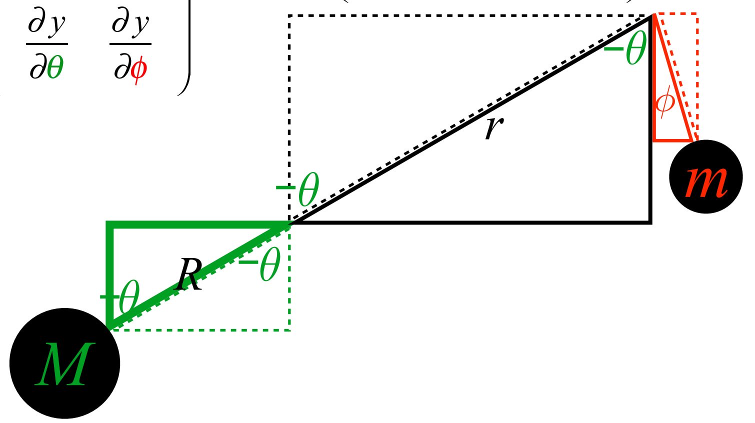
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{array}{lllll} dW = F_X\, dX & = M\ddot{X}\,dX & = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi & = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\ \quad + F_Y\, dY & + M\ddot{Y}\,dY & + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi & + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\ \quad + F_x\, dx & + m\ddot{x}\,dx & + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi & + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\ \quad + F_y\, dy & + m\ddot{y}\,dy & + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi & + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi \end{array}$$

Assuming variables θ and ϕ are independent...

Set: $d\theta=1 \quad d\phi=0$

$$\begin{array}{ll} F_X\frac{\partial X}{\partial\theta} & = M\ddot{X}\frac{\partial X}{\partial\theta} \\ + F_Y\frac{\partial Y}{\partial\theta} & + M\ddot{Y}\frac{\partial Y}{\partial\theta} \\ + F_x\frac{\partial x}{\partial\theta} & + m\ddot{x}\frac{\partial x}{\partial\theta} \\ + F_y\frac{\partial y}{\partial\theta} & + m\ddot{y}\frac{\partial y}{\partial\theta} \end{array}$$

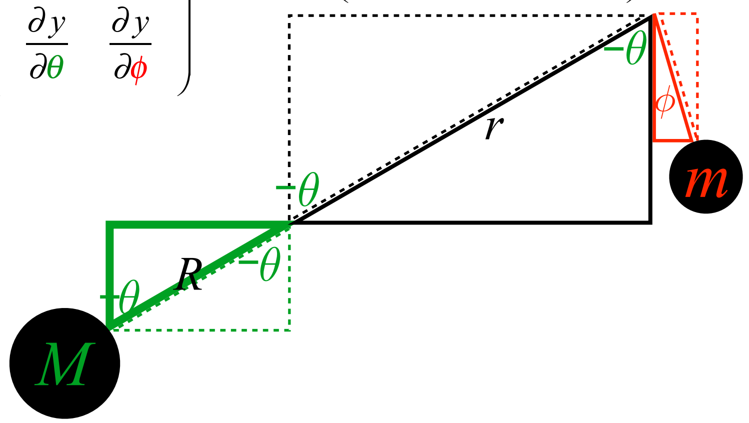
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial \theta}d\theta + F_X\frac{\partial X}{\partial \phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial \theta}d\theta + M\ddot{X}\frac{\partial X}{\partial \phi}d\phi \\
+F_Y\, dY &\quad + M\ddot{Y}\,dY &\quad + F_Y\frac{\partial Y}{\partial \theta}d\theta + F_Y\frac{\partial Y}{\partial \phi}d\phi &\quad + M\ddot{Y}\frac{\partial Y}{\partial \theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial \phi}d\phi \\
+F_x\, dx &\quad + m\ddot{x}\,dx &\quad + F_x\frac{\partial x}{\partial \theta}d\theta + F_x\frac{\partial x}{\partial \phi}d\phi &\quad + m\ddot{x}\frac{\partial x}{\partial \theta}d\theta + m\ddot{x}\frac{\partial x}{\partial \phi}d\phi \\
+F_y\, dy &\quad + m\ddot{y}\,dy &\quad + F_y\frac{\partial y}{\partial \theta}d\theta + F_y\frac{\partial y}{\partial \phi}d\phi &\quad + m\ddot{y}\frac{\partial y}{\partial \theta}d\theta + m\ddot{y}\frac{\partial y}{\partial \phi}d\phi
\end{aligned}$$

Assuming variables θ and ϕ are independent...

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial \theta} &= M\ddot{X}\frac{\partial X}{\partial \theta} \\
+F_Y\frac{\partial Y}{\partial \theta} &\quad + M\ddot{Y}\frac{\partial Y}{\partial \theta} \\
+F_x\frac{\partial x}{\partial \theta} &\quad + m\ddot{x}\frac{\partial x}{\partial \theta} \\
+F_y\frac{\partial y}{\partial \theta} &\quad + m\ddot{y}\frac{\partial y}{\partial \theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial \phi} &= M\ddot{X}\frac{\partial X}{\partial \phi} \\
+F_Y\frac{\partial Y}{\partial \phi} &\quad + M\ddot{Y}\frac{\partial Y}{\partial \phi} \\
+F_x\frac{\partial x}{\partial \phi} &\quad + m\ddot{x}\frac{\partial x}{\partial \phi} \\
+F_y\frac{\partial y}{\partial \phi} &\quad + m\ddot{y}\frac{\partial y}{\partial \phi}
\end{aligned}$$

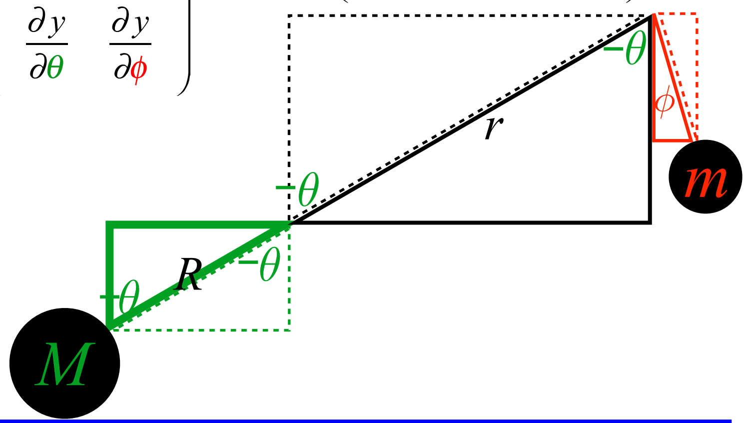
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\,dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X}\frac{\partial X}{\partial \theta} d\theta + M\ddot{X}\frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\,dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y}\frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\,dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x}\frac{\partial x}{\partial \theta} d\theta + m\ddot{x}\frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\,dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y}\frac{\partial y}{\partial \theta} d\theta + m\ddot{y}\frac{\partial y}{\partial \phi} d\phi$$

Lagrange trickery:

$$\ddot{X}\frac{\partial X}{\partial \theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial \theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial \theta}$$

(using $\frac{d}{dt}\left(\dot{X}U\right) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **A**

Set: $d\theta=1 \quad d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X}\frac{\partial X}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} \quad + m\ddot{x}\frac{\partial x}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} \quad + m\ddot{y}\frac{\partial y}{\partial \theta}$$

Set: $d\theta=0 \quad d\phi=1$

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X}\frac{\partial X}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} \quad + M\ddot{Y}\frac{\partial Y}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} \quad + m\ddot{x}\frac{\partial x}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} \quad + m\ddot{y}\frac{\partial y}{\partial \phi}$$

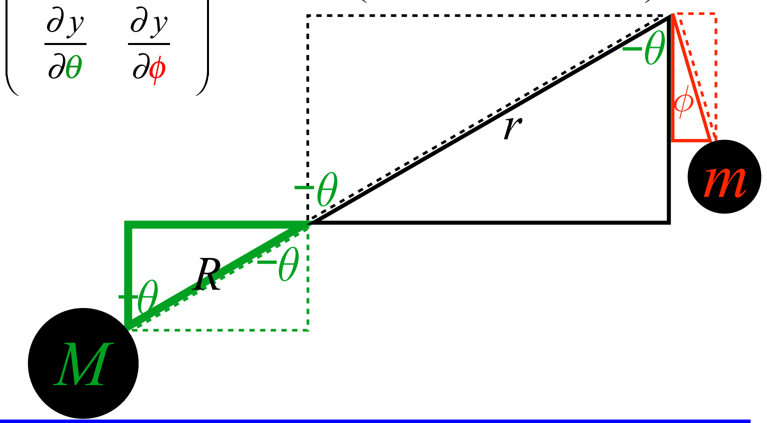
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC dθ and dφ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\, dY &\quad + M\ddot{Y}\,dY \quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\, dx &\quad + m\ddot{x}\,dx \quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\, dy &\quad + m\ddot{y}\,dy \quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Lagrange trickery:

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial \dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial \dot{X}}{\partial\theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **A**

STEP **B**

by *lemma 1*: $\frac{\partial \dot{X}}{\partial\dot{q}} = \frac{\partial X}{\partial q}$

and *lemma 2*: $\frac{\partial \dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad + m\ddot{x}\frac{\partial x}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad + m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad + m\ddot{x}\frac{\partial x}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad + m\ddot{y}\frac{\partial y}{\partial\phi}
\end{aligned}$$

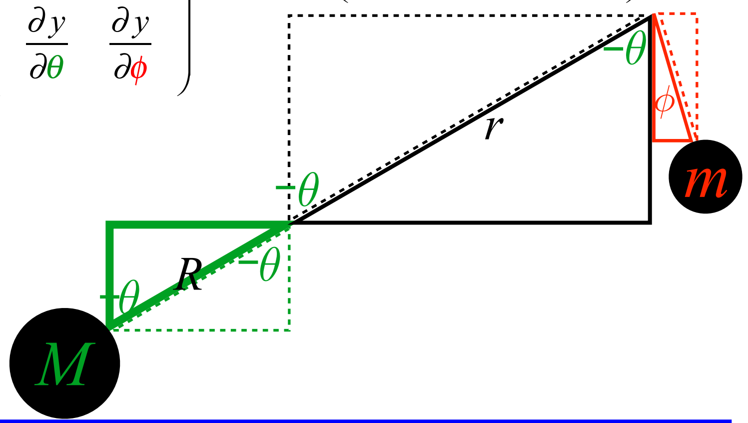
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot X\, dX + M\ddot Y\, dY + m\ddot x\, dx + m\ddot y\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot X\, dX &= F_X \frac{\partial X}{\partial\theta} d\theta + F_X \frac{\partial X}{\partial\phi} d\phi &= M\ddot X \frac{\partial X}{\partial\theta} d\theta + M\ddot X \frac{\partial X}{\partial\phi} d\phi \\
+ F_Y\, dY &+ M\ddot Y\, dY &+ F_Y \frac{\partial Y}{\partial\theta} d\theta + F_Y \frac{\partial Y}{\partial\phi} d\phi &+ M\ddot Y \frac{\partial Y}{\partial\theta} d\theta + M\ddot Y \frac{\partial Y}{\partial\phi} d\phi \\
+ F_x\, dx &+ m\ddot x\, dx &+ F_x \frac{\partial x}{\partial\theta} d\theta + F_x \frac{\partial x}{\partial\phi} d\phi &+ m\ddot x \frac{\partial x}{\partial\theta} d\theta + m\ddot x \frac{\partial x}{\partial\phi} d\phi \\
+ F_y\, dy &+ m\ddot y\, dy &+ F_y \frac{\partial y}{\partial\theta} d\theta + F_y \frac{\partial y}{\partial\phi} d\phi &+ m\ddot y \frac{\partial y}{\partial\theta} d\theta + m\ddot y \frac{\partial y}{\partial\phi} d\phi
\end{aligned}$$

Lagrange trickery:

$$\ddot X \frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot X \frac{\partial X}{\partial\theta}\right) - \dot X \frac{d}{dt}\frac{\partial X}{\partial\theta}$$

(using $\frac{d}{dt}\left(\dot X U\right) = \ddot X U + \dot X \dot U$)

STEP **A**

$$= \frac{d}{dt}\left(\dot X \frac{\partial \dot X}{\partial\dot\theta}\right) - \dot X \frac{\partial \dot X}{\partial\theta}$$

by lemma 1: $\frac{\partial X}{\partial q} = \frac{\partial \dot X}{\partial \dot q}$

STEP **B** *and lemma 2*: $\frac{\partial \dot X}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

$$= \frac{d}{dt}\left(\frac{\partial\left(\dot X^2/2\right)}{\partial\dot\theta}\right) - \frac{\partial\left(\dot X^2/2\right)}{\partial\theta}$$

STEP **C** (using $\frac{\partial\left(U^2/2\right)}{\partial q} = U\frac{\partial U}{\partial q}$)

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X \frac{\partial X}{\partial\theta} &= M\ddot X \frac{\partial X}{\partial\theta} \\
+ F_Y \frac{\partial Y}{\partial\theta} &+ M\ddot Y \frac{\partial Y}{\partial\theta} \\
+ F_x \frac{\partial x}{\partial\theta} &+ m\ddot x \frac{\partial x}{\partial\theta} \\
+ F_y \frac{\partial y}{\partial\theta} &+ m\ddot y \frac{\partial y}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X \frac{\partial X}{\partial\phi} &= M\ddot X \frac{\partial X}{\partial\phi} \\
+ F_Y \frac{\partial Y}{\partial\phi} &+ M\ddot Y \frac{\partial Y}{\partial\phi} \\
+ F_x \frac{\partial x}{\partial\phi} &+ m\ddot x \frac{\partial x}{\partial\phi} \\
+ F_y \frac{\partial y}{\partial\phi} &+ m\ddot y \frac{\partial y}{\partial\phi}
\end{aligned}$$

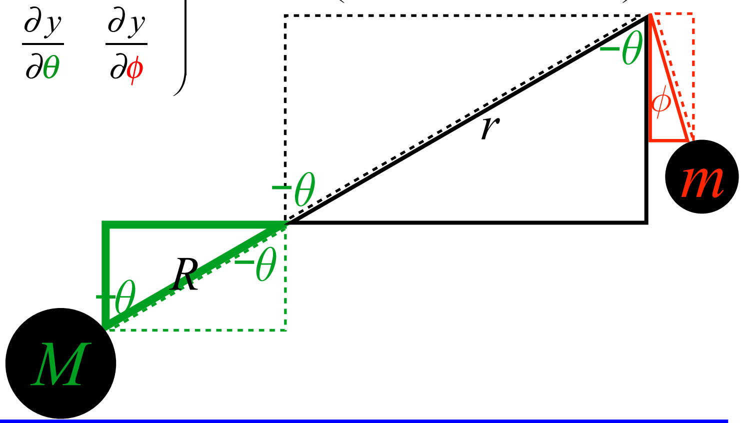
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\,dX &&= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &&= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\, dY &\;\; + M\ddot{Y}\,dY &&\;\; + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &&\;\; + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\, dx &\;\; + m\ddot{x}\,dx &&\;\; + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &&\;\; + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\, dy &\;\; + m\ddot{y}\,dy &&\;\; + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &&\;\; + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

Lagrange trickery:

STEP A

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta}$$

(using $\frac{d}{dt}\left(\dot{X}U\right) = \ddot{X}U + \dot{X}\dot{U}$)

STEP B

$$= \frac{d}{dt}\left(\dot{X}\frac{\partial \dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial \dot{X}}{\partial\theta}$$

by lemma 1: $\frac{\partial X}{\partial q} = \frac{\partial \dot{X}}{\partial \dot{q}}$

and lemma 2: $\frac{\partial \dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

STEP C

$$= \frac{d}{dt}\left(\frac{\partial\left(\dot{X}^2/2\right)}{\partial\dot{\theta}}\right) - \frac{\partial\left(\dot{X}^2/2\right)}{\partial\theta}$$

(using $\frac{\partial\left(U^2/2\right)}{\partial q} = U\frac{\partial U}{\partial q}$)

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\;\; + M\ddot{Y}\frac{\partial Y}{\partial\theta} \;\; + \frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\;\; + m\ddot{x}\frac{\partial x}{\partial\theta} \;\; + \frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\;\; + m\ddot{y}\frac{\partial y}{\partial\theta} \;\; + \frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\;\; + M\ddot{Y}\frac{\partial Y}{\partial\phi} \;\; + \frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\;\; + m\ddot{x}\frac{\partial x}{\partial\phi} \;\; + \frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\;\; + m\ddot{y}\frac{\partial y}{\partial\phi} \;\; + \frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\phi}
\end{aligned}$$

Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

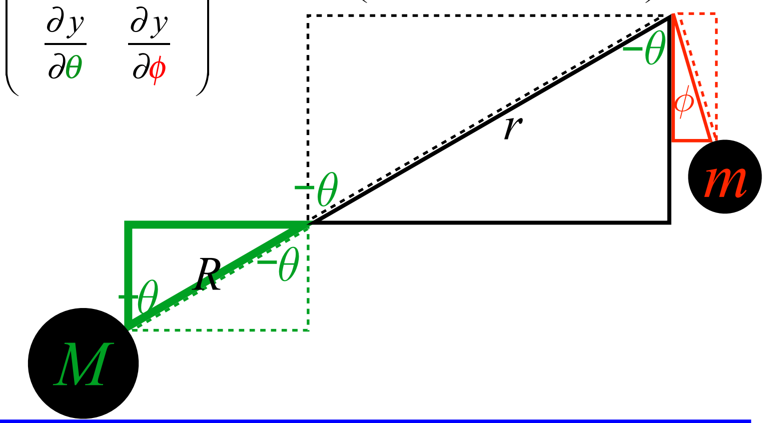
Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\,dX = F_X\frac{\partial X}{\partial \theta}d\theta + F_X\frac{\partial X}{\partial \phi}d\phi = M\ddot{X}\frac{\partial X}{\partial \theta}d\theta + M\ddot{X}\frac{\partial X}{\partial \phi}d\phi$$

$$+F_Y\, dY \quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial \theta}d\theta + F_Y\frac{\partial Y}{\partial \phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial \theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial \phi}d\phi$$

$$+F_x\, dx \quad +m\ddot{x}\,dx \quad +F_x\frac{\partial x}{\partial \theta}d\theta + F_x\frac{\partial x}{\partial \phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial \theta}d\theta + m\ddot{x}\frac{\partial x}{\partial \phi}d\phi$$

$$+F_y\, dy \quad +m\ddot{y}\,dy \quad +F_y\frac{\partial y}{\partial \theta}d\theta + F_y\frac{\partial y}{\partial \phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial \theta}d\theta + m\ddot{y}\frac{\partial y}{\partial \phi}d\phi$$



Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ*

Set: $d\theta=1 \quad d\phi=0$

$$F_X\frac{\partial X}{\partial \theta} = M\ddot{X}\frac{\partial X}{\partial \theta} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$

$$+F_Y\frac{\partial Y}{\partial \theta} \quad +M\ddot{Y}\frac{\partial Y}{\partial \theta} \quad +\frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$

$$+F_x\frac{\partial x}{\partial \theta} \quad +m\ddot{x}\frac{\partial x}{\partial \theta} \quad +\frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$

$$+F_y\frac{\partial y}{\partial \theta} \quad +m\ddot{y}\frac{\partial y}{\partial \theta} \quad +\frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

Set: $d\theta=0 \quad d\phi=1$

$$F_X\frac{\partial X}{\partial \phi} = M\ddot{X}\frac{\partial X}{\partial \phi} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$

$$+F_Y\frac{\partial Y}{\partial \phi} \quad +M\ddot{Y}\frac{\partial Y}{\partial \phi} \quad +\frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$

$$+F_x\frac{\partial x}{\partial \phi} \quad +m\ddot{x}\frac{\partial x}{\partial \phi} \quad +\frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi}$$

$$+F_y\frac{\partial y}{\partial \phi} \quad +m\ddot{y}\frac{\partial y}{\partial \phi} \quad +\frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}$$

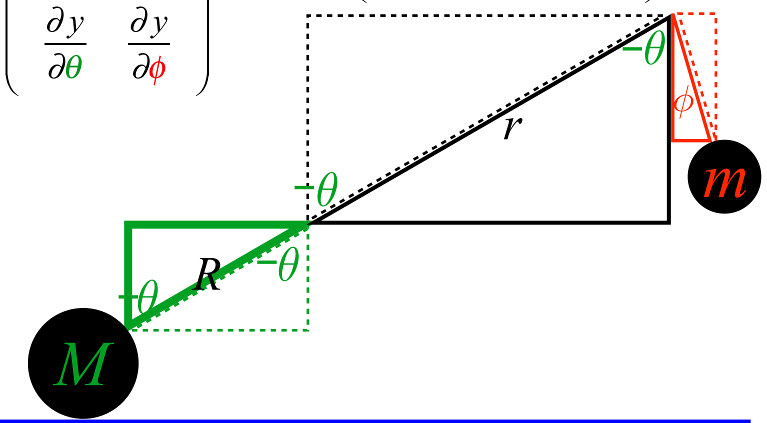
Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\,dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\,dY &\quad +M\ddot{Y}\,dY &\quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\,dx &\quad +m\ddot{x}\,dx &\quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &\quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\,dy &\quad +m\ddot{y}\,dy &\quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

STEP **D**

Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

Lagrange trickery:

$$\text{Let}: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Set: $d\theta=1 \quad d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} &= \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad +m\ddot{x}\frac{\partial x}{\partial\theta} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad +m\ddot{y}\frac{\partial y}{\partial\theta} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\theta}
\end{aligned}$$

Set: $d\theta=0 \quad d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} &= \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\phi} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad +m\ddot{x}\frac{\partial x}{\partial\phi} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad +m\ddot{y}\frac{\partial y}{\partial\phi} &\quad +\frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\phi}
\end{aligned}$$

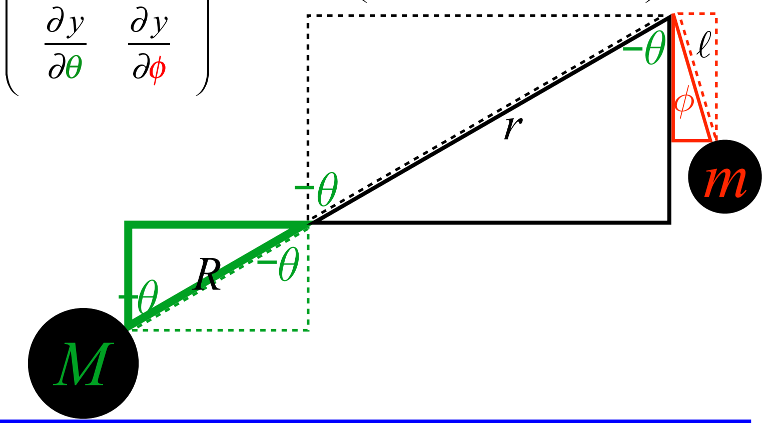
Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns:(Using GCC dθ and dφ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and φ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$Let: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Set: $d\theta=1 \quad d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} \quad + m\ddot{x} \frac{\partial x}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} \quad + m\ddot{y} \frac{\partial y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

Set: $d\theta=0 \quad d\phi=1$

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} \quad + M\ddot{Y} \frac{\partial Y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} \quad + m\ddot{x} \frac{\partial x}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} \quad + m\ddot{y} \frac{\partial y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}$$

Review of Lagrangian equation derivation (Elementary trebuchet)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

→ *Lagrangian force equation*

Canonical momentum and γ_{mn} tensor

Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

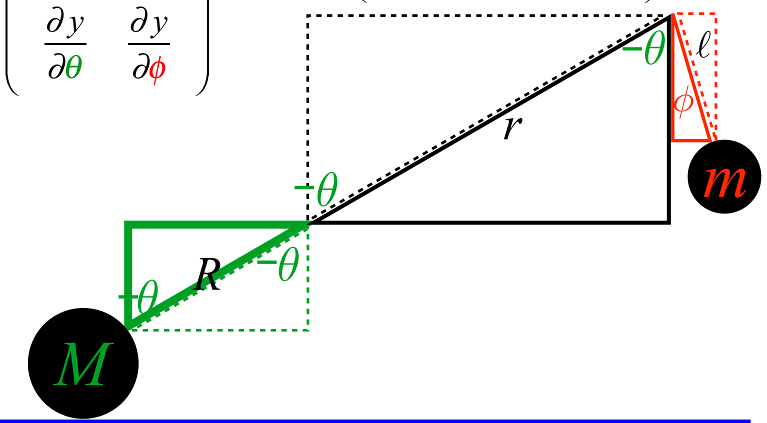
Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX \;=\; M\ddot{X}\, dX \;=\; F_X \frac{\partial X}{\partial\theta} d\theta + F_X \frac{\partial X}{\partial\phi} d\phi \;=\; M\ddot{X}\frac{\partial X}{\partial\theta} d\theta + M\ddot{X}\frac{\partial X}{\partial\phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial\theta} d\theta + F_Y \frac{\partial Y}{\partial\phi} d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta} d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial\theta} d\theta + F_x \frac{\partial x}{\partial\phi} d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta} d\theta + m\ddot{x}\frac{\partial x}{\partial\phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial\theta} d\theta + F_y \frac{\partial y}{\partial\phi} d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta} d\theta + m\ddot{y}\frac{\partial y}{\partial\phi} d\phi$$



Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X \frac{\partial X}{\partial\theta} + F_Y \frac{\partial Y}{\partial\theta} + F_x \frac{\partial x}{\partial\theta} + F_y \frac{\partial y}{\partial\theta}$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X \frac{\partial X}{\partial\phi} + F_Y \frac{\partial Y}{\partial\phi} + F_x \frac{\partial x}{\partial\phi} + F_y \frac{\partial y}{\partial\phi}$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

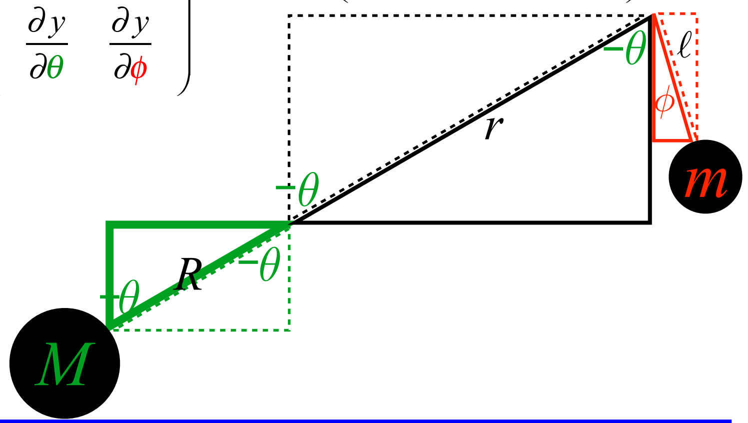
Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_y\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\,dX = M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\,dY \quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\,dx \quad +m\ddot{x}\,dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\,dy \quad +m\ddot{y}\,dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$\text{Let}: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$\text{Let}: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

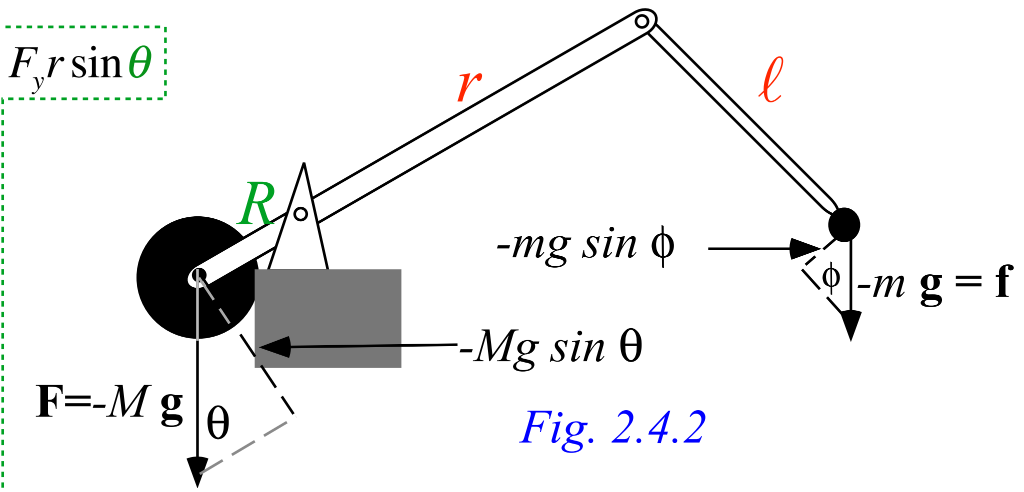


Fig. 2.4.2

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_θ gravity given

$(F_X = 0,\ F_Y = -Mg)$

$(F_x = 0,\ F_y = -mg)$

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

These are competing torques on main beam R

Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\,dX \;=\; M\ddot{X}\,dX \;=\; F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi \;=\; M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\,dY \quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\,dx \quad +m\ddot{x}\,dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

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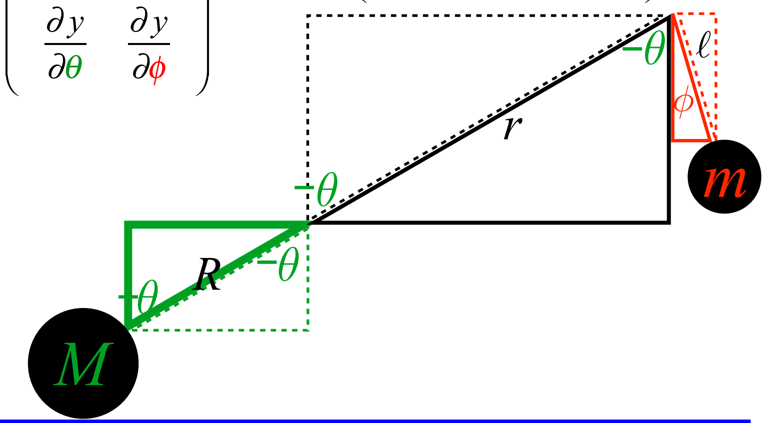

Lagrange trickery:

STEP **D** Add up first and last columns for each variable θ and ϕ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

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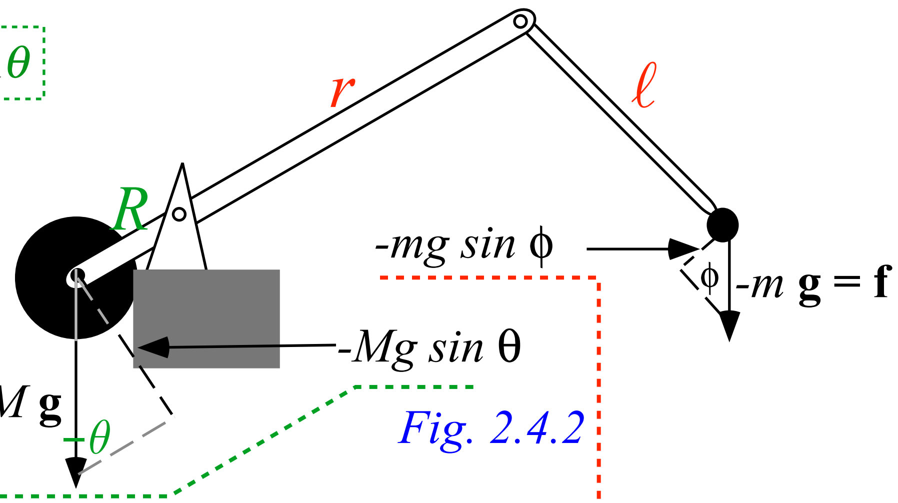

Fig. 2.4.2

Add F_θ gravity given

($F_X=0$, $F_Y=-Mg$)

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These are competing torques on main beam R... ... and a torque on throwing lever ℓ

Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$

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Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

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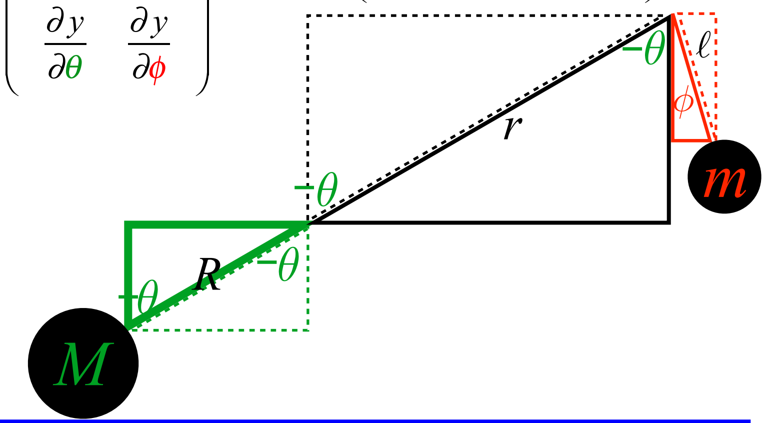

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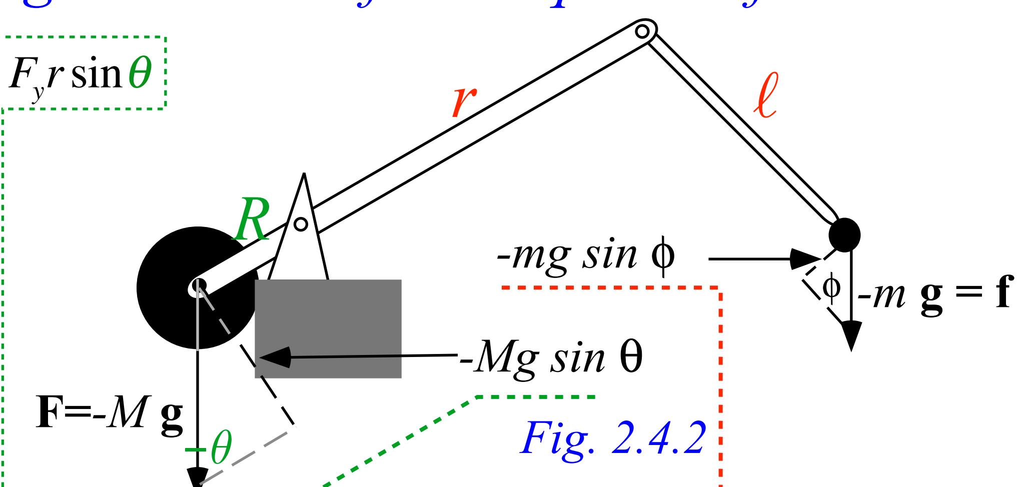

Fig. 2.4.2

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Q: Are there ± sign errors here?

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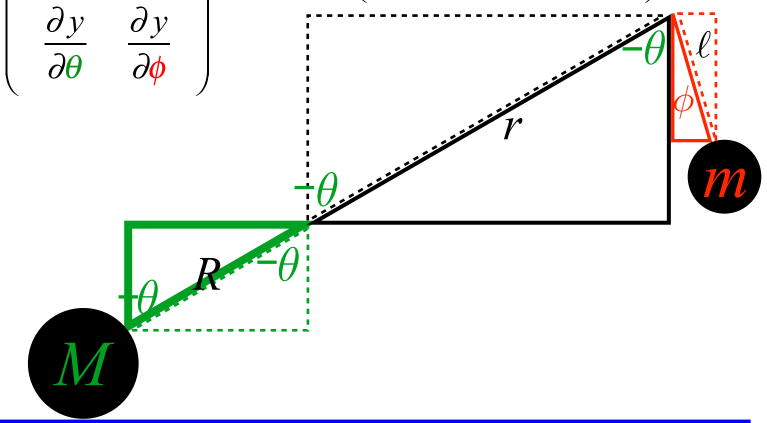

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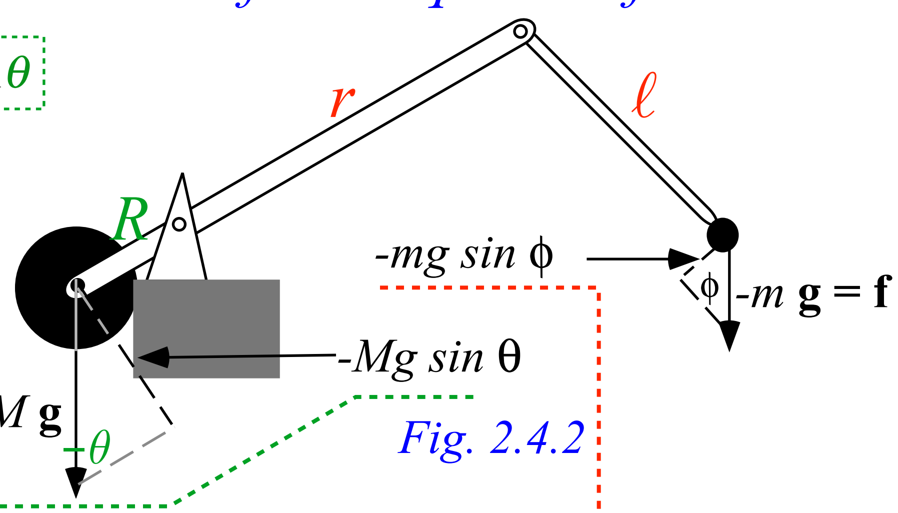

Fig. 2.4.2

Add F_θ gravity given
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$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR\sin\theta + mgr\sin\theta$$

Q: Are there ± sign errors here?
A: No. Beam in $-\theta$ position.

Add F_ϕ gravity given
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($F_x = 0$, $F_y = -mg$)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} = -mg\ell\sin\phi$$

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Review of Lagrangian equation derivation (Elementary trebuchet)

Coordinate geometry, kinetic energy, and dynamic metric tensor γ_{mn}

Basic force, work, and acceleration

Lagrangian force equation

→ *Canonical momentum and* γ_{mn} *tensor*

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

The γ_{mn} tensor/matrix formulation

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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Momentum γ_{mn}-matrix theorem: (proof on next page)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

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Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$Given: p_m = \frac{\partial T}{\partial \dot{q}^m} \quad where: \quad T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$Then: \quad p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

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$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (proof on next page)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

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Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$Total\ KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$Total\ KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: *(proof on next page)*

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

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Momentum γ_{mn}-tensor theorem: *(proof here)*

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$

$$Then:\quad p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

Canonical momentum and γ_{mn} tensor

Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

The γ_{mn} tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: γ_{mn} tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

Momentum γ_{mn}-matrix theorem: (proof on next page)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

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Momentum γ_{mn}-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

$$= \frac{1}{2}\gamma_{mn}\dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm} \qquad QED$$

Momentum γ_{mn}-matrix theorem: (proof)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot\theta} \\ \frac{\partial T}{\partial \dot\phi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial\dot\theta}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\frac{\partial}{\partial\dot\theta}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \\ \frac{\partial}{\partial\dot\phi}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\frac{\partial}{\partial\dot\phi}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\phi,\theta} \\ \gamma_{\theta,\phi} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \qquad QED$$

Equations of motion and force analysis (Mostly Unit 2.)

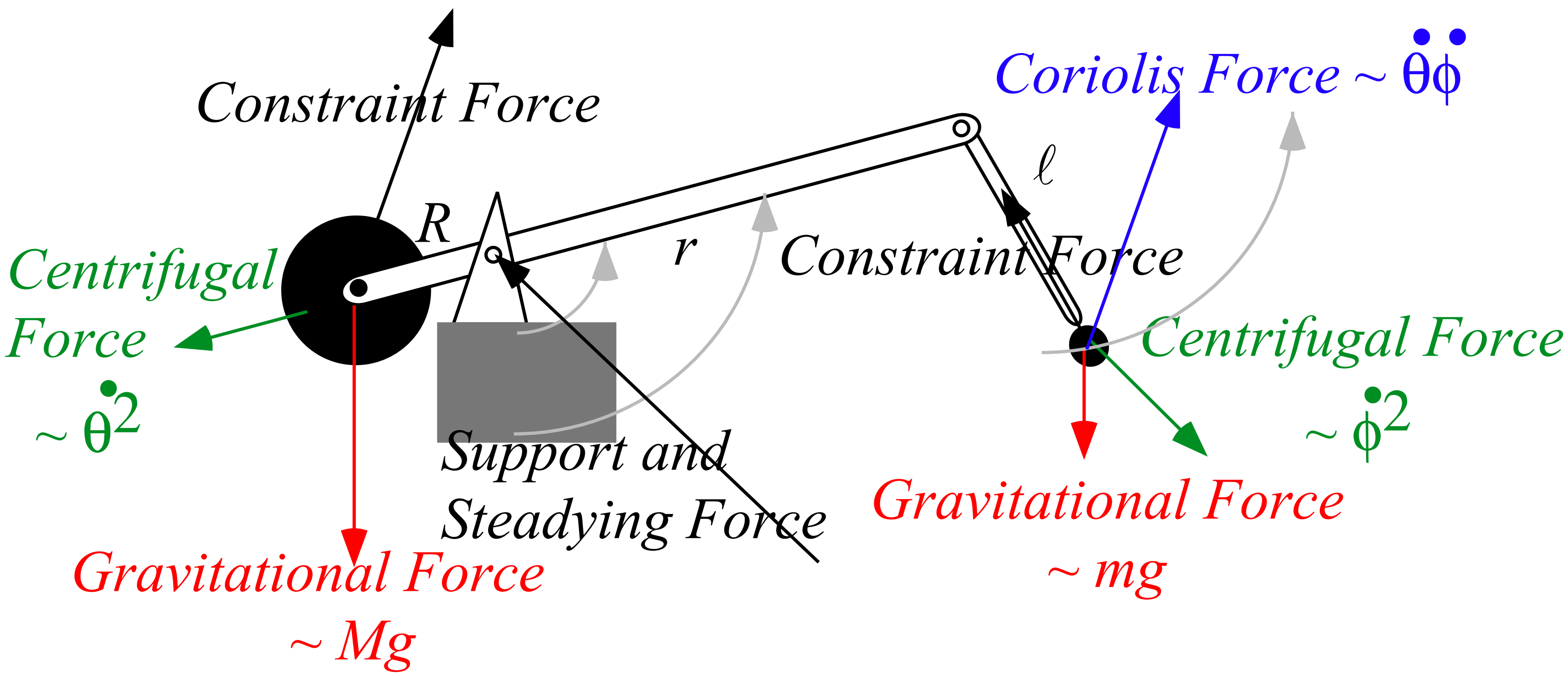
→ *Forces: total, genuine, potential, and/or fictitious*

Lagrange equation forms

Riemann equation forms

2nd-guessing Riemann? (More like Unit 3.)

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:	*Applied 'Real' Forces:*	*Constraint 'Internal' Forces:*
Coriolis	*Gravity*	*Stresses*
Centrifugal	*Stimuli*	*Support...*
	Friction...	*(Do not contribute. Do no work.)*

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot\theta} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

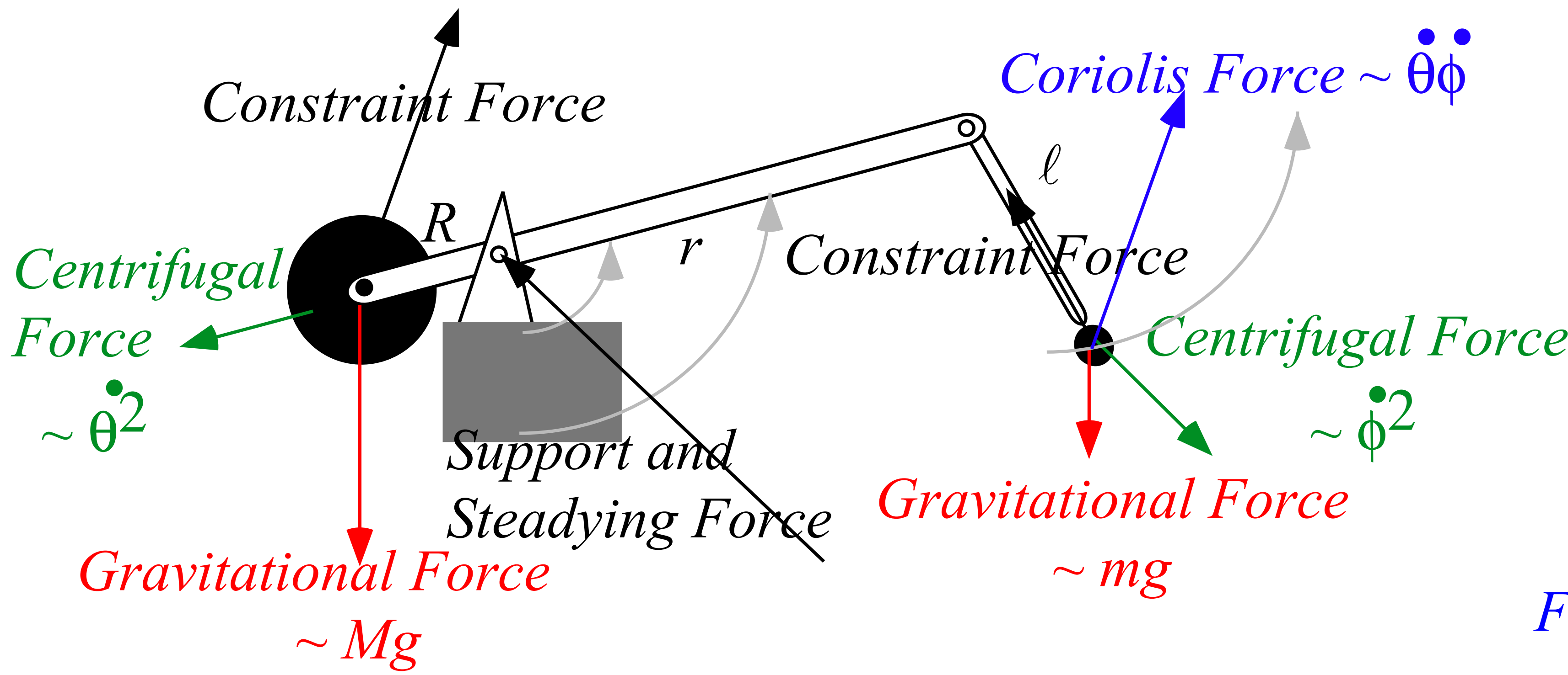
$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot\phi} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations

(derived on p. 26)

Fig. 2.5.2 (modified)

Forces: total, genuine, potential, and/or fictitious



For conservative forces

where: $F_\theta = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot{\phi}} = 0$

Acceleration and 'Fictitious' Forces: *Coriolis*, *Centrifugal*

Applied 'Real' Forces: *Gravity*, *Stimuli*, *Friction...*

Constraint 'Internal' Forces: *Stresses*, *Support...* *(Do not contribute. Do no work.)*

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

Lagrange Force equations
(derived on p. 26)

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations
$L = T - V$

Fig. 2.5.2 (modified)

Equations of motion and force analysis (Mostly Unit 2.)

- *Forces: total, genuine, potential, and/or fictitious*
- → *Lagrange equation force analysis*
- *Riemann equation force analysis*
- *2nd-guessing Riemann? (More like Unit 3.)*

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)\right)$$

[$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) + mr\ell\dot{\phi}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi)$$

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) + mr\ell\dot{\theta}\dot{\phi}\sin(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi)$$

$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta - \phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) + mr\ell\dot{\theta}(\dot{\theta} - \dot{\phi})\sin(\theta - \phi)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta - \phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta - \phi) + mr\ell\dot{\theta}^2\sin(\theta - \phi)$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

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$$\dot{p}_\theta = \frac{d}{dt}p_\theta = \frac{d}{dt}\left(\left(MR^2+mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right)$$ [$\dot{M}, \dot{R}, \dot{m}, \dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

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Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2+mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

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$$= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

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[$\dot{M}$, $\dot{R}$, $\dot{m}$, $\dot{r}$, and $\dot{\ell}$ are (thankfully) zero]

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

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$$\dot{p}_\phi = \frac{d}{dt}p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

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$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial\theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial\phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

gravity forces F_μ from p.30

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

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$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) \quad = F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

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$$= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

gravity forces F_μ from p.30

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

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$$\dot{p}_\theta = \frac{d}{dt}p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}]$$

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)} - mr\ell\dot{\phi}^2\sin(\theta-\phi) \quad = F_\theta + \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

$$\dot{p}_\phi = \frac{d}{dt}p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) - \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)} + mr\ell\dot{\theta}^2\sin(\theta-\phi) \quad = F_\phi - \cancel{mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)}$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

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$$= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

gravity forces F_μ from p.30

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

Dot means *total* differentiation

Everything that can move contributes. (Very easy to miss a term!)

$$\dot{p}_\theta = \frac{d}{dt} p_\theta = \frac{d}{dt}\left(\left(MR^2 + mr^2\right)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)\right) \quad [\dot{M}, \dot{R}, \dot{m}, \dot{r}, \text{ and } \dot{\ell} \text{ are (thankfully) zero}]$$

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$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

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Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

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$$= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

gravity forces F_μ from p.30

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

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Lagrange equation force analysis

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$$= \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) + mr\ell\dot{\phi}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

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$$\dot{p}_\phi = \frac{d}{dt} p_\phi = \frac{d}{dt}\left(m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)\right)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}(\dot{\theta}-\dot{\phi})\sin(\theta-\phi)$$

$$= m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) \quad = F_\phi = -mg\ell\sin\phi$$

Set equal to real (*gravity*) force F_μ plus *fictitious force* $\partial T/\partial q^\mu$ terms

$$\dot{p}_\theta = F_\theta + \frac{\partial T}{\partial \theta} = F_\theta + \frac{\partial}{\partial \theta}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= F_\theta + mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

$$\dot{p}_\phi = F_\phi + \frac{\partial T}{\partial \phi} = F_\phi + \frac{\partial}{\partial \phi}\left(\frac{1}{2}\left(MR^2 + mr^2\right)\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\phi}^2 - mr\ell\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right)$$

$$= F_\phi - mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$

gravity forces F_μ from p.30

$$F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$F_\phi = -mg\ell\sin\phi$$

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

$$\dot{p}_{\theta} = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta - \phi) - mr\ell\dot{\phi}^2\sin(\theta - \phi) = F_{\theta} = -MgR\sin\theta + mgr\sin\theta$$

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Equations of motion and force analysis (Mostly Unit 2.)

- *Forces: total, genuine, potential, and/or fictitious*
- *Lagrange equation force analysis*
- → *Riemann equation force analysis*
- *2nd-guessing Riemann? (More like Unit 3.)*

Lagrange equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

$$\dot{p}_\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

$$\dot{p}_{\theta} = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_{\theta} = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_{\phi} = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_{\phi} = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_{\theta} \\ F_{\phi} \end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

$$\dot{p}_{\theta} = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_{\theta} = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_{\phi} = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_{\phi} = -mg\ell\sin\phi$$

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This uses the γ_{mn} tensor :

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$$

$$\dot{p}_\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor :

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

Riemann equation force analysis

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = \dot{p}_{\mu} - \frac{\partial T}{\partial q^{\mu}} = F_{\mu}$$

$$\dot{p}_{\theta} = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_{\theta} = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_{\phi} = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_{\phi} = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_{\theta} \\ F_{\phi} \end{pmatrix} = \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

This uses the γ_{mn} tensor :

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_{\theta} \\ \dot{p}_{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_{\theta} + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_{\phi} - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$

$$\dot{p}_\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2 + mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor :

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\dot{p}_\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix}\dot{p}_\theta \\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\ddot{\theta} \\ \ddot{\phi}\end{pmatrix} - \begin{pmatrix}mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix} = \begin{pmatrix}F_\theta \\ F_\phi\end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix} = \begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}$

$$= \begin{pmatrix}-MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi\end{pmatrix}$$

$$\begin{pmatrix}\dot{p}_\theta \\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\ddot{\theta} \\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

... and apply it...

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1} = \frac{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{p}_\theta \\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\ddot{\theta} \\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Riemann equation form

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\dot{p}_\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor :

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Riemann equation form

Gravity-free case:

$F_\theta = 0 = F_\phi$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu$...

$$\dot{p}_\theta = \left(MR^2 + mr^2\right)\ddot{\theta} - mr\ell\ddot{\phi}\cos(\theta-\phi) - mr\ell\dot{\phi}^2\sin(\theta-\phi) = F_\theta = -MgR\sin\theta + mgr\sin\theta$$

$$\dot{p}_\phi = m\ell^2\ddot{\phi} - mr\ell\ddot{\theta}\cos(\theta-\phi) + mr\ell\dot{\theta}^2\sin(\theta-\phi) = F_\phi = -mg\ell\sin\phi$$

In matrix form:

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \left(MR^2+mr^2\right) & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} - \begin{pmatrix} mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ -mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix} = \begin{pmatrix} F_\theta \\ F_\phi \end{pmatrix}$$

This uses the γ_{mn} tensor : $\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$

$$= \begin{pmatrix} -MgR\sin\theta + mgr\sin\theta \\ -mg\ell\sin\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Need to invert the γ_{mn}-matrix...

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} = \frac{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]}$$

"Super-Inertia" I_S

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi) \end{pmatrix}$$

Riemann equation form

Gravity-free case:

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$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2+mr^2 \end{pmatrix}\begin{pmatrix} \dot{\phi}^2 \\ -\dot{\theta}^2 \end{pmatrix} mr\ell\sin(\theta-\phi)$$

Equations of motion and force analysis (Mostly Unit 2.)

- *Forces: total, genuine, potential, and/or fictitious*
- *Lagrange equation force analysis*
- *Riemann equation force analysis*
- → *2nd-guessing Riemann? (More like Unit 3.)*

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu$ ...

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$Let: (\theta-\phi) = -\frac{\pi}{2}$ so: $I_S = m\ell^2\left[MR^2 + mr^2\right]$ and let: $\omega \equiv \dot{\theta} = \dot{\phi}$

$$I_S\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = I_S\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} -\omega^2 \\ \omega^2 \end{pmatrix} mr\ell$$

$$\begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} -\dot{\phi}^2 \\ \dot{\theta}^2 \end{pmatrix} mr\ell = \frac{\begin{pmatrix} m\ell^2 & 0 \\ 0 & MR^2 + mr^2 \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\right]} \begin{pmatrix} -mr\ell\omega^2 \\ mr\ell\omega^2 \end{pmatrix} = \begin{pmatrix} \frac{-mr\ell\omega^2}{MR^2 + mr^2} \\ \omega^2 r/\ell \end{pmatrix}$$

Trying to 2nd-guess Riemann results

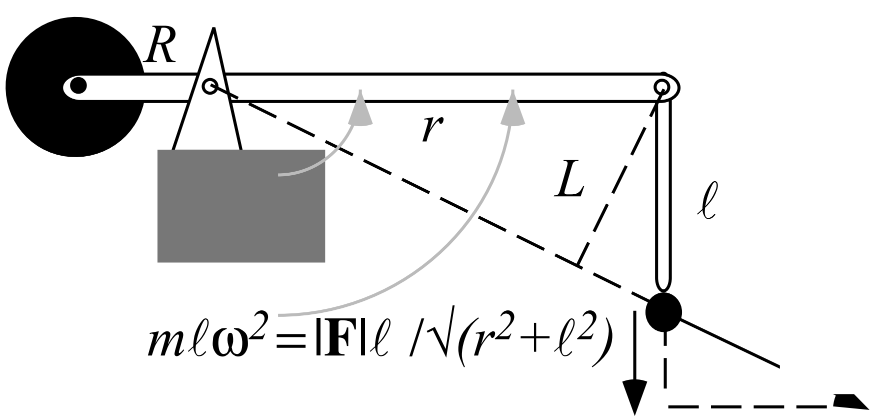

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi}$, $\theta = \frac{-\pi}{2}$, $\phi = 0$)

Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{p}_\theta\\ \dot{p}_\phi\end{pmatrix} = \begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}F_\theta + mr\ell\dot{\phi}^2\sin(\theta-\phi)\\ F_\phi - mr\ell\dot{\theta}^2\sin(\theta-\phi)\end{pmatrix}$$

Riemann equation form

Gravity-free case:

$F_\theta = 0 = F_\phi$

$$I_S\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = I_S\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}^{-1}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix} mr\ell\sin(\theta-\phi) = \begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}\dot{\phi}^2\\ -\dot{\theta}^2\end{pmatrix} mr\ell\sin(\theta-\phi$$

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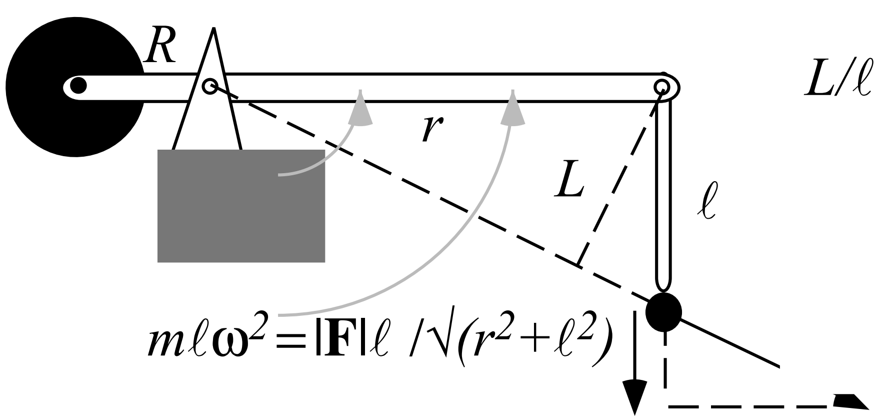
Trying to 2nd-guess Riemann results

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L = r\cdot\ell/\sqrt{(r^2+\ell^2)}$.



$L/\ell = r/\sqrt{(r^2+\ell^2)}$

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Riemann equation force analysis $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\mu} - \frac{\partial T}{\partial q^\mu} = \dot{p}_\mu - \frac{\partial T}{\partial q^\mu} = F_\mu$ becomes $\gamma^{\mu\nu}\dot{p}_\mu = \ddot{q}^\nu \ldots$

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} \dot{p}_\theta \\ \dot{p}_\phi \end{pmatrix} = \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}^{-1} \begin{pmatrix} F_\theta + mr\ell\dot{\phi}^2 \sin(\theta-\phi) \\ F_\phi - mr\ell\dot{\theta}^2 \sin(\theta-\phi) \end{pmatrix}$$

Riemann equation form

Gravity-free case:

$F_\theta = 0 = F_\phi$

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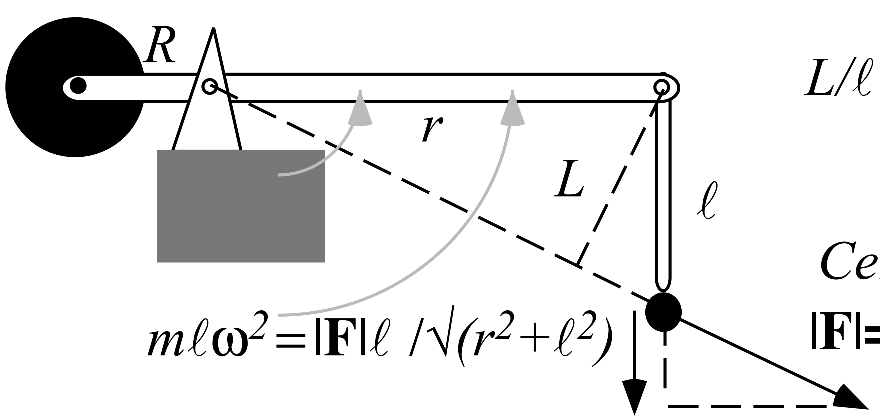
Trying to 2nd-guess Riemann results

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.

This is the rate of change of ϕ-angular momentum around the pivot at the top of ℓ.



$L/\ell = r/\sqrt{(r^2+\ell^2)}$

Centrifugal Force

$|\mathbf{F}| = m\omega^2\sqrt{(r^2+\ell^2)}$

$$m\ell^2\ddot{\phi} = FL = m\omega^2\sqrt{r^2+\ell^2}\,\frac{r\ell}{\sqrt{r^2+\ell^2}} = m\omega^2 r\ell$$

or: $\ddot{\phi} = FL/m\ell^2 = \omega^2 r/\ell$

Fig. 2.5.1 Centrifugal force for a particular state of motion ($\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0$)

Trying to 2nd-guess Riemann results (contd.)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.
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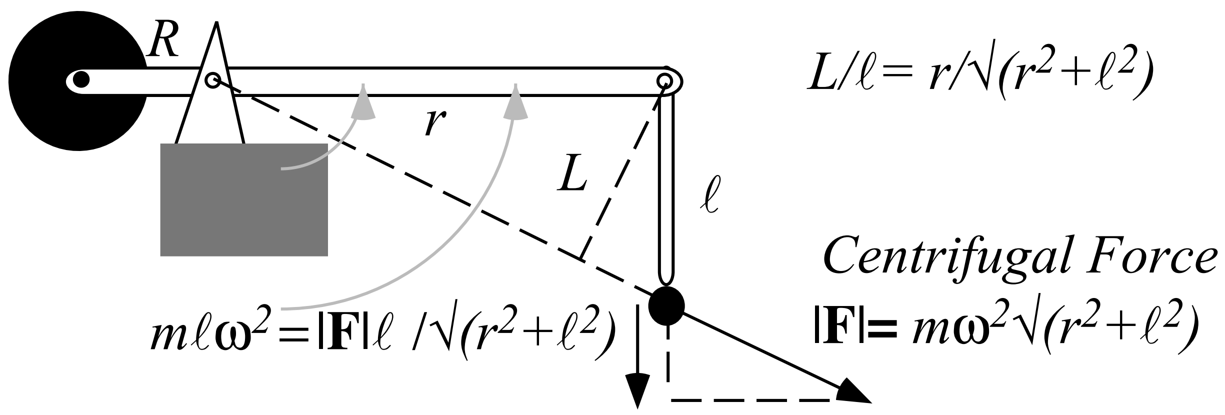

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2nd-guessing Riemann: $\begin{pmatrix}\ddot{\theta}\\ \ddot{\phi}\end{pmatrix} = \begin{pmatrix}\dfrac{-mr\ell\omega^2}{MR^2+mr^2}\\ \omega^2 r/\ell\end{pmatrix}$

Fig. 2.5.1 Centrifugal force for a particular state of motion $\left(\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0\right)$

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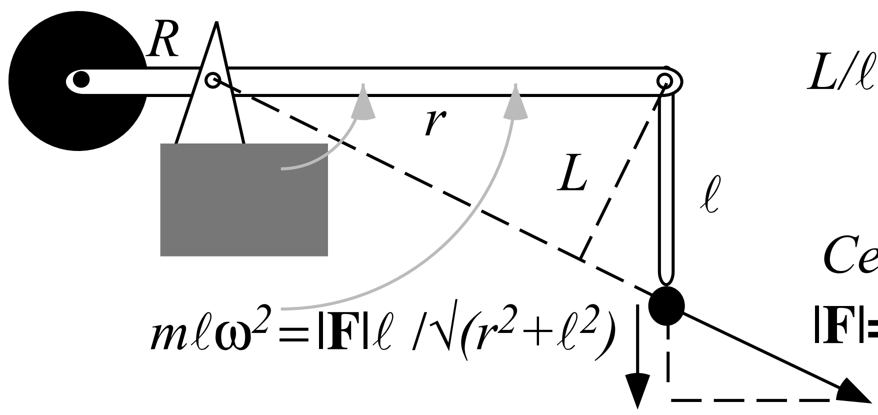

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Fig. 2.5.1 Centrifugal force for a particular state of motion $\left(\omega \equiv \dot{\theta} = \dot{\phi},\ \theta = \frac{-\pi}{2},\ \phi = 0\right)$

It may seem paradoxical that the θ-coordinate for main r-arm feels any torque or acceleration at all.
Indeed, if the device is rigid there can be none since the centrifugal force has no moment;
(Its line of action hits the θ-axis of the R-arm.)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.
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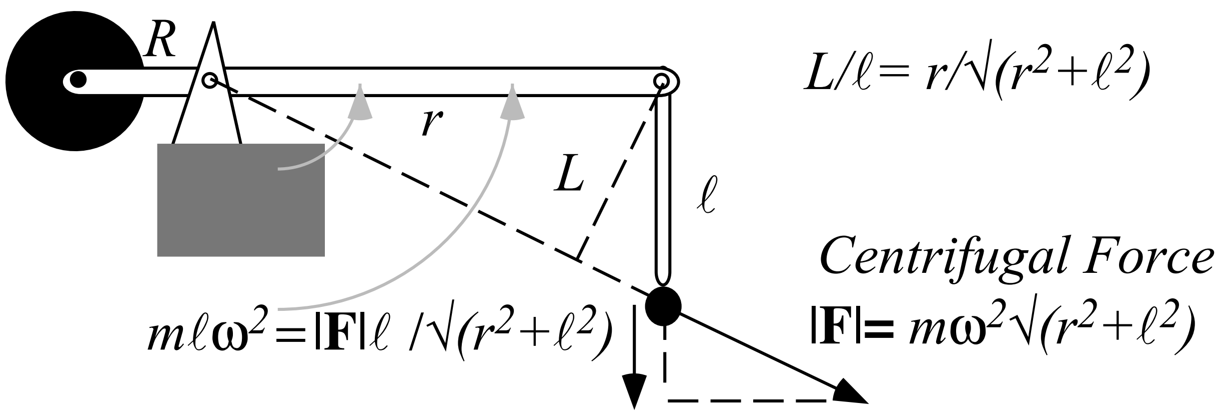

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Trying to 2nd-guess Riemann results (contd.)

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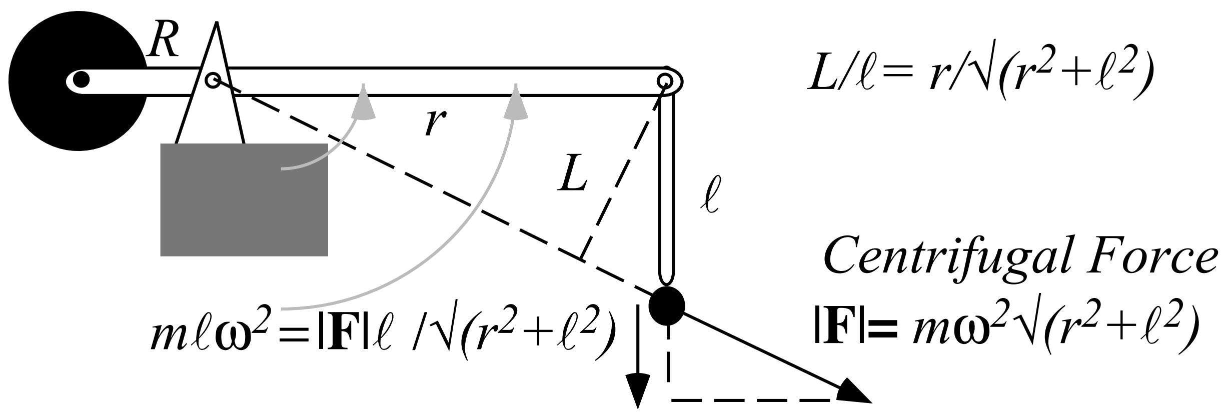

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This causes a negative torque $-mr\ell\,\omega^2$ on the big r-arm.
It reduces θ-angular momentum to exactly cancel the rate of increase in ϕ-momentum.

$$\left(MR^2+mr^2\right)\ddot{\theta} = -m\omega^2 r\ell$$

Trying to 2nd-guess Riemann results (contd.)

The ϕ–torque on mass m on leg ℓ due to centrifugal force is force times *moment* arm $L=r\cdot\ell/\sqrt{(r^2+\ell^2)}$.
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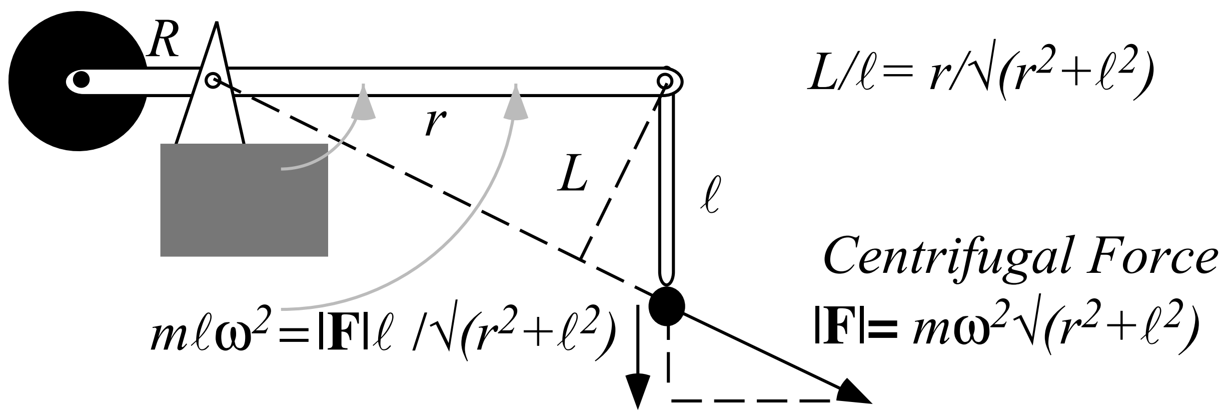

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$$\left(MR^2+mr^2\right)\ddot{\theta} = -m\omega^2 r\ell$$

Note the time derivative of total momentum is zero if outside torques are zero.(twirling skater analogy)

$$\dot{p}_\theta + \dot{p}_\phi = 0,\ \text{if}\ F_\theta = 0 = F_\phi$$