Analysis of 1D 2-Body Collisions
(Ch. 3 and Ch. 4 of Unit 1)

Review of elastic Kinetic Energy ellipse geometry

The X2 Superball pen launcher
Perfectly elastic “ka-bong” velocity amplification effects (Faux-Flubber)

Geometry of X2 launcher bouncing in box
Independent Bounce Model (IBM)
Geometric optimization and range-of-motion calculation(s)
Integration of \((V_1, V_2)\) data to space-time plots \((y_1(t), t)\) and \((y_2(t), t)\) plots
Integration of \((V_1, V_2)\) data to space-space plots \((y_1, y_2)\)

Multiple collisions calculated by matrix operator products
Matrix or tensor algebra of 1-D 2-body collisions

Ellipse rescaling-geometry and reflection-symmetry analysis
Rescaling KE ellipse to circle
How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics in Ch. 12
Review of elastic Kinetic Energy ellipse geometry

Elastic Kinetic Energy ellipse (KE = 7,250)

- \( a = \sqrt{\frac{2 \cdot KE}{M_{SUV}}} = 60.21 \)
- \( b = \sqrt{\frac{2 \cdot KE}{m_{VW}}} = 120.42 \)

Inelastic Kinetic Energy ellipse (IE = 6,250)

- \( a = \sqrt{\frac{2 \cdot IE}{M_{SUV}}} = 55.9 \)
- \( b = \sqrt{\frac{2 \cdot IE}{m_{VW}}} = 111.8 \)

Momentum \( p_{Total} = 250 \)

\( m_{VW} = 1 \)

Fig. 3.1 a in Unit 1

Fig. 3.1 b in Unit 1
The X-2 Pen launcher and Superball Collision Simulator*

**ballpoint pen**

\[ M_2 = 10 \text{g} \]

**Superball**

\[ M_1 = 70 \text{gm} \]

The X-2 pen-launcher

**Superball penetration depth**

\[ d = \frac{r^2}{2R} \]

**bounce plate**

\[ M_0 = 10 \text{kg} \]

*Simulator Website: [http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html](http://www.uark.edu/ua/modphys/testing/markup/BounceItWeb.html)
Fig. 4.1 and Fig. 4.3 in Unit 1

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

(Still Bigger BANG!) (Bigger BANG!)

1st bang: 
M₁ off floor

Fig. 4.4a-b in Unit 1

(a) Bang-1 

1st bang: mass (M₀) vs. mass (M₁)

This 1st bang is a floor-bounce of 
M₁ off very massive plate/Earth M₀

ballpoint pen 
M₂=10gm

Superball penetration 
depth 

M₁=70gm

R R R

d = \frac{r^2}{2R}

bounce plate 
M₀=10kg

The X-2 pen-launcher
1st bang: $M_1$ off floor
2nd bang: $m_2$ off $M_1$
The X-2 pen-launcher

Fig. 4.1 and Fig. 4.3 in Unit 1

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

3rd bang: m₂ off ceiling

(b) 2-Bang Model

1st bang: M₁ off floor

2nd bang: m₂ off M₁

This 1st bang is a floor-bounce of M₁ off very massive plate/Earth M₀

Bang₁(₀₁)
INIT point at (-1.0,-1.0)

Bang₁(₀₁)
FINAL point (+1.0,+1.0)

Ballpoint pen
M₂ = 10gm

Superball
M₁ = 70gm

Superball penetration depth
d = \frac{v^2}{2R}

Bounce plate
M₀ = 10kg

M₁ mirror reflection thru m₂ axis

1st bang: mass (M₀) vs. mass (M₁)

M₁ Velocity axis
V_{sym1}

m₁ Velocity axis
V_{sym2}
Fig. 4.1 and Fig. 4.3 in Unit 1

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

1st bang: $M_1$ off floor

2nd bang: $m_2$ off $M_1$

3rd bang: $m_2$ off ceiling

Fig. 4.4a-b in Unit 1

(a) Bang-1

1st bang:

mass ($M_0$) vs. mass ($M_1$)

This 1st bang is a floor-bounce of $M_1$ off very massive plate/Earth $M_0$

(b) $M_1 << M_0$

FINAL (Elastic)

INITIAL

FINAL

(Totally Inelastic)

Very skinny Energy ellipse for $M_0 >> M_1$

1st bang $M_1$ off floor “skinny-ellipse”
Fig. 4.1 and Fig. 4.3 in Unit 1

(a) Super-elastic 2nd-body bounce
(b) 2-Bang Model
(c) n-Body Supernova Superballs

1st bang: $M_1$ off floor
2nd bang: $m_2$ off $M_1$

Later: (a) $M_1 \gg M_2$
FINAL (Totally Inelastic)

Fig. 4.2a in Unit 1 (slightly modified)

1st bang $M_1$ off floor “skinny-ellipse”

Very skinny Energy ellipse for $M_0 \gg M_1$
**Fig. 4.1 and Fig. 4.3 in Unit 1**

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

(Still Bigger BANG!)

(Bigger BANG!)

**Fig. 4.4a-b in Unit 1**

1st bang: 
$M_1$ off floor 

2nd bang: 
$m_2$ off $M_1$

Later: 
(a) $M_1 \gg M_2$

(b) $M_1 < < M_0$

**Fig. 4.2a in Unit 1** (slightly modified)

1st bang $M_1$ off floor “skinny-ellipse”

**Fig. 4.2b in Unit 1** (slightly modified)

Very skinny Energy ellipse for $M_0 > > M_1$
Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)
Geometric optimization and range-of-motion calculation(t)
Integration of $(V_1, V_2)$ data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots
Integration of $(V_1, V_2)$ data to space-space plots $(y_1, y_2)$
The X-2 pen-launcher

Superball penetration depth

\[ d = \frac{r^2}{2R} \]

M₁ = 70gm

M₀ = 10kg

ballpoint pen
M₂ = 10gm

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

(Bigger BANG!)

(Bigger BANG!)

(Still Bigger BANG!)

This 1st bang is a floor-bounce of M₁ off very massive plate/Earth M₀

M₁ mirror reflection thru m₂ axis

M₁ Velocity axis

Vₘ₁

M₂ Velocity axis

Vₘ₂

(m,0)

(0,0)

(0.5,2.5)

(1.0,0.0)

(0.0,1.0)

(M₁)

(M₀)

(M₀)

Bang-2

Bang-1

INNIT point at (-1.0,-1.0)

FINAL point (+1.0,-1.0)

FINAL point (0.5,2.5)

INIT point at (0.0,0.0)

COM-point at (0.75,0.75)

2nd bang: (M₁) vs. (M₂)

1st bang: mass (M₀) vs. mass (M₁)

Fig. 4.1 and Fig. 4.3 in Unit 1

Fig. 4.4a-b in Unit 1

Bang-2 (12)

Bang-1 (01)

Bang-1 (01)

Bang-2 (12)
Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)

Geometric optimization and range-of-motion calculation(s)

Integration of $(V_1, V_2)$ data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of $(V_1, V_2)$ data to space-space plots $(y_1, y_2)$
Fig. 4.1 and Fig. 4.3 in Unit 1

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

 STILL BIGGER BANG (Bigger BANG?)

Bang-2(12) FINAL points

Line CPL is elastic collision final pt. locus for different momentum slopes or mass ratios $M_1:M_2$

$L$ is 15:1

$P$ is 7:1

$C$ is 4:1

START at (1.0,-1.0)

Fig. 4.5a in Unit 1

ballpoint pen $M_2=10\text{gm}$

The X-2 pen-launcher

Superball penetration depth $d = \frac{R^2}{2R}$

bounce plate $M_0=10\text{kg}$

104x500
Fig. 4.1 and Fig. 4.3 in Unit 1

(a) Super-elastic 2nd-body bounce

(b) 2-Bang Model

(c) n-Body Supernova Superballs

(Still Bigger BANG!)

(Bigger BANG!)

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The X-2 pen-launcher

Superball penetration depth

\[ d = \frac{r^2}{2R} \]

- Ballpen
- \( M_2 = 10 \text{ gm} \)
- Superball
- \( M_1 = 70 \text{ gm} \)
- Bounce plate \( M_0 = 10 \text{ kg} \)

---

Line CPL is elastic collision final pt. locus for different momentum slopes or mass ratios \( M_1 : M_2 \)

- \( L \) is 15:1
- \( P \) is 7:1
- \( C \) is 4:1

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\( V \) is 1::0 or \( \infty :: 1 \)

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Fig. 4.5a-b in Unit 1

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- \( V_2 \)
- \( M_2 \) Velocity axis
- \( M_1 \) Velocity axis
- \( U_2 \)
- \( U_1 \)
Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)
Geometric optimization and range-of-motion calculation(s)
Integration of $(V_1, V_2)$ data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots
Integration of $(V_1, V_2)$ data to space-space plots $(y_1, y_2)$
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_y_2$ vs. $V_y_1$ Plot

$V_y_2 = -0.5$ means $M_2$ is somewhere on some path of slope -0.5

$V_y_1 = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

Position $y$ vs. Time $t$ Plot
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_{y2}$ vs. $V_{y1}$ Plot

Height $y$-axis

Position $y$ vs. Time $t$ Plot

Time $t$-axis

- $V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5
- $V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

Ceiling at $y = 7.1$

Floor at $y = 0$

slope $-0.5/1 = -0.5$

slope $1/1 = +1$
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_{y2}$ vs. $V_{y1}$ Plot

Position $y$ vs. Time $t$ Plot

$V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope $-0.5$

$V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope $+1.0$

Until you specify initial conditions $y_0(t_0)$...
...you don’t know which $v_y$-lines to use
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_{y2}$ vs. $V_{y1}$ Plot

$V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5

$V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

Fig. 4.6a-b in Unit 1
**Geometric “Integration” (Converting Velocity data to Spacetime)**

**Position y vs. Time t Plot**

- **Height y-axis**
  - Ceiling at $y=7.1$
- **Time t-axis**
  - Floor at $y=0$

**Velocity $V_{y2}$ vs. $V_{y1}$ Plot**

- $V_{y2}=-0.5$ means $M_2$ is somewhere on some path of slope -0.5
- $V_{y1}=+1.0$ means $M_1$ is somewhere on some path of slope +1.0

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**Fig. 4.6a-b in Unit 1**

**Plot of Bang-1$_{(01)}$**

- Initial conditions $y_1(0)$ and $y_2(0)$
- Bang-1$_{(01)}$ Bounces (-1,-1) to (+1,-1)

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Until you specify initial conditions $y_0(t_0)$...

...you don’t know which $v_y$-lines to use
Geometric “Integration” (Converting Velocity data to Spacetime)

In Unit 1

**Fig. 4.6a-b**

(a) $V_{y2}$ vs. $V_{y1}$ Plot of Bang-1 $(01)$

(b) $y$ vs. $t$ Plot of Bang-1 $(01)$

- $V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5
- $V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

Until you specify initial conditions $y_0(t_0)$...

...you don’t know which $v_y$-lines to use
**Geometric “Integration” (Converting Velocity data to Spacetime)**

*Velocity $V_{y2}$ vs. $V_{y1}$ Plot*

- $V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5.
- $V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0.

*Position $y$ vs. Time $t$ Plot*

**Fig. 4.6a-b in Unit 1**

(a) $V_{y2} vs. V_{y1}$ Plot of Bang-1 $(01)$

- Initial conditions $y_1(0)$ ...
- ...and $y_2(0)$

(b) $y$ vs. $t$ Plot of Bang-1 $(01)$

- $y_2(0) = 3$
- $y_1(0) = 1$
- Bang-1 $(01)$
- Position $(y=0, t=1)$

Until you specify initial conditions $y_0(t_0)$...

...you don’t know which $v_y$-lines to use.
Geometric “Integration” (Converting Velocity data to Spacetime)

Velocity $V_{y2}$ vs. $V_{y1}$ Plot

$V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5

$V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

---

Fig. 4.6a-b in Unit 1

(a) $V_{y2}$ vs. $V_{y1}$ Plot of Bang-1 $(01)$

(V$_{y1}$, V$_{y2}$) = (-1.0, -1.0) Bounces (-1,-1) to (+1,-1)

Initial conditions $y_1(0)$ ...and $y_2(0)$

(b) $y$ vs. $t$ Plot of Bang-1 $(01)$

$y_1(0) = 1$

$y_2(0) = -1$

$y(0,t) = 1$

$y(1,t) = 2$

---

Position $y$ vs. Time $t$ Plot

Height

$y$-axis

Ceiling at $y = 7$

Floor at $y = 0$

---

Until you specify initial conditions $y_0(t_0)$...

...you don’t know which $v_y$-lines to use

---
**Geometric “Integration” (Converting Velocity data to Spacetime)**

**Fig. 4.6a-b**

*in Unit 1*

(a) $V_{y2}$ vs. $V_{y1}$ Plot of $\mathcal{B}ang-1_{(01)}$

$V_{y2} = -0.5$ means $M_2$ is somewhere on some path of slope -0.5

$V_{y1} = +1.0$ means $M_1$ is somewhere on some path of slope +1.0

(b) $y$ vs. $t$ Plot of $\mathcal{B}ang-1_{(01)}$

Initial conditions $y_1(0)$ and $y_2(0)$

Bang-1$_{(01)}$ Bounces (-1,-1) to (+1,-1)

Until you specify initial conditions $y_0(t_0)$...

...you don’t know which $v_y$-lines to use

Ceiling at $y=7$

Floor at $y=0$

Bang-2$_{(12)}$

Position

Initial conditions:

$y_2(0) = 3$

$y_1(0) = 1$

$y_1(t) = 1$

$(y=0, t=1)$

$(y=1, t=2)$

Time $t$-axis

Height $y$-axis
Geometric “Integration” (Converting Velocity data to Spacetime)
**Geometric “Integration” (Converting Velocity data to Spacetime)**

**(a)**

- 2.0
- 1.0
- 0.0
- 1.0

- \( V_{y2} \)
- \( V_{y1} \)

- Bang-3(20)
- Bang-2(12)
- Bang-1(01)

- Start at (-1.0, -1.0)

**(b)**

- **Height y**
- Bang-3(20)
- Bang-2(12)
- Bang-1(01)

- Ceiling at \( y = 7 \)
- \( M_2 \) slope +2.5
- \( M_1 \) slope +0.5

- \( y = 3 \)
- \( y = 1 \)

**(c)**

- 2.0
- 1.0
- 0.0
- 1.0

- \( V_{y2} \)
- \( V_{y1} \)

- Bang-7(12)
- Bang-6(12)
- Bang-5(20)
- Bang-4(12)
- Bang-3(20)

- Bang-2(12)
- Bang-1(01)

- Start at (-1.0, -1.0)

**(d)**

- **y**
- Bang-3(20)
- Bang-5(20)
- Bang-8(20)

- Ceiling at \( y = 7 \)

- 

- \( y = 3 \)
- \( y = 1 \)

- Floor at \( y = 0 \)

- Bang-4(12)
- Bang-6(12)
- Bang-9(12)

- Bang-2(12)
- Bang-1(01)

- Time t:

**Kinetic Energy Ellipse**

\[
KE = \frac{1}{2} M_1 v_1^2 + \frac{1}{2} M_2 v_2^2 = \frac{1}{2} + \frac{7}{2} = 4
\]

\[
1 = \frac{v_1^2}{2KE / M_1} + \frac{v_2^2}{2KE / M_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}
\]

Fig. 4.7a-d in Unit 1
Geometric “Integration” (Converting Velocity data to Spacetime)

Kinetic Energy Ellipse

\[ KE = \frac{1}{2} M_1 v_1^2 + \frac{1}{2} M_2 v_2^2 = \frac{7}{2} + \frac{1}{2} = 4 \]

\[ 1 = \frac{v_1^2}{2KE / M_1} + \frac{v_2^2}{2KE / M_2} = \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} \]

Ellipse radius 1

\[ a_1 = \sqrt{2KE / M_1} \]

Ellipse radius 2

\[ a_2 = \sqrt{2KE / M_1} \]

Fig. 4.7a-d in Unit 1
**Geometric “Integration” (Converting Velocity data to Spacetime)**

**Kinetic Energy Ellipse**

\[ KE = \frac{1}{2} M_1 V_1^2 + \frac{1}{2} M_2 V_2^2 = \frac{7}{2} + \frac{1}{2} = 4 \]

\[ 1 = \frac{V_1^2}{2KE / M_1} + \frac{V_2^2}{2KE / M_2} = \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} \]

**Ellipse radius 1**

\[ a_1 = \sqrt{2KE / M_1} = \sqrt{2KE / 7} = \sqrt{8/7} = 1.07 \]

**Ellipse radius 2**

\[ a_2 = \sqrt{2KE / M_1} = \sqrt{2KE / 1} = \sqrt{8/1} = 2.83 \]

Fig. 4.7a-d in Unit 1
Geometric “Integration” (Converting Velocity data to Spacetime)

Fig. 4.8
in Unit 1

(a) \( t = \frac{25}{E = 3.9999964} \)

(b) \( t = \frac{278}{E = 3.99995} \)

(c) \( t = \infty \) \( E = 4 \)

Fig. 4.9
in Unit 1

Ergodic Fill-in at \( t = \infty \)
Geometry of X2 launcher bouncing in box

Independent Bounce Model (IBM)

Geometric optimization and range-of-motion calculation(t)

Integration of $(V_1, V_2)$ data to space-time plots $(y_1(t), t)$ and $(y_2(t), t)$ plots

Integration of $(V_1, V_2)$ data to space-space plots $(y_1, y_2)$
Geometric “Integration” (Converting Velocity data to Space-space trajectory)

Fig. 4.11
in Unit 1

Step-0: At starting position \(y(0) = (1, 3)\) draw initial velocity \(v(0) = (-1, -1)\) line.

Step-1: Extend \(v(0)\) line to floor point \(y(0) = (0,?)\) and draw \(B(01)\) velocity \(v(1) = (1, -1)\) line. (Find \(v(1)\) using V-V plot.)

Step-2: Extend \(v(1)\) line to collision point \(y(0) = (?, ?)\) and draw \(B(12)\) velocity \(v(2) = (0.5, 2.5)\). (Find \(v(2)\) using V-V plot.)
Fig. 4.11
in Unit 1

Step-2: Extend \( \mathbf{v}(2) \) line to ceiling point \( \mathbf{y}(3)=(?,7.1) \) and draw \( \text{Bang-3}_1 \) velocity \( \mathbf{v}(3)=(1,-1) \) line. (Find \( \mathbf{v}(3) \) using \( V-V \) plot.)

Step-3: Extend \( \mathbf{v}(3) \) line to collision point \( \mathbf{y}(4)=(?,?) \) and draw \( \text{Bang-4}_1 \) velocity \( \mathbf{v}(4)=(0.5,2.5) \). (Find \( \mathbf{v}(4) \) using \( V-V \) plot.)

Step-4: Extend \( \mathbf{v}(4) \) line to ceiling point \( \mathbf{y}(4)=(?,7.1) \) and draw \( \text{Bang-5}_1 \) velocity \( \mathbf{v}(5)=(1,-1) \) line. (Find \( \mathbf{v}(5) \) using \( V-V \) plot.)

Step-5: Extend \( \mathbf{v}(5) \) line to collision point \( \mathbf{y}(6)=(?,?) \) and draw \( \text{Bang-6}_1 \) velocity \( \mathbf{v}(6)=(0.5,2.5) \). (Find \( \mathbf{v}(6) \) using \( V-V \) plot.)
Geometric “Integration” (Converting Velocity data to Space-time trajectory)

Example with masses: $m_1=49$ and $m_2=1$

Fig. 5.1 in Unit 1
Geometric “Integration” (Converting Velocity data to Space-time trajectory)

Example with masses: $m_1 = 49$ and $m_2 = 1$

**Kinetic Energy Ellipse**

\[
KE = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 = \frac{49}{2} + \frac{1}{2} = 25
\]

\[
1 = \frac{V_1^2}{2KE/m_1} + \frac{V_2^2}{2KE/m_2} = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}
\]

**Ellipse radius 1**

\[
a_1 = \sqrt{2KE/M_1} = \sqrt{2\times 49} = 7.07
\]

**Ellipse radius 2**

\[
a_2 = \sqrt{2KE/m_2} = \sqrt{2\times 1} = 1.01
\]

Fig. 5.1 in Unit 1
Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions
“Mass-bang” matrix $\mathbf{M}$, “Floor-bang” matrix $\mathbf{F}$, “Ceiling-bang” matrix $\mathbf{C}$.
Geometry and algebra of “ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ v^{COM} = \frac{v^{FIN} + v^{IN}}{2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \]

Gives \( v^{FIN} \) in terms of \( v^{IN} \)... 

\[
\begin{pmatrix}
v_1^{FIN} \\
v_2^{FIN}
\end{pmatrix} = \begin{pmatrix} 2V^{COM} - v_1^{IN} \\ 2V^{COM} - v_2^{IN} \end{pmatrix} = \begin{pmatrix} \frac{2m_1v_1^{IN} + m_2v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\ \frac{2m_1v_1^{IN} + m_2v_2^{IN}}{m_1 + m_2} - v_2^{IN} \end{pmatrix}
\]

Finally as a matrix operation: 

\[ v^{FIN} = M \cdot v^{IN} \ldots \]

\[
M = \begin{pmatrix}
m_1 - m_2 & 2m_2 \\
2m_1 & m_2 - m_1
\end{pmatrix}
\]

Let: \( m_1 = 49 \) and \( m_2 = 1 \)

\[ M = \begin{pmatrix}
0.96 & 0.04 \\
0.96 & 1.96 - 0.96
\end{pmatrix}
\]

Define a “rotation” \( R \) as group product:

\[ R = C \cdot M = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \cdot M = \begin{pmatrix}
0.96 & 0.04 \\
0.96 & 1.96 - 0.96
\end{pmatrix}
\]
Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

"Mass-bang" matrix $\mathbf{M}$, "Floor-bang" matrix $\mathbf{F}$, "Ceiling-bang" matrix $\mathbf{C}$.

Geometry and algebra of “ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ V_{\text{COM}} = \frac{v^{FIN} + v^{IN}}{2} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \]

Gives \( v^{FIN} \) in terms of \( v^{IN} \):

\[
\begin{pmatrix}
  v_1^{FIN} \\
  v_2^{FIN}
\end{pmatrix} =
\begin{pmatrix}
  2 V_{\text{COM}} - v_1^{IN} \\
  2 V_{\text{COM}} - v_2^{IN}
\end{pmatrix} =
\begin{pmatrix}
  2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_1^{IN} \\
  2 \frac{m_1 v_1^{IN} + m_2 v_2^{IN}}{m_1 + m_2} - v_2^{IN}
\end{pmatrix}
\]

Finally as a matrix operation:

\[ v^{FIN} = M \cdot v^{IN} \]

Matrix operations include...

**Floor-bang** \( F \) of \( m_1 \):

\[
F = \begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix}
\]

**Mass-bang** \( M \) of \( m_1 \) and \( m_2 \):

\[
M = \begin{pmatrix}
  \frac{m_1 - m_2}{m_1 + m_2} & \frac{2 m_2}{m_1 + m_2} \\
  \frac{m_1 + m_2}{m_2 - m_1} & \frac{m_2 - m_1}{m_1 + m_2}
\end{pmatrix}
\]

**Ceiling-bang** \( C \) of \( m_2 \):

\[
C = \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}
\]

\[ M = \begin{pmatrix}
  0.96 & 0.04 \\
  1.96 & -0.96
\end{pmatrix} \]

Let: \( m_1 = 49 \) and \( m_2 = 1 \)
Multiple collisions calculated by matrix operator products

Matrix or tensor algebra of 1-D 2-body collisions

“Mass-bang” matrix $\mathbf{M}$, “Floor-bang” matrix $\mathbf{F}$, “Ceiling-bang” matrix $\mathbf{C}$.

Geometry and algebra of “ellipse-Rotation” group product: $\mathbf{R} = \mathbf{C} \cdot \mathbf{M}$
Multiple Collisions by Matrix Operator Products

T-Symmetry & Momentum Axioms give:

\[ v^{\text{COM}} = \frac{v^{\text{FIN}} + v^{\text{IN}}}{2} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \]

Gives \( v^{\text{FIN}} \) in terms of \( v^{\text{IN}} \):

\[
\begin{pmatrix}
  v^{\text{FIN}}_1 \\
  v^{\text{FIN}}_2
\end{pmatrix}
= 
\begin{pmatrix}
  2V^{\text{COM}} - v^{\text{IN}}_1 \\
  2V^{\text{COM}} - v^{\text{IN}}_2
\end{pmatrix}
= 
\begin{pmatrix}
  \frac{2m_1v^{\text{IN}}_1 + m_2v^{\text{IN}}_2 - v^{\text{IN}}_1}{m_1 + m_2} \\
  \frac{2m_1v^{\text{IN}}_1 + m_2v^{\text{IN}}_2 - v^{\text{IN}}_2}{m_1 + m_2}
\end{pmatrix}
= 
\begin{pmatrix}
  m_1v^{\text{IN}}_1 + 2m_2v^{\text{IN}}_2 - 2m_1v^{\text{IN}}_1 \\
  2m_1v^{\text{IN}}_1 + m_2v^{\text{IN}}_2 - m_1v^{\text{IN}}_2
\end{pmatrix}
= 
\begin{pmatrix}
  m_1 - m_2 \\
  2m_2
\end{pmatrix}
\begin{pmatrix}
  v^{\text{IN}}_1 \\
  v^{\text{IN}}_2
\end{pmatrix}
\]

Finally as a matrix operation:

\[ v^{\text{FIN}} = M \cdot v^{\text{IN}} \]

Matrix operations include...

Floor-bang \( F \) of \( m_1 \):

\[ F = \begin{pmatrix}
  -1 & 0 \\
  0 & 1
\end{pmatrix} \]

Mass-bang \( M \) of \( m_1 \) and \( m_2 \):

\[ M = \begin{pmatrix}
  \frac{m_1 - m_2}{m_1 + m_2} & \frac{2m_2}{m_1 + m_2} \\
  \frac{2m_1}{m_1 + m_2} & \frac{m_2 - m_1}{m_1 + m_2}
\end{pmatrix} \]

Ceiling-bang \( C \) of \( m_2 \):

\[ C = \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix} \]

Let: \( m_1 = 49 \) and \( m_2 = 1 \)

\[ M = \begin{pmatrix}
  0.96 & 0.04 \\
  1.96 & -0.96
\end{pmatrix} \]

Define "ellipse-Rotation" \( R \) as group product:

\[ R = C \cdot M = \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix} \begin{pmatrix}
  0.96 & 0.04 \\
  1.96 & -0.96
\end{pmatrix} = \begin{pmatrix}
  0.96 & 0.04 \\
  -1.96 & 0.96
\end{pmatrix} \]
The text on the image is mathematical and involves matrices and vectors. The focus is on the calculation and interpretation of these elements, specifically involving operations and transformations represented by matrices. The text refers to a process of ellipse rotation and provides a set of equations and results. The final result includes a group product involving matrices and vectors, indicating a transformation process.
\[
\begin{align*}
\text{Ellipse-Rotation } \text{ group product: } R &= \mathbf{C} \cdot \mathbf{M} \\
\end{align*}
\]
Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics in Ch. 12
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \), \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \), \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

\[
\begin{pmatrix}
  v_1^{FIN_1} \\
v_2^{FIN_1}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
m_1 - m_2 & 2m_2 \\
2m_1 & m_2 - m_1
\end{pmatrix} \begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
\]

becomes:

\[
\begin{pmatrix}
v_1^{FIN_1} / \sqrt{m_1} \\
v_2^{FIN_1} / \sqrt{m_2}
\end{pmatrix} = \frac{1}{M} \begin{pmatrix}
m_1 - m_2 & 2m_2 \\
2m_1 & m_2 - m_1
\end{pmatrix} \begin{pmatrix}
v_1 / \sqrt{m_1} \\
v_2 / \sqrt{m_2}
\end{pmatrix}
\]
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: $V_1 = v_1 \cdot \sqrt{m_1}$, $V_2 = v_2 \cdot \sqrt{m_1}$, symmetrize: $KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2$

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2 m_2 \\ 2 m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$ becomes:

$$\begin{pmatrix} v_1^{FIN} / \sqrt{m_1} \\ v_2^{FIN} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2 m_2 \\ 2 m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}$$

or:

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{M} \cdot \mathbf{V}$$

or:

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ -2\sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{\bar{V}}$$

or:

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2 \sqrt{m_1 m_2} \\ 2 \sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{M} \cdot \mathbf{\bar{V}}$$

or:

$$\begin{pmatrix} v_1^{FIN} \\ v_2^{FIN} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2 \sqrt{m_1 m_2} \\ -2 \sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \mathbf{C} \cdot \mathbf{M} \cdot \mathbf{\bar{V}}$$
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \), \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

or \[ \begin{pmatrix} v_{1\text{FIN}} \\ v_{2\text{FIN}} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \]

becomes: \[ \begin{pmatrix} v_{1\text{FIN}} / \sqrt{m_1} \\ v_{2\text{FIN}} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 / \sqrt{m_1} \\ v_2 / \sqrt{m_2} \end{pmatrix} \]

or: \[ \begin{pmatrix} v_{1\text{FIN}} \\ v_{2\text{FIN}} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2\sqrt{m_1 m_2} \\ 2\sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mathbf{M} \cdot \mathbf{\bar{V}} \]

Then collisions become \textit{reflections} \( \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \) and double-collisions become \textit{rotations} \( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \)

where: \( \cos \theta = \frac{m_1 - m_2}{m_1 + m_2} \) and: \( \sin \theta = \frac{2 \sqrt{m_1 m_2}}{m_1 + m_2} \)

with: \( \left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left( \frac{2 \sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1 \)
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \), \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

\[
\begin{pmatrix}
  v_{1\text{FIN}} \\
  v_{2\text{FIN}}
\end{pmatrix}
= \frac{1}{M}
\begin{pmatrix}
  m_1 - m_2 & 2m_2 \\
  2m_1 & m_2 - m_1
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
\]

becomes:

\[
\begin{pmatrix}
  v_{1\text{FIN}} / \sqrt{m_1} \\
  v_{2\text{FIN}} / \sqrt{m_2}
\end{pmatrix}
= \frac{1}{M}
\begin{pmatrix}
  m_1 - m_2 & 2m_2 \\
  2m_1 & m_2 - m_1
\end{pmatrix}
\begin{pmatrix}
  v_1 / \sqrt{m_1} \\
  v_2 / \sqrt{m_2}
\end{pmatrix}
\]

or:

\[
\begin{pmatrix}
  v_{1\text{FIN}} \\
  v_{2\text{FIN}}
\end{pmatrix}
= \frac{1}{M}
\begin{pmatrix}
  m_1 - m_2 & 2\sqrt{m_1 m_2} \\
  2\sqrt{m_1 m_2} & m_2 - m_1
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
\]

\[= M \cdot \vec{V} \], or:

\[
\begin{pmatrix}
  v_{1\text{FIN}} \\
  v_{2\text{FIN}}
\end{pmatrix}
= \frac{1}{M}
\begin{pmatrix}
  m_1 - m_2 & 2\sqrt{m_1 m_2} \\
  -2\sqrt{m_1 m_2} & m_1 - m_2
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
\]

Then collisions become reflections \( \begin{pmatrix}
  \cos \theta & \sin \theta \\
  \sin \theta & -\cos \theta
\end{pmatrix} \) and double-collisions become rotations \( \begin{pmatrix}
  \cos \theta & \sin \theta \\
  -\sin \theta & \cos \theta
\end{pmatrix} \)

where:

\[
\cos \theta \equiv \frac{m_1 - m_2}{m_1 + m_2}
\]

and:

\[
\sin \theta \equiv \frac{2\sqrt{m_1 m_2}}{m_1 + m_2}
\]

with:

\[
\left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left( \frac{2\sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1
\]

\[
\frac{m_1 - m_2}{m_1 + m_2} = \frac{48}{50}
\]

\[
\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} = \frac{14}{50}
\]

\[
\theta = 16.26^\circ
\]

Fig. 5.2a-c (revised)
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1}, \quad V_2 = v_2 \cdot \sqrt{m_1}, \) symmetrize: \( KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2 \)

\[
\begin{pmatrix} v_{1}^{FIN} \\ v_{2}^{FIN} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2m_1 & m_2 - m_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
\]

becomes:

\[
\begin{pmatrix} v_{1}^{FIN} / \sqrt{m_1} \\ v_{2}^{FIN} / \sqrt{m_2} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & 2m_2 \\ 2 \sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix} \begin{pmatrix} V_1 / \sqrt{m_1} \\ V_2 / \sqrt{m_2} \end{pmatrix}
\]

or:

\[
\begin{pmatrix} v_{1}^{FIN} \\ v_{2}^{FIN} \end{pmatrix} = \frac{1}{M} \begin{pmatrix} m_1 - m_2 & -2 \sqrt{m_1 m_2} \\ 2 \sqrt{m_1 m_2} & m_1 - m_2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}
\]

Then collisions become reflections \( \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \) and double-collisions become rotations \( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \)

where:

\[
\cos \theta = \frac{m_1 - m_2}{m_1 + m_2}
\]

and:

\[
\sin \theta = \frac{2 \sqrt{m_1 m_2}}{m_1 + m_2}
\]

with:

\[
\left( \frac{m_1 - m_2}{m_1 + m_2} \right)^2 + \left( \frac{2 \sqrt{m_1 m_2}}{m_1 + m_2} \right)^2 = 1
\]

\[
\frac{m_1 - m_2}{m_1 + m_2} = \frac{48}{50}
\]

\[
\frac{2 \sqrt{m_1 m_2}}{m_1 + m_2} = \frac{14}{50}
\]

\[\theta = 16.26^\circ\]

Fig. 5.2a-c (revised)

Note: If \( m_1 \cdot m_2 \) is perfect-square, then \( \theta \)-triangle is rational \( (3^2 + 4^2 = 5^2, \ etc.) \)
Ellipse rescaling geometry and reflection symmetry analysis

Convert to rescaled velocity: \( V_1 = v_1 \cdot \sqrt{m_1} \), \( V_2 = v_2 \cdot \sqrt{m_1} \), symmetrize: 
\[
KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} V_1^2 + \frac{1}{2} V_2^2
\]

Then collisions become reflections and double-collisions become rotations where:

\[
\cos \theta \equiv \frac{m_1 - m_2}{m_1 + m_2} \quad \text{and} \quad \sin \theta \equiv \frac{2 \sqrt{m_1 m_2}}{m_1 + m_2}
\]

Fig. 5.2a-c (revised)

\[ \theta = 16.26^\circ \]

\[ \frac{m_1 - m_2}{m_1 + m_2} = \frac{48}{50} \]

\[ \frac{2 \sqrt{m_1 m_2}}{m_1 + m_2} = \frac{14}{50} \]

slope: \( \frac{\sqrt{m_2}}{\sqrt{m_1}} = -0.49 \)
\[
\theta = 16.26^\circ
\]

\[
\frac{m_1 - m_2}{m_1 + m_2} = \frac{48}{50}
\]

\[
\frac{2\sqrt{m_1 m_2}}{m_1 + m_2} = \frac{14}{50}
\]

\[
\sqrt{\frac{m_2}{m_1}} = -7
\]

\[
\sqrt{\frac{m_1}{m_2}} = 4.9
\]

Fig. 5.2a-c
(revised)
Ellipse rescaling-geometry and reflection-symmetry analysis

Rescaling KE ellipse to circle

How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics in Ch. 12
What ellipse rescaling leads to...(in Ch. 9-12)

How this relates to Lagrangian, \( L = L(v_1, v_2) \)
and Hamiltonian mechanics in Ch. 12

(a) Lagrangian \( L = L(v_1, v_2) \)

Collision line and COM tangent slope
\[ = \frac{-m_1}{m_2} = -16 \]

COM Bisector
slope = 1/1

(b) Hamiltonian \( H = H(p_1, p_2) \)

COM Bisector slope
\[ = \frac{m_2}{m_1} = 1/16 \]

Collision line and COM tangent slope
\[ = \frac{-1}{1} \]

velocity \( v_1 \) rescaled to momentum: \( p_1 = m_1 v_1 \)
velocity \( v_2 \) rescaled to momentum: \( p_2 = m_2 v_2 \)
What ellipse rescaling leads to...(in Ch. 9-12)

How this relates to Lagrangian, \( L_{\text{Lagr}}(v_1, v_2) \) and Hamiltonian mechanics in Ch. 12

(a) Lagrangian \( L = L(v_1, v_2) \)

\[ \text{velocity } v_1 \text{ rescaled to momentum: } p_1 = m_1 v_1 \]
\[ \text{velocity } v_2 \text{ rescaled to momentum: } p_2 = m_2 v_2 \]

Collision line and COM tangent slope
\( \frac{\sqrt{m_1}}{\sqrt{m_2}} = -16 \)

COM Bisector slope
\( \frac{\sqrt{m_1}}{\sqrt{m_2}} = 1/1 \)

Lagrangian \( L(v_1, v_2) = KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \)

rescaled to

Hamiltonian \( H(p_1, p_2) = KE = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \)

(c) Hamiltonian \( H = H(p_1, p_2) \)

Collision line and COM tangent slope
\( \frac{\sqrt{m_1}}{\sqrt{m_2}} = -1/16 \)

COM Bisector slope
\( \frac{\sqrt{m_1}}{\sqrt{m_2}} = 1/16 \)

\[ p_1 = m_1 v_1 \]
\[ p_2 = m_2 v_2 \]
What ellipse rescaling leads to... (in Ch. 9-12)

How this relates to Lagrangian, l’Etrangian, and Hamiltonian mechanics in Ch. 12

(a) Lagrangian \( L = L(v_1, v_2) \)

Collision line and COM tangent slope
\[ = -\frac{m_1}{m_2} = -16 \]

COM Bisector slope
\[ = \frac{\sqrt{m_1}}{\sqrt{m_2}} = 1/4 \]

(b) Estrangian \( E = E(V_1, V_2) \)

Collision line and COM tangent slope
\[ = -\frac{\sqrt{m_1}}{\sqrt{m_2}} = -4 \]

COM Bisector slope
\[ = \frac{\sqrt{m_2}}{\sqrt{m_1}} = 1/4 \]

(c) Hamiltonian \( H = H(p_1, p_2) \)

Collision line and COM tangent slope
\[ = -\frac{1}{1} \]

COM Bisector slope
\[ = \frac{m_2}{m_1} = 1/16 \]