Lecture 8

Thur. 9.17.2015

Kepler Geometry of IHO (Isotropic Harmonic Oscillator) Elliptical Orbits

(Ch. 9 and Ch. 11 of Unit 1)

Review of IHO orbital phasor "clock" dynamics in uniform-body with two "movie" examples

Constructing 2D IHO orbits using Kepler anomaly plots
- *Mean-anomaly and eccentric-anomaly geometry*
- *Calculus and vector geometry of IHO orbits*
- *A confusing introduction to Coriolis-centrifugal force geometry* *(Derived better in Ch. 12)*

Some Kepler's "laws" for all central (isotropic) force F(r) fields
- *Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$* *(Derived here)*
- *Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$* *(Derived in Unit 5)*
- *Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$* *(Derived here)*
- *Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$* *(Derived in Unit 5)*

Introduction to dual matrix operator contact geometry (based on IHO orbits)
- *Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ vs.inverse form ellipse $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$*
  - *Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)*
  - *Q-Ellipse tangents $\mathbf{r}'$ normal to dual Q^{-1}-ellipse position $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)*
- *Operator geometric sequences and eigenvectors*
- *Alternative scaling of matrix operator geometry*
  - *Vector calculus of tensor operation*

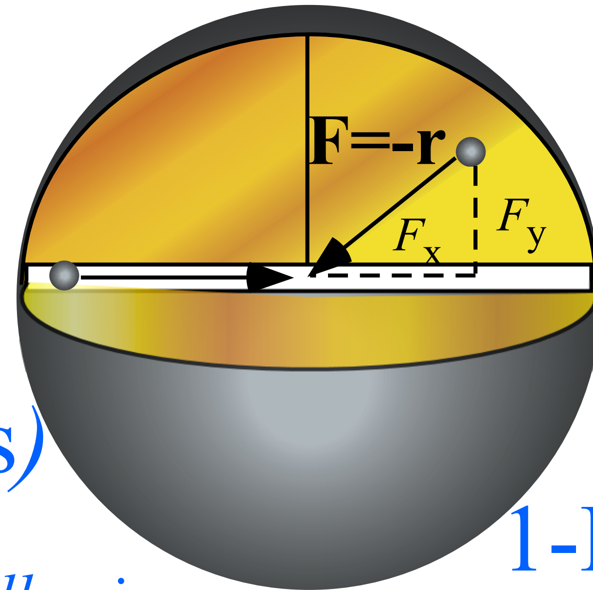
Q:Where is this headed? A: Lagrangian-Hamiltonian duality

Link ⇒ BoxIt simulation of IHO orbits
Link → IHO orbital time rates of change
Link → IHO Exegesis Plot

→ *Review of IHO orbital phasor "clock" dynamics in uniform-body with two "movie" examples*

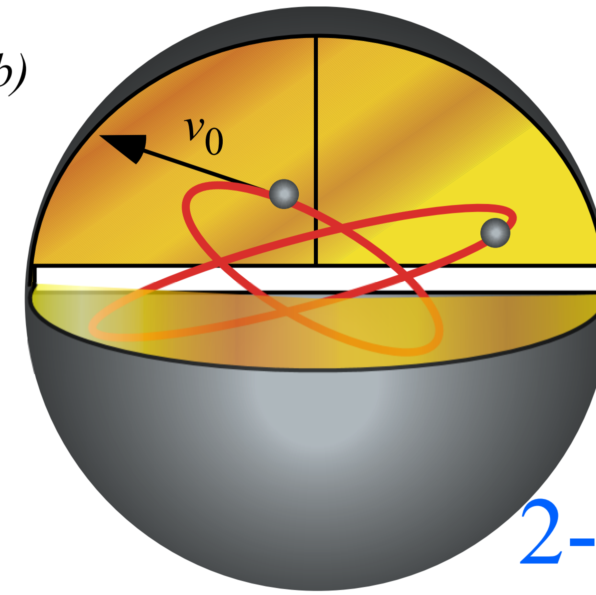
Review of IHO orbital phase dynamics in uniform-body

(a)



1-D

(b)



Unit 1
Fig. 9.10

2-D or 3-D

(Paths are *always* 2-D ellipses if viewed right!)

I.H.O. Force law

F =-x (1-Dimension)

F =-r (2 or 3-Dimensions)

Each dimension x, y, or z obeys the following:

$$Total\ E = KE + PE = \frac{1}{2}mv^2 + U(x) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = const.$$

Equations for x-motion [x(t) and v_x=v(t)] are given first. They apply as well to dimensions [y(t) and v_y=v(t)] and [z(t) and v_z=v(t)] in the ideal isotropic case.

$$1 = \frac{mv^2}{2E} + \frac{kx^2}{2E} = \left(\frac{v}{\sqrt{2E/m}}\right)^2 + \left(\frac{x}{\sqrt{2E/k}}\right)^2$$

$$1 = \frac{mv^2}{2E} + \frac{kx^2}{2E} = (\cos\theta)^2 + (\sin\theta)^2$$

Another example of the old "scale-a-circle" trick...

velocity: *position:* *angular velocity:*

$$Let:\ \textbf{(1)}\ v = \sqrt{2E/m}\cos\theta, \quad and:\ \textbf{(2)}\ x = \sqrt{2E/k}\sin\theta \qquad def.\ \textbf{(3)}\ \omega = \frac{d\theta}{dt}$$

velocity:

$$\sqrt{\frac{2E}{m}}\cos\theta = v = \frac{dx}{dt} = \frac{d\theta}{dt}\frac{dx}{d\theta} = \omega\frac{dx}{d\theta} = \omega\sqrt{\frac{2E}{k}}\cos\theta$$

by **(1)** by def. **(3)** by **(2)**

by def. **(3)**

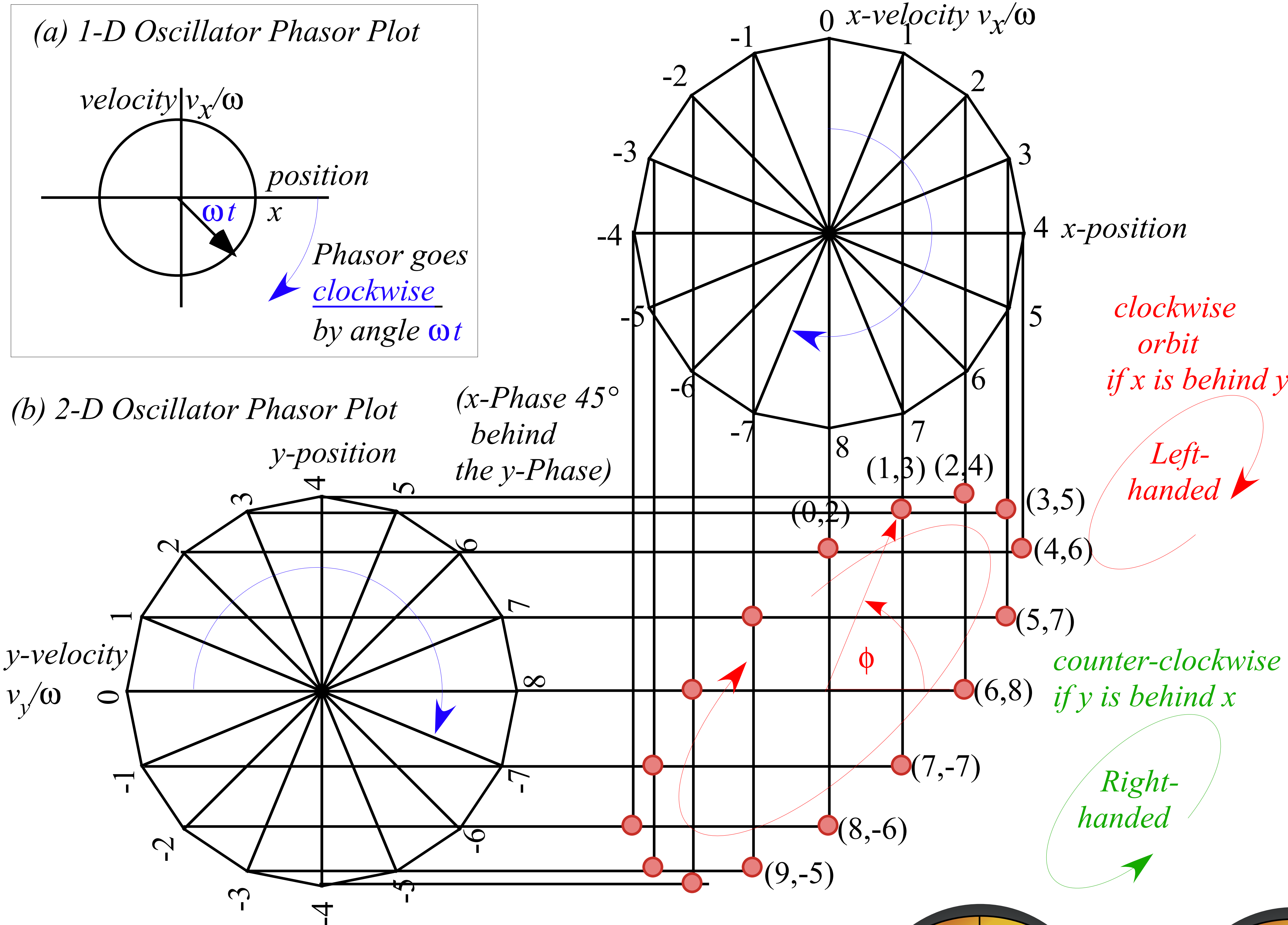
$$\omega = \frac{d\theta}{dt} = \sqrt{\frac{k}{m}}$$

divide this by **(1)**

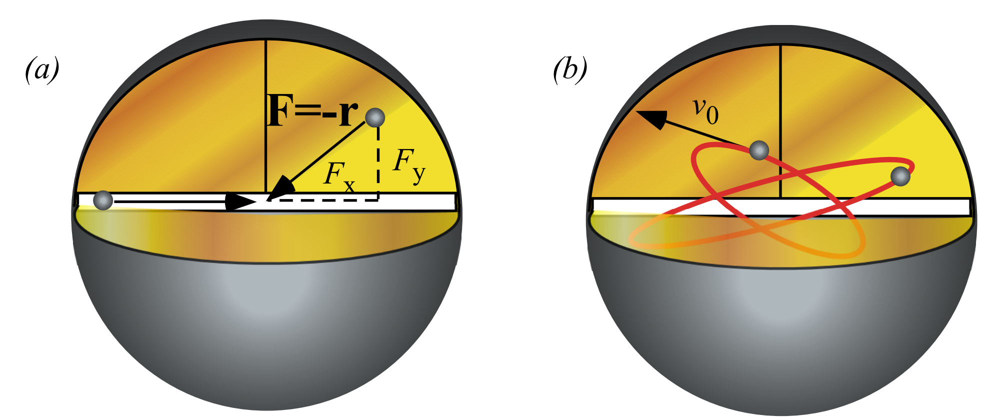
by integration given constant ω:

$$\theta = \int \omega \cdot dt = \omega \cdot t + \alpha$$

Review of IHO orbital phase dynamics in uniform-body

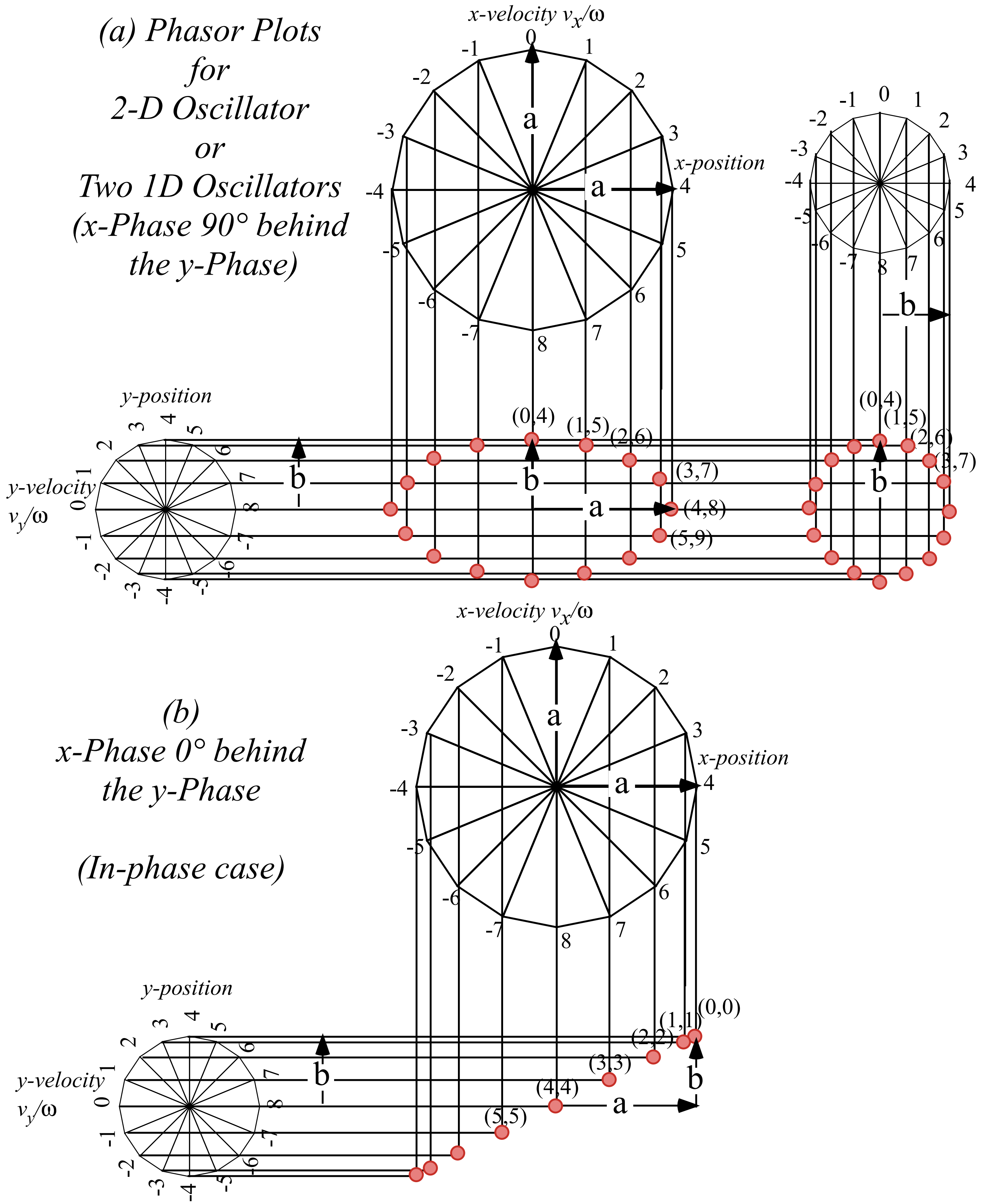


Unit 1
Fig. 9.10



Review of IHO *orbital phase dynamics* in uniform-body

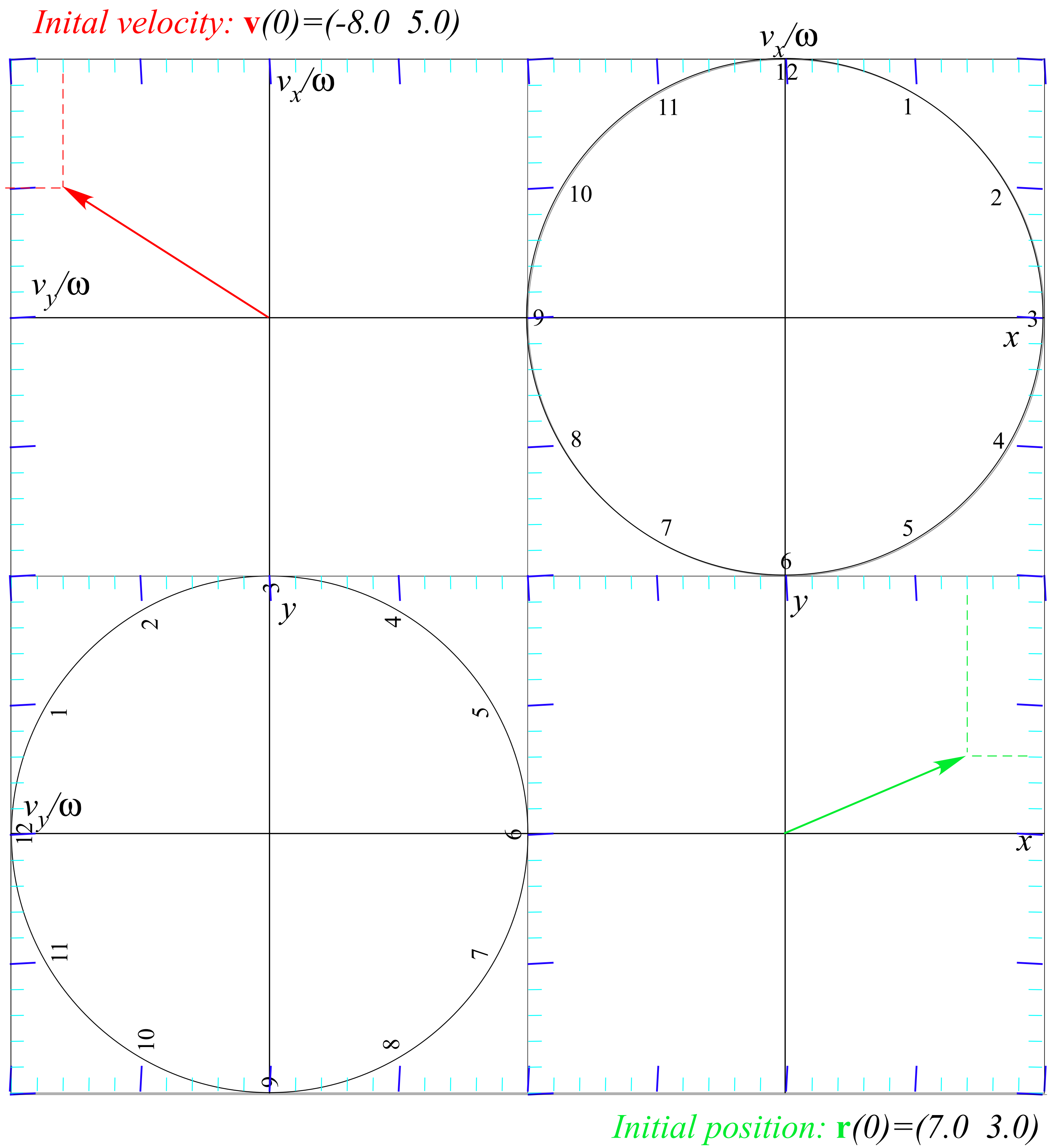
Unit 1
Fig. 9.12



These are more generic examples with radius of x-phasor differing from that of the y-phasor.

Review of IHO orbital phasor "clock" dynamics in uniform-body with two "movie" examples

Review of IHO orbital phasor "clock" dynamics in uniform-body



[BoxIt simulation of U(2) orbits](http://www.uark.edu/ua/modphys/markup/BoxItWeb.html)
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Review of IHO orbital phasor "clock" dynamics in uniform-body

phase lag:

$\Delta\alpha = \alpha_x - \alpha_y = 30°$

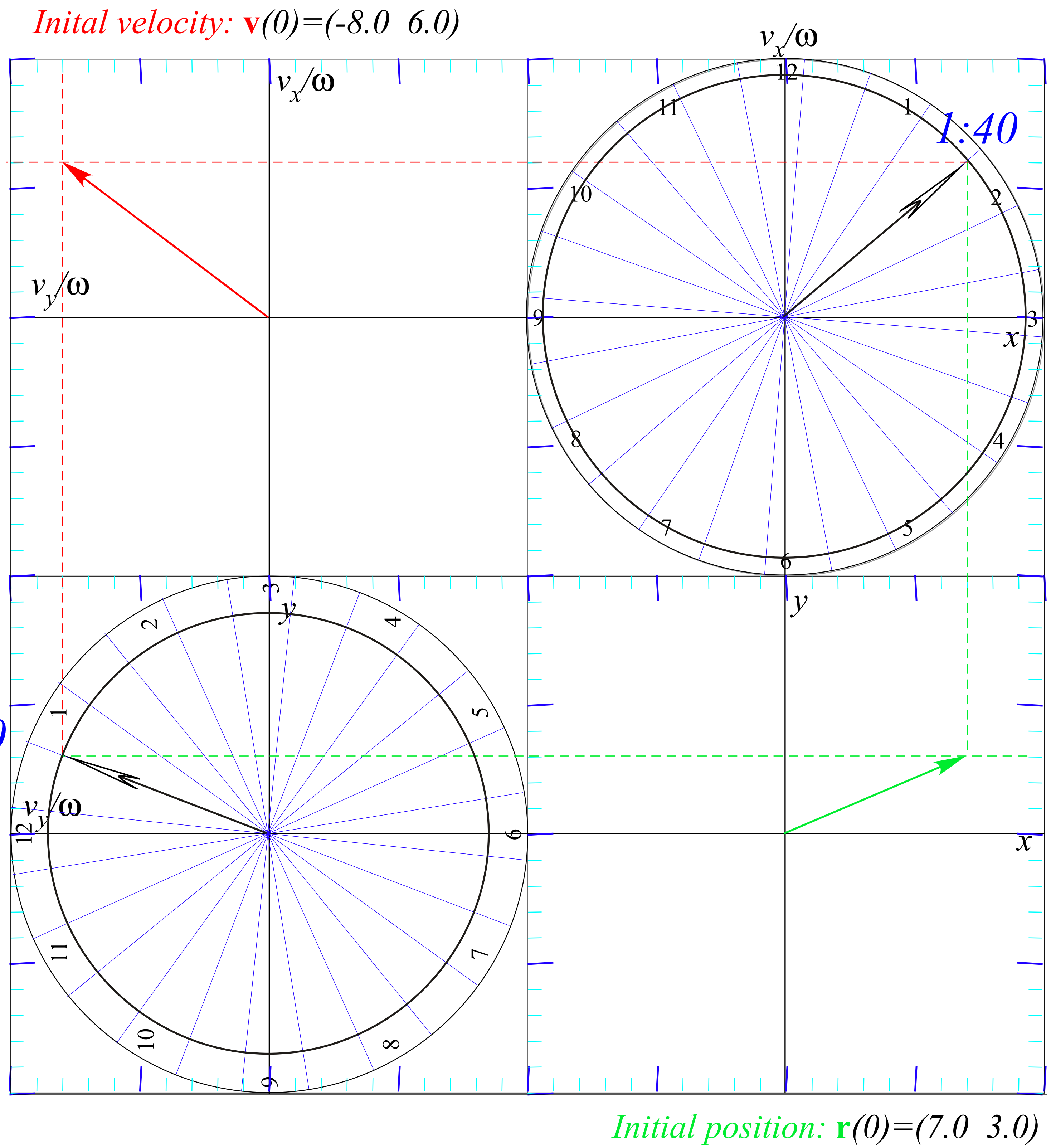
1 minute orbit (5 seconds for second hand)

or

1 hour orbit (5 minutes for minute hand)

or

12 hour orbit (1 hour for hour hand)



Arbitrary initial position $\mathbf{r}(0)=(x(0),y(0))$

and initial velocity $\mathbf{v}(0)=(v_x(0),v_y(0))$

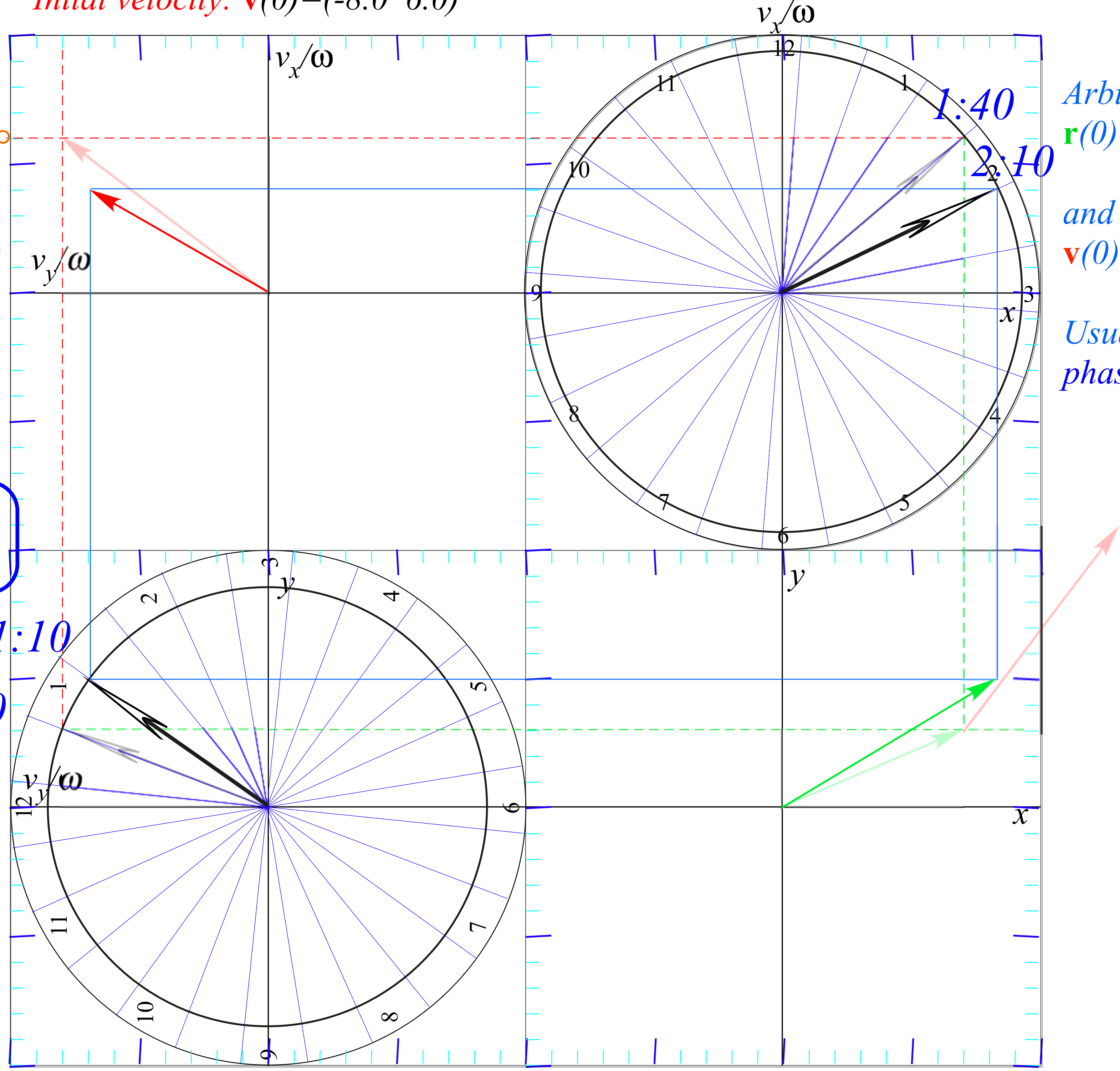
Usually have x and y phasor circles of unequal size

Review of IHO orbital phasor "clock" dynamics in uniform-body

Inital velocity: **v**(0)=(-8.0 6.0)

phase lag:
$\Delta\alpha = \alpha_x - \alpha_y = 30°$

1 minute orbit (5 seconds for second hand)
or
1 hour orbit (5 minutes for minute hand)
or
12 hour orbit (1 hour for hour hand)



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r(0)=(x(0),y(0))

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Usually have x and y
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Initial position: **r**(0)=(7.0 3.0)

Review of IHO orbital phasor "clock" dynamics in uniform-body

Inital velocity: $\mathbf{v}(0)=(-8.0\ \ 6.0)$

phase lag:

$\Delta\alpha=\alpha_x-\alpha_y=30°$

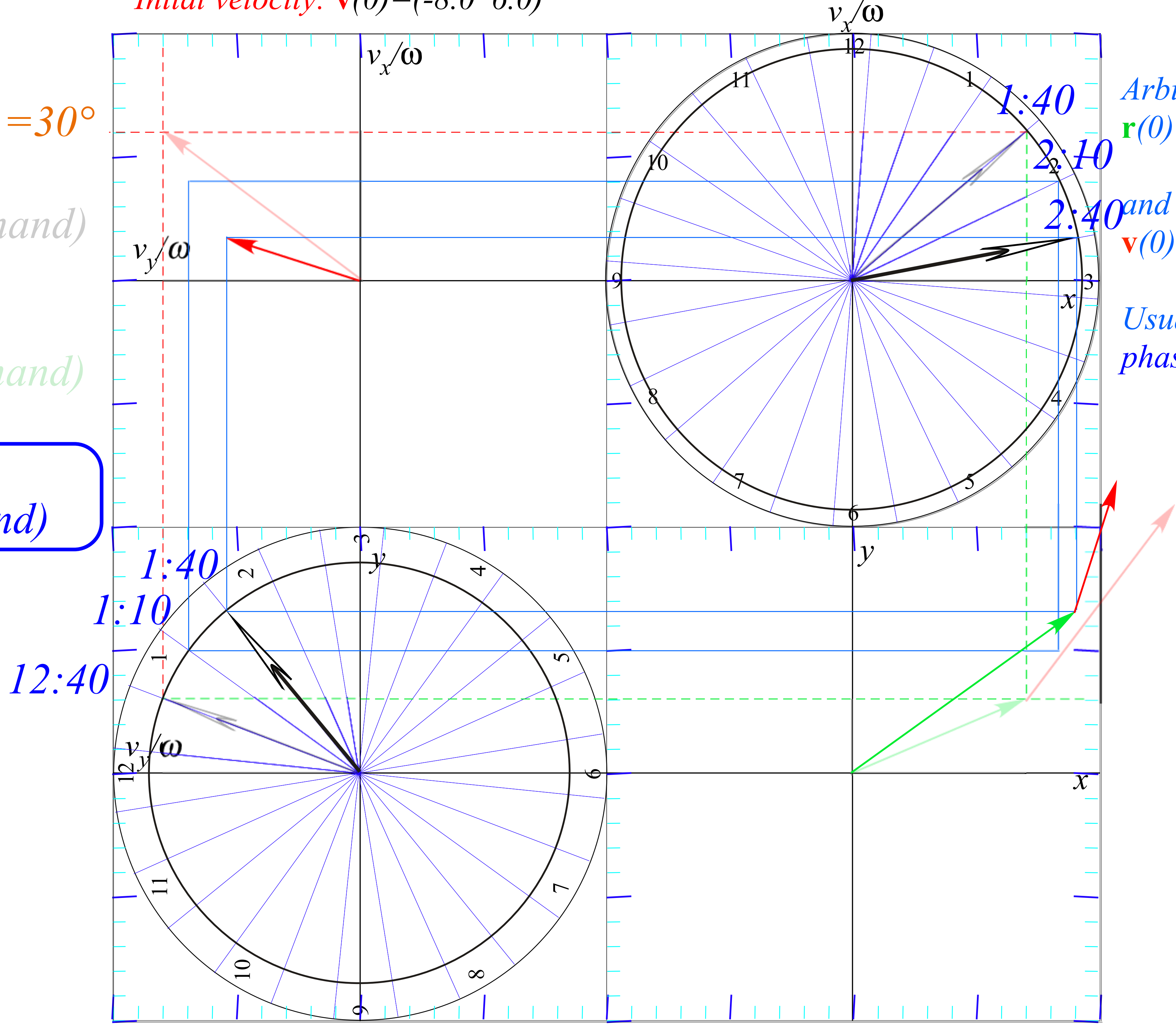
1 minute orbit *(5 seconds for second hand)*

or

1 hour orbit *(5 minutes for minute hand)*

or

12 hour orbit *(1 hour for hour hand)*



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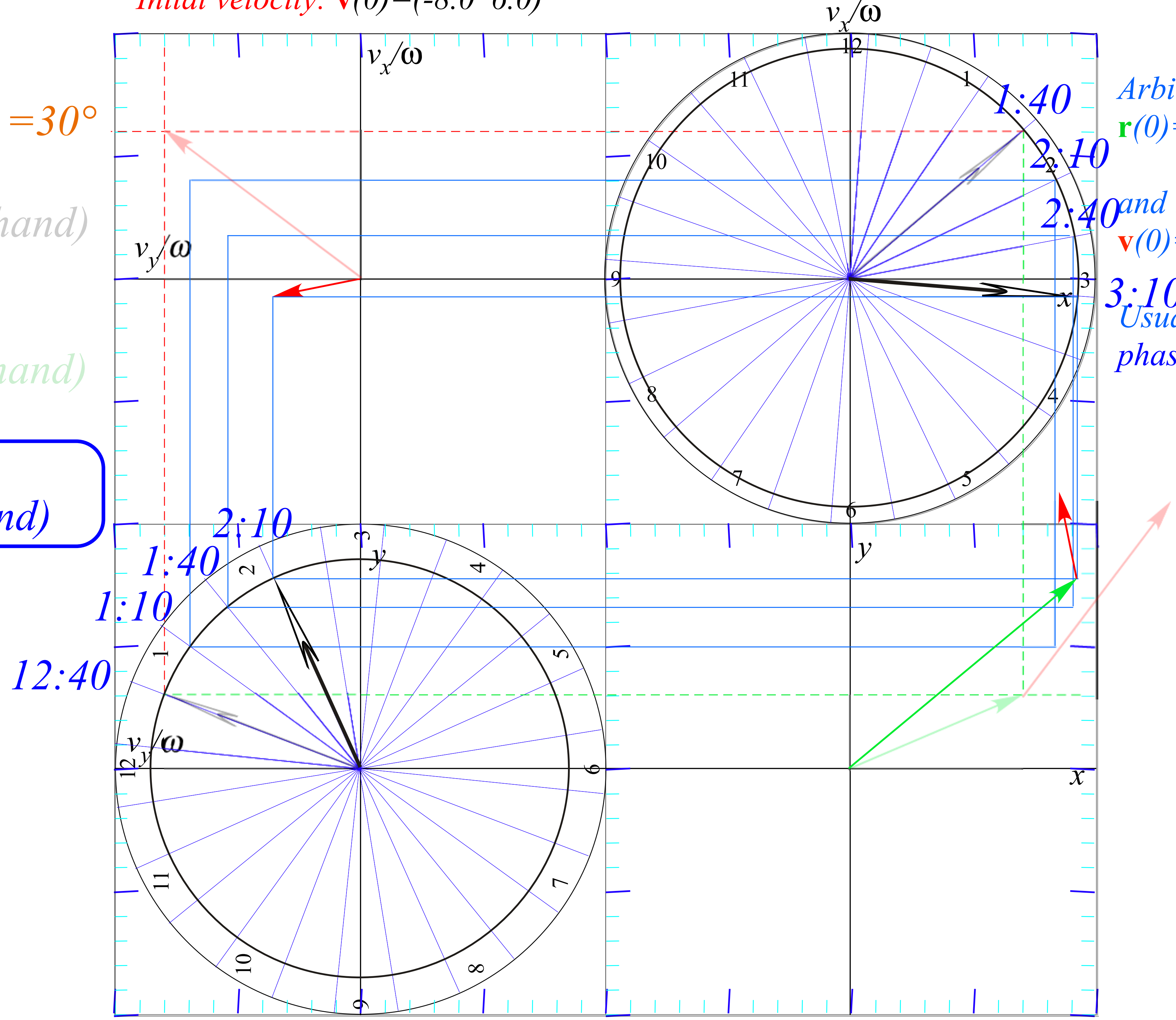

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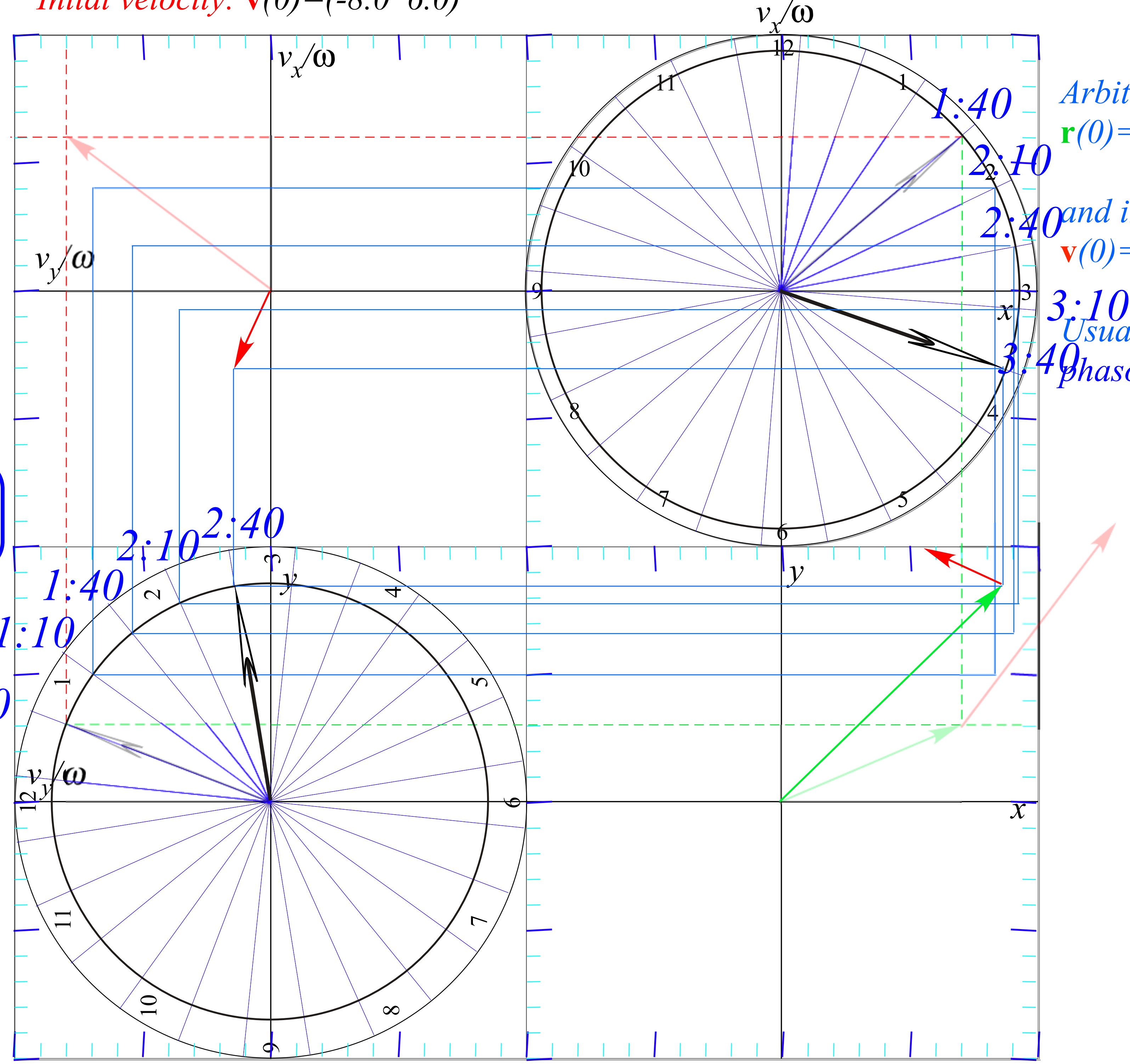

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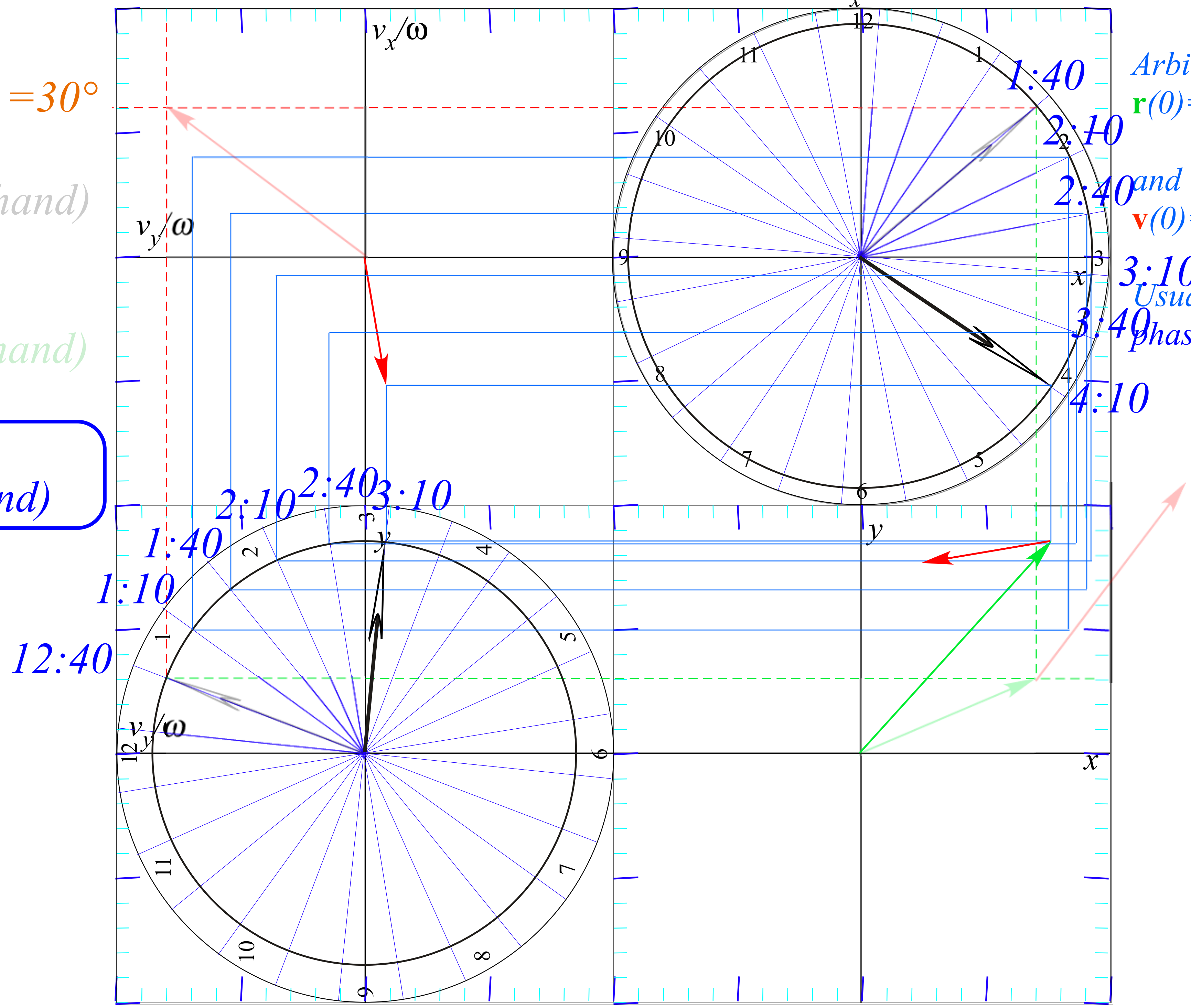

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or

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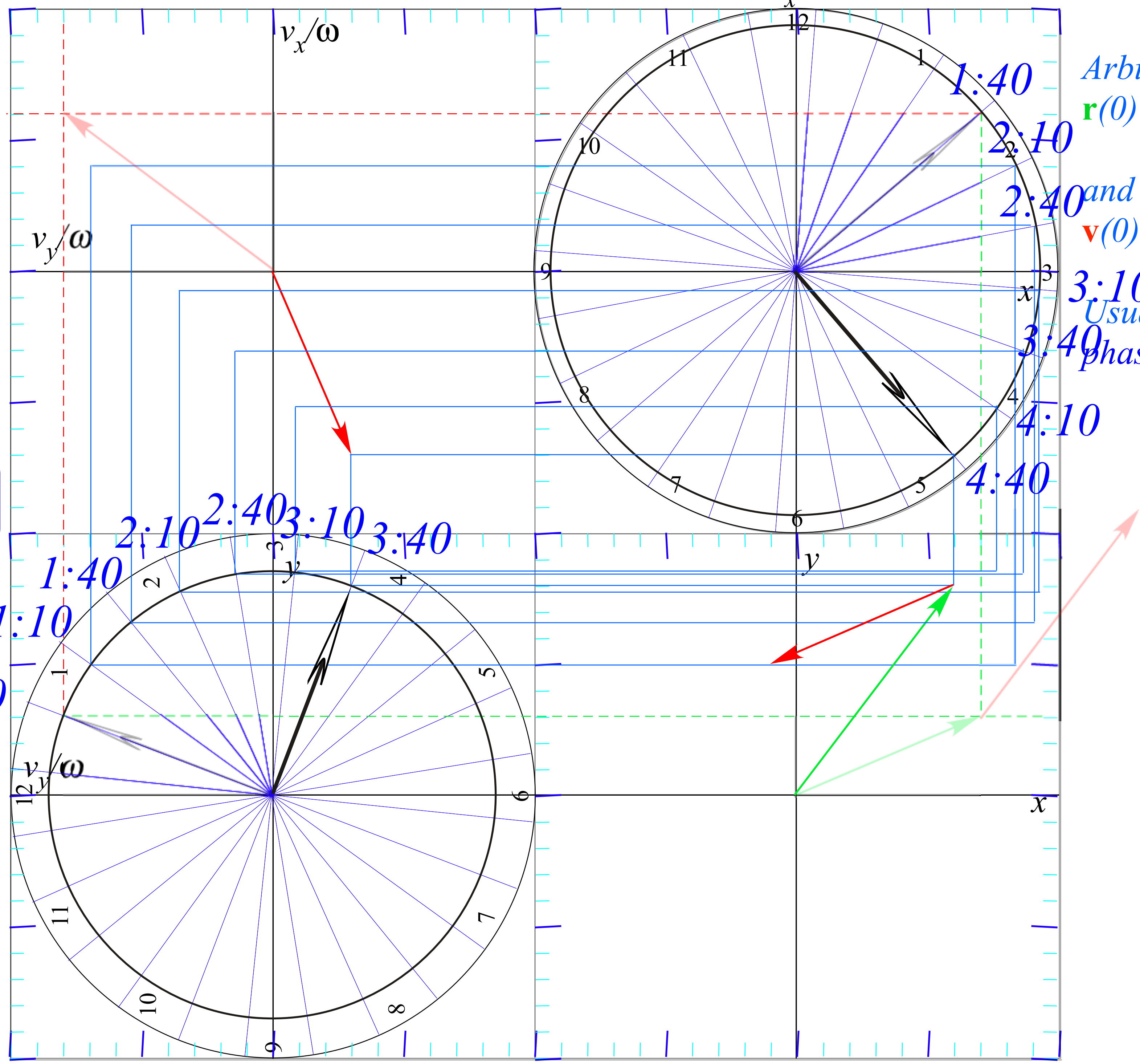
or

12 hour orbit (1 hour for hour hand)



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Review of IHO orbital phasor "clock" dynamics in uniform-body

phase lag:
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1 minute orbit *(5 seconds for second hand)*
or
1 hour orbit *(5 minutes for minute hand)*
or
12 hour orbit *(1 hour for hour hand)*

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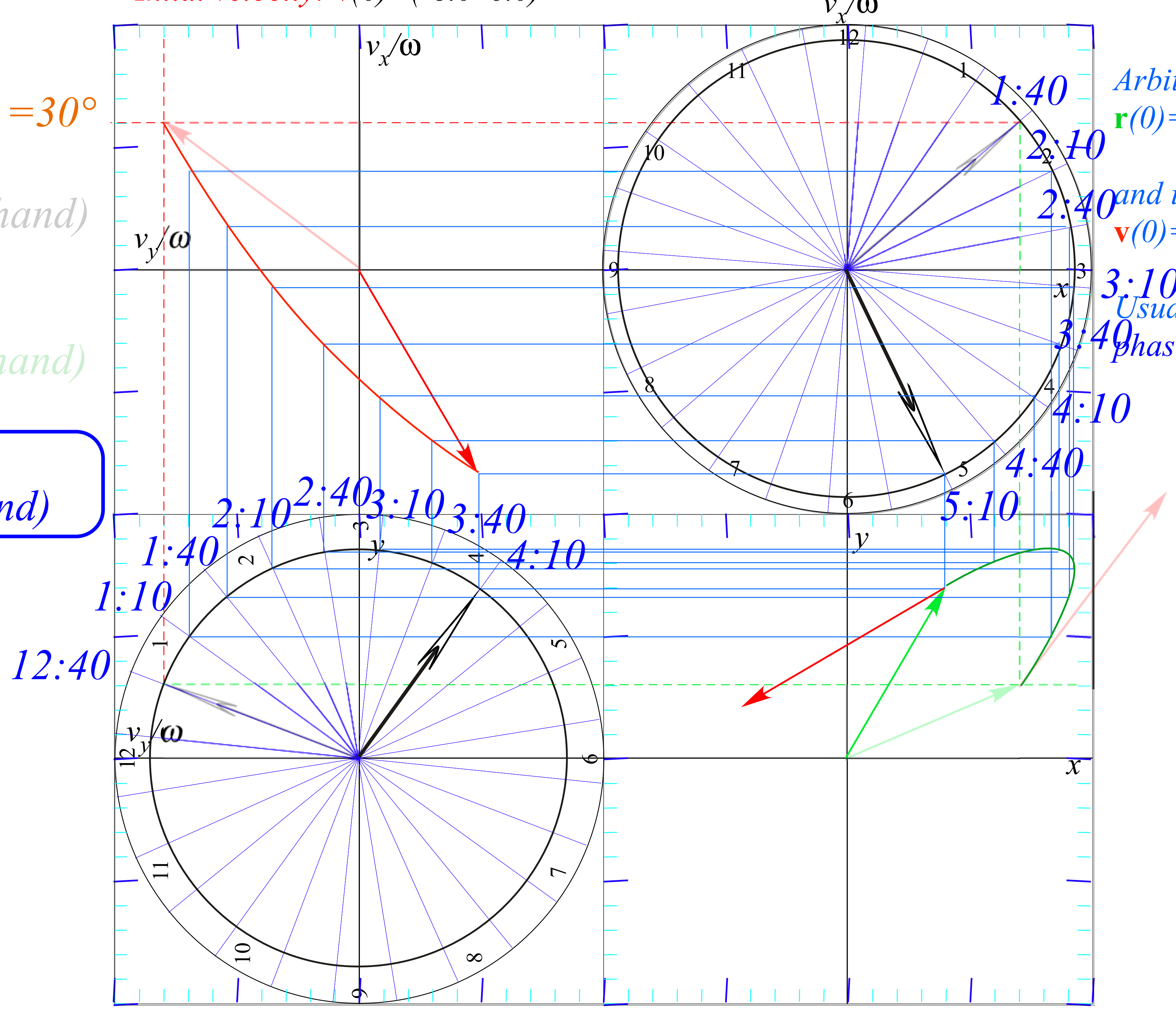

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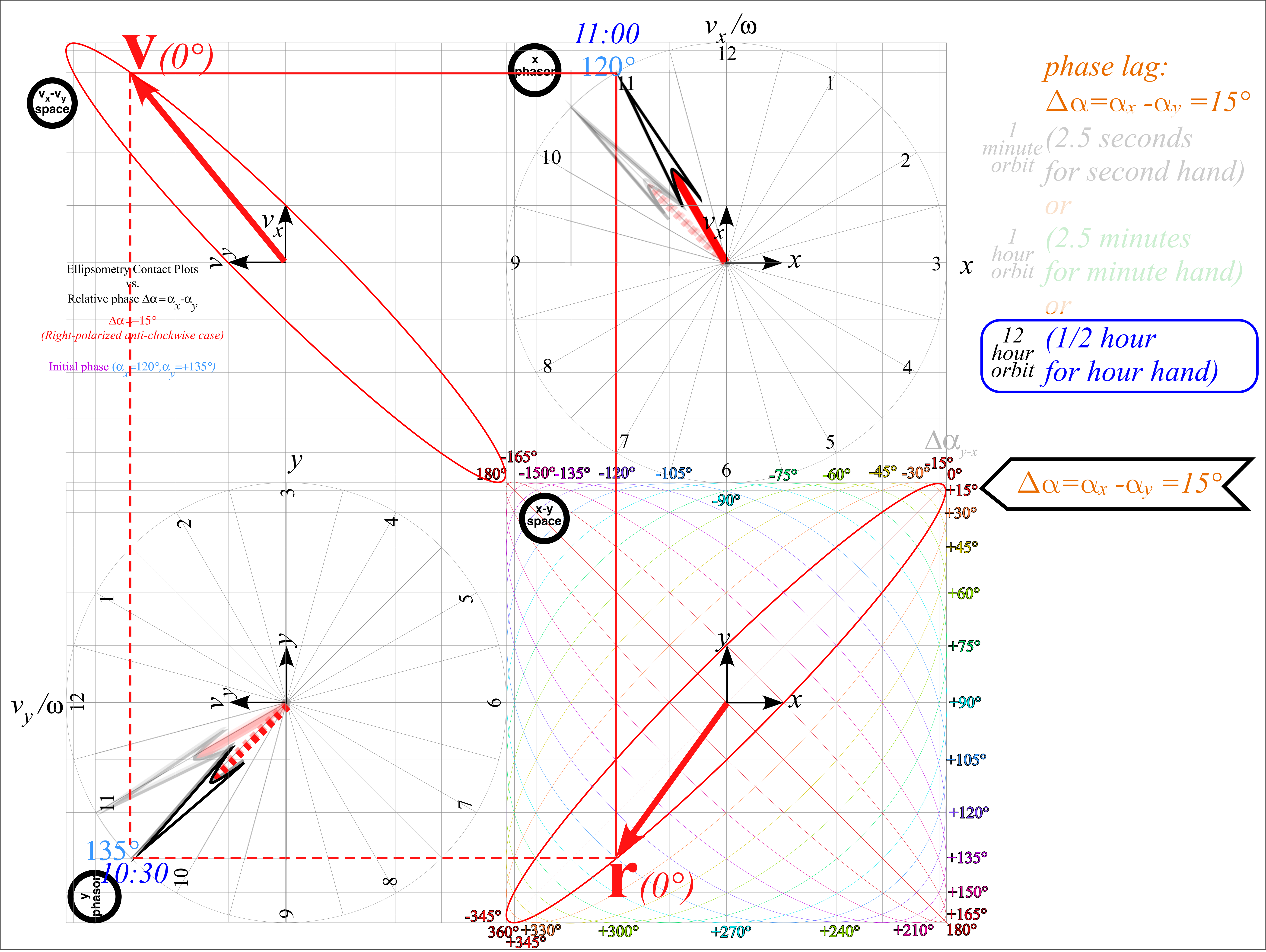
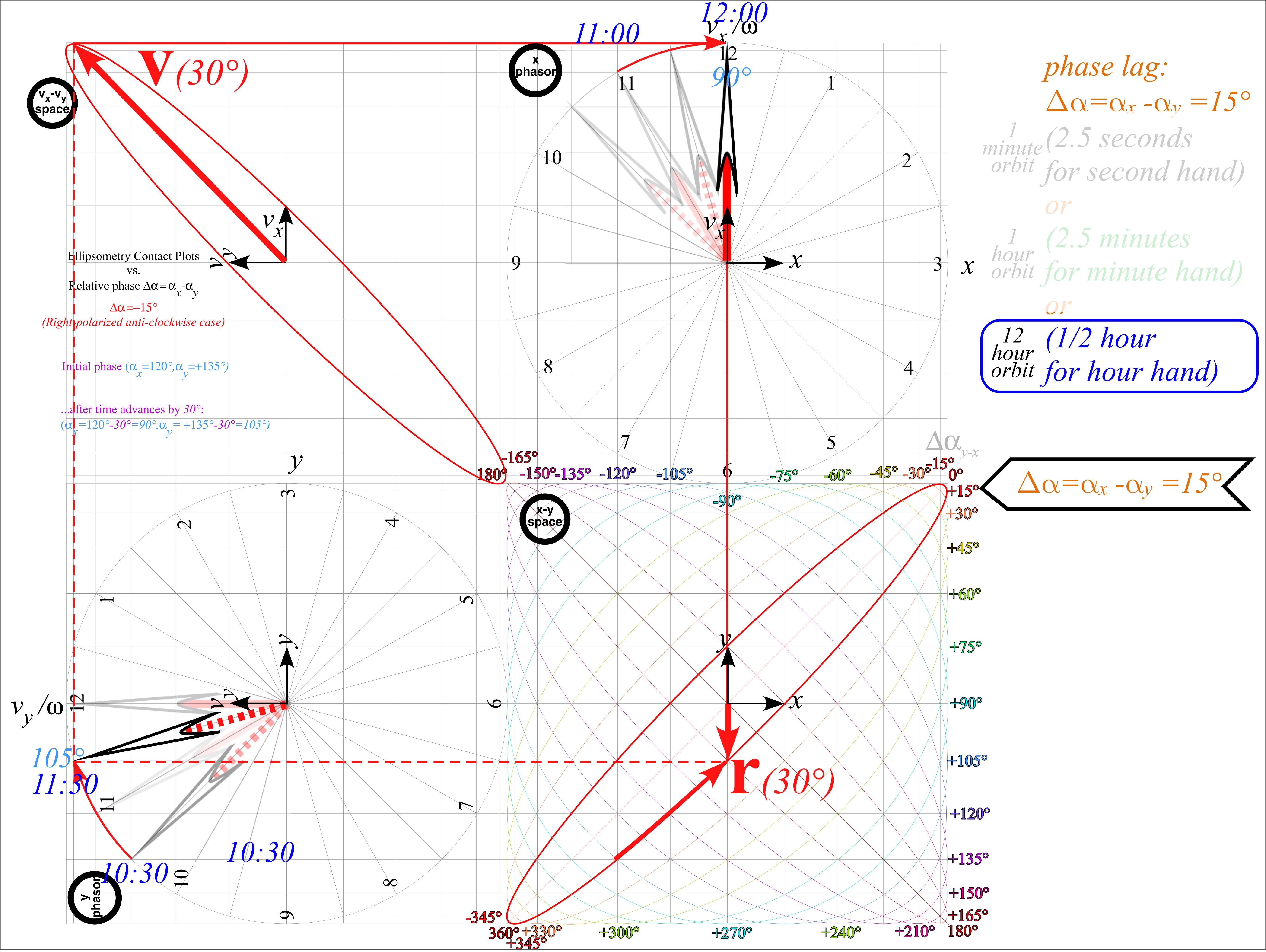
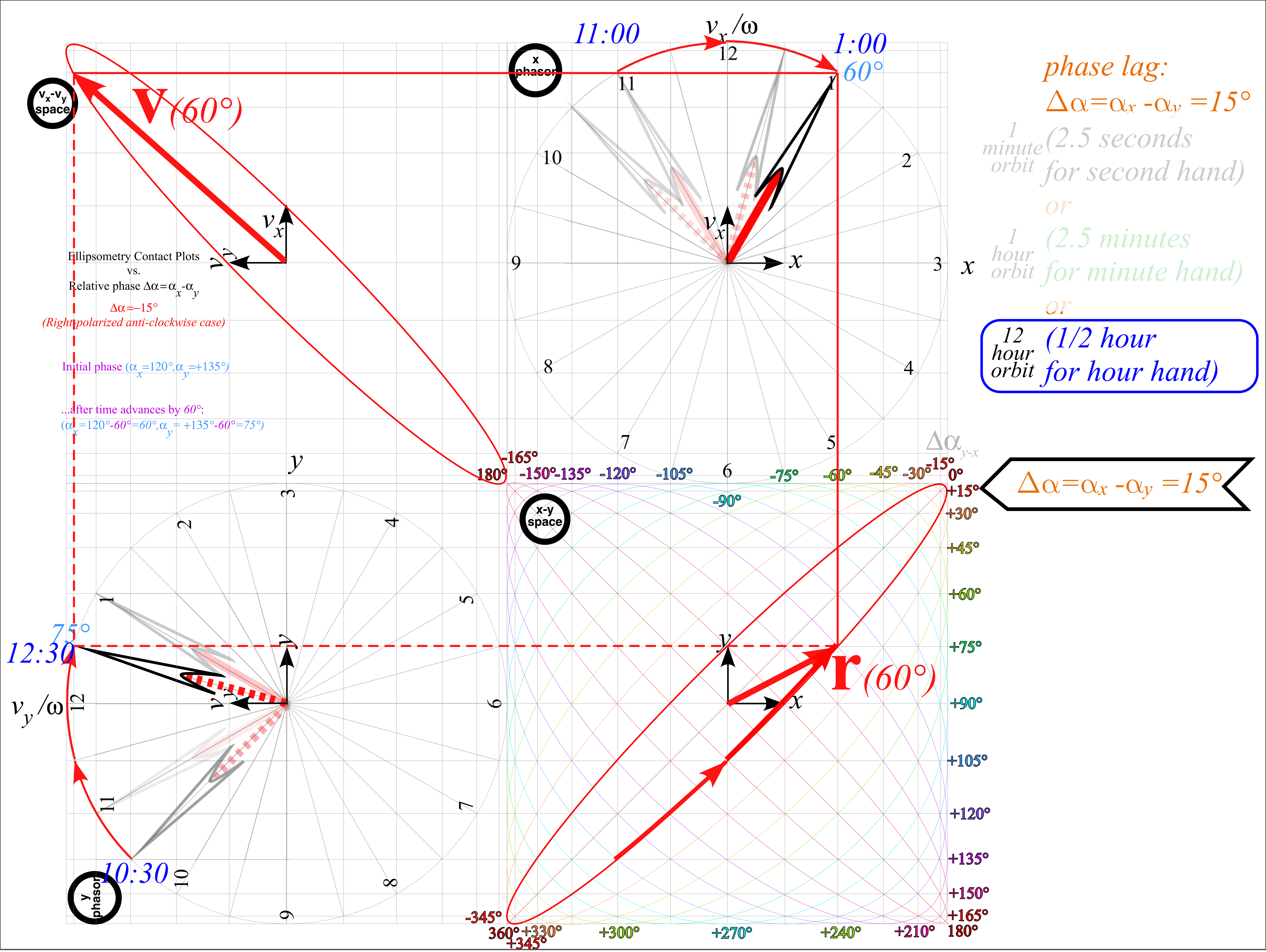
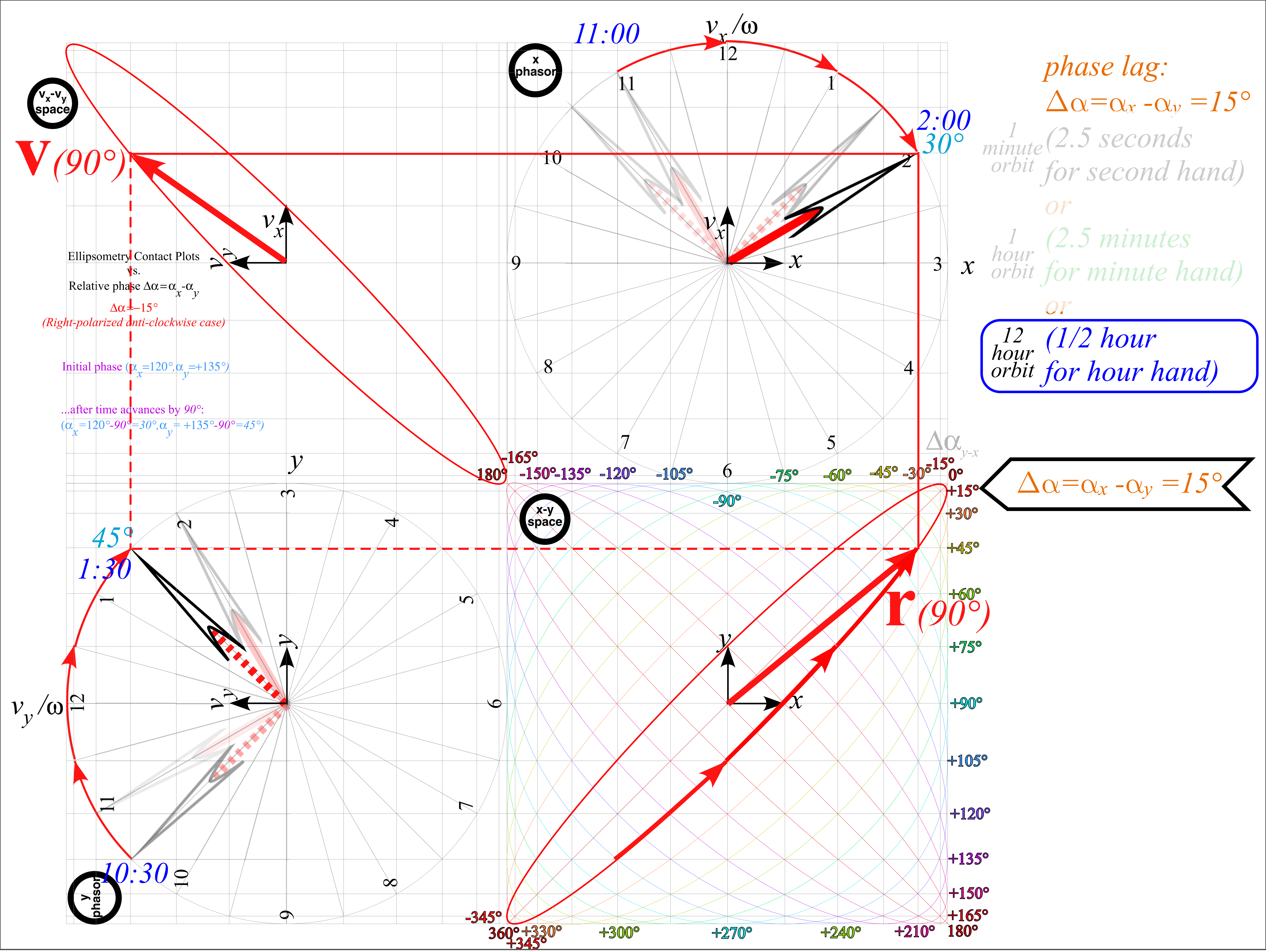
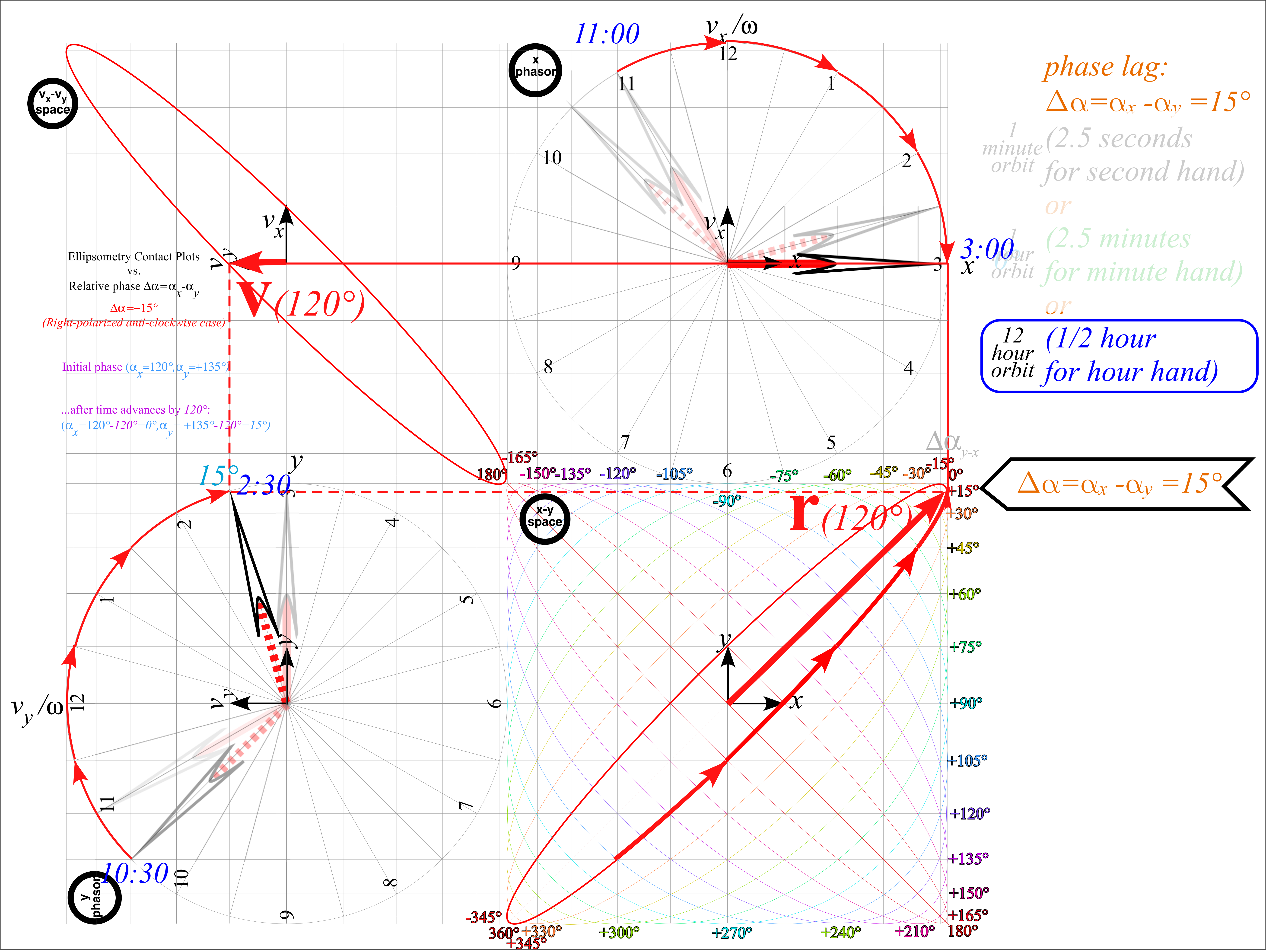

V(0°)
v_x-v_y space
x phasor
11:00
120°
v_x/ω
phase lag:
$\Delta\alpha=\alpha_x-\alpha_y=15°$
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Ellipsometry Contact Plots vs. Relative phase $\Delta\alpha=\alpha_x-\alpha_y$
$\Delta\alpha=-15°$
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
$\Delta\alpha_{y-x}$
$\Delta\alpha=\alpha_x-\alpha_y=15°$
x-y space
v_y/ω
135°
10:30
y phasor
r(0°)

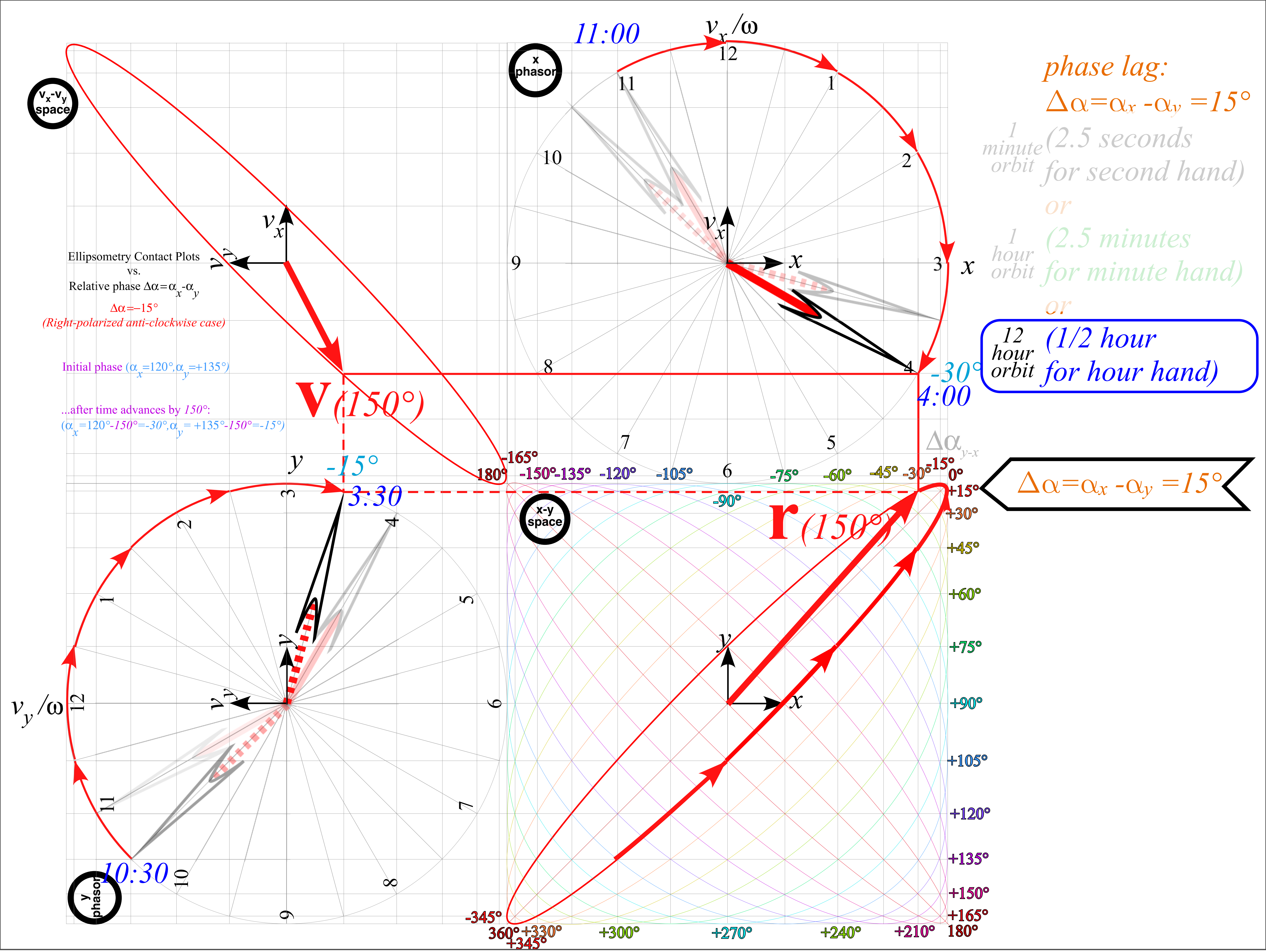

Ellipsometry Contact Plots
vs.
Relative phase Δα=α_x-α_y
Δα=−15°
(Right polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 30°:
(α_x=120°-30°=90°,α_y= +135°-30°=105°)
V(30°)
r(30°)
phase lag:
Δα=α_x -α_y =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Δα=α_x -α_y =15°
v_x-v_y space
x phasor
x-y space
y phasor
11:00
12:00
11:30
10:30
105°
90°


Ellipsometry Contact Plots
vs.
Relative phase Δα=αx-αy
Δα=−15°
(Right polarized anti-clockwise case)
Initial phase (αx=120°,αy=+135°)
...after time advances by 60°:
(αx=120°-60°=60°,αy= +135°-60°=75°)
vx-vy space
x phasor
y phasor
x-y space
V(60°)
r(60°)
11:00
1:00
60°
12:30
75°
10:30
vx/ω
vy/ω
phase lag:
Δα=αx -αy =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Δα=αx -αy =15°
Δαy-x
0° -15° -30° -45° -60° -75° -90° -105° -120° -135° -150° -165° 180°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°
+210° +240° +270° +300° +330° +345° 360° -345°

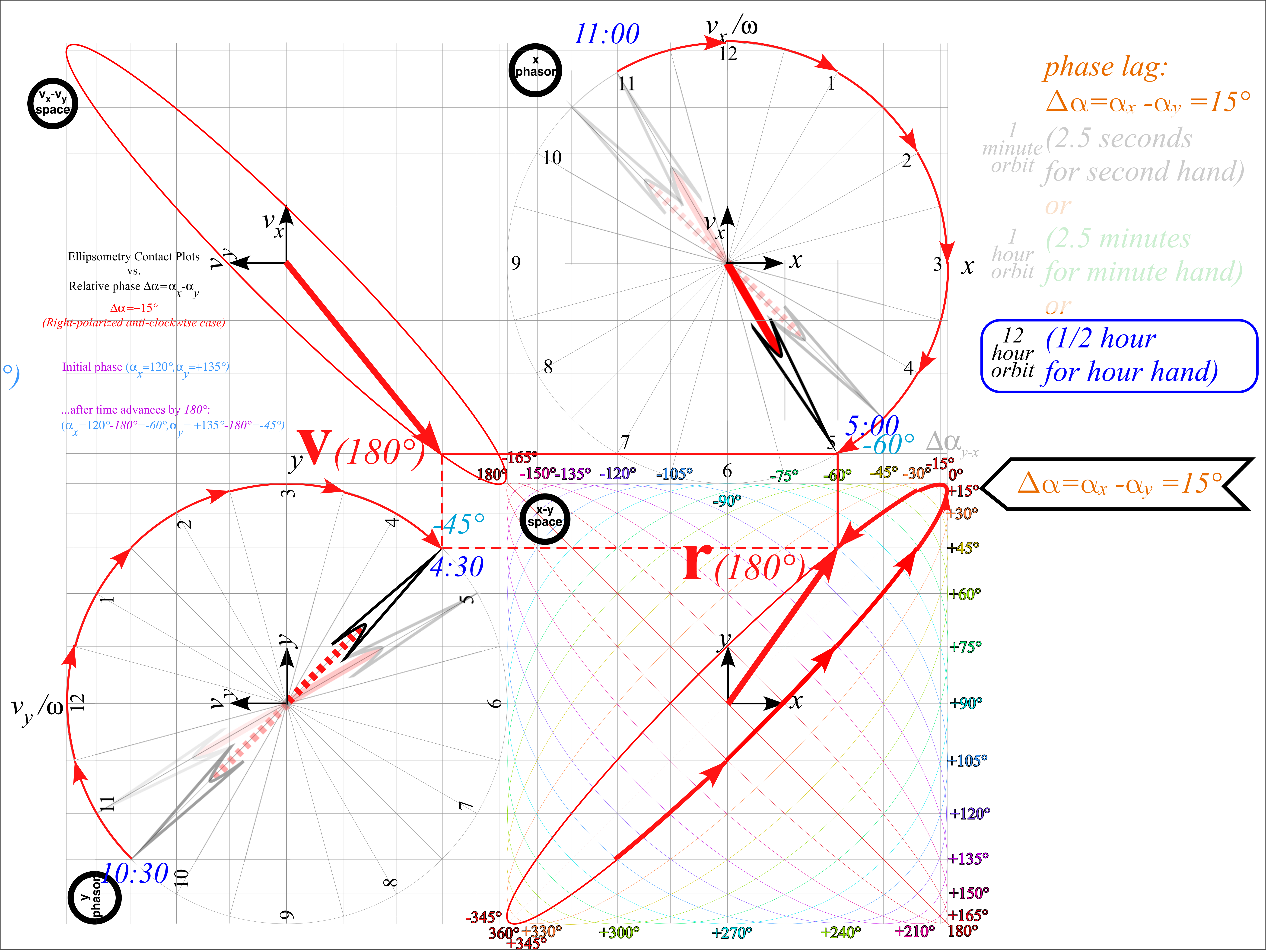

Ellipsometry Contact Plots
vs.
Relative phase Δα=α_x-α_y
Δα=−15°
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 90°:
(α_x=120°-90°=30°,α_y=+135°-90°=45°)
v_x-v_y space
x phasor
y phasor
x-y space
V(90°)
r(90°)
11:00
2:00
30°
45°
1:30
10:30
phase lag:
Δα=α_x -α_y =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Δα=α_x -α_y =15°
Δα_y-x

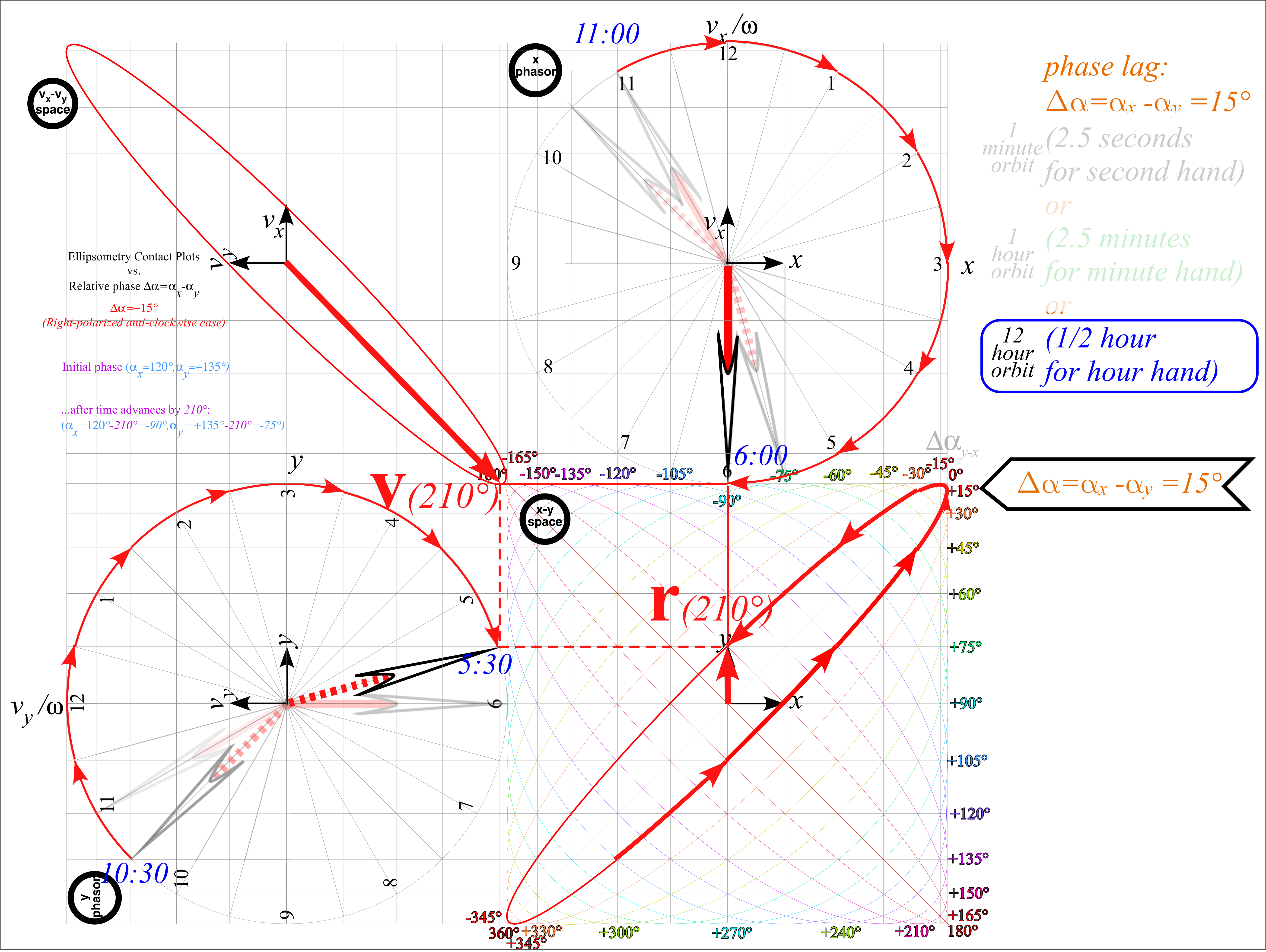


Ellipsometry Contact Plots
vs.
Relative phase Δα=α_x-α_y
Δα=−15°
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 120°:
(α_x=120°-120°=0°,α_y=+135°-120°=15°)
phase lag:
Δα=α_x -α_y =15°
1 minute orbit
(2.5 seconds for second hand)
or
1 hour orbit
(2.5 minutes for minute hand)
or
12 hour orbit
(1/2 hour for hour hand)
Δα=α_x -α_y =15°
v_x-v_y space
x phasor
y phasor
x-y space
v_x/ω
v_y/ω
11:00
3:00
2:30
10:30
15°
V(120°)
r(120°)
Δα_y-x

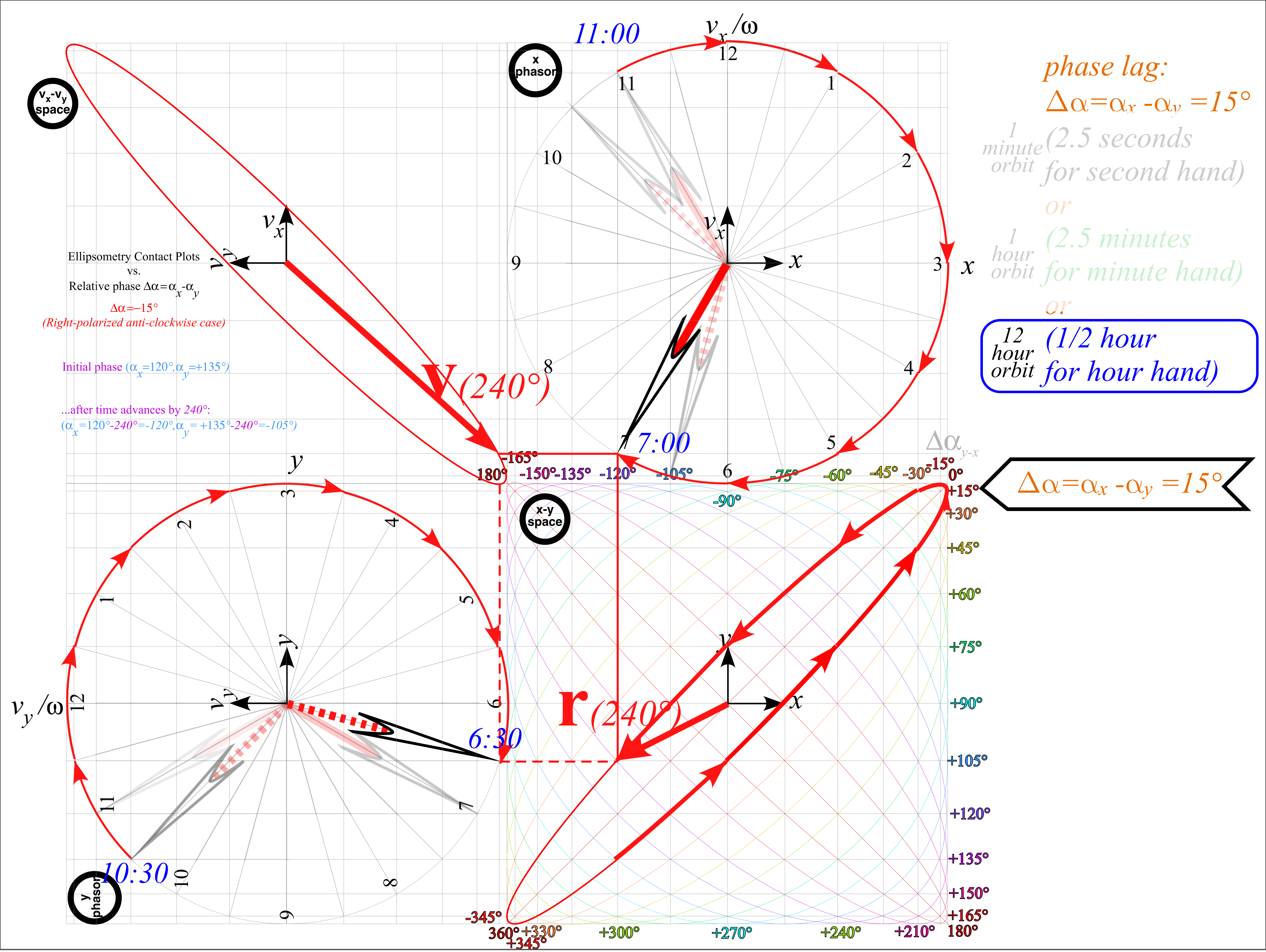


Ellipsometry Contact Plots
vs.
Relative phase Δα=αx-αy
Δα=−15°
(Right-polarized anti-clockwise case)
Initial phase (αx=120°,αy=+135°)
...after time advances by 150°:
(αx=120°-150°=-30°,αy= +135°-150°=-15°)
vx-vy space
x phasor
y phasor
x-y space
11:00
4:00
3:30
10:30
V(150°)
r(150°)
-30°
-15°
phase lag:
Δα=αx -αy =15°
1 minute orbit
(2.5 seconds for second hand)
or
1 hour orbit
(2.5 minutes for minute hand)
or
12 hour orbit
(1/2 hour for hour hand)
Δα=αx -αy =15°

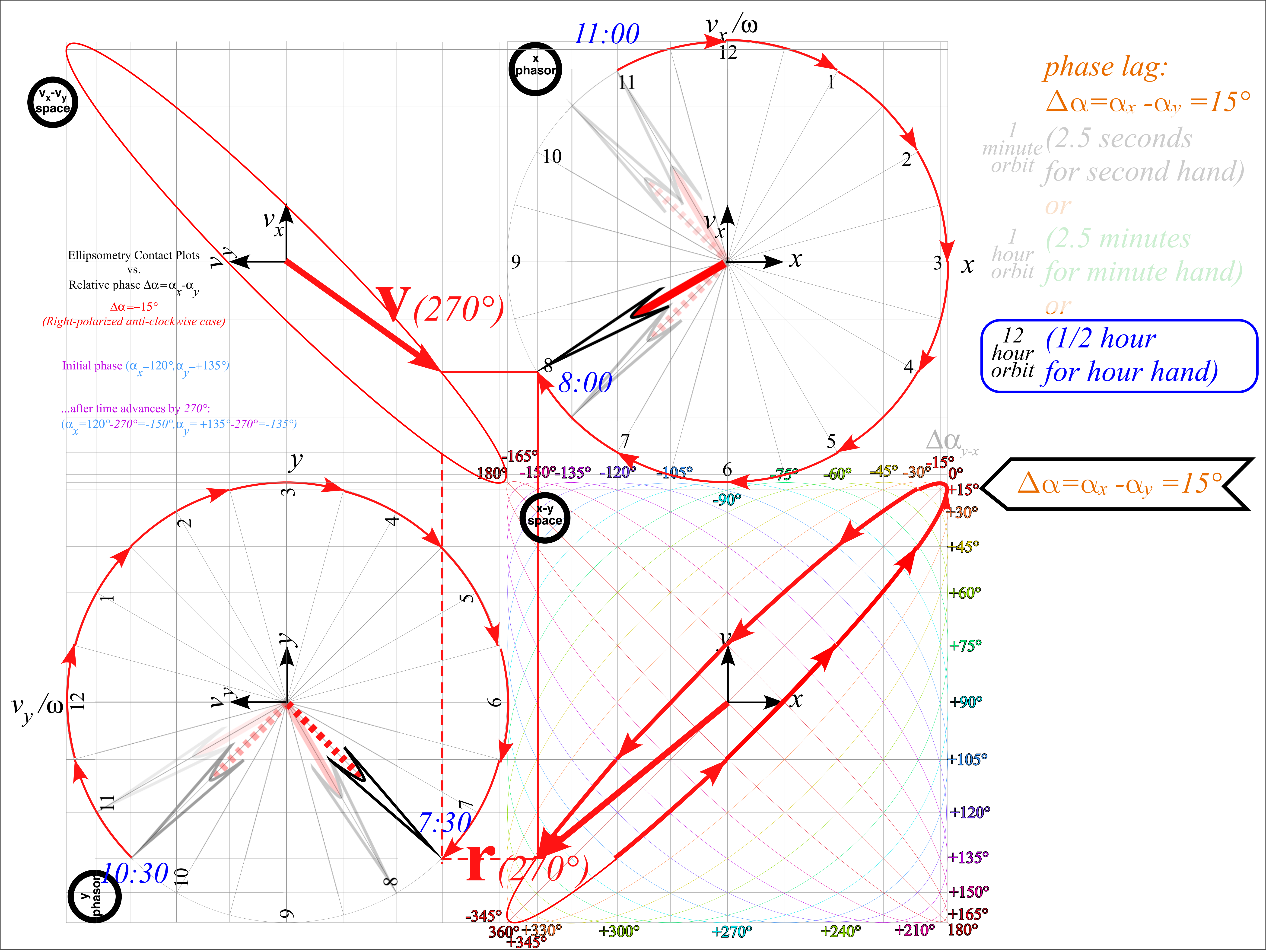

Ellipsometry Contact Plots
vs.
Relative phase Δα=αx-αy
Δα=−15°
(Right-polarized anti-clockwise case)
Initial phase (αx=120°,αy=+135°)
...after time advances by 180°:
(αx=120°-180°=-60°,αy = +135°-180°=-45°)
vx-vy space
x phasor
y phasor
x-y space
11:00
5:00
4:30
10:30
vx/ω
vy/ω
V(180°)
r(180°)
-60°
-45°
Δαy-x
phase lag:
Δα=αx -αy =15°
1 minute orbit
(2.5 seconds for second hand)
or
1 hour orbit
(2.5 minutes for minute hand)
or
12 hour orbit
(1/2 hour for hour hand)
Δα=αx -αy =15°
180° -165° -150° -135° -120° -105° -90° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°
-345° 360° +345° +330° +300° +270° +240° +210°



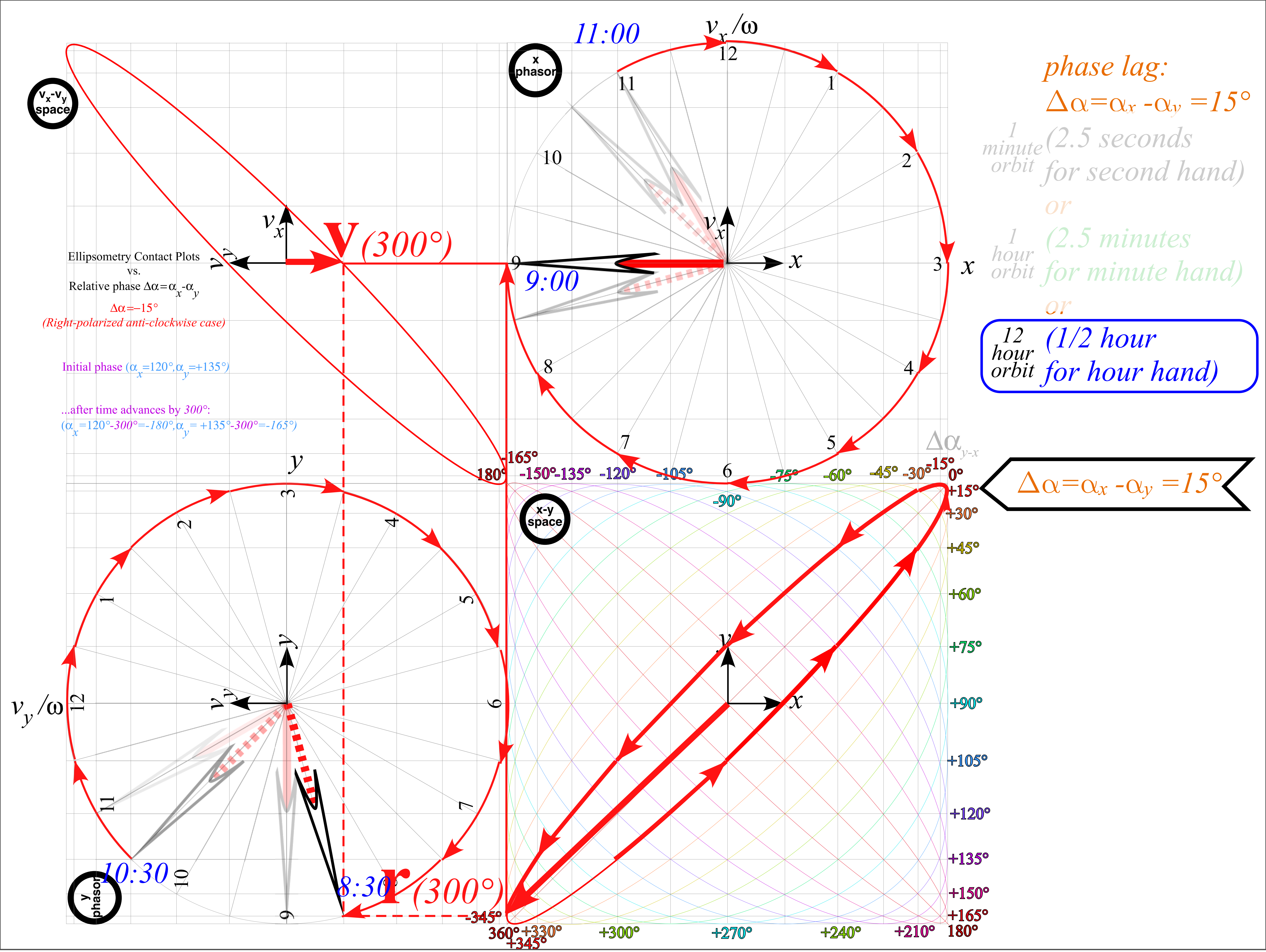
Ellipsometry Contact Plots
vs.
Relative phase Δα=α_x-α_y
Δα=-15°
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 210°:
(α_x=120°-210°=-90°,α_y=+135°-210°=-75°)
phase lag:
Δα=α_x -α_y =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Δα=α_x -α_y =15°
v_x-v_y space
x phasor
y phasor
x-y space
11:00
6:00
5:30
10:30
V(210°)
r(210°)



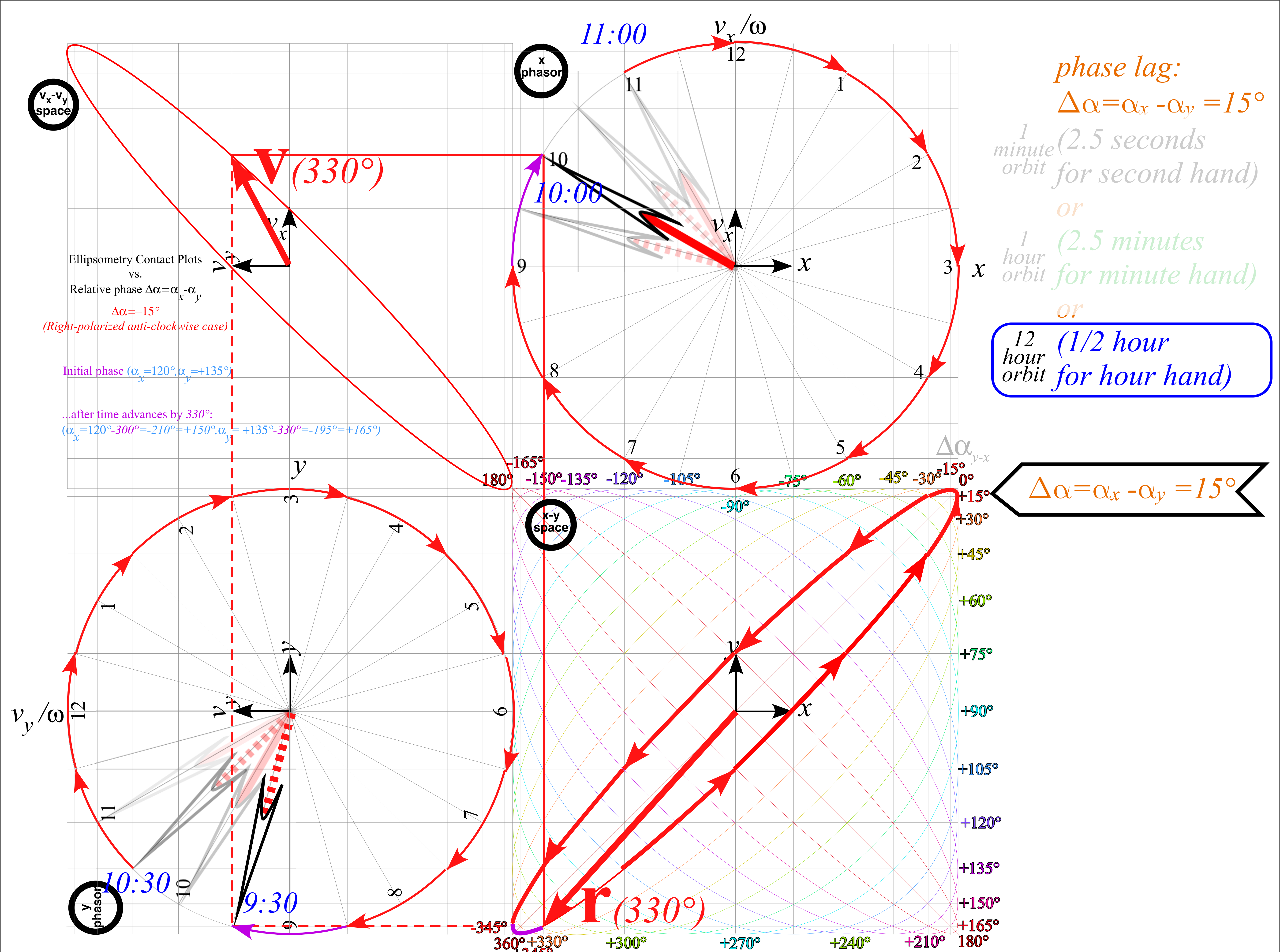
Ellipsometry Contact Plots vs. Relative phase Δα=α_x-α_y
Δα=-15°
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 240°:
(α_x=120°-240°=-120°,α_y=+135°-240°=-105°)
phase lag:
Δα=α_x -α_y =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Δα=α_x -α_y =15°
V(240°)
r(240°)
11:00
7:00
6:30
10:30
v_x-v_y space
x phasor
y phasor
x-y space



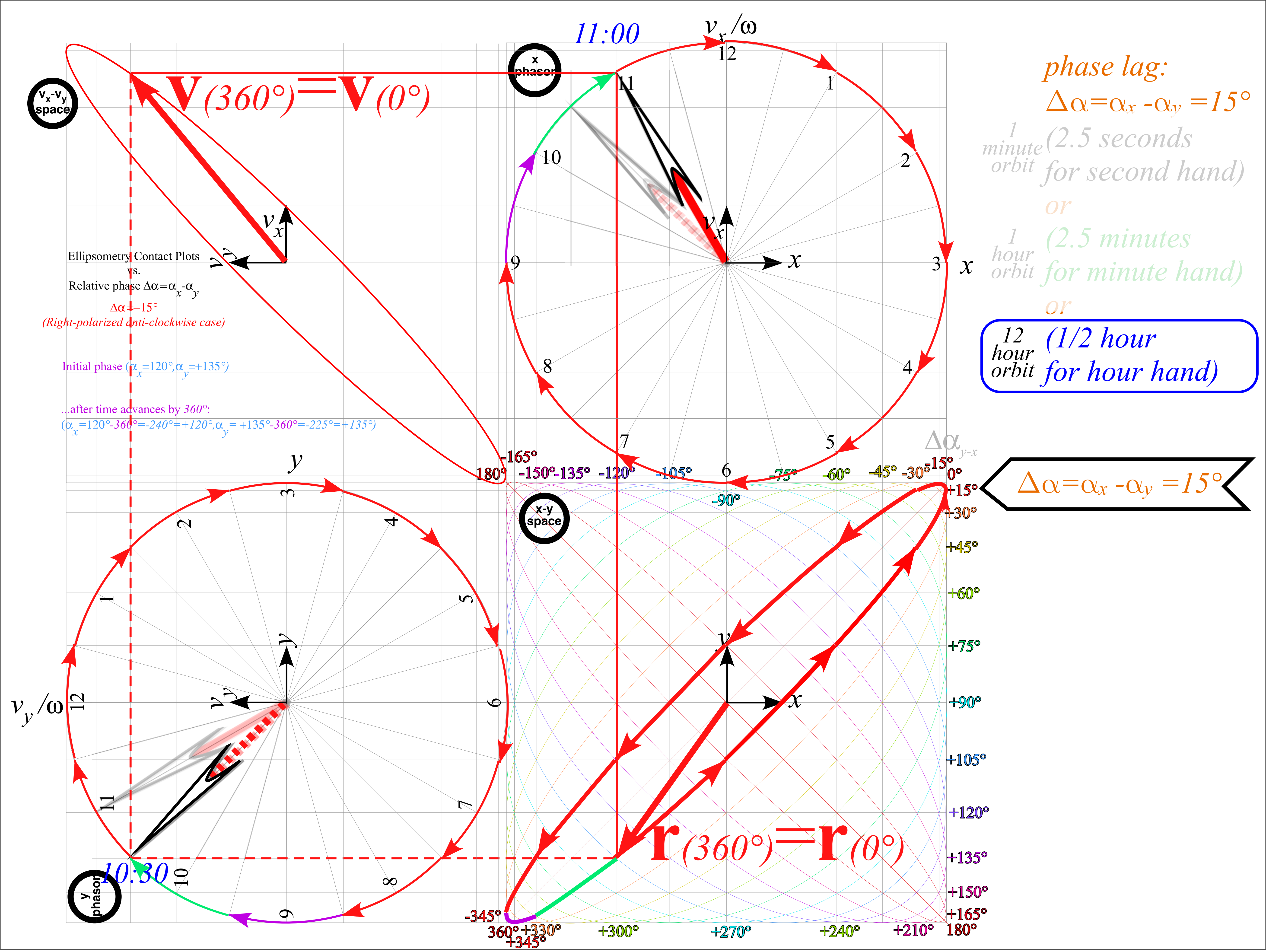
Ellipsometry Contact Plots
vs.
Relative phase Δα=αx-αy
Δα=-15°
(Right-polarized anti-clockwise case)
Initial phase (αx=120°,αy=+135°)
...after time advances by 270°:
(αx=120°-270°=-150°,αy = +135°-270°=-135°)
vx-vy space
x phasor
y phasor
x-y space
11:00
8:00
7:30
10:30
V(270°)
r(270°)
phase lag:
Δα=αx -αy =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Δα=αx -αy =15°



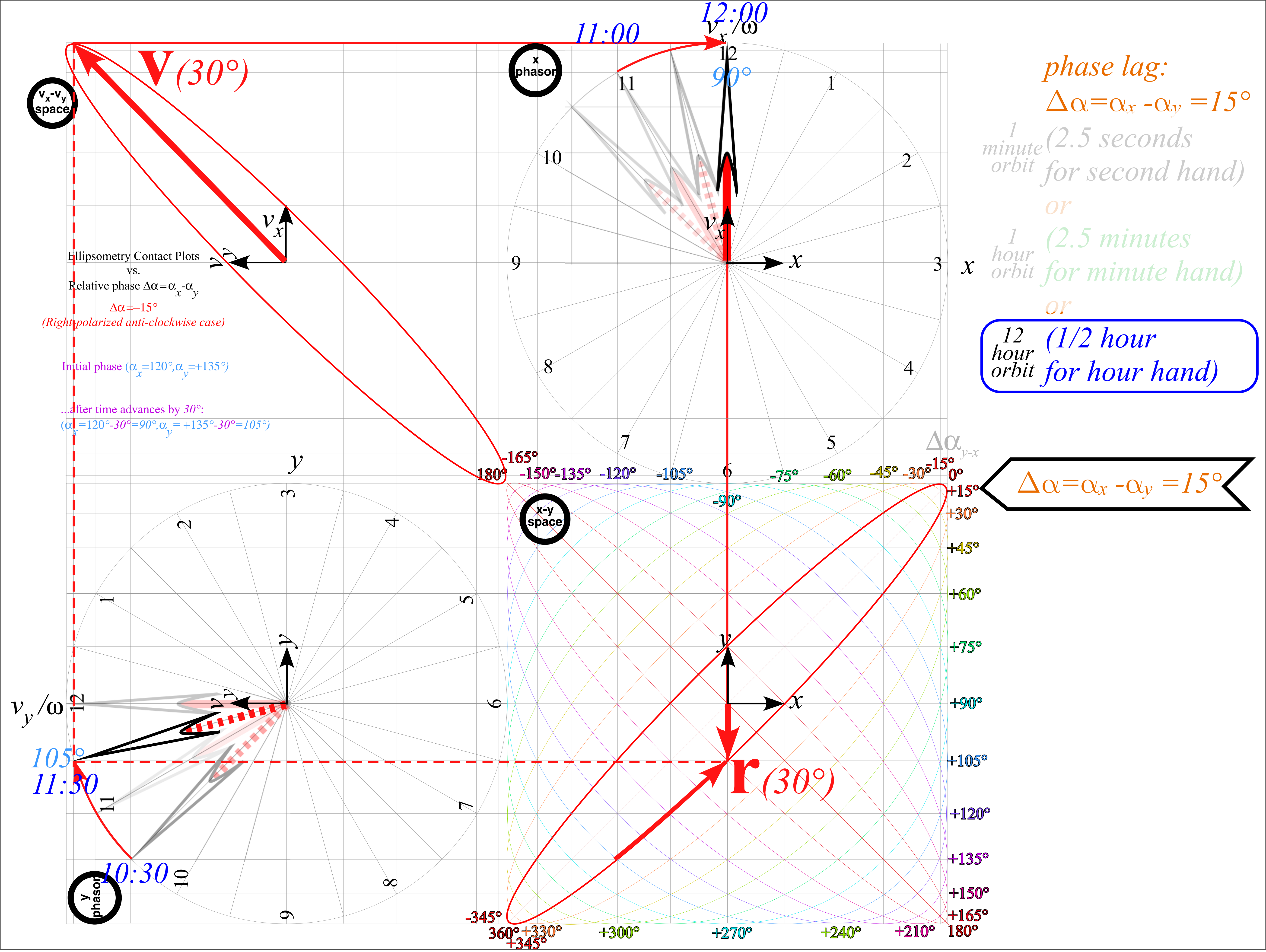
Ellipsometry Contact Plots
vs.
Relative phase Δα=αx-αy
Δα=-15°
(Right-polarized anti-clockwise case)
Initial phase (αx=120°,αy=+135°)
...after time advances by 300°:
(αx=120°-300°=-180°,αy = +135°-300°=-165°)
phase lag:
Δα=αx -αy =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Δα=αx -αy =15°
V(300°)
r(300°)
11:00
9:00
10:30
8:30
vx-vy space
x phasor
y phasor
x-y space

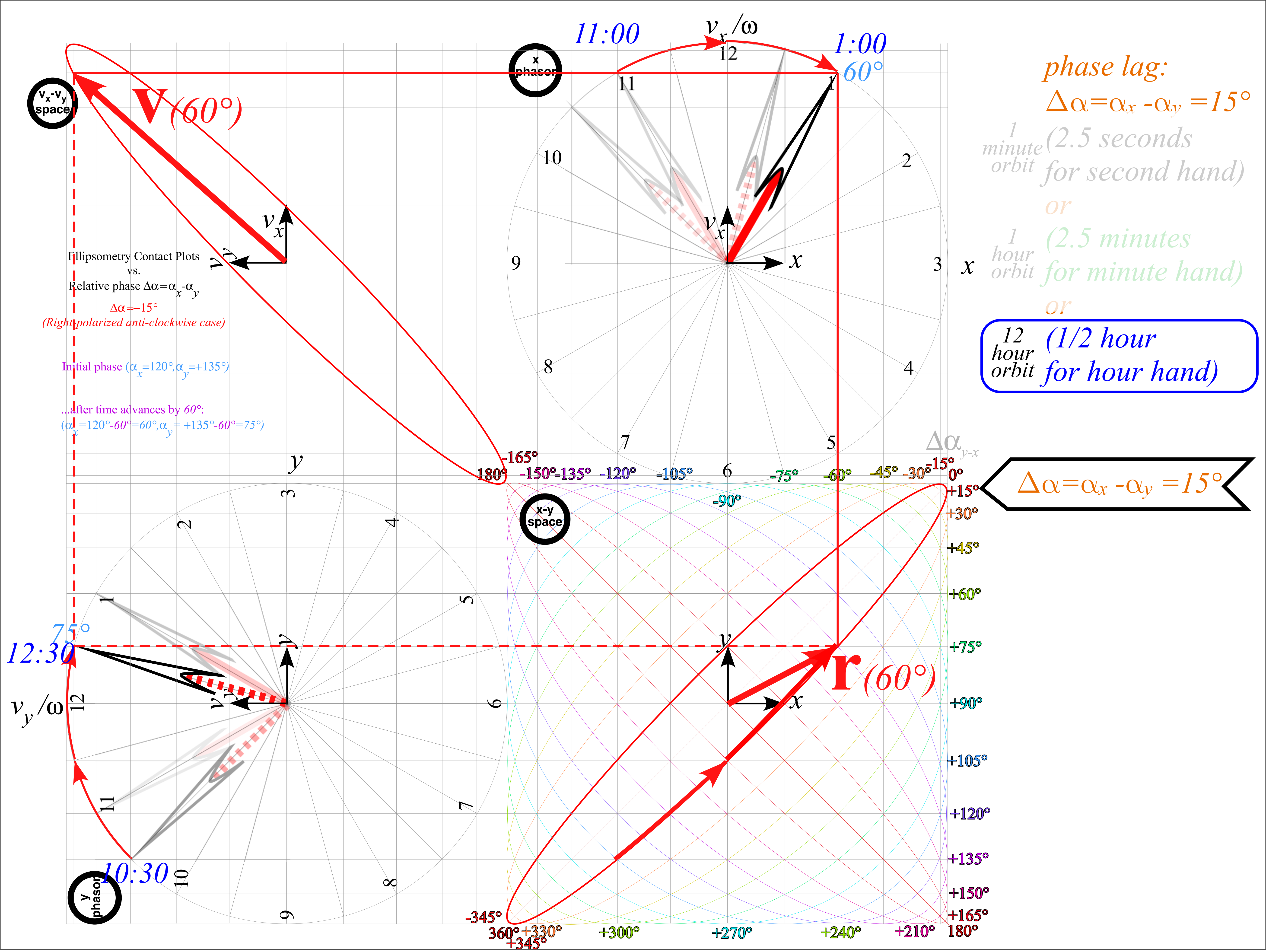


Ellipsometry Contact Plots
vs.
Relative phase $\Delta\alpha=\alpha_x-\alpha_y$
$\Delta\alpha=-15°$
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 330°:
(α_x=120°-300°=-210°=+150°,α_y= +135°-330°=-195°=+165°)
v_x-v_y space
x phasor
y phasor
x-y space
V (330°)
r (330°)
11:00
10:00
10:30
9:30
phase lag:
$\Delta\alpha=\alpha_x-\alpha_y=15°$
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
$\Delta\alpha=\alpha_x-\alpha_y=15°$
$\Delta\alpha_{y-x}$

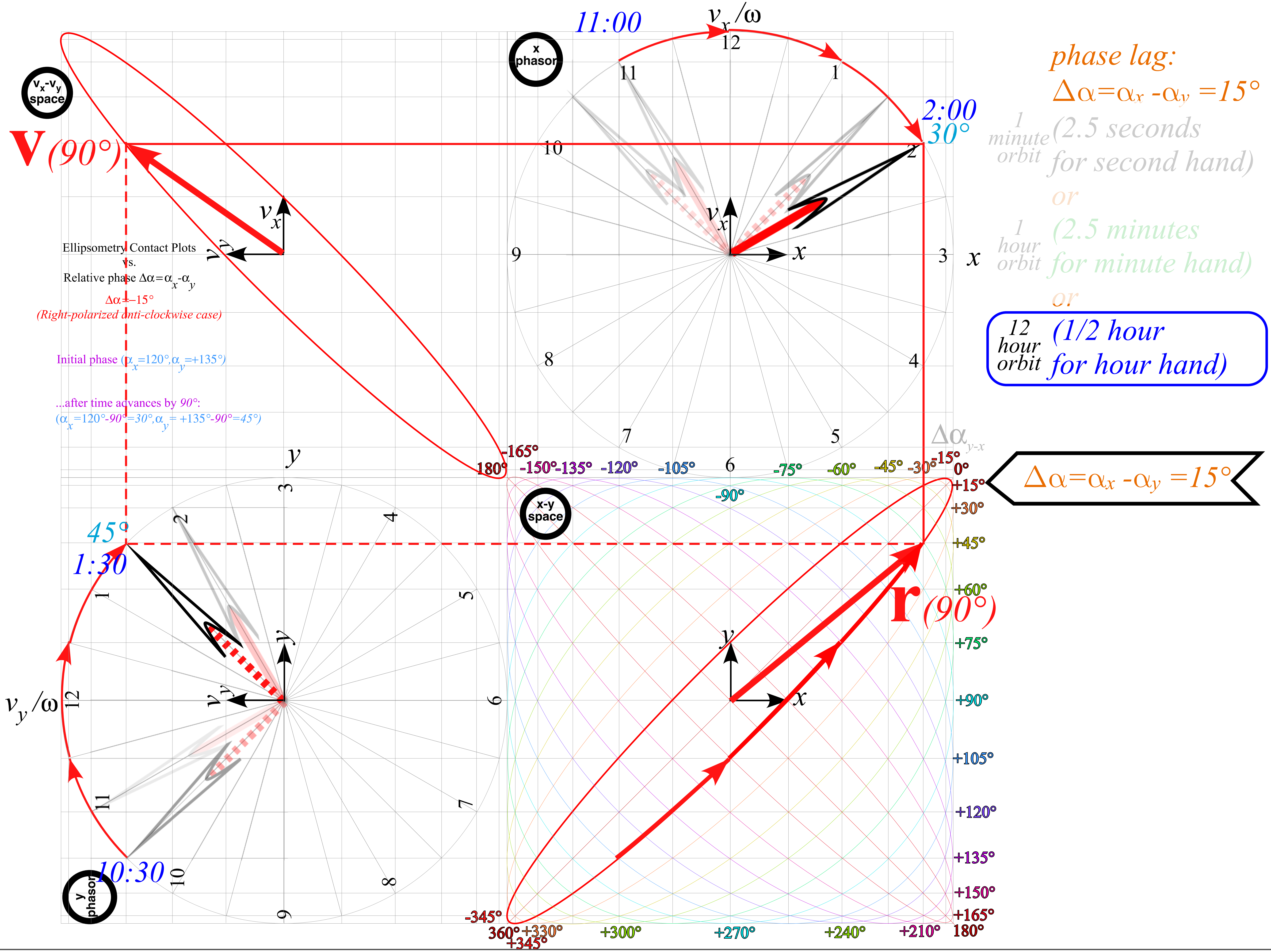


11:00
v_x/ω
x phasor
v_x-v_y space
V(360°)=V(0°)
phase lag:
Δα=αx -αy =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Ellipsometry Contact Plots vs. Relative phase Δα=αx-αy
Δα=-15°
(Right-polarized anti-clockwise case)
Initial phase (αx=120°,αy=+135°)
...after time advances by 360°:
(αx=120°-360°=-240°=+120°,αy=+135°-360°=-225°=+135°)
Δα=αx -αy =15°
x-y space
v_y/ω
y phasor
10:30
r(360°)=r(0°)


V(30°)
11:00
12:00
v_x/ω
90°
x phasor
v_x-v_y space
phase lag:
$\Delta\alpha=\alpha_x-\alpha_y=15°$
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Ellipsometry Contact Plots vs. Relative phase $\Delta\alpha=\alpha_x-\alpha_y$
$\Delta\alpha=-15°$
(Right-polarized anti-clockwise case)
Initial phase ($\alpha_x=120°$, $\alpha_y=+135°$)
...after time advances by 30°:
($\alpha_x=120°-30°=90°$, $\alpha_y=+135°-30°=105°$)
$\Delta\alpha_{y-x}$
$\Delta\alpha=\alpha_x-\alpha_y=15°$
x-y space
r(30°)
v_y/ω
105°
11:30
10:30
y phasor



Ellipsometry Contact Plots
vs.
Relative phase Δα=α_x-α_y
Δα=−15°
(Right polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 60°:
(α_x=120°-60°=60°,α_y=+135°-60°=75°)
phase lag:
Δα=α_x -α_y =15°
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Δα=α_x -α_y =15°
V(60°)
r(60°)
v_x-v_y space
x phasor
y phasor
x-y space
11:00
1:00
60°
12:30
75°
10:30
v_x/ω
v_y/ω
Δα_y-x



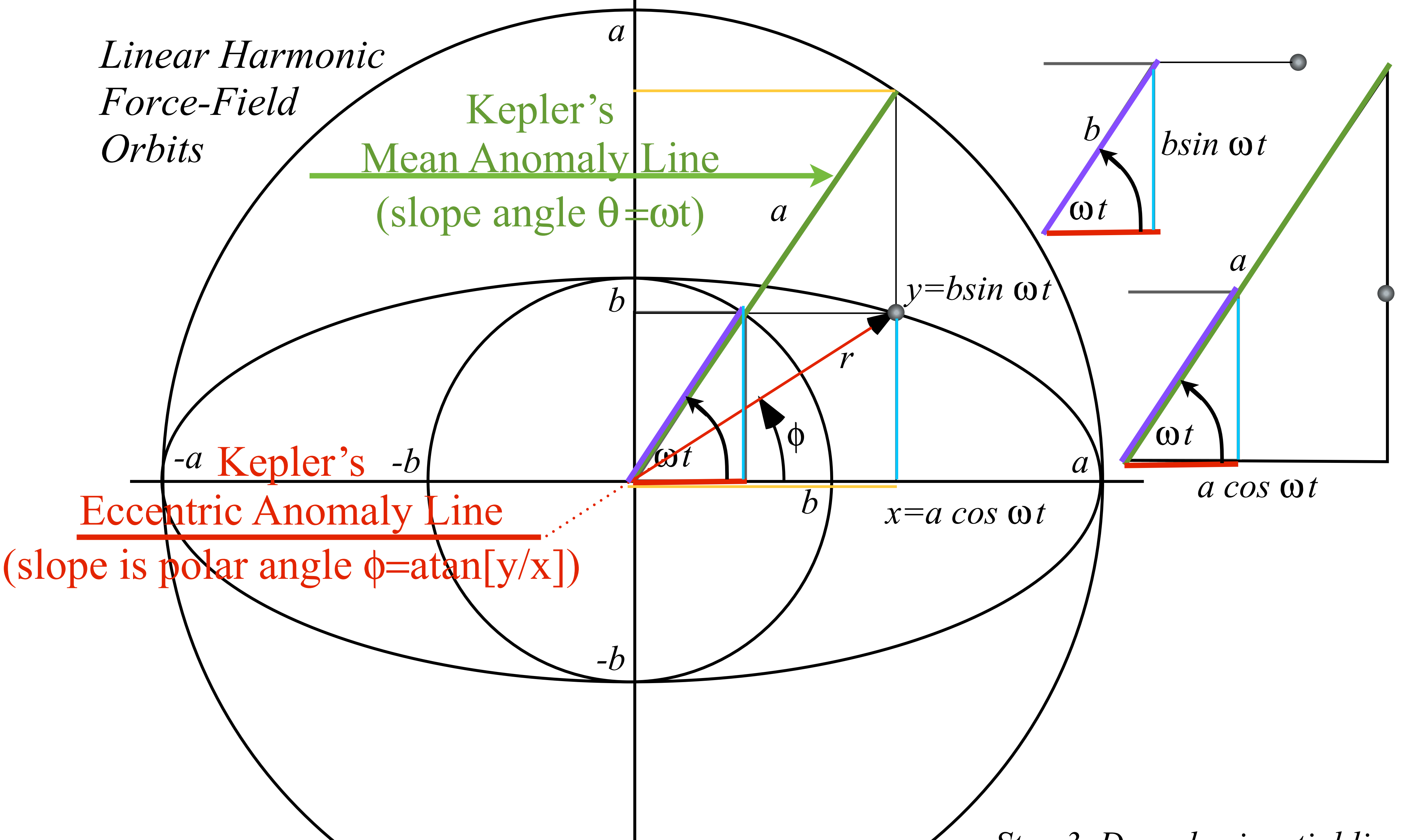
11:00
v_x/ω
x phasor
v_x-v_y space
V(90°)
2:00
30°
phase lag:
$\Delta\alpha=\alpha_x-\alpha_y=15°$
1 minute orbit (2.5 seconds for second hand)
or
1 hour orbit (2.5 minutes for minute hand)
or
12 hour orbit (1/2 hour for hour hand)
Ellipsometry Contact Plots vs. Relative phase $\Delta\alpha=\alpha_x-\alpha_y$
$\Delta\alpha=-15°$
(Right-polarized anti-clockwise case)
Initial phase (α_x=120°,α_y=+135°)
...after time advances by 90°:
(α_x=120°-90°=30°,α_y=+135°-90°=45°)
$\Delta\alpha_{y-x}$
$\Delta\alpha=\alpha_x-\alpha_y=15°$
x-y space
45°
1:30
r(90°)
v_y/ω
10:30
y phasor

Constructing 2D IHO orbits using Kepler anomaly plots

→ *Mean-anomaly and eccentric-anomaly geometry*

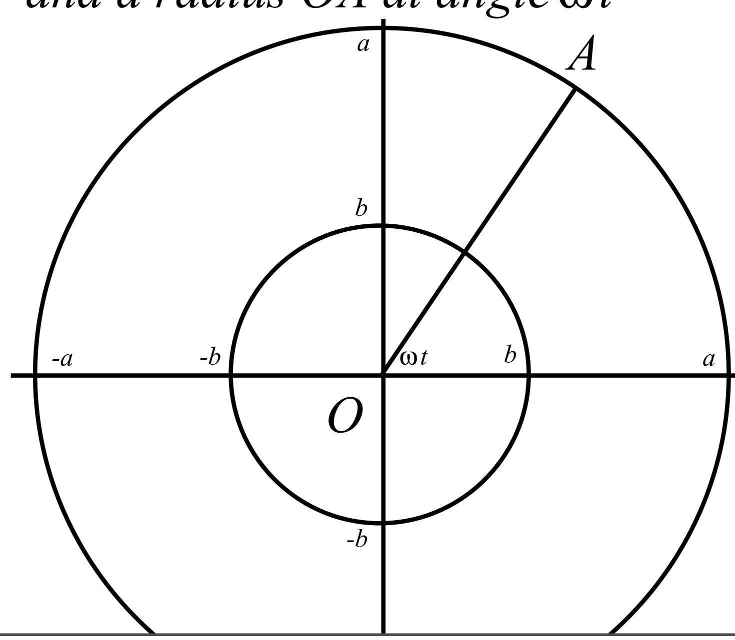
Calculus and vector geometry of IHO orbits

A confusing introduction to Coriolis-centrifugal force geometry *(Derived better in Ch. 12)*

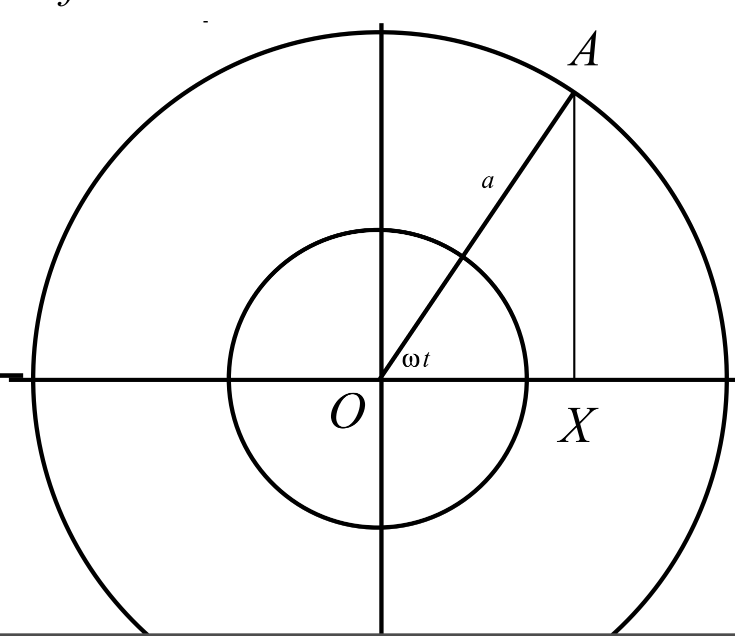


Unit 1
Fig. 11.1
(top 2/3's)

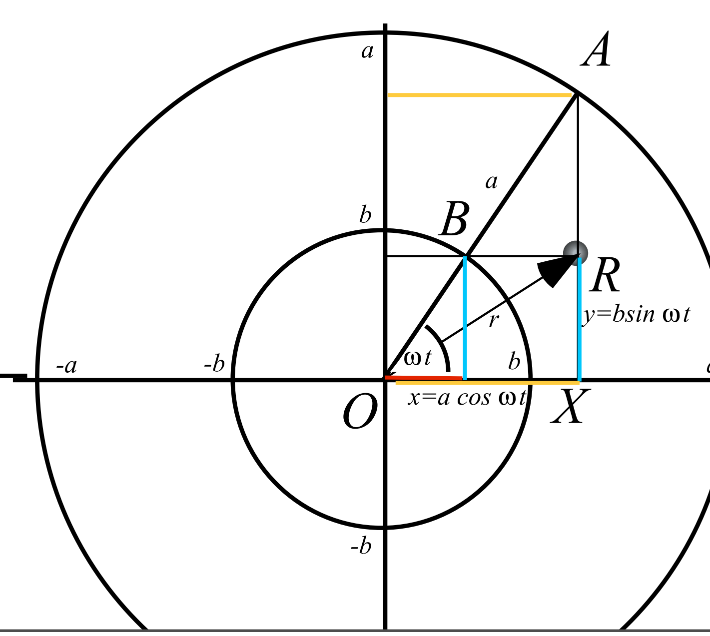
Step 1. Draw concentric circles of radius a and b and a radius OA at angle ωt

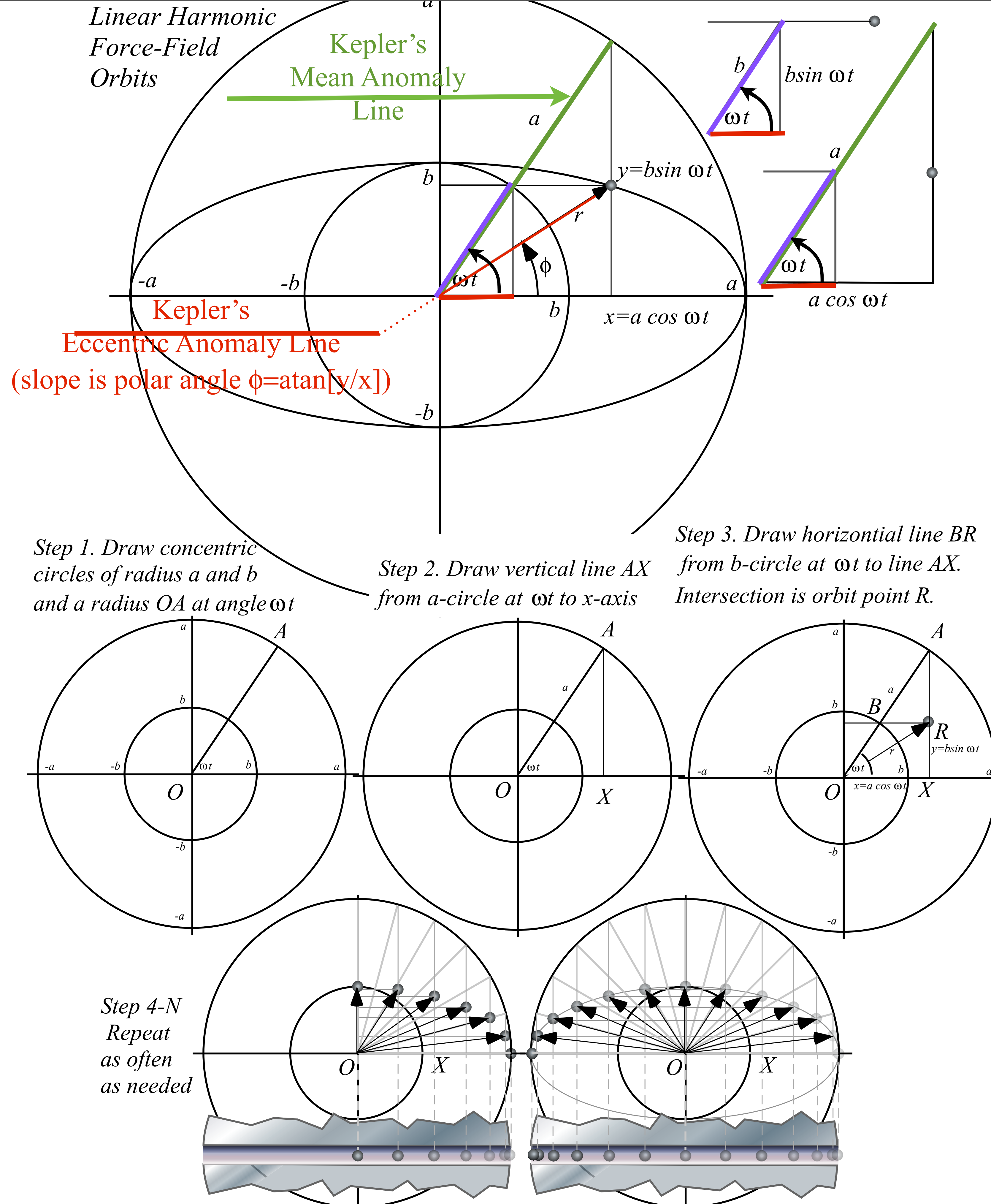


Step 2. Draw vertical line AX from a-circle at ωt to x-axis



Step 3. Draw horizontial line BR from b-circle at ωt to line AX. Intersection is orbit point R.





Unit 1
Fig. 11.1

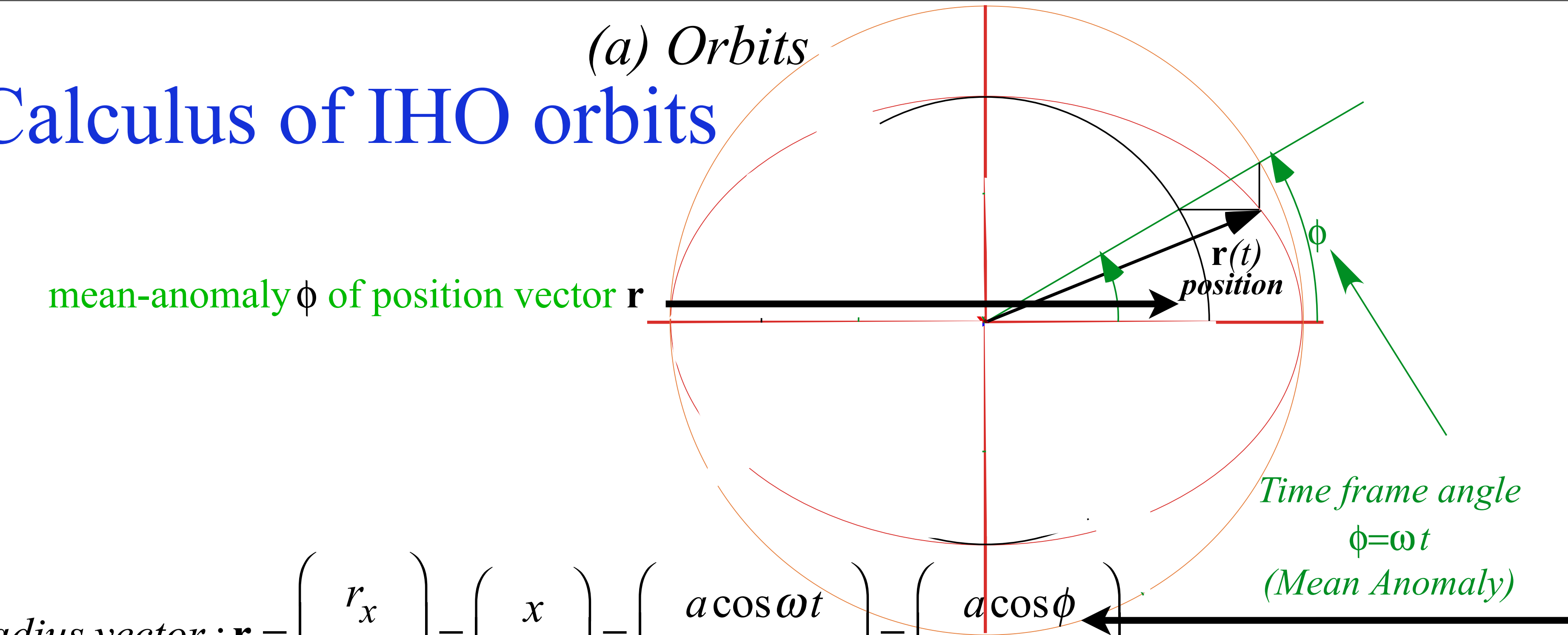
Constructing 2D IHO orbits using Kepler anomaly plots

Mean-anomaly and eccentric-anomaly geometry

→ *Calculus and vector geometry of IHO orbits*

A confusing introduction to Coriolis-centrifugal force geometry *(Derived better in Ch. 12)*

Calculus of IHO orbits

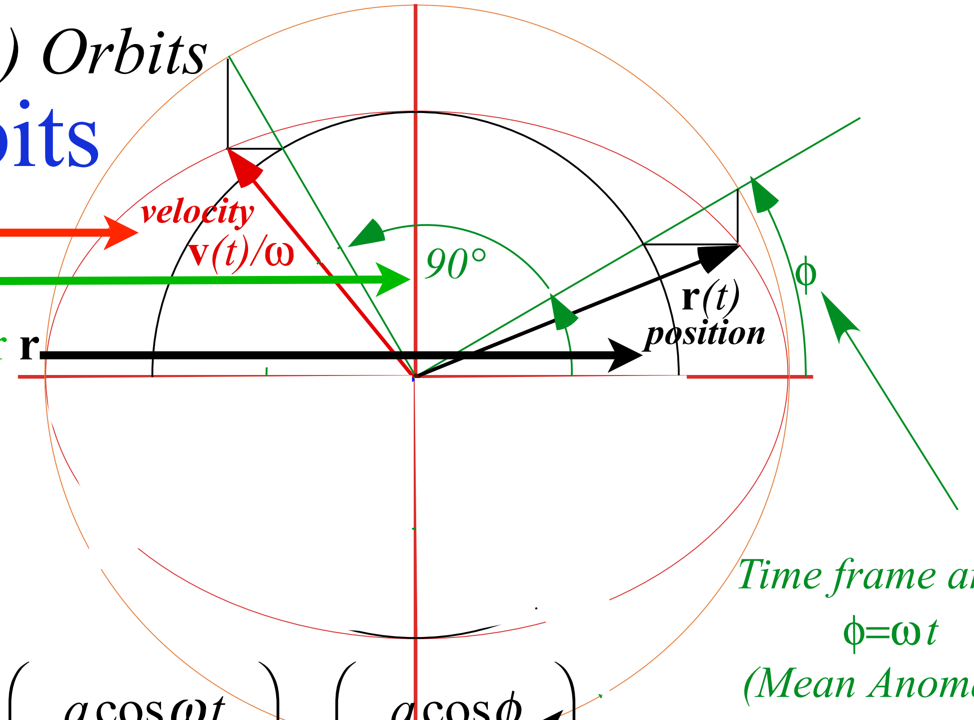


$$radius\ vector: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

Unit 1
Fig. 11.5

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly ϕ of position vector **r**



Unit 1
Fig. 11.5

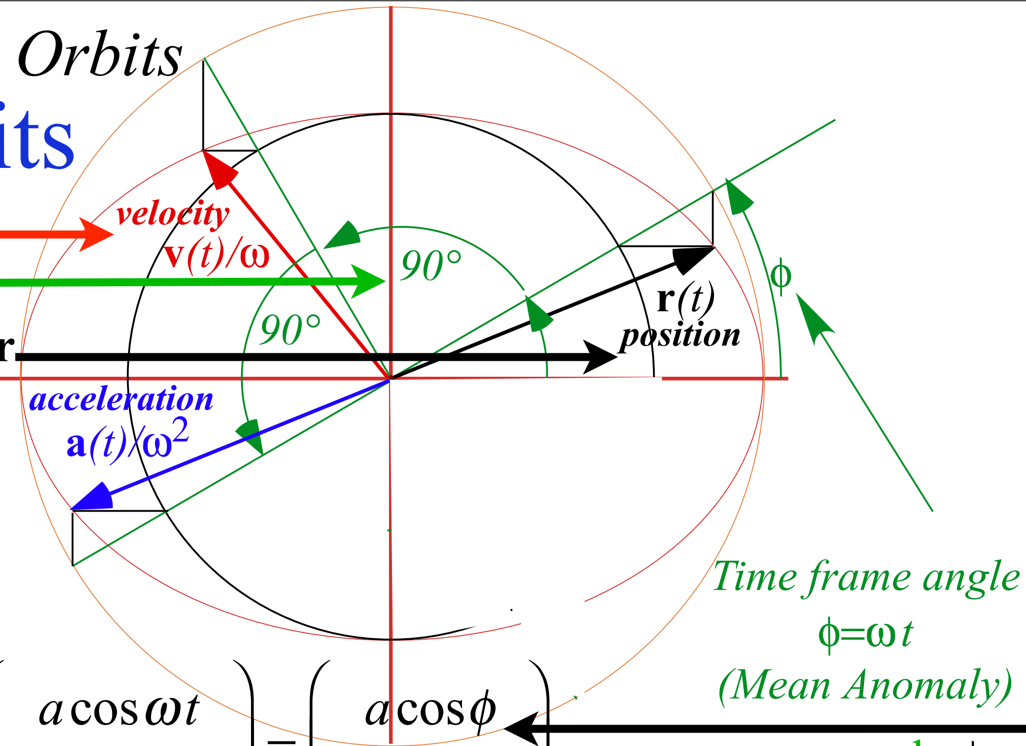
$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly ϕ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**



$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly φ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi + \frac{\pi}{2}\right) \\ b\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

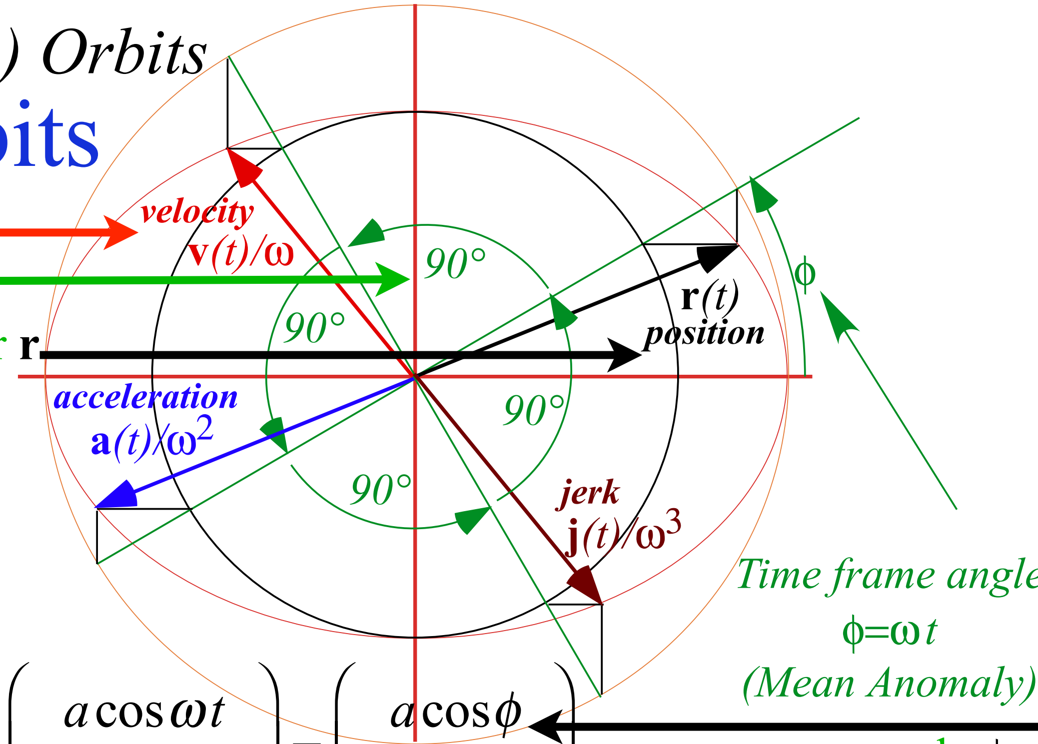
m.a. φ+π/2 of vector **v** rotated by
another π/2 is *m.a.* of vector **a**

$$\textit{acceleration or force vector}: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi + \frac{2\pi}{2}\right) \\ b\sin\left(\phi + \frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

Calculus of IHO orbits

To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly ϕ of position vector **r**



$$\textit{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly ϕ of position vector **r** rotated by π/2 or 90° is *m.a.* of vector **v**

$$\textit{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi+\frac{\pi}{2}\right) \\ b\sin\left(\phi+\frac{\pi}{2}\right) \end{pmatrix} (\textit{for } \omega = 1)$$

m.a. $\phi+\pi/2$ of vector **v** rotated by another π/2 is *m.a.* of vector **a**

$$\textit{acceleration or force vector}: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi+\frac{2\pi}{2}\right) \\ b\sin\left(\phi+\frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

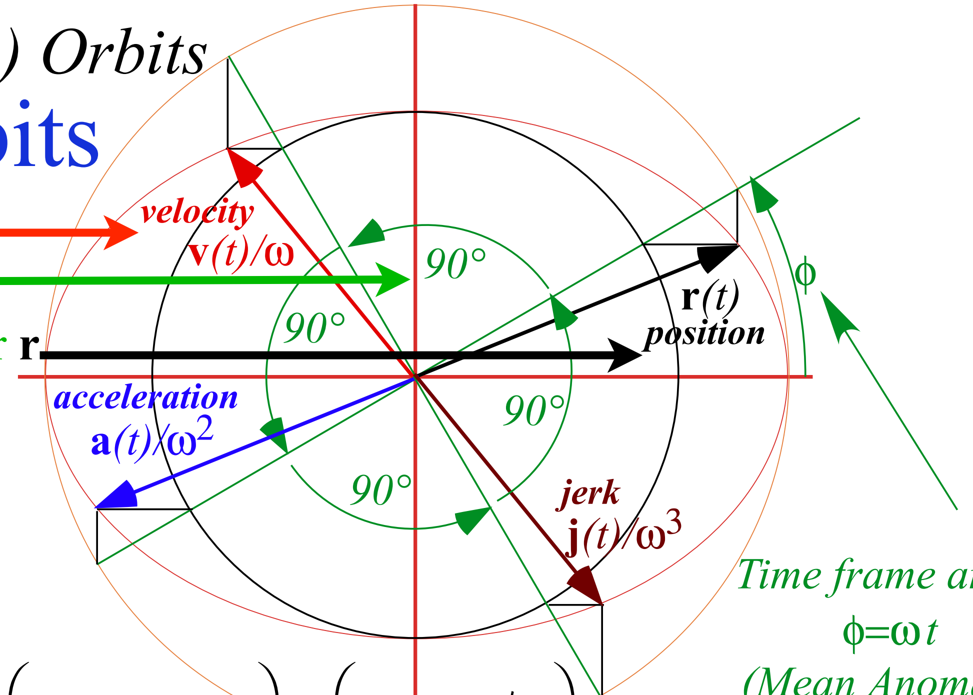
$$\textit{jerk or change of acceleration}: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi+\frac{3\pi}{2}\right) \\ b\sin\left(\phi+\frac{3\pi}{2}\right) \end{pmatrix}$$

...and so forth...

Unit 1
Fig. 11.5

Calculus of IHO orbits

(a) Orbits



To make velocity vector **v**
just rotate by π/2 or 90°
the mean-anomaly φ of position vector **r**

Time frame angle
φ=ω*t*
(Mean Anomaly)

mean-anomaly φ of position vector **r**
rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

$$radius\ vector: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

$$velocity\ vector: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi+\frac{\pi}{2}\right) \\ b\sin\left(\phi+\frac{\pi}{2}\right) \end{pmatrix} (for\ \omega=1)$$

m.a. φ+π/2 of vector **v** rotated by
another π/2 is *m.a.* of vector **a**

$$acceleration\ or\ force\ vector: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi+\frac{2\pi}{2}\right) \\ b\sin\left(\phi+\frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

$$jerk\ or\ change\ of\ acceleration: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi+\frac{3\pi}{2}\right) \\ b\sin\left(\phi+\frac{3\pi}{2}\right) \end{pmatrix}$$

...and so forth...

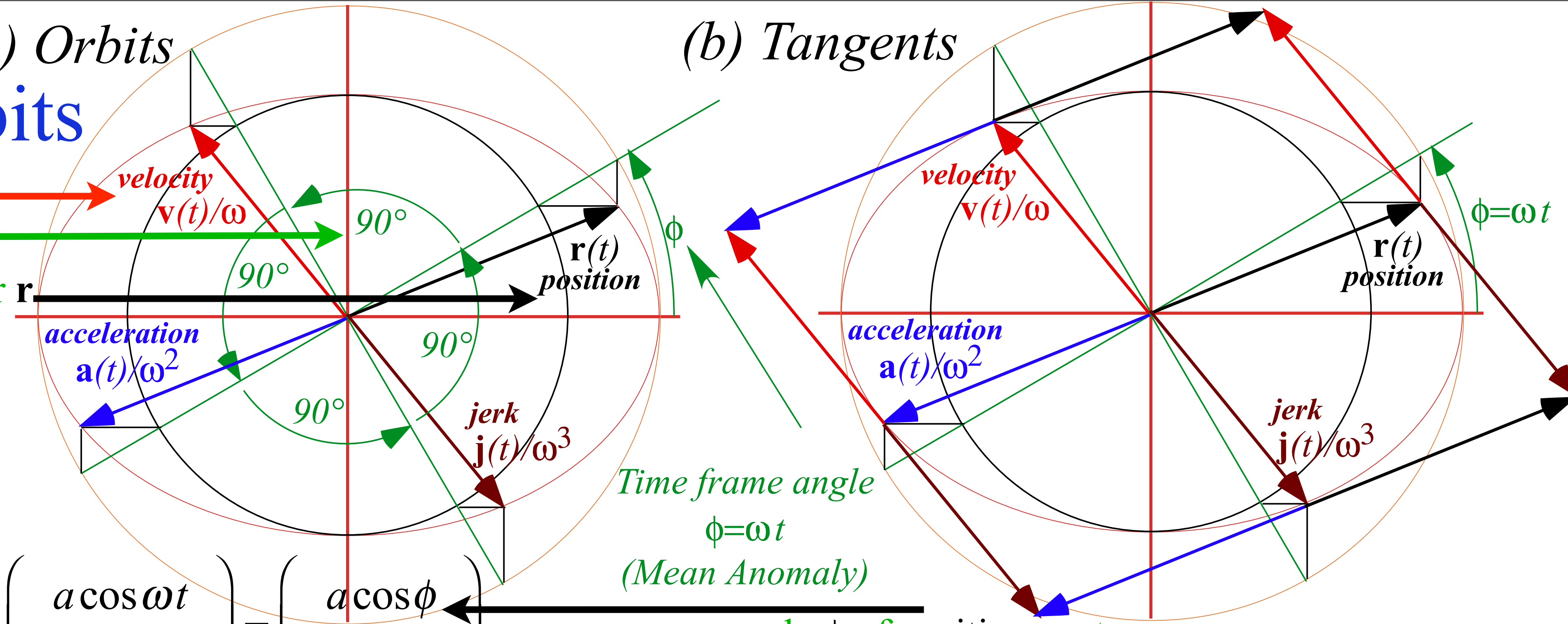
$$inauguration\ or\ change\ of\ jerk: \mathbf{i} = \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \begin{pmatrix} +a\omega^4\cos\omega t \\ +b\omega^4\sin\omega t \end{pmatrix} = \frac{d\mathbf{j}}{dt} = \dot{\mathbf{j}} = \ddot{\mathbf{a}} = \dddot{\mathbf{v}} = \ddddot{\mathbf{r}} = \frac{d^4\mathbf{r}}{dt^4} = \begin{pmatrix} a\cos\left(\phi+\frac{4\pi}{2}\right) \\ b\sin\left(\phi+\frac{4\pi}{2}\right) \end{pmatrix}$$

...and so on...
...But, now it
repeats after 4
t-derivatives

Calculus of IHO orbits

To make velocity vector **v** just rotate by π/2 or 90° the mean-anomaly ϕ of position vector **r**

Link ⇒ BoxIt simulation of IHO orbits



$$\text{radius vector}: \mathbf{r} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} = \begin{pmatrix} a\cos\phi \\ b\sin\phi \end{pmatrix}$$

mean-anomaly ϕ of position vector **r** rotated by π/2 or 90° is *m.a.* of vector **v**

Unit 1
Fig. 11.5

Link → IHO Exegesis Plot

Link → IHO orbital time rates of change

$$\text{velocity vector}: \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{pmatrix} a\cos\left(\phi+\frac{\pi}{2}\right) \\ b\sin\left(\phi+\frac{\pi}{2}\right) \end{pmatrix} (\text{for } \omega = 1)$$

m.a. ϕ+π/2 of vector **v** rotated by another π/2 is *m.a.* of vector **a**

$$\text{acceleration or force vector}: \frac{\mathbf{F}}{m} = \mathbf{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} -a\omega^2\cos\omega t \\ -b\omega^2\sin\omega t \end{pmatrix} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} a\cos\left(\phi+\frac{2\pi}{2}\right) \\ b\sin\left(\phi+\frac{2\pi}{2}\right) \end{pmatrix}$$

or change of velocity

$$\text{jerk or change of acceleration}: \mathbf{j} = \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} +a\omega^3\sin\omega t \\ -b\omega^3\cos\omega t \end{pmatrix} = \frac{d\mathbf{a}}{dt} = \dot{\mathbf{a}} = \ddot{\mathbf{v}} = \dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \begin{pmatrix} a\cos\left(\phi+\frac{3\pi}{2}\right) \\ b\sin\left(\phi+\frac{3\pi}{2}\right) \end{pmatrix}$$

...and so forth...

$$\text{inauguration or change of jerk}: \mathbf{i} = \begin{pmatrix} i_x \\ i_y \end{pmatrix} = \begin{pmatrix} +a\omega^4\cos\omega t \\ +b\omega^4\sin\omega t \end{pmatrix} = \frac{d\mathbf{j}}{dt} = \dot{\mathbf{j}} = \ddot{\mathbf{a}} = \dddot{\mathbf{v}} = \ddddot{\mathbf{r}} = \frac{d^4\mathbf{r}}{dt^4} = \begin{pmatrix} a\cos\left(\phi+\frac{4\pi}{2}\right) \\ b\sin\left(\phi+\frac{4\pi}{2}\right) \end{pmatrix}$$

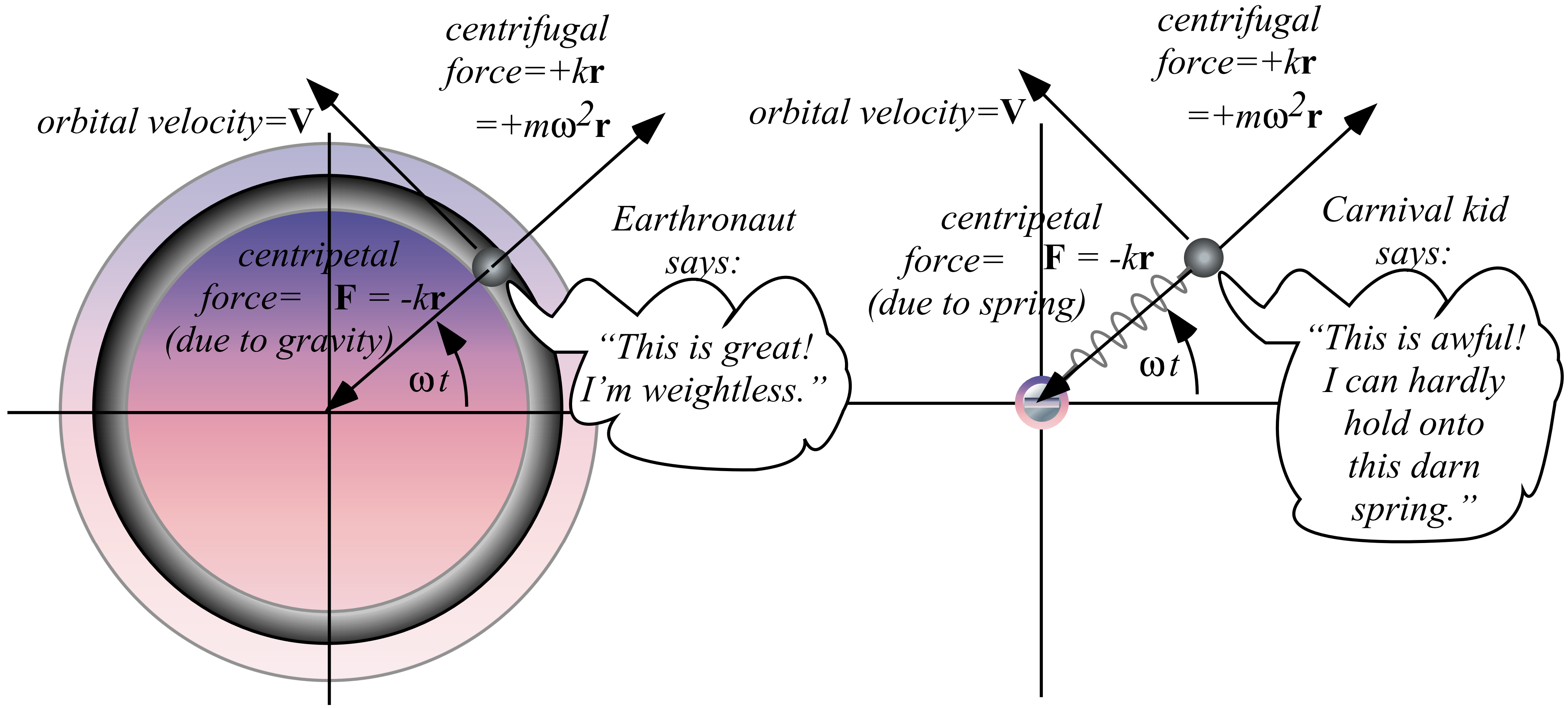
...and so on...
...But, now it repeats after 4 *t*-derivatives

Constructing 2D IHO orbits using Kepler anomaly plots

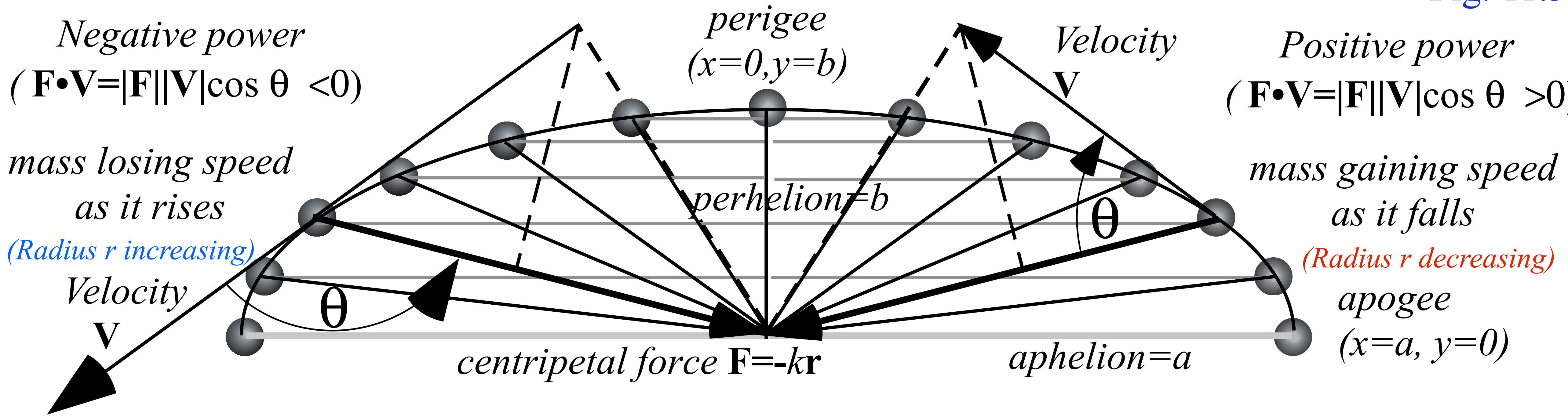
Mean-anomaly and eccentric-anomaly geometry

Calculus and vector geometry of IHO orbits

→ *A confusing introduction to Coriolis-centrifugal force geometry* *(Derived better in Ch. 12)* *(a) "Earthronaut" orbiting tunnel inside Earth* *(b) "Carnival kid" orbiting in space attached to a spring*

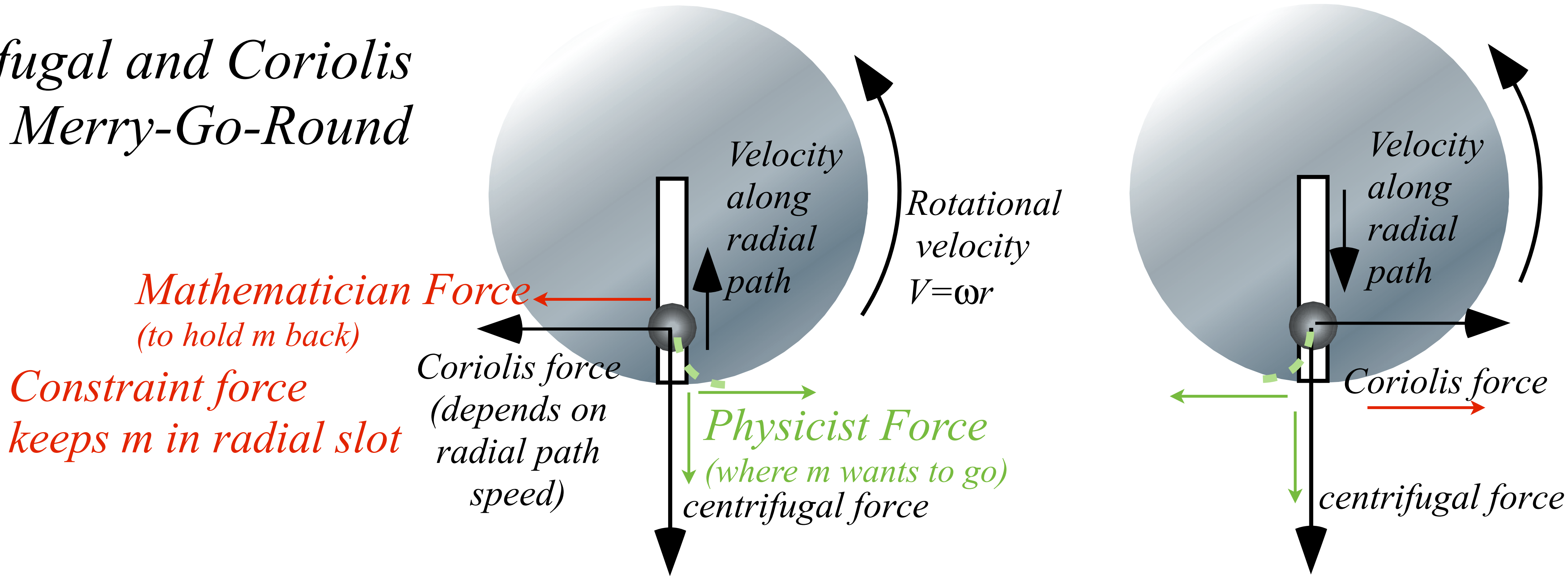


Unit 1 Fig. 11.2

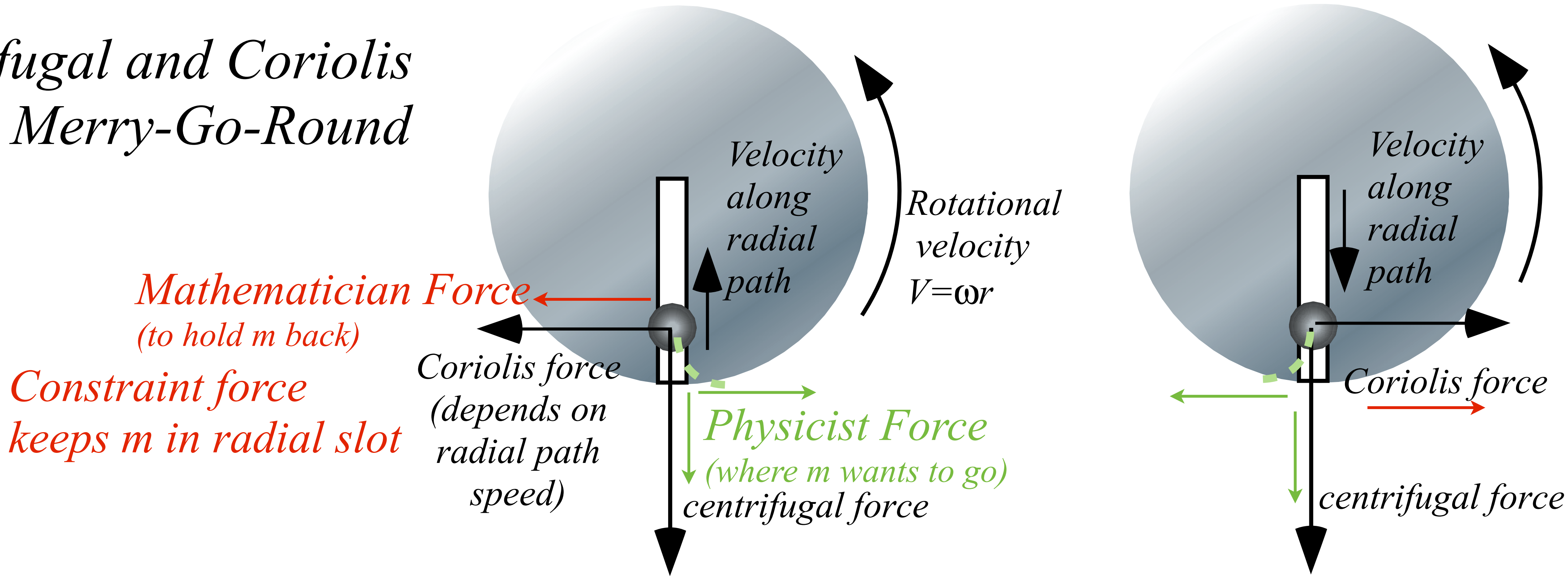


Unit 1 Fig. 11.3

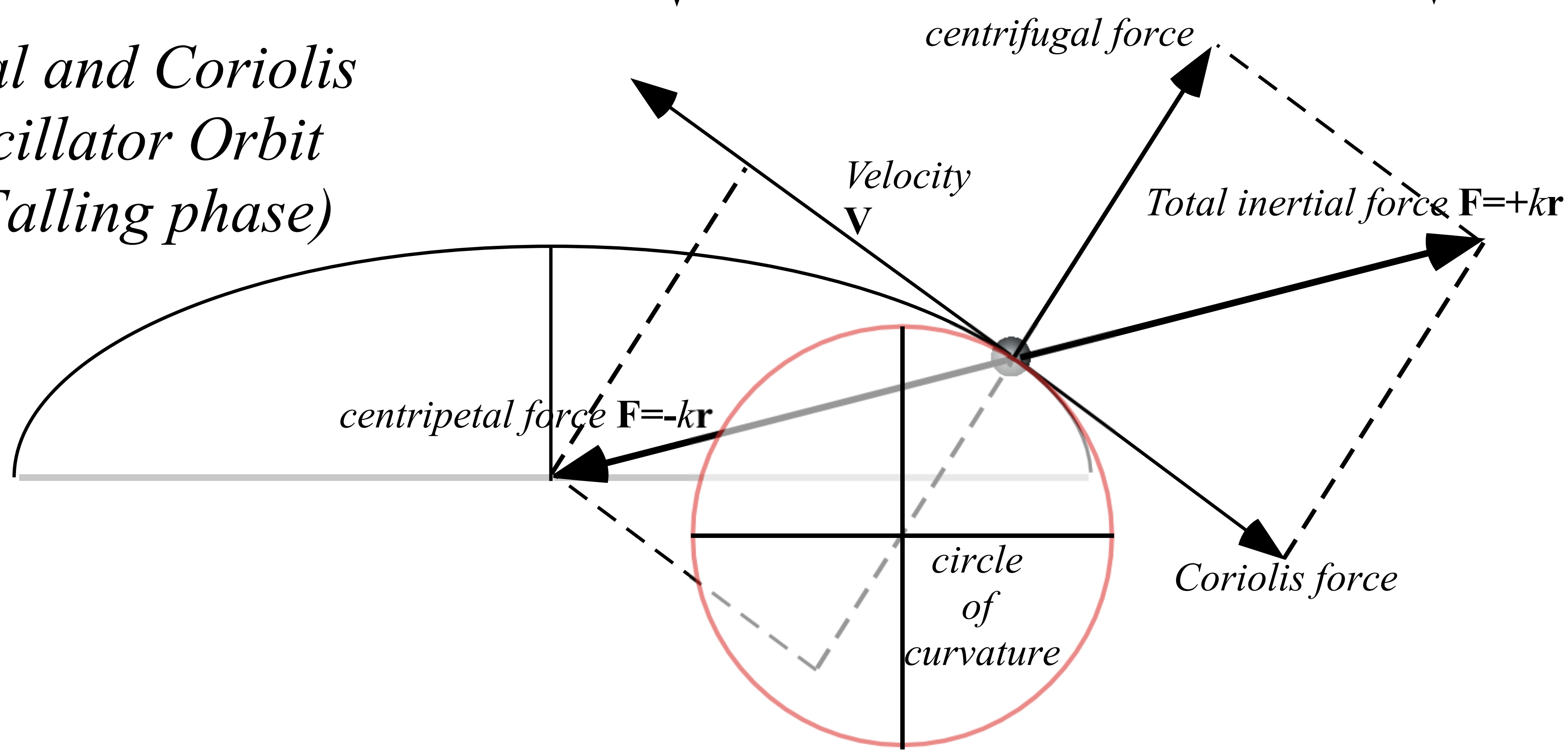
(a) Centrifugal and Coriolis Forces on Merry-Go-Round



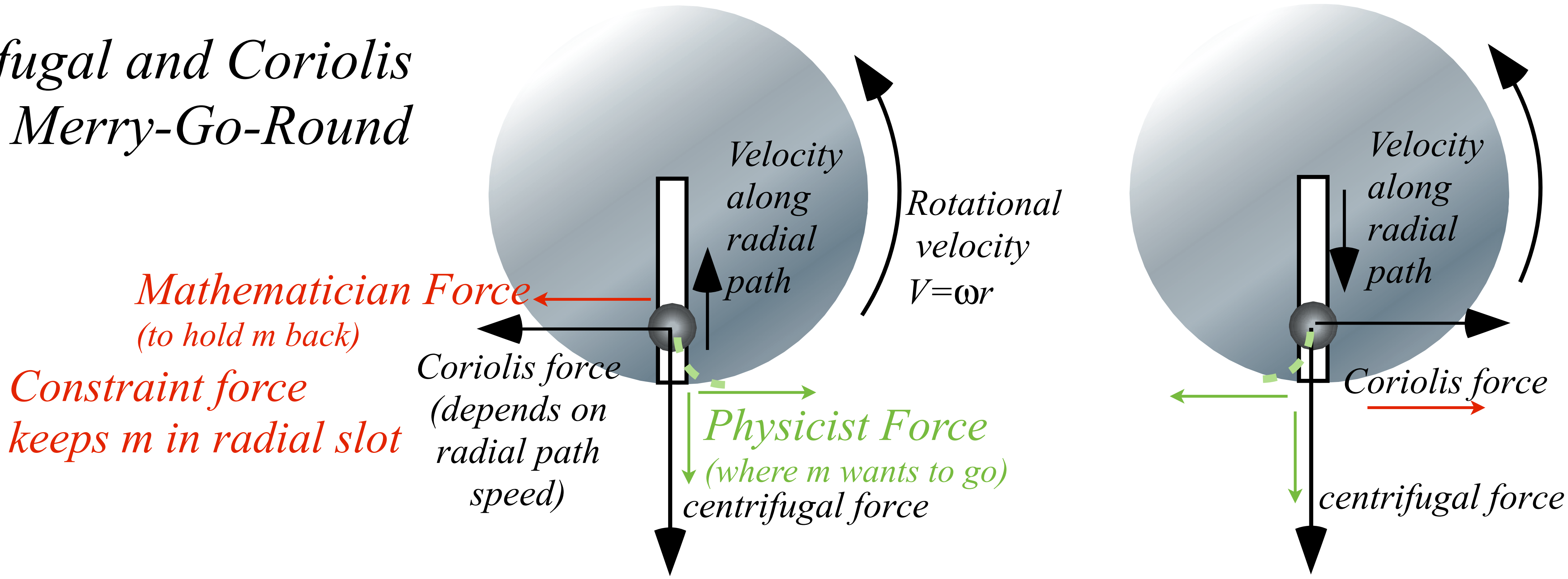
(a) Centrifugal and Coriolis Forces on Merry-Go-Round



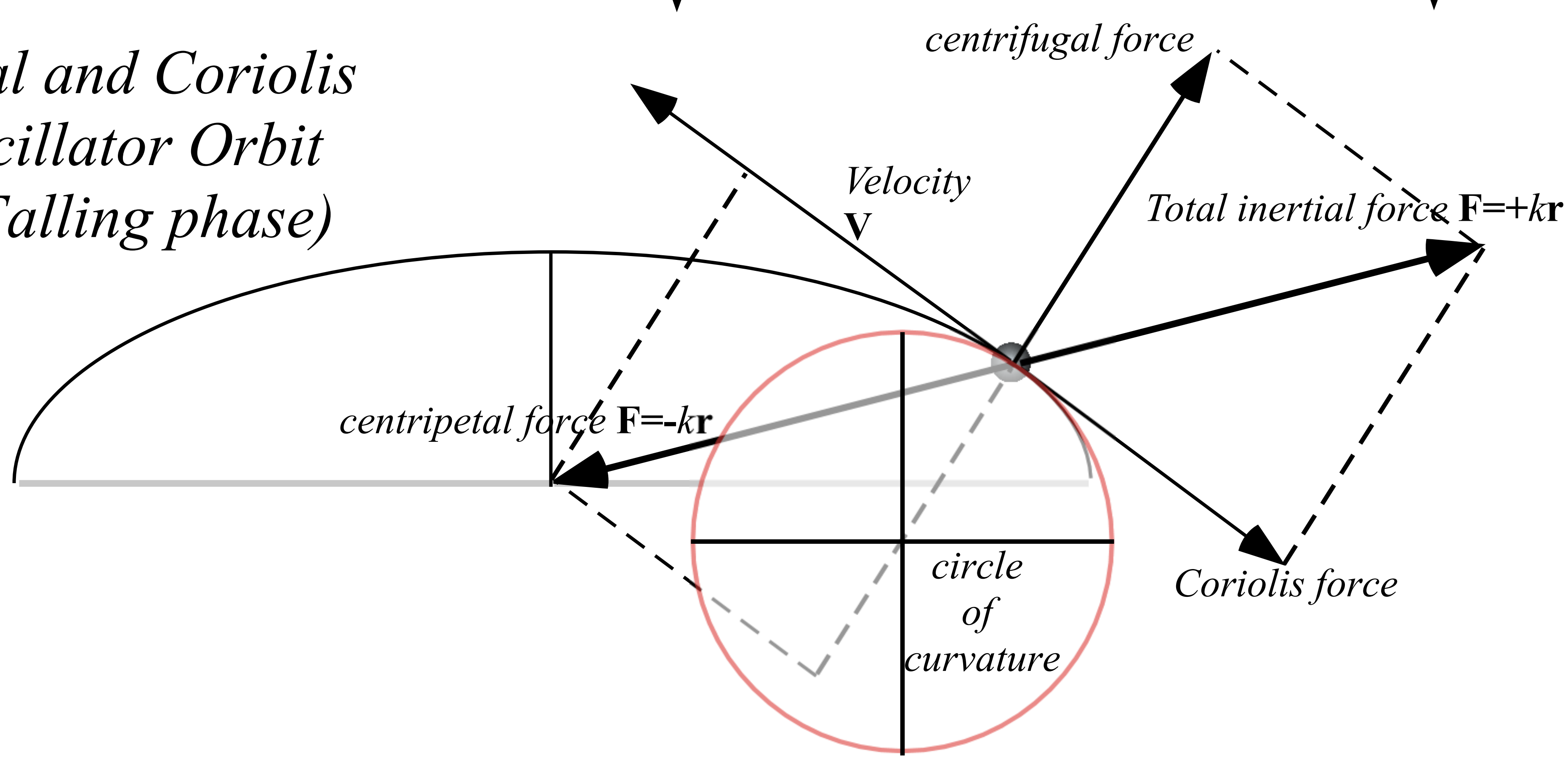
(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)



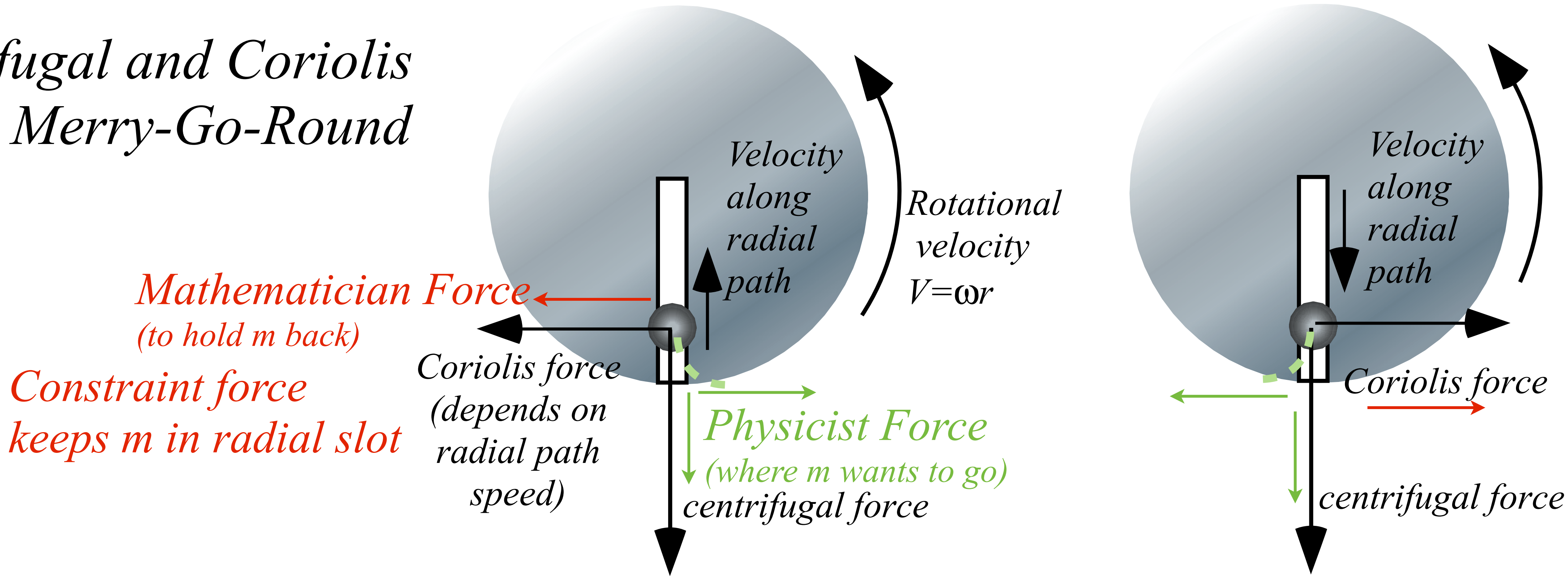
(a) Centrifugal and Coriolis Forces on Merry-Go-Round



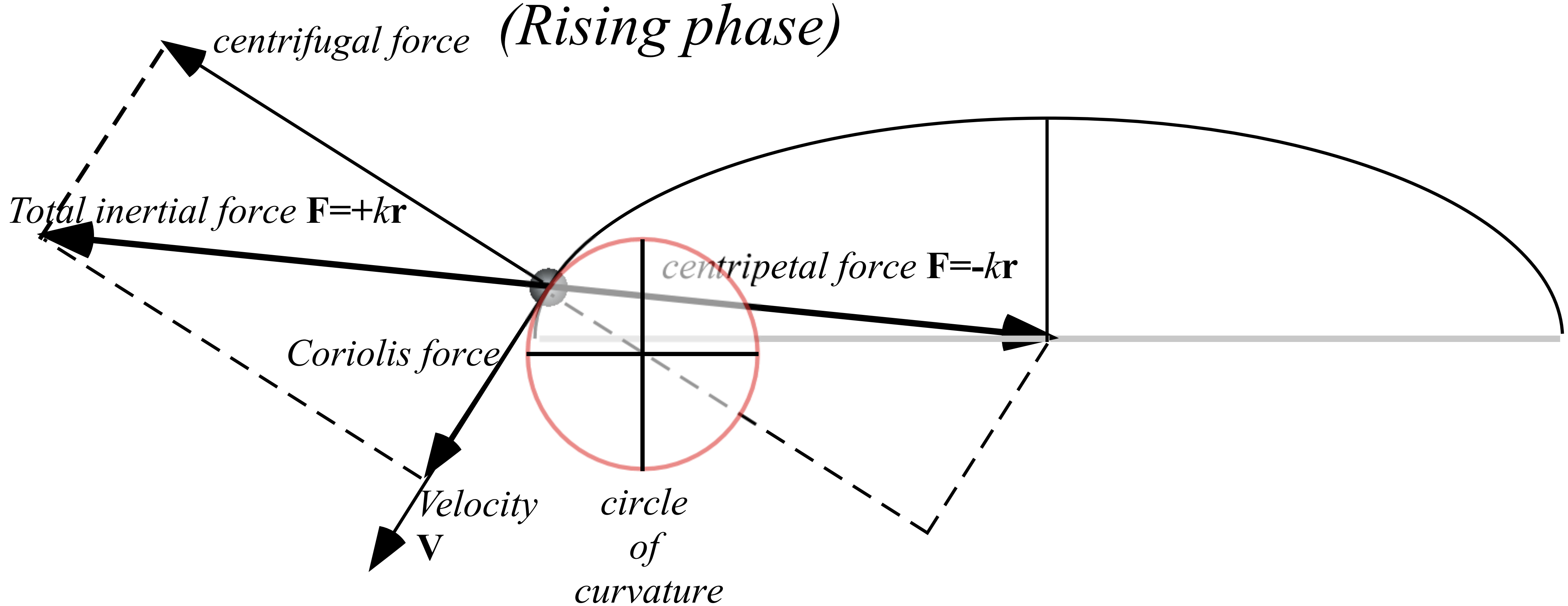
(b) Centrifugal and Coriolis Forces on Oscillator Orbit (Falling phase)

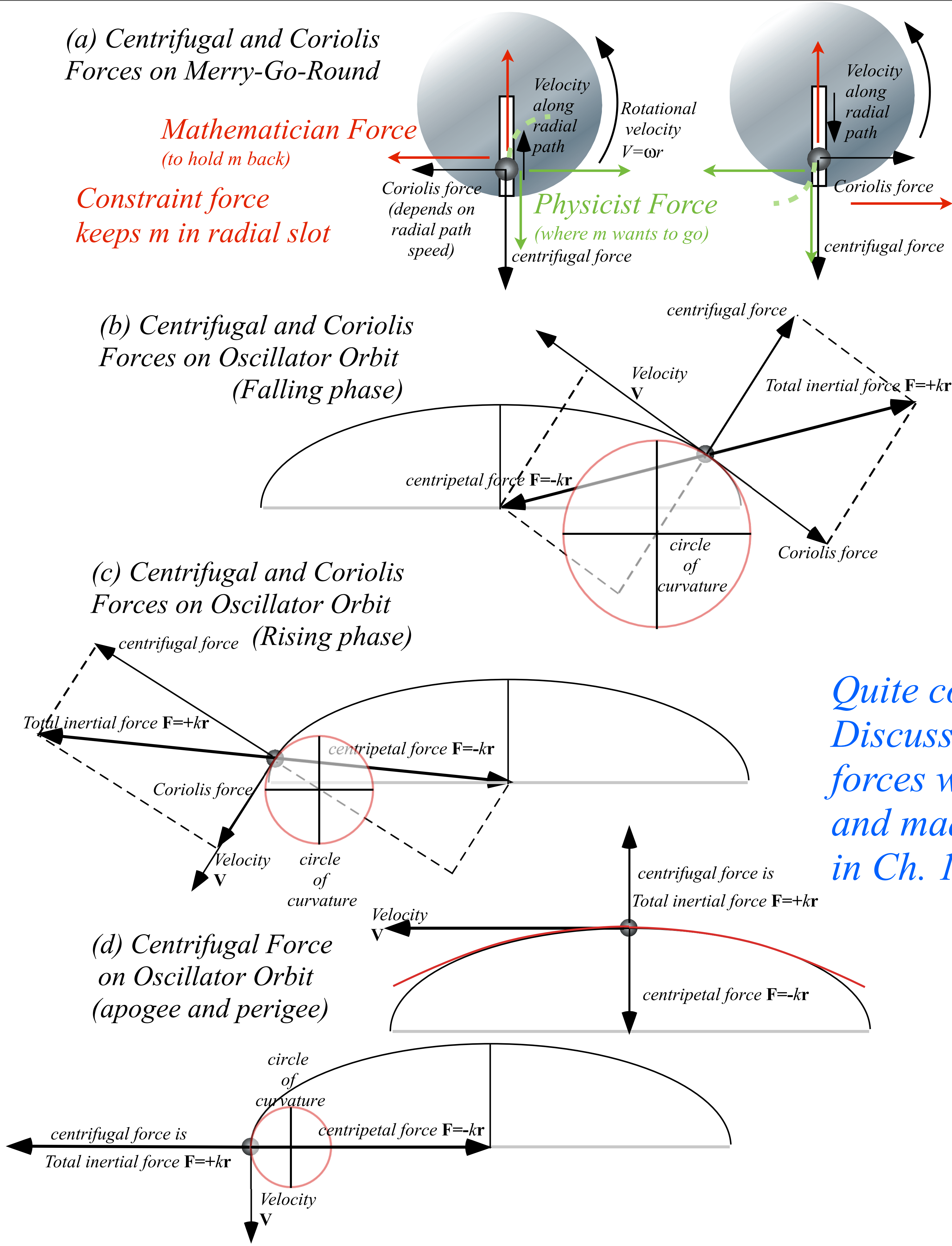


(a) Centrifugal and Coriolis Forces on Merry-Go-Round



(c) Centrifugal and Coriolis Forces on Oscillator Orbit (Rising phase)





Unit 1
Fig. 11.4
a-d

Quite confusing?
Discussion of Coriolis forces will be done more elegantly and made more physically intuitive in Ch. 12 of Unit1 and in Unit 6.

Some Kepler's "laws" for <u>all</u> central (isotropic) force F(r) fields

→ *Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$* *(Derived here)*

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ *(Derived in Unit 5)*

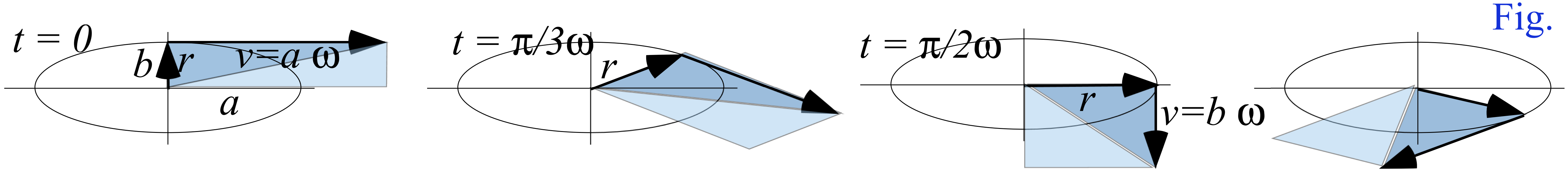
Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$ *(Derived here)*

Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ *(Derived in Unit 5)*

Some Kepler's "laws" for central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 8: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = a\cos\omega t\cdot(b\omega\cos\omega t) - b\sin\omega t\cdot(-a\omega\sin\omega t) = ab\cdot\omega(\cos^2\omega t+\sin^2\omega t)$$

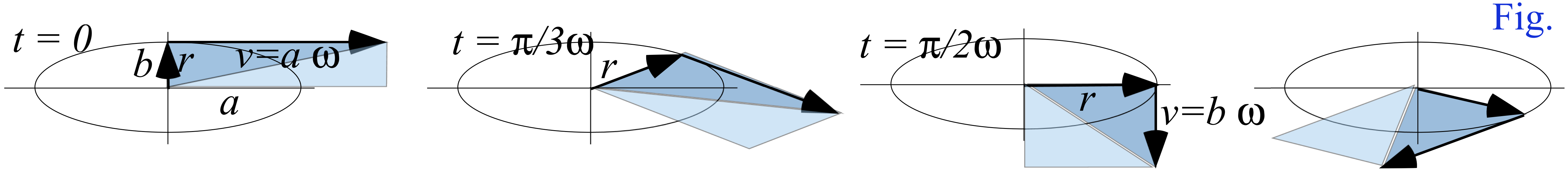
✓ *for IHO*

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix} \qquad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix}$$

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 8: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1
Fig. 11.8



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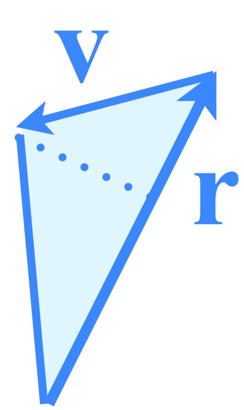
✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega$$

✓ *for IHO*

$$|\mathbf{r}\times\mathbf{v}| = r\cdot v\cdot \sin\measuredangle_r^v$$



Some Kepler's "laws" that apply to any central (isotropic) force F(r)

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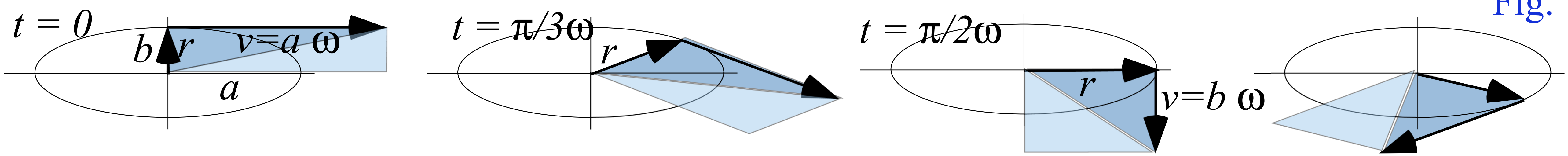
Unit 1
Fig. 11.8



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✓ *for IHO*

2. *Angular momentum* $\mathbf{L} = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,|\mathbf{r}\times\mathbf{v}| = m(r_x v_y - r_y v_x) = m\cdot ab\cdot\omega$$

✓ *for IHO*

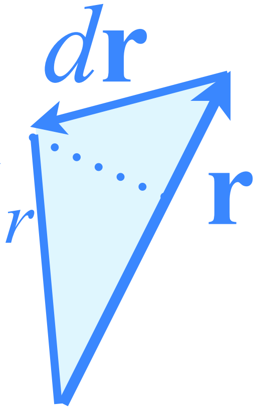
3. Equal area is swept by radius vector in each equal time interval T

$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{2m}\int_0^T dt = \frac{L}{2m}T$$

by 2.

✓ *for IHO*

$$|\mathbf{r}\times d\mathbf{r}| = r\cdot dr\cdot \sin\measuredangle_r^{dr}$$



Some Kepler's "laws" that apply to any central (isotropic) force F(r)

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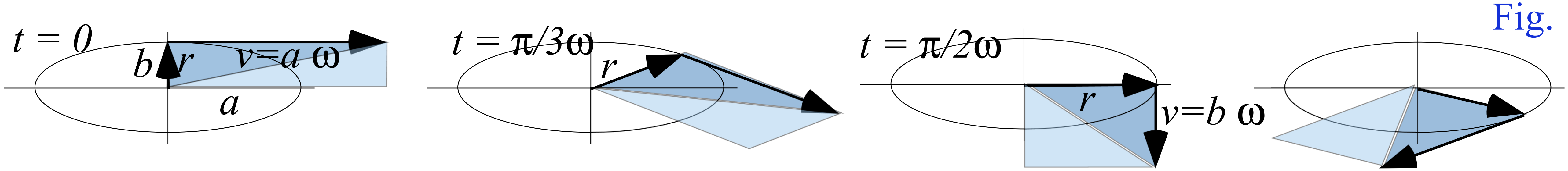
Unit 1
Fig. 11.8



1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

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✓ *for IHO*

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,\mathbf{r}\times\mathbf{v} = m\left(r_x v_y - r_y v_x\right) = m\cdot ab\cdot\omega = m\cdot ab\cdot\frac{2\pi}{\tau}$$

✓ *for IHO*

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$$A_T = \int_0^T \frac{\mathbf{r}\times d\mathbf{r}}{2} = \int_0^T \frac{\mathbf{r}\times\frac{d\mathbf{r}}{dt}}{2}dt = \int_0^T \frac{\mathbf{r}\times\mathbf{v}}{2}dt = \frac{L}{2m}\int_0^T dt = \frac{L}{2m}T$$

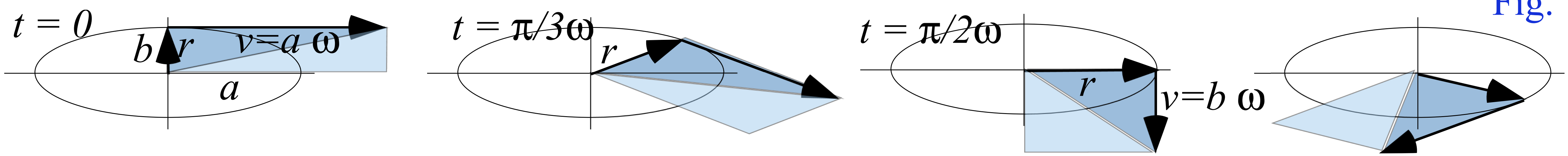
✓ *for IHO*

In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ $(= ab\cdot\pi$ for ellipse orbit$)$

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

...and certainly apply to the IHO: F(r)=-k·r with U(r)=k·r²/2 (Recall from Lecture 8: $k = Gm\frac{4\pi}{3}\rho_{\oplus}$)

Unit 1 Fig. 11.8



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In one period: $\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L}$ the area is: $A_\tau = \frac{L\tau}{2m}$ $(= ab \cdot \pi$ for ellipse orbit)

(Recall from Lecture 7: $\omega = \sqrt{k/m} = \sqrt{G\rho_{\oplus} 4\pi / 3}$)

Some Kepler's "laws" for all central (isotropic) force F(r) fields

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ *(Derived here)*

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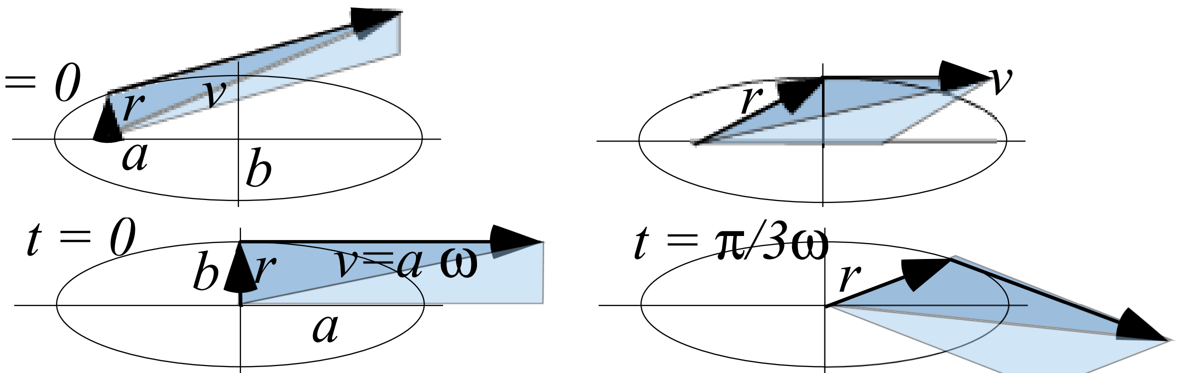
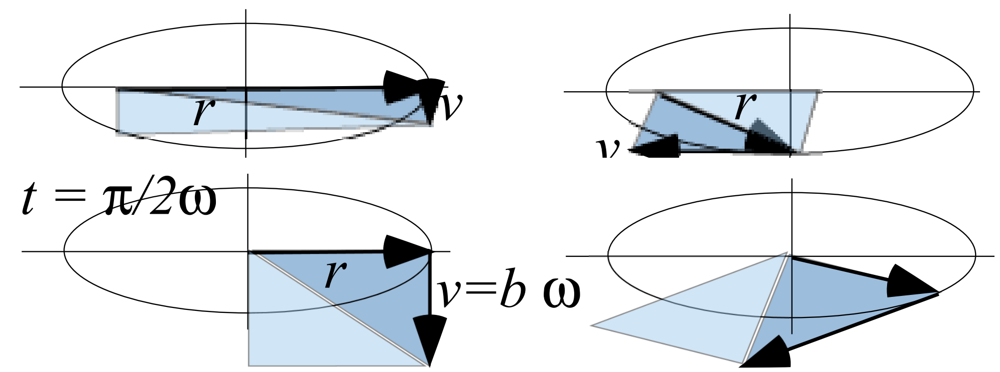
Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$

Coulomb:

IHO:





1. Area of triangle $\measuredangle_{\mathbf{r}}^{\mathbf{v}} = \mathbf{r}\times\mathbf{v}/2$ is constant

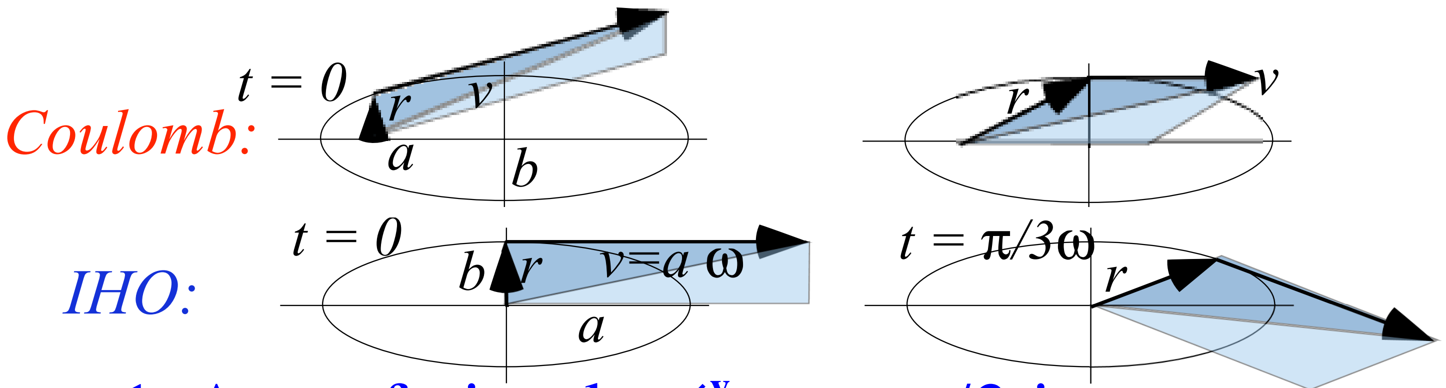
$$\mathbf{r}\times\mathbf{v} = r_x v_y - r_y v_x = \begin{cases} ab\cdot\sqrt{G\rho_{\oplus}4\pi/3} & \text{for IHO} \\ a^{-1/2}b\sqrt{GM_{\oplus}} & \text{for Coul.} \end{cases}$$

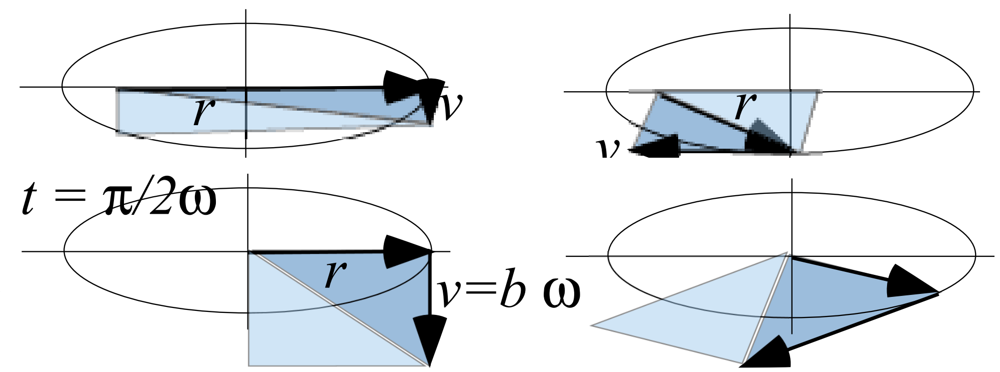
✓ *for IHO*

(Derived in Unit 5) ✓ *for Coul.*

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$





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✓ *for IHO*

(Derived in Unit 5) ✓ *for Coul.*

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

$$L = m\,\mathbf{r}\times\mathbf{v} = m\left(r_x v_y - r_y v_x\right) = \begin{cases} m\cdot ab\cdot\sqrt{G\rho_{\oplus}4\pi/3} & \text{for IHO} \\ m\cdot a^{-1/2}b\sqrt{GM_{\oplus}} & \text{for Coul.} \end{cases}$$

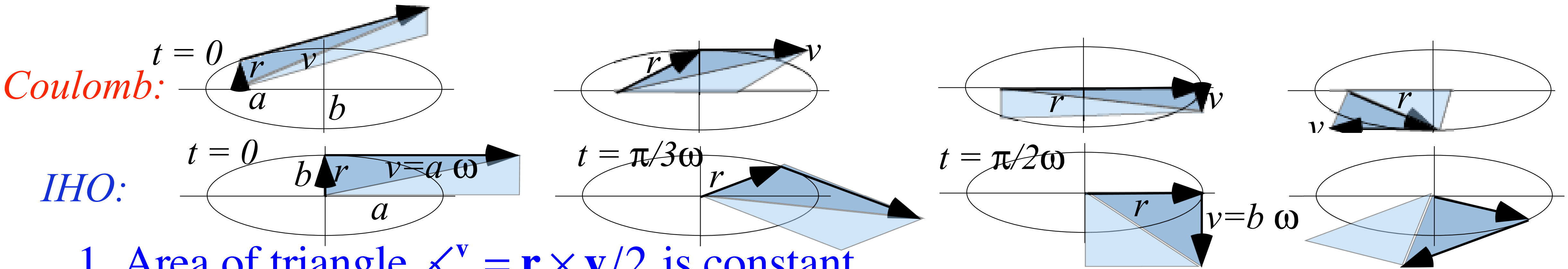
(... in Unit 5)

✓ *for IHO*

✓ *for Coul.*

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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(Derived in Unit 5) (Derived in Unit 5) ✓ for IHO ✓ for Coul.

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(... in Unit 5) ✓ for IHO ✓ for Coul.

3. Equal area is swept by radius vector in each equal time interval T

In one period:

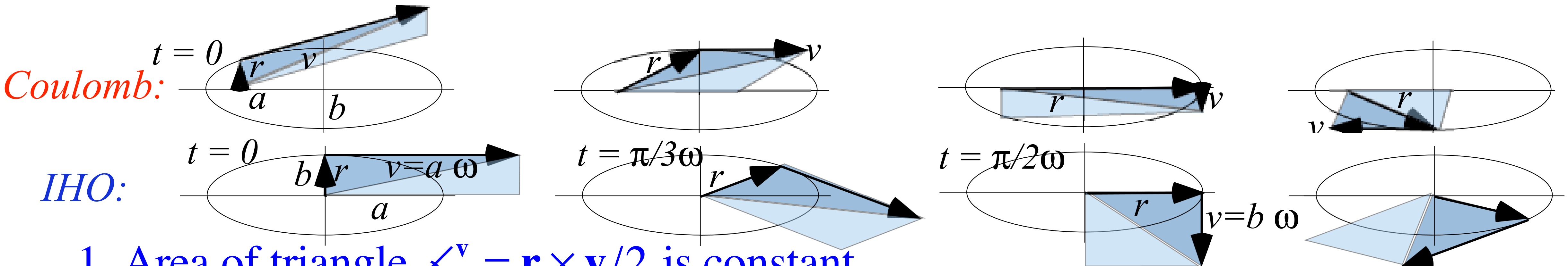
$$\tau = \frac{1}{\upsilon} = \frac{2\pi}{\omega} = \frac{2mA_\tau}{L} = \frac{2m\cdot ab\cdot\pi}{L} = \begin{cases} \dfrac{2m\cdot ab\cdot\pi}{m\cdot ab\cdot\sqrt{G\rho_\oplus 4\pi/3}} \\ \dfrac{2m\cdot ab\cdot\pi}{m\cdot a^{-1/2}b\sqrt{GM_\oplus}} \end{cases}$$

Applies to any central F(r)

Applies to IHO and Coulomb

Some Kepler's "laws" that apply to any central (isotropic) force F(r)

Apply to IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ and Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$



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(Derived in Unit 5) (Derived in Unit 5) ✓ for IHO ✓ for Coul.

2. *Angular momentum* $L = m\,\mathbf{r}\times\mathbf{v}$ is conserved

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$\frac{2\pi}{\omega}$: *Applies to any central F(r)*

$\frac{2m\cdot ab\cdot\pi}{L}$: *Applies to IHO and Coulomb*

(not a function of *a* or *b*) — that is ω_{IHO}

(not a function of *b*) — that is ω_{Coul}

Some Kepler's "laws" for all central (isotropic) force F(r) fields

Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$ *(Derived here)*

Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$ *(Derived in Unit 5)*

→ *Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$* *(Derived here)*

Total energy E=KE+PE invariance of Coulomb: $F(r)=-GMm/r^2$ *(Derived in Unit 5)*

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total energy= $KE + PE$ is constant

$$KE + PE = \frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$= \frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}kr_x^2 + \frac{1}{2}kr_y^2$$

$$= \frac{1}{2}m(-a\omega\sin\omega t)^2 + \frac{1}{2}m(b\omega\cos\omega t)^2 + \frac{1}{2}k(a\cos\omega t)^2 + \frac{1}{2}k(b\sin\omega t)^2$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ b\omega\cos\omega t \end{pmatrix} \qquad \begin{pmatrix} r_x \\ r_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\cos\omega t \\ b\sin\omega t \end{pmatrix}$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=*KE*+*PE* is constant

$$KE + PE = \frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$= \frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}kr_x^2 + \frac{1}{2}kr_y^2$$

$$= \frac{1}{2}m(-a\omega\sin\omega t)^2 + \frac{1}{2}m(b\omega\cos\omega t)^2 + \frac{1}{2}k(a\cos\omega t)^2 + \frac{1}{2}k(b\sin\omega t)^2$$

$$= \frac{1}{2}ma^2\omega^2(\sin^2\omega t) + \frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2 + \frac{1}{2}ka^2(\cos^2\omega t) + \frac{1}{2}kb^2(\sin^2\omega t)$$

$$= \frac{1}{2}m\omega^2(a^2 + b^2) \qquad Given: k = m\omega^2$$

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}kr_x^2 + \frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2 + \frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t) + \frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t) + \frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$$Given: k=m\omega^2$$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2) \quad \text{since: } \omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3} \quad \text{or: } m\omega^2=k$$

Some Kepler's "laws" for <u>all</u> central (isotropic) force $F(r)$ fields

- *Angular momentum invariance of IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$* *(Derived here)*
- *Angular momentum invariance of Coulomb: $F(r)=-GMm/r^2$ with $U(r)=-GMm\cdot/r$* *(Derived in Unit 5)*
- *Total energy E=KE+PE invariance of IHO: $F(r)=-k\cdot r$* *(Derived here)*
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Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: F(r)=-k·r with U(r)=k·r²/2

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE=\frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v}+\frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$=\frac{1}{2}mv_x^2+\frac{1}{2}mv_y^2+\frac{1}{2}kr_x^2+\frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$=\frac{1}{2}m\omega^2(a^2+b^2)$$

$$Given: k=m\omega^2$$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_\oplus 4\pi/3}$ or: $m\omega^2=k$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of b)

Kepler laws involve ∡-momentum conservation in isotropic force F(r)

Now consider orbital energy conservation of the IHO: $F(r)=-k\cdot r$ with $U(r)=k\cdot r^2/2$

Total *IHO* energy=$KE+PE$ is constant

$$KE+PE = \frac{1}{2}\mathbf{v}\bullet\mathbf{M}\bullet\mathbf{v} + \frac{1}{2}\mathbf{r}\bullet\mathbf{K}\bullet\mathbf{r}$$

$$=\frac{1}{2}\begin{pmatrix} v_x & v_y \end{pmatrix}\bullet\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\bullet\begin{pmatrix} v_x \\ v_y \end{pmatrix}+\begin{pmatrix} r_x & r_y \end{pmatrix}\bullet\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\bullet\begin{pmatrix} r_x \\ r_y \end{pmatrix}$$

$$= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}kr_x^2 + \frac{1}{2}kr_y^2$$

$$=\frac{1}{2}m(-a\omega\sin\omega t)^2+\frac{1}{2}m(b\omega\cos\omega t)^2+\frac{1}{2}k(a\cos\omega t)^2+\frac{1}{2}k(b\sin\omega t)^2$$

$$=\frac{1}{2}ma^2\omega^2(\sin^2\omega t)+\frac{1}{2}mb^2\omega^2(\cos^2\omega t)^2+\frac{1}{2}ka^2(\cos^2\omega t)+\frac{1}{2}kb^2(\sin^2\omega t)$$

$$= \frac{1}{2}m\omega^2(a^2+b^2) \qquad Given: k=m\omega^2$$

$$E=KE+PE=\frac{1}{2}m\omega^2(a^2+b^2)=\frac{1}{2}k(a^2+b^2)$$ since: $\omega=\sqrt{\frac{k}{m}}=\sqrt{G\rho_{\oplus}4\pi/3}$ or: $m\omega^2=k$

We'll see that the Coul. orbits are simpler: (*like the period*...not a function of *b*)

$$E=KE+PE=\frac{1}{2}mv^2{}_x+\frac{1}{2}mv^2{}_y-\frac{k}{r}=\frac{1}{2}mv^2{}_x+\frac{1}{2}mv^2{}_y-\frac{GM_{\oplus}m}{r}=-\frac{GM_{\oplus}m}{a}$$

Introduction to dual matrix operator contact geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$)

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ ($\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$)

Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\overbrace{\begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r}\ =1} = 1 = \overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}} \bullet \overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

A inverse matrix Q^{-1} generates an ellipse by **p**•Q^{-1}•**p**=*1 called* inverse *or* dual *ellipse:*

$$\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix} \bullet \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \bullet \begin{pmatrix} p_x \\ p_y \end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}\ =1} = 1 = \overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}} \bullet \overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}} = a^2 p_x^2 + b^2 p_y^2$$

Quadratic forms and tangent contact geometry of their ellipses

A matrix Q that generates an ellipse by **r**•*Q*•**r**=*1 is called* positive-definite (*if* **r**•*Q*•**r** *always >0*)

$$\overbrace{\begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix}}^{\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} \quad = 1} = 1 = \overbrace{\begin{pmatrix} x & y \end{pmatrix}}^{\mathbf{r}} \bullet \overbrace{\begin{pmatrix} \frac{x}{a^2} \\ \frac{y}{b^2} \end{pmatrix}}^{\mathbf{Q}\bullet\mathbf{r}=\mathbf{p}} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Defined mapping between ellipses

A inverse matrix Q^{-1} *generates an ellipse by* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$ *called* inverse *or* dual *ellipse:*

$$\overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix} \bullet \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \bullet \begin{pmatrix} p_x \\ p_y \end{pmatrix}}^{\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} \quad = 1} = 1 = \overbrace{\begin{pmatrix} p_x & p_y \end{pmatrix}}^{\mathbf{p}} \bullet \overbrace{\begin{pmatrix} a^2 p_x \\ b^2 p_y \end{pmatrix}}^{\mathbf{Q}^{-1}\bullet\mathbf{p}=\mathbf{r}} = a^2 p_x^2 + b^2 p_y^2$$

Introduction to dual matrix operator contact geometry (based on IHO orbits)

→ *Quadratic form ellipse* $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$*)*

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*

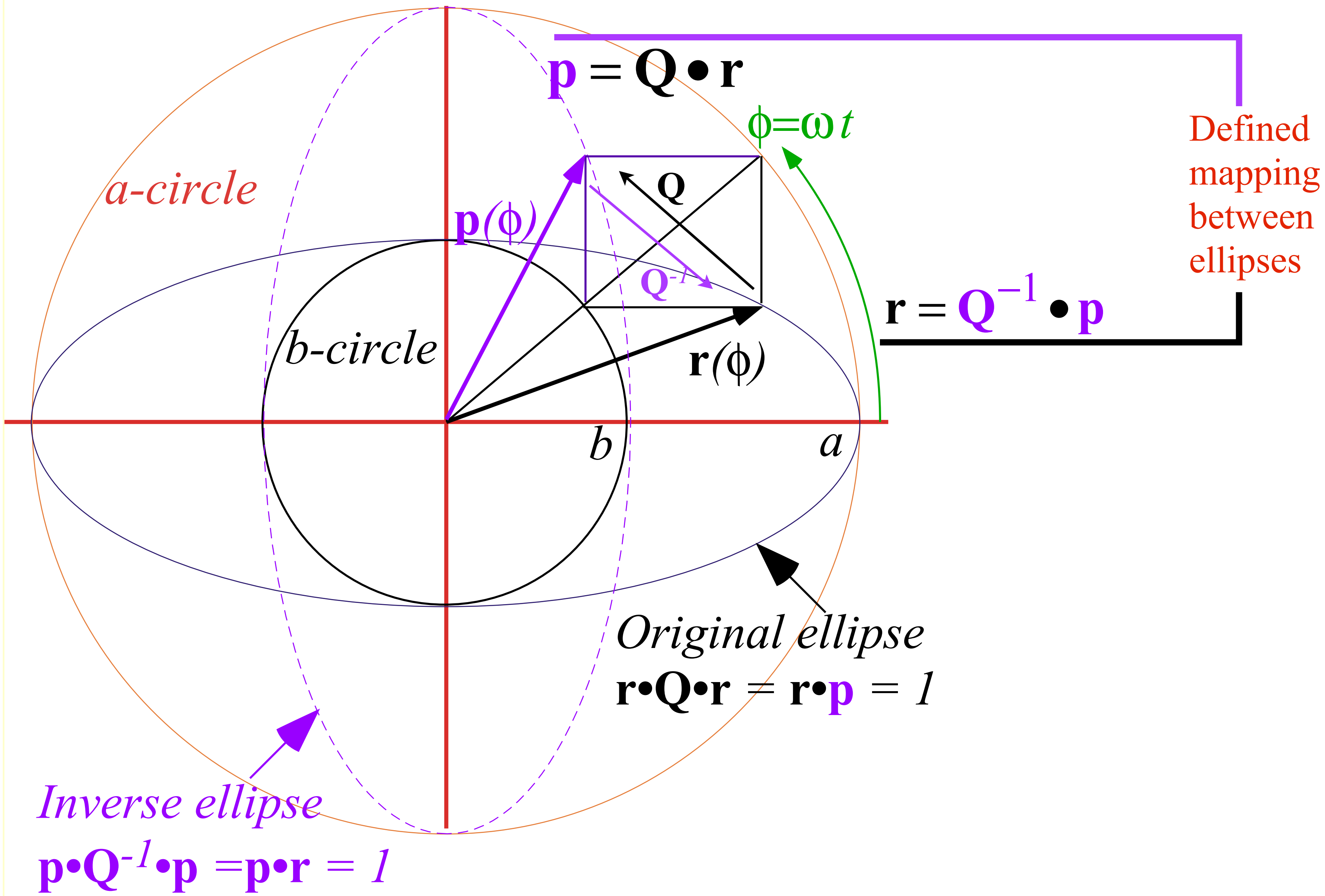
Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

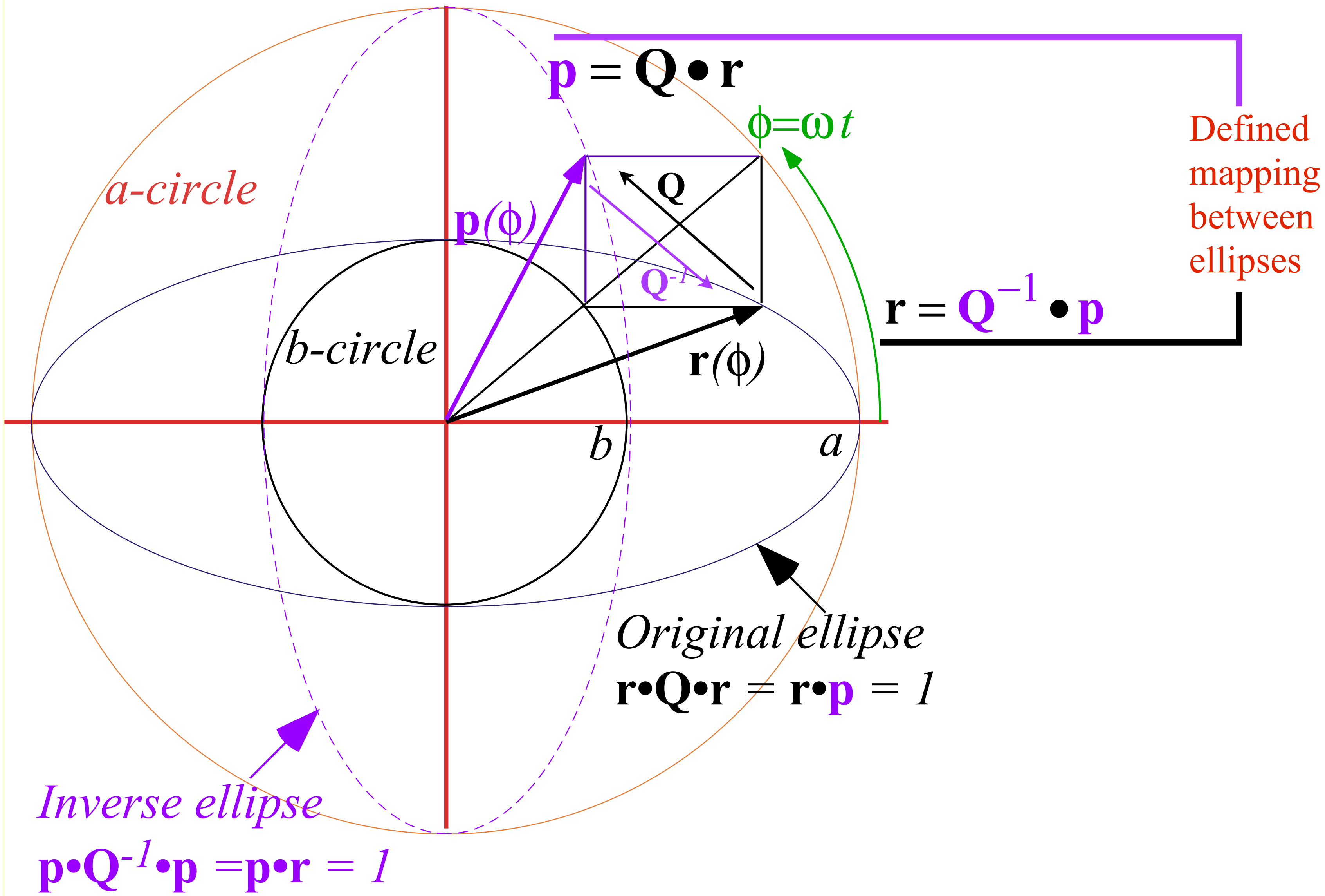
(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



(a) Quadratic form ellipse and
Inverse quadratic form ellipse

based on
Unit 1
Fig. 11.6



Here plot of **p**-ellipse is re-scaled by scalefactor $S=a \cdot b$

p-ellipse x-radius$=1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius$=1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

→ *Duality norm relations (* $\mathbf{r}\bullet\mathbf{p}=1$*)*

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*

Operator geometric sequences and eigenvectors

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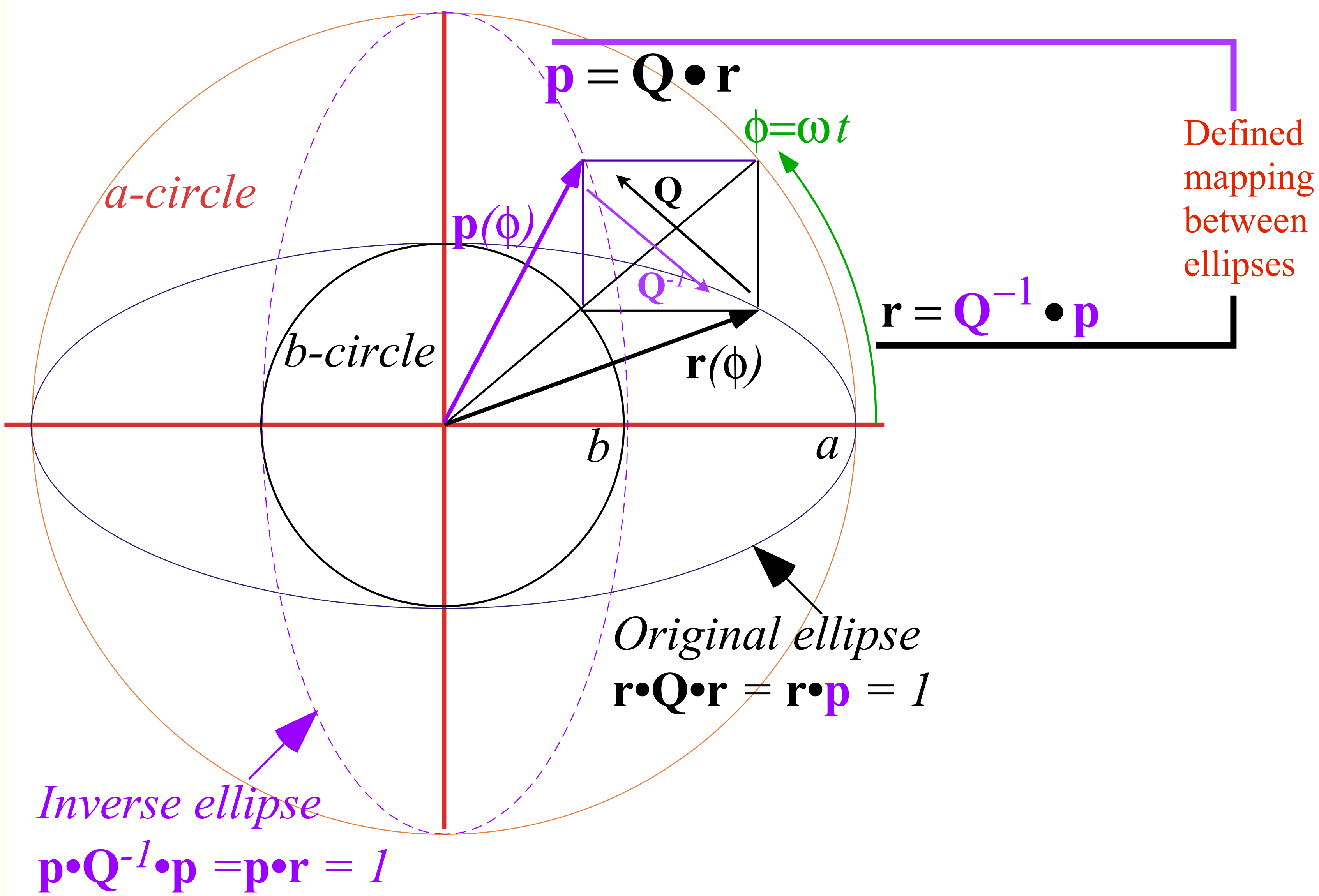
(a) Quadratic form ellipse and Inverse quadratic form ellipse

based on Unit 1 Fig. 11.6



Quadratic form $\mathbf{r} \bullet \mathbf{Q} \bullet \mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p} \bullet \mathbf{Q}^{-1} \bullet \mathbf{p} = 1 = \mathbf{p} \bullet \mathbf{r}$

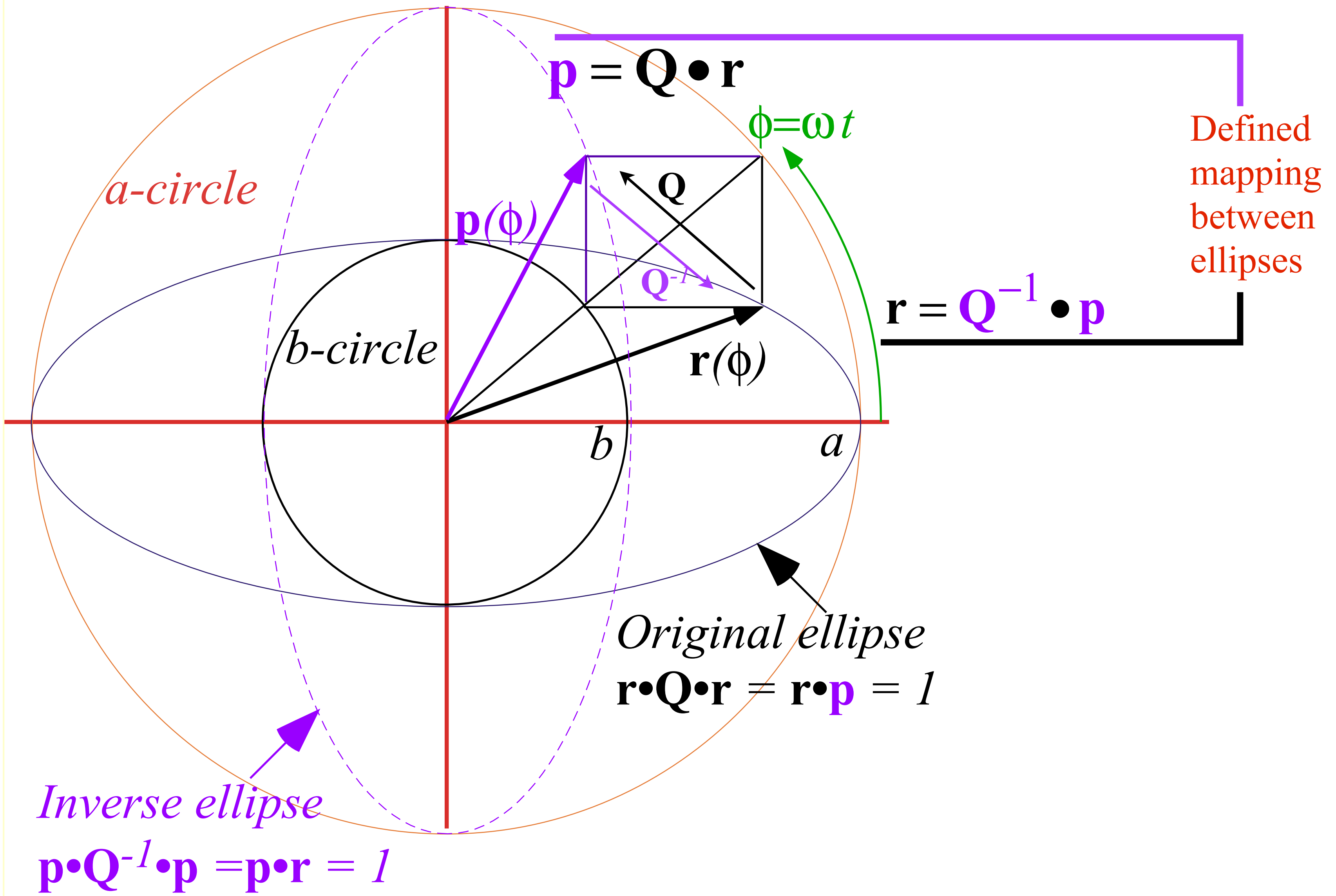
Here plot of **p**-ellipse is re-scaled by scalefactor $S = a \cdot b$

p-ellipse x-radius$=1/a$ plotted at: $S(1/a) = b$ (=1 for a=2, b=1)

p-ellipse y-radius$=1/b$ plotted at: $S(1/b) = a$ (=2 for a=2, b=1)

(a) Quadratic form ellipse and *Inverse quadratic form ellipse*

based on Unit 1 Fig. 11.6



Quadratic form $\mathbf{r}\bullet\mathbf{Q}\bullet\mathbf{r} = 1$ *has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p} = 1 = \mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p} = \mathbf{Q}\bullet\mathbf{r} = \overbrace{\begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}}^{\mathbf{Q}} \bullet \overbrace{\begin{pmatrix} x \\ y \end{pmatrix}}^{\mathbf{r}} = \overbrace{\begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}}^{\mathbf{p}} = \begin{pmatrix} (1/a)\cos\phi \\ (1/b)\sin\phi \end{pmatrix} \quad \text{where: } \begin{matrix} x = r_x = a\cos\phi = a\cos\omega t \\ y = r_y = b\sin\phi = b\sin\omega t \end{matrix} \quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r} = 1}$$

Here plot of **p**-ellipse is re-scaled by scalefactor $S = a \cdot b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

Link ⇒ BoxIt simulation of IHO orbits
Link → IHO orbital time rates of change
Link → IHO Exegesis Plot

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

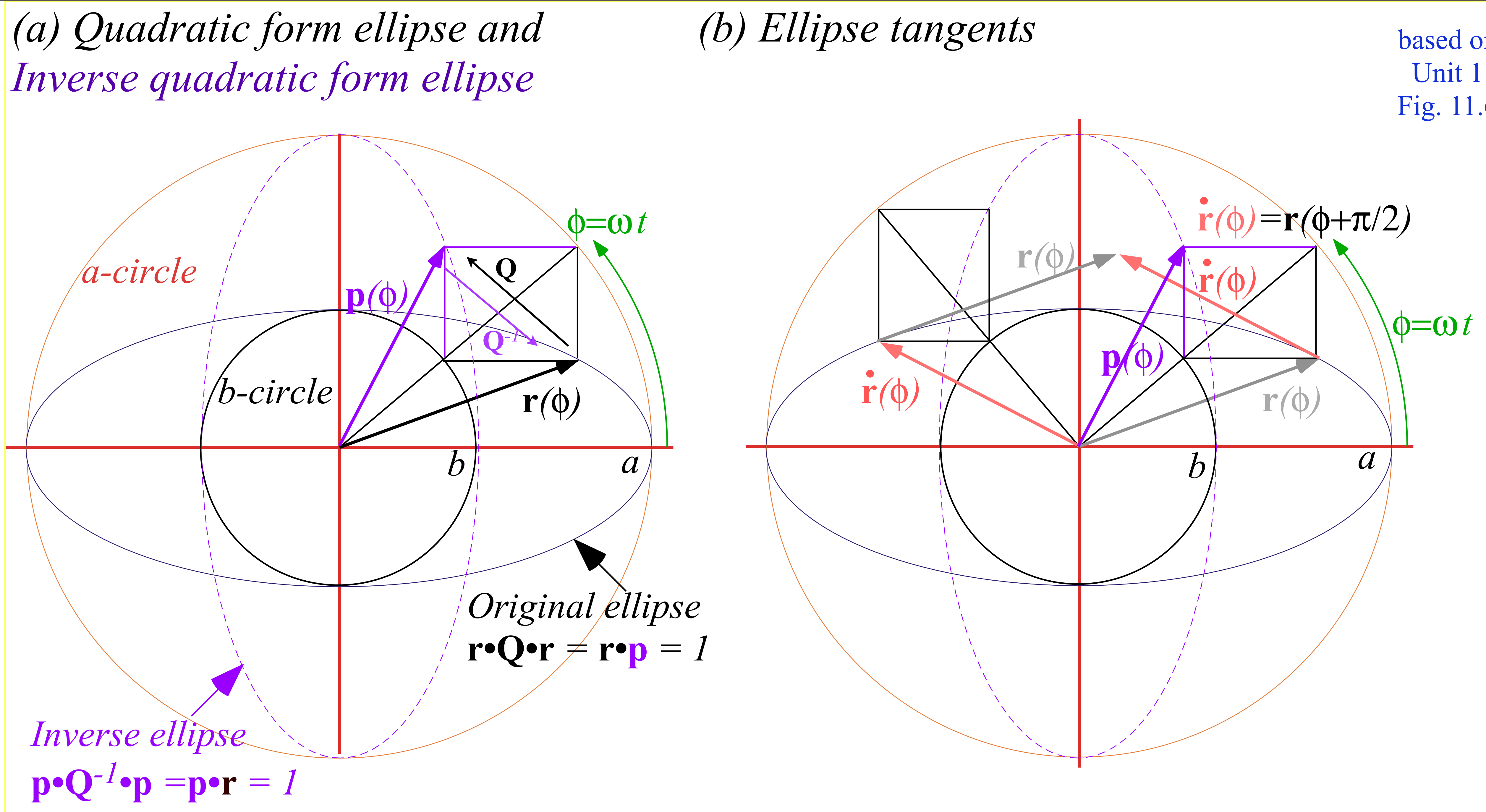
Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$*)*

→ *Q-Ellipse tangents* $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*

Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation



Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q**$^{-1}$**•p** *=1=* **p•r**

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad\text{where: }\begin{matrix}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi\ \ =b\sin\omega t\end{matrix}\quad\text{so: }\boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

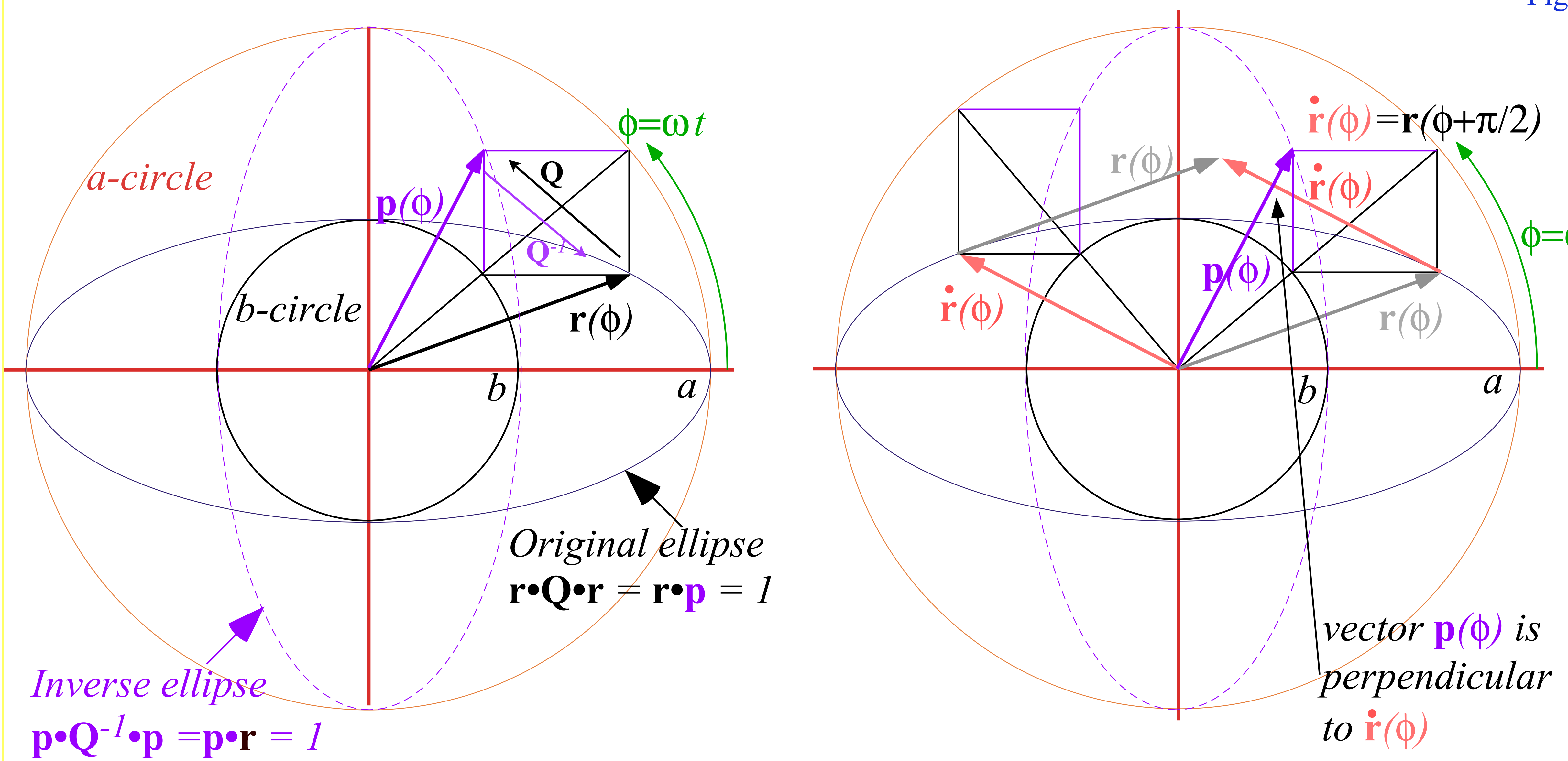
Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

p-ellipse x-radius=1/a plotted at: $S(1/a)=b$ (=1 for a=2, b=1)

p-ellipse y-radius=1/b plotted at: $S(1/b)=a$ (=2 for a=2, b=1)

(a) *Quadratic form ellipse and* *Inverse quadratic form ellipse* (b) *Ellipse tangents*

based on Unit 1 Fig. 11.6



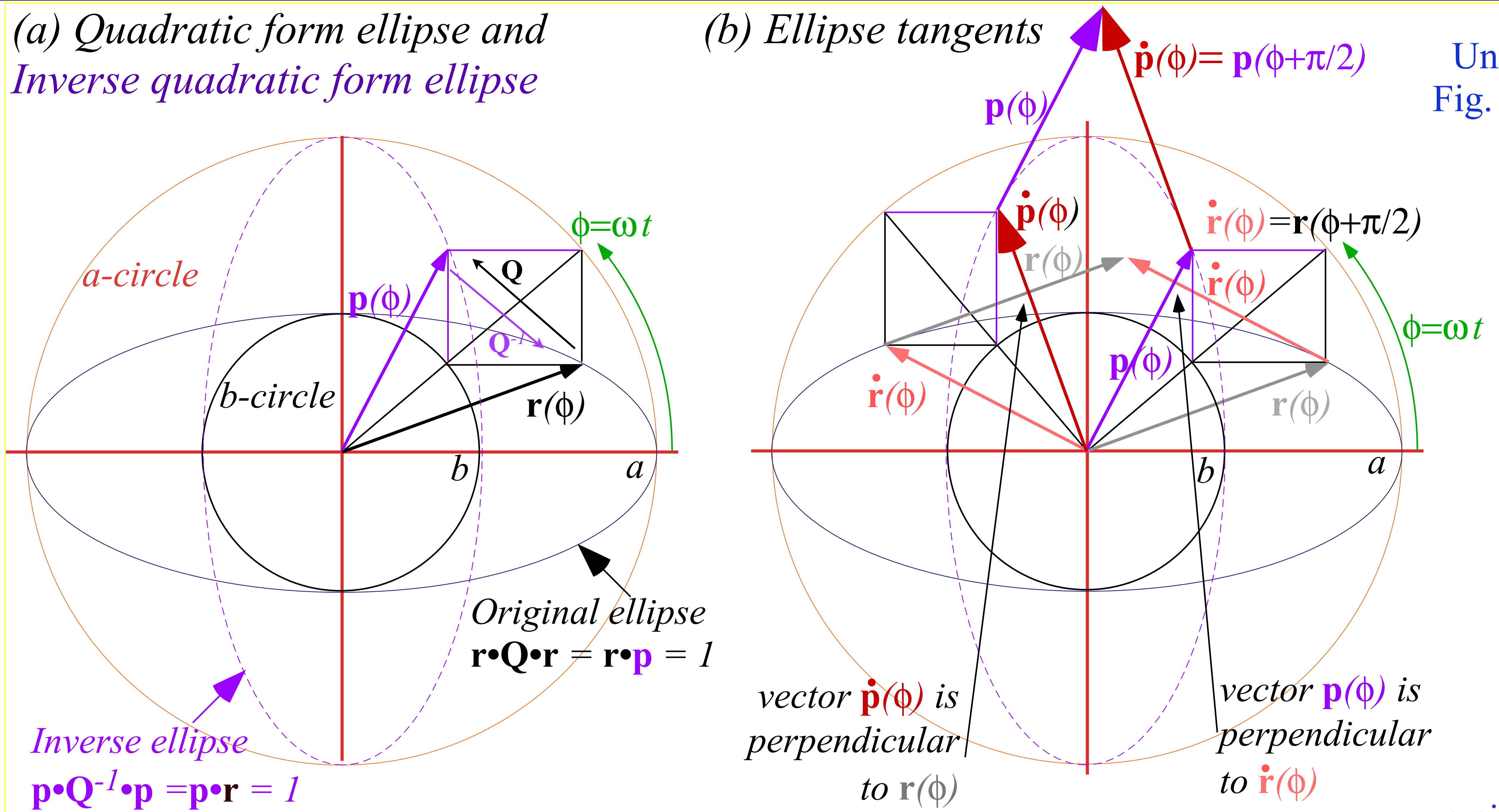
Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* $\mathbf{p}\bullet\mathbf{Q}^{-1}\bullet\mathbf{p}=1=\mathbf{p}\bullet\mathbf{r}$

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{array}{l}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi\ =b\sin\omega t\end{array}\quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

p *is perpendicular to velocity* $\mathbf{v}=\dot{\mathbf{r}}$ *, a mutual orthogonality*

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot{r}_x & \dot{r}_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{array}{l}\dot{r}_x=-a\sin\phi\\ \dot{r}_y=b\cos\phi\end{array}\quad \text{and: } \begin{array}{l}p_x=(1/a)\cos\phi\\ p_y=(1/b)\sin\phi\end{array}$$

Unit 1
Fig. 11.6



Quadratic form **r•Q•r** *=1 has mutual duality relations with inverse form* **p•Q⁻¹•p** *=1* — *unit mutual projection*

$$\mathbf{p}=\mathbf{Q}\bullet\mathbf{r}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\bullet\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}=\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{matrix}x=r_x=a\cos\phi=a\cos\omega t\\ y=r_y=b\sin\phi\ =b\sin\omega t\end{matrix}\quad \text{so: } \boxed{\mathbf{p}\bullet\mathbf{r}=1}$$

$\mathbf{p}$ is perpendicular to velocity $\mathbf{v}=\dot{\mathbf{r}}$ *, a mutual orthogonality. So is* $\mathbf{r}$ *perpendicular to* $\dot{\mathbf{p}}$*:* $\boxed{\dot{\mathbf{p}}\bullet\mathbf{r}=0}$

$$\boxed{\dot{\mathbf{r}}\bullet\mathbf{p}=0}=\begin{pmatrix}\dot{r}_x & \dot{r}_y\end{pmatrix}\bullet\begin{pmatrix}p_x\\ p_y\end{pmatrix}=\begin{pmatrix}-a\sin\phi & b\cos\phi\end{pmatrix}\bullet\begin{pmatrix}(1/a)\cos\phi\\ (1/b)\sin\phi\end{pmatrix}\quad \text{where: } \begin{matrix}\dot{r}_x=-a\sin\phi\\ \dot{r}_y=b\cos\phi\end{matrix}\quad \text{and: } \begin{matrix}p_x=(1/a)\cos\phi\\ p_y=(1/b)\sin\phi\end{matrix}$$

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

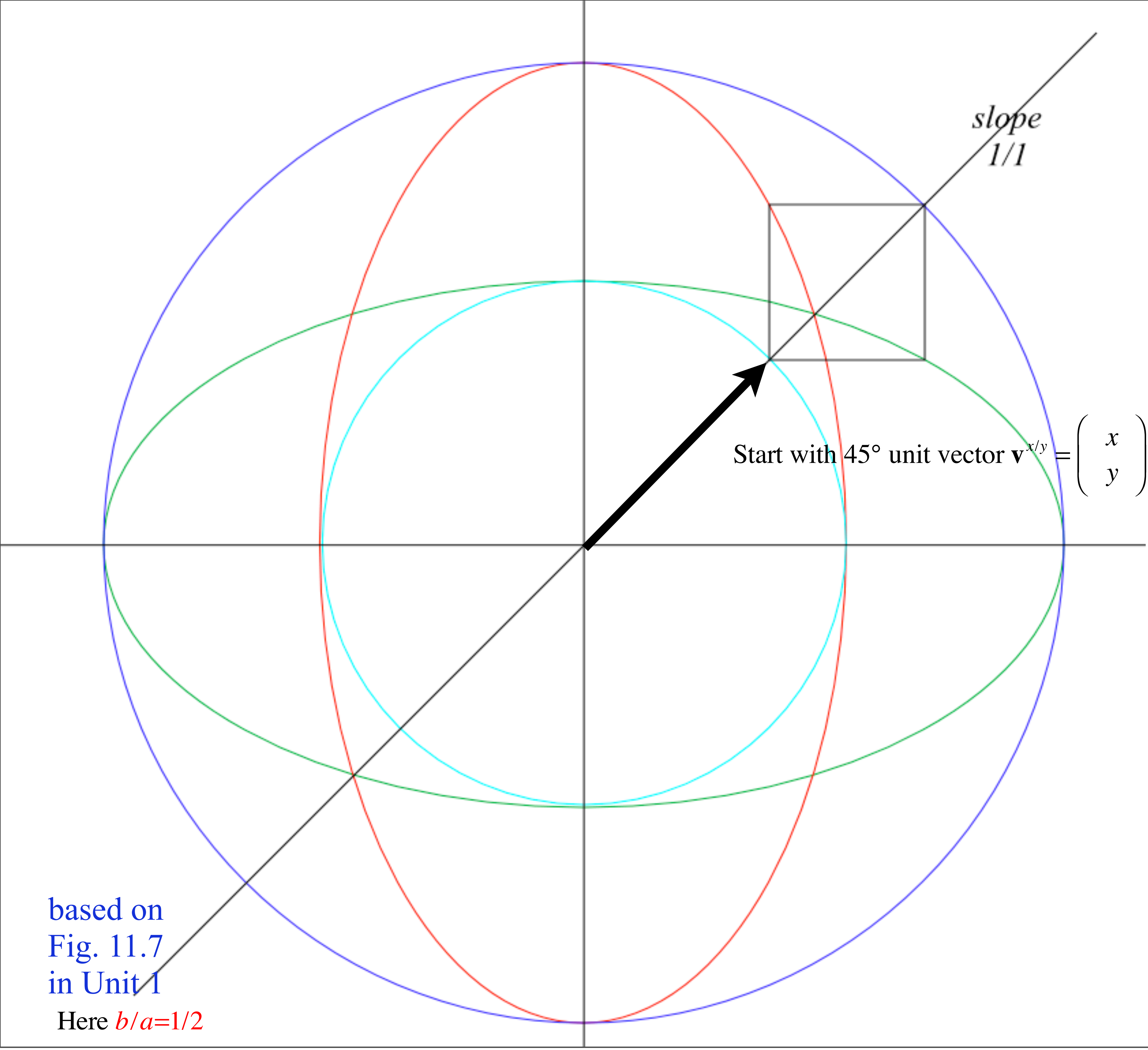
Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$*)*

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*

→ *Operator geometric sequences and eigenvectors*

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation



Start with 45° unit vector $\mathbf{v}^{x/y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$

based on
Fig. 11.7
in Unit 1

Here $b/a = 1/2$

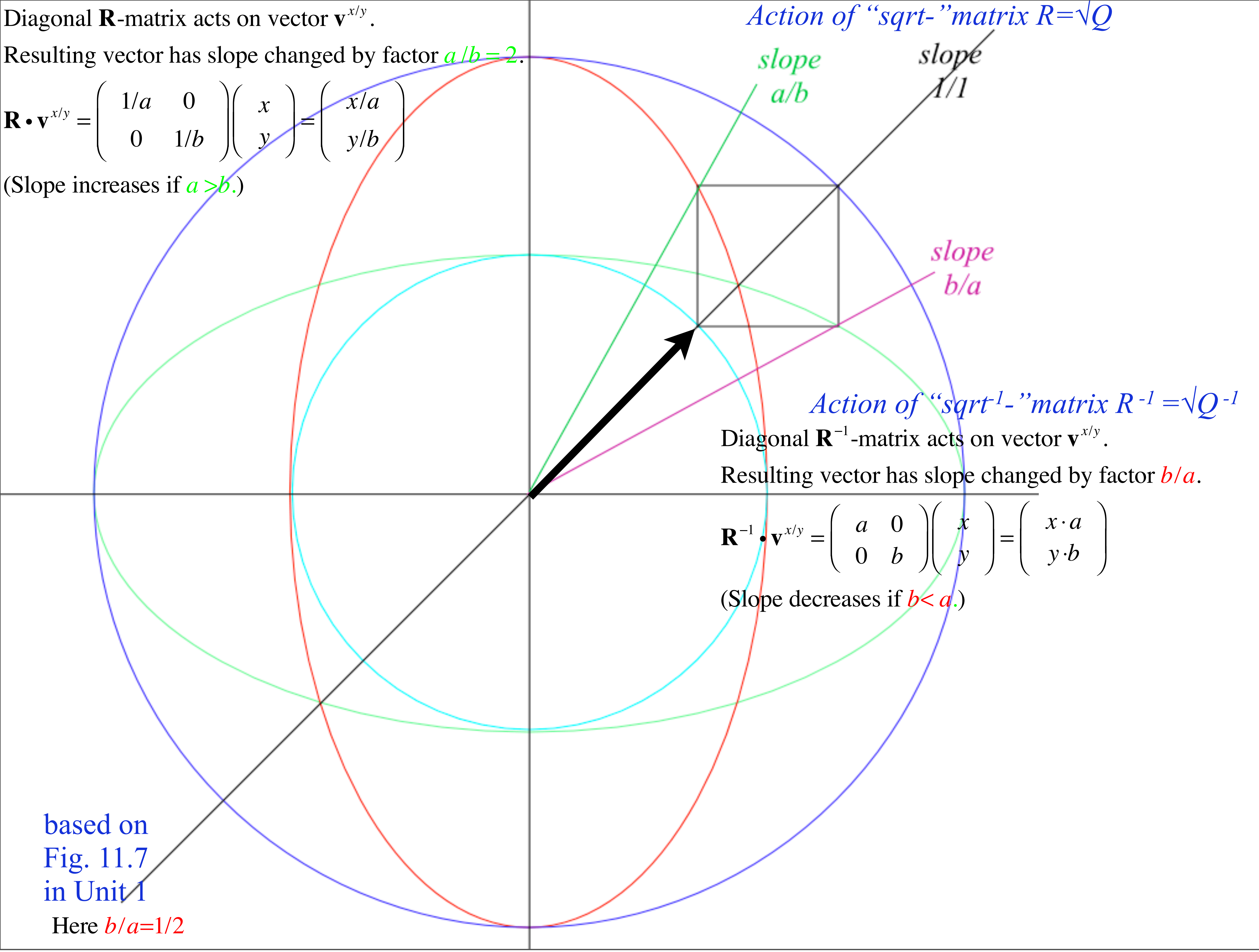
Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(Slope increases if $a>b$.)

Action of "sqrt-"matrix $R=\sqrt{Q}$



Action of "sqrt^{-1}-"matrix $R^{-1}=\sqrt{Q}^{-1}$

Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor b/a.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cdot a\\ y\cdot b\end{pmatrix}$$

(Slope decreases if $b<a$.)

based on
Fig. 11.7
in Unit 1

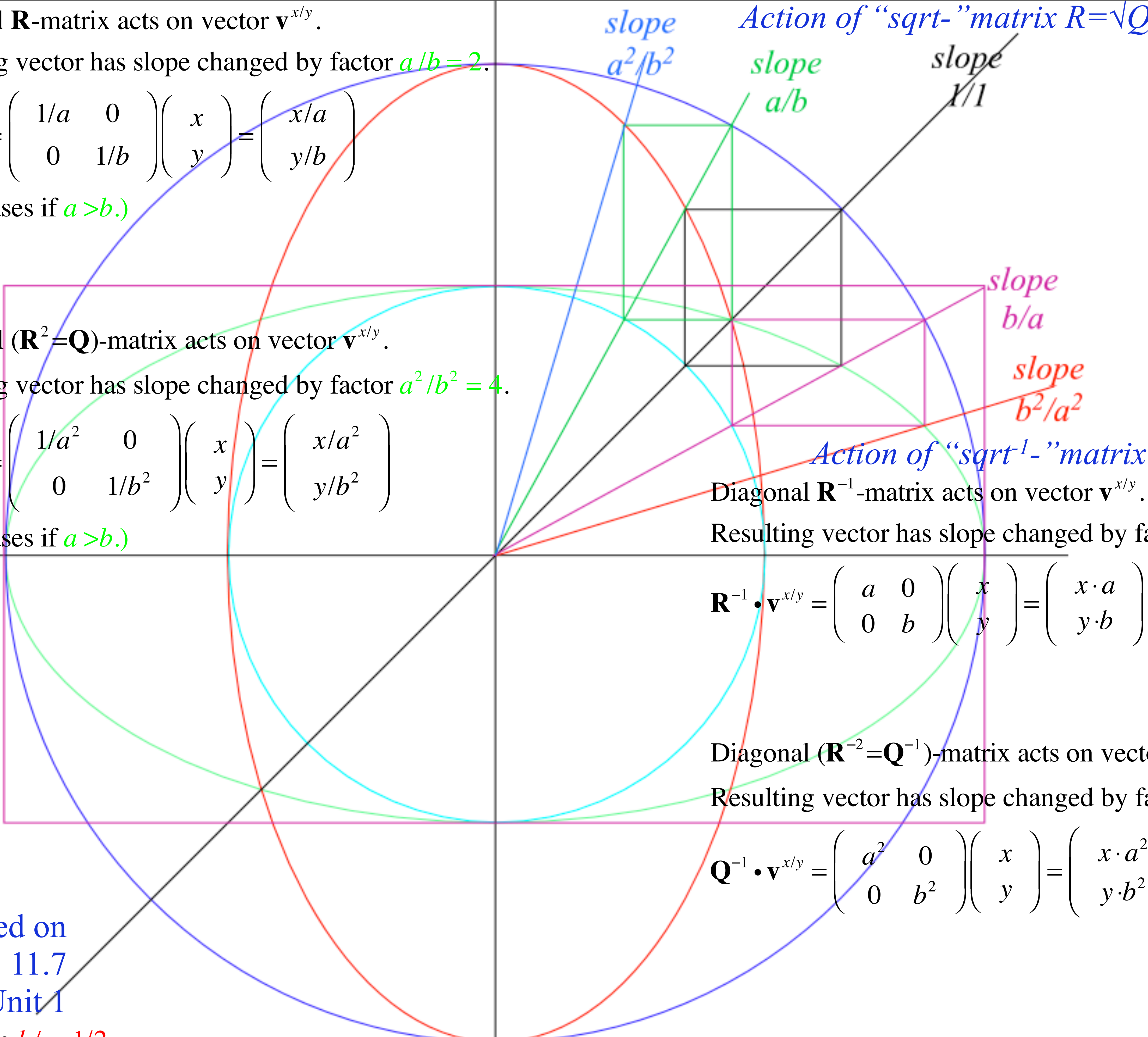
Here $b/a=1/2$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b=2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a & 0\\ 0 & 1/b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a\\ y/b\end{pmatrix}$$

(It increases if $a>b$.)



Diagonal ($\mathbf{R}^2$=**Q**)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2=4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x/a^2\\ y/b^2\end{pmatrix}$$

(It increases if $a>b$.)

Diagonal $\mathbf{R}^{-1}$-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor b/a=1/2.

$$\mathbf{R}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cdot a\\ y\cdot b\end{pmatrix}$$

Diagonal ($\mathbf{R}^{-2}$=$\mathbf{Q}^{-1}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor b^2/a^2=1/4.

$$\mathbf{Q}^{-1}\bullet\mathbf{v}^{x/y}=\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\cdot a^2\\ y\cdot b^2\end{pmatrix}$$

based on
Fig. 11.7
in Unit 1

Here b/a=1/2

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

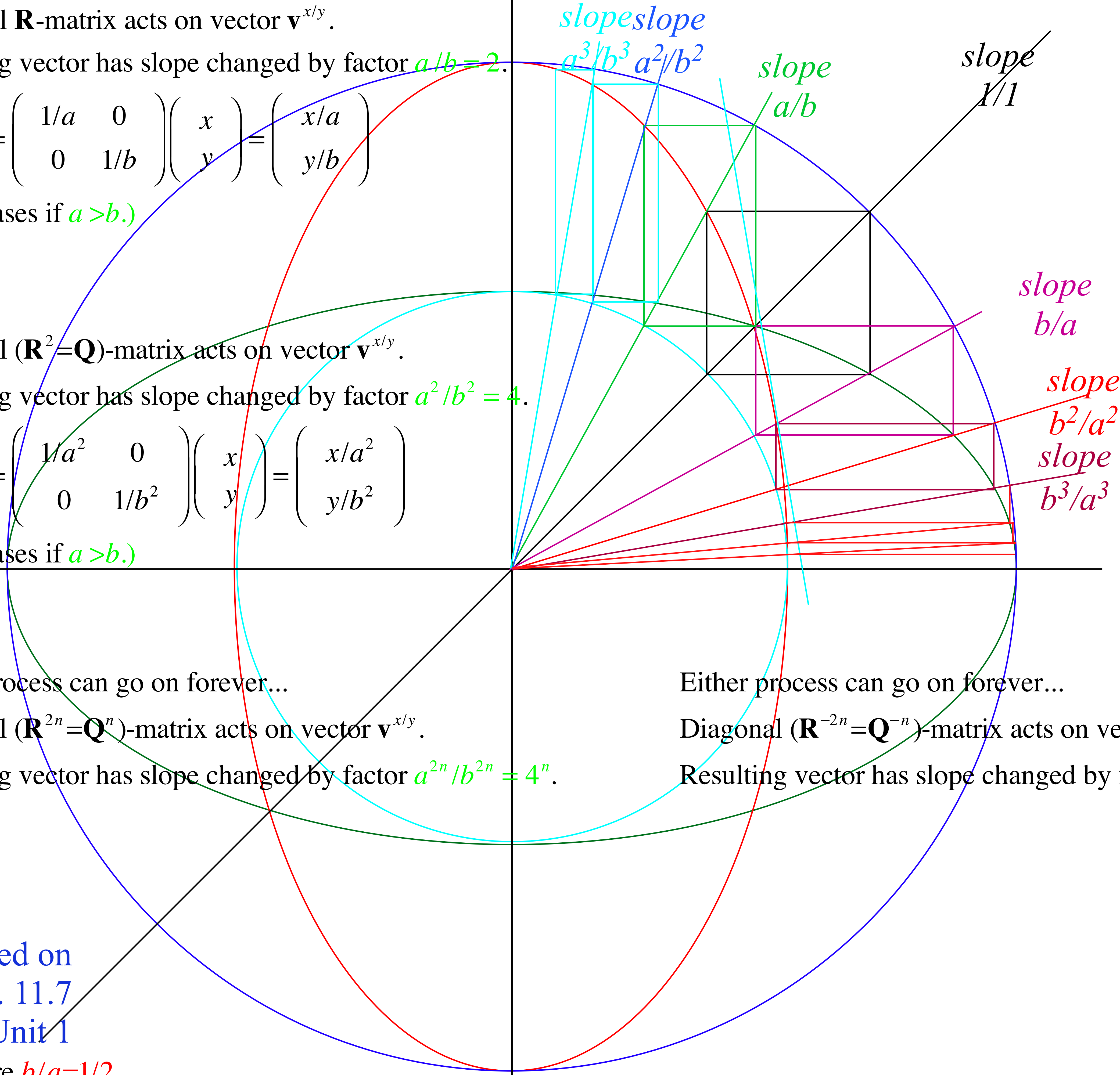
(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)



Either process can go on forever...

Diagonal ($\mathbf{R}^{2n} = \mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n} = \mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

based on Fig. 11.7 in Unit 1

Here $b/a = 1/2$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

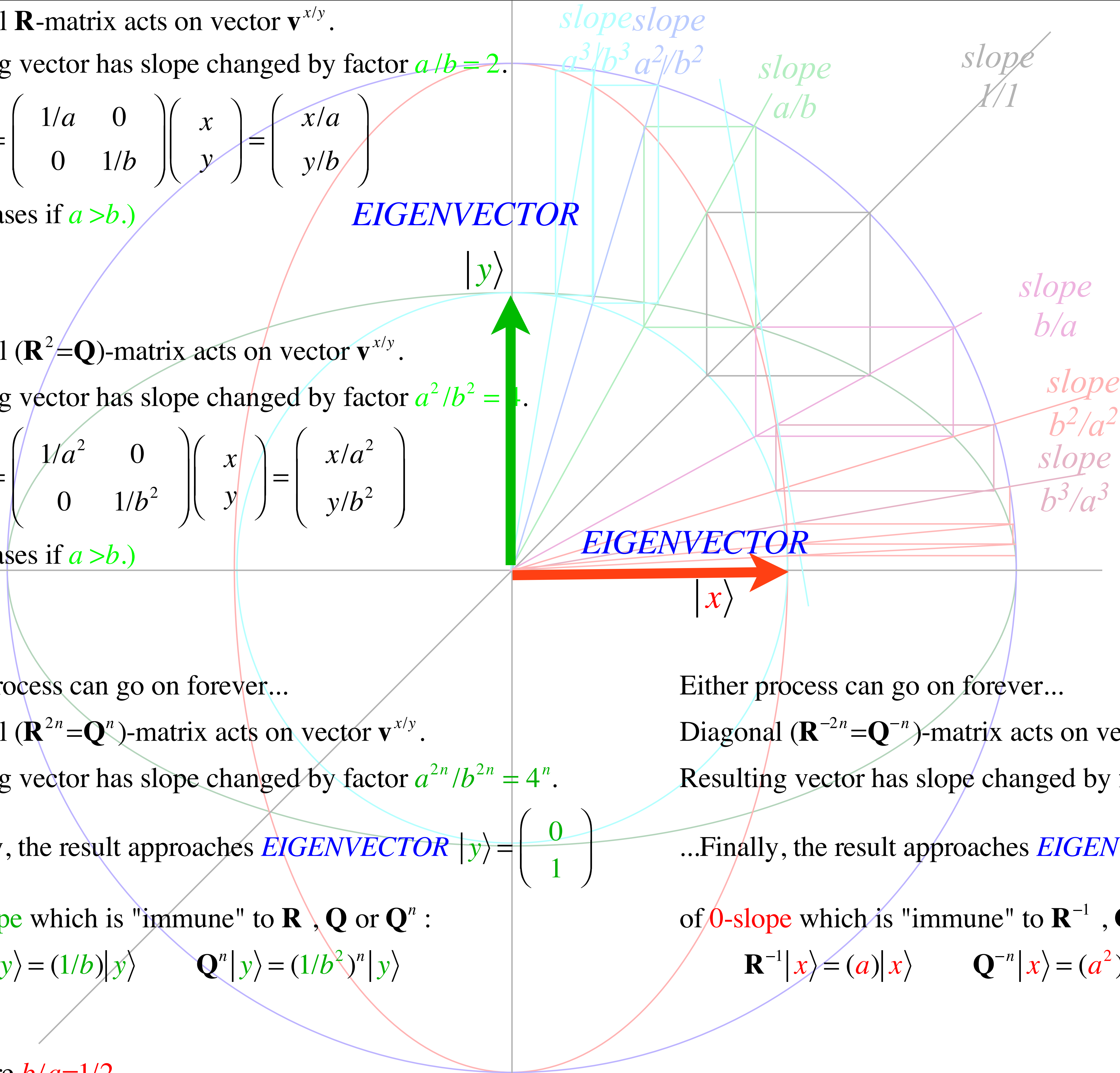
(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2 = \mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)



Either process can go on forever...

Diagonal ($\mathbf{R}^{2n} = \mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

$$\mathbf{R}|y\rangle = (1/b)|y\rangle \qquad \mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$$

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n} = \mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

$$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle \qquad \mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$$

Here $b/a = 1/2$

Here $b/a=1/2$

Diagonal **R**-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a/b = 2$.

$$\mathbf{R}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a \\ y/b \end{pmatrix}$$

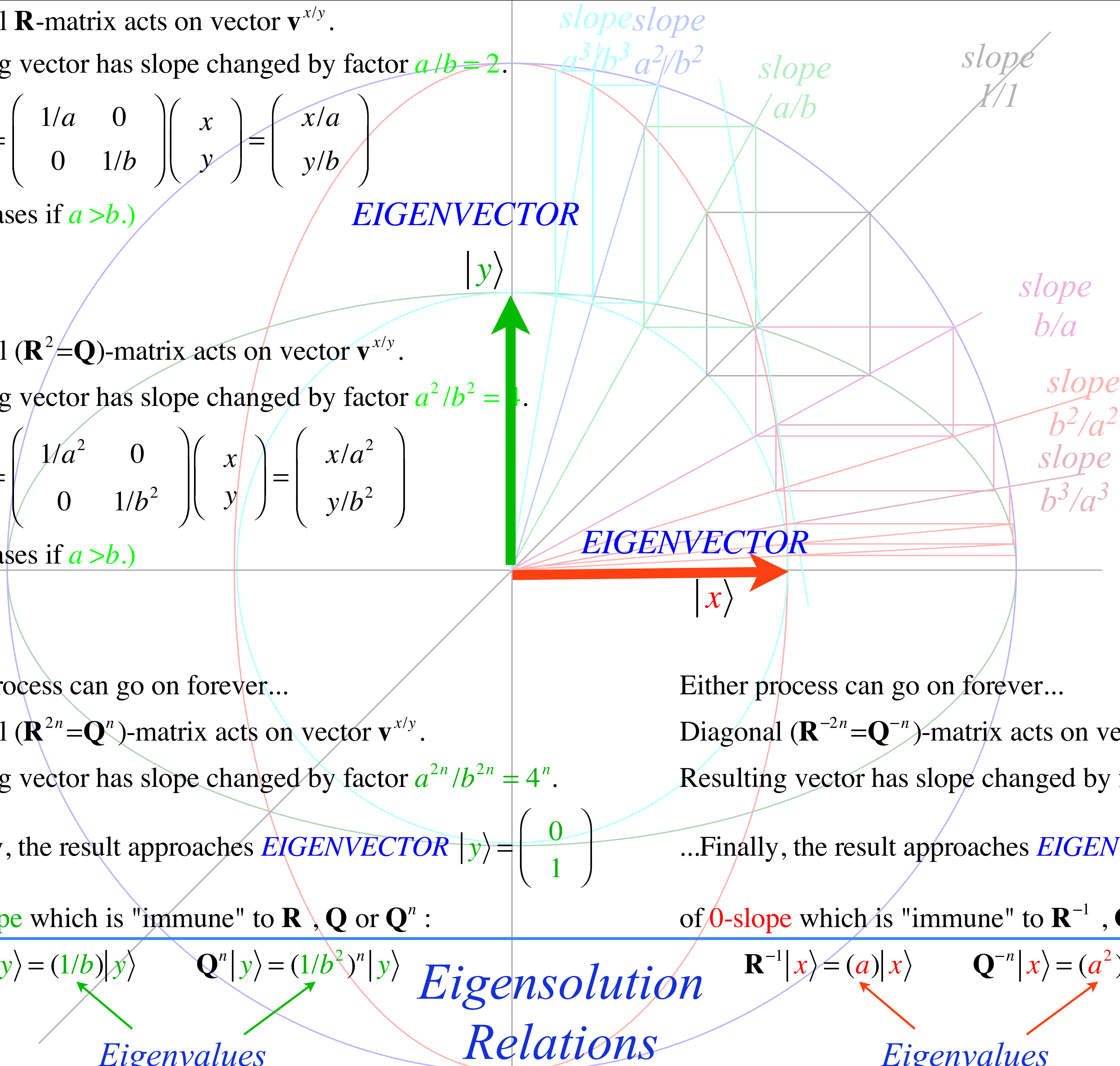
(It increases if $a > b$.)

Diagonal ($\mathbf{R}^2=\mathbf{Q}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^2/b^2 = 4$.

$$\mathbf{Q}\bullet\mathbf{v}^{x/y} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/a^2 \\ y/b^2 \end{pmatrix}$$

(It increases if $a > b$.)



Either process can go on forever...

Diagonal ($\mathbf{R}^{2n}=\mathbf{Q}^n$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $a^{2n}/b^{2n} = 4^n$.

...Finally, the result approaches *EIGENVECTOR* $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

of ∞-slope which is "immune" to **R** , **Q** or $\mathbf{Q}^n$:

Either process can go on forever...

Diagonal ($\mathbf{R}^{-2n}=\mathbf{Q}^{-n}$)-matrix acts on vector $\mathbf{v}^{x/y}$.

Resulting vector has slope changed by factor $b^{2n}/a^{2n} = 4^{-n}$.

...Finally, the result approaches *EIGENVECTOR* $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

of 0-slope which is "immune" to $\mathbf{R}^{-1}$, $\mathbf{Q}^{-1}$ or $\mathbf{Q}^{-n}$:

Eigensolution Relations

$\mathbf{R}|y\rangle = (1/b)|y\rangle$ $\quad$ $\mathbf{Q}^n|y\rangle = (1/b^2)^n|y\rangle$

Eigenvalues

$\mathbf{R}^{-1}|x\rangle = (a)|x\rangle$ $\quad$ $\mathbf{Q}^{-n}|x\rangle = (a^2)^n|x\rangle$

Eigenvalues

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$*)*

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*

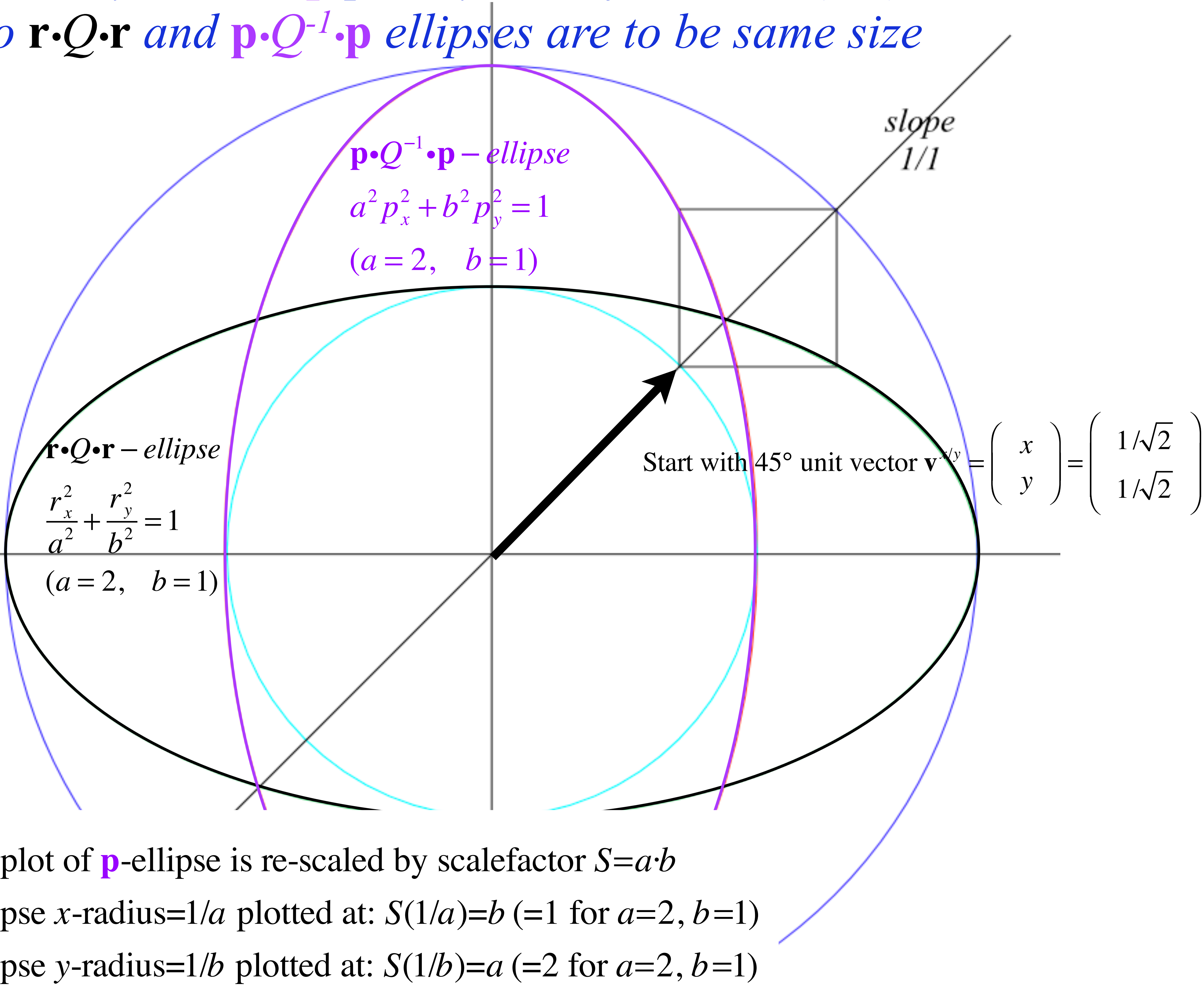
Operator geometric sequences and eigenvectors

→ *Alternative scaling of matrix operator geometry*

Vector calculus of tensor operation

You may rescale **p**-*plot by scale factor* $S=(a\cdot b)$ *so* $\mathbf{r}\cdot Q\cdot\mathbf{r}$ *and* $\mathbf{p}\cdot Q^{-1}\cdot\mathbf{p}$ *ellipses are to be same size*

Here $b/a=1/2$



Here plot of **p**-ellipse is re-scaled by scalefactor $S=a\cdot b$

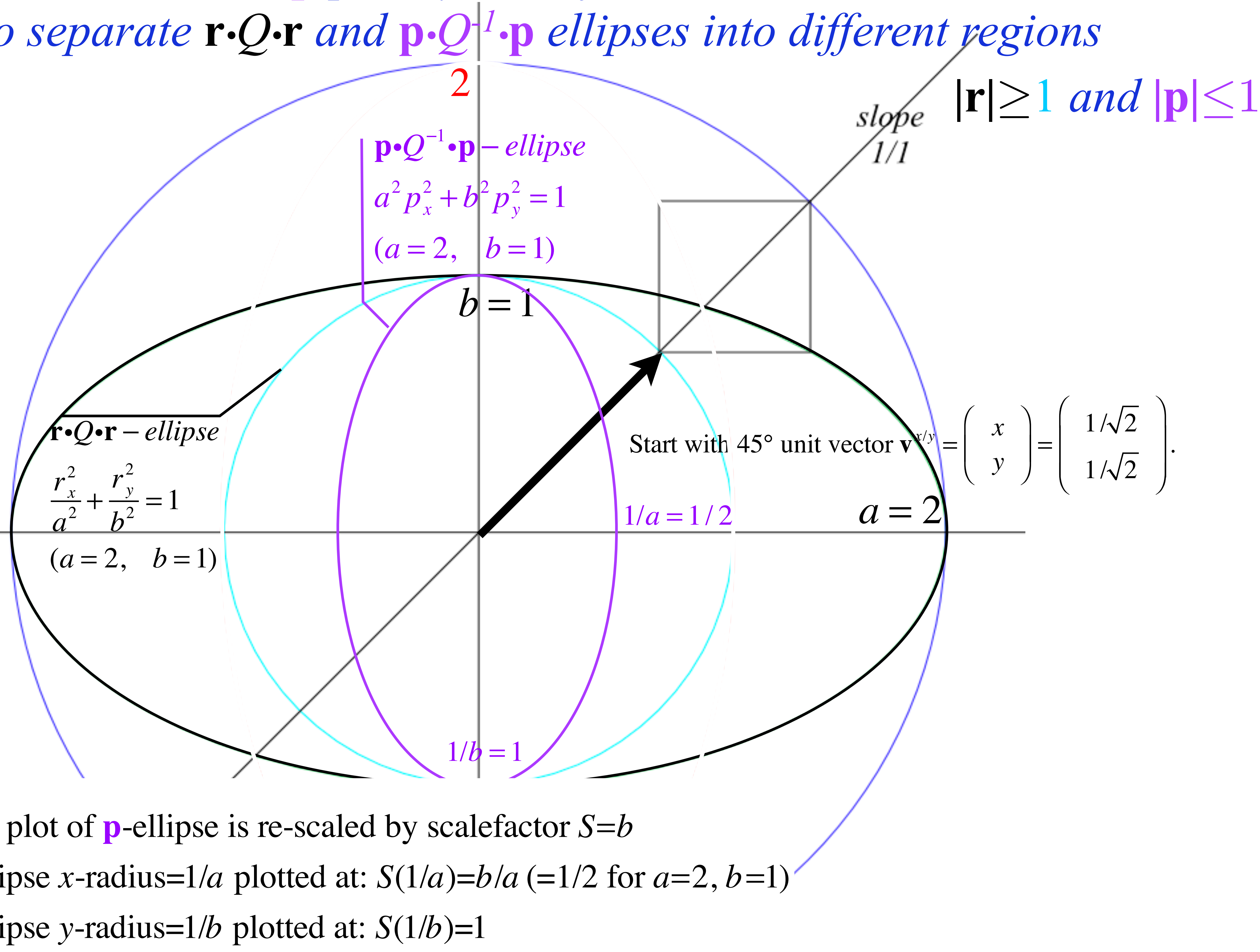
p-ellipse x-radius$=1/a$ plotted at: $S(1/a)=b$ ($=1$ for $a=2$, $b=1$)

p-ellipse y-radius$=1/b$ plotted at: $S(1/b)=a$ ($=2$ for $a=2$, $b=1$)

..or else rescale **p***-plot by scale factor S=b*

to separate **r**$\cdot Q \cdot$**r** *and* **p**$\cdot Q^{-1} \cdot$**p** *ellipses into different regions*

Here b/a=1/2



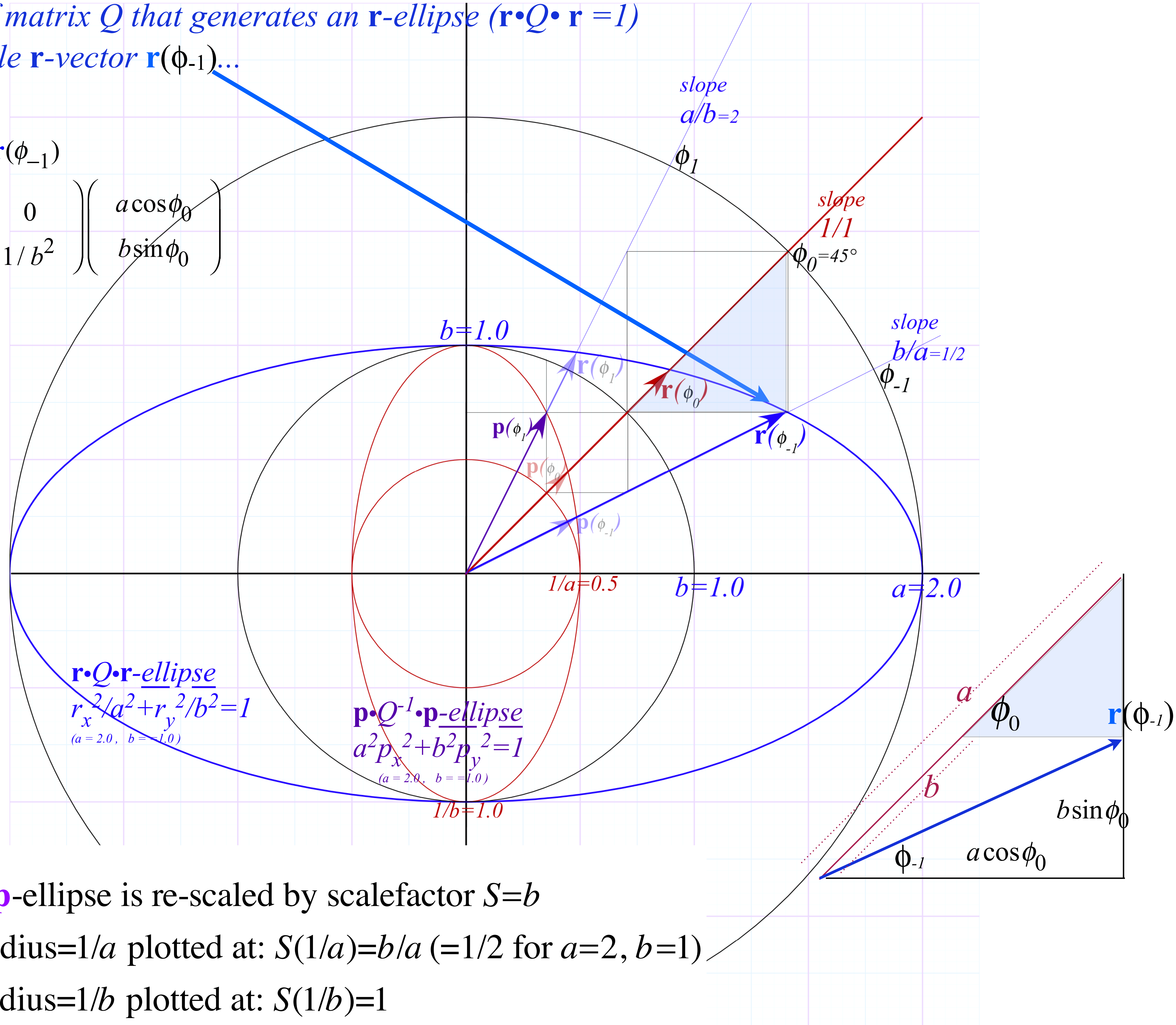
Here plot of **p**-ellipse is re-scaled by scalefactor $S=b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b/a$ (=1/2 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)$=1

Action of matrix Q that generates an **r***-ellipse (***r***•Q•* **r** *=1)*
on a single **r***-vector* **r**(ϕ_{-1})...

$$\mathbf{p}(\phi_1) = \mathbf{Q} \cdot \mathbf{r}(\phi_{-1})$$
$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

Here plot of **p**-ellipse is re-scaled by scalefactor $S=b$

p-ellipse x-radius=$1/a$ plotted at: $S(1/a)=b/a$ (=1/2 for a=2, b=1)

p-ellipse y-radius=$1/b$ plotted at: $S(1/b)=1$

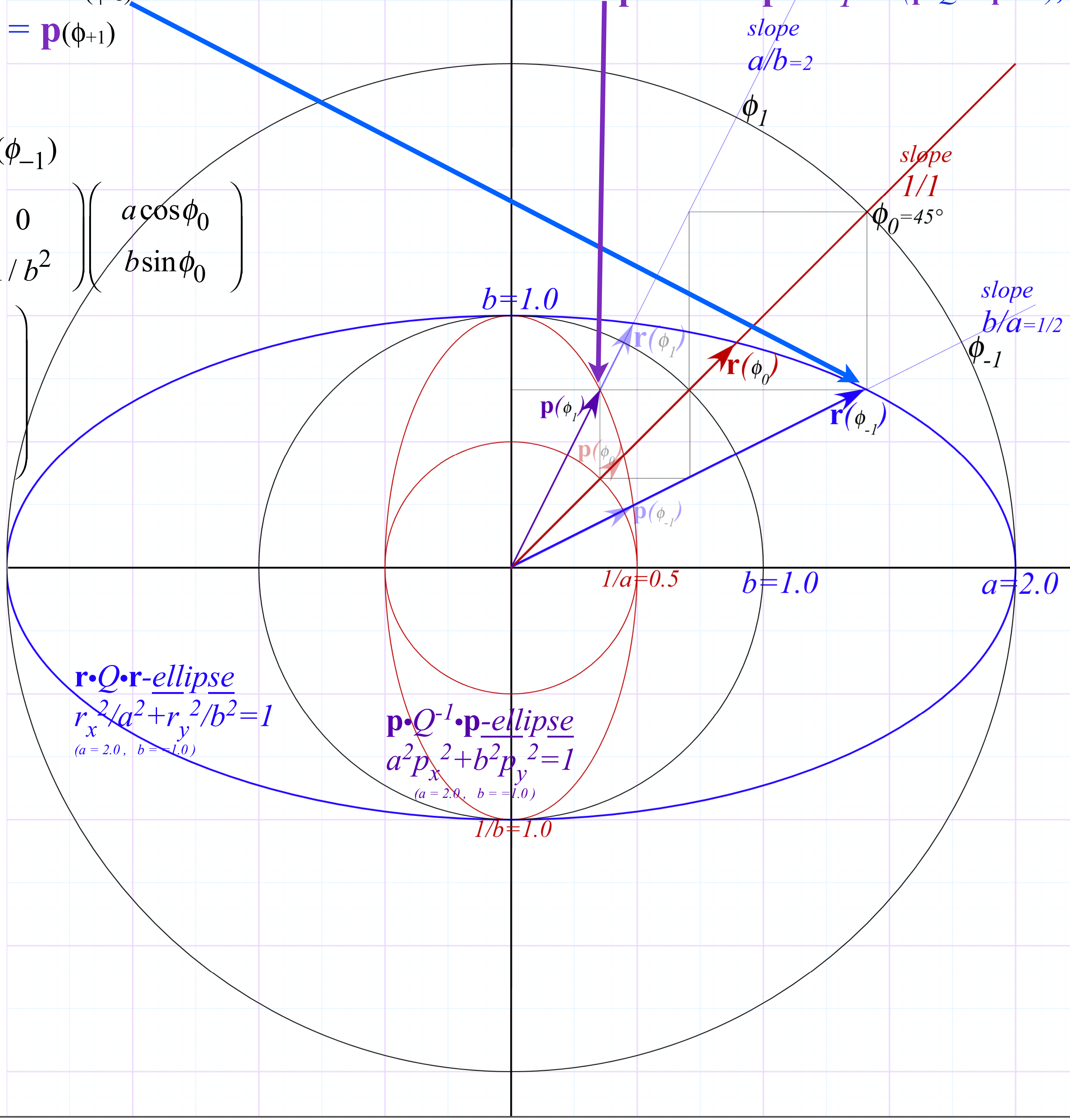
Action of matrix Q that generates an **r**-*ellipse* (**r**•*Q*•**r** =*1*)
on a single **r**-*vector* **r**(ϕ_{-1})... *is to rotate it to a new vector* **p** *on the* **p**-*ellipse* (**p**•Q^{-1}• **p** =*1*),
that is, Q•**r**(ϕ_{-1}) = **p**(ϕ_{+1})

$$\mathbf{p}(\phi_1) = \mathbf{Q} \bullet \mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix} \begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

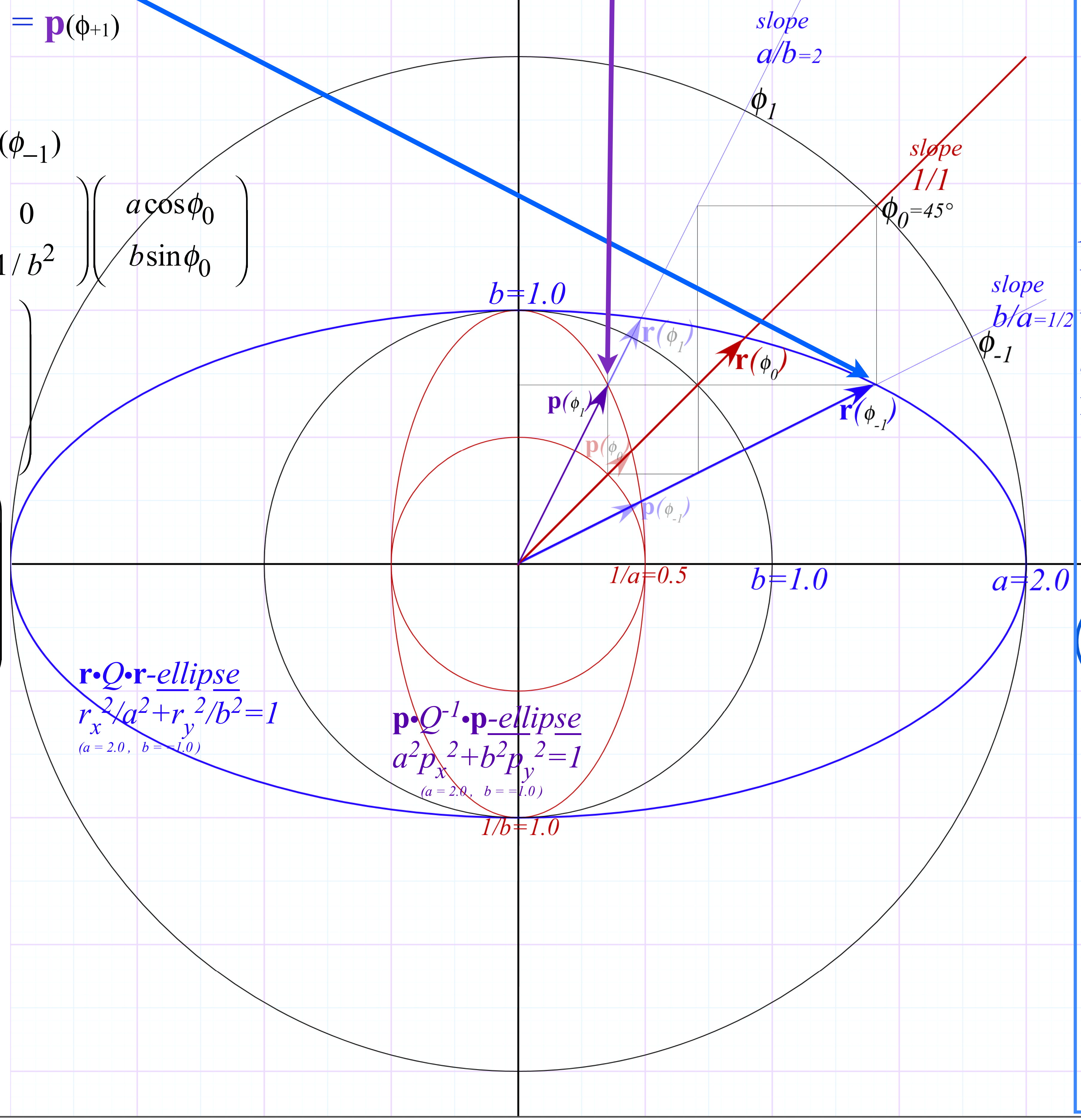
Action of matrix Q that generates an **r***-ellipse* (**r**•*Q*•**r** *=1*)
on a single **r***-vector* **r**(ϕ_{-1})*... is to rotate it to a new vector* **p** *on the* **p***-ellipse* (**p**•Q^{-1}• **p** *=1*),
that is, *Q*•**r**(ϕ_{-1}) = **p**(ϕ_{+1})

$$\mathbf{p}(\phi_1) = \mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$= \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/b^2 \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}\cos\phi_0 \\ \frac{1}{b}\sin\phi_0 \end{pmatrix}$$

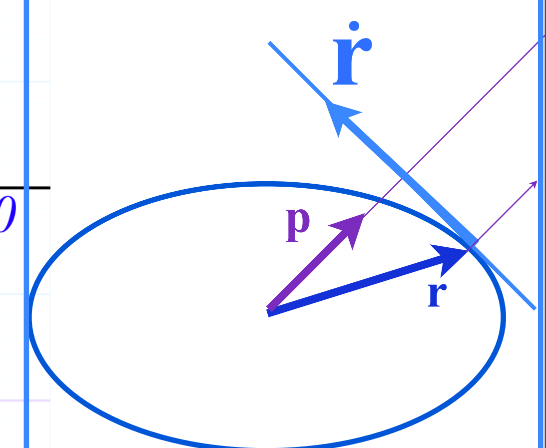
$$= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{2}} \\ \frac{1}{1}\frac{1}{\sqrt{2}} \end{pmatrix}$$



Variation of
Fig. 11.7
in Unit 1

Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* **ṙ** *to its* **r**•*Q*•**r***-ellipse.*



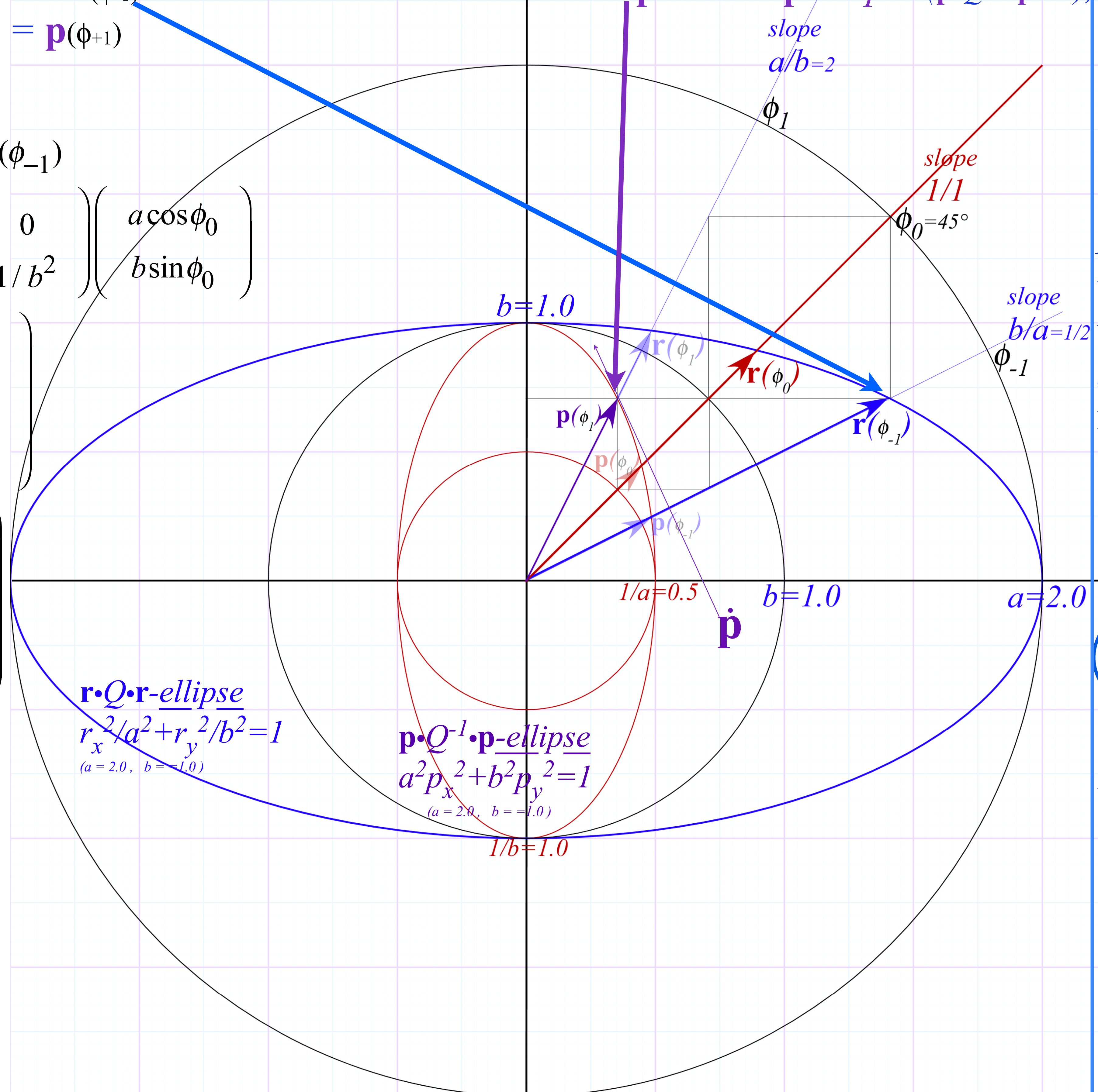
Action of matrix Q that generates an **r***-ellipse* (**r**•*Q*•**r** *=1)* *on a single* **r***-vector* **r**(ϕ-1)*... is to rotate it to a new vector* **p** *on the* **p***-ellipse* (**p**•Q^{-1}• **p** *=1),* *that is,* *Q*•**r**(ϕ-1) = **p**(ϕ+1)

$$\mathbf{p}(\phi_1)=\mathbf{Q}\bullet\mathbf{r}(\phi_{-1})$$

$$=\begin{pmatrix}1/a^2 & 0\\ 0 & 1/b^2\end{pmatrix}\begin{pmatrix}a\cos\phi_0\\ b\sin\phi_0\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{a}\cos\phi_0\\ \frac{1}{b}\sin\phi_0\end{pmatrix}$$

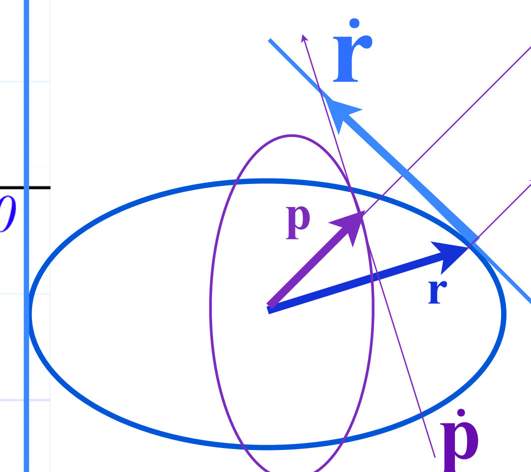
$$=\begin{pmatrix}\frac{1}{2}\frac{1}{\sqrt{2}}\\ \frac{1}{1}\frac{1}{\sqrt{2}}\end{pmatrix}$$



Variation of Fig. 11.7 in Unit 1

Key points of matrix geometry:

Matrix Q maps any vector **r** *to a new vector* **p** *normal to the tangent* **ṙ** *to its* **r**•*Q*•**r***-ellipse.*



Matrix Q^{-1} *maps* **p** *back to* **r** *that is normal to the tangent* **ṗ** *to its* **p**• Q^{-1}• **p***-ellipse.*

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

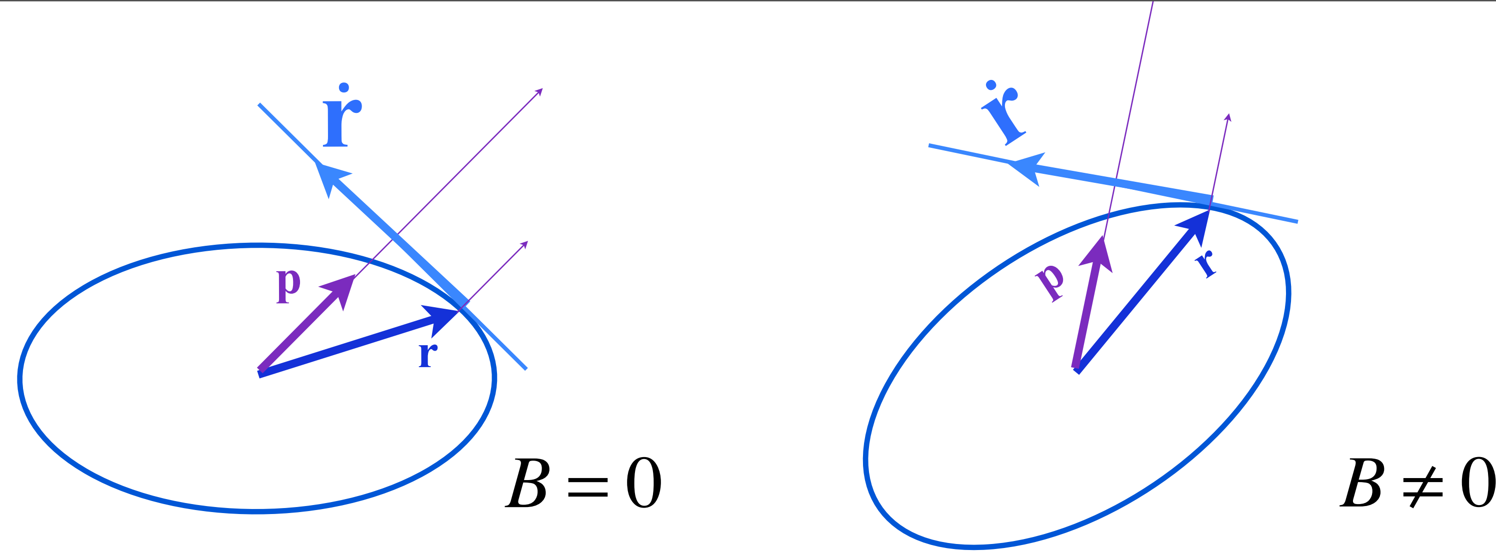
Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$*)*

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*

Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

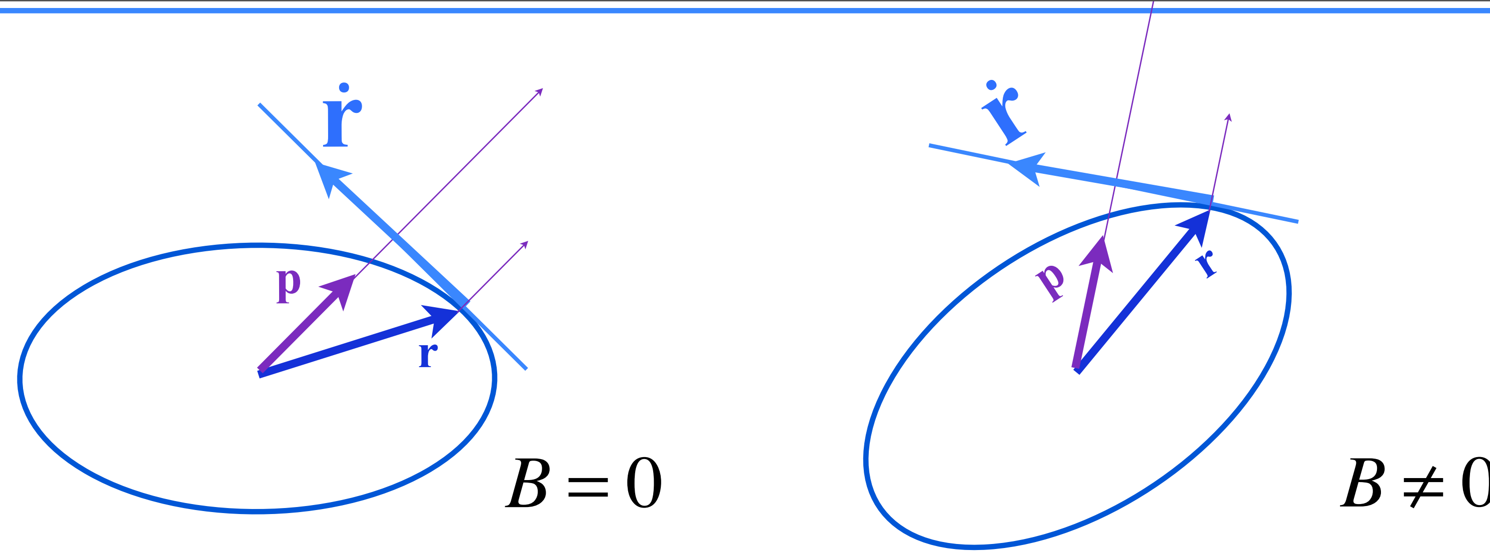
→ *Vector calculus of tensor operation*



Derive matrix "normal-to-ellipse" geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse $1 = \mathbf{r} \bullet Q \bullet \mathbf{r} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A & B \\ B & D \end{pmatrix} \bullet \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \bullet \begin{pmatrix} A \cdot x + B \cdot y \\ B \cdot x + D \cdot y \end{pmatrix} = A \cdot x^2 + 2B \cdot xy + D \cdot y^2 = 1$



Derive matrix "normal-to-ellipse" geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

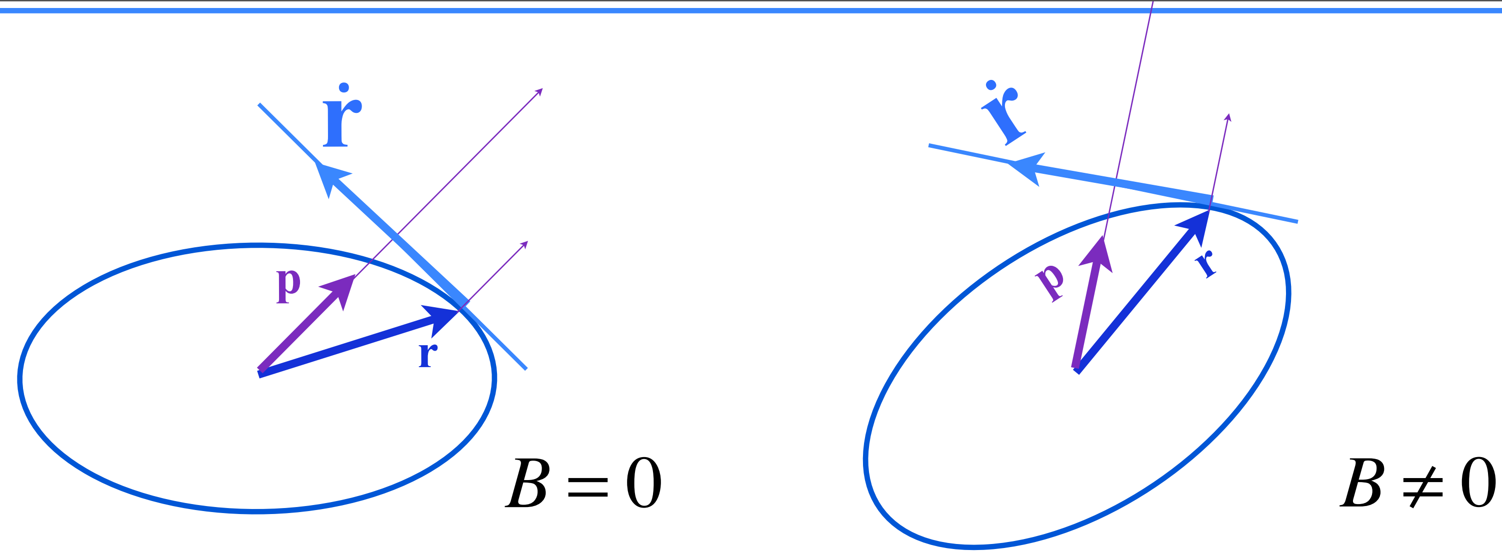
define the ellipse $1=\mathbf{r}\bullet Q\bullet\mathbf{r} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A\cdot x+B\cdot y \\ B\cdot x+D\cdot y \end{pmatrix} = A\cdot x^2+2B\cdot xy+D\cdot y^2=1$

Compare operation by Q on vector **r** *with* *vector derivative or gradient of* **r**•*Q*•**r**

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\cdot x+B\cdot y \\ B\cdot x+D\cdot y \end{pmatrix}$$

$$\frac{\partial}{\partial\mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r}) = \nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2+2B\cdot xy+D\cdot y^2\right) = \begin{pmatrix} 2A\cdot x+2B\cdot y \\ 2B\cdot x+2D\cdot y \end{pmatrix}$$



Derive matrix "normal-to-ellipse" geometry by vector calculus:

Let matrix $Q = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$

define the ellipse $1 = \mathbf{r}\bullet Q\bullet\mathbf{r} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix}\bullet\begin{pmatrix} A\cdot x + B\cdot y \\ B\cdot x + D\cdot y \end{pmatrix} = A\cdot x^2 + 2B\cdot xy + D\cdot y^2 = 1$

Compare operation by Q on vector $\mathbf{r}$ *with vector derivative or gradient of* $\mathbf{r}\bullet Q\bullet\mathbf{r}$

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix}\bullet\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\cdot x + B\cdot y \\ B\cdot x + D\cdot y \end{pmatrix}$$

$$\frac{\partial}{\partial\mathbf{r}}(\mathbf{r}\bullet Q\bullet\mathbf{r}) = \nabla(\mathbf{r}\bullet Q\bullet\mathbf{r})$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\left(A\cdot x^2 + 2B\cdot xy + D\cdot y^2\right) = \begin{pmatrix} 2A\cdot x + 2B\cdot y \\ 2B\cdot x + 2D\cdot y \end{pmatrix}$$

Very simple result:

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{\mathbf{r}\bullet Q\bullet\mathbf{r}}{2}\right) = \nabla\left(\frac{\mathbf{r}\bullet Q\bullet\mathbf{r}}{2}\right) = Q\bullet\mathbf{r}$$

Introduction to dual matrix operator geometry (based on IHO orbits)

Quadratic form ellipse $\mathbf{r}\bullet Q\bullet\mathbf{r}=1$ *vs.inverse form ellipse* $\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}=1$

Duality norm relations ($\mathbf{r}\bullet\mathbf{p}=1$*)*

Q-Ellipse tangents $\mathbf{r}'$ *normal to dual* Q^{-1}*-ellipse position* $\mathbf{p}$ *(* $\mathbf{r}'\bullet\mathbf{p}=0=\mathbf{r}\bullet\mathbf{p}'$*)*

(Still more) Operator geometric sequences and eigenvectors

Alternative scaling of matrix operator geometry

Vector calculus of tensor operation

Action of "sqrt-"matrix $R=\sqrt{Q}$ on a single **r***-vector* **r**(ϕ_{-1})*... is to rotate it to* **u***-circle (***u•u** *=1), that is, R•***r**(ϕ_{-1}) = **u** *=(const.)***r**(ϕ_0)

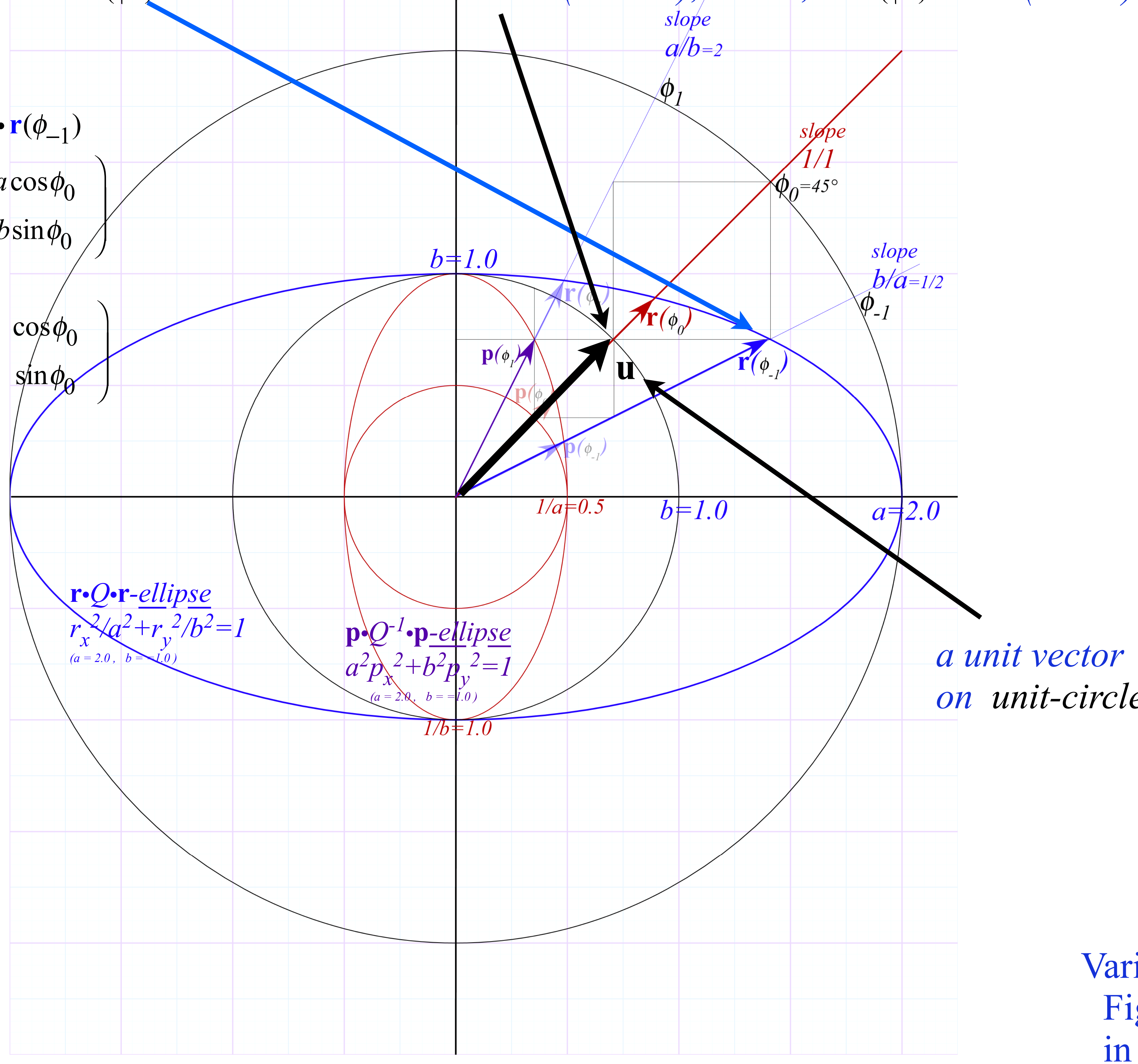
(R generates another ellipse **r•***R***•r** *=1 not shown)*

$$\mathbf{u} = \sqrt{\mathbf{Q}}\bullet\mathbf{r}(\phi_{-1}) = \mathbf{R}\bullet\mathbf{r}(\phi_{-1})$$

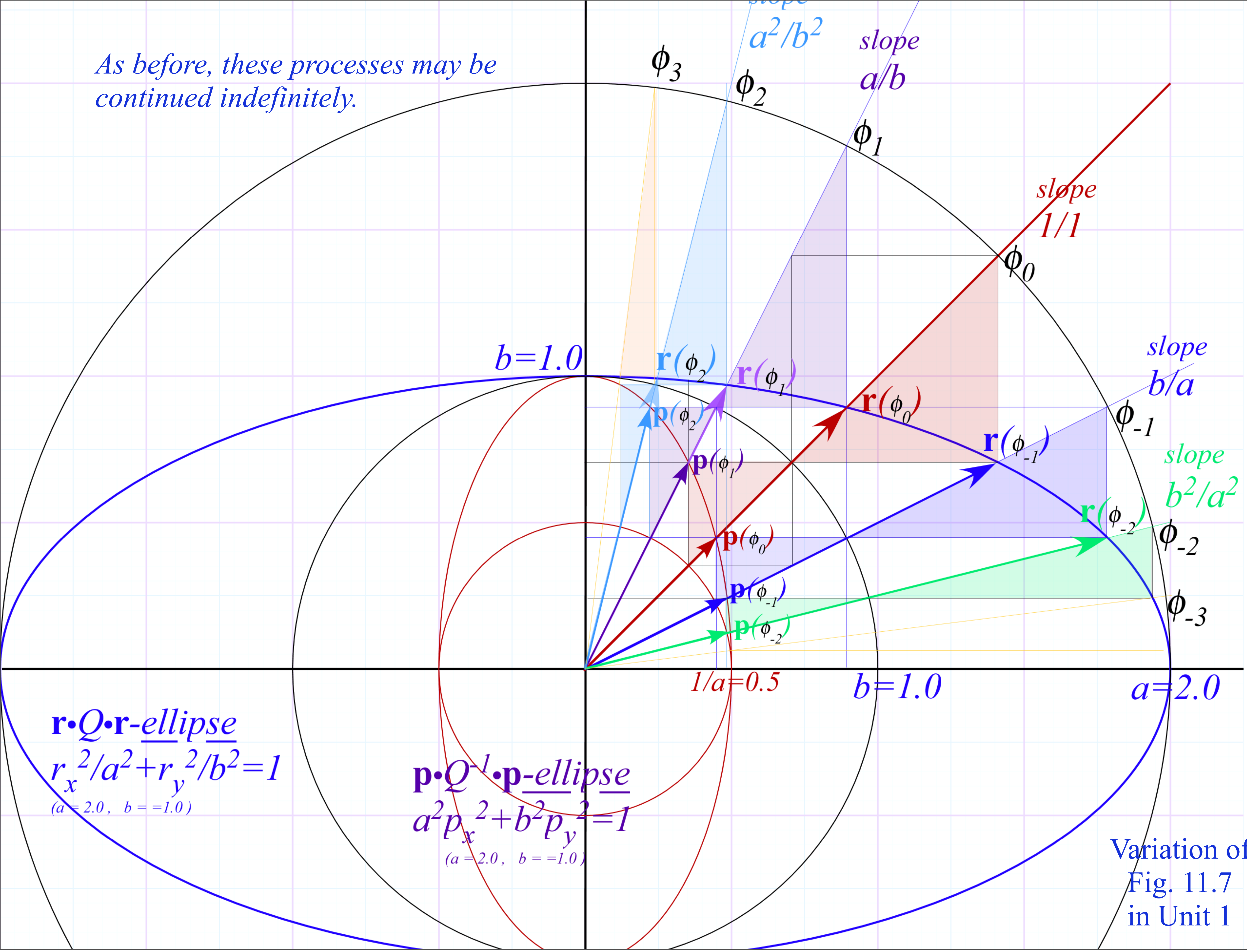
$$= \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix}\begin{pmatrix} a\cos\phi_0 \\ b\sin\phi_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{a}a\cos\phi_0 \\ \frac{1}{b}b\sin\phi_0 \end{pmatrix} = \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix}$$

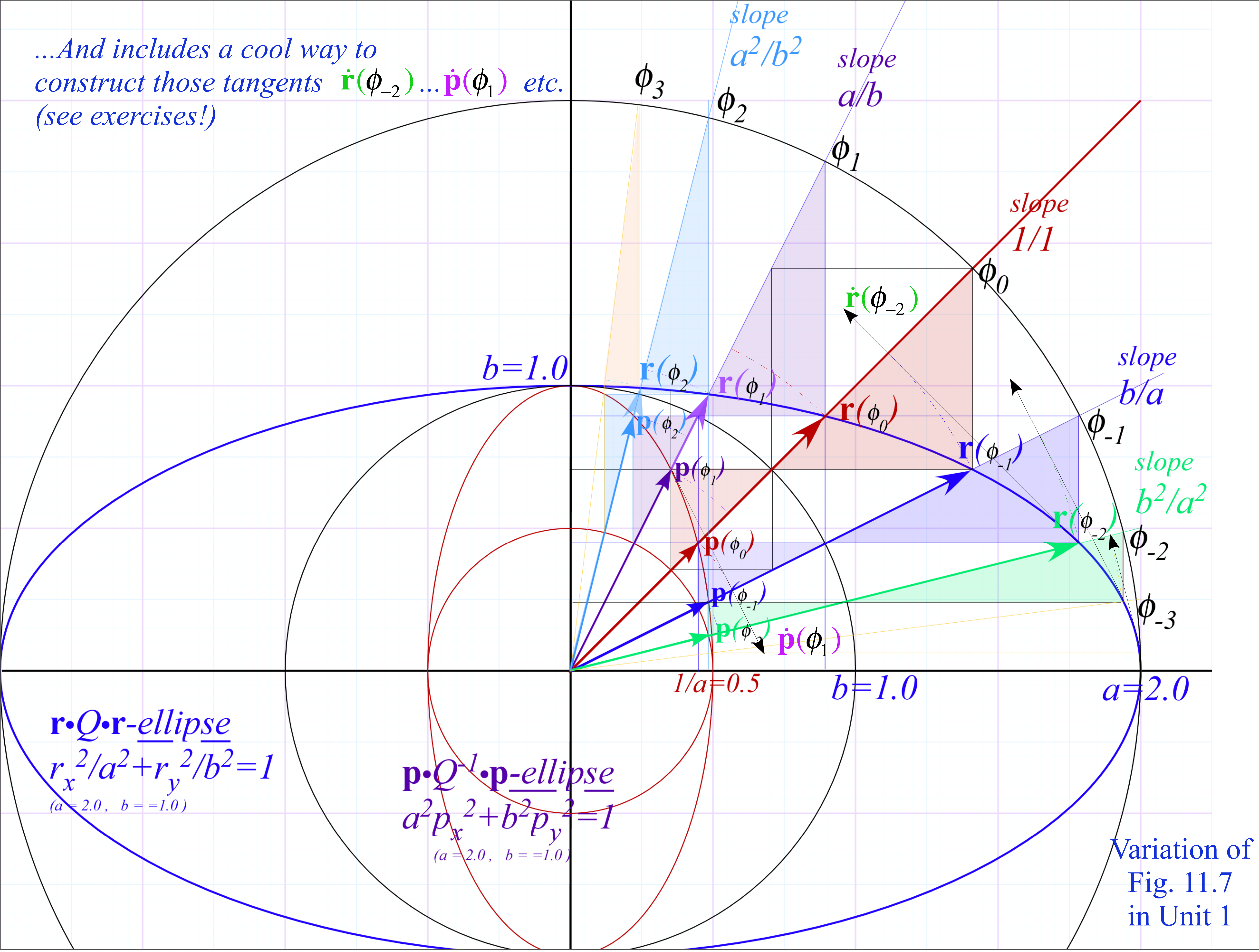
$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



Variation of Fig. 11.7 in Unit 1


As before, these processes may be continued indefinitely.
slope a^2/b^2
slope a/b
slope 1/1
slope b/a
slope b^2/a^2
ϕ_3
ϕ_2
ϕ_1
ϕ_0
ϕ_{-1}
ϕ_{-2}
ϕ_{-3}
b=1.0
$\mathbf{r}(\phi_2)$
$\mathbf{r}(\phi_1)$
$\mathbf{r}(\phi_0)$
$\mathbf{r}(\phi_{-1})$
$\mathbf{r}(\phi_{-2})$
$\mathbf{p}(\phi_2)$
$\mathbf{p}(\phi_1)$
$\mathbf{p}(\phi_0)$
$\mathbf{p}(\phi_{-1})$
$\mathbf{p}(\phi_{-2})$
1/a=0.5
b=1.0
a=2.0
$\mathbf{r}\bullet Q\bullet\mathbf{r}$-ellipse
$r_x^2/a^2+r_y^2/b^2=1$
(a = 2.0 , b = =1.0)
$\mathbf{p}\bullet Q^{-1}\bullet\mathbf{p}$-ellipse
$a^2p_x^2+b^2p_y^2=1$
(a = 2.0 , b = =1.0)
Variation of Fig. 11.7 in Unit 1

...And includes a cool way to construct those tangents $\dot{\mathbf{r}}(\phi_{-2})\ldots\dot{\mathbf{p}}(\phi_1)$ *etc. (see exercises!)*



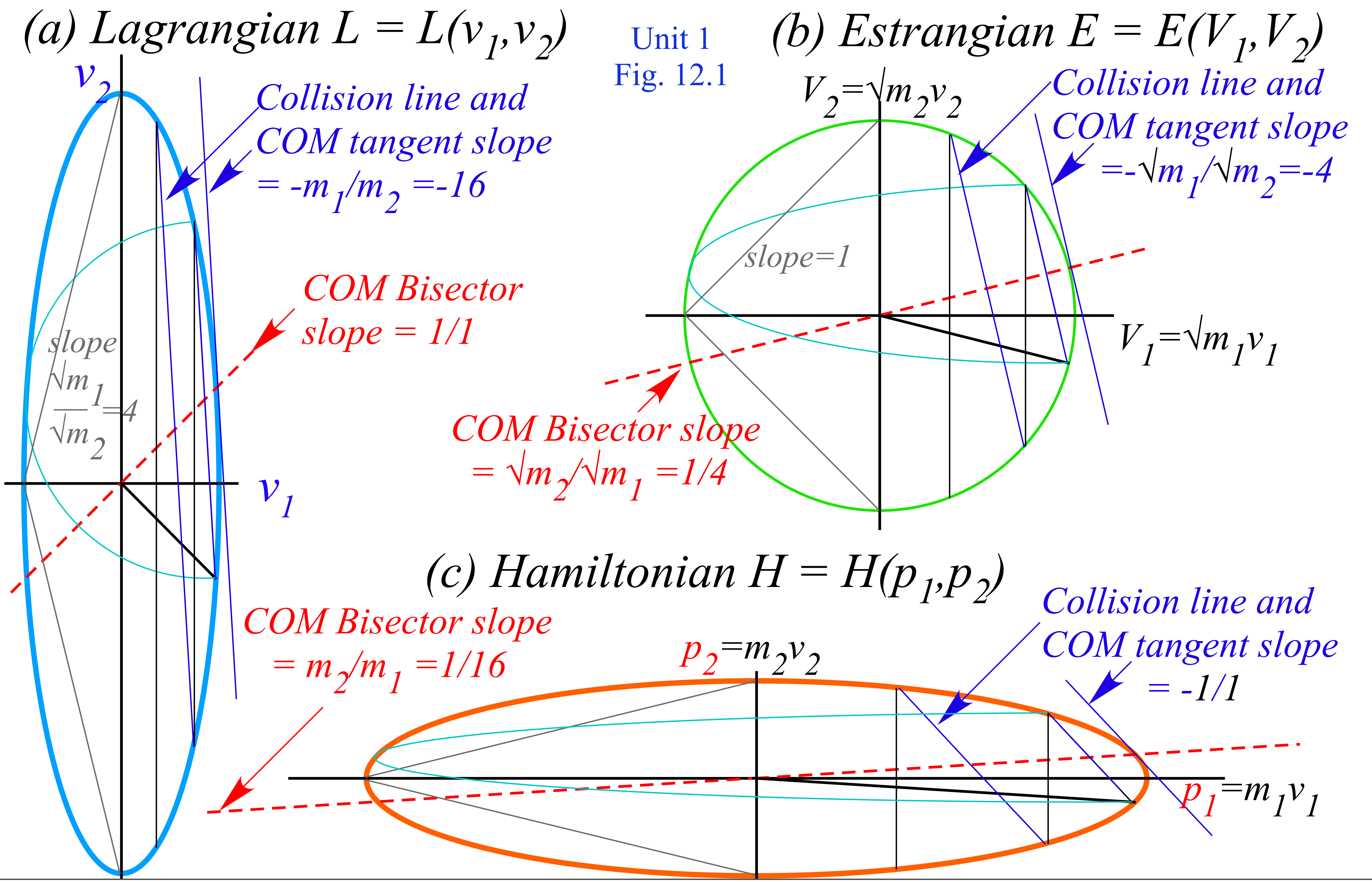
Variation of Fig. 11.7 in Unit 1

Q:Where is this headed? *A: Lagrangian-Hamiltonian duality*

Preview of Lecture 9

The R and Q matrix transformations are like the mechanics rescaling matrices $\sqrt{\mathbf{M}}$ *and* $\mathbf{M}$:

Like $Q=R^2$: $\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \mathbf{R}^2$ *Like* $\sqrt{Q}=R$: $\sqrt{\mathbf{M}} = \begin{pmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{pmatrix} = \mathbf{R}$ *Like* $Q^{-1}=R^{-2}$: $\mathbf{M}^{-1} = \begin{pmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{pmatrix} = \mathbf{R}^{-2}$



Unit 1
Fig. 12.2