Lecture 29
Tue. 12.08.2015

Formerly Lect. 23 for Unit 3

Classical Constraints: Comparing various methods (Ch. 9 of Unit 3)

Some Ways to do constraint analysis

- *Way 1. Simple constraint insertion*
- *Way 2. GCC constraint webs*
  - *Find covariant force equations*
  - *Compare covariant vs. contravariant forces*

Other Ways to do constraint analysis

- *Way 3. OCC constraint webs*
  - *Sketch of atomic-Stark orbit parabolic OCC analysis*
    - *Classical Hamiltonian separability*
- *Way 4. Lagrange multipliers*
  - *Lagrange multiplier as eigenvalues*
  - *Multiple multipliers*
  - *"Non-Holonomic" multipliers*

Cycloid-like curves for rolling constraints

- *Quickest intra-planetary subways*

Some Ways to do constraint analysis

→ *Way 1. Simple constraint insertion*

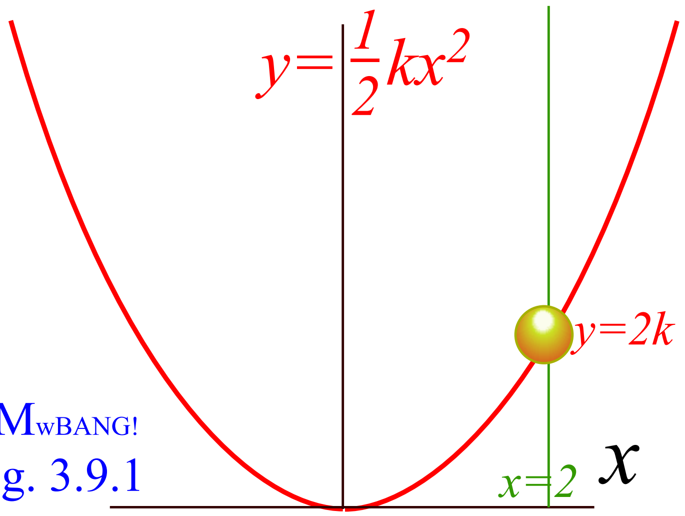
Way 2. GCC constraint webs

Find covariant force equations

Compare covariant vs. contravariant forces

Ways to analyze a particle m constrained to parabola $y=\frac{1}{2}kx^2$ on (x,y)-plane with gravitational potential $V(\mathbf{r})=mgy$.

(a) Constrained motion



CMwBANG! Fig. 3.9.1

Way 1. Lagrangian has the constraint(s) simply inserted.

$$L=\frac{1}{2}(m\dot{x}^2+m\dot{y}^2)-mgy$$

Let: $y=\frac{1}{2}kx^2$ and: $\dot{y}=kx\dot{x}$

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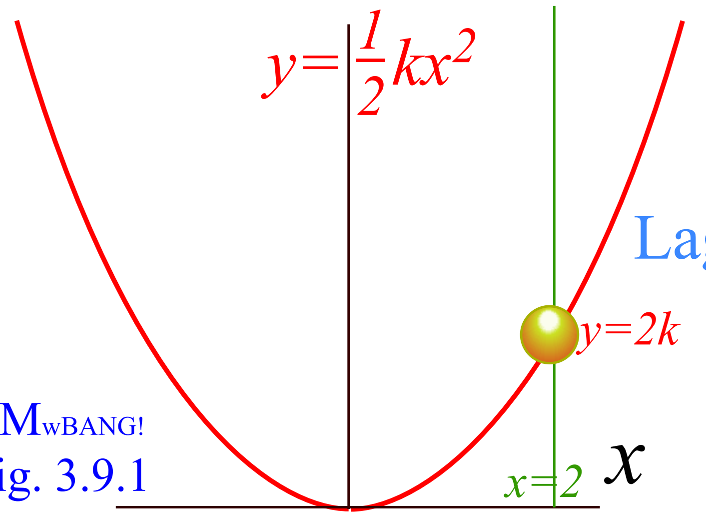

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Lagrangian then has one dimension x, one momentum p_x, and one force f_x.

$$L=\frac{1}{2}(m\dot{x}^2+m(kx\dot{x})^2)-m\frac{1}{2}gkx^2 \qquad p_x=\frac{\partial L}{\partial \dot{x}} \qquad f_x=\frac{\partial L}{\partial x}$$

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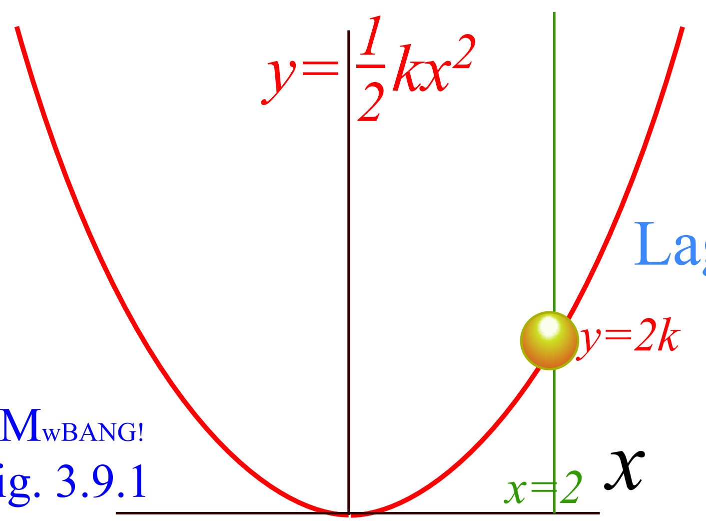

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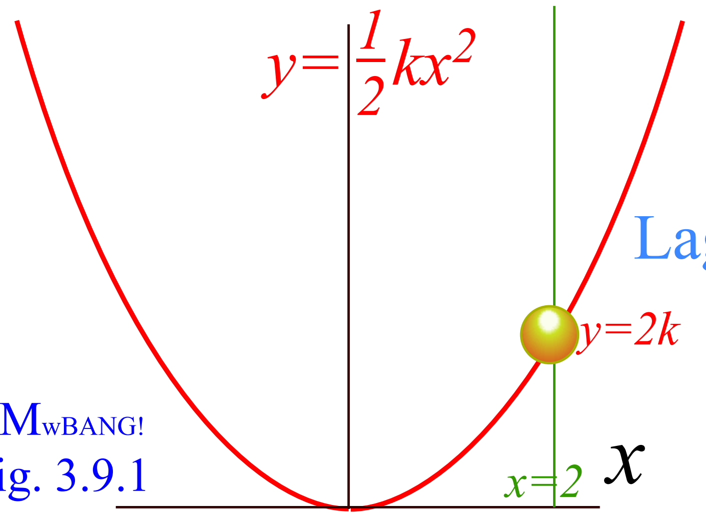

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$$p_x=\frac{\partial L}{\partial \dot{x}}$$

$$=m(\dot{x}+k^2x^2\dot{x})$$

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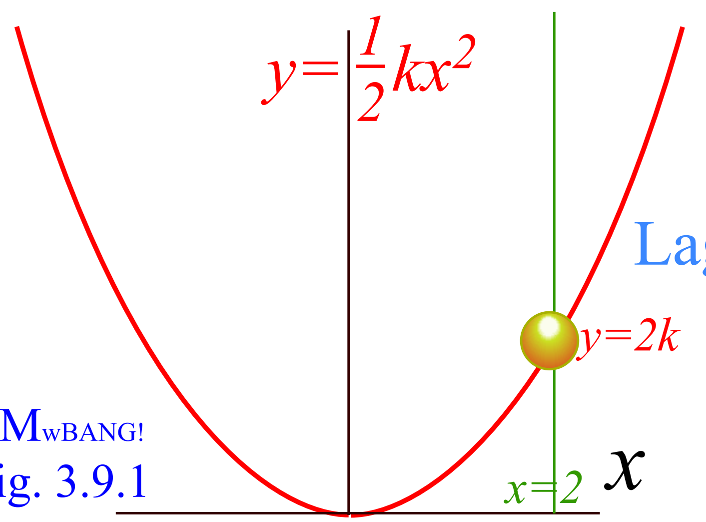

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$$p_x=\frac{\partial L}{\partial \dot{x}}=m(\dot{x}+k^2x^2\dot{x})$$

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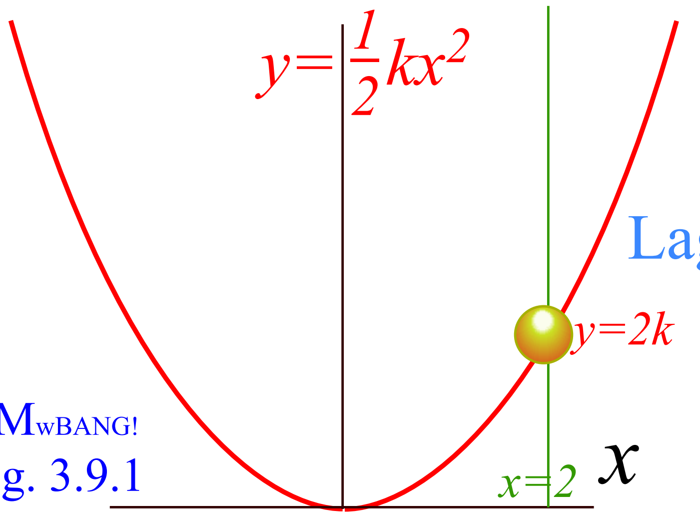

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}$$

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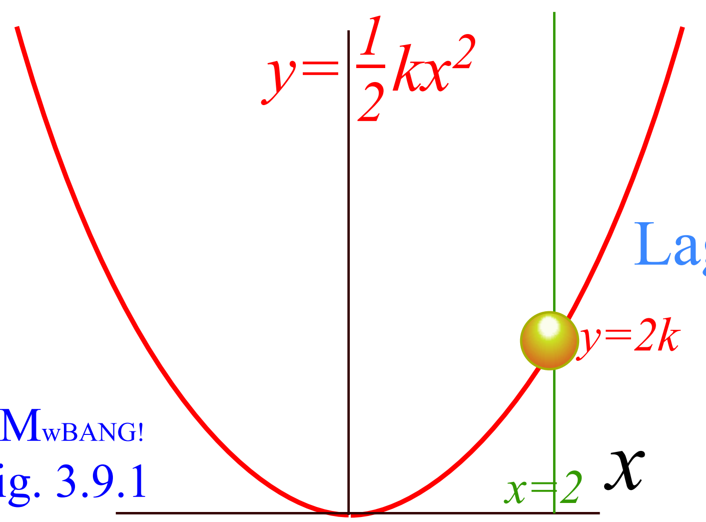

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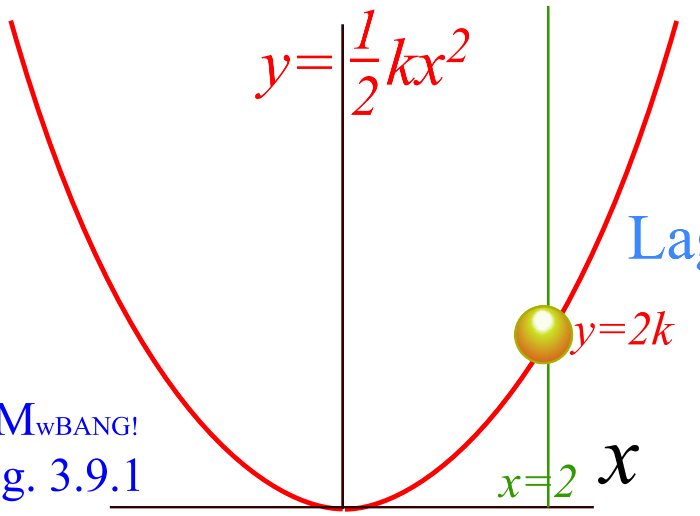

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$$\dot{p}_x=m(1+k^2x^2)\ddot{x}\qquad=-mk^2x\dot{x}^2-mgkx$$

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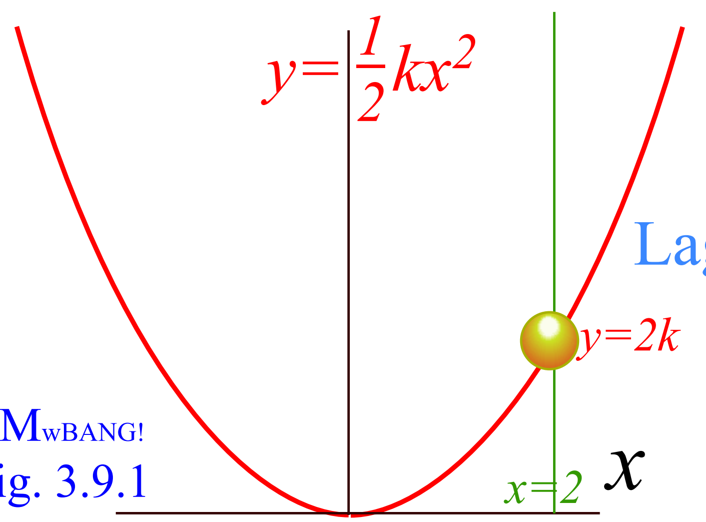

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x} \qquad =-mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

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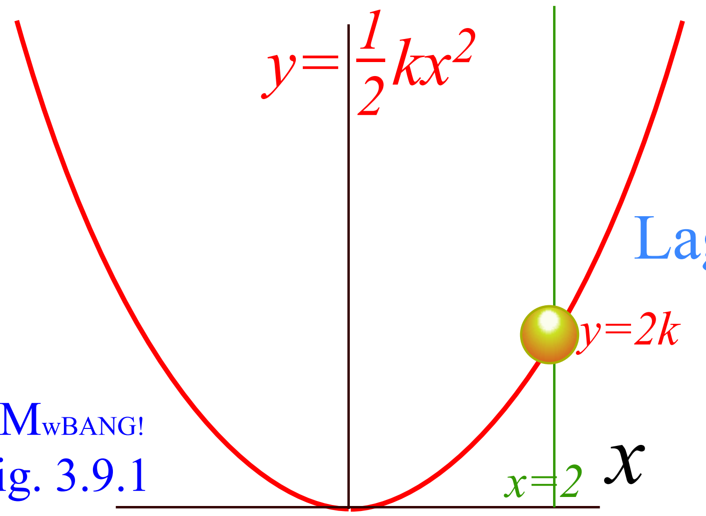

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$$p_x=\frac{\partial L}{\partial \dot{x}} = m(\dot{x}+k^2x^2\dot{x})$$

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Lagrange equation $\dot{p}_x=f_x=\frac{\partial L}{\partial x}$ gives oscillator $\ddot{x}=-K(x,\dot{x})x$ with "spring factor" K:

$$\dot{p}_x=m(\ddot{x}+k^2x^2\ddot{x}+2k^2x\dot{x}^2)=\frac{\partial L}{\partial x}=m(k^2x\dot{x}^2-gkx)$$

$$m(1+k^2x^2)\ddot{x} = -mk^2x\dot{x}^2-mgkx=-m(k\dot{x}^2-g)kx$$

$$\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx$$

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

→ *Way 2. GCC constraint webs*

Find covariant force equations

Compare covariant vs. contravariant forces

(a) Constrained motion

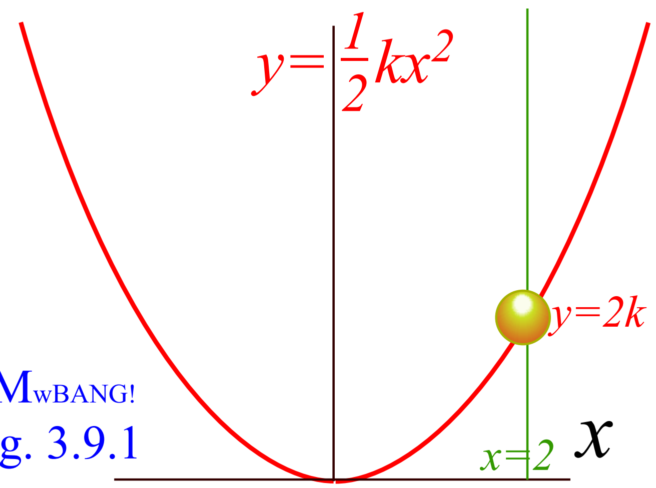

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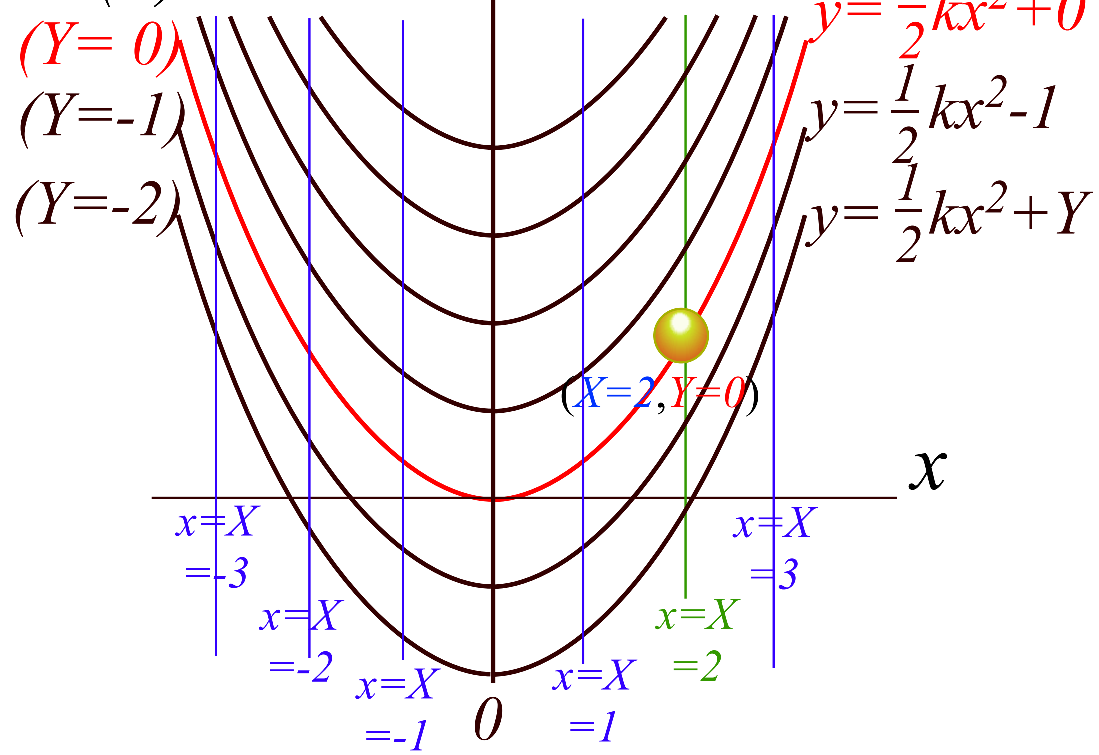
Fig. 3.9.1

$$x = X$$
$$y=\frac{1}{2}kx^2+Y$$

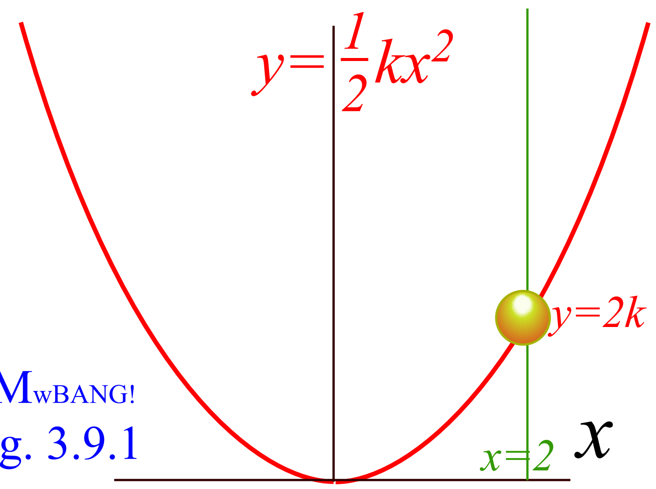
Cartesian (x,y) transform to GCC (X,Y)

(b) GCC constraint web



Incorporate the constraint curve $y=\frac{1}{2}kx^2$ into any matching GCC web.

(a) Constrained motion



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$x = X$

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Cartesian (x,y) transform to GCC (X,Y)

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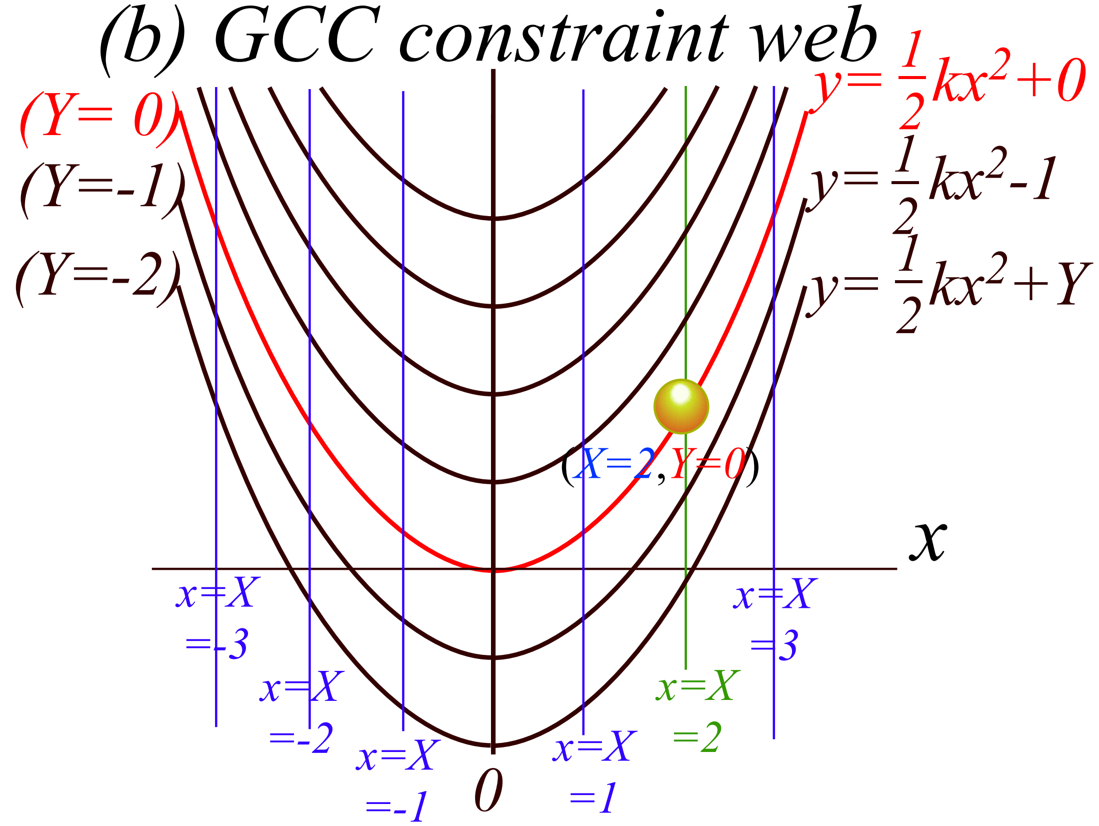

Incorporate the constraint curve $y=½kx^2$ into any matching GCC web.

$x=q^1=X$ $\qquad$ $y=½kx^2 + q^2 = kX^2/2+Y$

we define shorthand:

$X \equiv q^1$ and $Y \equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

Way 2. *GCC constraint webs.*

(a) Constrained motion

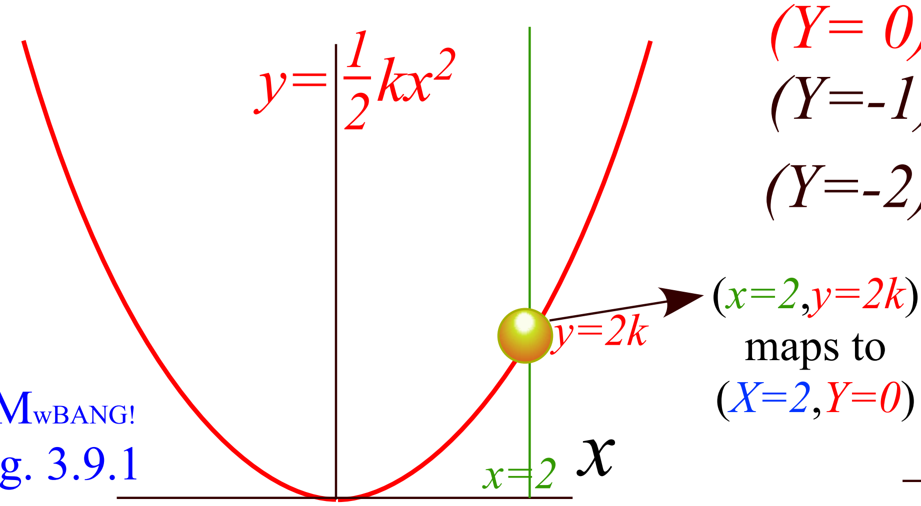

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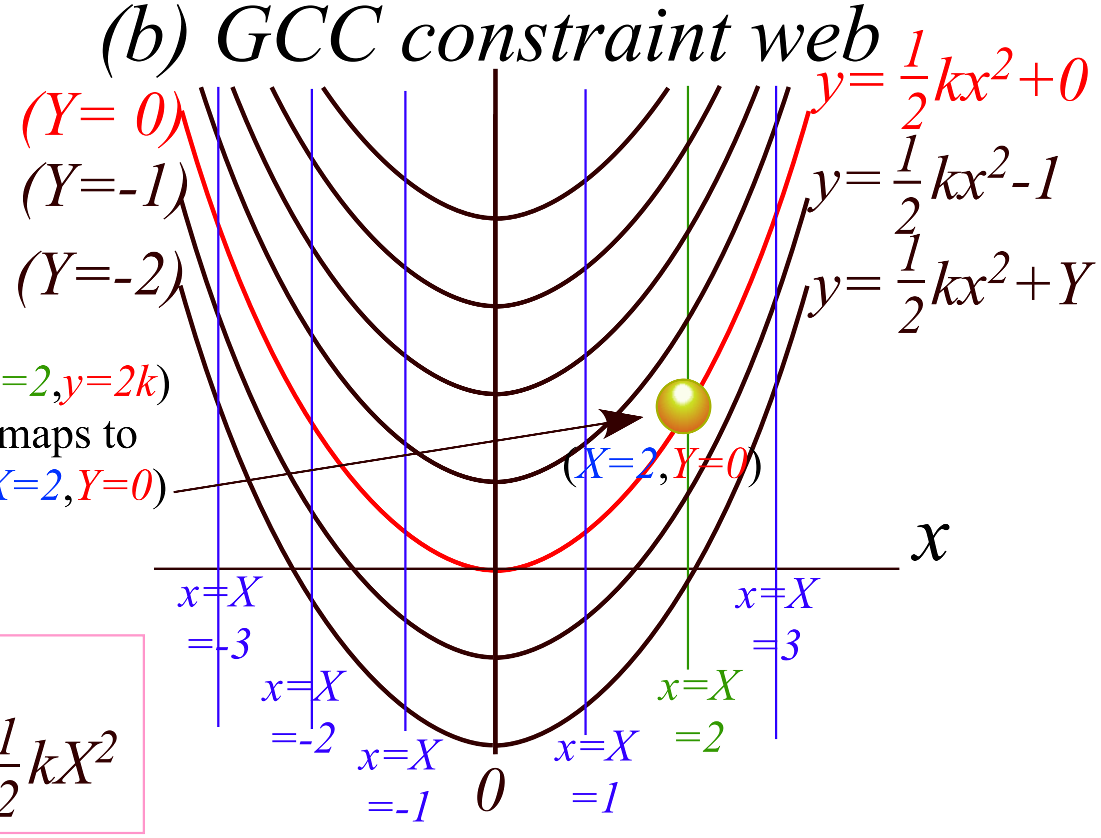
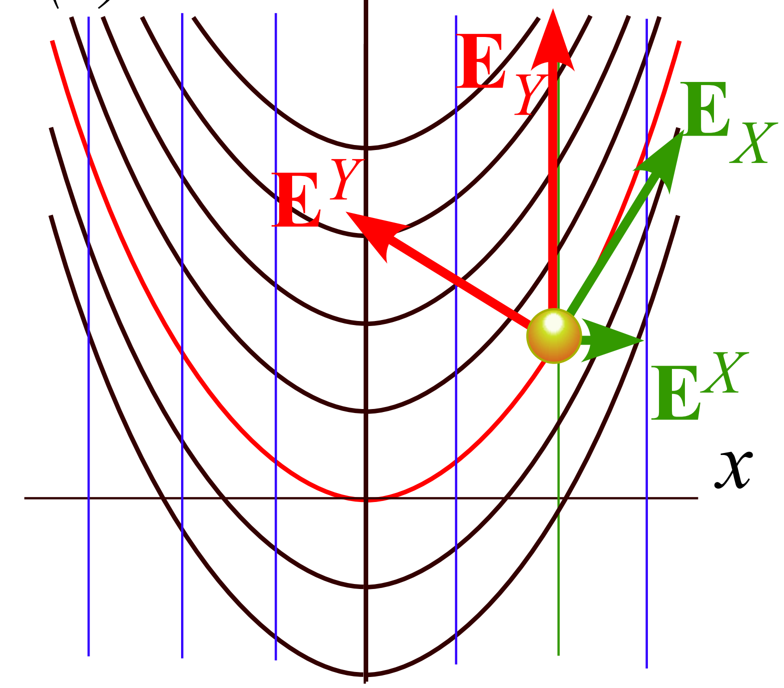
Fig. 3.9.1

(b) GCC constraint web



(c) GCC **E***-vectors*



$x = X$
$y=\frac{1}{2}kx^2+Y$

Cartesian (x,y) transform to GCC (X,Y)

$X = x$
$Y=y-\frac{1}{2}kX^2$

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{queer}^{Indices}$

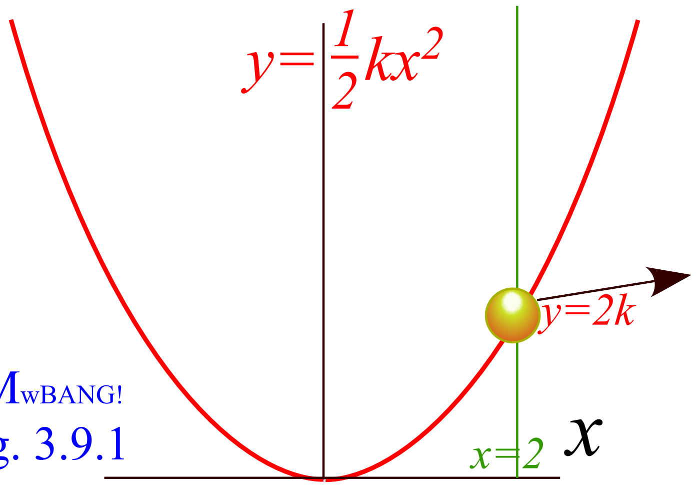
Incorporate the constraint curve $y=\frac{1}{2}kx^2$ into any matching GCC web.

$$x=q^1=X \qquad y=\tfrac{1}{2}kx^2+q^2=kX^2/2+Y$$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J=\begin{pmatrix} \frac{\partial x}{\partial X}=1 & \frac{\partial x}{\partial Y}=0 \\ \frac{\partial y}{\partial X}=+kx & \frac{\partial y}{\partial Y}=1 \end{pmatrix} \qquad \mathbf{E}_X=\begin{pmatrix}1\\kx\end{pmatrix},\ \mathbf{E}_Y=\begin{pmatrix}0\\1\end{pmatrix}$$

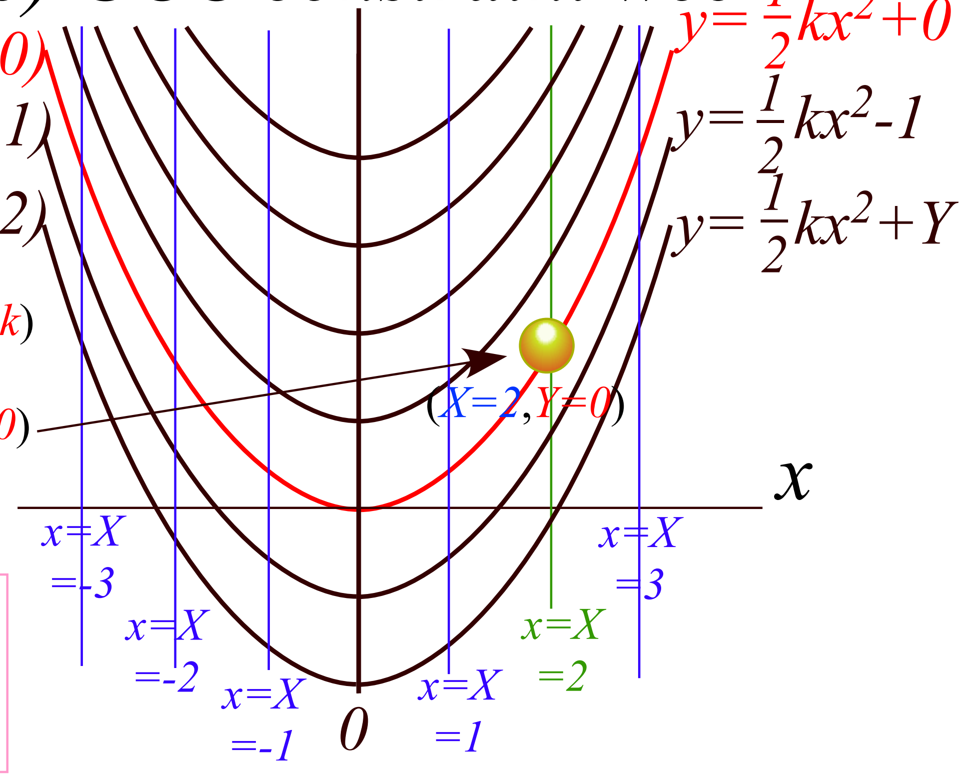
(a) Constrained motion



$(x=2, y=2k)$ maps to $(X=2, Y=0)$

CMwBANG! Fig. 3.9.1

(b) GCC constraint web



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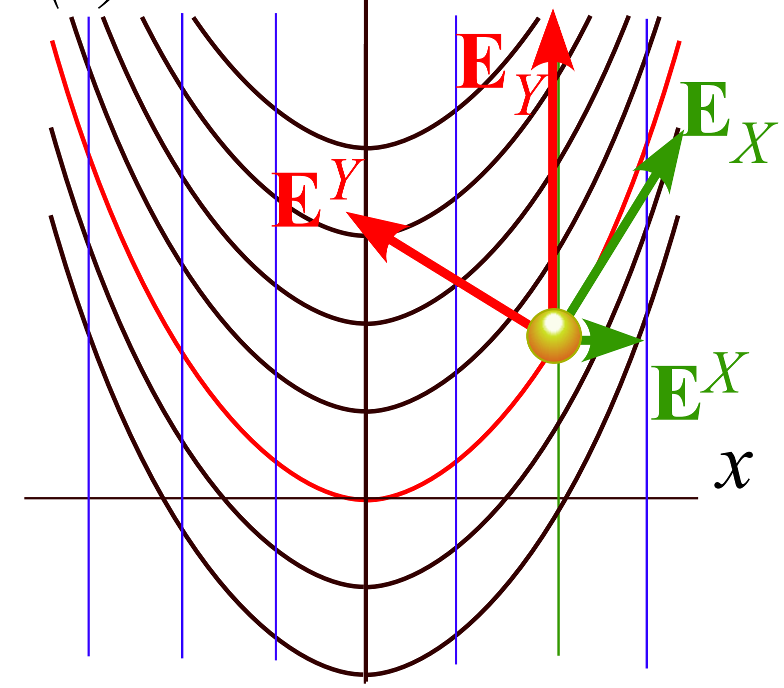

$x = X$ $y=\frac{1}{2}kx^2+Y$	*Cartesian (x,y) transform to GCC (X,Y)*	$X=x$ $Y=y-\frac{1}{2}kX^2$

we define shorthand: $X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{queer}^{Indices}$

Incorporate the constraint curve $y=\frac{1}{2}kx^2$ into any matching GCC web.

$x=q^1=X \qquad y=\frac{1}{2}kx^2+q^2=kX^2/2+Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

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Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \mathbf{E}^X=\begin{pmatrix}1 & 0\end{pmatrix},\ \mathbf{E}^Y=\begin{pmatrix}-kx & 1\end{pmatrix}$$

(a) Constrained motion

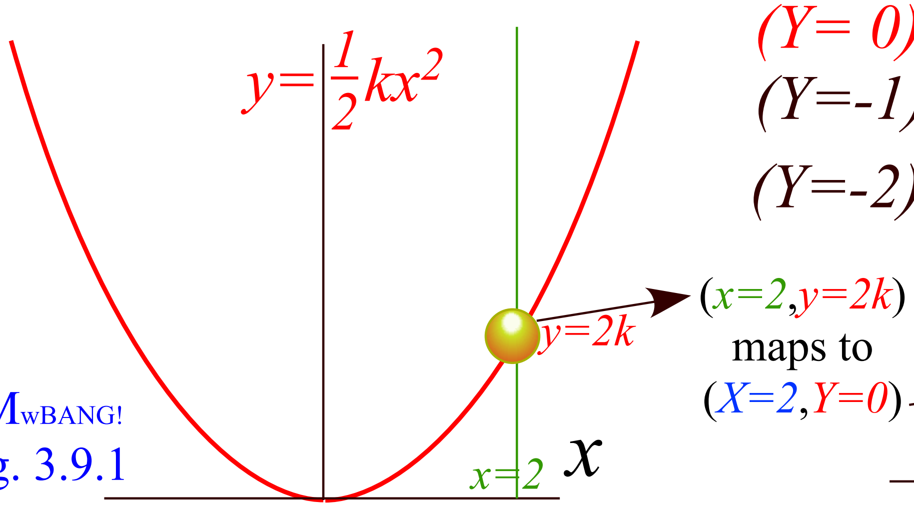

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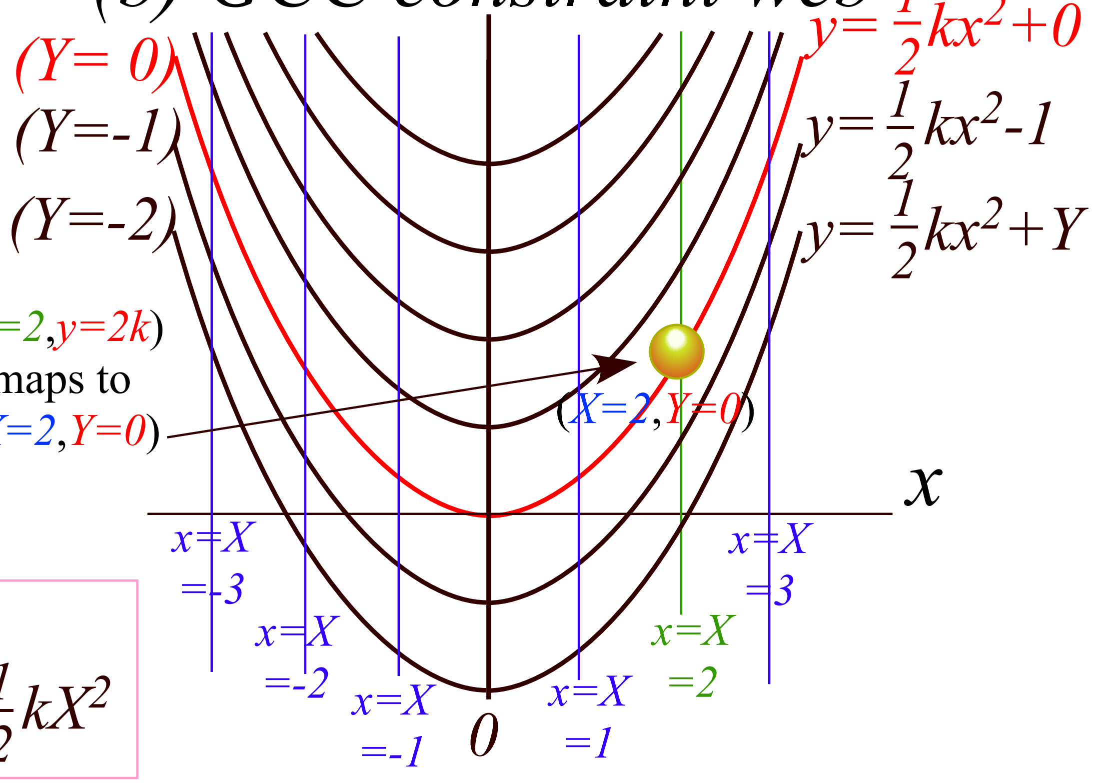
Fig. 3.9.1

(b) GCC constraint web



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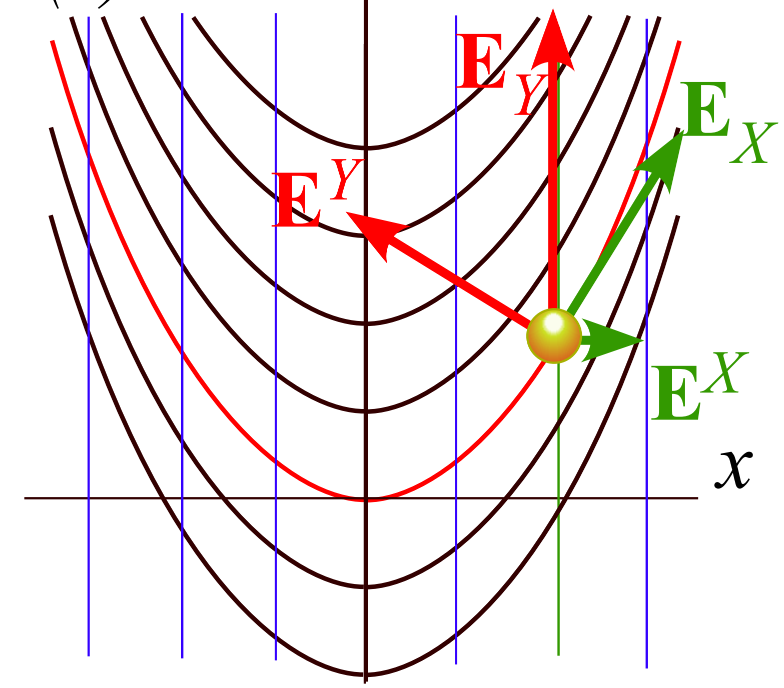

$x = X$

$y=\frac{1}{2}kx^2+Y$

Cartesian (x,y) transform to GCC (X,Y)

$X = x$

$Y=y-\frac{1}{2}kX^2$

we define shorthand:

$X\equiv q^1$ and $Y\equiv q^2$ to avoid writing $q_{queer}^{Indices}$

Incorporate the constraint curve $y=\frac{1}{2}kx^2$ into any matching GCC web.

$x=q^1=X \qquad y=\frac{1}{2}kx^2+q^2= kX^2/2+Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

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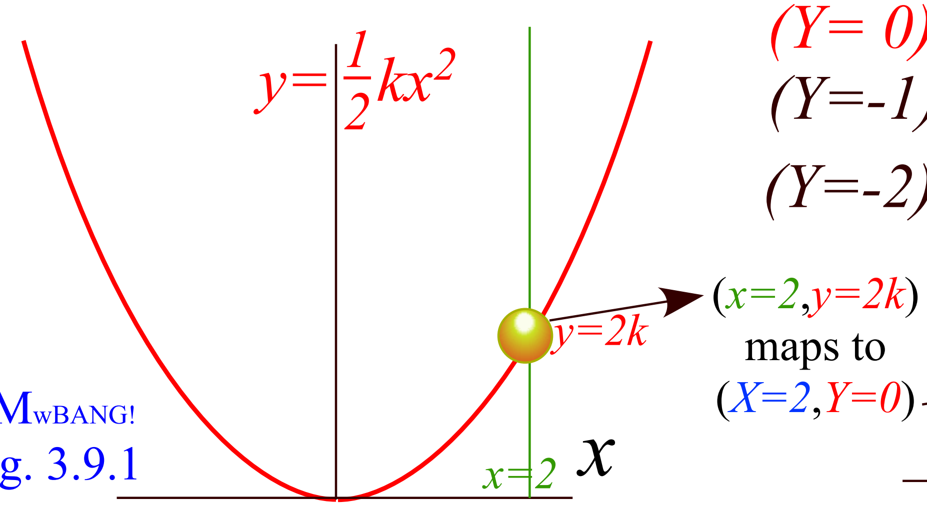
$$\begin{pmatrix} \frac{\partial X}{\partial x}=1 & \frac{\partial X}{\partial y}=0 \\ \frac{\partial Y}{\partial x}=-kx & \frac{\partial Y}{\partial y}=1 \end{pmatrix}=K \qquad \mathbf{E}^X=\begin{pmatrix}1 & 0\end{pmatrix},\ \mathbf{E}^Y=\begin{pmatrix}-kx & 1\end{pmatrix}$$

Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ +kx & 1\end{pmatrix}\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} \qquad \begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix}=\begin{pmatrix}1 & 0\\ -kx & 1\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$$

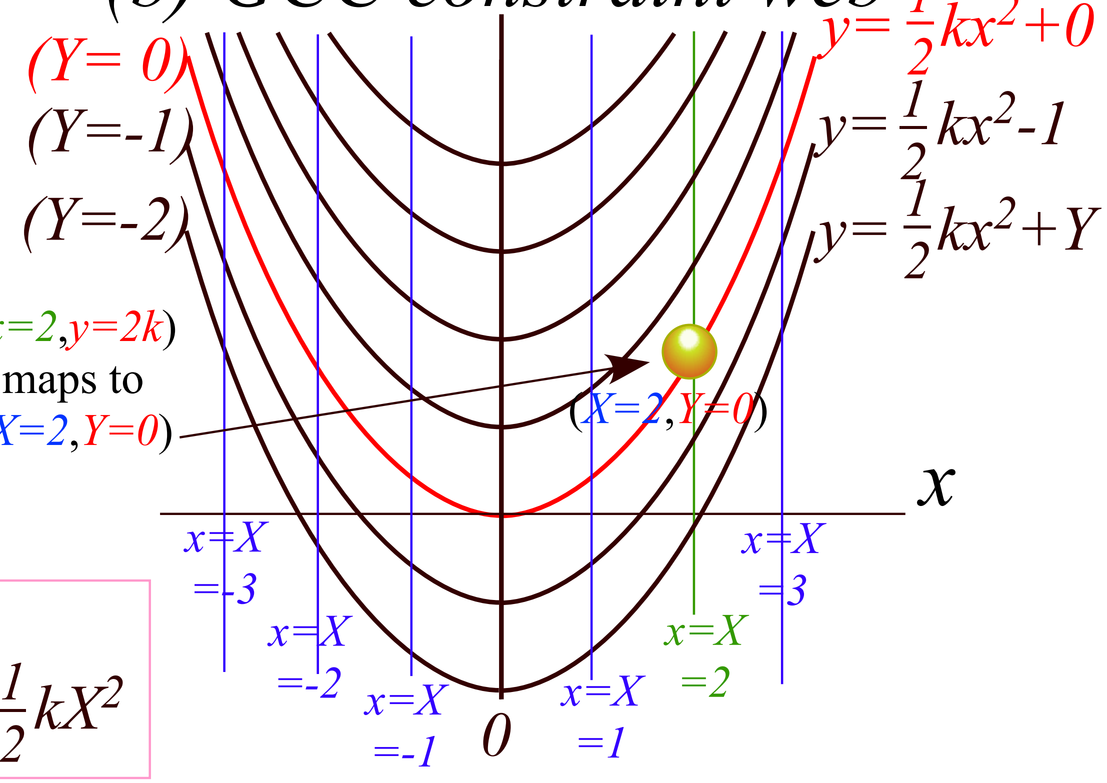
Way 2. GCC constraint webs.

(a) Constrained motion

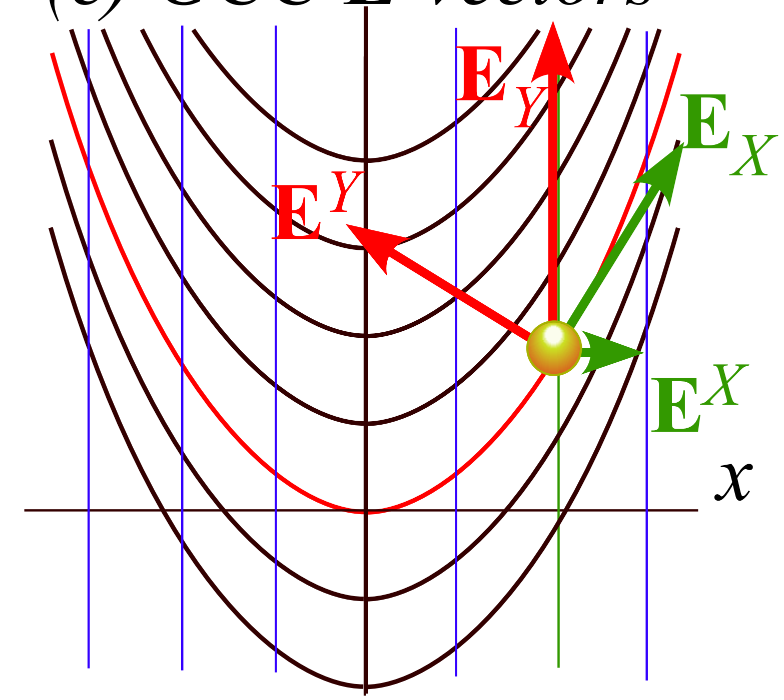


CMwBANG! Fig. 3.9.1

(b) GCC constraint web



(c) GCC E-vectors



$x = X$ $y = \frac{1}{2}kx^2 + Y$	*Cartesian (x,y) transform to GCC (X,Y)*	$X = x$ $Y = y - \frac{1}{2}kX^2$

we define shorthand:

$X \equiv q^1$ and $Y \equiv q^2$ to avoid writing $q_{ueer}{}^{Indices}$

Incorporate the constraint curve $y=½kx^2$ into any matching GCC web.

$x = q^1 = X$ $\qquad$ $y = ½kx^2 + q^2 = kX^2/2 + Y$

Find: Covariant $\mathbf{E}_k$ in columns of Jacobian J matrix

$$J = \begin{pmatrix} \frac{\partial x}{\partial X} = 1 & \frac{\partial x}{\partial Y} = 0 \\ \frac{\partial y}{\partial X} = +kx & \frac{\partial y}{\partial Y} = 1 \end{pmatrix} \qquad \mathbf{E}_X = \begin{pmatrix} 1 \\ kx \end{pmatrix}, \; \mathbf{E}_Y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Contravariant $\mathbf{E}^k$ in rows of Kajobian K

$$\begin{pmatrix} \frac{\partial X}{\partial x} = 1 & \frac{\partial X}{\partial y} = 0 \\ \frac{\partial Y}{\partial x} = -kx & \frac{\partial Y}{\partial y} = 1 \end{pmatrix} = K \qquad \begin{aligned} \mathbf{E}^X &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\ \mathbf{E}^Y &= \begin{pmatrix} -kx & 1 \end{pmatrix} \end{aligned}$$

Find: 1st coordinate differentials and velocity relations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ +kx & 1 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} \qquad \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -kx & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

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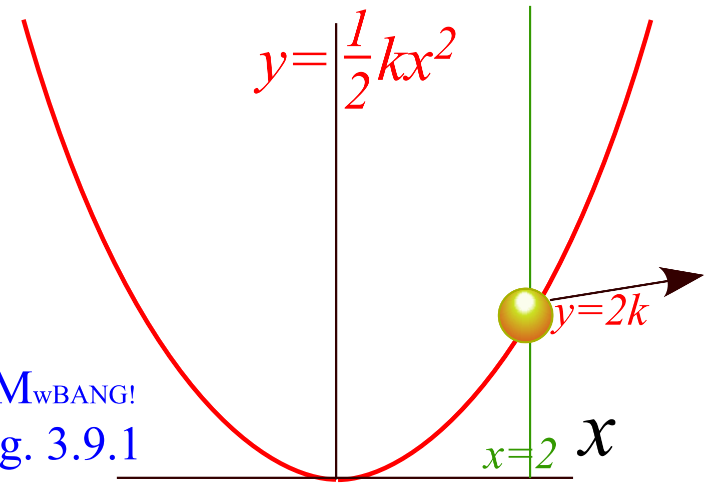
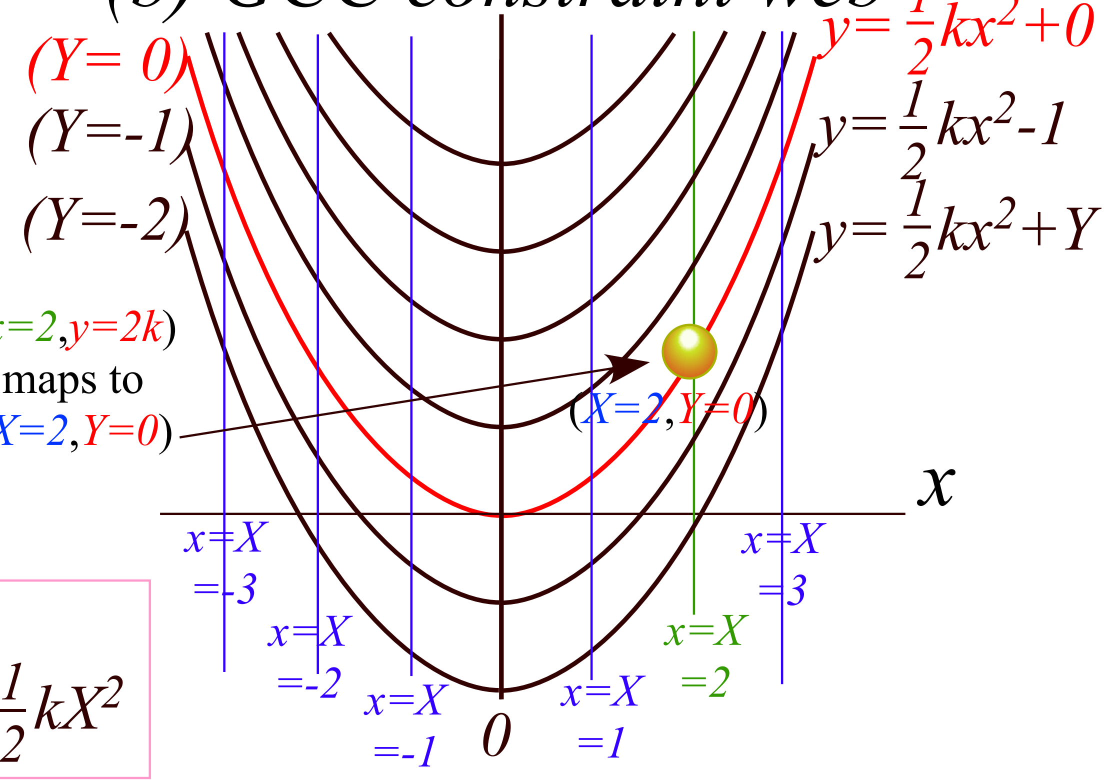
(a) Constrained motion

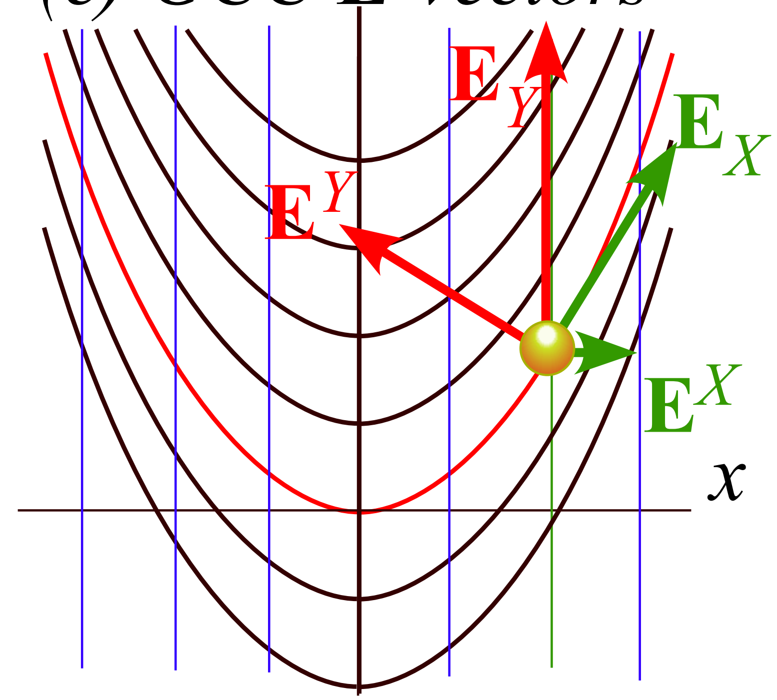


CMwBANG!
Fig. 3.9.1

(b) GCC constraint web



(c) GCC **E**-vectors



$x = X$
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Cartesian (x,y) transform to GCC (X,Y)

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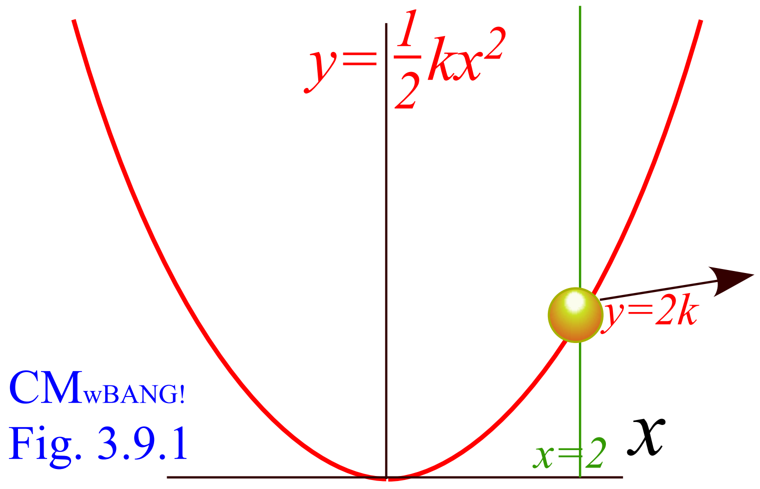
$$\frac{1}{m}\begin{pmatrix}\mathbf{E}^X\bullet\mathbf{E}^X & \mathbf{E}^X\bullet\mathbf{E}^Y\\\mathbf{E}^Y\bullet\mathbf{E}^Y & \mathbf{E}^Y\bullet\mathbf{E}^Y\end{pmatrix}=\begin{pmatrix}\gamma^{XX} & \gamma^{XY}\\\gamma^{YX} & \gamma^{YY}\end{pmatrix}=\frac{1}{m}\begin{pmatrix}1 & -kx\\-kx & 1+k^2x^2\end{pmatrix}$$

(Need contra-γ for Hamilton or Riemann equations)

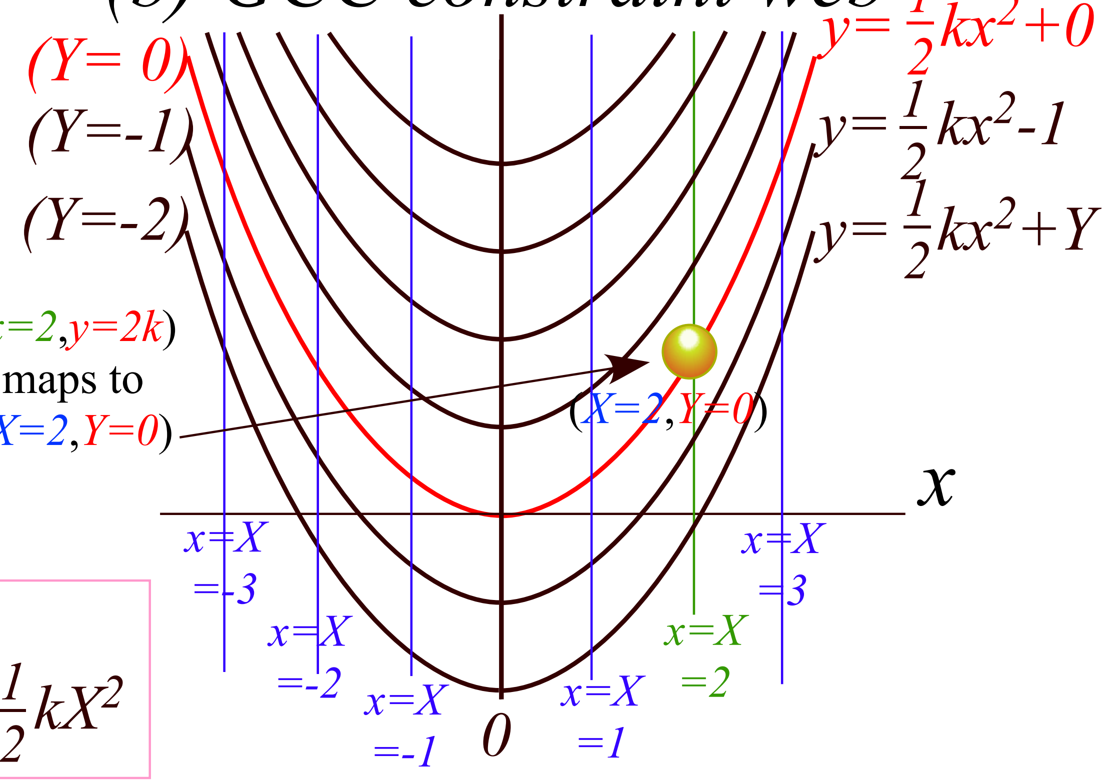
Way 2. GCC constraint webs.

(a) Constrained motion

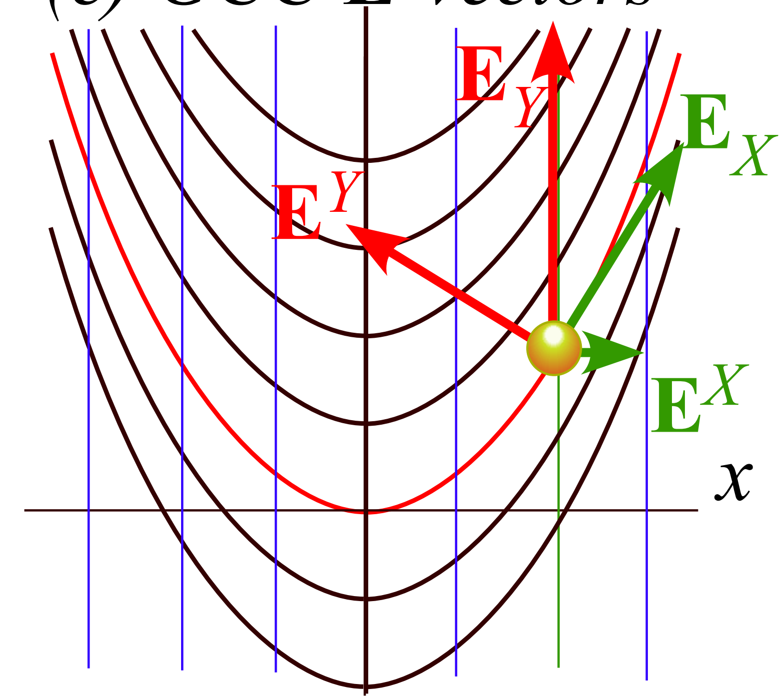


CMwBANG!
Fig. 3.9.1

(b) GCC constraint web



(c) GCC **E**-vectors



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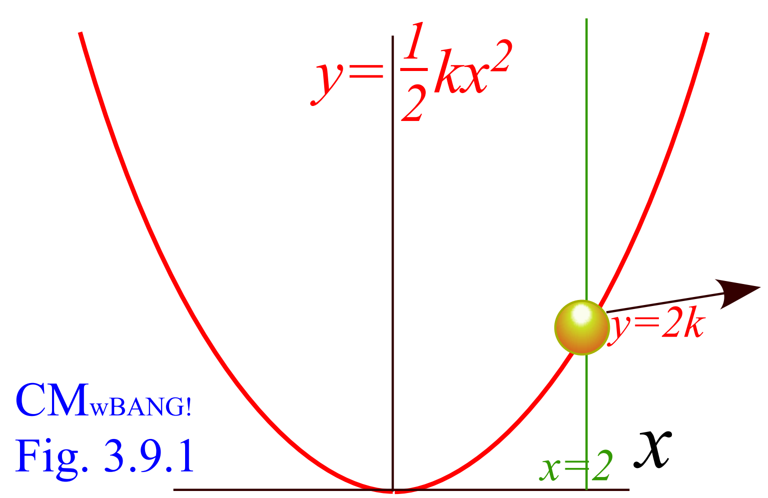
$$\frac{1}{m} \begin{pmatrix} \mathbf{E}^X \bullet \mathbf{E}^X & \mathbf{E}^X \bullet \mathbf{E}^Y \\ \mathbf{E}^Y \bullet \mathbf{E}^X & \mathbf{E}^Y \bullet \mathbf{E}^Y \end{pmatrix} = \begin{pmatrix} \gamma^{XX} & \gamma^{XY} \\ \gamma^{YX} & \gamma^{YY} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} 1 & -kx \\ -kx & 1 + k^2x^2 \end{pmatrix}$$

(Need contra-γ for Hamilton or Riemann equations)

Find: Kinetic energy:

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}(\gamma_{XX}\dot{X}^2 + 2\gamma_{XY}\dot{X}\dot{Y} + \gamma_{YY}\dot{Y}^2) = m\left[\frac{1}{2}(1 + k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2\right]$$

(a) Constrained motion



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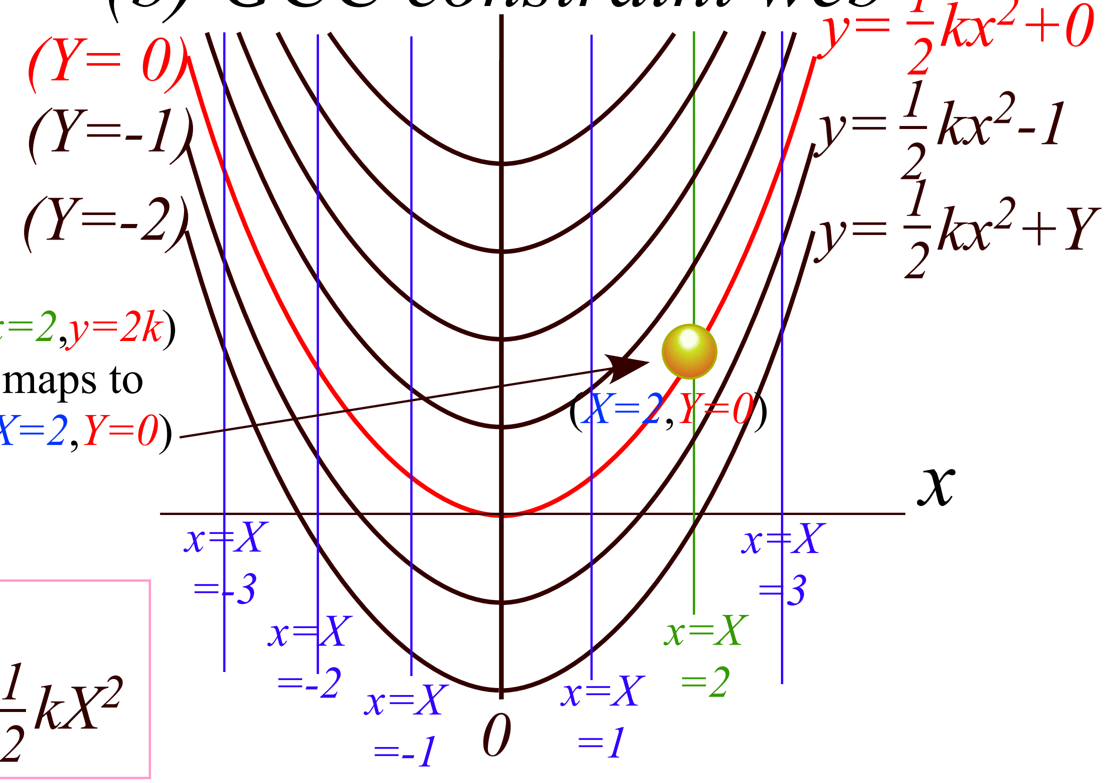

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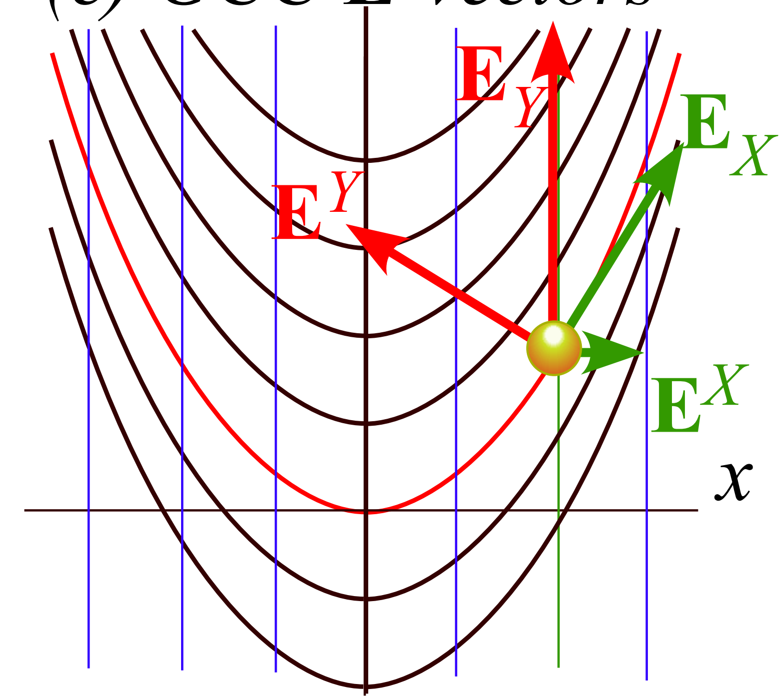

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...and Lagrangian: $L=T-V=m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2+kX\dot{X}\dot{Y}+\frac{1}{2}\dot{Y}^2-gY-\frac{gk}{2}X^2\right]$ $\qquad$ $V=mgy=mg(Y+kX^2/2)$

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

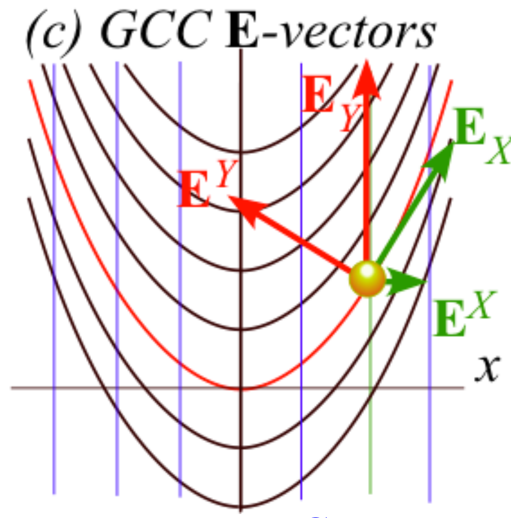
→ *Find covariant force equations*

Compare covariant vs. contravariant forces

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m \underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{(metric\ \gamma_{AB})} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$



CMwBANG!
Fig. 3.9.1

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

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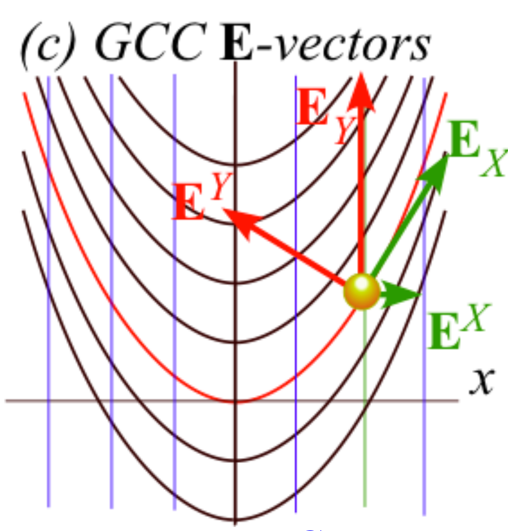

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(metric γ_{AB})

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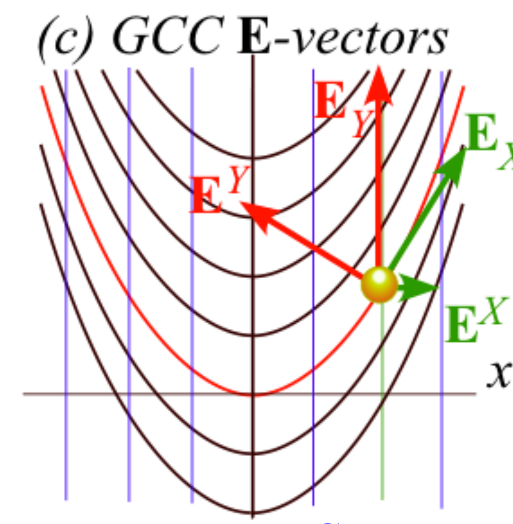

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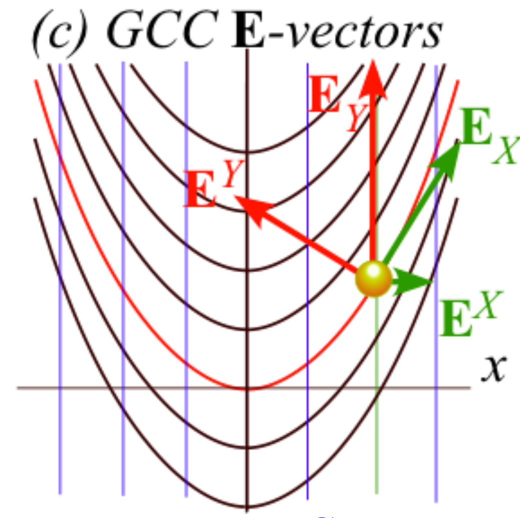

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$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m \frac{d}{dt} \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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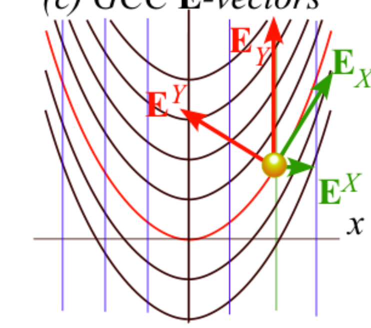
$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m \begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix} \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m \begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m \begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



CMwBANG!
Fig. 3.9.1

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{\text{cov}}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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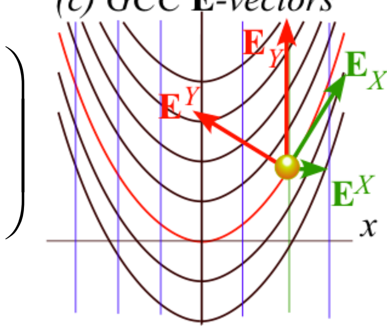
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



CMwBANG!
Fig. 3.9.1

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{\text{(metric } \gamma_{AB})}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\underbrace{\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}}_{\text{(metric } \gamma_{AB})}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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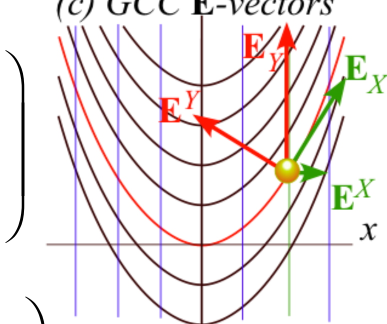
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(c) GCC **E**-vectors



CMwBANG! Fig. 3.9.1

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1^{st} Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2^{nd} Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{\text{cov}}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

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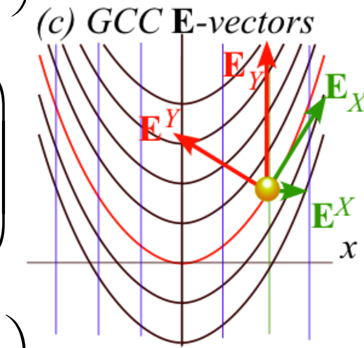
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



CMwBANG! Fig. 3.9.1

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F^m_{con} is zero until we turn on constraint $Y=_{const.}$)

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1[st] Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2[nd] Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

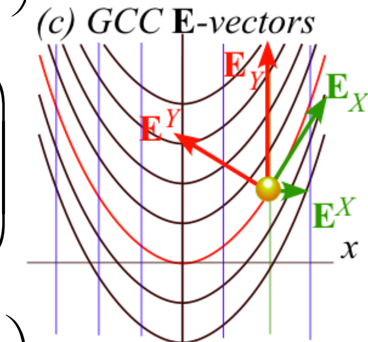
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

CMwBANG! Fig. 3.9.1

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB}) (inverse of γ_{AB})

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1st Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2nd Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov} = 0 = F_Y^{cov}$)

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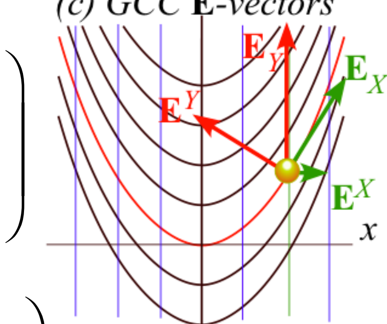
(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



CMwBANG! Fig. 3.9.1

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB}) *(inverse of γ_{AB})*

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} \cancel{1} & \cancel{-kX} \\ \cancel{-kX} & \cancel{1+k^2X^2} \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

Find: Lagrange equations from Lagrangian $L = T - V = m\left[\frac{1}{2}(1+k^2X^2)\dot{X}^2 + kX\dot{X}\dot{Y} + \frac{1}{2}\dot{Y}^2 - gY - \frac{gk}{2}X^2\right]$

(metric γ_{AB})

$$\begin{pmatrix} p_X \\ p_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{X}} \\ \frac{\partial L}{\partial \dot{Y}} \end{pmatrix}$$

(1^{st} Lagrange equations) $p_m = \frac{\partial L}{\partial \dot{q}^m}$

(metric γ_{AB})

$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = \frac{d}{dt}\left[m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix}\right] = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix}$$

(2^{nd} Lagrange equations) $\dot{p}_m = \frac{\partial L}{\partial q^m} + F_m^{cov}$

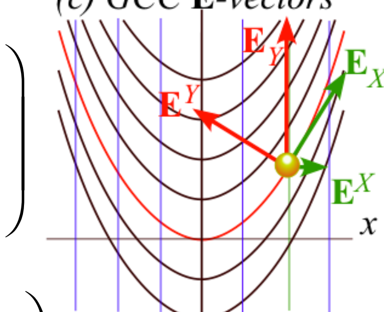
$$\begin{pmatrix} \dot{p}_X \\ \dot{p}_Y \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\frac{d}{dt}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} + m\frac{d}{dt}\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial X} \\ \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix}$$

No constraints added yet to these equations (only gravity in L) so covariant force F_m^{cov} is zero. ($F_X^{cov}=0=F_Y^{cov}$)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} 2k^2X\dot{X} & k\dot{X} \\ k\dot{X} & 0 \end{pmatrix}\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} - m\begin{pmatrix} k^2X\dot{X}^2 + k\dot{X}\dot{Y} - gkX \\ -g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

(cancel)

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} 1+k^2X^2 & kX \\ kX & 1 \end{pmatrix}\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + m\begin{pmatrix} k^2X\dot{X}^2 + gkX \\ k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

CMwBANG! Fig. 3.9.1

Use γ^{AB} to get contra-(Riemann) equations. (Contra-force F_{con}^m is zero until we turn on constraint $Y=_{const.}$)

(inverse of γ_{AB}) (inverse of γ_{AB}) (cancelations)

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 1 & -kX \\ -kX & 1+k^2X^2 \end{pmatrix}\begin{pmatrix} kX(k\dot{X}^2+g) \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

$$\frac{1}{m}\begin{pmatrix} 1 & -kX \\ -kX & 1+k^2x^2 \end{pmatrix}\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} + \begin{pmatrix} 0 \\ k\dot{X}^2+g \end{pmatrix} = \begin{pmatrix} \ddot{X} \\ \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{con}^X \\ F_{con}^Y \end{pmatrix}$$

$\ddot{x} = 0 = \ddot{X}$

Some Ways to do constraint analysis

Way 1. Simple constraint insertion

Way 2. GCC constraint webs

Find covariant force equations

→ *Compare covariant vs. contravariant forces*

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

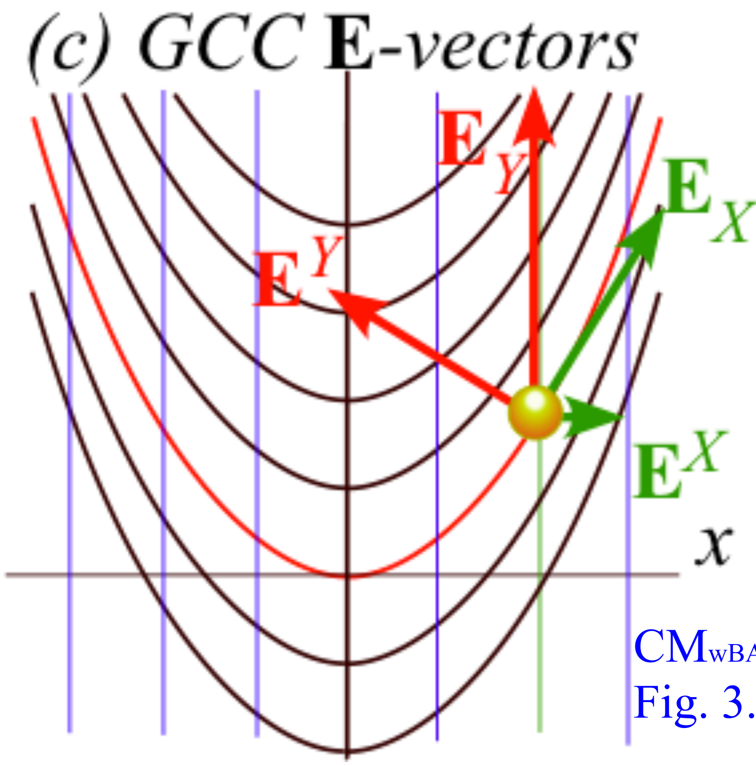
Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$



CMwBANG! Fig. 3.9.1

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$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

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Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y = const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

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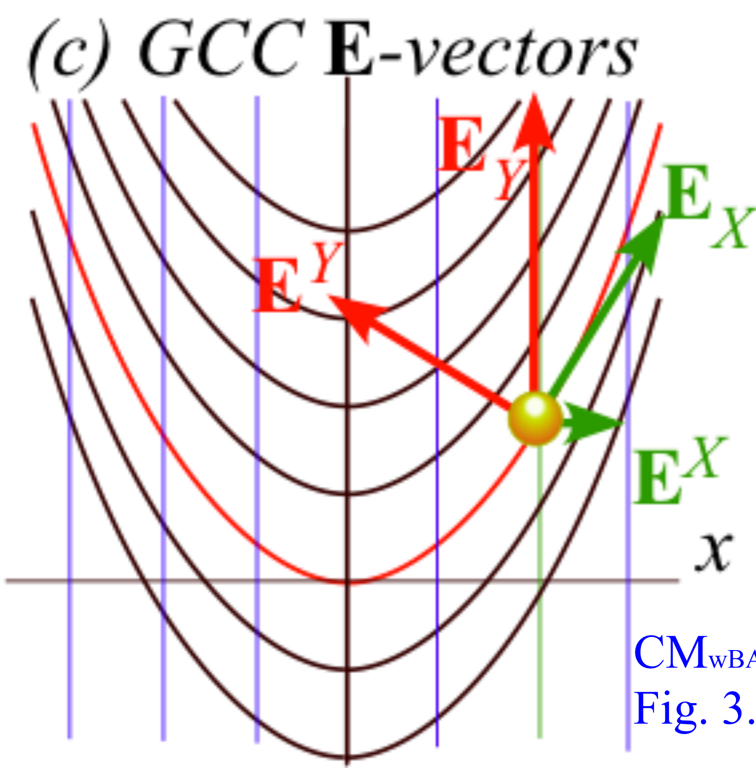

CMwBANG! Fig. 3.9.1

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

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$$\mathbf{F}(Y = const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

$$= 0 \cdot \nabla X + F_Y^{cov}\nabla Y$$

$$= 0 \cdot \mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

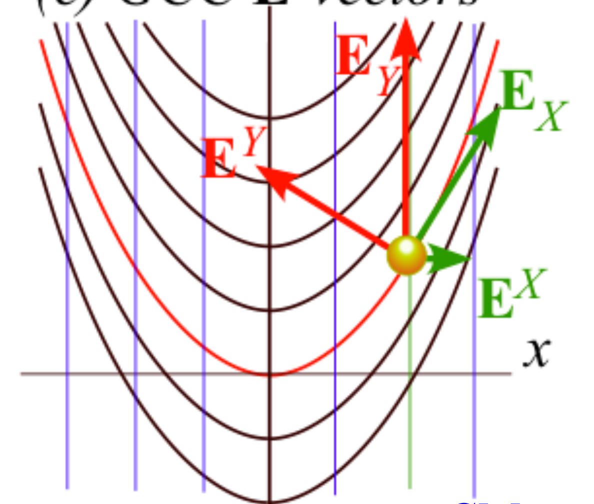
Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

(c) GCC **E***-vectors*



$\dot{Y} = 0 = \ddot{Y}$

CMwBANG! Fig. 3.9.1

General case repeated from p.34

$$\begin{pmatrix} \dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y} \end{pmatrix} = m\begin{pmatrix} (1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g \end{pmatrix} = \begin{pmatrix} F_X^{cov} \\ F_Y^{cov} \end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

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Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
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So constraint requirements in covariant equations are $F_X^{cov}=0$ and $F_Y^{cov}\neq 0$. (with: $\dot{Y}=0=\ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X}+0+k^2X\dot{X}^2+gkX\\ kX\ddot{X}+0+k\dot{X}^2+g\end{pmatrix}=\begin{pmatrix}0\\ F_Y^{cov}\end{pmatrix} \rightarrow \ddot{X}=-\frac{k^2X\dot{X}^2+gkX}{1+k^2X^2}=-\frac{k\dot{X}^2+g}{1+k^2X^2}kX$$

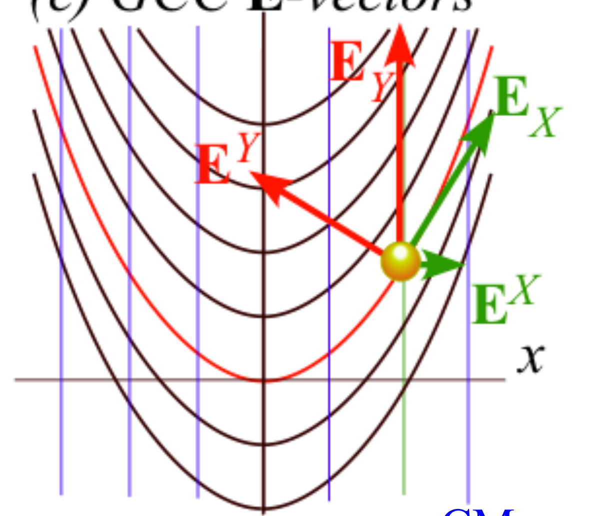
Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

(c) GCC **E***-vectors*



$\dot{Y}=0=\ddot{Y}$

CMwBANG! Fig. 3.9.1

FINALLY ! We get the Way 1. solution of p.12

Recall: $x\equiv X$

$$\ddot{X}\equiv\ddot{x}=\frac{-k\dot{x}^2-g}{1+k^2x^2}kx$$

General case repeated from p.34

$$\begin{pmatrix}\dot{p}_X-\frac{\partial L}{\partial X}\\ \dot{p}_Y-\frac{\partial L}{\partial Y}\end{pmatrix}=m\begin{pmatrix}(1+k^2X^2)\ddot{X}+kX\ddot{Y}+k^2X\dot{X}^2+gkX\\ kX\ddot{X}+\ddot{Y}+k\dot{X}^2+g\end{pmatrix}=\begin{pmatrix}F_X^{cov}\\ F_Y^{cov}\end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

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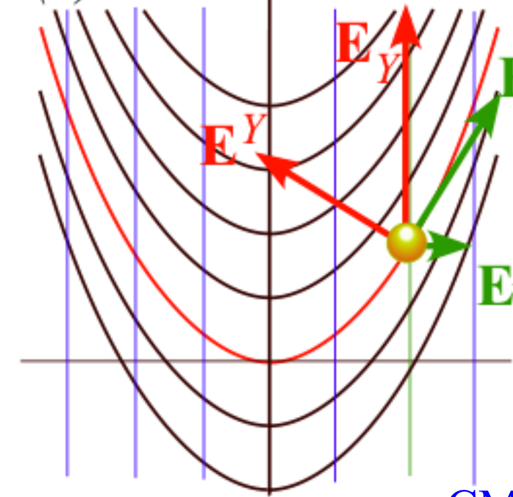
*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

Frictionless constraint of mass m by parabola $Y=const.$ is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

(c) GCC **E***-vectors*



$\dot{Y} = 0 = \ddot{Y}$

CMwBANG! Fig. 3.9.1

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix} \rightarrow \ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2+g}{1+k^2X^2}kX$$

$$\mathbf{F} = F_Y^{cov}\,\mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

General case repeated from p.34

$$\begin{pmatrix}\dot{p}_X - \frac{\partial L}{\partial X} \\ \dot{p}_Y - \frac{\partial L}{\partial Y}\end{pmatrix} = m\begin{pmatrix}(1+k^2X^2)\ddot{X} + kX\ddot{Y} + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + \ddot{Y} + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}F_X^{cov} \\ F_Y^{cov}\end{pmatrix}$$

Constraint force components are covariant

Frictionless constraint forces have *covariant* components F_B^{cov}

$$\mathbf{F} = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y = F_X^{cov}\nabla X + F_Y^{cov}\nabla Y$$

*(F_A are coefficients of **normal** vectors $\mathbf{E}^A$)*

Frictional force components are contravariant

Frictional or driving forces have *contravariant* components F_{con}^A

$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial\mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial\mathbf{r}}{\partial Y}$$

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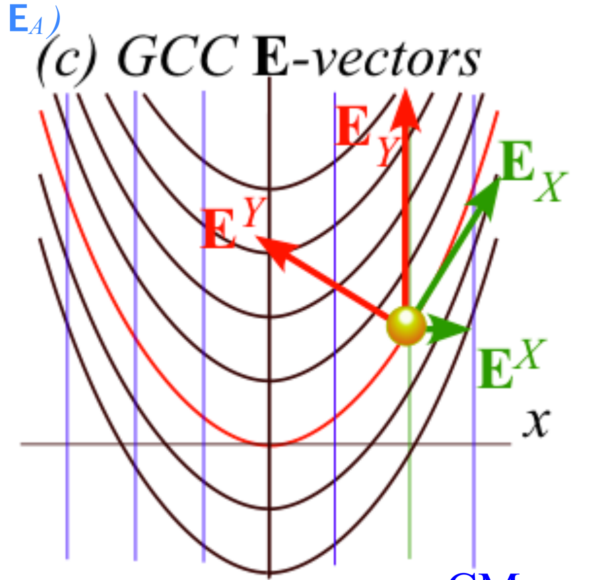
$$\mathbf{F}(Y=const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$
$$= 0\cdot\nabla X + F_Y^{cov}\nabla Y$$
$$= 0\cdot\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$



$\dot{Y} = 0 = \ddot{Y}$

CMwBANG! Fig. 3.9.1

So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix} \rightarrow \ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

$$\mathbf{F} = \quad F_Y^{cov} \quad \mathbf{E}^Y$$
$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$
$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

Recall: $x \equiv X$

$$\ddot{X} \equiv \boxed{\ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx}$$

Constraint force components are covariant

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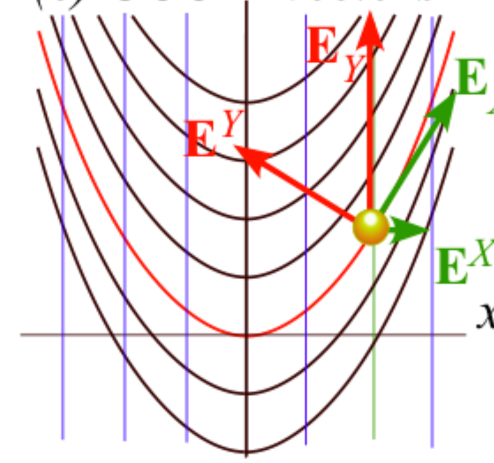
$$\mathbf{F} = F_{con}^X\mathbf{E}_X + F_{con}^Y\mathbf{E}_Y = F_{con}^X\frac{\partial \mathbf{r}}{\partial X} + F_{con}^Y\frac{\partial \mathbf{r}}{\partial Y}$$

*(F^A are coefficients of **tangent** vectors $\mathbf{E}_A$)*

(c) GCC **E***-vectors*



$$\dot{Y} = 0 = \ddot{Y}$$

CMwBANG! Fig. 3.9.1

Frictionless constraint of mass *m* by parabola *Y=const.* is normal to parabola (along its gradient ∇Y.)

$$\mathbf{F}(Y = const.) = F_X^{cov}\mathbf{E}^X + F_Y^{cov}\mathbf{E}^Y$$

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So constraint requirements in covariant equations are $F_X^{cov} = 0$ and $F_Y^{cov} \neq 0$. (with: $\dot{Y} = 0 = \ddot{Y}$).

$$m\begin{pmatrix}(1+k^2X^2)\ddot{X} + 0 + k^2X\dot{X}^2 + gkX \\ kX\ddot{X} + 0 + k\dot{X}^2 + g\end{pmatrix} = \begin{pmatrix}0 \\ F_Y^{cov}\end{pmatrix}$$

$$\ddot{X} = -\frac{k^2X\dot{X}^2 + gkX}{1+k^2X^2} = -\frac{k\dot{X}^2 + g}{1+k^2X^2}kX$$

$$\mathbf{F} = \quad F_Y^{cov} \quad \mathbf{E}^Y$$

$$= m(kX\ddot{X} + 0 + k\dot{X}^2 + g)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

$$= m\left(\frac{-kX(k\dot{X}^2+g)}{1+k^2X^2} + \frac{(k\dot{X}^2+g)(1+k^2X^2)}{1+k^2X^2}\right)\begin{pmatrix}-kX \\ 1\end{pmatrix}$$

$$\begin{pmatrix}F_x \\ F_y\end{pmatrix} = \left(=\begin{pmatrix}0 \\ mk\dot{X}^2 + mg\end{pmatrix}\right)_{at:X=0}$$

Centripetal force mkv^2+mg *(what roller-coaster rider feels at bottom)*

Recall: $x \equiv X$

$$\ddot{X} \equiv \ddot{x} = \frac{-k\dot{x}^2 - g}{1+k^2x^2}kx$$

$$-g = \ddot{y} = \tfrac{d^2}{dt^2}\left(\tfrac{1}{2}kX^2 + Y\right)$$

$$= k\dot{X}^2 + kX\ddot{X} + \ddot{Y}\,(= k\dot{X}^2 + \ddot{Y} \text{ for } \ddot{X} = 0)$$

Other Ways to do constraint analysis

→ *Way 3. OCC constraint webs*

Sketch of atomic-Stark orbit parabolic OCC analysis

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

"Non-Holonomic" multipliers

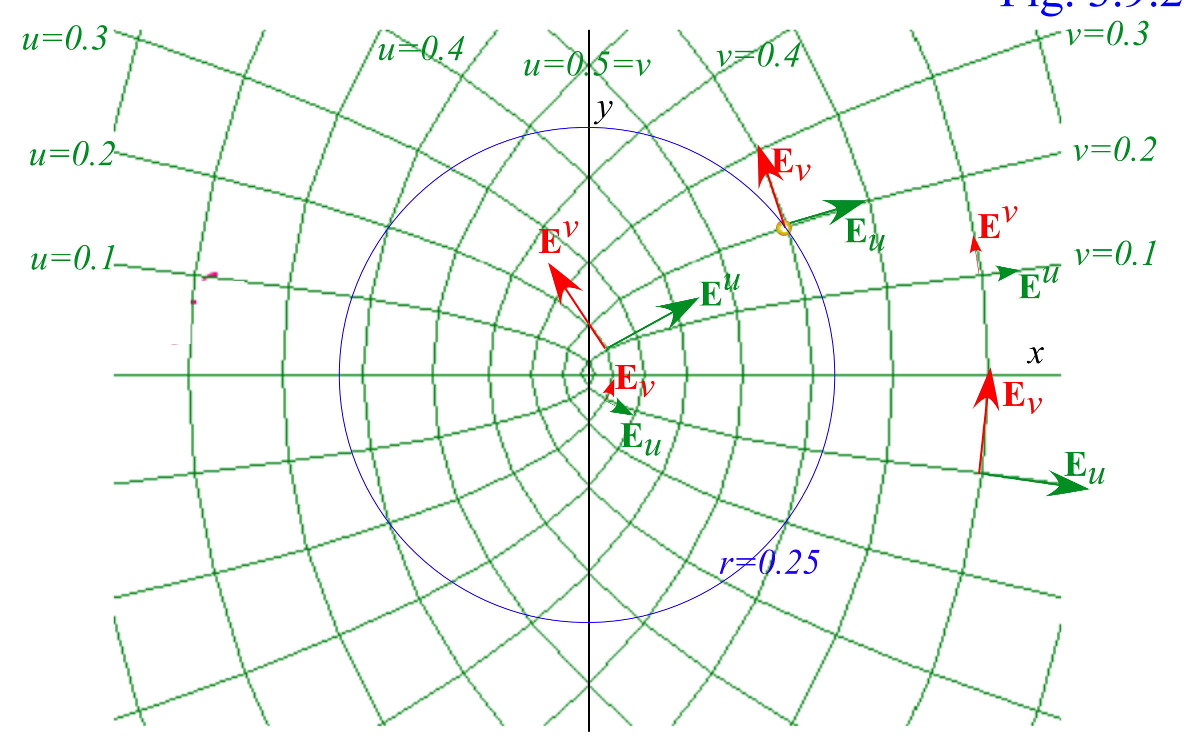
Way 3. Parabolic OCC approach

Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

Expansion of z and then absolute square $|z|^2$ give relations between Cartesian (x,y) and OCC(u,v)

$$z = x+iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2+y^2 = (u^2+v^2)^2 = u^4+v^4+2u^2v^2$$

CMwBANG!
Fig. 3.9.2



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$$z = x + iy = (u + iv)^2 = u^2 - v^2 + i2uv$$

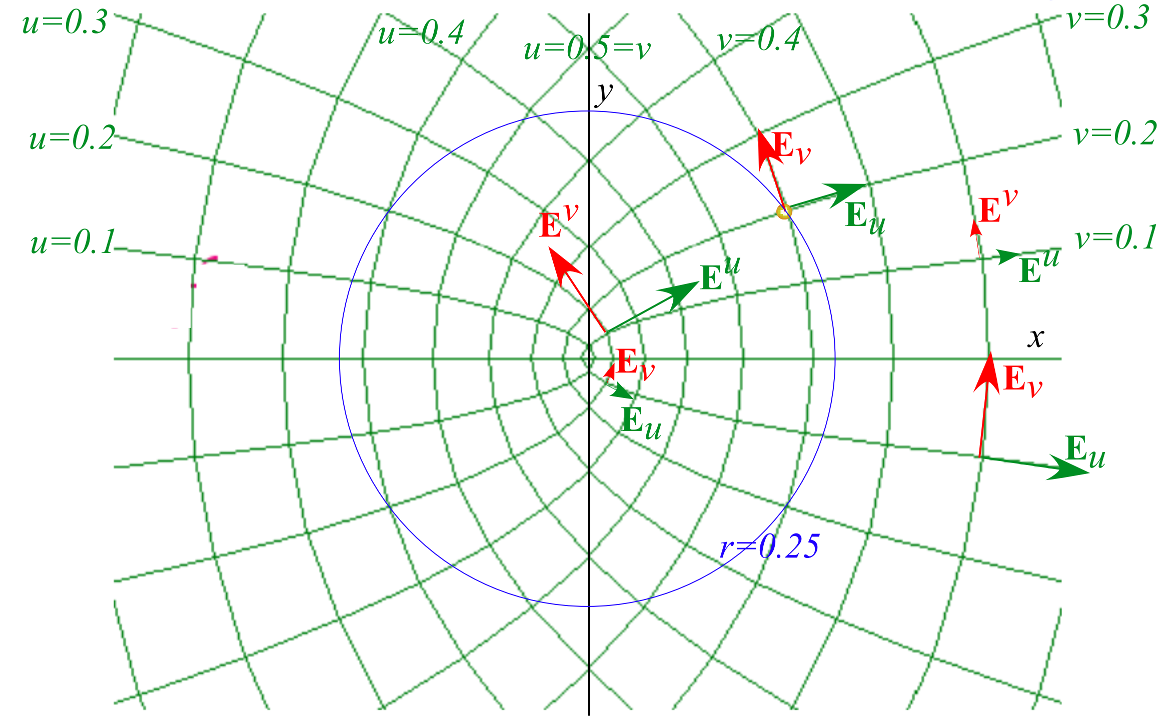
$$r^2 = z * z = x^2 + y^2 = (u^2 + v^2)^2 = u^4 + v^4 + 2u^2v^2$$

$$x = u^2 - v^2$$

$$y = 2uv$$

$$r = u^2 + v^2$$

CMwBANG! Fig. 3.9.2



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$$x = u^2 - v^2$$

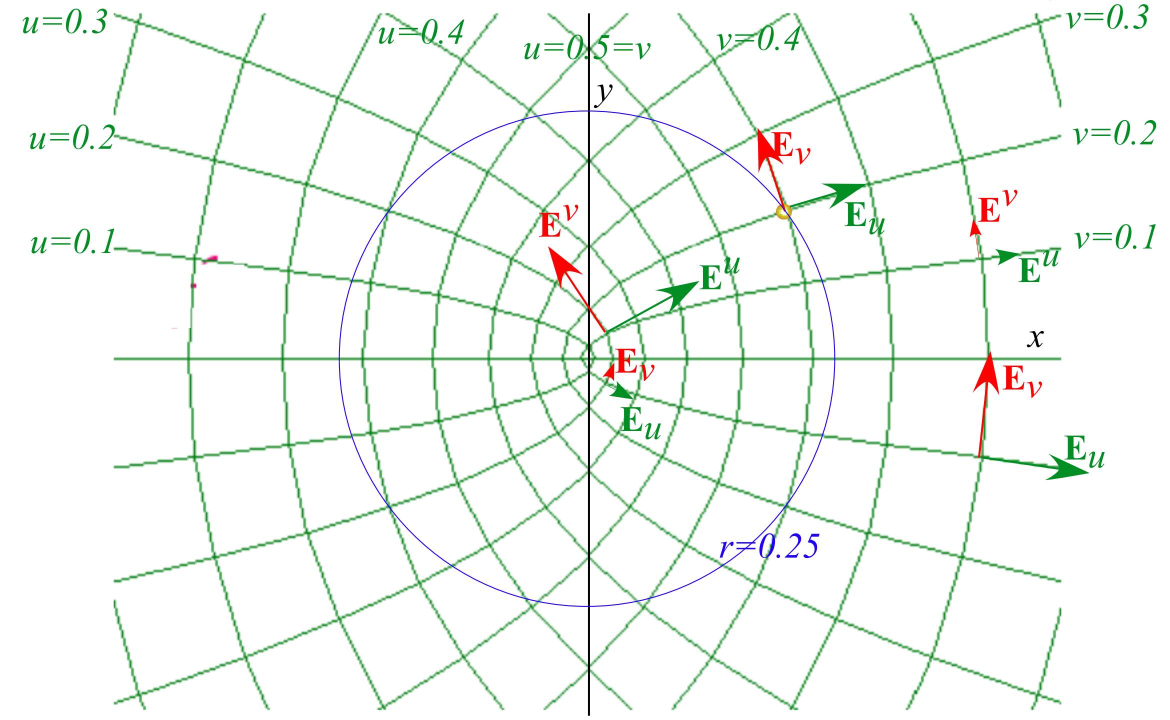
$$y = 2uv$$

$$r = u^2 + v^2$$

$$2u^2 = r + x = \sqrt{x^2 + y^2} + x$$

$$2v^2 = r - x = \sqrt{x^2 + y^2} - x$$

CMwBANG! Fig. 3.9.2



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Complex function $z=w^2$ or its inverse $w=z^{1/2}$ of complex variables $z=x+iy$ and $w=u+iv$.

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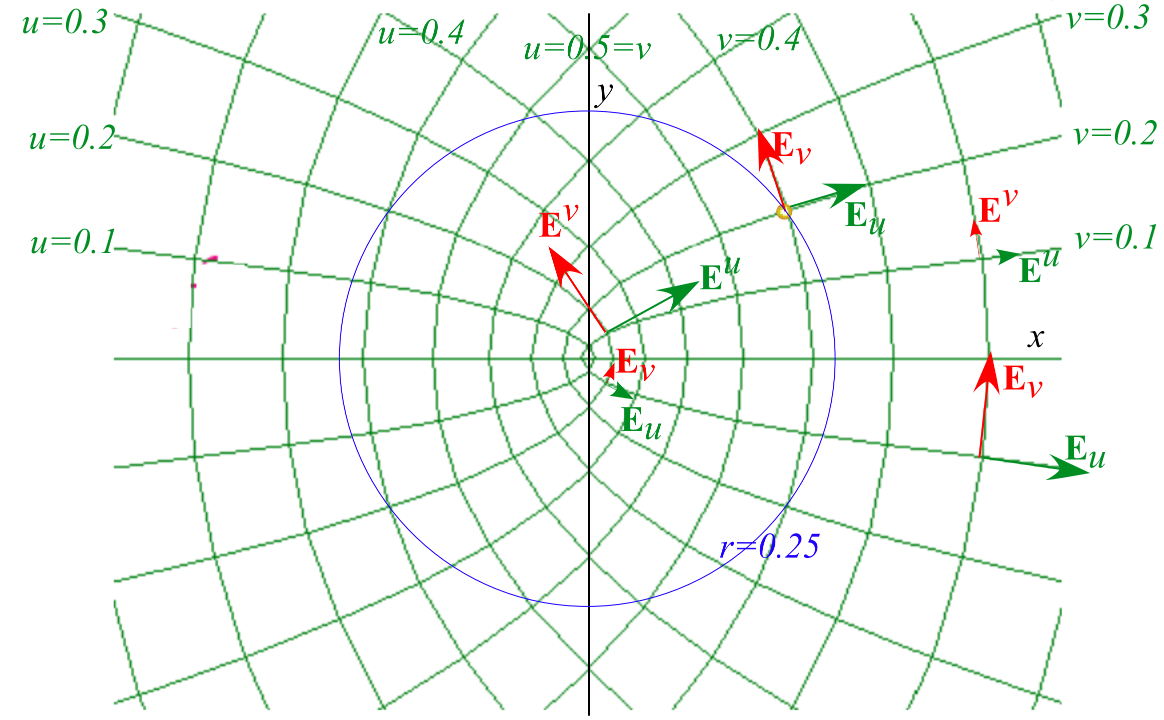
$$z = x + iy = (u+iv)^2 = u^2 - v^2 + i2uv \qquad r^2 = z^*z = x^2 + y^2 = (u^2+v^2)^2 = u^4 + v^4 + 2u^2v^2$$

CMwBANG!
Fig. 3.9.2



$$x = u^2 - v^2$$

$$y = 2uv \qquad 2u^2 = r + x = \sqrt{x^2+y^2} + x$$

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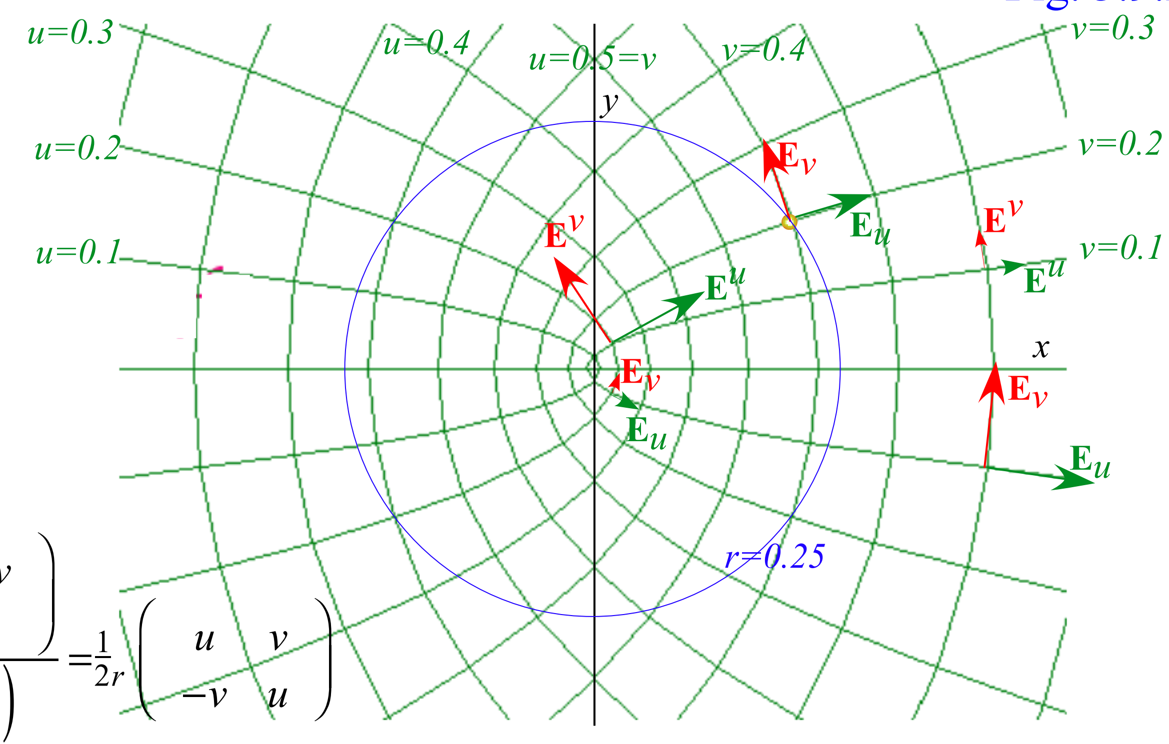

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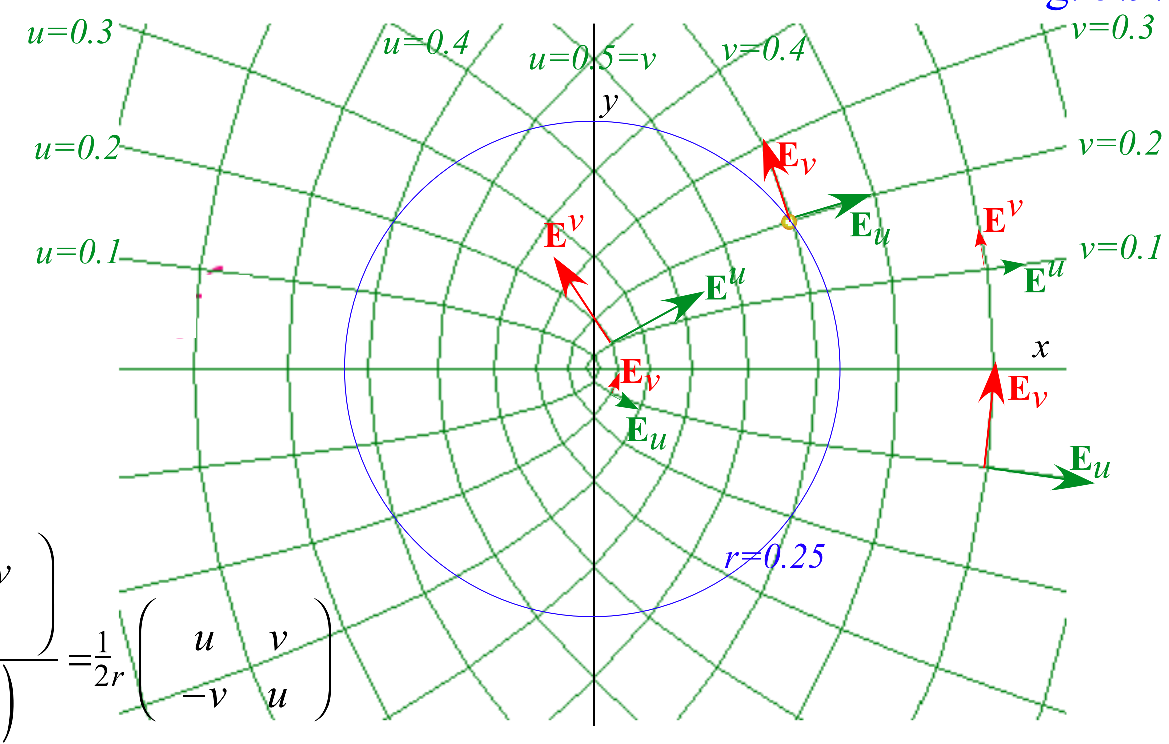

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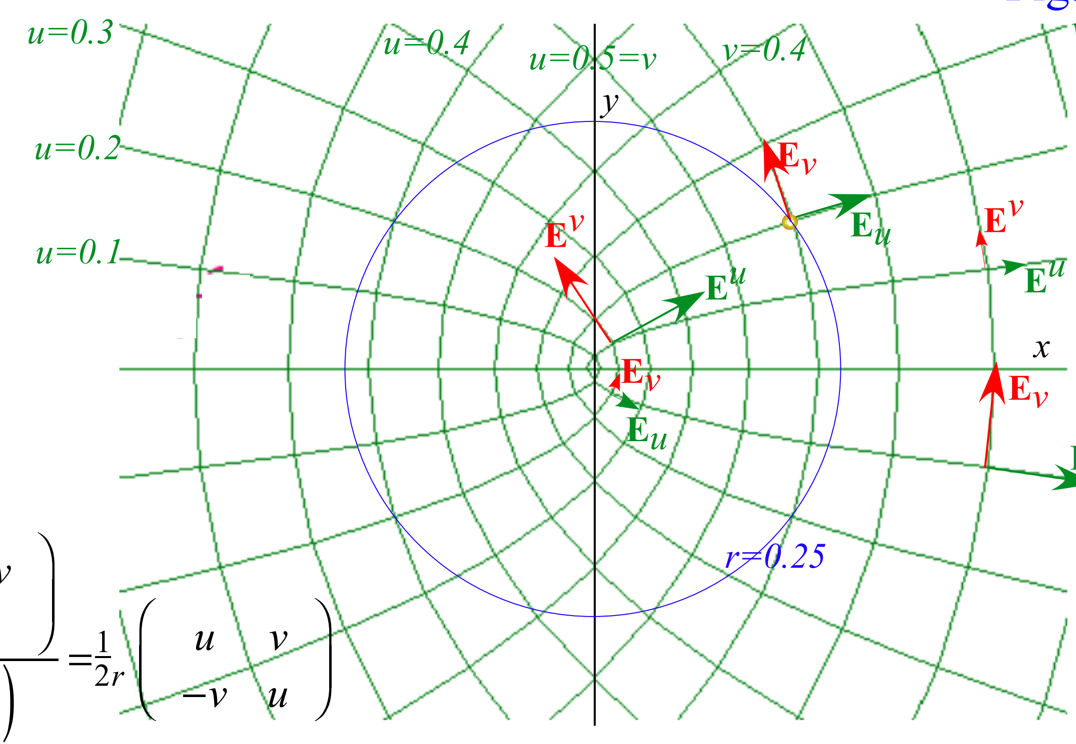

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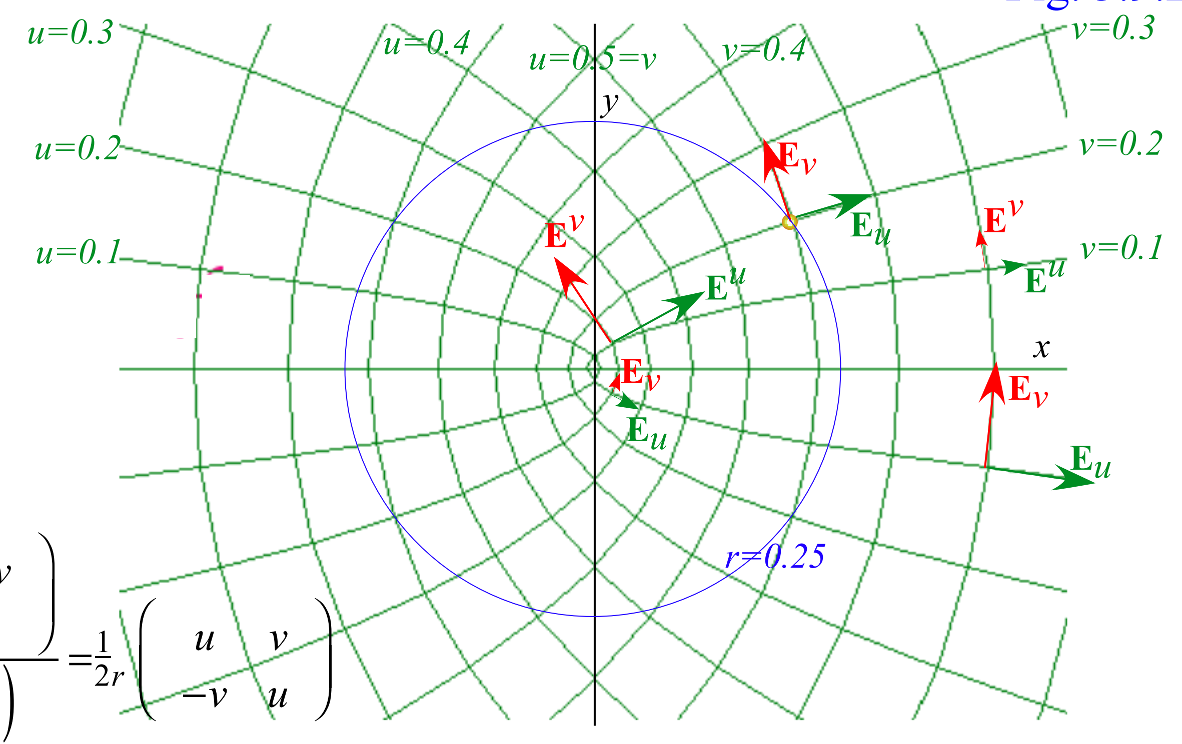

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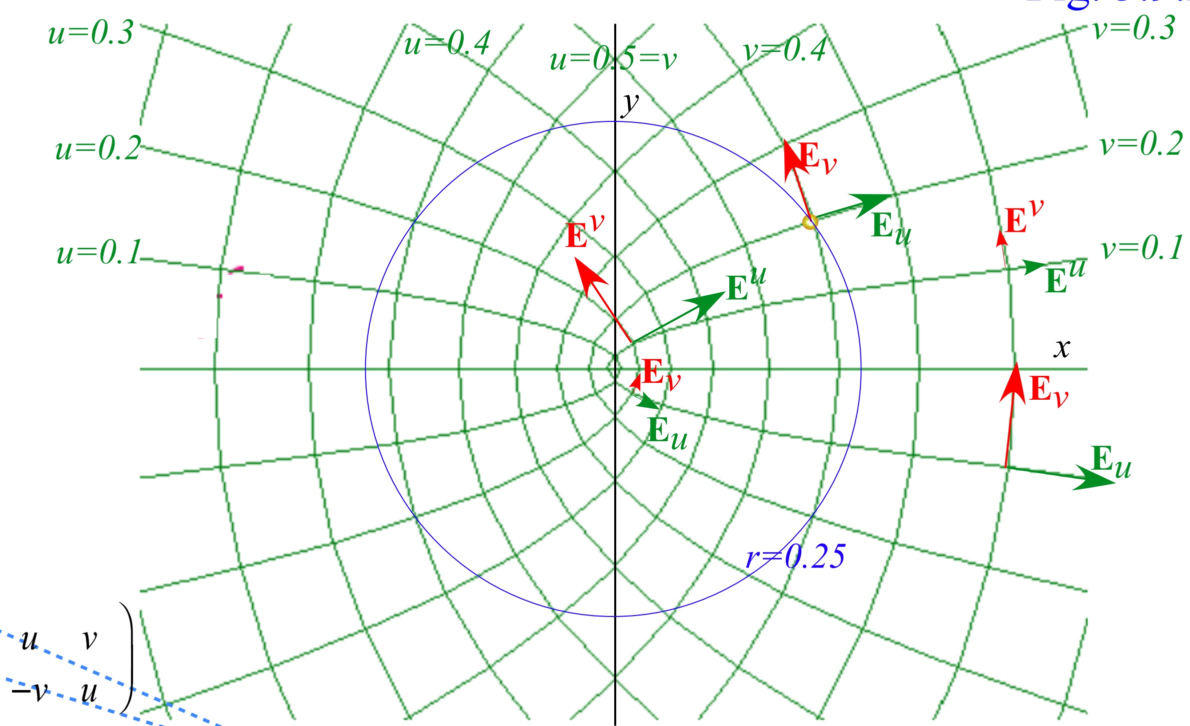

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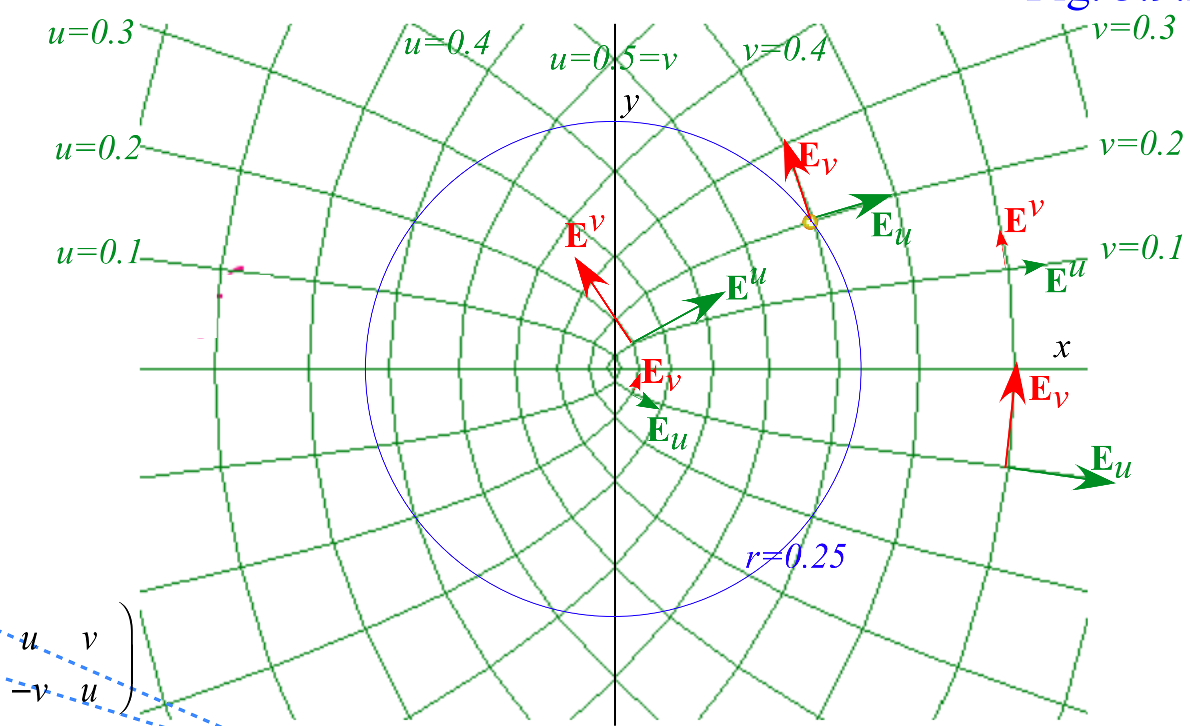

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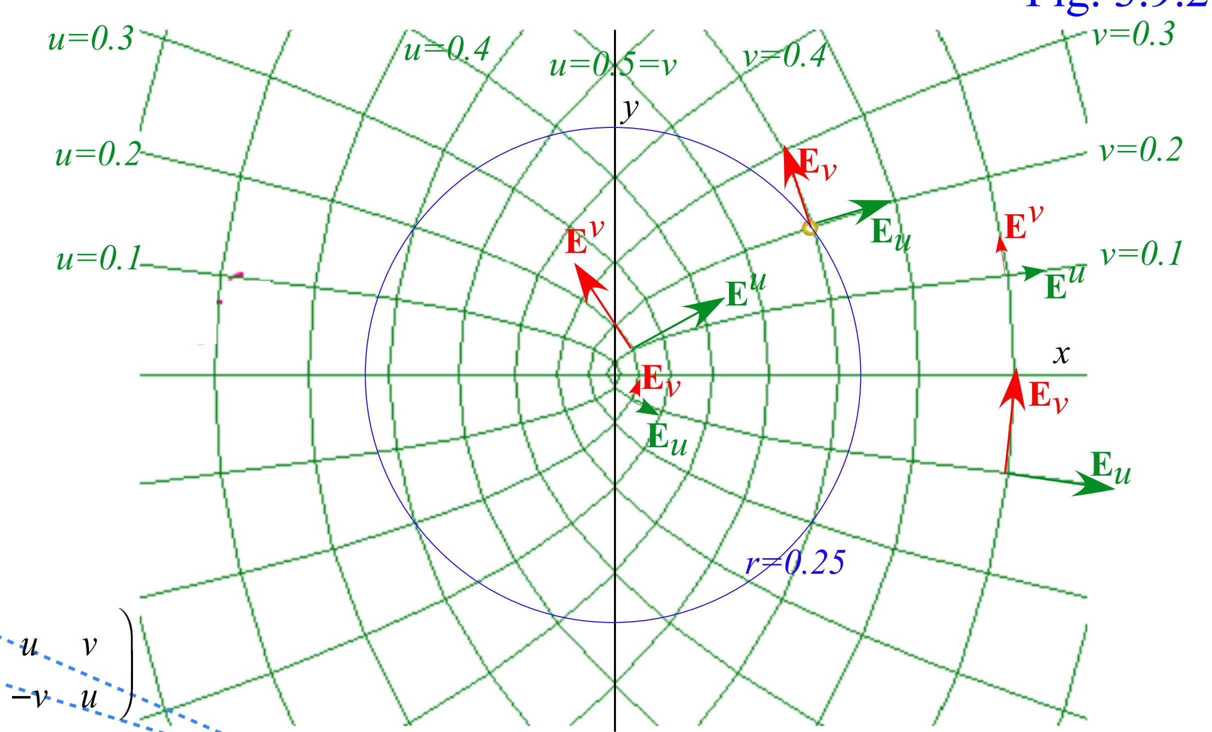

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Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

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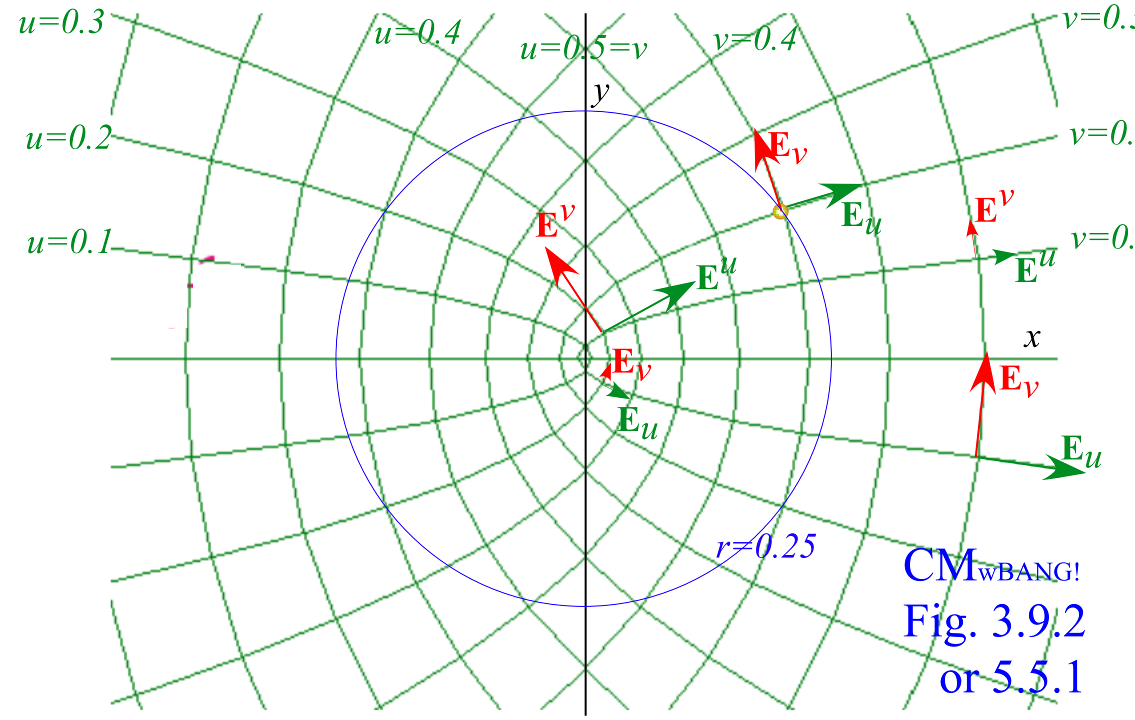

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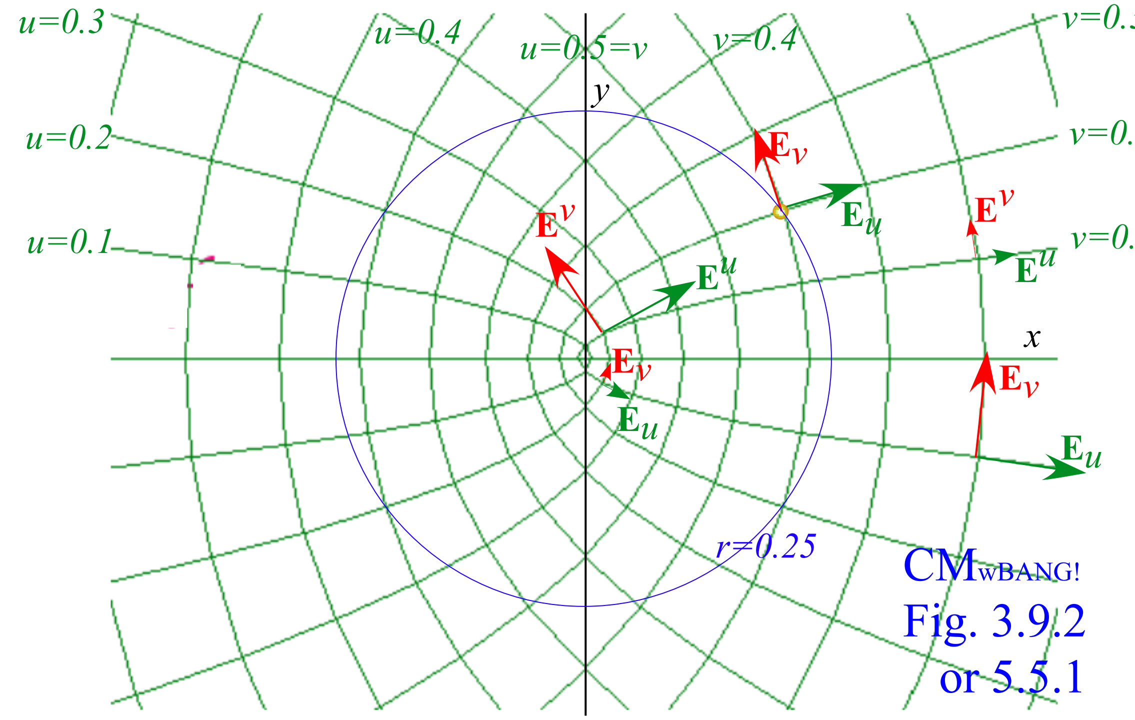

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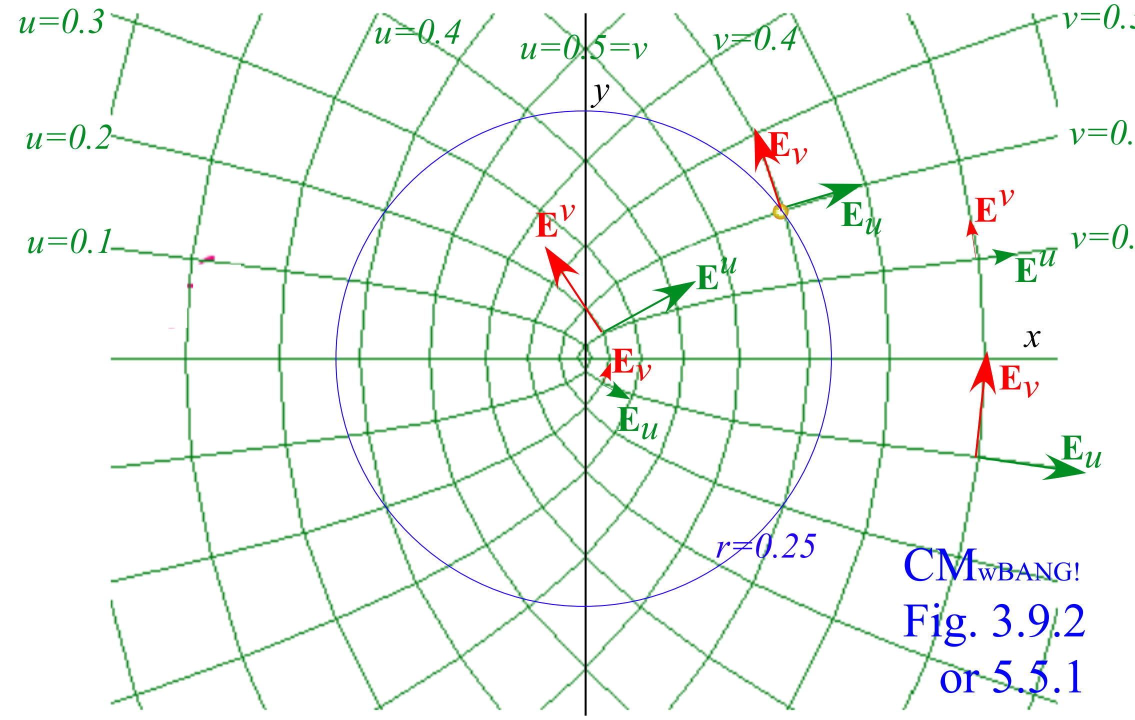

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$$V = \varepsilon x + k/r$$

Stark-Coulomb potential

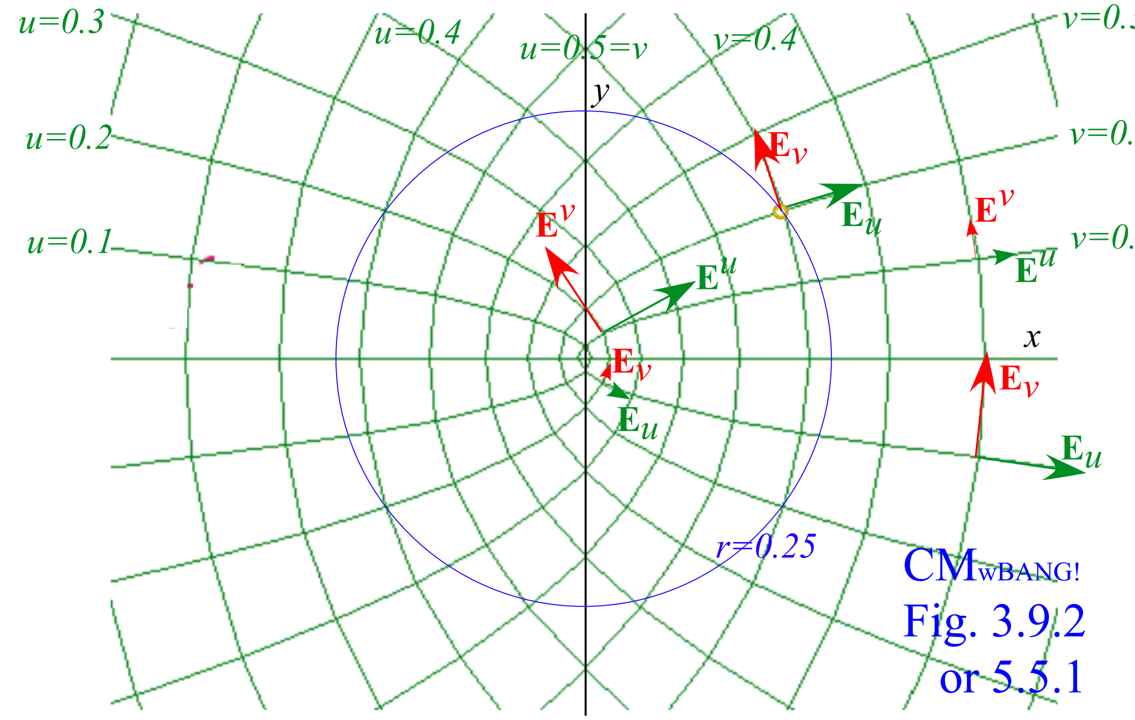
For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and ***separable*** into u and v parts.

$$4(u^2+v^2)E = \frac{1}{2m}(p_u^2+p_v^2) + 4(u^4-v^4)\varepsilon + 4k \quad \text{for: } H=E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2-v^2) + \frac{k}{u^2+v^2}$$

Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0 = \frac{1}{2m}p_u^2 - 4Eu^2 + 4\varepsilon u^4 \quad + \quad \frac{1}{2m}p_v^2 - 4Ev^2 - 4\varepsilon v^4 + 4k \quad = \quad h_u \quad + \quad h_v$$

Stark orbit parabolic OCC analysis



CMwBANG!
Fig. 3.9.2
or 5.5.1

Metric $g_{uv}=\mathbf{E}_u \bullet \mathbf{E}_v$ and g^{uv} are diagonal. Lagrangian L uses $g_{uv}=\delta_{uv}4r$. Hamiltonian H uses $g^{uv}=\delta^{uv}/4r$.

$$L = \tfrac{m}{2}(g_{ab}\dot{q}^a\dot{q}^b) - V = \tfrac{m}{2}(g_{uu}\dot{u}^2 + g_{vv}\dot{v}^2) - V = 2m(\dot{u}^2+\dot{v}^2)(u^2+v^2) - V$$

$$H = \tfrac{1}{2m}(g^{ab}p_a p_b) + V = \tfrac{1}{2m}(g^{uu}p_u^2 + g^{vv}p_v^2) + V = \frac{p_u^2+p_v^2}{8m(u^2+v^2)} + V$$

$$V = \varepsilon x + k/r$$

Stark-Coulomb potential

For a *Stark-Coulomb potential* Hamiltonian *(H=E)* is constant and *separable* into u and v parts.

$$4(u^2+v^2)E = \tfrac{1}{2m}(p_u^2+p_v^2) + 4(u^4-v^4)\varepsilon + 4k \quad \text{for: } H=E \text{ and: } V = \varepsilon x + \frac{k}{r} = \varepsilon(u^2-v^2) + \frac{k}{u^2+v^2}$$

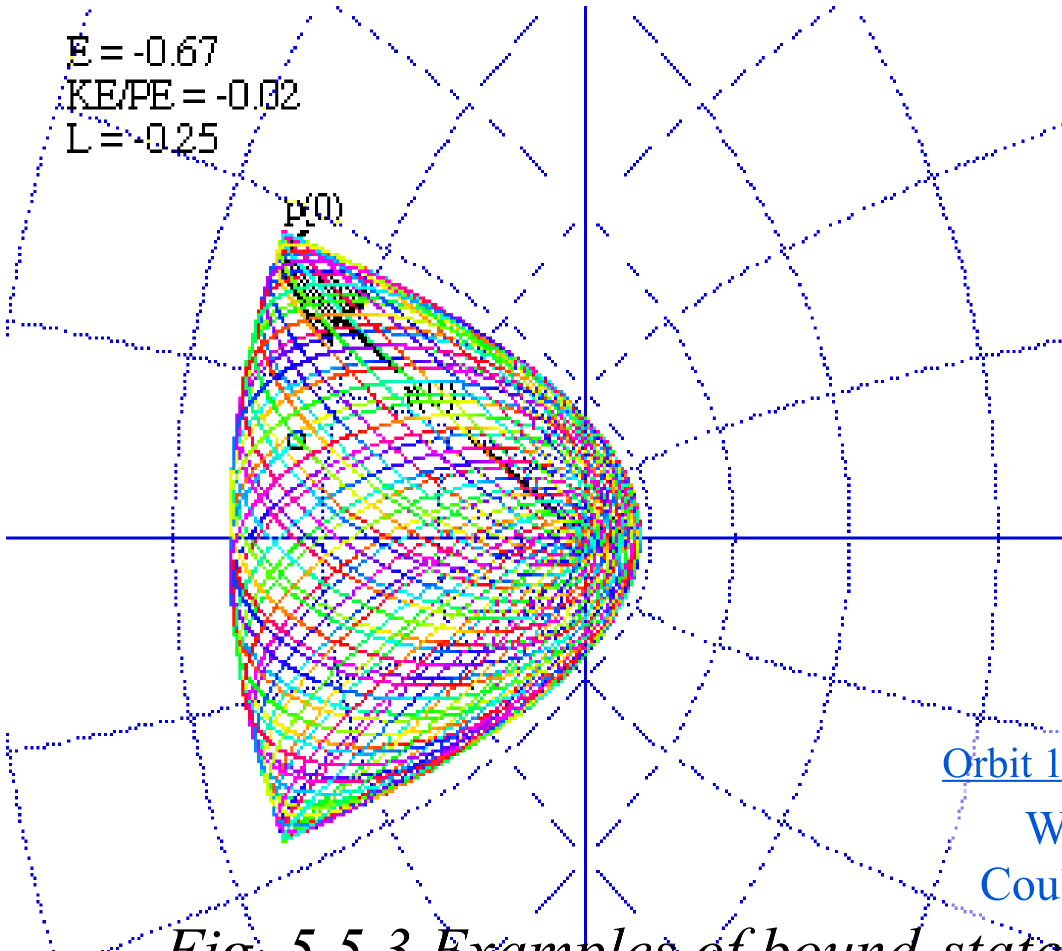
Each sub-Hamiltonian part h_u and h_v is a constant. Together they sum to zero total energy $0=h_u+h_v$.

$$0 = \tfrac{1}{2m}p_u^2 - 4Eu^2 + 4\varepsilon u^4 \quad + \quad \tfrac{1}{2m}p_v^2 - 4Ev^2 - 4\varepsilon v^4 + 4k \quad = \quad h_u \quad + \quad h_v$$

Zero Stark-field $(\varepsilon=0)$ gives h_u or h_v harmonic oscillation if $E<0$. It's unstable or anharmonic otherwise.

$$\dot{p}_u = -\frac{\partial h_u}{\partial u} = -8Eu + 16\varepsilon u^3 \qquad \dot{u} = \frac{\partial h_u}{\partial p_u} = p_u/m \qquad \dot{p}_v = -\frac{\partial h_v}{\partial v} = -8Ev - 16\varepsilon v^3 \qquad \dot{v} = \frac{\partial h_v}{\partial p_v} = p_v/m$$

Stark orbit parabolic OCC analysis



[Orbit 1](#)   [Orbit 2](#)

Web Simulations

CoulIt Stark-Coulonb

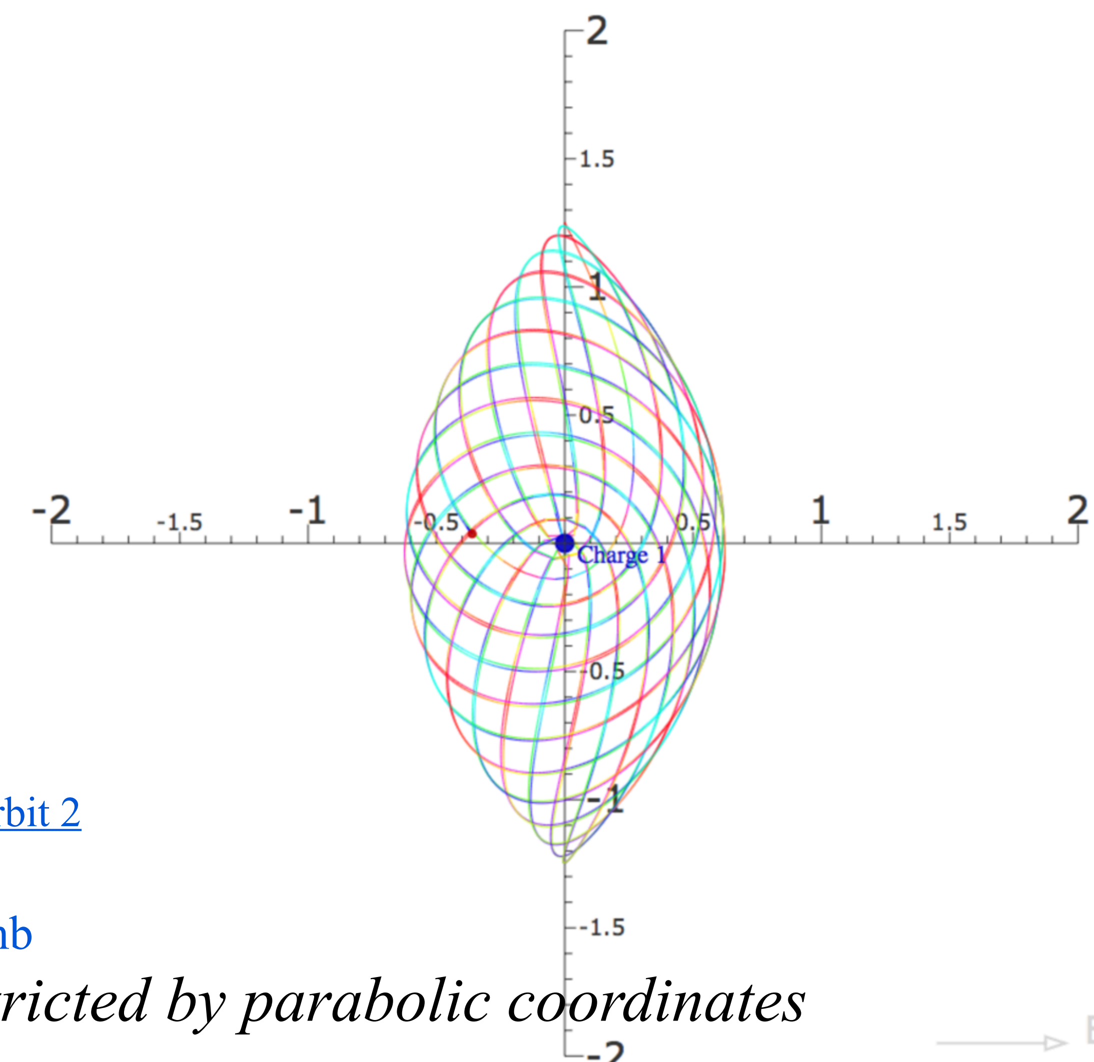


Fig. 5.5.3 Examples of bound-state motion restricted by parabolic coordinates

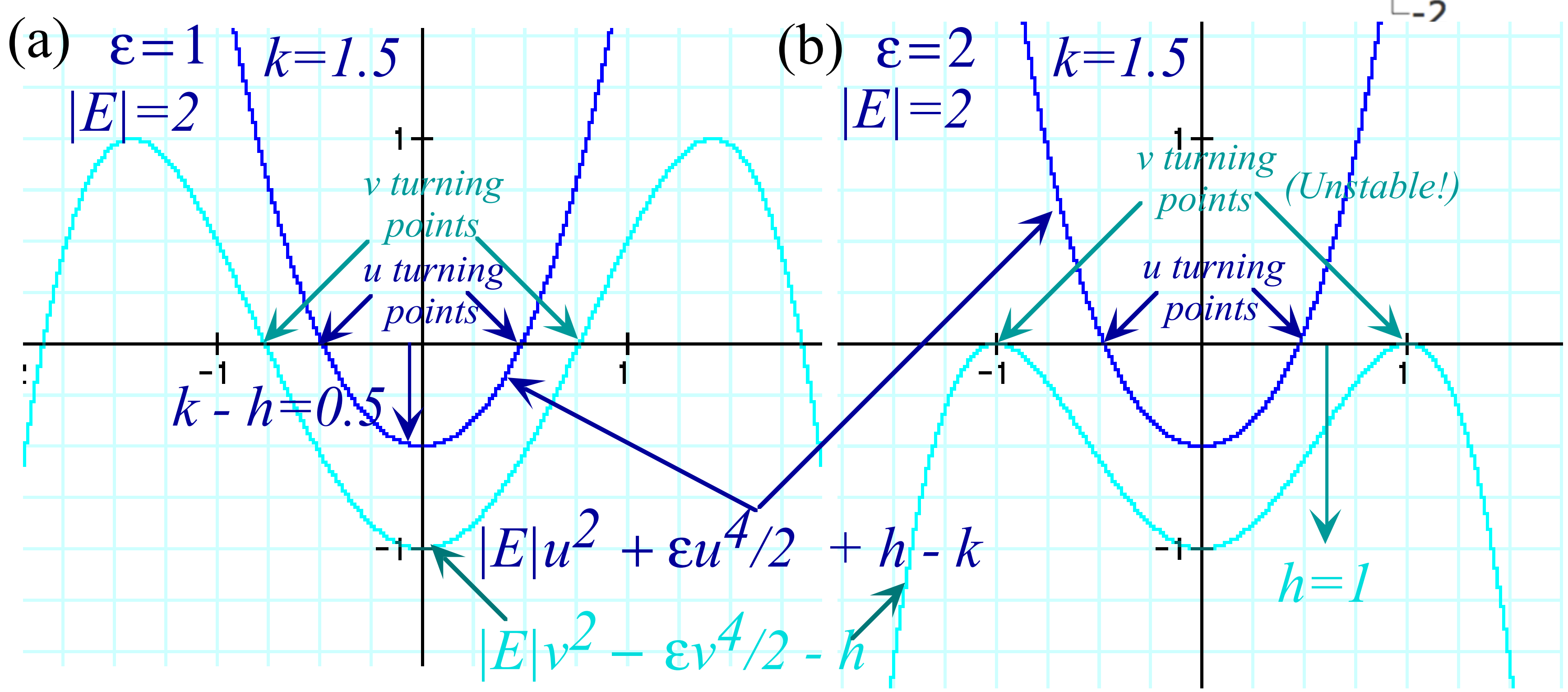
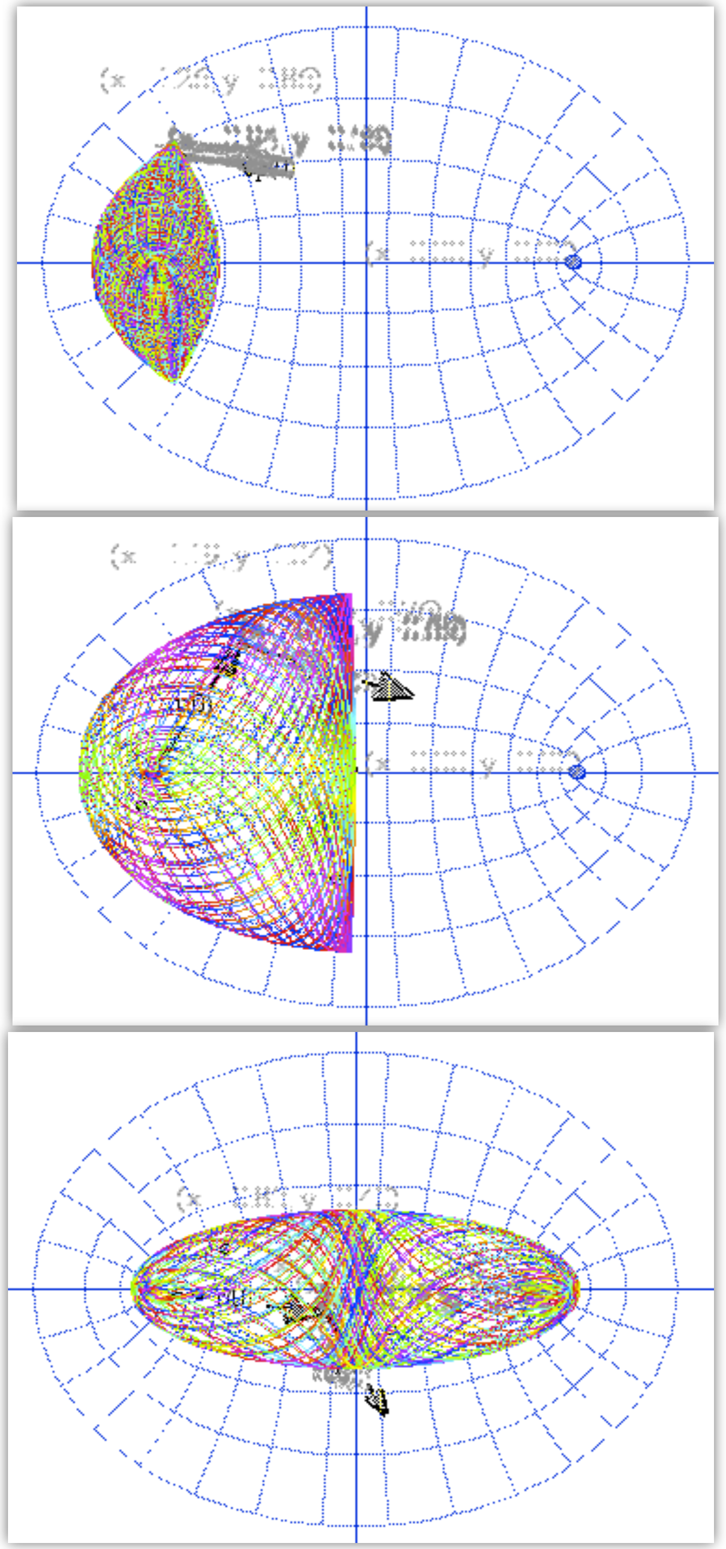
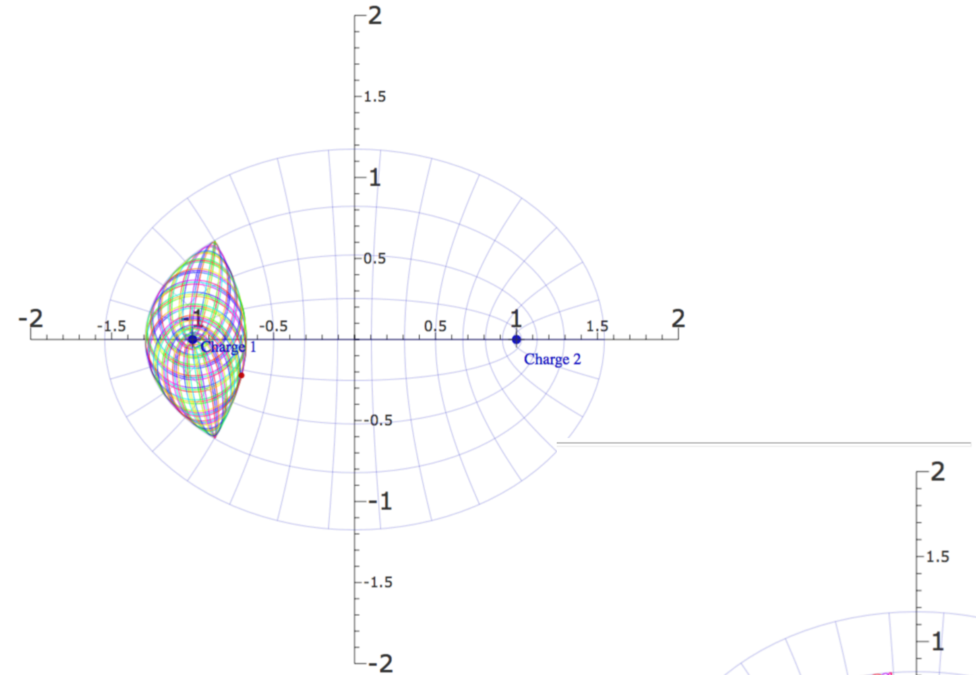


Fig. 5.5.2 Effective potentials for parabolic coordinates

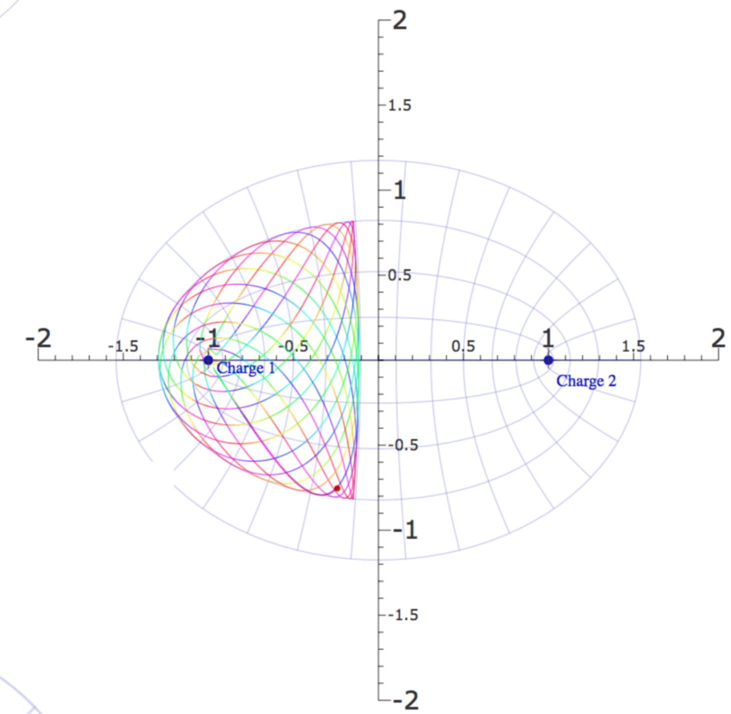
Hs+-ion orbit elliptic-hyperbolic OCC bound trajectories



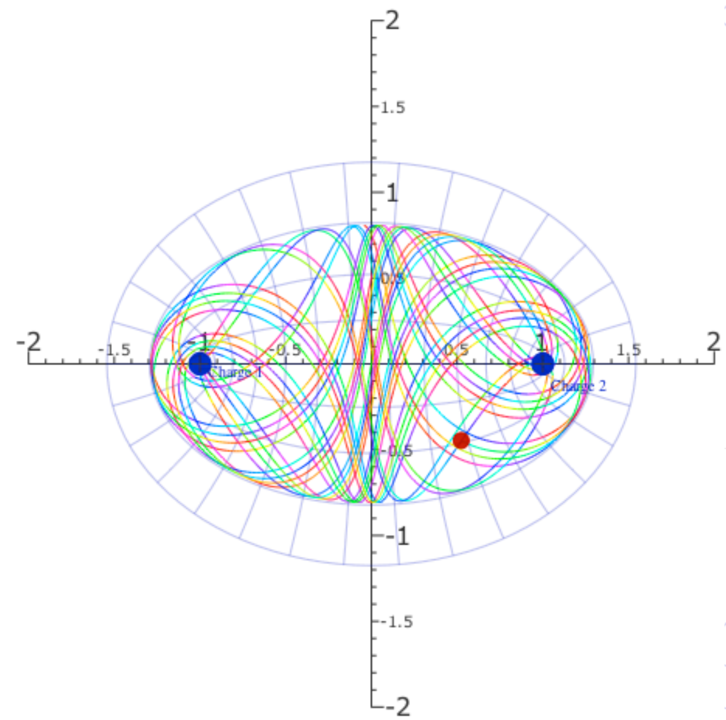
Web Simulations
CoulIt H_2^+



Orbit 1: Localized on C_1

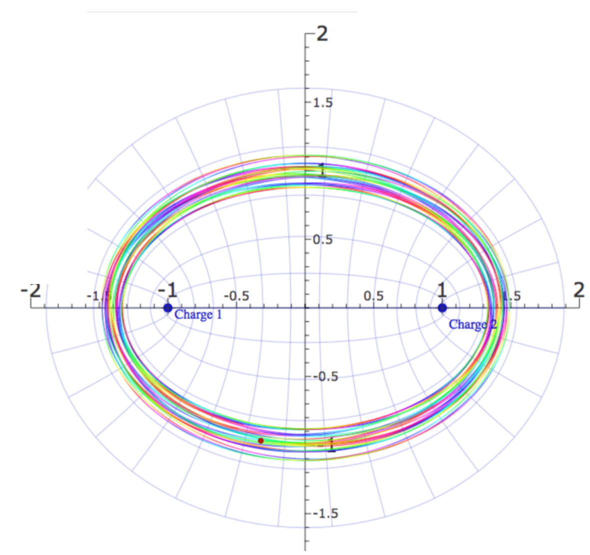


Orbit 2: Less localized on C_1



Orbit 3a: Sharing C_1 and C_2

Orbit 3b: Sharing C_1 and C_2



Orbit 4: Quasi-Stable Elliptical

CMwBANG!
Fig. 5.5.4

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Sketch of atomic-Stark orbit parabolic OCC analysis

Classical Hamiltonian separability

→ *Way 4. Lagrange multipliers*

Lagrange multiplier as eigenvalues

Multiple multipliers

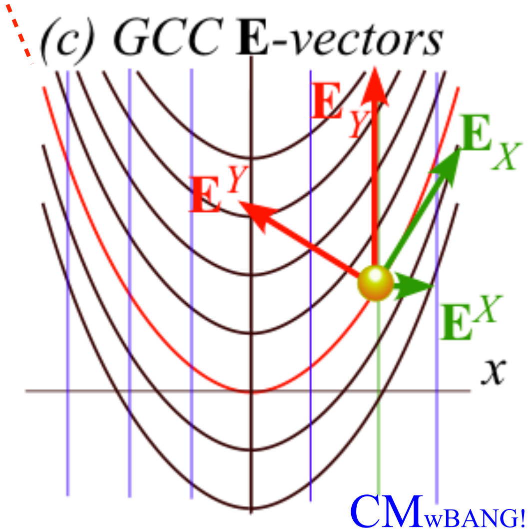
"Non-Holonomic" multipliers

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=\frac{1}{2}kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)



CMwBANG! Fig. 3.9.1

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=½kx^2$ is defined as follows.

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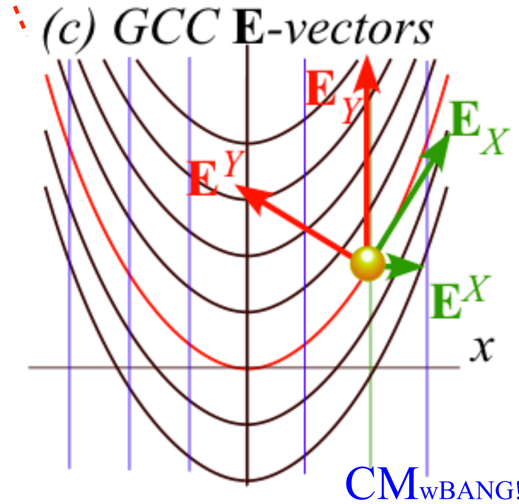
(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . ($\mathbf{F} \propto \nabla c^1$)



CMwBANG!
Fig. 3.9.1

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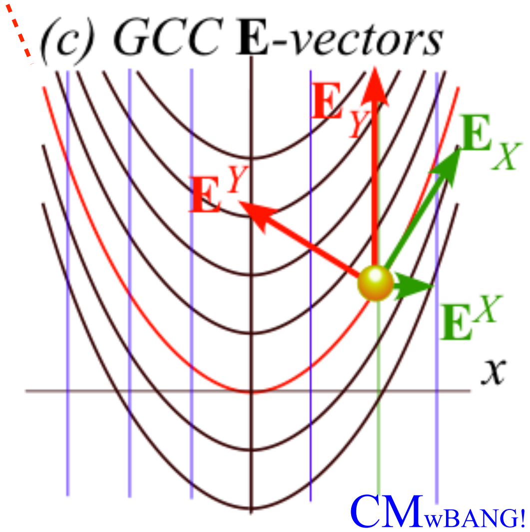

CMwBANG! Fig. 3.9.1

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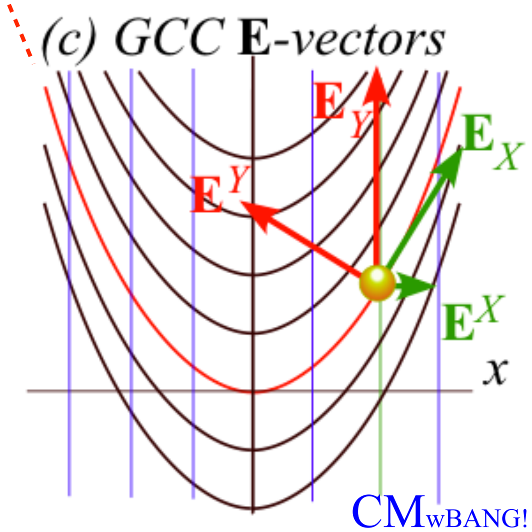
Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.



CMwBANG!
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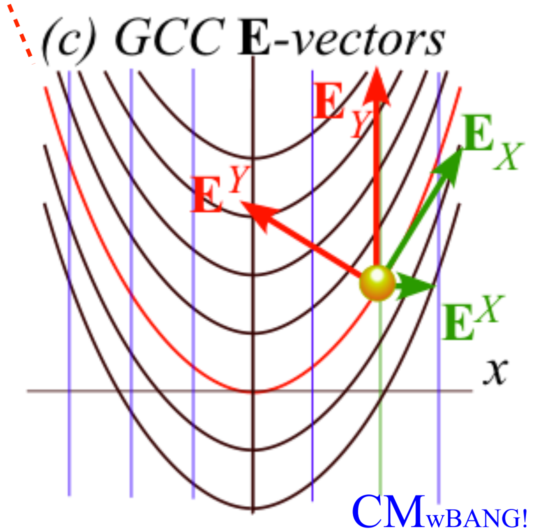

CMwBANG! Fig. 3.9.1

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It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

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Lagrange multiplier or λ-method. The constraining parabola $y=\frac{1}{2}kx^2$ is defined as follows.

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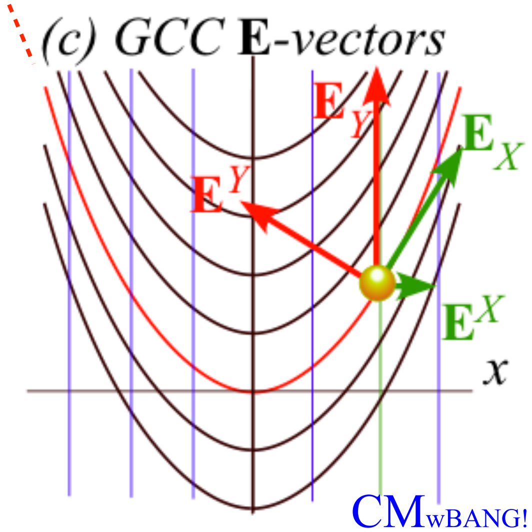

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(Back to "Stupid-Parabolic" GCC)

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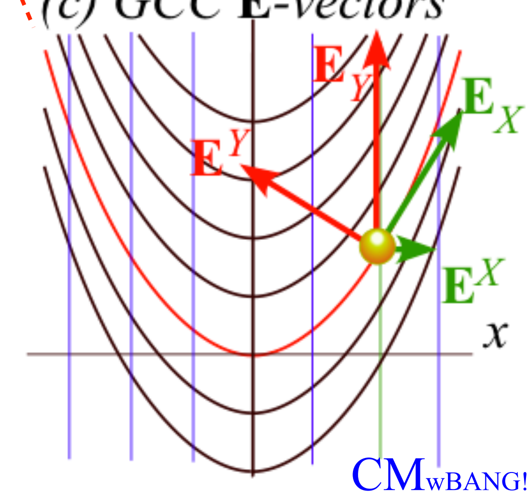

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$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ mg \end{pmatrix}$$

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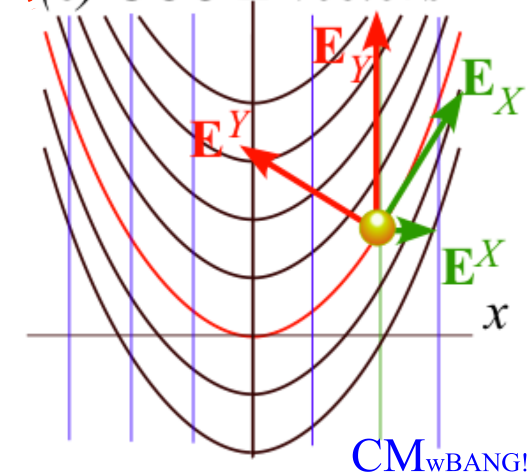
(c) GCC **E**-vectors



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Constraint function $y=\frac{1}{2}kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$

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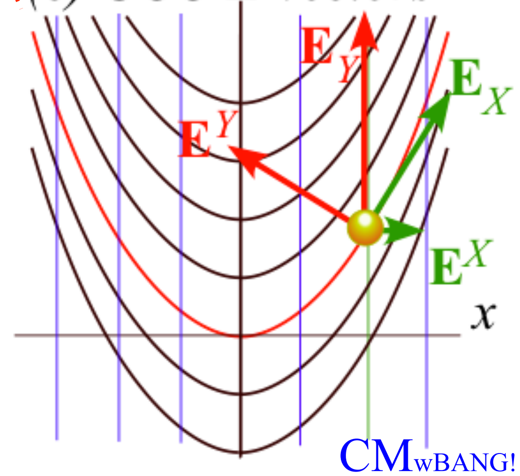

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Constraint function $y=½kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$ Now solve for multiplier λ.

Lagrange multiplier approaches

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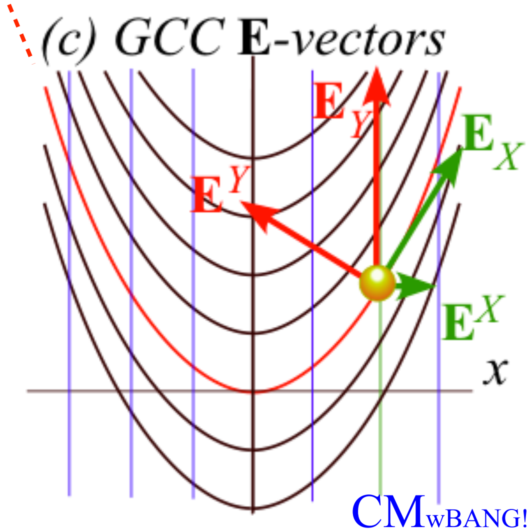

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$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

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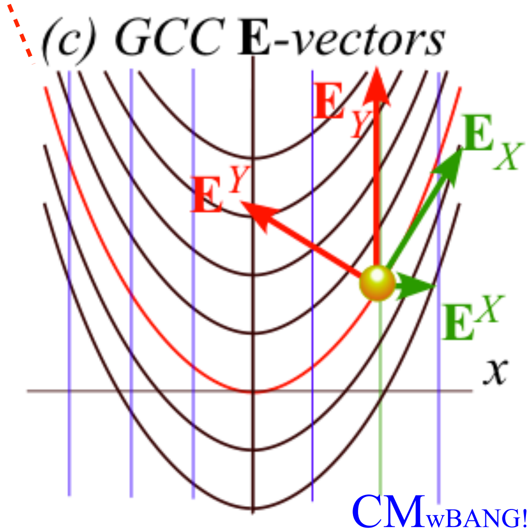

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$$\lambda = m(-k\dot{x}^2 - kx\ddot{x} - g)$$

Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=\frac{1}{2}kx^2$ is defined as follows.

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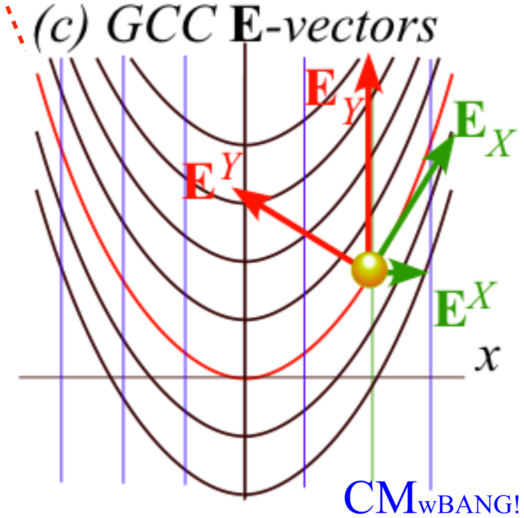

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$$mk(\dot{x}^2 + x\ddot{x}) = -\lambda - mg$$

Constraint function $y=\frac{1}{2}kx^2$ has derivatives $\dot{y} = kx\dot{x}$ and $\ddot{y} = k(\dot{x}^2 + x\ddot{x})$. Now solve for multiplier λ.

$$\lambda = -m(k\dot{x}^2 + kx\ddot{x} + g)$$

Then the λ function gives the new constrained *x*-equation of motion.

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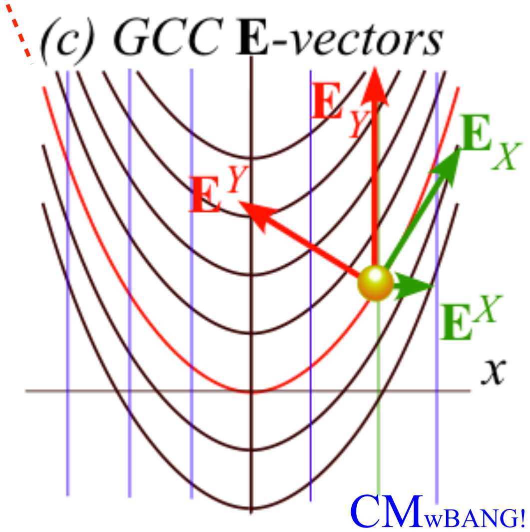
Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=\frac{1}{2}kx^2$ is defined as follows.

$$c^1 = \frac{1}{2}kx^2 - y = 0$$

(Back to "Stupid-Parabolic" GCC)

Imagine this is a coordinate line. Its normal constraining force **F** is along its c^1-gradient . ($\mathbf{F} \propto \nabla c^1$)



CMwBANG! Fig. 3.9.1

$$\mathbf{F} = \lambda \nabla c^1 = \lambda \nabla(\tfrac{1}{2}kx^2 - y) = \lambda \begin{pmatrix} \frac{\partial c^1}{\partial x} \\ \frac{\partial c^1}{\partial y} \end{pmatrix} = \lambda \begin{pmatrix} kx \\ -1 \end{pmatrix}$$

Proportionality factor $\lambda = F_1^c$ is a *Lagrange multiplier*.

It is like a covariant constraint component F_1^c of a contravariant vector $\mathbf{E}^1 = \nabla c^1$ that arises if $c^1(x,y) = const.$ was a coordinate line causing a constraint force $\mathbf{F} = F_1^c \nabla c^1$.

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Lagrange multiplier approaches

Lagrange multiplier or λ-method. The constraining parabola $y=½kx^2$ is defined as follows.

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(Back to "Stupid-Parabolic" GCC)

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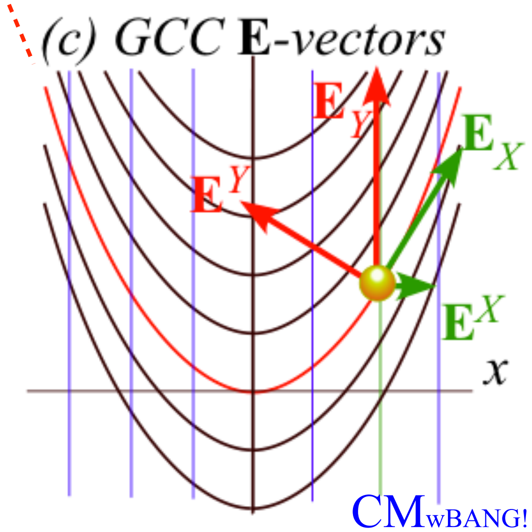

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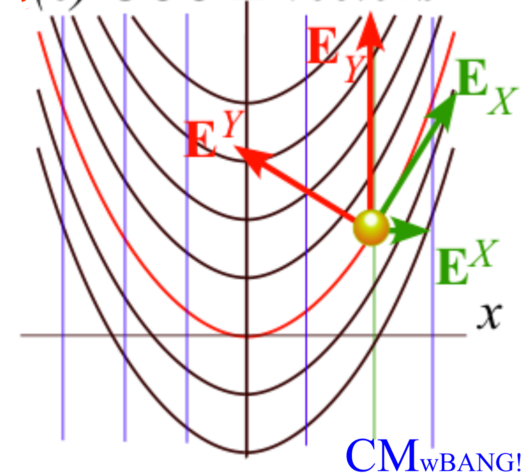
(c) GCC **E**-vectors



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$$(1 + k^2x^2)\ddot{x} = (-k\dot{x}^2 - g)kx$$

(Same equation as on p.12)

$$\ddot{x} = \frac{-k\dot{x}^2 - g}{1 + k^2x^2}kx$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Sketch of atomic-Stark orbit parabolic OCC analysis

Classical Hamiltonian separability

Way 4. Lagrange multipliers

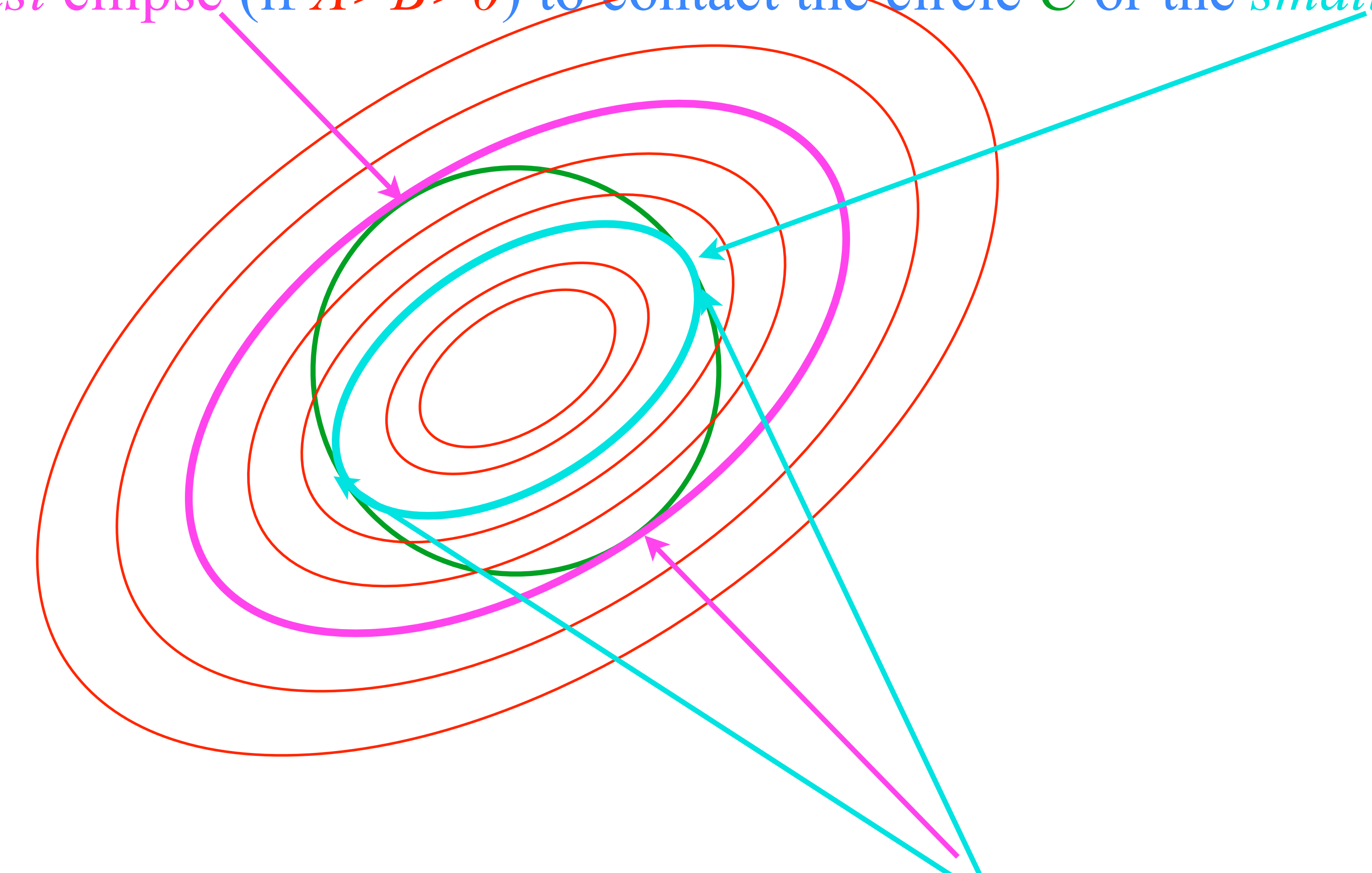
→ *Lagrange multiplier as eigenvalues*

Multiple multipliers

"Non-Holonomic" multipliers

Lagrange multiplier basics

Suppose you need to find maximum of $H=(Ax^2+Bxy+Ay^2)/2$ subject to constraint: $C=(x^2+y^2)/2=const.$
By geometry you are finding the *largest* ellipse (if $A>B>0$) to contact the circle C or the *smallest*.



The contact points satisfy gradient proportionality equations:

$$\nabla H = \lambda \cdot \nabla C$$

$$\begin{pmatrix} \partial_x H \\ \partial_y H \end{pmatrix} = \lambda \cdot \begin{pmatrix} \partial_x C \\ \partial_y C \end{pmatrix}$$

$$\begin{pmatrix} Ax + By \\ Bx + Dy \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Extreme cases occur only at *contact points*

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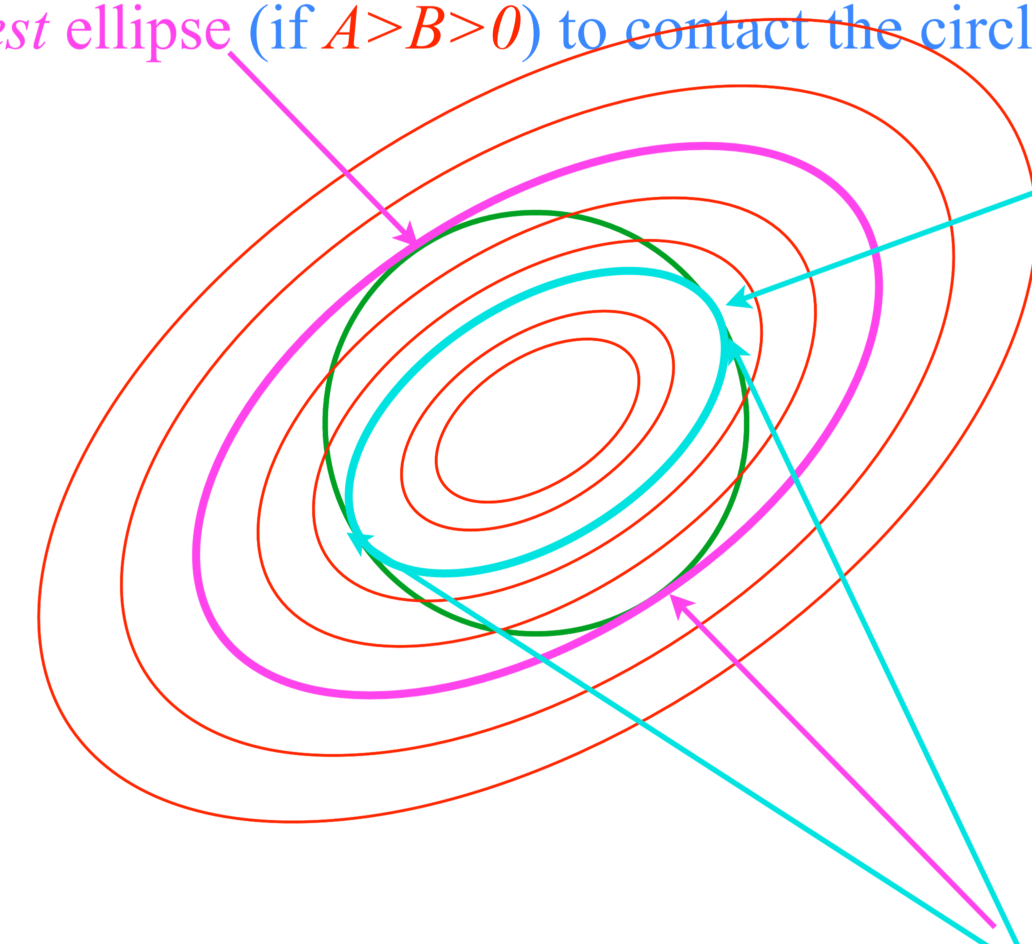

Extreme cases occur only at *contact points*

This amounts to a λ*-eigenvalue-eigenvector* equation

$$\begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(More about this in Units 4-6)

(Perhaps, this is why we often label eigenvalues λ with a Greek “L”)

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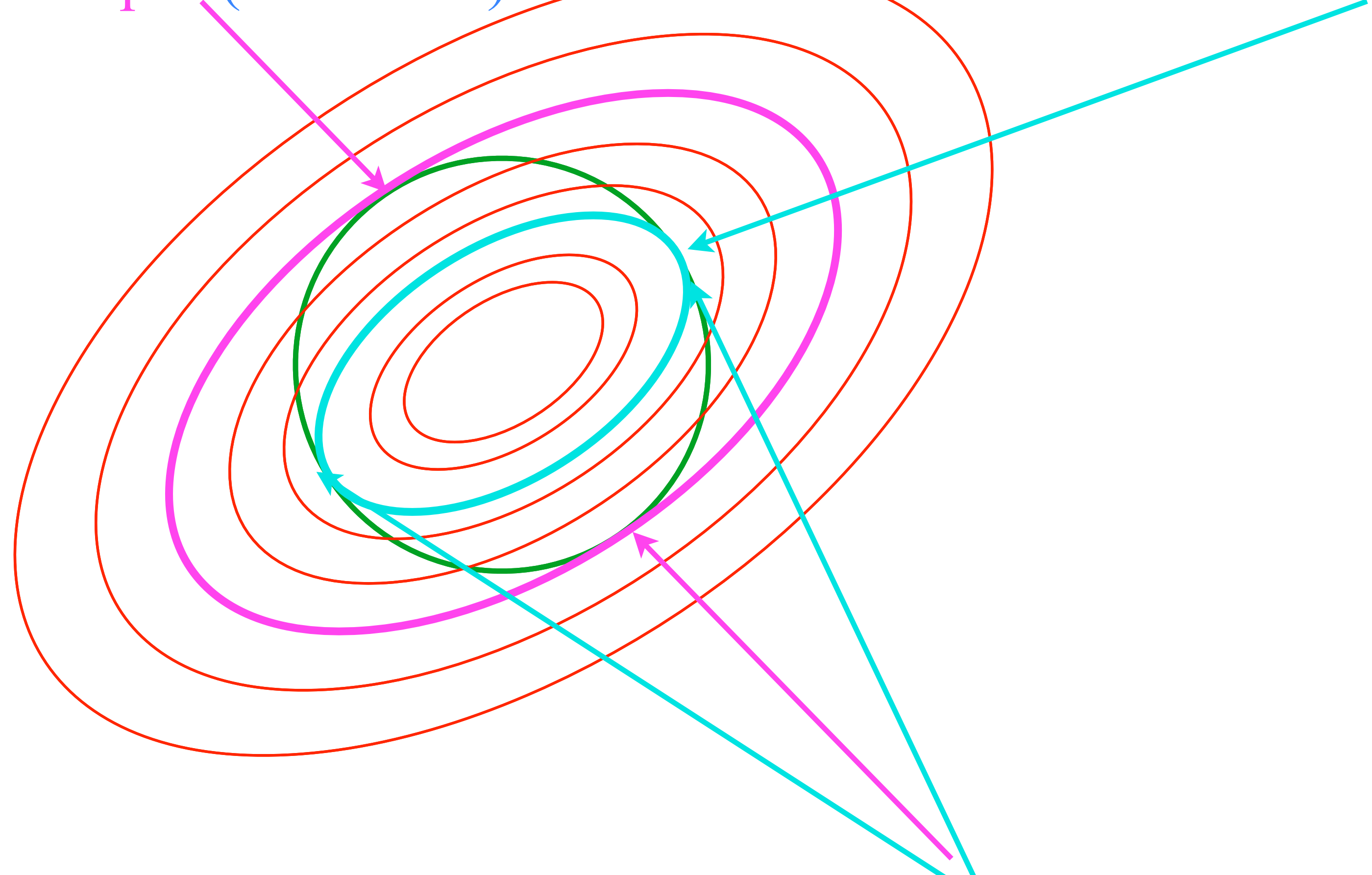

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Eigenvalues λ are *extreme* matrix "**own**"-values $\langle\psi|M|\psi\rangle$ subject *Norm-constraint* $\langle\psi|\psi\rangle=1$

***Eigen** - LEO Online German Dictionary*

Other Ways to do constraint analysis

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Sketch of atomic-Stark orbit parabolic OCC analysis

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Lagrange multiplier as eigenvalues

→ *Multiple multipliers*

"Non-Holonomic" multipliers

Lagrange multipliers also work for constraints $c(q^k) = const.$ that cut across GCC lines.
It is only necessary to express the gradient of $c(q^k)$ in terms of the GCC using chainsaw sum rule.

$$\nabla c = \frac{\partial c}{\partial x^j}\hat{\mathbf{e}}^j = \frac{\partial c}{\partial q^k}\mathbf{E}^k \qquad \frac{\partial c}{\partial q^k} = \frac{\quad}{\partial q^k}\frac{\partial c}{\quad} = \frac{\partial x^j}{\partial q^k}\frac{\partial c}{\partial x^j} = \frac{\partial \mathbf{r}}{\partial q^k}\bullet\frac{\partial c}{\partial \mathbf{r}} = \mathbf{E}_k\bullet\nabla c$$

Then the Lagrange equations for each GCC q^k will share a λ-multiplier on its c-gradient component.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda\frac{\partial c}{\partial q^1} \\ \lambda\frac{\partial c}{\partial q^2} \\ \vdots \end{pmatrix} \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\frac{\partial c}{\partial q^k}$$

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Two or more constraints $c^1(q^k) = const.,\ c^2(q^k) = const.,\cdots$ add two or more λ_γ terms to the equations.

$$\begin{pmatrix} \dot{p}_1 - \frac{\partial L}{\partial q^1} \\ \dot{p}_2 - \frac{\partial L}{\partial q^2} \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1\frac{\partial c^1}{\partial q^1} \\ \lambda_1\frac{\partial c^1}{\partial q^2} \\ \vdots \end{pmatrix} + \begin{pmatrix} \lambda_2\frac{\partial c^2}{\partial q^1} \\ \lambda_2\frac{\partial c^2}{\partial q^2} \\ \vdots \end{pmatrix} + \ldots \qquad \dot{p}_k - \frac{\partial L}{\partial q^k} = \lambda\gamma\frac{\partial c^\gamma}{\partial q^k}$$

Other Ways to do constraint analysis

Way 3. OCC constraint webs

Preview of atomic-Stark orbits

Classical Hamiltonian separability

Way 4. Lagrange multipliers

Lagrange multiplier as eigenvalues

Multiple multipliers

→ *"Non-Holonomic" multipliers*

Constraints may be determined by differential relations that are not integrable.
Lagrange methods use differentials and do not need integral c^γ surface functions.

Integral constraint differentials

$$0 = dc^1 = \frac{\partial c^1}{\partial q^1} dq^1 + \frac{\partial c^1}{\partial q^2} dq^2 + \ldots$$

$$0 = dc^2 = \frac{\partial c^2}{\partial q^1} dq^1 + \frac{\partial c^2}{\partial q^2} dq^2 + \ldots$$

$$\vdots \qquad \vdots$$

General differential constraint relations

$$0 = C^1_1 dq^1 + C^1_2 dq^2 + \ldots$$

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$$\vdots \qquad \vdots$$

Constrained equations of motion

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If a differential can't be integrated to give a constraint function it's called a *non-holonomic constraint*.
I guess that means that integrable ones are *holonomic*. (But why do we need the bigger words?)
A requirement for integrability (or "holonomicty") is that double differentials are symmetric.

$$\frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial^2 c^\gamma}{\partial q^k \partial q^j}$$

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Force components $F^\gamma_k = \frac{\partial c^\gamma}{\partial q^k} = C^\gamma_k$ must satisfy *reciprocity relations* to be gradients of a c^γ function.

Integral constraint differentials

$$\frac{\partial F^\gamma_k}{\partial q^j} = \frac{\partial^2 c^\gamma}{\partial q^j \partial q^k} = \frac{\partial F^\gamma_j}{\partial q^k}$$

General differential constraint relations

$$\frac{\partial C^\gamma_k}{\partial q^j} \quad \textit{may or may not be} \quad \frac{\partial C^\gamma_j}{\partial q^k}$$

Cycloid-like curves for rolling constraints

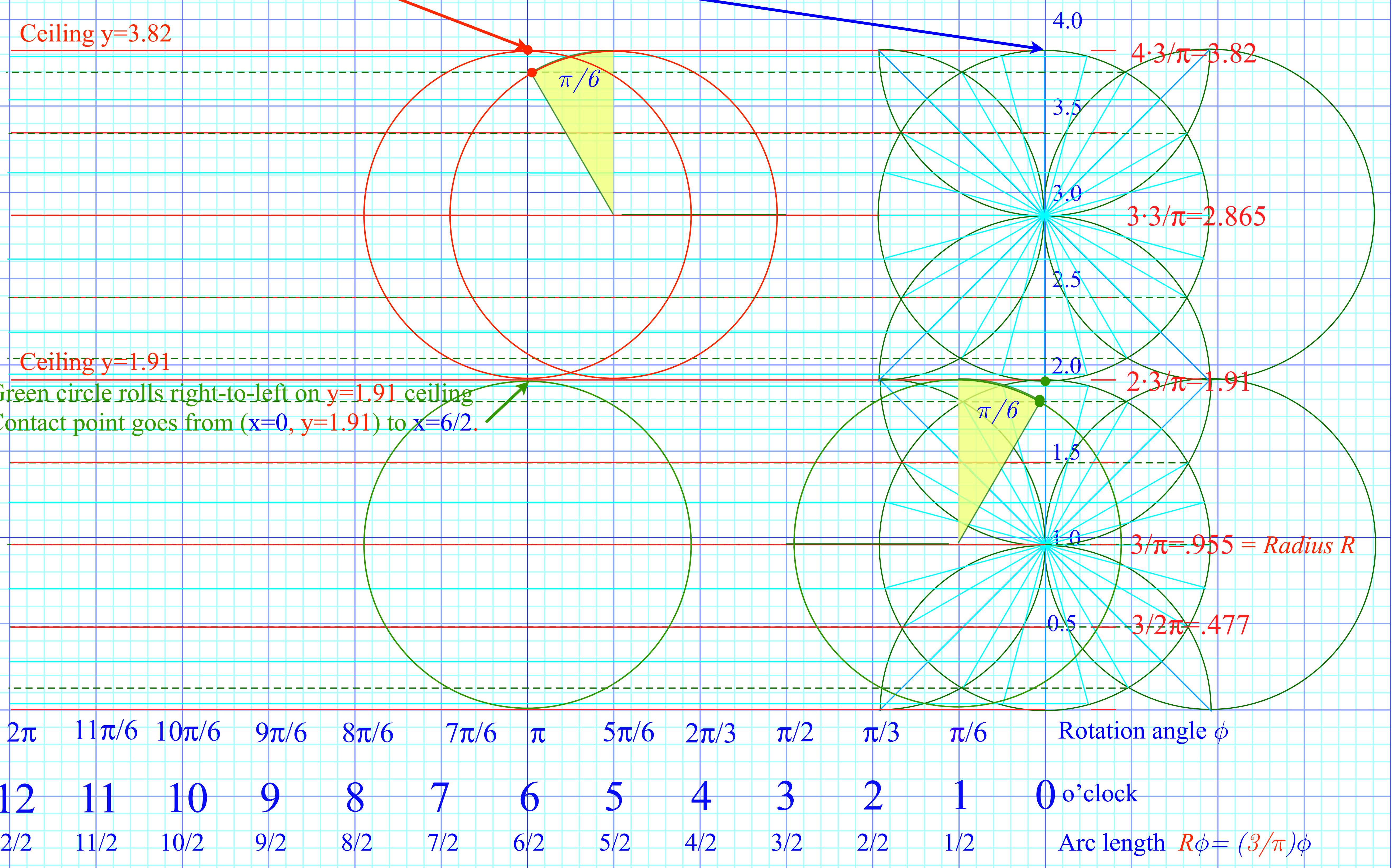
Cycloid-like curves for rolling constraints

First: A regular cycloid construction

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled -out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. Diameter is $2R=6/\pi=1.91$

Red circle rolls left-to-right on y=3.82 ceiling
Contact point goes from (x=6/2, y=3.82) to x=0.

Green circle rolls right-to-left on y=1.91 ceiling
Contact point goes from (x=0, y=1.91) to x=6/2.



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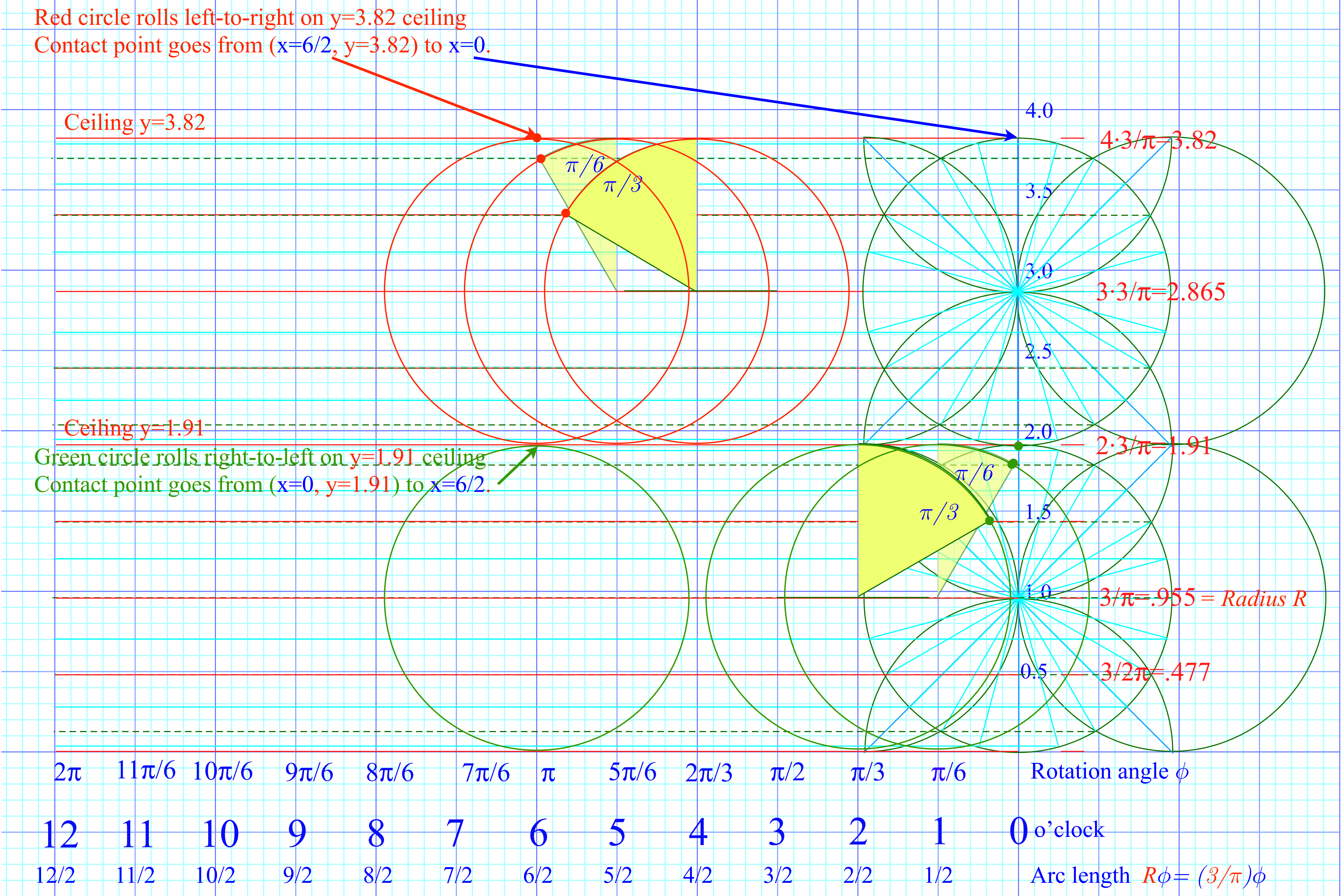
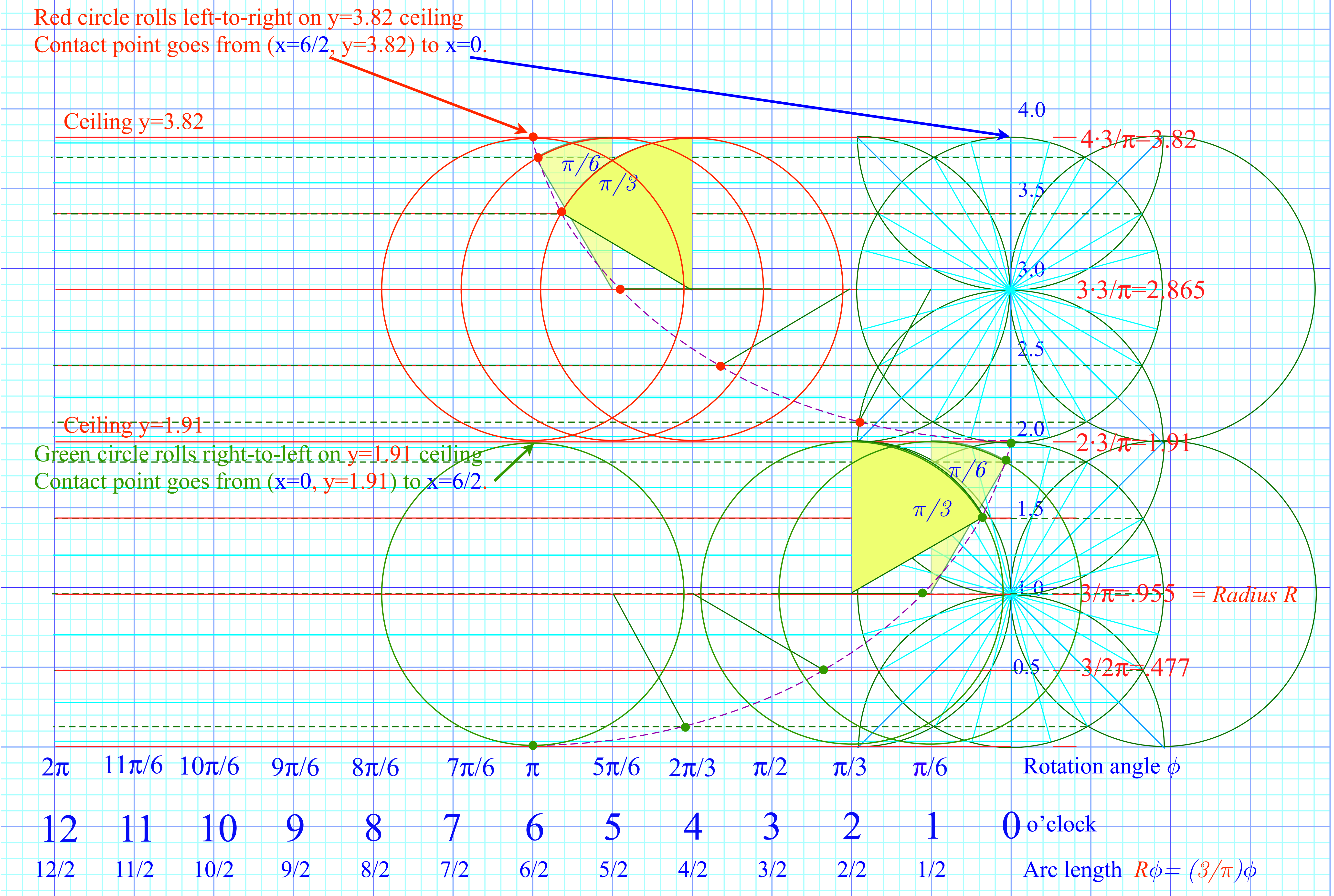
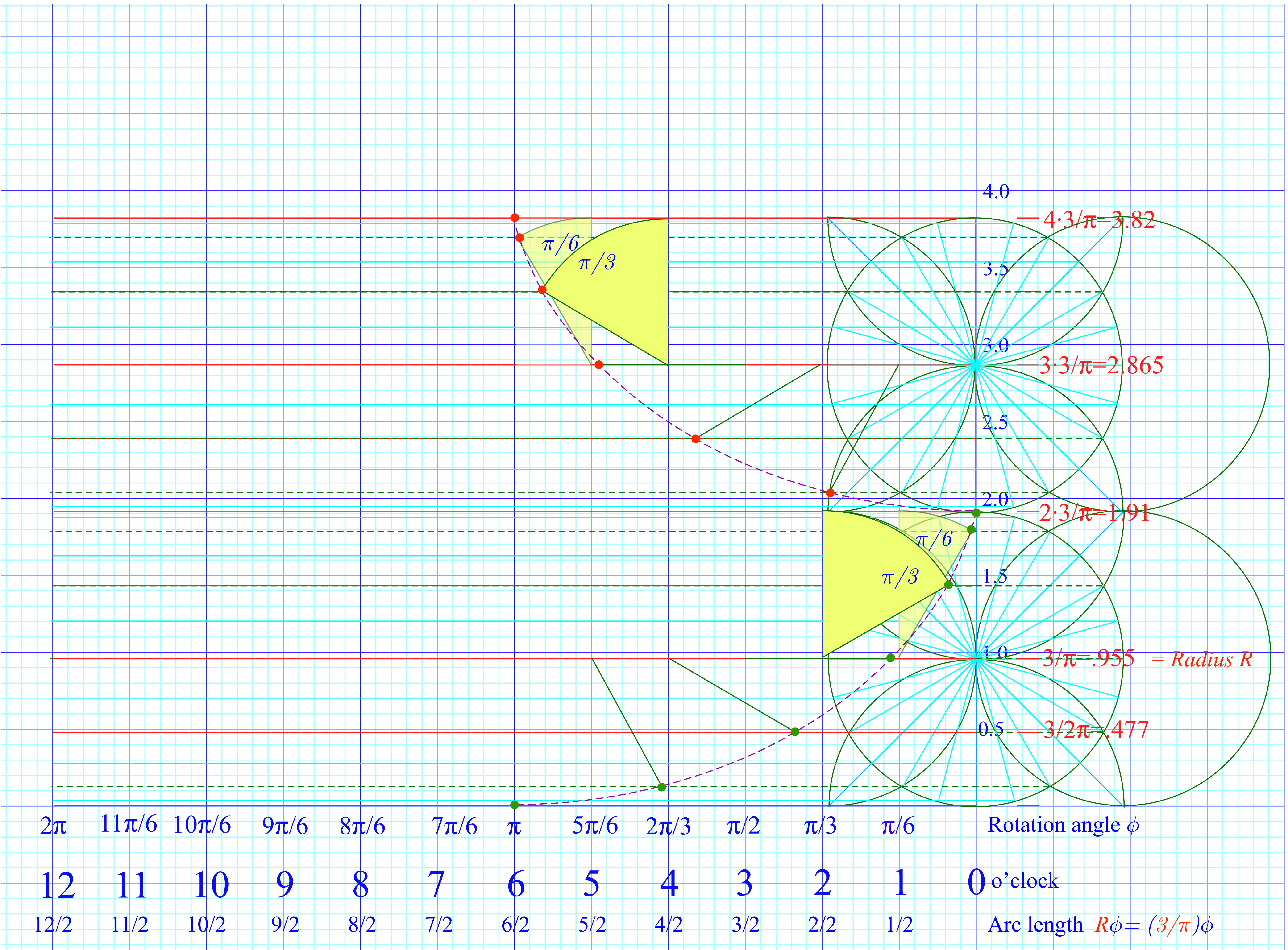
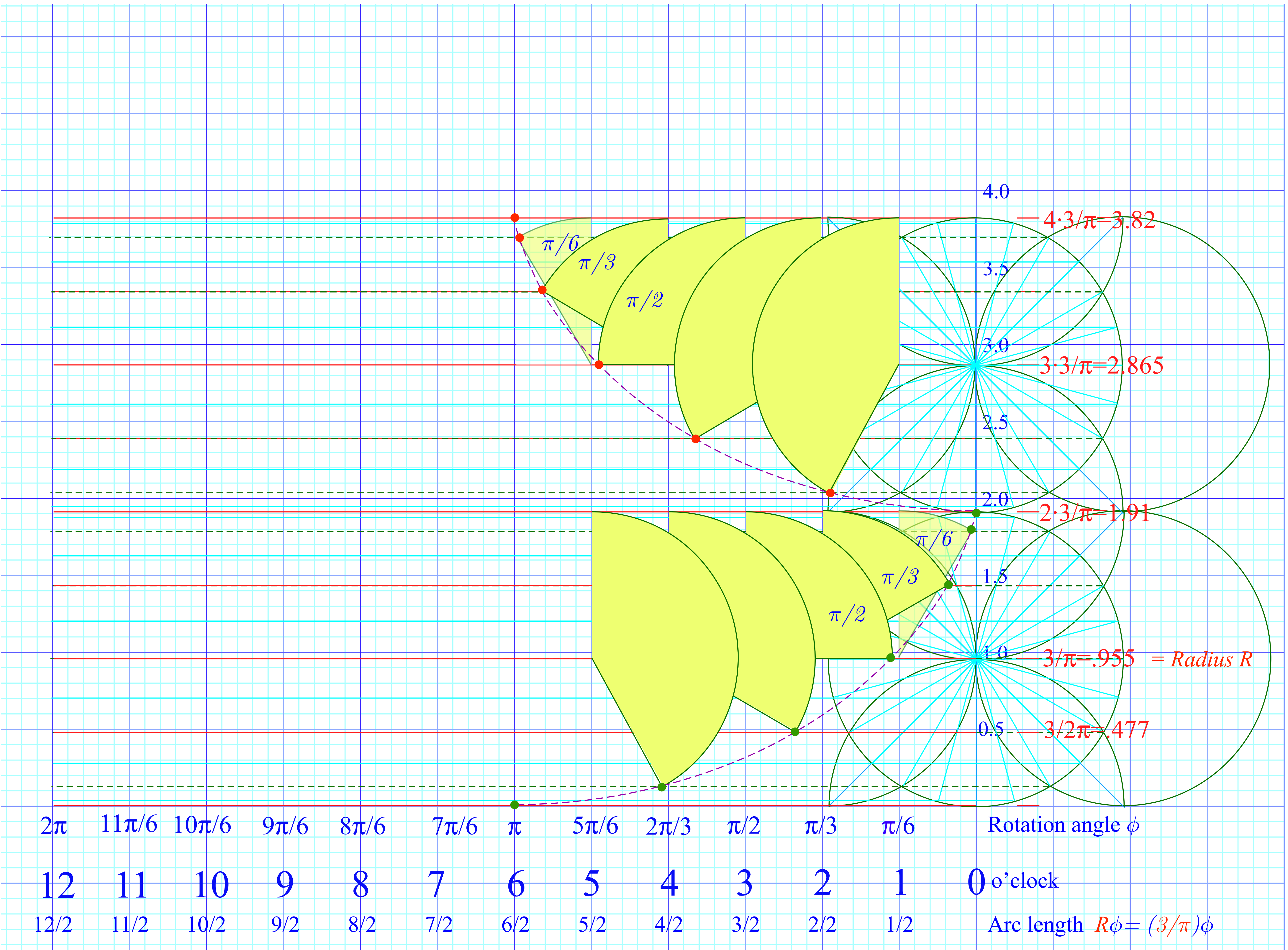


Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi=m\pi/n$ is a rational length of rolled -out circumference $R\phi=(3/\pi)m\pi/n=3m/n$. Diameter is $2R=6/\pi=1.91$




4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
4·3/π=3.82
3·3/π=2.865
2·3/π=1.91
3/π=.955 = Radius R
3/2π=.477
π/6
π/3
π/6
π/3
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6 Rotation angle φ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2 Arc length Rφ= (3/π)φ


4.0
3.5
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π/6
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2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6 Rotation angle ϕ
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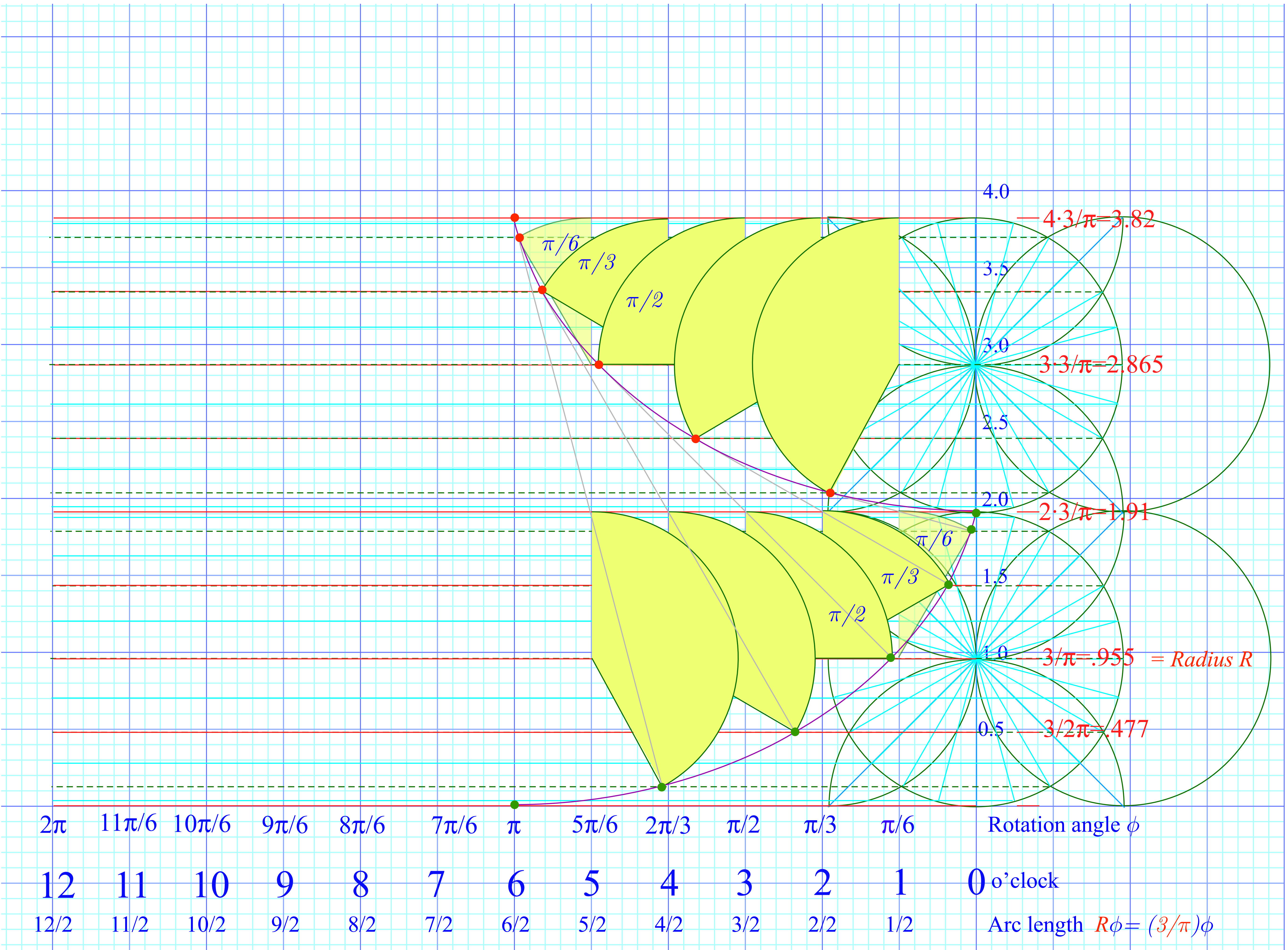

4.0
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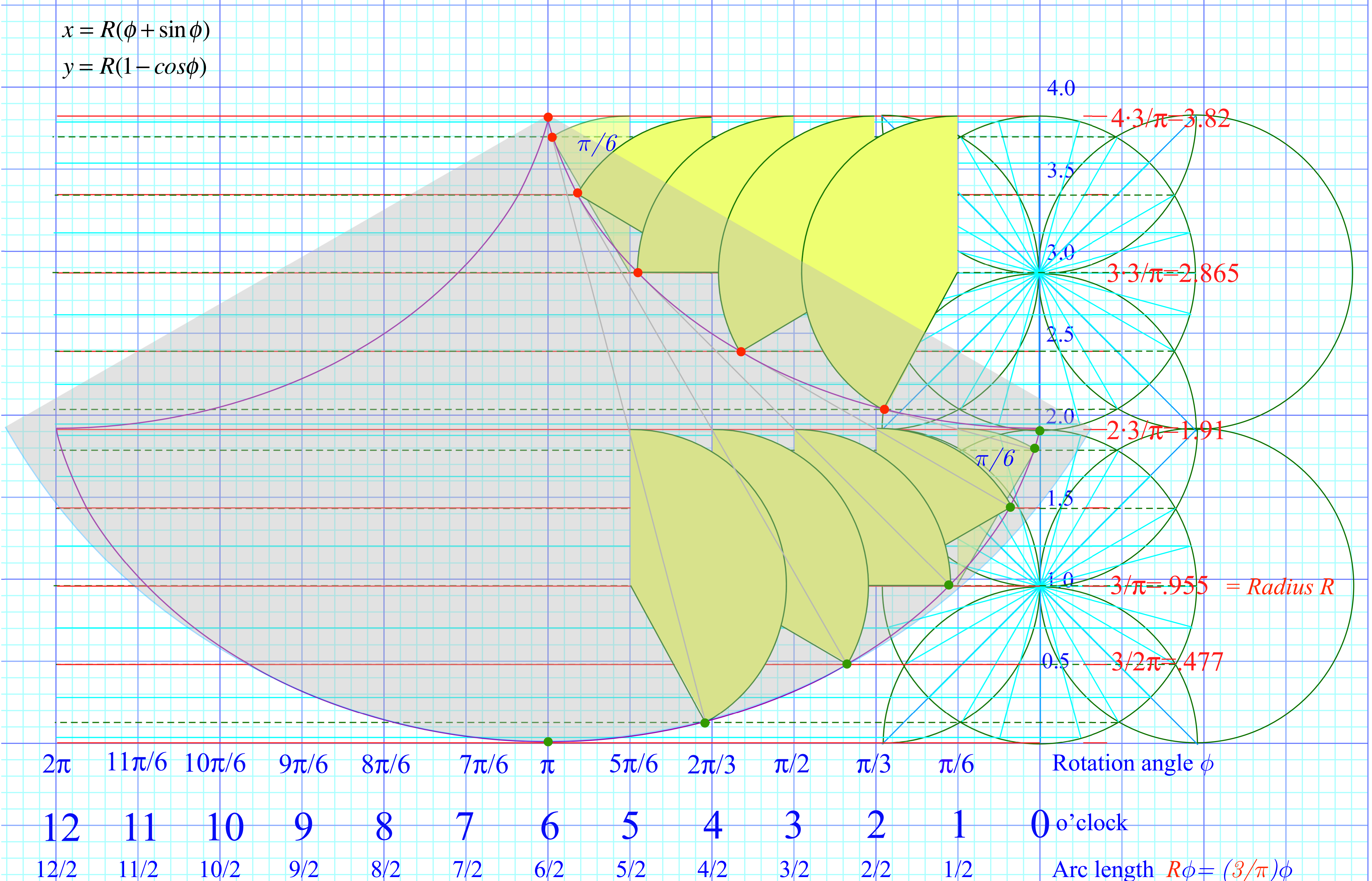
$x = R(\phi + \sin\phi)$
$y = R(1 - cos\phi)$
4.0
3.5
3.0
2.5
2.0
1.5
1.0
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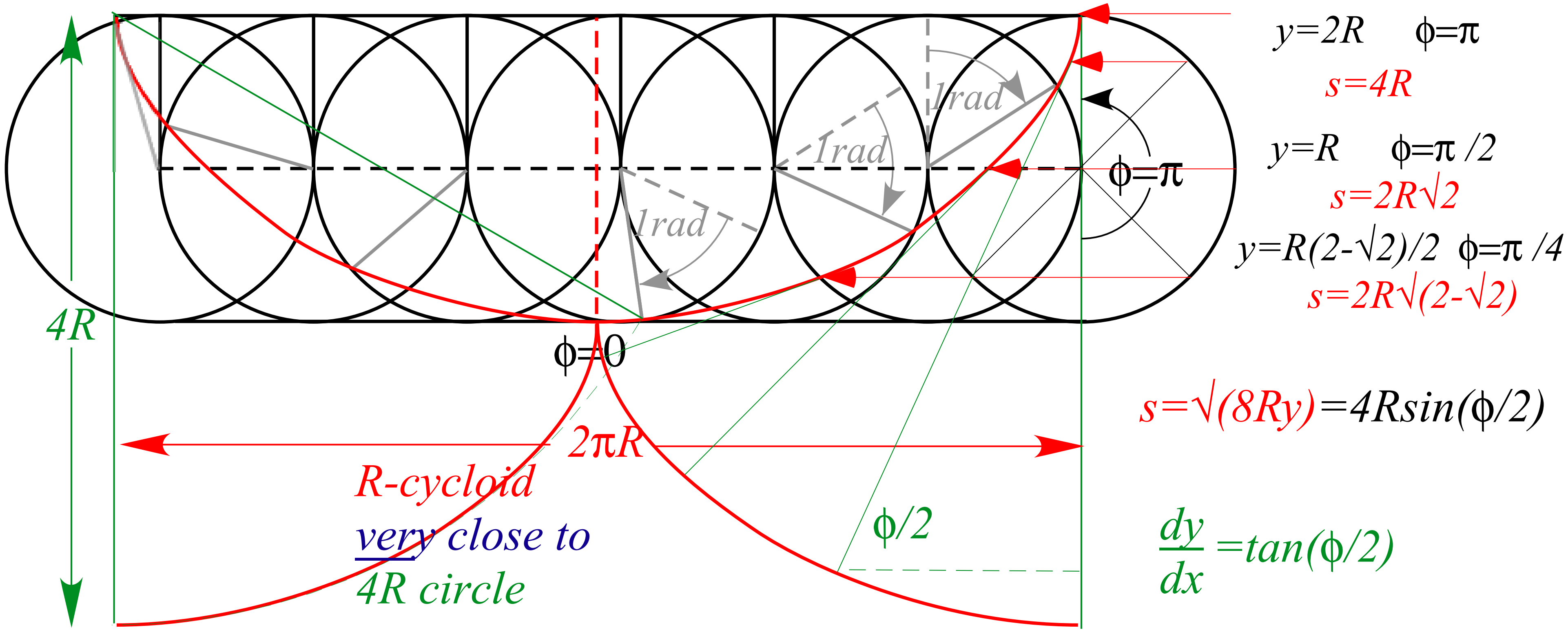
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2 Arc length $R\phi = (3/\pi)\phi$

$$x = R(\phi + \sin\phi) \qquad dx = R(1+\cos\phi)d\phi$$

$$y = R(1 - \cos\phi) \qquad dy = R\sin\phi \, d\phi$$

$$ds^2 = dx^2 + dy^2 = 2R^2(1+\cos\phi)d\phi^2 = 4R^2\cos^2\frac{\phi}{2}d\phi^2$$

$$ds = 2R\cos\frac{\phi}{2}d\phi \quad \text{or: } s = \int ds = 4R\sin\frac{\phi}{2} = 4R\sqrt{\frac{1-\cos\phi}{2}} = \sqrt{8Ry} = 4R(if: y=2R)$$



Cycloid Lagrangian $L = mR^2(1+\cos\phi)\dot{\phi}^2 - mgR(1-\cos\phi)$ gives: $p_\phi = 2mR^2(1+\cos\phi)\dot{\phi}$

and equation of motion

$$\ddot{\phi} = \frac{(R\dot{\phi}^2 - g)\sin\phi}{2R(1+\cos\phi)} = (R\dot{\phi}^2 - g)\frac{2\sin\phi/2\cos\phi/2}{4R\cos^2\phi/2} = \frac{(R\dot{\phi}^2 - g)}{2R}\tan\frac{\phi}{2} \qquad \text{Note: } \tan\frac{\phi}{2}\xrightarrow[\phi\to\pm\pi]{}\pm\infty$$

Time diff.eq.: $\dot{s}^2 = 2gy_0 - 2gy = 2g\frac{s_0^2 - s^2}{8R}$ integrates to: $t = \int dt = \sqrt{\frac{4R}{g}}\int\frac{ds}{\sqrt{s_0^2 - s^2}} = \sqrt{\frac{4R}{g}}\sin^{-1}\frac{s}{s_0}$ $+const.$

Arc length oscillates: $s = s_0\sin(\omega t - const.)$ at frequency $\omega = \sqrt{\frac{g}{4R}}$ of an $\ell=4R$ pendulum.

The rolling ϕ-angle time behavior $s = 4R\sin\frac{\phi}{2} = s_0\sin(\omega t - const.)$ is: $\frac{\phi}{2} = \sin^{-1}\left[\frac{s_0}{4R}\sin(\omega t - const.)\right]$

If initial value s_0 is maximum $s_0=4R$ then $\phi(t) = 2\omega t - const.$ has constant angular velocity $\dot{\phi} = 2\omega$ for $-\pi/2<\phi<\pi/2$.

Cycloid-like curves for rolling constraints

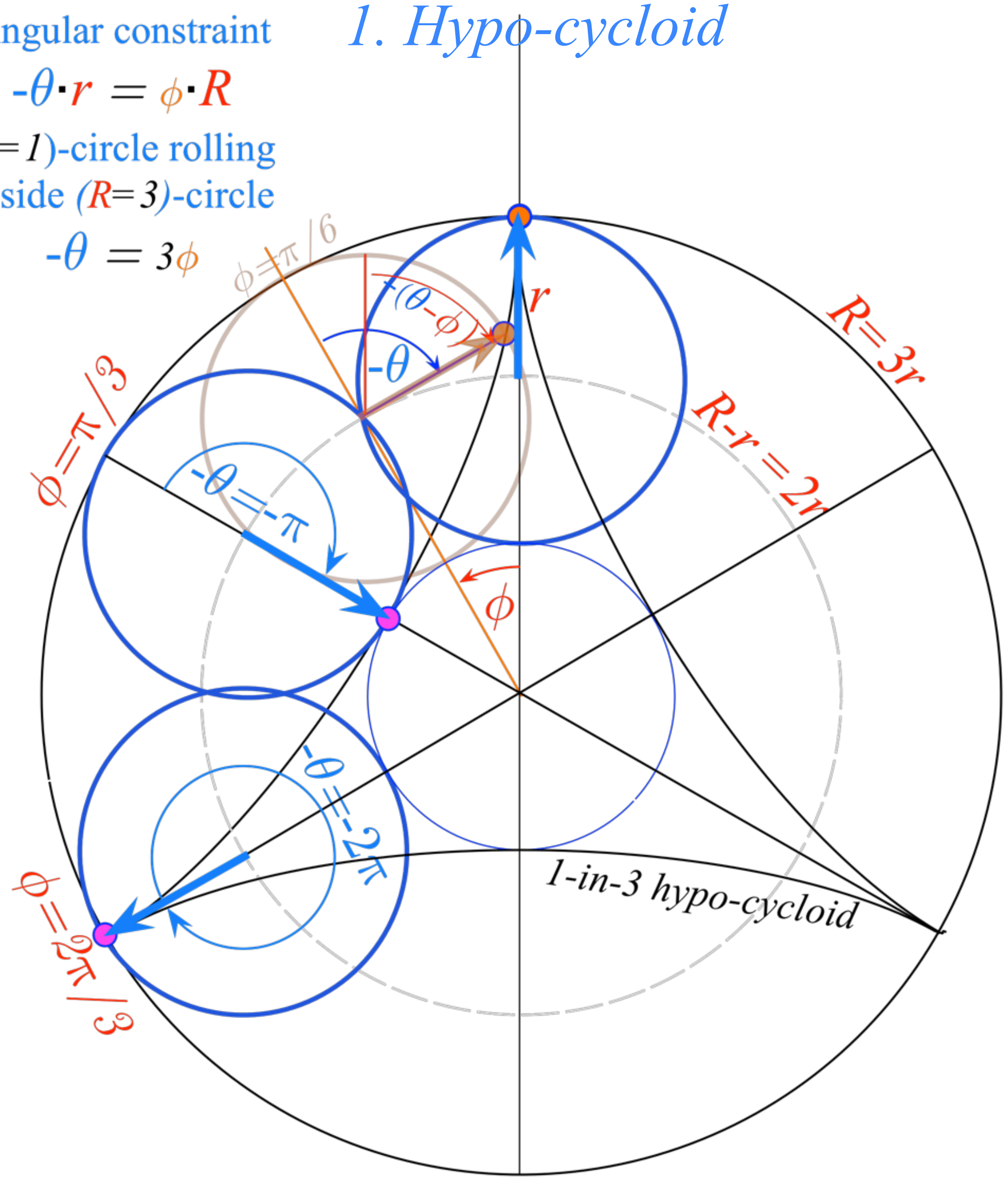
Angular constraint

$-\theta \cdot r = \phi \cdot R$

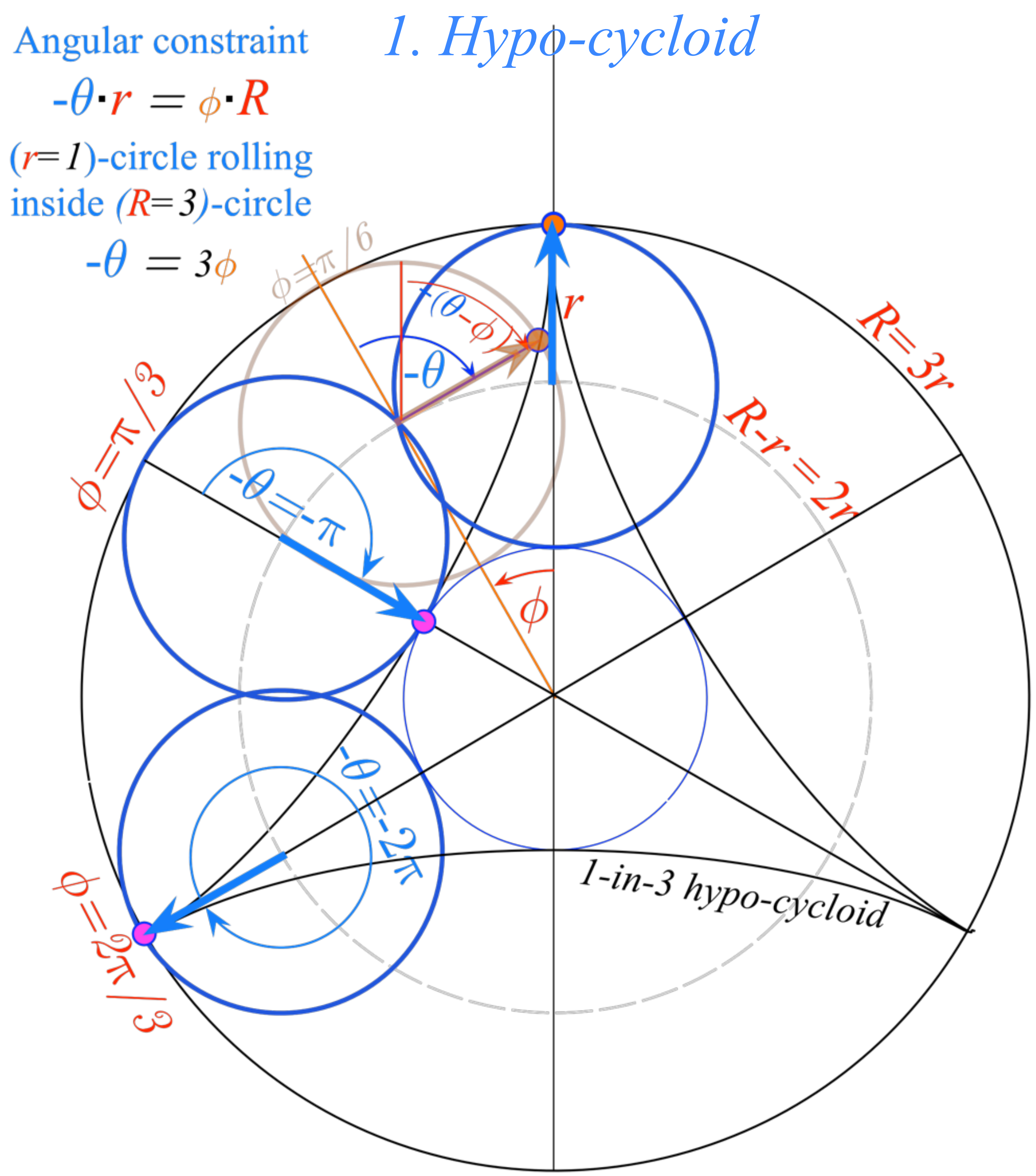
(r=1)-circle rolling inside (R=3)-circle

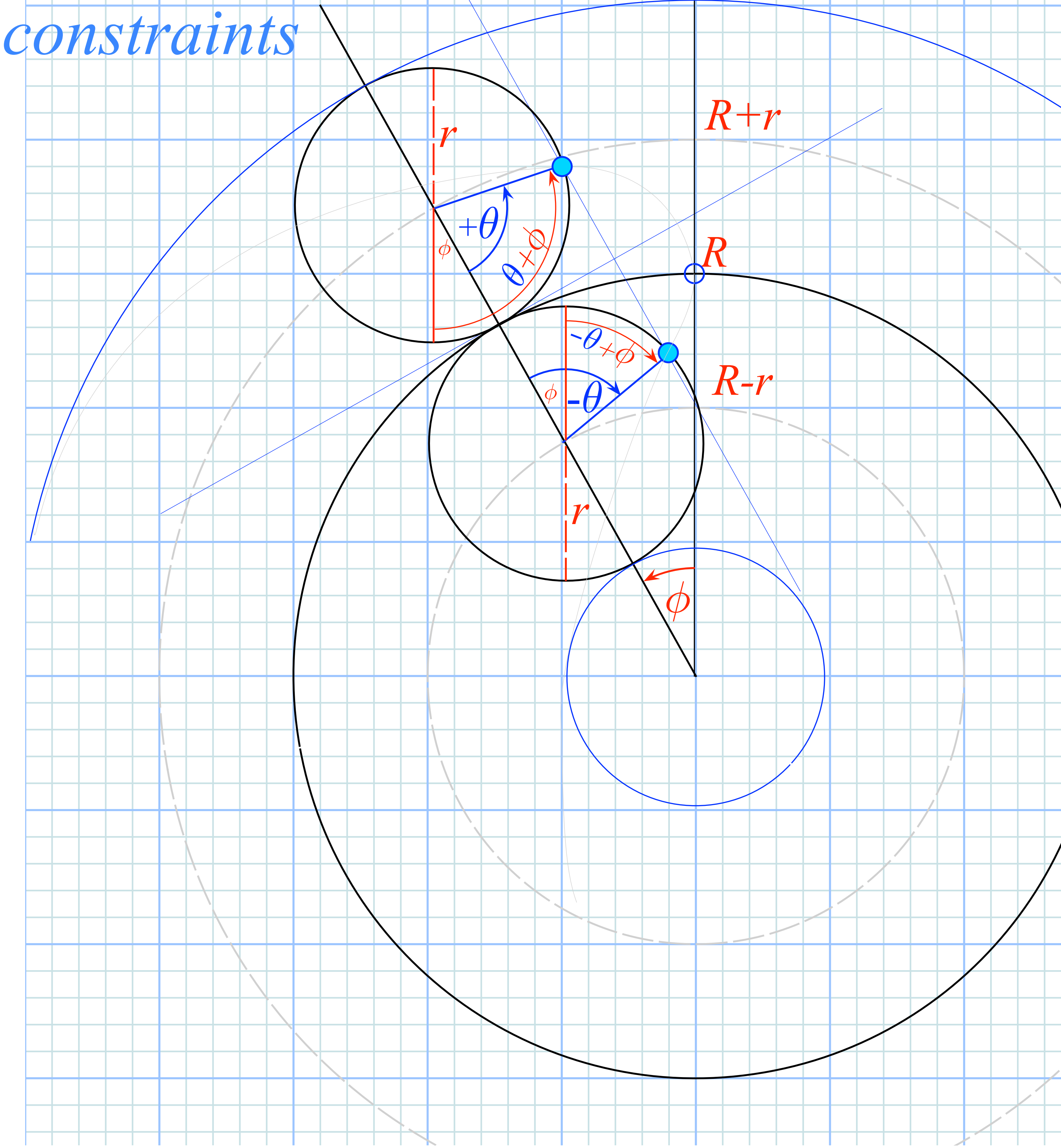
$-\theta = 3\phi$

1. Hypo-cycloid



Cycloid-like curves for rolling constraints





Cycloid-like curves for rolling constraints

Angular constraint
$-\theta \cdot r = \phi \cdot R$
($r=1$)-circle rolling inside ($R=3$)-circle
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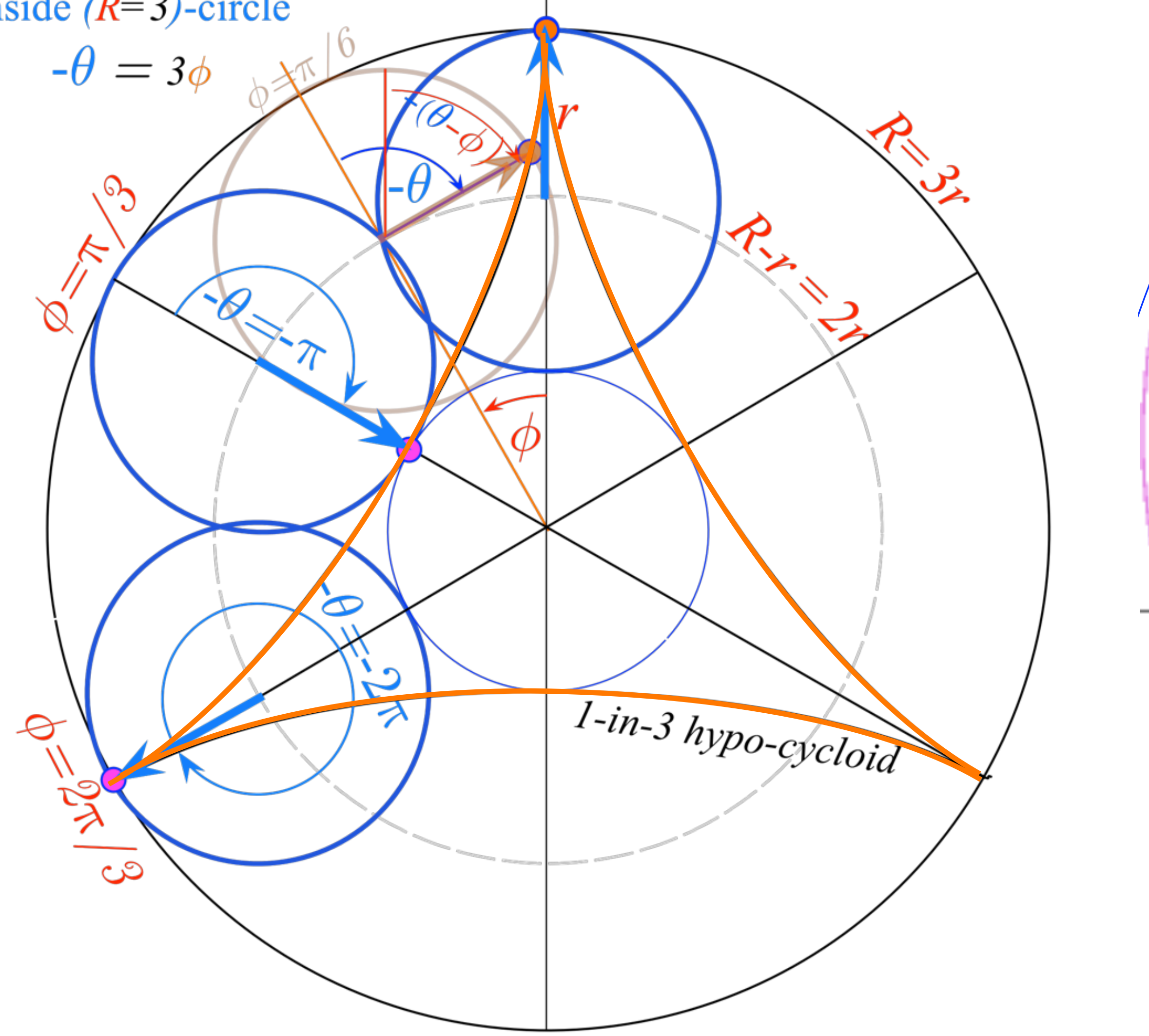
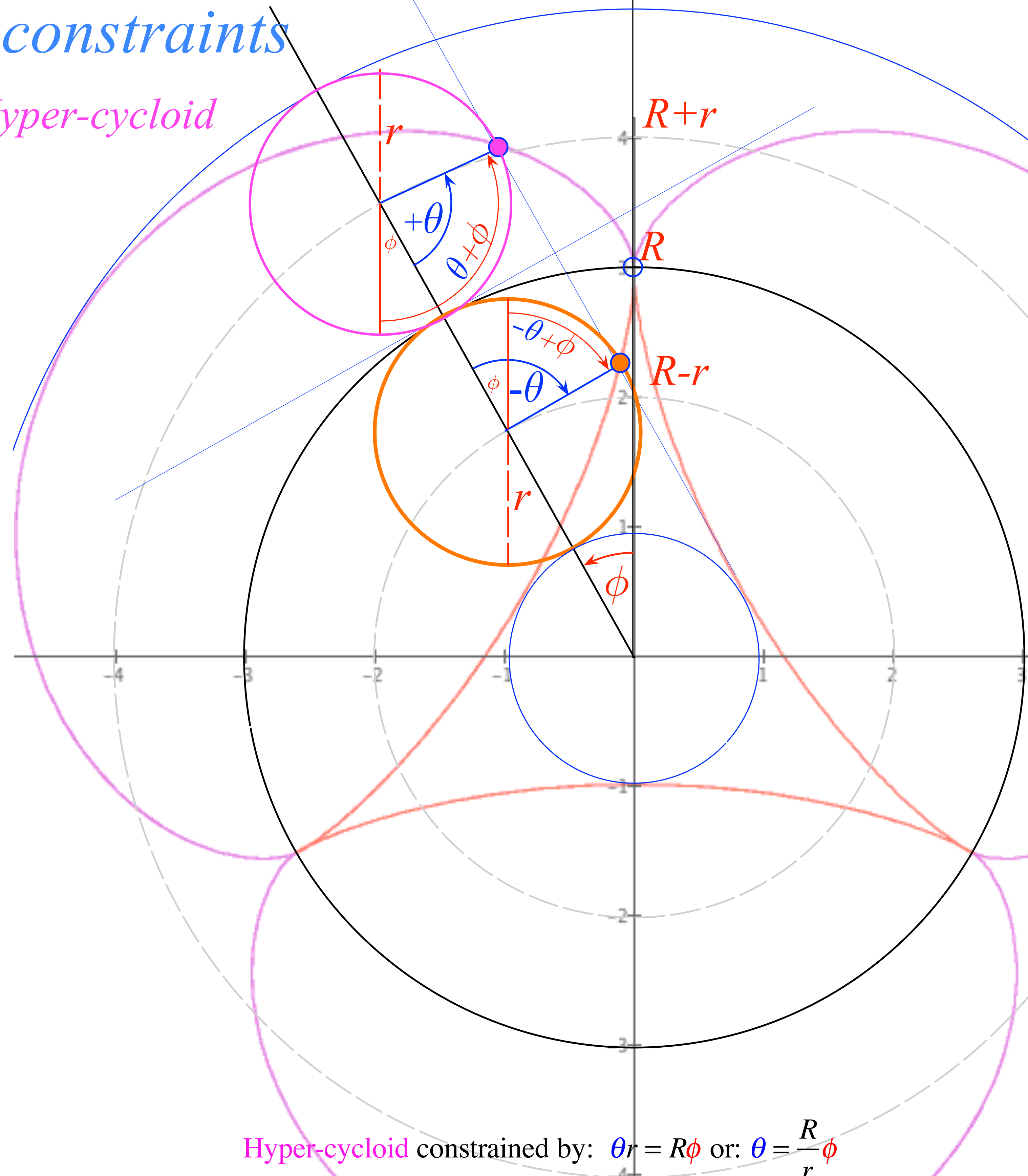


2. Hyper-cycloid



Hypo-cycloid constrained by: $-\theta r = -R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R-r)\sin\phi + r\sin(\theta\text{-}\phi) = r\left[-\left(\frac{R}{r}-1\right)\sin\phi + \sin\left(\frac{R}{r}-1\right)\phi\right]$$

$$y = (R-r)\cos\phi + r\cos(\theta\text{-}\phi) = r\left[\left(\frac{R}{r}-1\right)\cos\phi + \cos\left(\frac{R}{r}-1\right)\phi\right]$$

Hyper-cycloid constrained by: $\theta r = R\phi$ or: $\theta = \frac{R}{r}\phi$

$$x = -(R+r)\sin\phi + r\sin(\theta+\phi) = r\left[-\left(\frac{R}{r}+1\right)\sin\phi + \sin\left(\frac{R}{r}+1\right)\phi\right]$$

$$y = (R+r)\cos\phi - r\cos(\theta+\phi) = r\left[\left(\frac{R}{r}+1\right)\cos\phi - \cos\left(\frac{R}{r}+1\right)\phi\right]$$

Cycloid-like curves for rolling constraints

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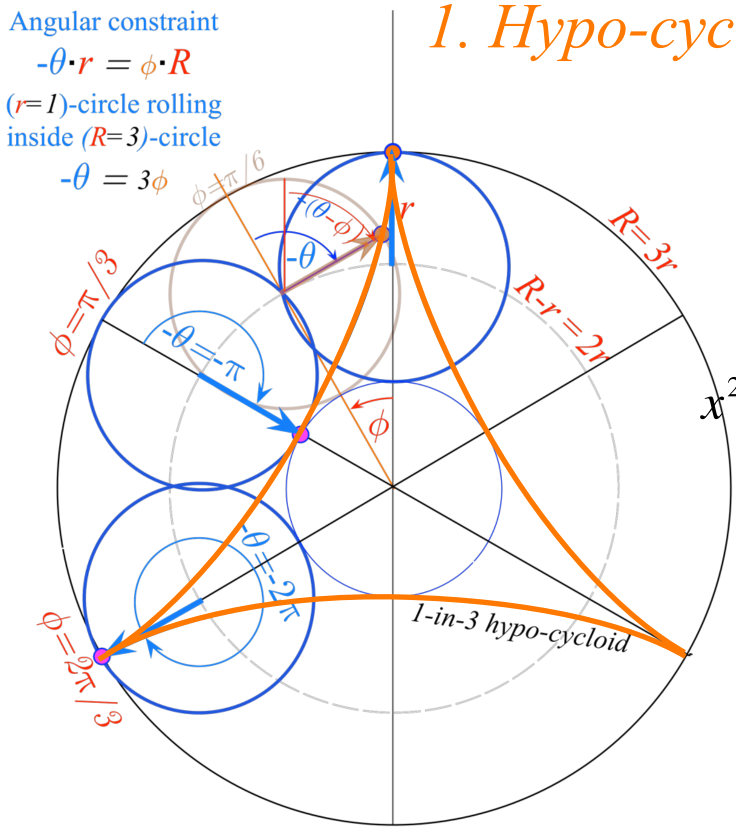

2. Hyper-cycloid

Hyper-cycloid constrained by $A=\dfrac{R}{r}+1,\ \theta=(A-1)\phi=\dfrac{R}{r}\phi$

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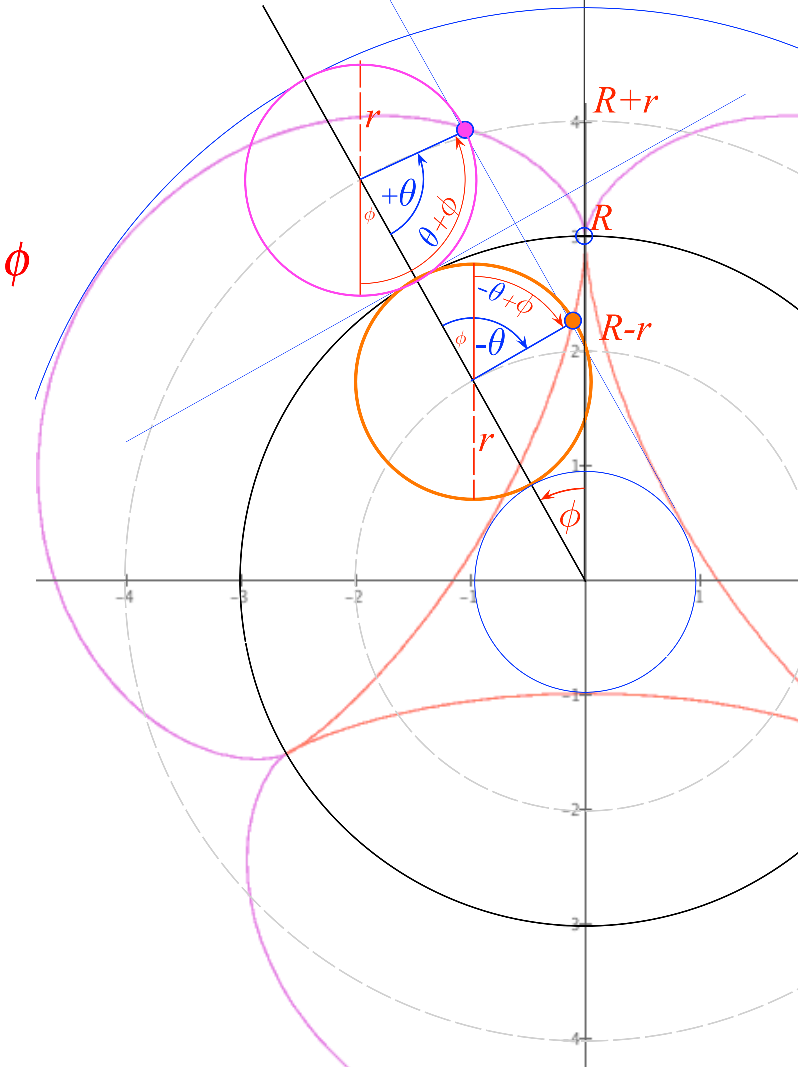
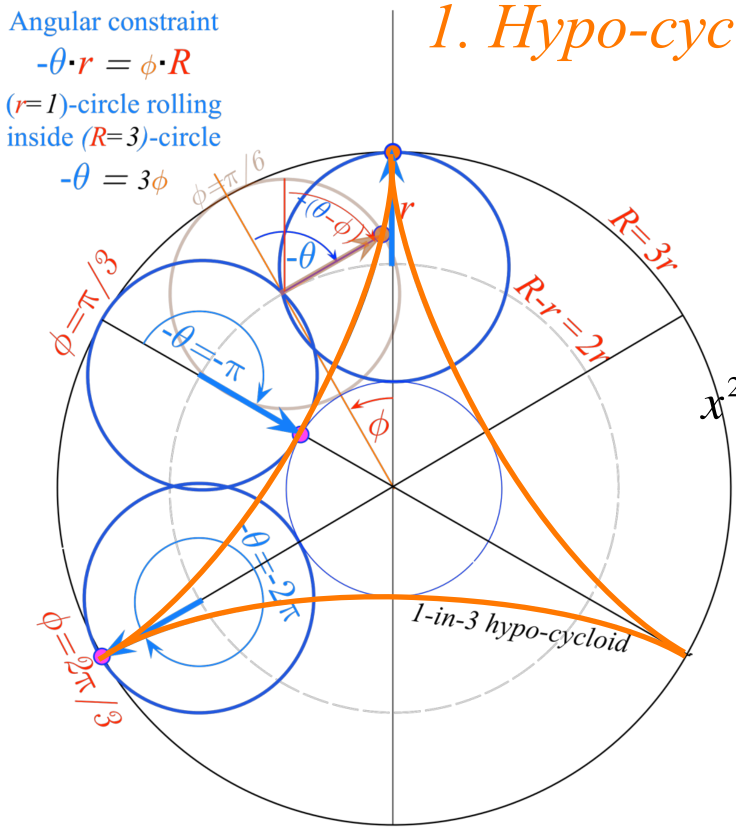


Cycloid-like curves for rolling constraints



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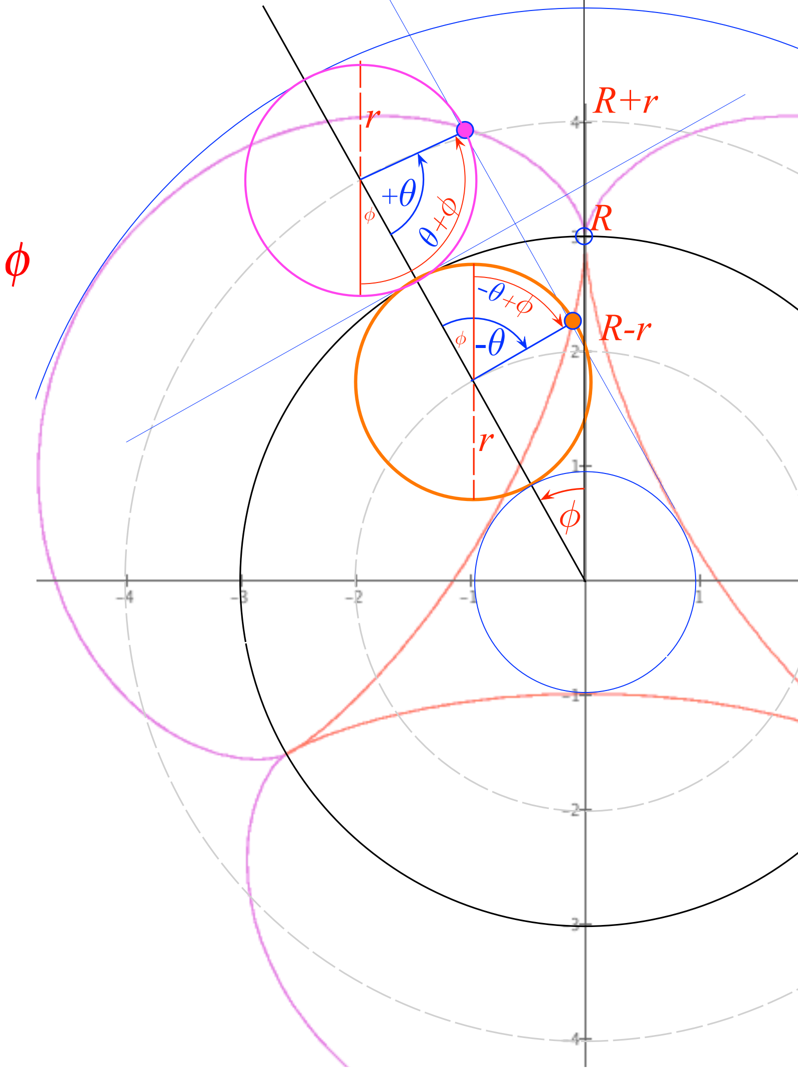
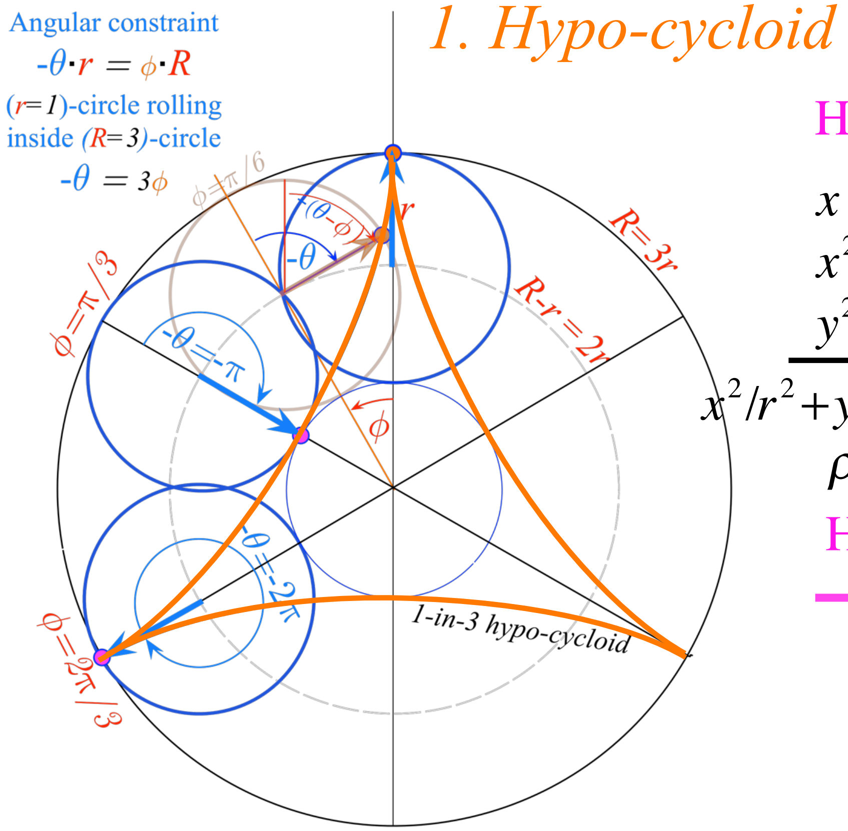
$$\rho^2/r^2=A^2-2A\cos(A-1)\phi+1$$

Hyper-cycloid radius ρ:



Cycloid-like curves for rolling constraints



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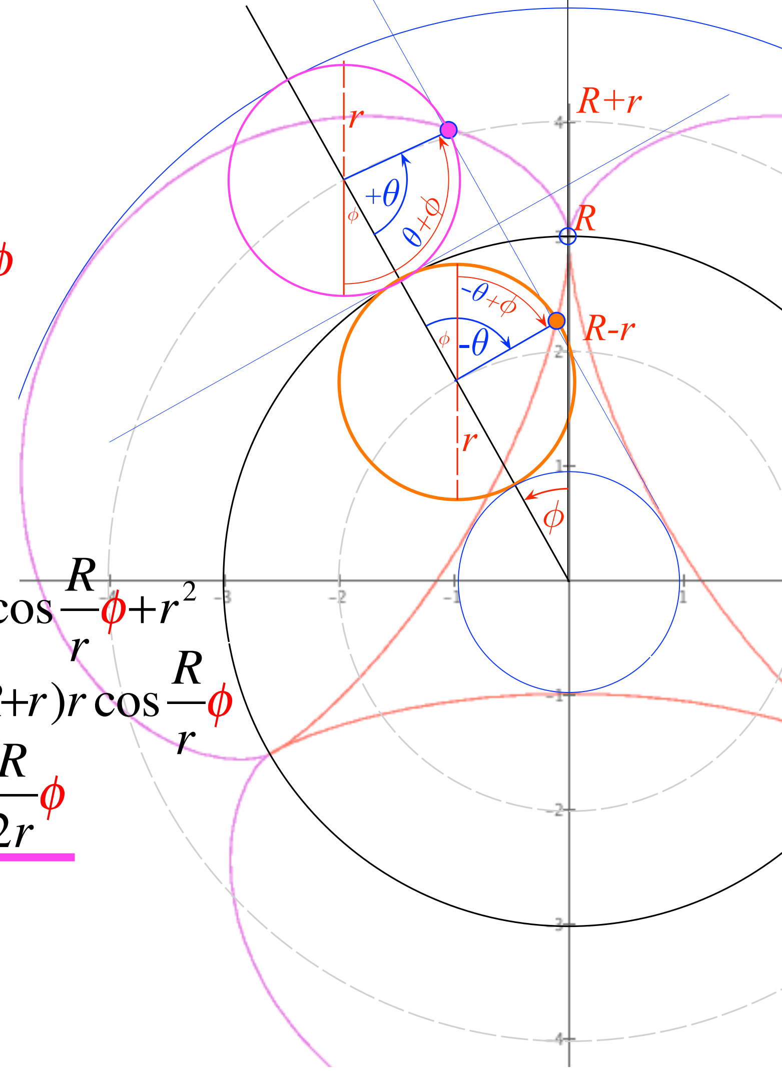
Hyper-cycloid radius ρ:

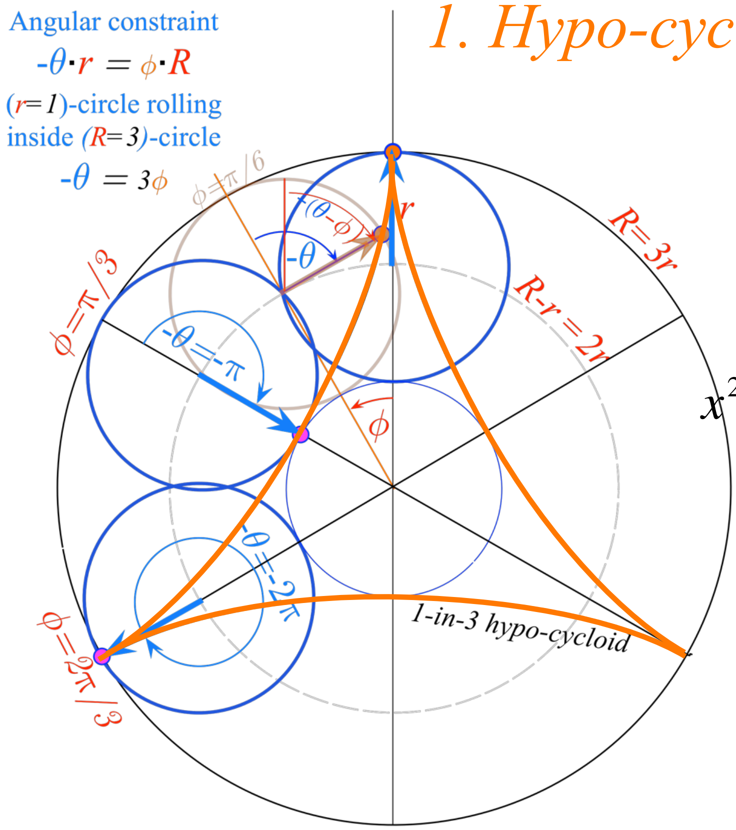
$$\rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$$

$$=R^2+2(R+r)r-2(R+r)r\cos\frac{R}{r}\phi$$

$$\rho^2=R^2+4(R+r)r\sin^2\frac{R}{2r}\phi$$



Cycloid-like curves for rolling constraints



1. Hypo-cycloid

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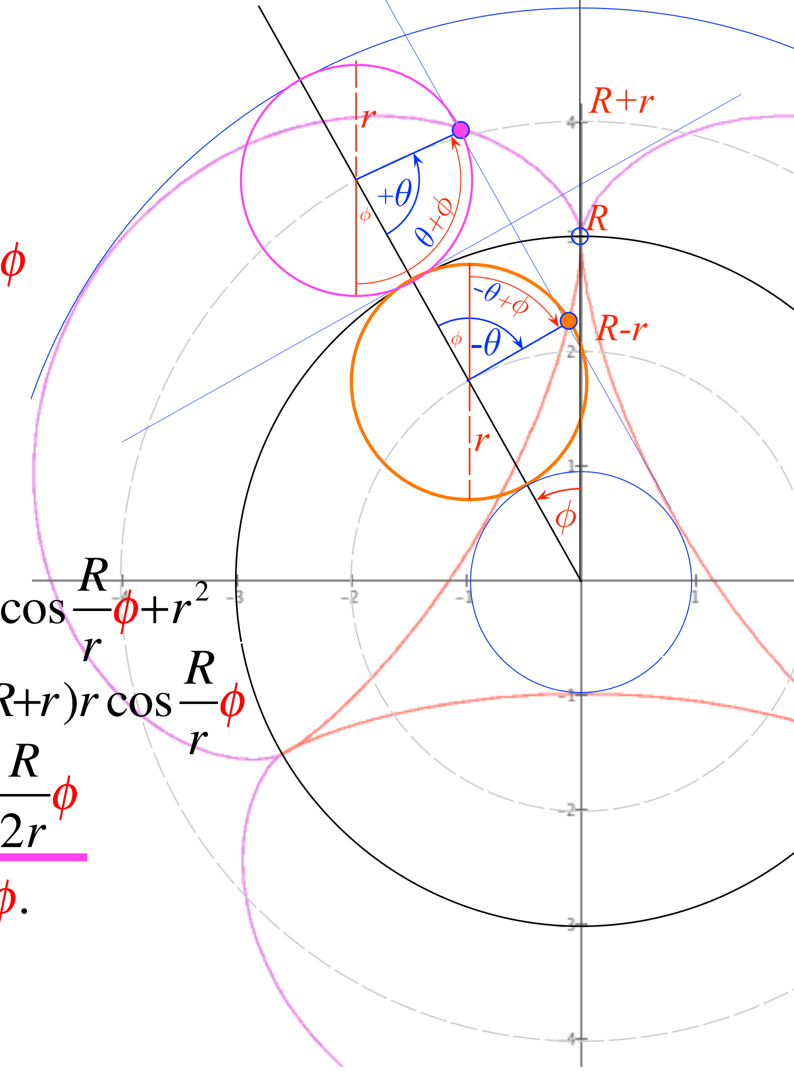
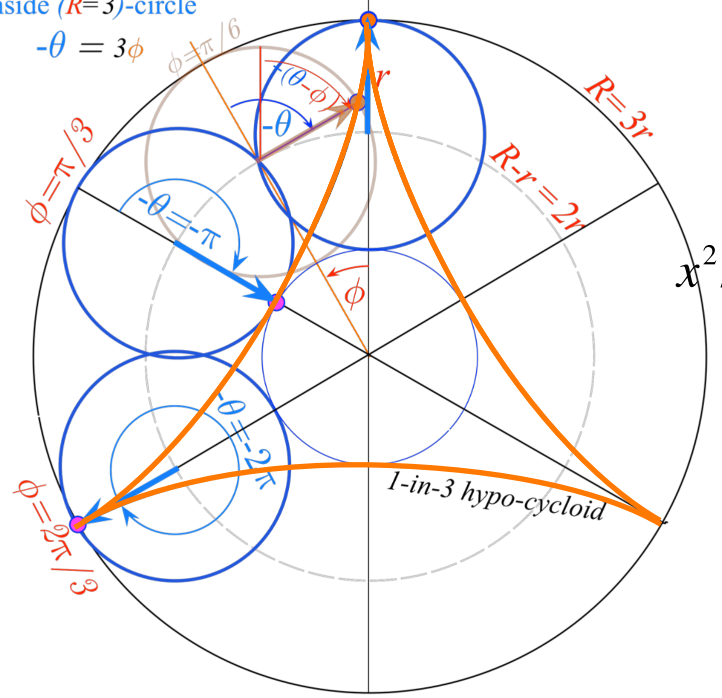
Hyper-cycloid velocity

$$\dot{x}/r=-A\dot{\phi}\cos\phi+A\dot{\phi}\cos A\phi, \qquad \dot{y}/r=-A\dot{\phi}\sin\phi+A\dot{\phi}\sin A\phi.$$



Cycloid-like curves for rolling constraints

1. Hypo-cycloid



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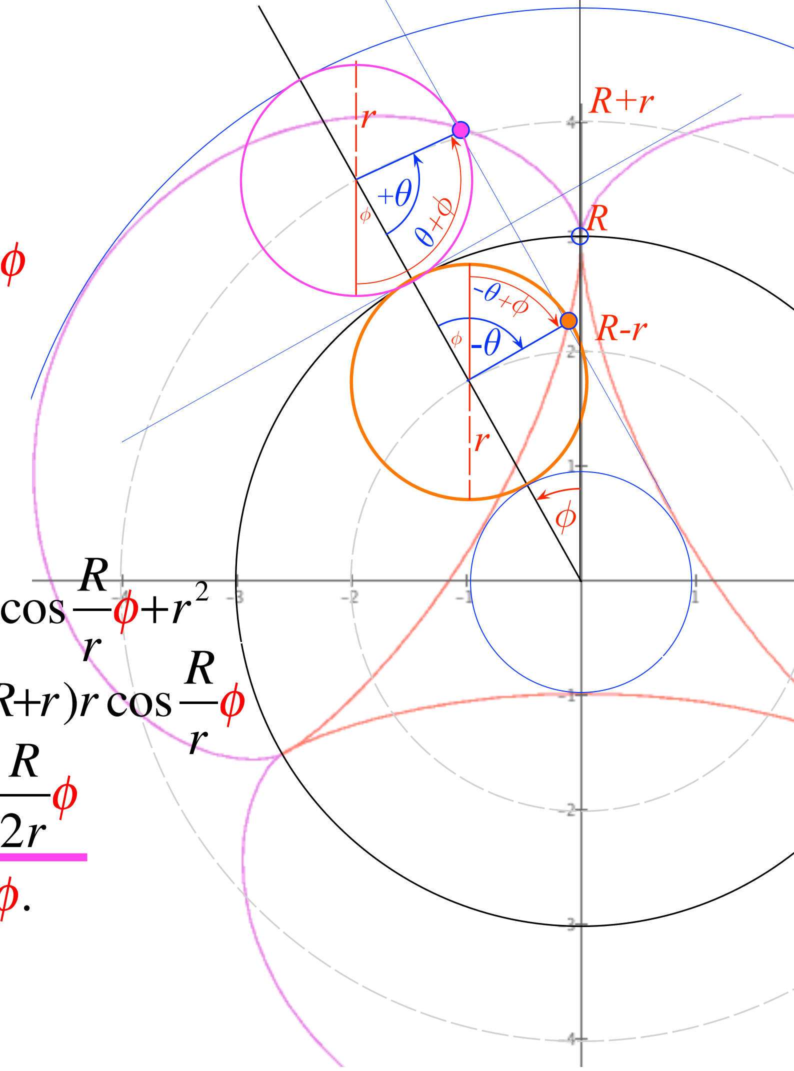
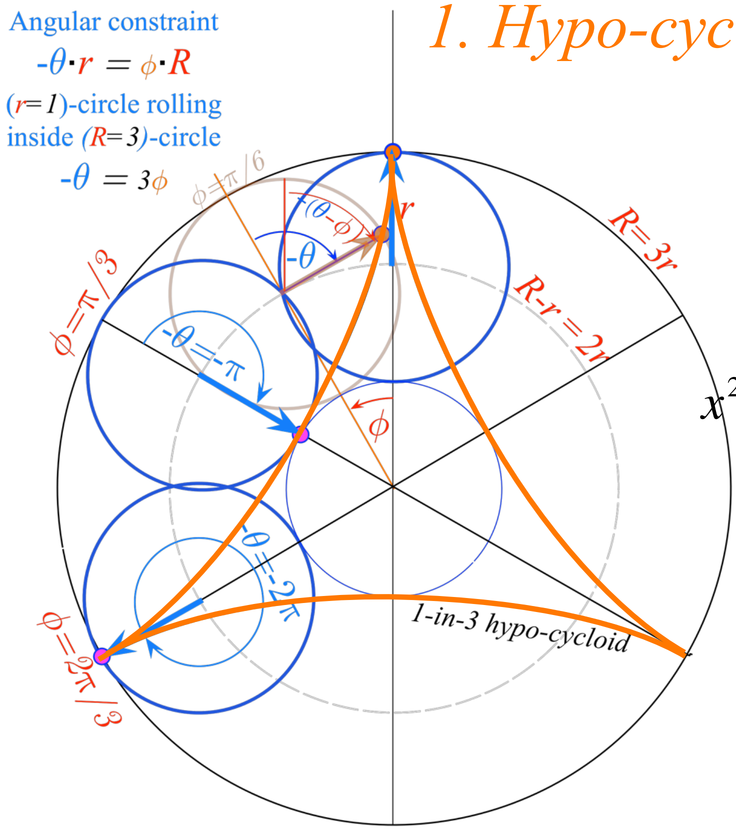


Cycloid-like curves for rolling constraints



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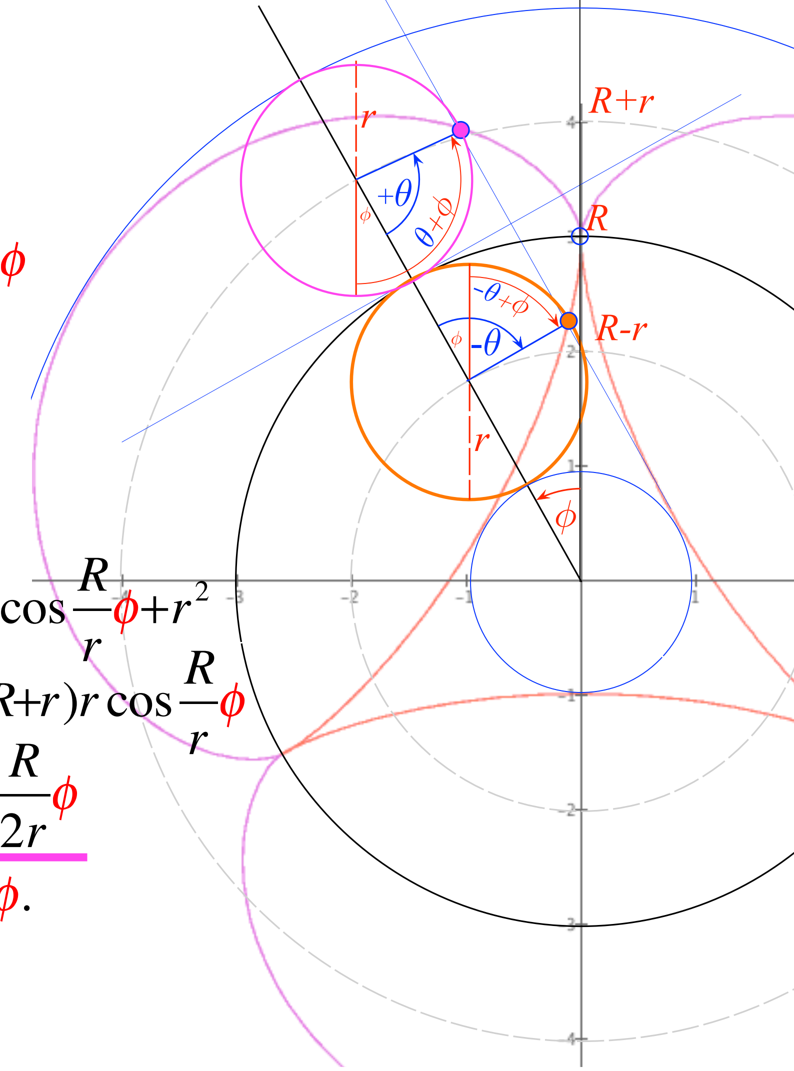
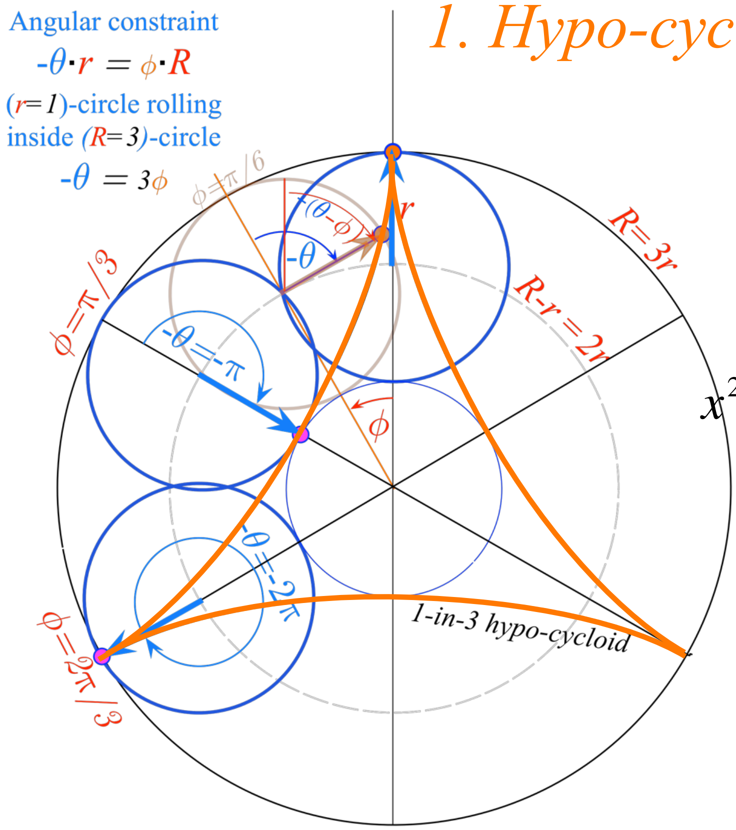
$\dot x^2/r^2+\dot y^2/r^2=2A^2\dot\phi^2(1-\cos\phi\cos A\phi+\sin\phi\sin A\phi)=2A^2\dot\phi^2(1-\cos(A-1)\phi)$



Cycloid-like curves for rolling constraints



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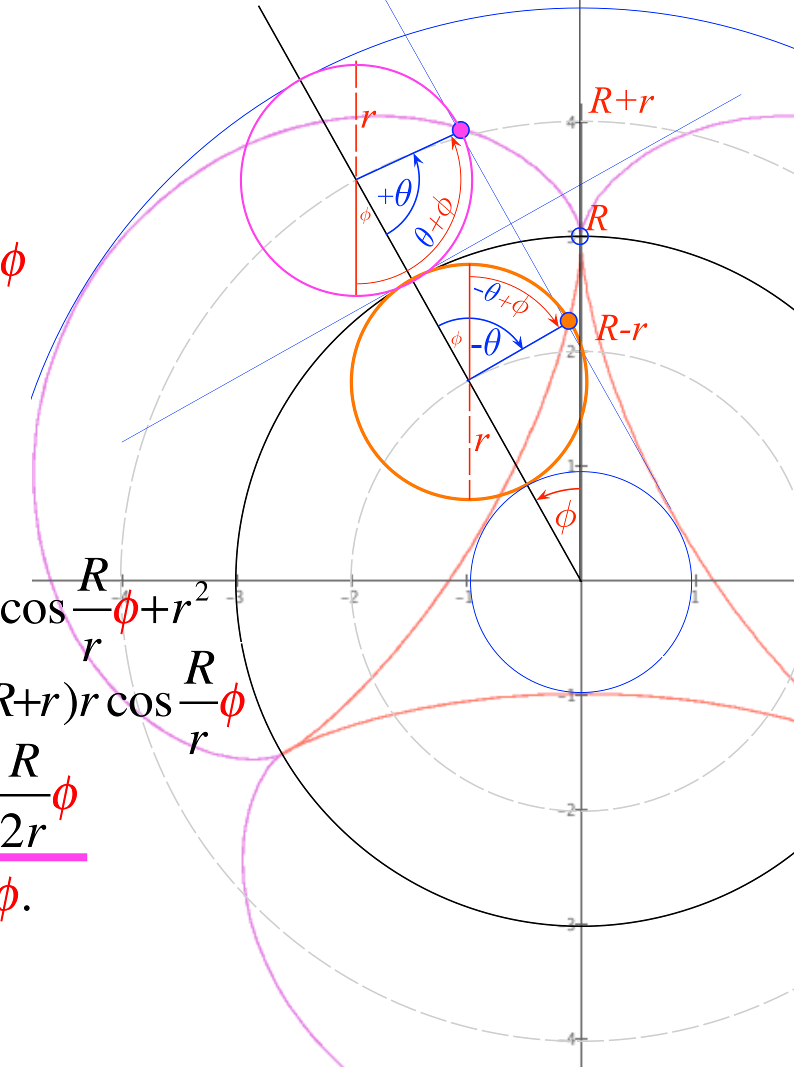
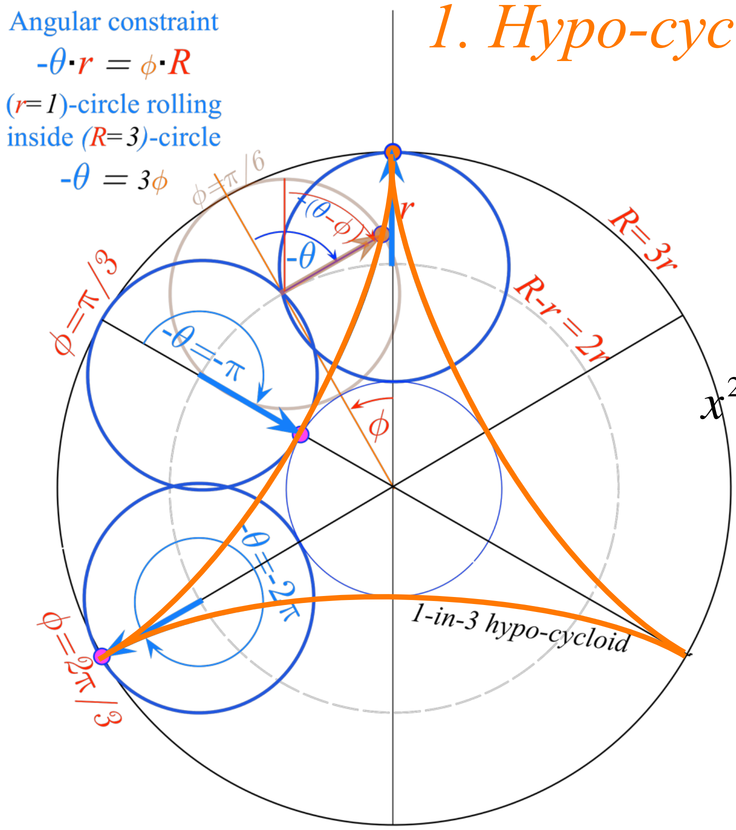
$$\dot{x}^2/r^2+\dot{y}^2/r^2=2A^2\dot{\phi}^2(1-\cos\phi\cos A\phi+\sin\phi\sin A\phi)=2A^2\dot{\phi}^2(1-\cos(A-1)\phi)$$

$$\dot{\rho}^2=\dot{x}^2+\dot{y}^2=2A^2r^2\dot{\phi}^2(1-\cos(A-1)\phi)$$

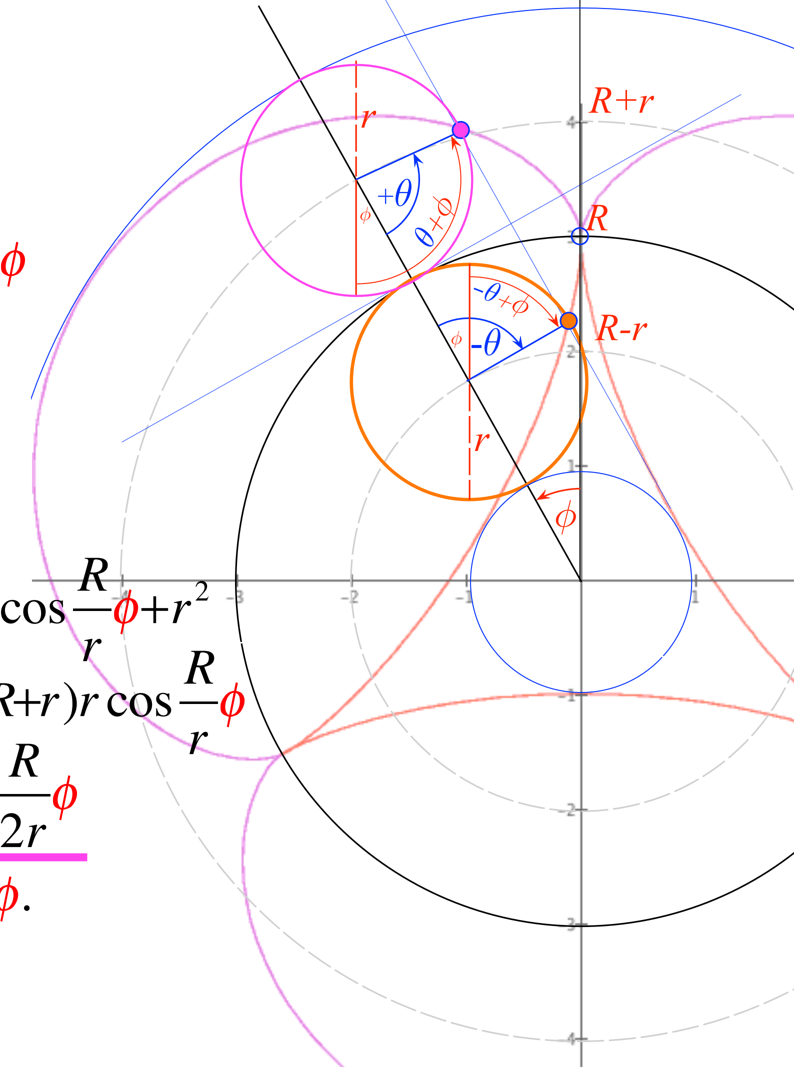


Cycloid-like curves for rolling constraints



1. Hypo-cycloid

2. Hyper-cycloid



Hyper-cycloid constrained by $A=\frac{R}{r}+1$, $\theta=(A-1)\phi=\frac{R}{r}\phi$

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$\rho^2/r^2=A^2-2A\cos(A-1)\phi+1$

$\rho^2=(R+r)^2-2(R+r)r\cos\frac{R}{r}\phi+r^2$

Hyper-cycloid radius ρ:

$=R^2+2(R+r)r-2(R+r)r\cos\frac{R}{r}\phi$

$\rho^2=R^2+4(R+r)r\sin^2\frac{R}{2r}\phi$

Hyper-cycloid velocity

$\dot{x}/r=-A\dot{\phi}\cos\phi+A\dot{\phi}\cos A\phi,\qquad \dot{y}/r=-A\dot{\phi}\sin\phi+A\dot{\phi}\sin A\phi.$

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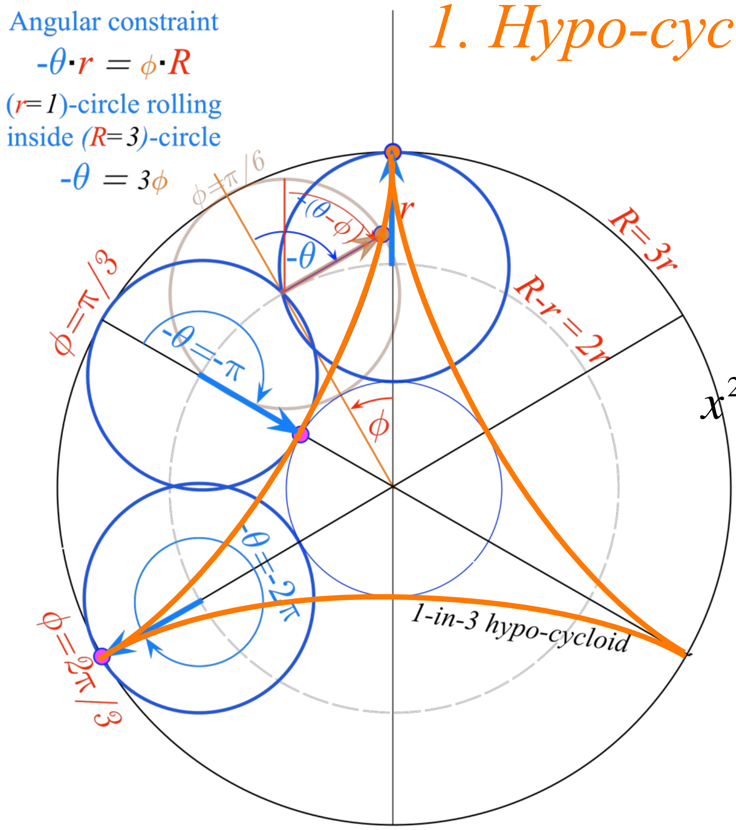
$\dot{\rho}^2=\dot{x}^2+\dot{y}^2=2A^2r^2\dot{\phi}^2(1-\cos(A-1)\phi)=2(R+r)^2\dot{\phi}^2(1-\cos\frac{R}{r}\phi)$

Cycloid-like curves for rolling constraints



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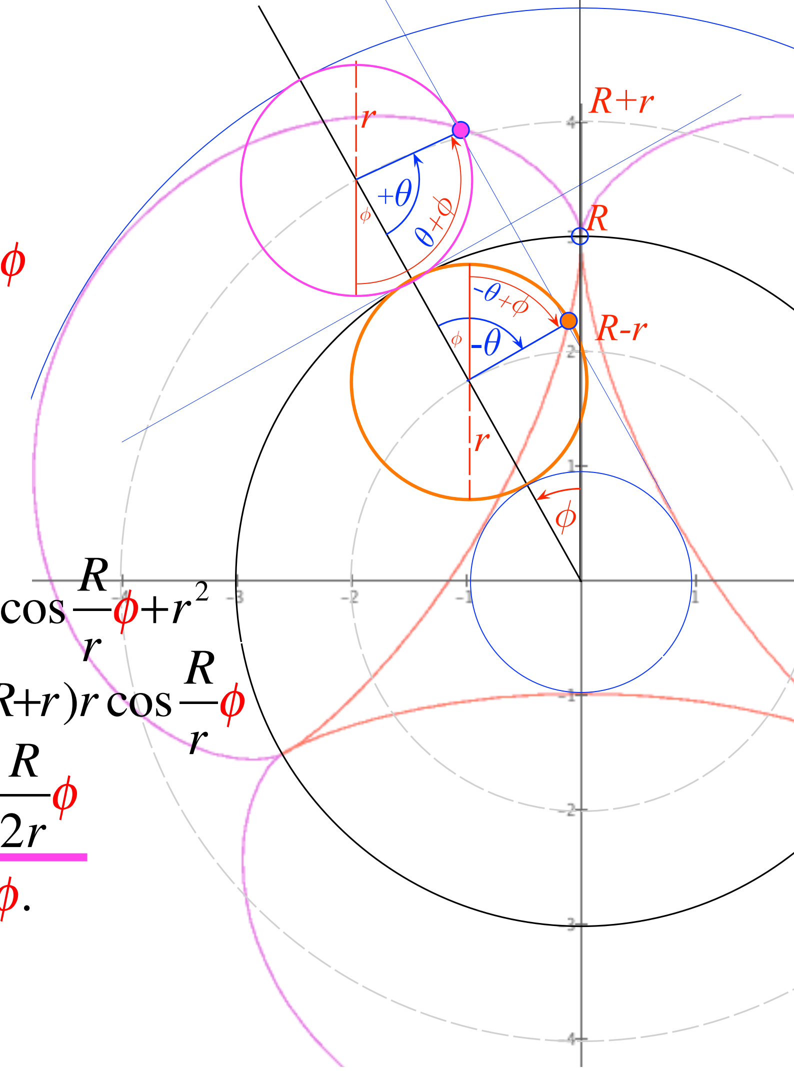

Cycloid-like curves for rolling constraints

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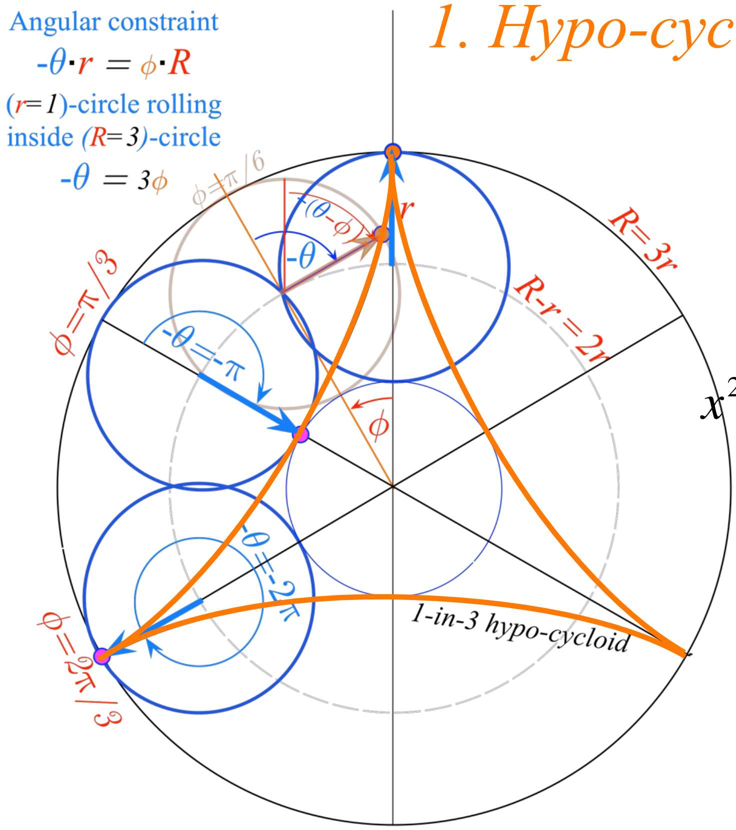

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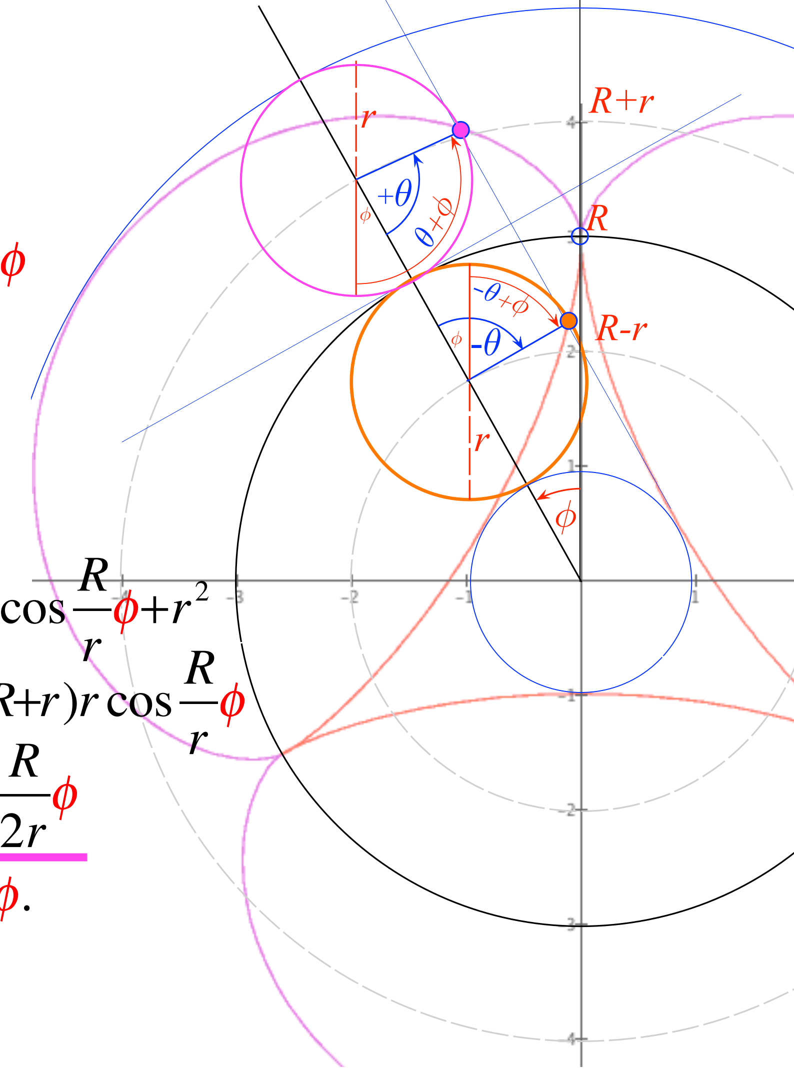

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Cycloid-like curves for rolling constraints

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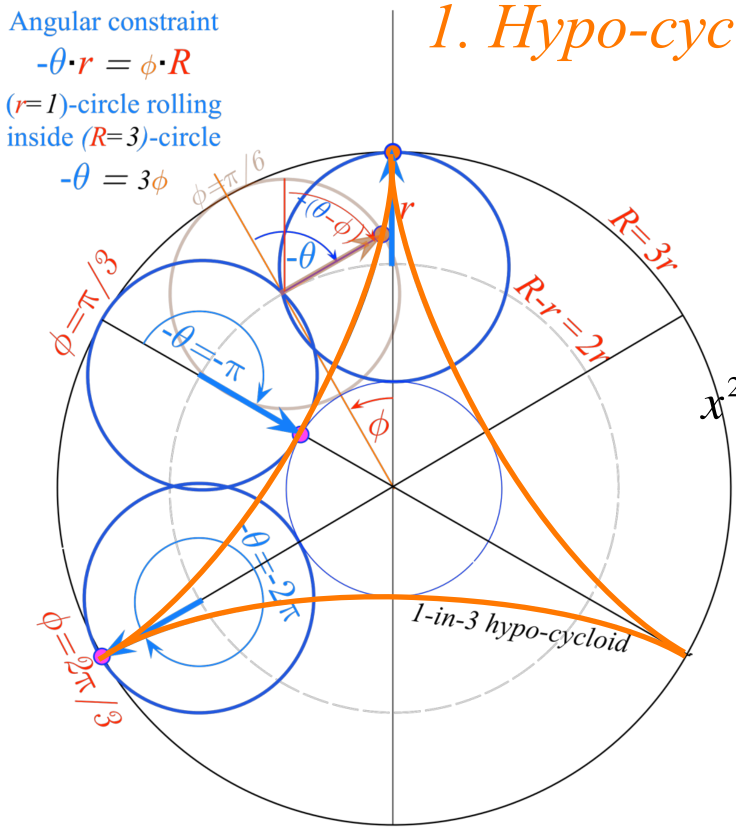

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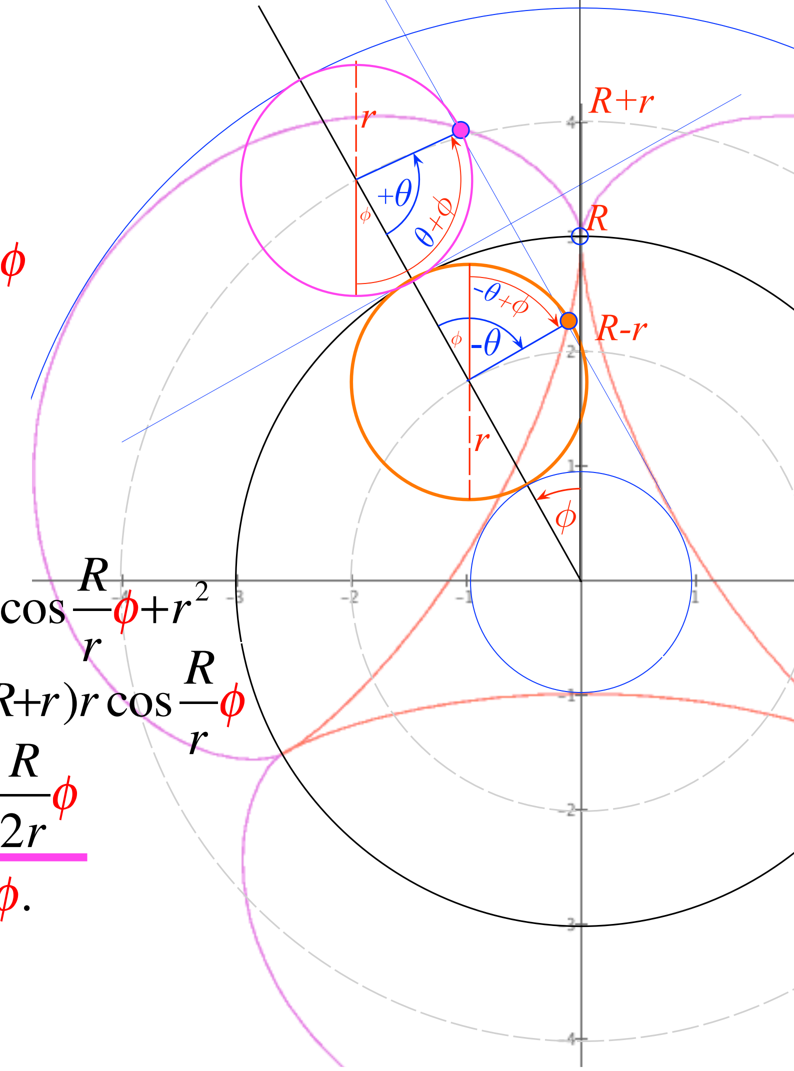

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Hyper-cycloid energy and dynamics

based on: $E=\frac{1}{2}m\dot{\rho}^2-\frac{1}{2}m\omega_{\odot}^2\rho^2=const.$ with a repulsive PE: $V(\rho)=-\frac{1}{2}m\omega_{\odot}^2\rho^2$

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Cycloid-like curves for rolling constraints

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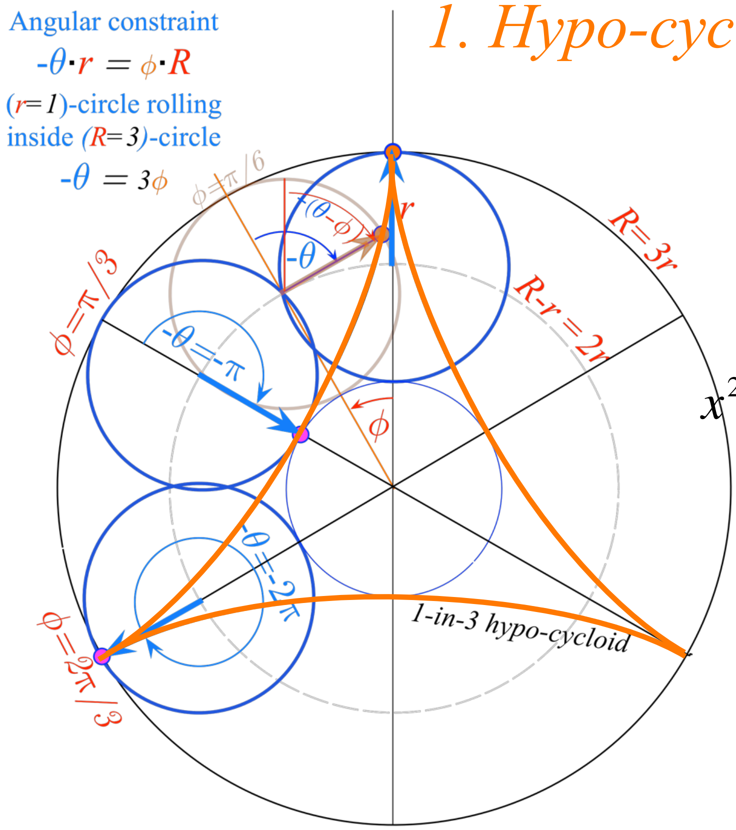

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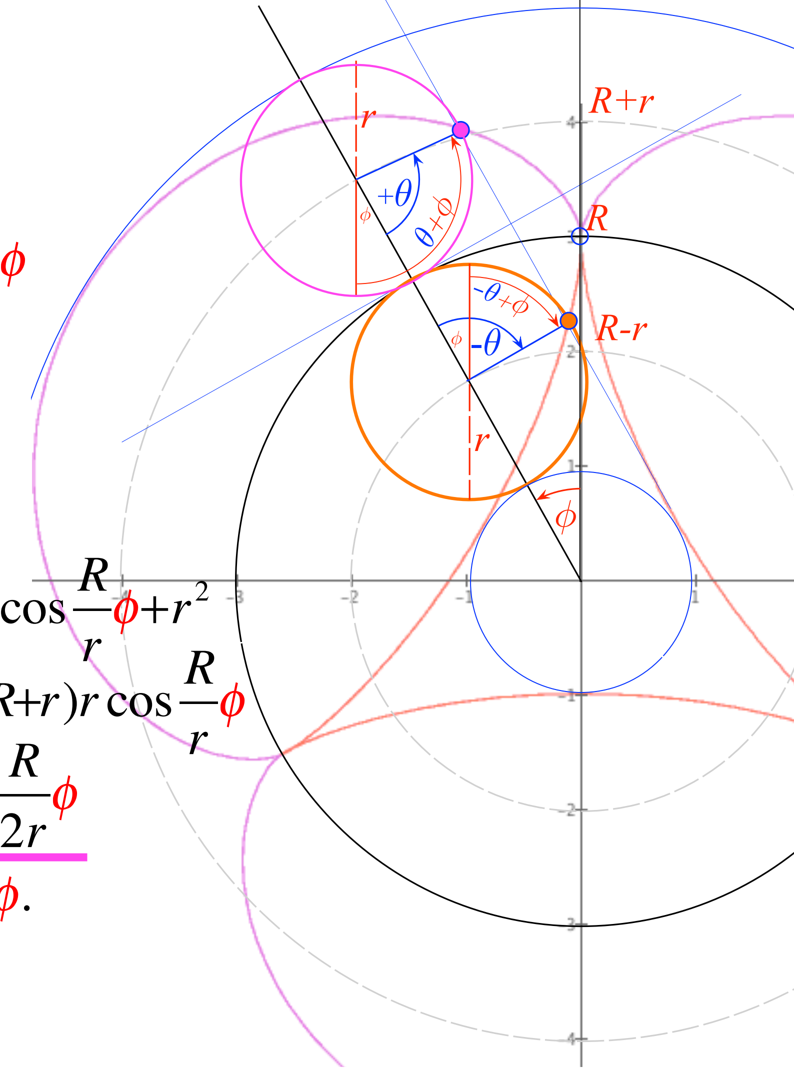

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Cycloid-like curves for rolling constraints

Angular constraint
$-\theta \cdot r = \phi \cdot R$
($r=1$)-circle rolling
inside ($R=3$)-circle
$-\theta = 3\phi$

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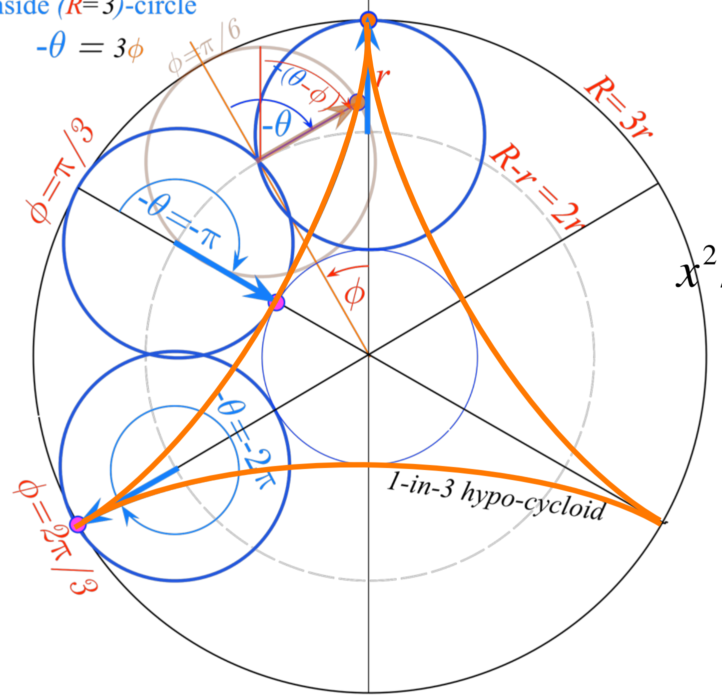

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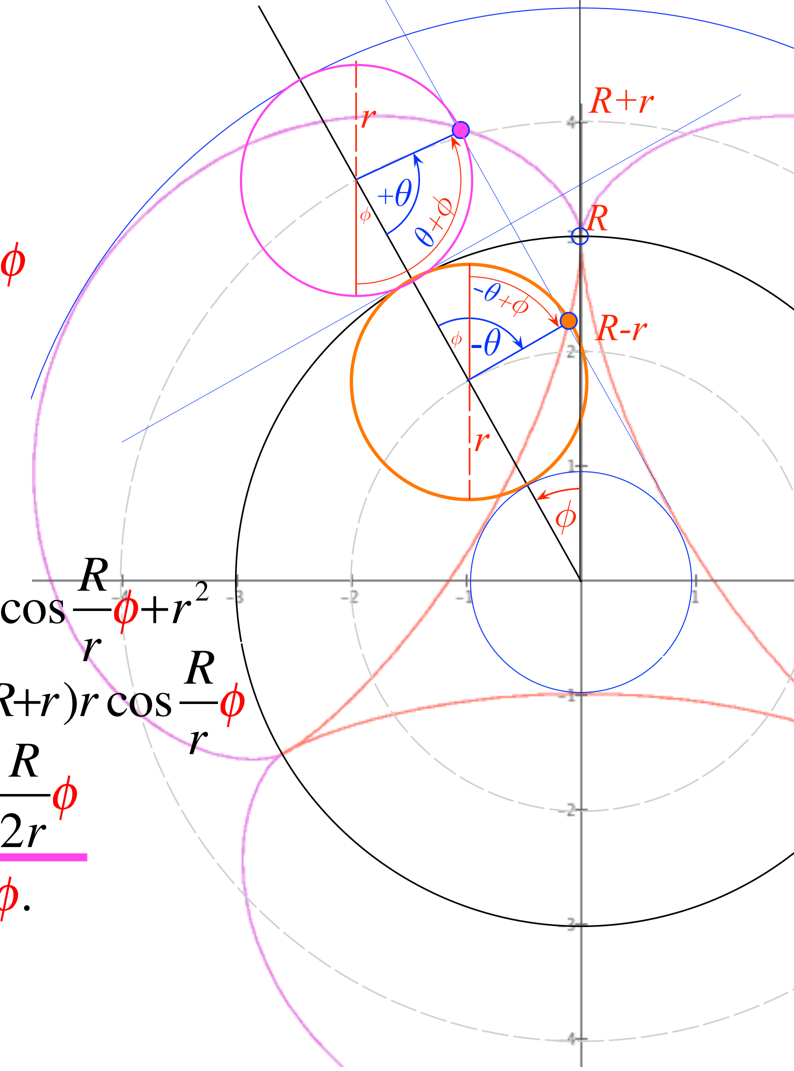

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Cycloid-like curves for rolling constraints

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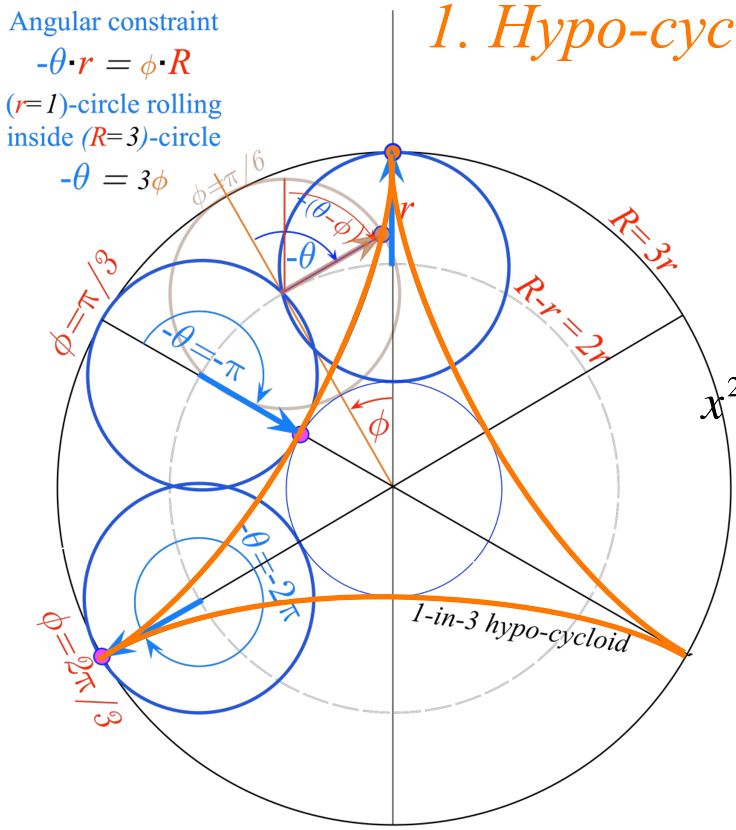

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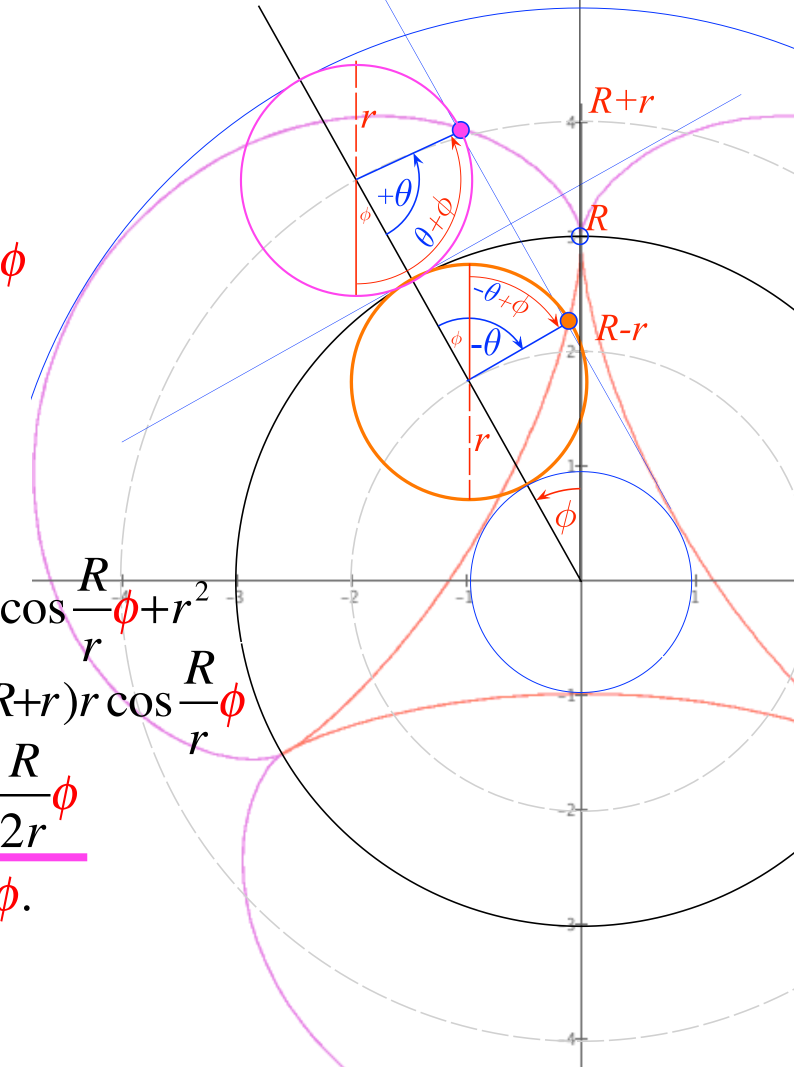

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Cycloid-like curves for rolling constraints

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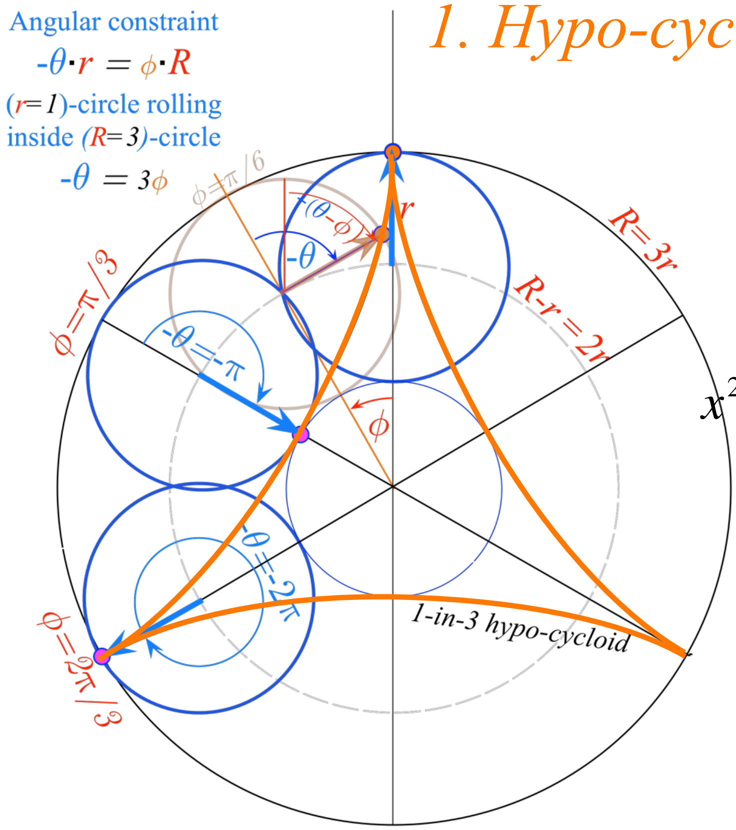

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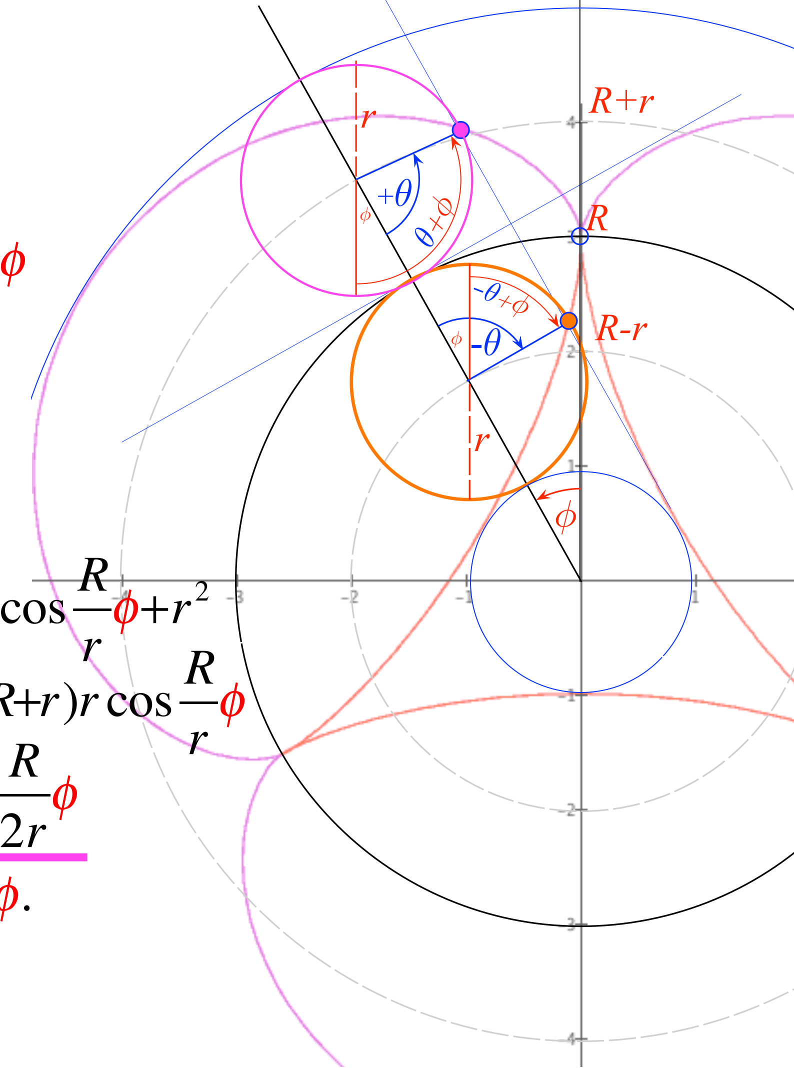
$$\dot{x}^2/r^2=A^2\dot{\phi}^2\cos^2\phi-2A^2\dot{\phi}^2\cos\phi\cos A\phi+A^2\dot{\phi}^2\cos^2 A\phi$$

$$\dot{y}^2/r^2=A^2\dot{\phi}^2\sin^2\phi-2A^2\dot{\phi}^2\sin\phi\sin A\phi+A^2\dot{\phi}^2\sin^2 A\phi$$



$$\dot{x}^2/r^2+\dot{y}^2/r^2=2A^2\dot{\phi}^2(1-\cos\phi\cos A\phi+\sin\phi\sin A\phi)=2A^2\dot{\phi}^2(1-\cos(A-1)\phi)$$

$$\dot{\rho}^2=\dot{x}^2+\dot{y}^2=2A^2r^2\dot{\phi}^2(1-\cos(A-1)\phi)=2(R+r)^2\dot{\phi}^2(1-\cos\frac{R}{r}\phi)=4(R+r)^2\dot{\phi}^2\sin^2\frac{R}{2r}\phi$$

Hyper-cycloid energy and dynamics

based on: $E=\frac{1}{2}m\dot{\rho}^2-\frac{1}{2}m\omega_\odot^2\rho^2=const.$ with a repulsive PE: $V(\rho)=-\frac{1}{2}m\omega_\odot^2\rho^2$

Start with 100% Potential Energy ($\dot{\rho}_0=0$) at $\rho_0=R$: $\frac{2E_0}{m}=\dot{\rho}_0^2-\omega_\odot^2\rho_0^2=-\omega_\odot^2R^2=const.$

Then at any time t: $\dot{\rho}_t^2-\omega_\odot^2\rho_t^2=-\omega_\odot^2R^2=const.$ (Constant total energy)

$$4(R+r)^2\dot{\phi}^2\sin^2\frac{R}{2r}\phi-\omega_\odot^2\left[\cancel{R^2}+4(R+r)r\sin^2\frac{R}{2r}\phi\right]=-\cancel{\omega_\odot^2R^2}=const.$$

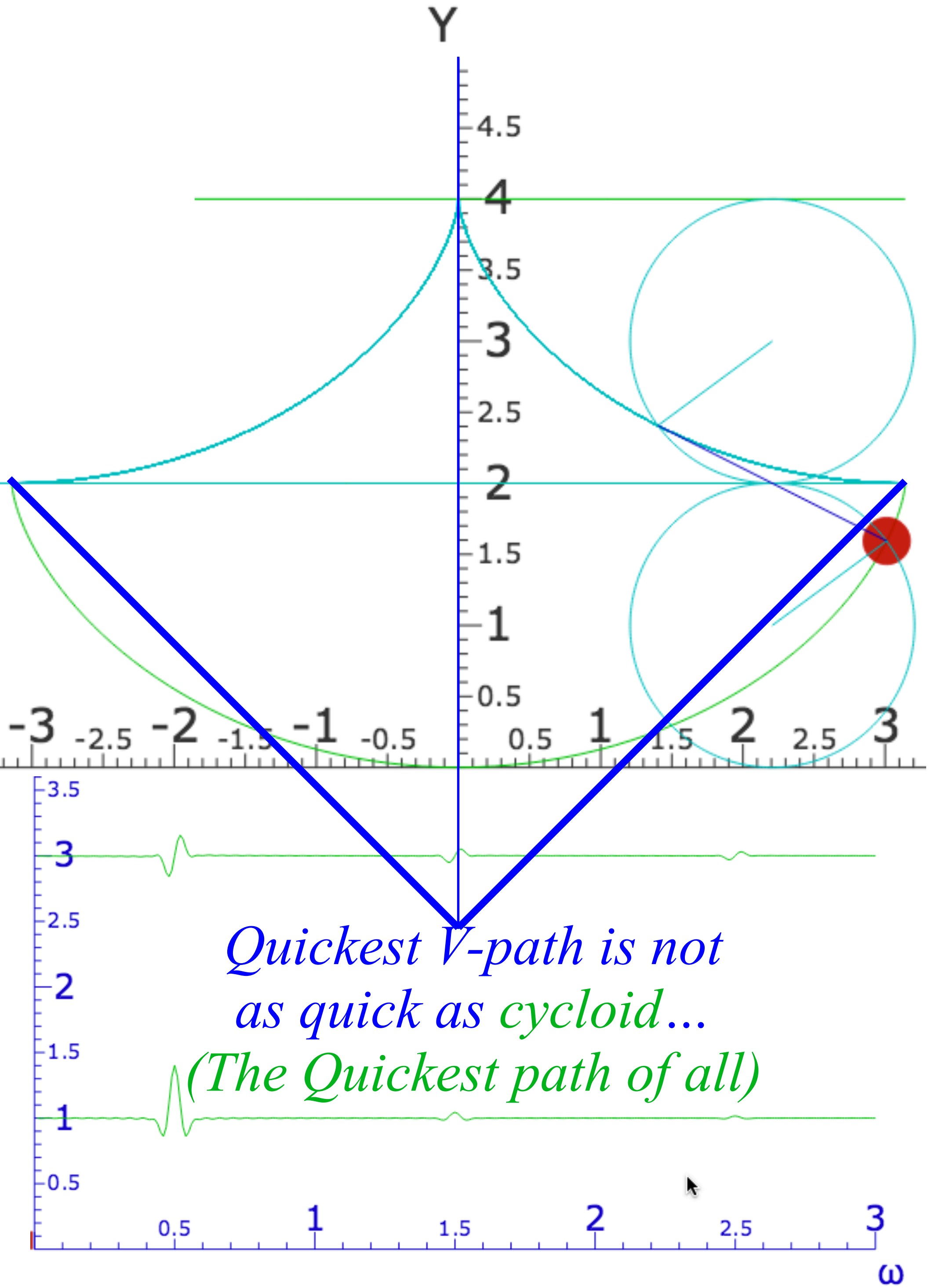
$$(R+r)\dot{\phi}^2\qquad -\omega_\odot^2\left[r\right]=0$$

Results in hyper-circle orbiting at constant $\dot{\phi}=\omega_\odot\sqrt{\frac{r}{R+r}}=\frac{\omega_\odot}{\sqrt{A}}$

...and turning at constant $\dot{\theta}=\frac{R}{r}\dot{\phi}=\omega_\odot\frac{R}{\sqrt{r(R+r)}}$

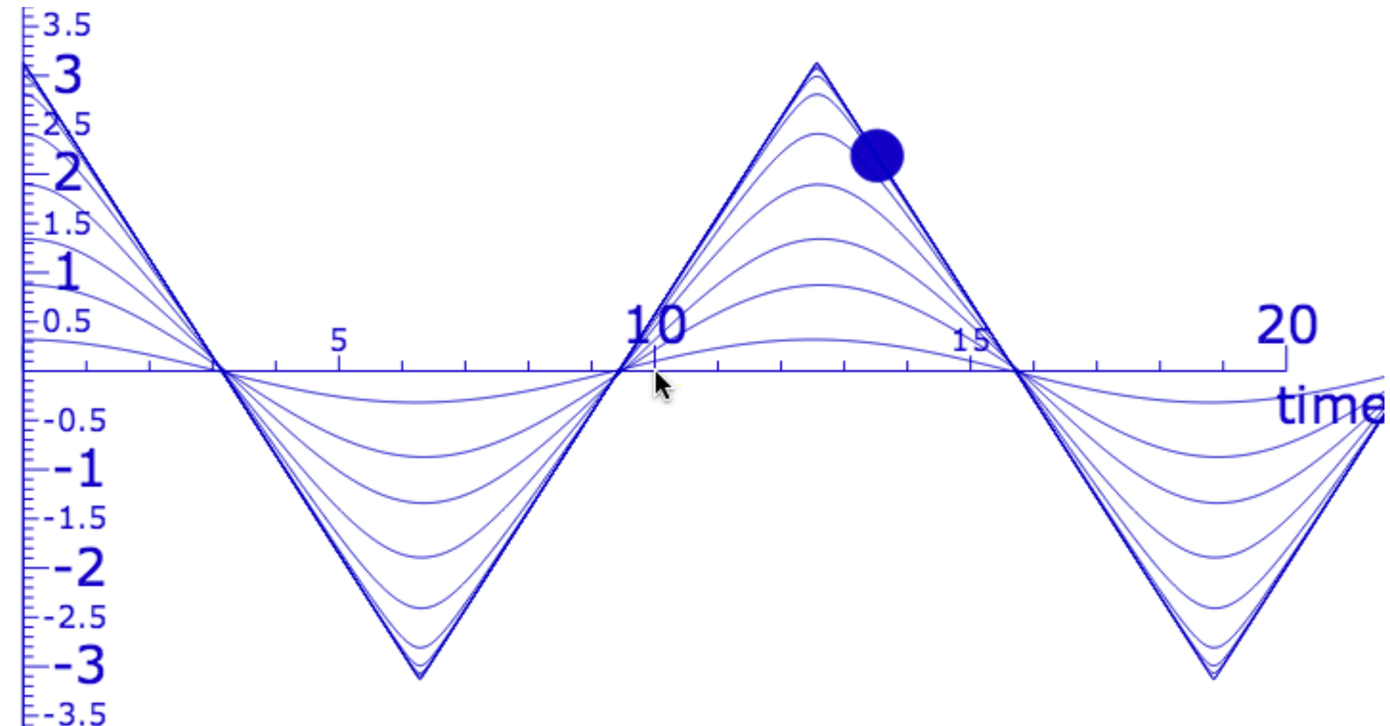
Cycloid-like curves for rolling constraints

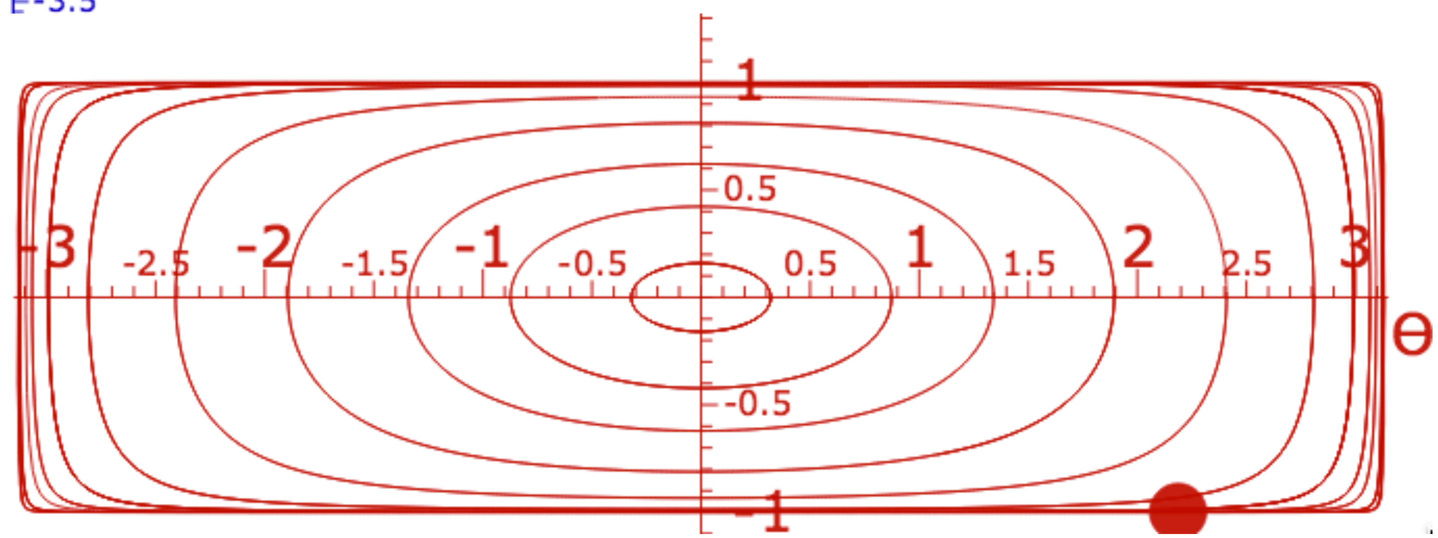
Quickest intra-planetary subways

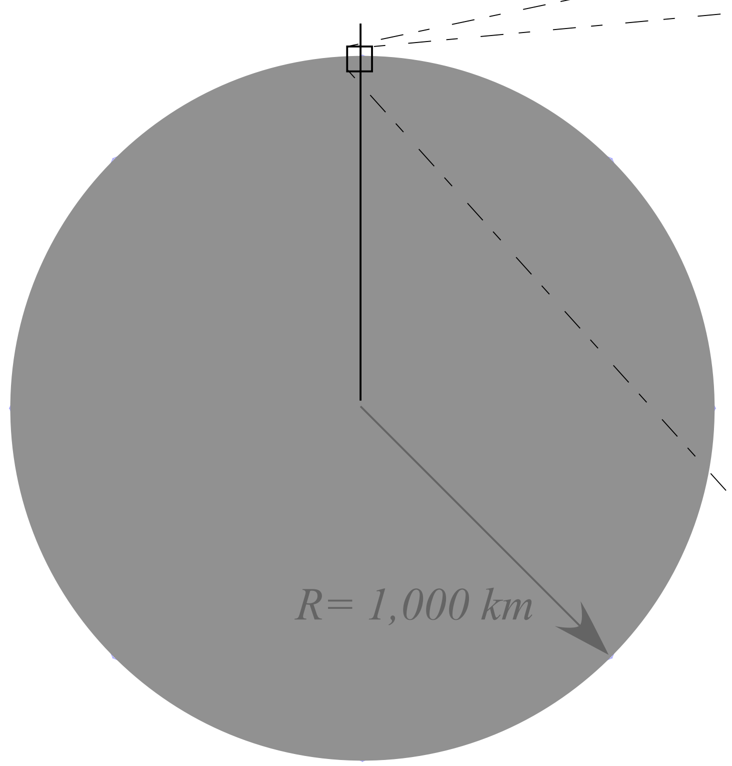


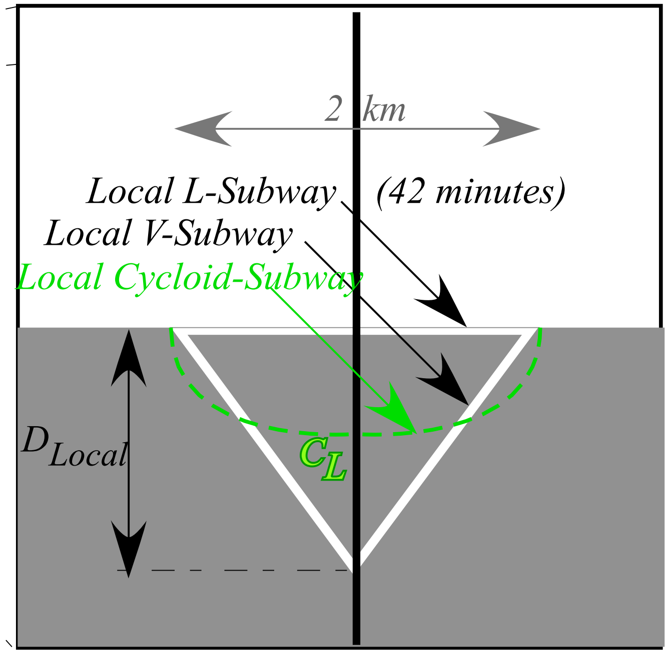
Quickest V-path is not as quick as cycloid…
(The Quickest path of all)

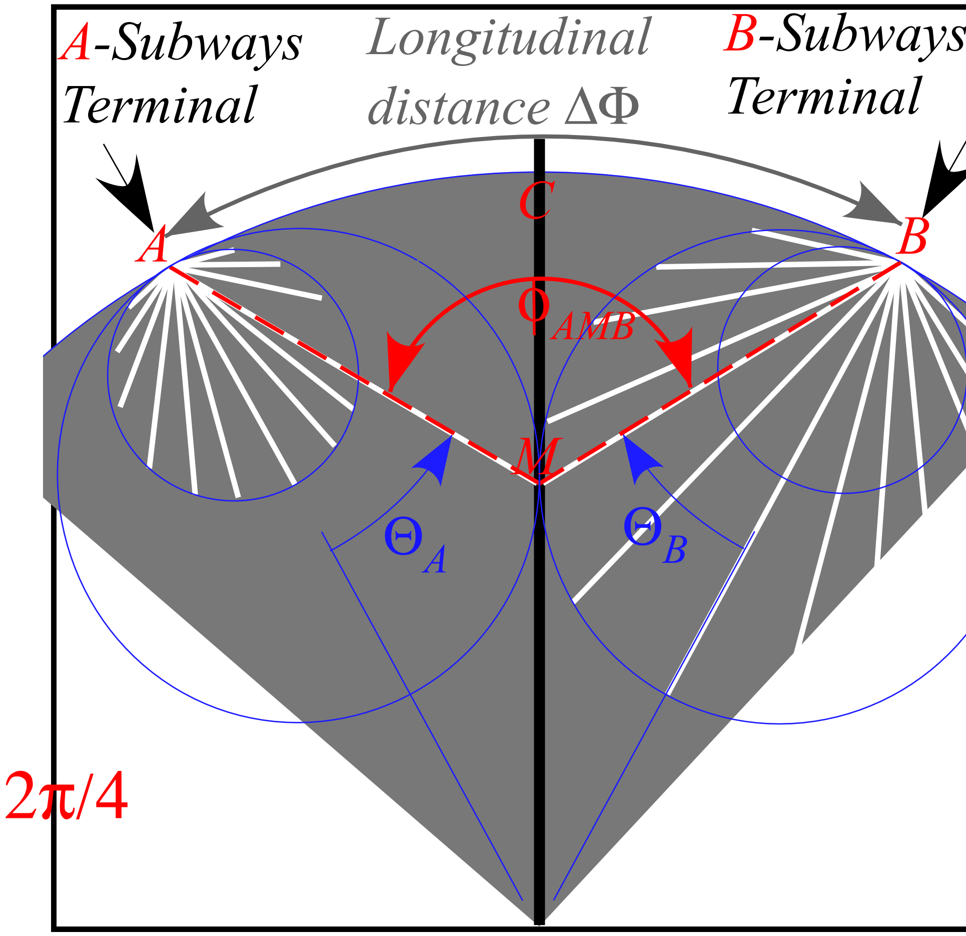
http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html

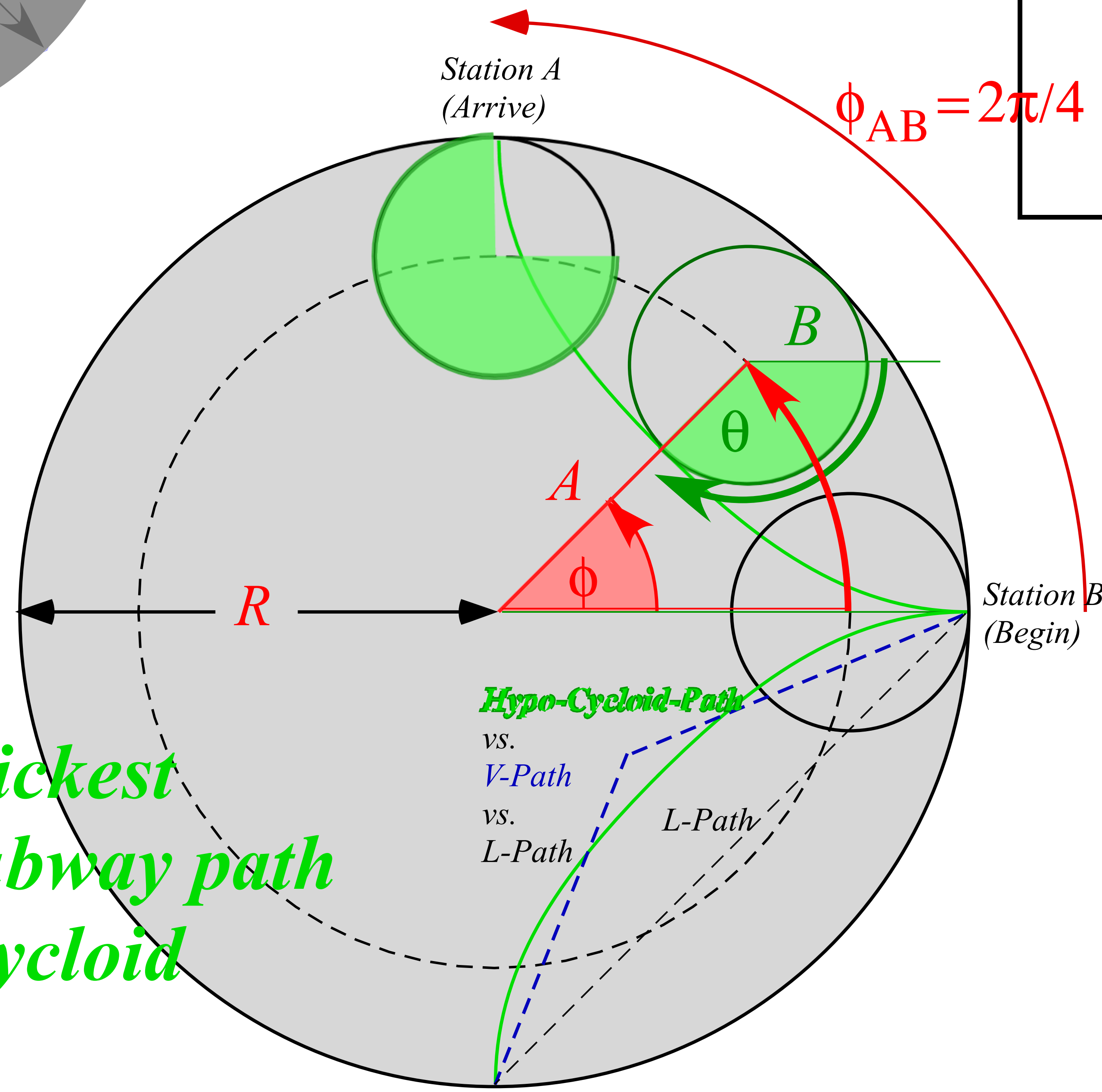












The very quickest planetary subway path is a Hypo-Cycloid