Dynamics of Potentials and Force Fields
(Ch. 7 and Ch. 8 of Unit 1)

(From Lect 5.) A lesson in geometry of fractions and fractals: Ford Circles and Farey Sums

Potential energy geometry of Superballs and related things
Thales geometry and “Sagittal approximation”
Geometry and dynamics of single ball bounce
Examples: (a) Constant force (like kidee pool) (b) Linear force (like balloon)
Some physics of dare-devil-divers
Non-linear force (like superball-floor or ball-bearing-anvil)
Geometry and dynamics of 2-ball bounce (again with feeling)
The parable of RumpCo. vs CrapCorp.
The story of USC pre-meds visiting Whammo Manufacturing Co.

Geometry and dynamics of 3-ball bounce
A story of Stirling Colgate (Palmolive) and core-collapse supernovae
Other bangings-on: The western buckboard and Newton’s balls

Crunch energy geometry of freeway crashes and related things
Crunch energy played backwards: This really is “Rocket-Science”
A Thales construction for momentum-energy
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If superball was a balloon its bounce force law would be linear \( F = -k \cdot x \) (Hooke Law)

\[
F_{\text{balloon}}(x) = P \cdot A = P \cdot \pi r^2
\approx P \cdot \pi 2Rx
\]

Thales' geometry and "Sagittal\(^\dagger\)" approx.

\(^\dagger\) "bow"
Potential Energy Geometry of Superballs and Related things

If superball was a balloon its bounce force law would be linear $F = -k \cdot x$ (Hooke Law)

$$F_{\text{balloon}}(x) = P \cdot A = P \cdot \pi r^2$$

$$\approx P \cdot \pi 2Rx = P \cdot 2\pi Rx \quad \text{(Hooke spring constant } k \text{)}$$

$$= kx$$

Thales' geometry and "Sagittal†" approx.

† "bow"
Potential Energy Geometry of Superballs and Related things

(a) If superball was a balloon its bounce force law would be linear \( F = -k \cdot x \) (Hooke Law)

\[
F_{\text{balloon}}(x) = P \cdot A = P \cdot \pi r^2 \\
\approx P \cdot \pi 2Rx = P \cdot 2\pi Rx \quad \text{(Hooke spring constant } k) \\
= kx
\]

(b) Instead superball force law depends on bulk volume modulus and is non-linear \( F \sim x^p \) ? (Power Law?)

\[
Volume(X) = \int_0^X \pi r^2 \, dx = \int_0^X \pi x(2R - x) \, dx = \int_0^X 2R \pi x \, dx - \int_0^X \pi x^2 \, dx = R\pi X^2 - \frac{\pi X^3}{3} \approx \begin{cases} R\pi X^2 & \text{(for } X \ll R) \\
\frac{4}{3} \pi R^3 & \text{(for } X = 2R) \end{cases}
\]

It also depends on velocity \( \dot{x} = \frac{dx}{dt} \). Adiabatic differs from Isothermal as shown by “Project-Ball*”

Potential energy geometry of Superballs and related things

Thales geometry and “Sagittal approximation”

Geometry and dynamics of single ball bounce  (See Simulation)

Examples: (a) Constant force (like kidee pool) (b) Linear force (like balloon)
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Other hangings-on: The western buckboard and Newton’s balls
(a) Drop height
(Zero kinetic energy)

Force is weight mg only

Height h

(b) Maximum kinetic energy
(Zero total force)

Floor force balances weight mg

Penetration $x_{\text{static}}$

(c) Maximum penetration
(Zero kinetic energy again)

Force is maximum

Maximum penetration $x_{\text{max}}$

Details of each case follows in simulation

Unit 1
Fig. 7.2

Total energy $E = mgh$

Total potential energy curve $U(x) + mgY$

Equilibrium penetration $x_{\text{static}}$

Total Force curve $F(x) + mg$

Maximum Force $F(x_{\text{max}})$

Maximum penetration $x_{\text{max}}$

Maximum kinetic energy
(Zero total force)

Floor force balances weight mg

Penetration $x_{\text{static}}$

Maximum Force $F(x_{\text{max}})$

Maximum penetration $x_{\text{max}}$

Total energy $E$

Details of each case follows in simulation
Main Control Panel

- Let mouse set: \((x,y,Vx,Vy)\)
- Let mouse set force: \(F(t)\)
- Plot solid paths
- Plot dotted paths
- Plot no paths
- Plot \(V_1\) vs. \(V_2\)
- Plot \(Y_1(t), Y_2(t), \ldots\)
- Plot PE of \(m_1\) vs. \(Y_1\)
- Plot \(Y_2\) vs. \(Y_1\)
- Plot user defined i.e. \(Y_1\) vs. \(Y_2\)
- Balls initially falling
- Balls initially fixed
- No preset initial values

Number of masses

- 1 Ball

Collision friction (Viscosity)

- \(0 \times 10^\text{8} \, \text{g}^{-1} \)

Acceleration of gravity

- \(0.5 \, 100\times\text{cm/s}^2\)

Initial gap between balls

- \(5.45 \times 10^\text{6} \, \text{g}^{-1} \)

Force power law exponent

- 1

Force Constant

- 500

Canvas Aspect Ratio - W/H i.e. 0.75 & 1.0

- 0.75

This is linear setting
(increase for non-linear)

\[ y \text{ Max } = \]

\[ y \text{ Min } = \]

\[ \text{Max } x \text{ PE plot } = \]

\[ T \text{ Max } = \]

\[ F\text{-Vector scale } = \]

\[ V_{2y} \text{ Max } = \]

\[ V_{2y} \text{ Min } = \]

\[ \text{Error step } = \]

\[ m_1 = 1 \times 10^\text{6} \]

\[ V_{10} = 0 \times 10^\text{6} \]

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html

(See Simulations)
(a) Drop height $h$
(Zero kinetic energy)

Total Force curve $F(x) + mg$

Total potential energy curve $U(x) + mgY$

Force is weight $mg$ only

Display of Force vector using similar triangle construction based on the slope of potential curve.

http://www.uark.edu/ua/modphys/markup/BounceItWeb.html
Floor force balances weight mg

(b) Maximum kinetic energy
(Zero total force)

Total potential energy curve
$U(x) + mgY$

Equilibrium penetration $x_{static}$

Total Force curve $F(x) + mg$

Force is zero

Kinetic energy $KE$

Total energy $E$

$E = KE$
(c) Maximum penetration
(Zero kinetic energy again)

Display of Force vector using similar triangle construction based on the slope of potential curve.
(a) Drop height \( h \) (Zero kinetic energy)

(b) Maximum kinetic energy (Zero total force)

(c) Maximum penetration (Zero kinetic energy again)

Display of Force vector using similar triangle construction based on the slope of potential curve.
**Unit 1**

**Fig. 7.5**

**Force** \( F(x) \) and **Potential** \( U(x) \) for a soft heavy non-linear superball

\[
\begin{align*}
F_{total}(y) &= -Mg + F_{ball}(y) \\
U_{total}(y) &= -Mgx + U_{ball}(y)
\end{align*}
\]

**Total Energy** \( E = Mg\text{h} \)

\[
\begin{align*}
U_{total}(y_{max}) &= \int_{y_{static}}^{y_{max}} F_{total}(y) \, dy + \int_{y_{static}}^{y_{h}} F_{total}(y) \, dy + U(h) = U(h) = E
\end{align*}
\]

\[
F(x) = -\frac{dU(x)}{dx}
\]
**Force** $F(x)$ and **Potential** $U(x)$ for soft heavy non-linear superball

*Unit 1*  
*Fig. 7.5*

\[
F_{\text{total}}(y) = -Mg + F_{\text{ball}}(y)
\]

\[
U_{\text{total}}(y) = -Mgx + U_{\text{ball}}(y)
\]

Total Energy $E = Mg$  

\[
U_{\text{total}}(y_{\text{max}}) = \int_{y_{\text{static}}}^{y_{\text{max}}} F_{\text{total}}(y) \, dy + \int_{y_{\text{static}}}^{h} F_{\text{total}}(y) \, dy + U(h) = U(h) = E
\]

**Work** $W = \int F(x) \, dx = \text{Energy acquired} = \text{Area of } F(x) = -U(x)$  

\[
F(x) = -\frac{dU(x)}{dx}
\]
Force $F(x)$ and Potential $U(x)$ for soft heavy non-linear superball

$$F_{\text{total}}(y) = -Mg + F_{\text{ball}}(y)$$

$$U_{\text{total}}(y) = -Mgx + U_{\text{ball}}(y)$$

Total Energy $E = Mgh$

Unit 1
Fig. 7.5

$$F_{\text{total}}(h)$$

$$F_{\text{areas cancel}}$$

$$y_{\text{max}}$$

$$y_{\text{static}}$$

$$U_{\text{total}}(y_{\text{max}}) = \int_{y_{\text{static}}}^{y_{\text{max}}} F_{\text{total}}(y) \, dy + \int_{y=h}^{y_{\text{static}}} F_{\text{total}}(y) \, dy + U(h) = U(h) = E$$

Work $W = \int F(x) \, dx = \text{Energy acquired} = \text{Area of } F(x) = -U(x)$

$$F(x) = -\frac{dU(x)}{dx}$$

Impulse $P = \int F(t) \, dt = \text{Momentum acquired} = \text{Area of } F(t) = P(t)$

$$F(t) = \frac{dP(t)}{dt}$$
Potential energy geometry of Superballs and related things

Thales geometry and “Sagittal approximation”

Geometry and dynamics of single ball bounce

Examples: (a) Constant force (like kidee pool) (b) Linear force (like balloon)

Some physics of dare-devil-divers

Non-linear force (like superball-floor or ball-bearing-anvil)

Geometry and dynamics of 2-ball bounce (again with feeling)

The parable of RumpCo. vs CrapCorp.

The story of USC pre-meds visiting Whammo Manufacturing Co.

Geometry and dynamics of 3-ball bounce

A story of Stirling Colgate (Palmolive) and core-collapse supernovae

Other hangings-on: The western buckboard and Newton’s balls
Work = W = \int F(x) \, dx = \text{Energy acquired} = \text{Area of } F(x) = -U(x)

F(x) = -\frac{dU(x)}{dx}

Impulse = P = \int F(t) \, dt = \text{Momentum acquired} = \text{Area of } F(t) = P(t)

F(t) = \frac{dP(t)}{dt}
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See Simulation
(a) Force $F(Y)$ Units $Mg$ (N)  

(b) Potential $U(Y)$ Units of $MgY$ (J)  

(c) Force $F(Y)$ Units $Mg$ (N)  

(d) Potential $U(Y)$ Units of $MgY$ (J)  

(e) Geometry of Linear Force with Constant $Mg$ and Quadratic Potential  

$F(Y) = -kY - Mg$  

$U(Y) = (1/2)kY^2 + MgY$  

$F_{total} = F^{grav} + F^{target} = \begin{cases} 
-Mg & (y \geq 0) \\
-Mg - ky & (y < 0) 
\end{cases}$  

$U_{total} = U^{grav} + U^{target} = \begin{cases} 
MgY & (y \geq 0) \\
MgY + \frac{1}{2}ky^2 & (y < 0) 
\end{cases}$  

Fig. 7.4
Potential energy geometry of Superballs and related things

Thales geometry and "Sagittal approximation"

Geometry and dynamics of single ball bounce

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Unit 1
Fig. 7.5

Force $F(x)$ and Potential $U(x)$ for soft heavy non-linear superball

$F_{\text{total}}(y) = -Mg + F^{\text{ball}}(y)$

$U_{\text{total}}(y) = -Mgx + U^{\text{ball}}(y)$

Total Energy $E = Mg \cdot h$

$U_{\text{total}}(y_{\text{max}}) = \int_{y_{\text{static}}}^{y_{\text{max}}} F_{\text{total}}(y) \, dy + \int_{y = h}^{y_{\text{static}}} F_{\text{total}}(y) \, dy + U(h) = U(h) = E$

Work $W = \int F(x) \, dx =$ Energy acquired $= \text{Area of } F(x) = -U(x)$

Impulse $P = \int F(t) \, dt =$ Momentum acquired $= \text{Area of } F(t) = P(t)$

Important physics in this non-linear region!
Potential energy geometry of Superballs and related things

Thales geometry and “Sagittal approximation”

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Examples: (a) Constant force (like kidee pool) (b) Linear force (like balloon)

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Other hangings-on: The western buckboard and Newton’s balls
Unit 1
Fig. 7.6
Potential energy geometry of Superballs and related things
Thales geometry and “Sagittal approximation”
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Fig. 8.1a-c
Independent Bang Model
(IBM)
3-Body Geometry
Unit 1
Fig. 8.1b
Independent Bang Model
(IBM)
3-Body Geometry

(a) Quartic Force
\( F(y) = ky^4 \)

(m3 \(= 10 \text{ kg} \)

(m2 \(= 30 \text{ kg} \)

(m1 \(= 100 \text{ kg} \)

(b) Independent Collisions (Independent of Force Law)

(m3 \(= 10 \text{ kg} \)

(m2 \(= 30 \text{ kg} \)

(m1 \(= 100 \text{ kg} \)

(c) Linear Force
\( F(y) = ky \)

(m3 \(= 10 \text{ kg} \)

(m2 \(= 30 \text{ kg} \)

(m1 \(= 100 \text{ kg} \)
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A story of Stirling Colgate (Palmolive) and core-collapse supernovae

Other hangings-on: The western buckboard and Newton’s balls
A story of Stirling Colgate (Palmolive) and core-collapse supernovae

Source
http://hubblesite.org/newscenter/archive/releases/2007/10/image/a/

Author
NASA, ESA, P. Challis, and R. Kirshner (Harvard-Smithsonian Center for Astrophysics)

<table>
<thead>
<tr>
<th>Process</th>
<th>Main fuel</th>
<th>Main products</th>
<th>25 M\textsubscript{\odot} star$^6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Temperature (Kelvin)</td>
</tr>
<tr>
<td>hydrogen burning</td>
<td>hydrogen</td>
<td>helium</td>
<td>7$x$10\textsuperscript{7}</td>
</tr>
<tr>
<td>triple-alpha process</td>
<td>helium</td>
<td>carbon, oxygen</td>
<td>2$x$10\textsuperscript{8}</td>
</tr>
<tr>
<td>carbon burning process</td>
<td>carbon</td>
<td>Ne, Na, Mg, Al</td>
<td>8$x$10\textsuperscript{8}</td>
</tr>
<tr>
<td>neon burning process</td>
<td>neon</td>
<td>O, Mg</td>
<td>1.6$x$10\textsuperscript{9}</td>
</tr>
<tr>
<td>oxygen burning process</td>
<td>oxygen</td>
<td>Si, S, Ar, Ca</td>
<td>1.8$x$10\textsuperscript{9}</td>
</tr>
<tr>
<td>silicon burning process</td>
<td>silicon</td>
<td>nickel (decays into iron)</td>
<td>2.5$x$10\textsuperscript{9}</td>
</tr>
</tbody>
</table>
A story of Stirling Colgate (Palmolive) and core-collapse supernovae

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Core-burning nuclear fusion stages for a 25-solar mass star

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<tbody>
<tr>
<td></td>
<td>Temperature (Kelvin)</td>
<td>Density (g/cm(^3))</td>
<td>Duration</td>
</tr>
<tr>
<td>hydrogen burning</td>
<td>(7 \times 10^7)</td>
<td>10</td>
<td>(10^7) years</td>
</tr>
<tr>
<td>triple-alpha process</td>
<td>(2 \times 10^8)</td>
<td>2000</td>
<td>(10^6) years</td>
</tr>
<tr>
<td>carbon burning process</td>
<td>(8 \times 10^8)</td>
<td>(10^6)</td>
<td>(10^3) years</td>
</tr>
<tr>
<td>neon burning process</td>
<td>(1.6 \times 10^9)</td>
<td>(10^7)</td>
<td>3 years</td>
</tr>
<tr>
<td>oxygen burning process</td>
<td>(1.8 \times 10^9)</td>
<td>(10^7)</td>
<td>0.3 years</td>
</tr>
<tr>
<td>silicon burning process</td>
<td>(2.5 \times 10^9)</td>
<td>(10^8)</td>
<td>5 days</td>
</tr>
</tbody>
</table>

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Within a massive, evolved star (a) the onion-layered shells of elements undergo fusion, forming a nickel-iron core (b) that reaches Chandrasekhar mass and starts to collapse. The inner part of the core is compressed into neutrons (c), causing infalling material to bounce (d) and form an outward-propagating shock front (rod). The shock starts to stall (e), but it is re-invigorated by neutrino interaction. The surrounding material is blasted away (f), leaving only a degenerate remnant.
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Other hangings-on: The western buckboard and Newton’s balls
(a) \( m_k/m_{k+1} = 3 \)

(b) \( m_k/m_{k+1} = 7 \)

(c) Bouncing column
\( m_k/m_{k+1} = 1 \)

(d) Single pop-up
\( (0,1) \)

Unit 1
Fig. 8.2a-b
4-Body IBM Geometry
Fig. 8.2c-d
4-Equal-Body Geometry

"Shockwave" or pulse wave Dynamics

Opposite of continuous wave dynamics introduced in Unit 2
Crunch energy geometry of freeway crashes and related things

Crunch energy played backwards: This really is “Rocket-Science”
Of course, these examples neglect friction and “crunch-energy” losses.
Speeding car and five stationary cars

\[ V_{M(0)} = 60, \quad V_{m(1)} = 0 \]

\[ V_{M(0)} = 30 \]

\[ V_{M(01)} = 20 \]

\[ V_{M(012)} = 15 \]

\[ V_{M(0123)} = 12 \]

\[ V_{M(01234)} = 10 \]

Of course, these examples neglect friction and “crunch-energy” losses.

Five speeding cars and a stationary car

\[ V_{M(1)} = 60, \quad V_{m(0)} = 0 \]

\[ V_{M(0)} = 30 \]

\[ V_{M(10)} = 40 \]

\[ V_{M(210)} = 40 \]

\[ V_{M(310)} = 45 \]

\[ V_{M(4310)} = 48 \]

\[ V_{M(54310)} = 50 \]

Fig. 8.5
Pile-up:
One 60mph car hits
five standing cars

Fig. 8.6
Pile-up:
Five 60mph cars hit
one standing car
Of course, these examples neglect friction and “crunch-energy” losses.

(Fug-gedda-aboud-dit!!)

Unit 1

Fig. 8.5
Pile-up:
One 60mph car
hits
five standing cars

Fig. 8.6
Pile-up:
Five 60mph cars
hit
one standing cars

Fig. 8.7
Pile-up:
Five 60mph cars
hit
five standing cars
Crunch energy geometry of freeway crashes and related things

Crunch energy played backwards: This really is “Rocket-Science”
(a) Harmonic progression
1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, ...

(b) Harmonic series
1+1/2+1/3+1/4+1/5+1/6+1/7+...

Unit 1
Fig. 8.8a-b

Rocket Science!

\[ m \Delta v_7 + 3m \Delta V_M(7) = 0 \]

\[ m \Delta v_6 + 4m \Delta V_M(6) = 0 \]

\[ m \Delta v_5 + 5m \Delta V_M(5) = 0 \]

\[ m \Delta v_4 + 6m \Delta V_M(4) = 0 \]

\[ m \Delta v_3 + 7m \Delta V_M(3) = 0 \]

\[ m \Delta v_2 + 8m \Delta V_M(2) = 0 \]

\[ m \Delta v_1 + 9m \Delta V_M(1) = 0 \]

\[ m \Delta v_0 + 10m \Delta V_M(0) = 0 \]
By calculus: $M \cdot \Delta V = -v_e \cdot \Delta M$ or: $dV = -v_e \frac{dM}{M}$ Integrate: $\int_{V_{IN}}^{V_{FIN}} dV = -v_e \int_{M_{IN}}^{M_{FIN}} \frac{dM}{M}$

**The Rocket Equation:**

$$V_{FIN} - V_{IN} = -v_e \left[ \ln M_{FIN} - \ln M_{IN} \right] = v_e \left[ \ln \frac{M_{IN}}{M_{FIN}} \right]$$
A Thales construction for momentum-energy

(Made obsolete by Estrangian scaling to circular \((V_1, V_2)\) plots. Still, one has to construct \(\sqrt{m_1/m_2}\) slopes.)
This is a construction of the energy ellipse in a Largangian \((v_1,v_2)\) plot given the initial \((v_1,v_2)\).

The Estrangian \((V_1,V_2)\) plot makes the \((v_1,v_2)\) plot and this construction obsolete.

(Easier to just draw circle through initial \((V_1,V_2)\).)