

Analysis of 1D 2-Body Collisions II: Reflection groups

(Ch. 2 to Ch. 5 of Unit 1)

Review of air-track collision geometry

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of (V_1, V_2) data to space-space plots (y_1, y_2) Examples $(M_1=7, M_2=1)$ and $(M_1=49, M_2=1)$

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

What about that 2nd quadratic solution?

*“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

*Geometry and algebra of “ellipse-Rotation” group product: **R = C · M***

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics in Ch.12

Reflections in the clothing store: “It’s all done with mirrors!”

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$ and $(M_1=100, M_2=1)$

2019 Advanced Mechanics Running Reference Link Listing

[Web Resources - front page](#)
[UAF Physics UTube channel](#)

[Quantum Theory for the Computer Age](#)
[Principles of Symmetry, Dynamics, and Spectroscopy](#)
[Classical Mechanics with a Bang!](#)
[Modern Physics and its Classical Foundations](#)

[2017 Group Theory for QM](#)
[2018 Adv CM](#)
[2018 AMOP](#)
[2019 Advanced Mechanics](#)

Lecture #3

[BounceIt Superball Collision Web Simulations](#)

[With \$g=0\$ and 70:10 mass ratio](#)

[With non zero \$g\$, velocity dependent damping and mass ratio of 70:35](#)

[\$M_1=49, M_2=1\$ with Newtonian time plot](#)

[\$M_1=49, M_2=1\$ with \$V_2\$ vs \$V_1\$ plot](#)

[Example with friction](#)

[Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off](#)

[\$m_1:m_2=3:1\$ and \$\(v_1, v_2\) = \(1, 0\)\$ Comparison with Estrangian](#)

$m_1:m_2 = 3:1$ Dual plots

v_2 vs v_1 and V_2 vs V_1

[\(\$v_1, v_2\$ \)=\(1, 0.1\)](#)

[\(\$v_1, v_2\$ \)=\(1, 0\)](#)

[\(\$v_1, v_2\$ \)=\(1, -1\)](#)

$m_1:m_2 = 3:1$

y_2 vs y_1 plots

[\(\$v_1, v_2\$ \)=\(1, 0.1\)](#)

[\(\$v_1, v_2\$ \)=\(1, 0\)](#)

[\(\$v_1, v_2\$ \)=\(1, -1\)](#)

$m_1:m_2 = 3:1$

Estrangian plot V_2 vs V_1

[\(\$v_1, v_2\$ \)=\(0, 1\)](#)

$m_1:m_2 = 4:1$

$(v_1, v_2)=(1, 0)$

[v2 vs v1 plot](#)

[y2 vs y1 plot](#)

$m_1:m_2 = 100:1$ $(v_1, v_2)=(1, 0)$

[V2 vs V1 Estrangian plot](#)

[y2 vs y1 plot](#)

<<Under Construction>>

[Matrix Collision Simulator: \$M_1=49, M_2=1\$ \$V_2\$ vs \$V_1\$ plot](#)

Prior to Lecture #3

X2 paper: [Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 \(pdf\)](#)

Car Collision Web Simulator: <https://modphys.hosted.uark.edu/markup/CMMotionWeb.html>

Superball Collision Web Simulator: <https://modphys.hosted.uark.edu/markup/BounceItWeb.html>; with Scenarios: [1007](#)

[BounceIt web simulation with \$g=0\$ and 70:10 mass ratio](#)

[With non zero \$g\$, velocity dependent damping and mass ratio of 70:35](#)

Elastic Collision Dual Panel Space vs Space: [Space vs Time \(Newton\)](#), [Time vs. Space\(Minkowski\)](#)

Inelastic Collision Dual Panel Space vs Space: [Space vs Time \(Newton\)](#), [Time vs. Space\(Minkowski\)](#)

More Advanced QM and classical references on [end pages of this Lecture](#)

(after graph paper blanks)

Supplementary references and incidental interest items

[Web Resources - front page](#)

[UAF Physics UTube channel](#)

[Quantum Theory for the Computer Age](#)

[Principles of Symmetry, Dynamics, and Spectroscopy](#)

[Classical Mechanics with a Bang!](#)

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[2017 Group Theory for QM](#)

[2018 Adv CM](#)

[2018 AMOP](#)

[**2019 Advanced Mechanics**](#)

[AIP publications](#)


[AJP article on superball dynamics](#)

[AAPT summer reading](#)

More Advanced QM and classical references on [end pages of this Lecture](#)

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Review of air-track collision geometry

 *Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots*
Integration of (V_1, V_2) data to space-space plots (y_1, y_2) Examples $(M_1=7, M_2=1)$ and $(M_1=49, M_2=1)$

Geometric "Integration" (Converting Velocity data to Spacetime)

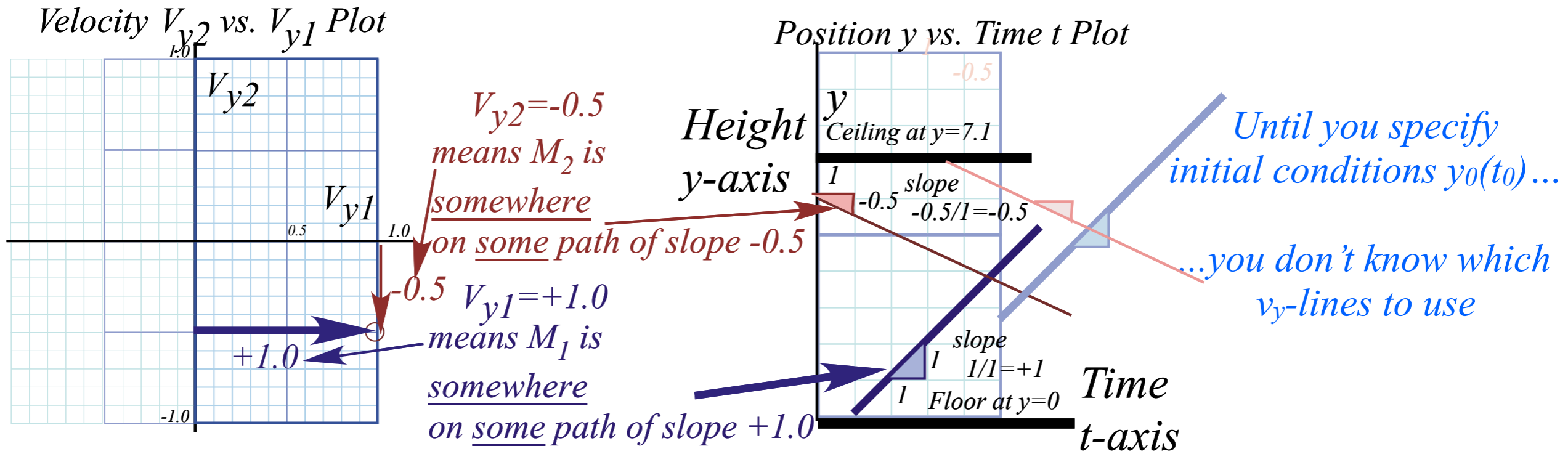
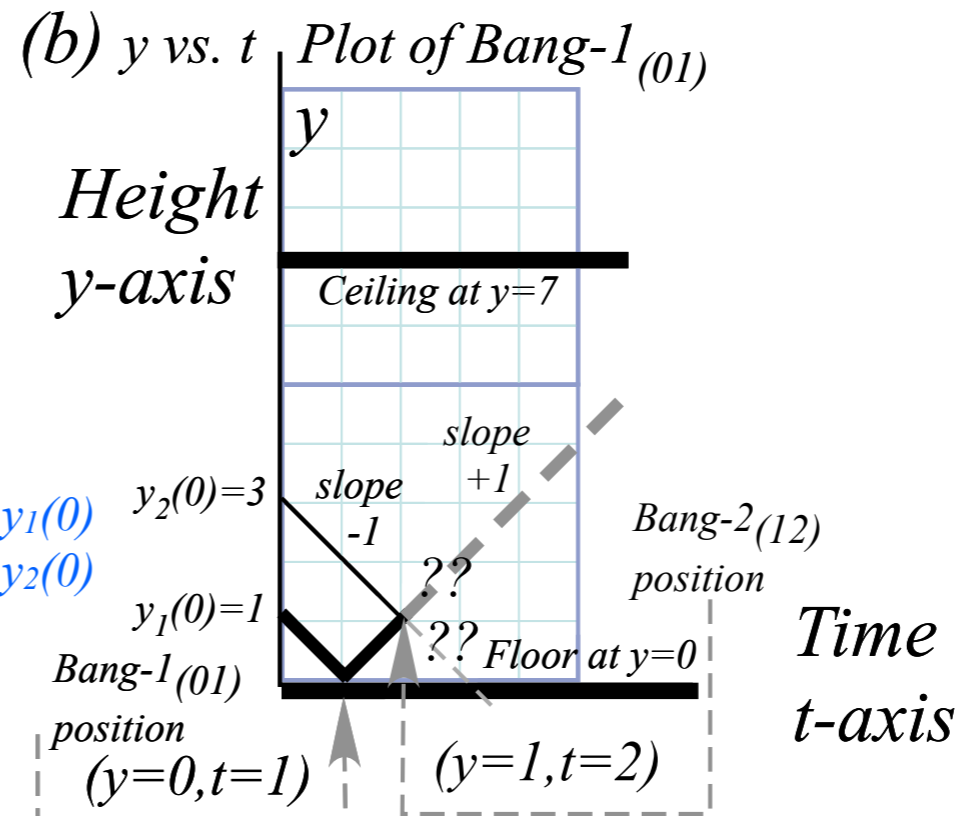
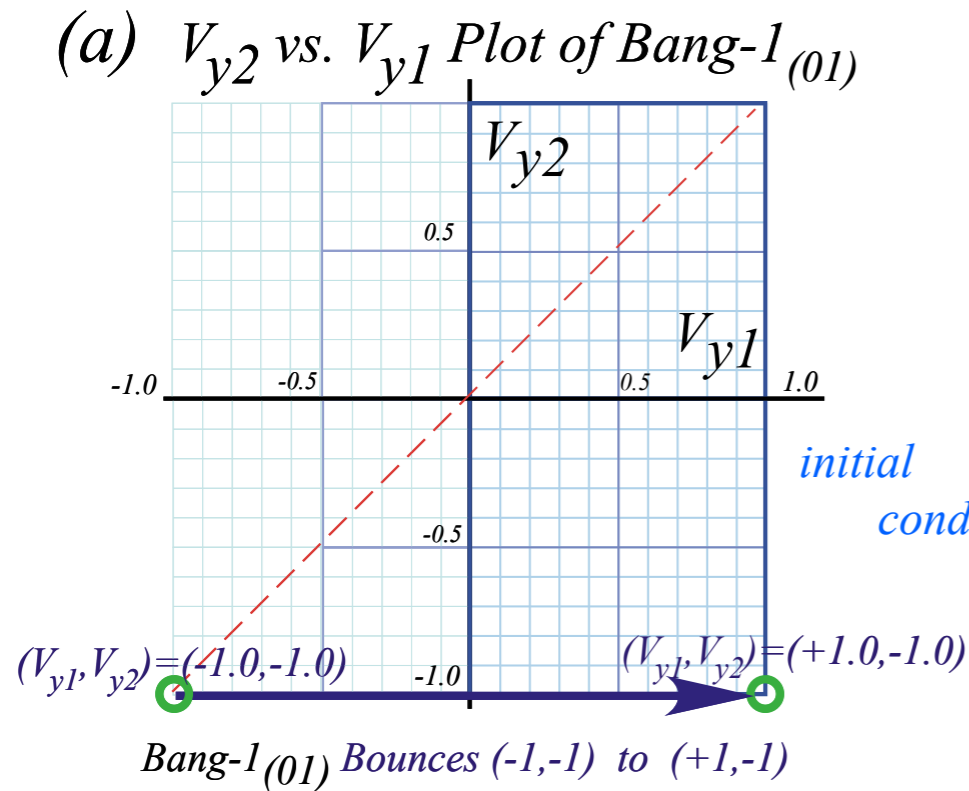
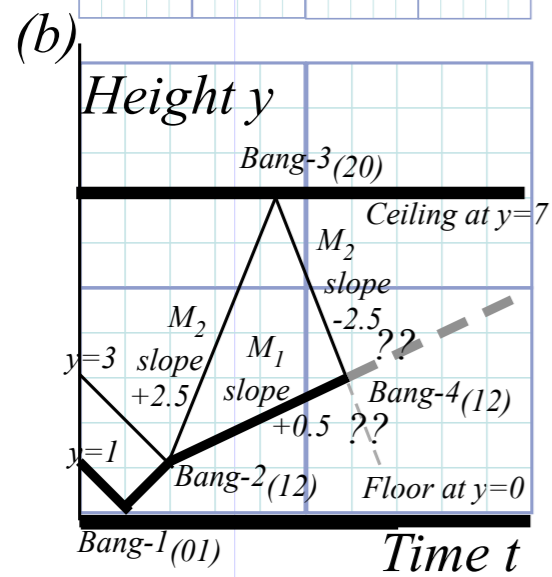
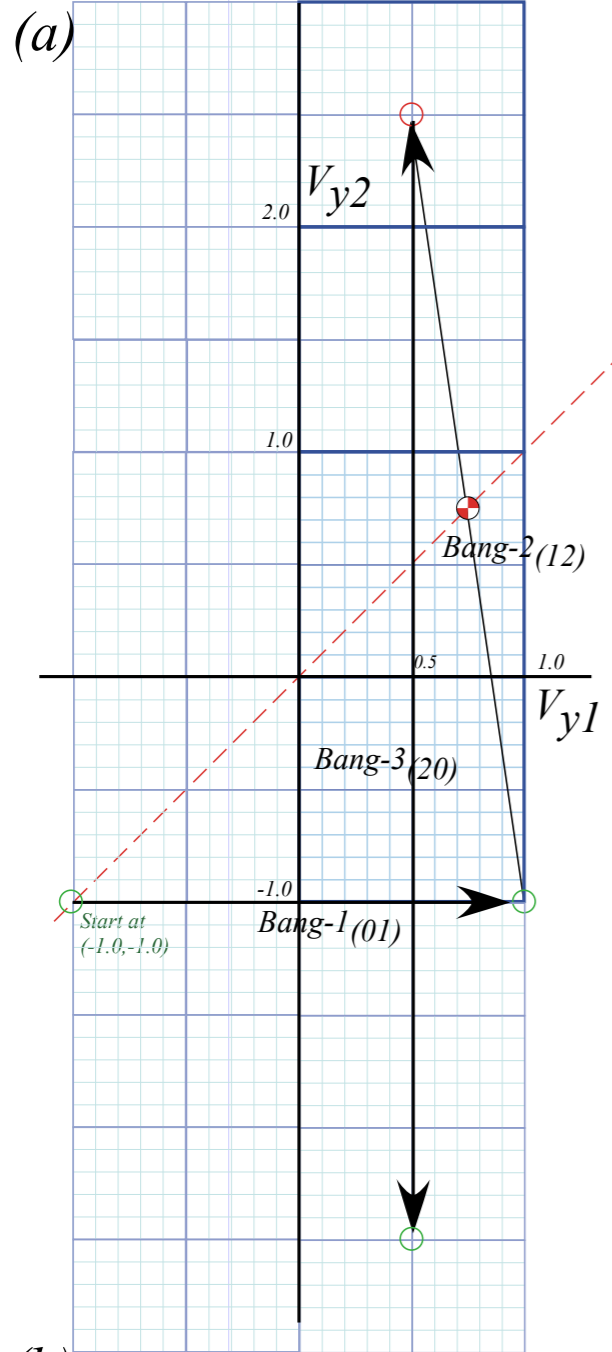


Fig. 4.6a-b
in Unit 1



Geometric "Integration" (Converting Velocity data to Spacetime)



Geometric "Integration" (Converting Velocity data to Spacetime)

Kinetic Energy Ellipse

$$KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE / M_1}$$

$$= \sqrt{2KE / 7}$$

$$= \sqrt{8/7}$$

$$= 1.07$$

Ellipse radius 2

$$a_2 = \sqrt{2KE / M_2}$$

$$= \sqrt{2KE / 1}$$

$$= \sqrt{8/1}$$

$$= 2.83$$

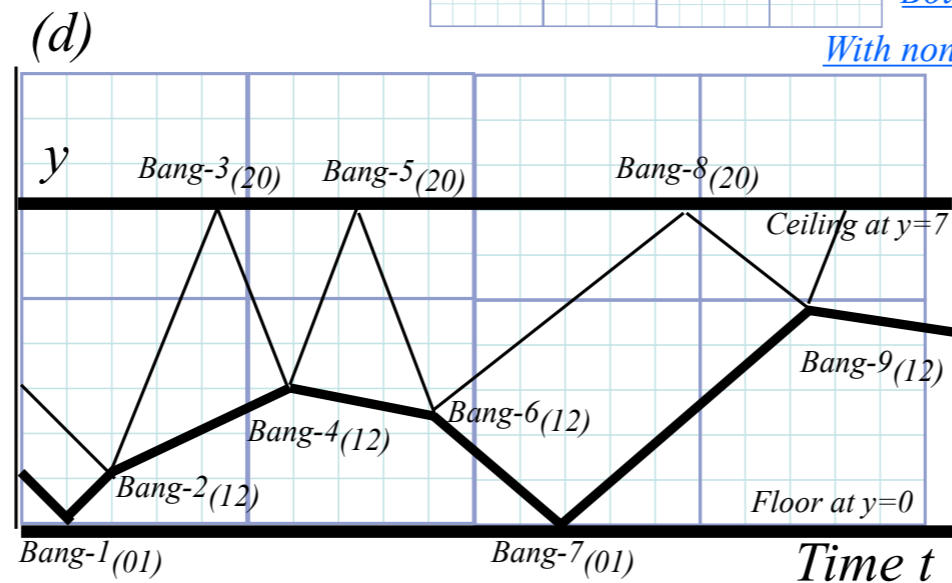
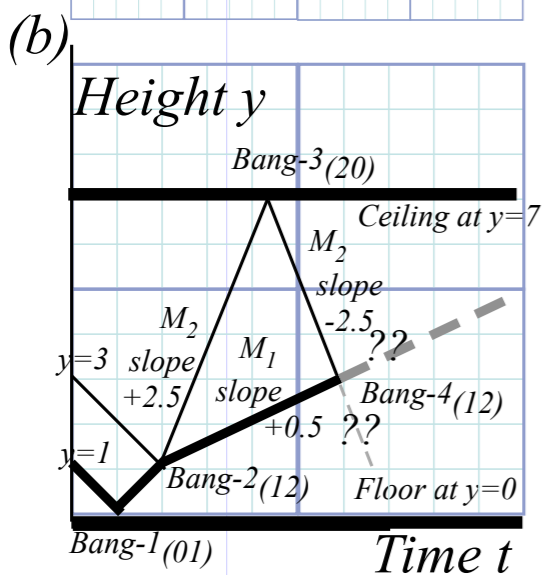
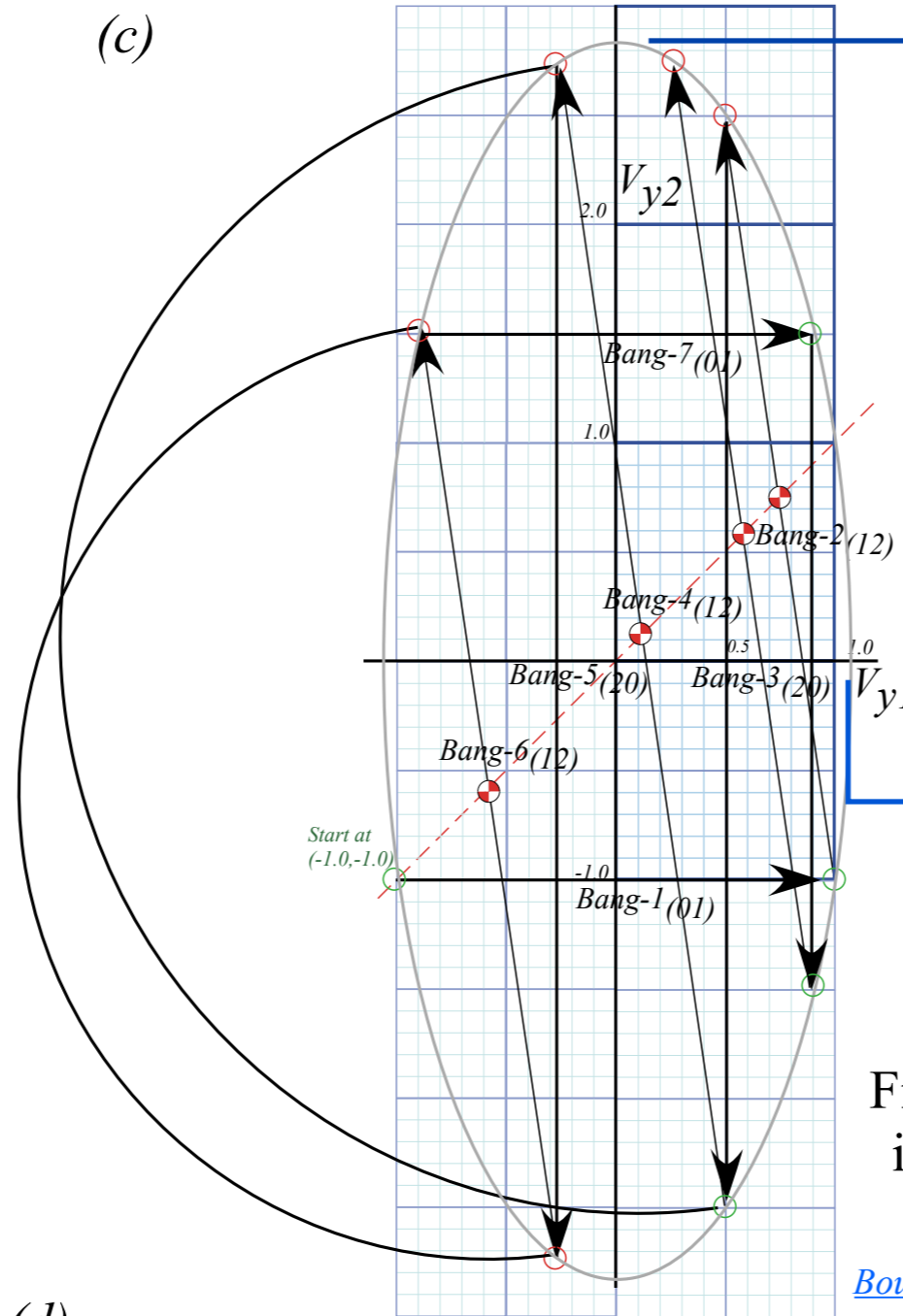
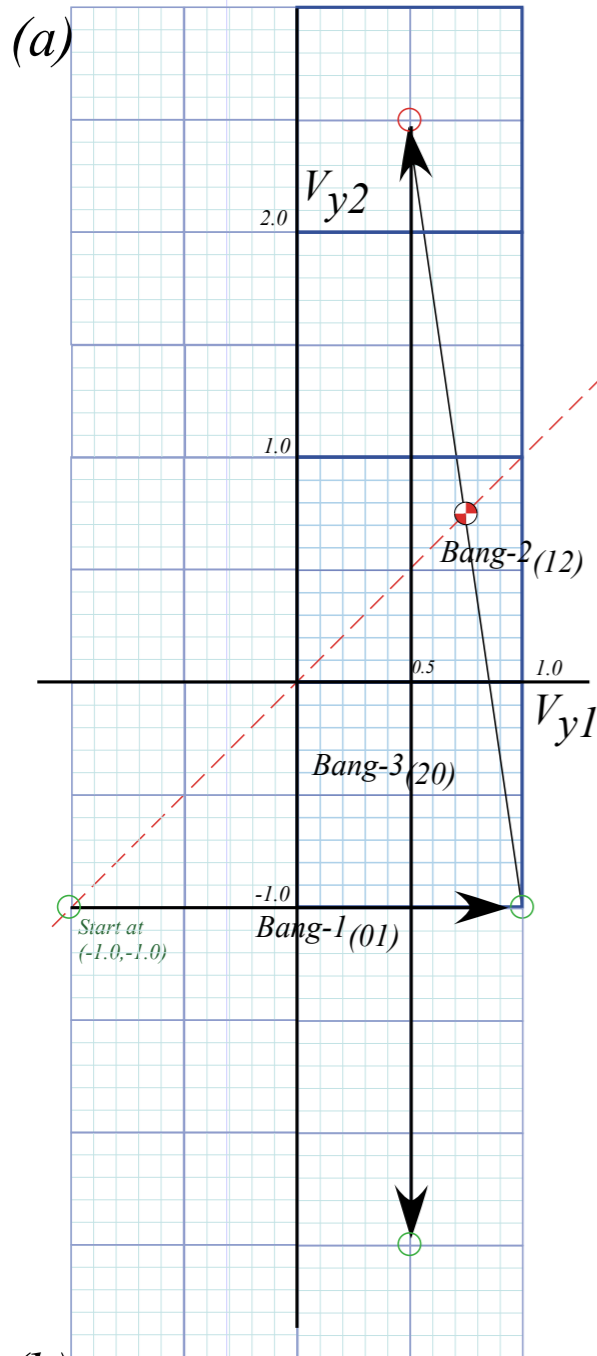



Fig. 4.7a-d
in Unit 1

Bouncelt web simulation with g=0 and 70:10 mass ratio

With non zero g, velocity dependent damping and mass ratio of 70:35

Review of air-track collision geometry

Integration of (V_1, V_2) data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

 *Integration of (V_1, V_2) data to space-space plots (y_1, y_2) Examples $(M_1=7, M_2=1)$ and $(M_1=49, M_2=1)$*

Geometric "Integration" (Converting Velocity data to Space-space trajectory)

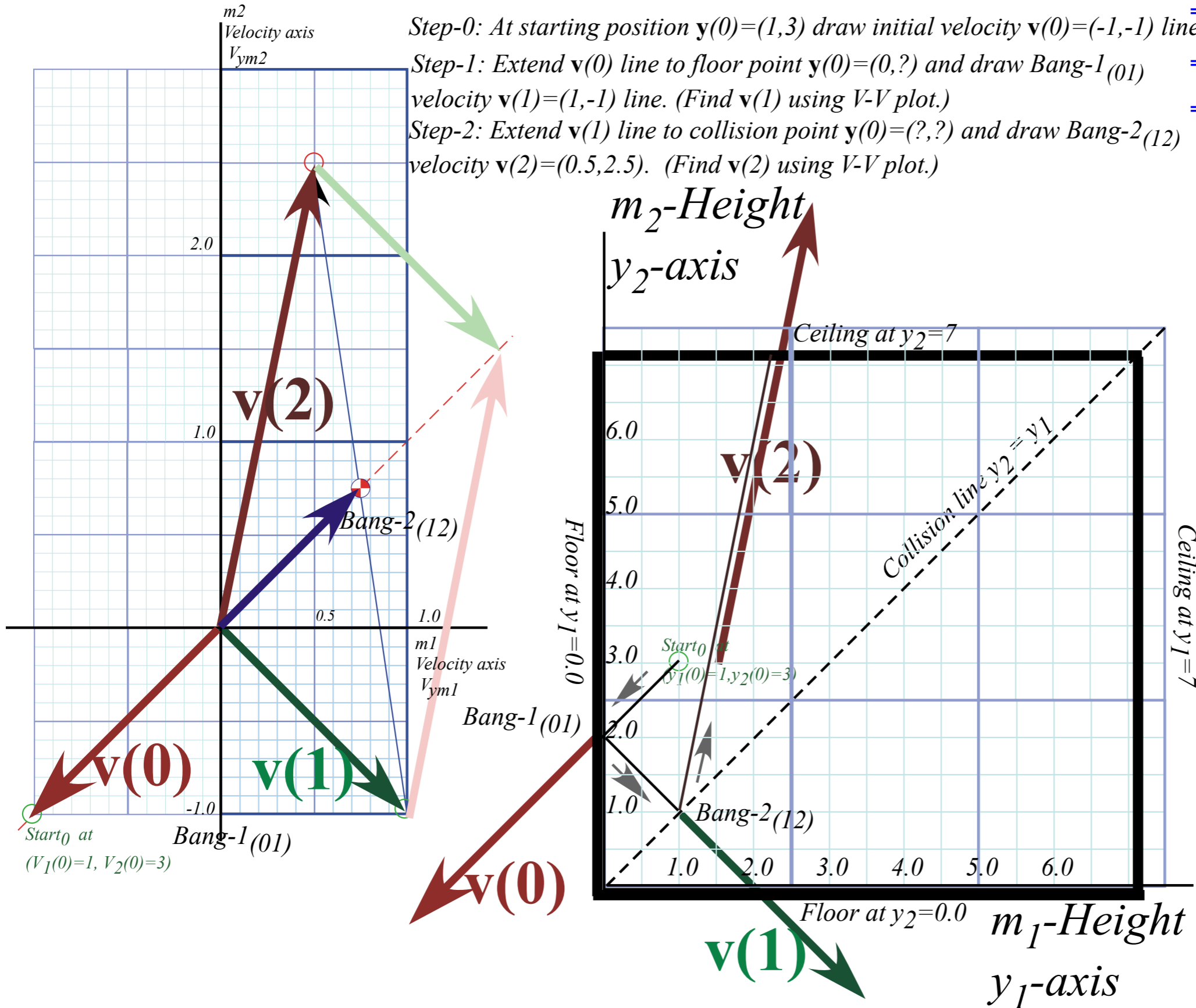
Fig. 4.11
in Unit 1

<u>Ellipse radius 1</u>	<u>Ellipse radius 2</u>
$a_1 = \sqrt{2KE/M_1}$	$a_2 = \sqrt{2KE/M_2}$
$= \sqrt{2KE/7}$	$= \sqrt{2KE/1}$
$= \sqrt{8/7}$	$= \sqrt{8/1}$
$= 1.07$	$= 2.83$

Step-0: At starting position $\mathbf{y}(0)=(1,3)$ draw initial velocity $\mathbf{v}(0)=(-1,-1)$ line.

Step-1: Extend $\mathbf{v}(0)$ line to floor point $\mathbf{y}(0)=(0,?)$ and draw Bang-1(01) velocity $\mathbf{v}(1)=(1,-1)$ line. (Find $\mathbf{v}(1)$ using V-V plot.)

Step-2: Extend $\mathbf{v}(1)$ line to collision point $\mathbf{y}(0)=(?,?)$ and draw Bang-2(12) velocity $\mathbf{v}(2)=(0.5,2.5)$. (Find $\mathbf{v}(2)$ using V-V plot.)



Geometric "Integration" (Converting Velocity data to Space-space trajectory)

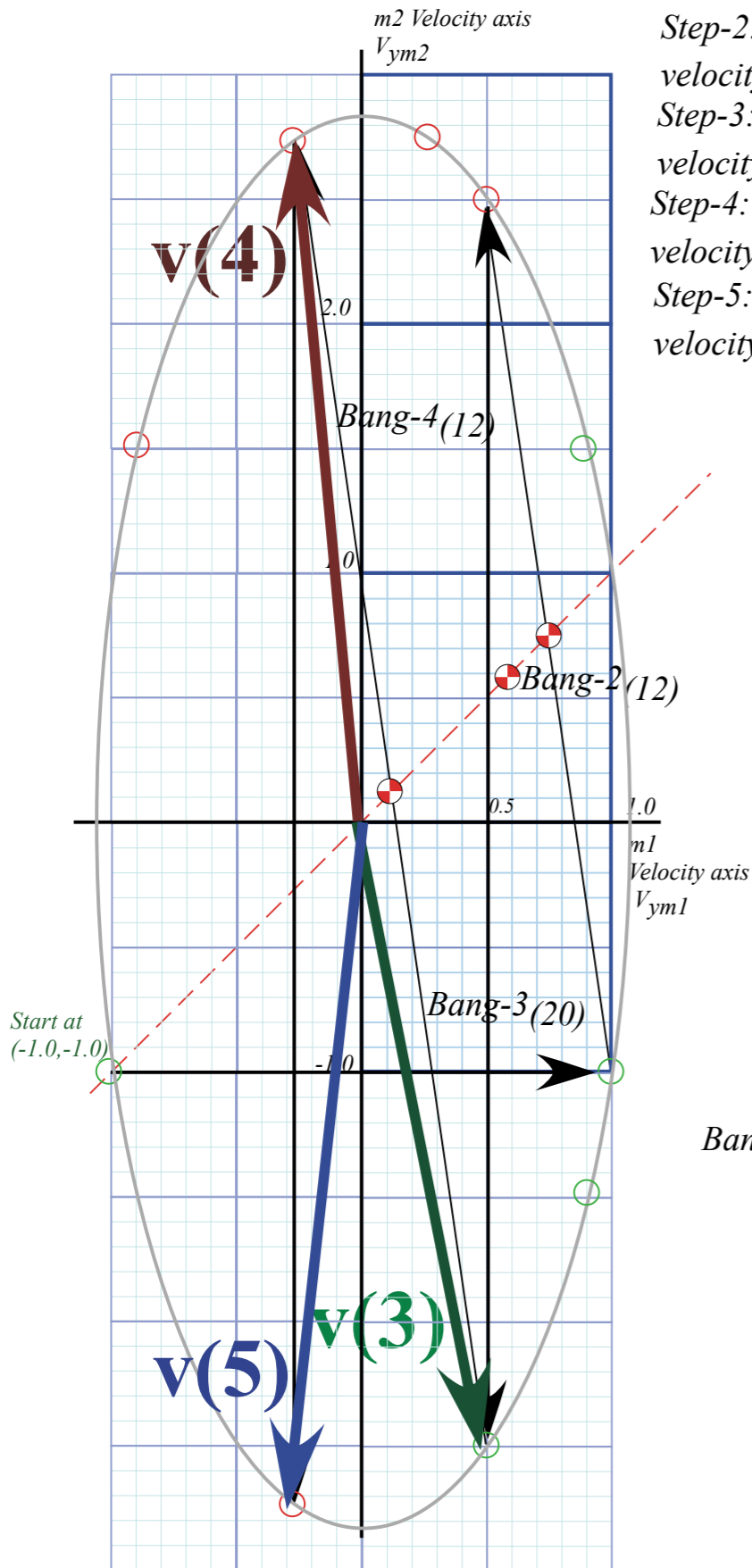
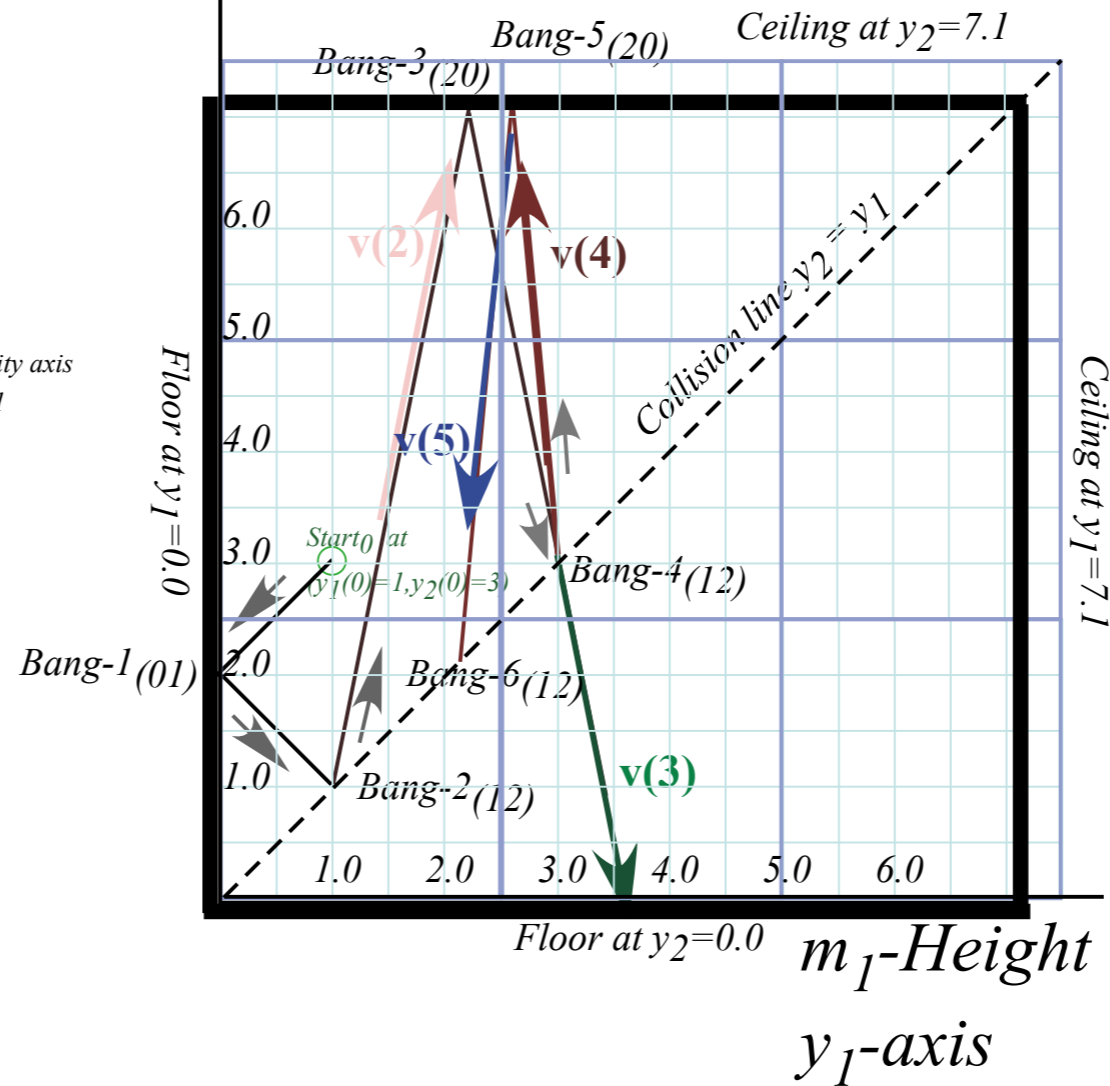
Fig. 4.11
in Unit 1

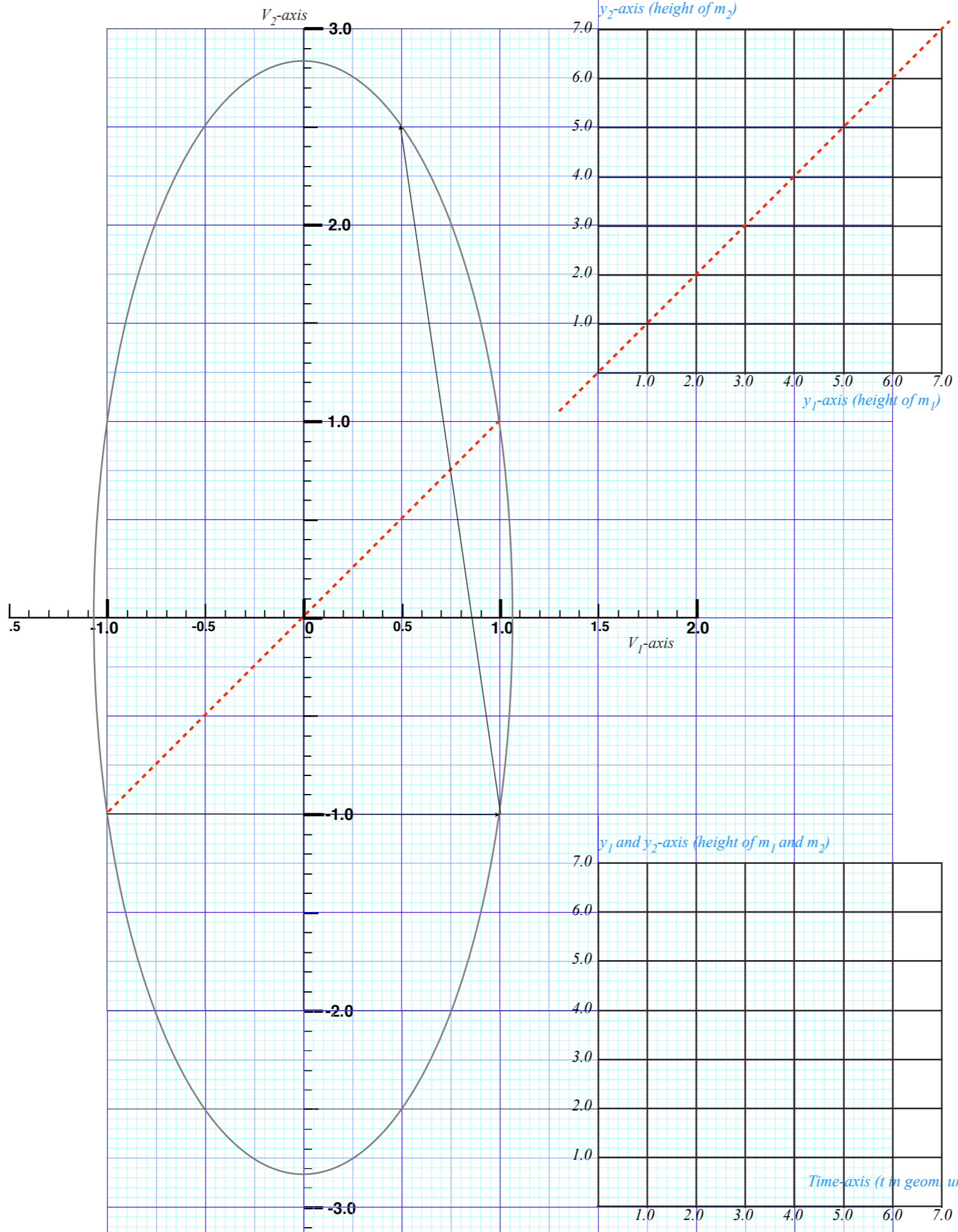
<u>Ellipse radius 1</u>	<u>Ellipse radius 2</u>
$a_1 = \sqrt{2KE/M_1}$	$a_2 = \sqrt{2KE/M_2}$
$= \sqrt{2KE/7}$	$= \sqrt{2KE/1}$
$= \sqrt{8/7}$	$= \sqrt{8/1}$
$= 1.07$	$= 2.83$

Step-2: Extend $v(2)$ line to ceiling point $y(3)=(?, 7.1)$ and draw Bang-3(20) velocity $v(3)=(1, -1)$ line. (Find $v(3)$ using V-V plot.)
 Step-3: Extend $v(3)$ line to collision point $y(4)=(?, ?)$ and draw Bang-4(12) velocity $v(4)=(0.5, 2.5)$. (Find $v(4)$ using V-V plot.)
 Step-4: Extend $v(4)$ line to ceiling point $y(4)=(?, 7.1)$ and draw Bang-5(20) velocity $v(5)=(1, -1)$ line. (Find $v(5)$ using V-V plot.)
 Step-5: Extend $v(5)$ line to collision point $y(6)=(?, ?)$ and draw Bang-6(12) velocity $v(6)=(0.5, 2.5)$. (Find $v(6)$ using V-V plot.)

m_2 -Height

y_2 -axis





<u>Ellipse radius 1</u>	<u>Ellipse radius 2</u>
$a_1 = \sqrt{2KE/M_1}$	$a_2 = \sqrt{2KE/M_2}$
$= \sqrt{2KE/7}$	$= \sqrt{2KE/1}$
$= \sqrt{8/7}$	$= \sqrt{8/1}$
$= 1.07$	$= 2.83$

Geometric "Integration" (Converting Velocity data to Spacetime)

Kinetic Energy Ellipse

$$KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4$$

$$1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$a_1 = \sqrt{2KE / M_1}$$

$$= \sqrt{2KE / 7}$$

$$= \sqrt{8/7}$$

$$= 1.07$$

Ellipse radius 2

$$a_2 = \sqrt{2KE / M_2}$$

$$= \sqrt{2KE / 1}$$

$$= \sqrt{8/1}$$

$$= 2.83$$

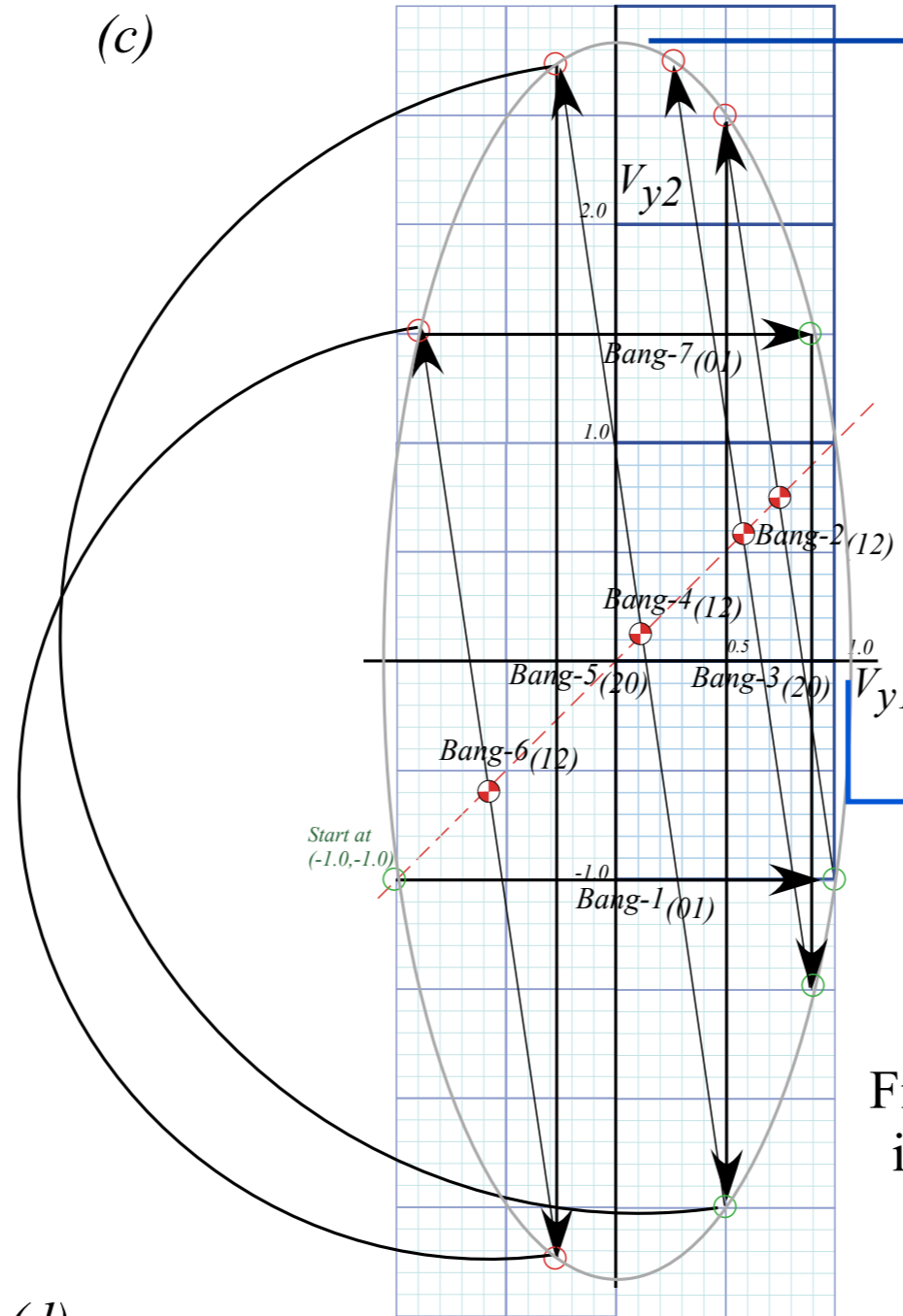
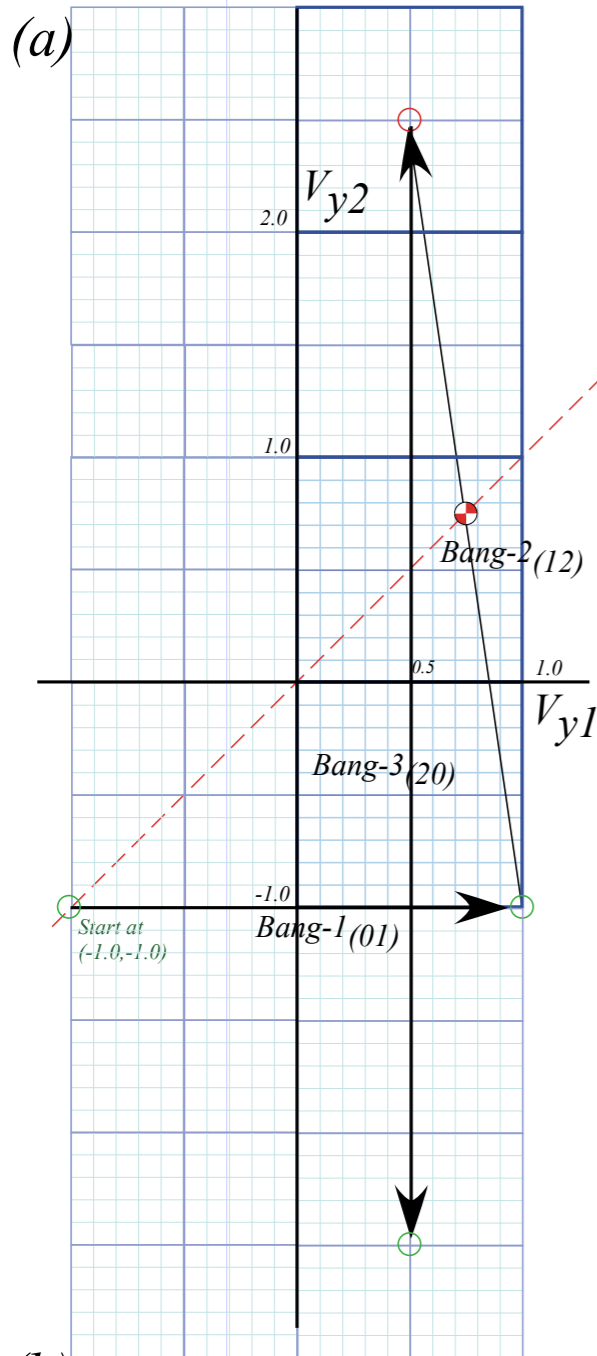
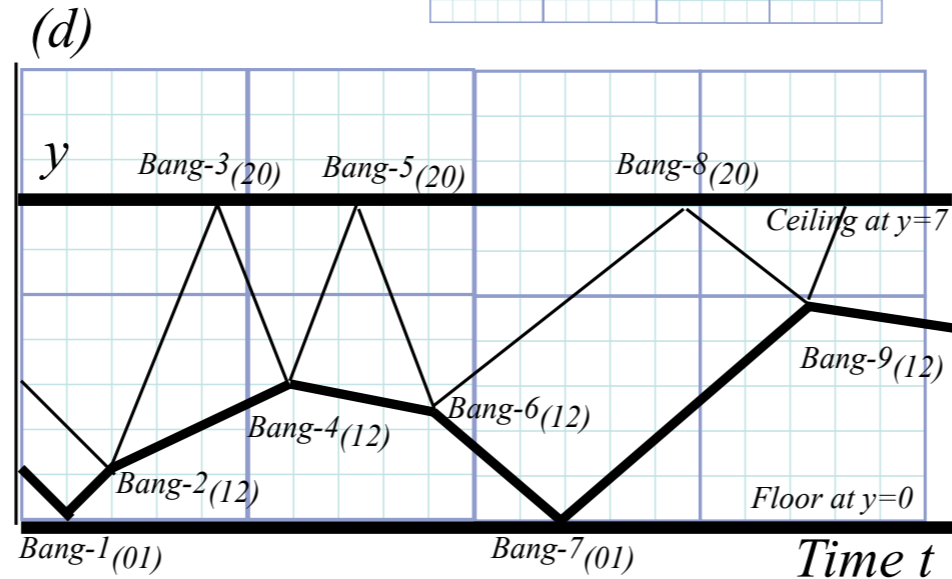
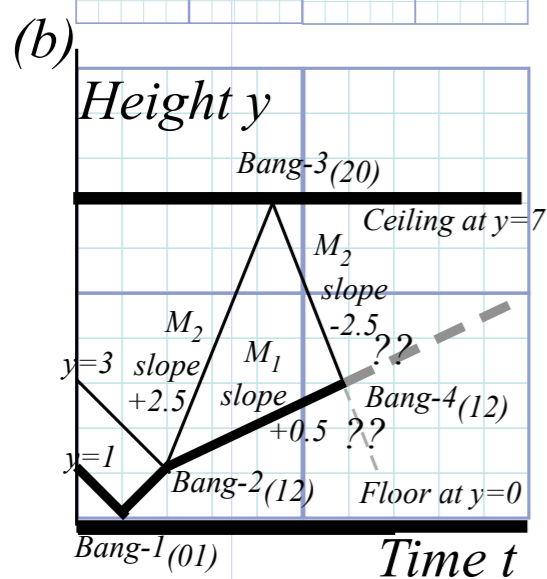
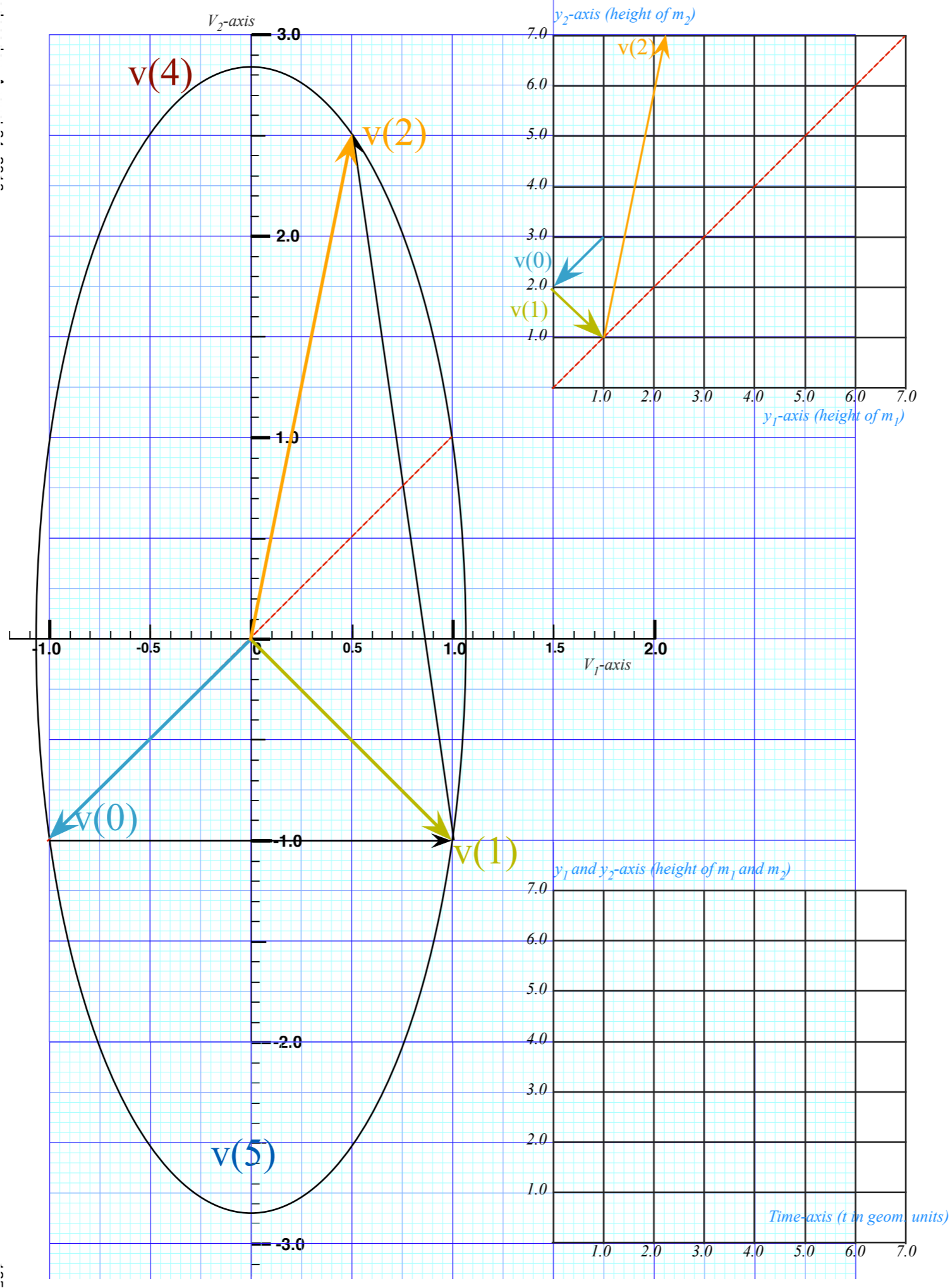
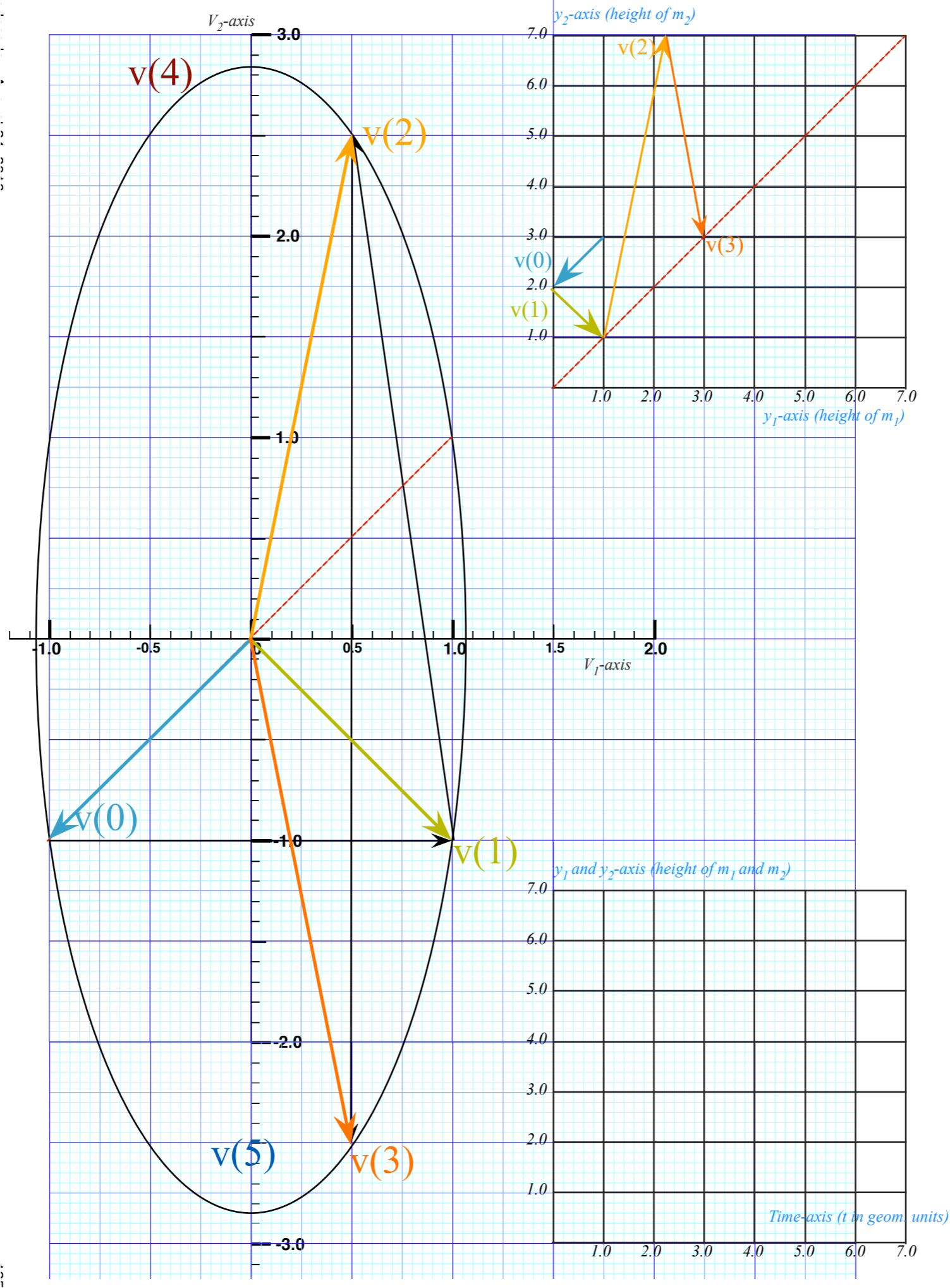


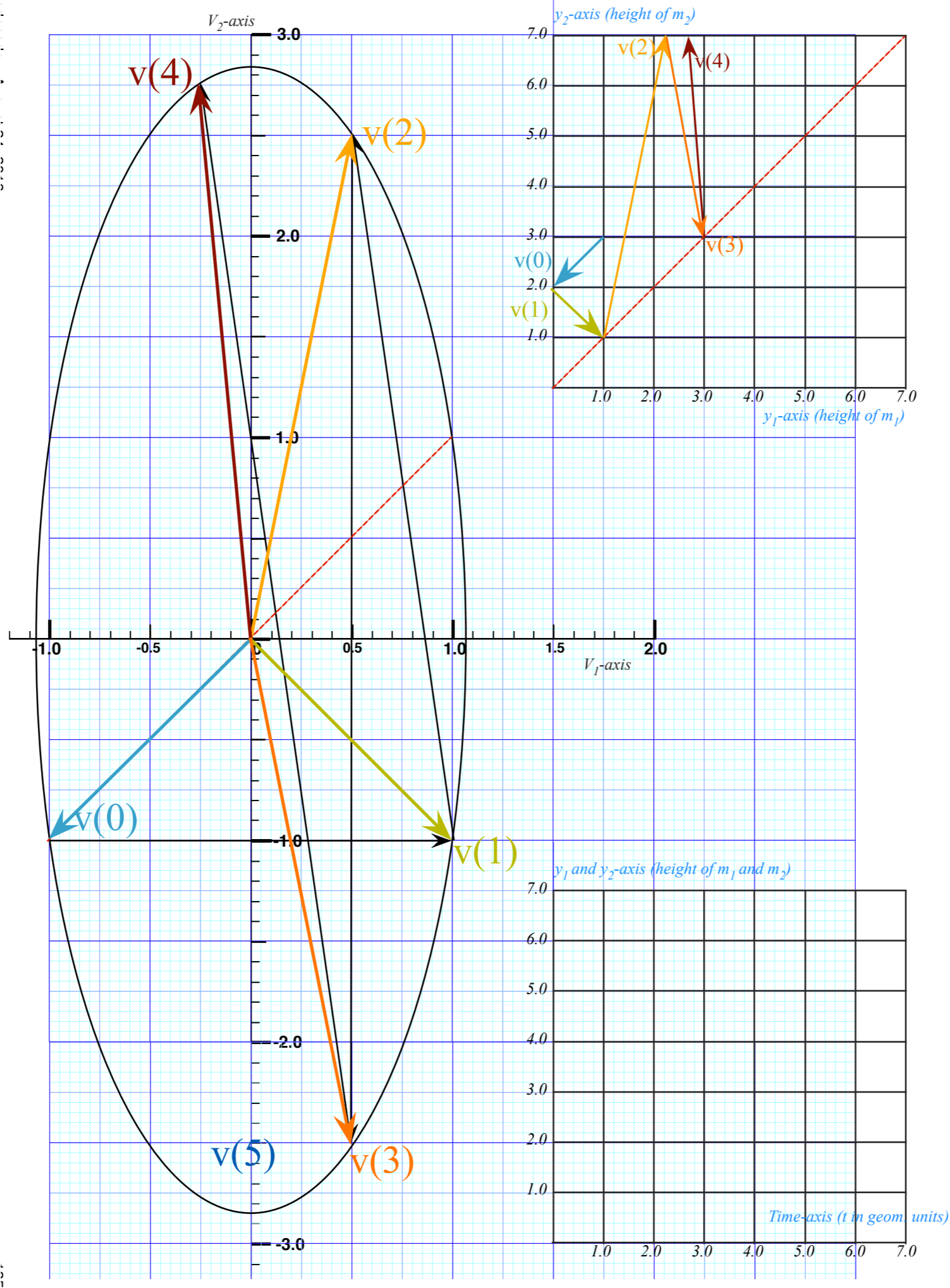
Fig. 4.7a-d
in Unit 1

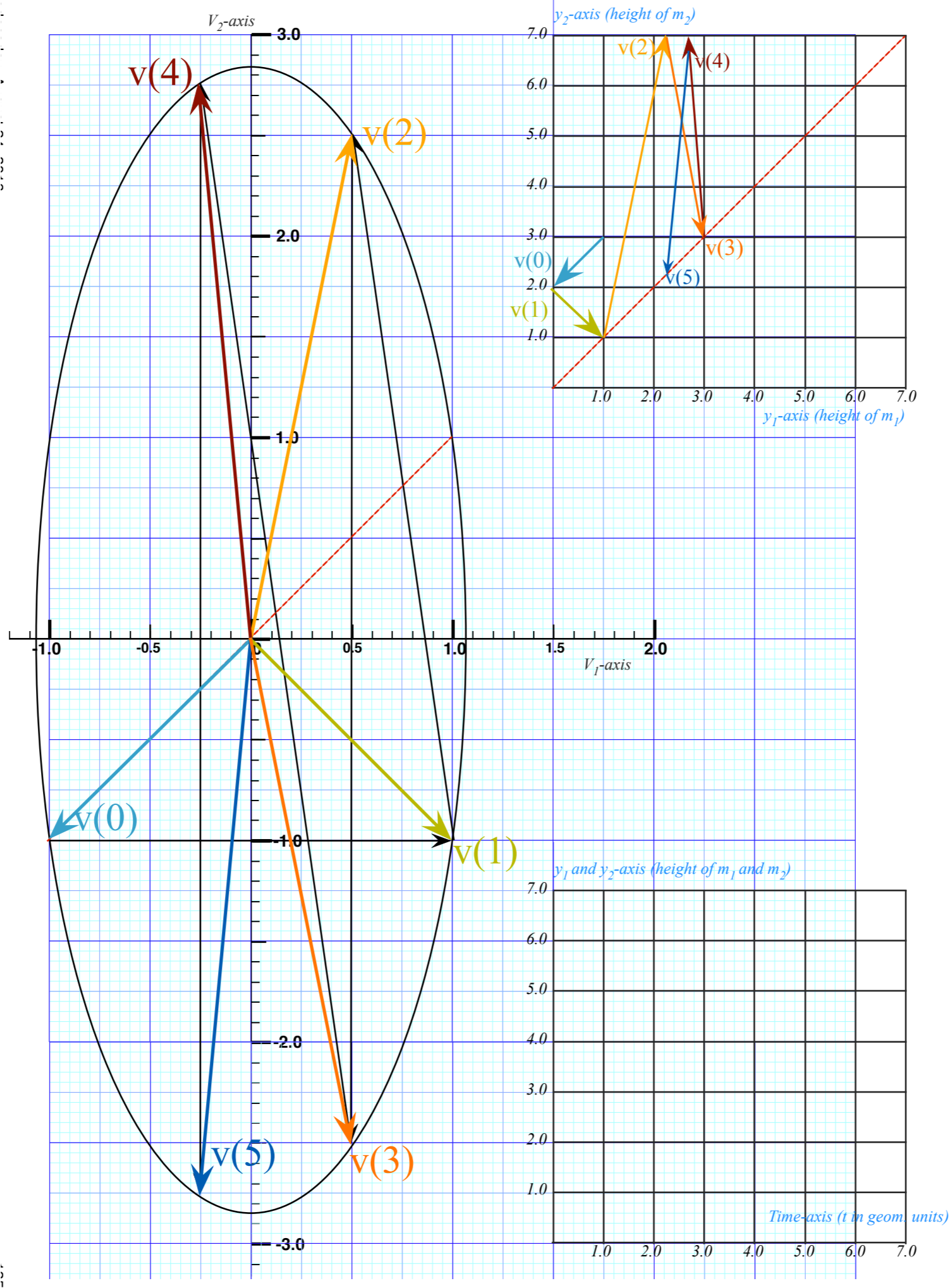


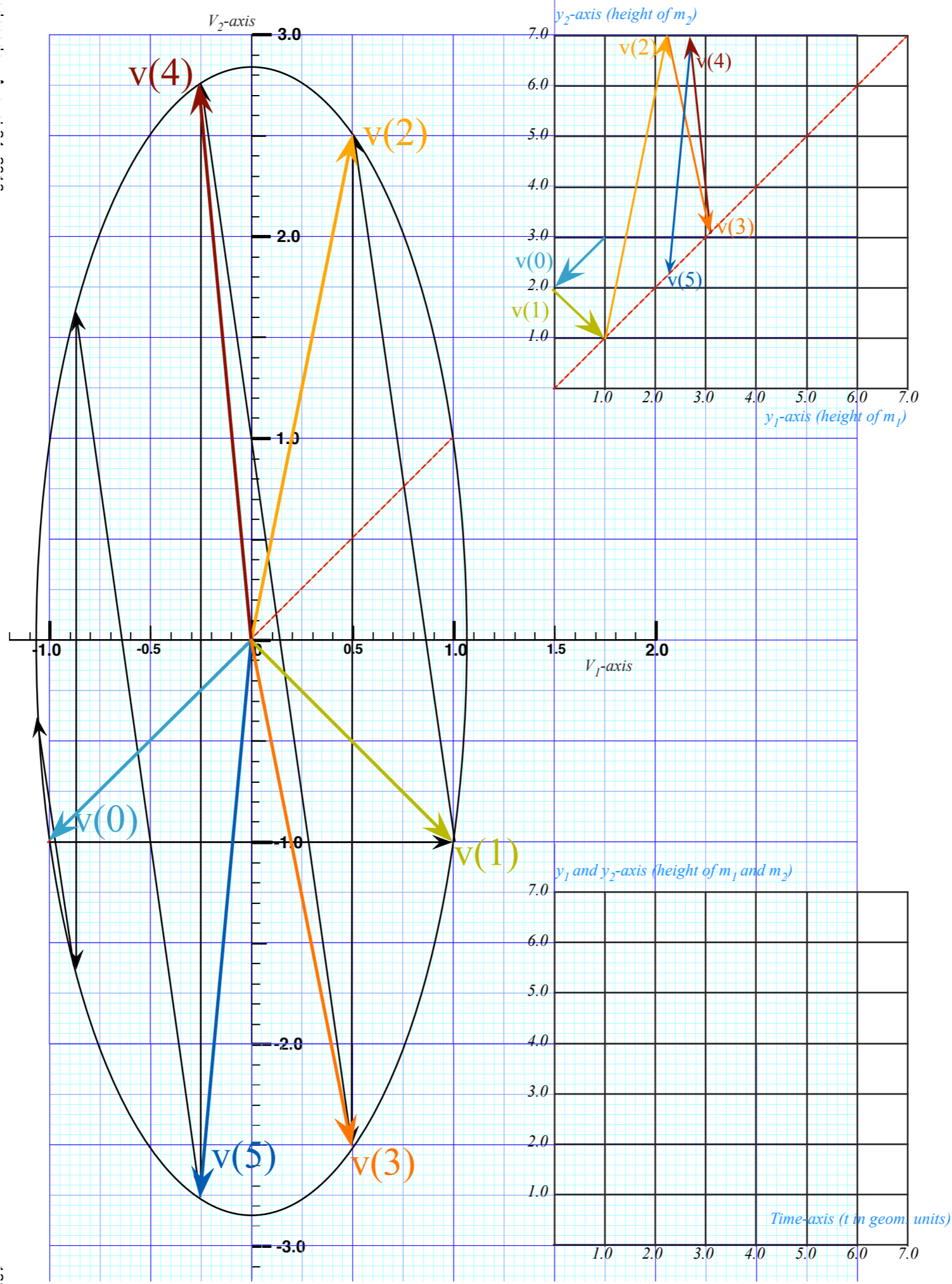
Collisions for
mass ratio
 $m_1:m_2 = 7:1$

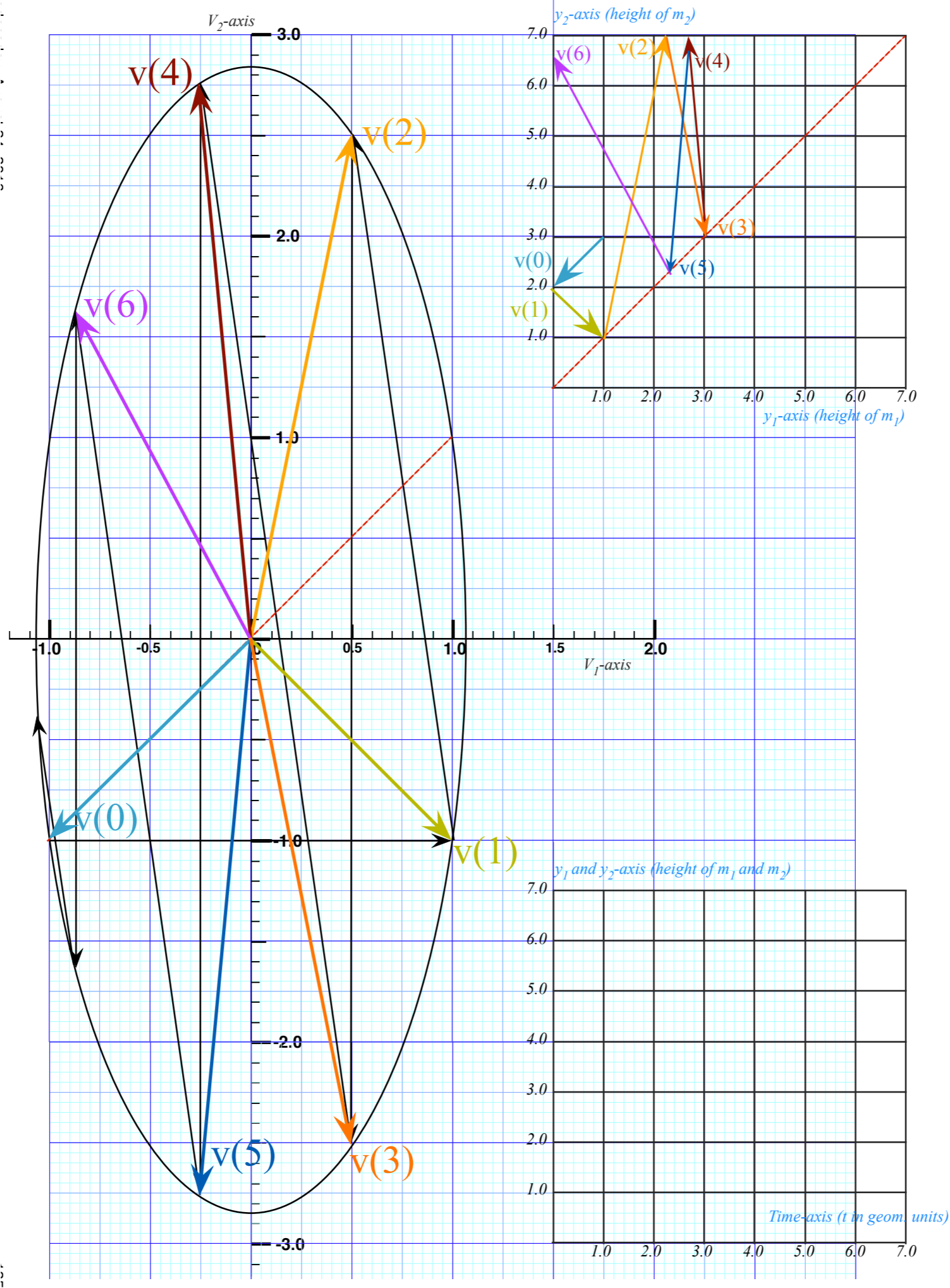












Review: Geometry of 1-D 2-body collisions (Example with masses: $m_1=49$ and $m_2=1$)

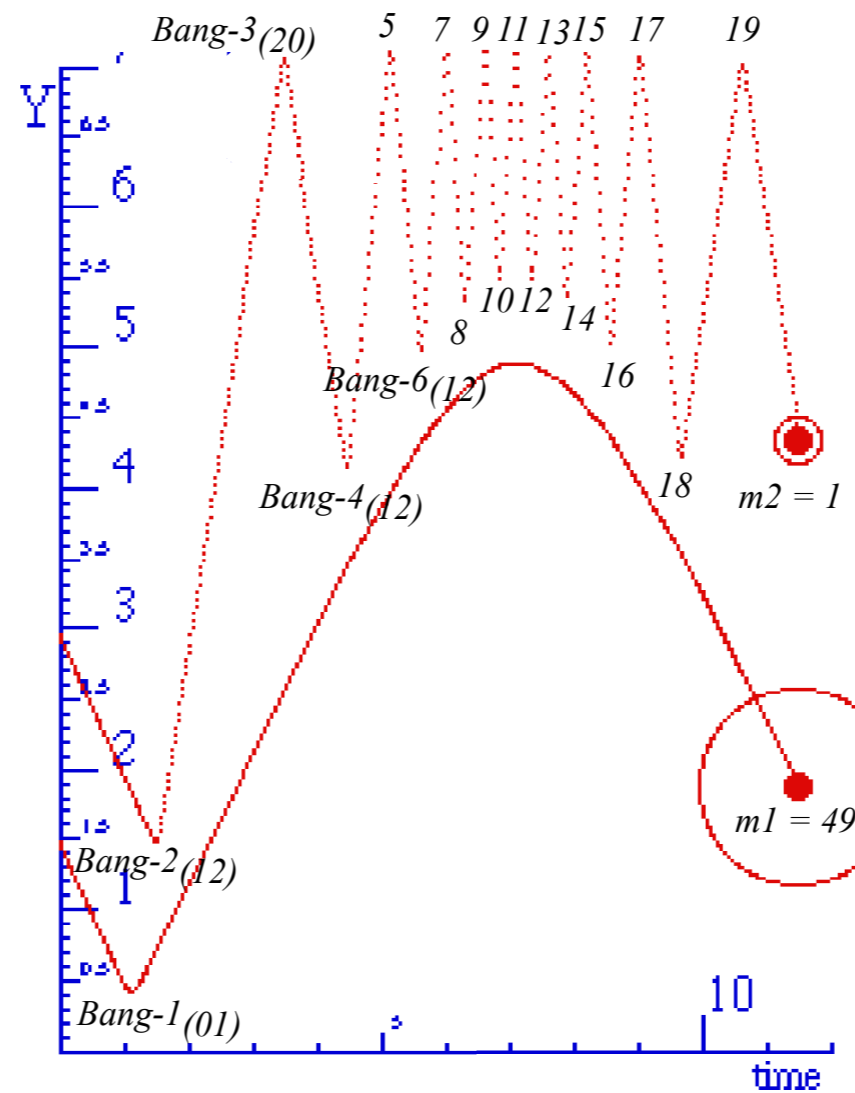


Fig. 5.1 *BouncIt Superball Collision Web Simulator: $M_1=49, M_2=1$ with Newtonian time plot in Unit 1*

BouncIt Superball Collision Web Simulator: $M_1=49, M_2=1$ with V_2 vs V_1 plot

Review: Geometry of 1-D 2-body collisions (Example with masses: $m_1=49$ and $m_2=1$)

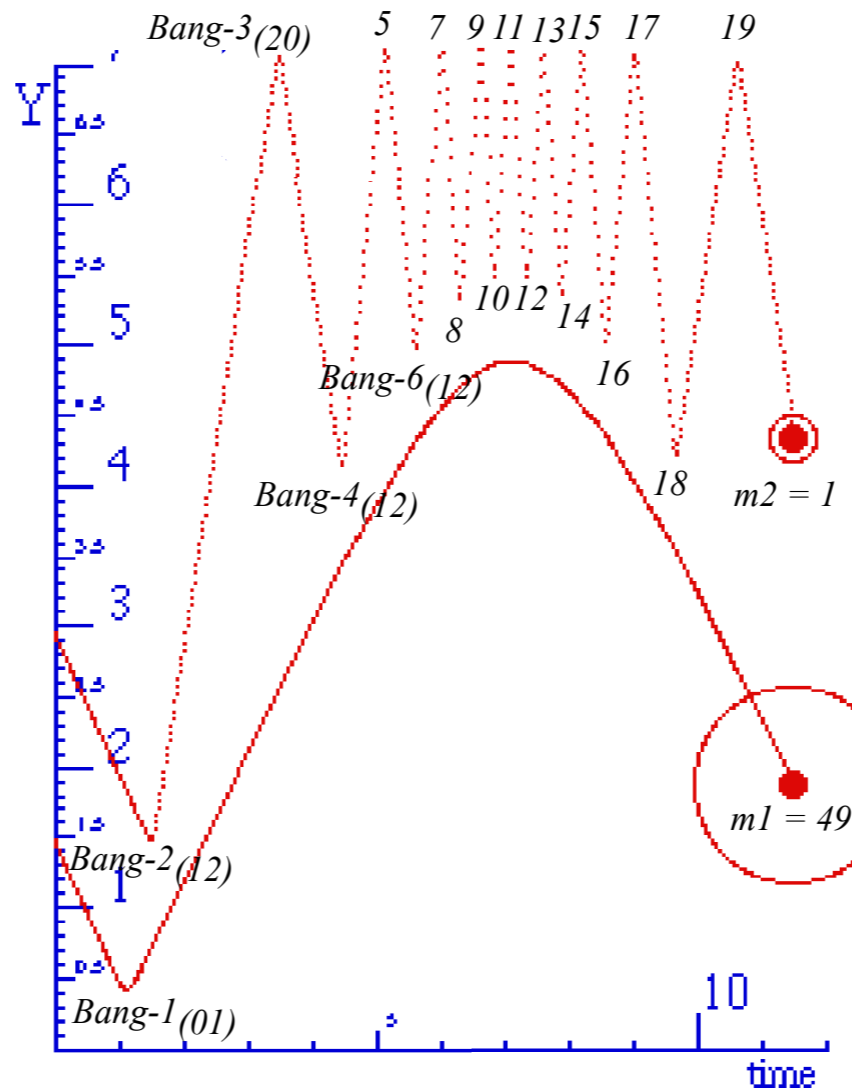
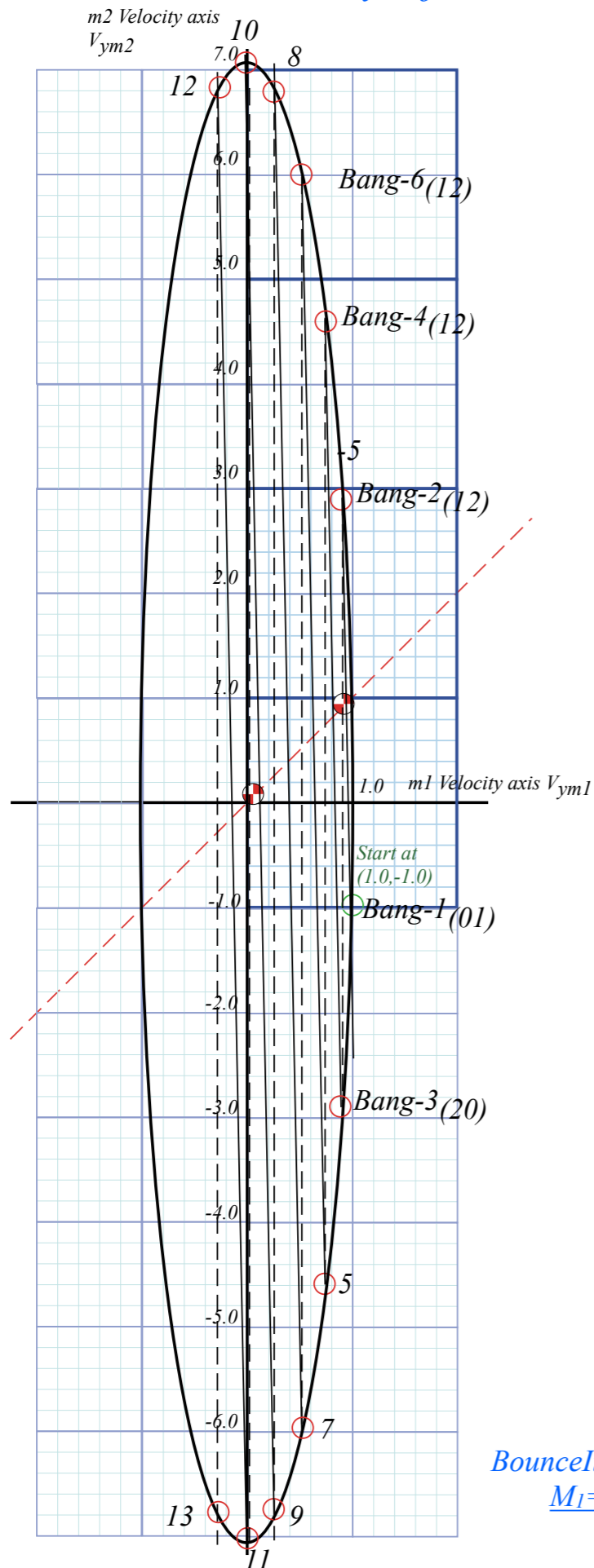
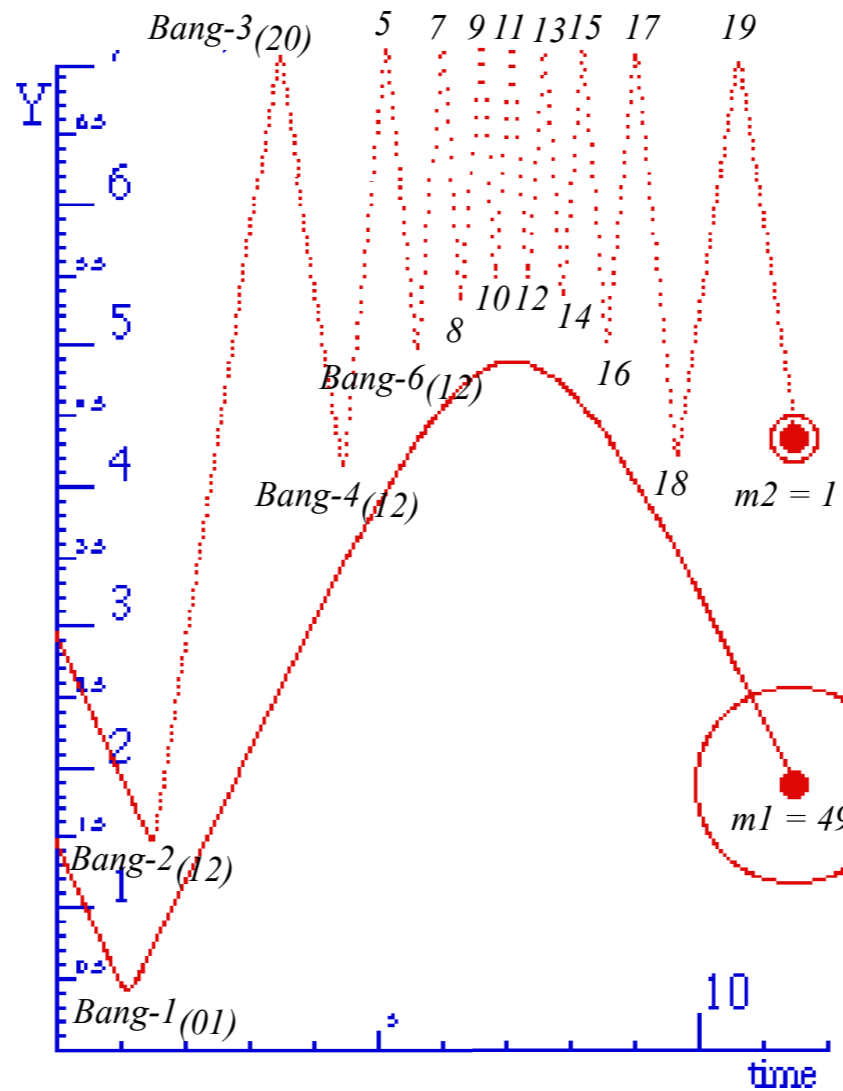
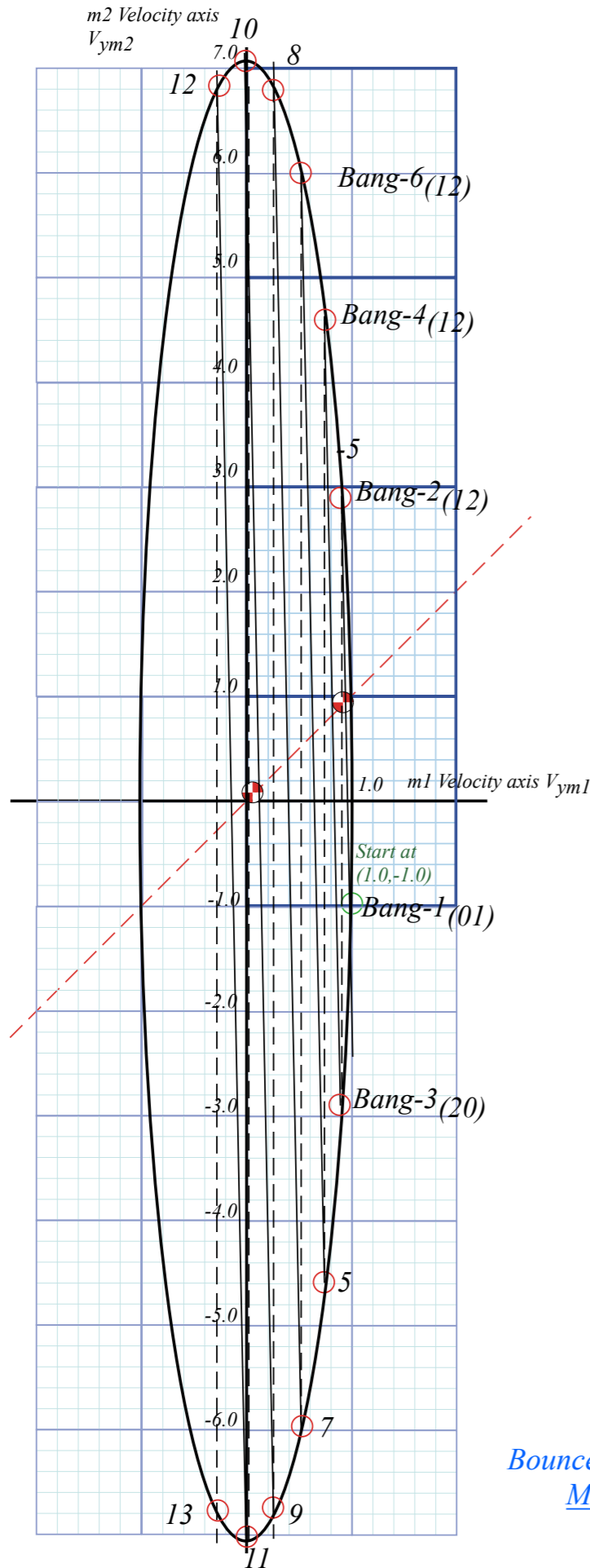


Fig. 5.1 [Bouncelt Superball Collision Web Simulator: \$M_1=49, M_2=1\$ with Newtonian time plot](#)
in Unit 1

[Bouncelt Superball Collision Web Simulator: \$M_1=49, M_2=1\$ with \$V_2\$ vs \$V_1\$ plot](#)

Review: Geometry of 1-D 2-body collisions (Example with masses: $m_1=49$ and $m_2=1$)



Kinetic Energy Ellipse

$$KE = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{49}{2} + \frac{1}{2} = 25$$

$$1 = \frac{V_1^2}{2KE/m_1} + \frac{V_2^2}{2KE/m_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$$

Ellipse radius 1

$$\begin{aligned} a_1 &= \sqrt{2KE/M_1} \\ &= \sqrt{2KE/49} \\ &= \sqrt{50/49} \\ &= 1.01 \end{aligned}$$

Ellipse radius 2

$$\begin{aligned} a_2 &= \sqrt{2KE/m_2} \\ &= \sqrt{2KE/1} \\ &= \sqrt{50/1} \\ &= 7.07 \end{aligned}$$

Fig. 5.1 *Bouncelt Superball Collision Web Simulator: $M_1=49, M_2=1$ with Newtonian time plot*

Bouncelt Superball Collision Web Simulator: $M_1=49, M_2=1$ with V_2 vs V_1 plot

Multiple collisions calculated by matrix operator products

 *Matrix or tensor algebra of 1-D 2-body collisions*

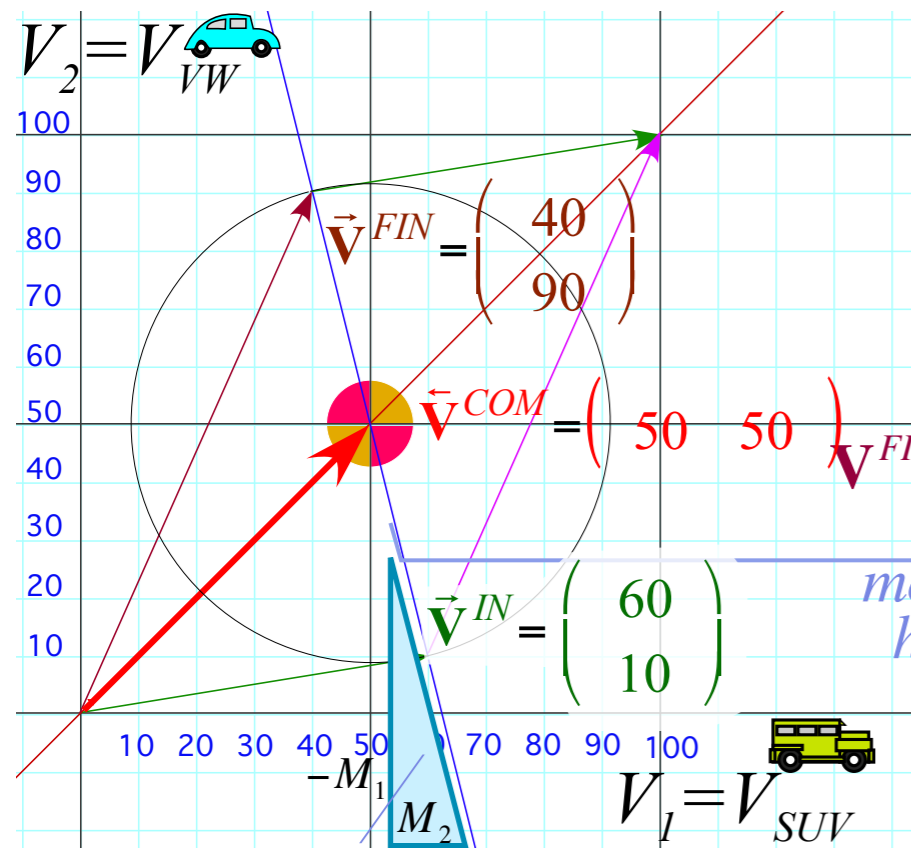
What about that 2nd quadratic solution?

*“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

*Geometry and algebra of “ellipse-Rotation” group product: **R= C•M***

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:
$$\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN} + \mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



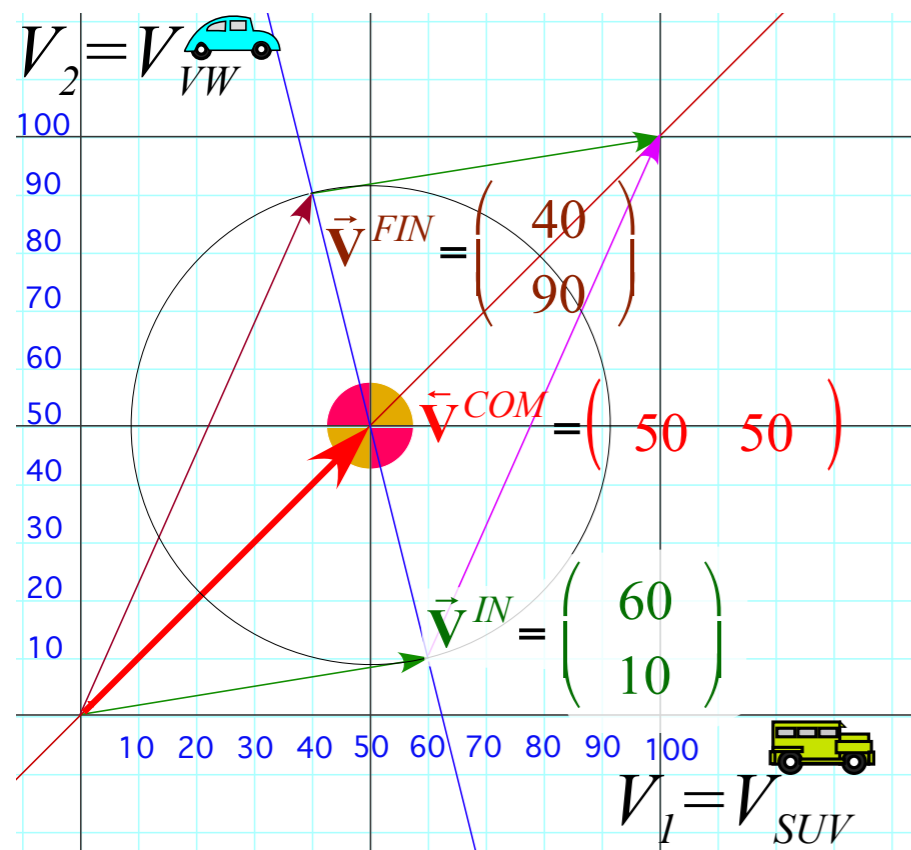
Points of vectors
and \mathbf{V}^{COM} and \mathbf{V}^{IN}
all lie on

*momentum - conservation line
having slope $-M_1 / M_1$*

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

$$\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN} + \mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} V^{COM} \\ V^{COM} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$



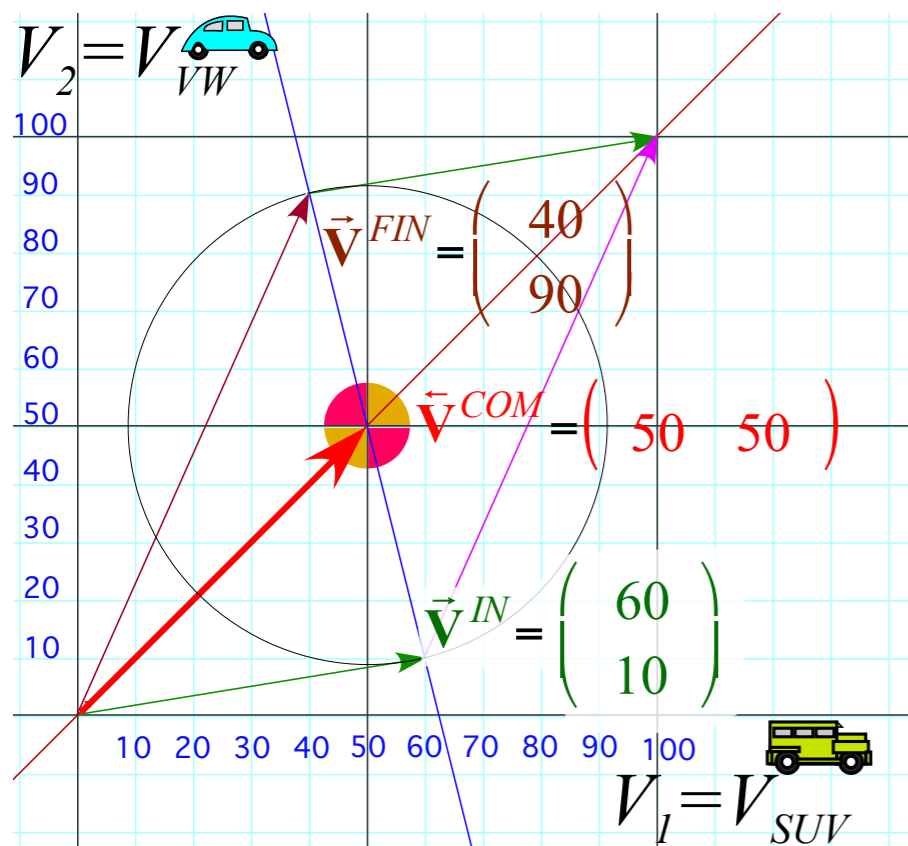
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

$$\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN} + \mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} V^{COM} \\ V^{COM} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Gives v^{FIN} in terms of v^{IN} ...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} =$$



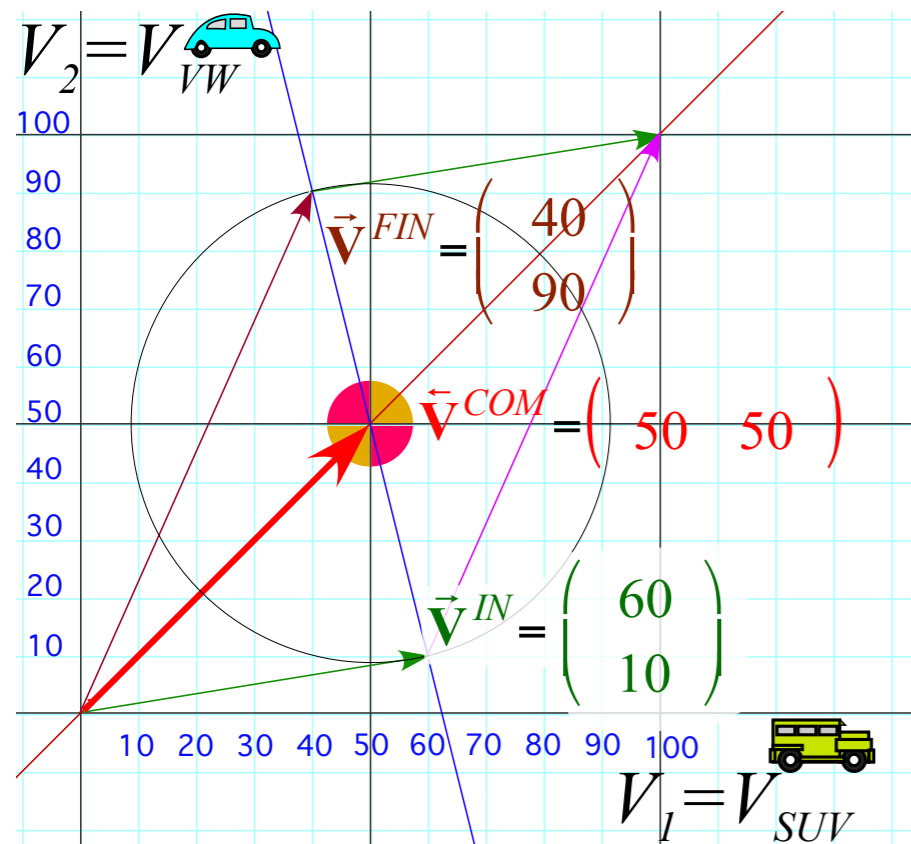
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

$$\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN} + \mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} V^{COM} \\ V^{COM} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Gives v^{FIN} in terms of v^{IN} ...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix}$$



Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

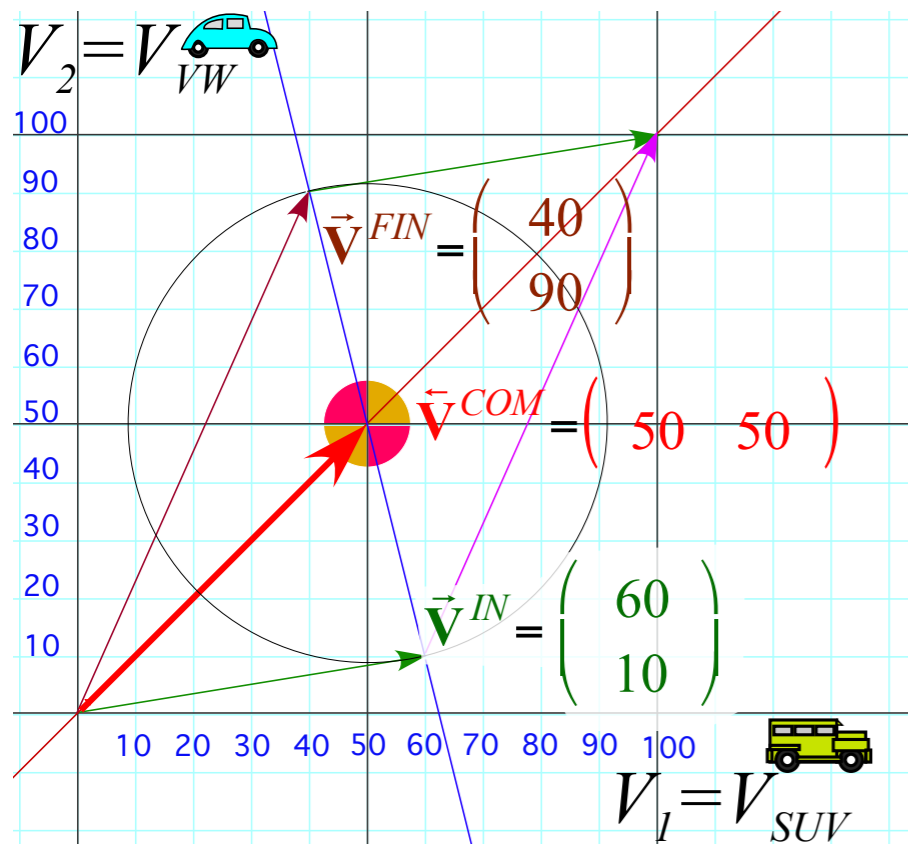
Gives \mathbf{v}^{FIN} in terms of \mathbf{v}^{IN} ...

$$\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN} + \mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} V^{COM} \\ V^{COM} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

$$= \begin{pmatrix} \frac{2m_1 v_1^{IN} + 2m_2 v_2^{IN} - m_1 v_1^{IN} - m_2 v_1^{IN}}{m_1 + m_2} \\ \frac{2m_1 v_1^{IN} + 2m_2 v_2^{IN} - m_1 v_2^{IN} - m_2 v_2^{IN}}{m_1 + m_2} \end{pmatrix}$$



Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

Gives \mathbf{v}^{FIN} in terms of \mathbf{v}^{IN} ...

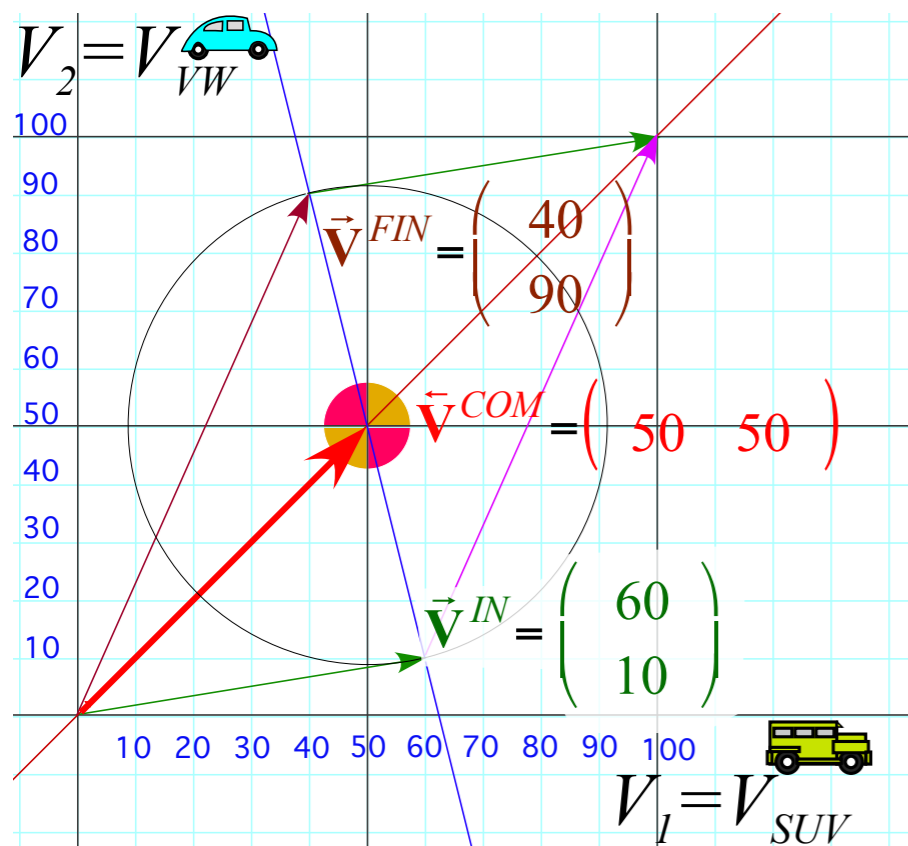
$$\mathbf{V}^{COM} = \frac{\mathbf{V}^{FIN} + \mathbf{V}^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} V^{COM} \\ V^{COM} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$.

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} \frac{2m_1 v_1^{IN} + 2m_2 v_2^{IN} - m_1 v_1^{IN} - m_2 v_1^{IN}}{m_1 + m_2} \\ \frac{2m_1 v_1^{IN} + 2m_2 v_2^{IN} - m_1 v_2^{IN} - m_2 v_2^{IN}}{m_1 + m_2} \end{pmatrix}$$


$$= \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$



Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

 *What about that 2nd quadratic solution?*

*“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

*Geometry and algebra of “ellipse-Rotation” group product: **R= C•M***

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

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What about that 2nd quadratic solution?

Linear formula $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$ gives just *one* solution to quadratic collision equations.

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

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Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

What about that 2nd quadratic solution?

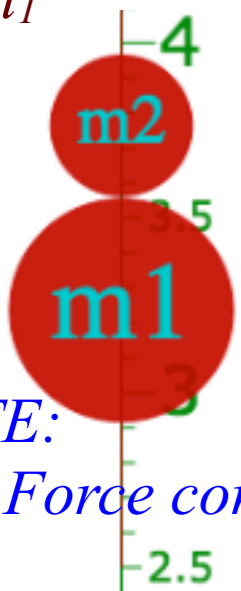
Linear formula $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$ gives just *one* solution to quadratic collision equations.

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

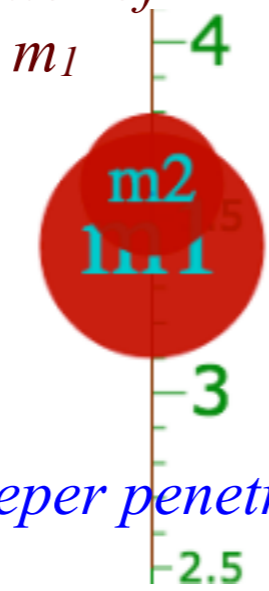
Q: What is the *second* solution and to what simple process would it correspond?

[Example with friction](#)

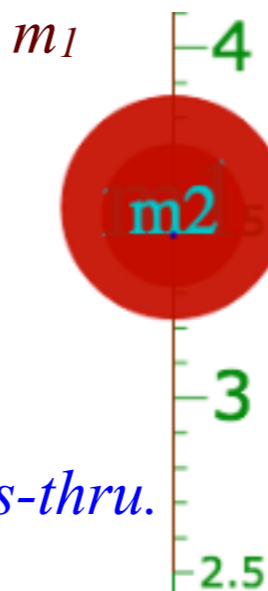
m_2
enters
 m_1



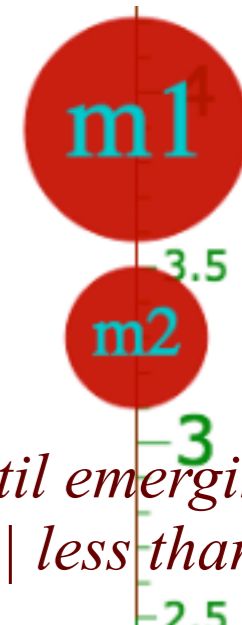
center of m_2
approaches
center of
 m_1



center of m_2
just past
center of
 m_1



...and quickly
accelerates
downward...



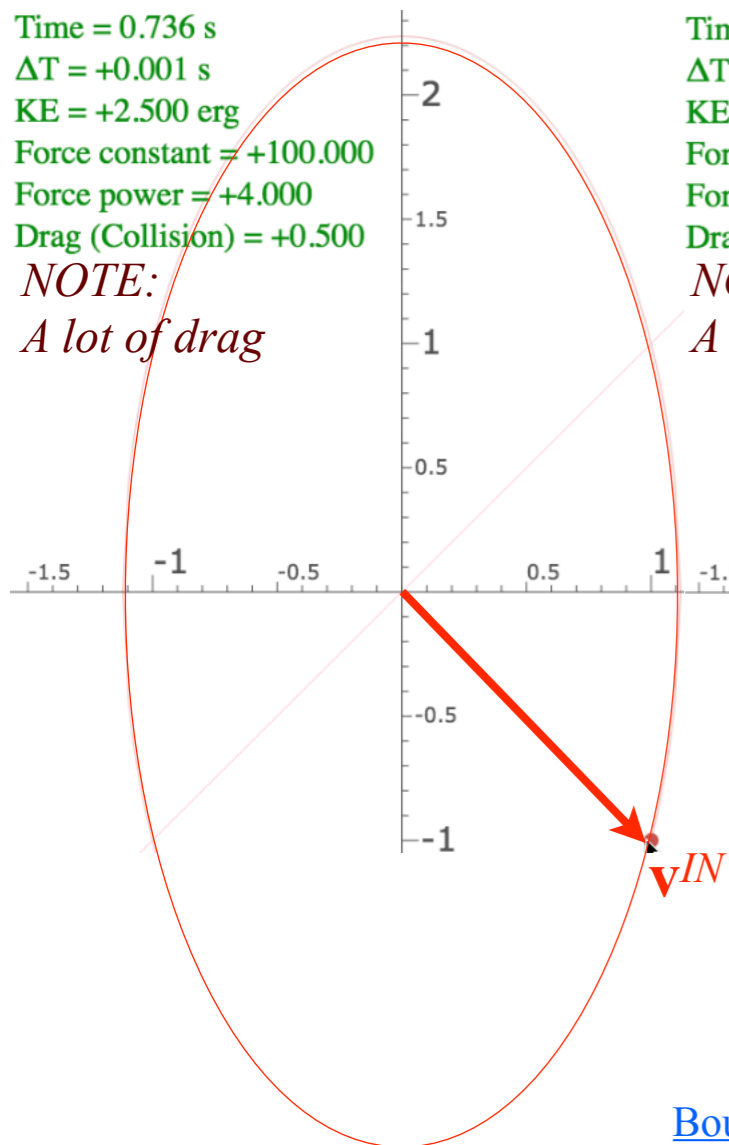
...thru drag until emerging from
 m_1 with $|v^{FIN}|$ less than $|v^{IN}|$

NOTE:

Low Force constant allows deeper penetration and pass-thru.

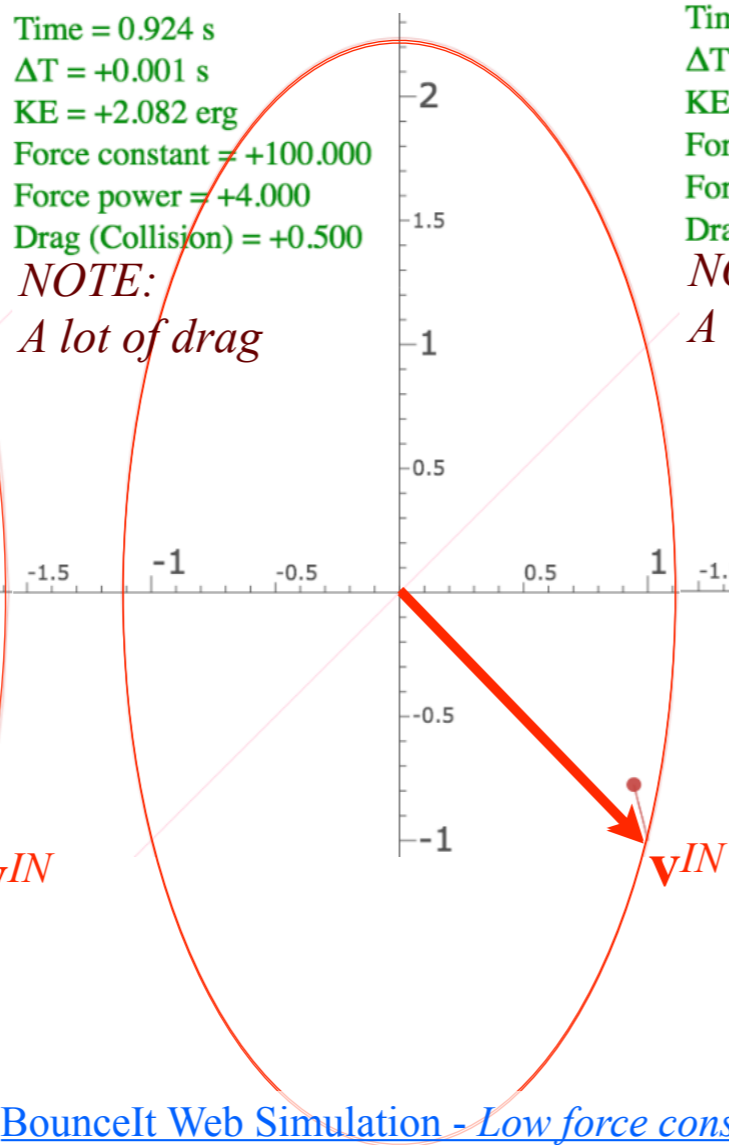
Time = 0.736 s
 $\Delta T = +0.001$ s
 KE = +2.500 erg
 Force constant = +100.000
 Force power = +4.000
 Drag (Collision) = +0.500

NOTE:
A lot of drag



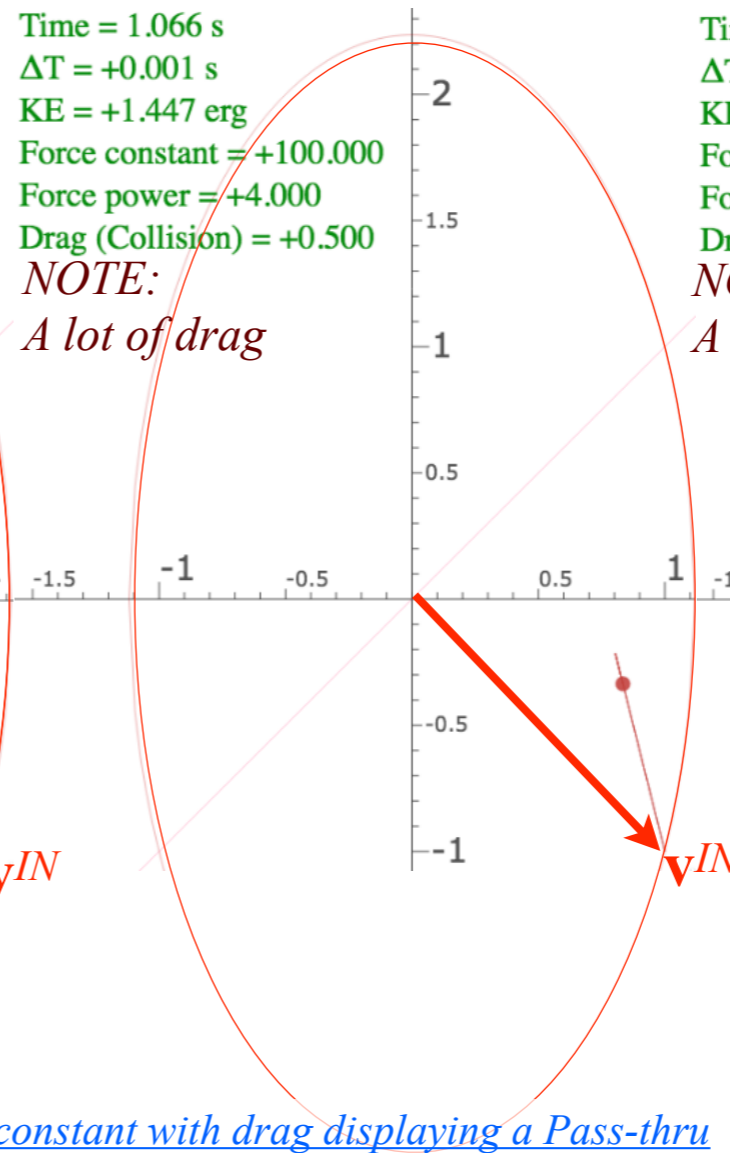
Time = 0.924 s
 $\Delta T = +0.001$ s
 KE = +2.082 erg
 Force constant = +100.000
 Force power = +4.000
 Drag (Collision) = +0.500

NOTE:
A lot of drag



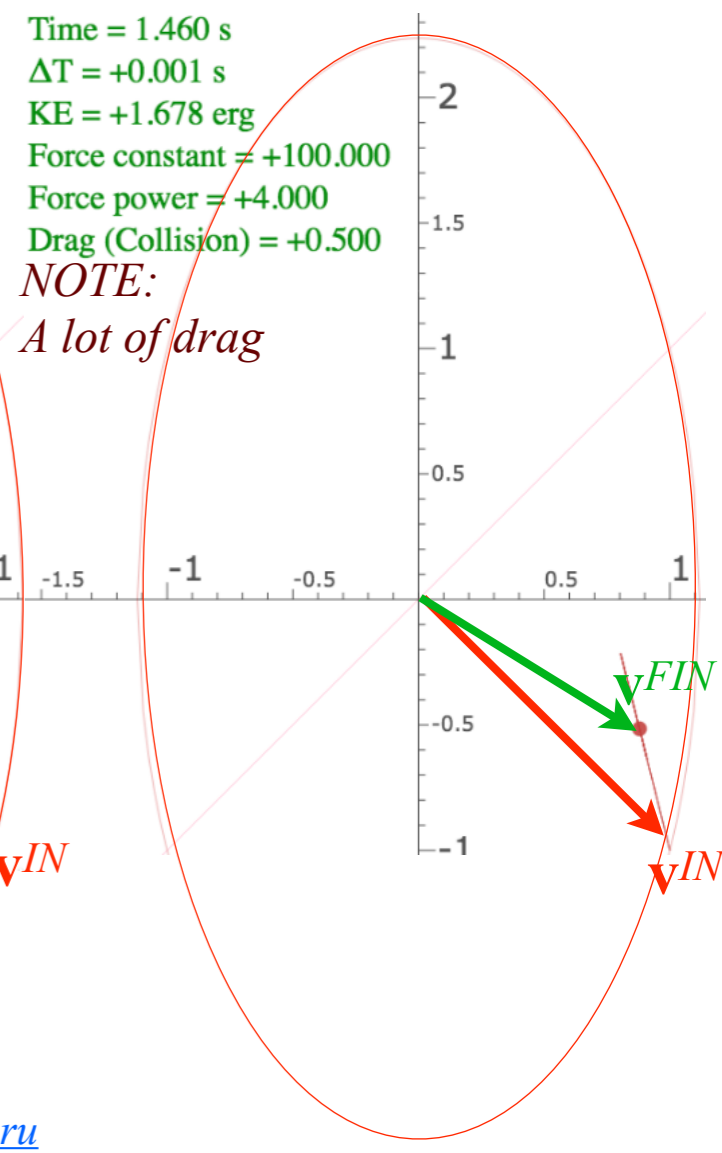
Time = 1.066 s
 $\Delta T = +0.001$ s
 KE = +1.447 erg
 Force constant = +100.000
 Force power = +4.000
 Drag (Collision) = +0.500

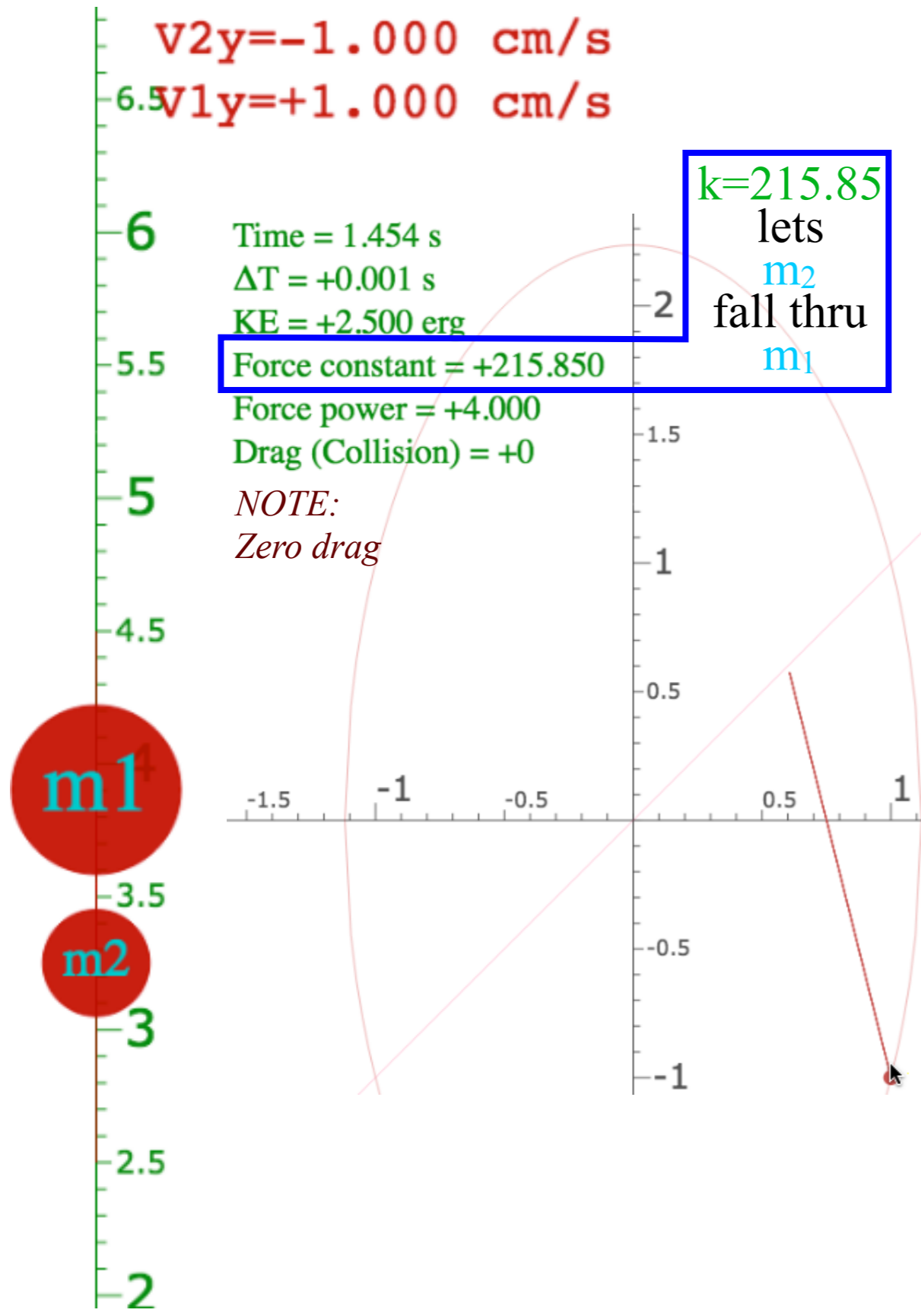
NOTE:
A lot of drag



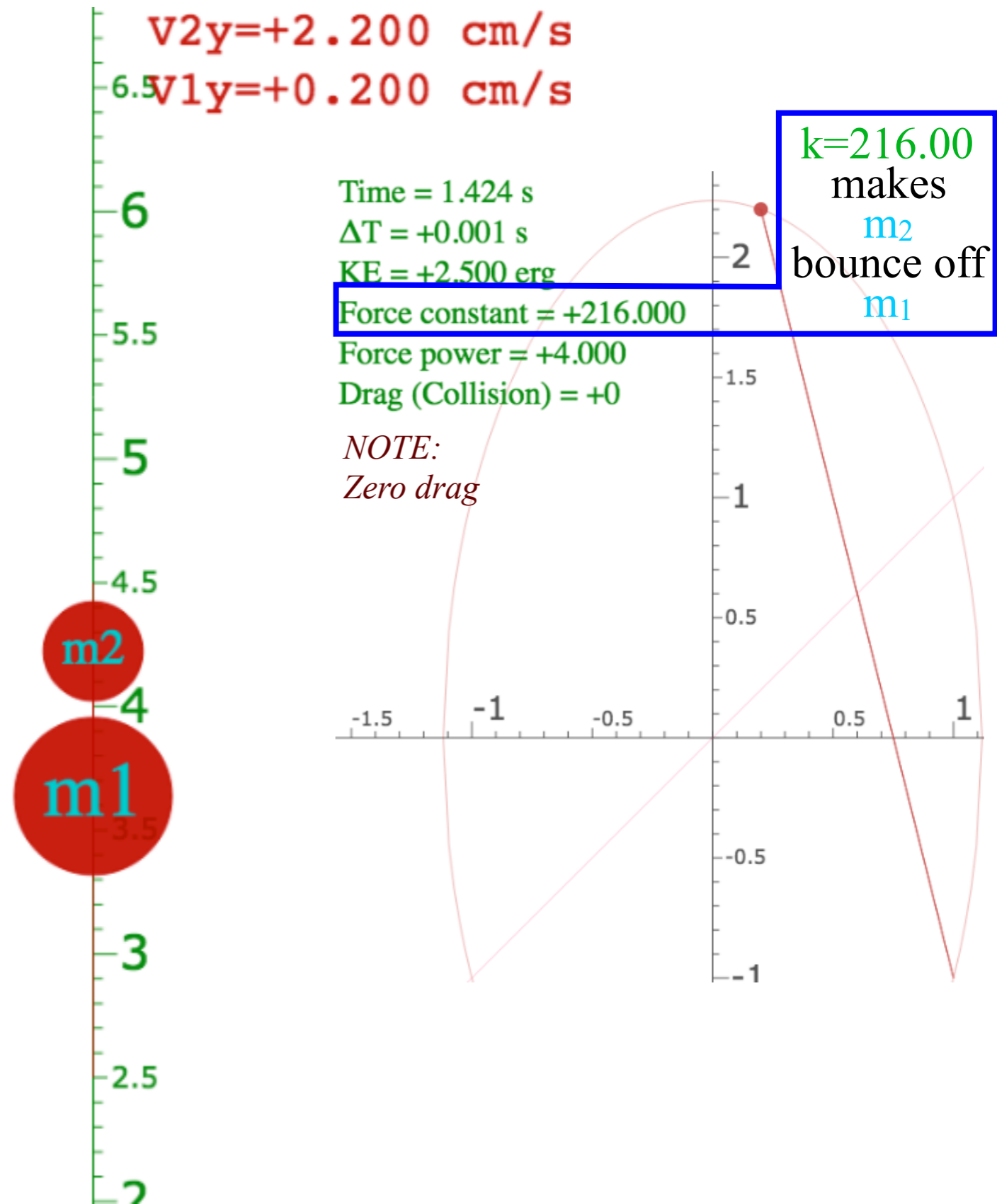
Time = 1.460 s
 $\Delta T = +0.001$ s
 KE = +1.678 erg
 Force constant = +100.000
 Force power = +4.000
 Drag (Collision) = +0.500

NOTE:
A lot of drag





Fall-Thru



Bounce-Off

Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

What about that 2nd quadratic solution?

 *“Mass-bang” matrix **M**, “Floor-bang” matrix **F**, “Ceiling-bang” matrix **C**.*

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Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give: $V^{COM} = \frac{V^{FIN} + V^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

Gives v^{FIN} in terms of v^{IN} ...

Finally as a matrix operation: $\mathbf{v}^{FIN} = \mathbf{M} \cdot \mathbf{v}^{IN}$...

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ 2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix} = \frac{\begin{pmatrix} m_1 v_1^{IN} - m_2 v_1^{IN} + 2m_2 v_2^{IN} \\ 2m_1 v_1^{IN} + m_2 v_2^{IN} - m_1 v_2^{IN} \end{pmatrix}}{m_1 + m_2} = \frac{\begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}}{m_1 + m_2}$$

Matrix operations include...

Floor-bang \mathbf{F} of m_I :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

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Matrix operations include...

Floor-bang \mathbf{F} of m_1 :

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:
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Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiple Collisions by Matrix Operator Products

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Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ and $m_2=1$

$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Review: Multiple Collisions by Matrix Operator Products

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Mass-bang \mathbf{M} of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ and $m_2=1$

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$$= \begin{pmatrix} \frac{2m_1 v_1^{IN} + 2m_2 v_2^{IN} - m_1 v_1^{IN} - m_2 v_1^{IN}}{m_1 + m_2} \\ \frac{2m_1 v_1^{IN} + 2m_2 v_2^{IN} - m_1 v_2^{IN} - m_2 v_2^{IN}}{m_1 + m_2} \end{pmatrix}$$

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix} \begin{pmatrix} v_1^{IN} \\ v_2^{IN} \end{pmatrix}$$

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Floor-bang \mathbf{F} of m_1 :

$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mass-bang \mathbf{M} of m_1 and m_2 :

$$\mathbf{M} = \begin{pmatrix} \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\ \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2} \end{pmatrix}$$

Ceiling-bang \mathbf{C} of m_2 :

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let: $m_1=49$ and $m_2=1$
$$\mathbf{M} = \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}$$

Define "ellipse-Rotation" \mathbf{R} as group product:
$$\mathbf{R} = \mathbf{C} \cdot \mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} = \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix}$$

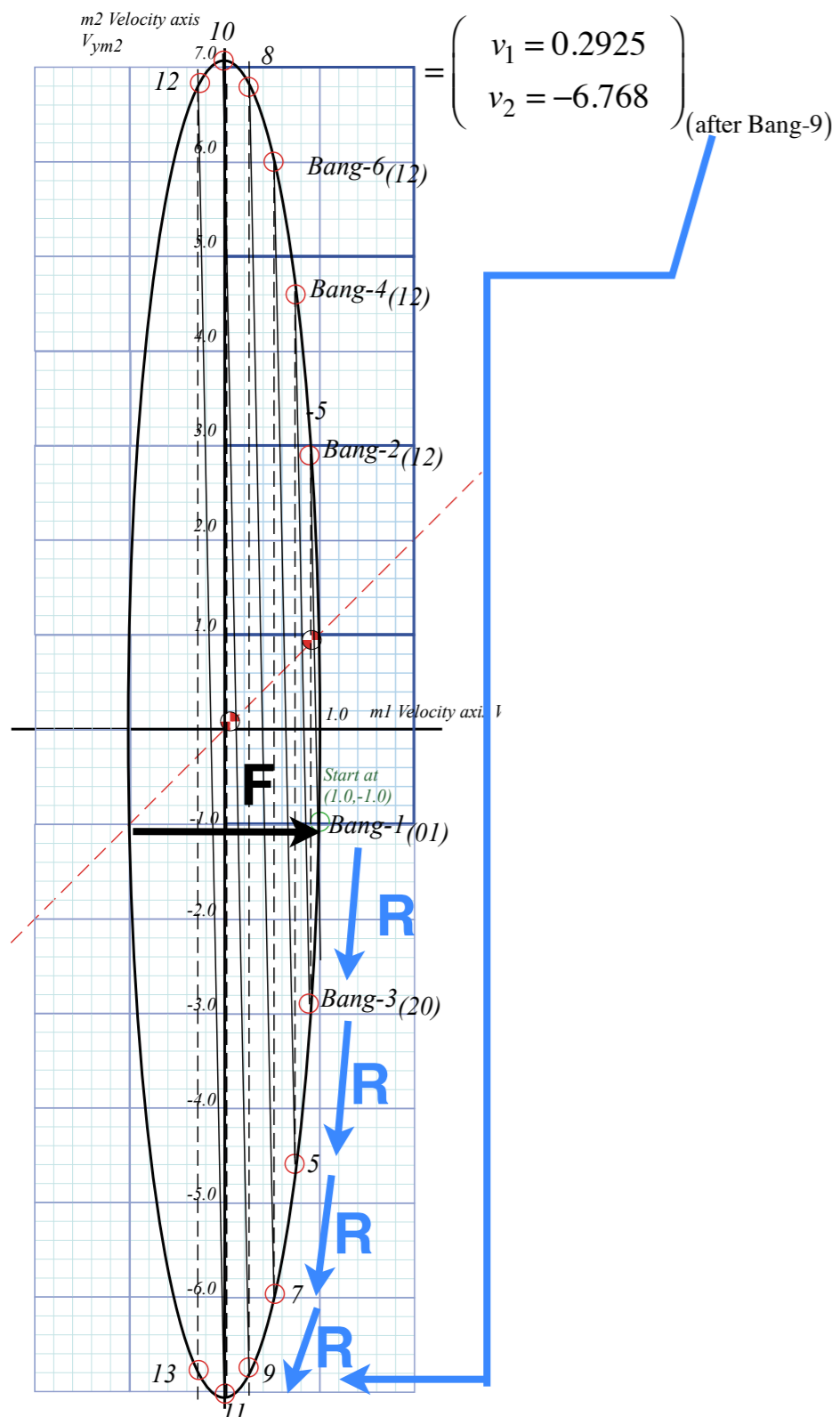
$$\begin{aligned}
 \left. \begin{array}{l} |FIN^9\rangle \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{array} \right\} &= \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \mathbf{F} \left. \begin{array}{l} |IN^0\rangle \\ v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{array} \right\}_{(\text{INITIAL } (0))} \\
 \left. \begin{array}{l} |FIN^9\rangle \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{array} \right\} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{F} |IN^0\rangle \\ v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{(\text{after Bang-1})}
 \end{aligned}$$

“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
\left. \begin{array}{l} |FIN^9\rangle \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{array} \right\} &= \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \underbrace{\mathbf{C} \cdot \mathbf{M}}_{\mathbf{R}} \cdot \mathbf{F} \left. \begin{array}{l} |IN^0\rangle \\ v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{array} \right\}_{(INITIAL (0))} \\
\left. \begin{array}{l} |FIN^9\rangle \\ v_1^{FIN-9} \\ v_2^{FIN-9} \end{array} \right\} &= \left(\begin{array}{cc} 0.96 & 0.04 \\ -1.96 & 0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ -1.96 & 0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ -1.96 & 0.96 \end{array} \right) \cdot \left(\begin{array}{cc} 0.96 & 0.04 \\ -1.96 & 0.96 \end{array} \right) \cdot \left. \begin{array}{l} \mathbf{F} |IN^0\rangle \\ v_1 = 1 \\ v_2 = -1 \end{array} \right\}_{(after Bang-1)} \\
&= \left. \begin{array}{l} v_1 = 0.2925 \\ v_2 = -6.768 \end{array} \right\}_{(after Bang-9)}
\end{aligned}$$

“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\mathbf{R}} \cdot \underbrace{\begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix}}_{\mathbf{R}} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{(\text{INITIAL } (0))} \\
 |FIN^9\rangle &= \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{(\text{after Bang-1})}
 \end{aligned}$$

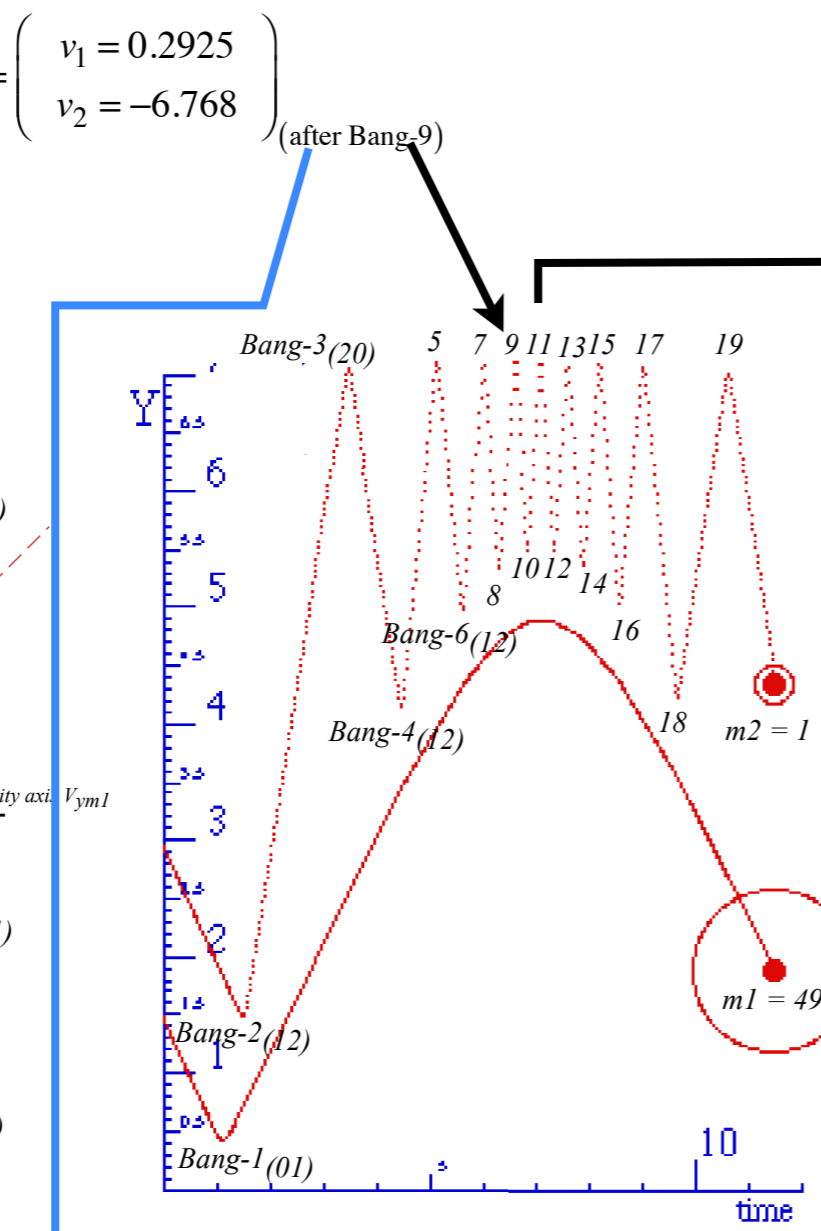
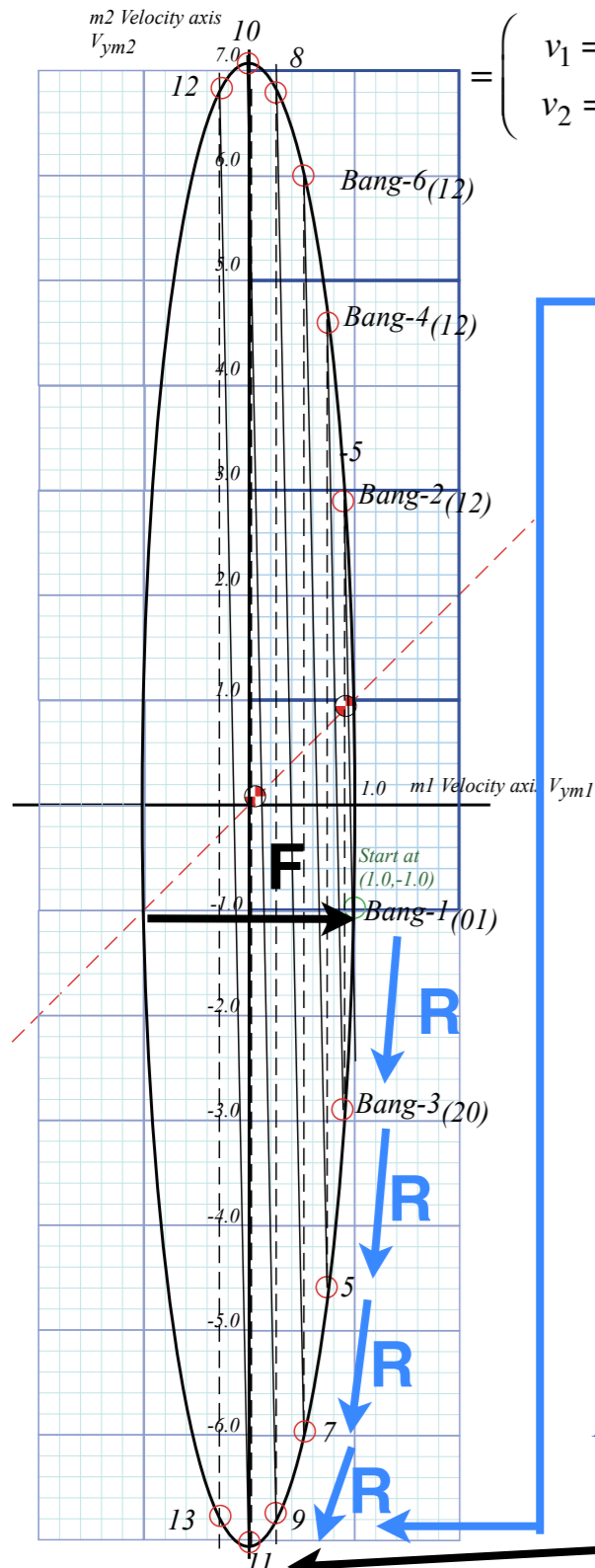


“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

Collisions for mass ratio $m_1:m_2 = 49:1$

Fig. 5.1a (revised)

$$\begin{aligned}
 |FIN^9\rangle &= \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ 1.96 & -0.96 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \begin{pmatrix} v_1^{IN} = -1 \\ v_2^{IN} = -1 \end{pmatrix}_{(INITIAL (0))} \\
 |FIN^9\rangle &= \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{F} |IN^0\rangle \\
 \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1 = 1 \\ v_2 = -1 \end{pmatrix}_{(after Bang-1)}
 \end{aligned}$$



“ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$

$$\begin{aligned}
 \begin{pmatrix} v_1^{FIN-11} \\ v_2^{FIN-11} \end{pmatrix} &= \begin{pmatrix} 0.96 & 0.04 \\ -1.96 & 0.96 \end{pmatrix} \cdot \begin{pmatrix} v_1^{FIN-9} \\ v_2^{FIN-9} \end{pmatrix} \\
 &= \begin{pmatrix} v_1 = 0.0100 \\ v_2 = -7.071 \end{pmatrix}_{(after Bang-11)}
 \end{aligned}$$

Collisions for mass ratio $m_1:m_2 = 49:1$

BouncIt Superball Collision Web Simulator: $M_1=49, M_2=1$ with Newtonian time plot

BouncIt Superball Collision Web Simulator: $M_1=49, M_2=1$ with V_2 vs V_1 plot

<<Under Construction>>
Matrix Collision Web Simulator:
 $M_1=49, M_2=1$ V_2 vs V_1 plot

Fig. 5.1a-b (revised)

Ellipse rescaling-geometry and reflection-symmetry analysis

 *Rescaling KE ellipse to circle*

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

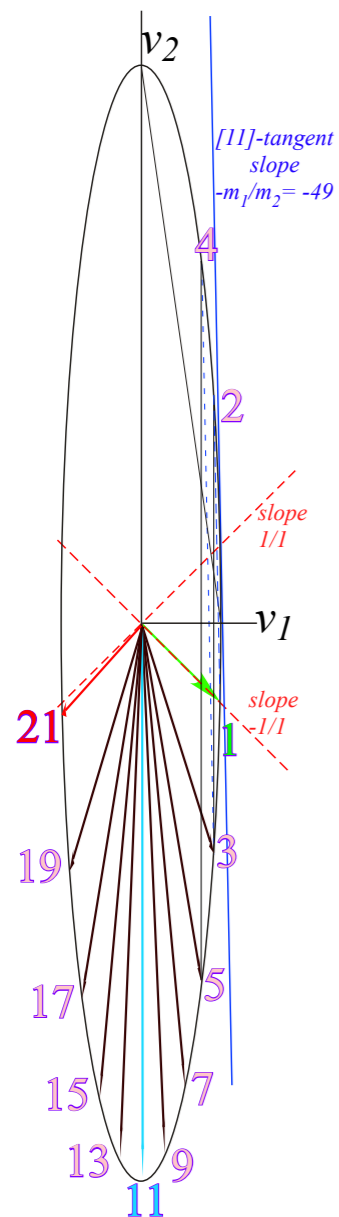
Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

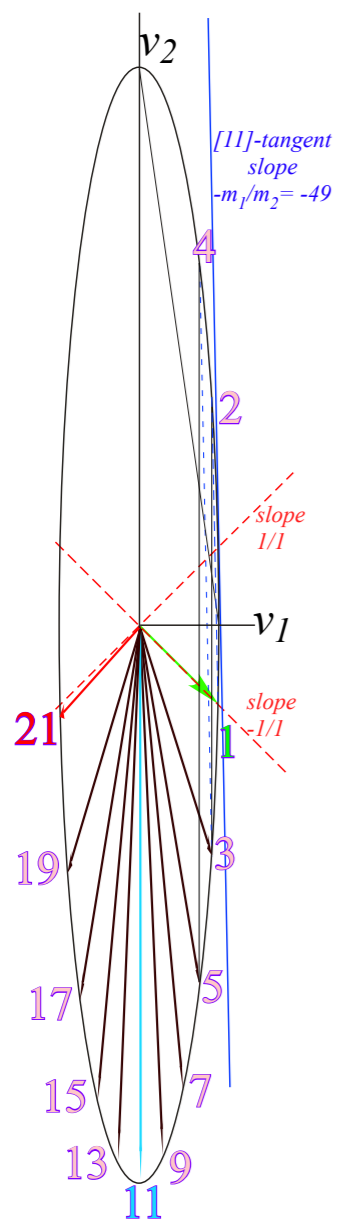


Collisions for
mass ratio
 $m_1:m_2 = 49:1$

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$



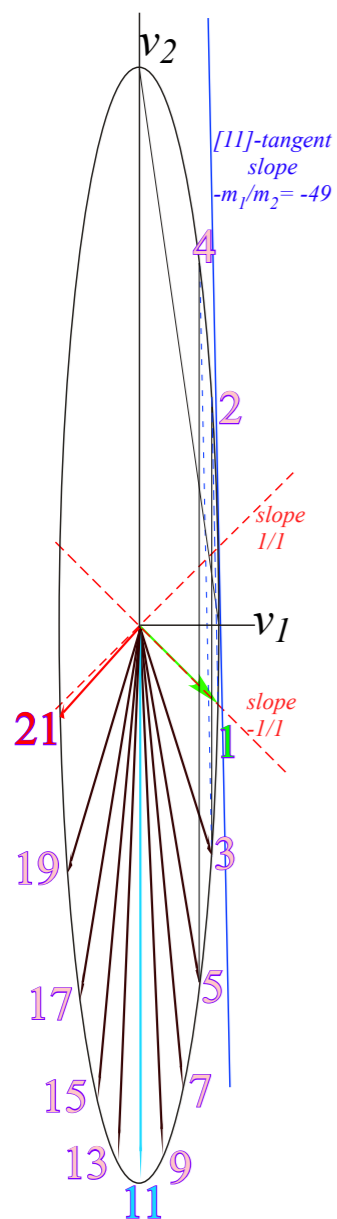
Collisions for
mass ratio
 $m_1:m_2 = 49:1$

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or:} \quad \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or:} \quad \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$



Collisions for
mass ratio
 $m_1:m_2 = 49:1$

Ellipse rescaling geometry and reflection symmetry analysis

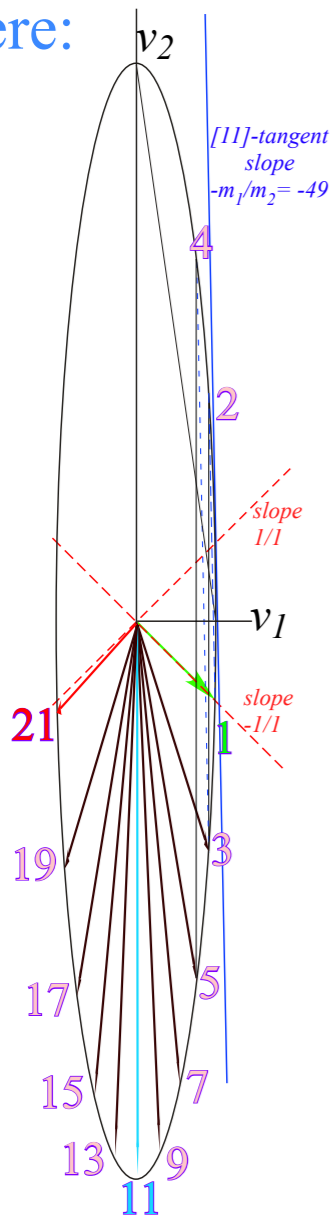
Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where: $\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$ and: $\sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$ with: $\left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$



Collisions for
mass ratio
 $m_1:m_2 = 49:1$

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

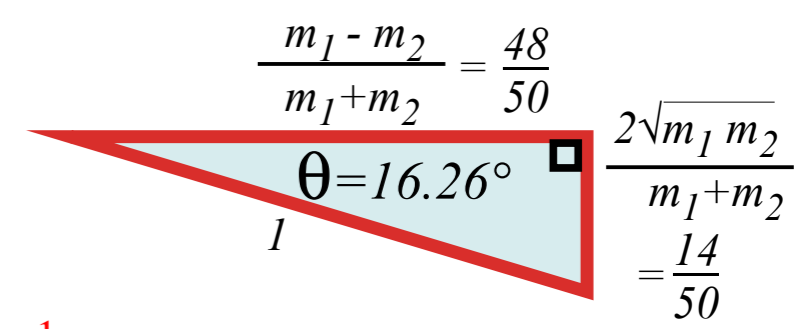
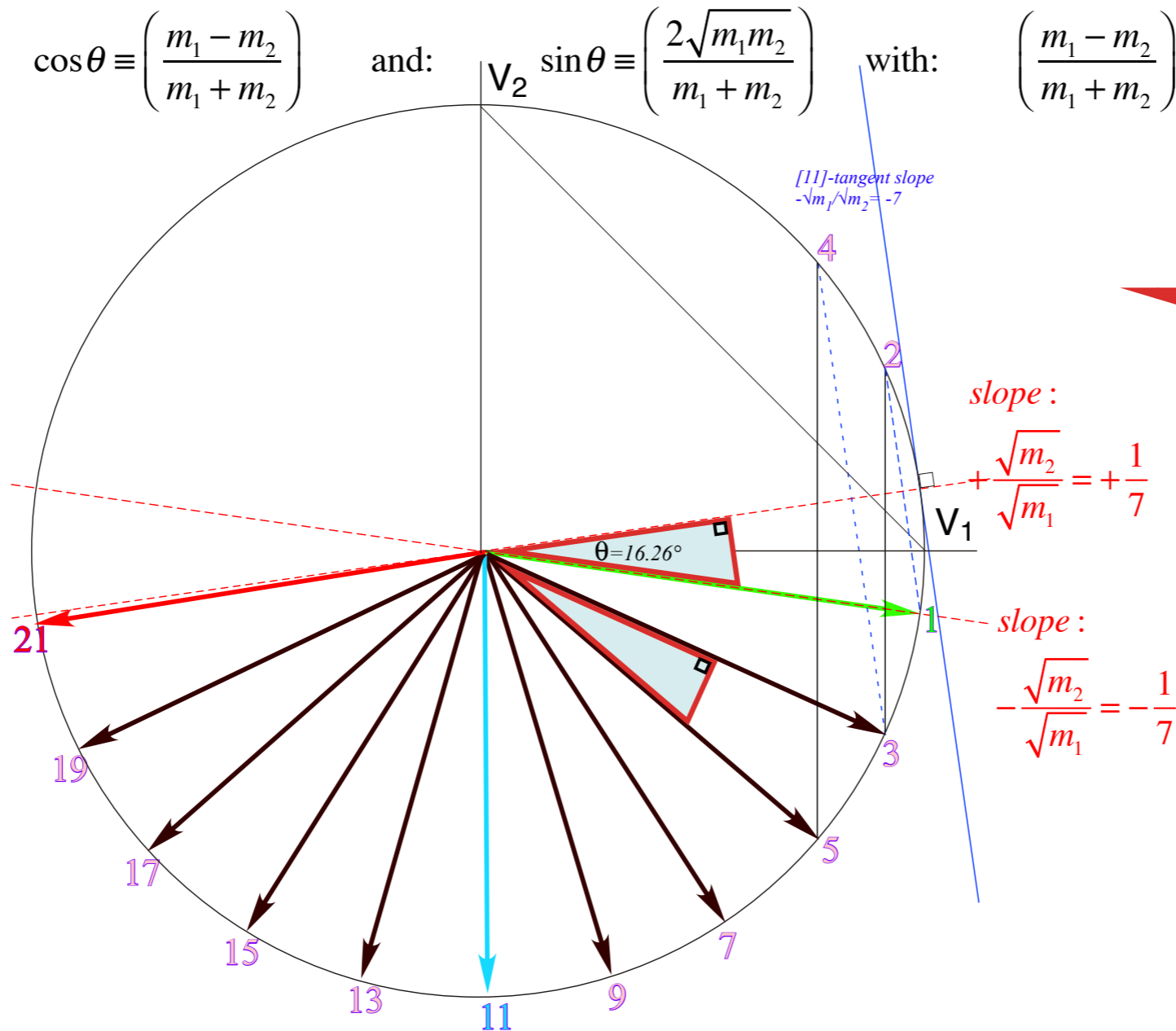
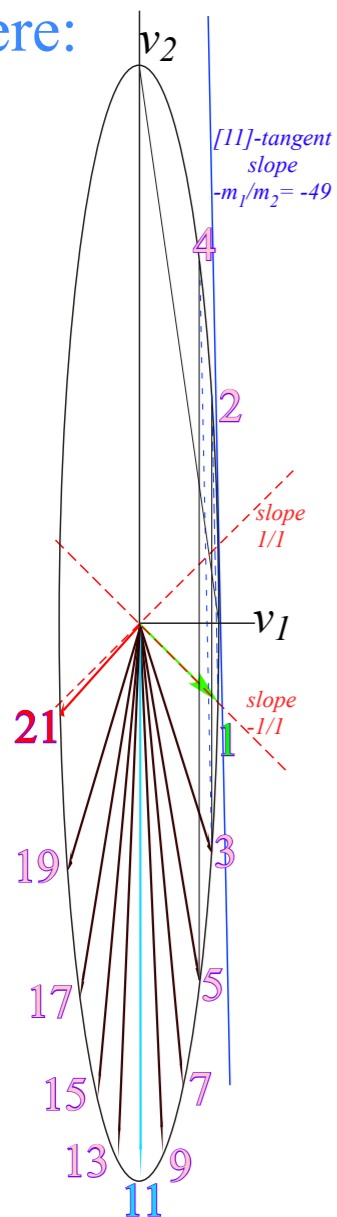
$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ becomes: } \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \text{ or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where:

$$\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right) \text{ and: } \sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right) \text{ with: } \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$$



Collisions for
mass ratio
 $m_1:m_2 = 49:1$

Fig. 5.2a-c
(revised)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

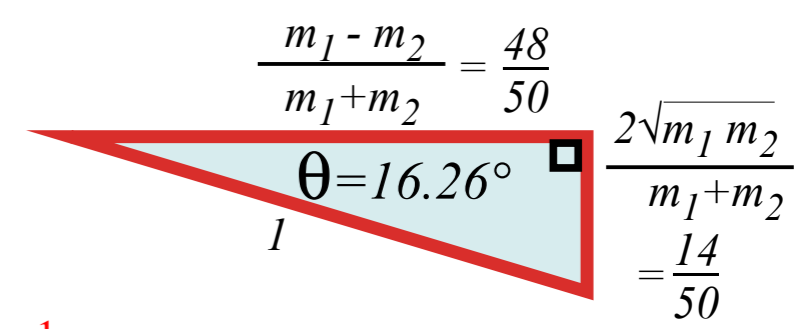
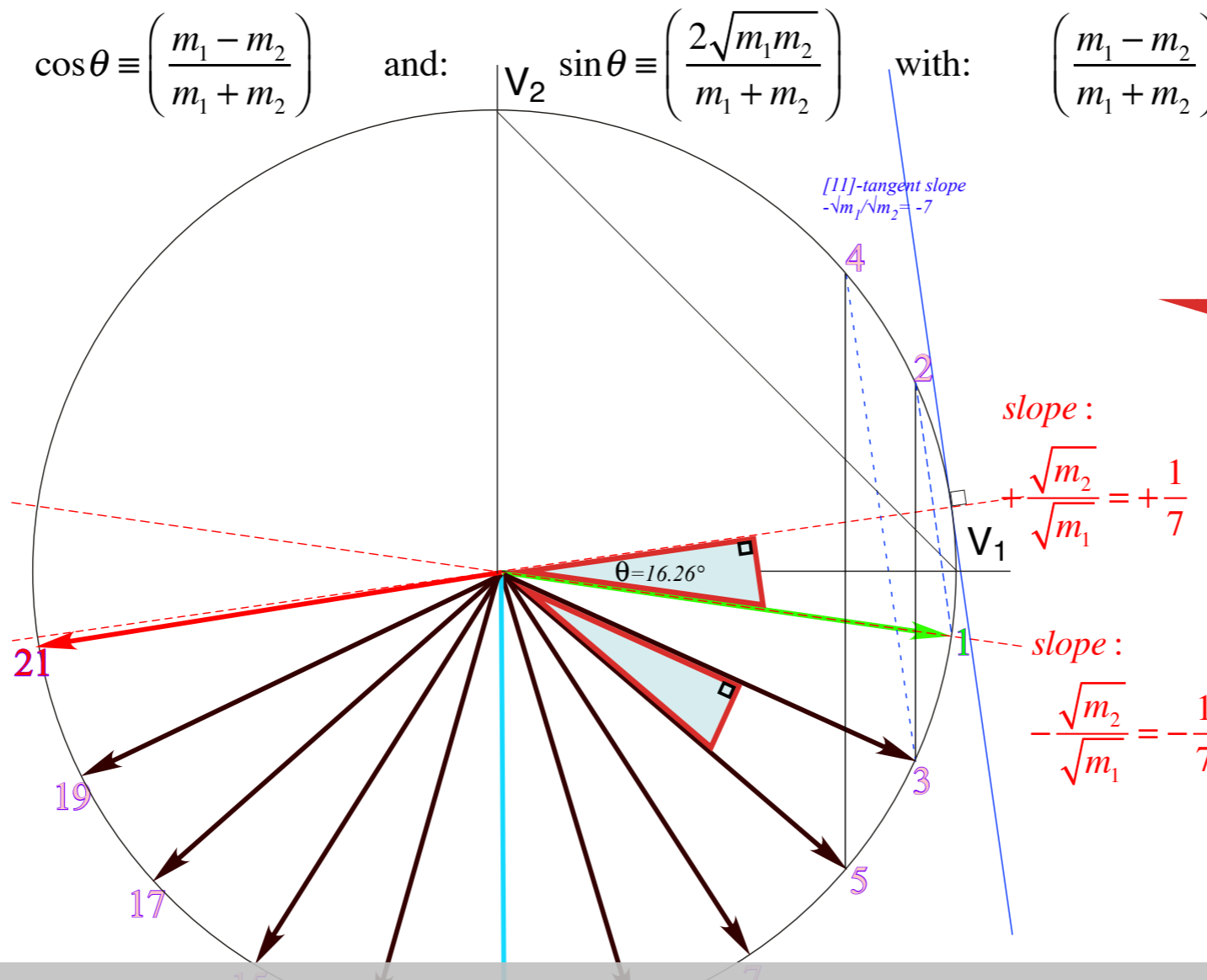
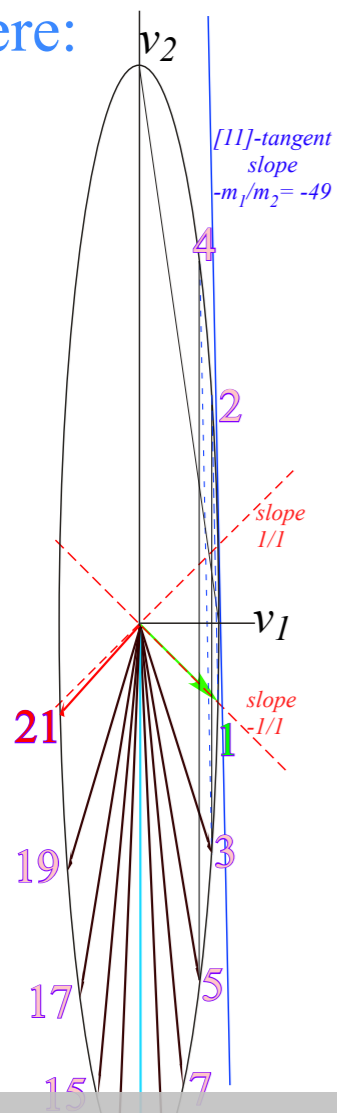
$$\begin{pmatrix} v_1^{FIN1} \\ v_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{becomes:} \quad \begin{pmatrix} V_1^{FIN1} / \sqrt{m_1} \\ V_2^{FIN1} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

$$\text{or: } \begin{pmatrix} V_1^{FIN1} \\ V_2^{FIN1} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{M} \cdot \vec{V}, \quad \text{or: } \begin{pmatrix} V_1^{FIN2} \\ V_2^{FIN2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \vec{V}$$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations* $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

where:

$$\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right) \quad \text{and:} \quad \sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right) \quad \text{with:} \quad \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$$



Collisions for mass ratio $m_1:m_2 = 49:1$

Fig. 5.2a-c (revised)

Note: If $m_1 \cdot m_2$ is perfect-square, then θ -triangle is rational ($3^2 + 4^2 = 5^2$, etc.)

Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_2}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

Then collisions become *reflections* $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and double-collisions become *rotations*

where:

$$\cos\theta \equiv \left(\frac{m_1 - m_2}{m_1 + m_2} \right) \quad \text{and:} \quad \sin\theta \equiv \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)$$

$$\text{with:} \quad \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left(\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1$$

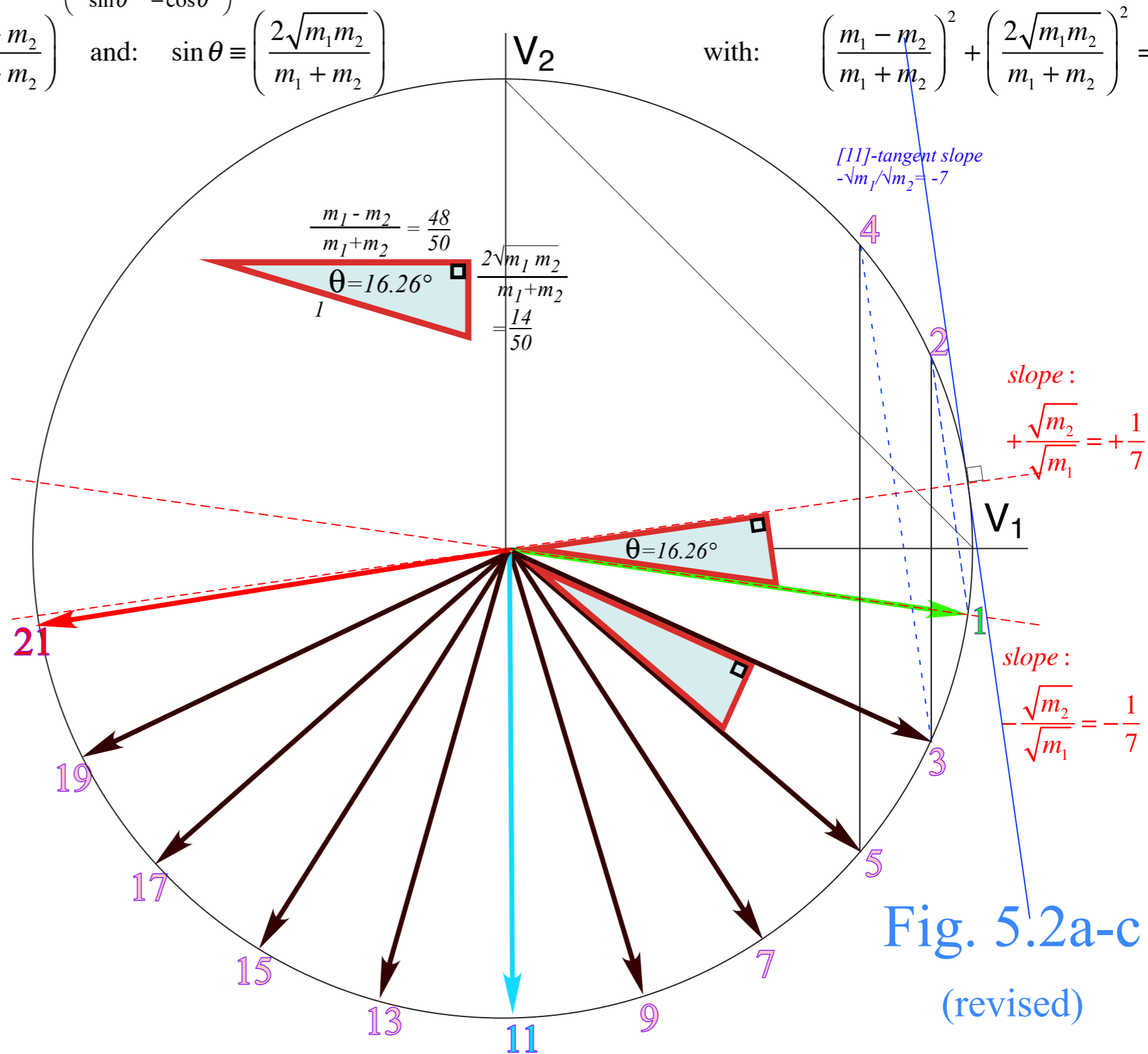
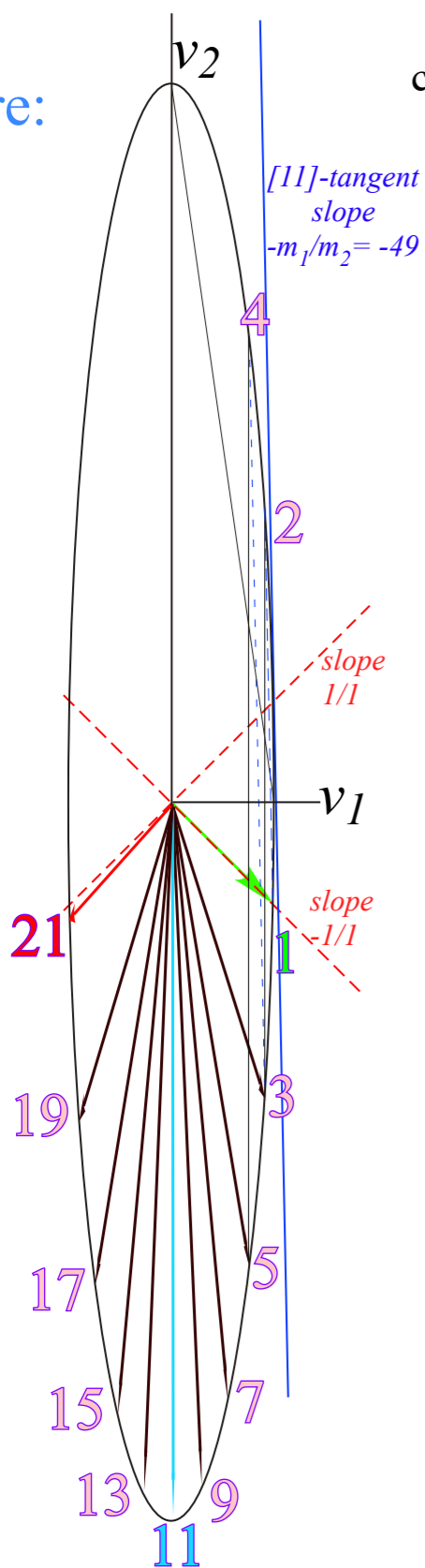


Fig. 5.2a-c
(revised)

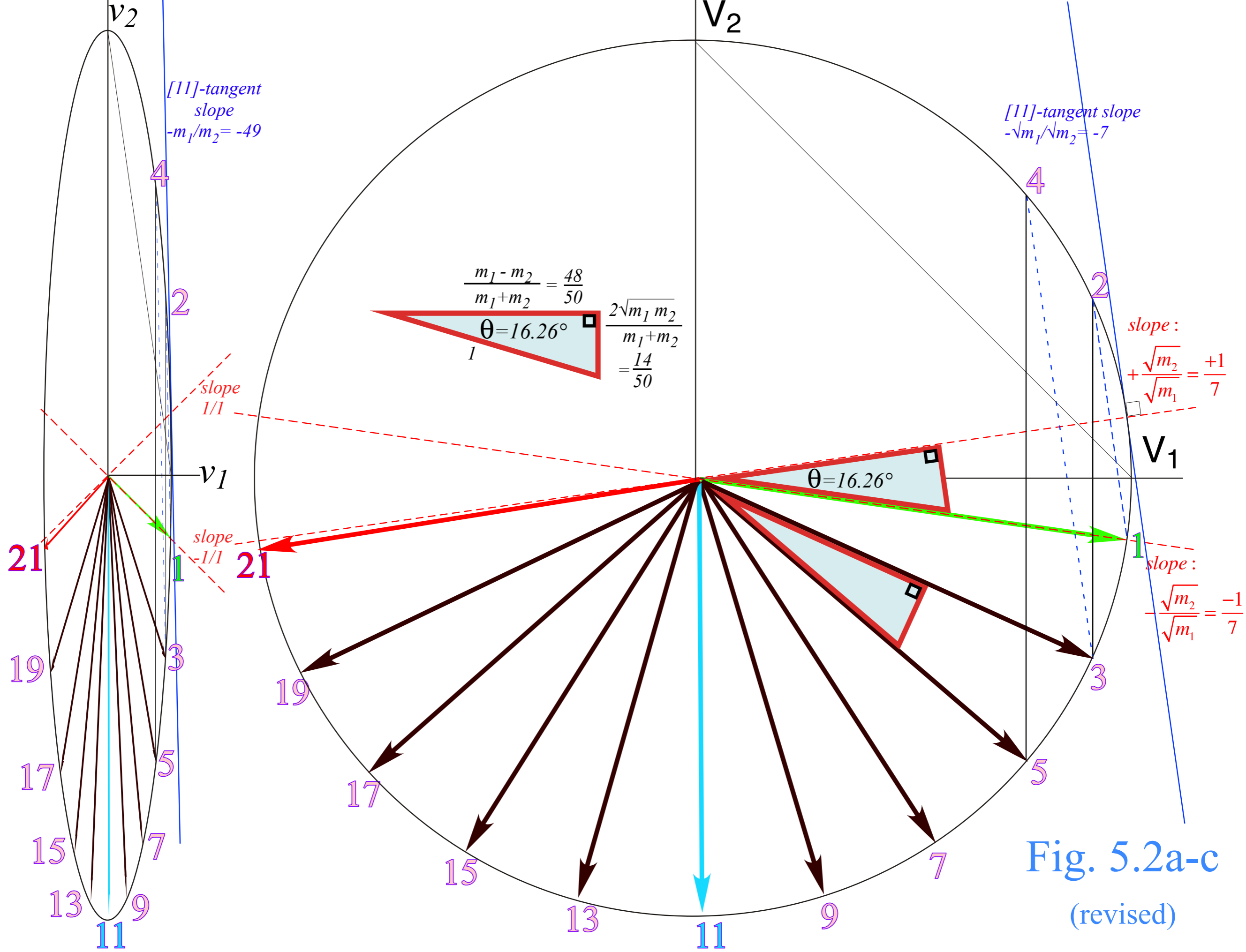


Fig. 5.2a-c
(revised)

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

 *How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on*

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

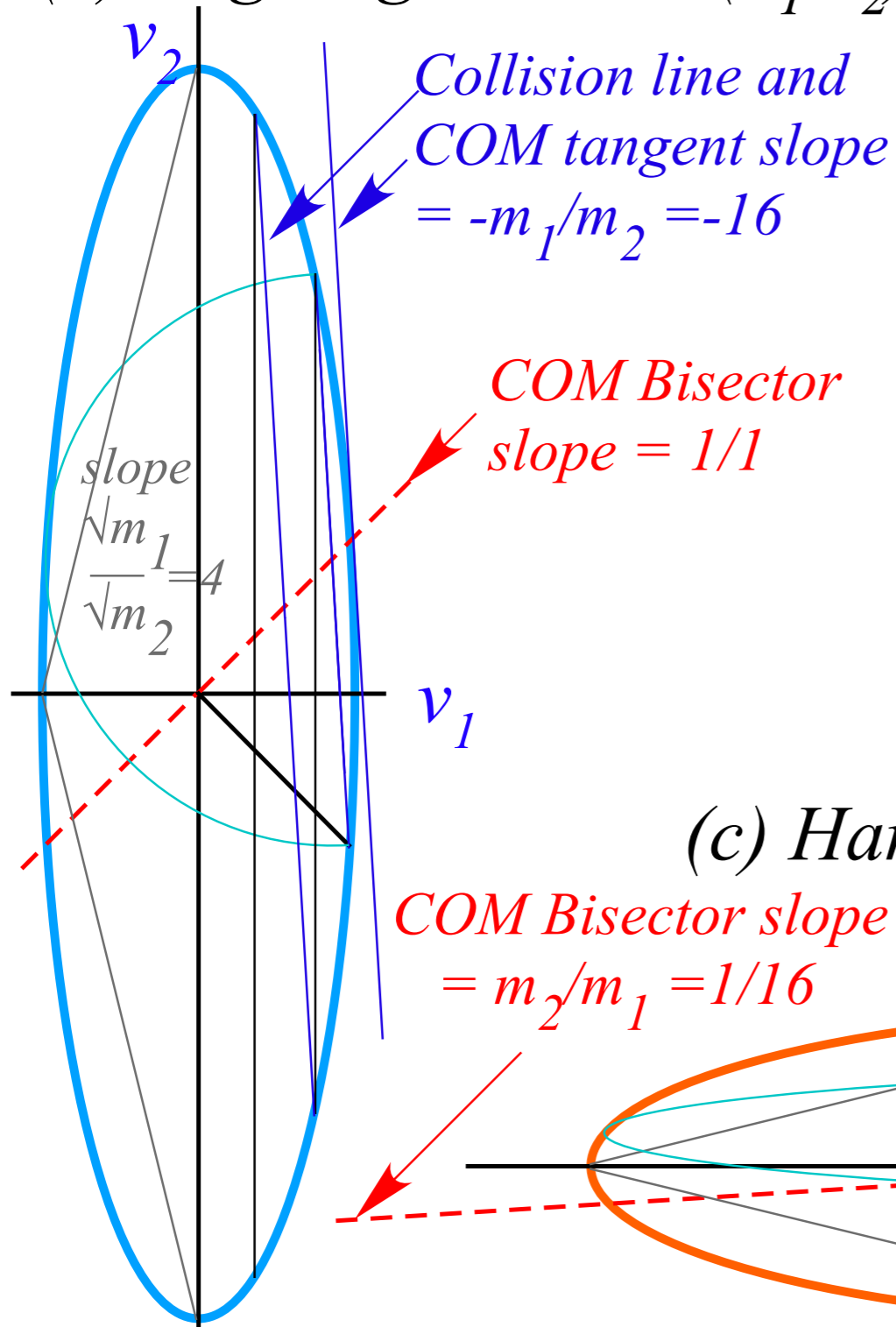
Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$

What ellipse rescaling leads to...

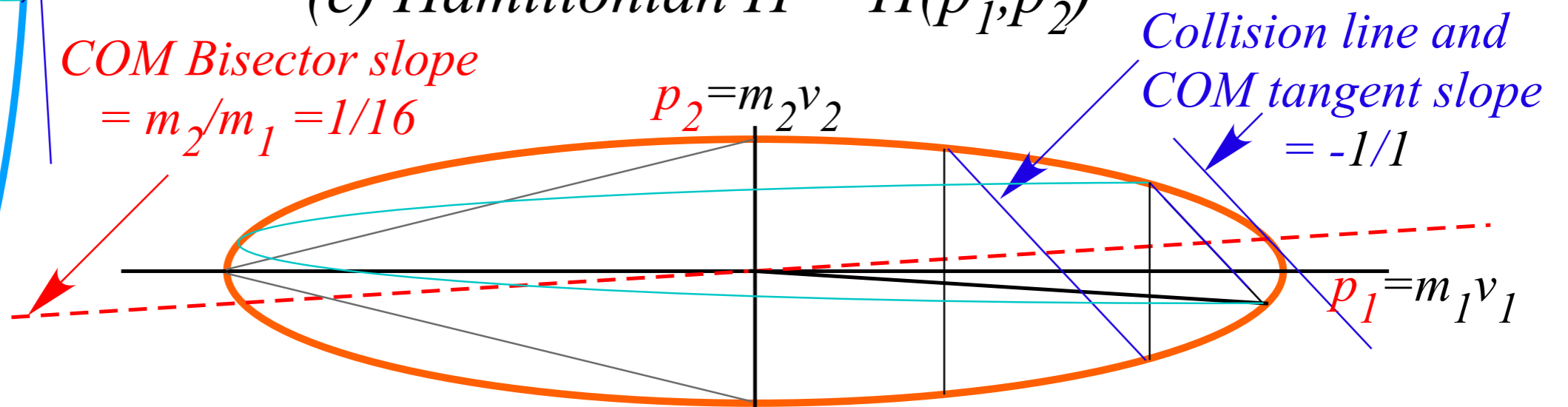
How this relates to *Lagrangian*, and *Hamiltonian* mechanics later on (Ch.12 of Unit 1)

(a) Lagrangian $L = L(v_1, v_2)$



velocity v_1 rescaled to *momentum*: $p_1 = m_1 v_1$
 velocity v_2 rescaled to *momentum*: $p_2 = m_2 v_2$

(c) Hamiltonian $H = H(p_1, p_2)$

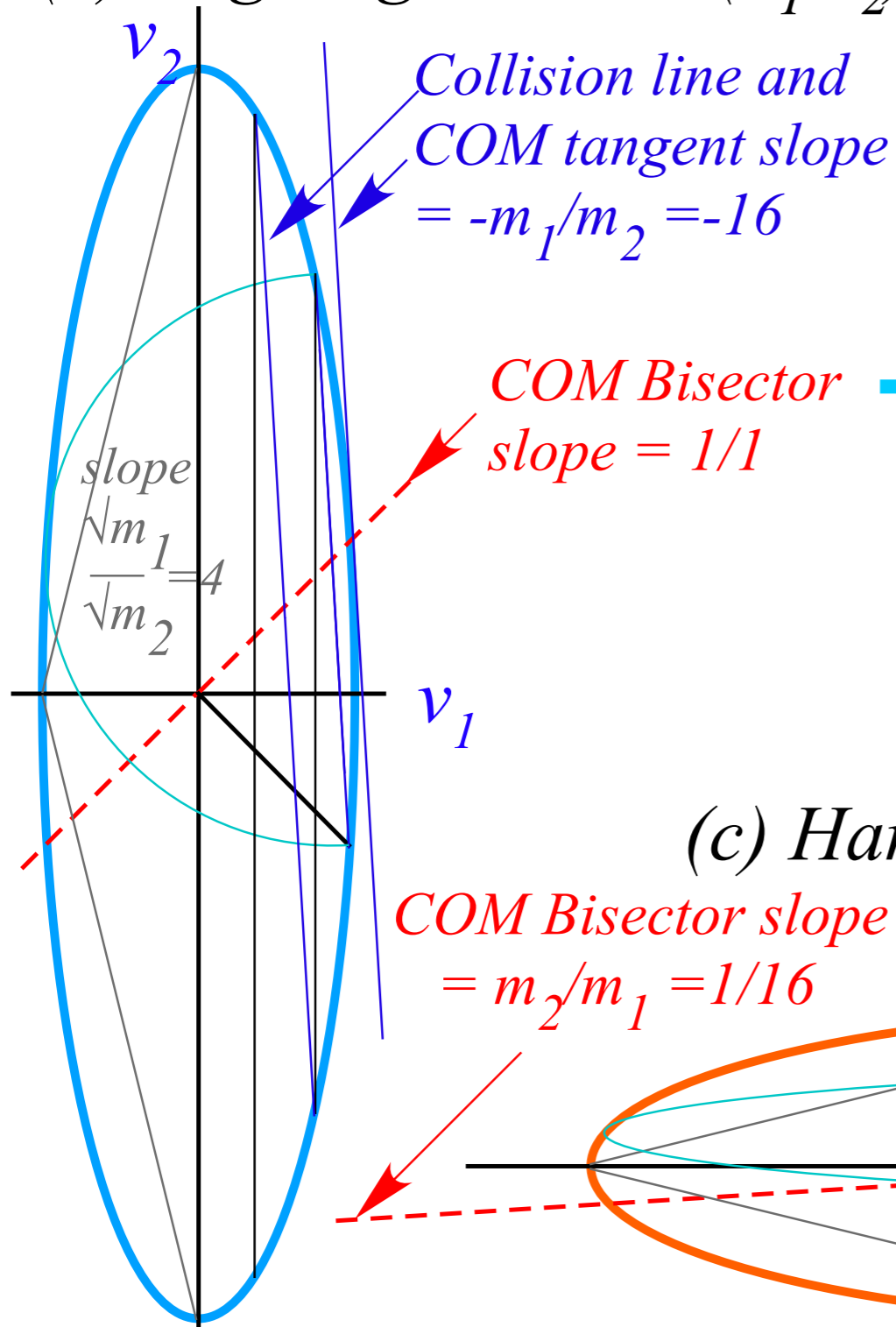


What ellipse rescaling leads to...

Fig. 12.1
(Unit 1)

How this relates to *Lagrangian*, and *Hamiltonian* mechanics later on (Ch.12 of Unit 1)

(a) Lagrangian $L = L(v_1, v_2)$



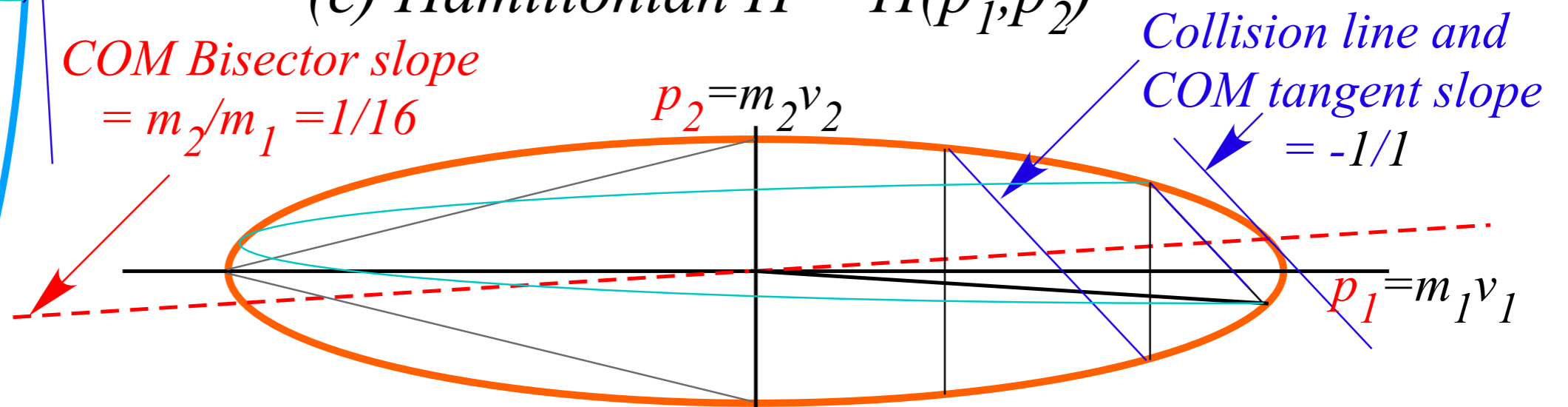
velocity v_1 rescaled to *momentum*: $p_1 = m_1 v_1$
 velocity v_2 rescaled to *momentum*: $p_2 = m_2 v_2$

Lagrangian $L(v_1, v_2) = KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

rescaled to

Hamiltonian $H(p_1, p_2) = KE = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$

(c) Hamiltonian $H = H(p_1, p_2)$

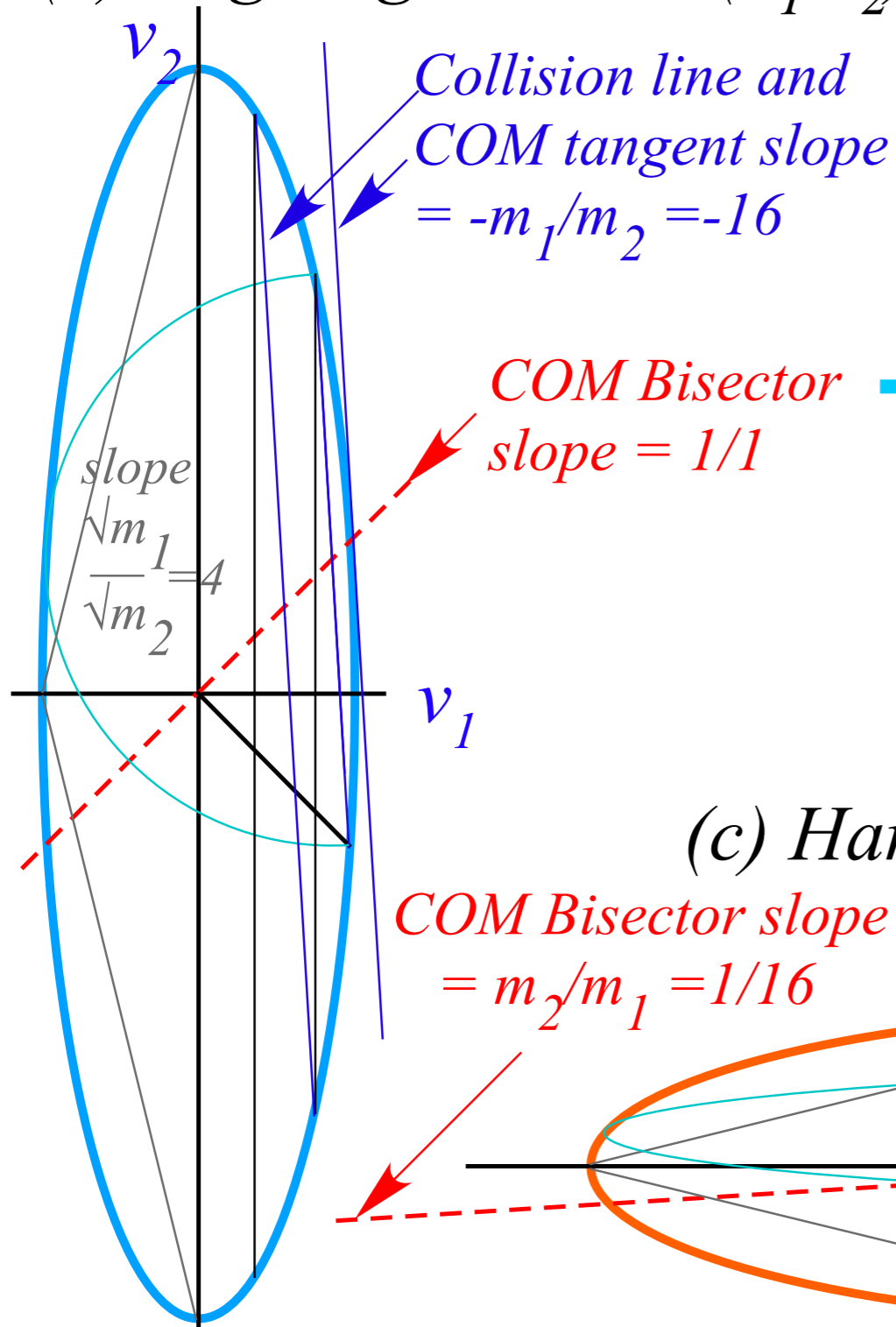


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How this relates to *Lagrangian*, and *Hamiltonian* mechanics later on (Ch.12 of Unit 1)

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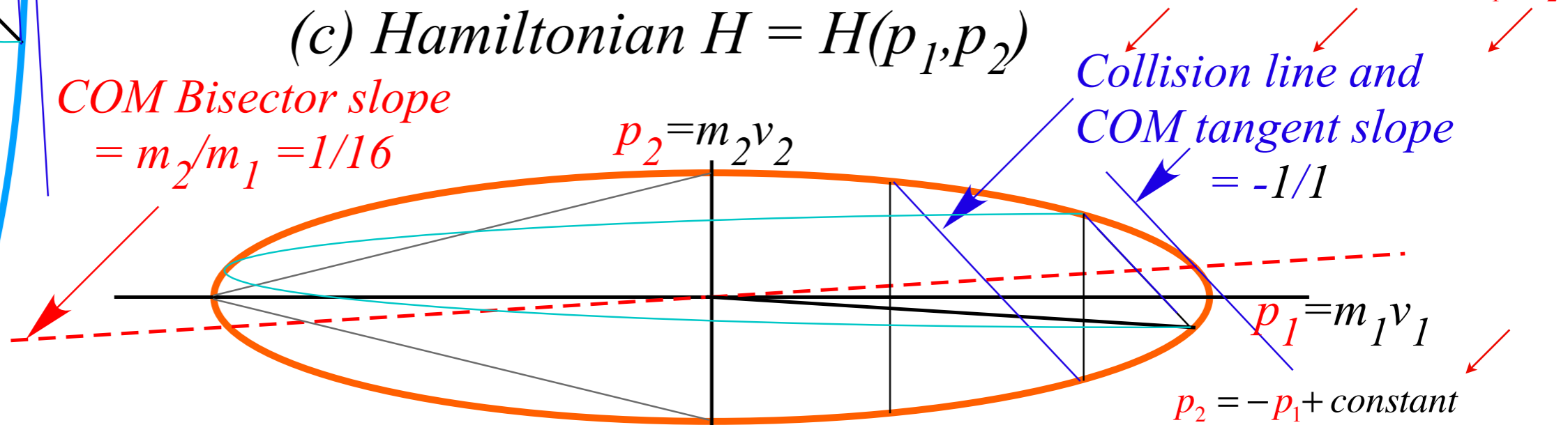
Lagrangian $L(v_1, v_2) = KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

rescaled to

Hamiltonian $H(p_1, p_2) = KE = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}$

(c) Hamiltonian $H = H(p_1, p_2)$

COM Bisector slope = $m_2/m_1 = 1/16$

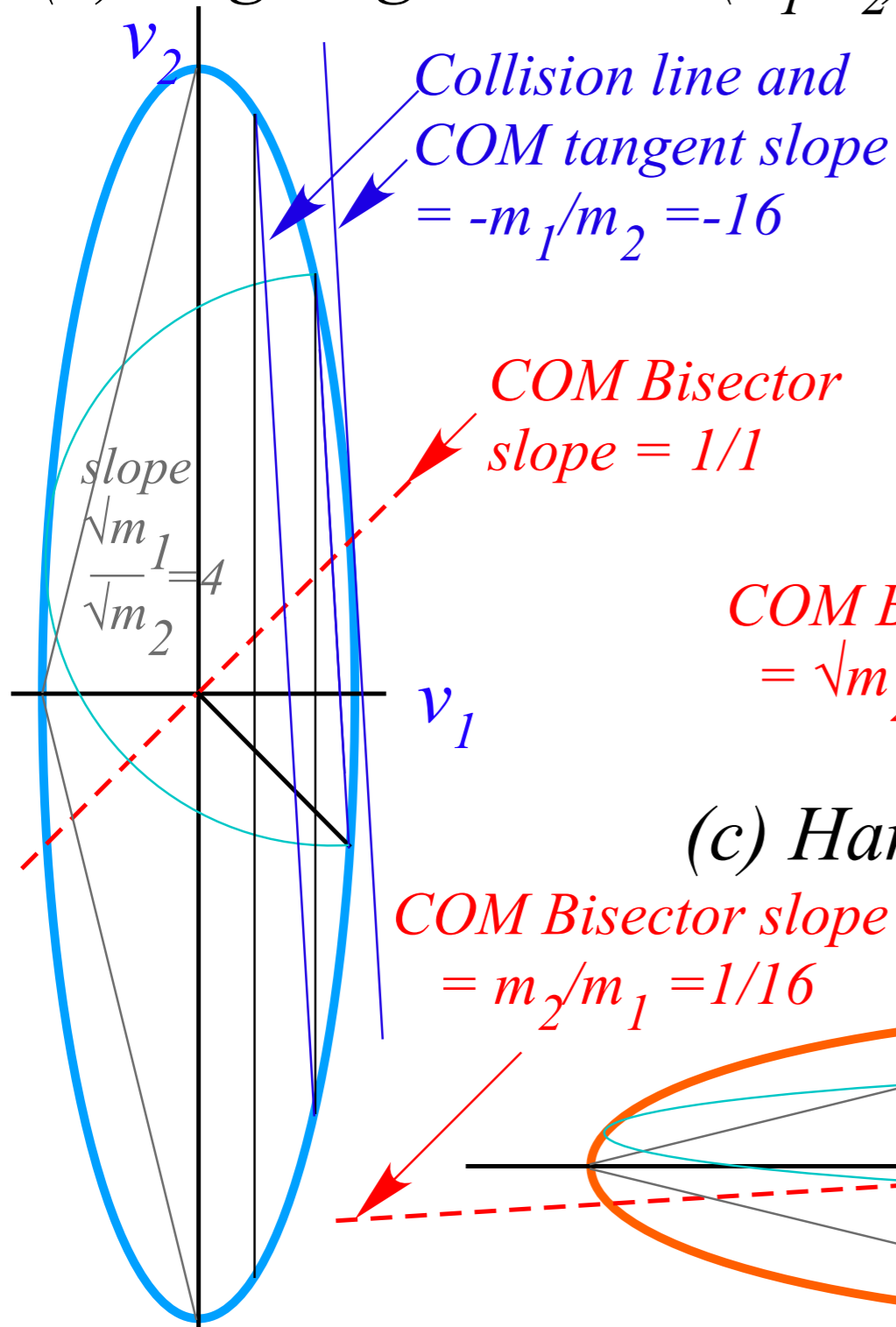


What ellipse rescaling leads to...

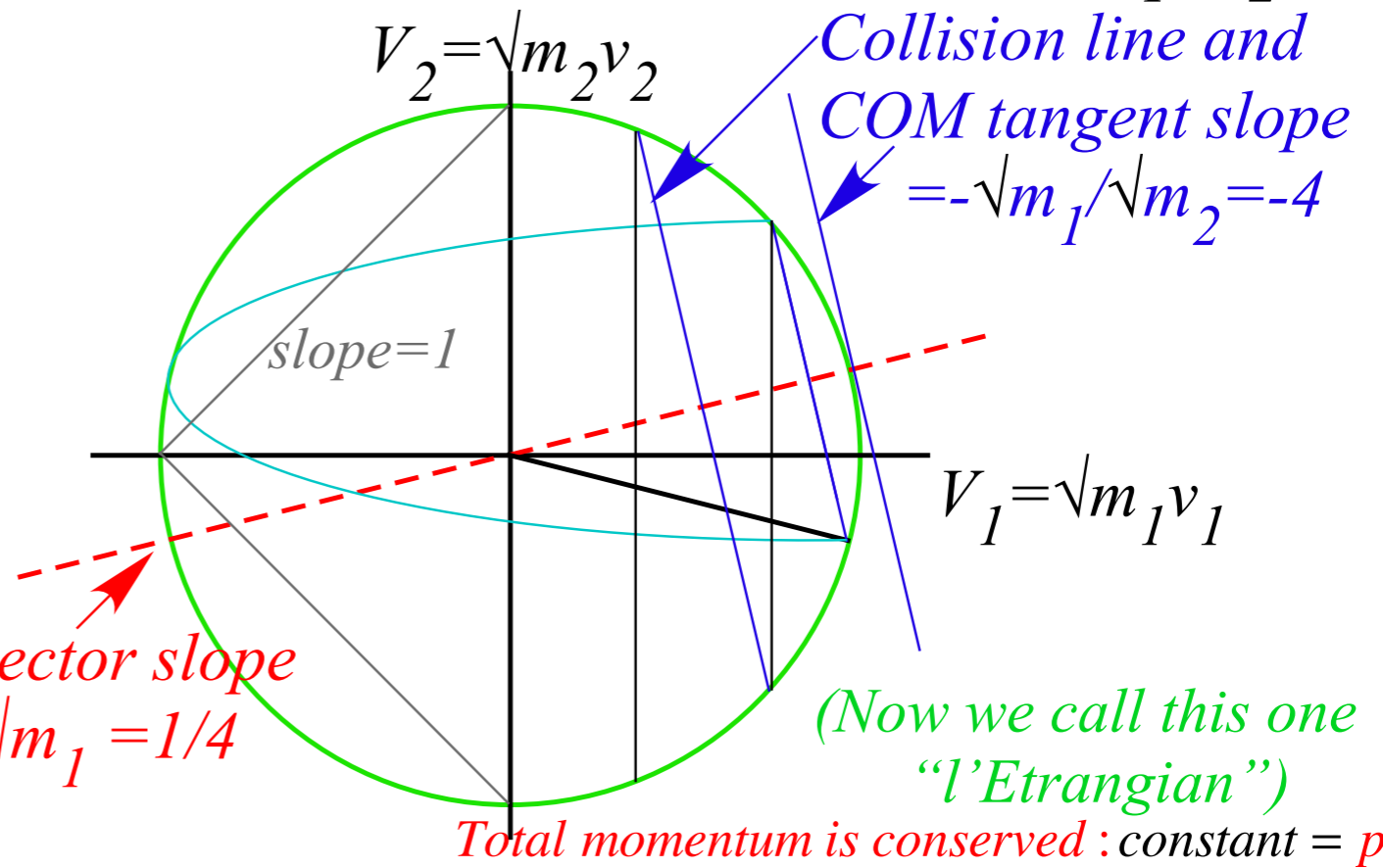
Fig. 12.1
(Unit 1)

How this relates to *Lagrangian*, *l'Etrangian*, and *Hamiltonian* mechanics later on (Ch.12 of Unit 1)

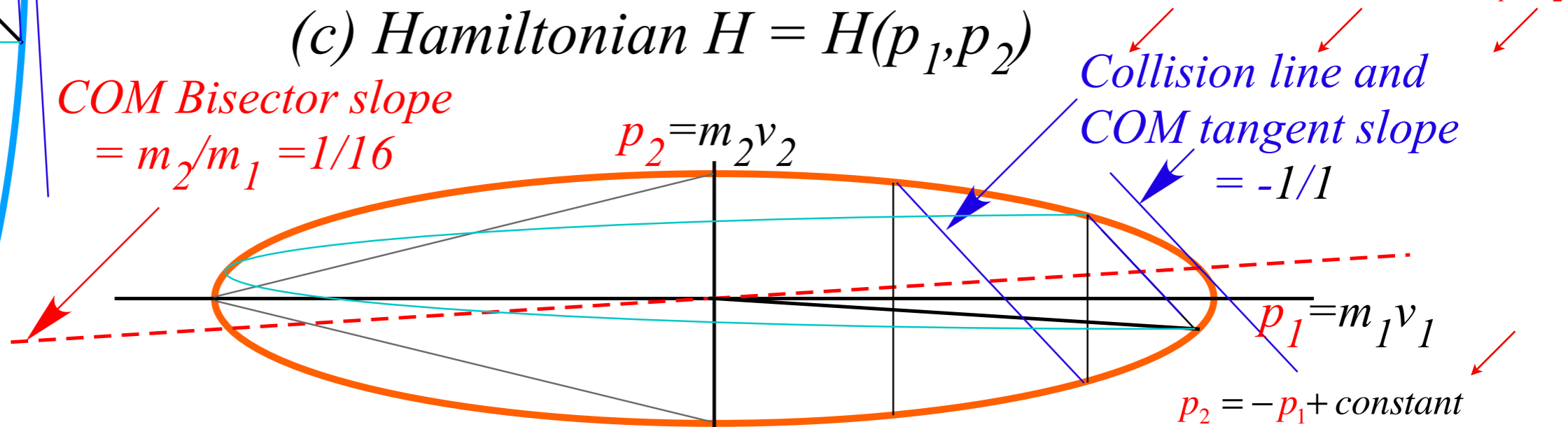
(a) Lagrangian $L = L(v_1, v_2)$



(b) Estrangian $E = E(V_1, V_2)$



(c) Hamiltonian $H = H(p_1, p_2)$



Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

 *Reflections in the clothing store: "It's all done with mirrors!"*

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

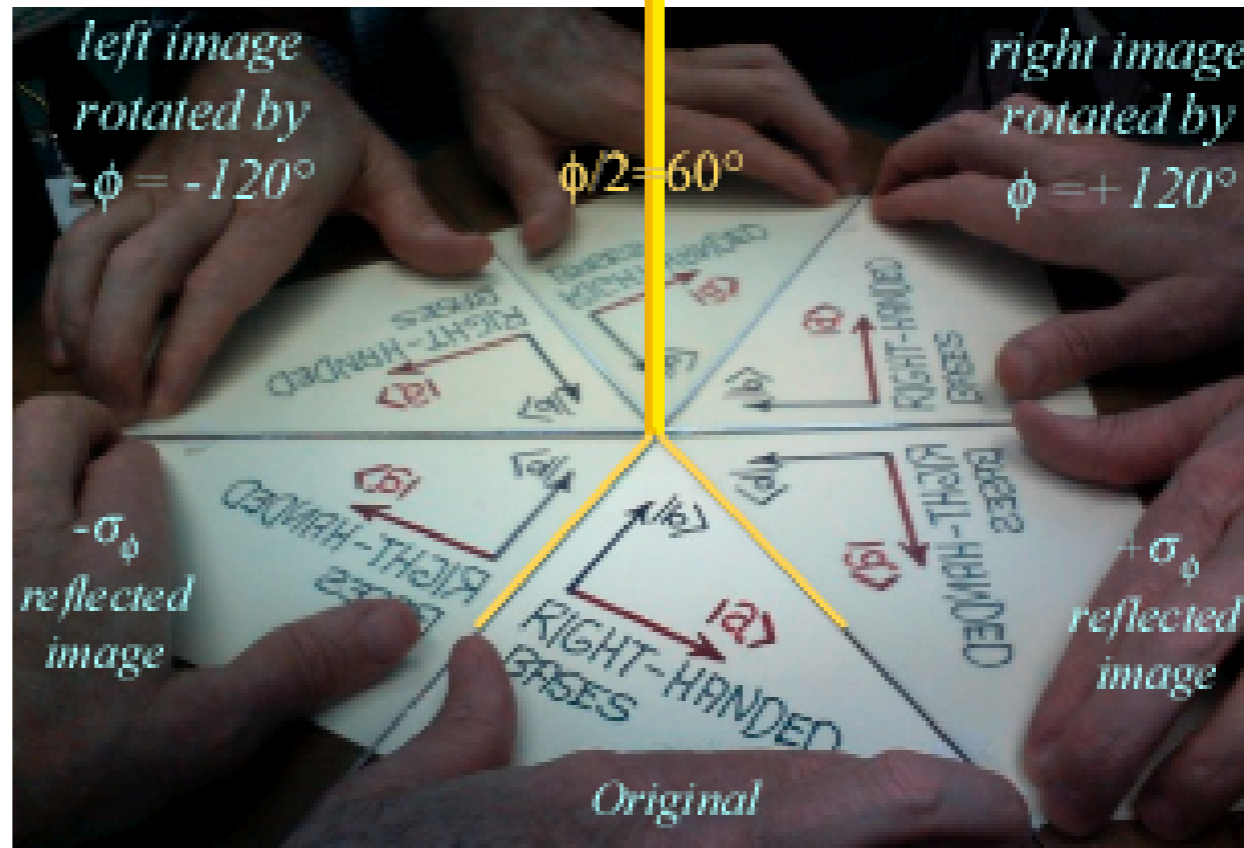
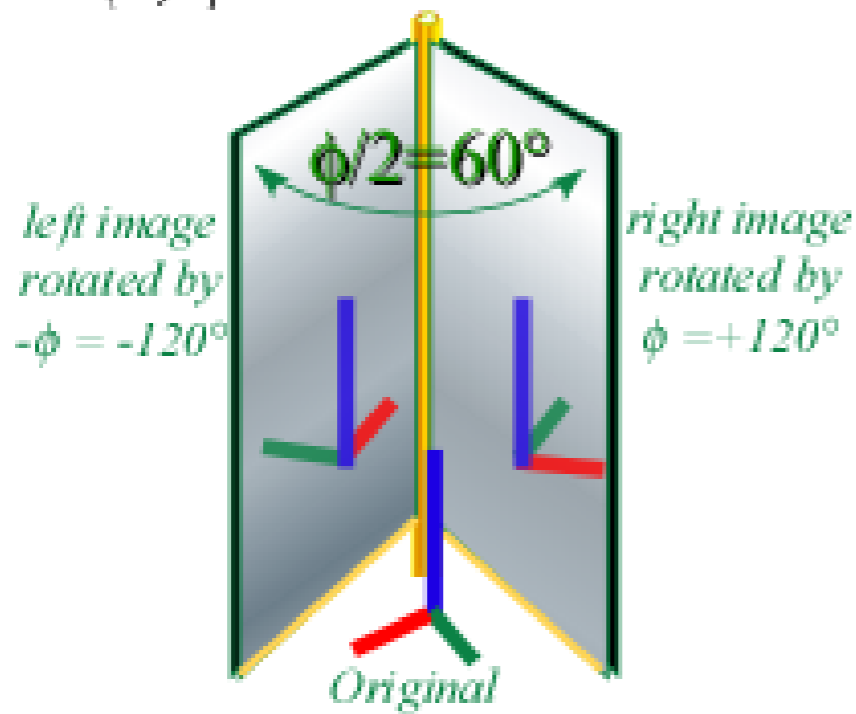
Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$

Reflections in clothing store mirrors

(a) $\phi = \pm 120^\circ$ rotations



(b) $\phi = \pm 180^\circ$ rotations

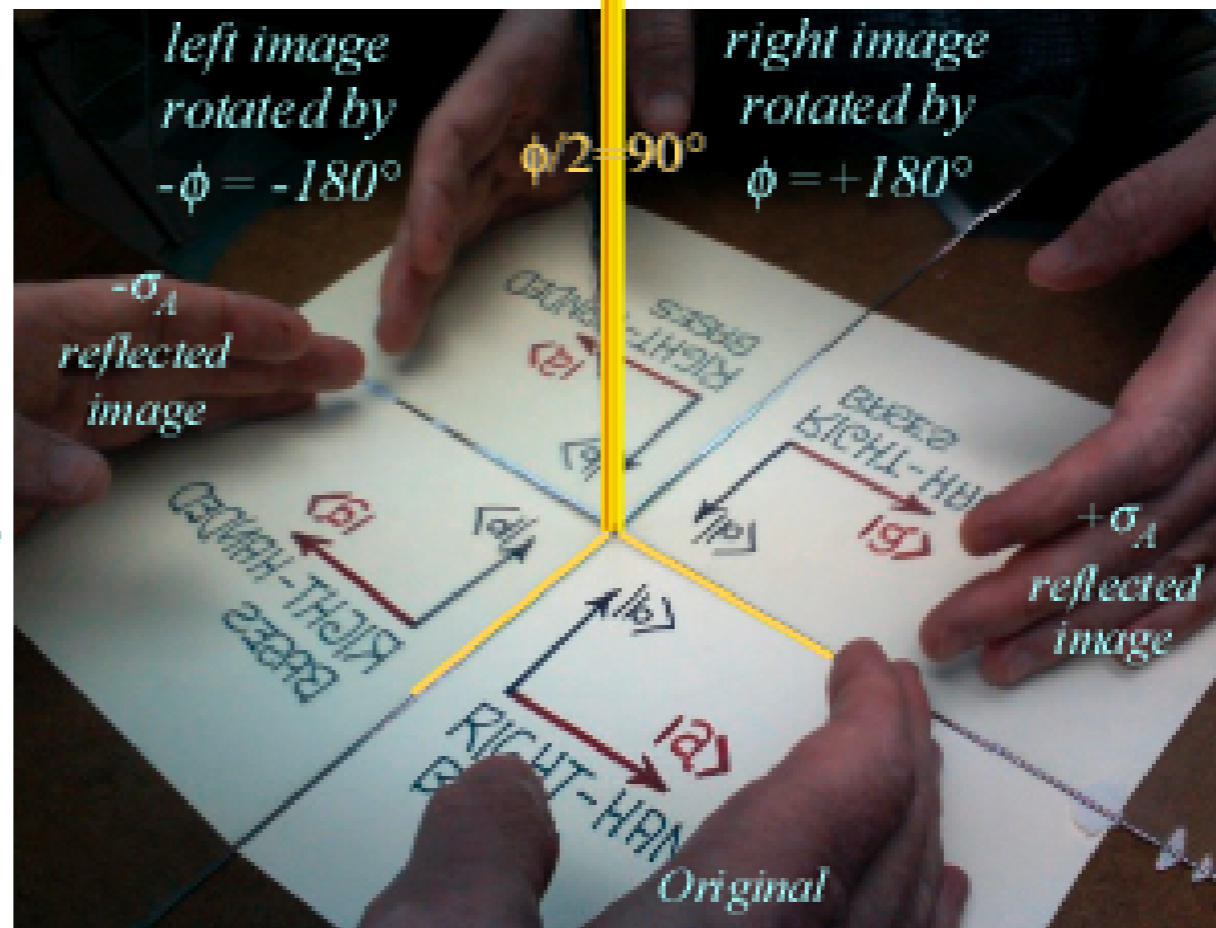
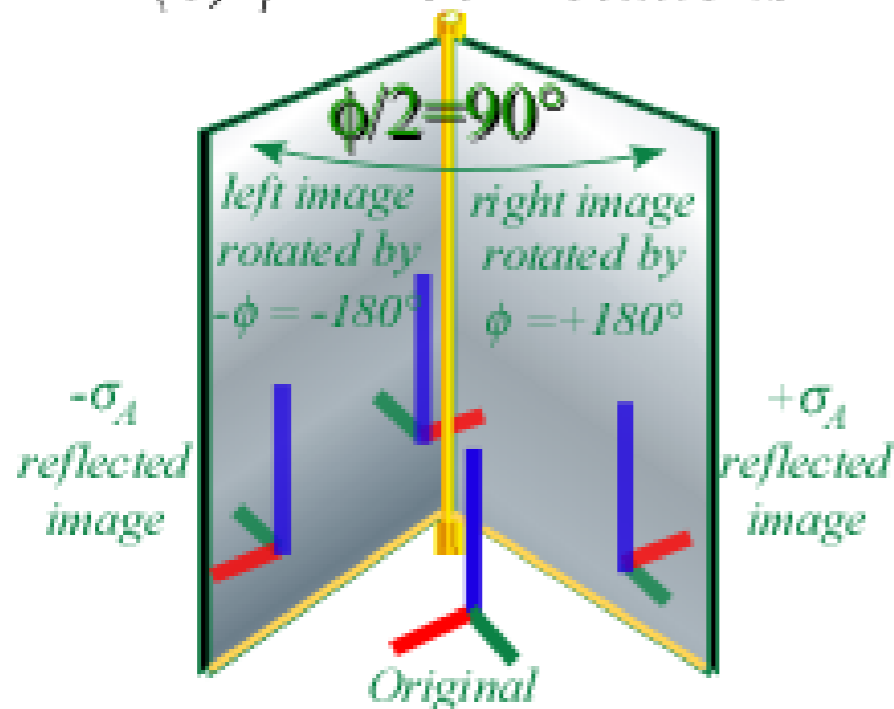


Fig. 5.4a-b

Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

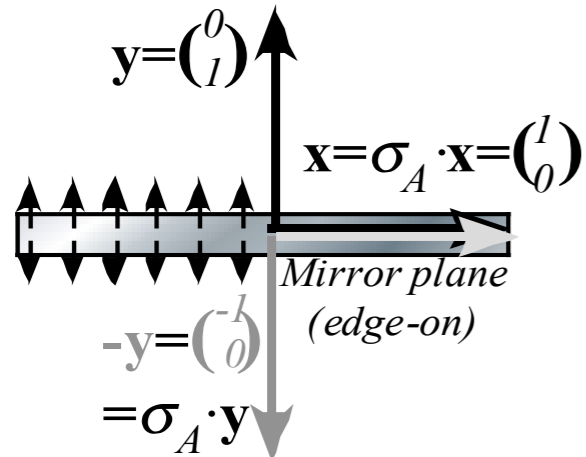


Fig.
5.3a-e

Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

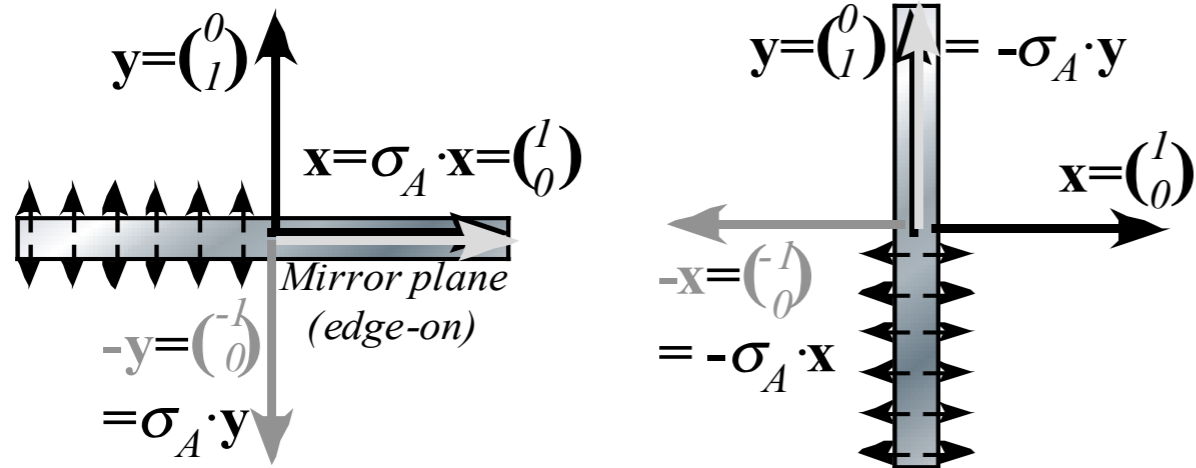


Fig.
5.3a-e

Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

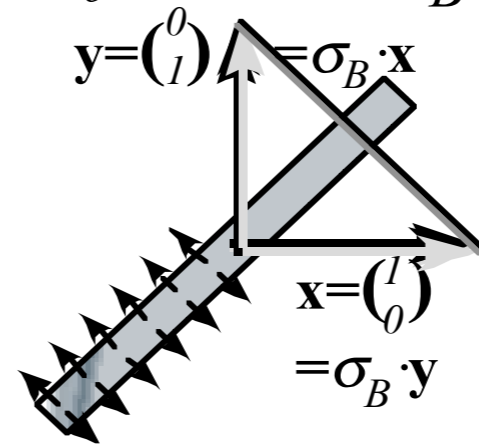
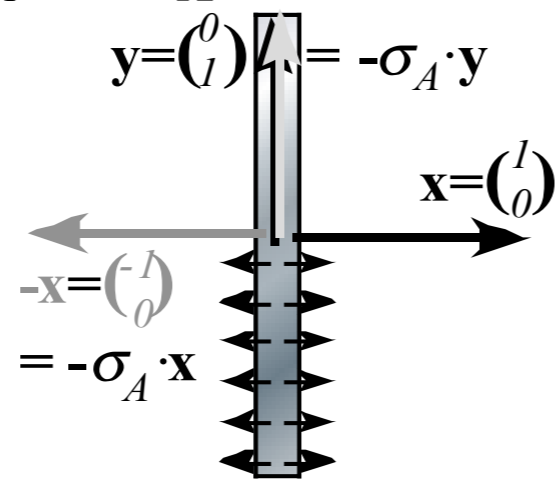
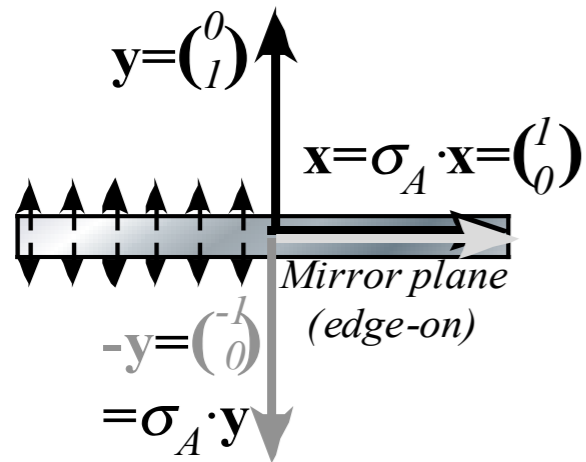
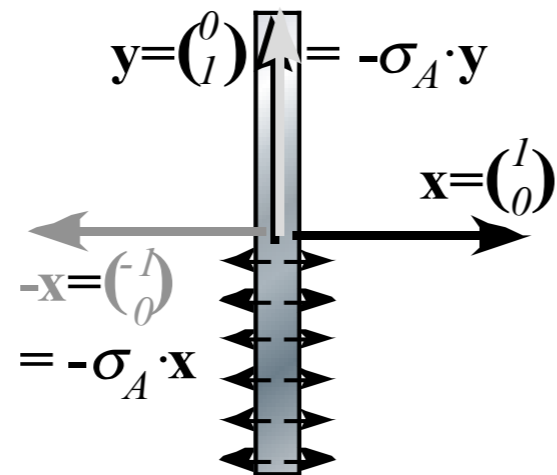
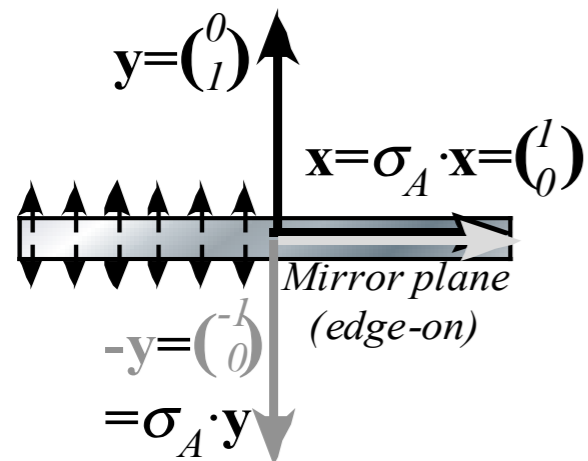


Fig.
5.3a-e

Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

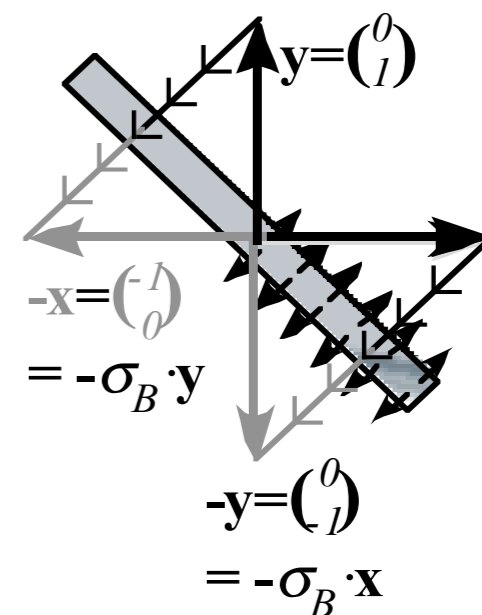
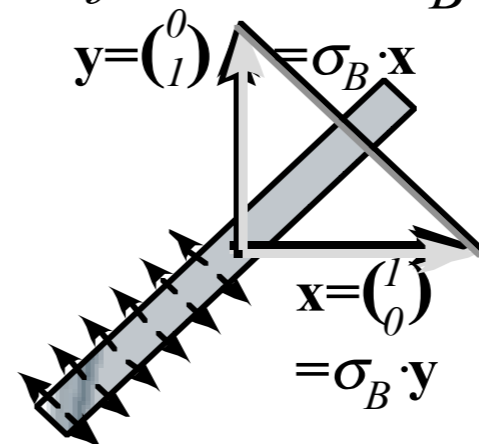
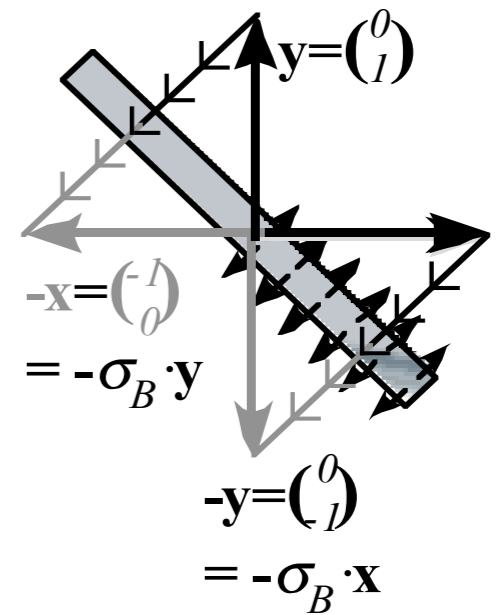
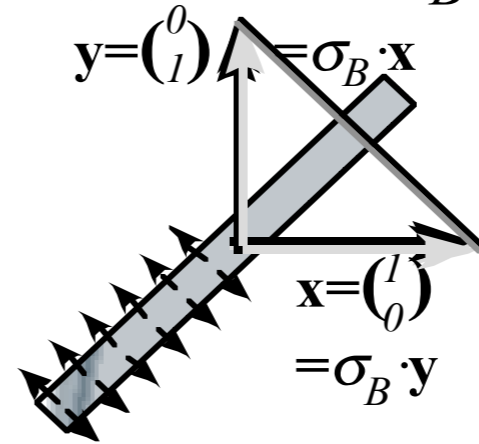
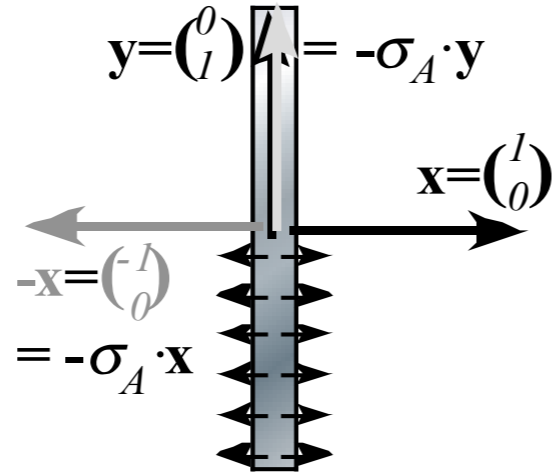
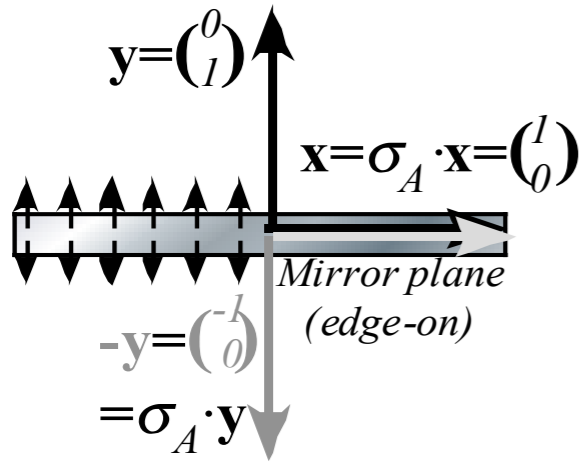


Fig.
5.3a-e

Symmetry: It's all done with mirrors!

(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

of \mathbf{x} -vector:

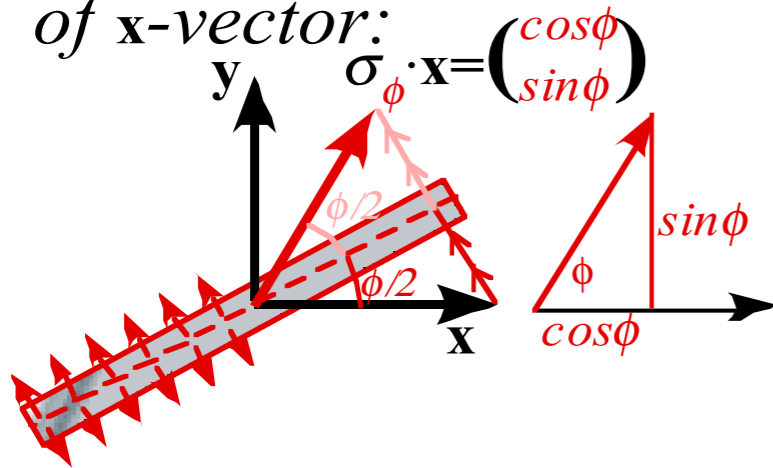
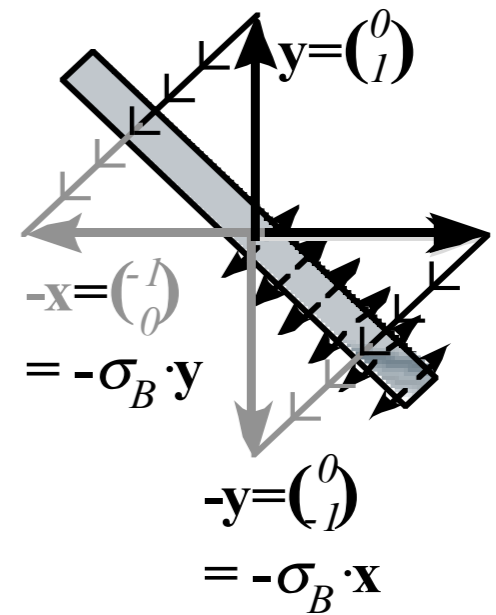
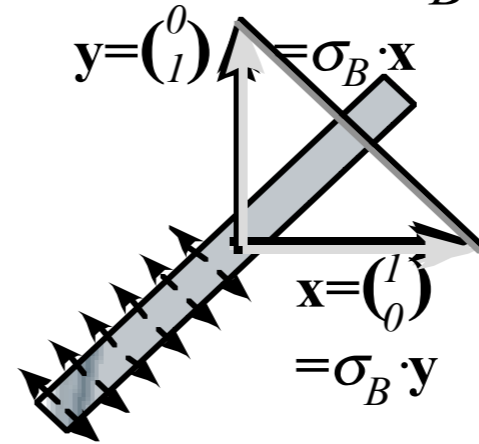
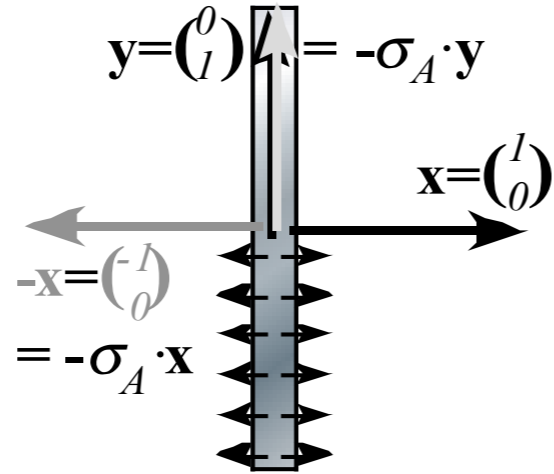
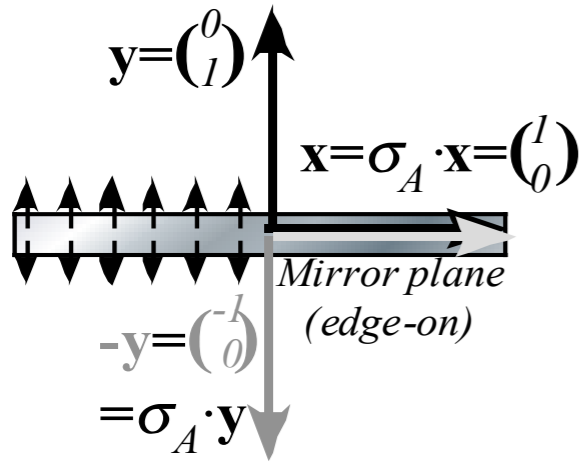


Fig.
5.3a-e

Symmetry: It's all done with mirrors!

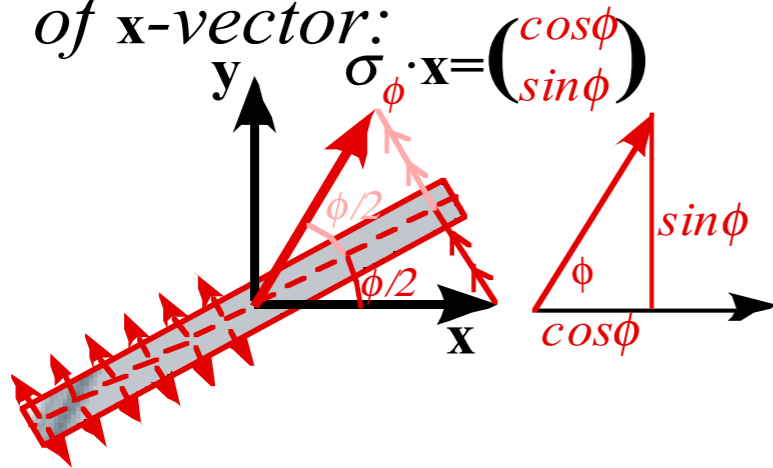
(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflections $\sigma_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $-\sigma_B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$



(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

of \mathbf{x} -vector:



...of \mathbf{y} -vector:

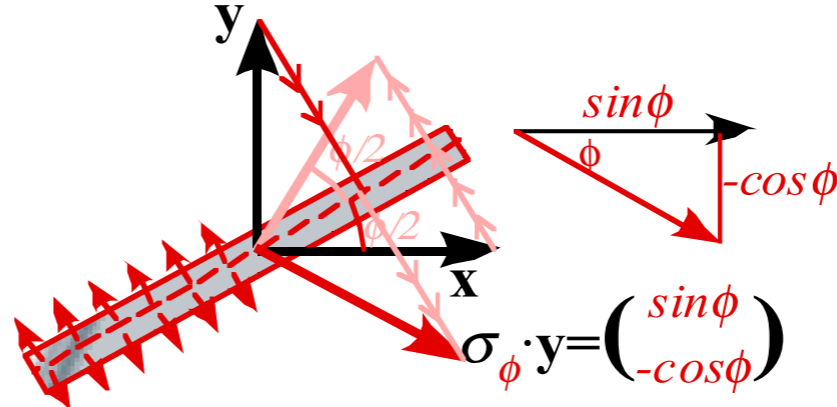
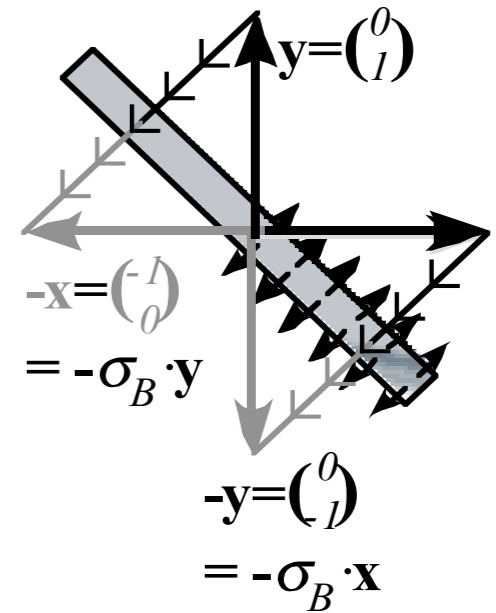
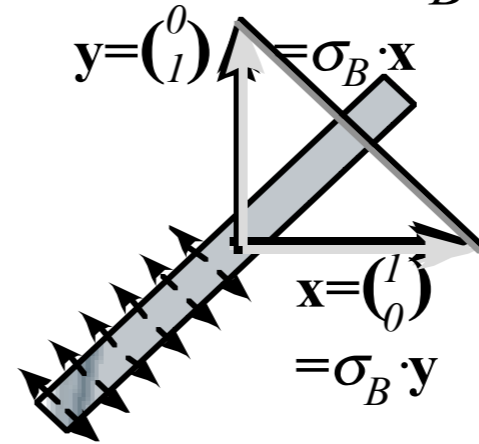
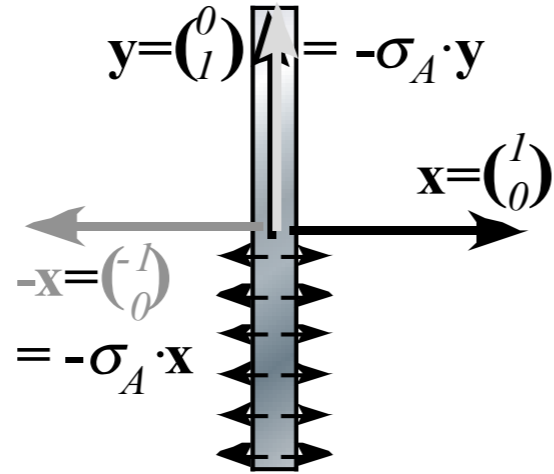
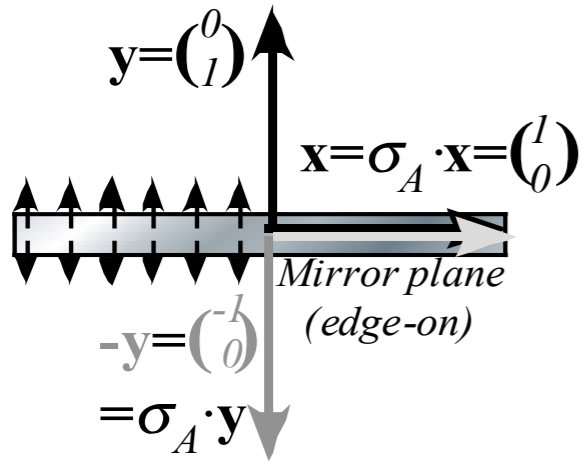


Fig. 5.3a-e

Symmetry: It's all done with mirrors!

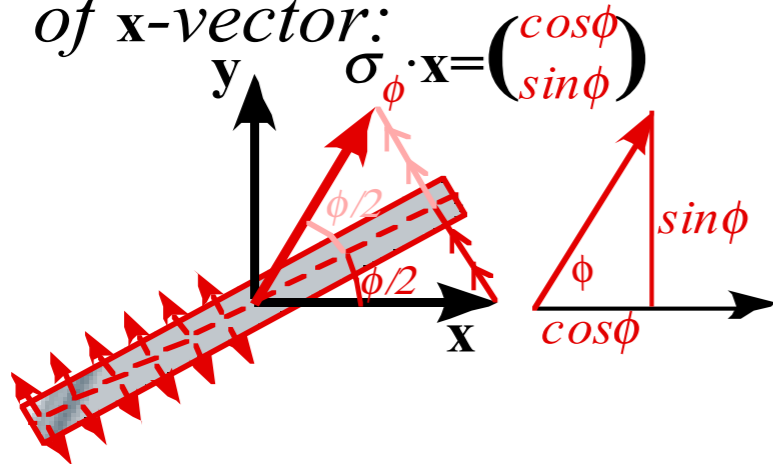
(a) Reflections $\sigma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $-\sigma_A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

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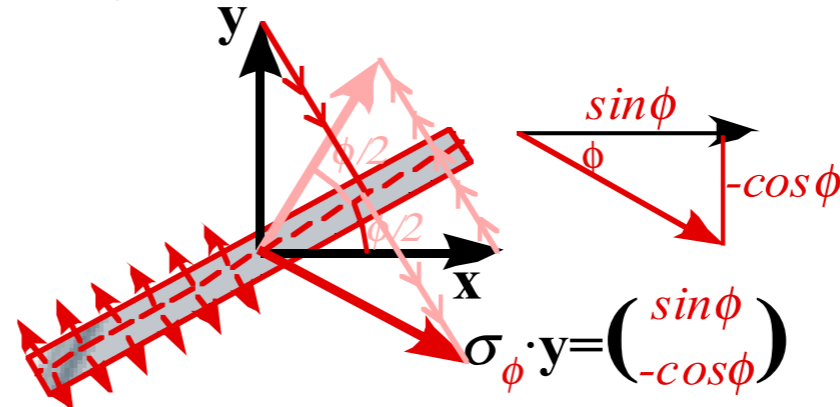


(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

of x -vector:



...of y -vector:



(d) Rotation: $R_{+\phi} = \sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

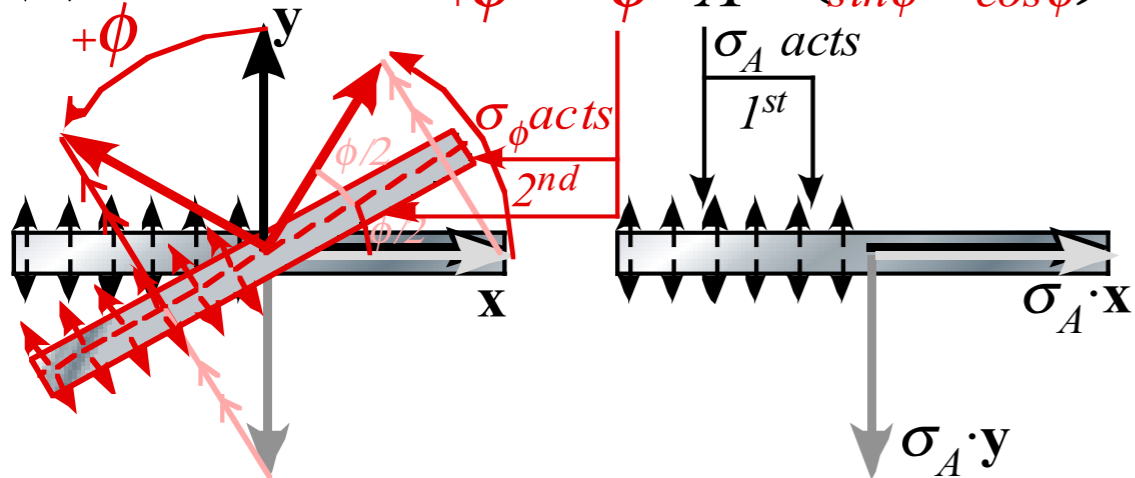
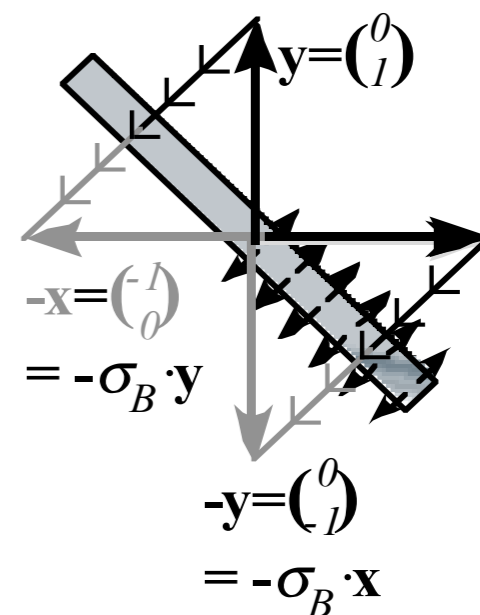
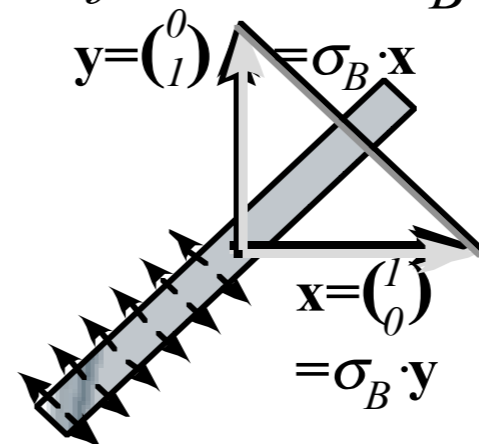
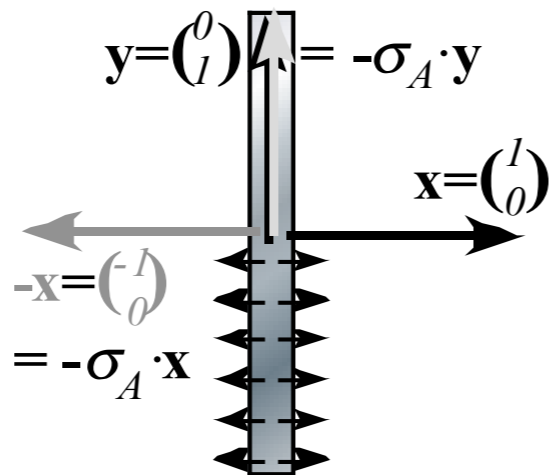
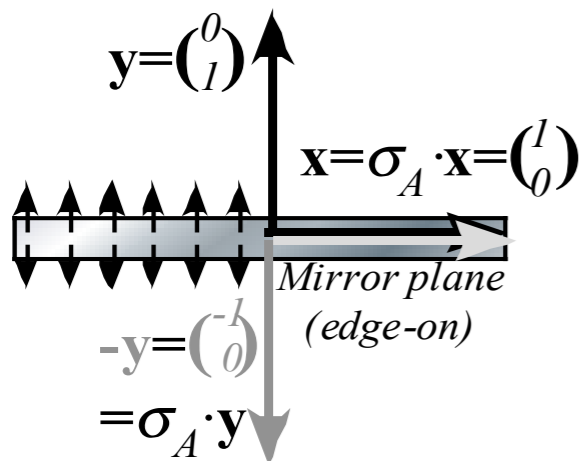


Fig. 5.3a-e

Symmetry: It's all done with mirrors!

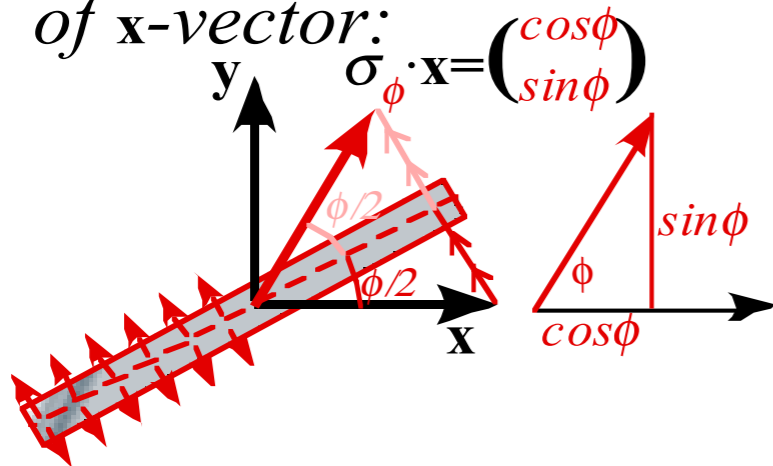
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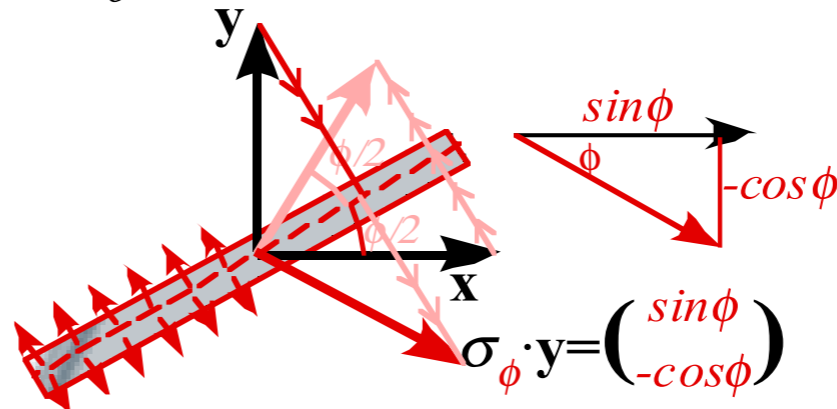


(c) σ_ϕ reflection $\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$

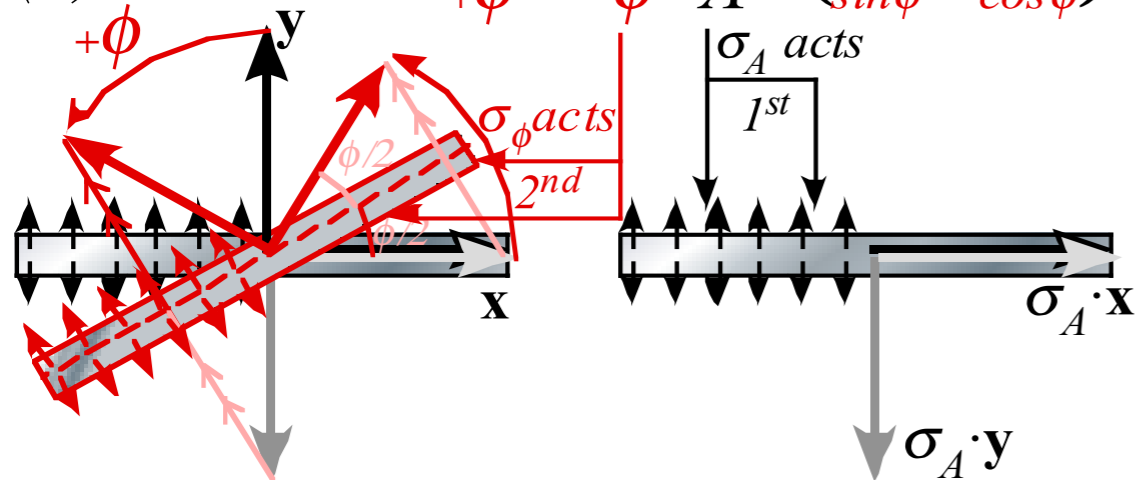
of x -vector:



...of y -vector:



(d) Rotation: $R_{+\phi} = \sigma_\phi \sigma_A = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$



(e) Rotation: $R_{-\phi} = \sigma_A \sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$

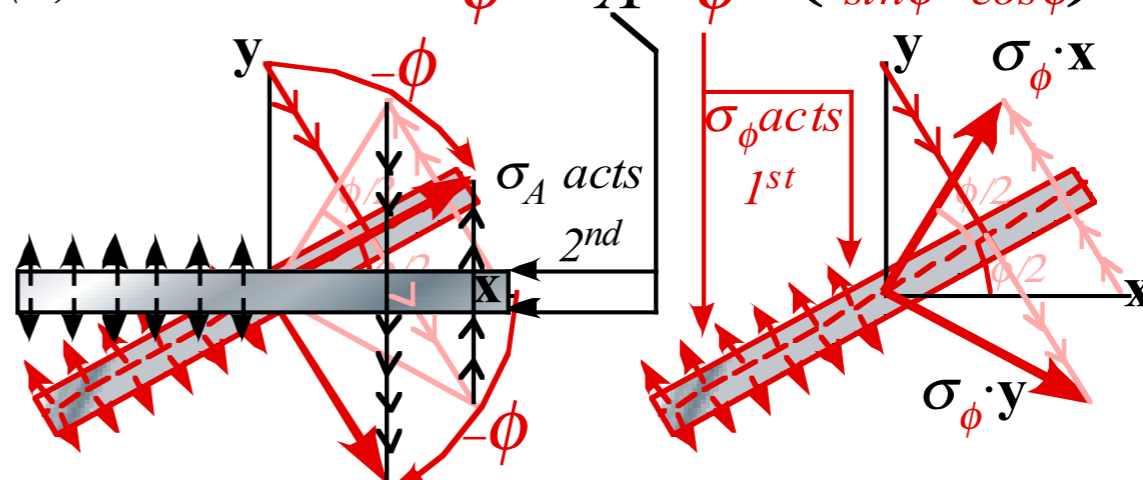


Fig. 5.3a-e

Why reflections underlie all symmetry analyses

They work in 1D, 2D, 3D,.....,ND

Product of odd number of reflections is a reflection

*... even number of reflections is a rotation (or unit-op **1**)*

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily

Waves patterns are non-rigid and reflect easily

Why reflections underlie all symmetry analyses

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Product of odd number of reflections is a reflection

*... even number of reflections is a rotation (or unit-op **1**)*

Product of rotations just give rotations

Classical objects are semi-rigid and rotate easily

Waves patterns are non-rigid and reflect easily

∴ ...wave reflections underlie modern physics

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: "It's all done with mirrors!"

 *Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)*

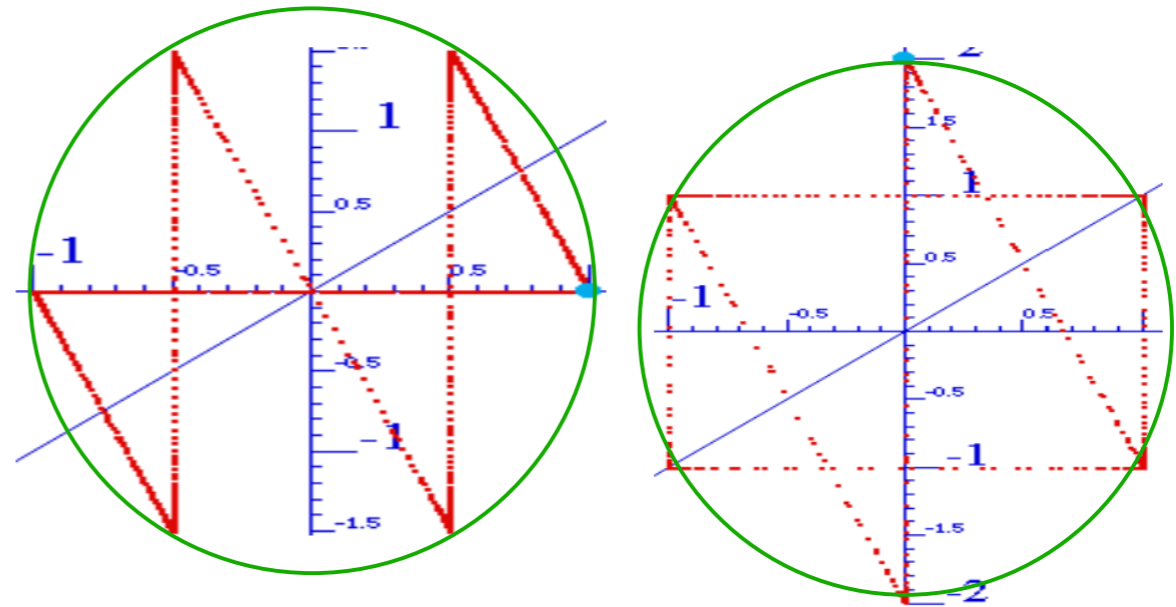
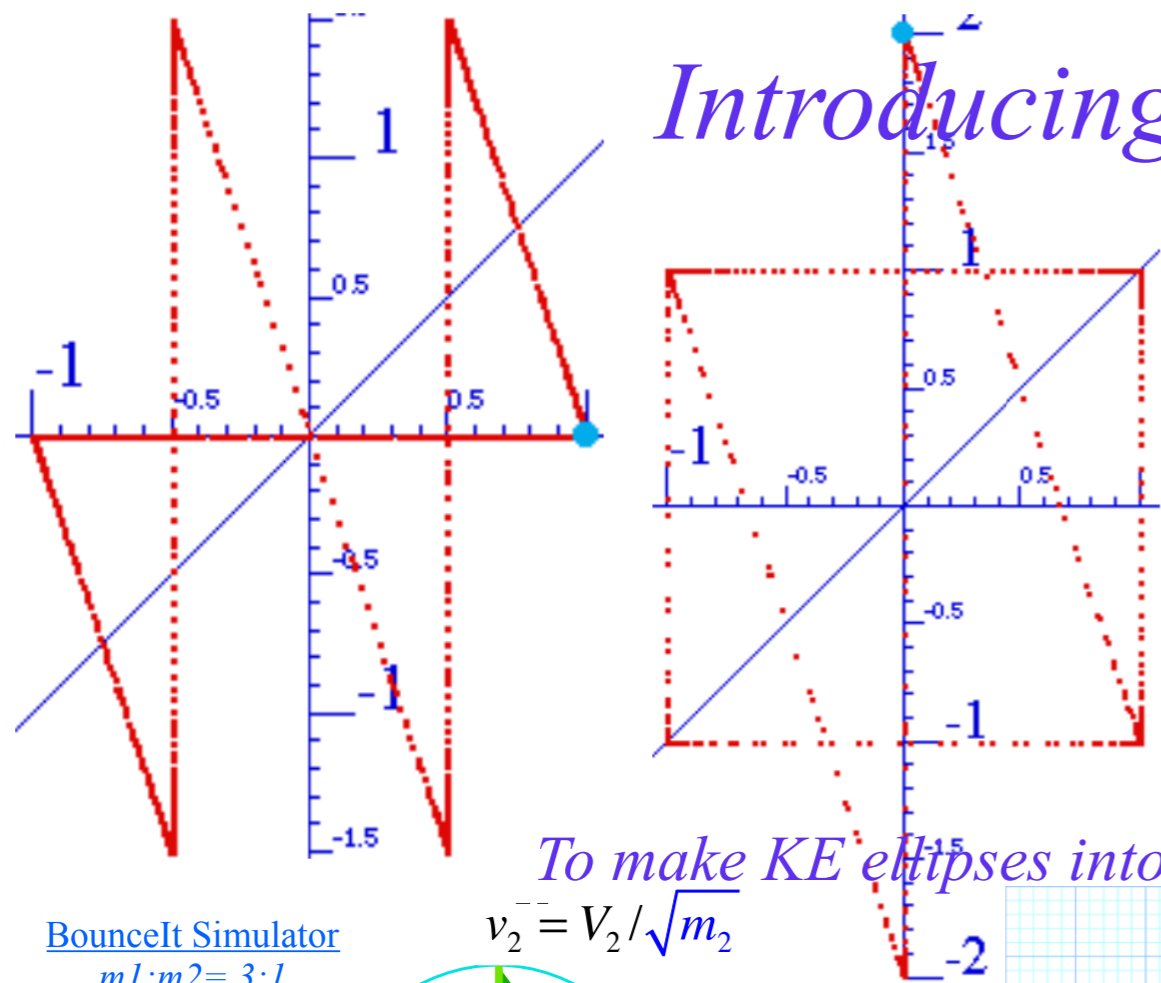
Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$

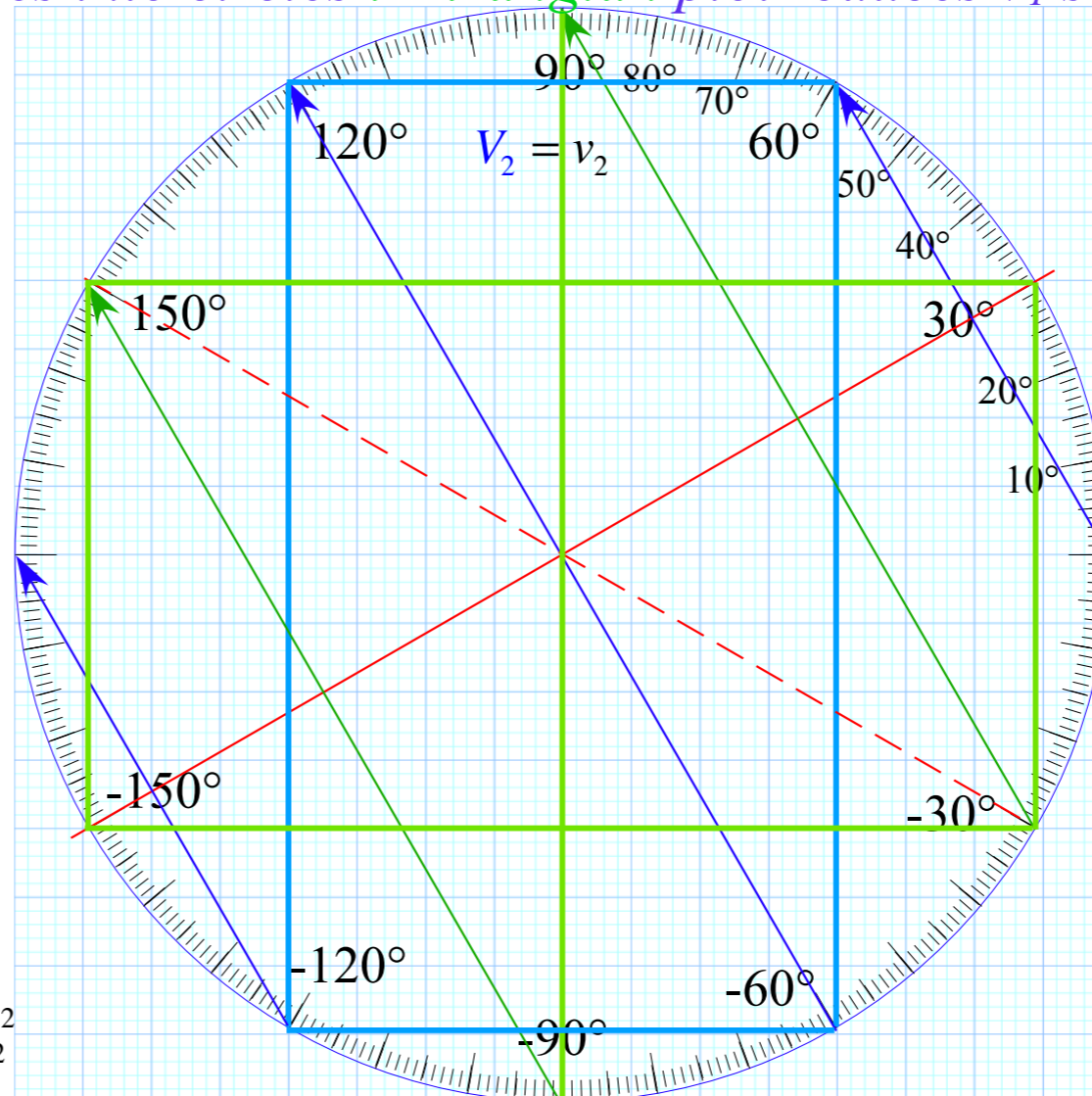
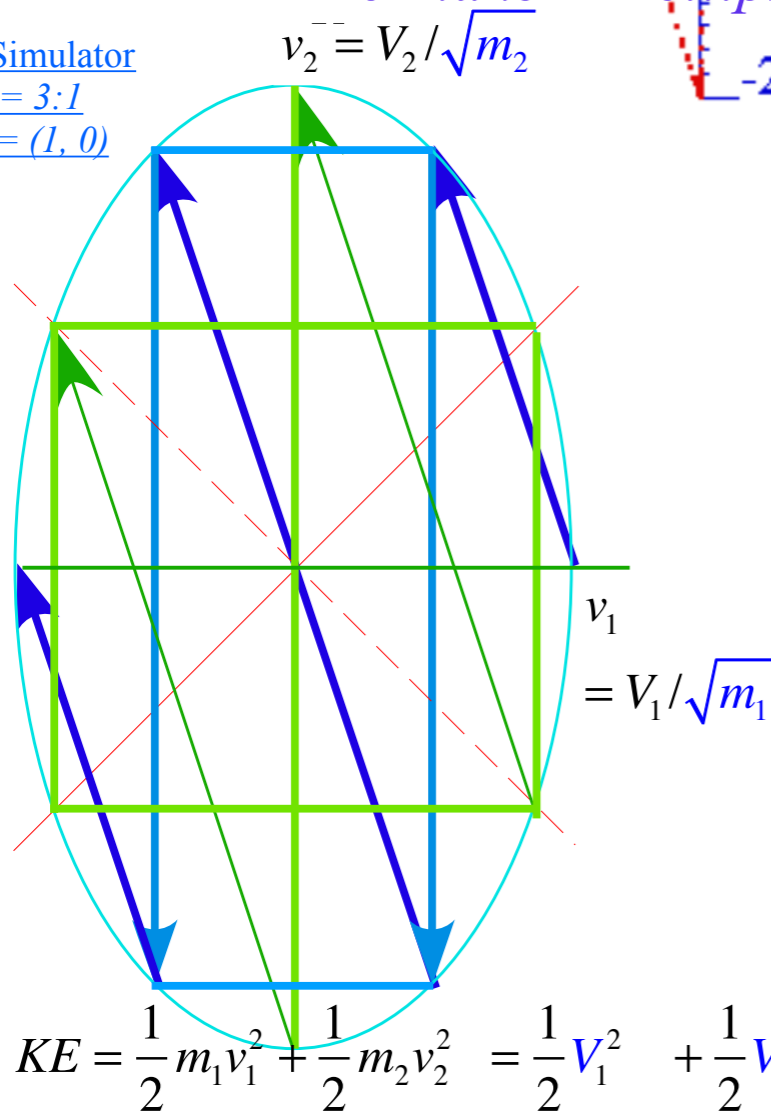
Introducing Symmetry Operators

Collisions for
mass ratio
 $m_1:m_2 = 3:1$



To make KE ellipses into circles l'Etrangian plot reduces v_1 scale by $1/\sqrt{m_1}$, etc.

BounceIt Simulator
 $m_1:m_2 = 3:1$
 $(v_1, v_2) = (1, 0)$



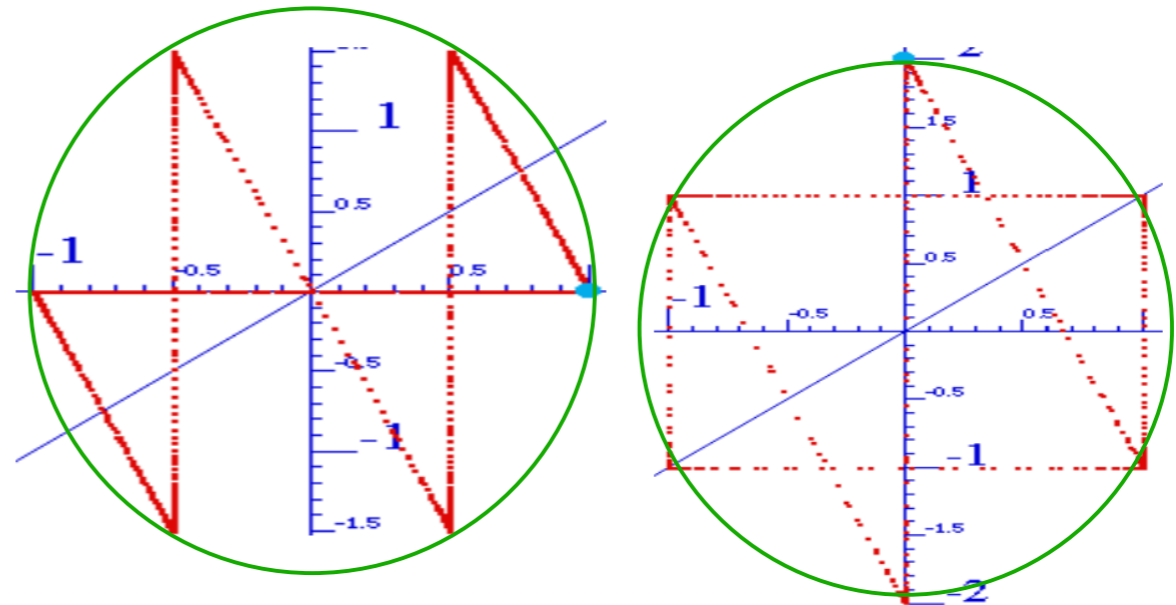
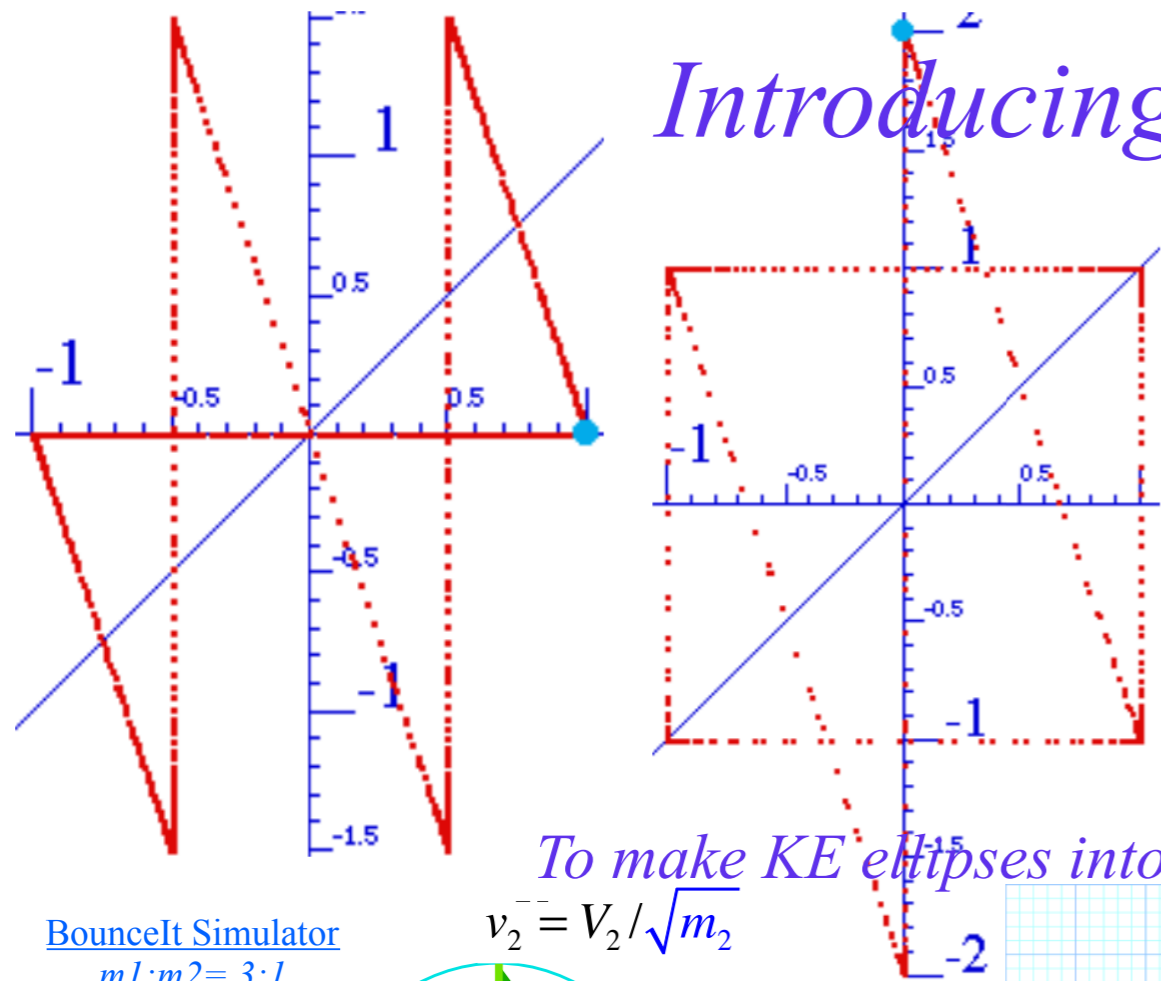
Here:
 $1/\sqrt{m_1} = 1/\sqrt{3} = 0.577$
 $1/\sqrt{m_2} = 1/\sqrt{1} = 1.0$

$$V_1 = v_1 \sqrt{m_1} = v_1 \sqrt{3}$$

BounceIt Web Simulator
 $m_1:m_2 = 3:1$ and $(v_1, v_2) = (1, 0)$
Comparison with Estrangian

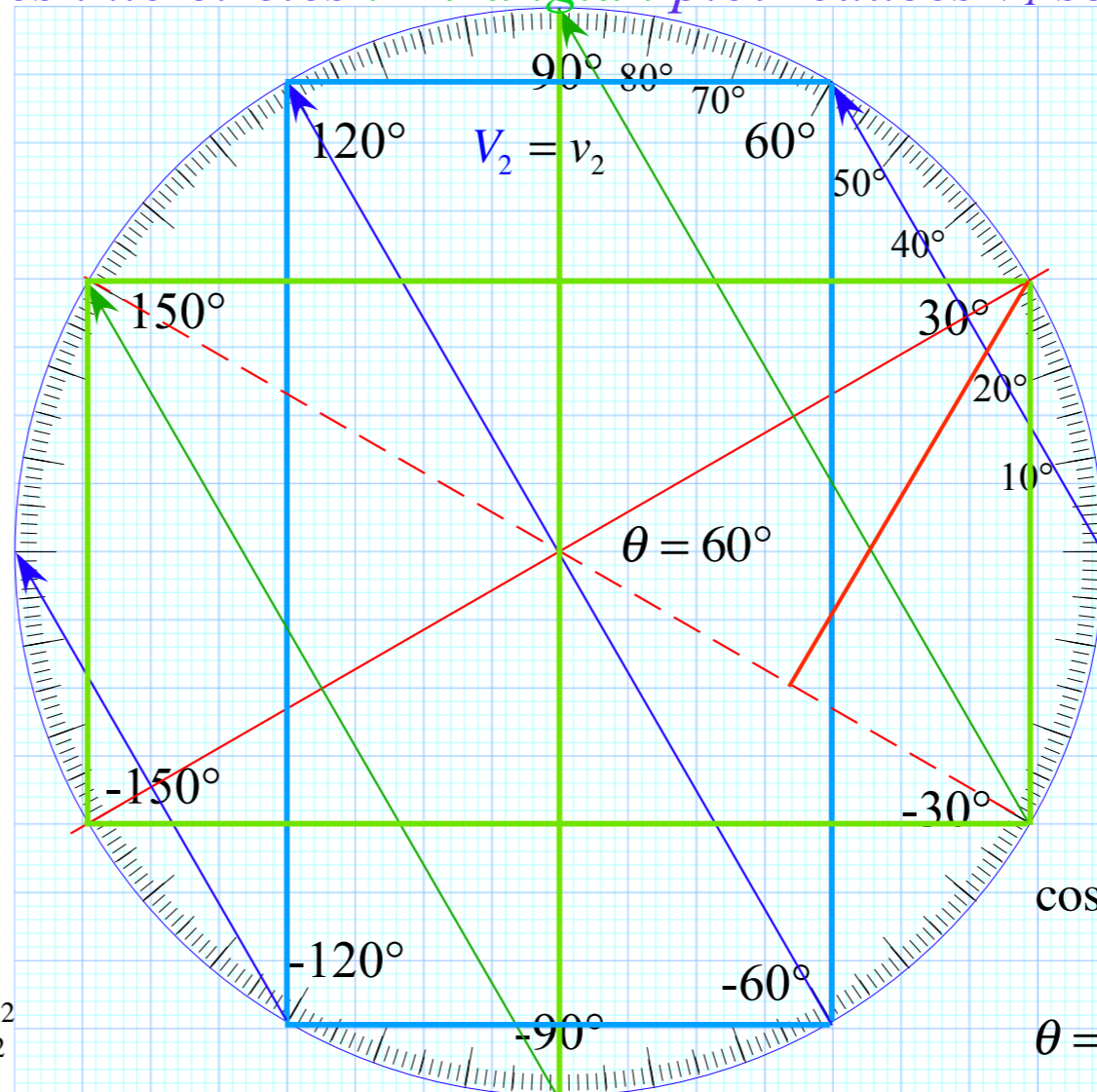
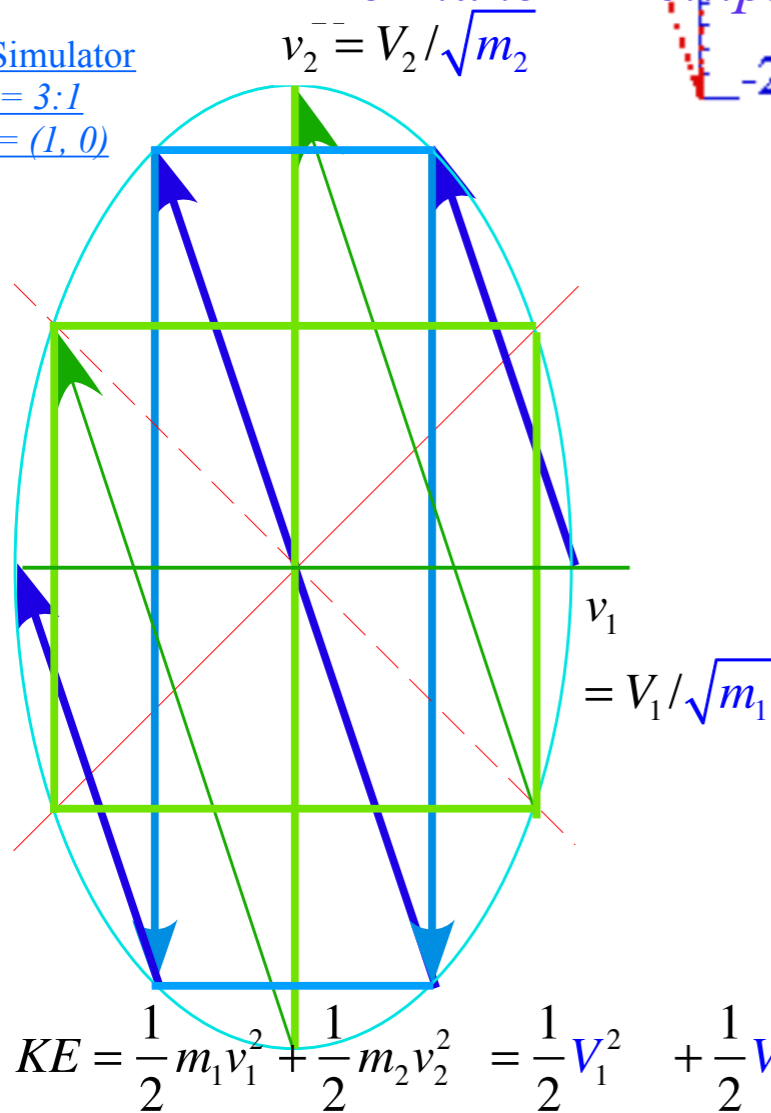
Introducing Symmetry Operators

Collisions for
mass ratio
 $m_1:m_2 = 3:1$



To make KE ellipses into circles *l'Etrangian* plot reduces v_1 scale by $1/\sqrt{m_1}$, etc.

BounceIt Simulator
 $m_1:m_2 = 3:1$
 $(v_1, v_2) = (1, 0)$



Here:
 $1/\sqrt{m_1} = 1/\sqrt{3} = 0.577$
 $1/\sqrt{m_2} = 1/\sqrt{1} = 1.0$

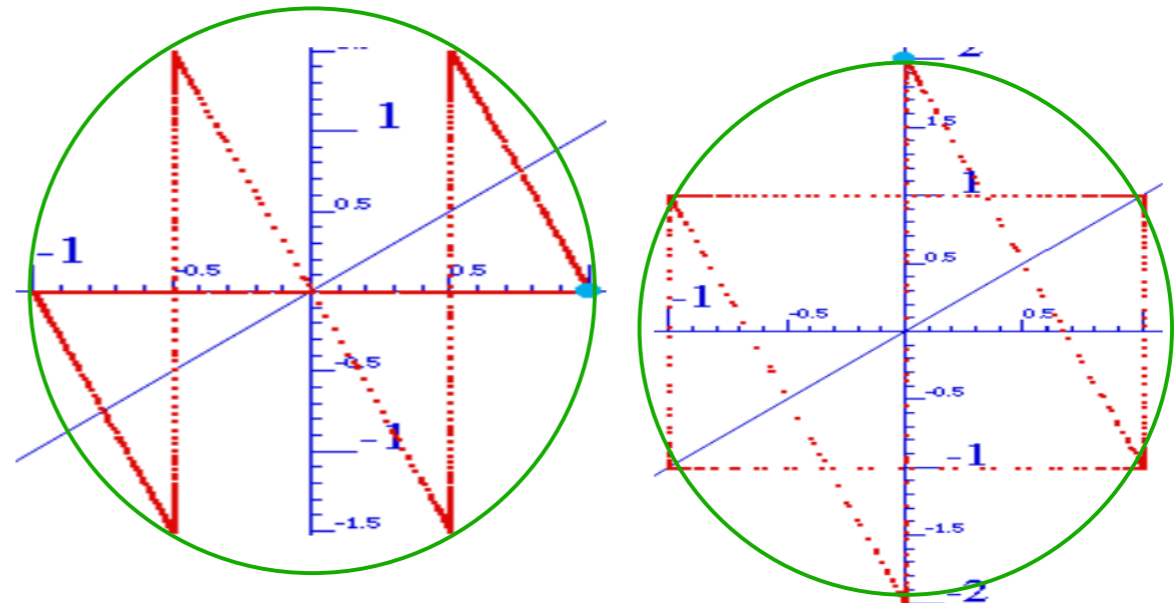
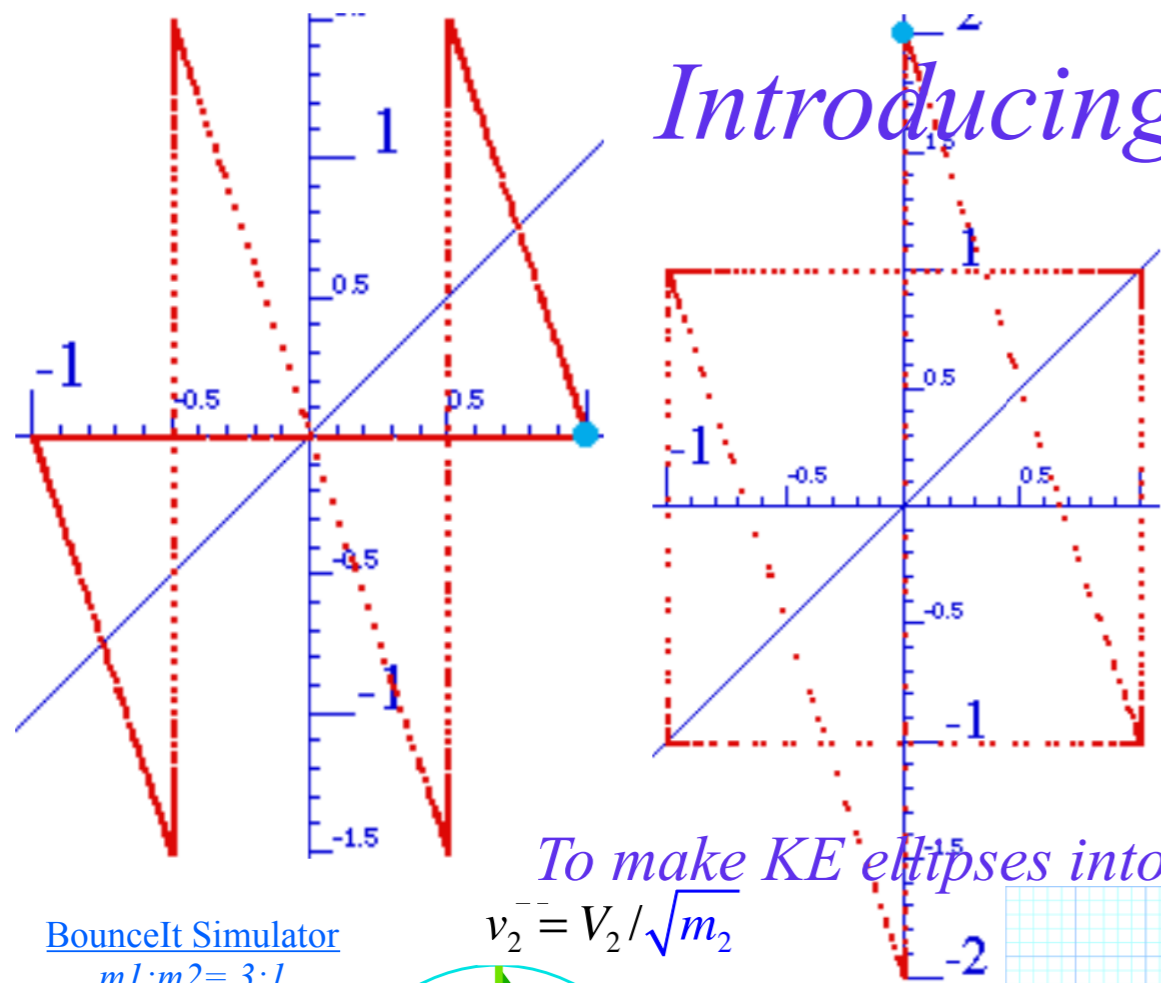
$$V_1 = v_1 \sqrt{m_1} = v_1 \sqrt{3}$$

$$\cos \theta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{m_1 - 1}{m_1 + 1} = \frac{2}{4} = \frac{1}{2}$$

$\theta = 60^\circ$ $m_1:m_2 = 3:1$ and $(v_1, v_2) = (1, 0)$
Comparison with Estrangian

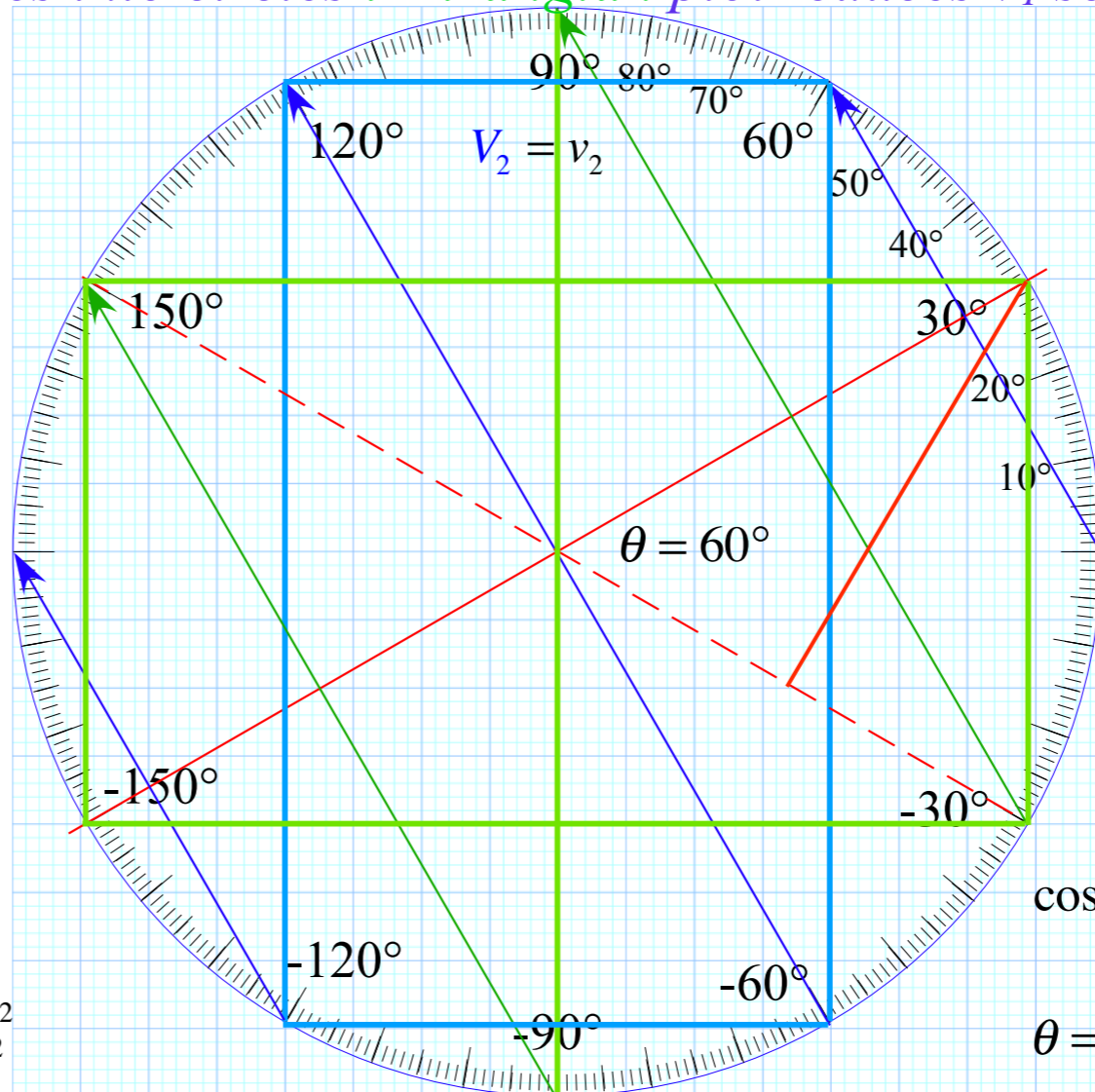
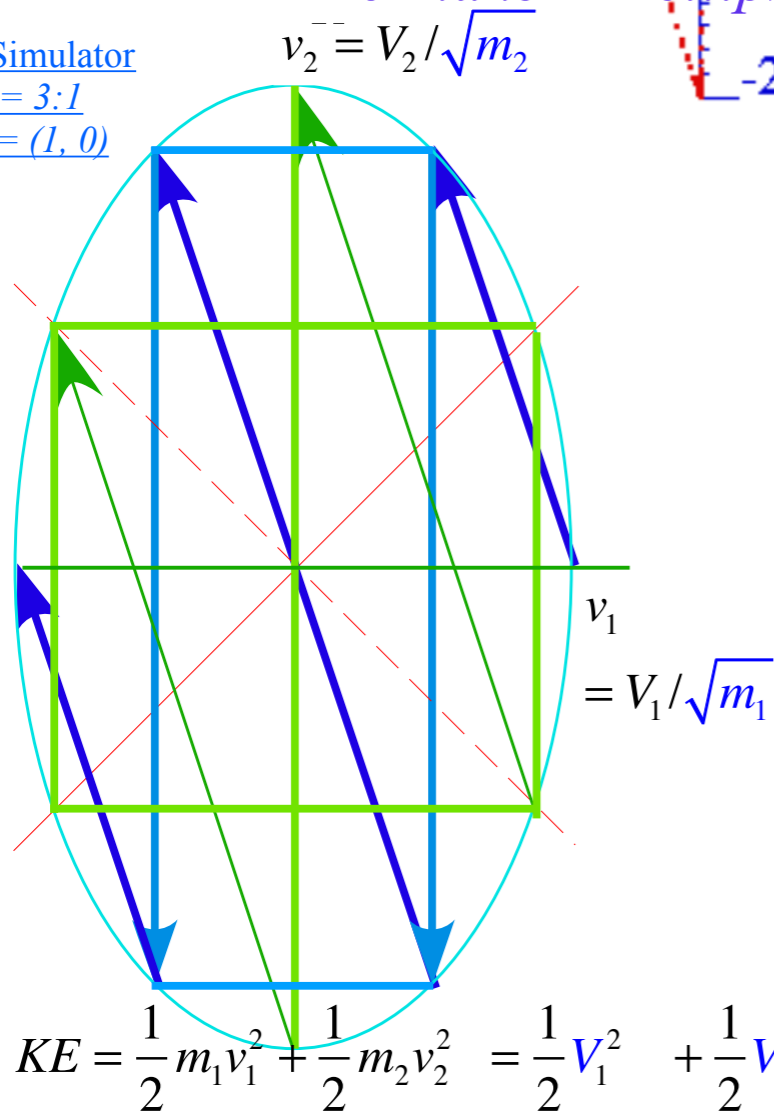
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 $m_1:m_2=3:1$
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Here:
 $1/\sqrt{m_1} = 1/\sqrt{3} = 0.577$
 $1/\sqrt{m_2} = 1/\sqrt{1} = 1.0$

$$\frac{m_1}{m_2} = \frac{1 + \cos\theta}{1 - \cos\theta} = \frac{3/2}{1/2} = 3$$

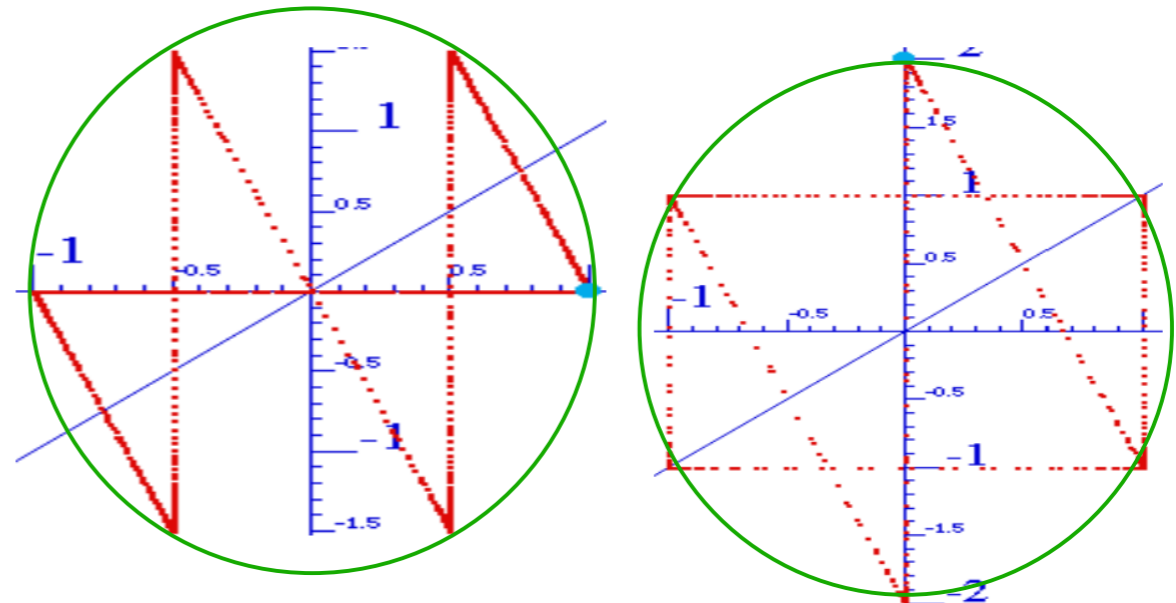
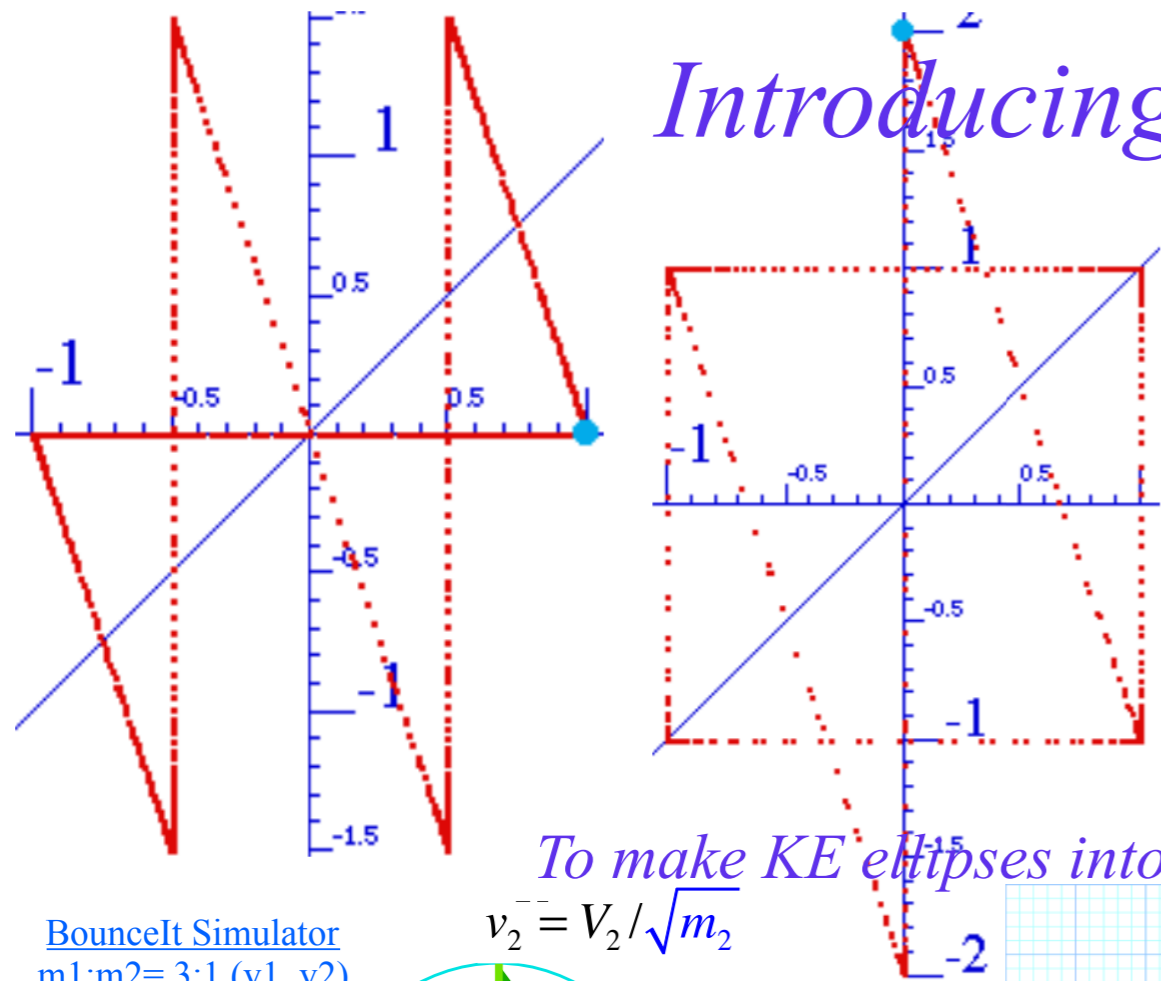
$$V_1 = v_1 \sqrt{m_1} = v_1 \sqrt{3}$$

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$\theta = 60^\circ$ [m1:m2=3:1 and \(v1, v2\) = \(1, 0\)](#)
[Comparison with Estrangian](#)

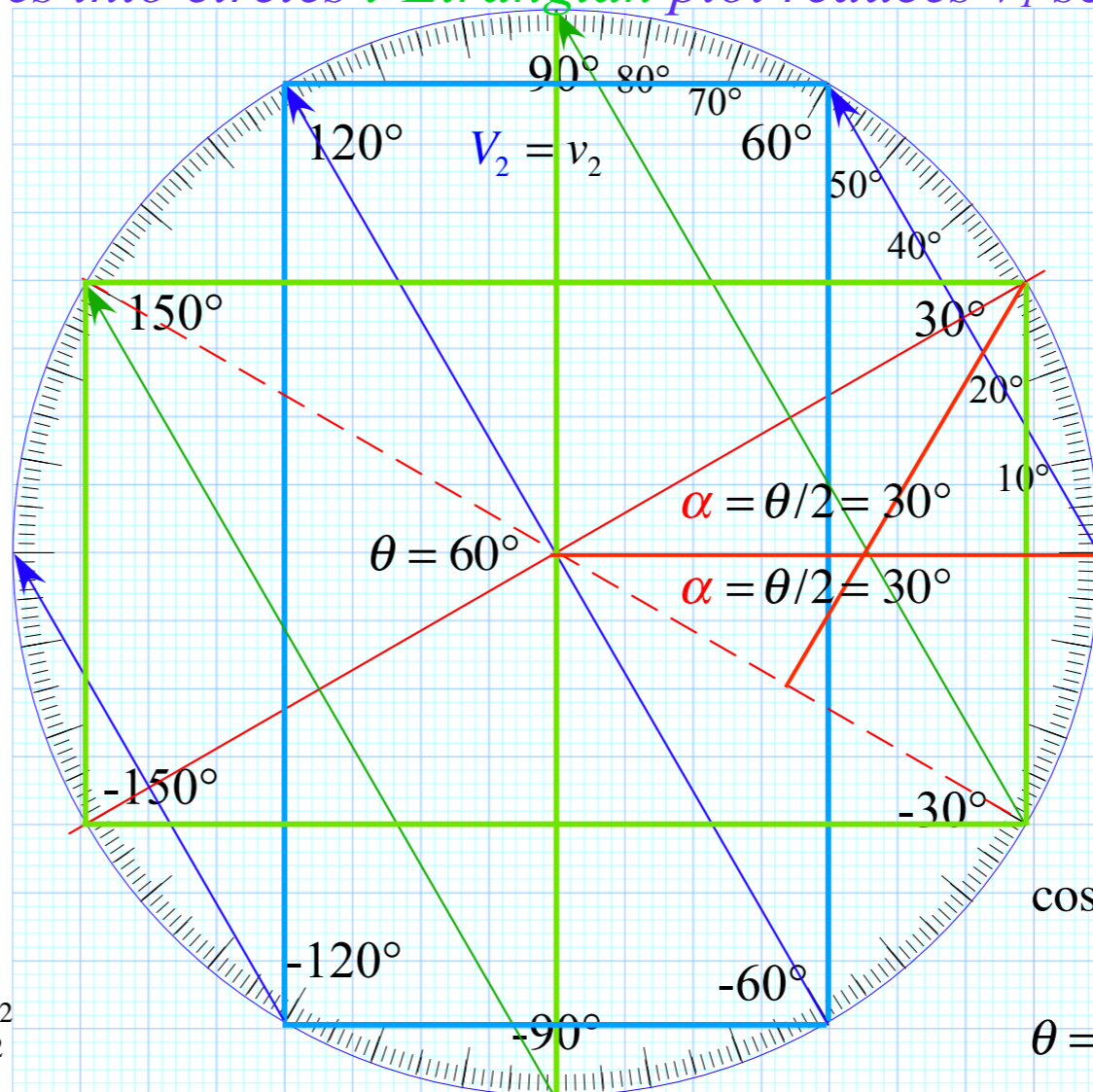
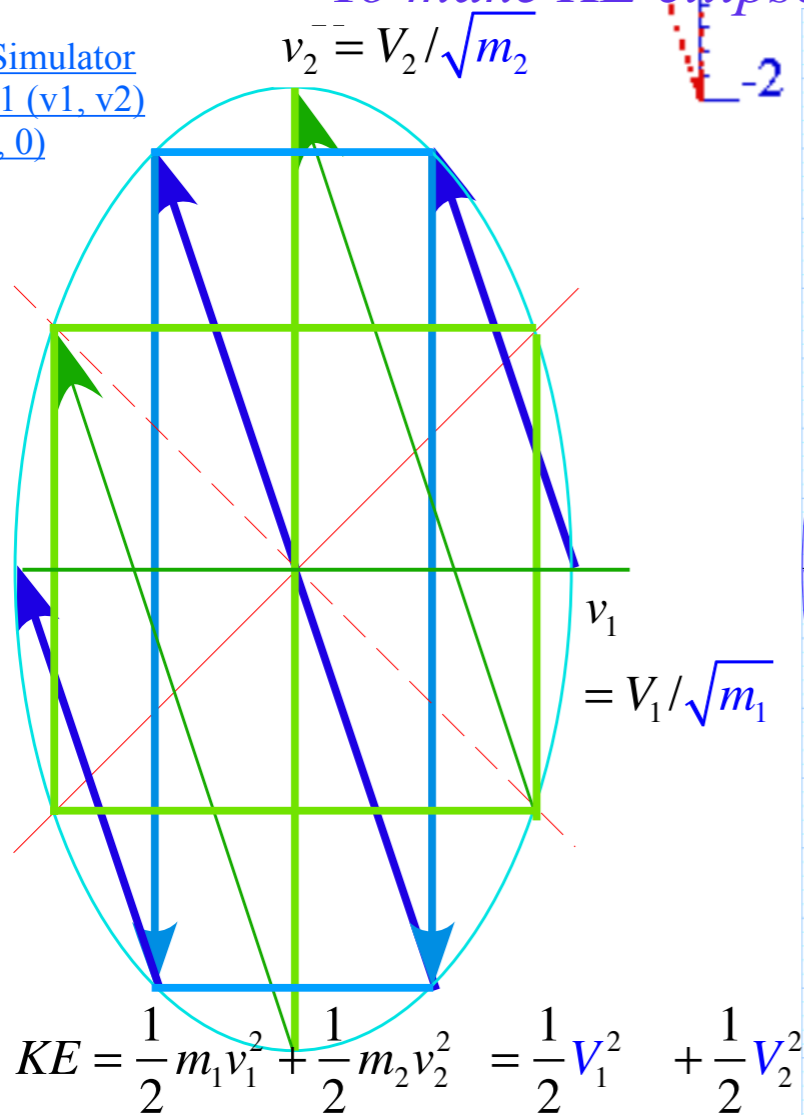
Introducing Symmetry Operators

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 $m_1:m_2 = 3:1$



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BounceIt Simulator
 $m_1:m_2 = 3:1$ (v_1, v_2)
 $= (1, 0)$



Here:

$$\frac{1}{\sqrt{m_1}} = \frac{1}{\sqrt{3}} = 0.577$$

$$\frac{1}{\sqrt{m_2}} = \frac{1}{\sqrt{1}} = 1.0$$

$$\frac{m_1}{m_2} = \frac{1 + \cos \theta}{1 - \cos \theta} = \frac{3/2}{1/2} = 3$$

$$V_1 = v_1 \sqrt{m_1} = v_1 \sqrt{3}$$

$$\frac{m_1}{m_2} = \frac{1 + \cos \theta}{1 - \cos \theta} = \frac{\sin^2 \alpha}{\cos^2 \alpha} = \tan^2 \alpha$$

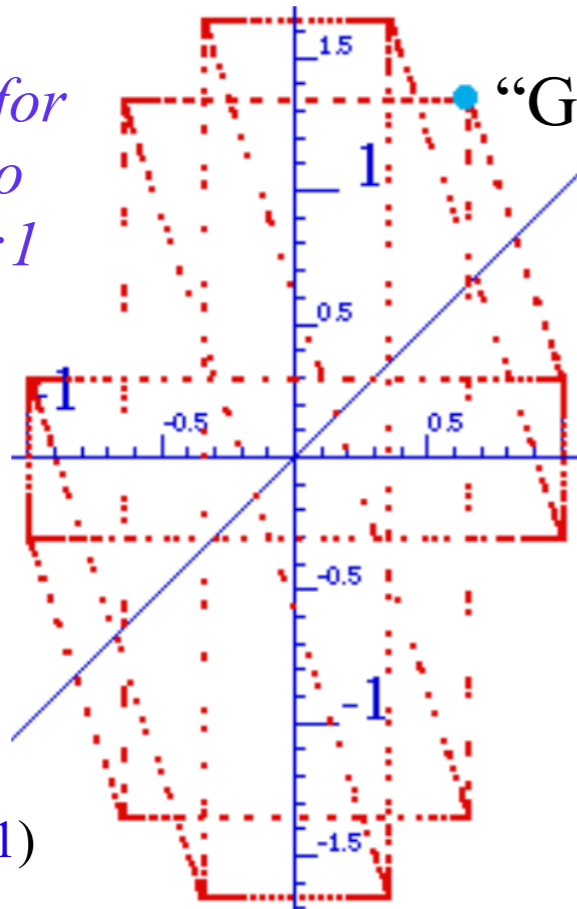
$$\alpha = \theta/2$$

$$\cos \theta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1} = \frac{2}{4} = \frac{1}{2}$$

$$\theta = 60^\circ$$

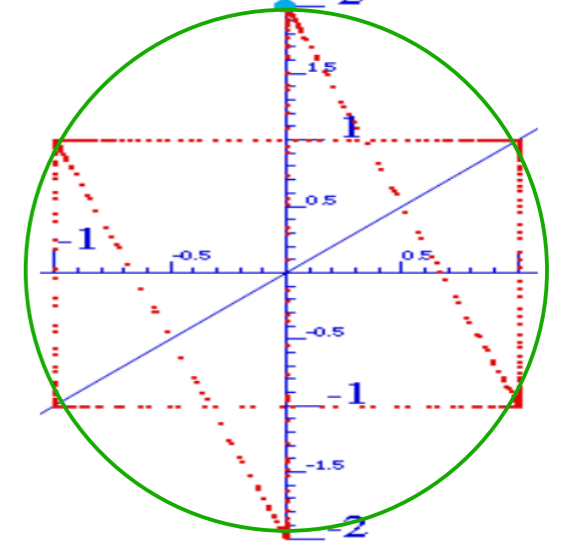
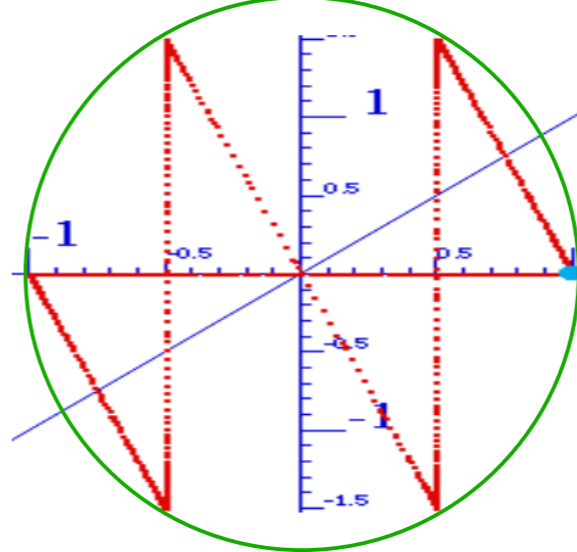
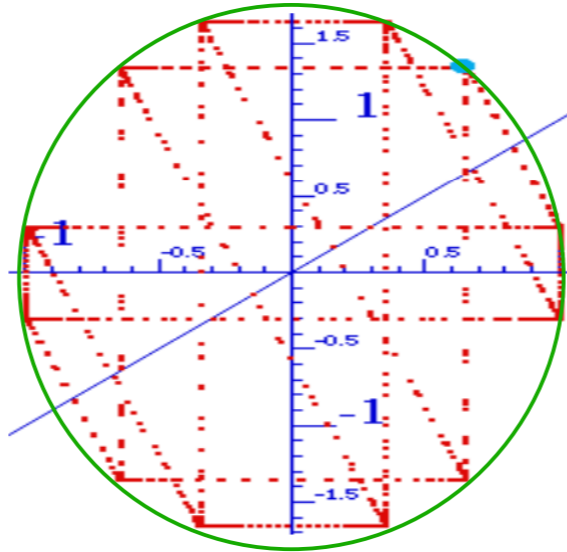
[m1:m2= 3:1 and \(v1, v2\) = \(1, 0\)](#)
[Comparison with Estrangian](#)

Collisions for
mass ratio
 $m_1:m_2=3:1$



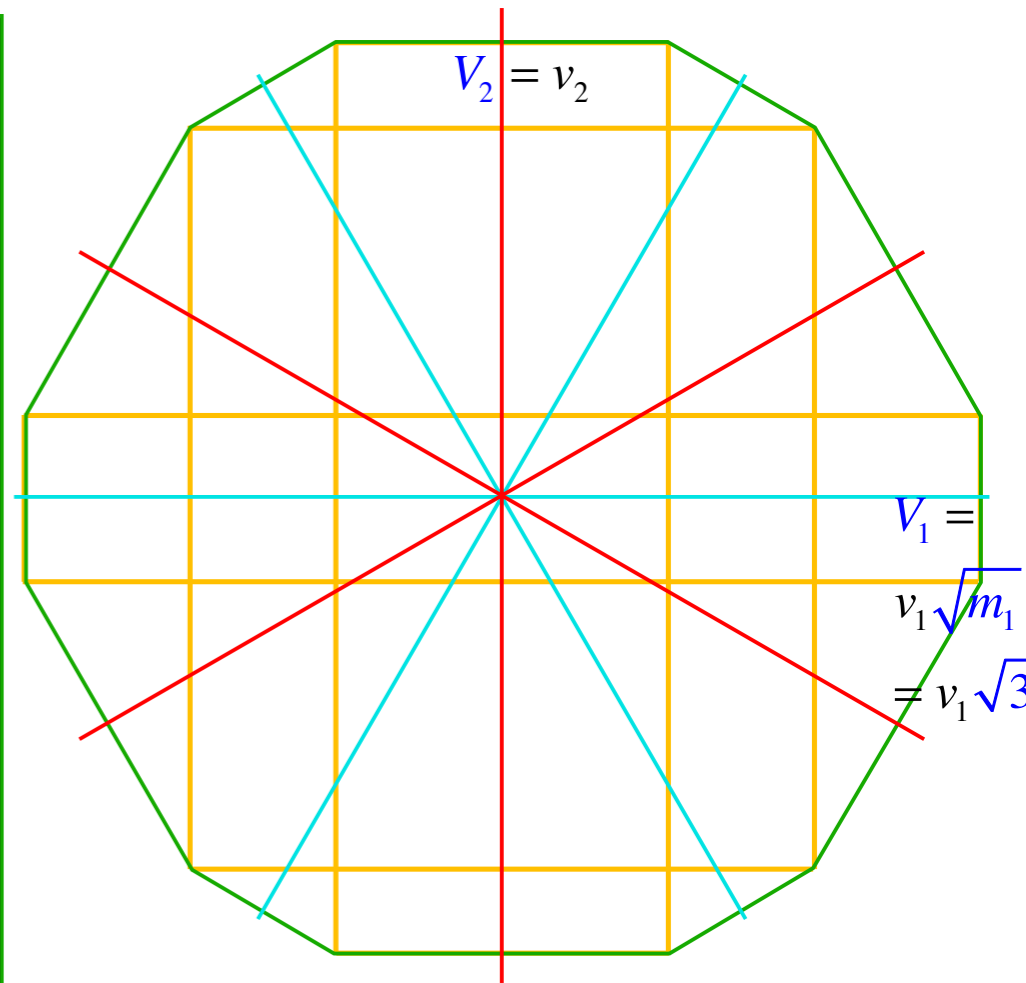
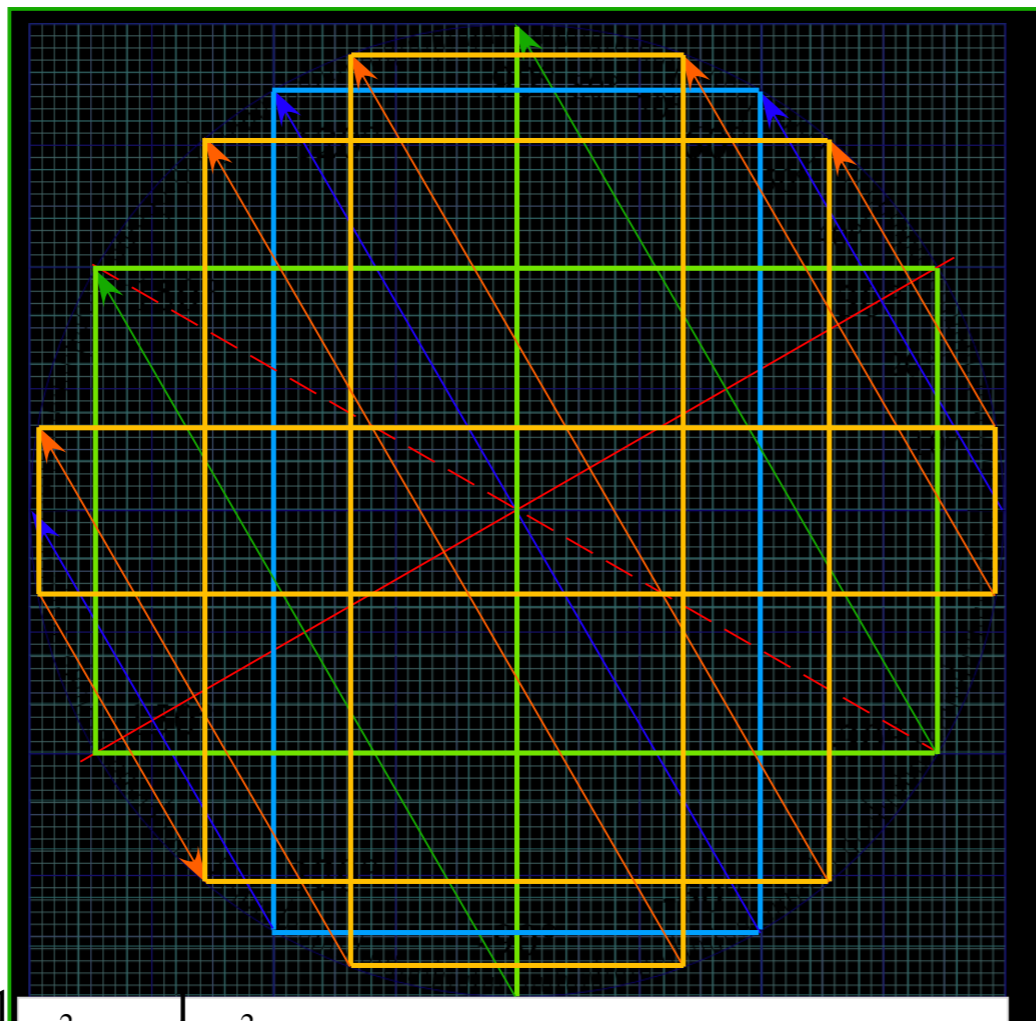
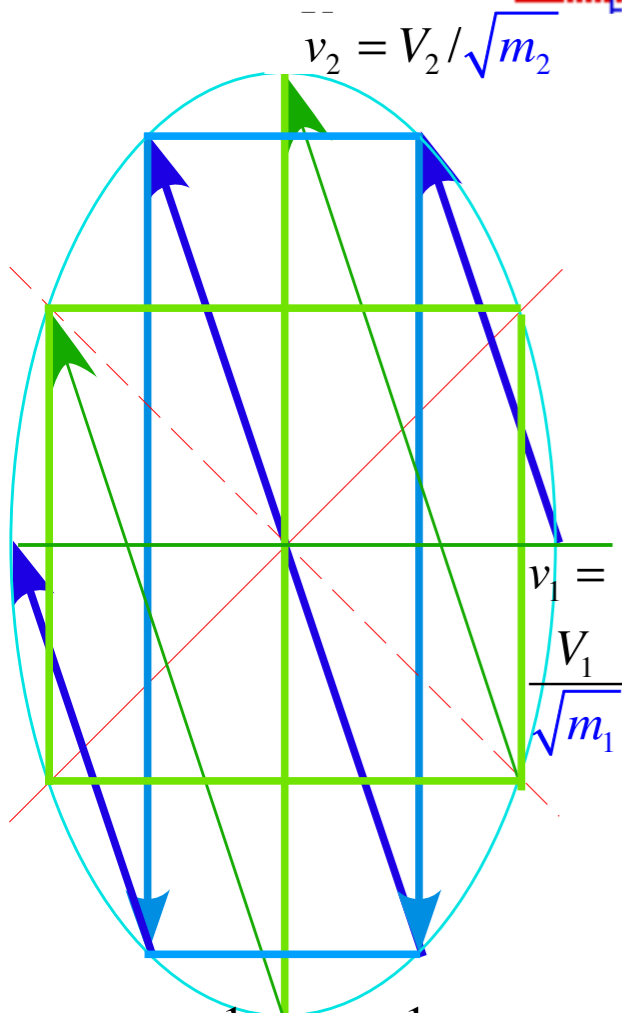
“Generic” initial velocity
 $(v_1=1.0, v_2=0.1)$

“Symmetric” initial velocity
 $(v_1=1, v_2=0)$ or $(v_1=1, v_2=-1)$



$m_1/m_2=(3)/(1)$

reduce v_1 scale by $1/\sqrt{m_1} = 1/\sqrt{3}=0.577$



$$KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$$

$m_1:m_2=3:1$ and $(v_1, v_2) = (1, 0)$
Comparison with Estrangian

Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: "It's all done with mirrors!"

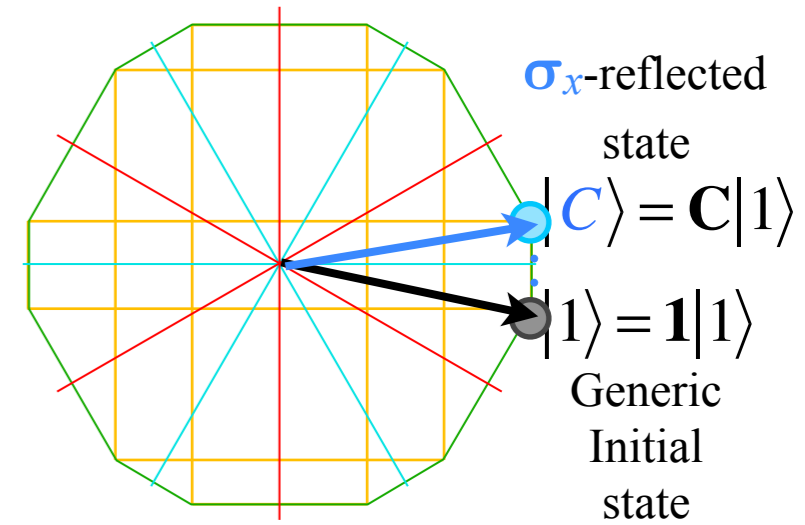
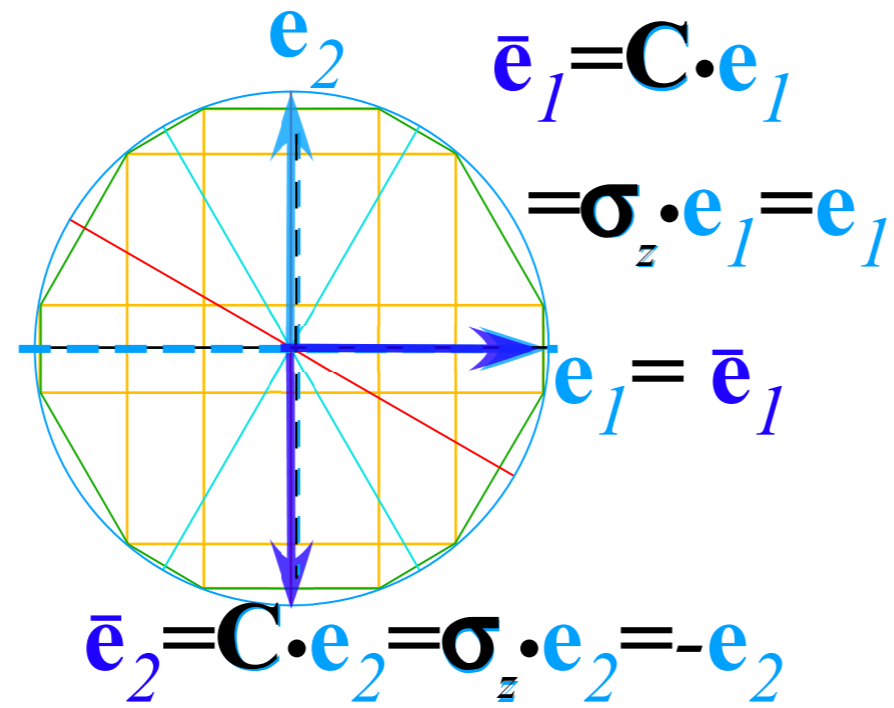
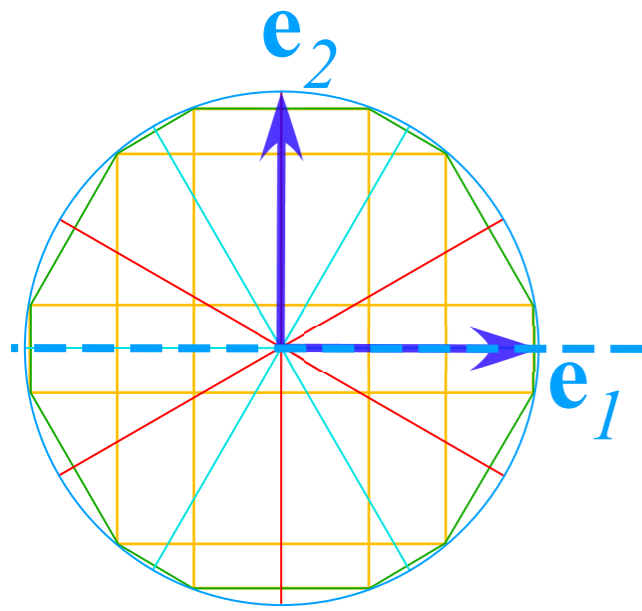
Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

 *Group multiplication and product table*

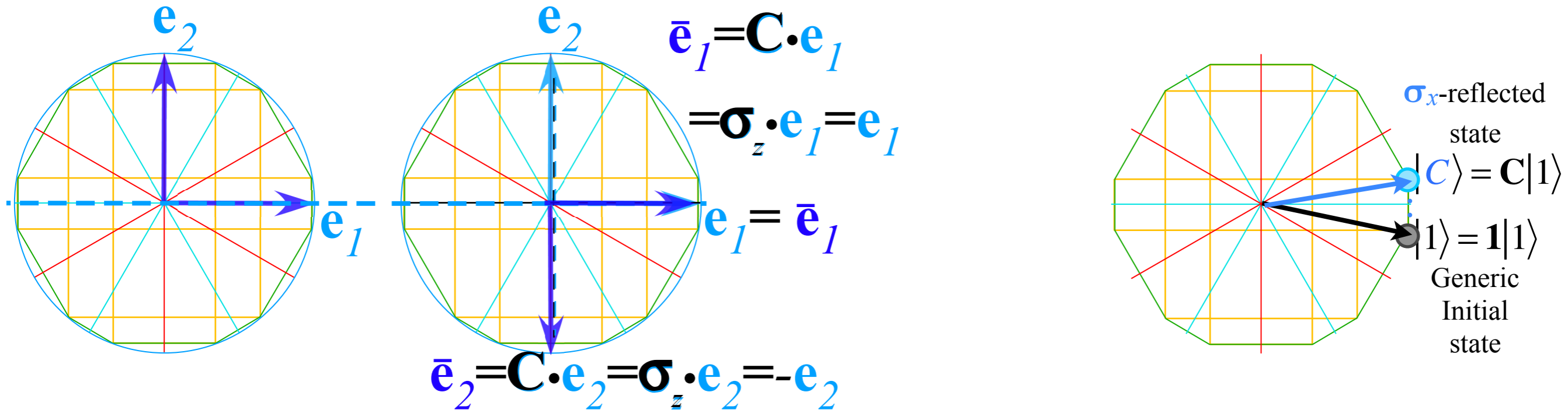
Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$

Effects of Ceiling Bang Matrix $\mathbf{C} = \boldsymbol{\sigma}_z = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{C} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{C} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{C} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{C} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



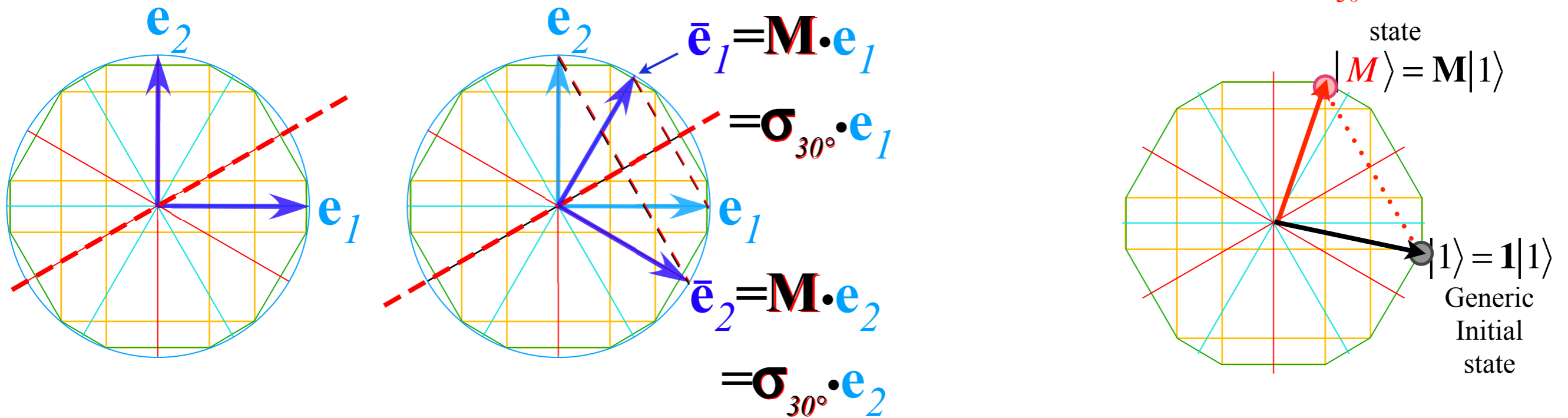
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Known as *matrix elements* or *components*

Known as *relative direction cosines*

Effects of Mass Bang Matrix $\mathbf{M} = \boldsymbol{\sigma}_{30^\circ} = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{M} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{M} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{M} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{M} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{pmatrix}$



$\boldsymbol{\sigma}_{30^\circ}$ -reflected state

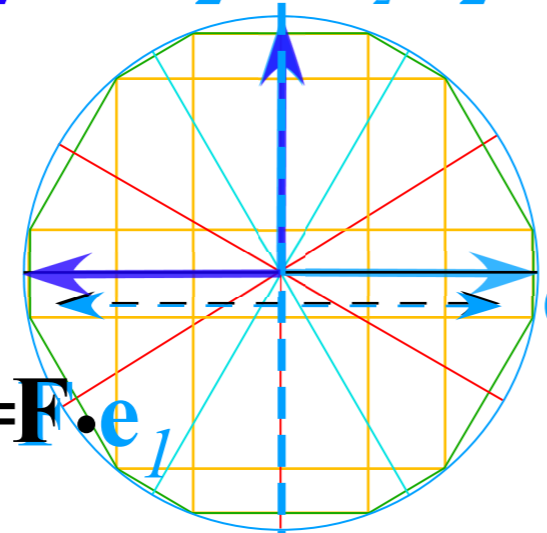
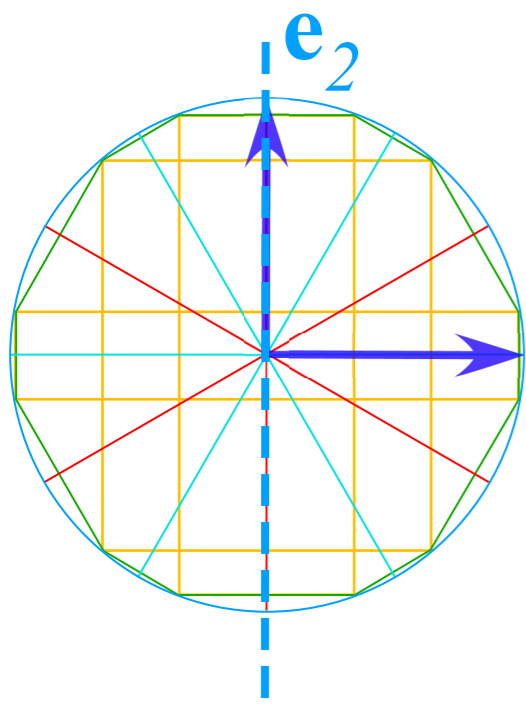
Generic Initial state

Effects of Floor Bang Matrix

$$\mathbf{F} = -\sigma_z = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{F} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{F} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{F} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{F} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

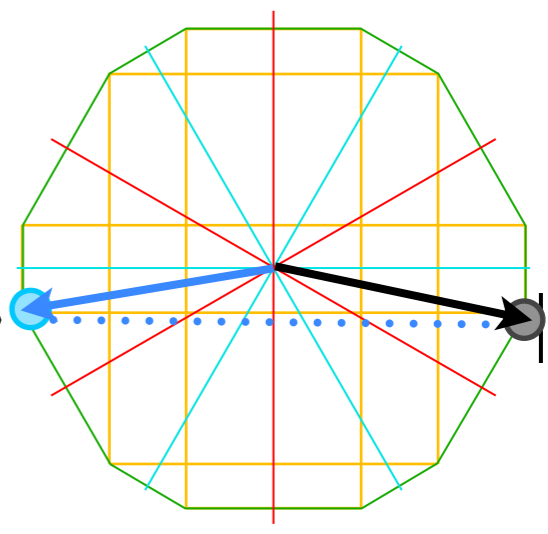
$$\bar{\mathbf{e}}_2 = \mathbf{F} \cdot \mathbf{e}_2 = -\sigma_z \cdot \mathbf{e}_2 = +\mathbf{e}_2$$

$$\bar{\mathbf{e}}_1 = \mathbf{F} \cdot \mathbf{e}_1 = -\sigma_z \cdot \mathbf{e}_1 = -\mathbf{e}_1$$



$-\sigma_z$ -reflected state

$$|F\rangle = \mathbf{F}|1\rangle$$



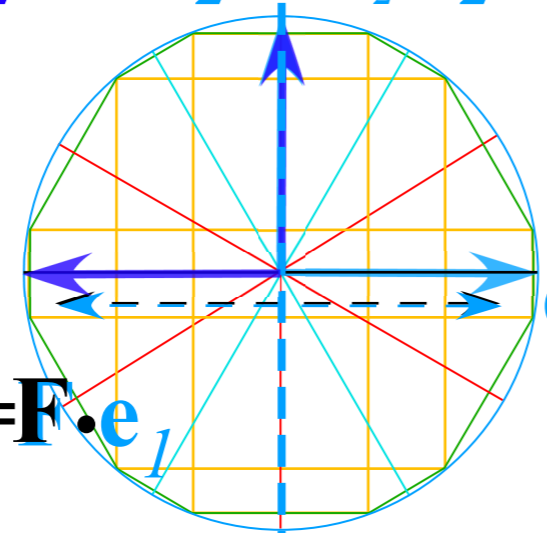
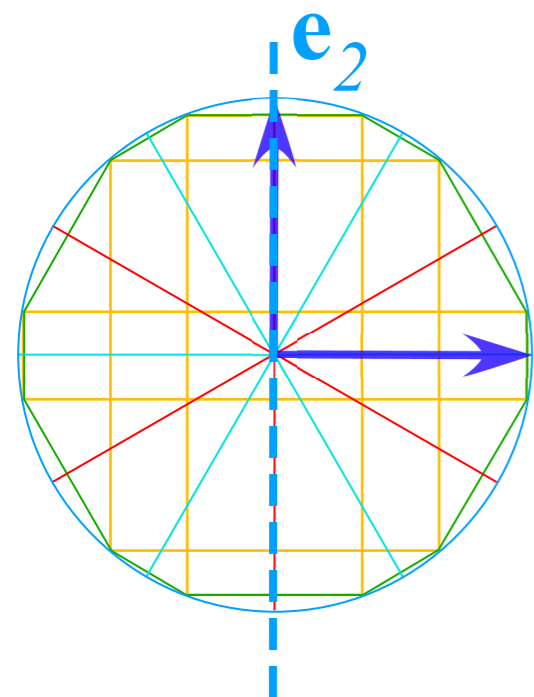
$|1\rangle = \mathbf{1}|1\rangle$
Generic Initial state

Effects of Floor Bang Matrix

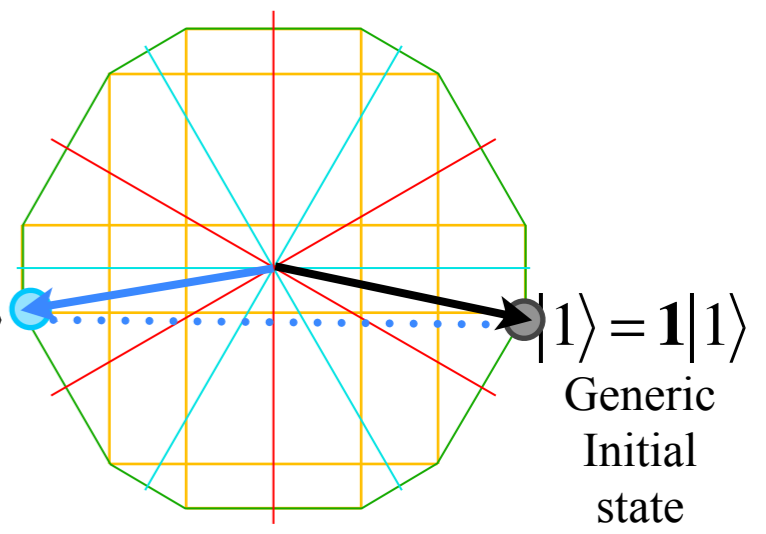
$$\mathbf{F} = -\sigma_z = \begin{pmatrix} \mathbf{e}_1 \cdot \mathbf{F} \cdot \mathbf{e}_1 & \mathbf{e}_1 \cdot \mathbf{F} \cdot \mathbf{e}_2 \\ \mathbf{e}_2 \cdot \mathbf{F} \cdot \mathbf{e}_1 & \mathbf{e}_2 \cdot \mathbf{F} \cdot \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_1 \cdot \bar{\mathbf{e}}_2 \\ \mathbf{e}_2 \cdot \bar{\mathbf{e}}_1 & \mathbf{e}_2 \cdot \bar{\mathbf{e}}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\bar{\mathbf{e}}_2 = \mathbf{F} \cdot \mathbf{e}_2 = -\sigma_z \cdot \mathbf{e}_2 = +\mathbf{e}_2$$

$$\bar{\mathbf{e}}_1 = \mathbf{F} \cdot \mathbf{e}_1 = -\sigma_z \cdot \mathbf{e}_1 = -\mathbf{e}_1$$

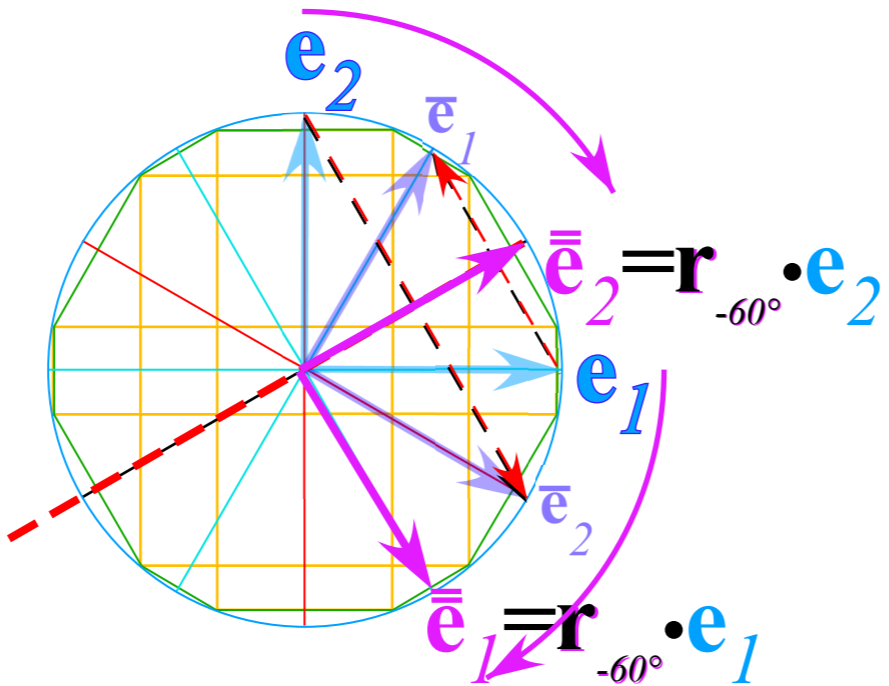
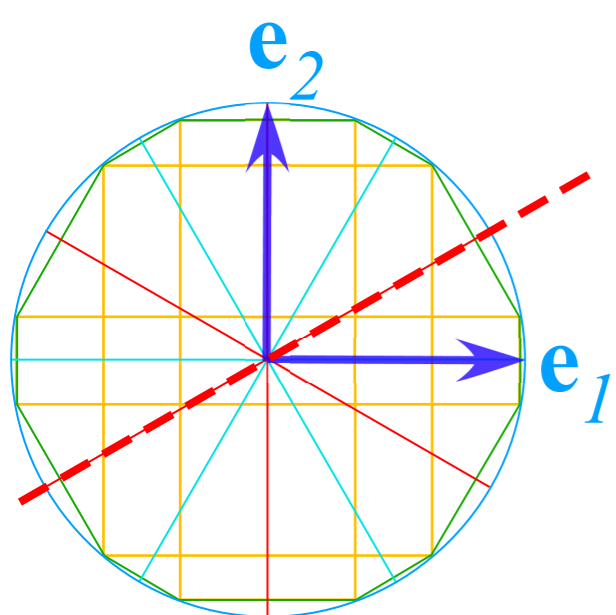


$-\sigma_z$ -reflected state
 $|F\rangle = \mathbf{F}|1\rangle$

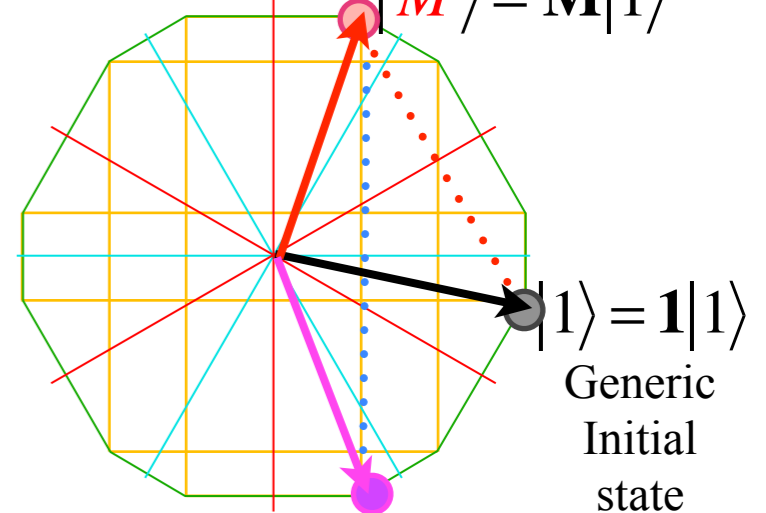


Effects of Ceiling \mathbf{C} after Bang \mathbf{M} :

$$\mathbf{r}_{-60^\circ} = \mathbf{C} \cdot \mathbf{M} = \sigma_z \cdot \sigma_{30^\circ}$$



σ_{30° -reflected state
 $|M\rangle = \mathbf{M}|1\rangle$



$$|r_{-60^\circ}\rangle = \mathbf{C} \cdot \mathbf{M}|1\rangle = \mathbf{r}_{-60^\circ}|1\rangle$$

σ_{30° σ_{30° -reflected state

is a \mathbf{r}_{-60° -rotated state

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Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

 *Group multiplication and product table* 

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$

D_6	1	r_{120}	\bar{r}_{120}	σ_{60}	$\bar{\sigma}_{60}$	σ_z	I	\bar{r}_{60}	r_{60}	$\bar{\sigma}_{30}$	σ_{30}	$\bar{\sigma}_z$
1	1											
\bar{r}_{120}		1										
r_{120}			1									
σ_{60}				1								
$\bar{\sigma}_{60}$					1							
σ_z						1					\bar{r}_{60}	
I							1					
r_{60}								1				
\bar{r}_{60}									1			
$\bar{\sigma}_{30}$										1		
σ_{30}						r_{60}					1	
$\bar{\sigma}_z$												1

Note: $\bar{r}_{60} = I r_{120} = r_{120} I = r_{-60}$ and: $I = r_{\pm 180}$
 $\bar{r}_{120} = I r_{60} = r_{60} I = r_{-120}$ and: $I^2 = 1$
 $\sigma_{60} = I \bar{\sigma}_{30} = \bar{\sigma}_{30} I$
 $\bar{\sigma}_{60} = I \sigma_{30} = \sigma_{30} I$
 $\bar{\sigma}_z = I \sigma_z = \sigma_z I$

Easy to make hexagonal (D_6) symmetry group table:

Example 1: Find $\sigma_{30^\circ} \cdot \sigma_{-60^\circ} = \underline{\hspace{2cm}}$?

Solution: Find σ_{30° -plane and state- $|\sigma_{-60^\circ}\rangle$

Operate former on latter to get: $\sigma_{30^\circ} |\sigma_{-60^\circ}\rangle = |\mathbf{I}\rangle$

That gives answer: $\sigma_{30^\circ} \cdot \sigma_{-60^\circ} = \mathbf{I}$.

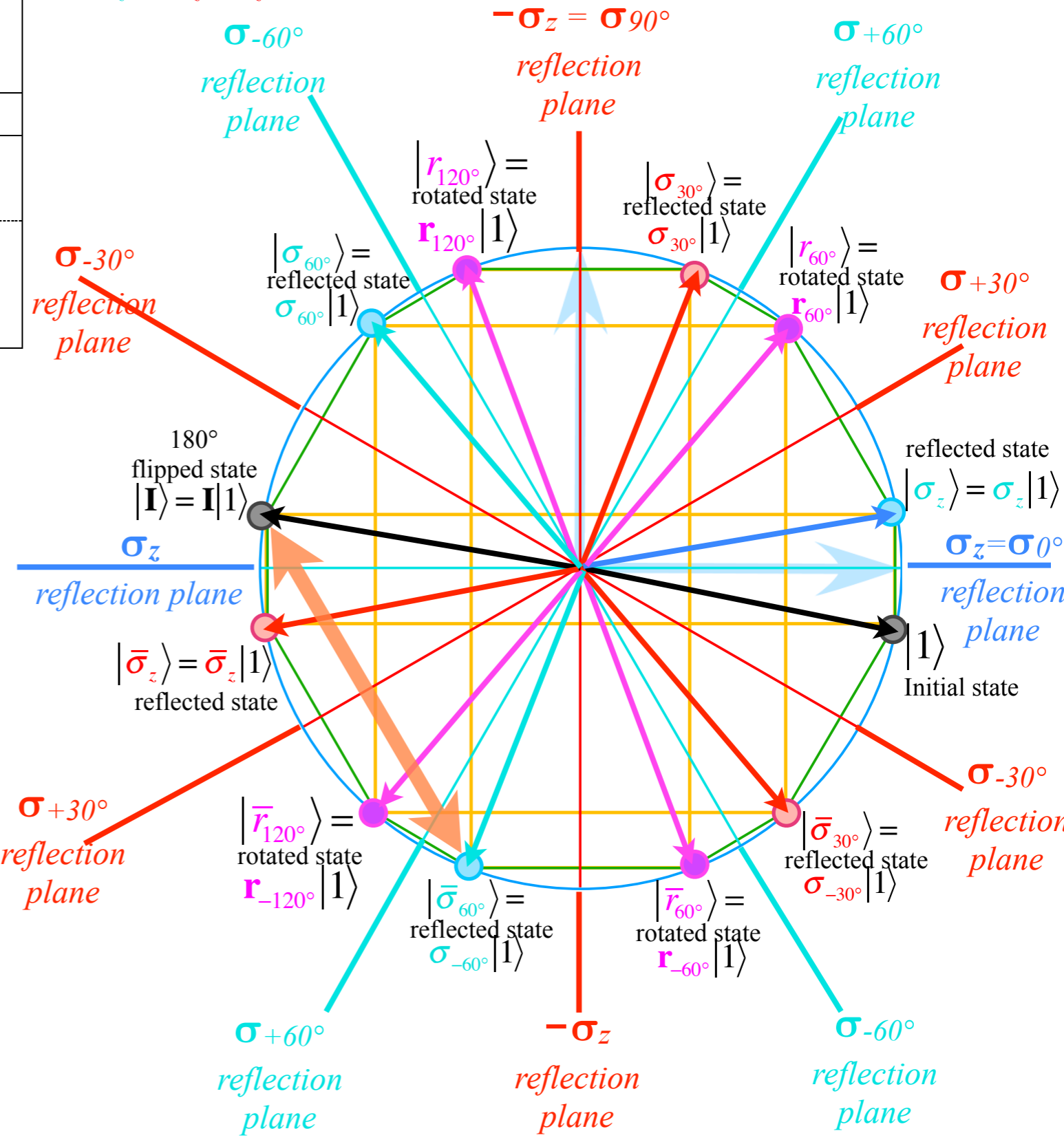
Rest of σ_{30° row follows:

^{11th} row	1	r_{120}	\bar{r}_{120}	σ_{60}	$\bar{\sigma}_{60}$	σ_z	I	\bar{r}_{60}	r_{60}	$\bar{\sigma}_{30}$	σ_{30}	$\bar{\sigma}_z$
σ_{30}	σ_{30}	$\bar{\sigma}_{30}$	$\bar{\sigma}_z$	\bar{r}_{60}	I	r_{60}	$\bar{\sigma}_{60}$	σ_{60}	σ_z	r_{120}	1	\bar{r}_{120}

Example 2: Find $r_{60^\circ} \cdot \sigma_{-60^\circ} = \underline{\hspace{2cm}}$?

Solution: Do r_{60° -rotation $r_{60^\circ} |\sigma_{-60^\circ}\rangle = |\sigma_{-30^\circ}\rangle$

That gives answer: $r_{60^\circ} \cdot \sigma_{-60^\circ} = \sigma_{-30^\circ}$



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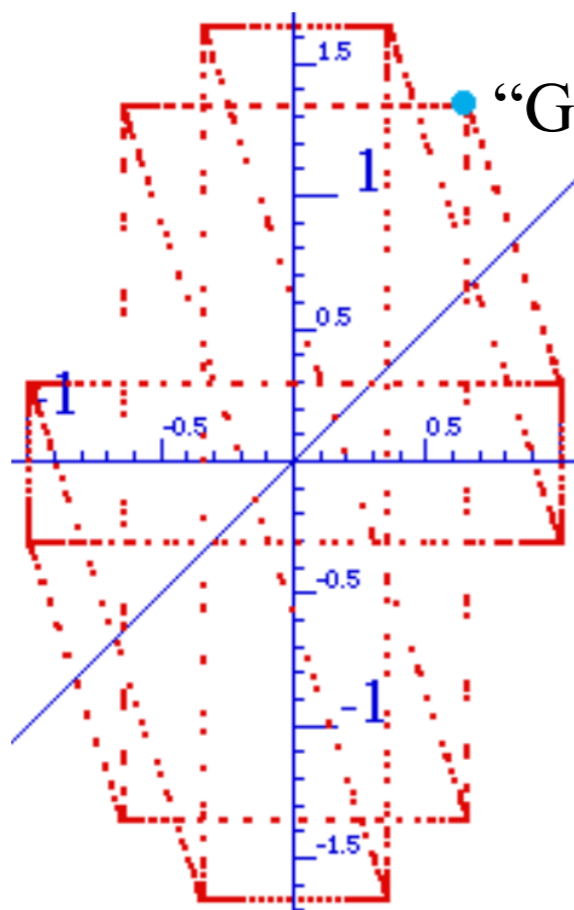
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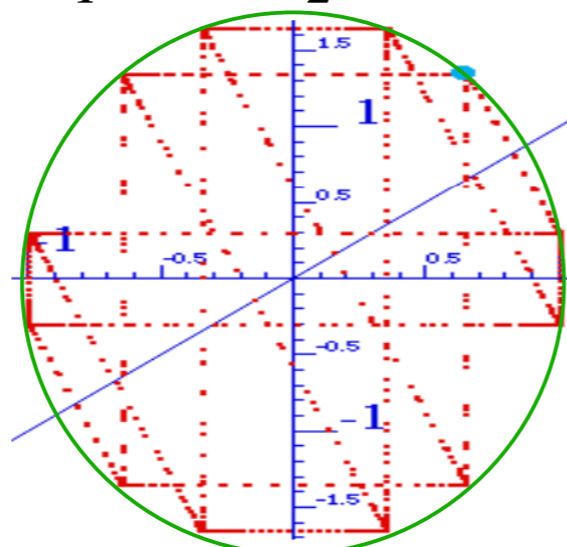
 *Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)*

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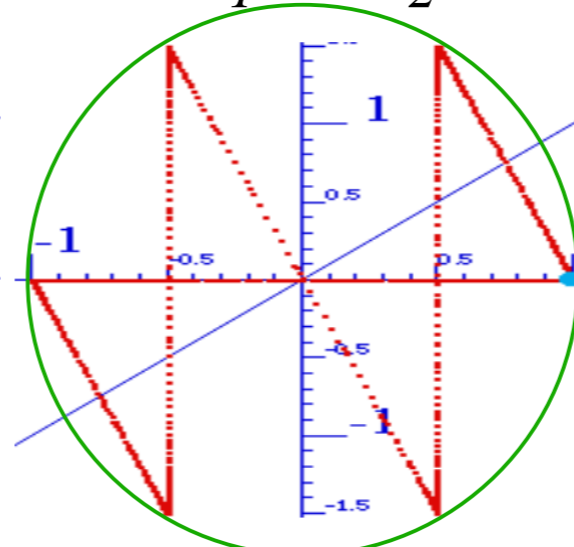
Collisions for
mass ratio
 $m_1:m_2 = 3:1$



“Generic” initial velocity
 $(v_1=1.0, v_2=0.1)$

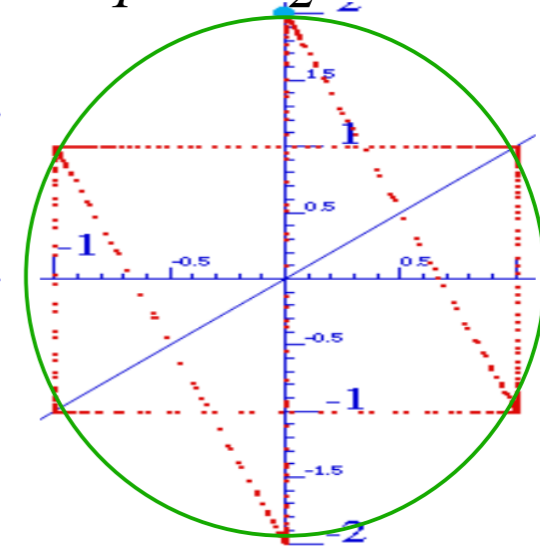


$(v_1, v_2) = (1, 0.1)$



“Symmetric” initial velocity
 $(v_1=1, v_2=0)$ or $(v_1=1, v_2=-1)$

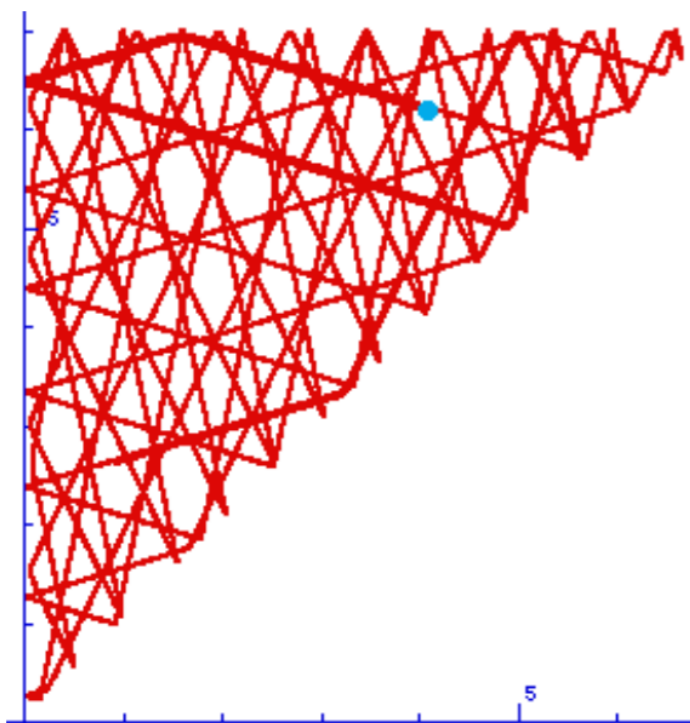
$(v_1, v_2) = (1, 0)$



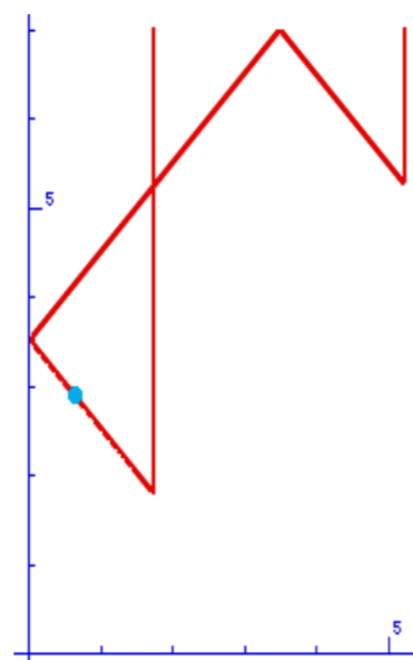
$(v_1, v_2) = (1, -1)$

Corresponding space-space (y_1, y_2) paths

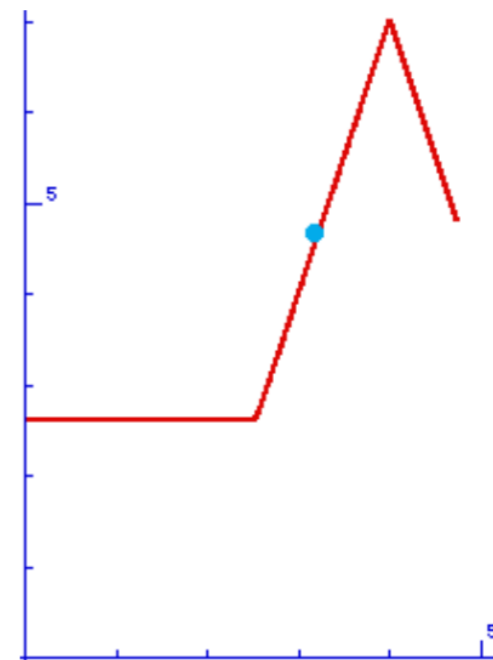
BounceIt
 $m_1:m_2 = 3:1$
Dual plots
 v_2 vs v_1 and V_2 vs V_1



$(v_1, v_2) = (1, 0.1)$



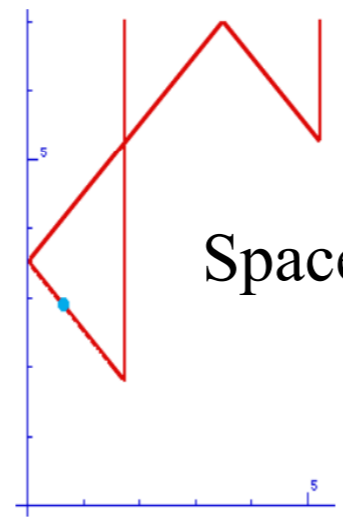
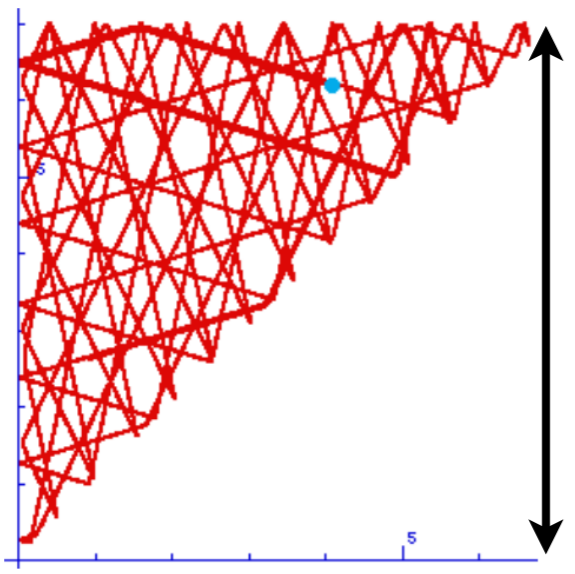
$(v_1, v_2) = (1, -1)$



$(v_1, v_2) = (1, 0)$

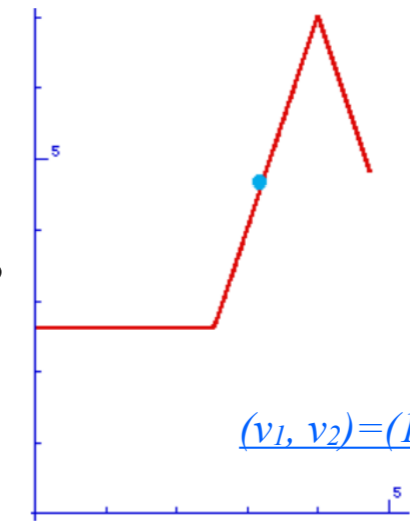
BounceIt
 $m_1:m_2 = 3:1$
 y_2 vs y_1 plots

*Collisions for
mass ratio
 $m_1:m_2=3:1$*



Space-space (y_1, y_2) paths

$(v_1, v_2)=(1, -1)$

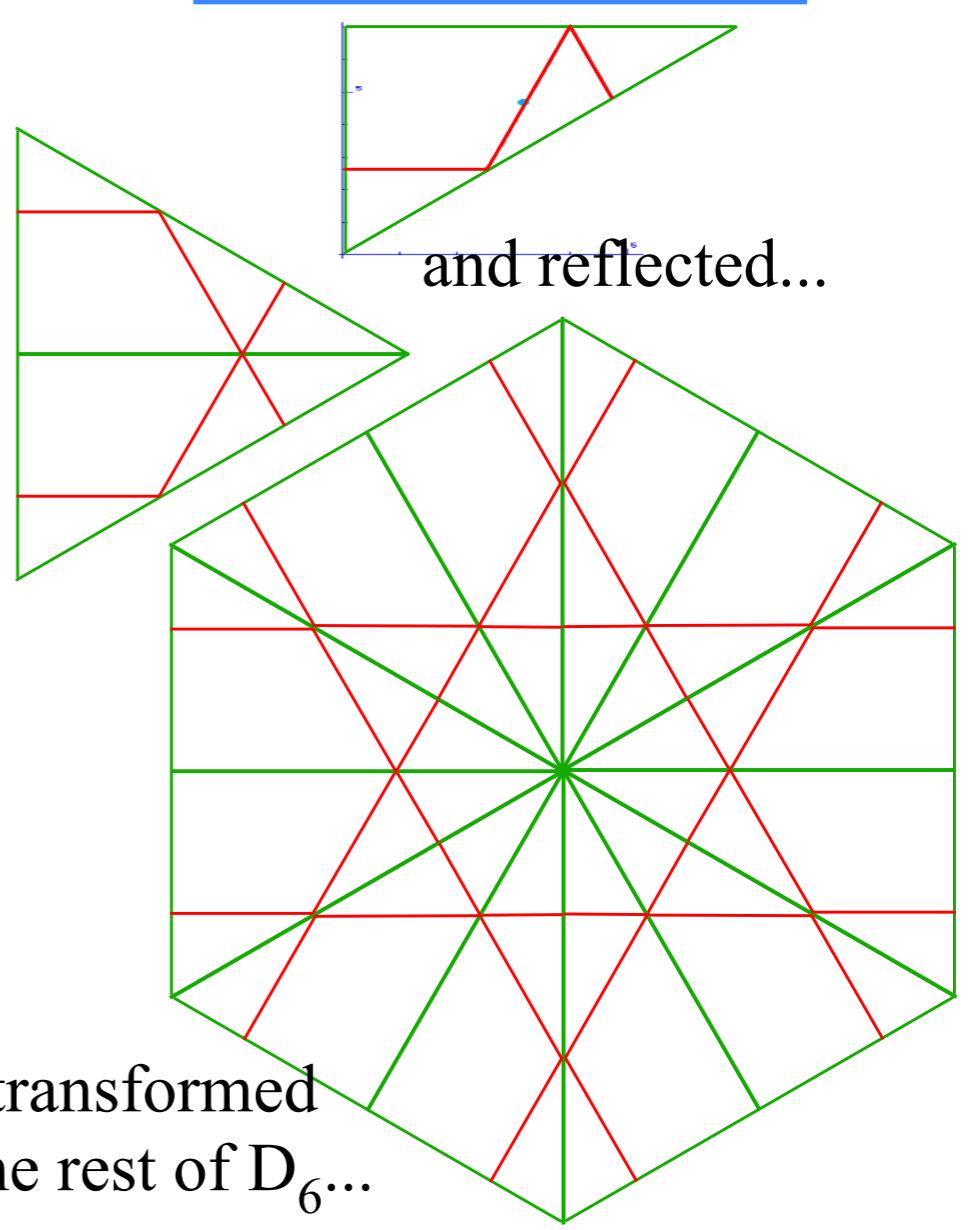
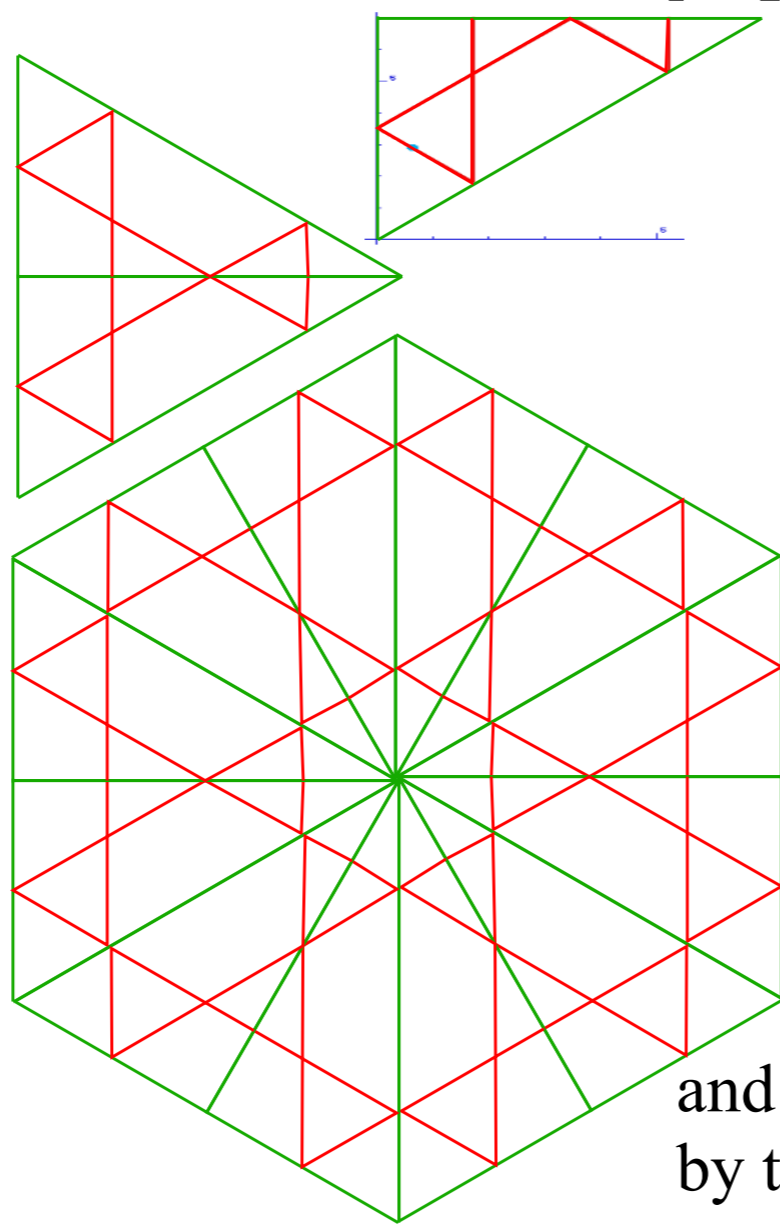
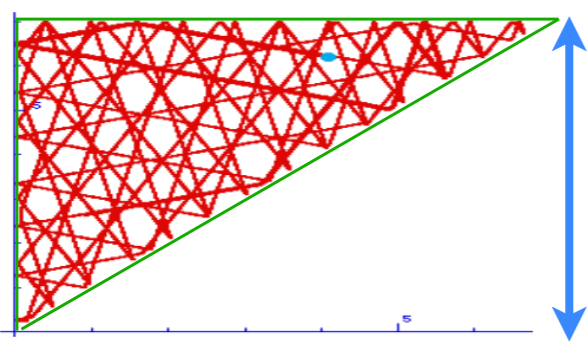


$(v_1, v_2)=(1, 0)$

BounceIt
Web Simulations
 $m_1:m_2=3:1$
 y_2 vs y_1 plots

Space-space (y_1, y_2) paths scaled down by $1/\sqrt{3}$...

*Scaled y down by
 $1/\sqrt{3}=0.577$*

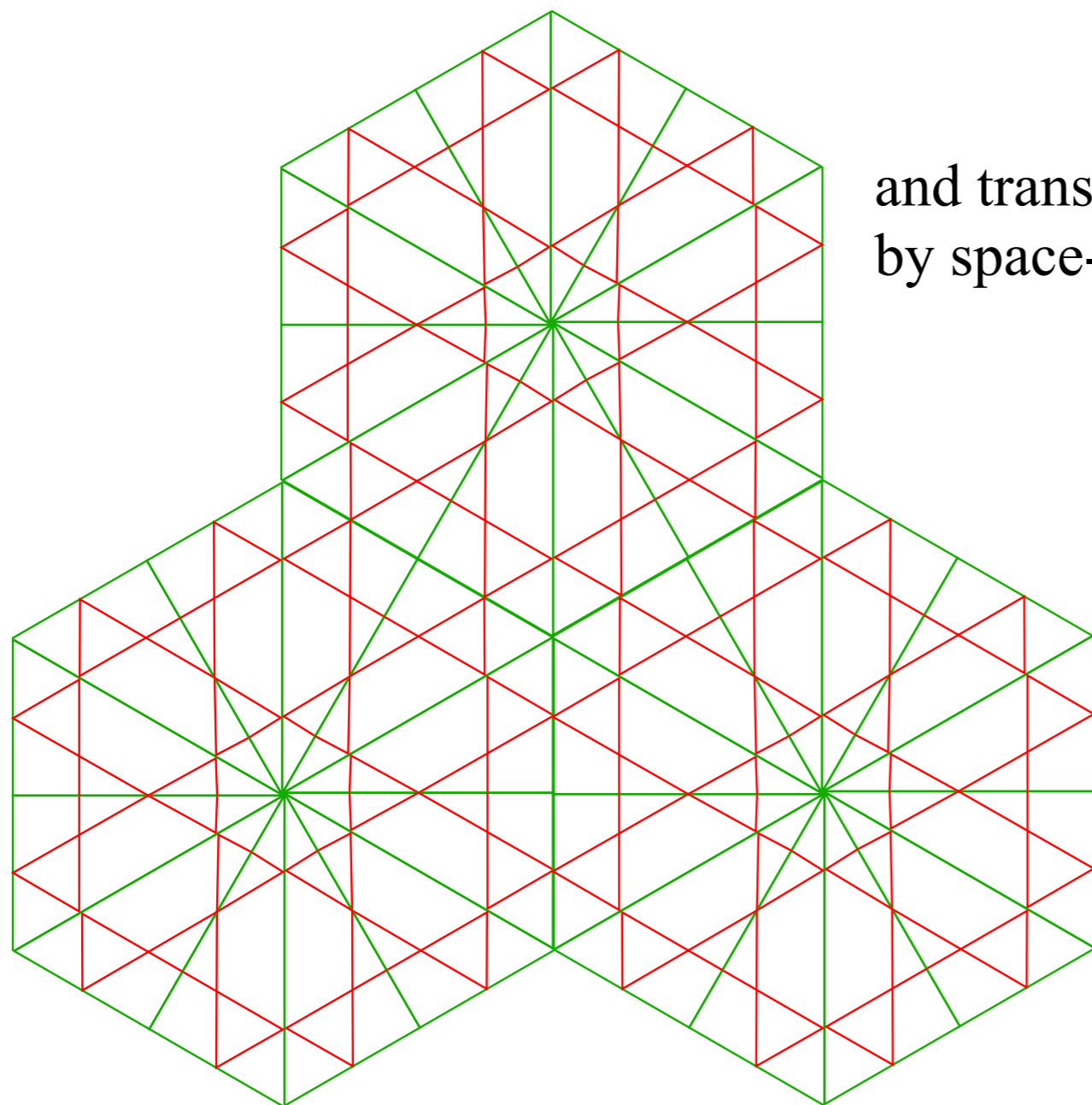


and reflected...

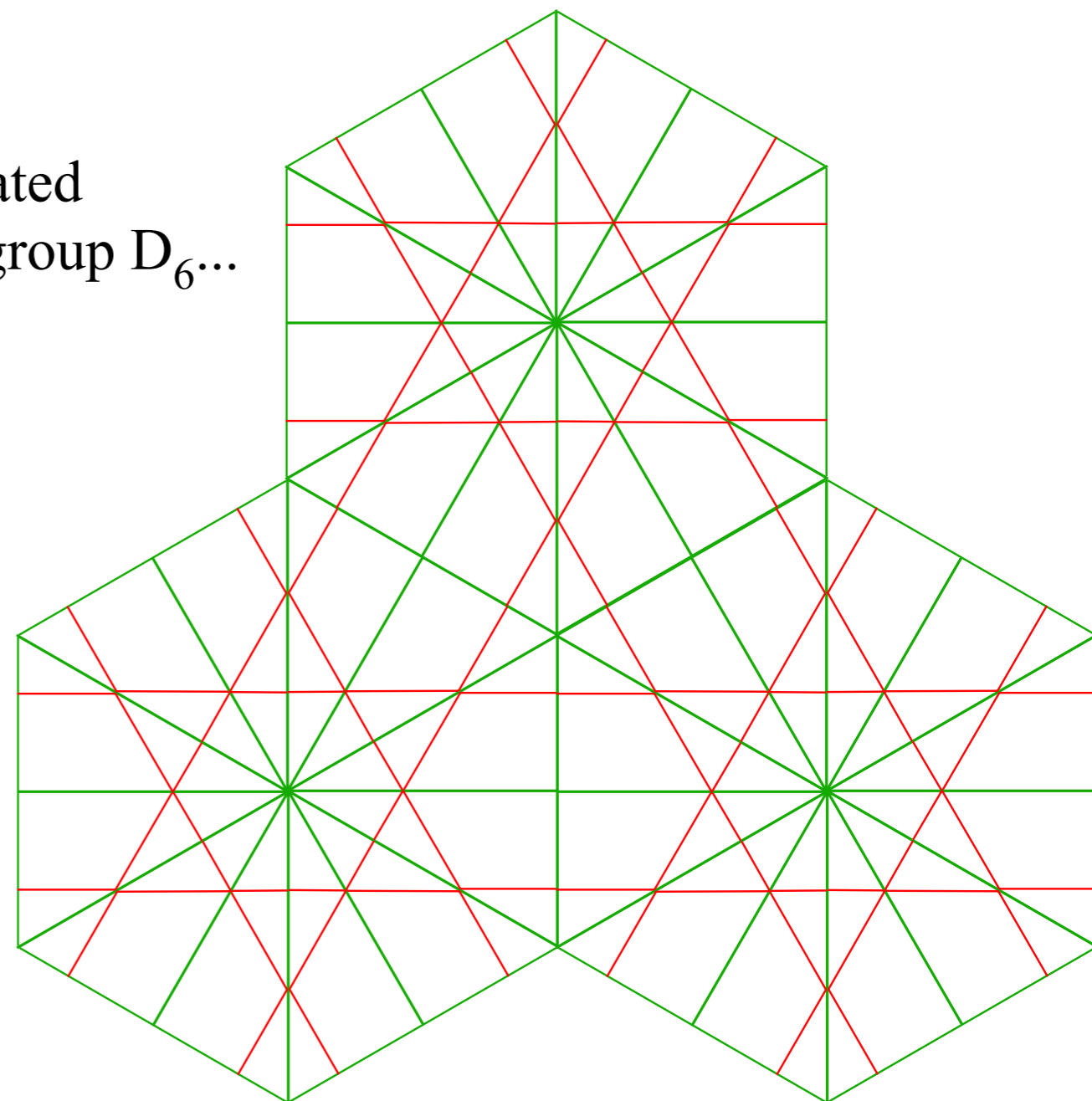
*..or could have scaled x up by
 $\sqrt{3}=1.732$*

and transformed
by the rest of D_6 ...

*Collisions for
mass ratio
 $m_1:m_2=3:1$*



and translated
by space-group D_6 ...



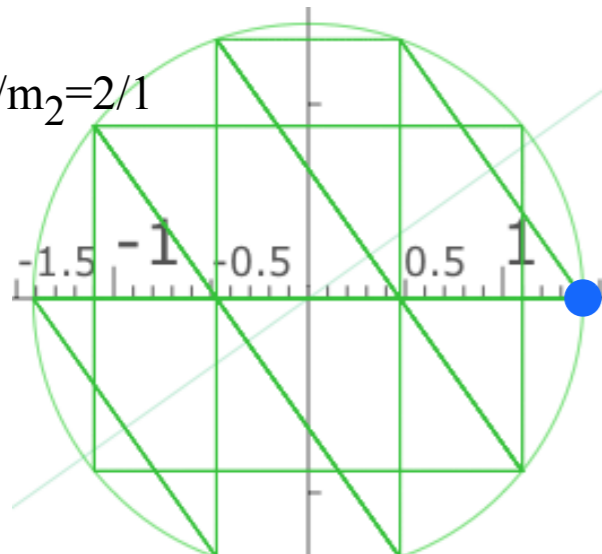
...they're just straight lines going forever.

Initial velocity $v_1=1, v_2=0$

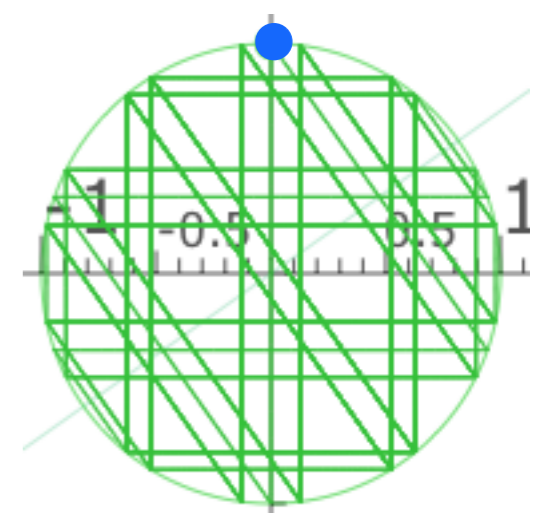
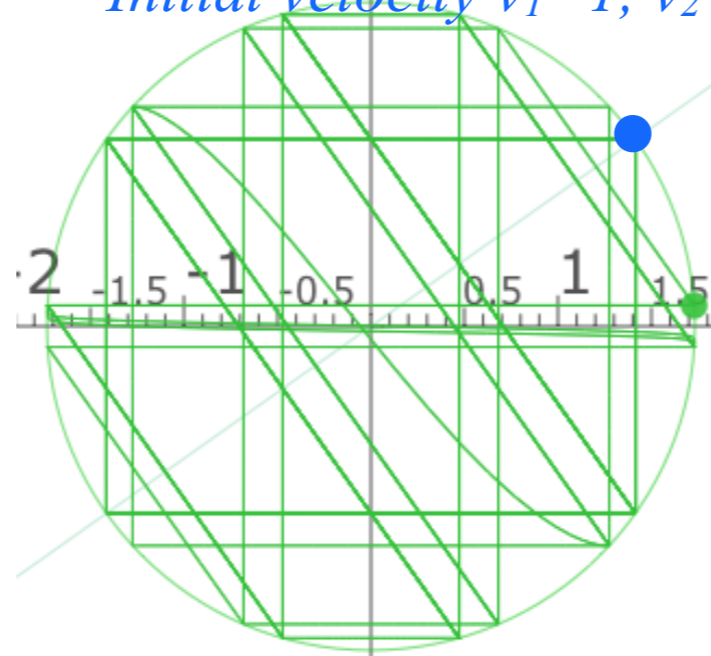
Initial velocity $v_1=1, v_2=1$

Initial velocity $v_1=0, v_2=1$

$M_1/m_2=2/1$



$\phi = \text{Acos}(M_1-m_2)/(M_1+m_2)$
 $= \text{Acos}(1/3) = 70.53^\circ$

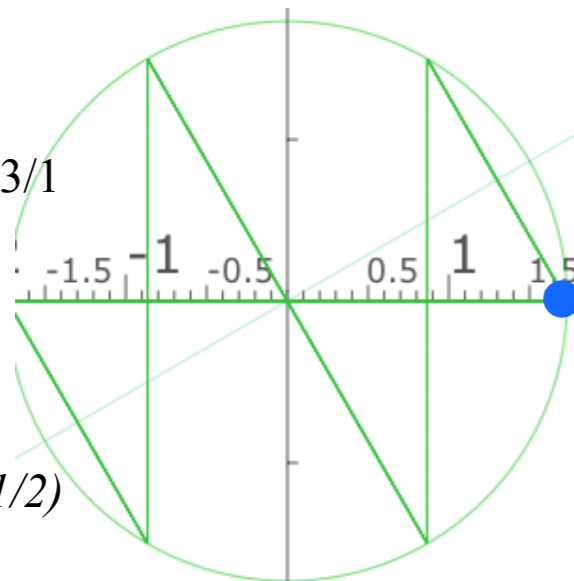


BounceIt Web Simulation

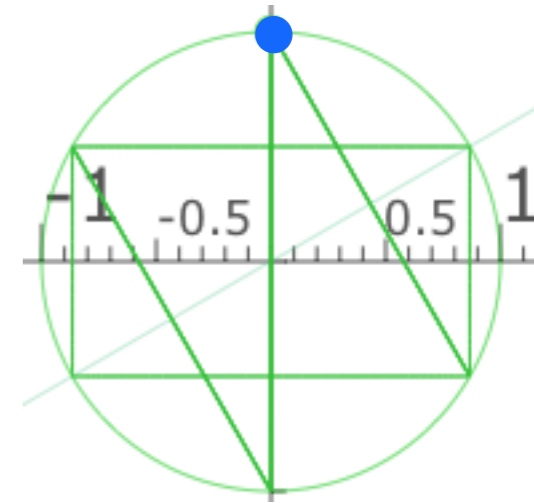
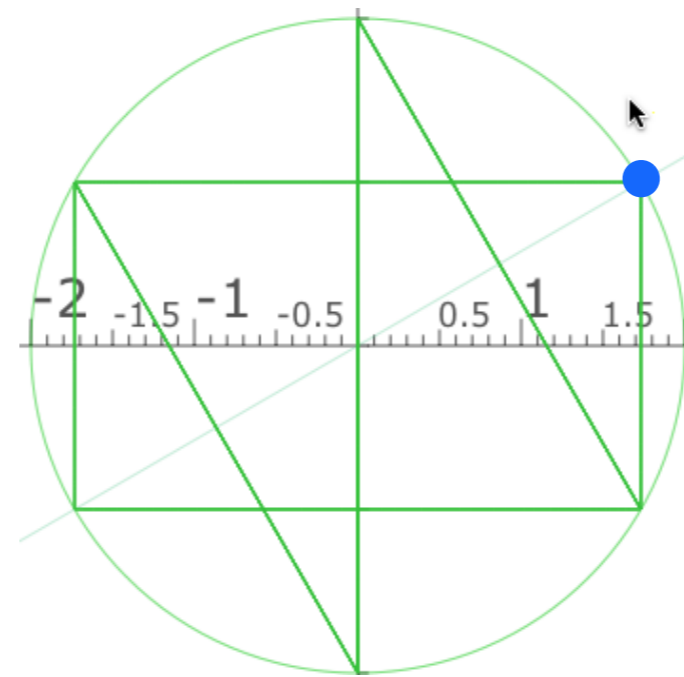
$m_1:m_2 = 3:1$ $(v_1, v_2) = (0, 1)$

Estrangian plot V_2 vs V_1

$M_1/m_2=3/1$

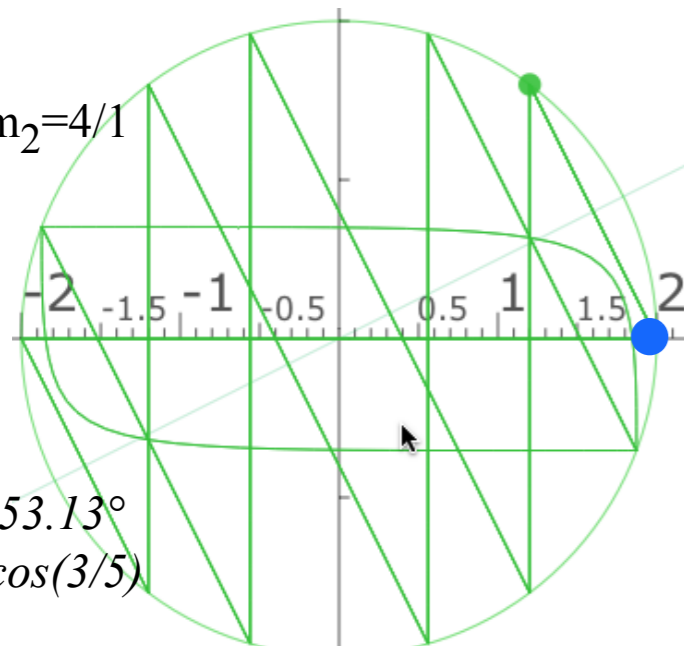


$\phi = 60^\circ$
 $= \text{Acos}(1/2)$

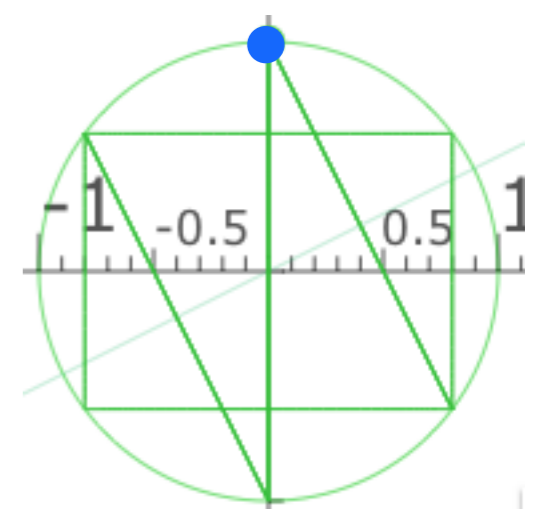
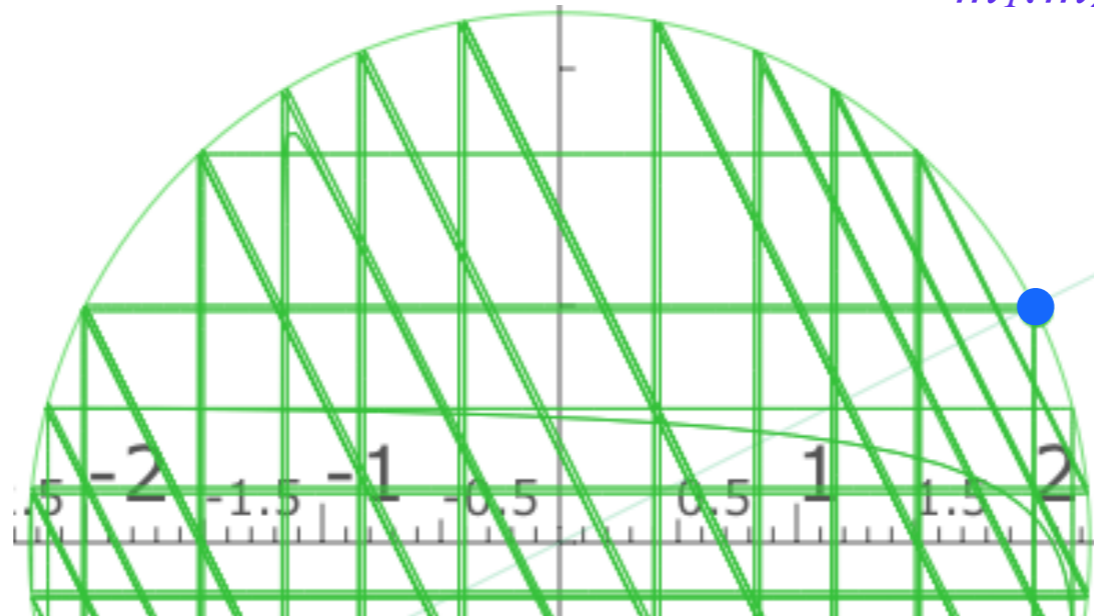


Collisions for
 mass ratio
 $m_1:m_2 = 3:1$

$M_1/m_2=4/1$



$\phi = 53.13^\circ$
 $= \text{Acos}(3/5)$



Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l'Etrangian, and Hamiltonian mechanics later on

Reflections in the clothing store: "It's all done with mirrors!"

Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$



Ellipse rescaling-geometry and reflection-symmetry analysis

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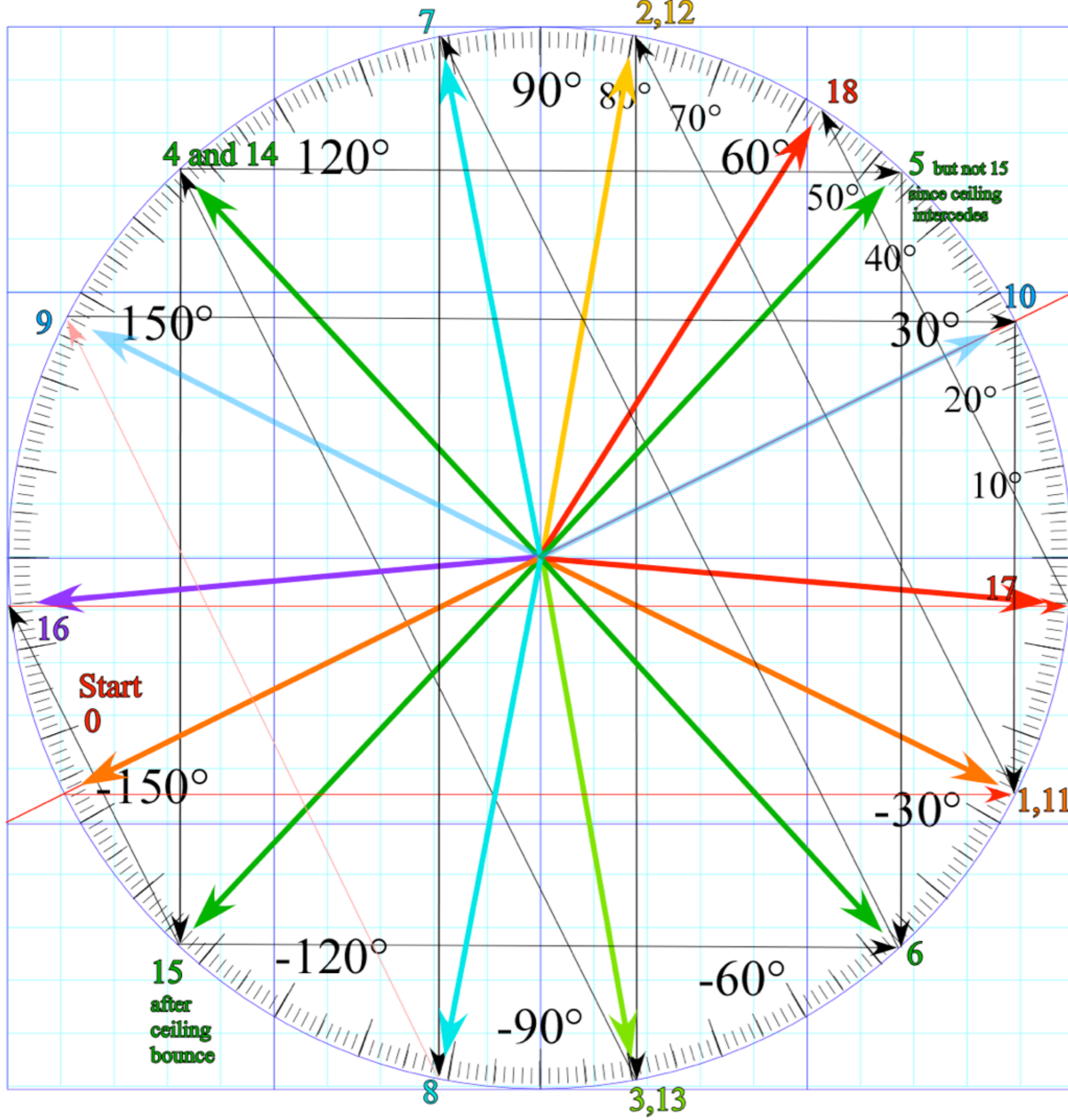
Introducing hexagonal symmetry $D_6 \sim C_{6v}$ (Resulting for $m_1/m_2=3$)

Group multiplication and product table

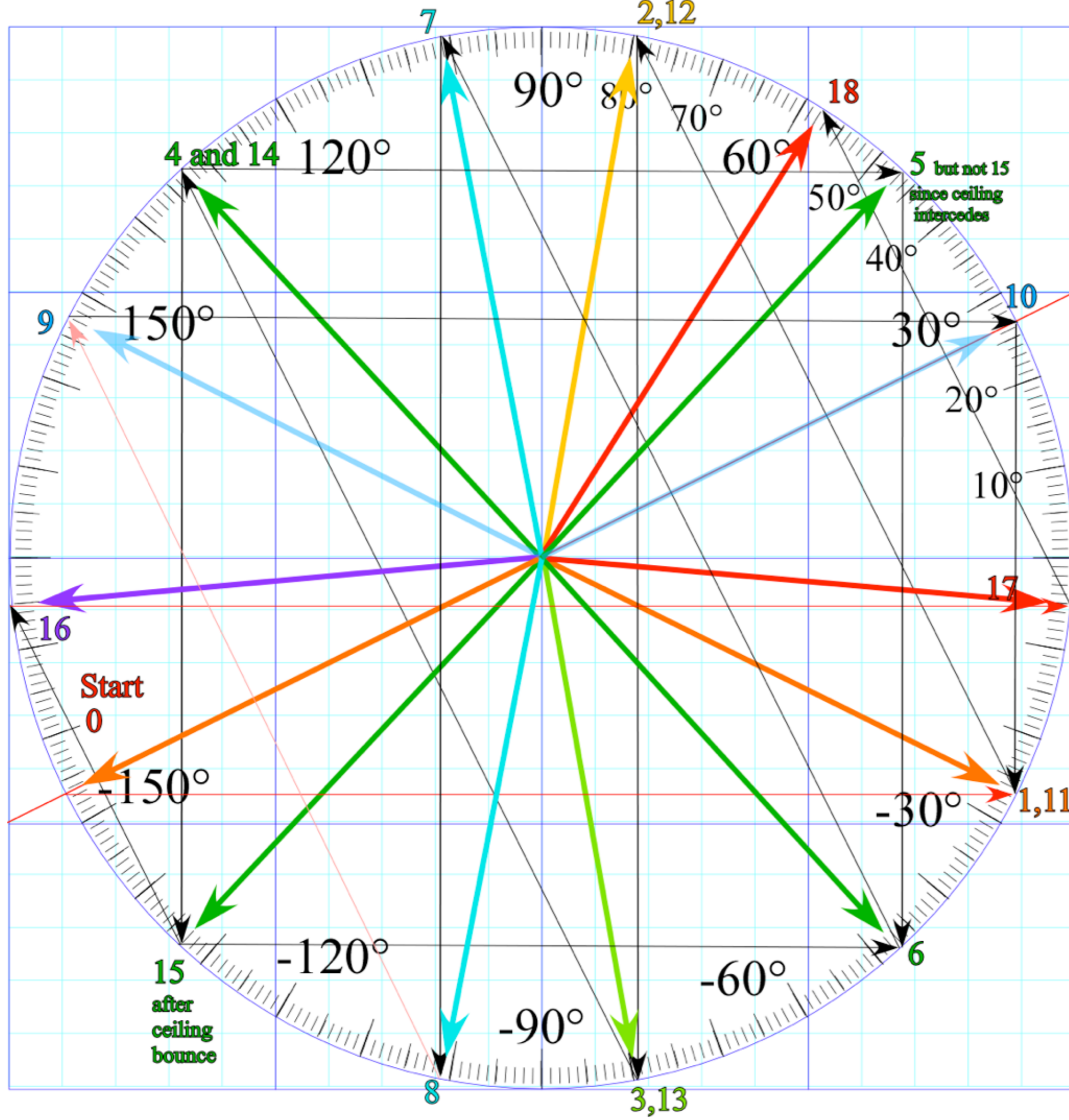
Classical collision paths with $D_6 \sim C_{6v}$ (Resulting from $m_1/m_2=3$)

Other not-so-symmetric examples: $m_1/m_2=4$ and $m_1/m_2=7$

*Collisions for
mass ratio
 $m_1:m_2=4:1$*



Collisions for
mass ratio
 $m_1:m_2=4:1$



$$\cos \theta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{m_1 - 1}{m_1 + 1} = \frac{3}{5} = 0.6$$

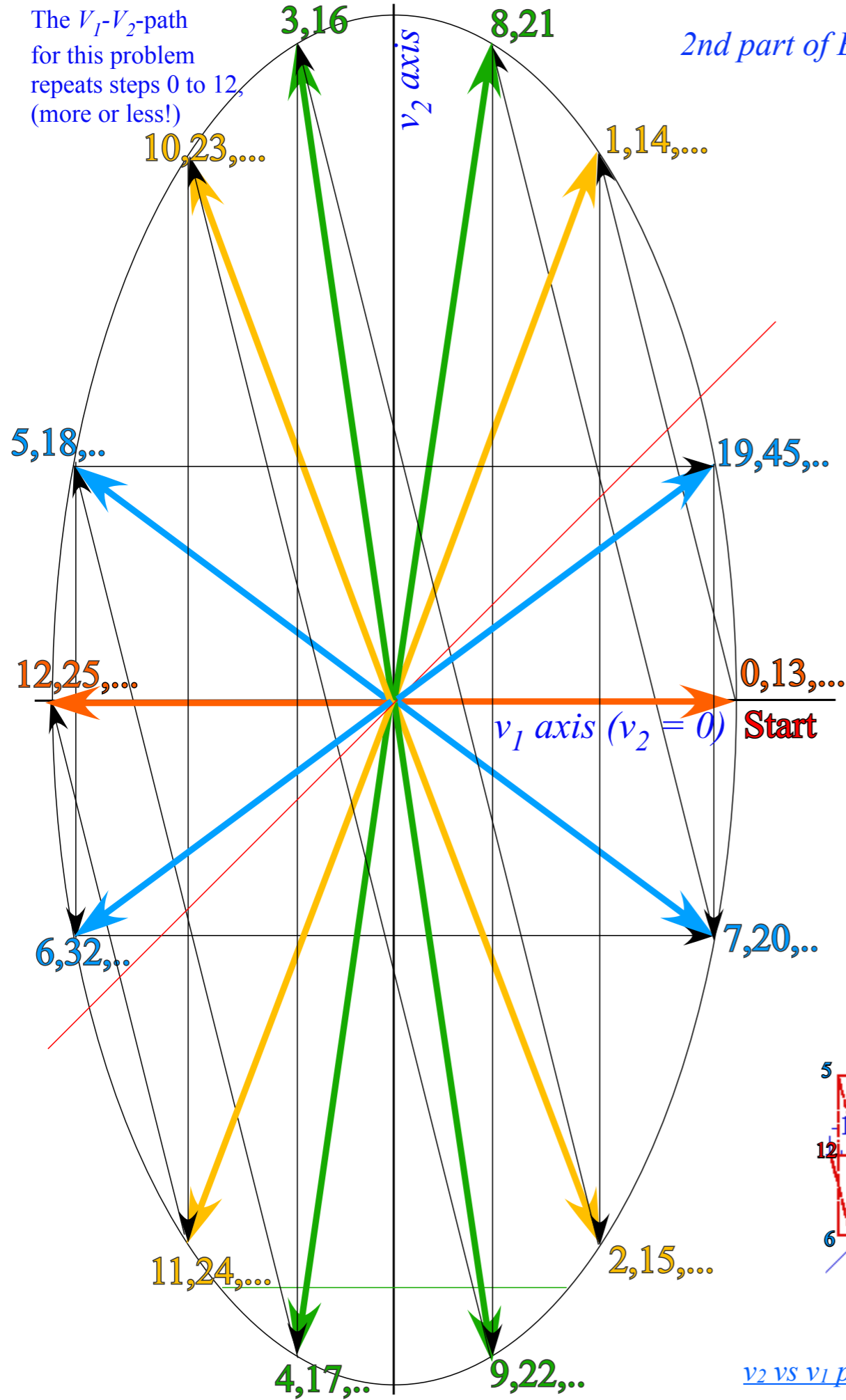
$$\theta = 53.13^\circ$$

$$\frac{m_1}{m_2} = \frac{1 + \cos \theta}{1 - \cos \theta} = \frac{\sin^2 \alpha}{\cos^2 \alpha} = \tan^2 \alpha$$

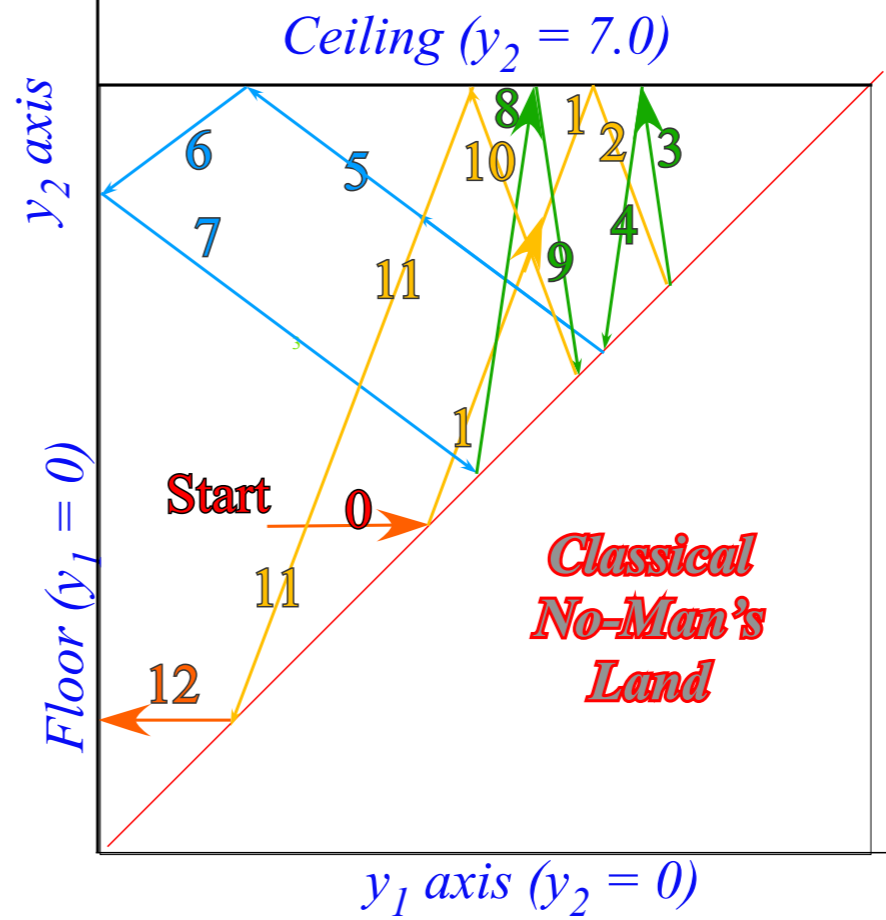
$$\alpha = \theta / 2 = 26.565^\circ$$

$$\frac{m_1}{m_2} = \frac{1 + \cos \theta}{1 - \cos \theta} = \frac{8/5}{2/5} = 4$$

The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)

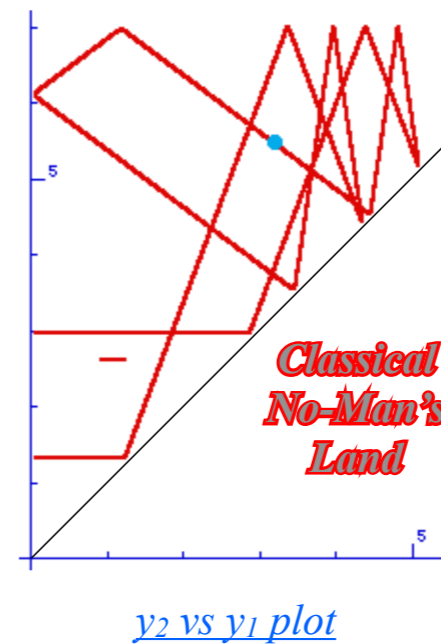
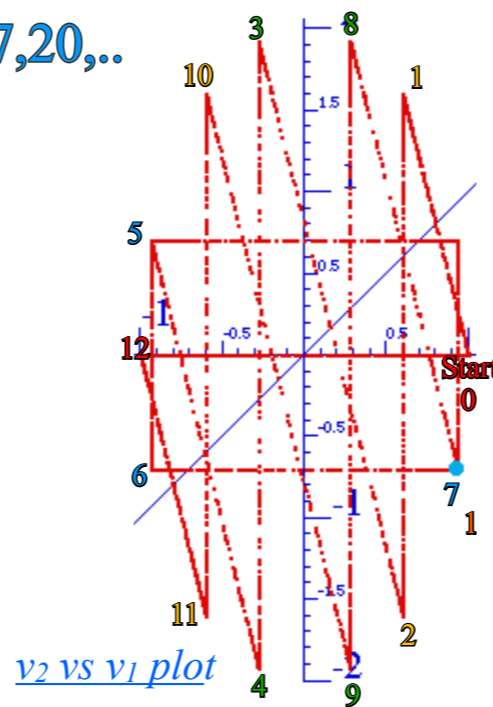


2nd part of Exercise 1.4.1 has pen-ball initial velocity values $v_1(0)=1$ and $v_2(0)=0$ at: $x_1(0)=1.5$ and $x_2(0)=3.0$

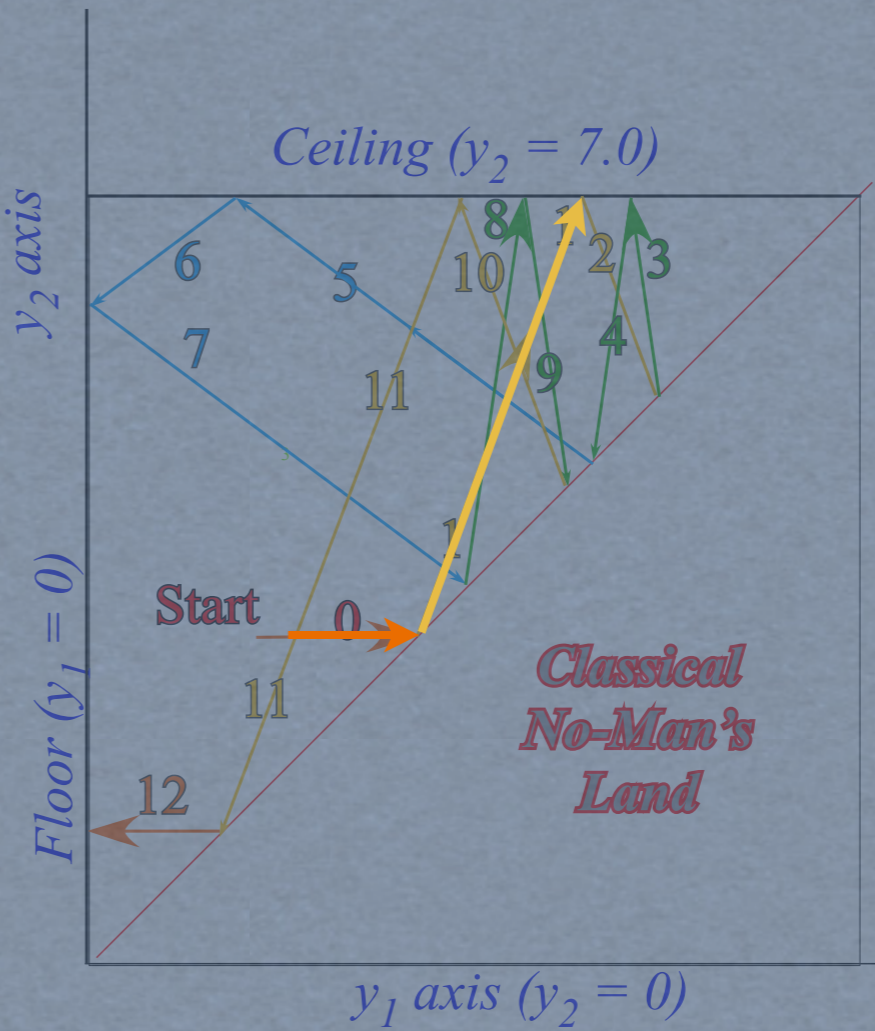
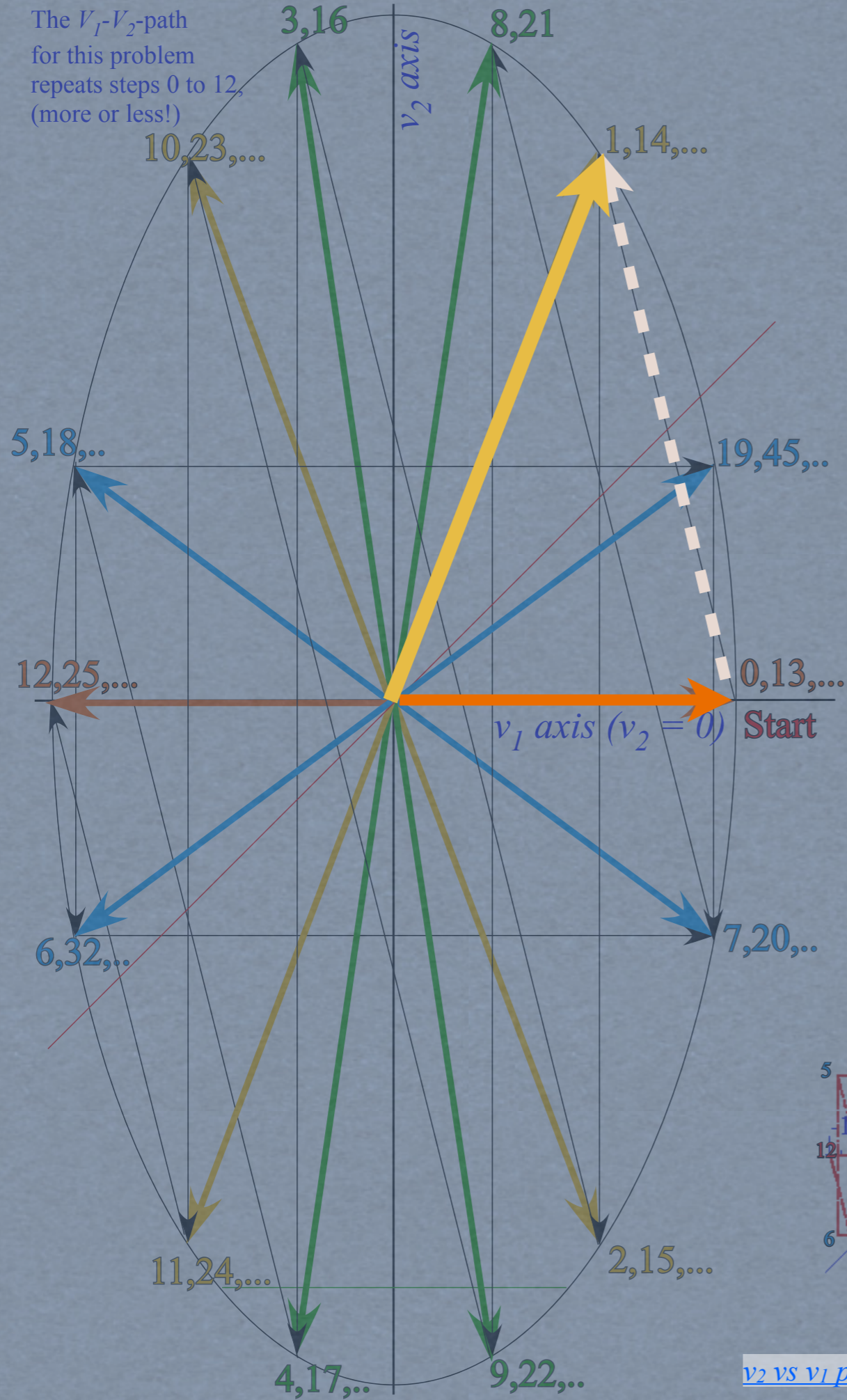


Collisions for mass ratio $m_1:m_2=4:1$

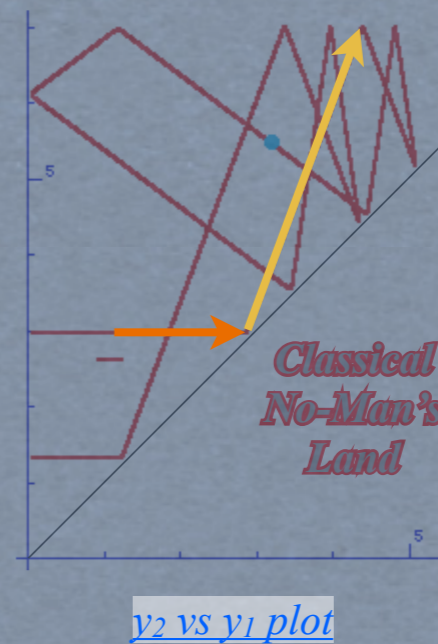
BounceIt Web Simulations
 $m_1:m_2=4:1$ (v_1, v_2)=(1, 0)



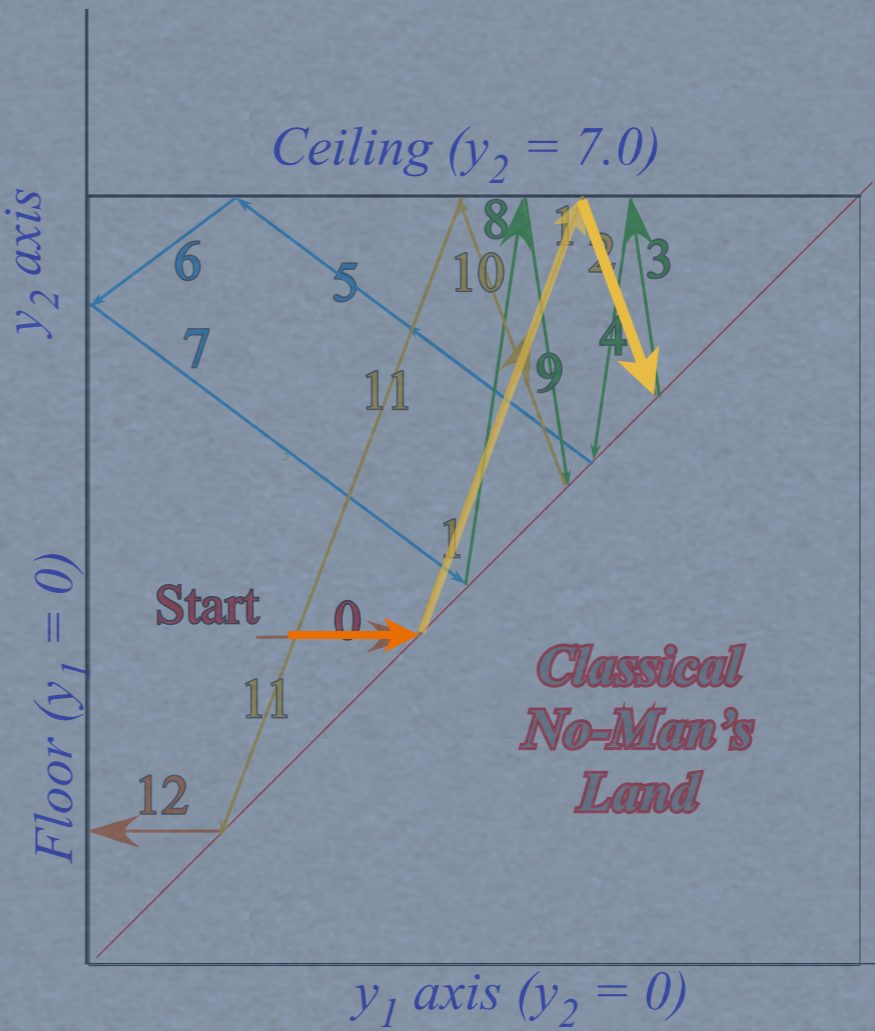
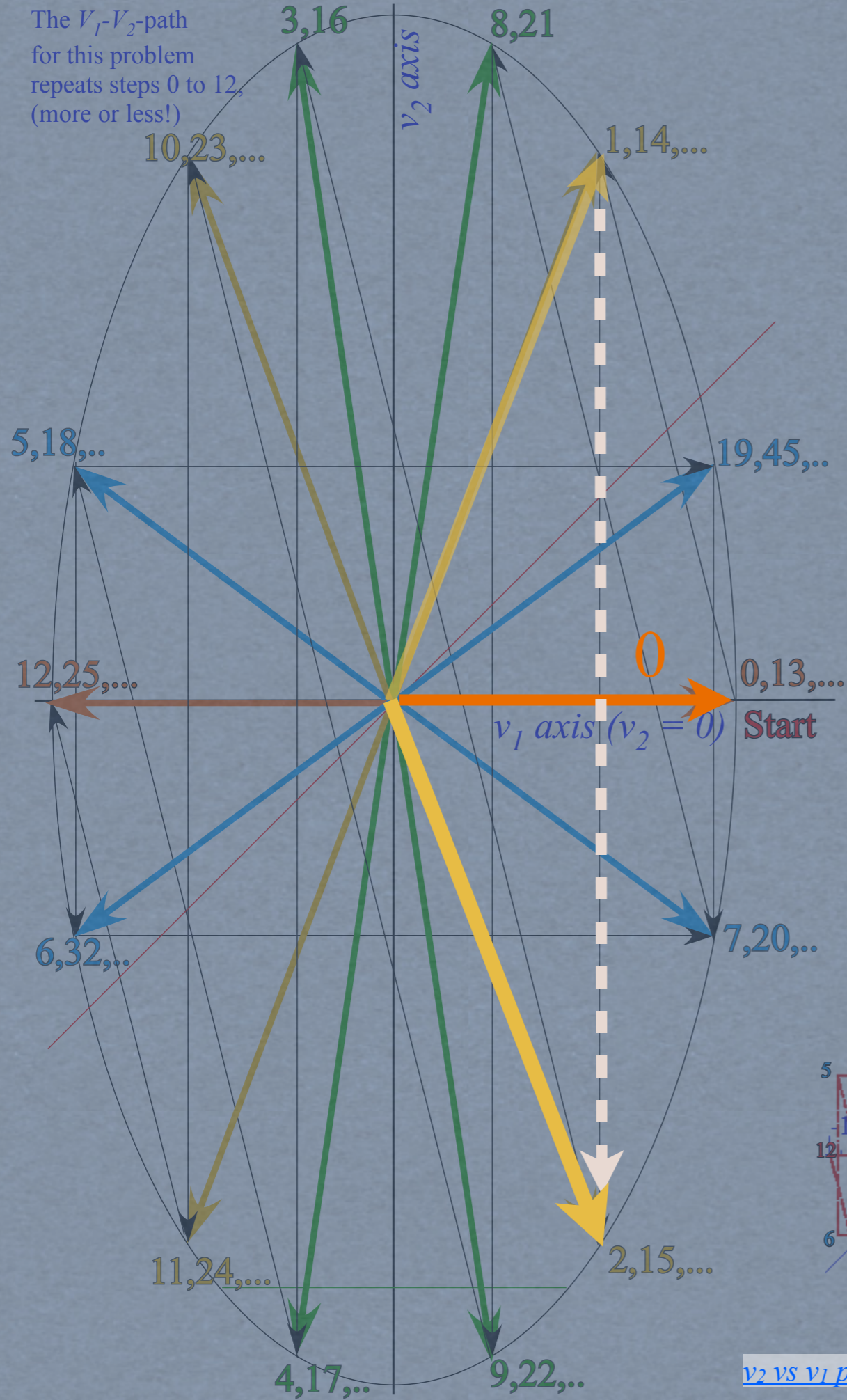
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



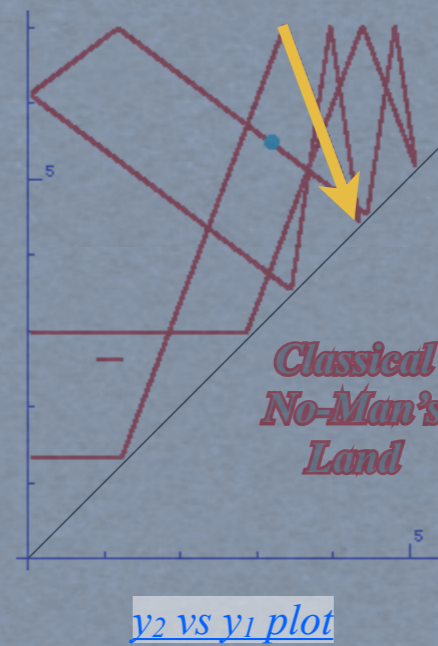
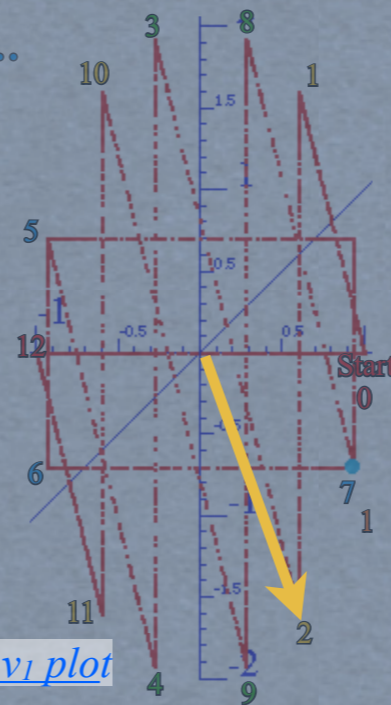
Simulations by *Bouncelt*



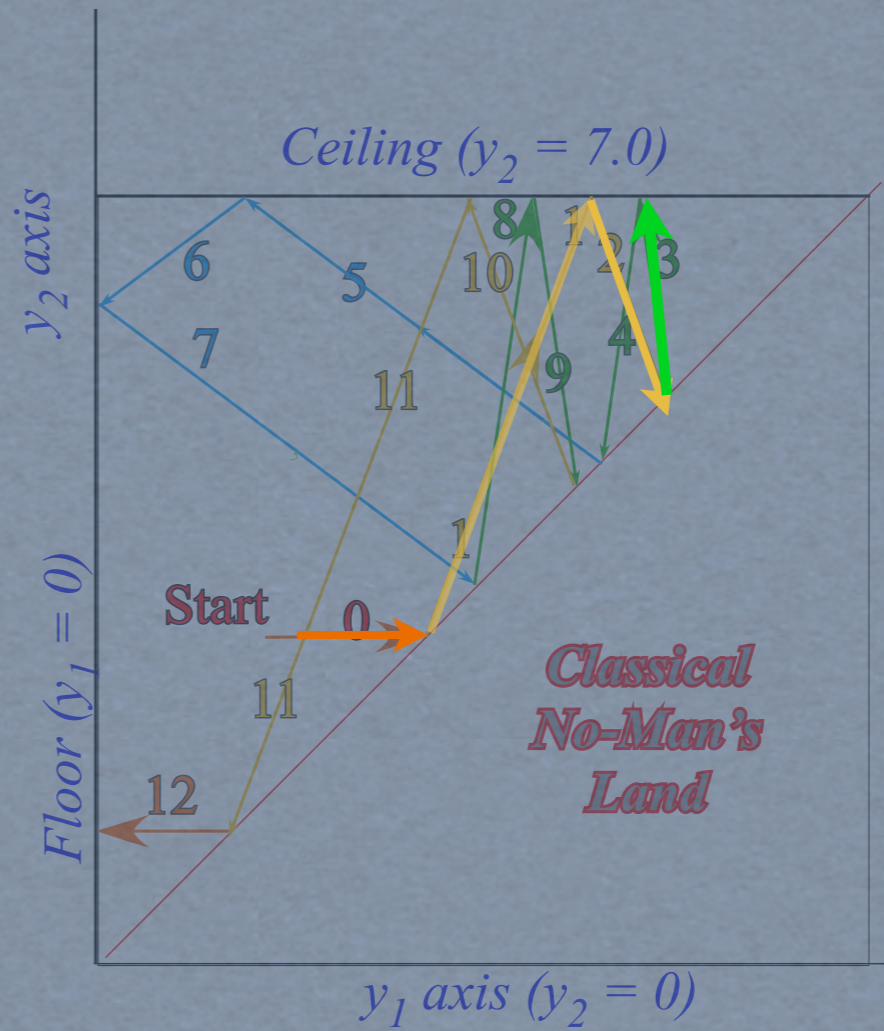
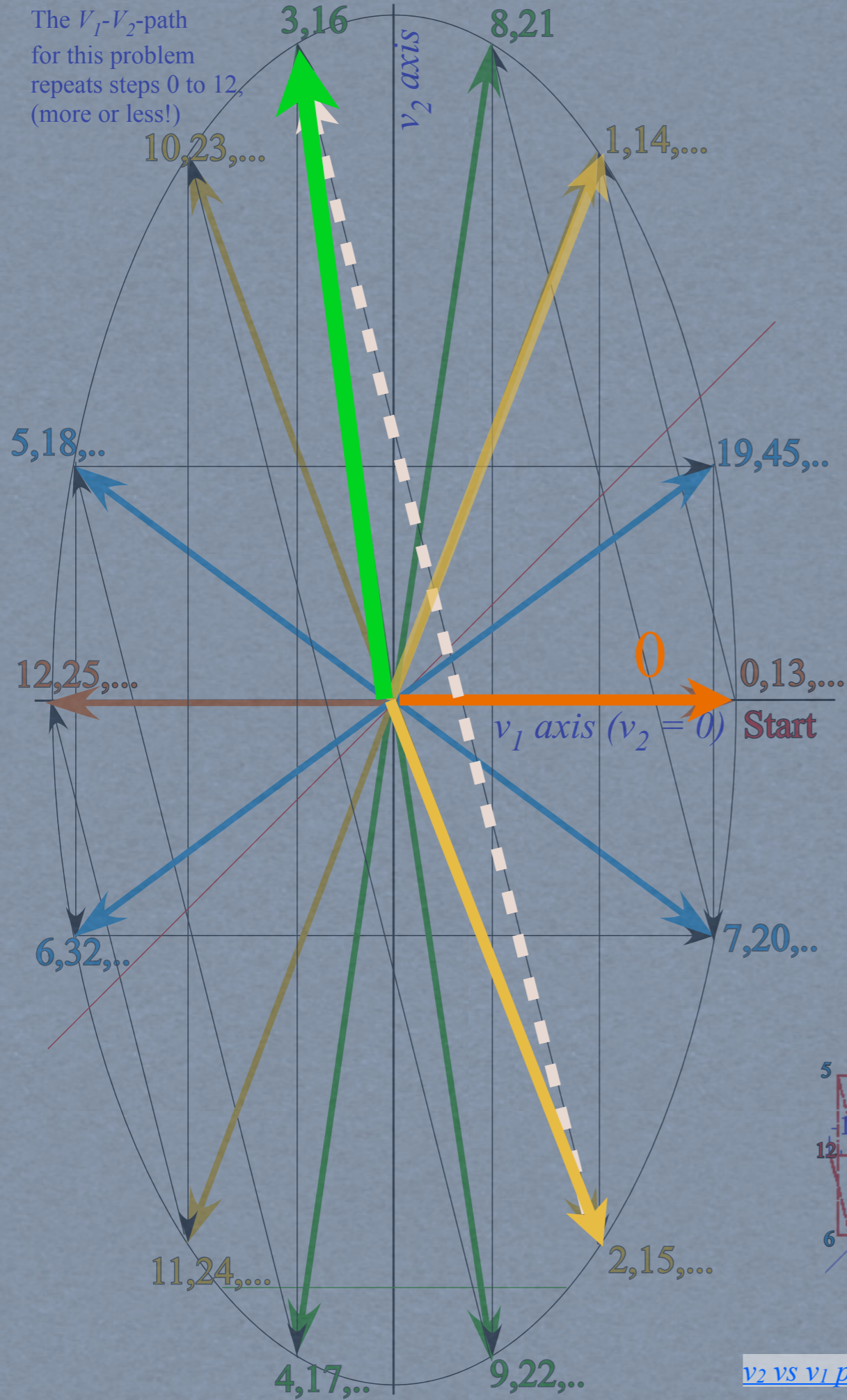
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



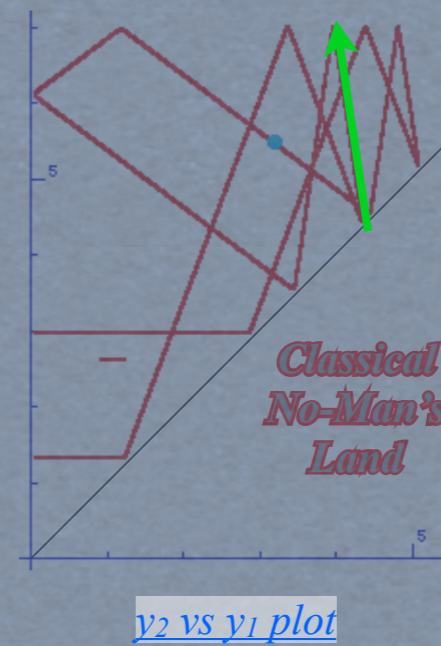
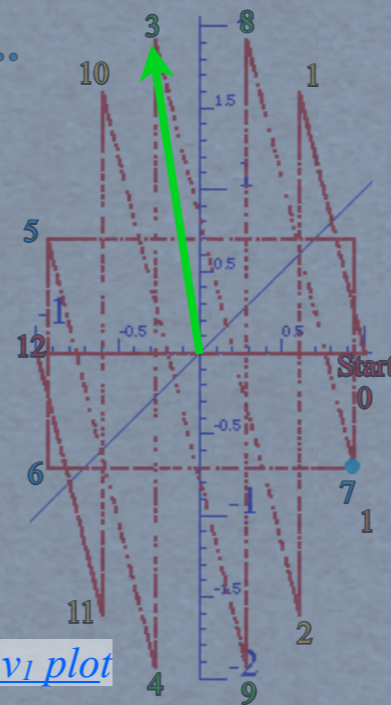
Simulations by *Bouncelt*



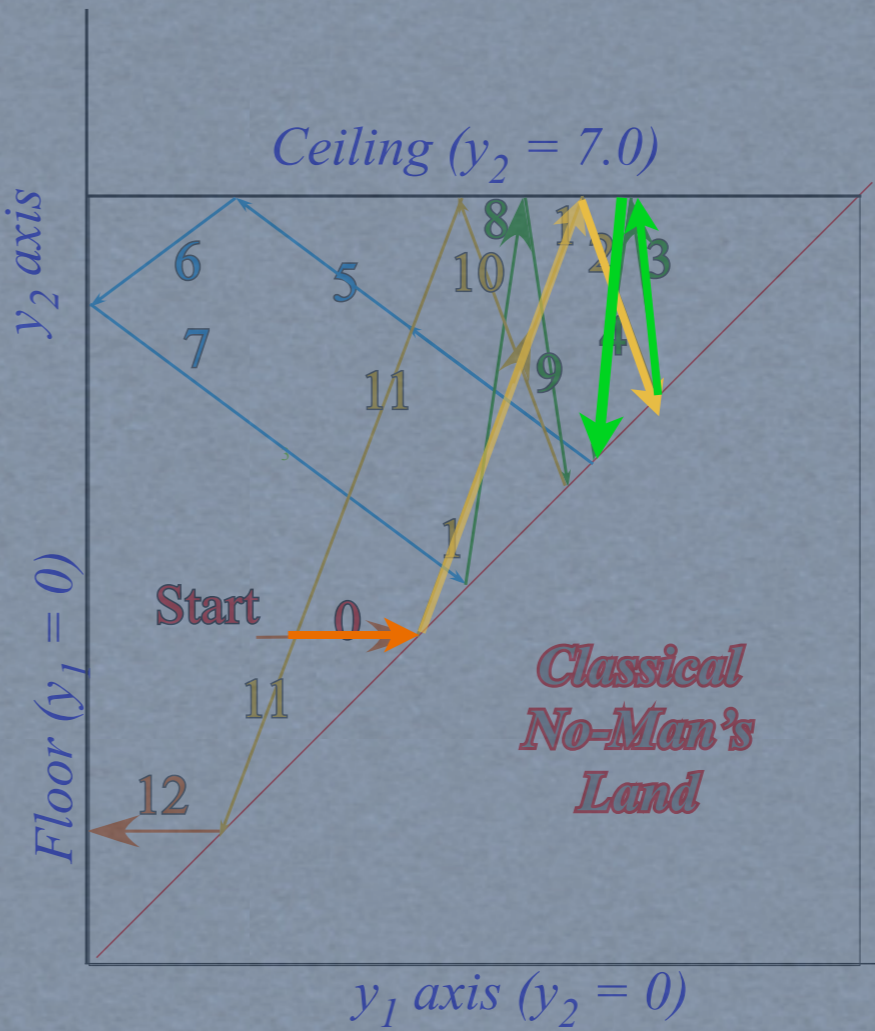
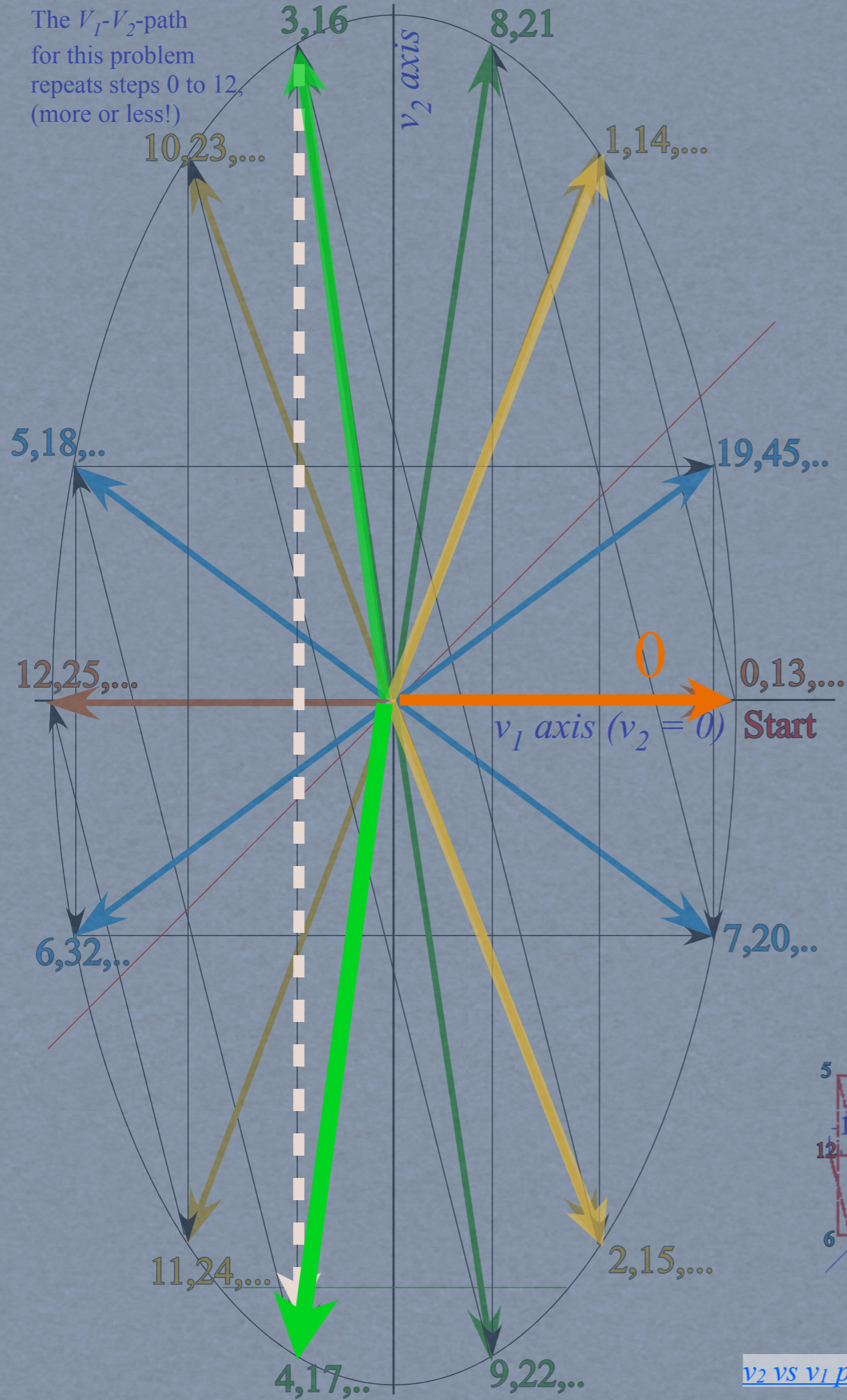
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



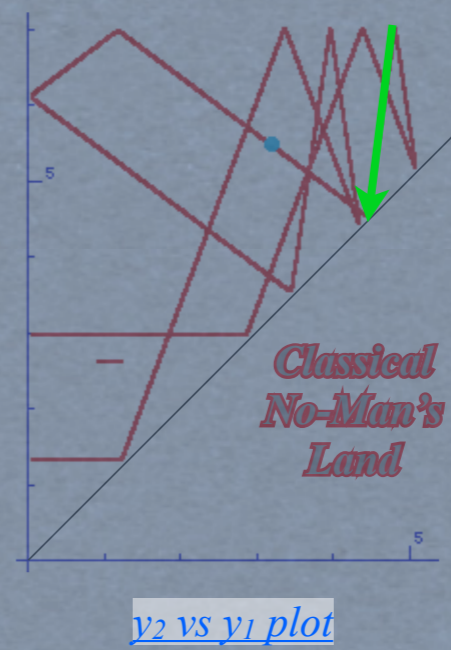
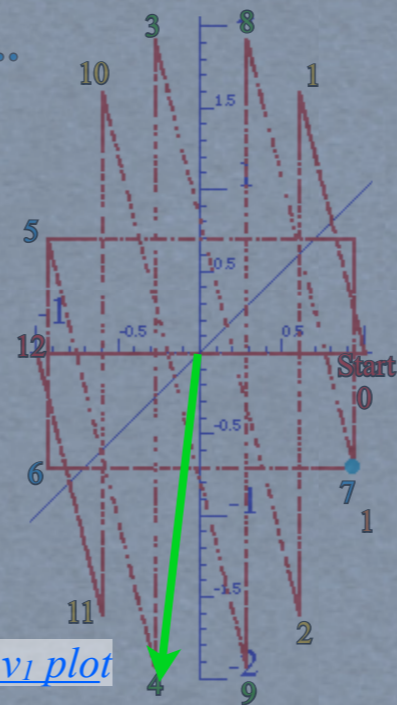
Simulations by *Bouncelt*



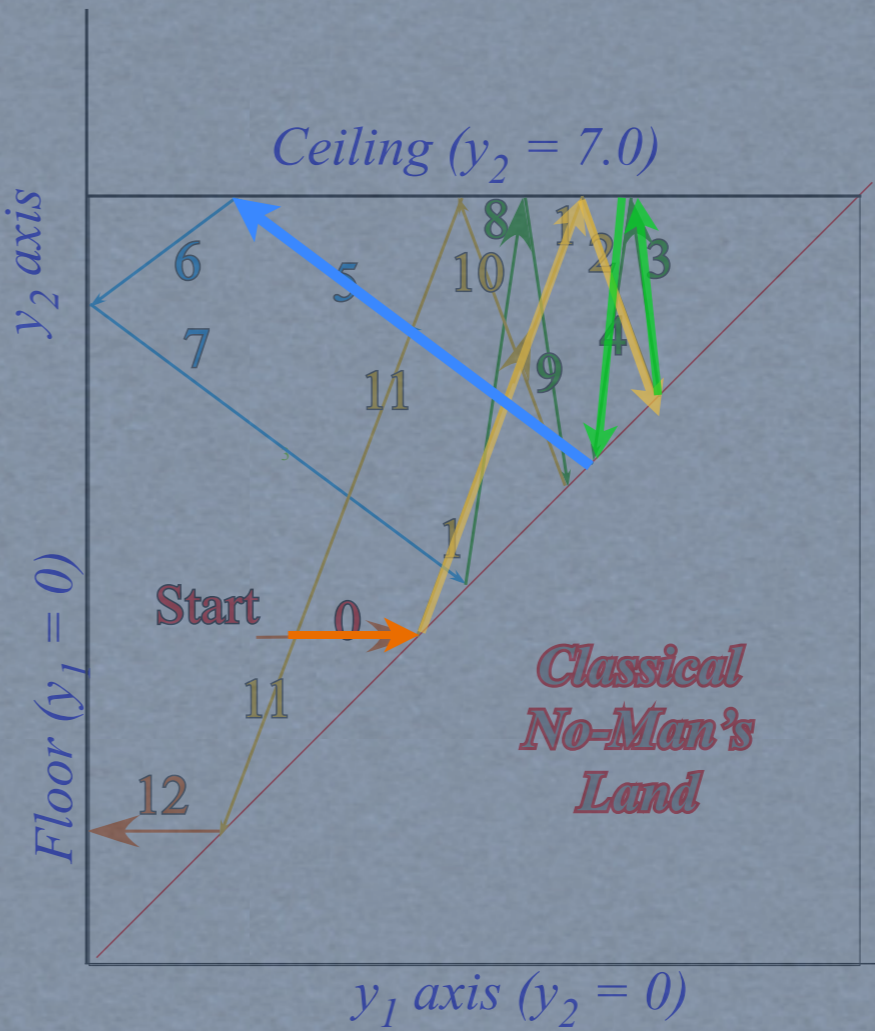
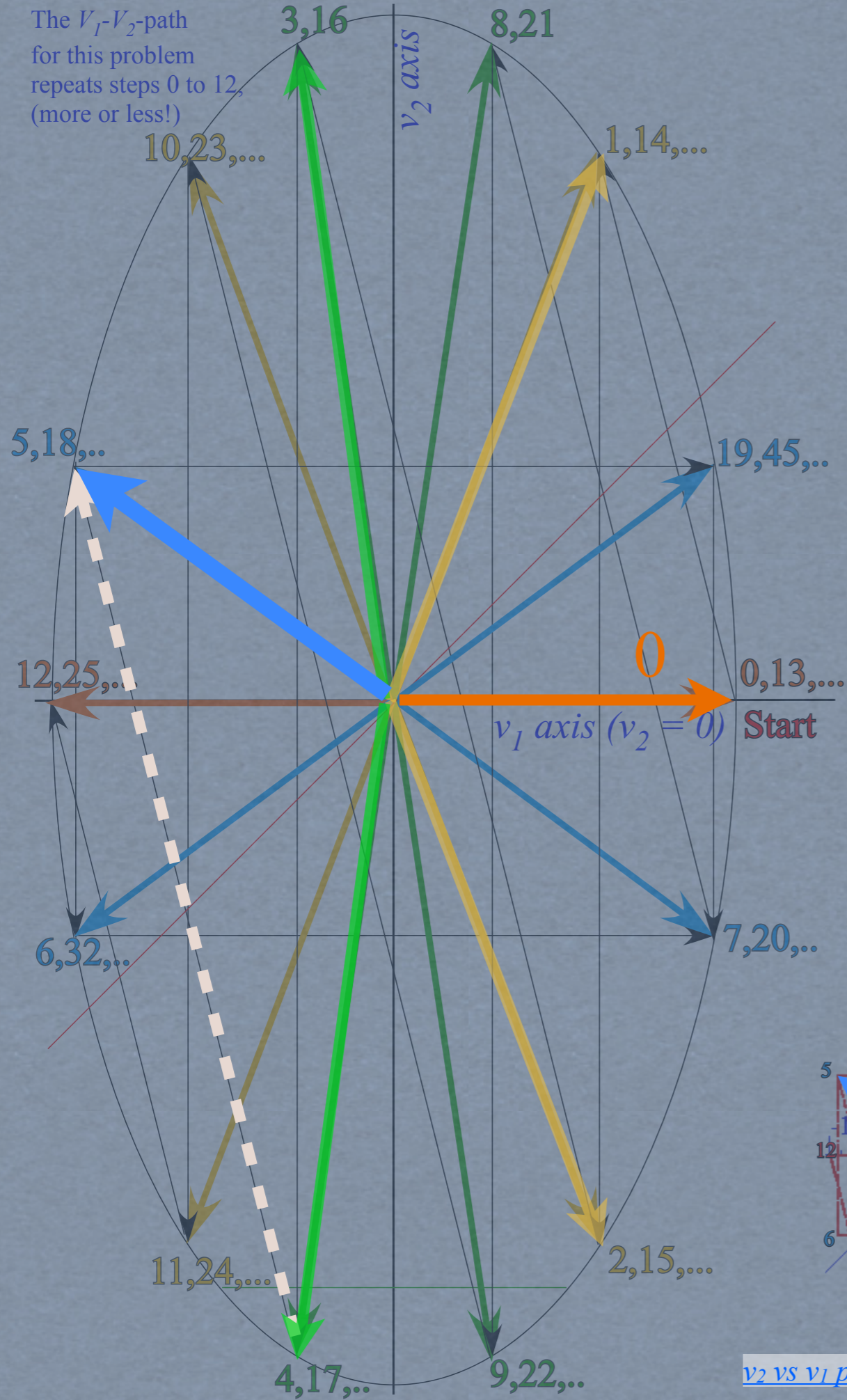
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



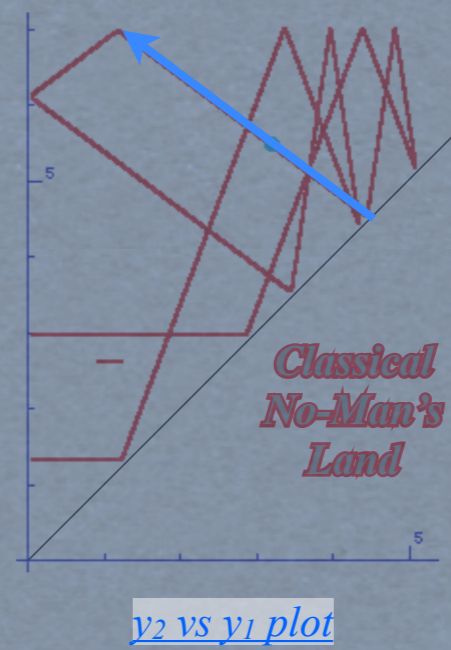
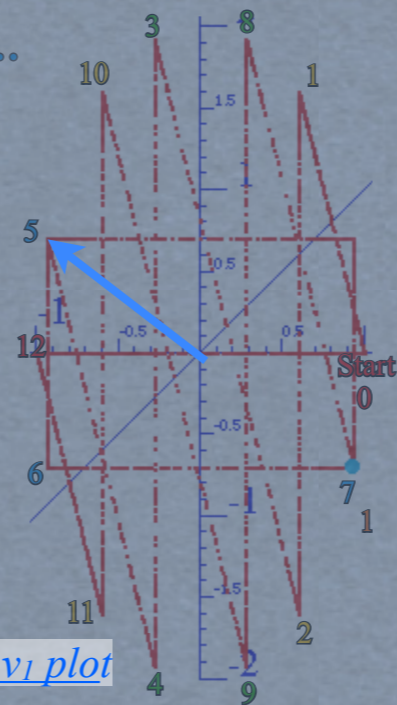
Simulations by *Bouncelt*



The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



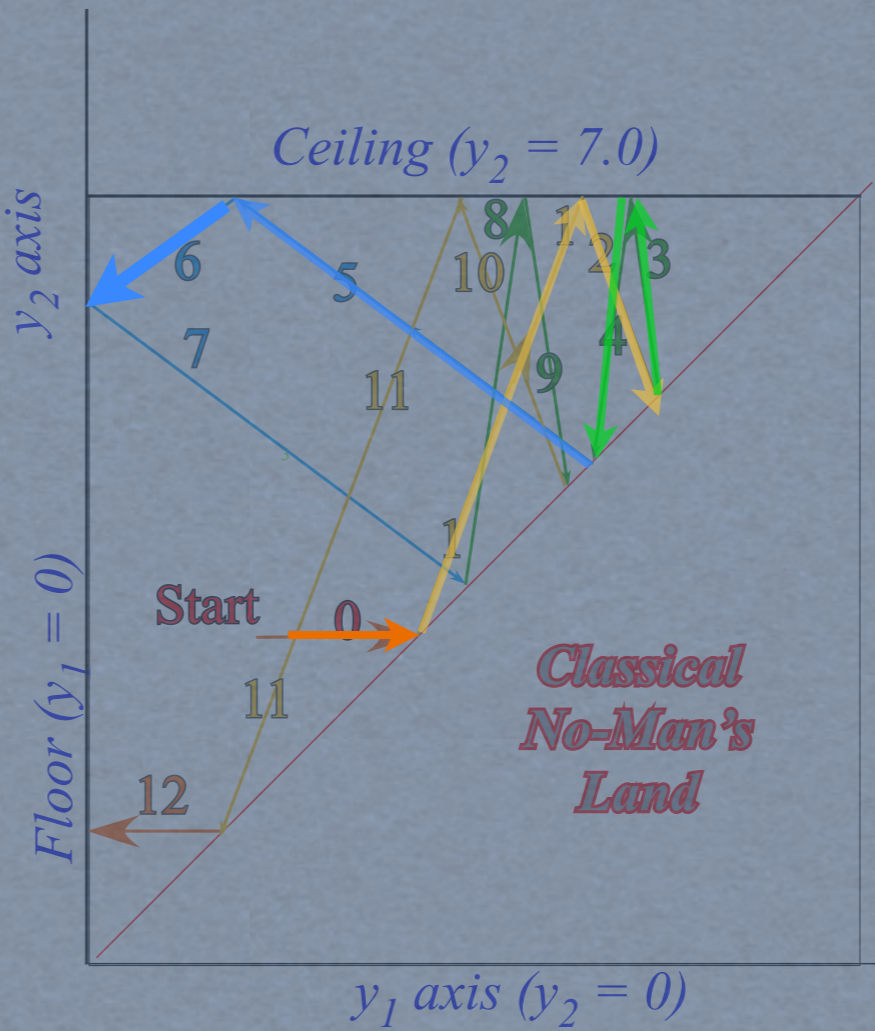
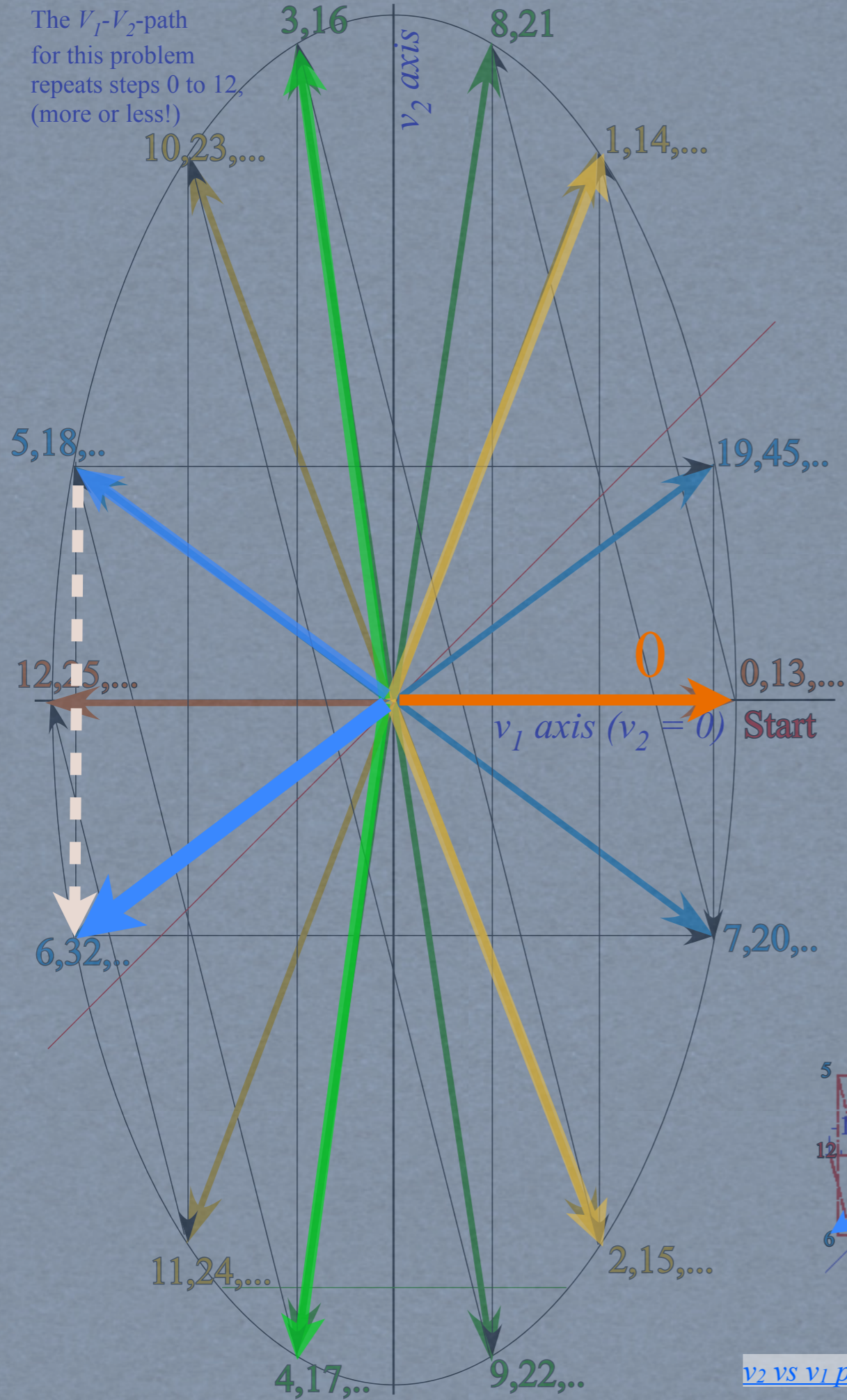
Simulations by *Bouncelt*



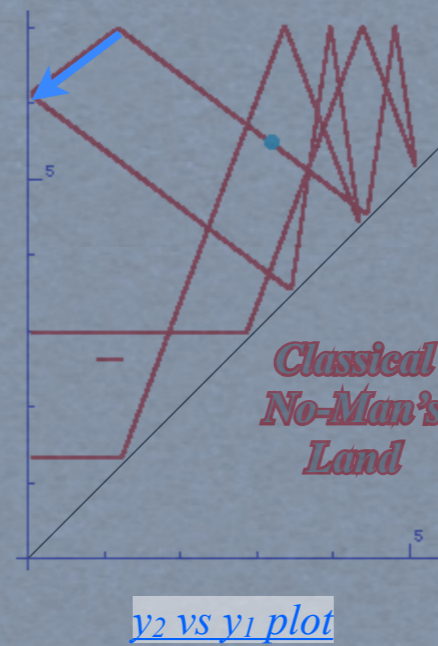
v2 vs v1 plot

y2 vs y1 plot

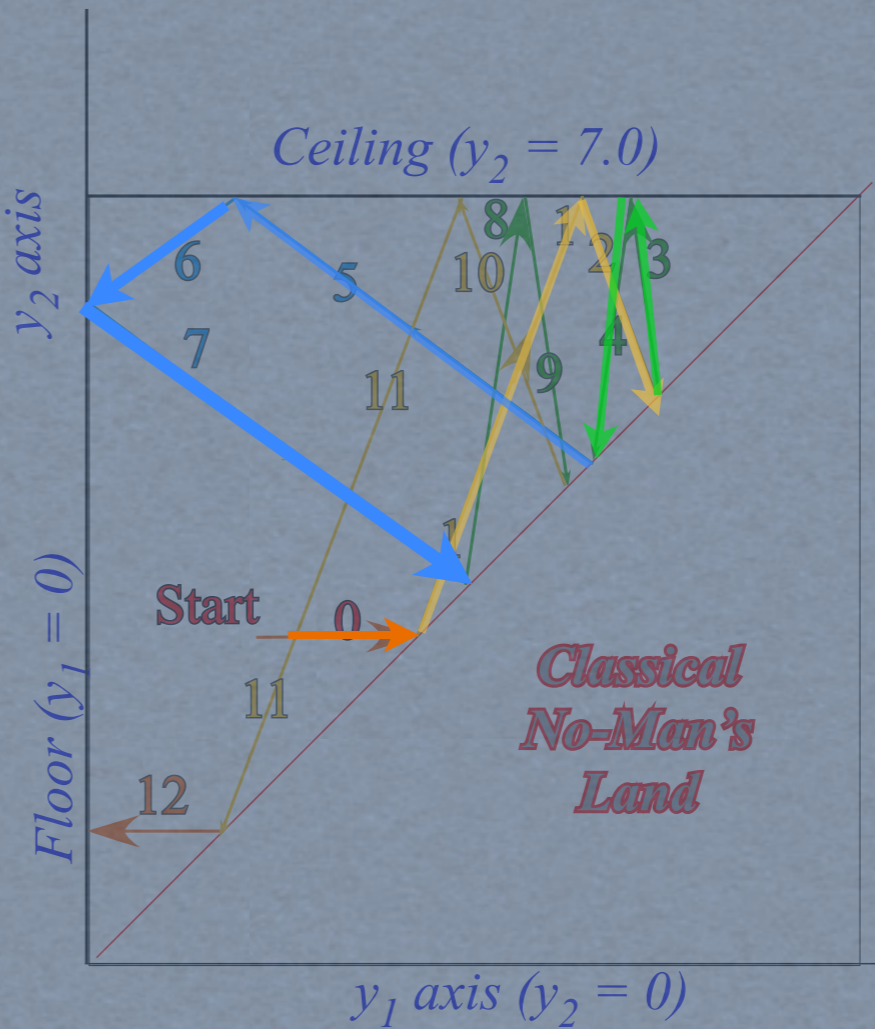
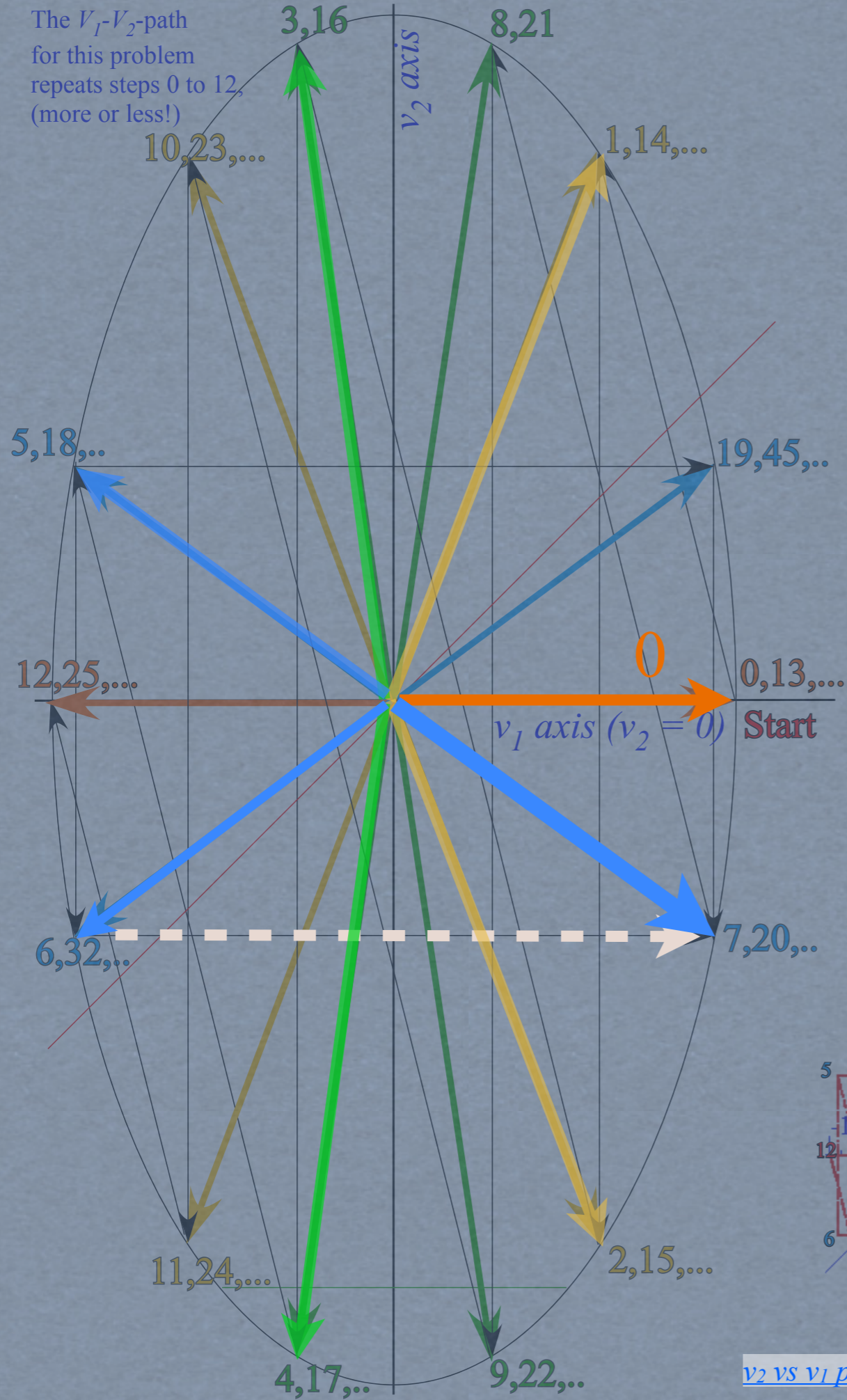
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



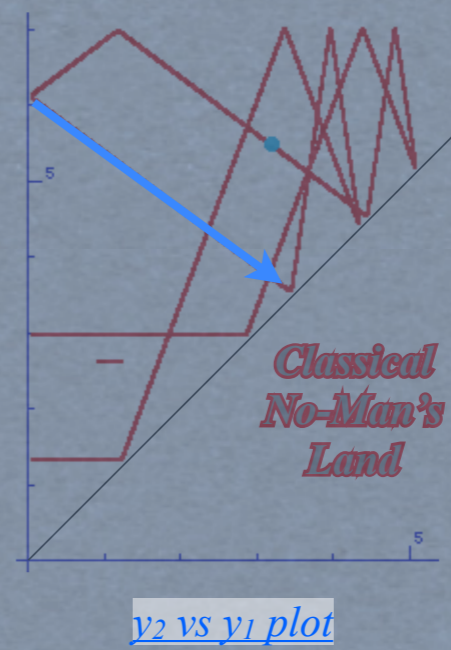
Simulations by *Bouncelt*



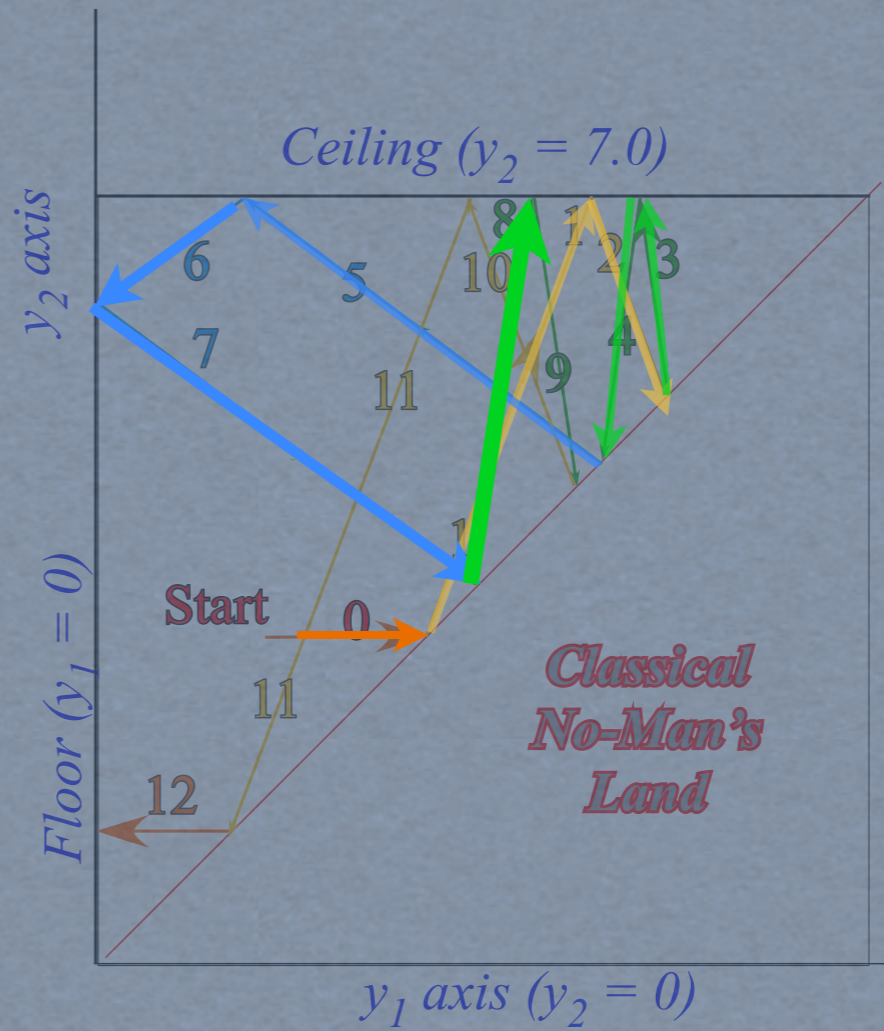
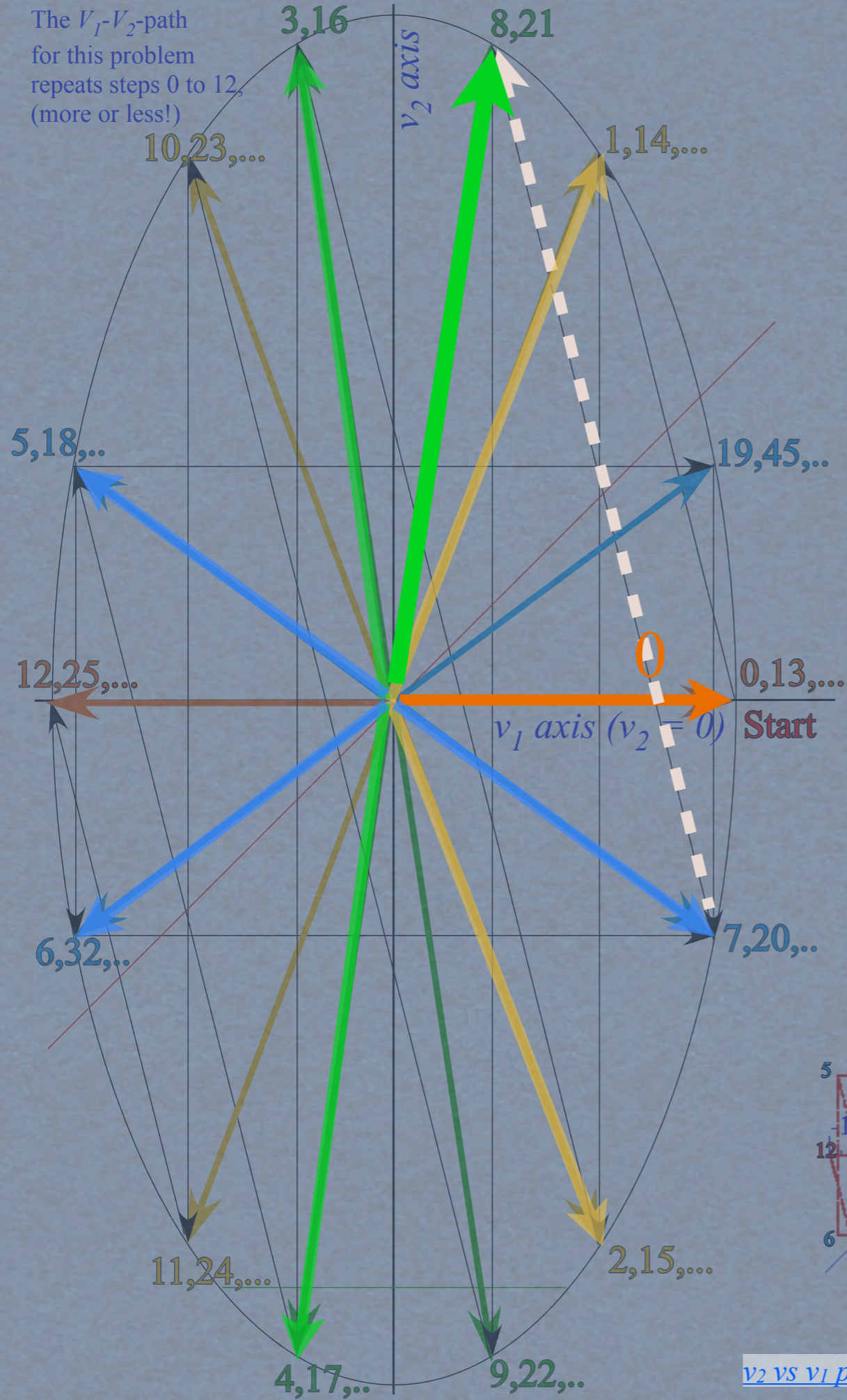
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



Simulations by Bouncelt



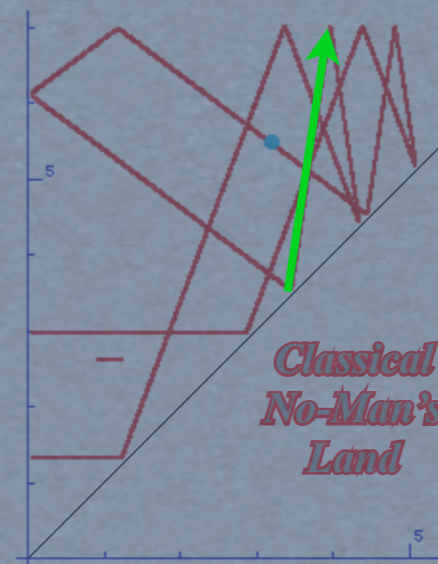
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



Simulations by *Bouncelt*

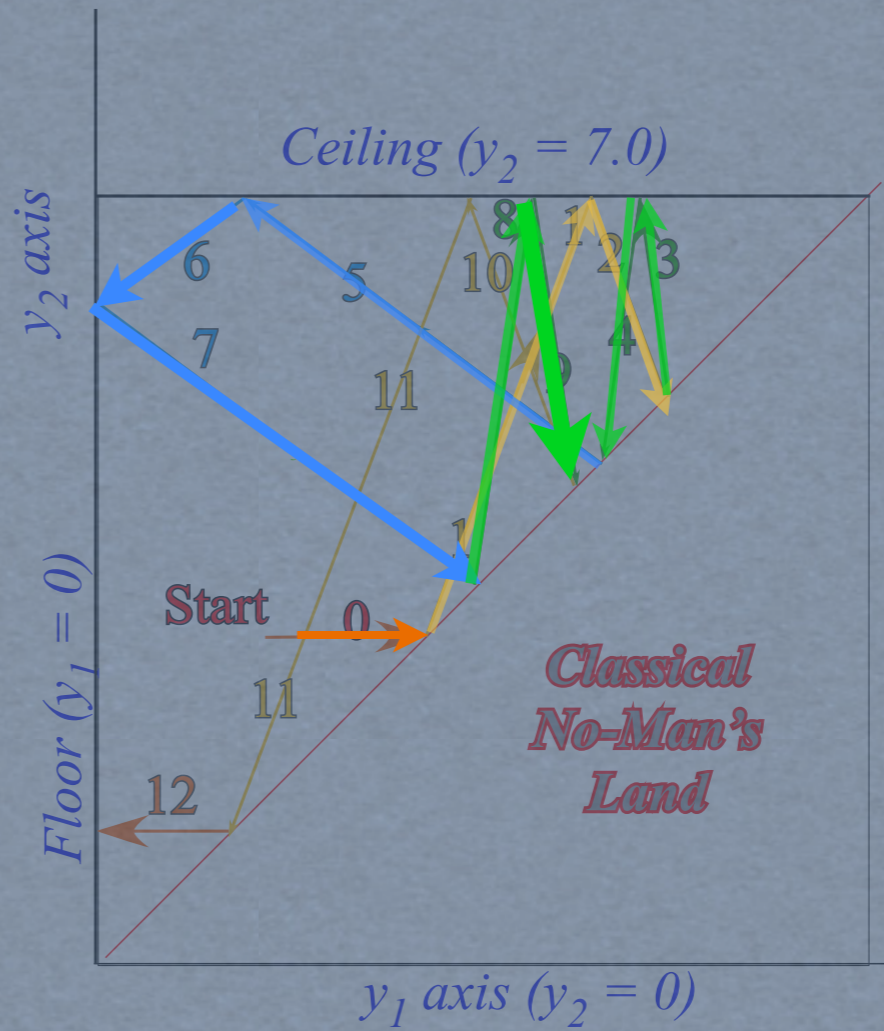
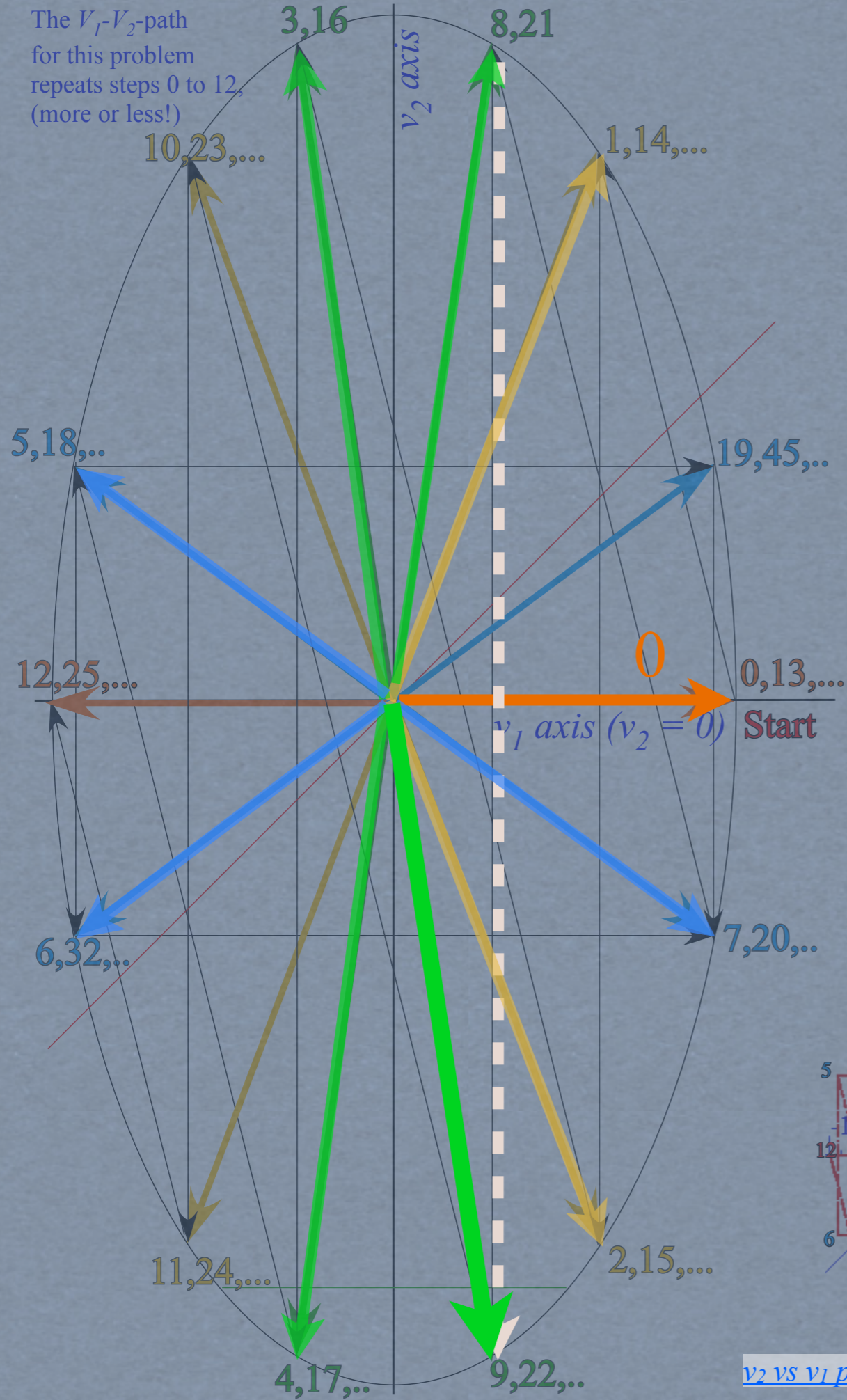


v_2 vs v_1 plot

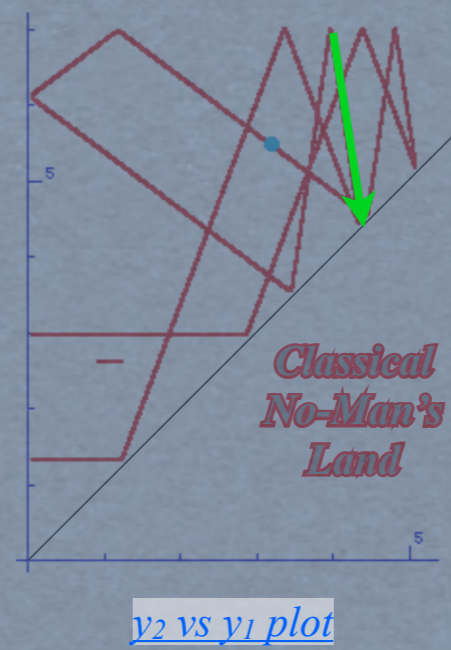
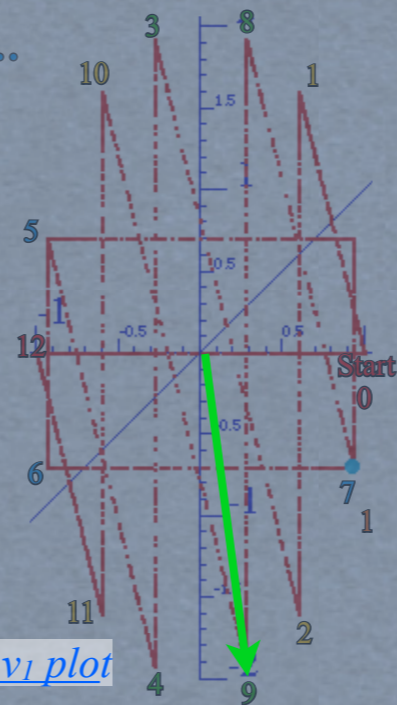


y_2 vs y_1 plot

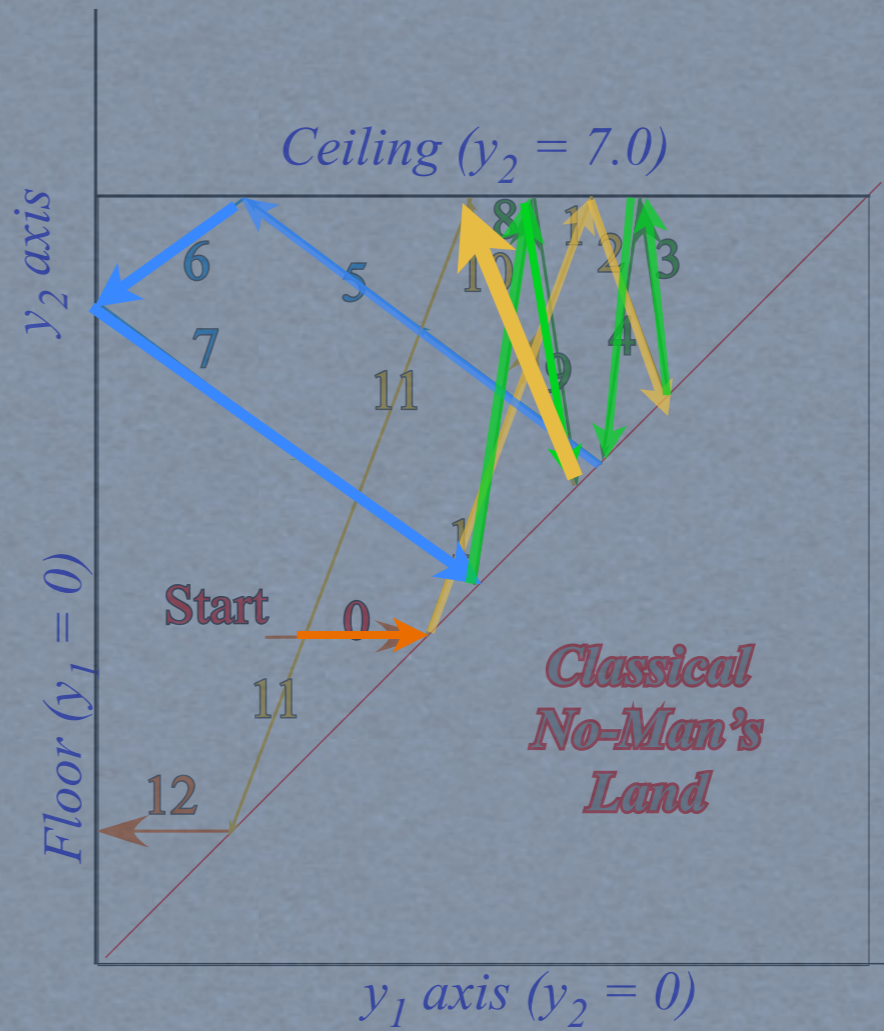
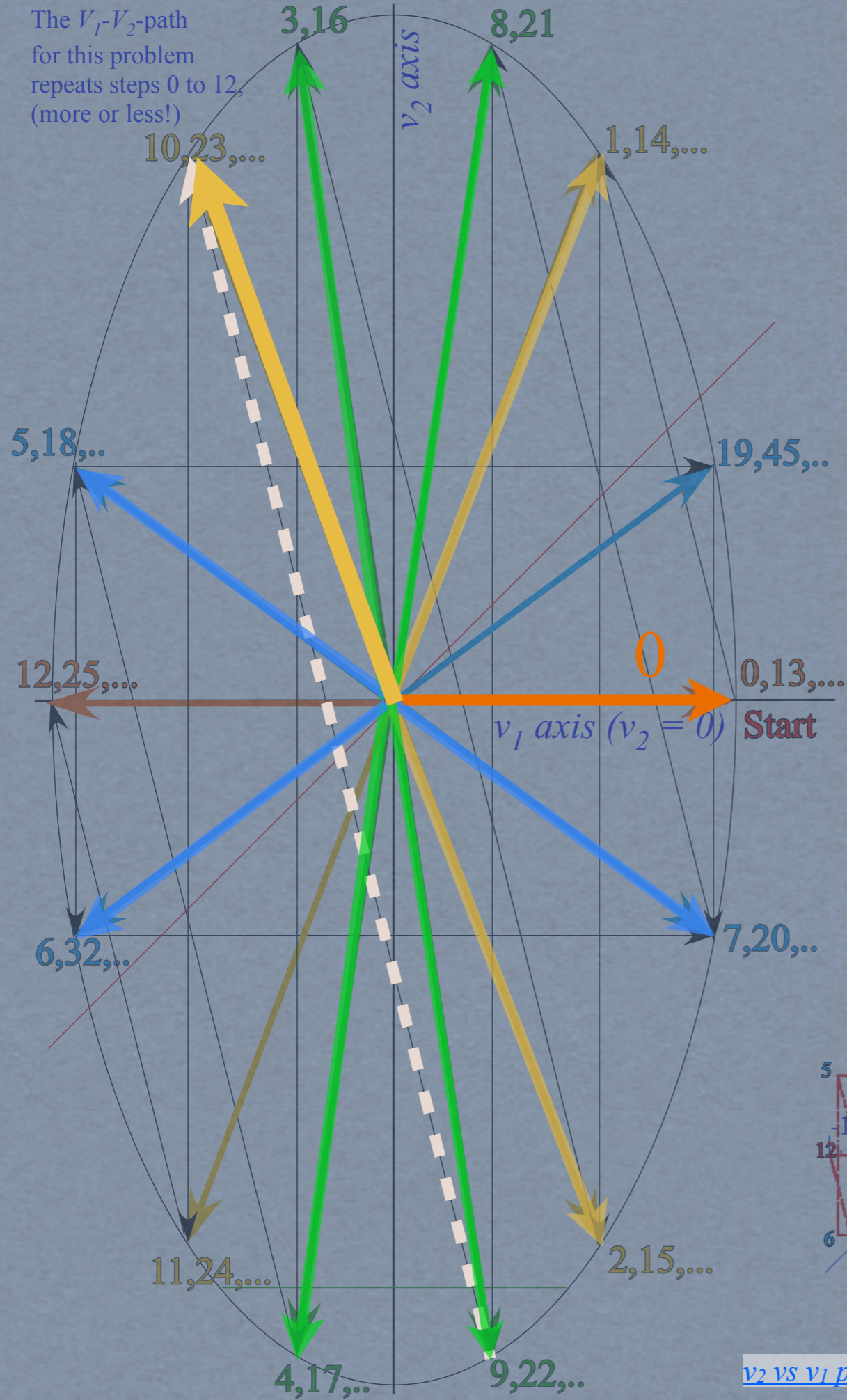
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



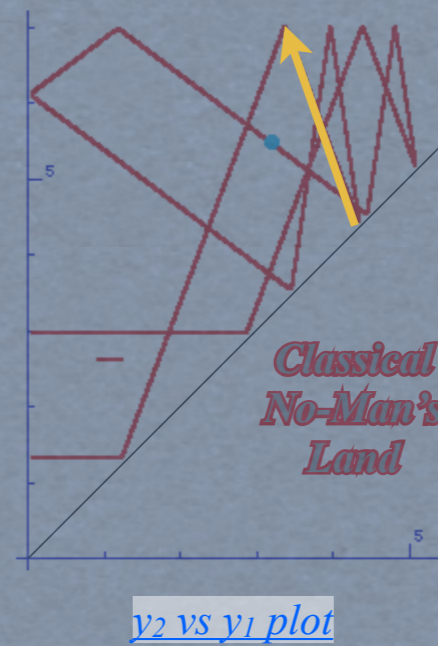
Simulations by *Bouncelt*



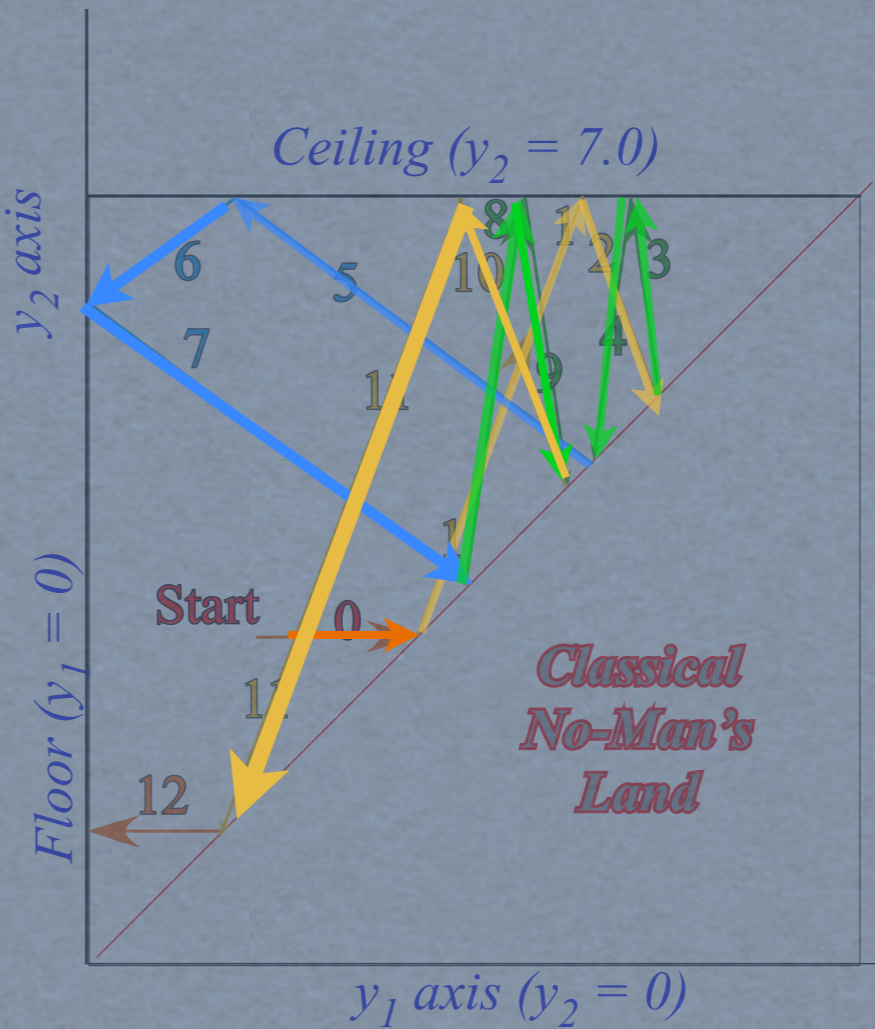
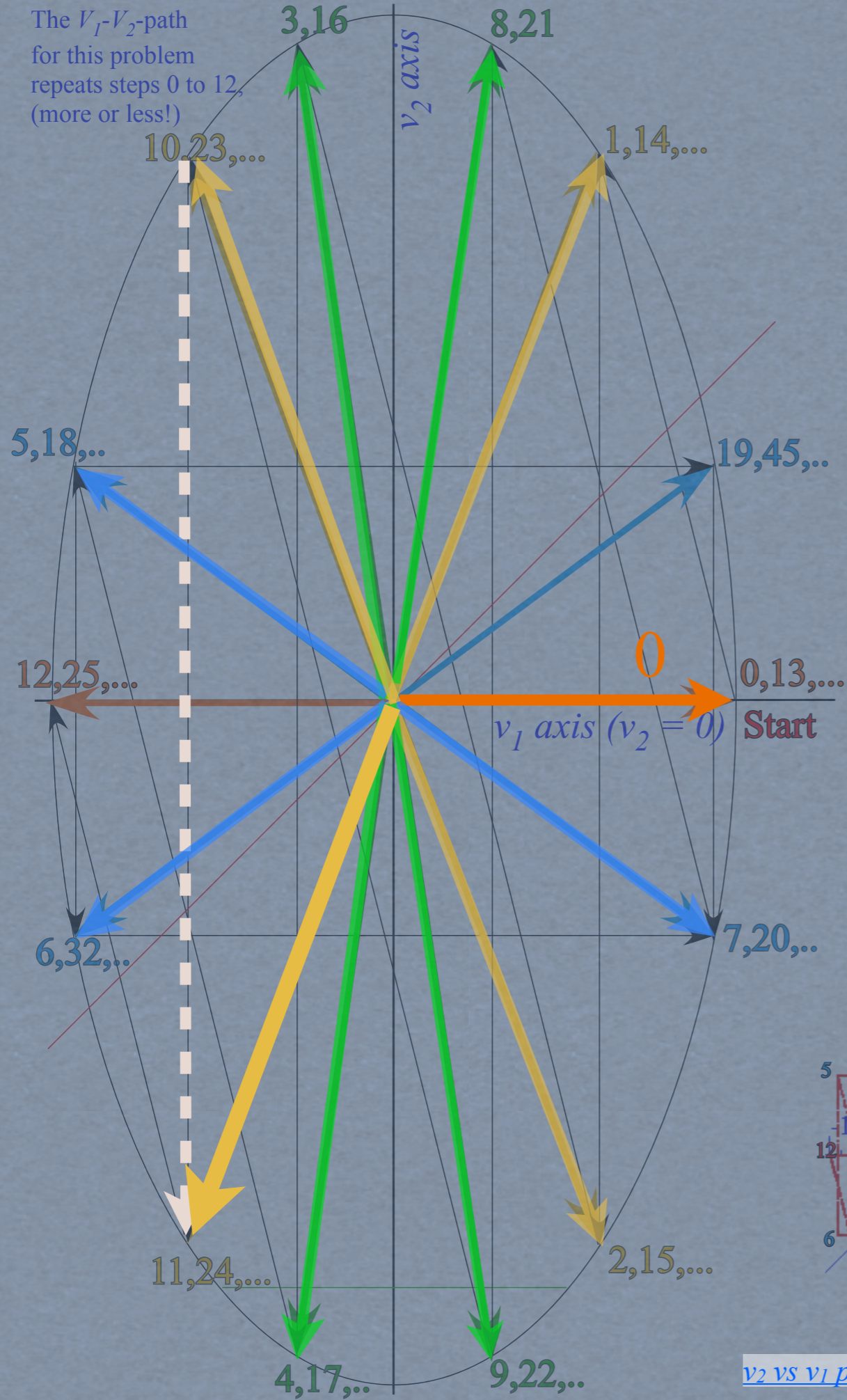
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



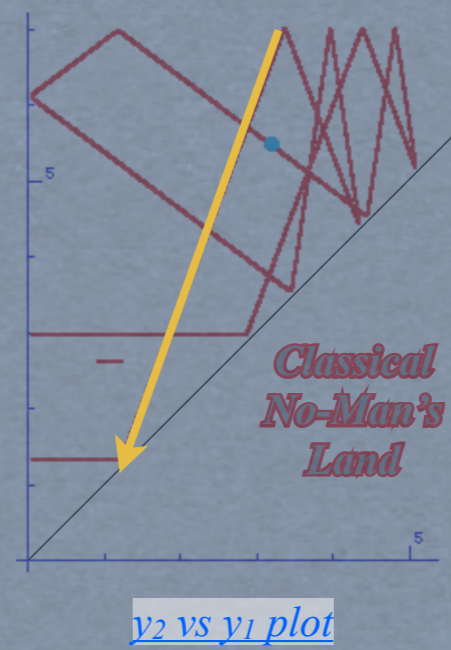
Simulations by *Bouncelt*



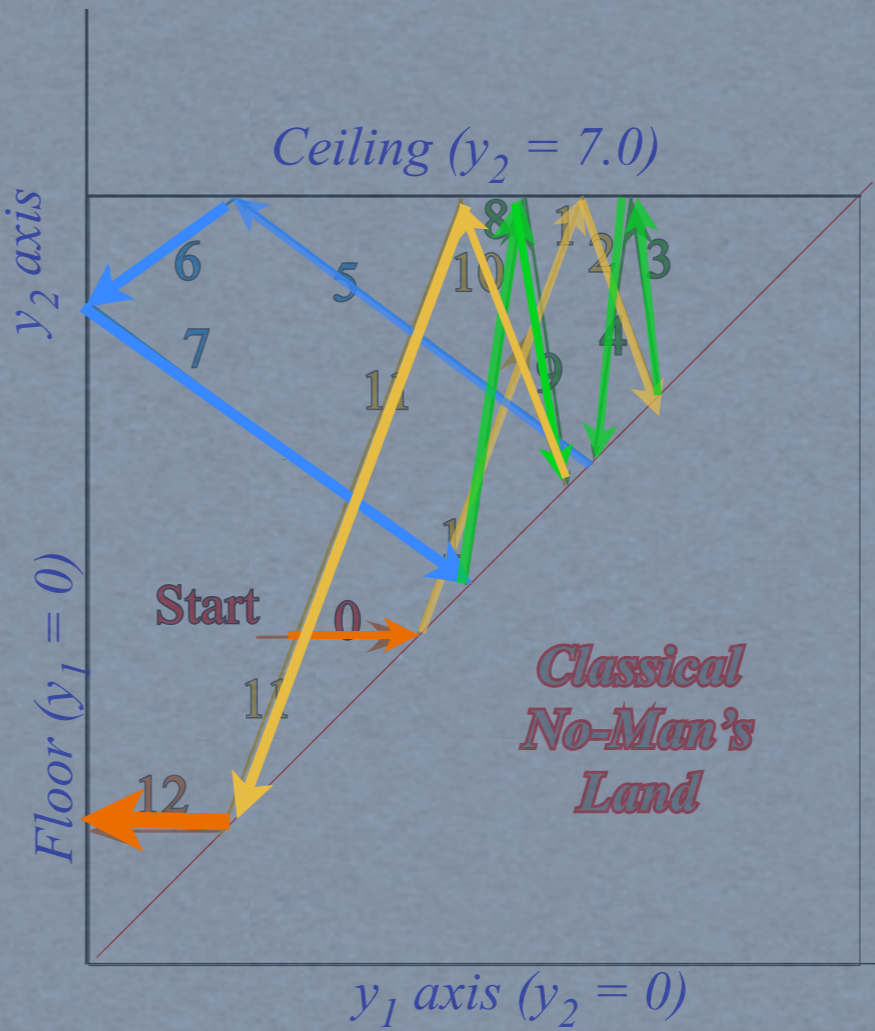
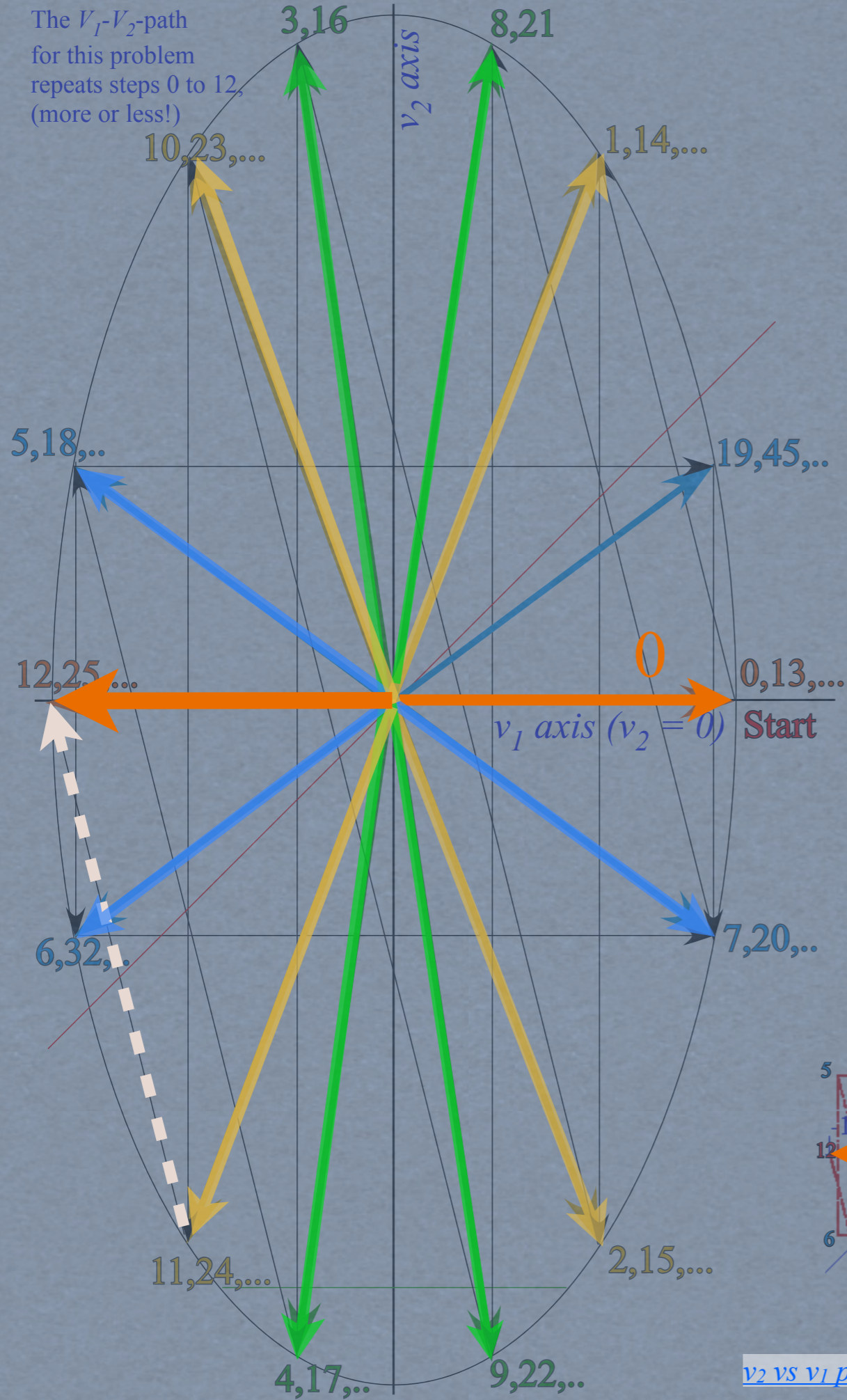
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



Simulations by *Bouncelt*

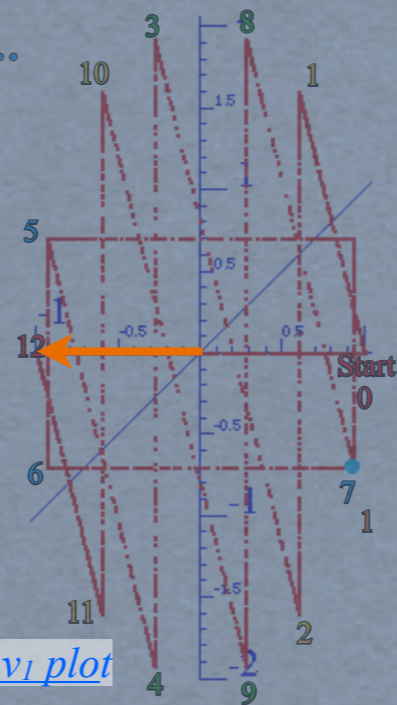


The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)

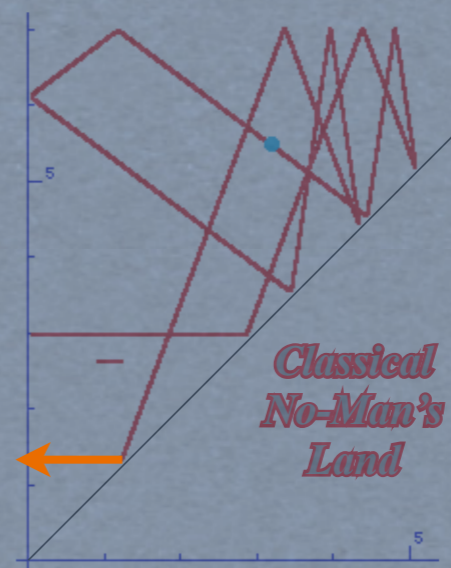


Classical No-Man's Land

Simulations by *Bouncelt*



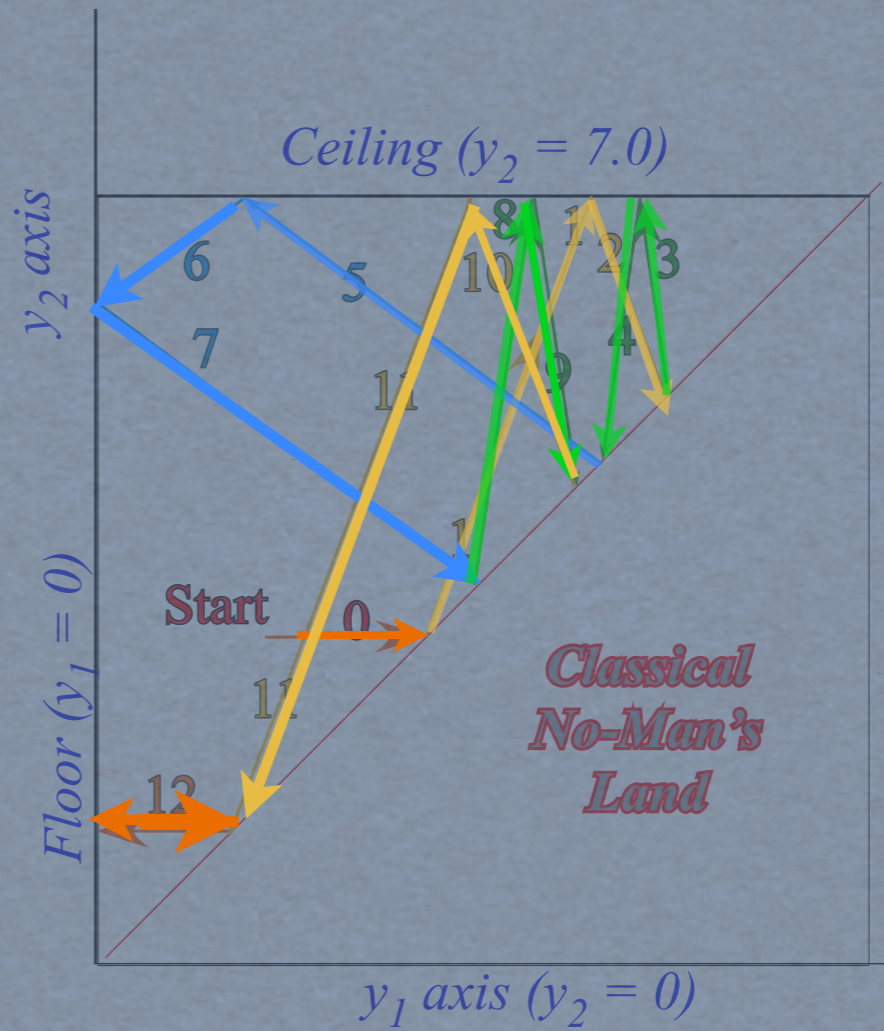
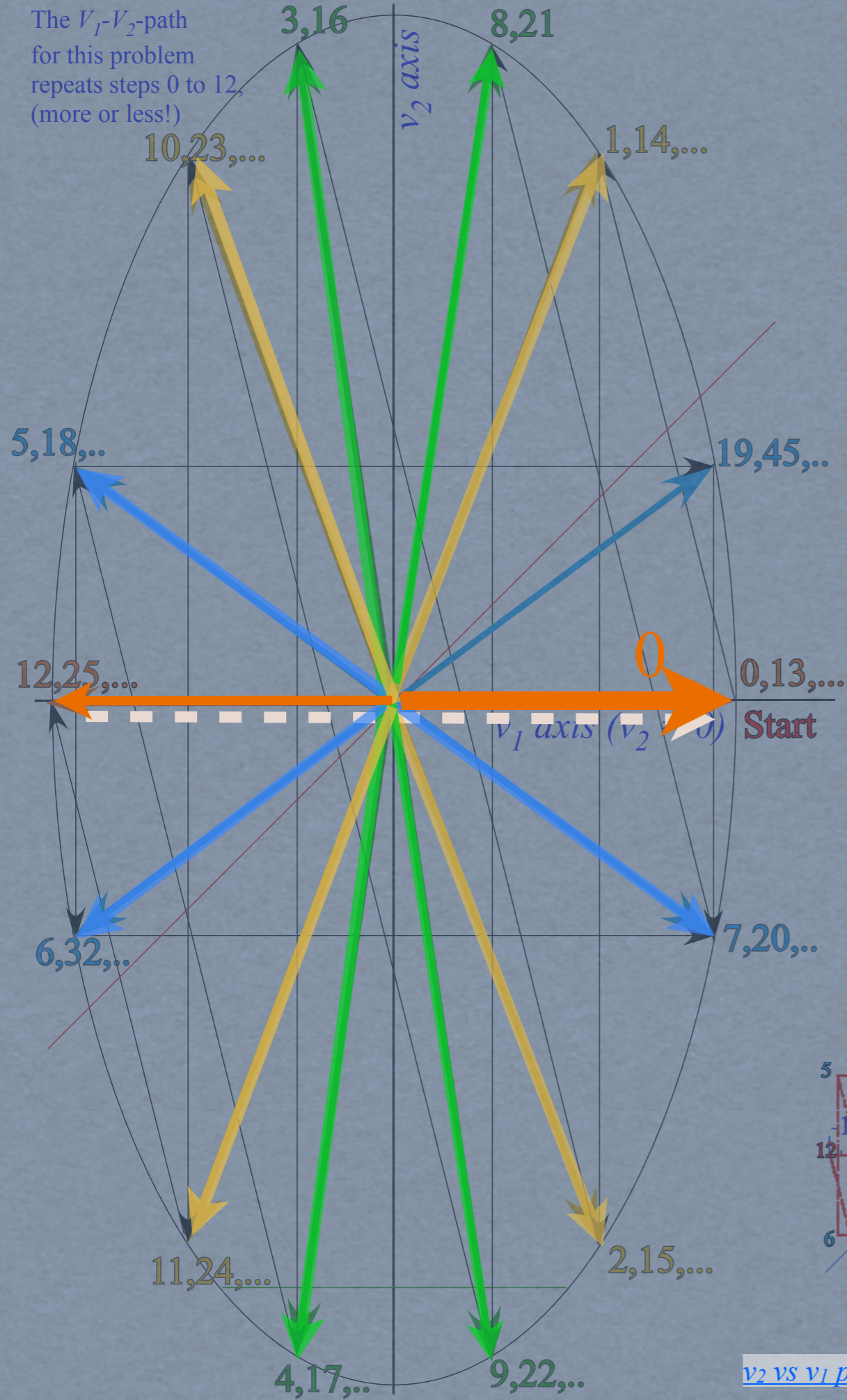
v2 vs v1 plot



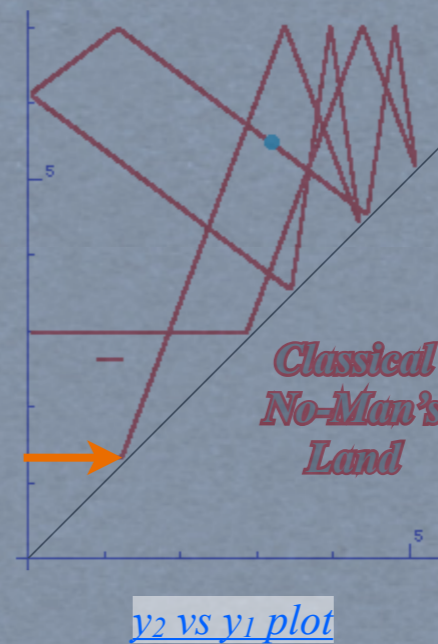
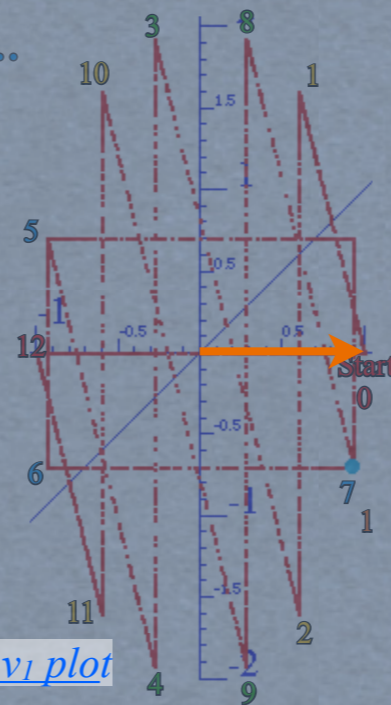
y2 vs y1 plot

Classical No-Man's Land

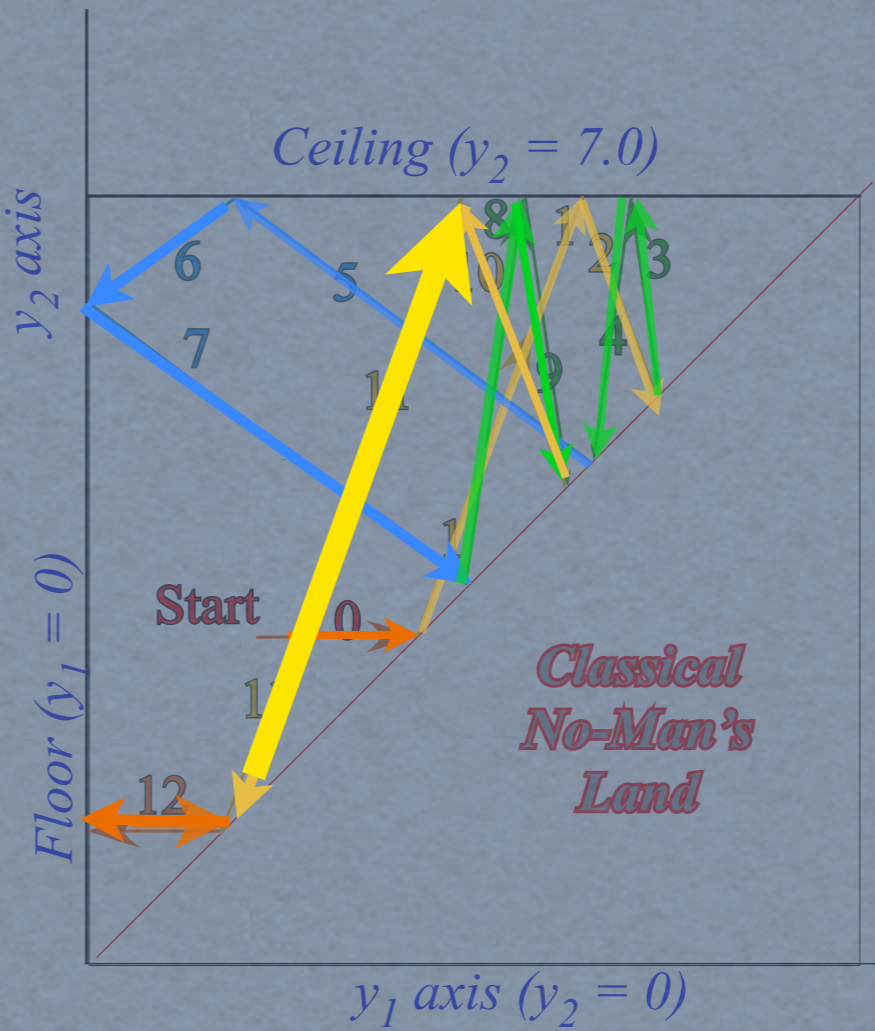
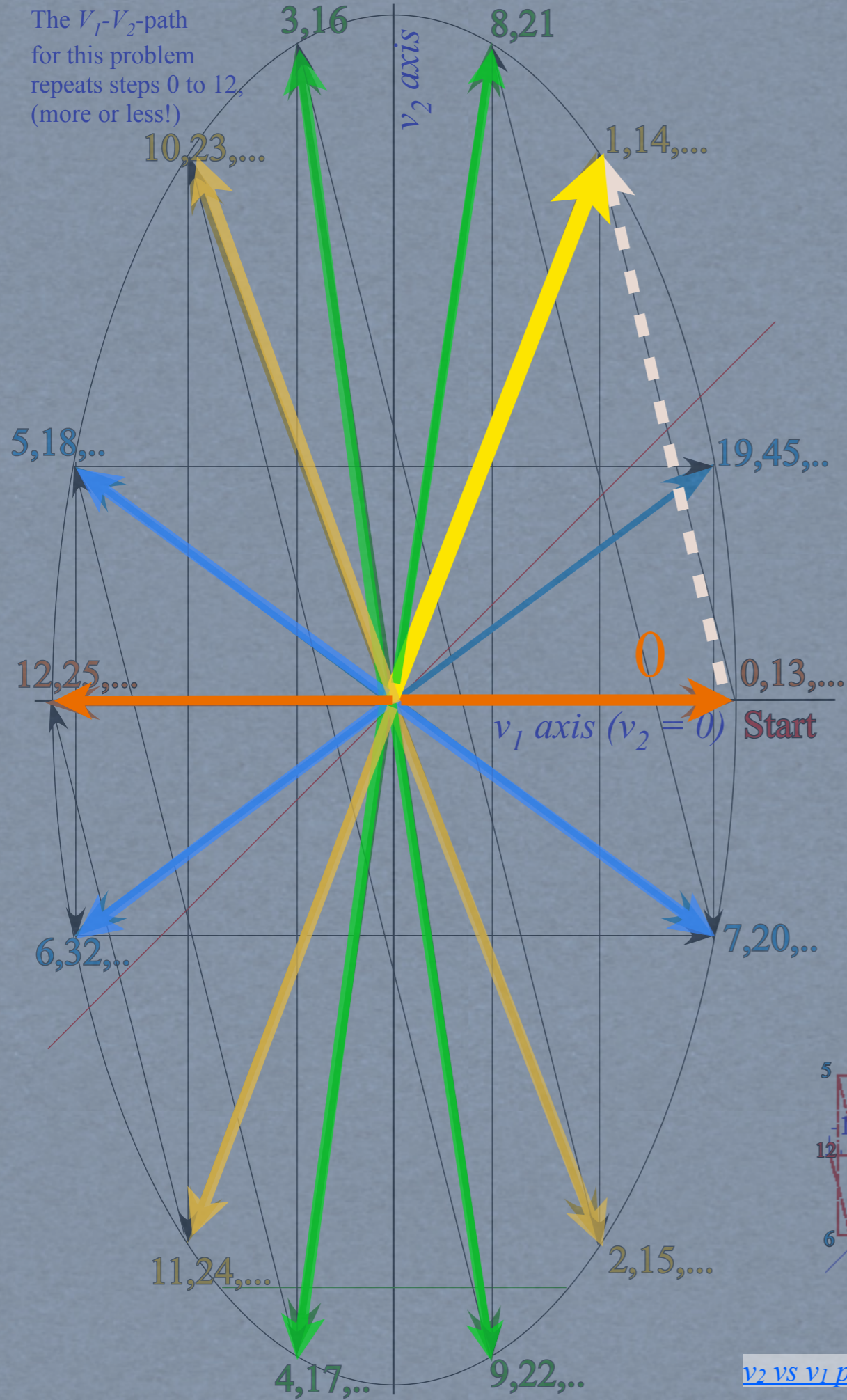
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



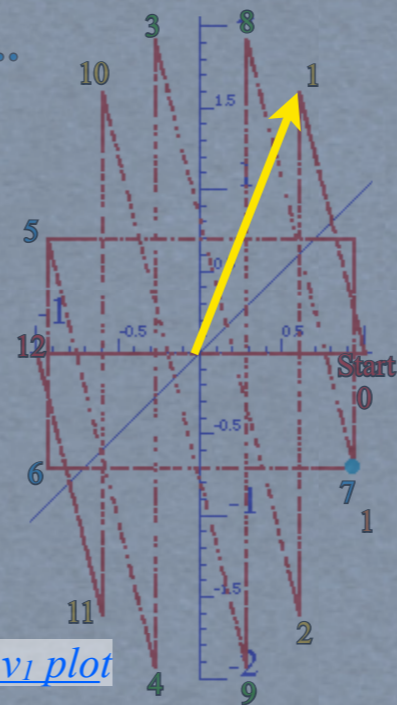
Simulations by *Bouncelt*



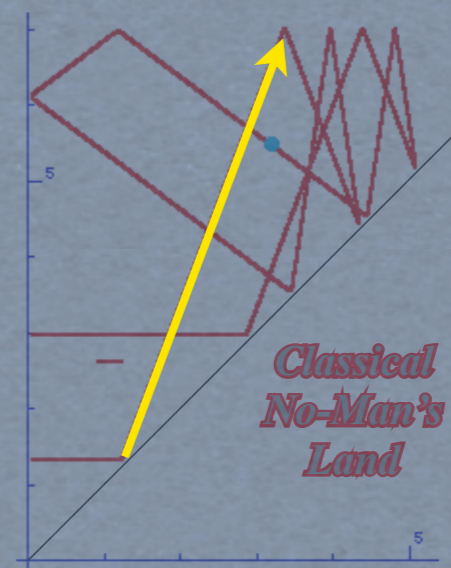
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



Simulations by *Bouncelt*

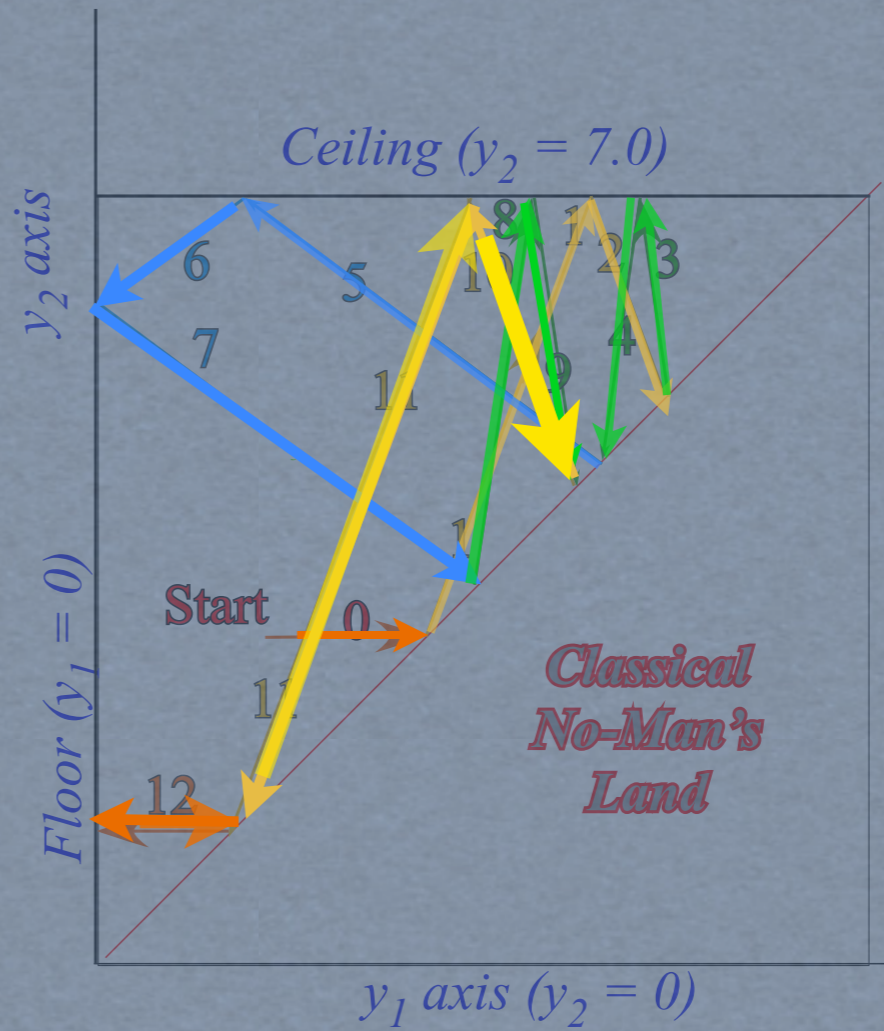
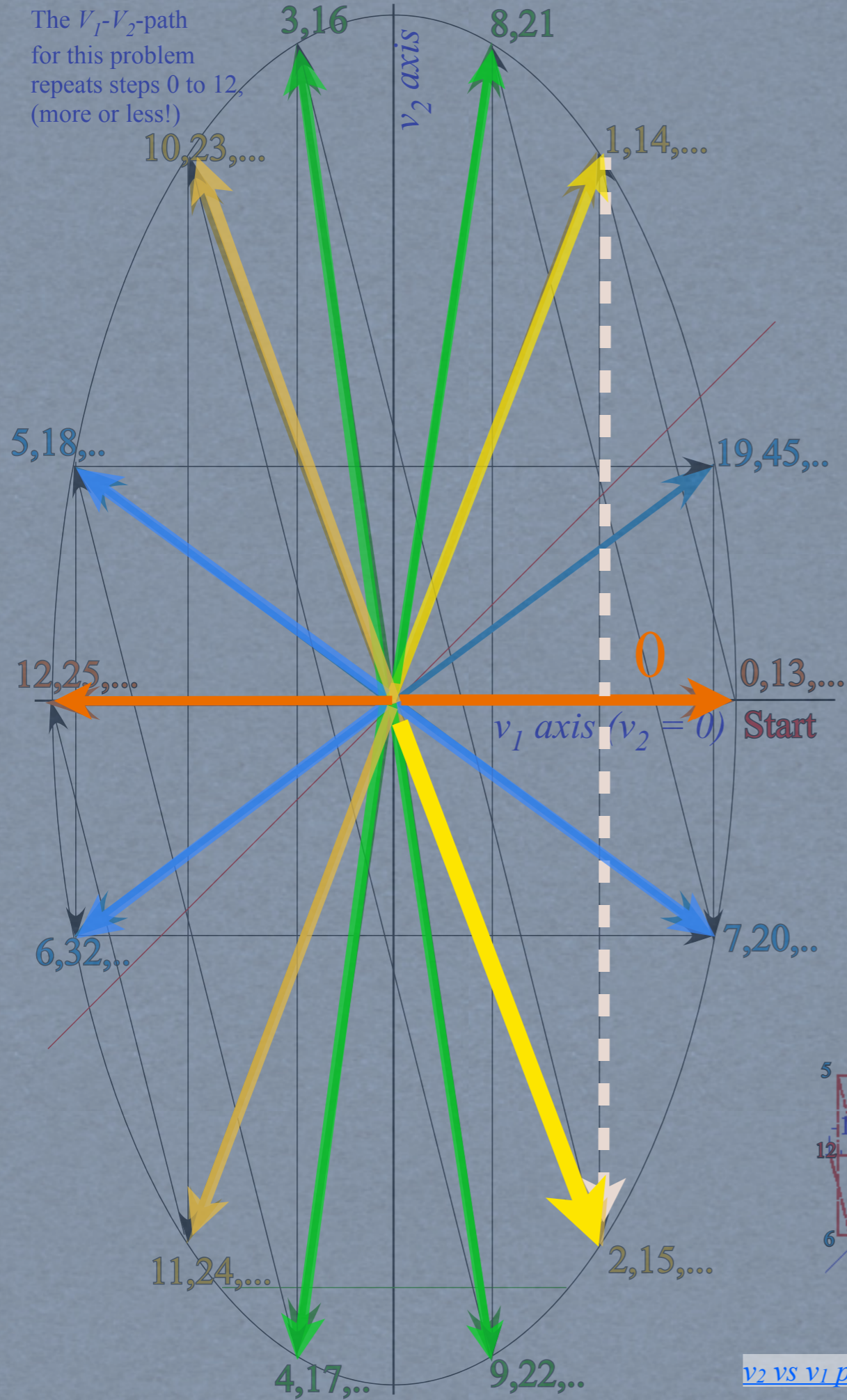


v2 vs v1 plot

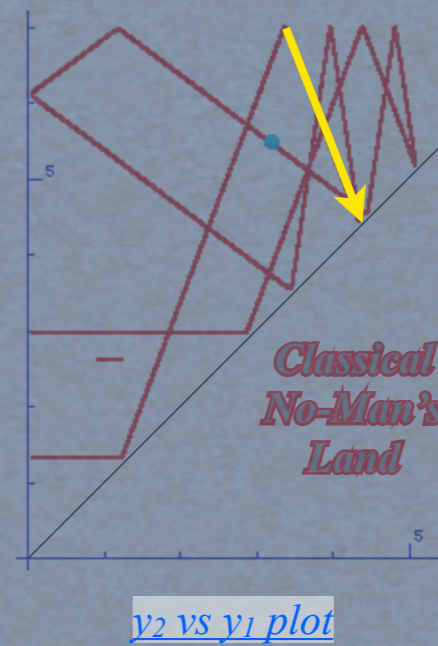


y2 vs y1 plot

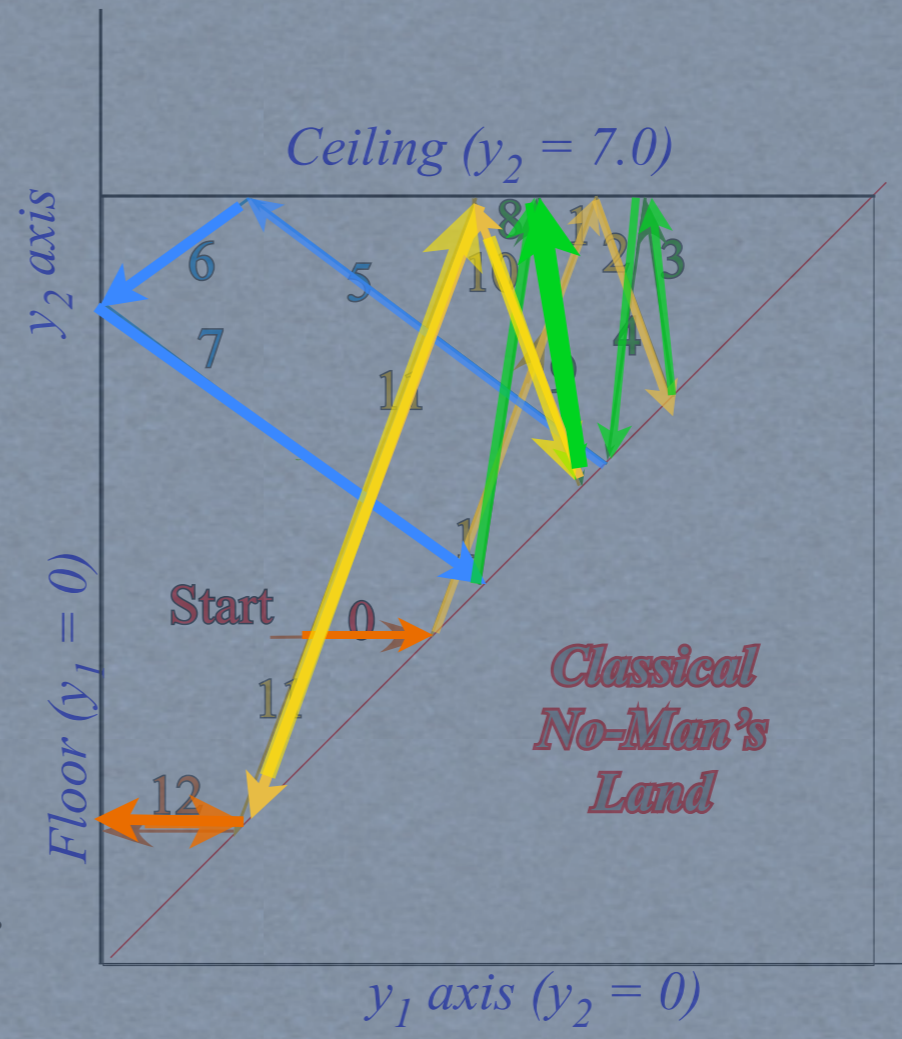
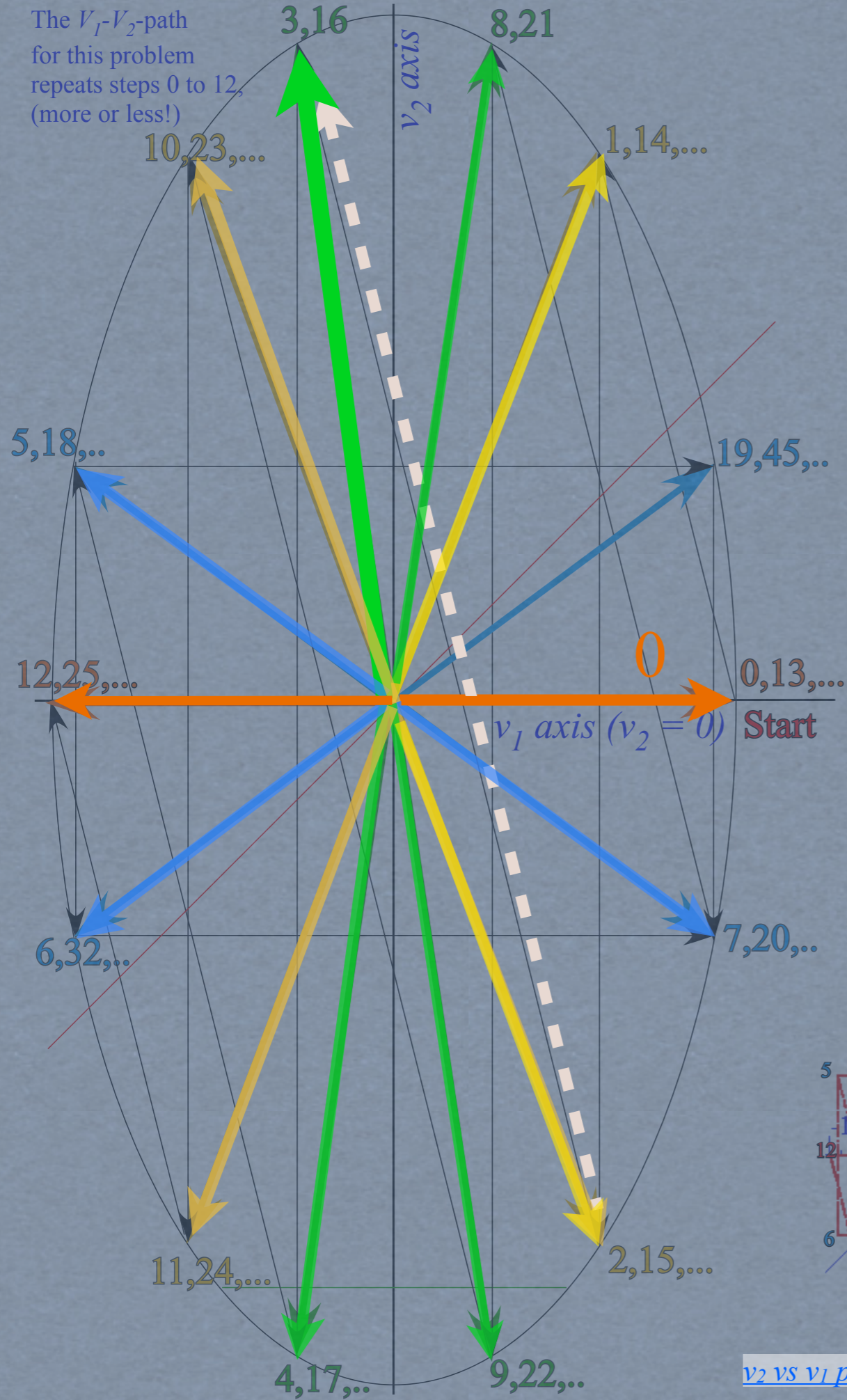
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



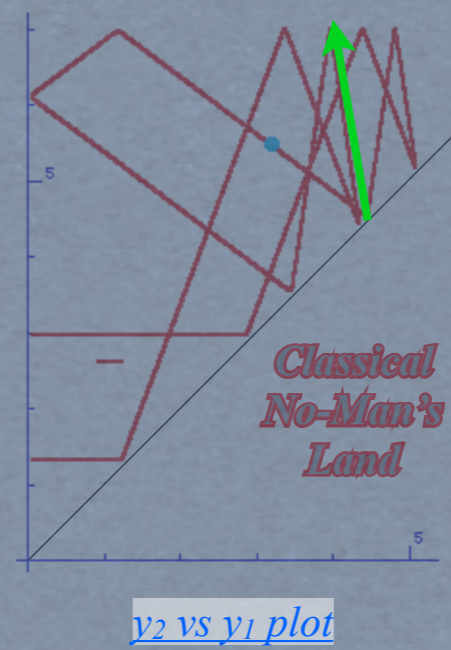
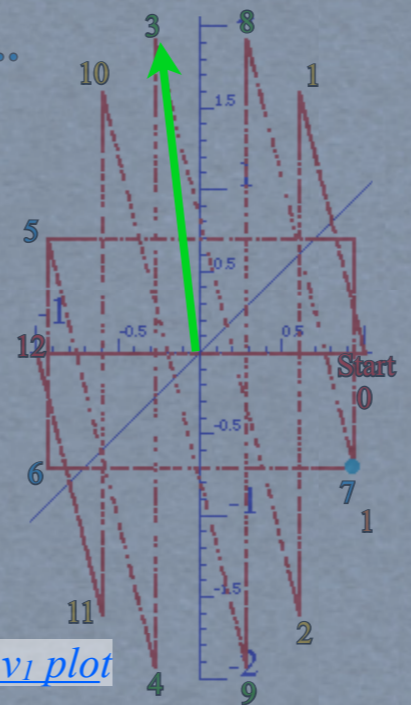
Simulations by Bouncelt



The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



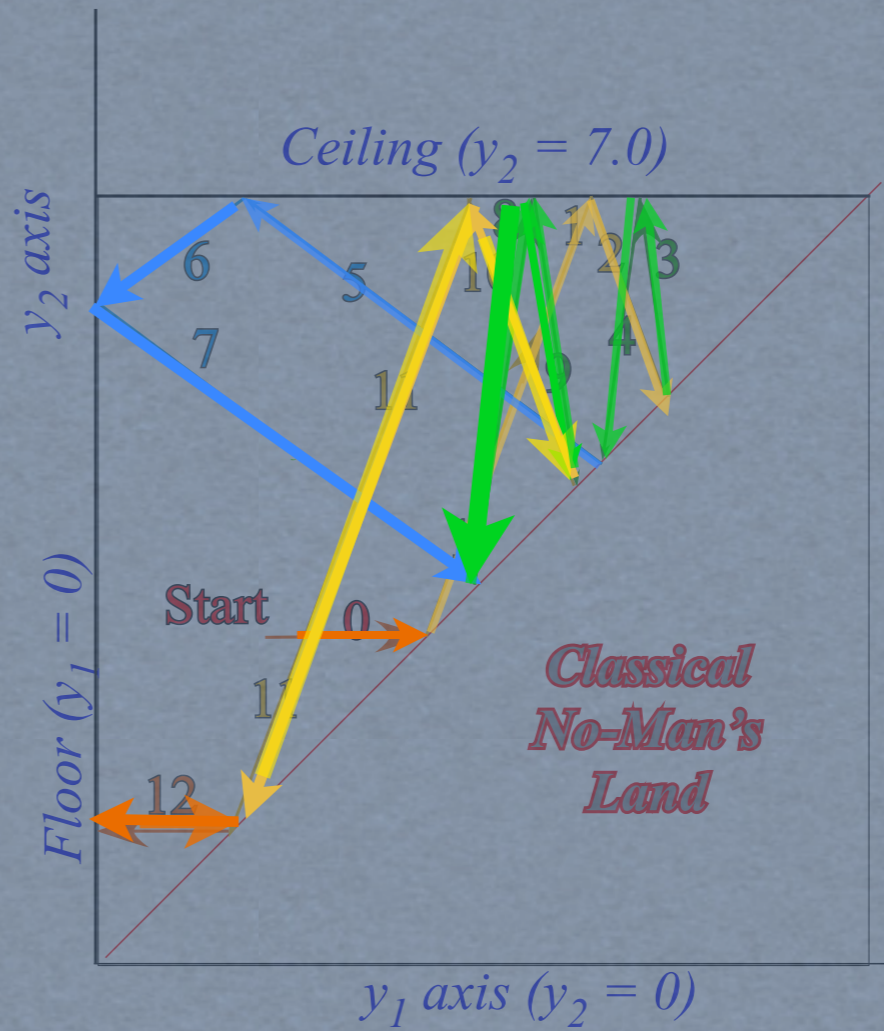
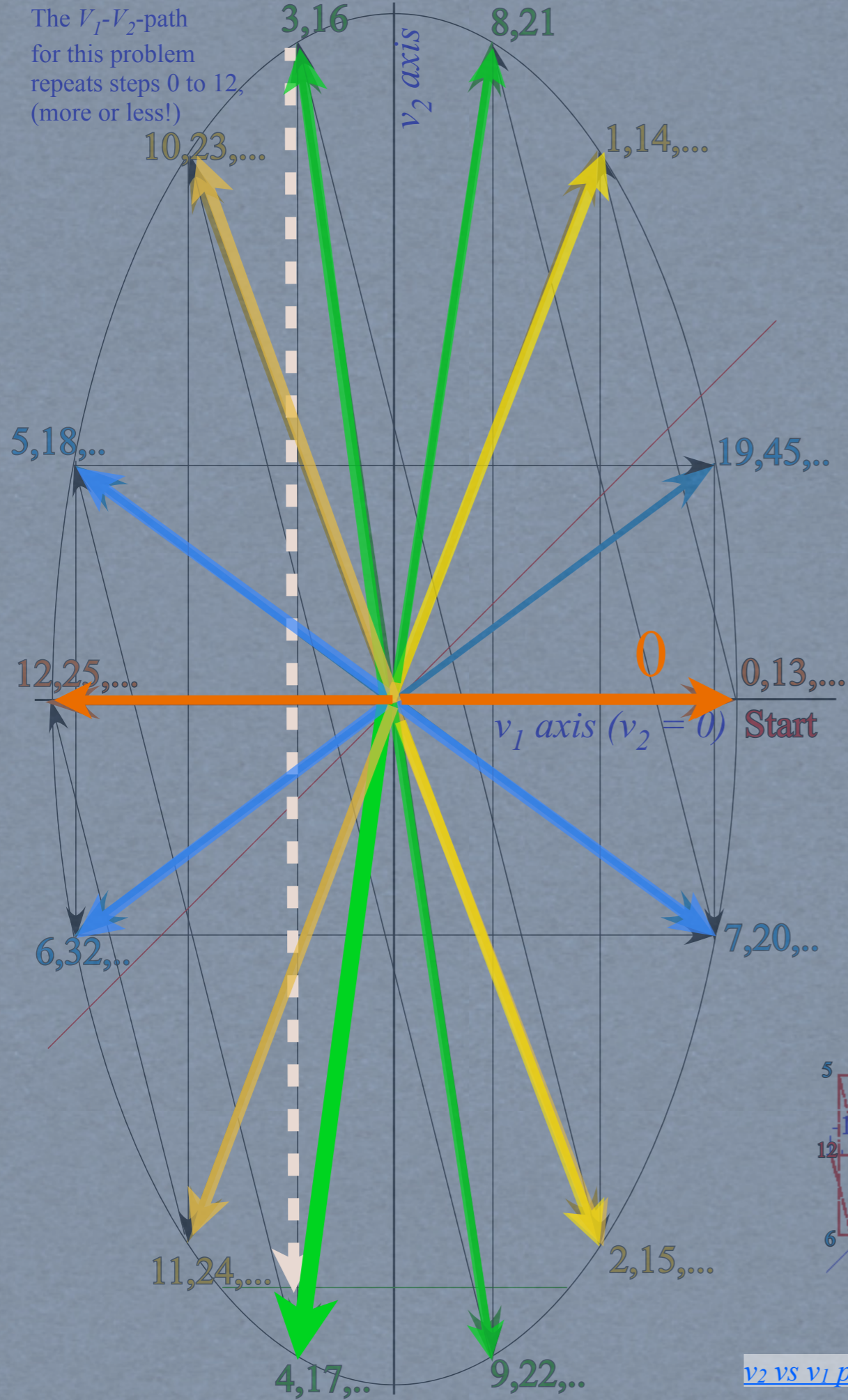
Simulations by Bouncelt



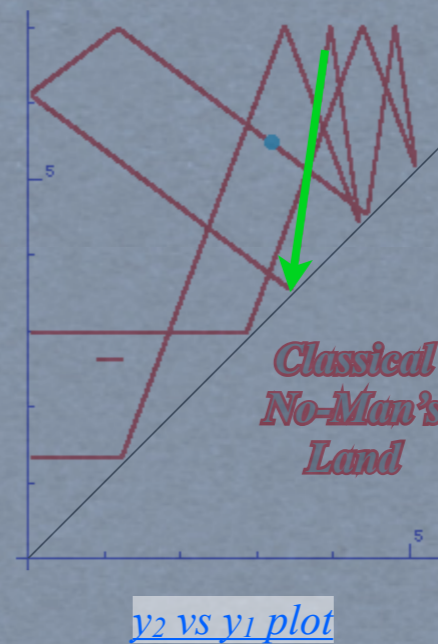
v2 vs v1 plot

y2 vs y1 plot

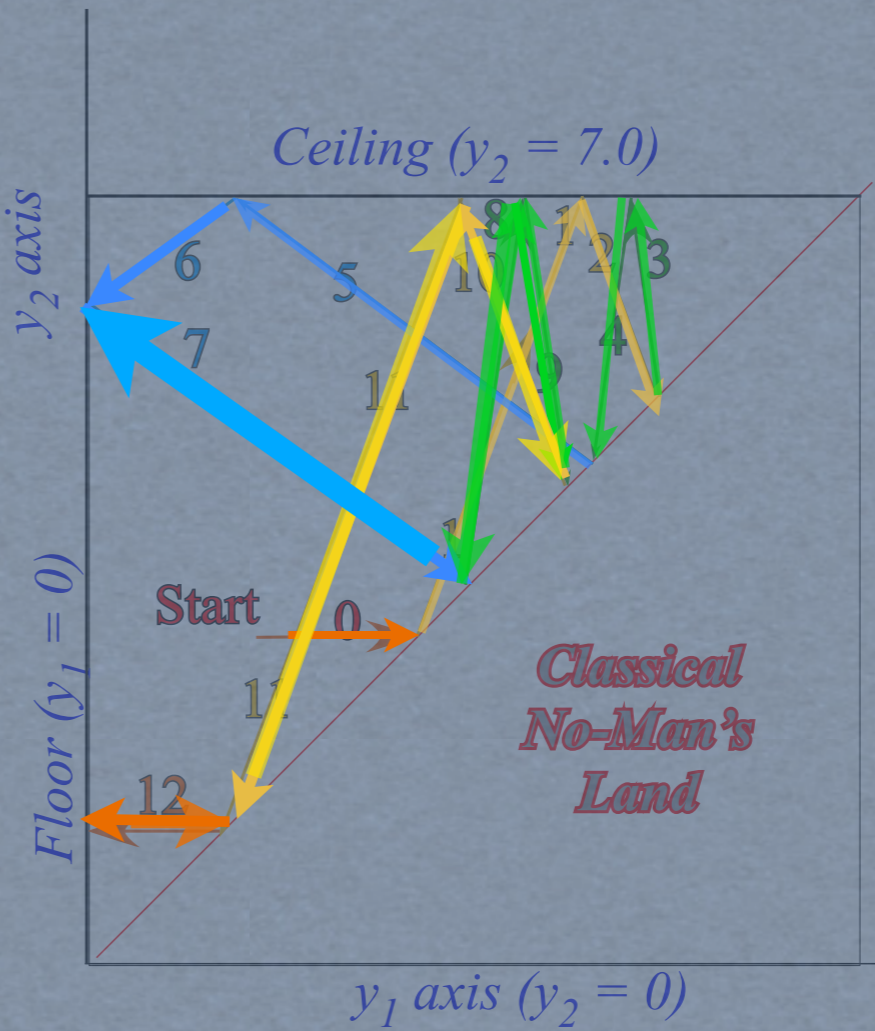
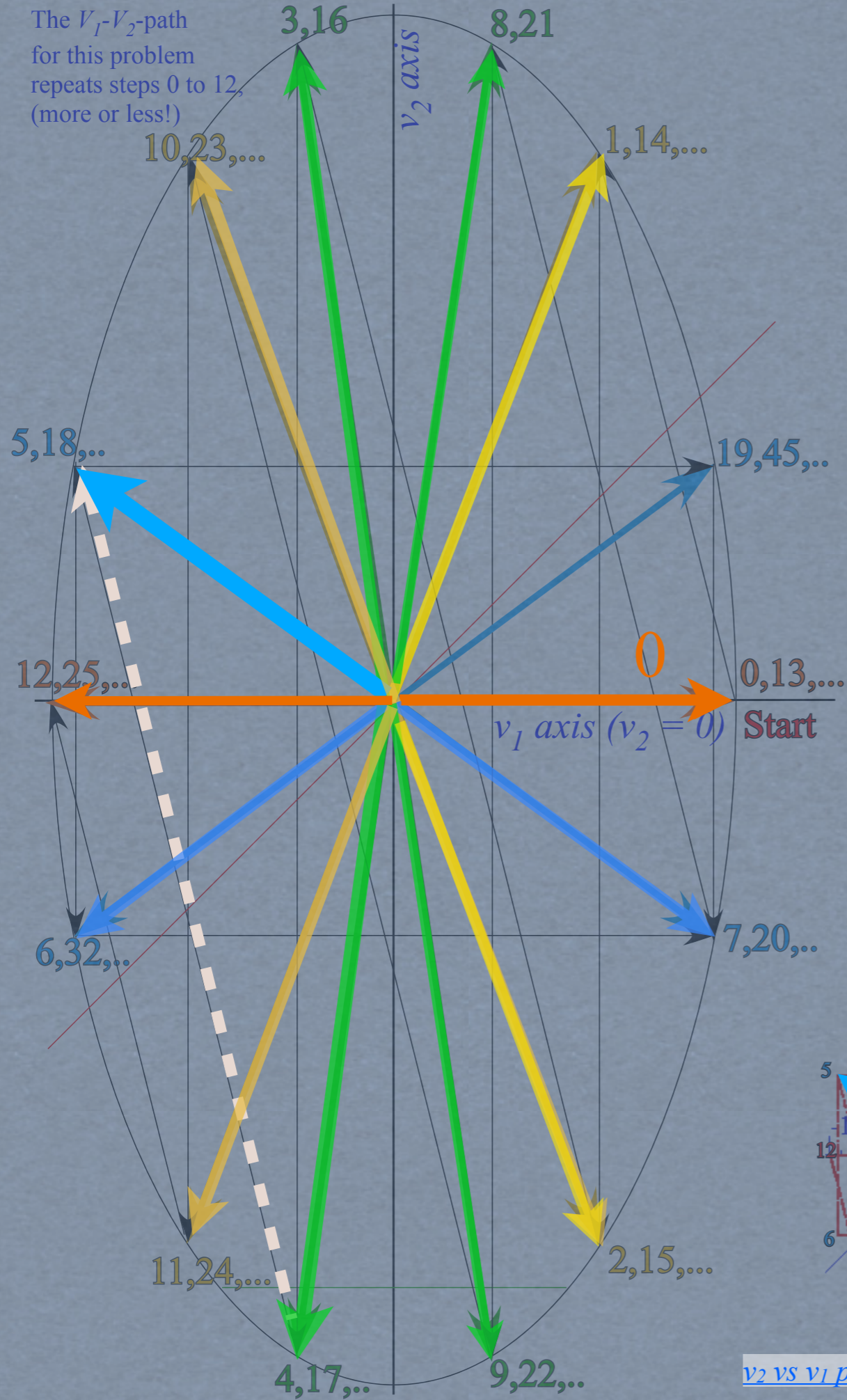
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



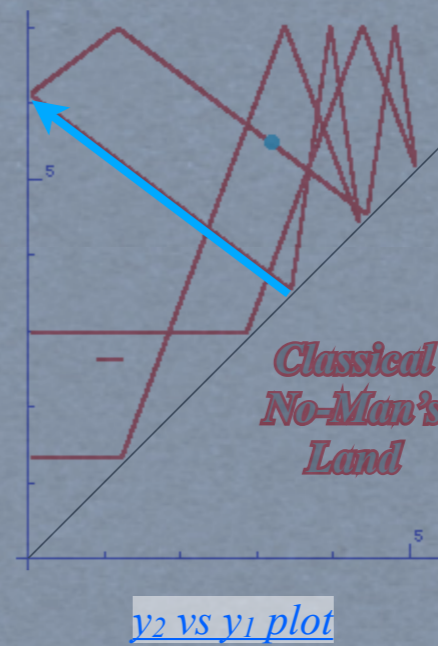
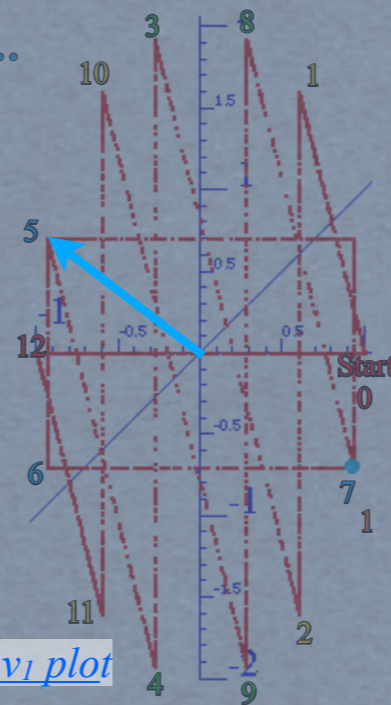
Simulations by *Bouncelt*



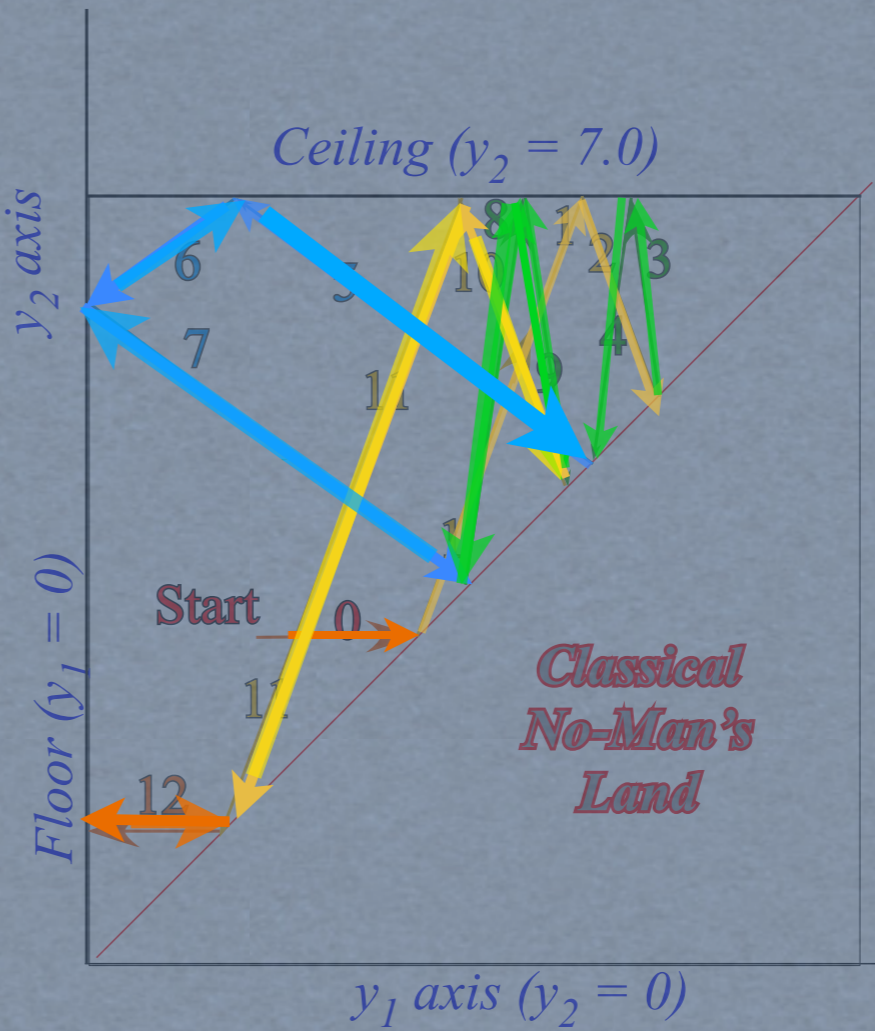
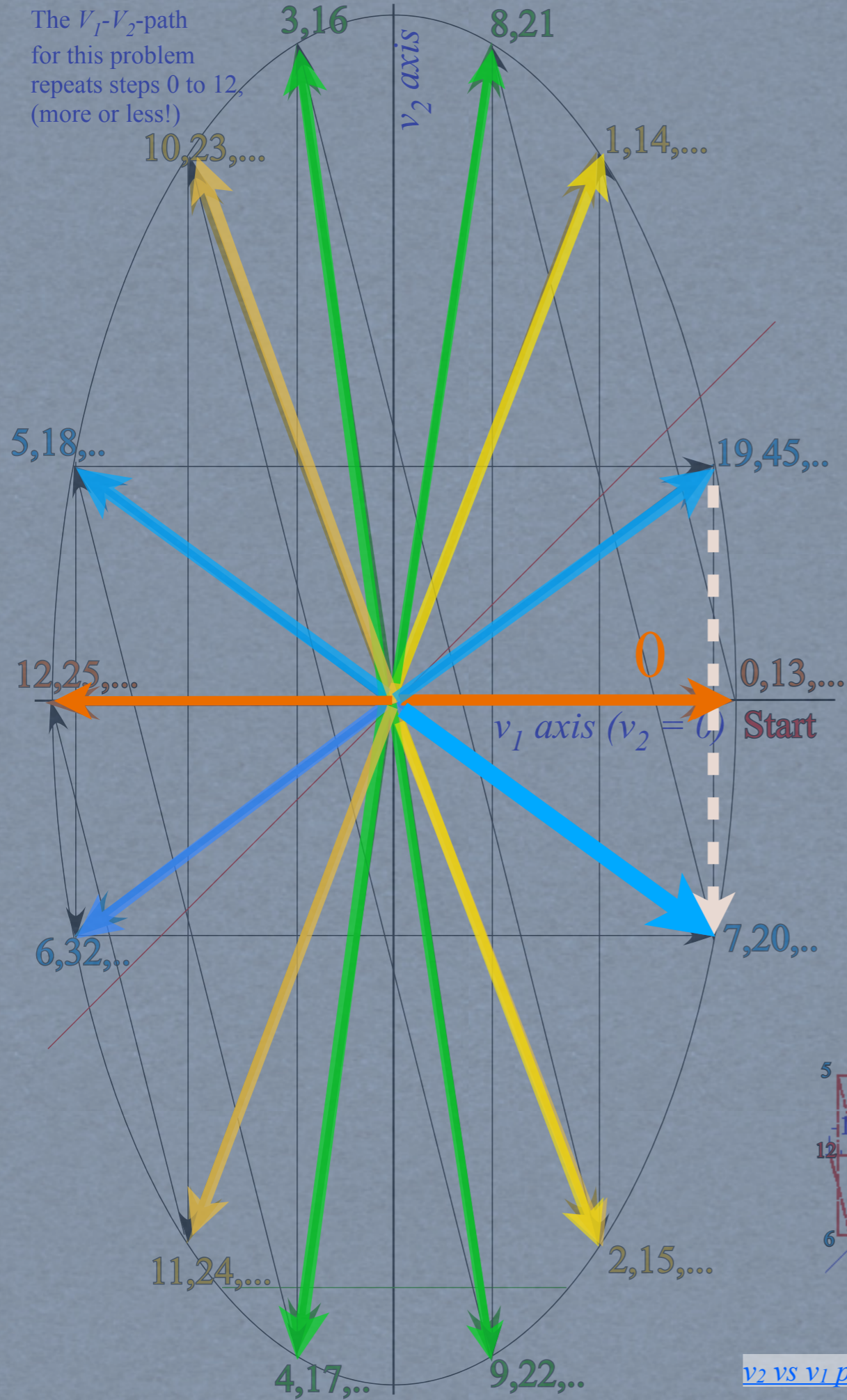
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



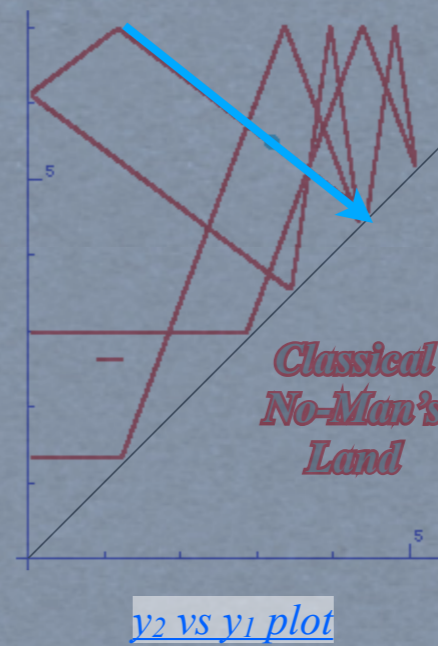
Simulations by *Bouncelt*



The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



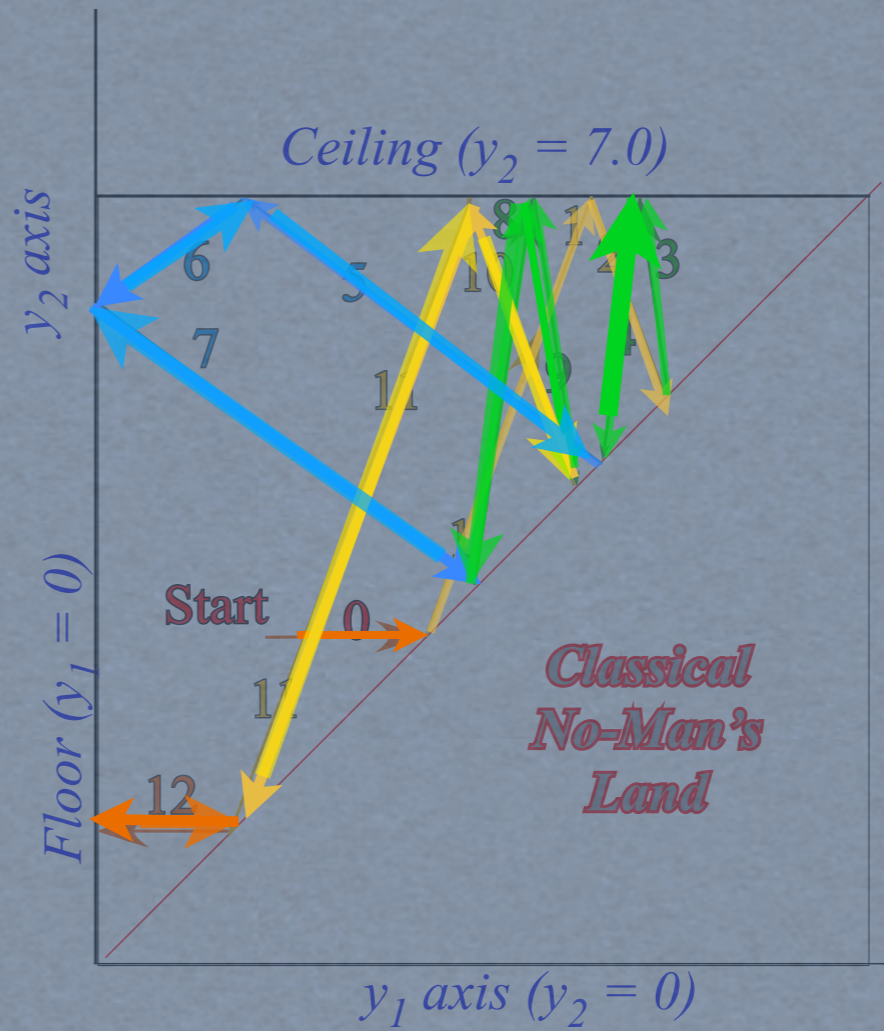
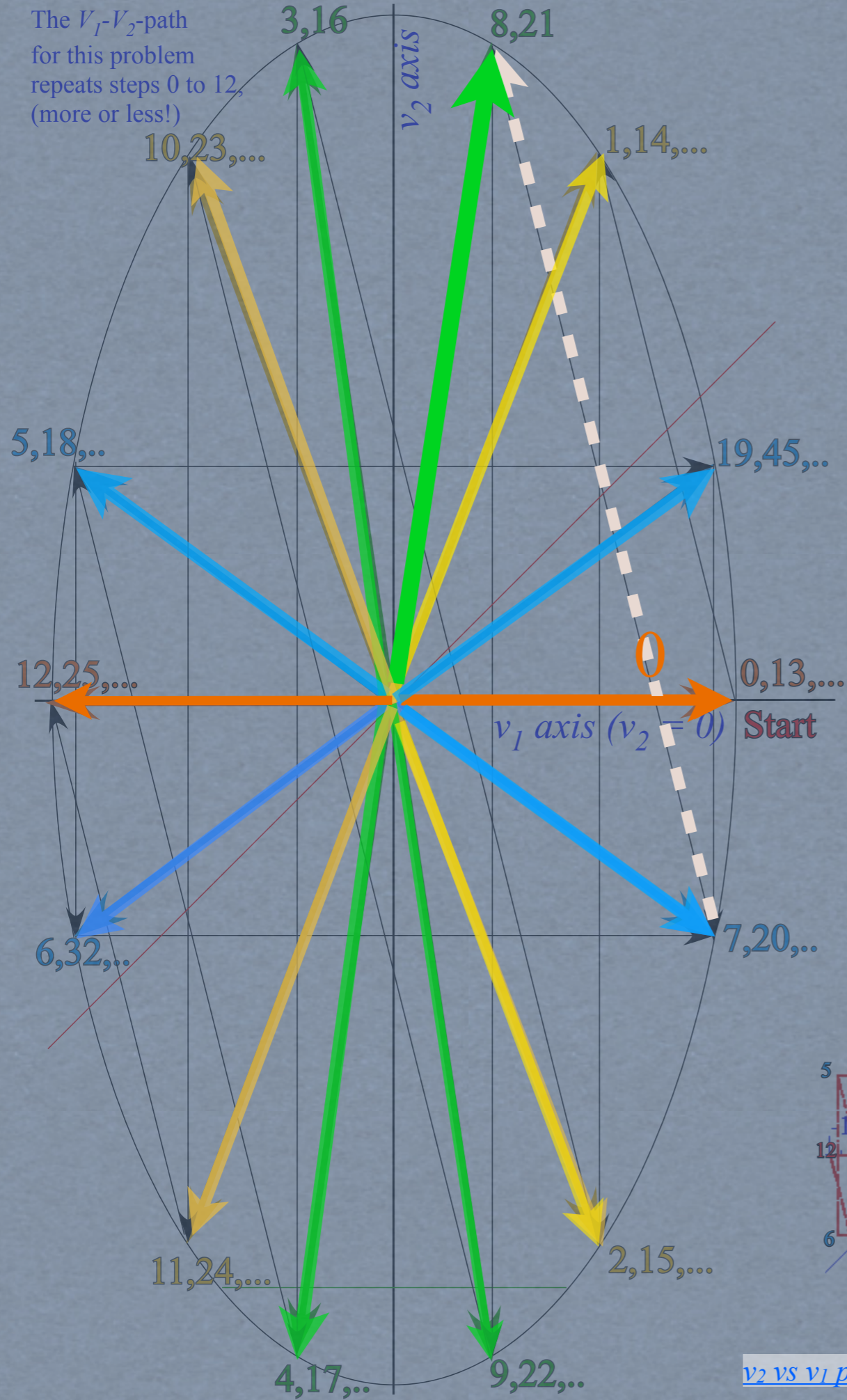
Simulations by *Bouncelt*



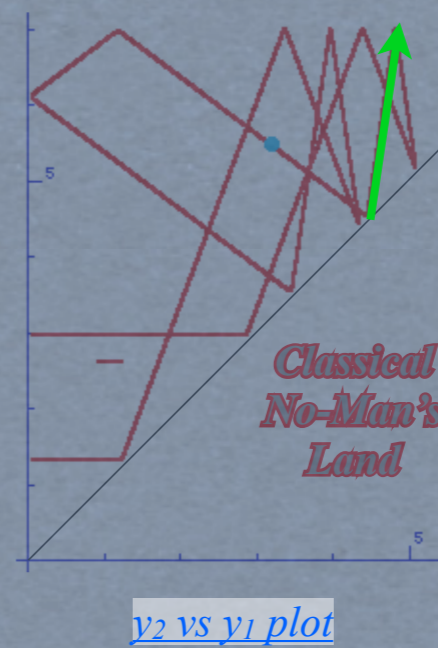
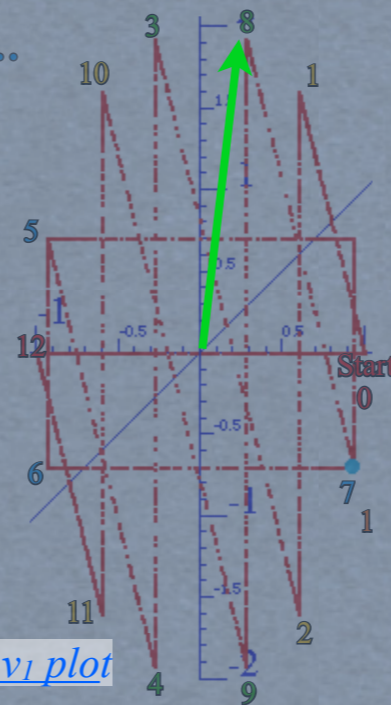
v2 vs v1 plot

y2 vs y1 plot

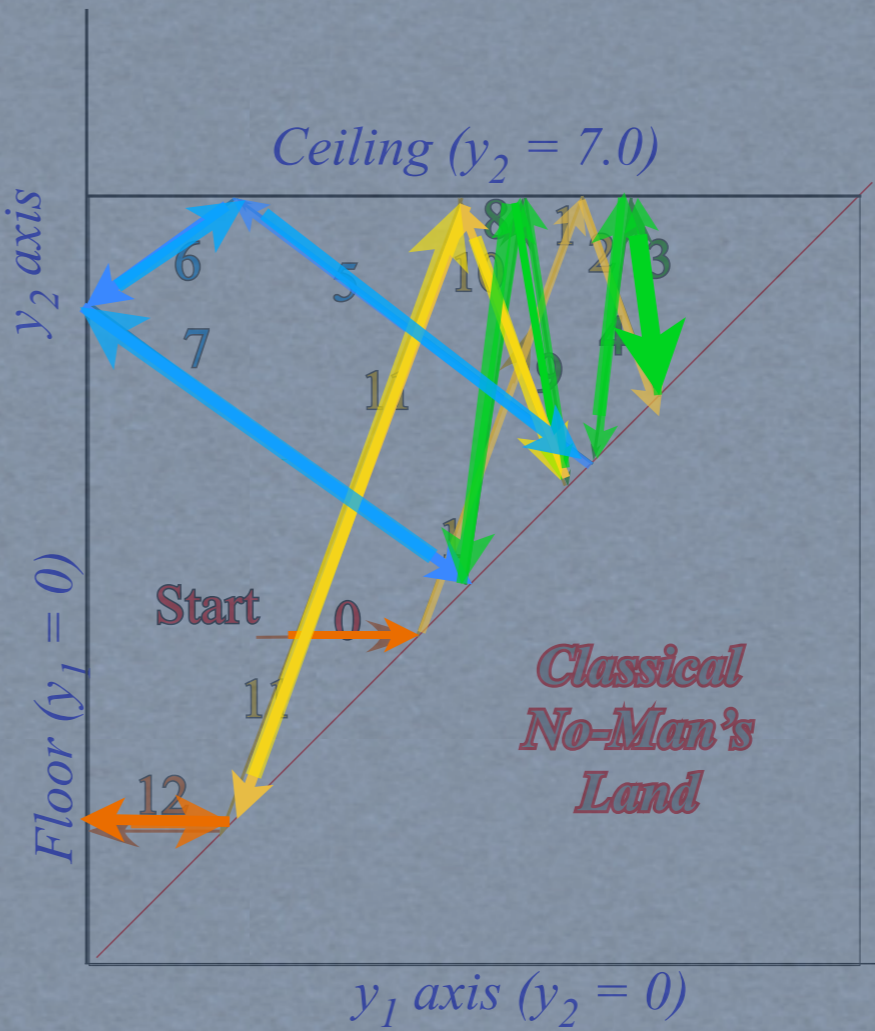
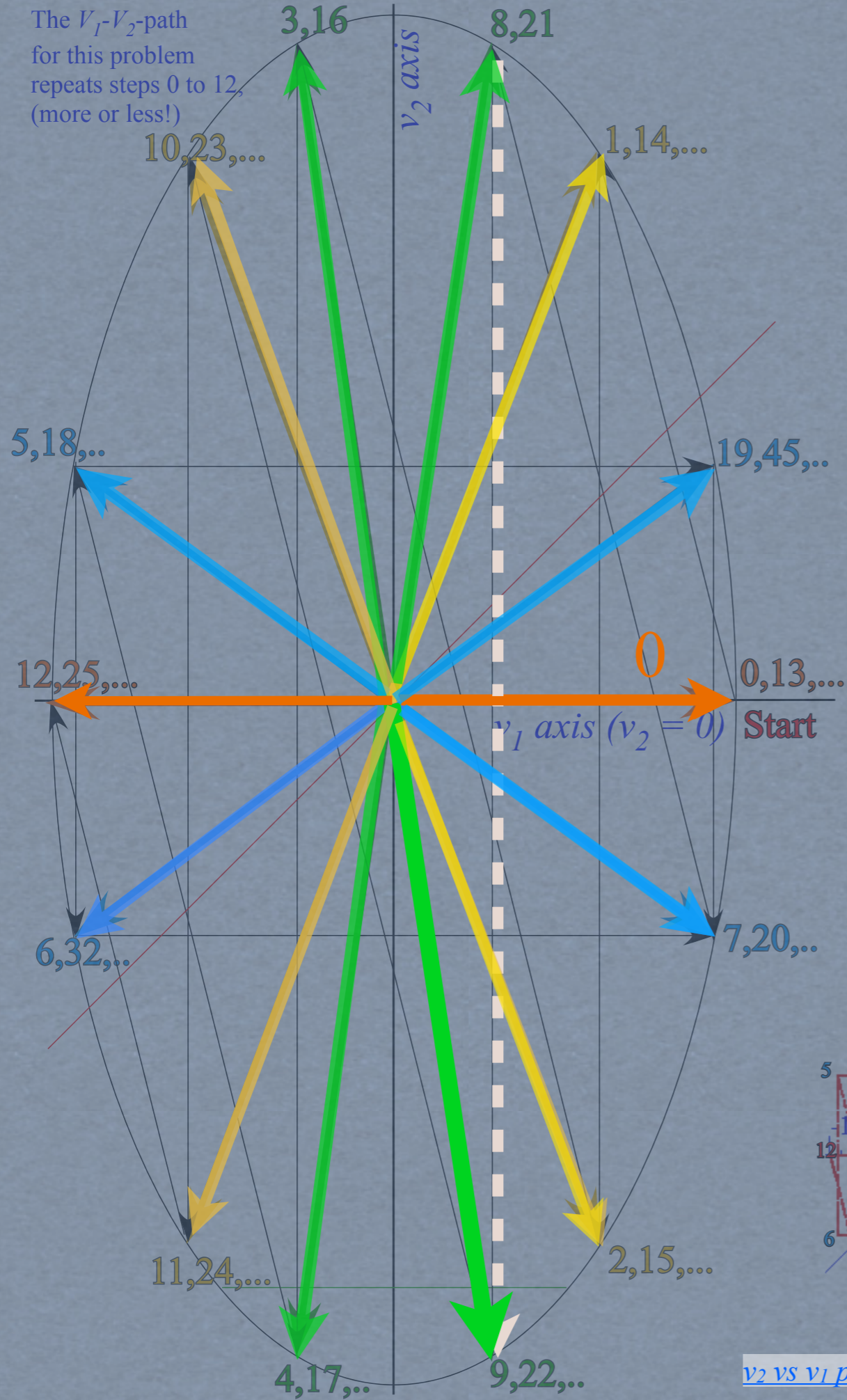
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



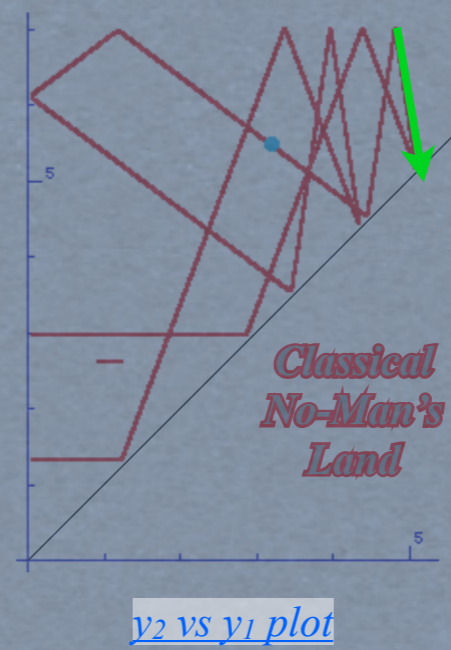
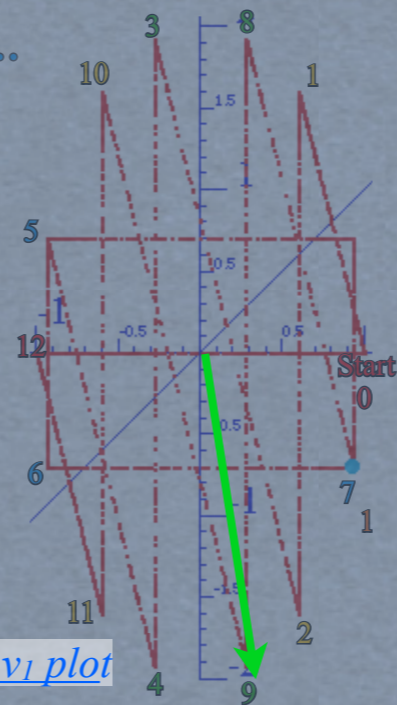
Simulations by *Bouncelt*



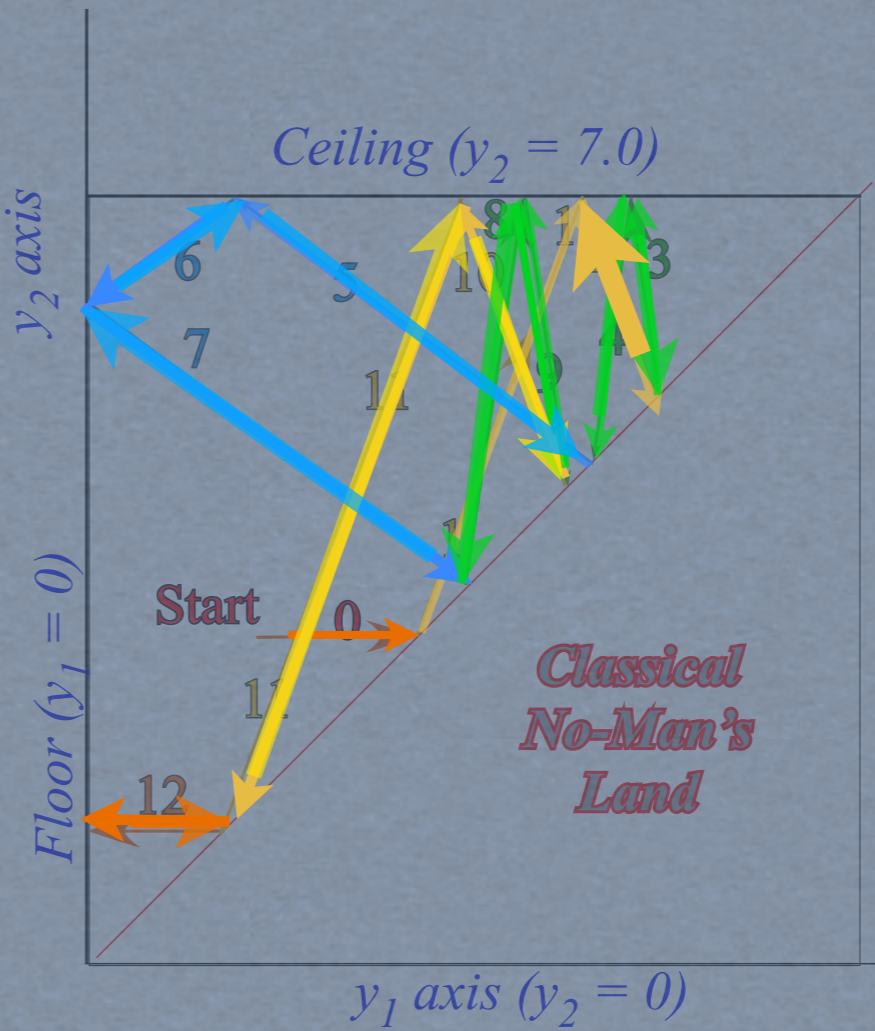
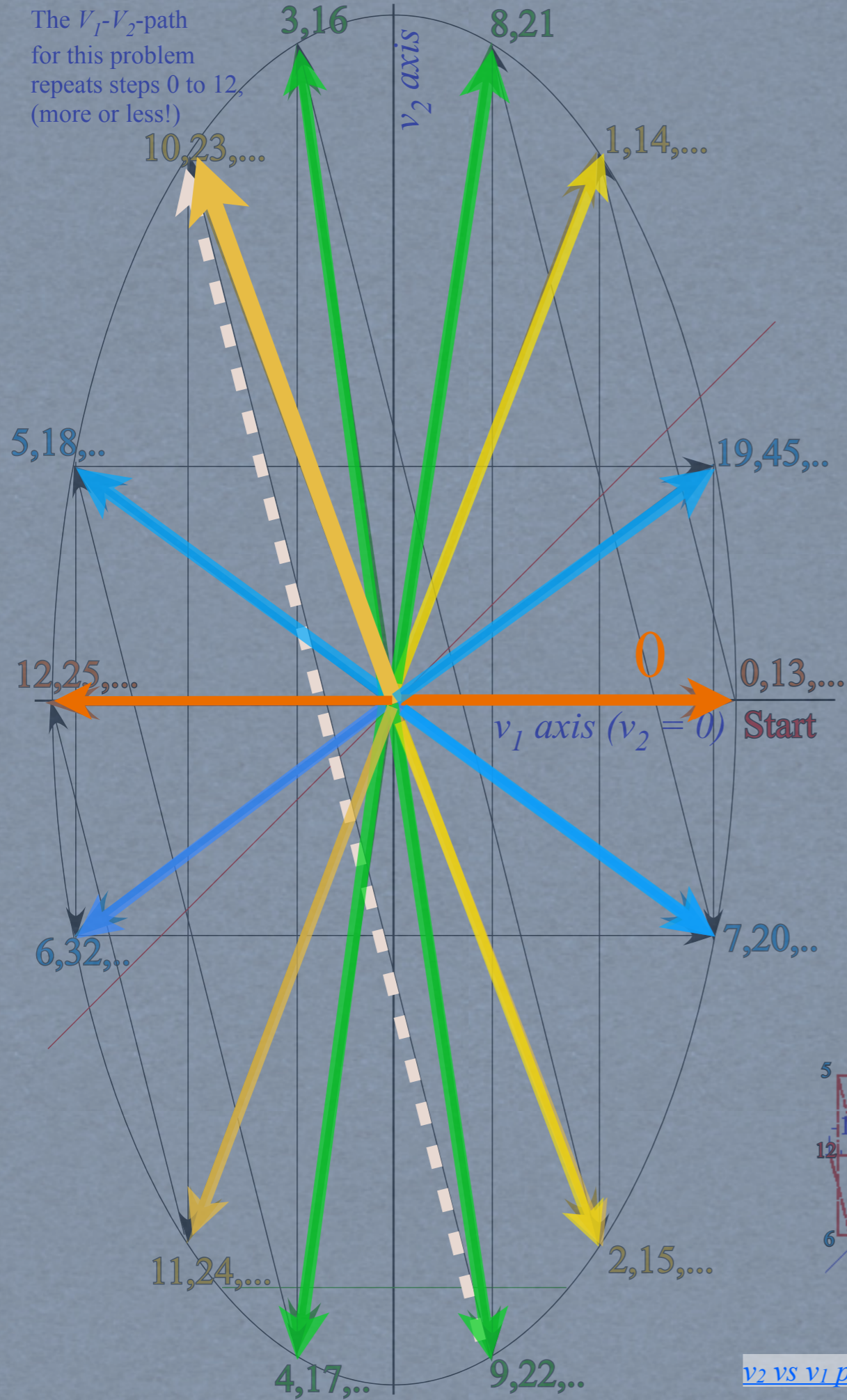
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



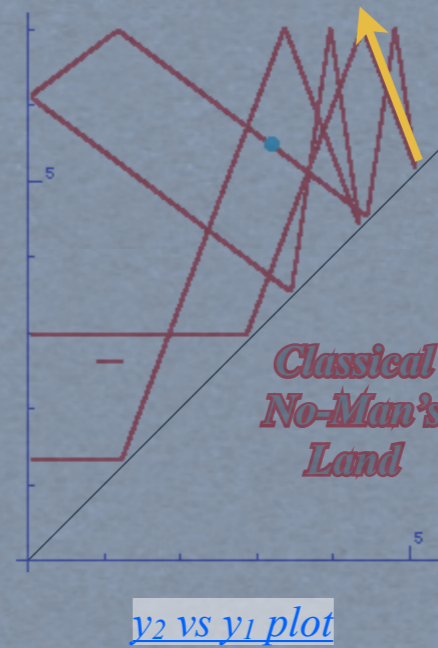
Simulations by *Bouncelt*



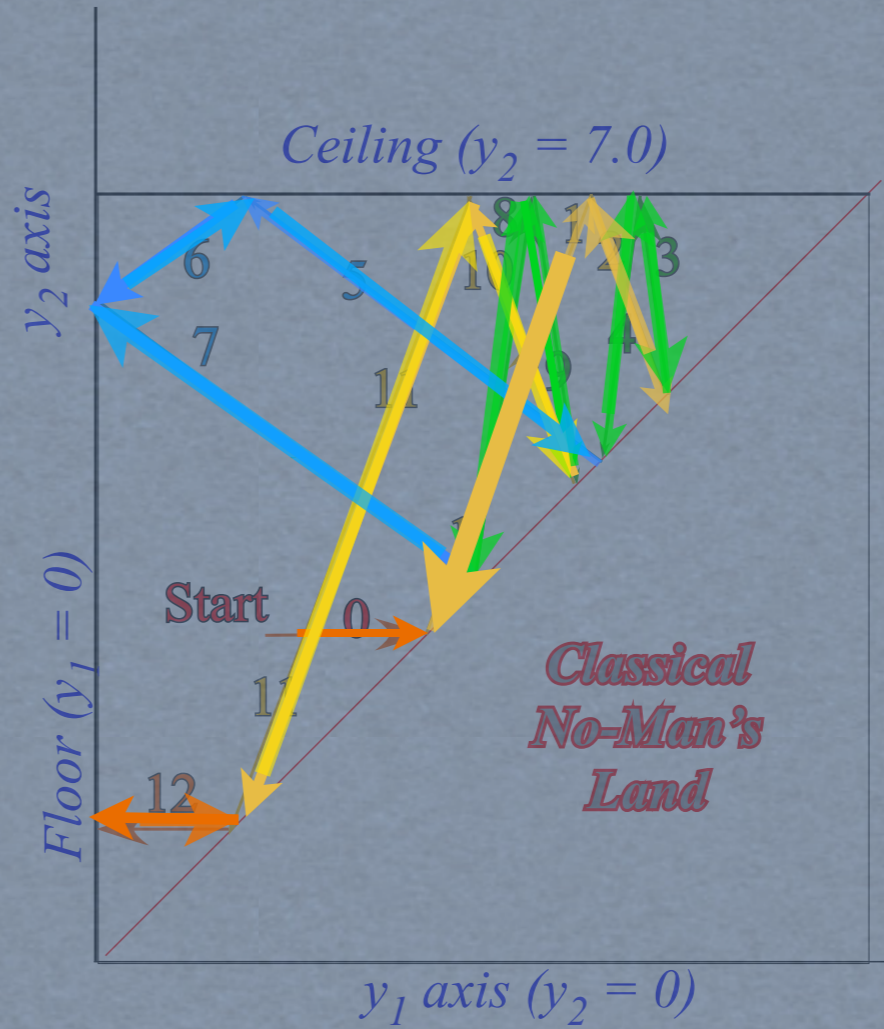
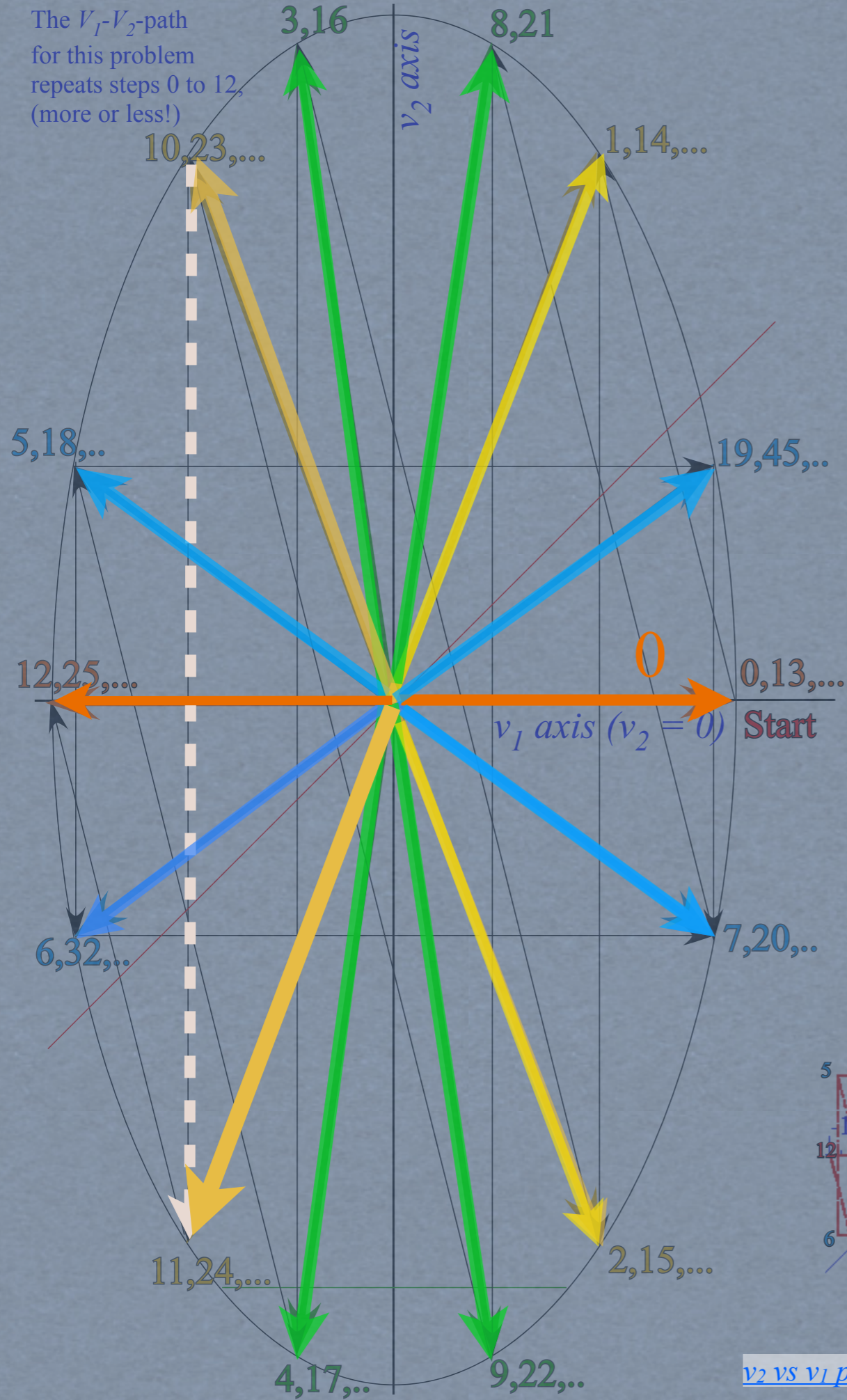
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



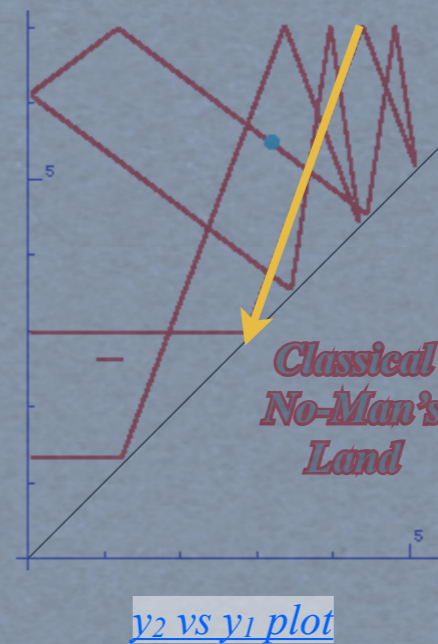
Simulations by *Bouncelt*



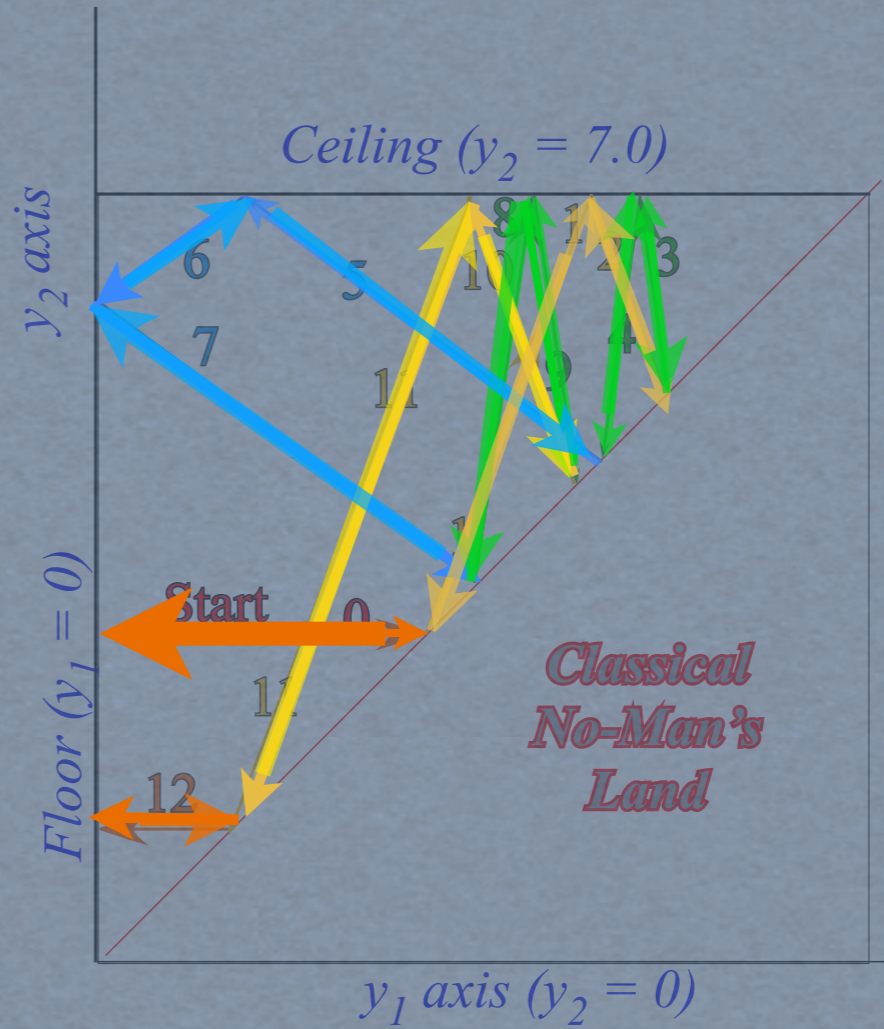
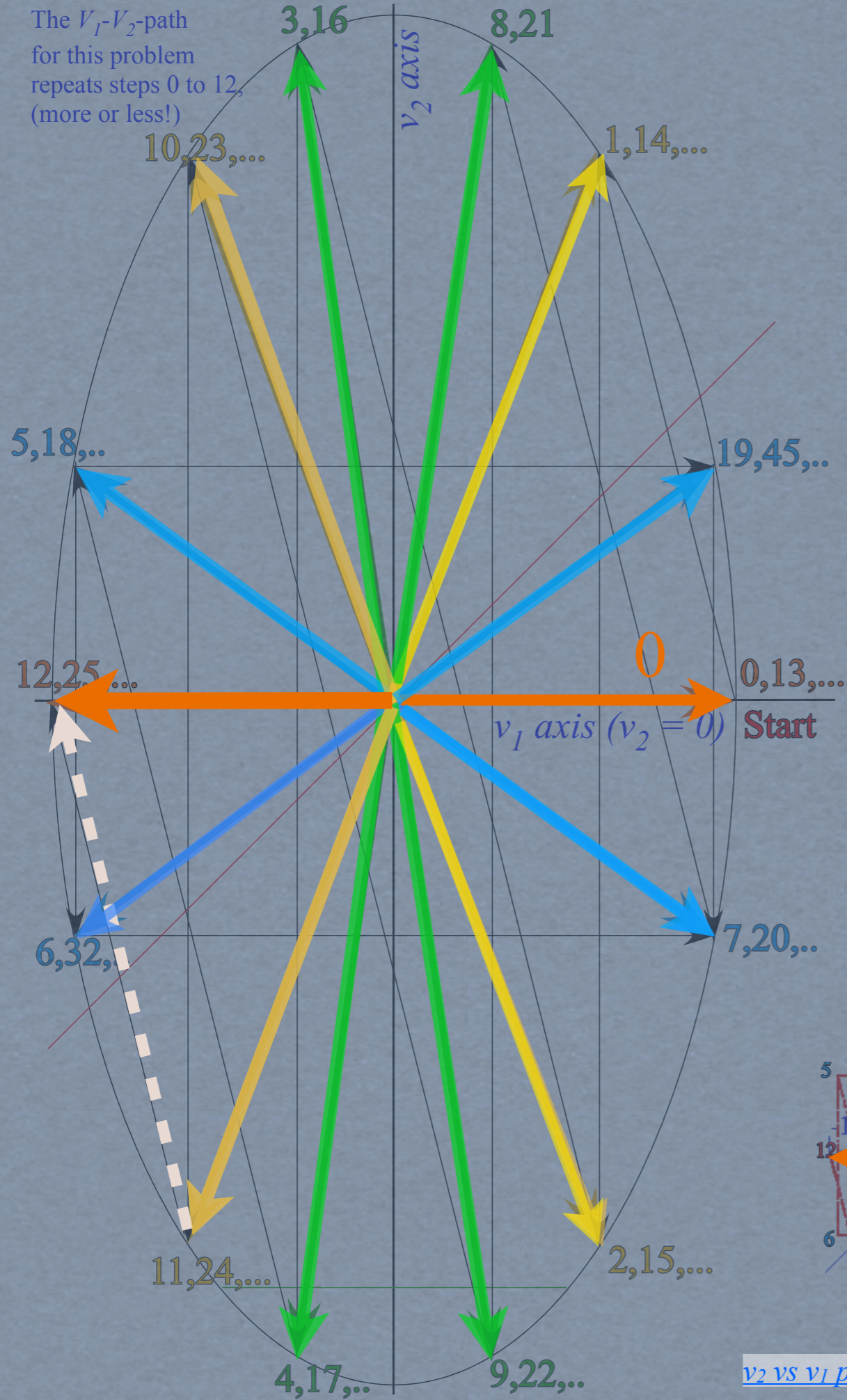
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



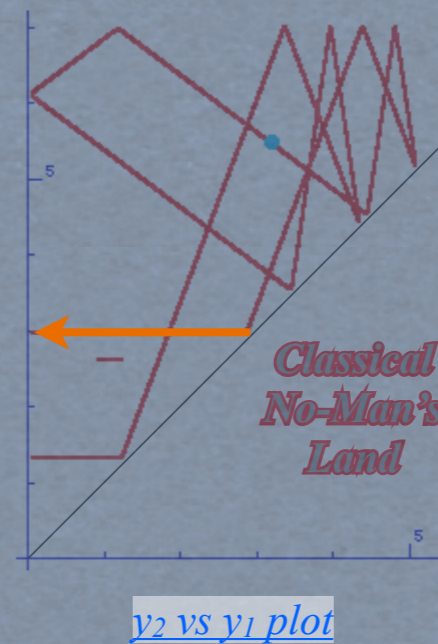
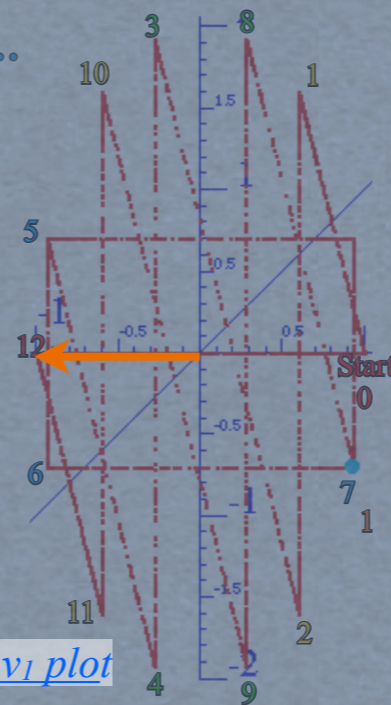
Simulations by *Bouncelt*



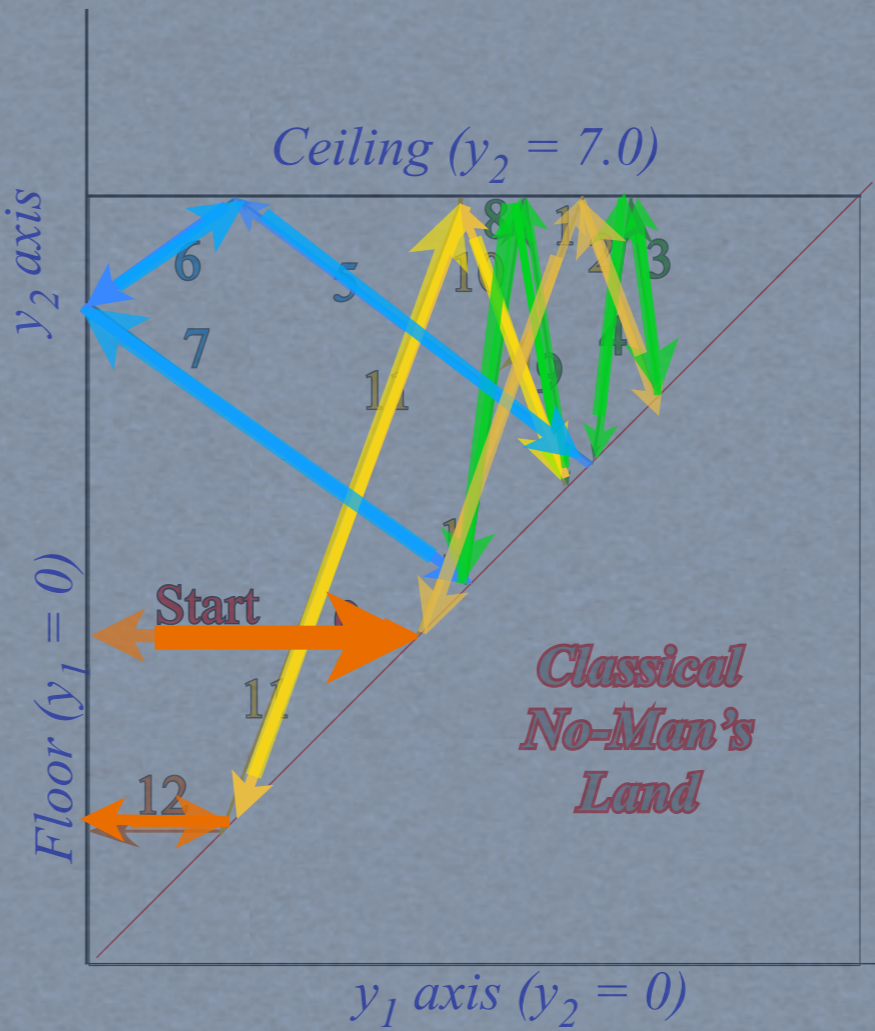
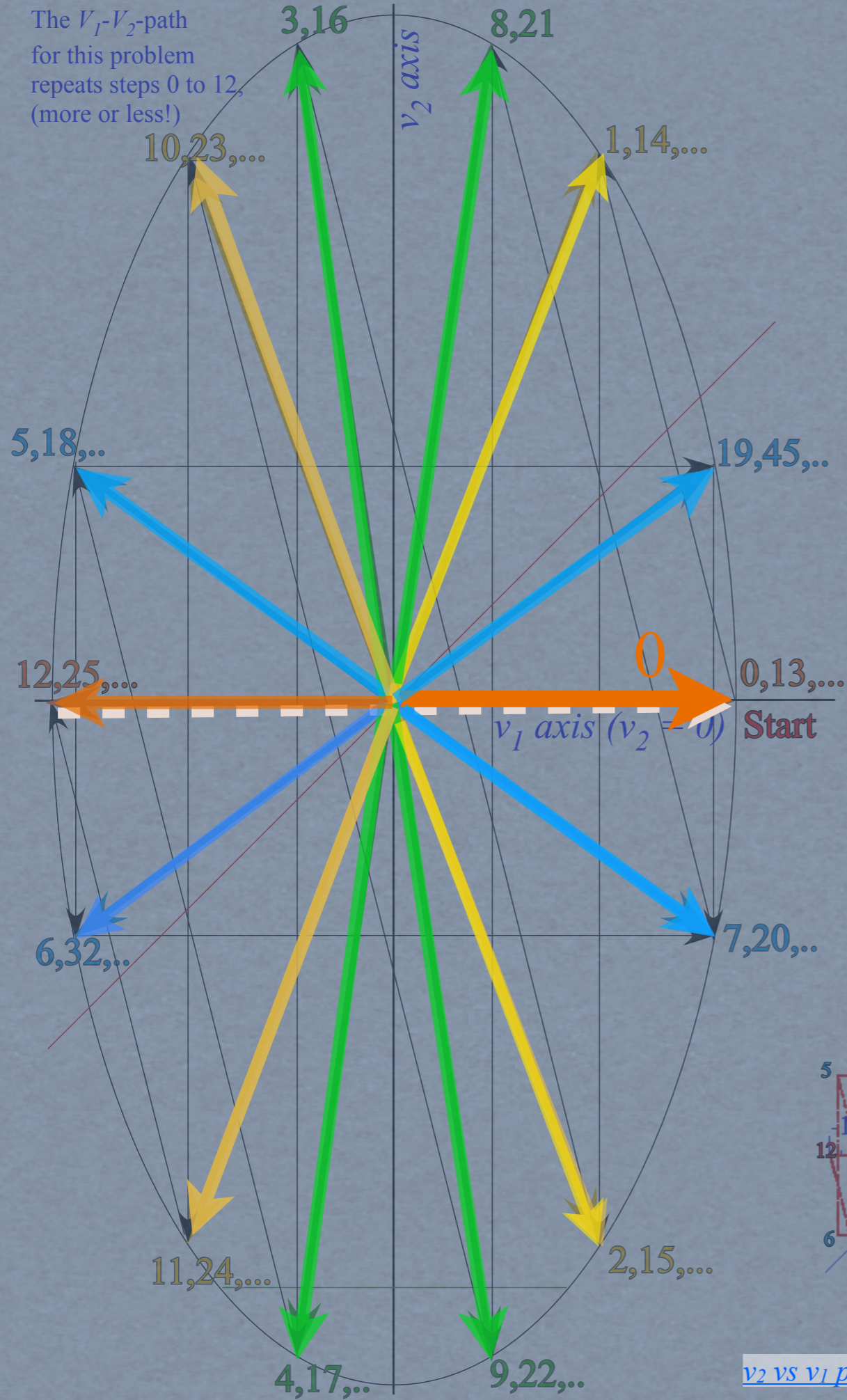
The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



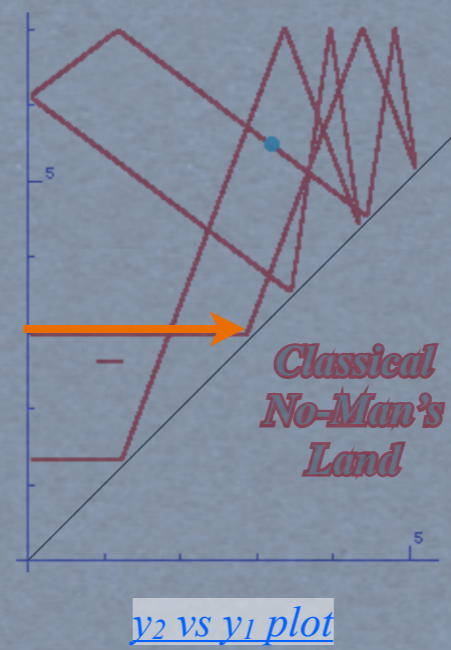
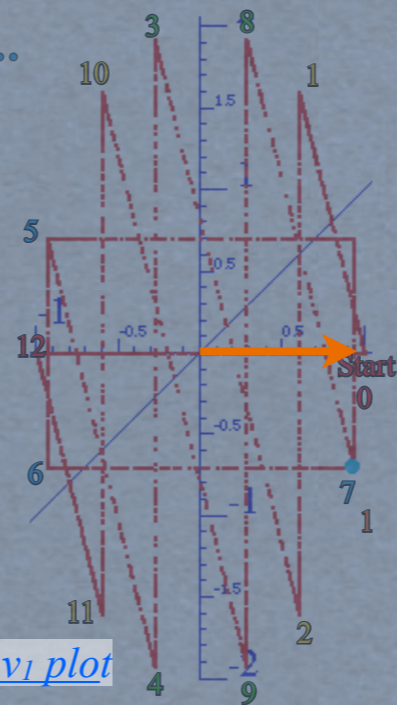
Simulations by Bouncelt



The V_1 - V_2 -path for this problem repeats steps 0 to 12, (more or less!)



Simulations by *Bouncelt*

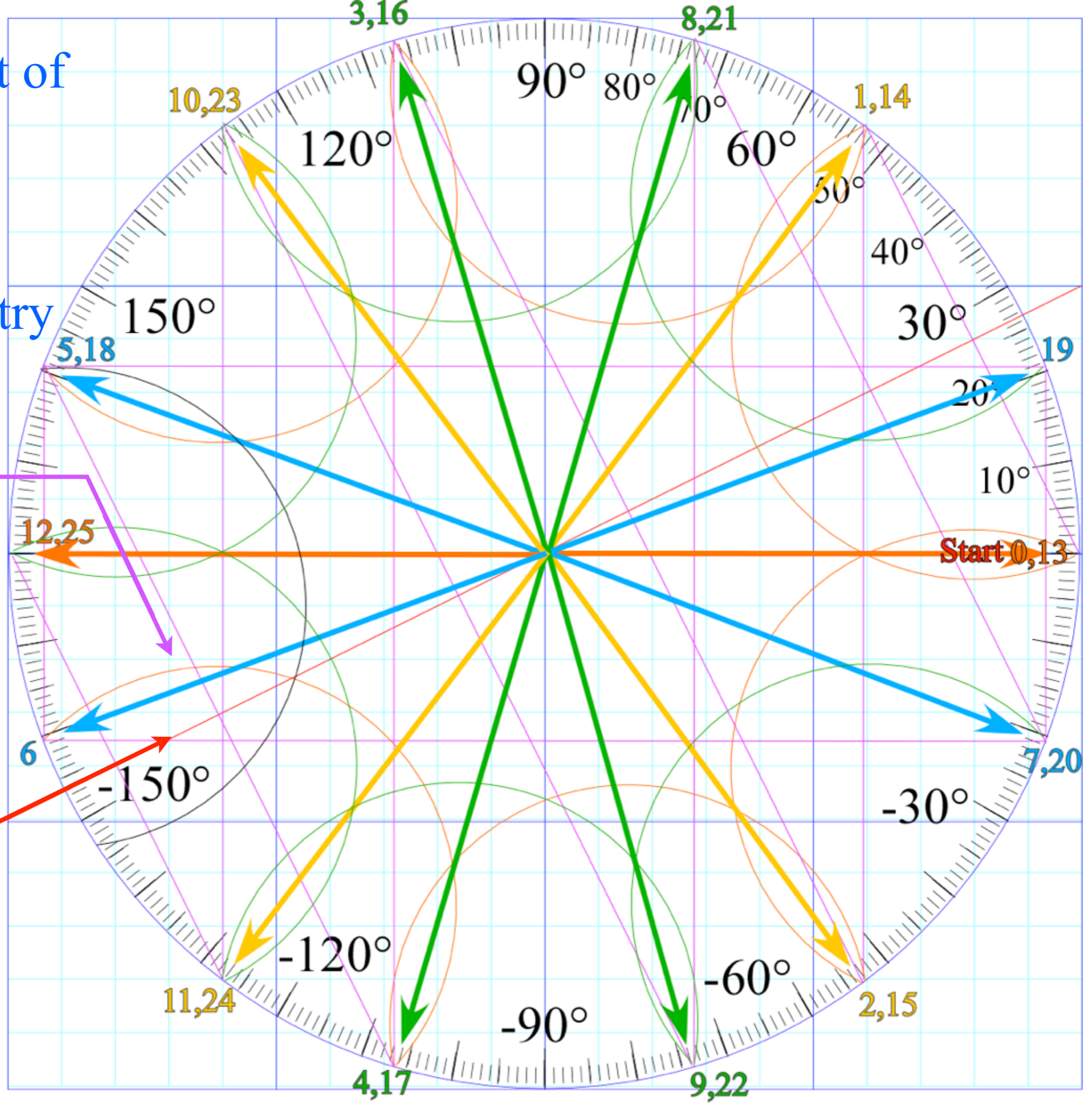


Classical No-Man's Land

Estrangian plot of
 $m_1/m_2=4/1$
 collision
 sequence
 shows symmetry
 (sort of)

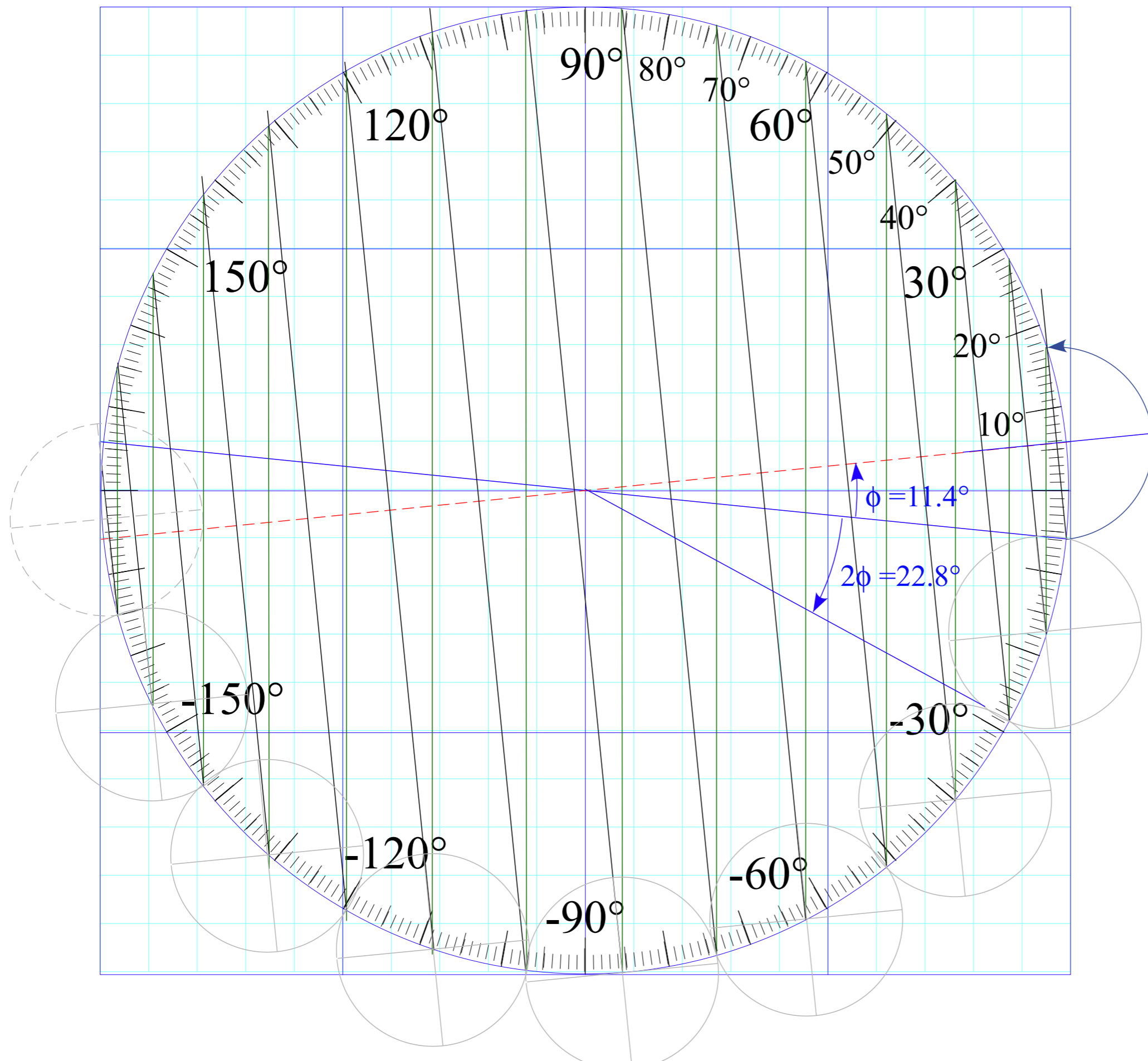
c.o.m. lines
 (cons. of mom.)
 have slope
 $-\sqrt{m_2}/\sqrt{m_1}=-2/1$

COM line
 has slope
 $\sqrt{m_2}/\sqrt{m_1}=1/2$

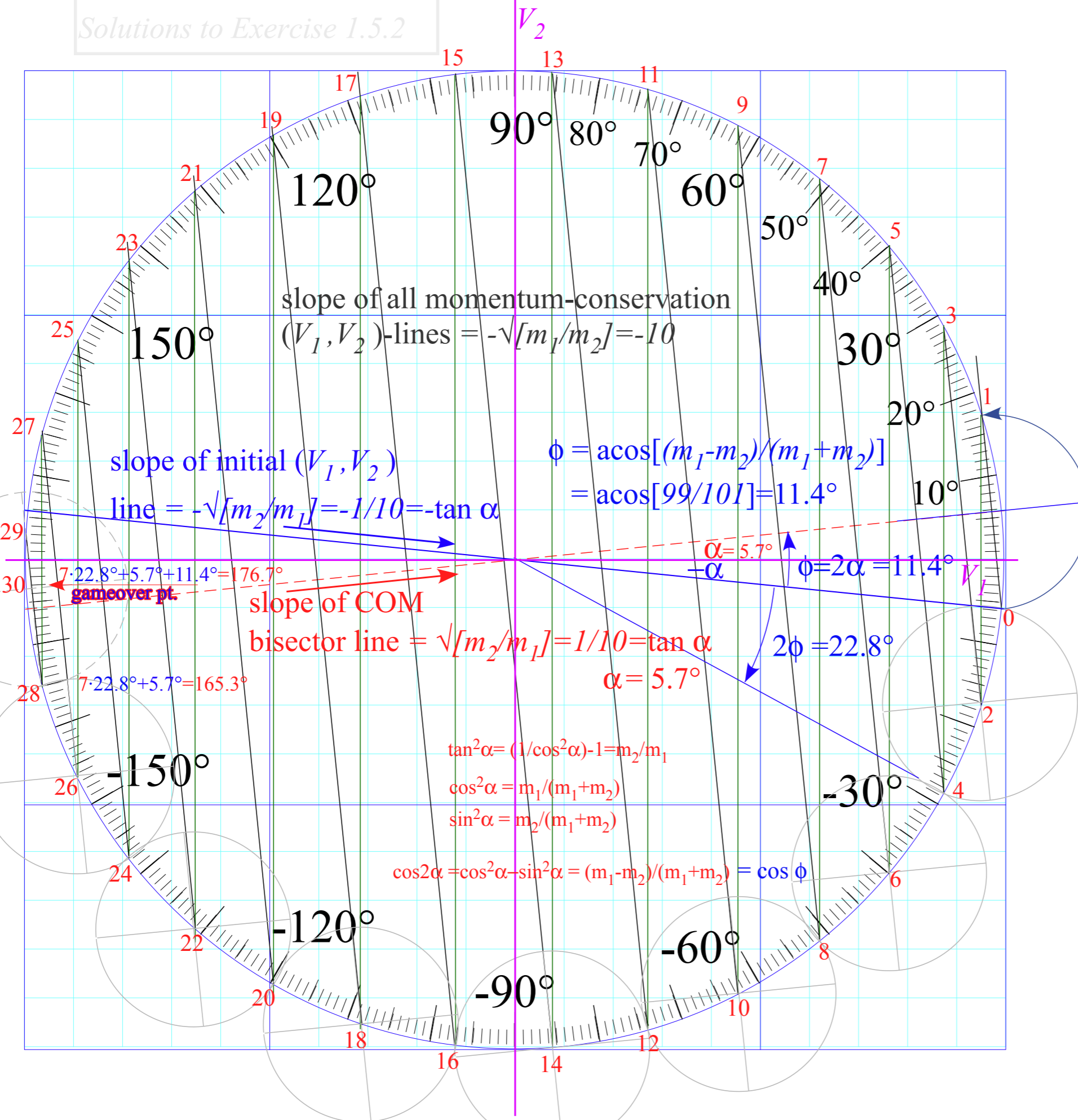


$$\phi = \arccos[(m_1 - m_2)/(m_1 + m_2)] = \arccos[99/101] = 11.4^\circ$$

*Collisions for
mass ratio
 $m_1:m_2 = 100:1$*



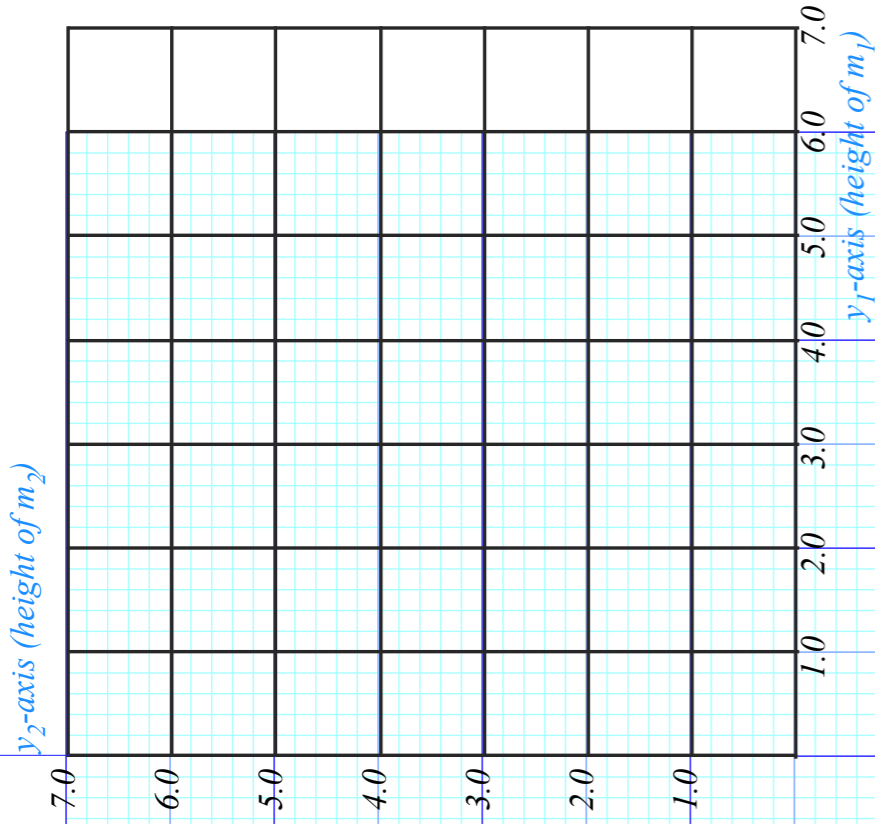
Collisions for
mass ratio
 $m_1:m_2 = 100:1$



BounceIt Web Simulations
 $m_1:m_2 = 100:1$ $(v_1, v_2) = (1, 0)$

[V2 vs V1 Estrangian plot](#)

[Supplementary: y2 vs y1 plot](#)



V_2 -axis

3.0

2.0

1.0

1.0

2.0

3.0

4.0

5.0

6.0

7.0

y₁-axis (height of m₁)

-1.0

-0.5

0

0.5

1.0

1.5

2.0

V_1 -axis

-1.0

-2.0

y₁ and y₂-axis (height of m₁ and m₂)

7.0

6.0

5.0

4.0

3.0

2.0

1.0

1.0

2.0

3.0

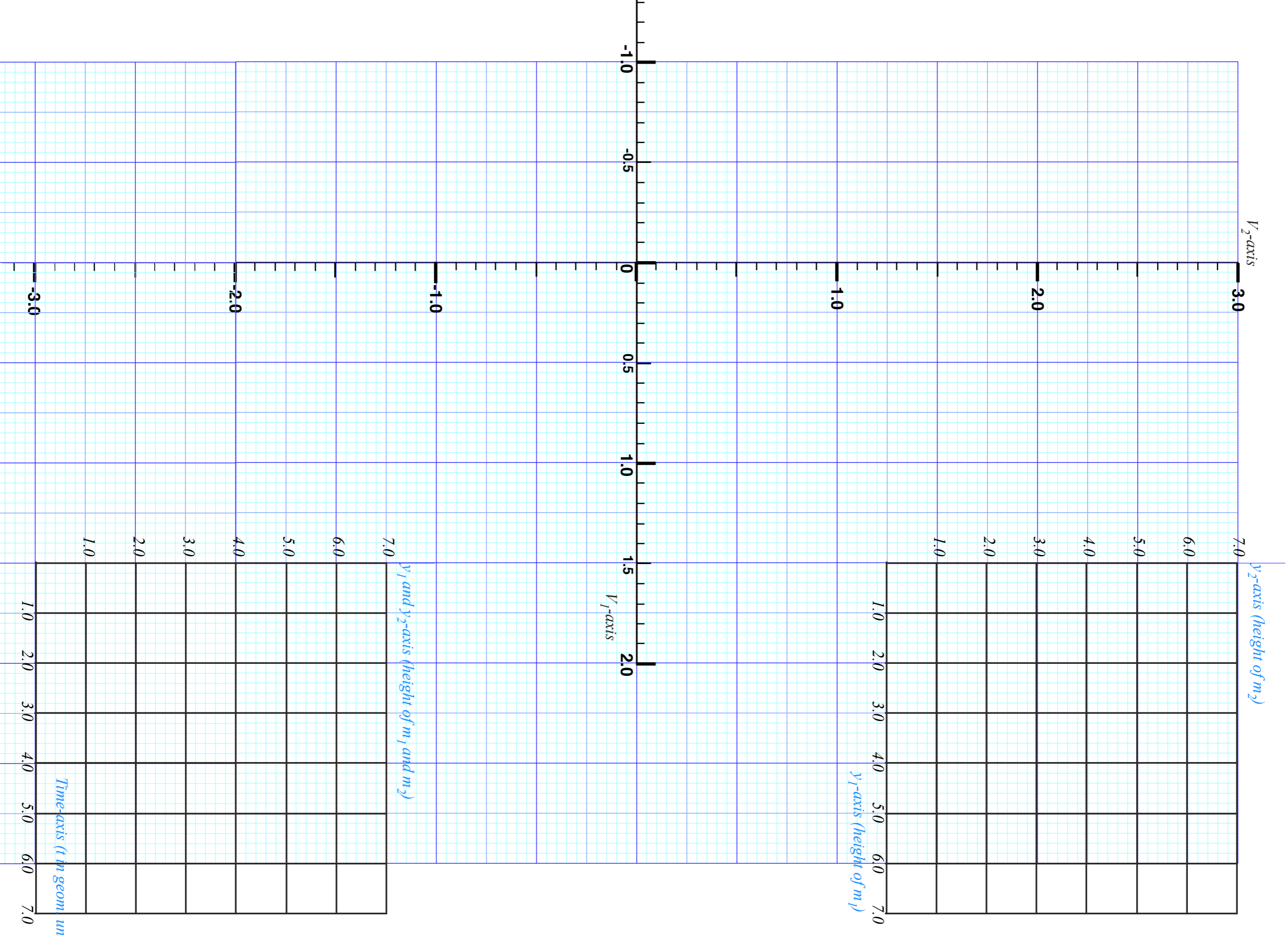
4.0

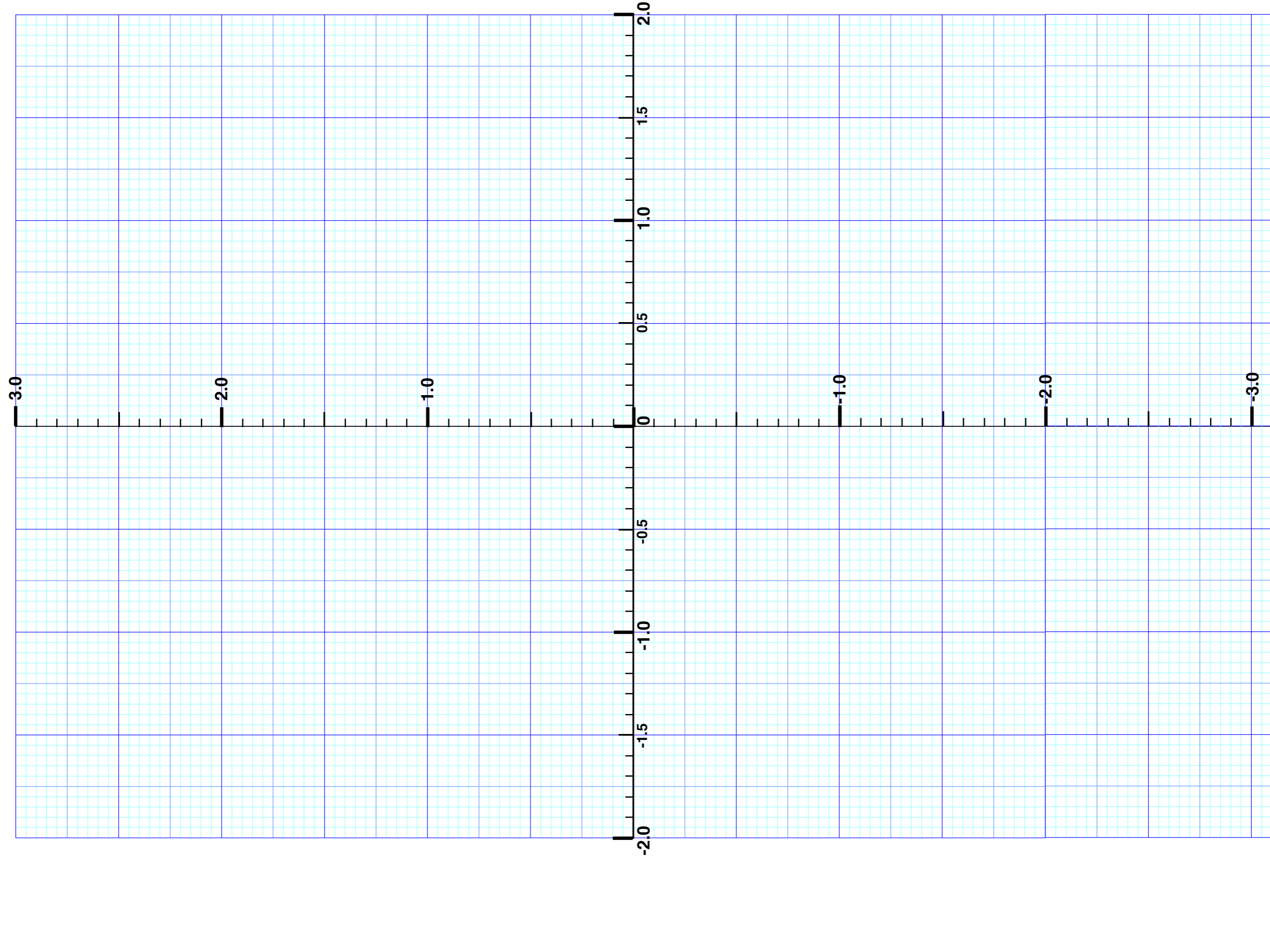
5.0

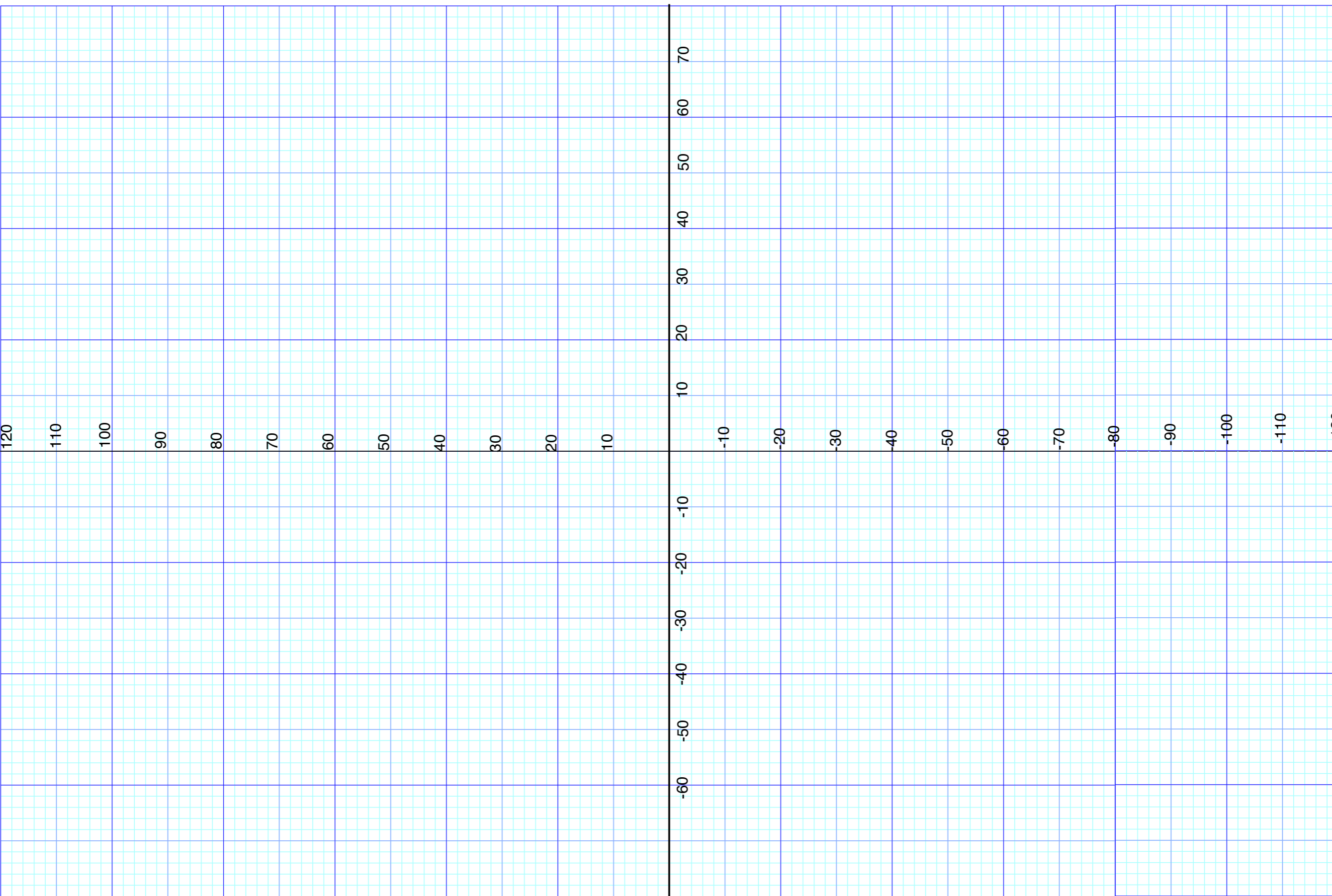
6.0

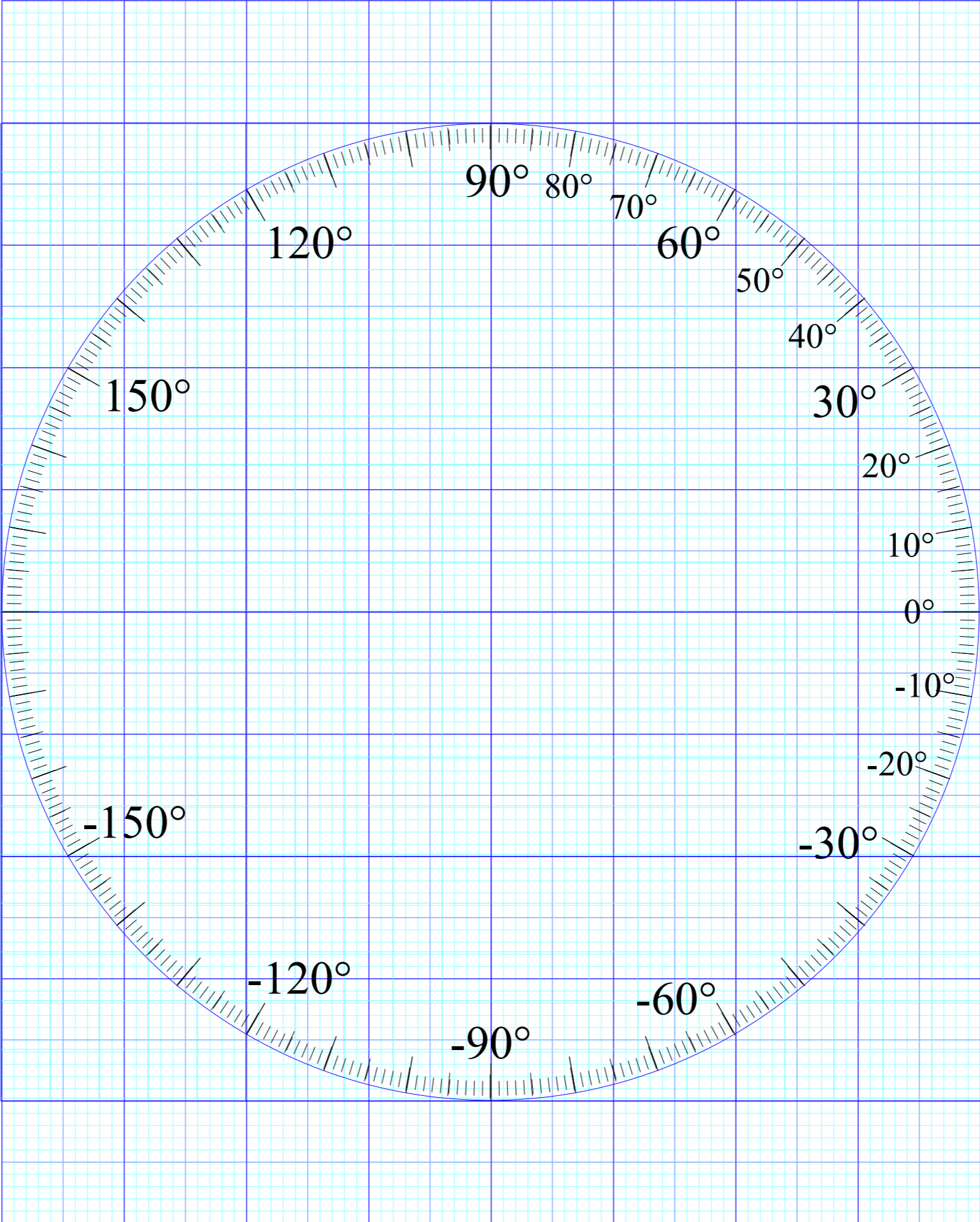
7.0

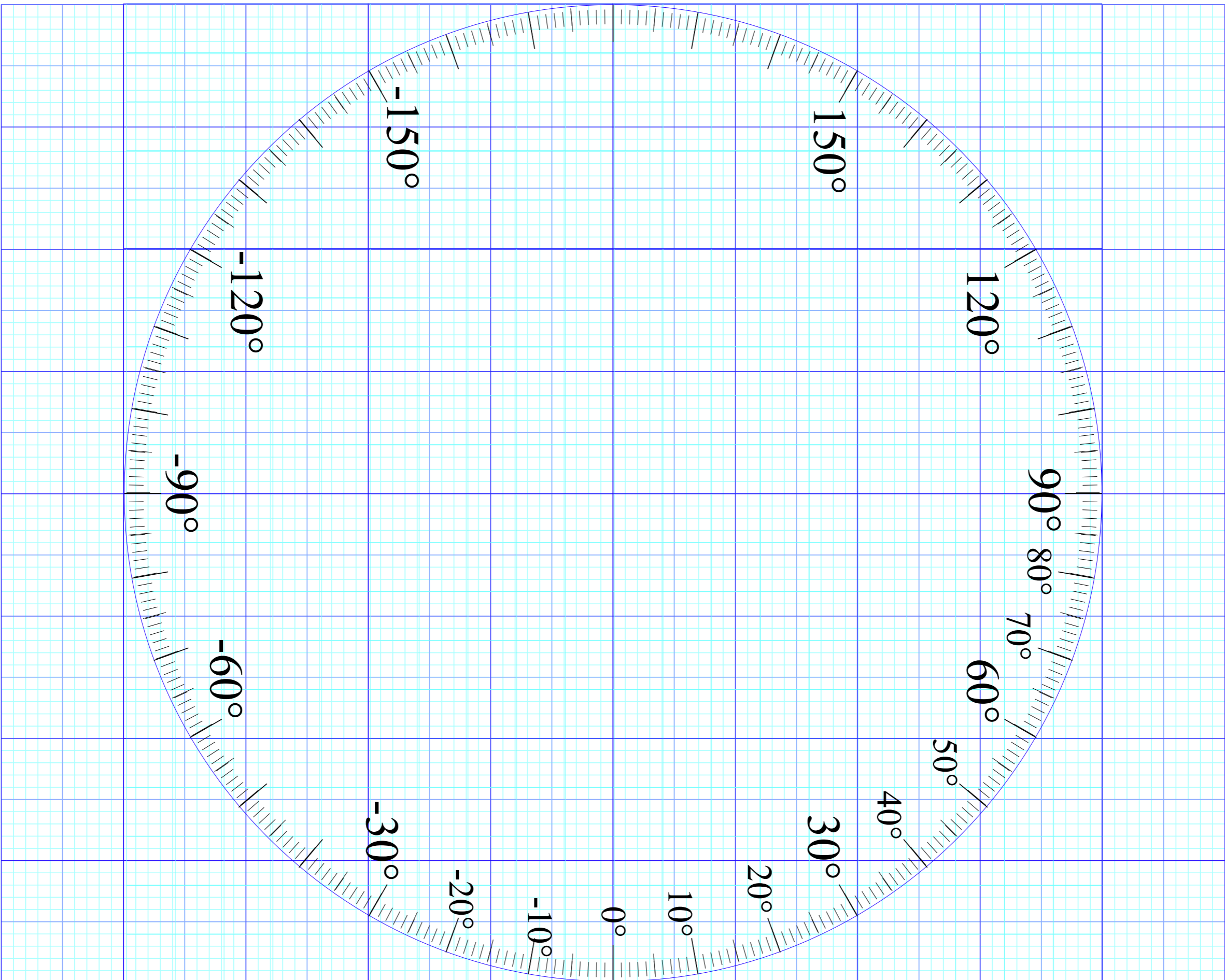
Time-axis (t in geom. units)











AMOP reference links (Updated list given on 2nd and 3rd pages of each class presentation)

[Web Resources - front page](#)

[Quantum Theory for the Computer Age](#)

[2014 AMOP](#)

[UAF Physics UTube channel](#)

[Principles of Symmetry, Dynamics, and Spectroscopy](#)

[2017 Group Theory for QM](#)

[Classical Mechanics with a Bang!](#)

[2018 AMOP](#)

[Modern Physics and its Classical Foundations](#)

[Representations Of Multidimensional Symmetries In Networks - harter-jmp-1973](#)

Alternative Basis for the Theory of Complex Spectra

[Alternative Basis for the Theory of Complex Spectra I - harter-pra-1973](#)

[Alternative Basis for the Theory of Complex Spectra II - harter-patterson-pra-1976](#)

[Alternative Basis for the Theory of Complex Spectra III - patterson-harter-pra-1977](#)

[Frame Transformation Relations And Multipole Transitions In Symmetric Polyatomic Molecules - RMP-1978](#)

[Asymptotic eigensolutions of fourth and sixth rank octahedral tensor operators - Harter-Patterson-JMP-1979](#)

[Rotational energy surfaces and high- J eigenvalue structure of polyatomic molecules - Harter - Patterson - 1984](#)

[Galloping waves and their relativistic properties - ajp-1985-Harter](#)

[Rovibrational Spectral Fine Structure Of Icosahedral Molecules - Cpl 1986 \(Alt Scan\)](#)

Theory of hyperfine and superfine levels in symmetric polyatomic molecules.

I) [Trigonal and tetrahedral molecules: Elementary spin-1/2 cases in vibronic ground states - PRA-1979-Harter-Patterson \(Alt scan\)](#)

II) [Elementary cases in octahedral hexafluoride molecules - Harter-PRA-1981 \(Alt scan\)](#)

Rotation-vibration spectra of icosahedral molecules.

I) [Icosahedral symmetry analysis and fine structure - harter-weeks-jcp-1989 \(Alt scan\)](#)

II) [Icosahedral symmetry, vibrational eigenfrequencies, and normal modes of buckminsterfullerene - weeks-harter-jcp-1989 \(Alt scan\)](#)

III) [Half-integral angular momentum - harter-reimer-jcp-1991](#)

[Rotation-vibration scalar coupling zeta coefficients and spectroscopic band shapes of buckminsterfullerene - Weeks-Harter-CPL-1991 \(Alt scan\)](#)

[Nuclear spin weights and gas phase spectral structure of ¹²C₆₀ and ¹³C₆₀ buckminsterfullerene -Harter-Reimer-Cpl-1992 - \(Alt1, Alt2 Erratum\)](#)

[Gas Phase Level Structure of C₆₀ Buckyball and Derivatives Exhibiting Broken Icosahedral Symmetry - reimer-diss-1996](#)

[Fullerene symmetry reduction and rotational level fine structure/ the Buckyball isotopomer ¹²C ¹³C₅₉ - jcp-Reimer-Harter-1997 \(HiRez\)](#)

[Wave Node Dynamics and Revival Symmetry in Quantum Rotors - harter - jms - 2001](#)

[Molecular Symmetry and Dynamics - Ch32-Springer Handbooks of Atomic, Molecular, and Optical Physics - Harter-2006](#)

Resonance and Revivals

I) [QUANTUM ROTOR AND INFINITE-WELL DYNAMICS - ISMSLi2012 \(Talk\) OSU knowledge Bank](#)

II) [Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 \(Talks\)](#)

III) [Quantum Resonant Beats and Revivals in the Morse Oscillators and Rotors - \(2013-Li-Diss\)](#)

[Resonance and Revivals in Quantum Rotors - Comparing Half-integer Spin and Integer Spin - Alva-ISMS-Ohio2013-R777 \(Talk\)](#)

[Molecular Eigensolution Symmetry Analysis and Fine Structure - IJMS-harter-mitchell-2013](#)

[Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-cpl-2013](#)

[QTCA Unit 10 Ch 30 - 2013](#)

[AMOP Ch 0 Space-Time Symmetry - 2019](#)

**In development - a web based A.M.O.P. oriented reference page, with thumbnail/previews, greater control over the information display, and eventually full on Apache-SOLR Index and search for nuanced, whole-site content/metadata level searching.*

AMOP reference links (Updated list given on 2nd and 3rd pages of each class presentation)

(Int.J.Mol.Sci, 14, 714(2013) p.755-774 ,

QTCA Unit 7 Ch. 23-26),

(PSDS - Ch. 5, 7)

[Int.J.Mol.Sci, 14, 714\(2013\),](#)

[QTCA Unit 8 Ch. 23-25,](#)

[QTCA Unit 9 Ch. 26,](#)

[PSDS Ch. 5,](#)

[PSDS Ch. 7](#)

Intro spin ½ coupling

[Unit 8 Ch. 24 p3](#)

Irrep Tensor building

[Unit 8 Ch. 25 p5.](#)

Intro 3-particle coupling.

[Unit 8 Ch. 25 p28.](#)

H atom hyperfine-B-level crossing

[Unit 8 Ch. 24 p15](#)

Irrep Tensor Tables

[Unit 8 Ch. 25 p12.](#)

Intro 3,4-particle Young Tableaus

[GrpThLect29 p42.](#)

Hyperf. theory [Ch. 24 p48.](#)

Hyperf. theory Ch. 24 p48.

[Deeper theory ends p53](#)

Wigner-Eckart tensor Theorem.

[Unit 8 Ch. 25 p17.](#)

Young Tableau Magic Formulae

[GrpThLect29 p46-48.](#)

Intro 2p3p coupling

[Unit 8 Ch. 24 p17.](#)

Tensors Applied to d,f-levels.

[Unit 8 Ch. 25 p21.](#)

Intro LS-jj coupling

[Unit 8 Ch. 24 p22.](#)

CG coupling derived (start)

[Unit 8 Ch. 24 p39.](#)

Tensors Applied to high J levels.

[Unit 8 Ch. 25 p63.](#)

CG coupling derived (formula)

[Unit 8 Ch. 24 p44.](#)

Lande' g-factor

[Unit 8 Ch. 24 p26.](#)

AMOP reference links (Updated list given on 2nd and 3rd and 4th pages of each class presentation)

Predrag Cvitanovic's: Birdtrack Notation, Calculations, and Simplification

[Chaos Classical and Quantum - 2018-Cvitanovic-ChaosBook](#)
[Group Theory - PUP Lucy Day - Diagrammatic notation - Ch4](#)
[Simplification Rules for Birdtrack Operators - Alcock-Zeilinger-Weigert-zeilinger-jmp-2017](#)
[Group Theory - Birdtracks Lies and Exceptional Groups - Cvitanovic-2011](#)
[Simplification rules for birdtrack operators- jmp-alcock-zeilinger-2017](#)
[Birdtracks for SU\(N\) - 2017-Keppeler](#)

Frank Rioux's: UMA method of vibrational induction

[Quantum Mechanics Group Theory and C60 - Frank Rioux - Department of Chemistry Saint Johns U](#)
[Symmetry Analysis for H2O- H2OGrpTheory- Rioux](#)
[Quantum Mechanics-Group Theory and C60 - JChemEd-Rioux-1994](#)
[Group Theory Problems- Rioux- SymmetryProblemsX](#)
[Comment on the Vibrational Analysis for C60 and Other Fullerenes Rioux-RSP](#)

Supplemental AMOP Techniques & Experiment

[Many Correlation Tables are Molien Sequences - Klee \(Draft 2016\)](#)
[High-resolution spectroscopy and global analysis of CF4 rovibrational bands to model its atmospheric absorption- carlos-Boudon-iqsrt-2017](#)
[Symmetry and Chirality - Continuous Measures - Avnir](#)

*

Special Topics & Colloquial References

[r-process nucleosynthesis from matter ejected in binary neutron star mergers-PhysRevD-Bovard-2017](#)

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