Lecture 27
Fri. 12.04.2015

Geometry and Symmetry of Coulomb Orbital Dynamics

(Ch. 2-4 of Unit 5 12.05.15)

*Review of **Eccentricity vector ε** and (ε,λ)-geometry of orbital mechanics*

Analytic geometry derivation of ε-construction

Connection formulas for (a,b) and (ε,λ) with (γ,R)

Review of lecture 26

Detailed ruler & compass construction of ε-vector and orbits

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

Properties of Coulomb trajectory families and envelopes

Graphical ε-development of orbits

Launch angle fixed-Varied launch energy

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes

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Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* $\mathbf{L}$

(Review of Lect. 26)

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* $\boldsymbol{S}$

$$S_A = \tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$
$$S_B = x_1p_1 + x_2p_2$$
$$S_C = x_1p_2 - x_2p_1$$

$\mathbf{A} = km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Generate symmetry groups: $U(2)\subset U(2)$

or: $R(3)\subset R(3)\times R(3)\subset O(4)$

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector $\mathbf{r}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r} = \frac{\mathbf{r}\bullet\mathbf{r}}{r} - \frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km} = r - \frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km} = r - \frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

...or of $\boldsymbol{\varepsilon}$ with momentum vector $\mathbf{p}$:

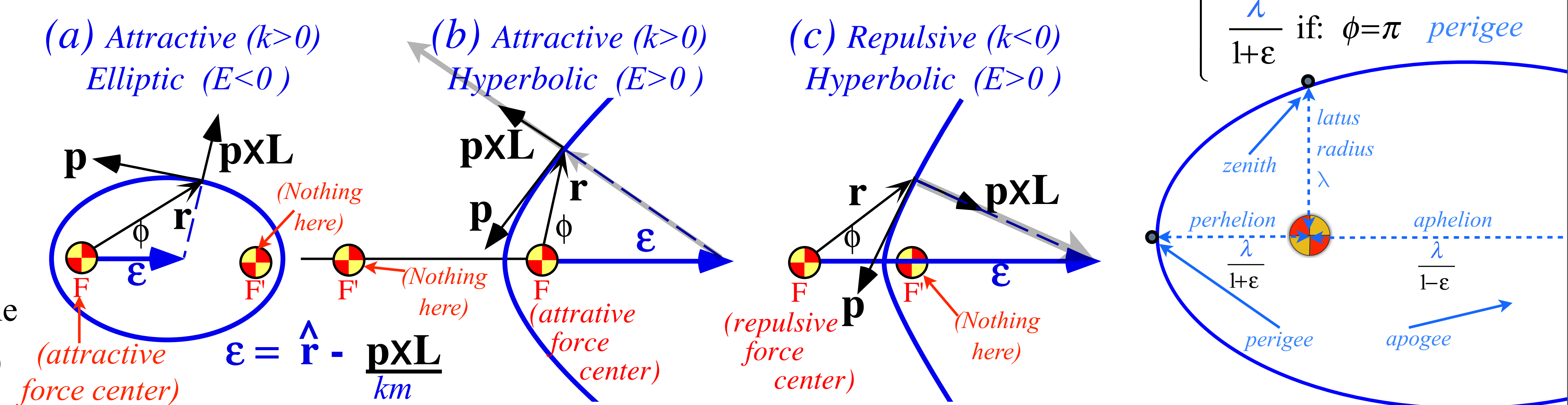
$$\boldsymbol{\varepsilon}\bullet\mathbf{p} = \frac{\mathbf{p}\bullet\mathbf{r}}{r} - \frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km} = \mathbf{p}\bullet\hat{\mathbf{r}} = p_r$$

Let angle ϕ be angle between $\boldsymbol{\varepsilon}$ and radial vector $\mathbf{r}$

$$\varepsilon\, r\cos\phi = r - \frac{L^2}{km} \qquad \text{or:} \qquad r = \frac{L^2/km}{1-\varepsilon\cos\phi}$$

For $\lambda = L^2/km$ that matches: $r = \dfrac{\lambda}{1-\varepsilon\cos\phi} = \begin{cases} \dfrac{\lambda}{1-\varepsilon} & \text{if: } \phi=0 \quad \textit{apogee} \\ \lambda & \text{if: } \phi=\dfrac{\pi}{2} \quad \textit{zenith} \\ \dfrac{\lambda}{1+\varepsilon} & \text{if: } \phi=\pi \quad \textit{perigee} \end{cases}$

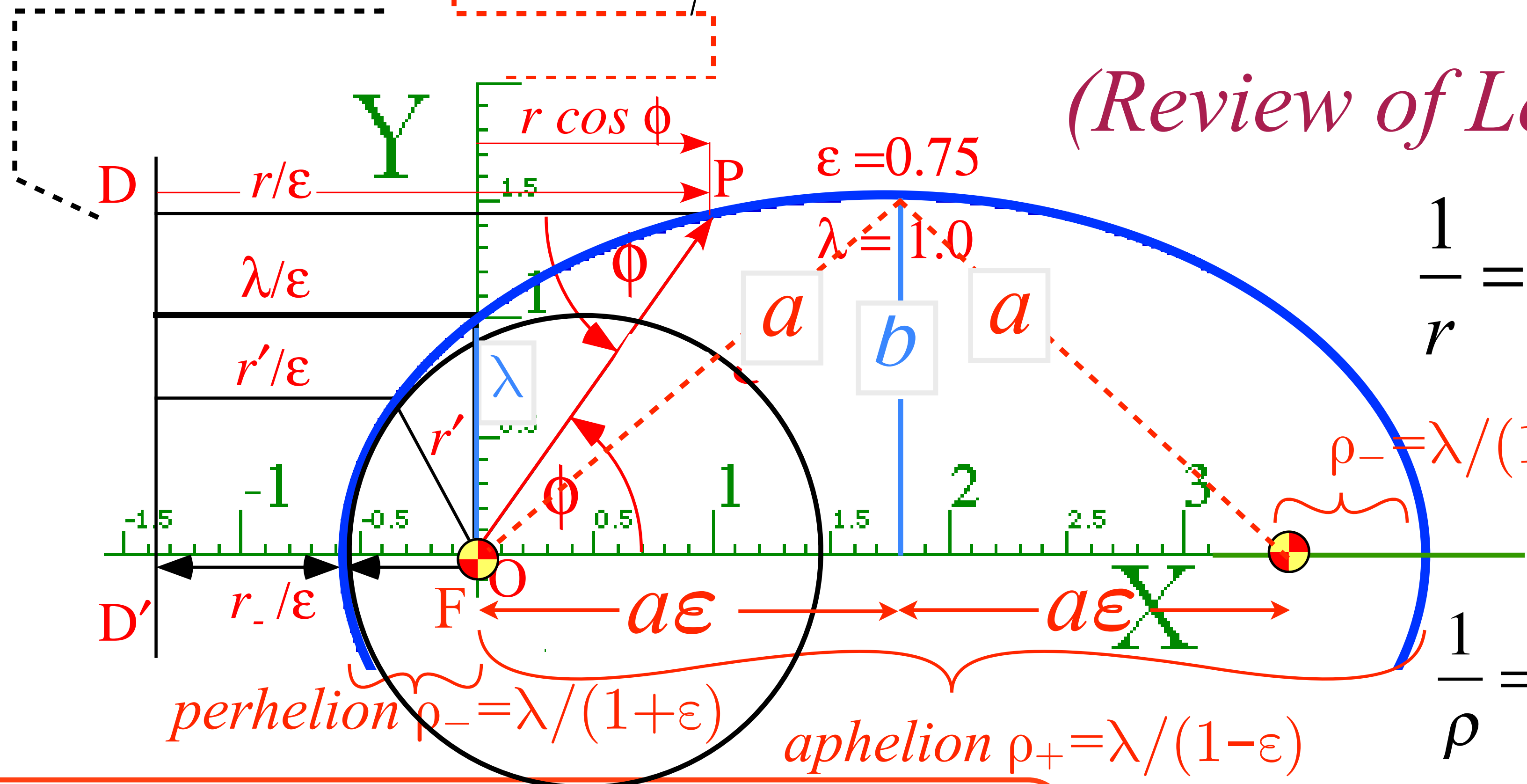
(Rotational momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)



(From Lecture 26 p. 48) *Geometry of Coulomb orbits (Let: r = ρ here)*

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$

(Review of Lect. 25)



$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by:

Defining eccentricity ε

Distance to Focal-point $= \varepsilon \cdot$ Distance to Directrix-line

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|$

Focal axis: $\rho_+ - \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/|1-\varepsilon^2|$

Minor radius: $b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon<1$)*

Minor radius: $b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon>1$)*

(x,y) *parameters*	*physical constants*	(r,ϕ) *parameters*
$a = \frac{k}{2E}$	$E = \frac{k}{2a}$	$\varepsilon = \sqrt{\frac{k^2 m + 2L^2 E}{k^2 m}} = \sqrt{1 \pm \frac{b^2}{a^2}}$
$b = \frac{L}{\sqrt{2m\lvert E\rvert}}$	$L = \sqrt{km\lambda}$	$\lambda = \frac{L^2}{km} = \frac{b^2}{a}$

$\varepsilon^2 = 1 - \frac{b^2}{a^2}$ (ellipse: $\varepsilon<1$) $\frac{b^2}{a^2} = \sqrt{1-\varepsilon^2}$

$\varepsilon^2 = 1 + \frac{b^2}{a^2}$ (hyperbola: $\varepsilon>1$) $\frac{b^2}{a^2} = \sqrt{\varepsilon^2 - 1}$

$\lambda = a(1-\varepsilon^2)$ *(ellipse: $\varepsilon<1$)*

$\lambda = a(\varepsilon^2-1)$ *(hyperb: $\varepsilon>1$)*

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➔ *Analytic geometry derivation of* ***ε****-construction*

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Detailed ruler & compass construction of ***ε-vector*** *and orbits*

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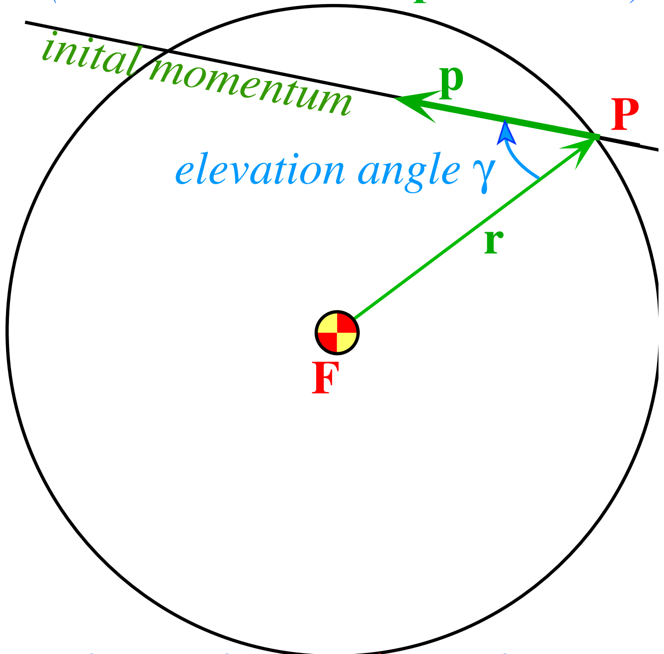
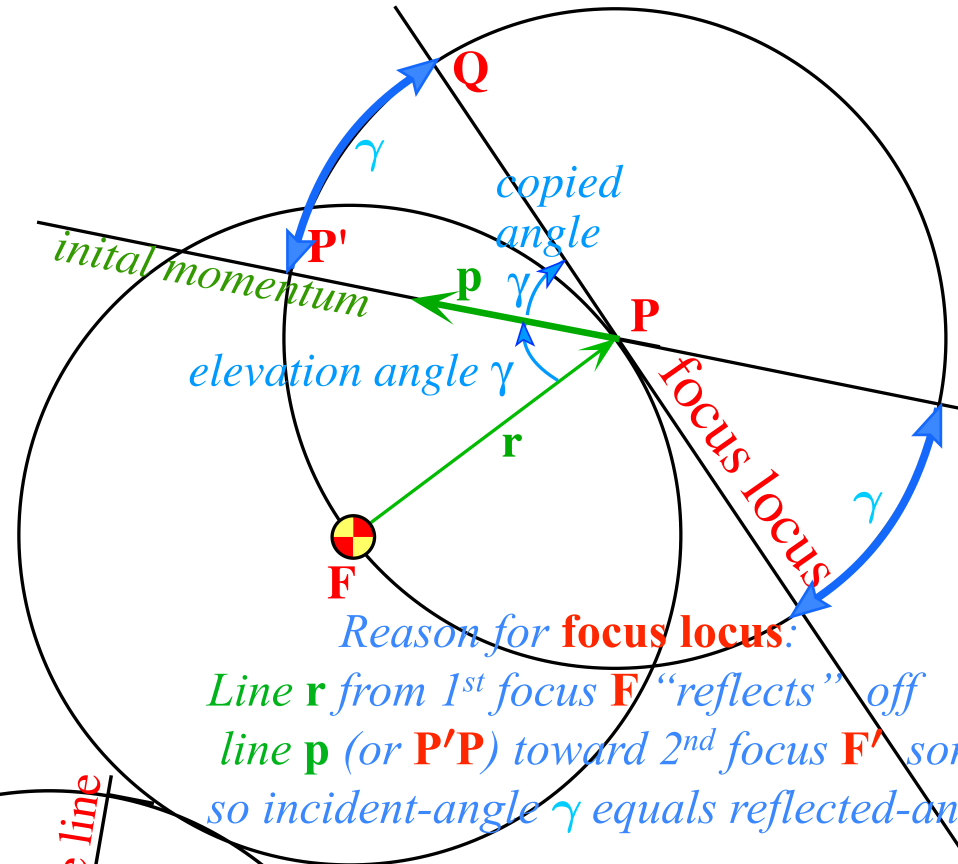
Launch optimization and orbit family envelopes

ε-vector and Coulomb orbit construction steps

Pick launch point P
(radius vector **r** *)*
and elevation angle γ from radius
(momentum initial **p** *direction)*

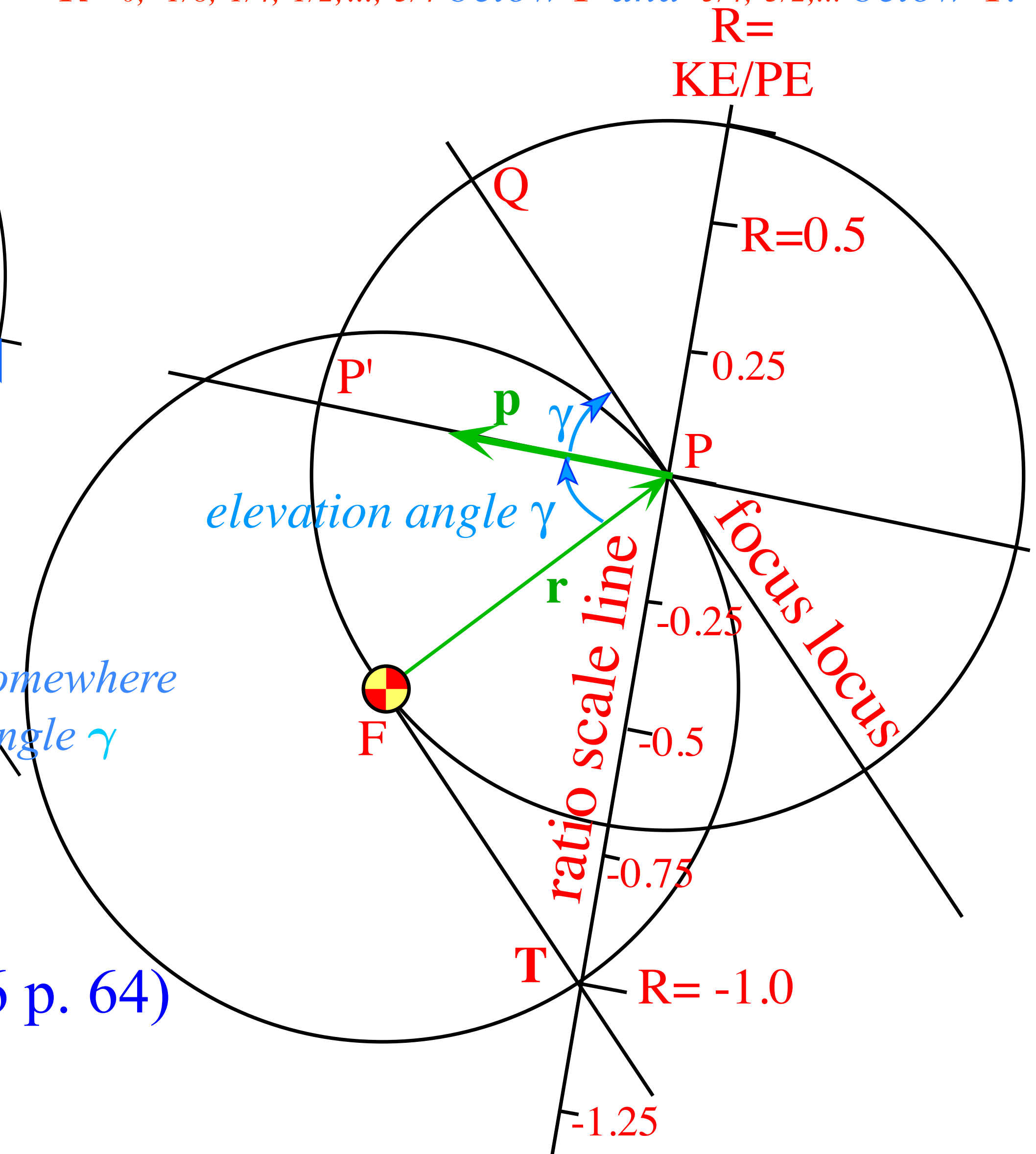


Copy F-center circle around launch point P
Copy elevation angle γ (∠FPP′) onto ∠P′PQ
Extend resulting line QPQ′ to make **focus locus**

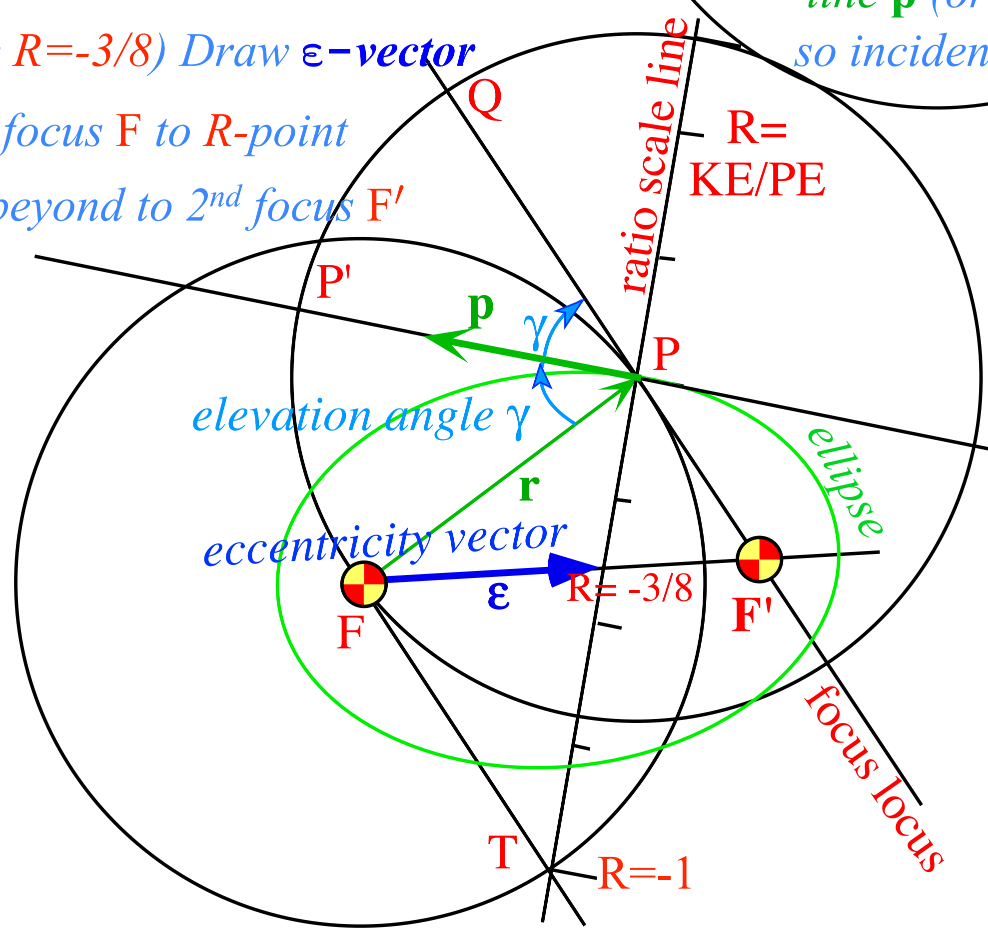


Reason for **focus locus***:*
Line **r** *from 1st focus* **F** *"reflects" off*
line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*
so incident-angle γ equals reflected-angle γ

Copy double angle 2γ (∠FPQ) onto ∠PFT
*Extend ∠PFT chord PT to make R-***ratio scale line**
Label chord PT with R=0 at P and R=-1.0 at T.
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above P and*
R=0, -1/8,-1/4,-1/2,...,-3/4 below P and -5/4,-3/2,... below T.



Pick initial R=KE/PE value
(here R=-3/8) Draw **ε−vector**
from focus F to R-point
and beyond to 2nd focus F′



(From Lecture 26 p. 64)

$$R = \frac{\text{Initial KE}}{\text{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

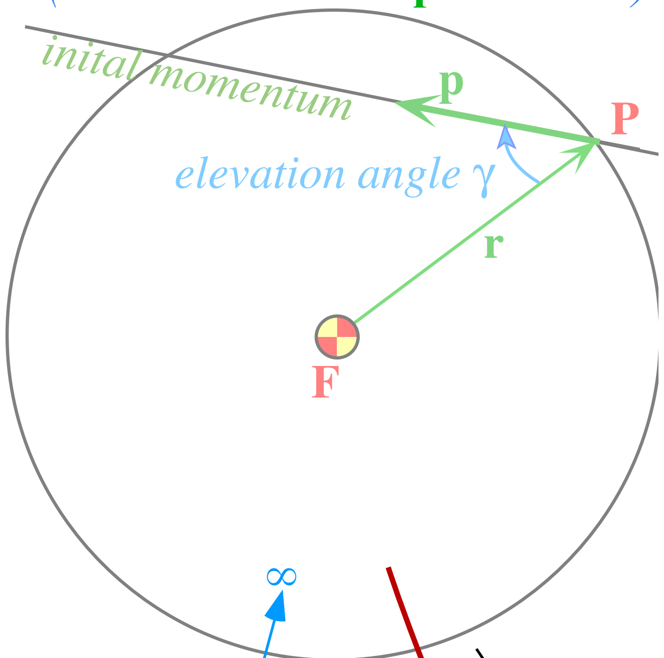
$$= \pm\left(\frac{\text{Initial velocity}}{\text{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

focus F and 2nd focus F′ allow final
construction of orbital trajectory.
Here it is an R=-3/8 ellipse.

(Detailed Analytic geometry of ***ε-vector*** *follows.)*

ε-vector and Coulomb orbit construction steps

Pick launch point P
(radius vector **r** *)*
and elevation angle γ *from radius*
(momentum initial **p** *direction)*



Copy F-center circle around launch point P
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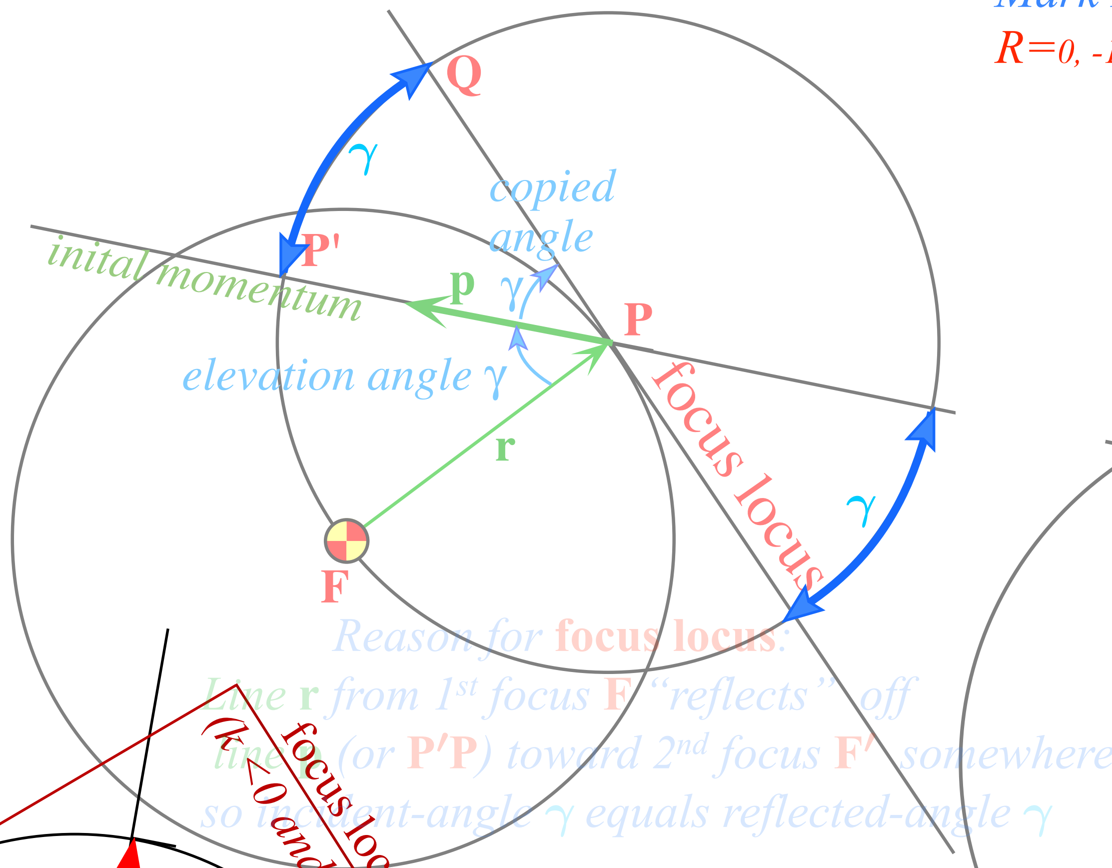
Extend resulting line QPQ′ *to make* **focus locus**



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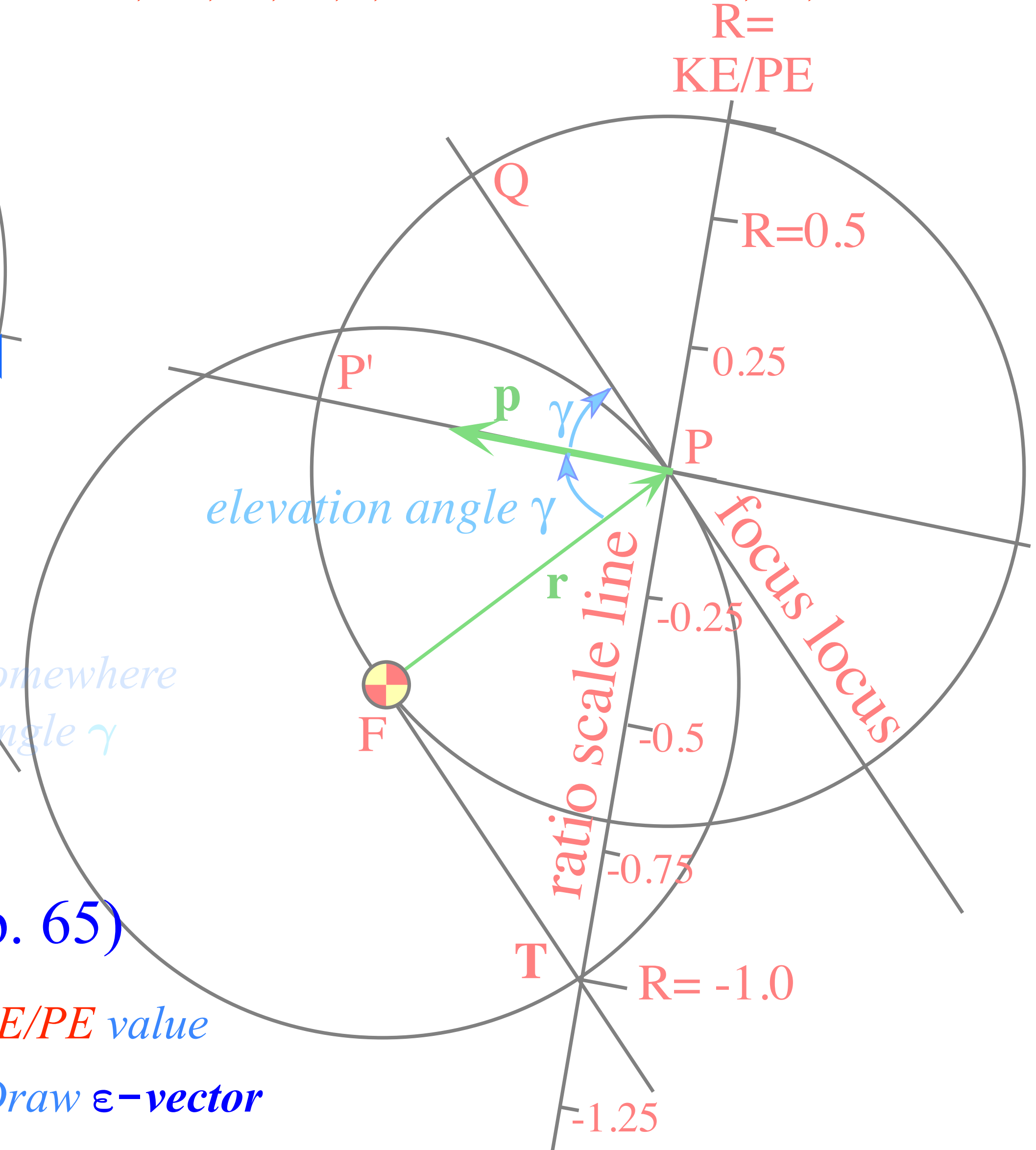
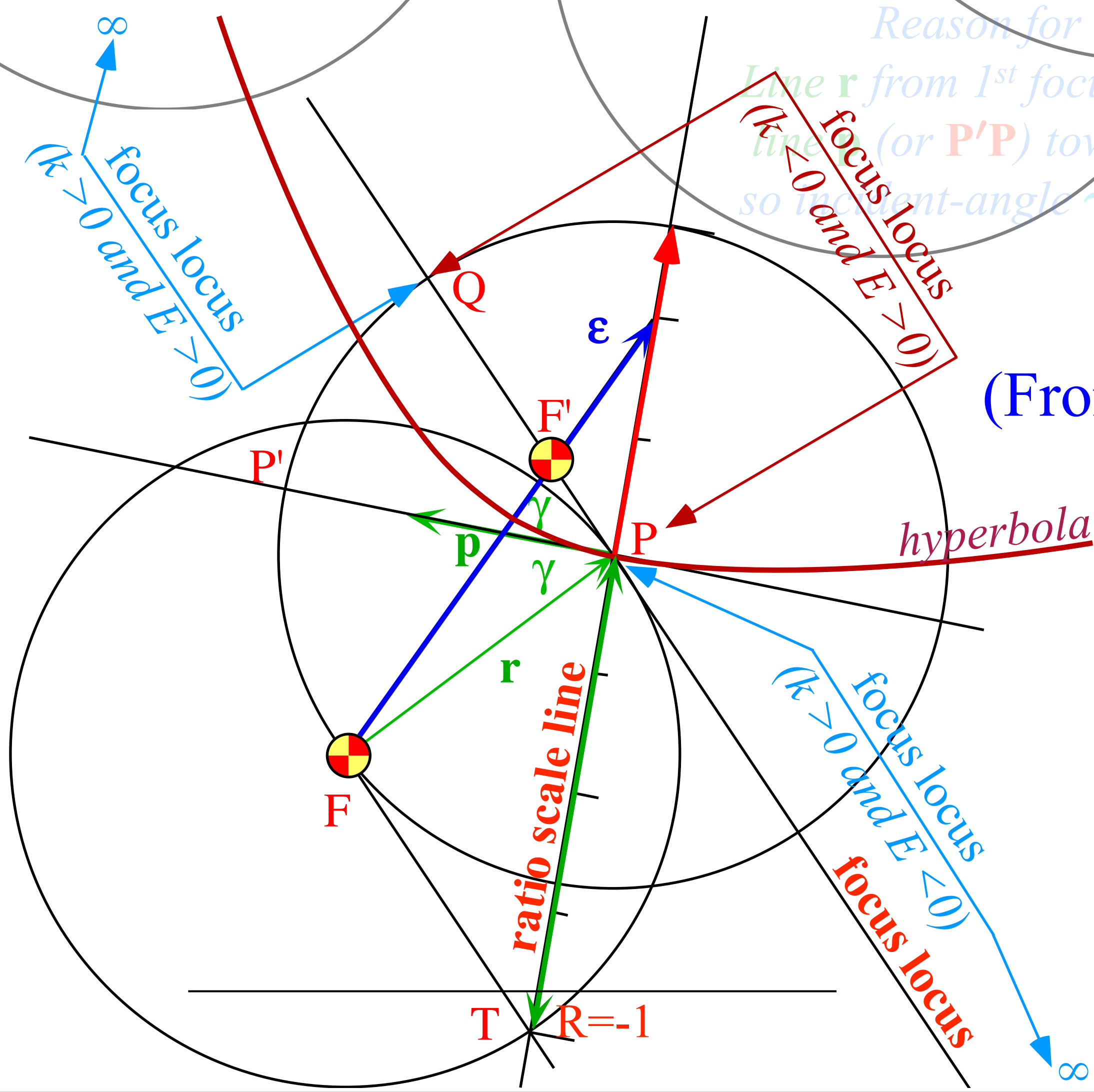
R=0, -1/8,-1/4,-1/2,...,-3/4 below P *and -5/4,-3/2,... below* T.





(From Lecture 26 p. 65)

Pick initial R=KE/PE value
(here R=+1/2) Draw ***ε−vector***
from focus F *to R-point*
(Here it intersects 2nd focus F′

> *focus* F *and 2nd focus* F′ *allow final*
> *construction of orbital trajectory.*
> *Here it is an R=+1/2 hyperbola.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

(Detailed Analytic geometry of ***ε-vector*** *follows.)*

Analytic geometry derivation of ε-constructions

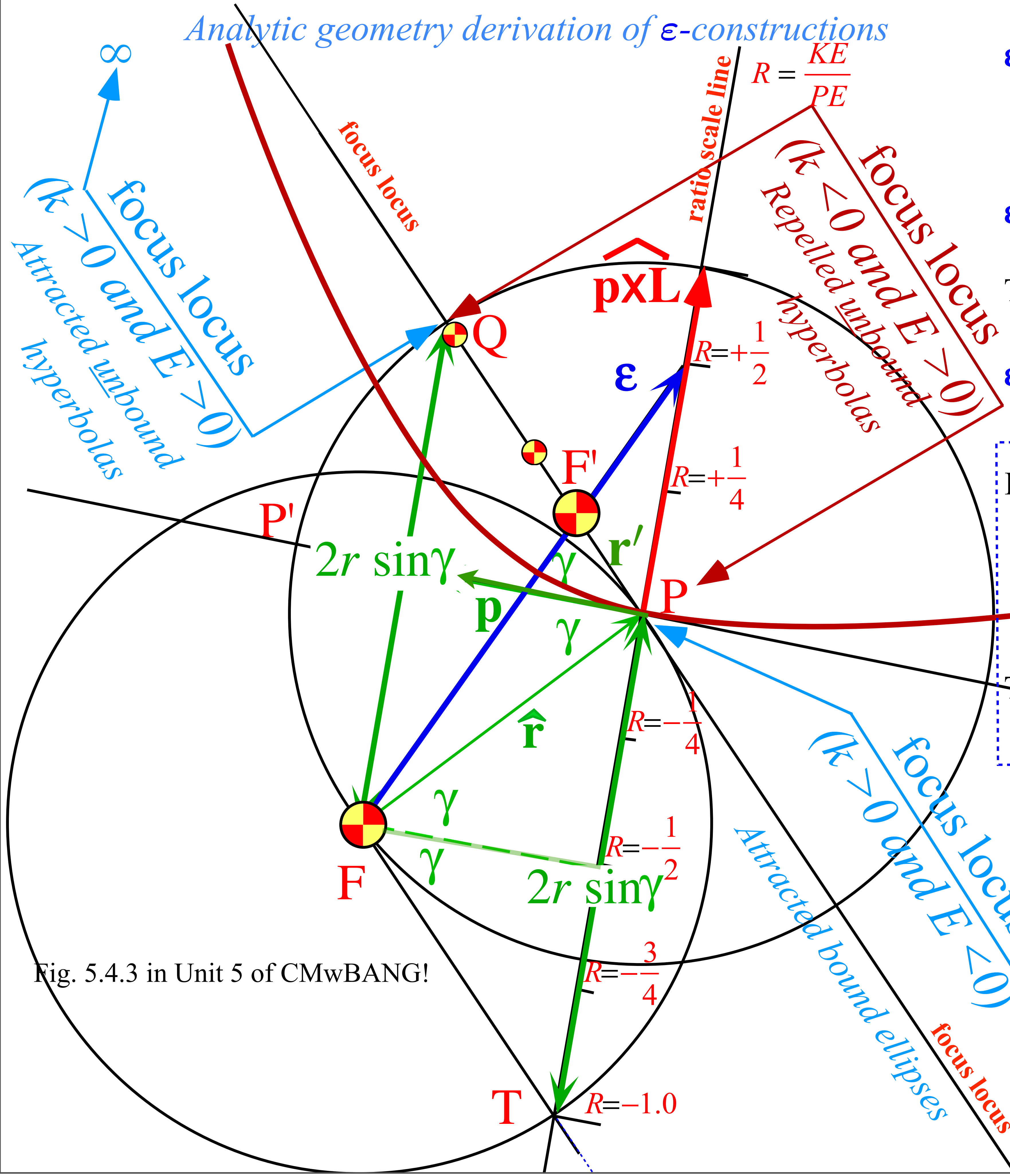


Fig. 5.4.3 in Unit 5 of CMwBANG!

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

For: $\gamma = 45°$ and: $R = +\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

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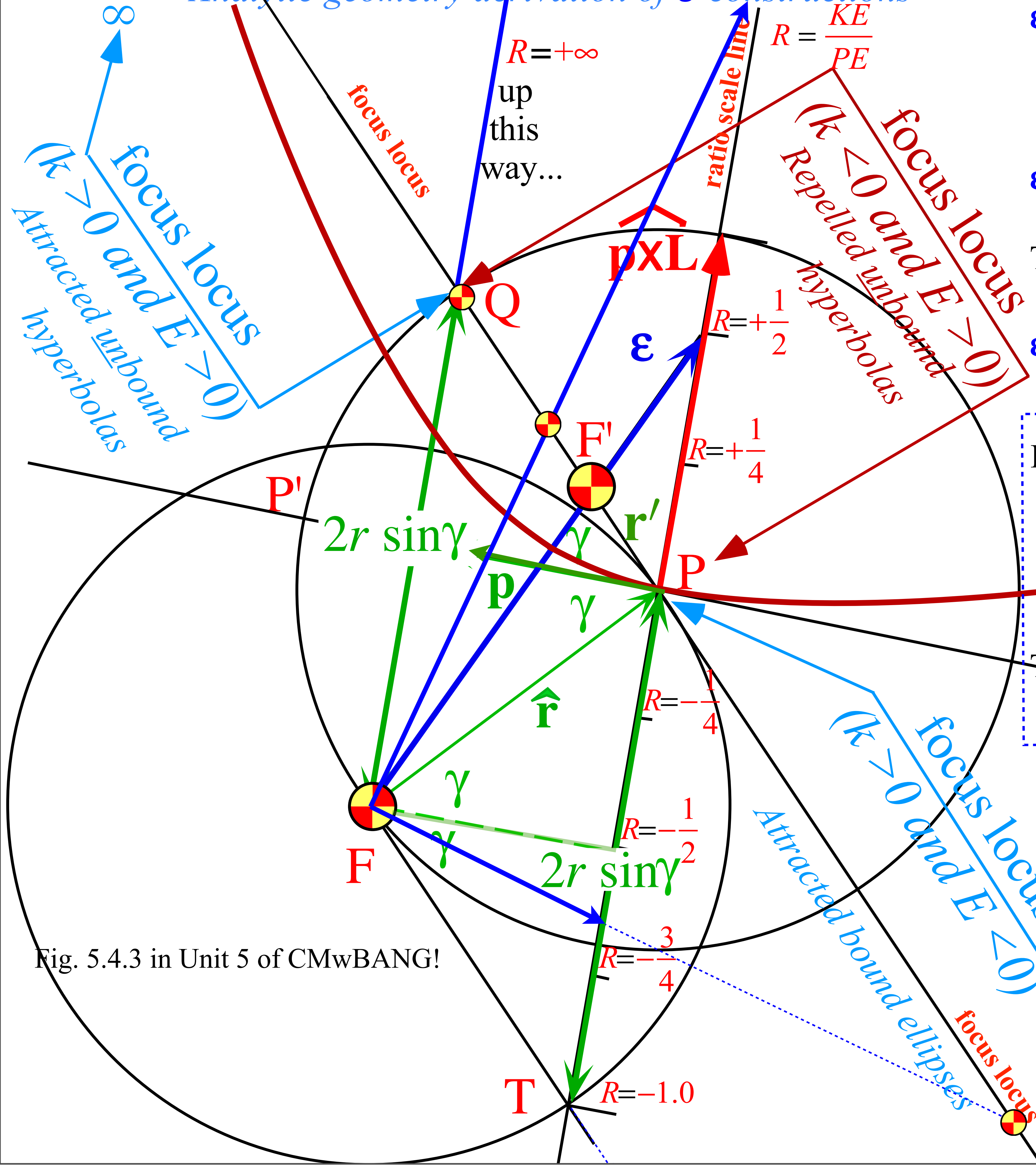

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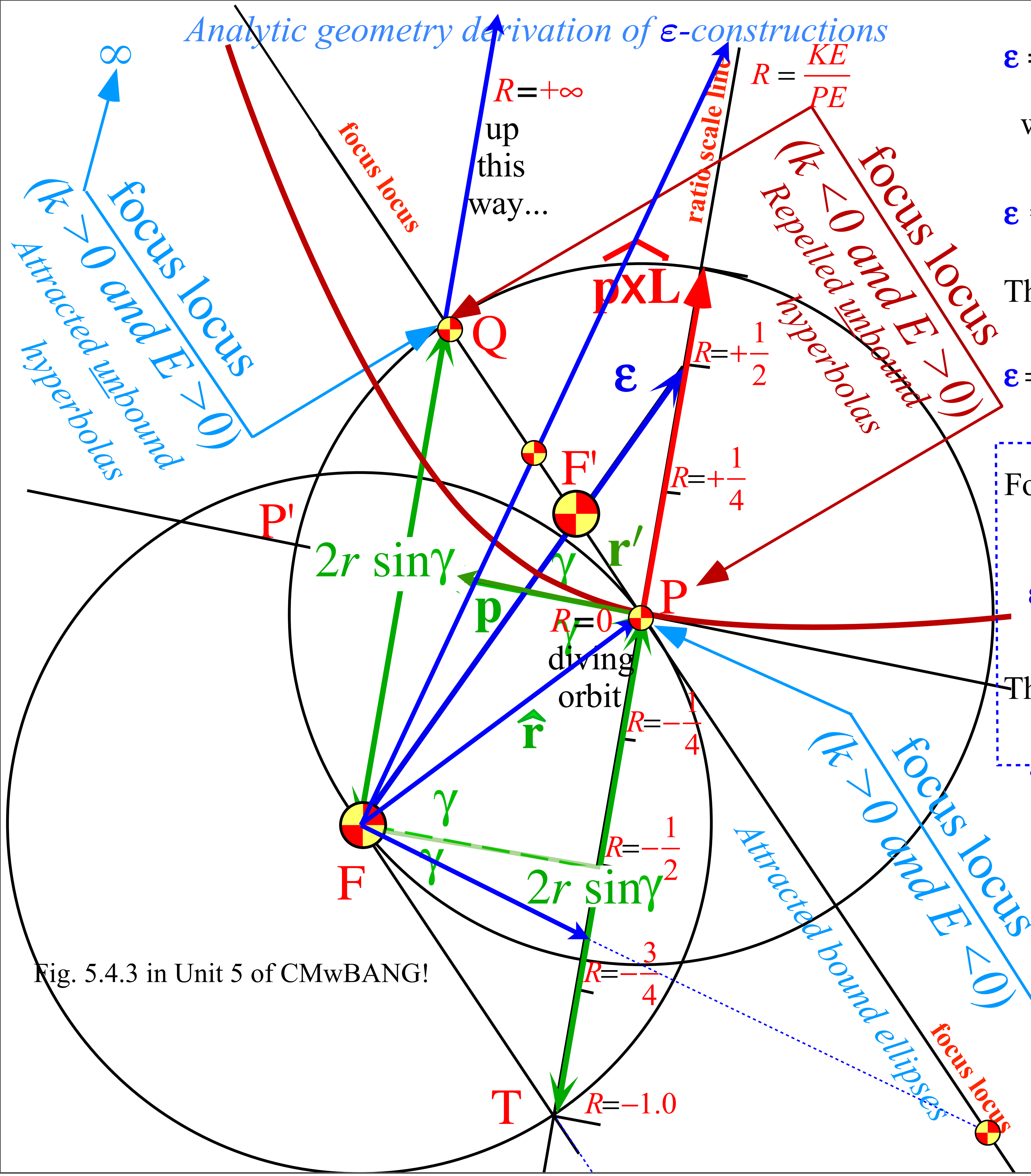

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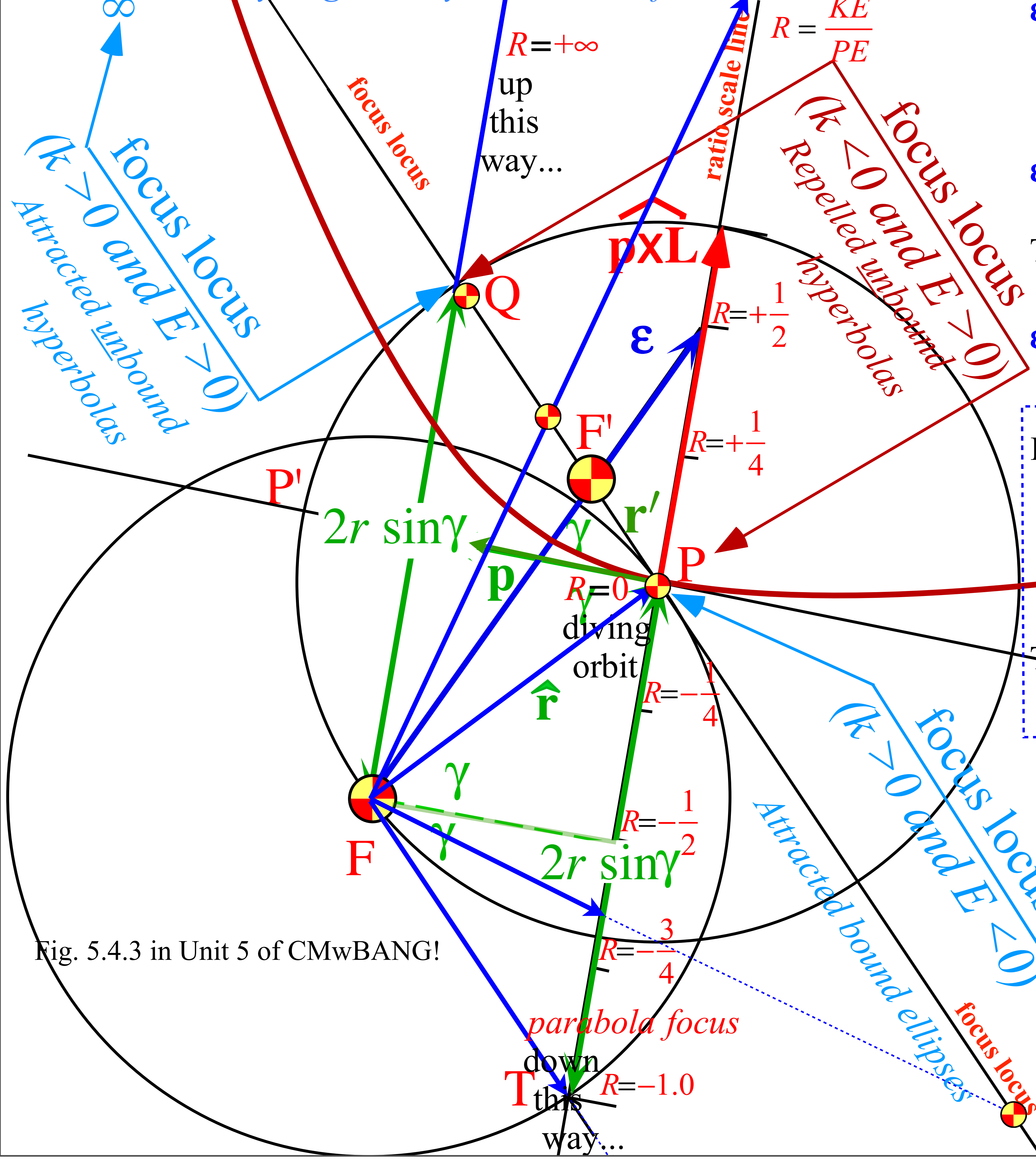

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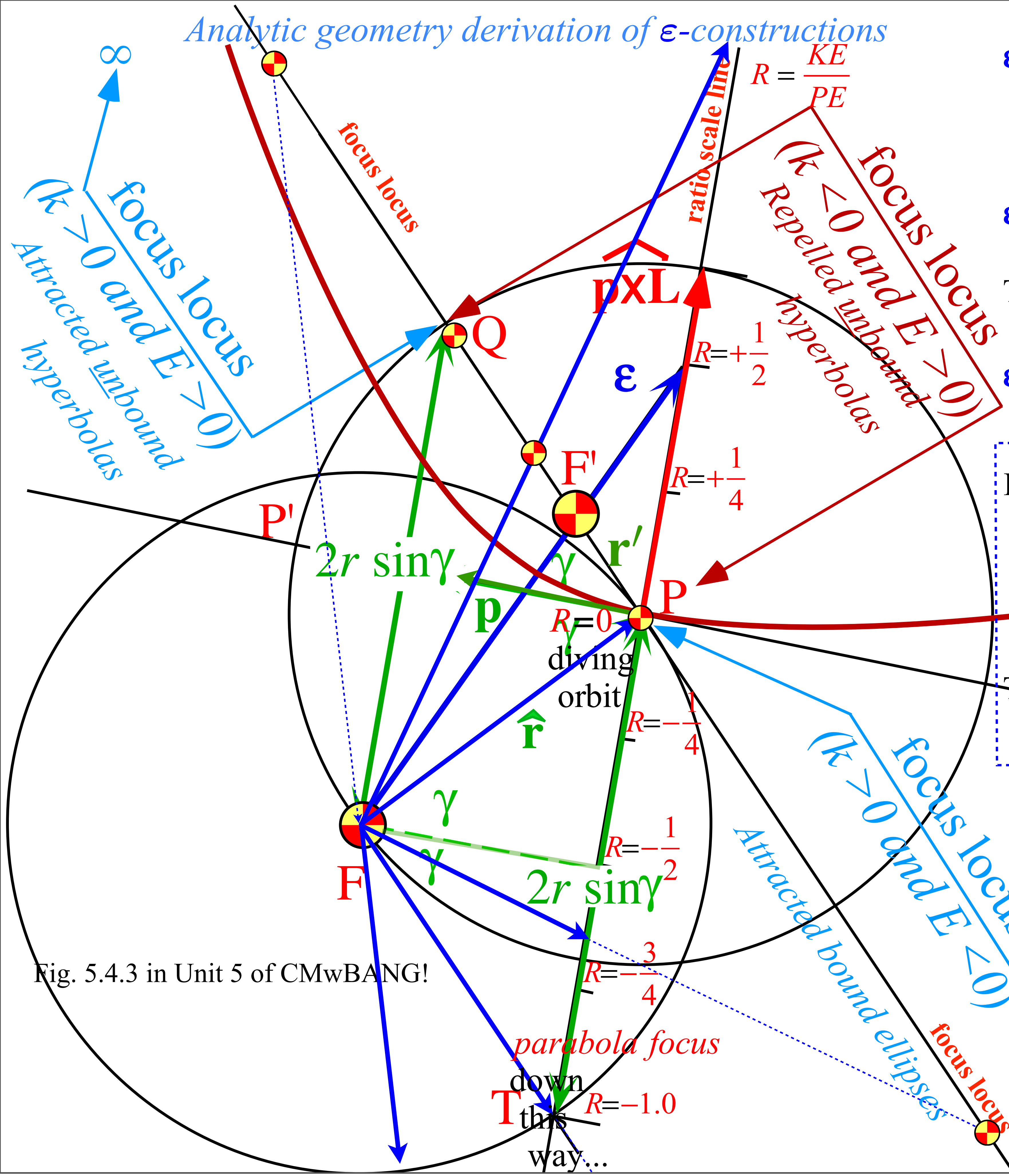

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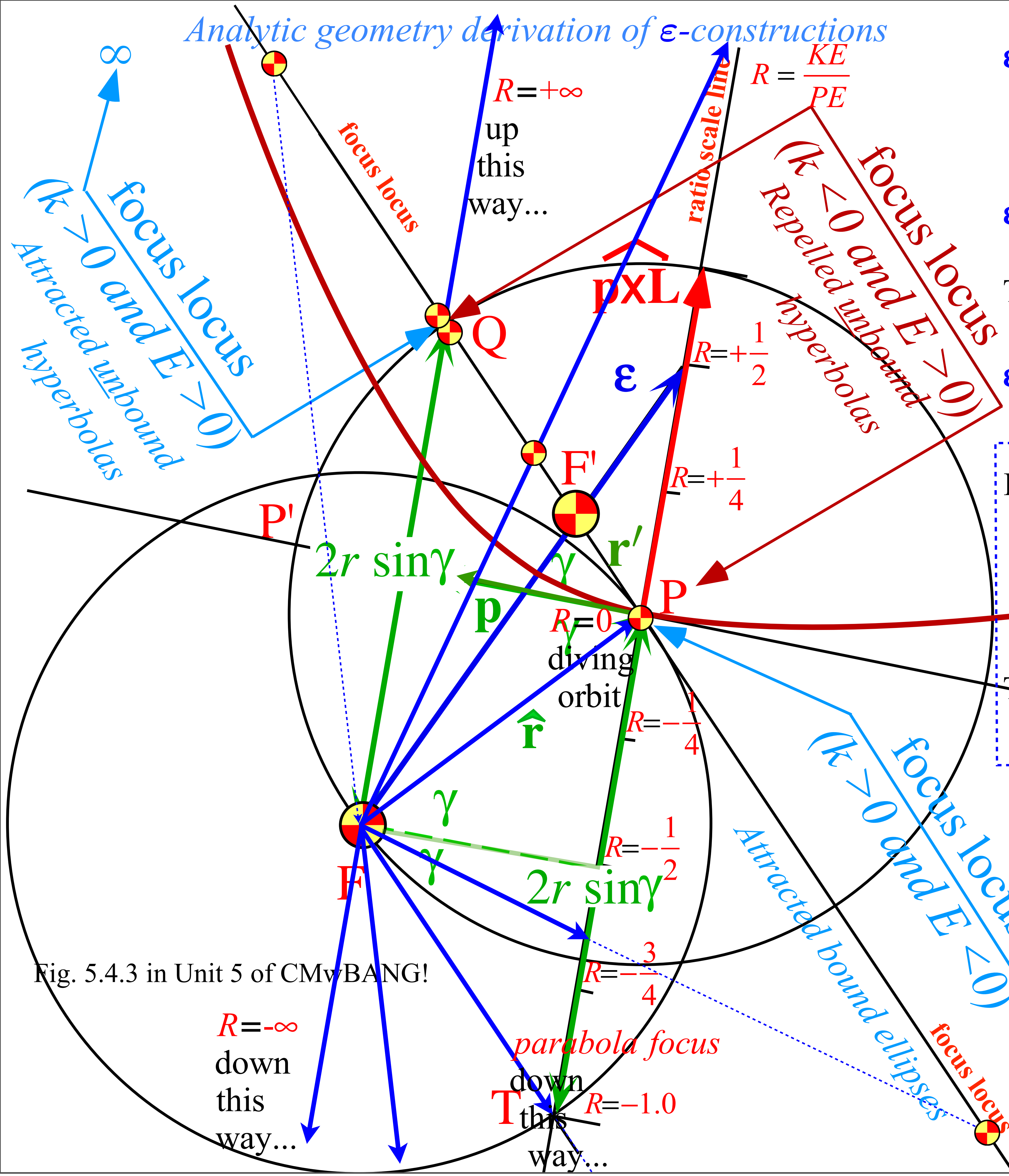

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Graphical $\boldsymbol{\varepsilon}$*-development of orbits*

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Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R=\frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$

$$=1-\frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon<1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=-\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)<0$$

(or: $-R^2>R$)
(or: $0>R>-1$)

$$=1+\frac{b^2}{a^2} \quad \text{for hyperbola} \quad (\varepsilon>1) \quad \text{where:} \quad 4R(R+1)\sin^2\gamma=+\frac{b^2}{a^2}=\varepsilon^2-1 \quad \text{implying:} \quad R(R+1)>0$$

(or: $-R^2<R$)
(or: $0<R<-1$)

Total $\frac{-k}{2a}=E=$ *energy* $= KE+PE$ relates ratio $R=\frac{KE}{PE}$ to individual radii a, b, and λ.

$$\frac{-k}{2a}=E=KE+PE=R\cdot PE+PE=(R+1)PE=(R+1)\frac{-k}{r} \quad \text{or:} \quad \frac{1}{2a}=(R+1)\frac{1}{r}=(R+1)$$

$$a=\frac{r}{2(R+1)}=\left(\frac{1}{2(R+1)} \text{ assuming \textbf{unit} initial radius } (r\equiv 1).\right)$$

$$4R(R+1)\sin^2\gamma=\mp\frac{b^2}{a^2} \quad \text{implies:} \quad 2\sqrt{\mp R(R+1)}\sin\gamma=\frac{b}{a} \quad \text{or:} \quad b=2a\sqrt{\mp R(R+1)}\sin\gamma$$

$$b=r\sqrt{\frac{\mp R}{R+1}}\sin\gamma\left(=\sqrt{\frac{\mp R}{R+1}}\sin\gamma \text{ assuming \textbf{unit} initial radius } (r\equiv 1)\right)$$

Latus radius is similarly related:

$$\lambda=\frac{b^2}{a}=\mp 2rR\sin^2\gamma$$

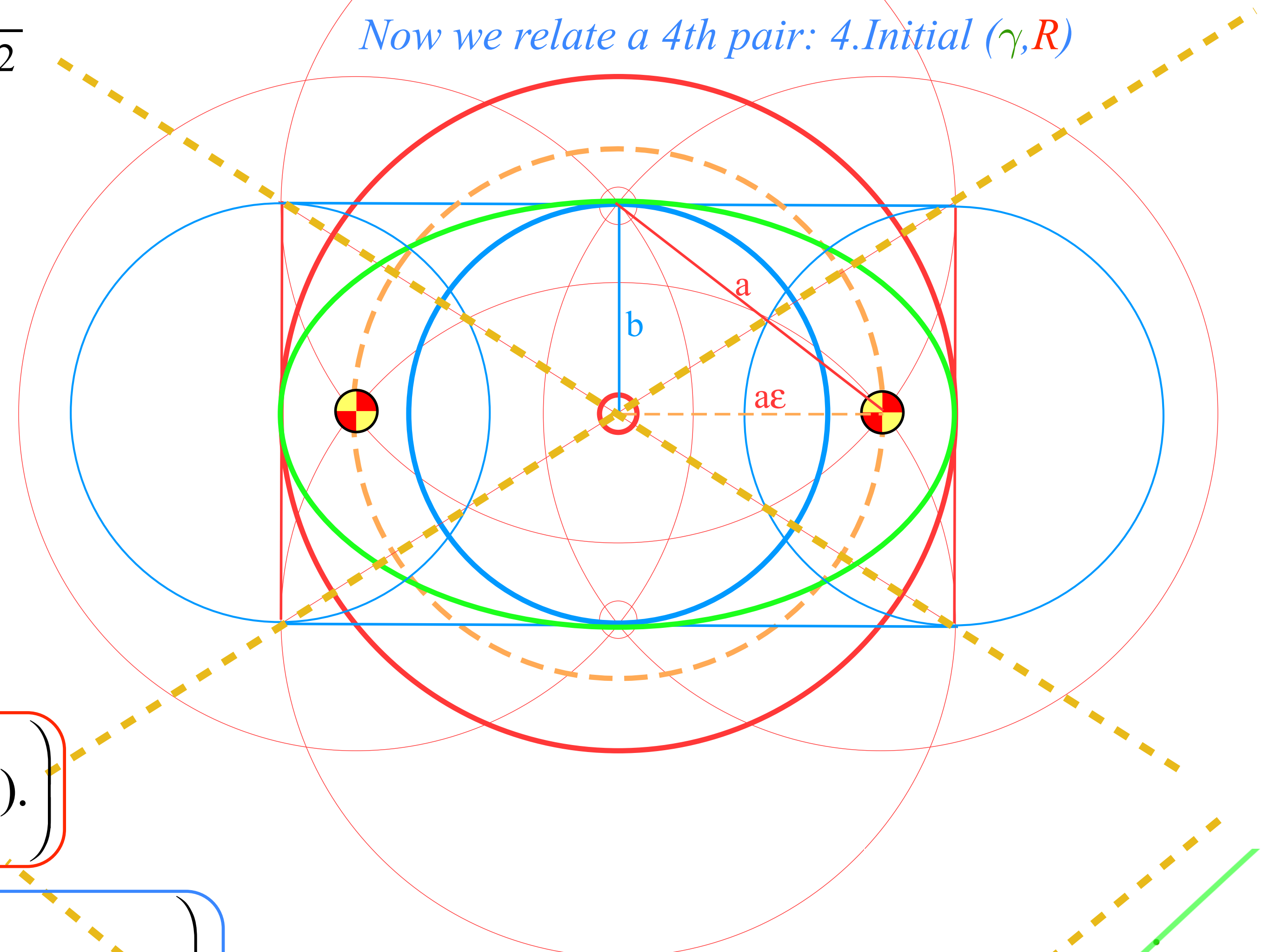
(Review of Lect. 26 p.107-108)

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:
1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)
Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R=\frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1+4R(R+1)\sin^2\gamma$$
$$= 1-\frac{b^2}{a^2}\text{ellipse}(\varepsilon<1) \qquad 4R(R+1)\sin^2\gamma = -\frac{b^2}{a^2}$$
$$= 1+\frac{b^2}{a^2}\text{hyperbola }(\varepsilon>1) \quad 4R(R+1)\sin^2\gamma = +\frac{b^2}{a^2}$$



$$a=\frac{r}{2(R+1)}=\left(\frac{1}{2(R+1)}\text{ assuming **unit** initial radius }(r\equiv 1).\right)$$

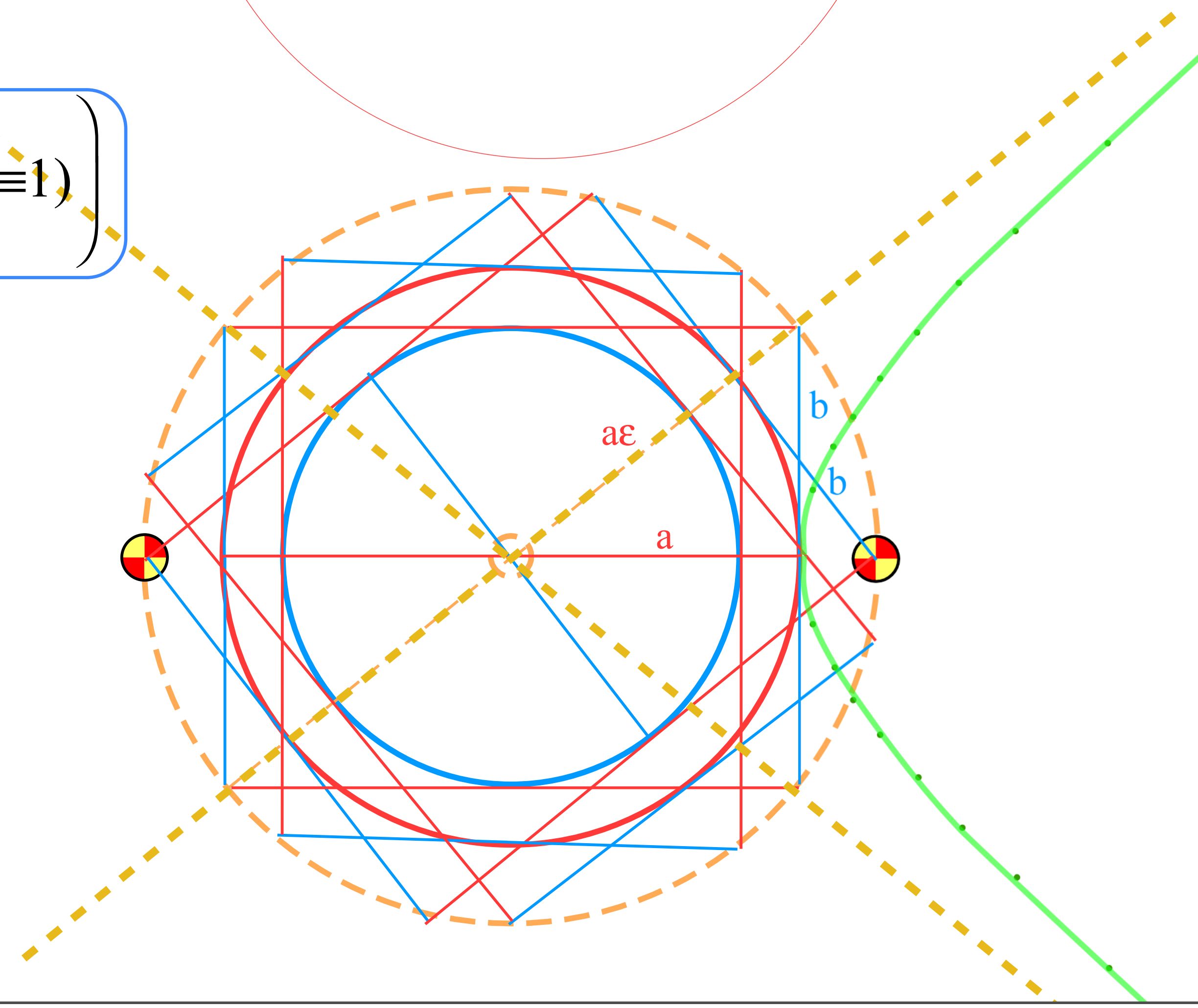
$$b=r\sqrt{\frac{\mp R}{R+1}}\sin\gamma\left(=\sqrt{\frac{\mp R}{R+1}}\sin\gamma\text{ assuming **unit** initial radius }(r\equiv 1)\right)$$

Latus radius is similarly related:

$$\lambda=\frac{b^2}{a}=\mp 2rR\sin^2\gamma$$

From ε^2 result (at top):

$$\frac{b}{a}=2\sqrt{\mp R(R+1)}\sin\gamma=\sqrt{\pm(1-\varepsilon^2)}$$



(Review of Lect. 26 p.107-108)

Review of ***Eccentricity vector*** $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Analytic geometry derivation of $\boldsymbol{\varepsilon}$*-construction*

Connection formulas for (a,b) and (ε,λ) *with* (γ,R)

Detailed ruler & compass construction of $\boldsymbol{\varepsilon}$***-vector*** *and orbits*

➔ *(R=-0.375 elliptic orbit)*

(R=+0.5 hyperbolic orbit)

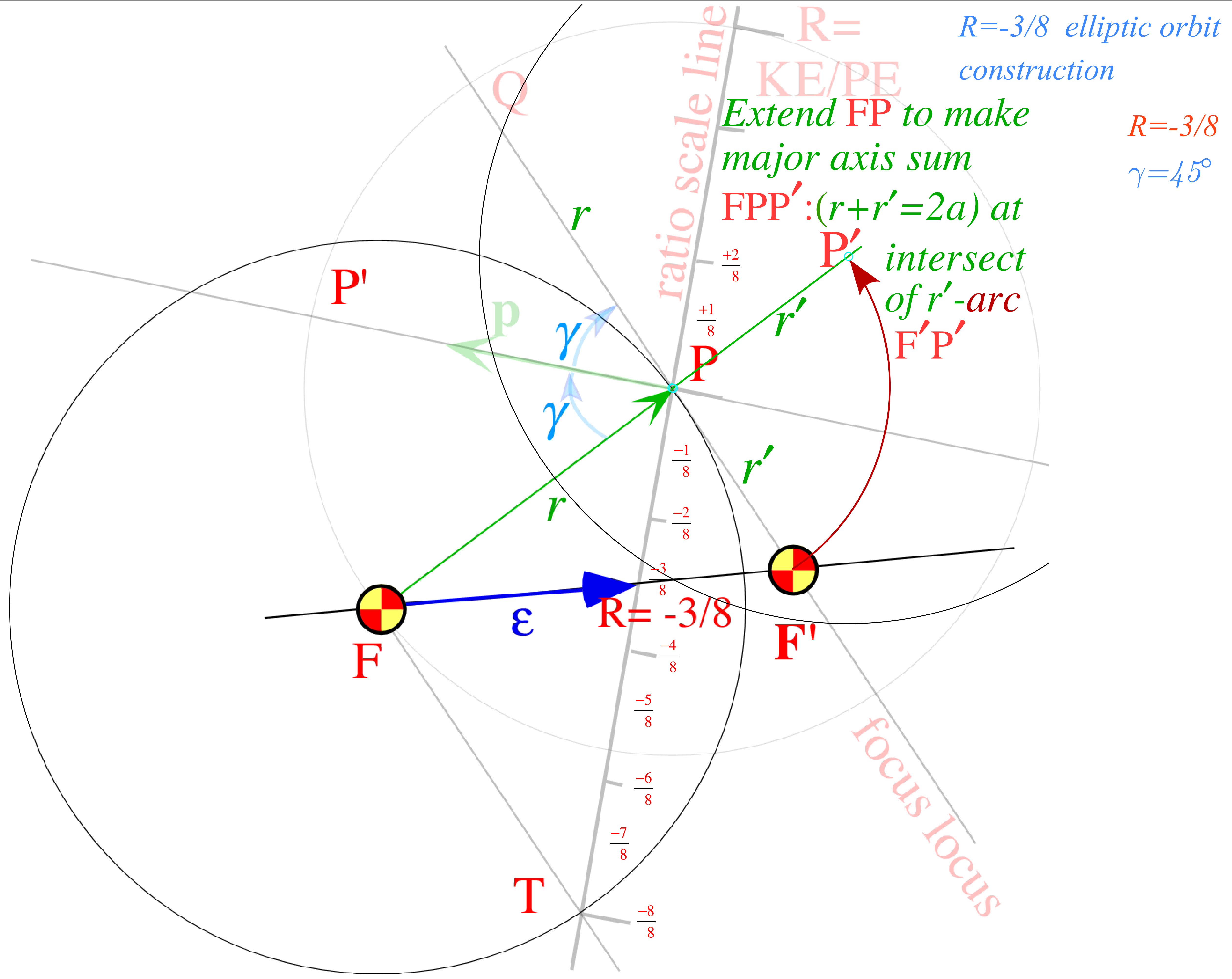
Properties of Coulomb trajectory families and envelopes

Graphical $\boldsymbol{\varepsilon}$*-development of orbits*

Launch angle fixed-Varied launch energy

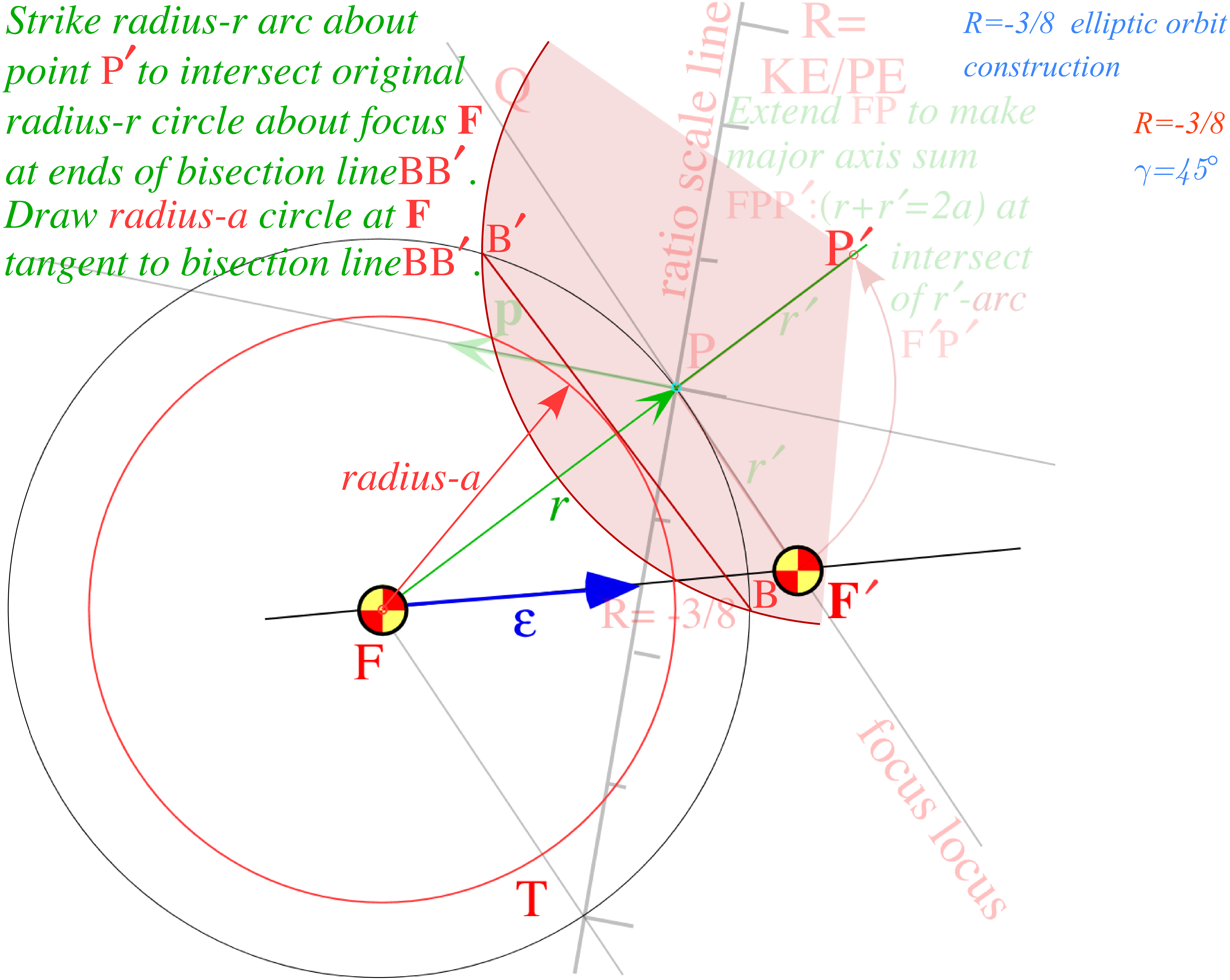
Launch energy fixed-Varied launch angle

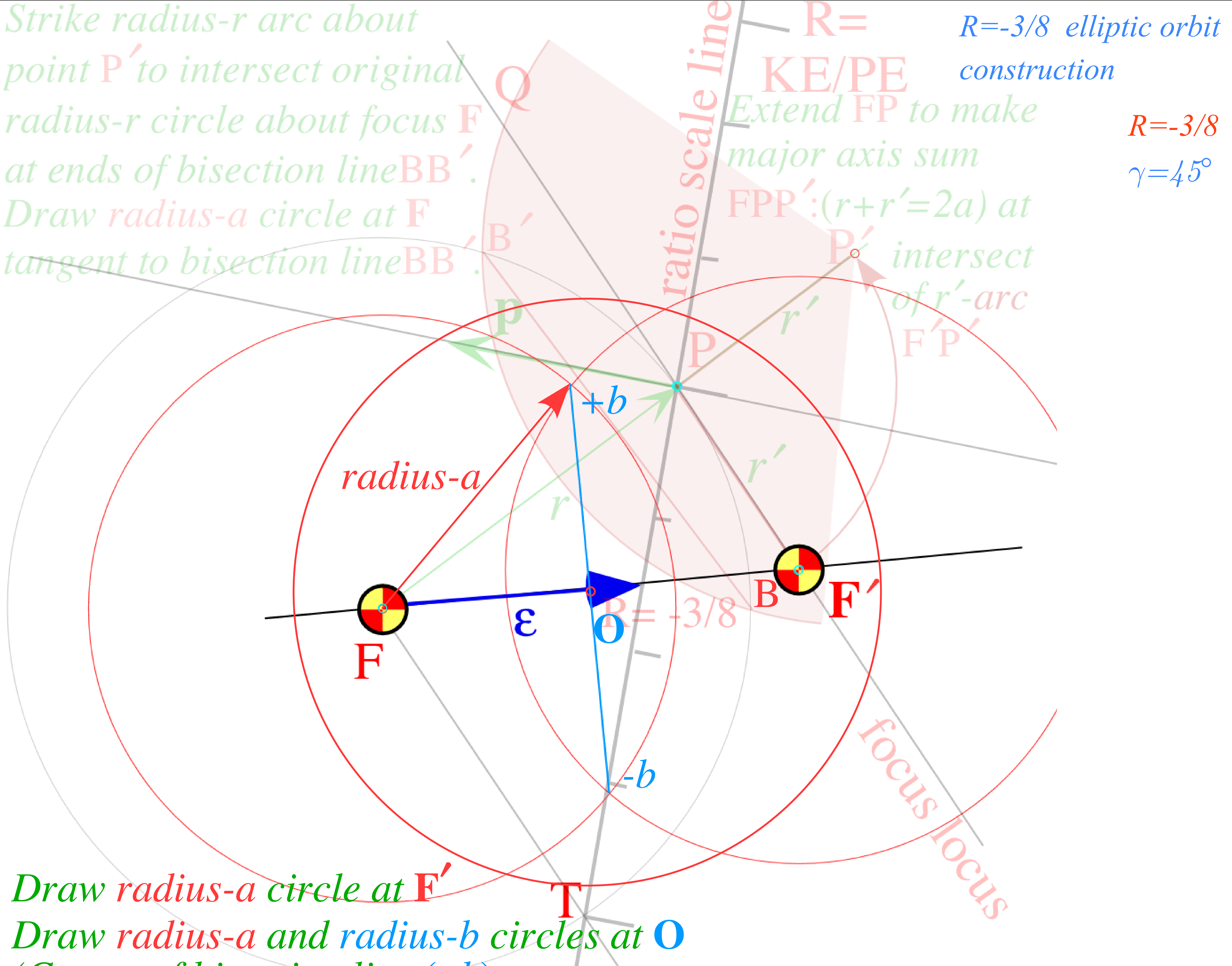
Launch optimization and orbit family envelopes


R=-3/8 elliptic orbit
construction
R=-3/8
γ=45°
R=
KE/PE
ratio scale line
Extend FP to make
major axis sum
FPP′:(r+r′=2a) at
P′
intersect
of r′-arc
F′P′
Q
r
P'
p
γ
γ
P
r′
r′
r
+2/8
+1/8
−1/8
−2/8
−3/8
R= -3/8
−4/8
−5/8
−6/8
−7/8
−8/8
ε
F
F'
T
focus locus

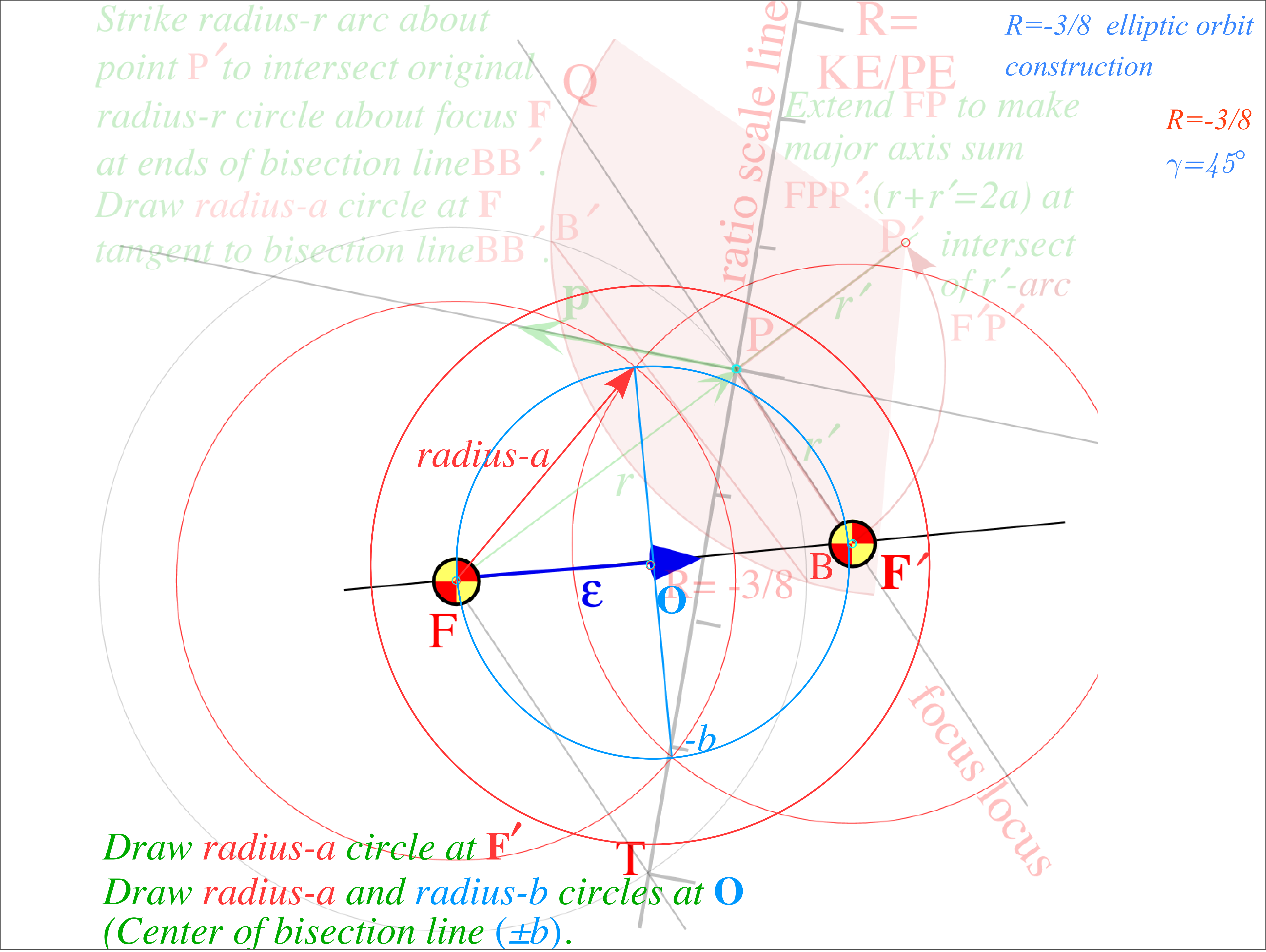
Strike radius-r arc about point P′ to intersect original radius-r circle about focus **F** *at ends of bisection line* BB′.
Draw *radius-a* *circle at* **F** *tangent to bisection line* BB′.

R=-3/8 elliptic orbit construction





Draw radius-a circle at **F′**
Draw radius-a and radius-b circles at **O**
(Center of bisection line (±b).



Strike radius-r arc about point P′ to intersect original radius-r circle about focus F at ends of bisection line BB′.
Draw radius-a circle at F tangent to bisection line BB′.
R= KE/PE
ratio scale line
Extend FP to make major axis sum FPP′:(r+r′=2a) at P′ intersect of r′-arc F′P′
R=-3/8 elliptic orbit construction
R=-3/8
γ=45°
Q
B′
p
P
r′
r
radius-a
ε
O
R= -3/8
B
F′
F
-b
T
focus locus
Draw radius-a circle at F′
Draw radius-a and radius-b circles at O
(Center of bisection line (±b).

R=-3/8 elliptic orbit construction

R=-3/8

$\gamma=45°$

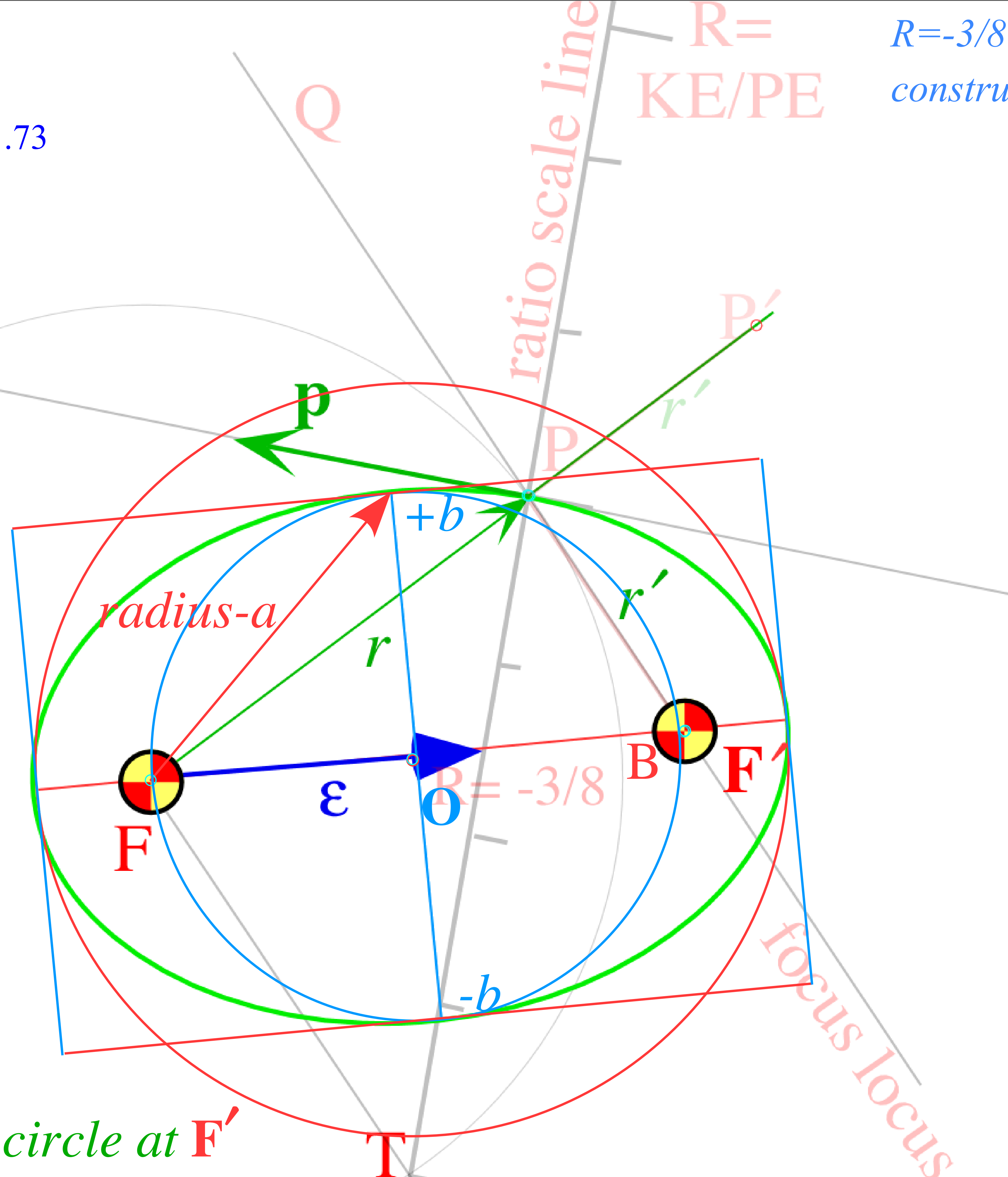
$$\varepsilon=\sqrt{1+4R(R+1)\sin^2\gamma}=\frac{\sqrt{34}}{8}=.73$$

$$a=\frac{1}{2(R+1)}=\frac{4}{5}$$

$$b=\sqrt{\frac{R}{R+1}}\sin\gamma=\sqrt{\frac{3}{10}}=.54$$

$$\lambda=\frac{b^2}{a}=2R\sin^2\gamma=\frac{3}{8}=.375$$

$$\frac{b}{a}=2\sqrt{R(R+1)}\sin\gamma=\tan 34°$$



Draw radius-a circle at **F′**

Draw radius-a and radius-b circles at **O**

(Center of bisection line (±b).) Do (a,b)-ellipse construction.

Review of ***Eccentricity vector ε*** *and (ε,λ)-geometry of orbital mechanics*

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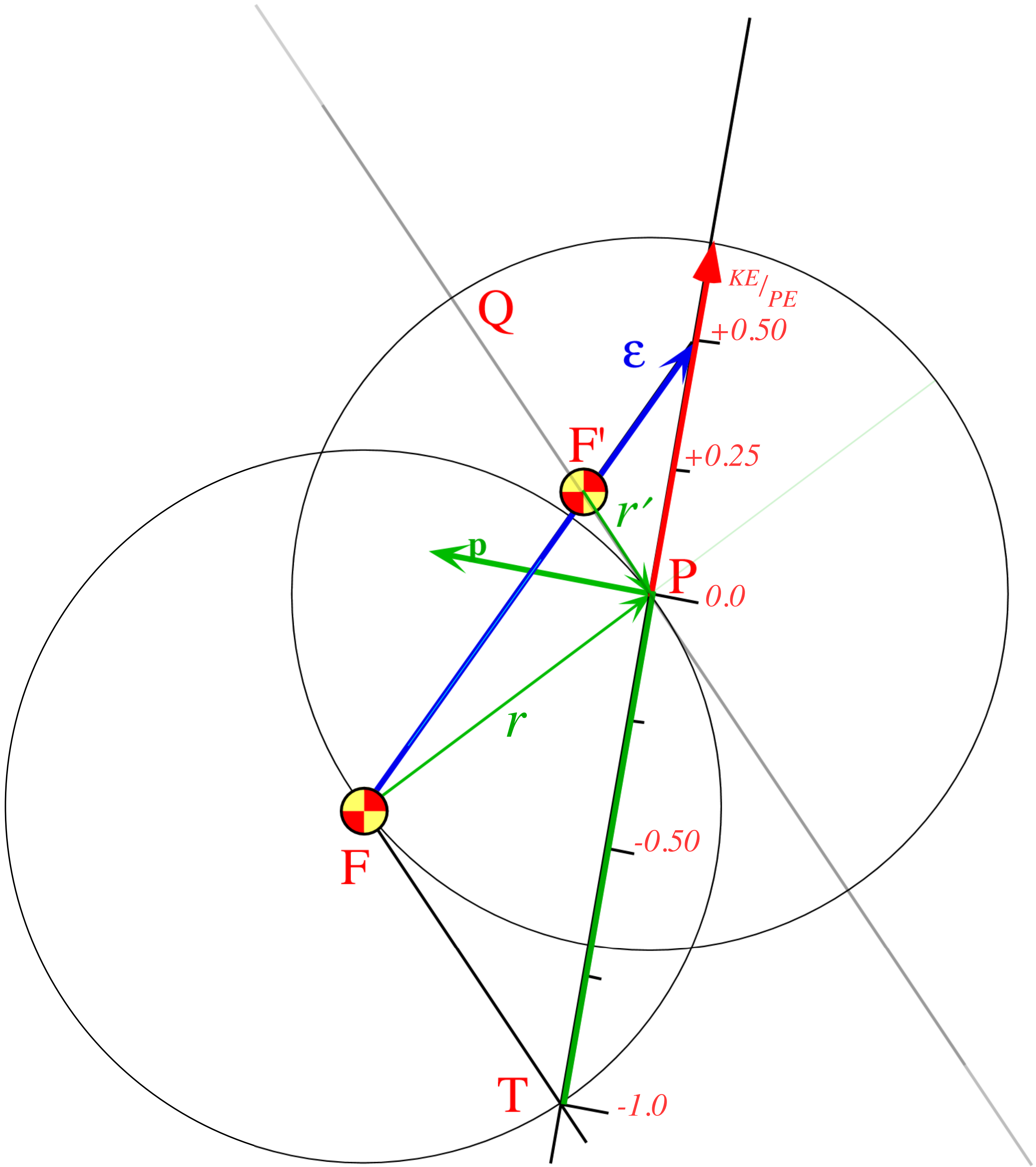
Major radius a is half of difference (r-r′)/2=a
Major diameter 2a needs to be centered on F-F′ focal axis

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



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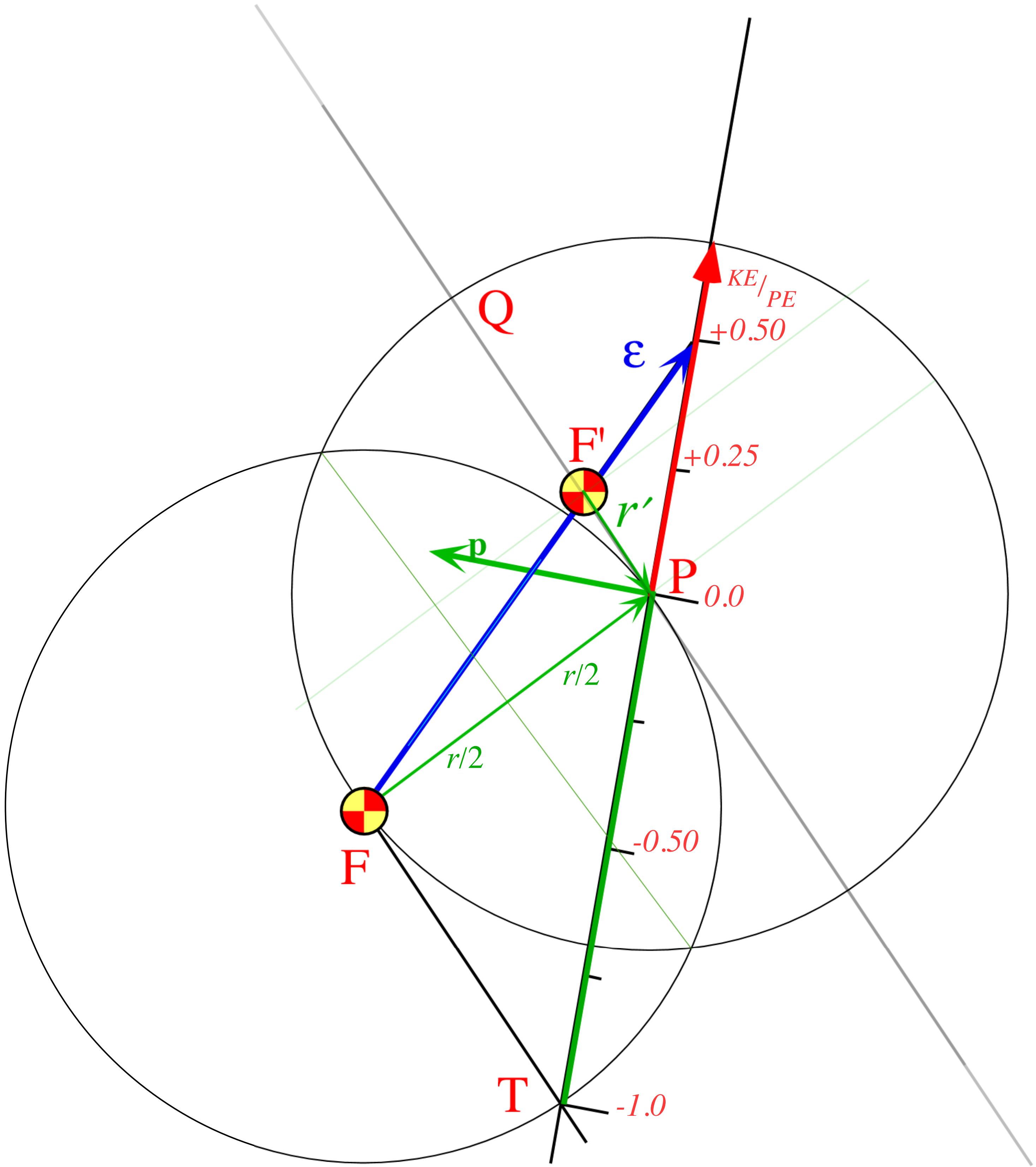
1. Bisect F-P radius r using F-P circle intersections to define r/2 sections.

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



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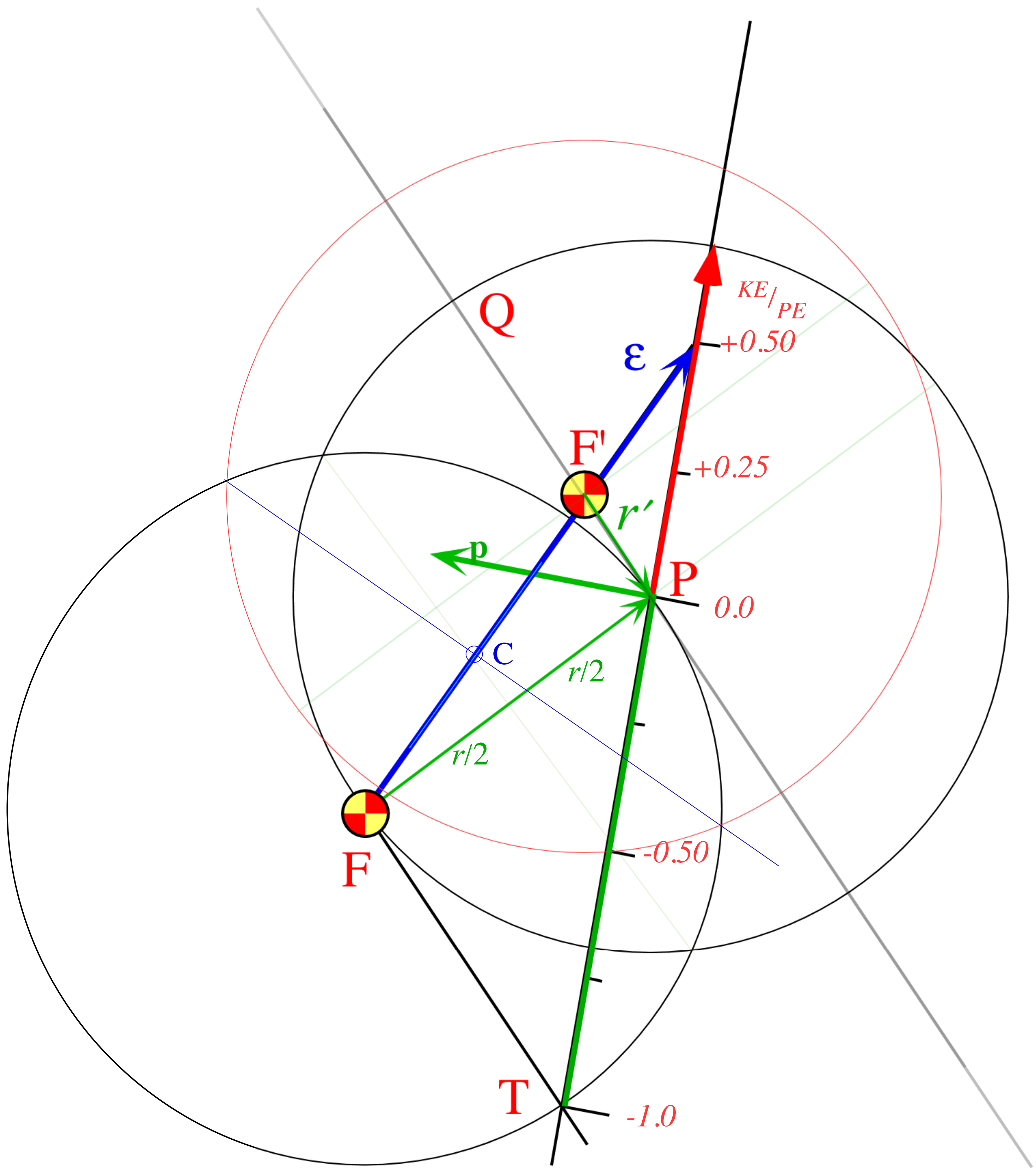
2. Bisect F-F′ focal axis using F-F′ circle intersections to locate orbit center C.

R=+1/2 hyperbolic orbit construction

R=+1/2

$\gamma=45°$



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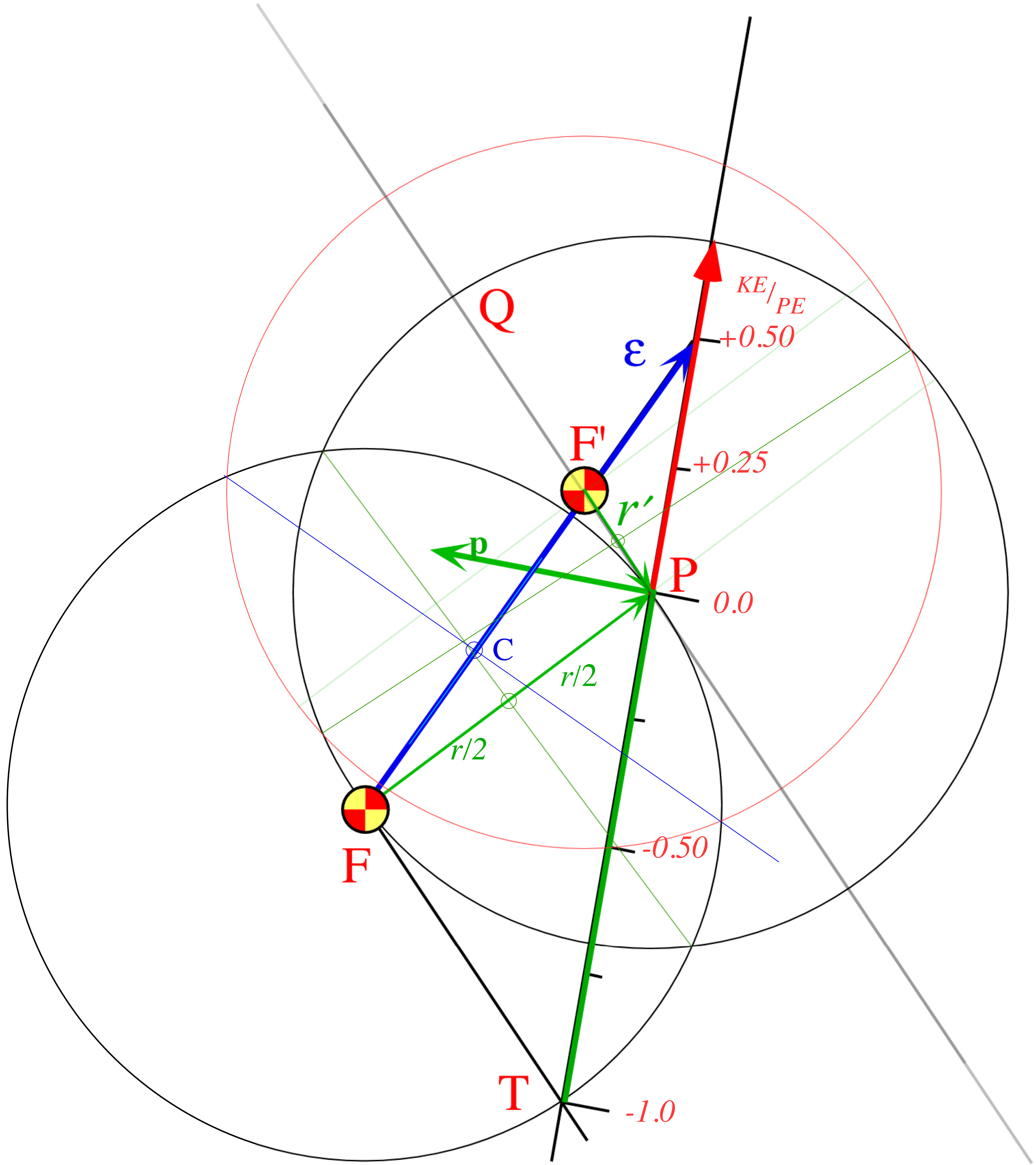
3. Bisect F′-P radius r′ using F′-P circle intersections.

R=+1/2

γ=45°



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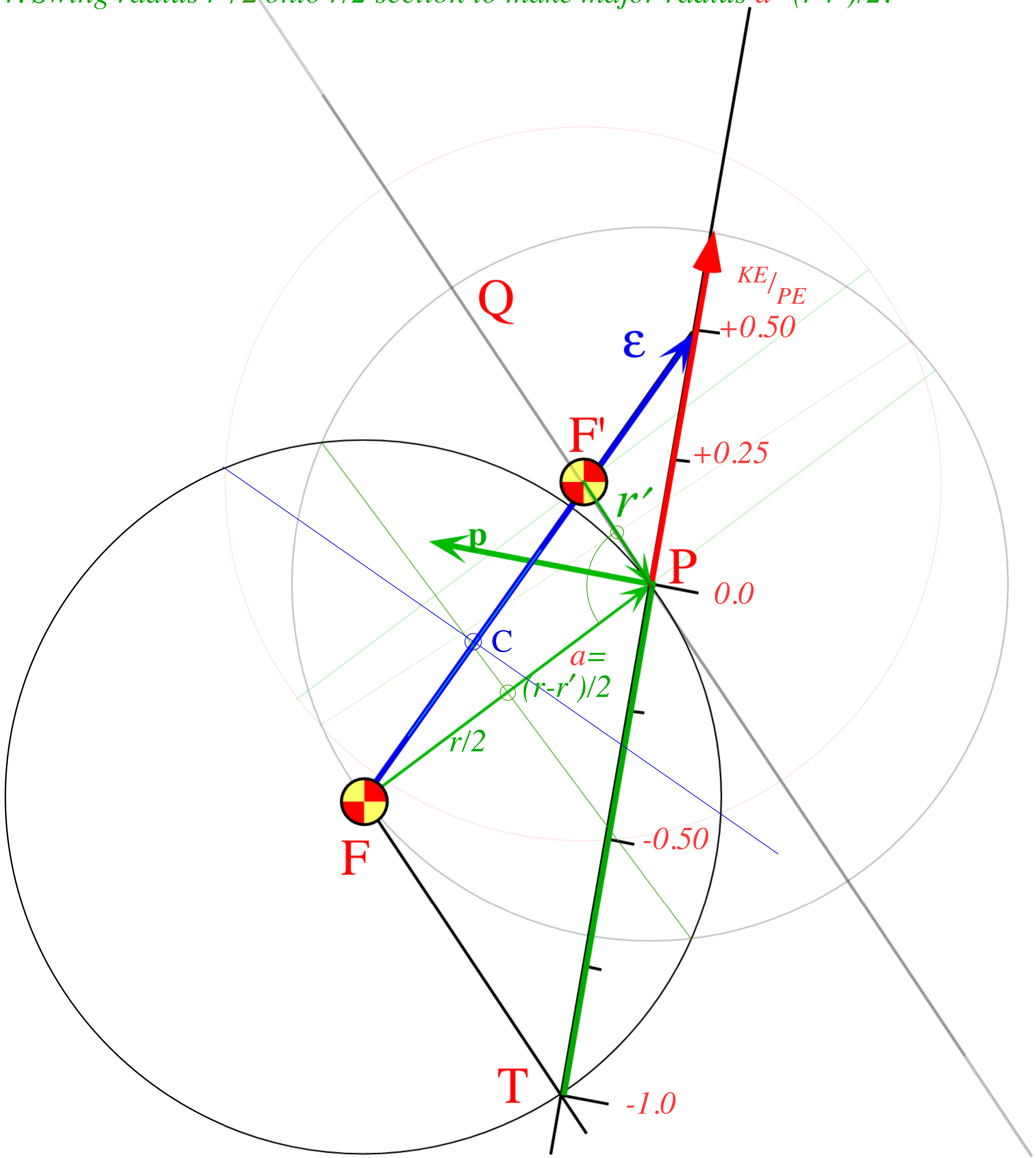
3. Bisect F′-P radius r′ using F′-P circle intersections.
4. Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.

R=+1/2

γ=45°



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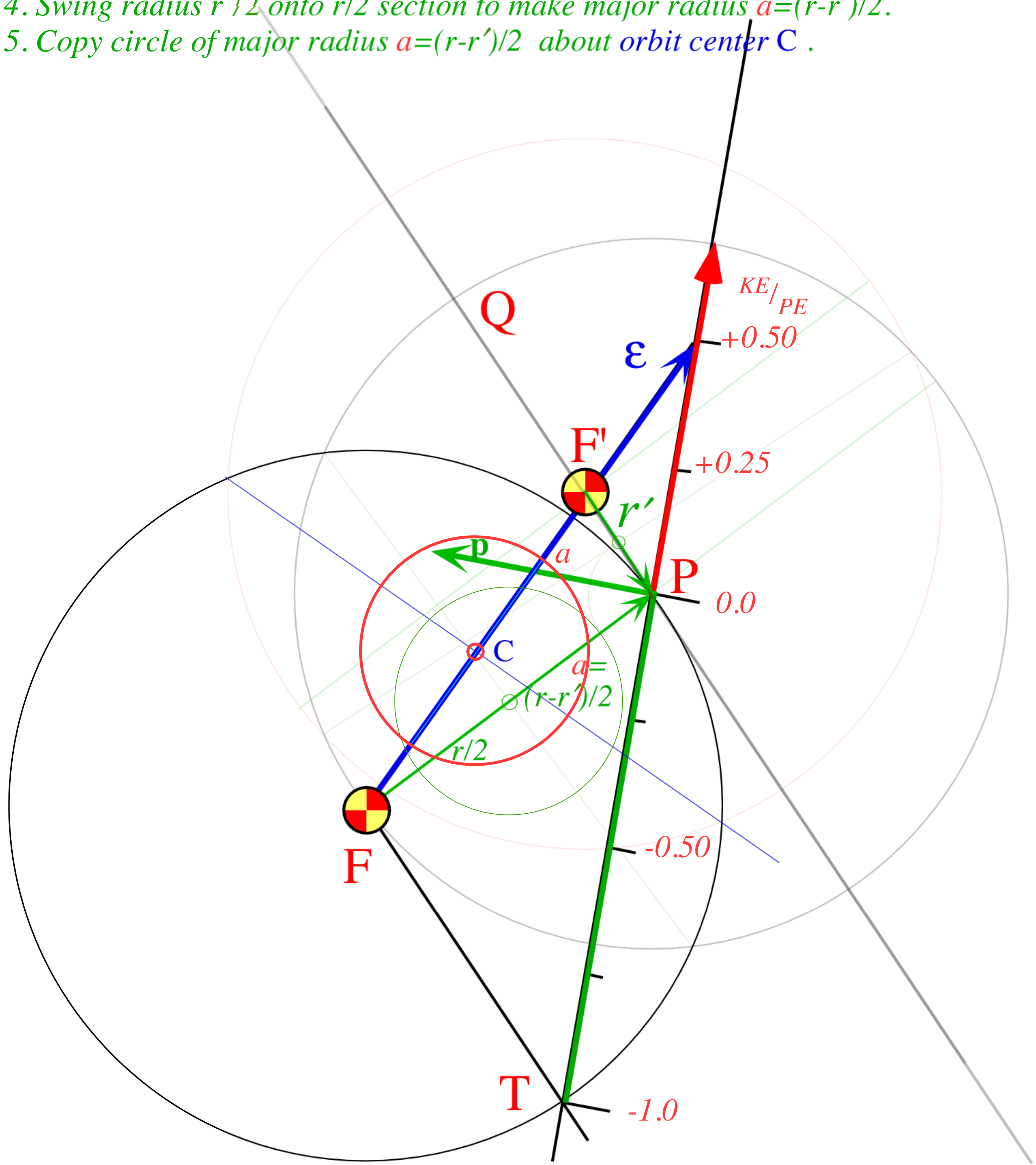
3. Bisect F′-P radius r′ using F′-P circle intersections.
4. Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.
5. Copy circle of major radius a=(r-r′)/2 about orbit center C .

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



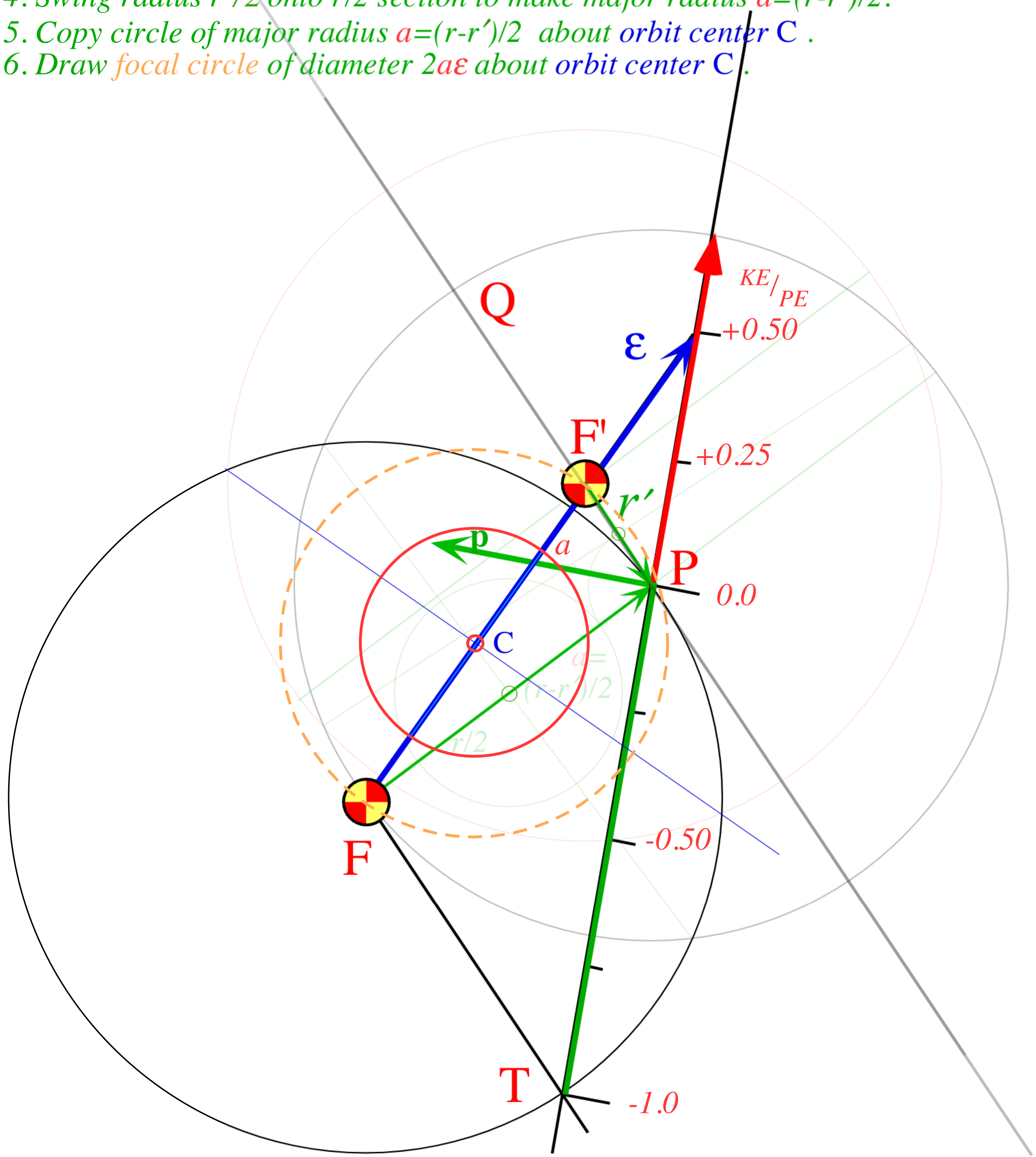
Major diameter 2a is difference (r-r′=2a).
Major radius a is half of difference (r-r′)/2=a
Major diameter 2a needs to be centered on F-F′ focal axis

1. *Bisect F-P radius r using F-P circle intersections to define r/2 sections.*
2. *Bisect F-F′ focal axis using F-F′ circle intersections to locate orbit center C.*
3. *Bisect F′-P radius r′ using F′-P circle intersections.*
4. *Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.*
5. *Copy circle of major radius a=(r-r′)/2 about orbit center C .*
6. *Draw focal circle of diameter 2aε about orbit center C .*

R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°



R=+1/2 hyperbolic orbit construction

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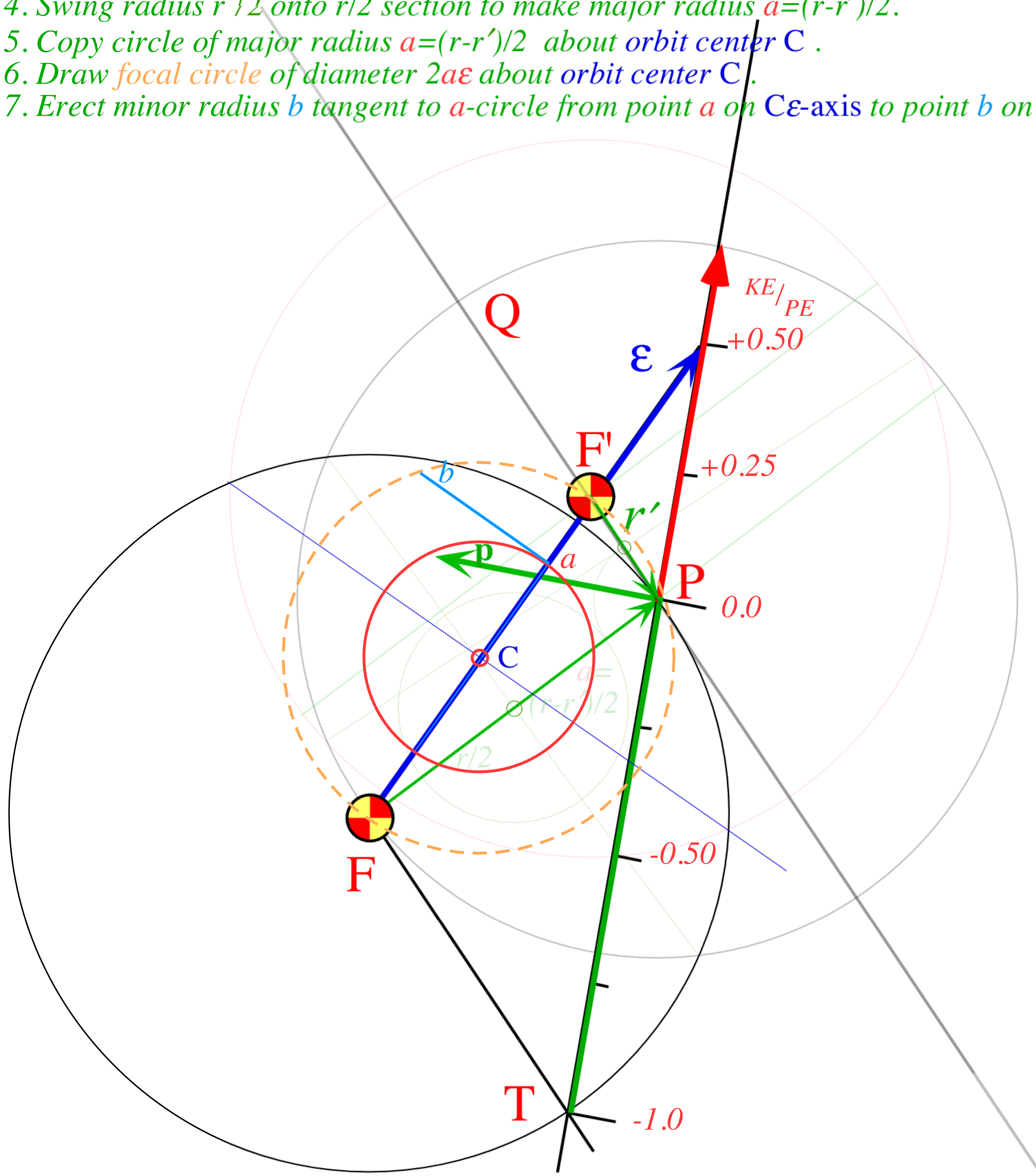
2. *Bisect F-F′ focal axis using F-F′ circle intersections to locate orbit center C.*
3. *Bisect F′-P radius r′ using F′-P circle intersections.*
4. *Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.*
5. *Copy circle of major radius a=(r-r′)/2 about orbit center C.*
6. *Draw focal circle of diameter 2aε about orbit center C.*
7. *Erect minor radius b tangent to a-circle from point a on Cε-axis to point b on focal circle.*

R=+1/2

γ=45°



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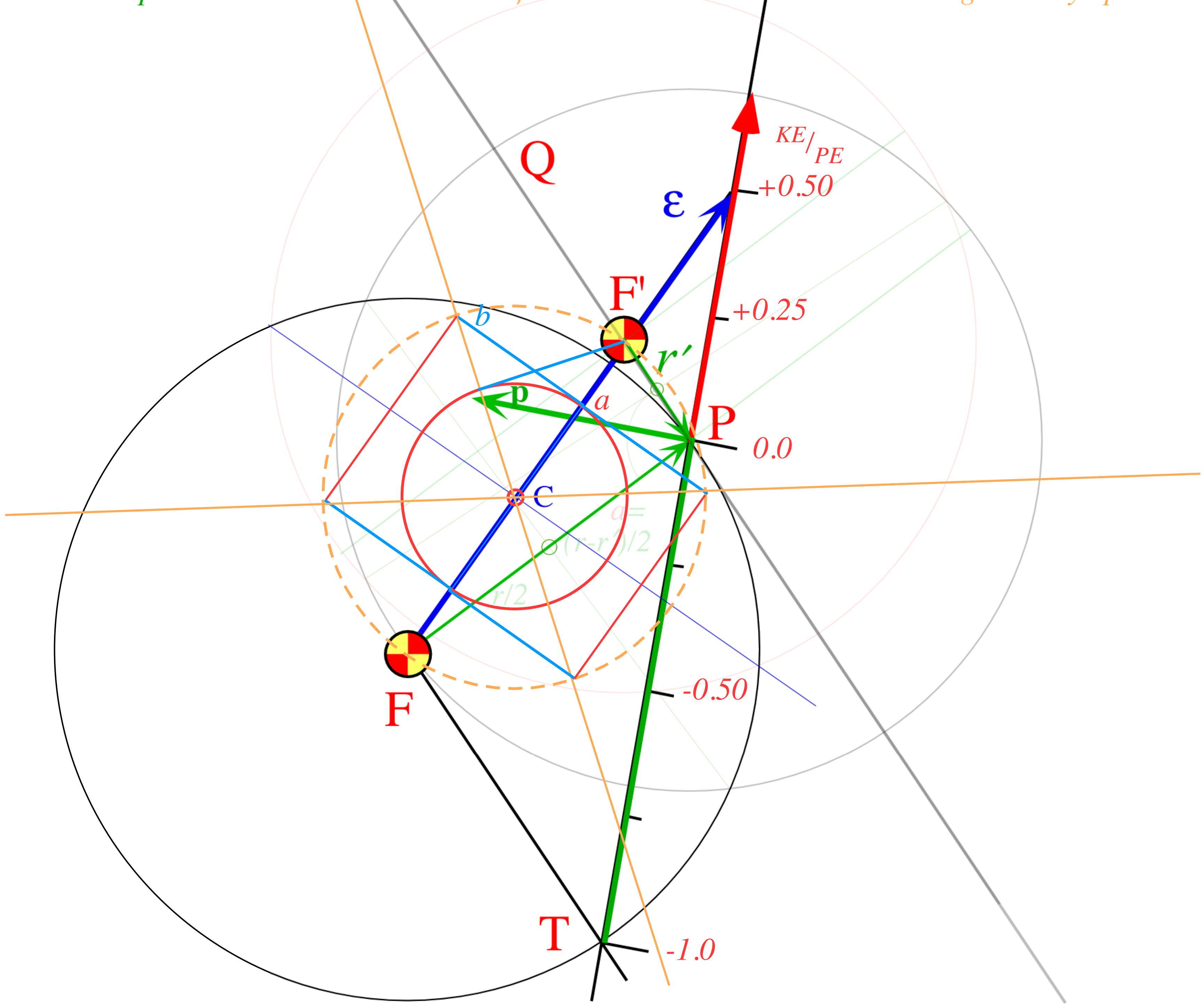
4. *Swing radius r′/2 onto r/2 section to make major radius a=(r-r′)/2.*
5. *Copy circle of major radius a=(r-r′)/2 about orbit center C .*
6. *Draw focal circle of diameter 2aε about orbit center C.*
7. *Erect minor radius b tangent to a-circle from point a on Cε-axis to point b on focal circle.*
8. *Complete orbit a-x-b box between focal circle and a-circle and its diagonal asymptotes.*

R=+1/2 hyperbolic orbit construction

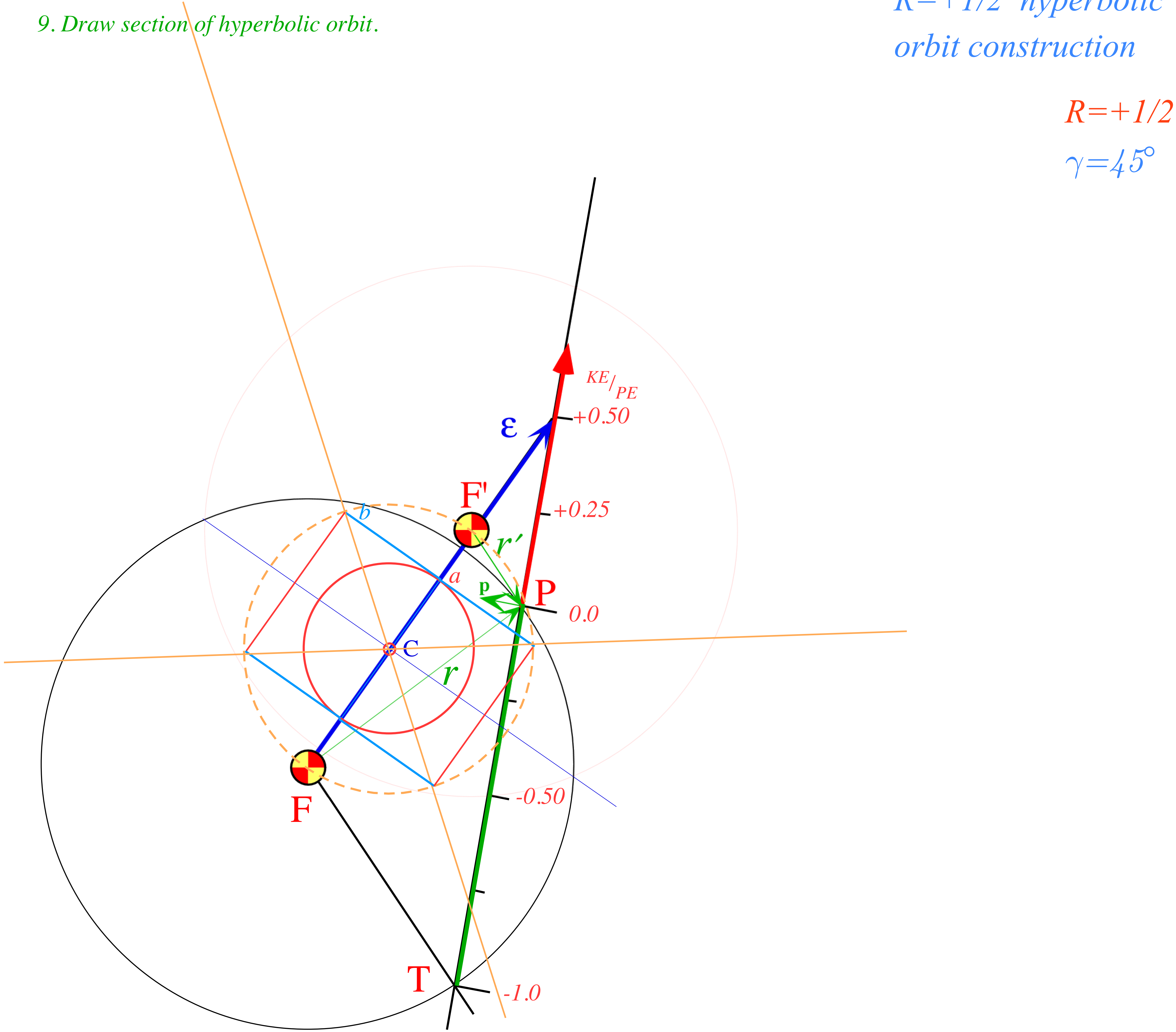
R=+1/2

$\gamma=45°$



R=+1/2 hyperbolic orbit construction

9. Draw section of hyperbolic orbit.



R=+1/2 hyperbolic orbit construction

R=+1/2

γ=45°

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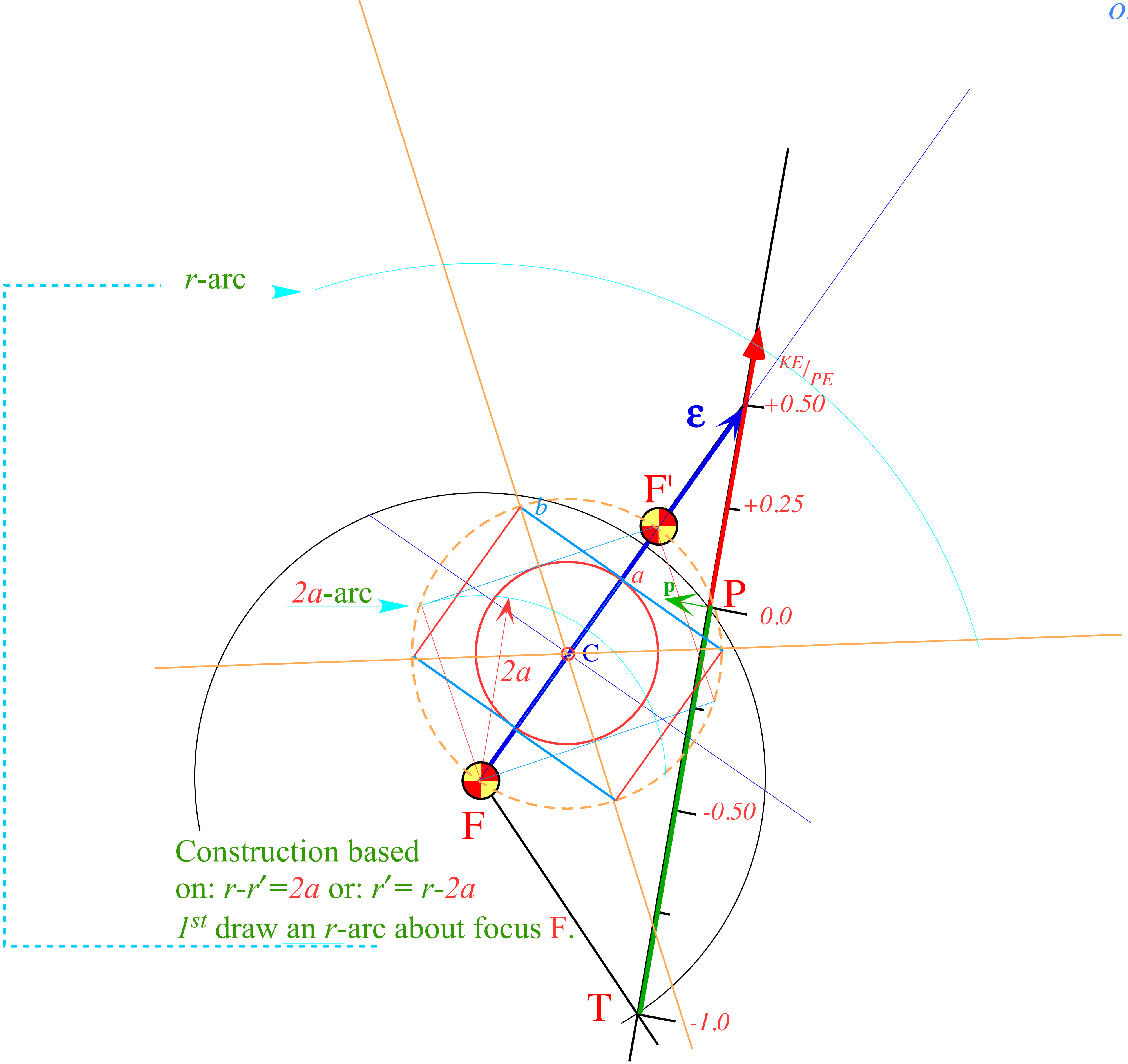

R=+1/2 hyperbolic orbit construction

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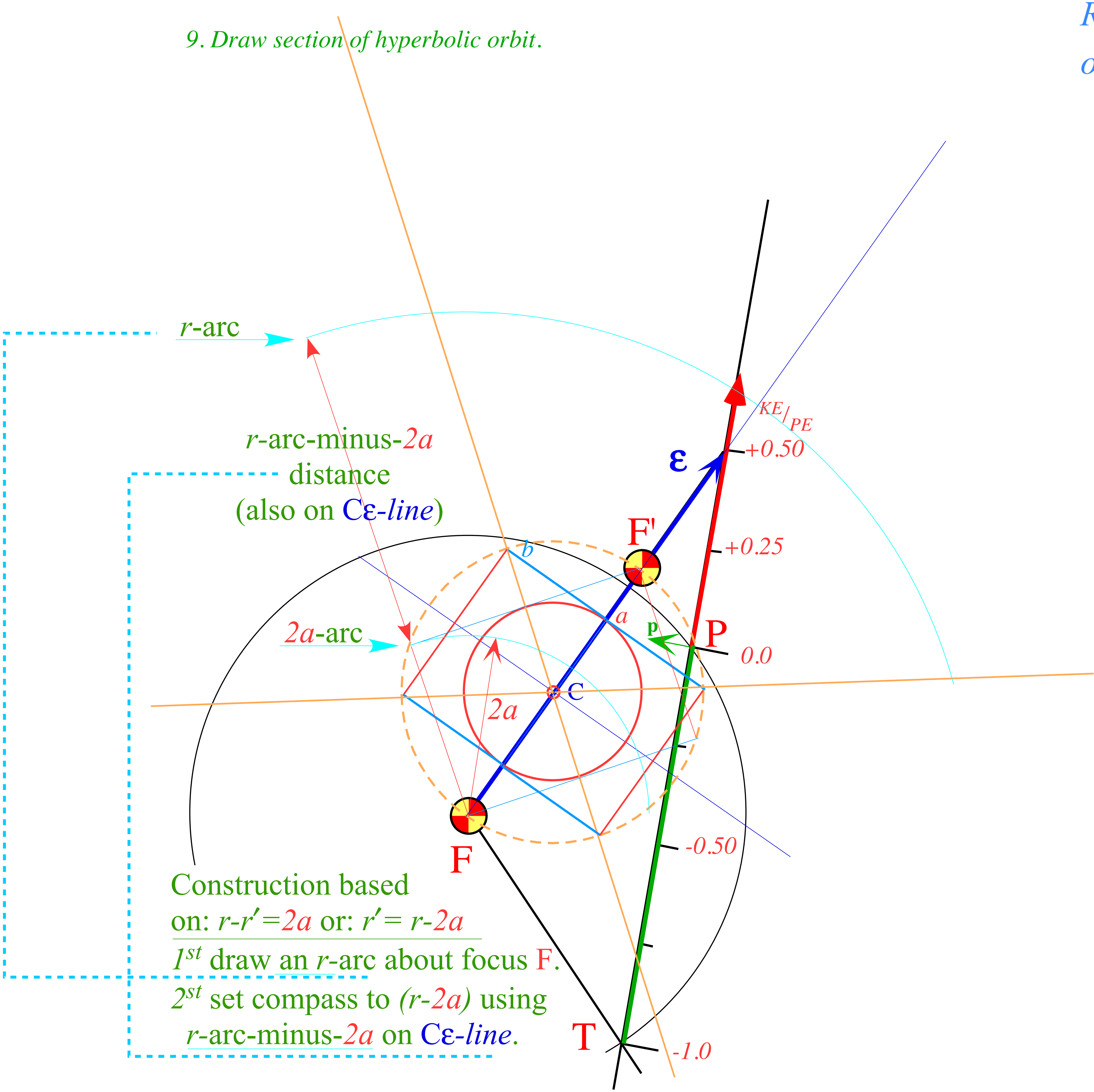

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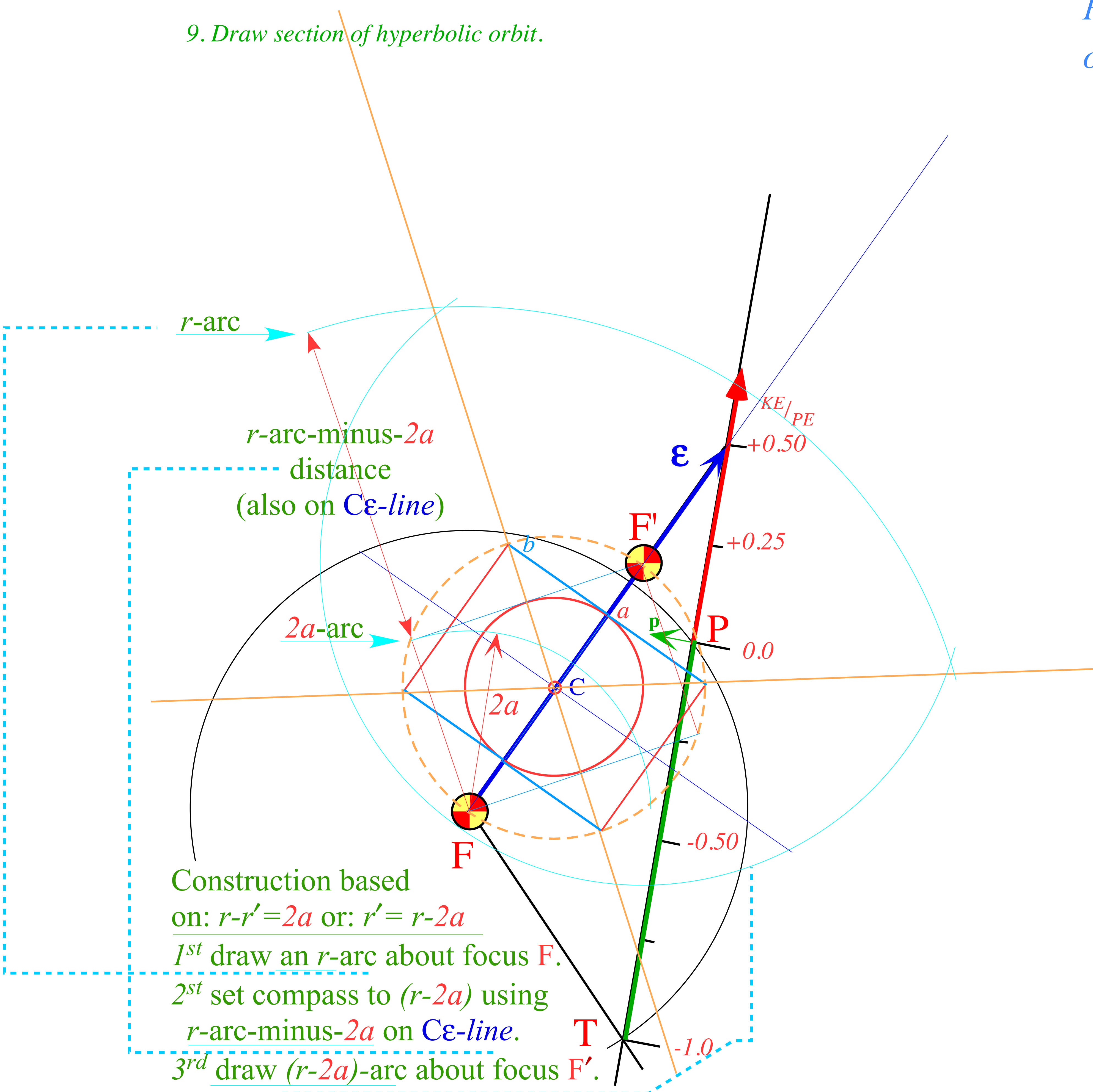


Construction based on: $r-r'=2a$ or: $r'= r-2a$

1st draw an r-arc about focus F.

2st set compass to (r-2a) using r-arc-minus-2a on Cε-line.

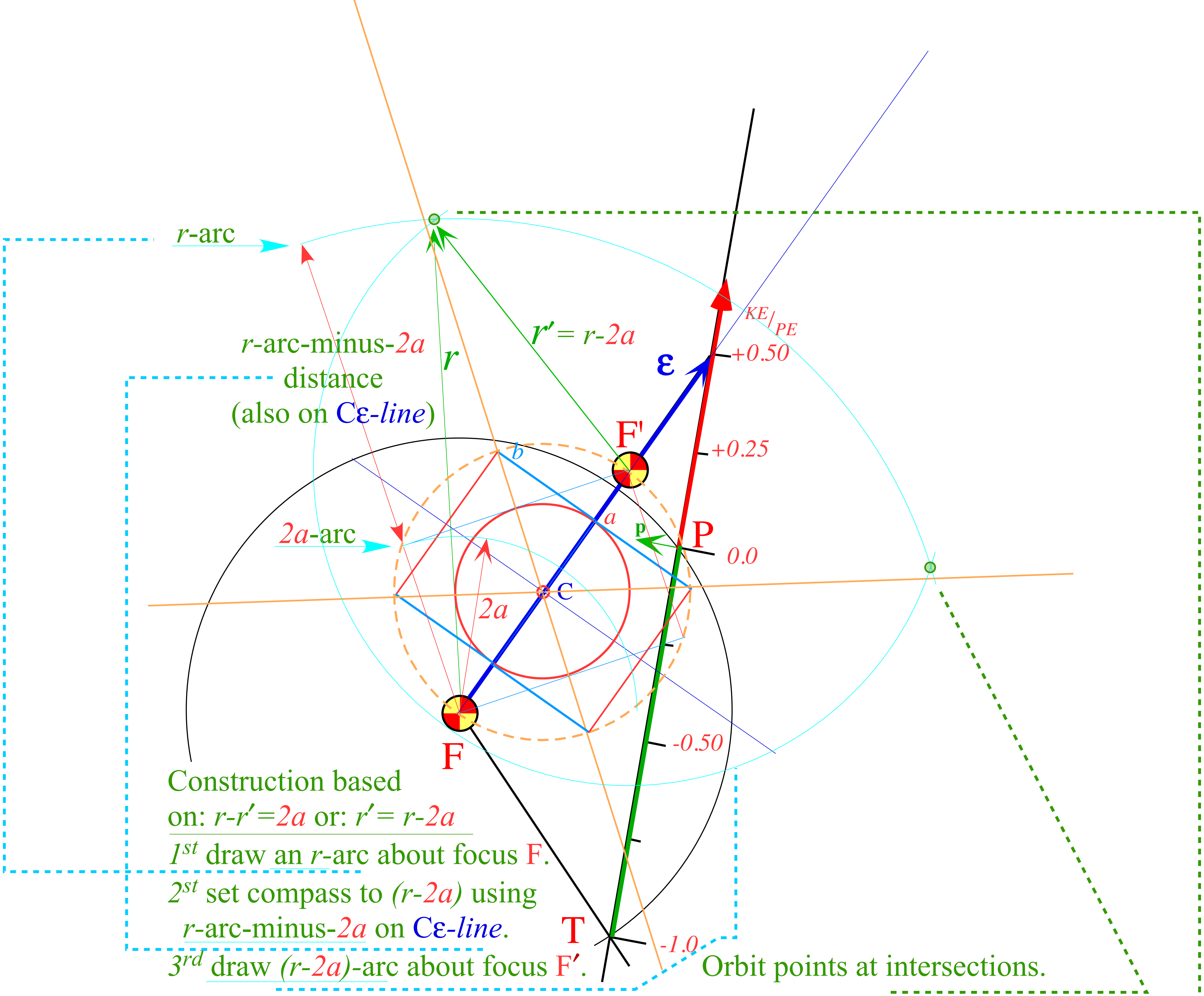
3rd draw (r-2a)-arc about focus F′.

R=+1/2 hyperbolic orbit construction

9. Draw section of hyperbolic orbit.

$R=+1/2$

$\gamma=45°$



R=+1/2

$\gamma=45°$

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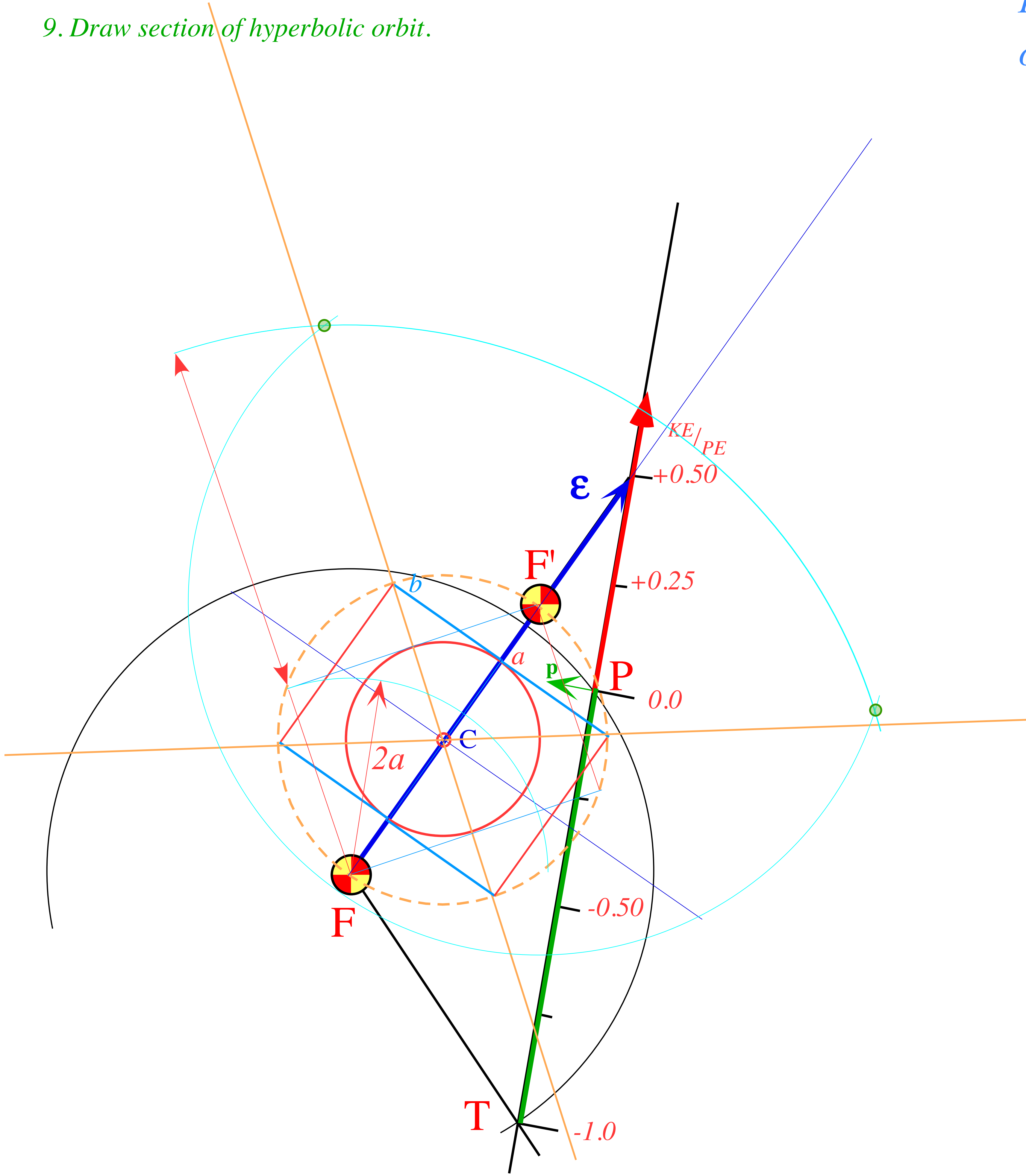
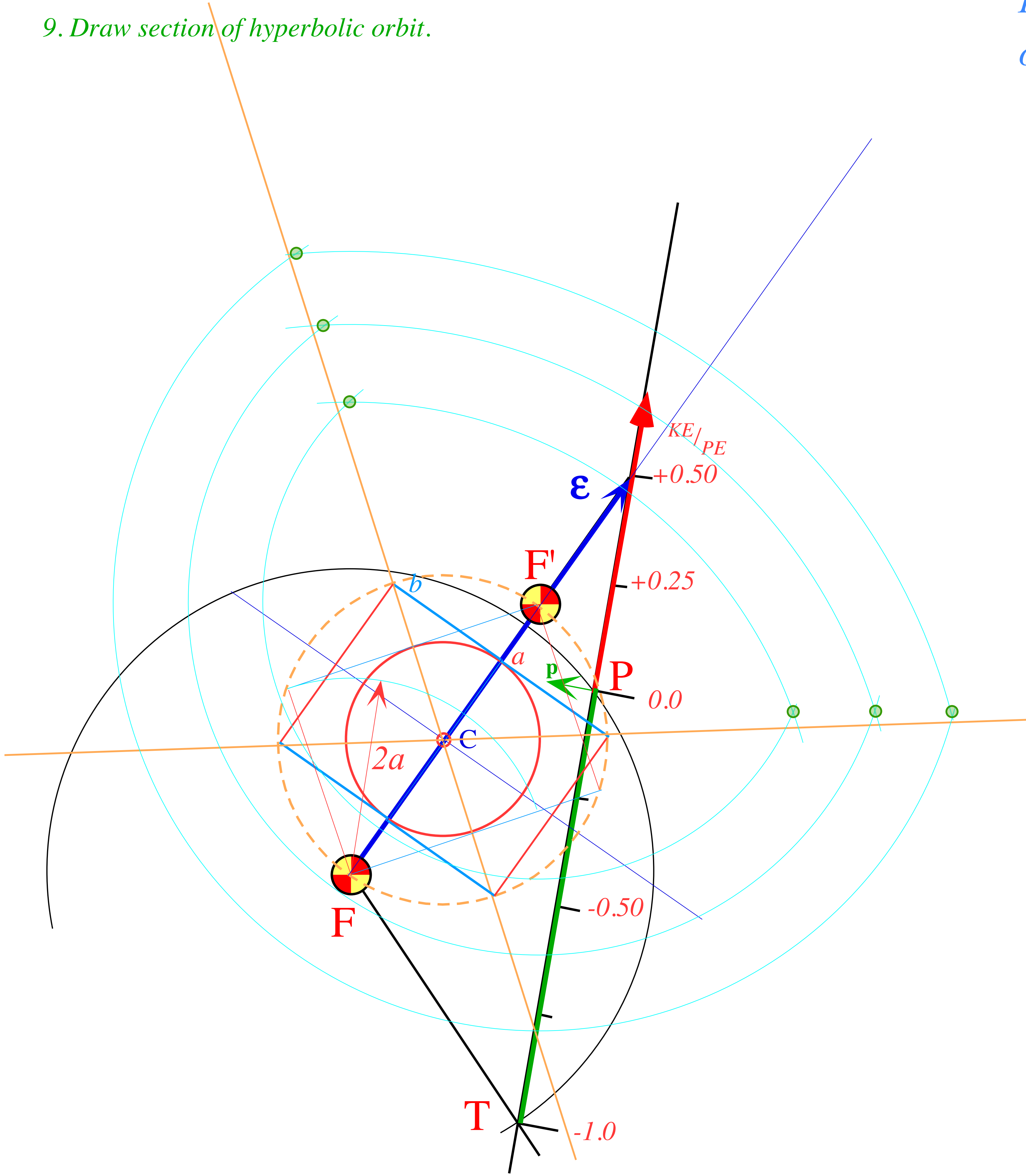


9. *Draw section of hyperbolic orbit.*

R=+1/2 hyperbolic orbit construction

R=+1/2

$\gamma=45°$



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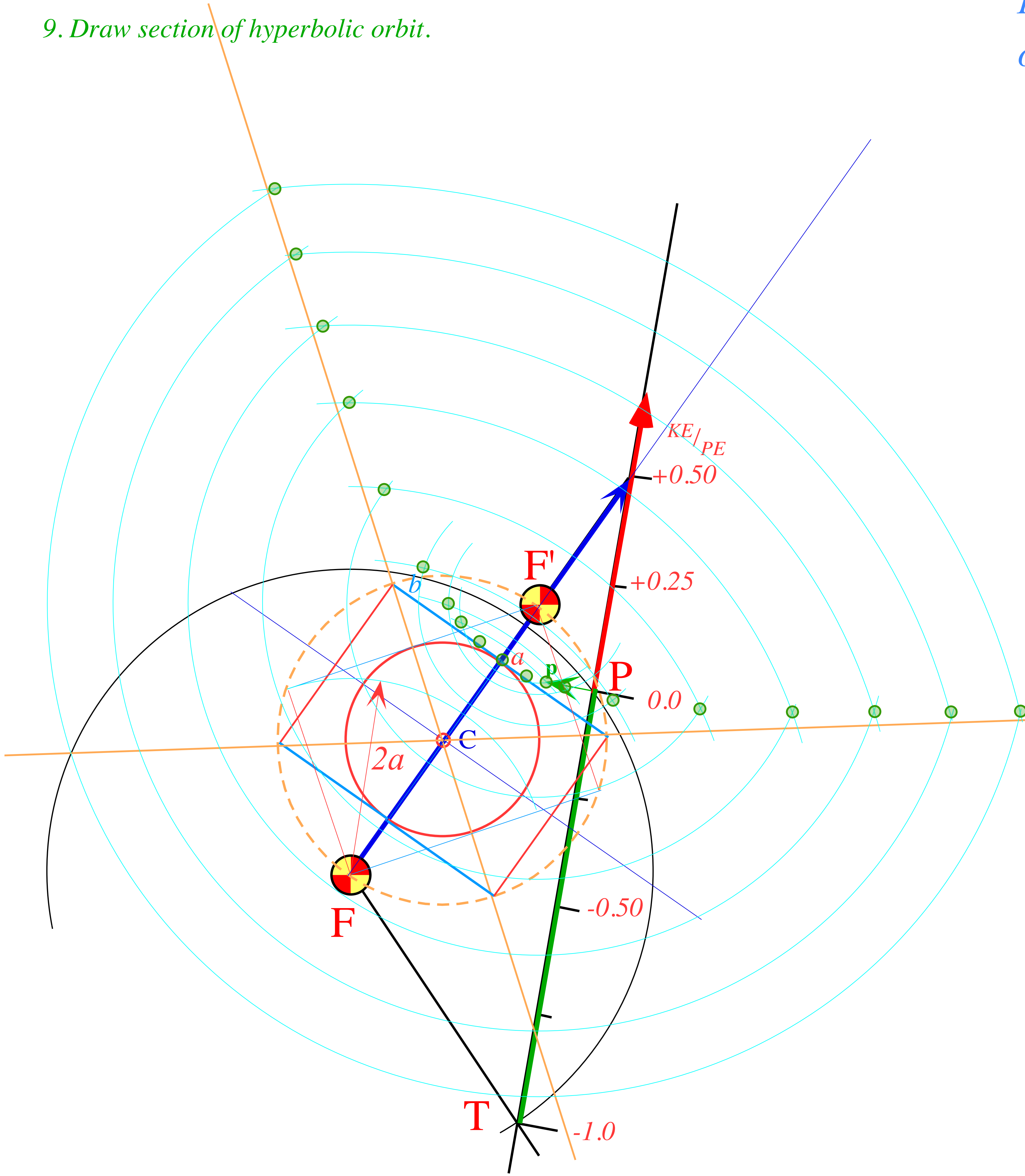
R=+1/2 hyperbolic orbit construction



R=+1/2 hyperbolic orbit construction

9. Draw section of hyperbolic orbit.

$$\varepsilon = \sqrt{1+4R(R+1)\sin^2\gamma} = \sqrt{\frac{3}{2}} = 1.58$$

$$a = \frac{1}{2(R+1)} = \frac{1}{3} = .33$$

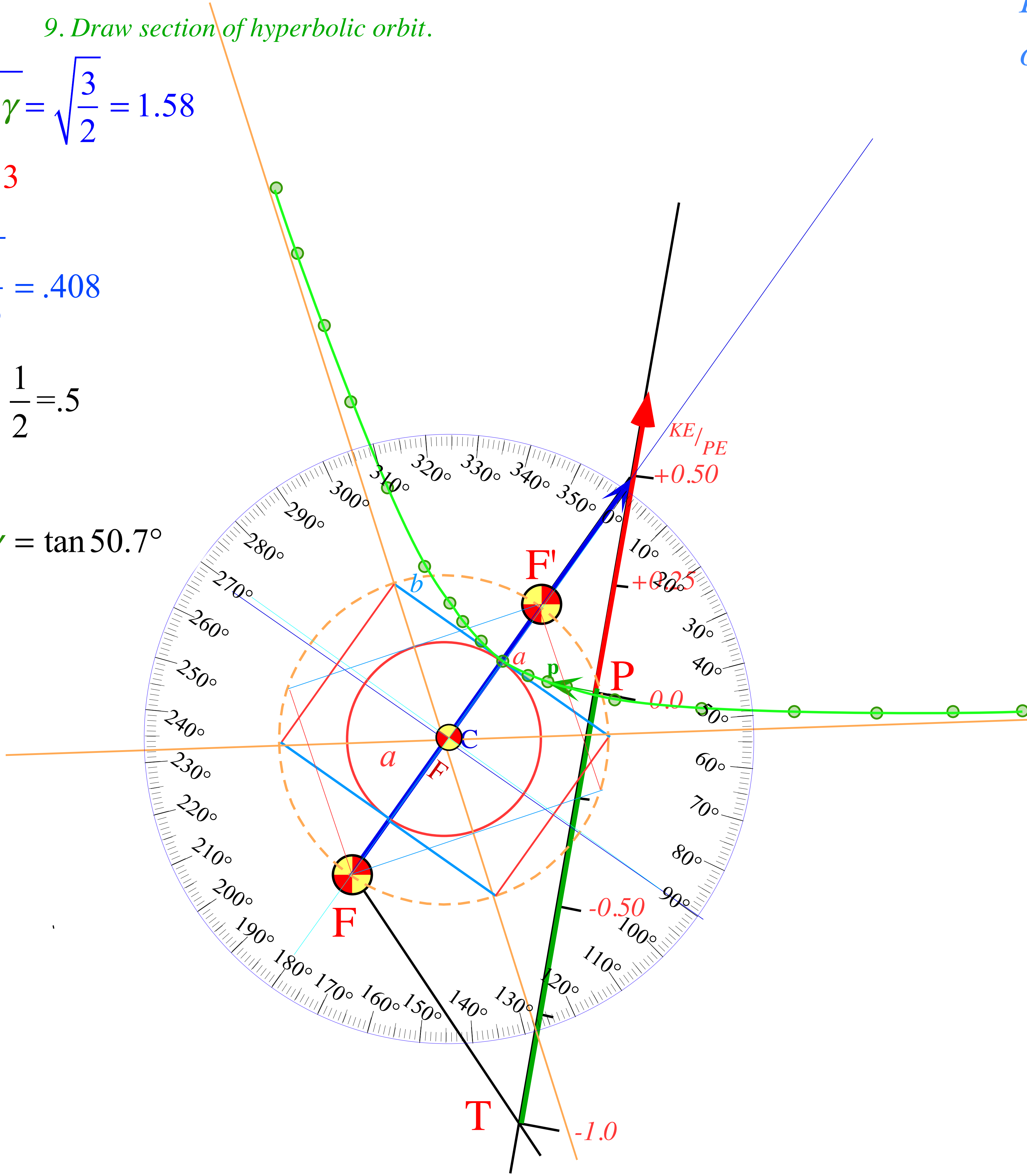
$$b = \sqrt{\frac{R}{R+1}}\sin\gamma = \sqrt{\frac{1}{6}} = .408$$

$$\lambda = \frac{b^2}{a} = 2R\sin^2\gamma = \frac{1}{2} = .5$$

$$\frac{b}{a} = 2\sqrt{R(R+1)}\sin\gamma = \tan 50.7°$$

R=+1/2

γ=45°



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*Detailed ruler & compass construction of **ε-vector** and orbits*

(R=-0.375 elliptic orbit)

(R=+0.5 hyperbolic orbit)

Properties of Coulomb trajectory families and envelopes

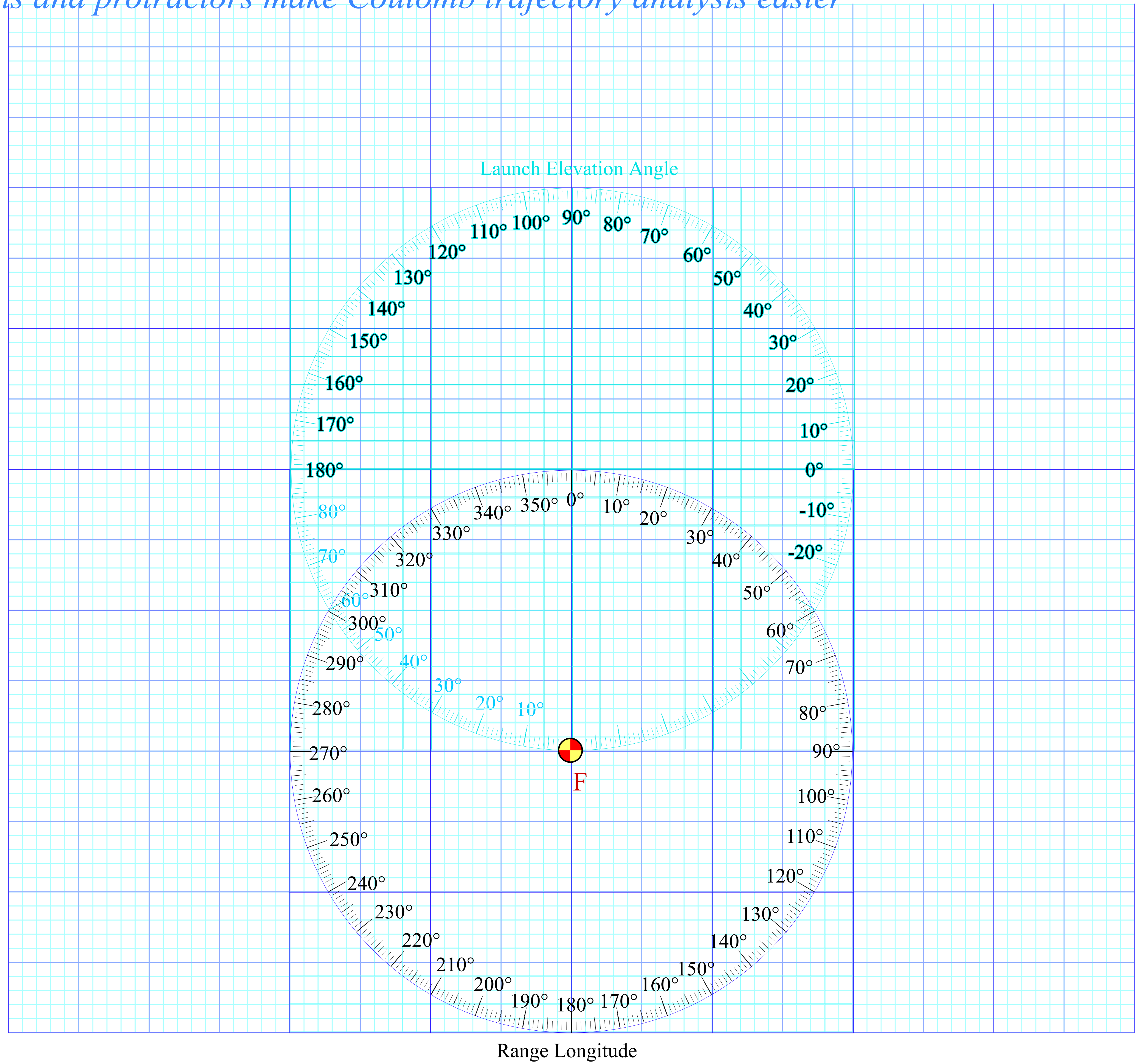
➔ *Graphical **ε**-development of orbits*

➔ *Launch angle fixed-Varied launch energy*

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes

Graphs and protractors make Coulomb trajectory analysis easier



Start with initial angle

$\alpha=20°$

(horiz. elev.)

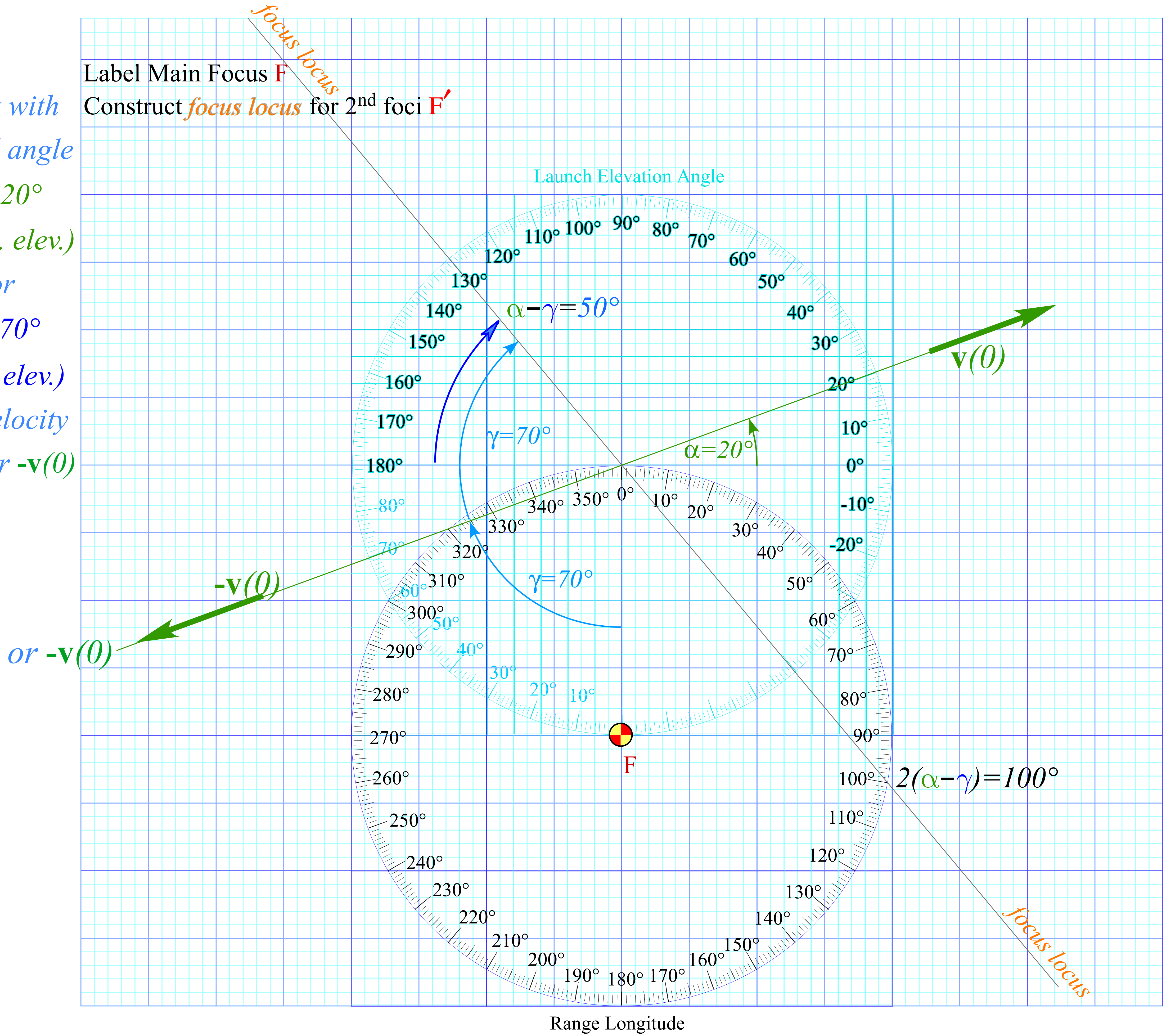
or

$\gamma=70°$

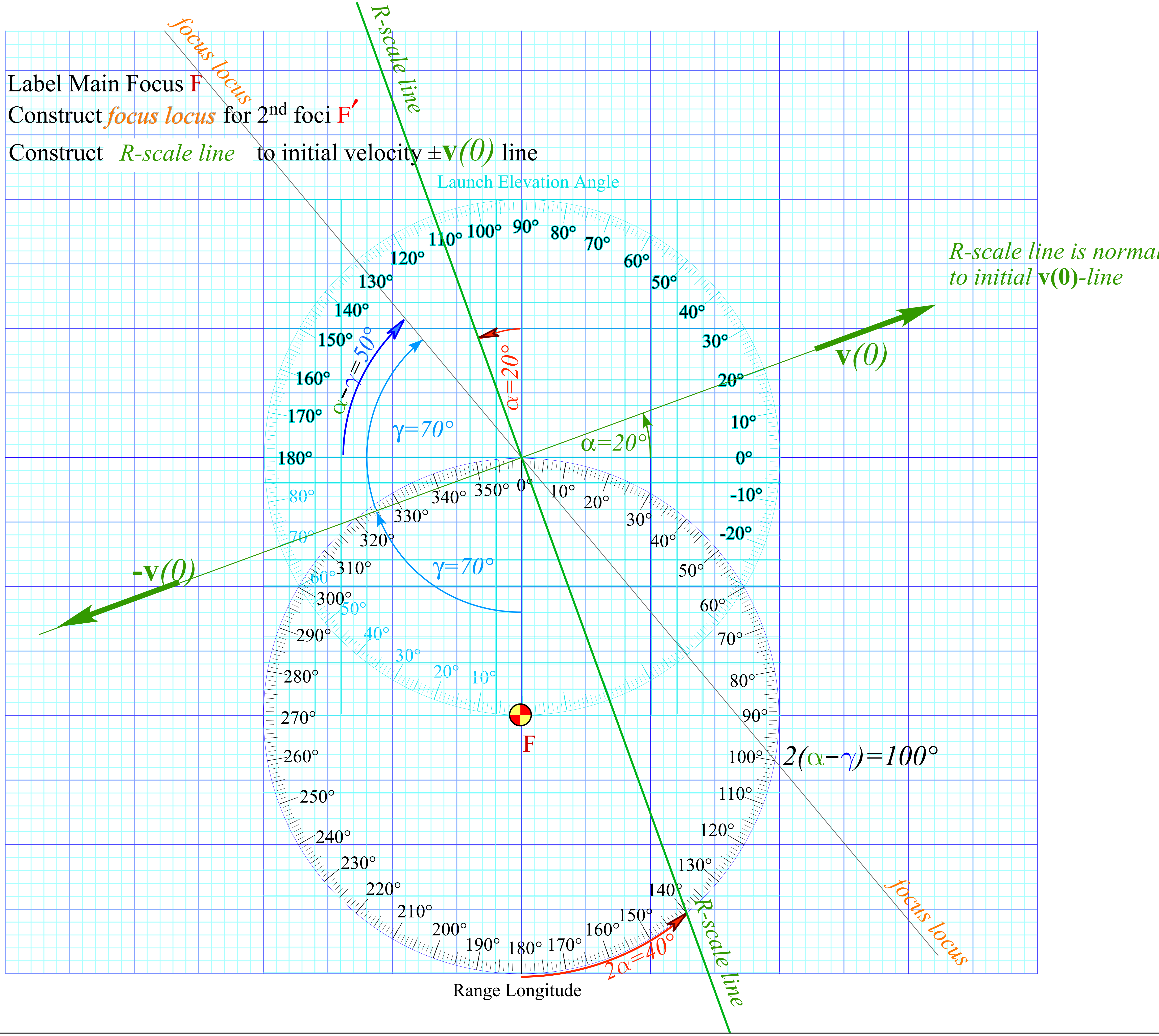
(rad. elev.)

for velocity

v(0) *or* -**v**(0)



Start with initial angle $\alpha=20°$ *(horiz. elev.)* *or* $\gamma=70°$ *(rad. elev.)* *for velocity* **v**(0) *or* **-v**(0)



Start with
initial angle
$\alpha=20°$
(horiz. elev.)
or
$\gamma=70°$
(rad. elev.)
for velocity
v(0) *or* -**v**(0)

or -**v**(0)

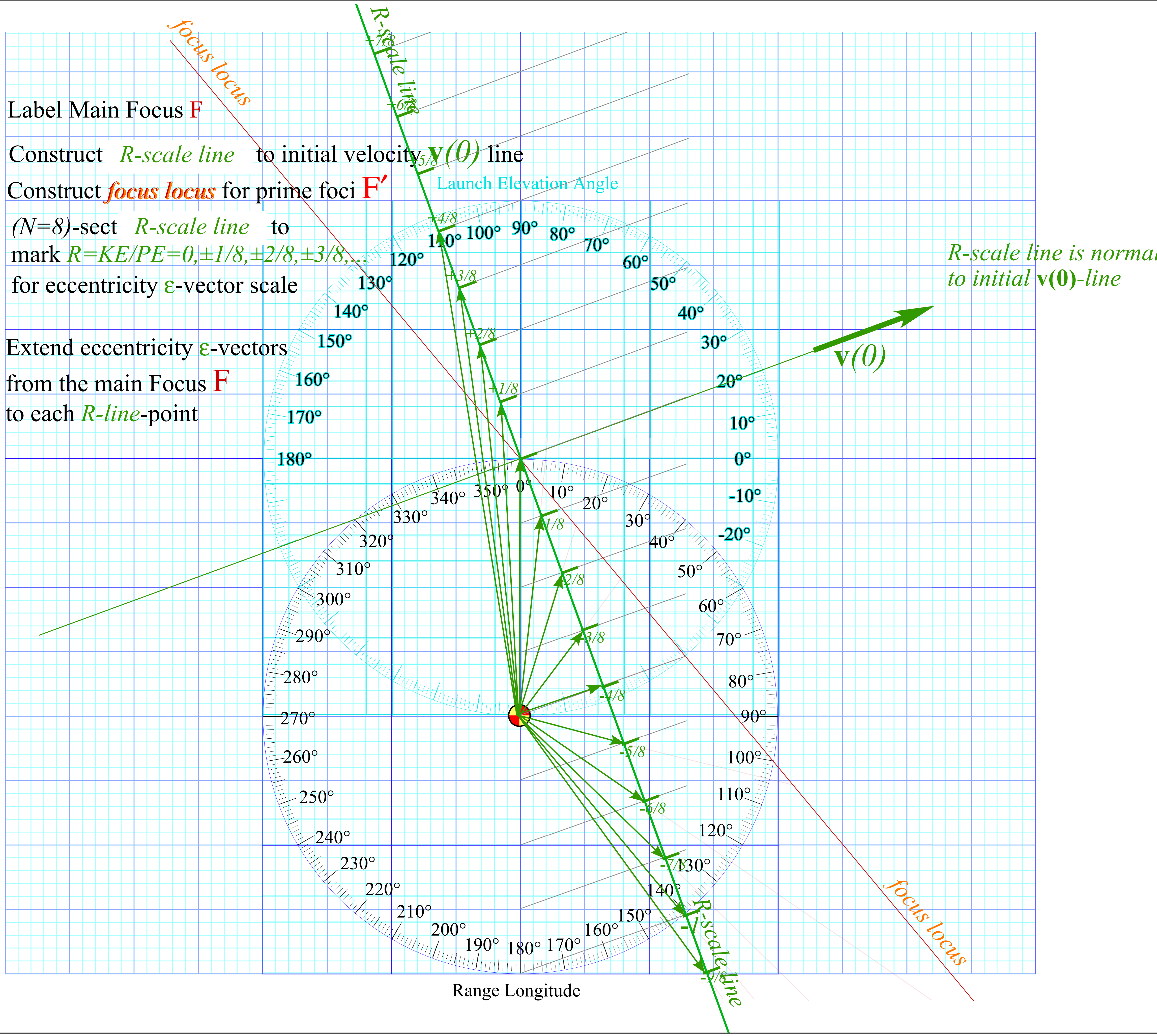
Label Main Focus F

Construct *R-scale line* to initial velocity **v**(0) line

Construct *focus locus* for prime foci F′

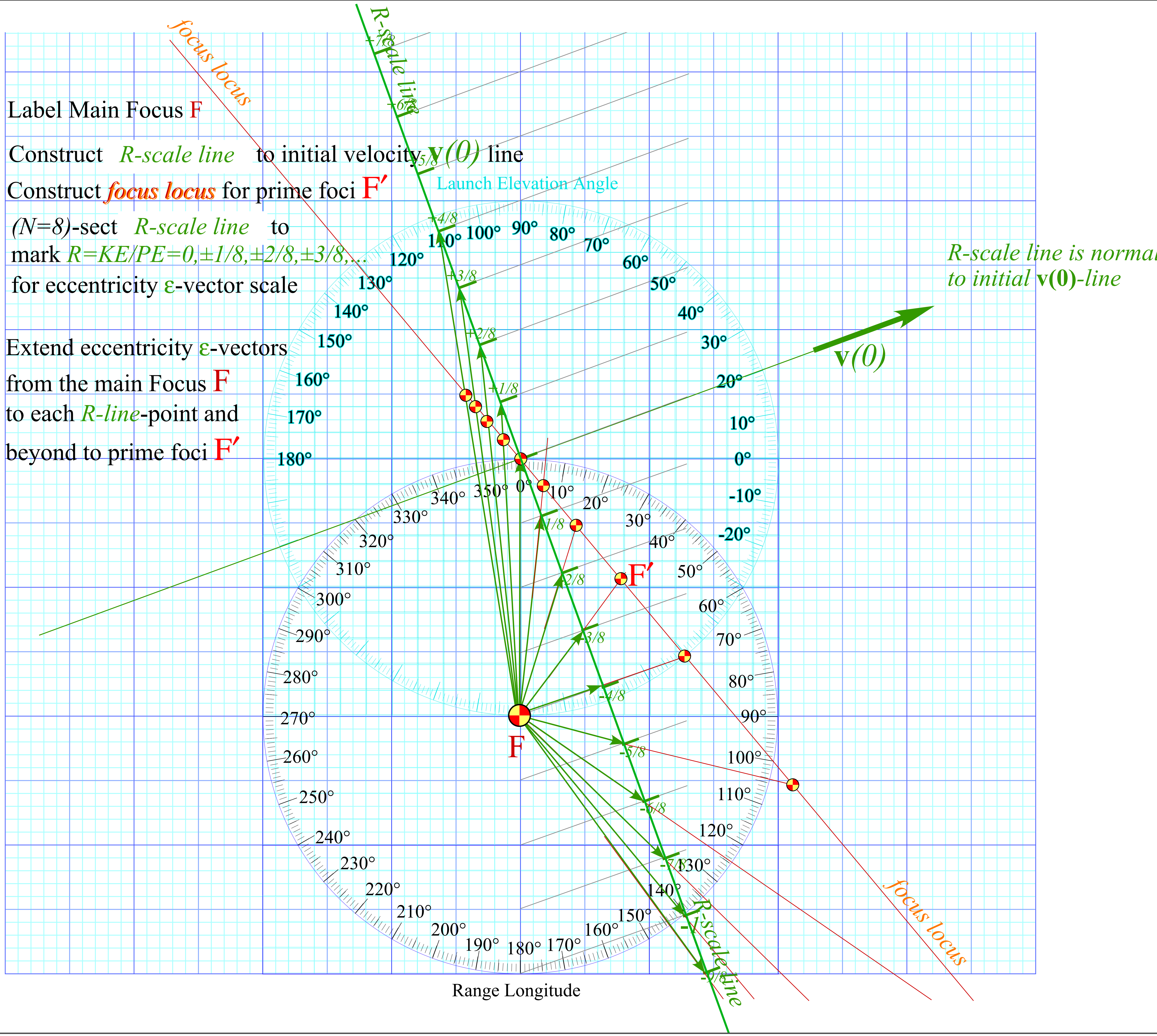
(N=8)-sect *R-scale line* to mark *R=KE/PE=0,±1/8,±2/8,±3/8,...* for eccentricity ε-vector scale

Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point

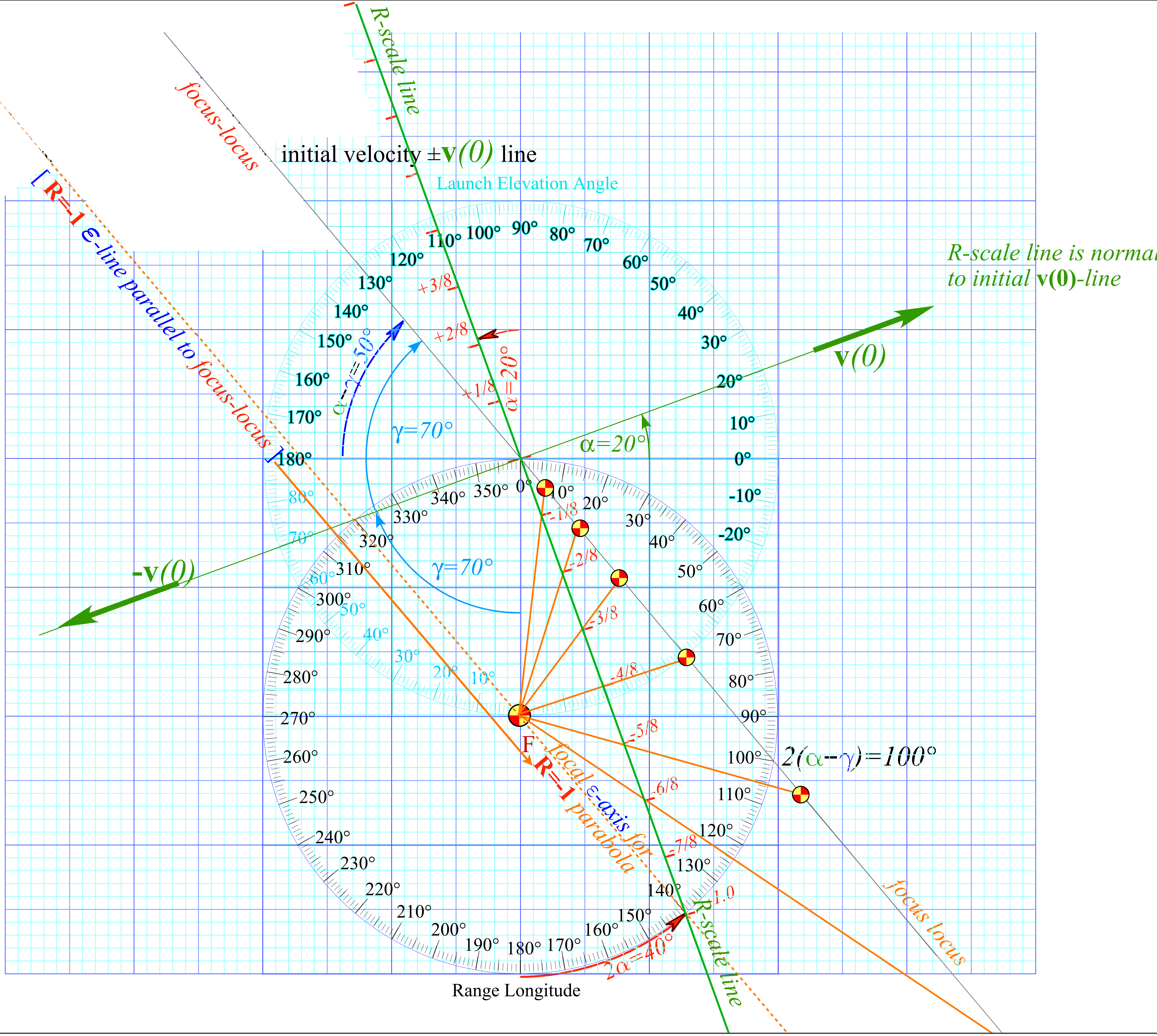


Start with initial angle
α=20° (horiz. elev.)
or
γ=70° (rad. elev.)
for velocity **v***(0) or* **-v***(0)*

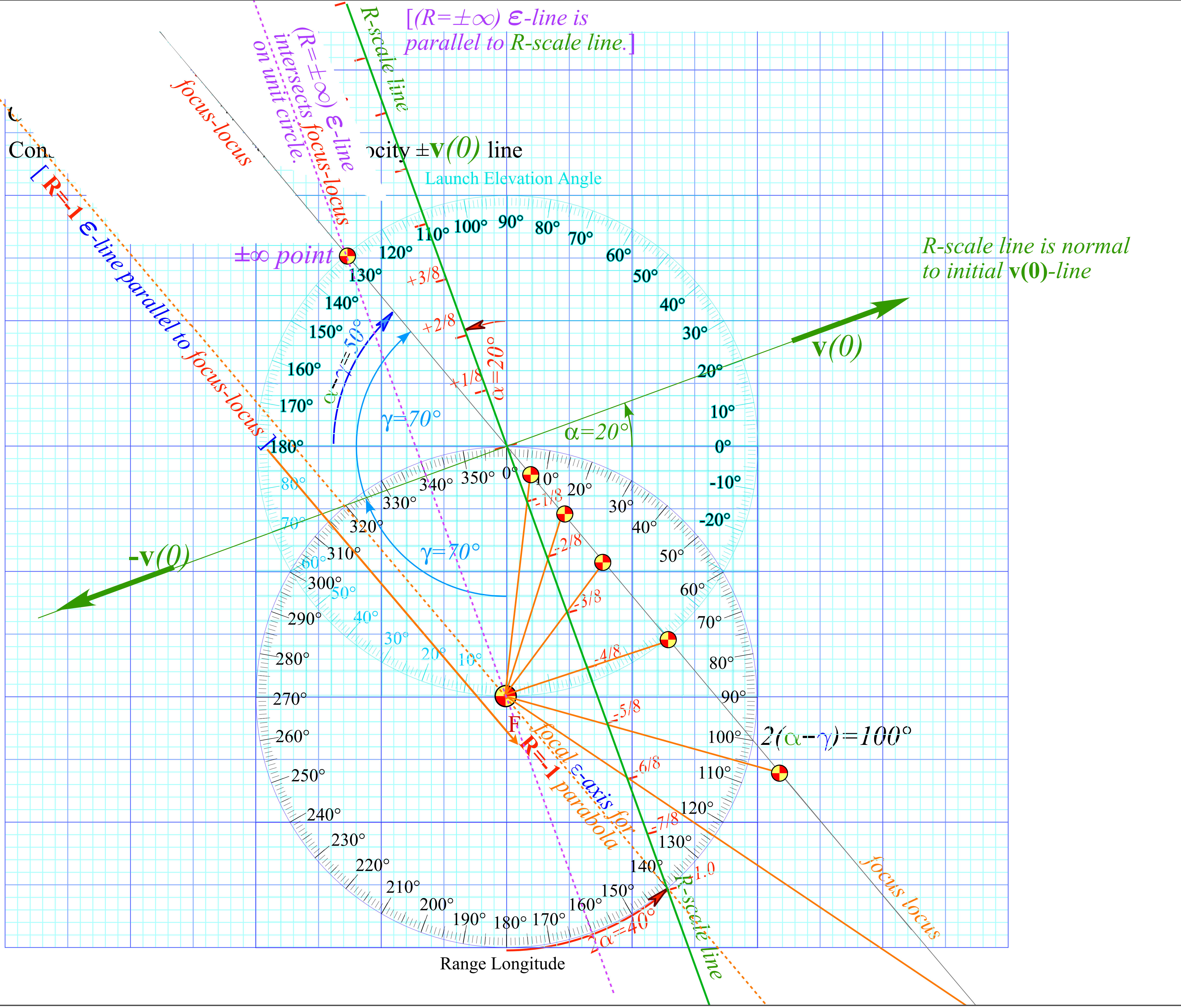
or **-v***(0)*



Start with initial angle $\alpha=20°$ *(horiz. elev.)* *or* $\gamma=70°$ *(rad. elev.)* *for velocity* **v***(0) or* **-v***(0)*

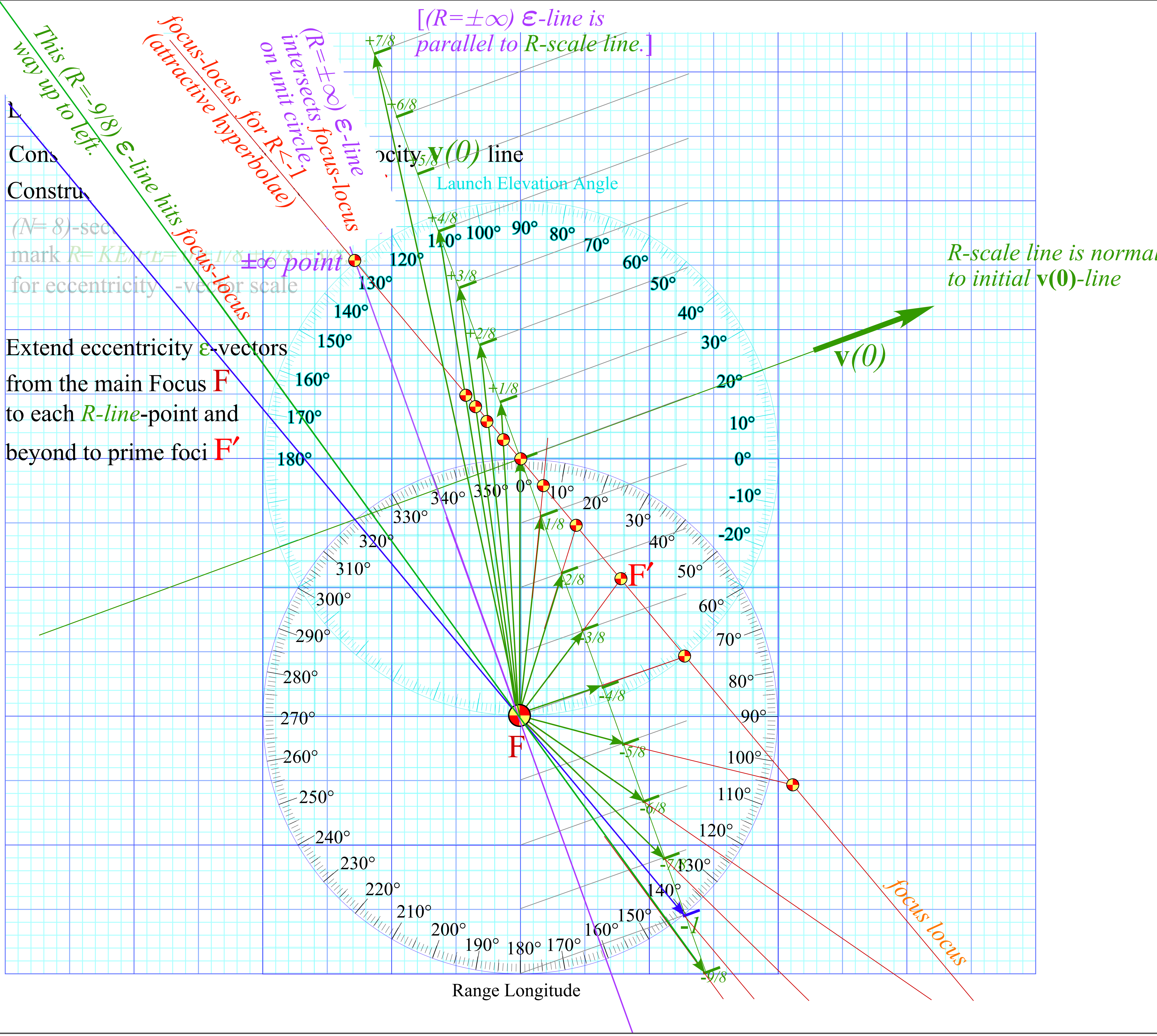


Start with initial angle $\alpha=20°$ *(horiz. elev.) or* $\gamma=70°$ *(rad. elev.) for velocity* $\mathbf{v}(0)$ *or* $-\mathbf{v}(0)$



This (R=-1) ...

Start with initial angle α=20° (horiz. elev.) or γ=70° (rad. elev.) for velocity **v**(0) or -**v**(0)



This (R=-1)

Start with initial angle

α=20°

(horiz. elev.)

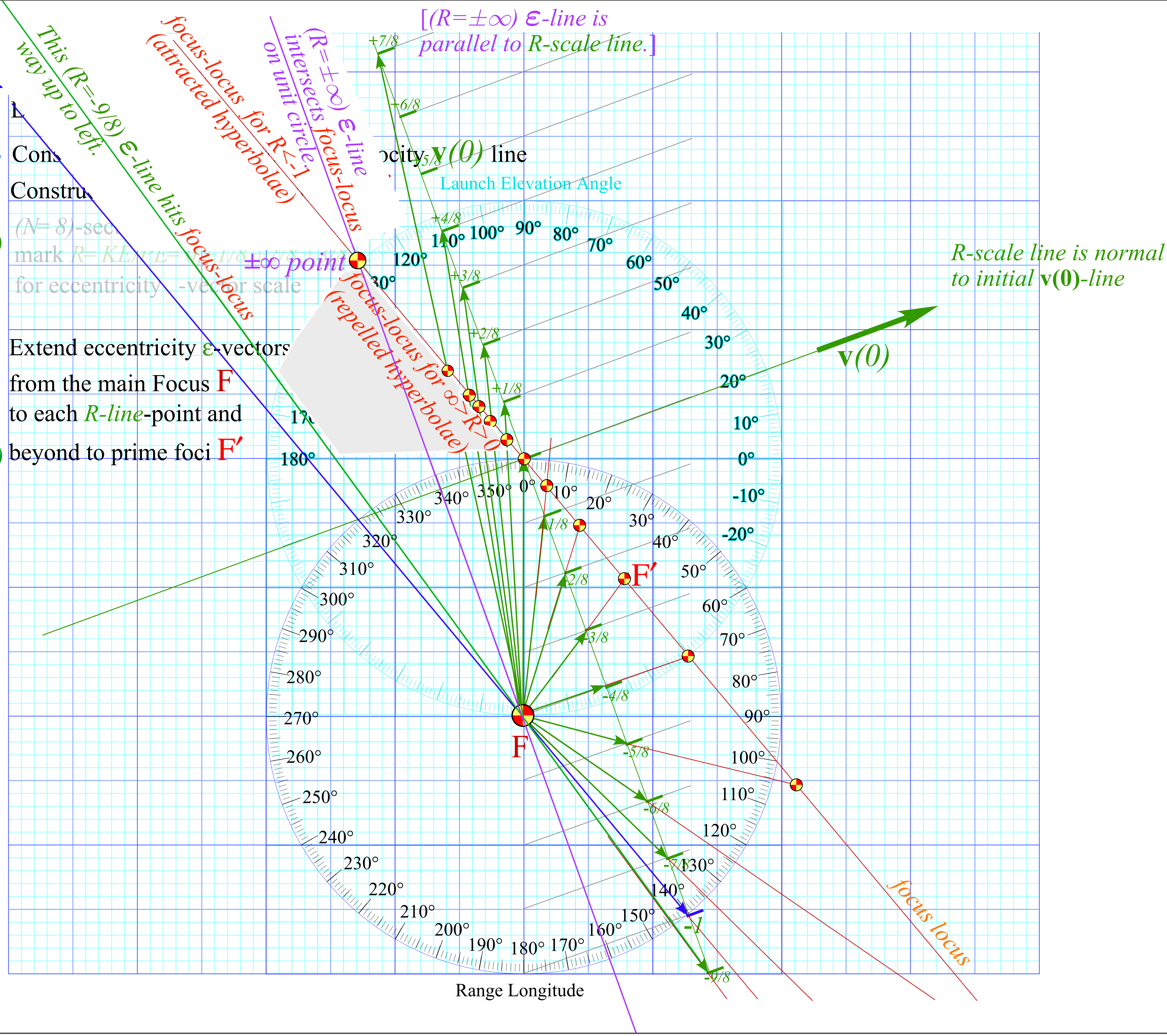
or

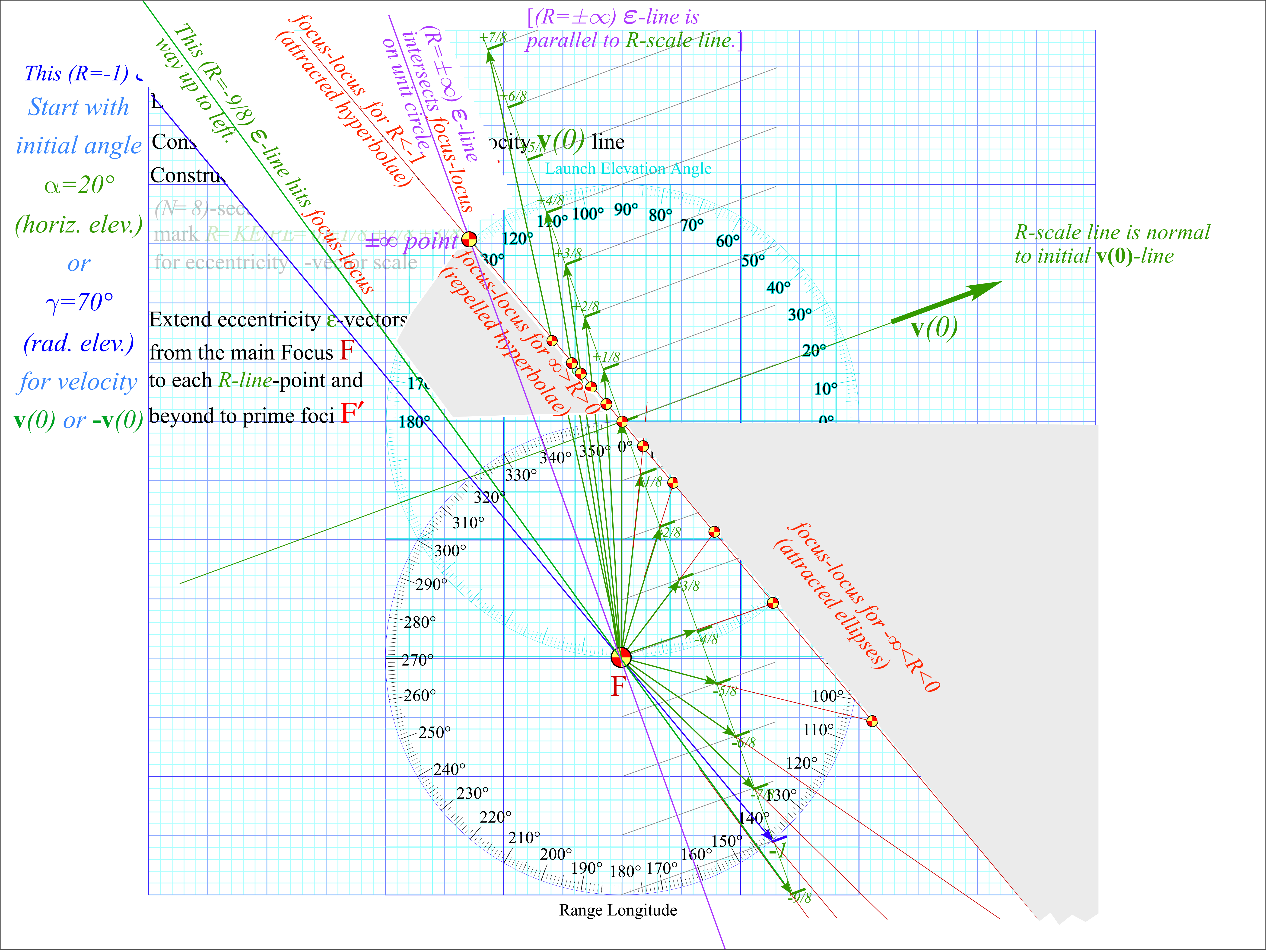
γ=70°

(rad. elev.)

for velocity

v(0) *or* -**v**(0)





This (R=-1)
Start with
initial angle
α=20°
(horiz. elev.)
or
γ=70°
(rad. elev.)
for velocity
v(0) or -v(0)
Constru
(N= 8)-sec
mark
for eccentricity -vector scale
Extend eccentricity ε-vectors
from the main Focus F
to each R-line-point and
beyond to prime foci F′
This (R=-9/8) ε-line hits focus-locus
way up to left.
focus-locus for R<-1
(attracted hyperbolae)
(R=±∞) ε-line
intersects focus-locus
on unit circle.
[(R=±∞) ε-line is
parallel to R-scale line.]
±∞ point
focus-locus for ∞>R>0
(repelled hyperbolae)
v(0) line
Launch Elevation Angle
R-scale line is normal
to initial v(0)-line
v(0)
focus-locus for -∞<R<0
(attracted ellipses)
F
Range Longitude

*Review of **Eccentricity vector ε** and (ε,λ)-geometry of orbital mechanics*

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(R=+0.5 hyperbolic orbit)

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➔ *Graphical **ε**-development of orbits*

➔ *Launch angle fixed-Varied launch energy*

Launch energy fixed-Varied launch angle

Launch optimization and orbit family envelopes

Start with initial velocity **v***(0)* *or* **-v***(0)*

Label Main Focus F

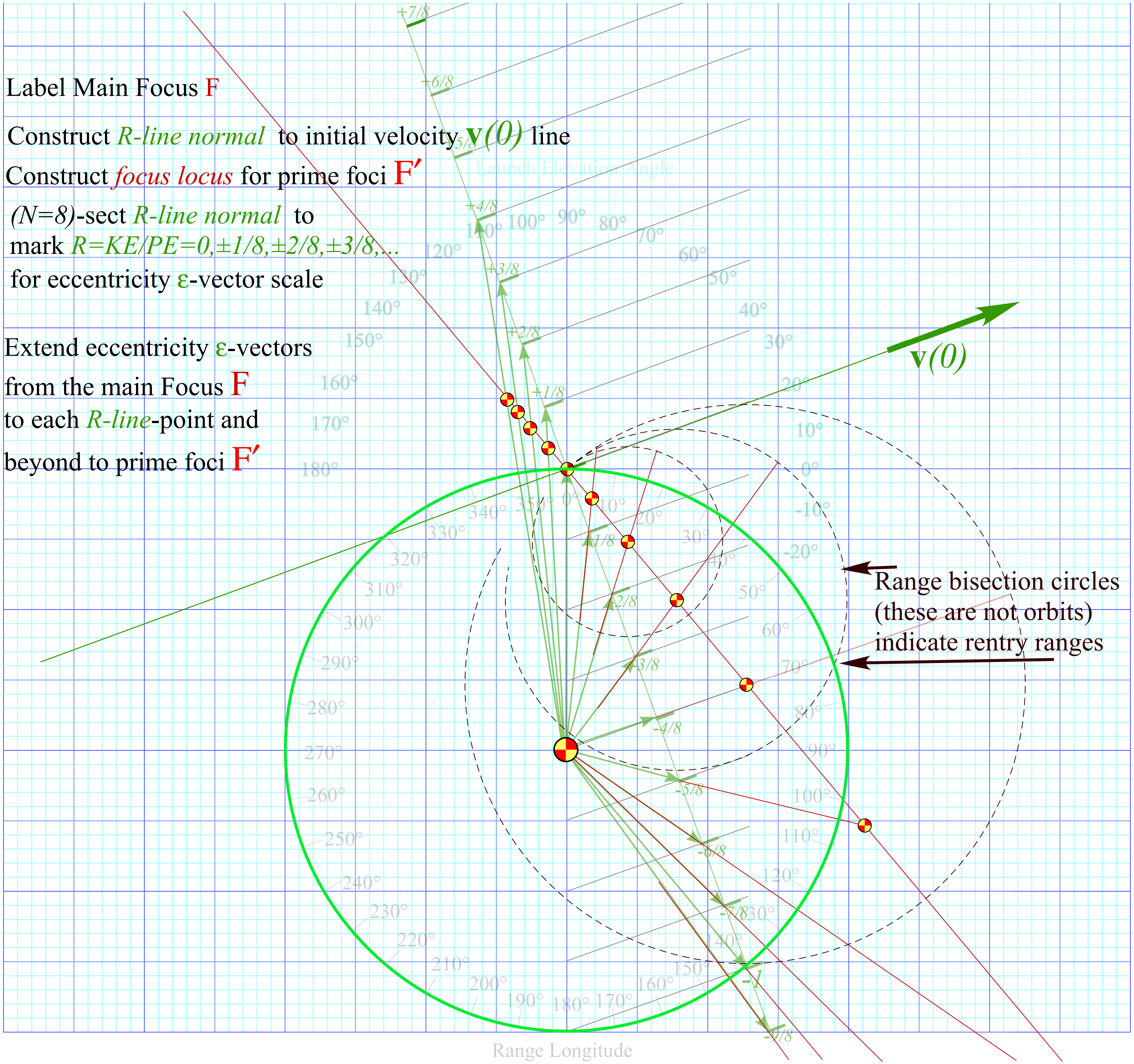
Construct *R-line normal* to initial velocity **v***(0)* line

Construct *focus locus* for prime foci F′

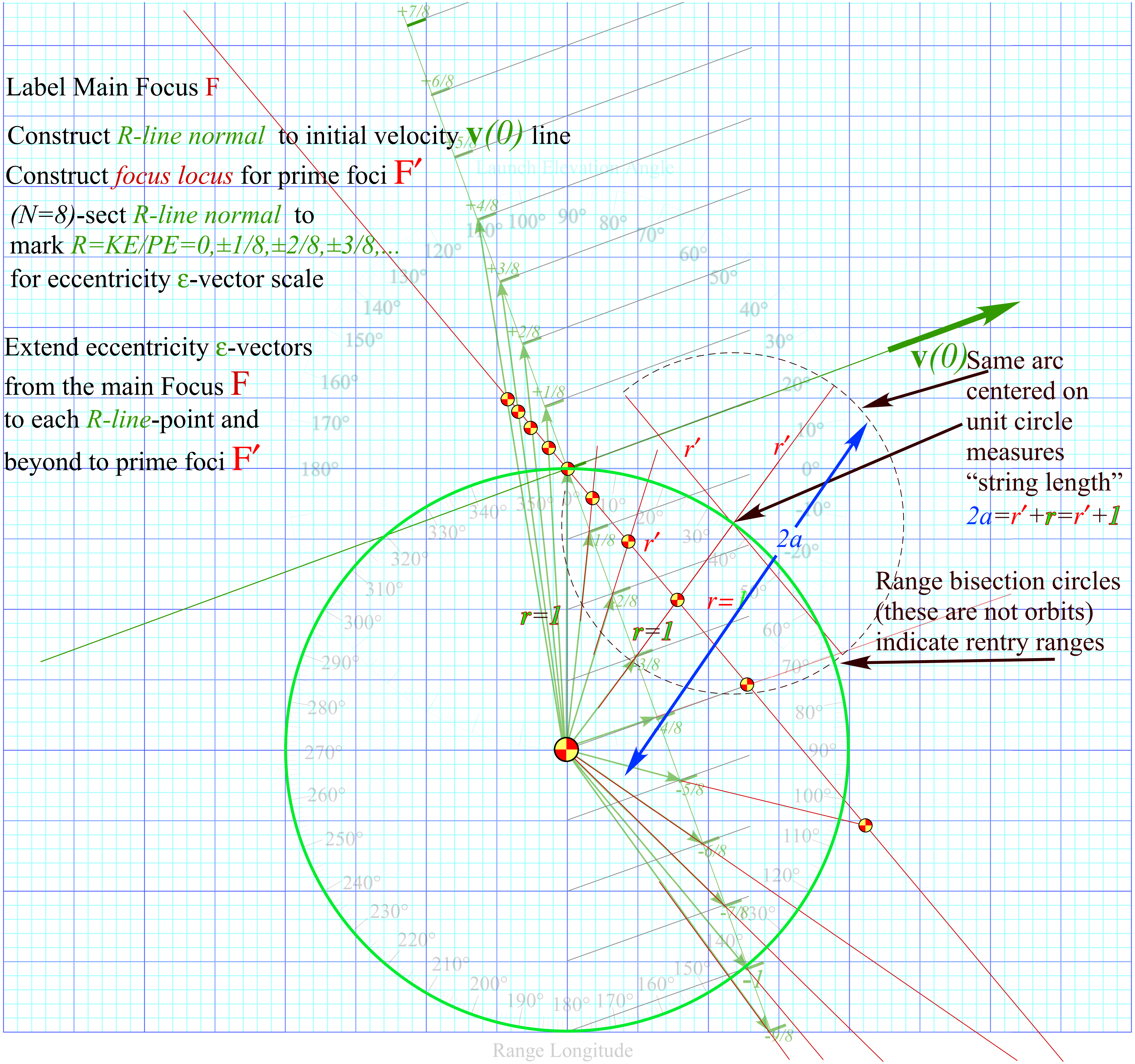
(N=8)-sect *R-line normal* to mark *R=KE/PE=0,±1/8,±2/8,±3/8,...* for eccentricity ε-vector scale

Extend eccentricity ε-vectors from the main Focus F to each *R-line*-point and beyond to prime foci F′

or **-v***(0)*

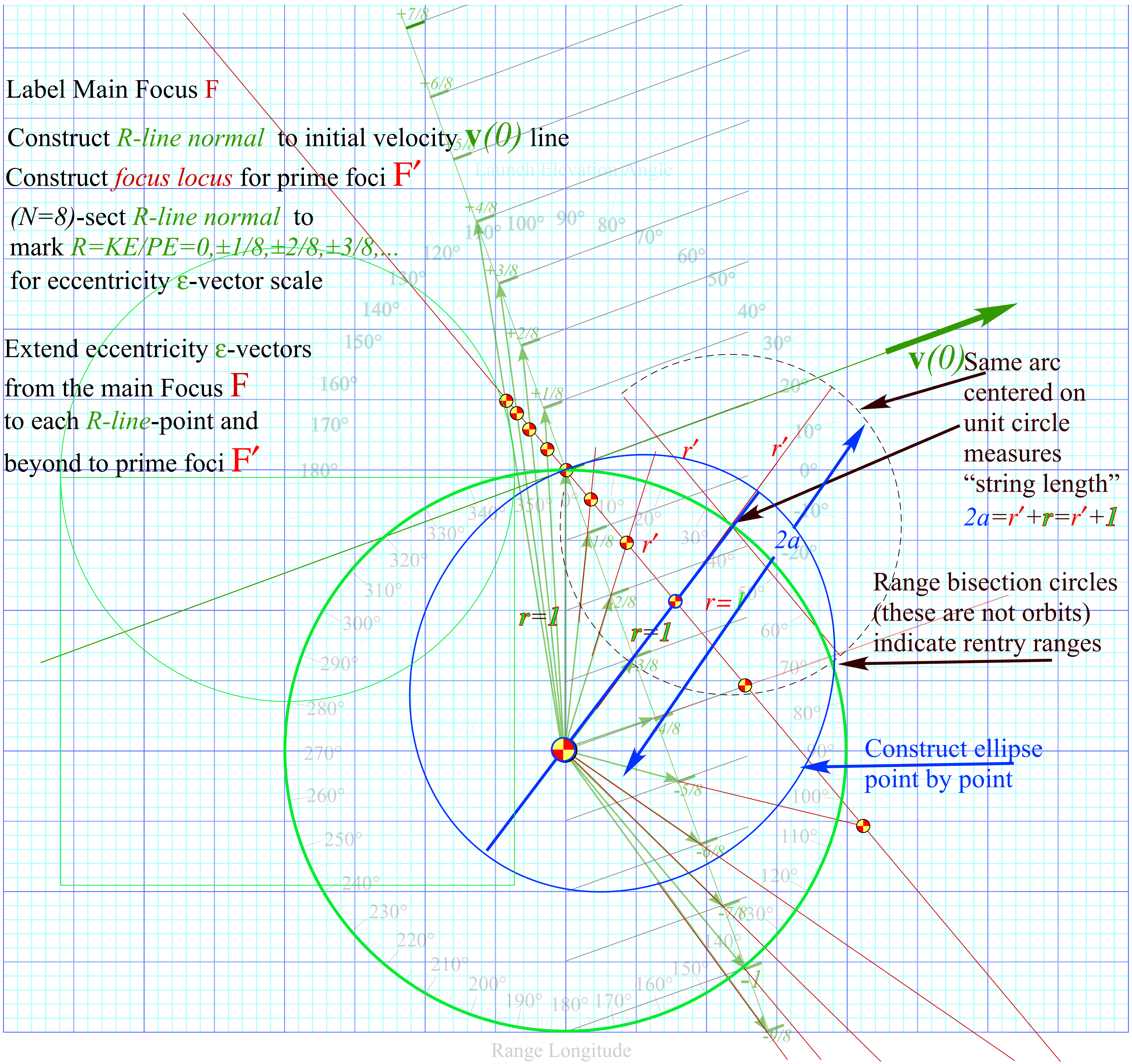


Start with initial velocity **v**(0) *or* **-v**(0)

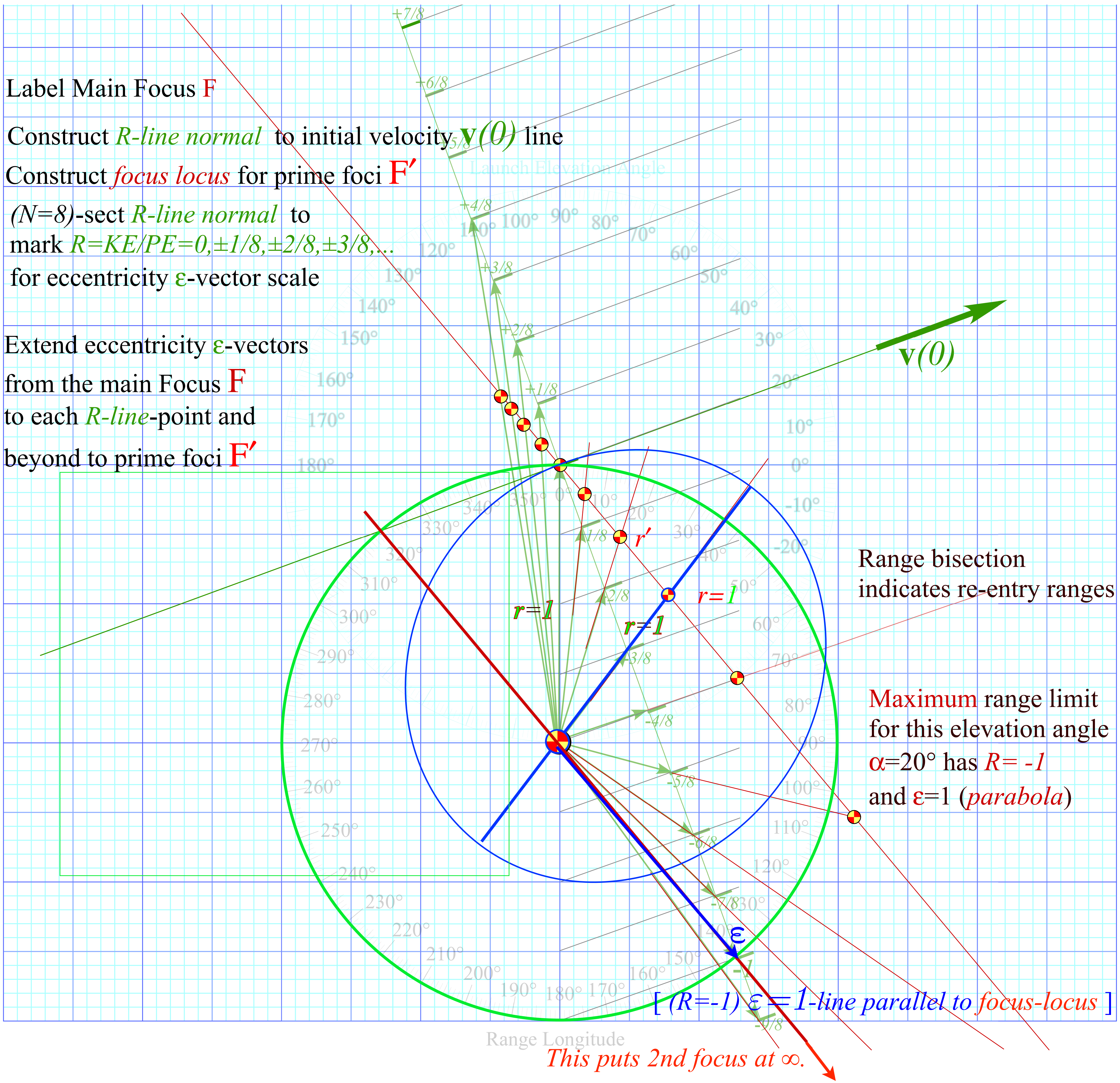


or **-v**(0)

Start with initial velocity **v***(0)* *or* **-v***(0)*



or **-v***(0)*


Label Main Focus F
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Construct focus locus for prime foci F′
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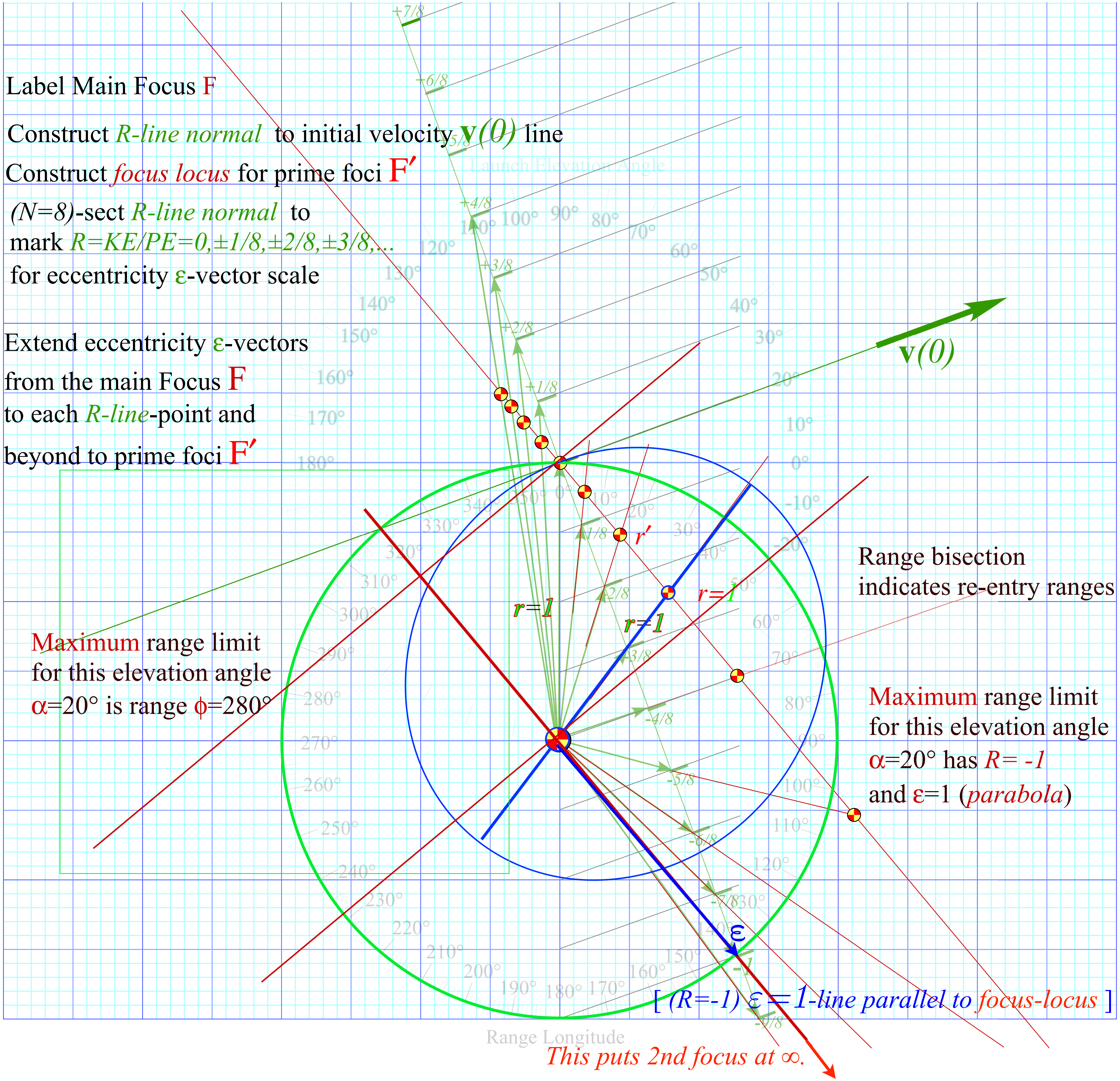
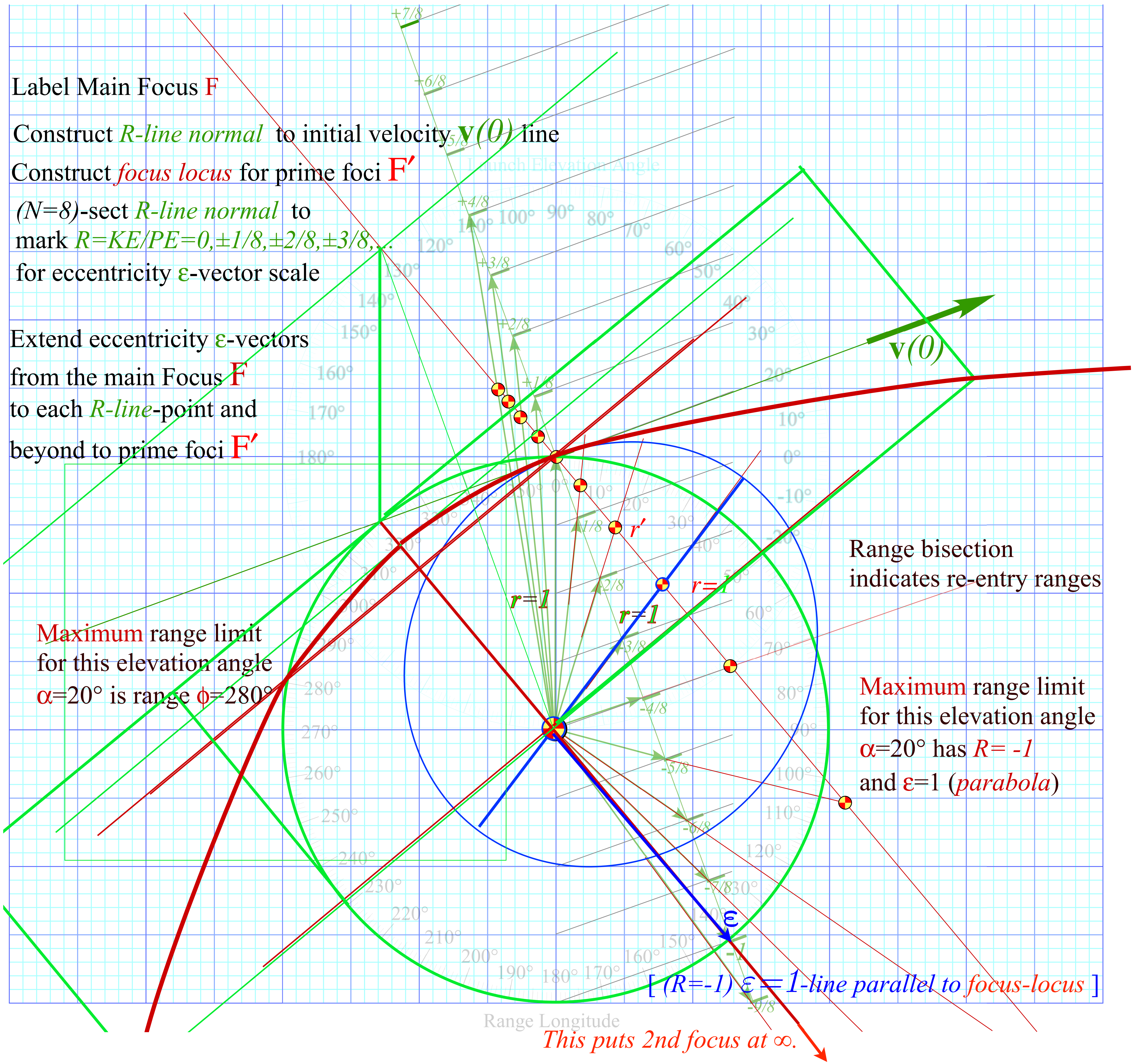
for eccentricity ε-vector scale
Extend eccentricity ε-vectors
from the main Focus F
to each R-line-point and
beyond to prime foci F′
Launch Elevation Angle
v(0)
r′
r=1
r=1
r=1
Range bisection
indicates re-entry ranges
Maximum range limit
for this elevation angle
α=20° has R= -1
and ε=1 (parabola)
ε
[(R=-1) ε =1-line parallel to focus-locus]
Range Longitude
This puts 2nd focus at ∞.



Label Main Focus F
Construct R-line normal to initial velocity v(0) line
Construct focus locus for prime foci F′
(N=8)-sect R-line normal to
mark R=KE/PE=0,±1/8,±2/8,±3/8,...
for eccentricity ε-vector scale
Extend eccentricity ε-vectors
from the main Focus F
to each R-line-point and
beyond to prime foci F′
Maximum range limit
for this elevation angle
α=20° is range ϕ=280°
Range bisection
indicates re-entry ranges
Maximum range limit
for this elevation angle
α=20° has R= -1
and ε=1 (parabola)
[(R=-1) ε=1-line parallel to focus-locus]
This puts 2nd focus at ∞.
v(0)
r=1
r′
ε
Launch Elevation Angle
Range Longitude
+7/8
+6/8
+5/8
+4/8
+3/8
+2/8
+1/8
-4/8
-5/8
-6/8
-7/8
-1
-9/8



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(N=8)-sect R-line normal to
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from the main Focus F
to each R-line-point and
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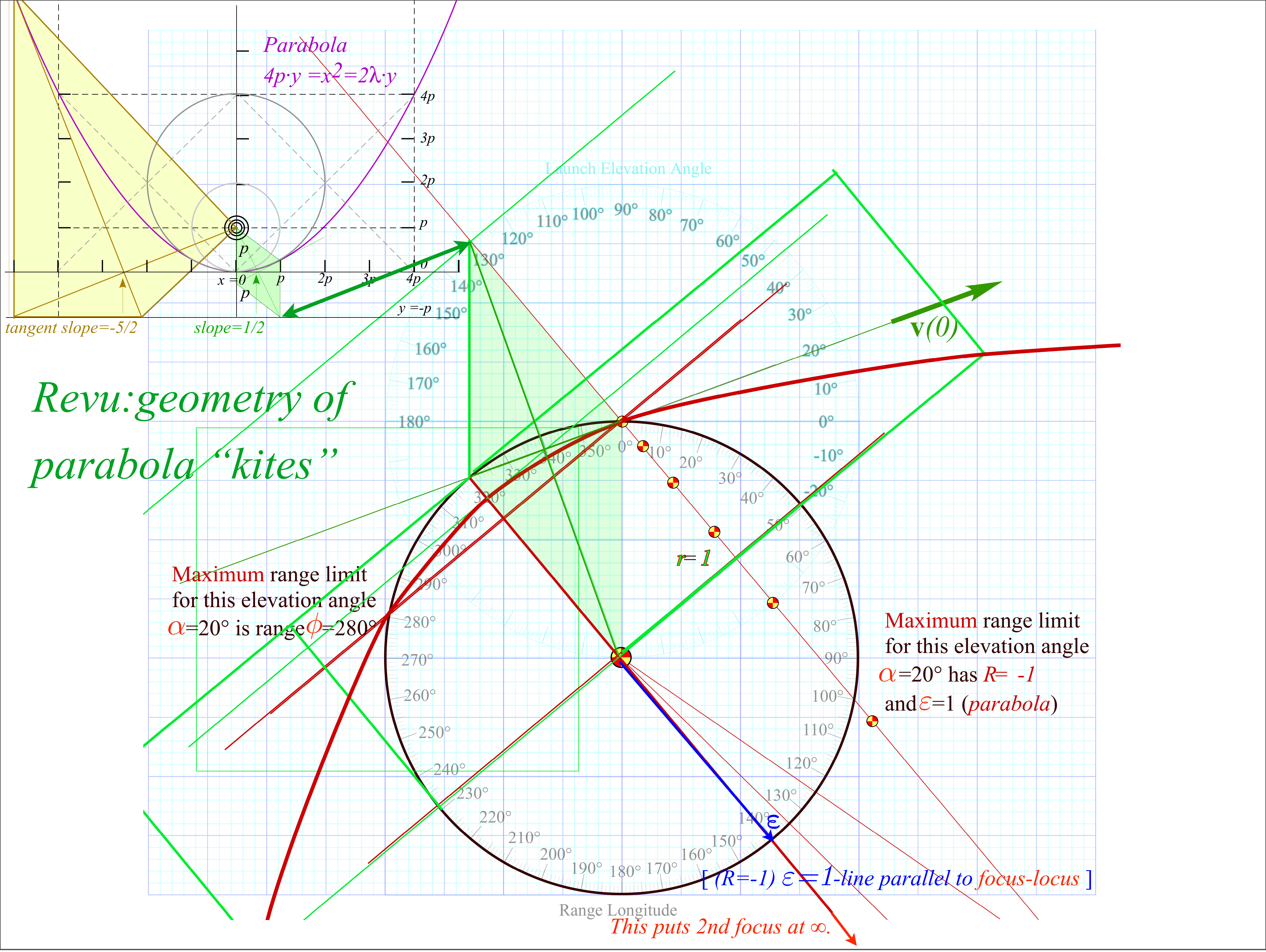
Maximum range limit
for this elevation angle
α=20° is range ϕ=280°
Range bisection
indicates re-entry ranges
Maximum range limit
for this elevation angle
α=20° has R= -1
and ε=1 (parabola)
v(0)
r′
r=1
r=1
r=1
ε
[(R=-1) ε=1-line parallel to focus-locus]
This puts 2nd focus at ∞.
Launch Elevation Angle
Range Longitude



Parabola
4p·y =x²=2λ·y
4p
3p
2p
p
x =0
y =-p
tangent slope=-5/2
slope=1/2
Launch Elevation Angle
Revu:geometry of
parabola "kites"
v(0)
r= 1
Maximum range limit
for this elevation angle
α=20° is range φ=280°
Maximum range limit
for this elevation angle
α=20° has R= -1
and ε=1 (parabola)
ε
[(R=-1) ε=1-line parallel to focus-locus]
Range Longitude
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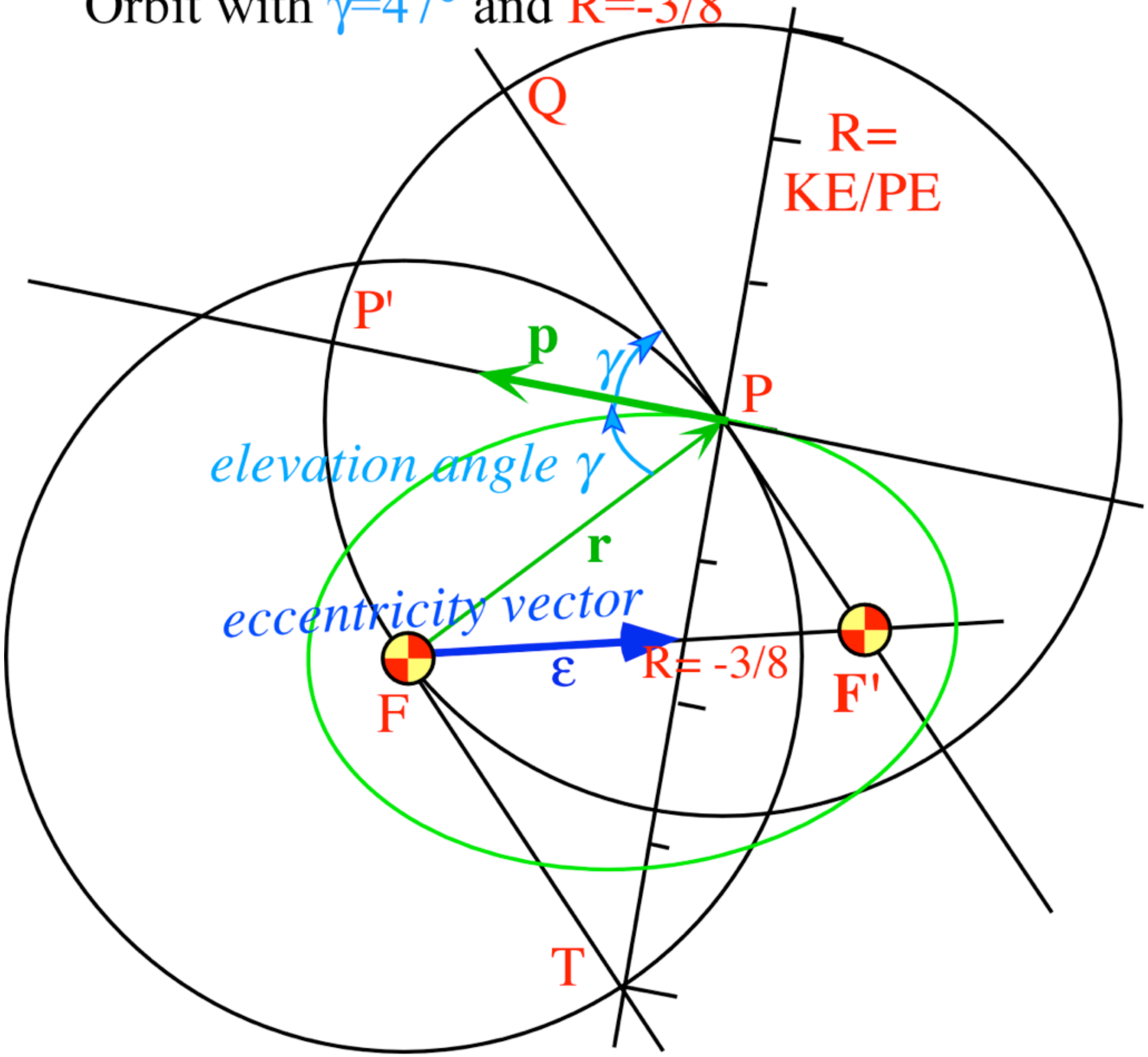
➡ *Graphical $\boldsymbol{\varepsilon}$-development of orbits*

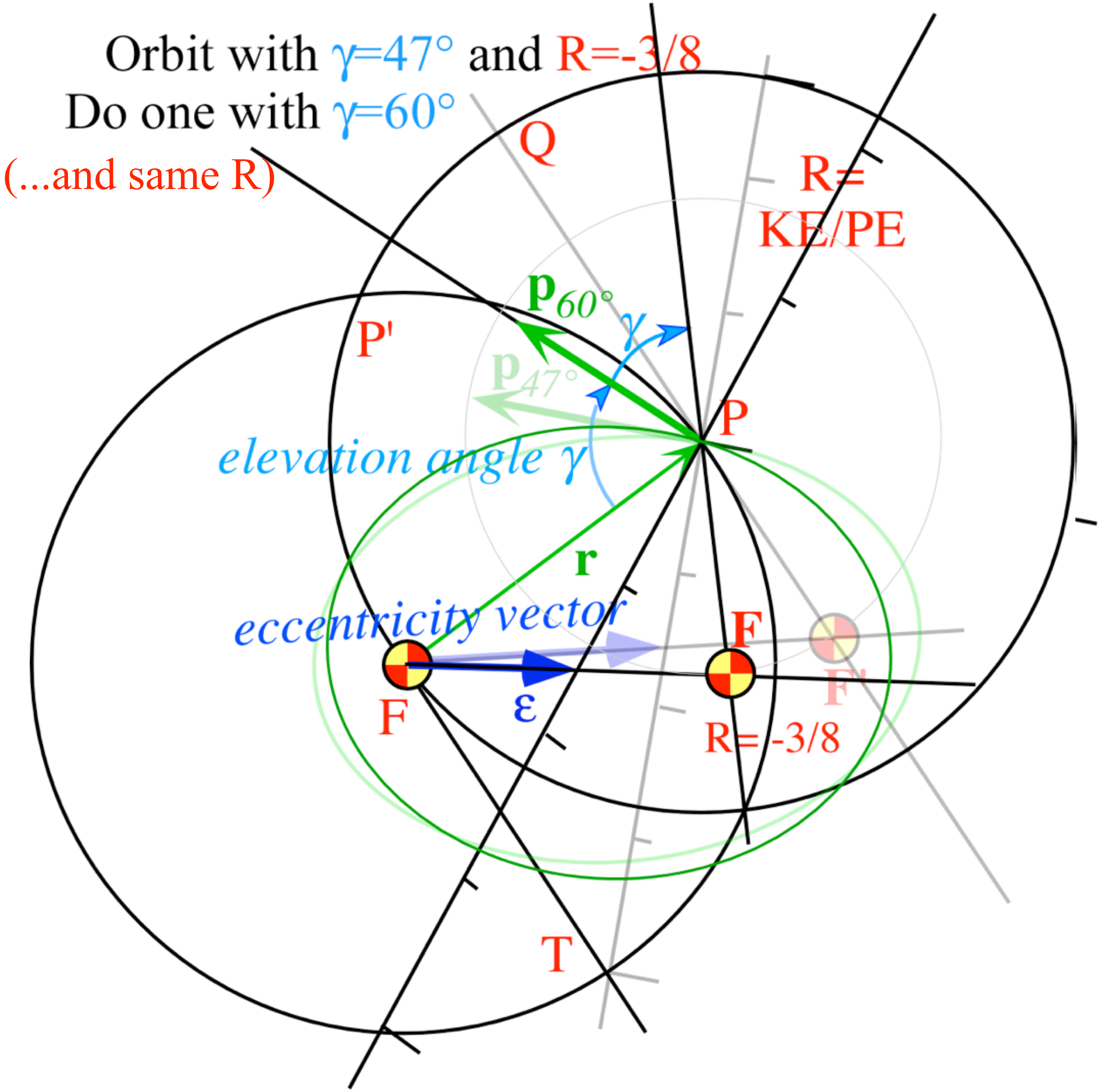
Launch angle fixed-Varied launch energy

➡ *Launch energy fixed-Varied launch angle*

Launch optimization and orbit family envelopes

Orbit with γ=47° and R=-3/8



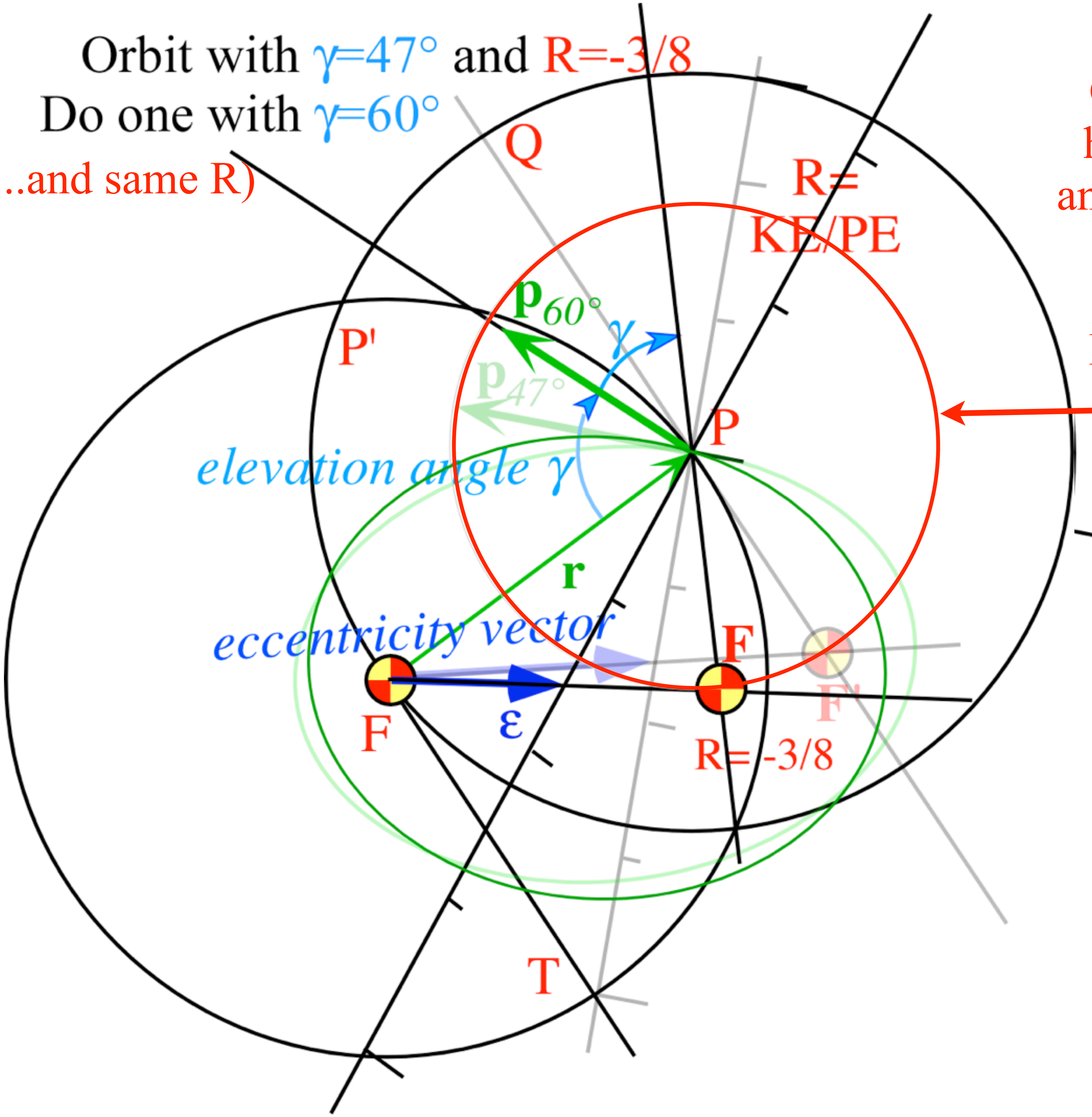


Orbit with γ=47° and R=-3/8
Do one with γ=60°
(...and same R)
Q
R=
KE/PE
P'
p60°
γ
p47°
P
elevation angle γ
r
eccentricity vector
F
F
F'
ε
R=-3/8
T

Orbit with γ=47° and R=-3/8

Do one with γ=60°

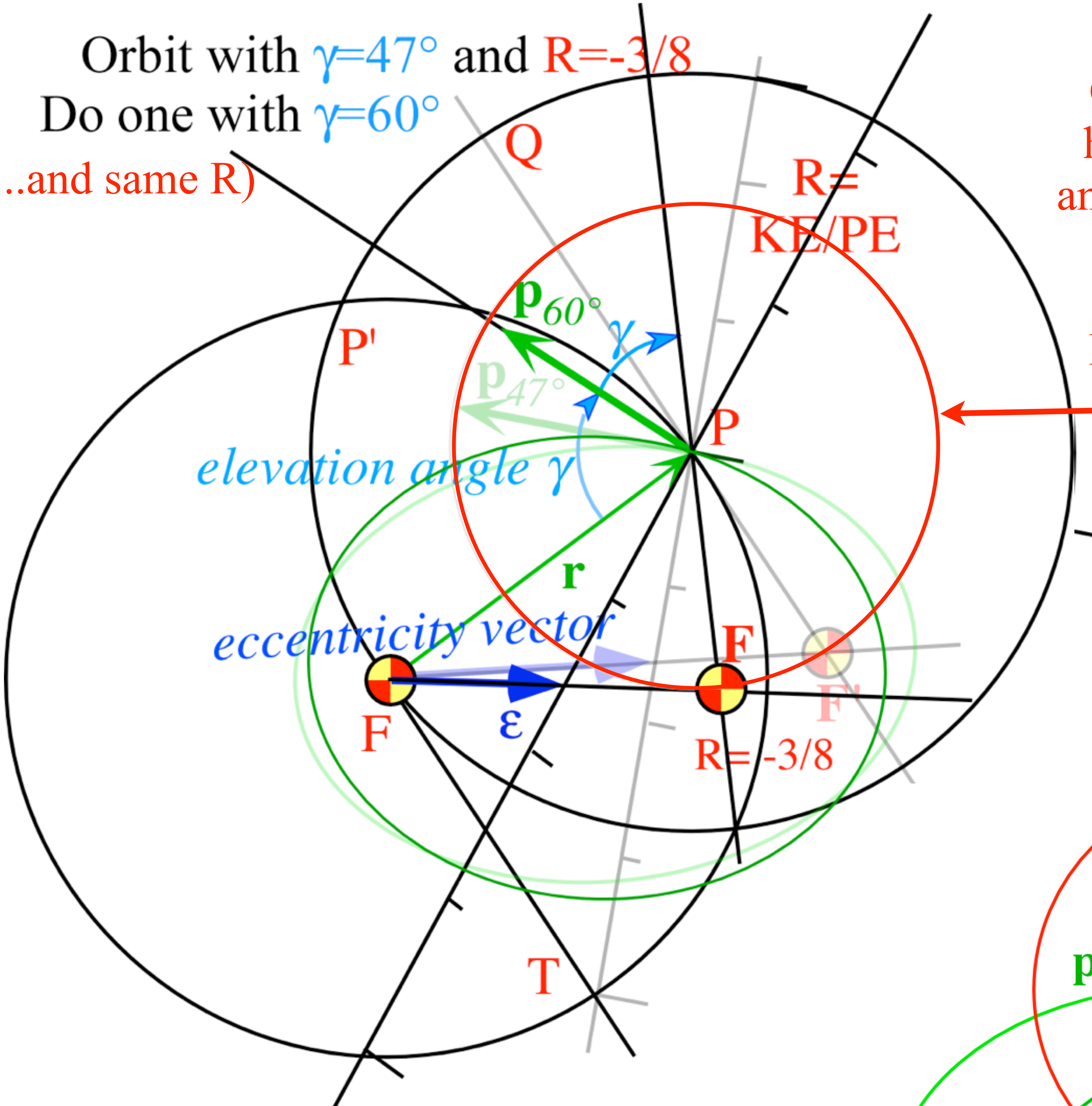
(...and same R)



Orbits with the same R
have the same energy E
and the same major radii *a*

Hence their foci lie on
a circle of radius *2a-r*
around launch point P

Orbit with γ=47° and R=-3/8
Do one with γ=60°
(...and same R)



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have the same energy E
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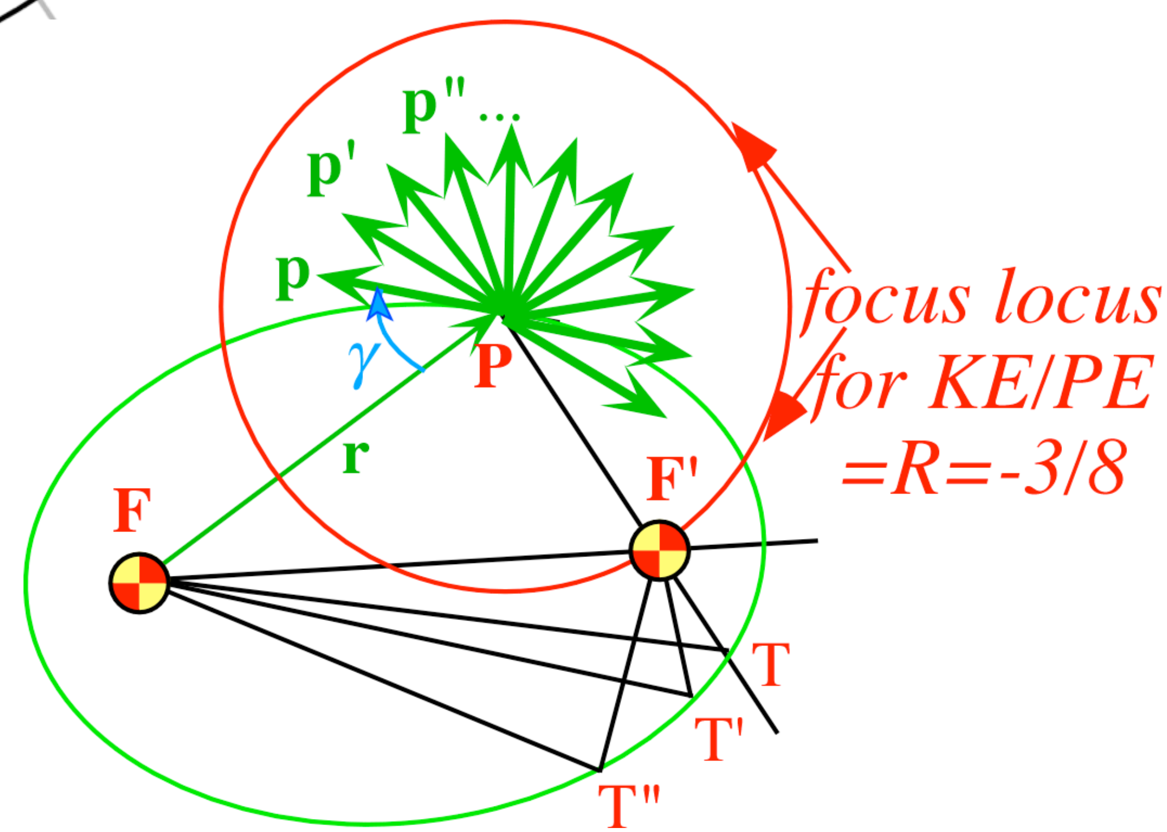
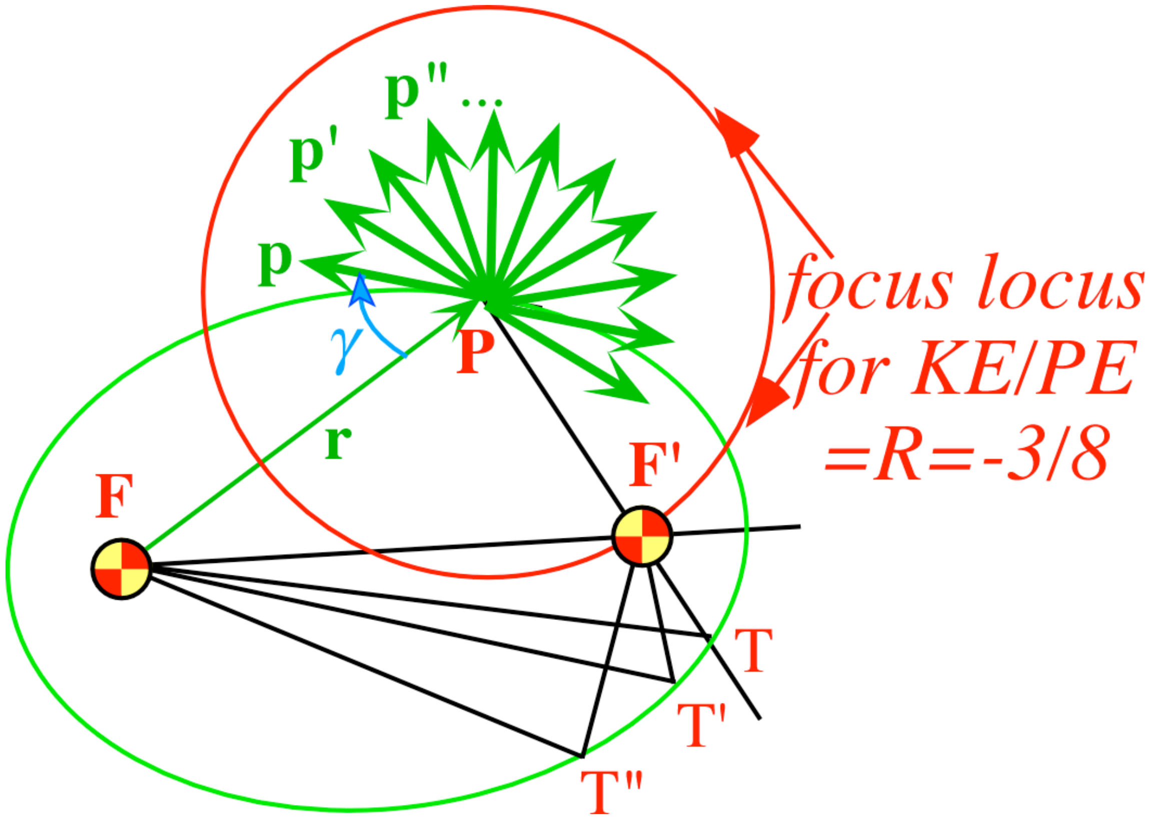
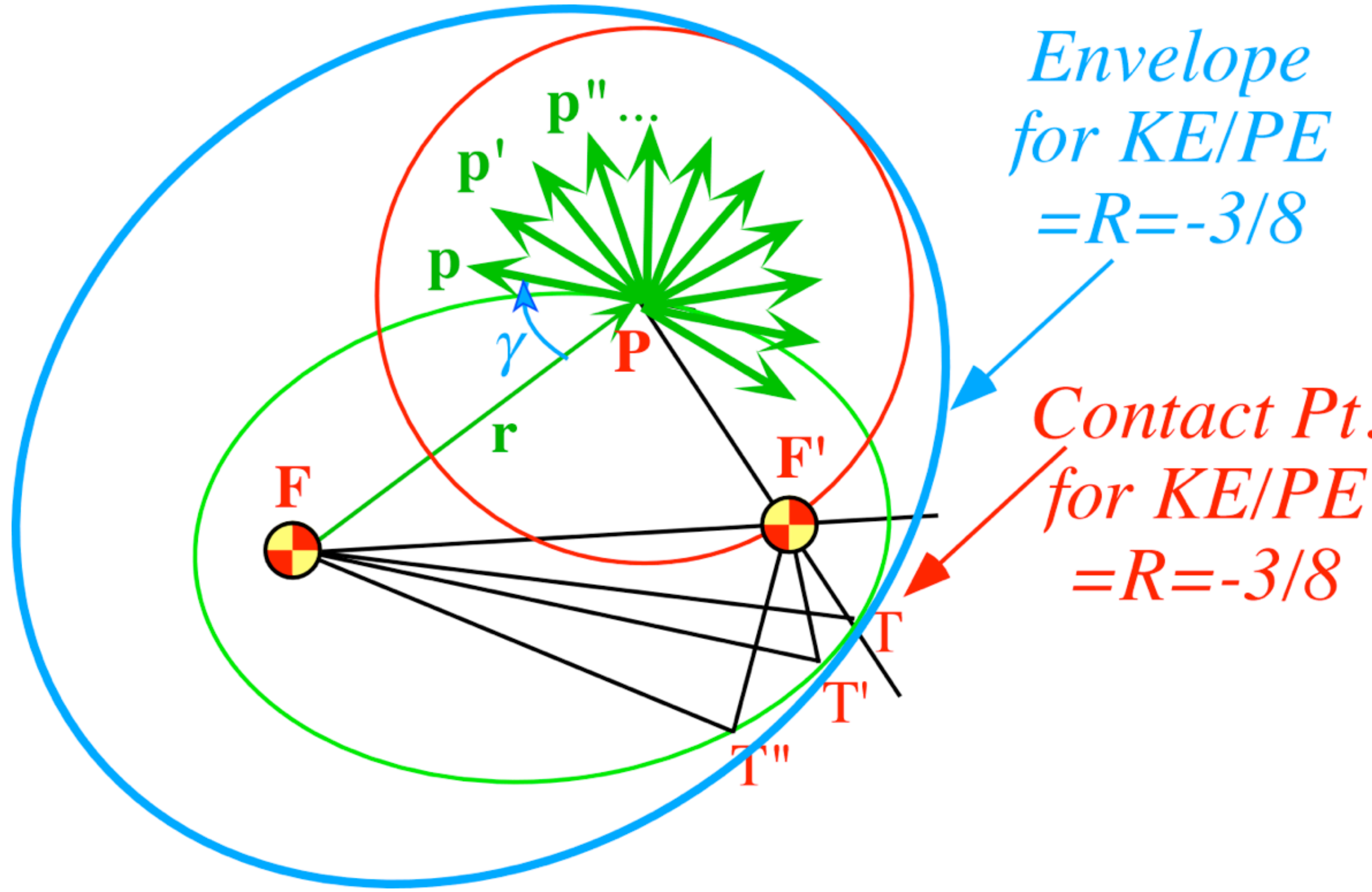
Hence their foci lie on
a circle of radius *2a-r*
around launch point P




p"...
p'
p
γ
P
r
F
F'
T
T'
T"
focus locus
for KE/PE
=R=-3/8


Envelope
for KE/PE
=R=-3/8
p"...
p'
p
γ
P
r
F
F'
Contact Pt.
for KE/PE
=R=-3/8
T
T'
T"

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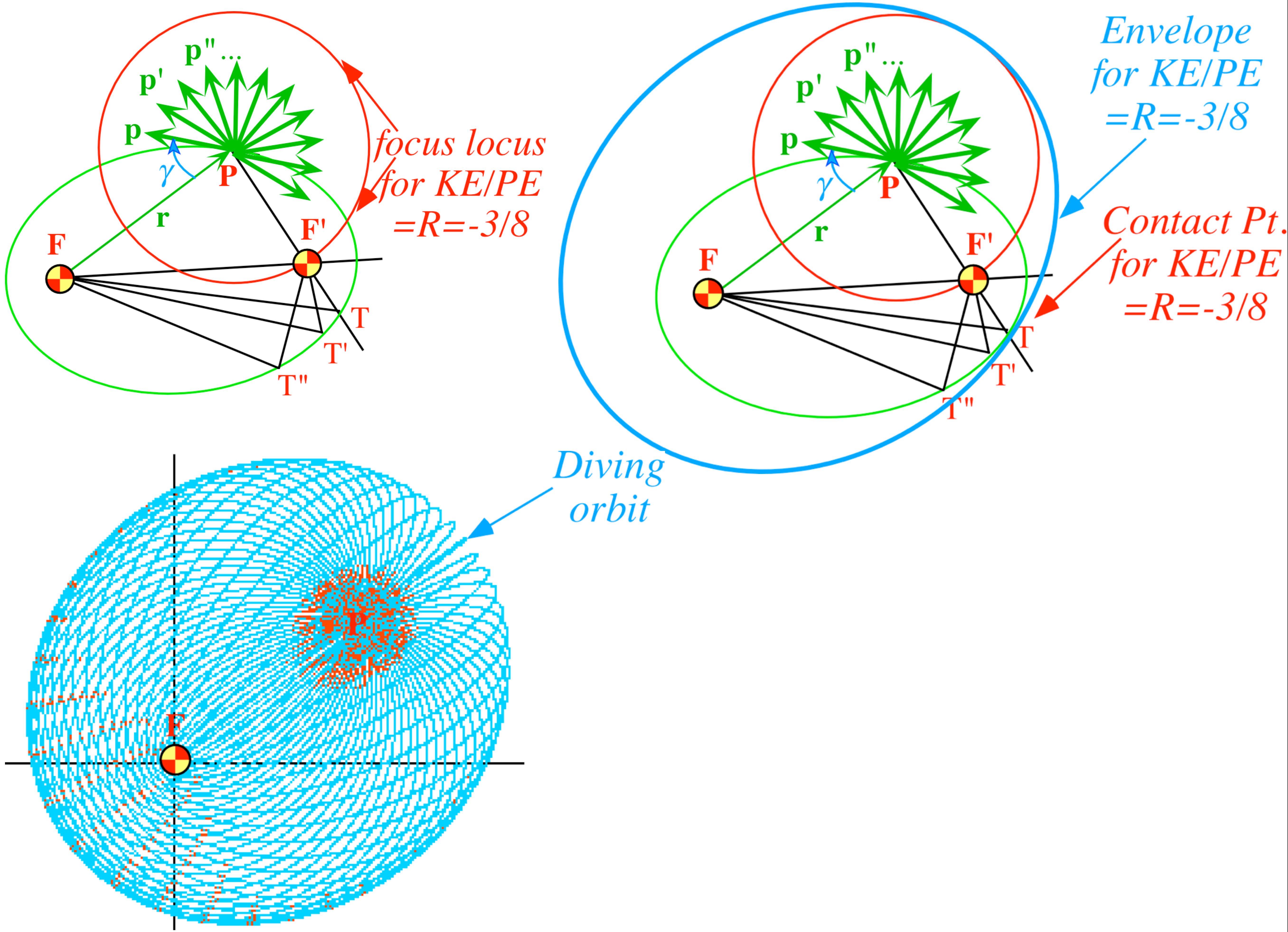

p"...
p'
p
γ
P
r
F
F'
T
T'
T"
focus locus
for KE/PE
=R=-3/8
Envelope
for KE/PE
=R=-3/8
Contact Pt.
for KE/PE
=R=-3/8
Diving
orbit

Coulomb envelope geometry

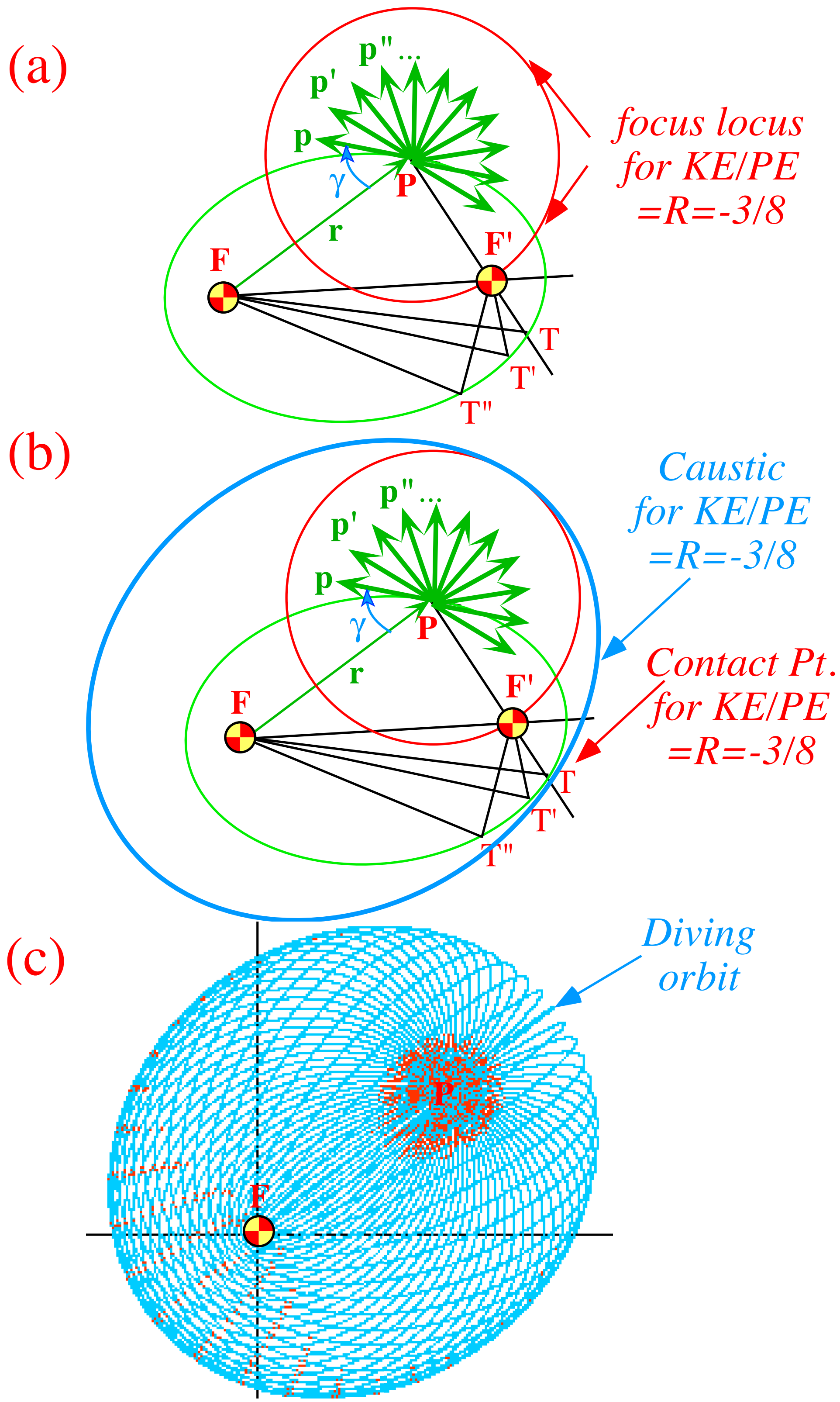


Fig. 5.4.4 in Unit 5 of CMwBANG!

Ideal comet "heads" or "tails" in solar wind

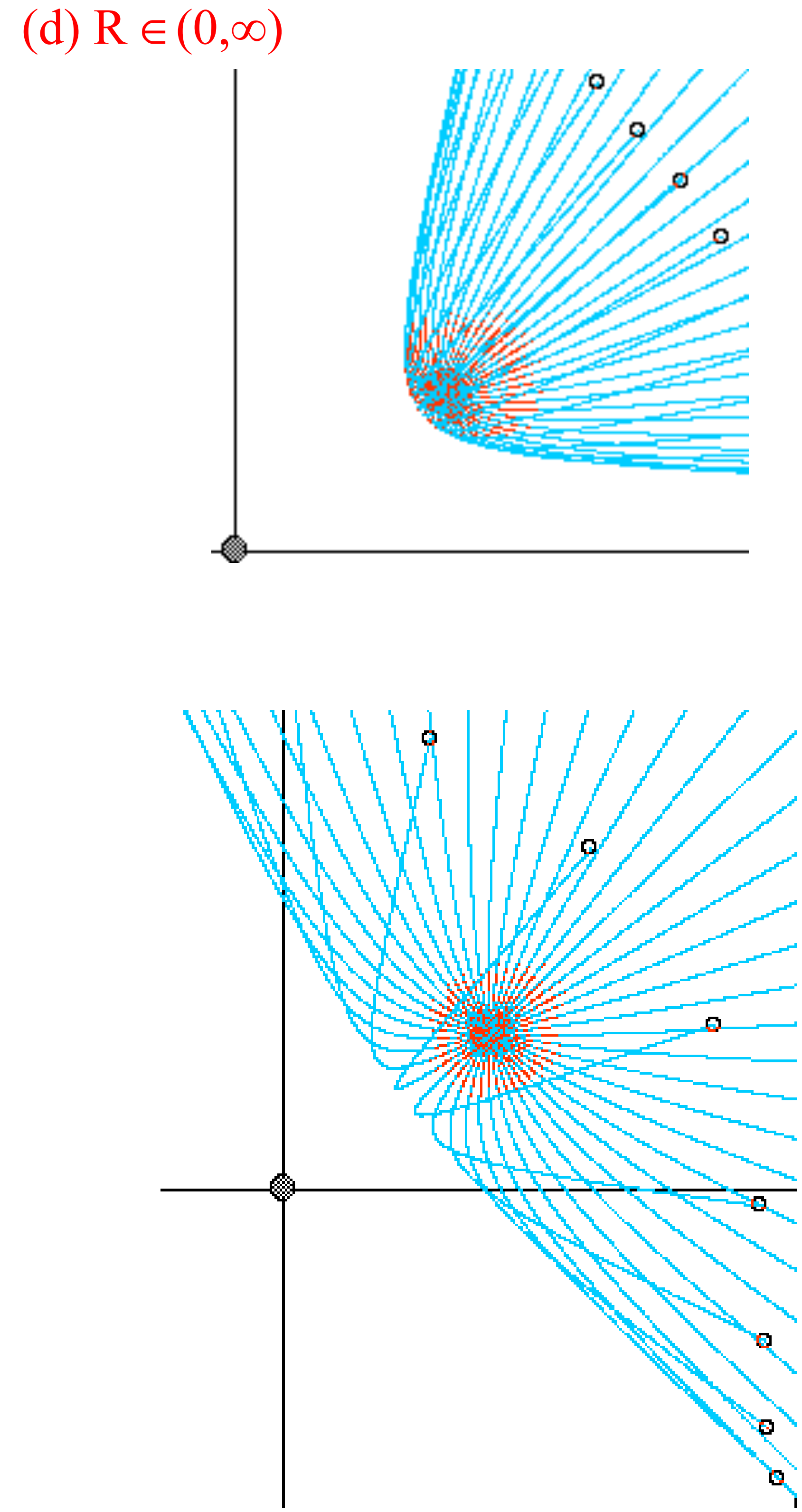
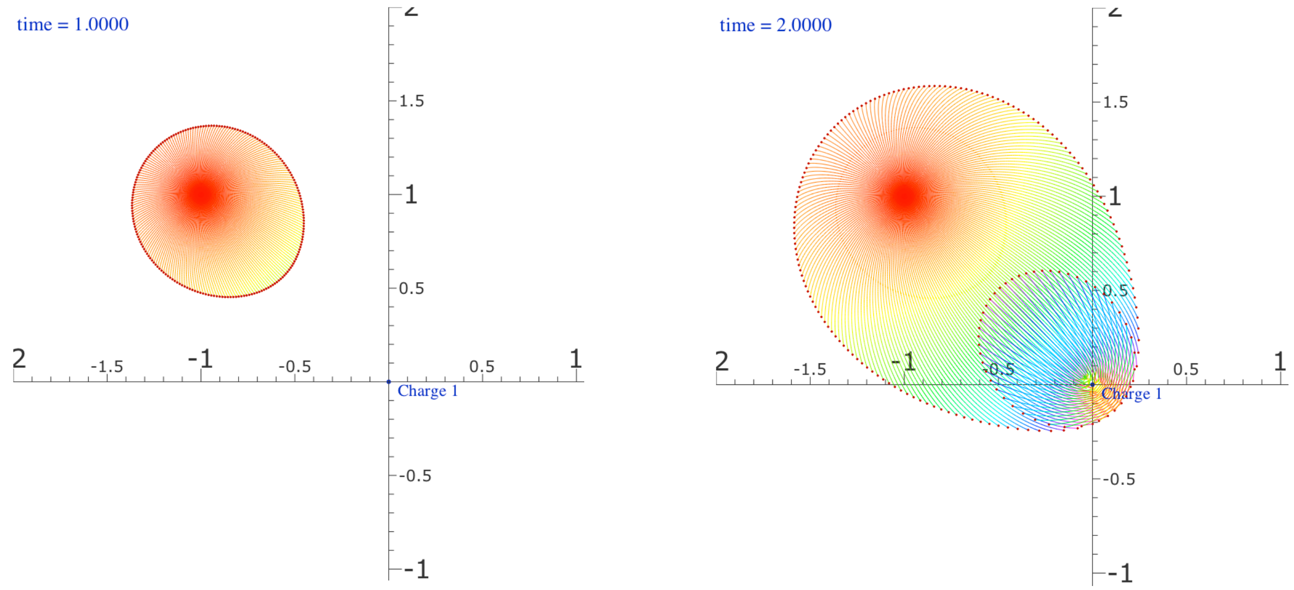
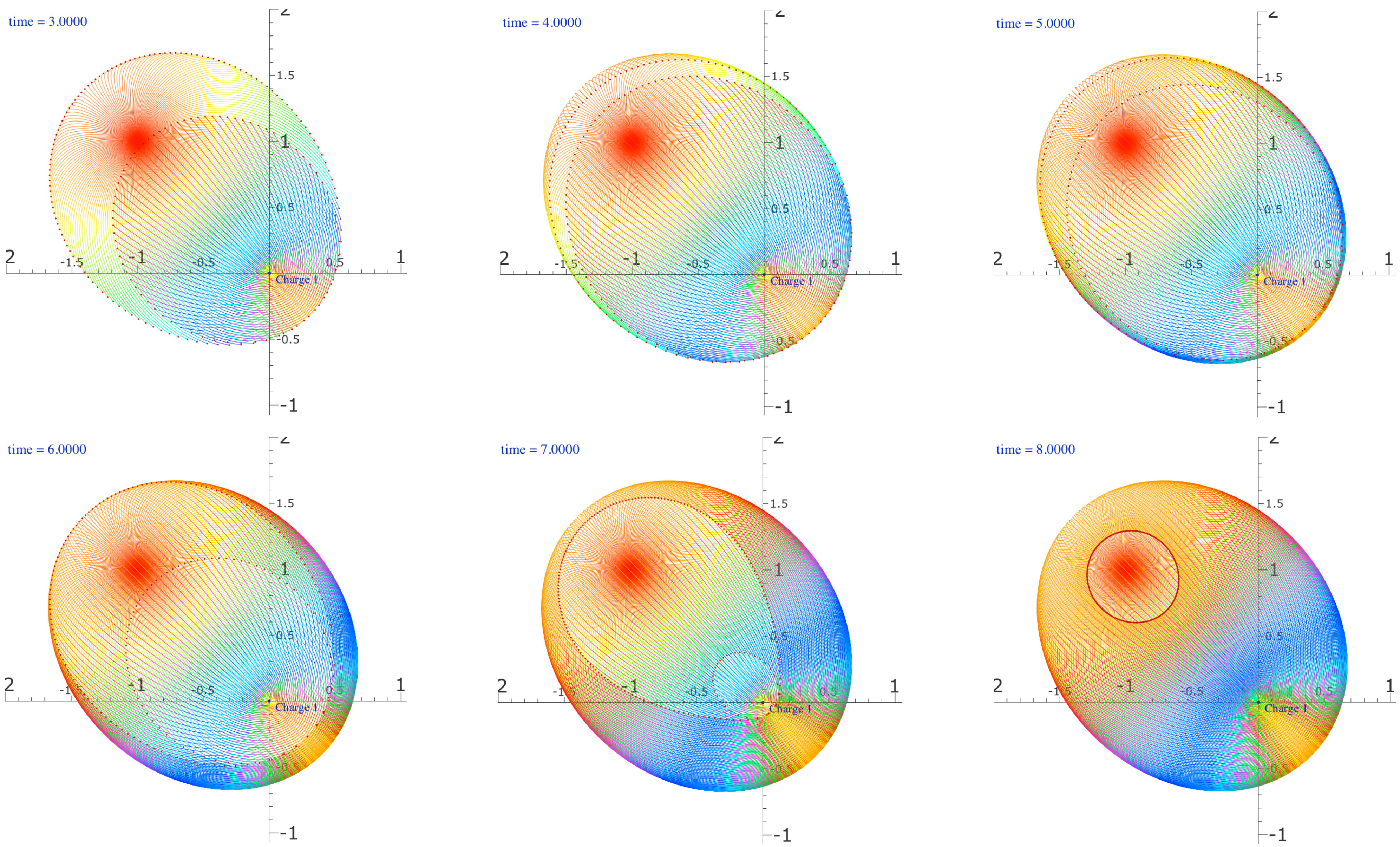
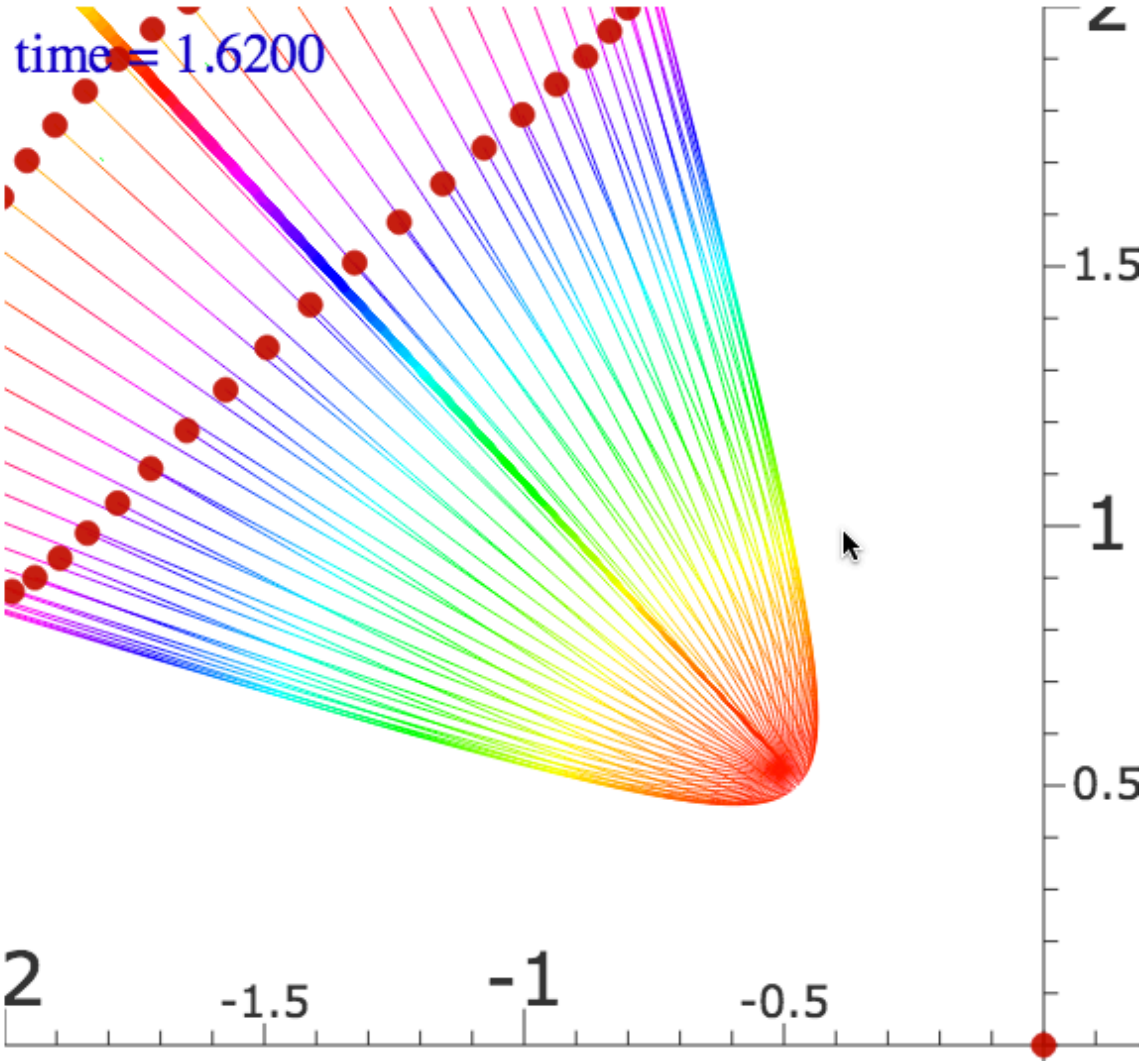


Fig. 5.4.5 in Unit 5 of CMwBANG!

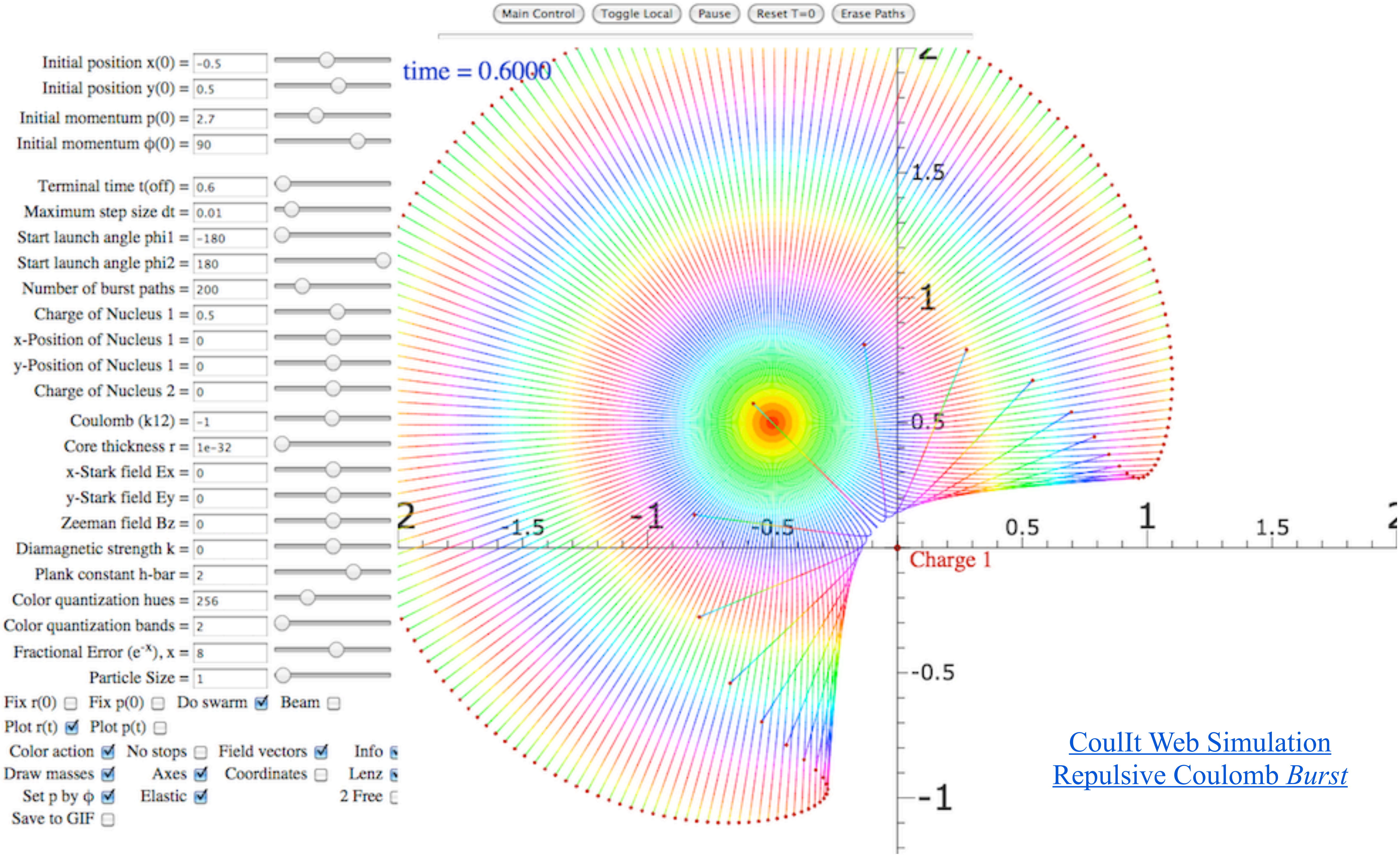
[CoulIt Web Simulation Attractive Coulomb *Burst*]



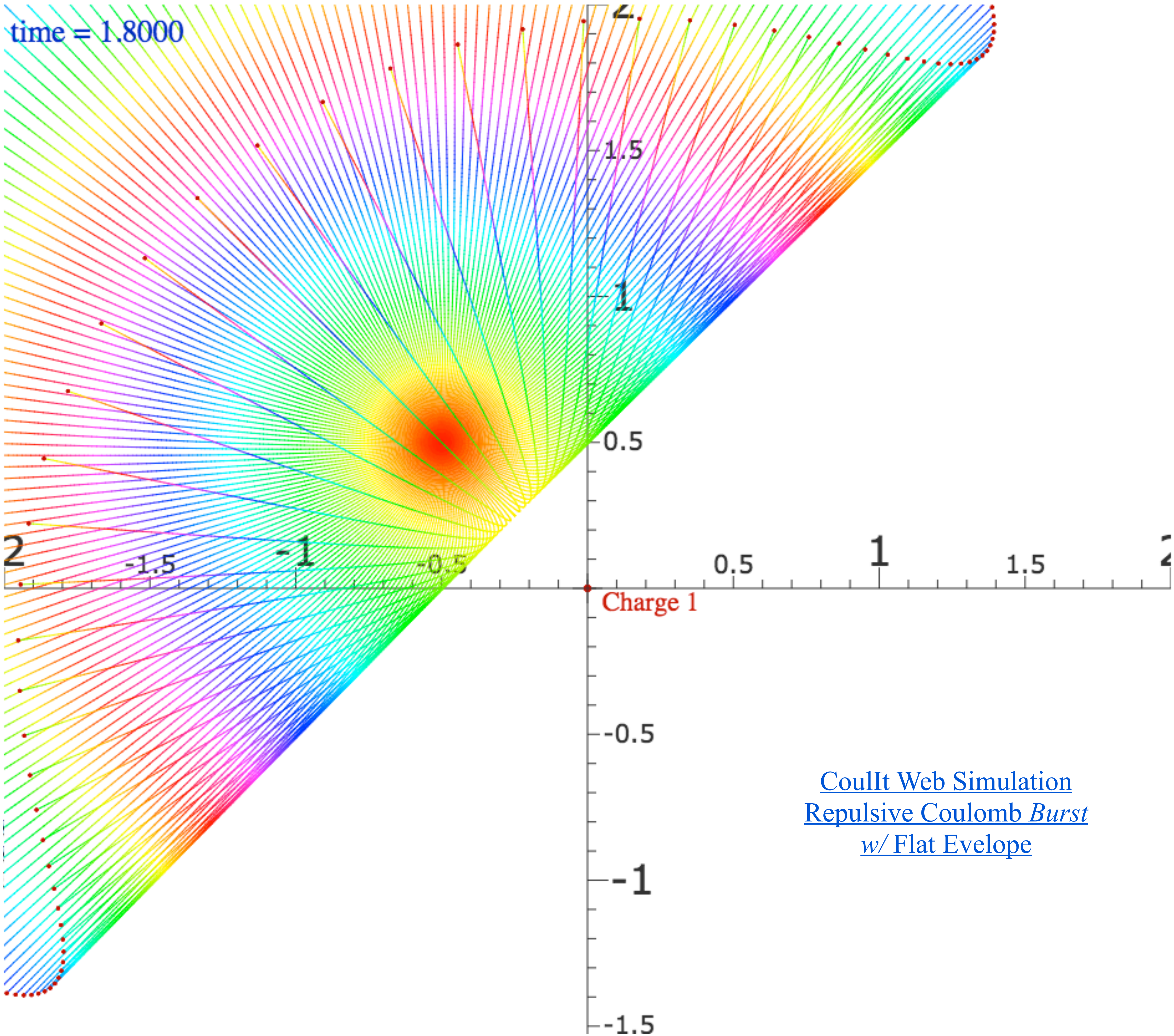




[CoulIt Web Simulation](#)
[Repulsive Coulomb *Burst - Tight*](#)



[CoulIt Web Simulation
Repulsive Coulomb *Burst*]



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w/ Flat Evelope]

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Launch energy fixed-Varied launch angle

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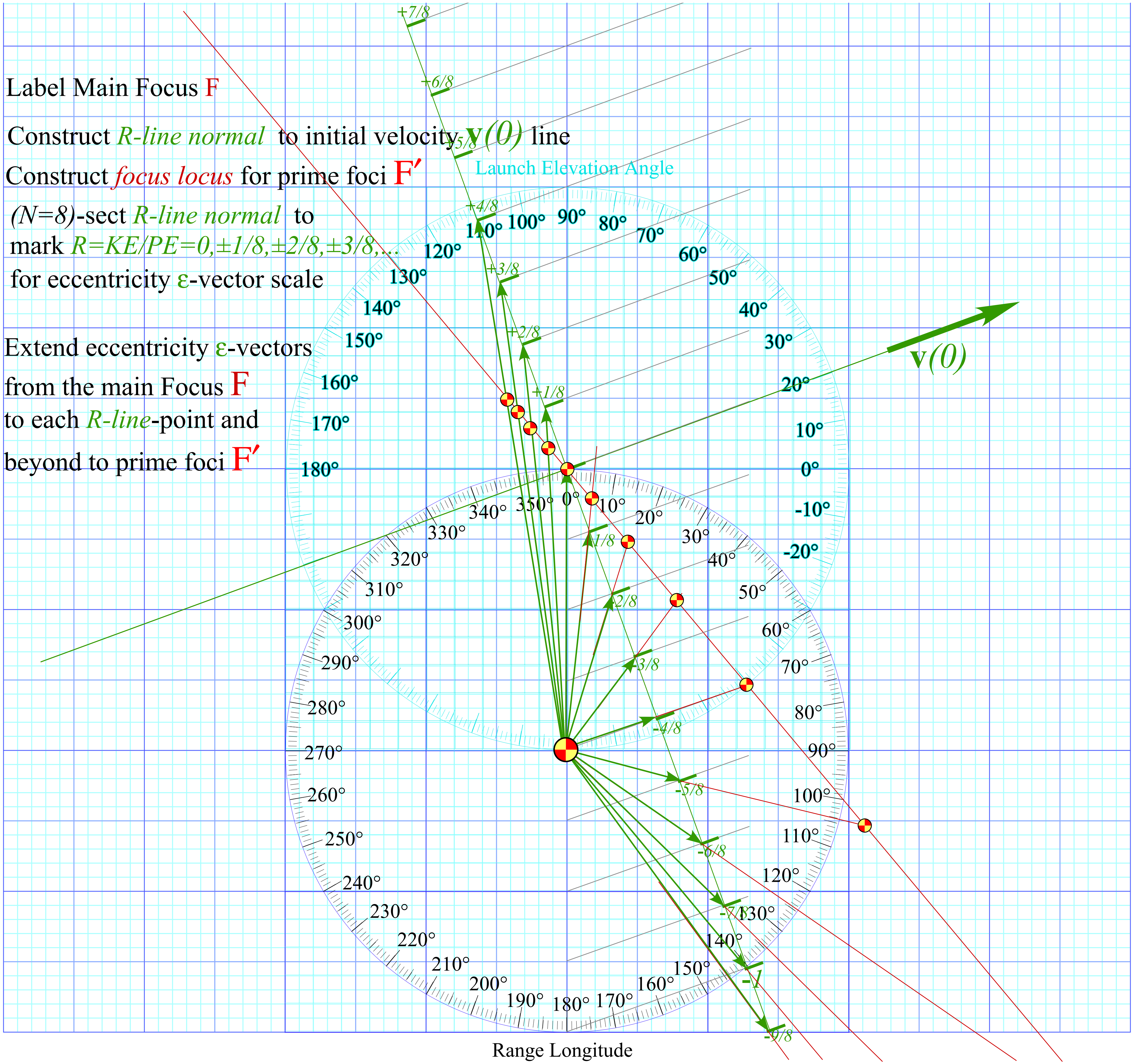
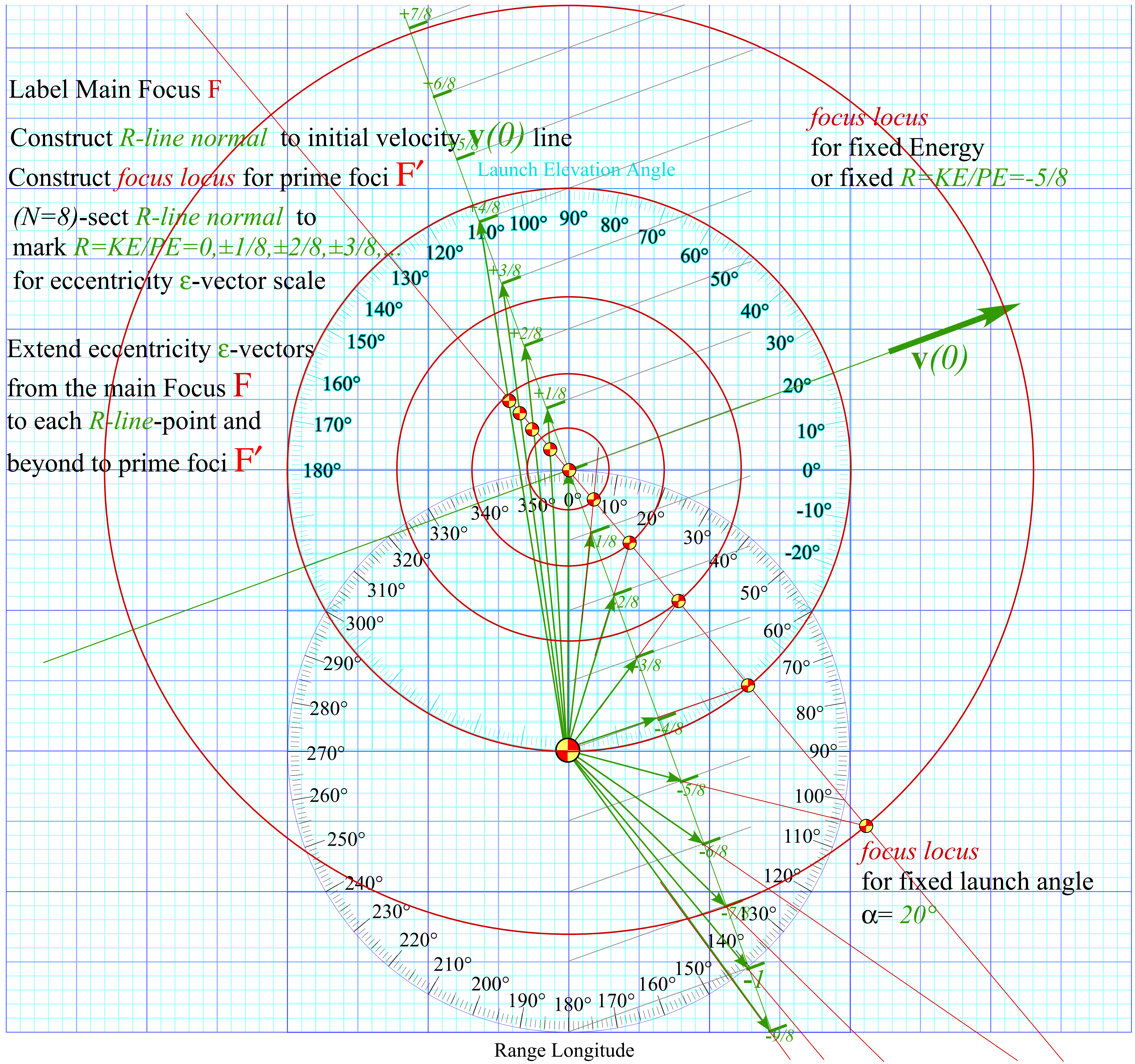


or **-v***(0)*



Label Main Focus F
Construct R-line normal to initial velocity v(0) line
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(N=8)-sect R-line normal to
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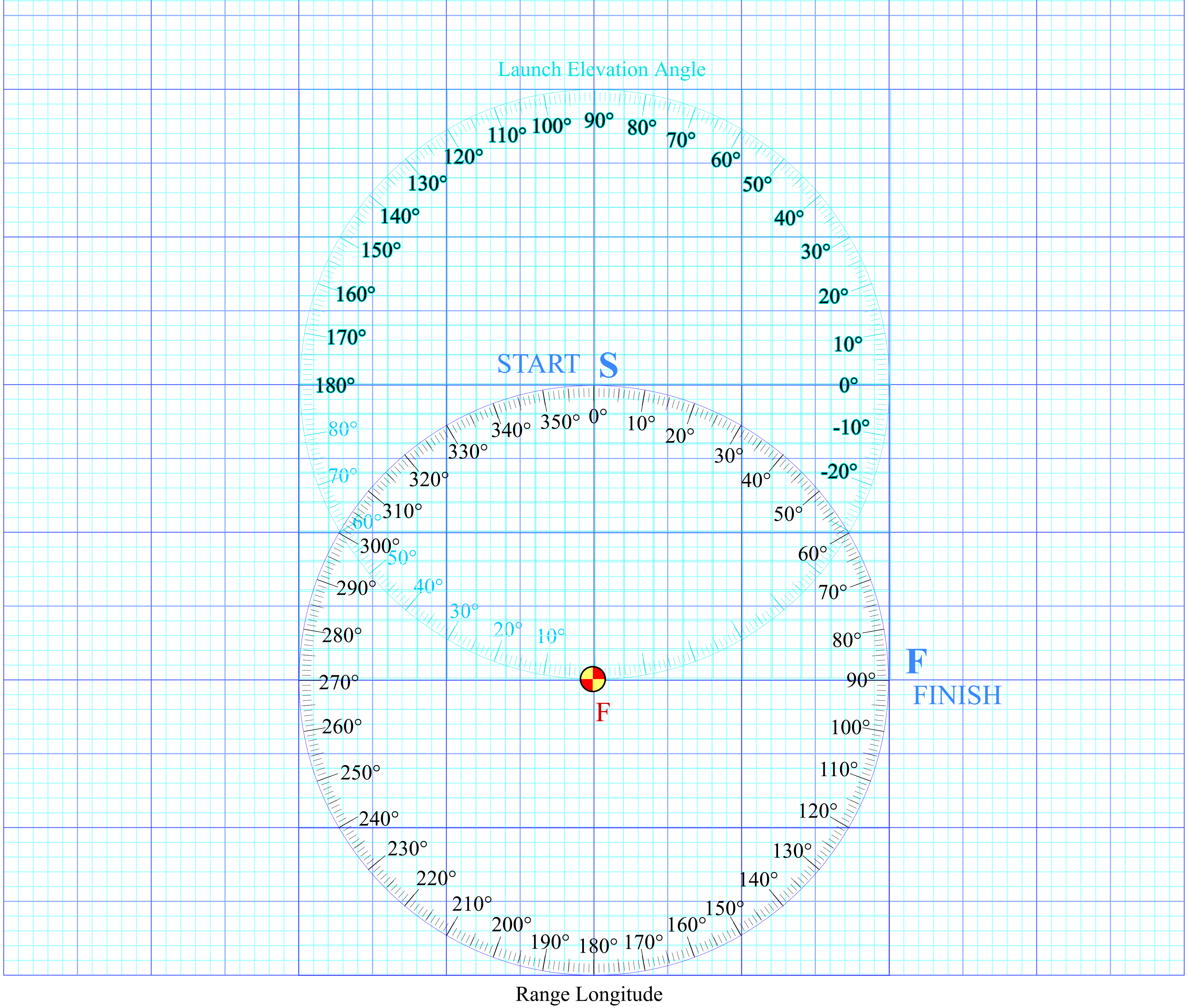
for eccentricity ε-vector scale
Extend eccentricity ε-vectors
from the main Focus F
to each R-line-point and
beyond to prime foci F′
focus locus
for fixed Energy
or fixed R=KE/PE=-5/8
Launch Elevation Angle
v(0)
focus locus
for fixed launch angle
α= 20°
+7/8
+6/8
+5/8
+4/8
+3/8
+2/8
+1/8
-1/8
-2/8
-3/8
-4/8
-5/8
-6/8
-7/8
-1
-9/8
Range Longitude

Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

Problem:

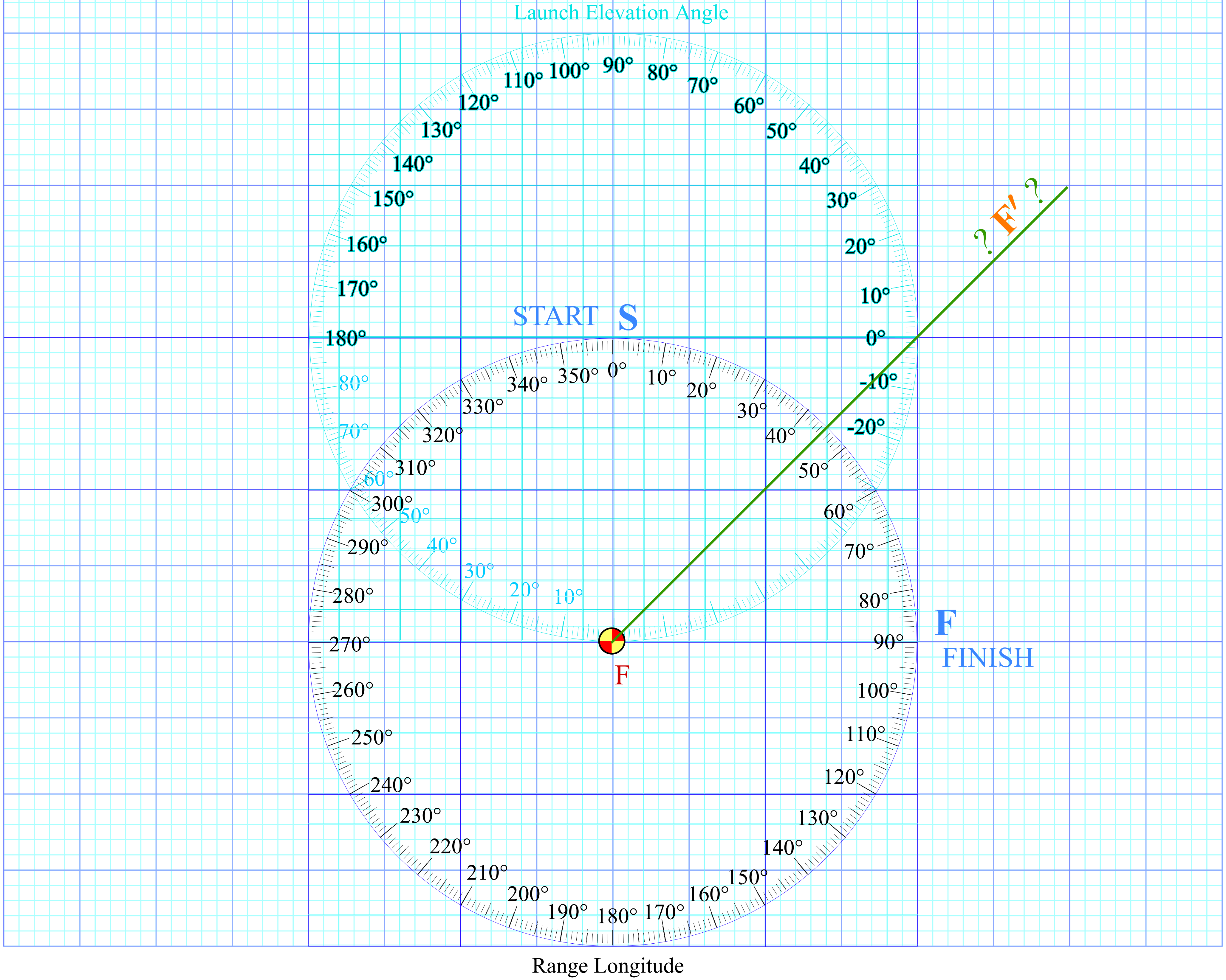
Find trajectory angle of minimum energy to fly 90° of arc (1/4 around planet)



Problem:

Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

Solution: Prime focus **F′** *lies on radial line that bisects longitude angle*

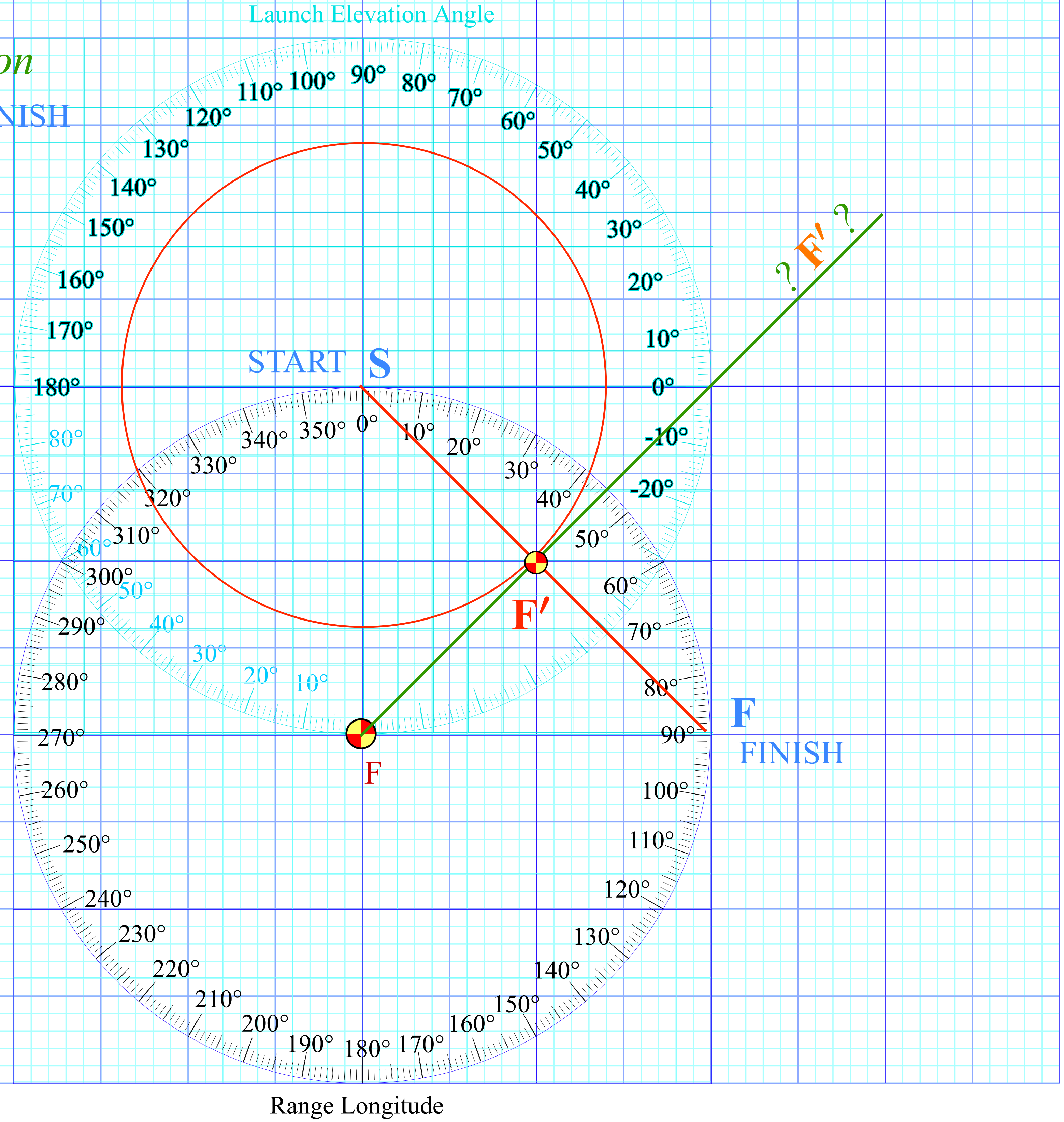


Problem:

Find trajectory angle of minimum energy to fly 90° of longitude (1/4 around planet)

Solution: Prime focus **F′** *lies on radial line that bisects longitude angle*

Optimal prime focus **F′** *lies on line connecting* START *and* FINISH *at tangent point of minimal energy circle* **SF′**.



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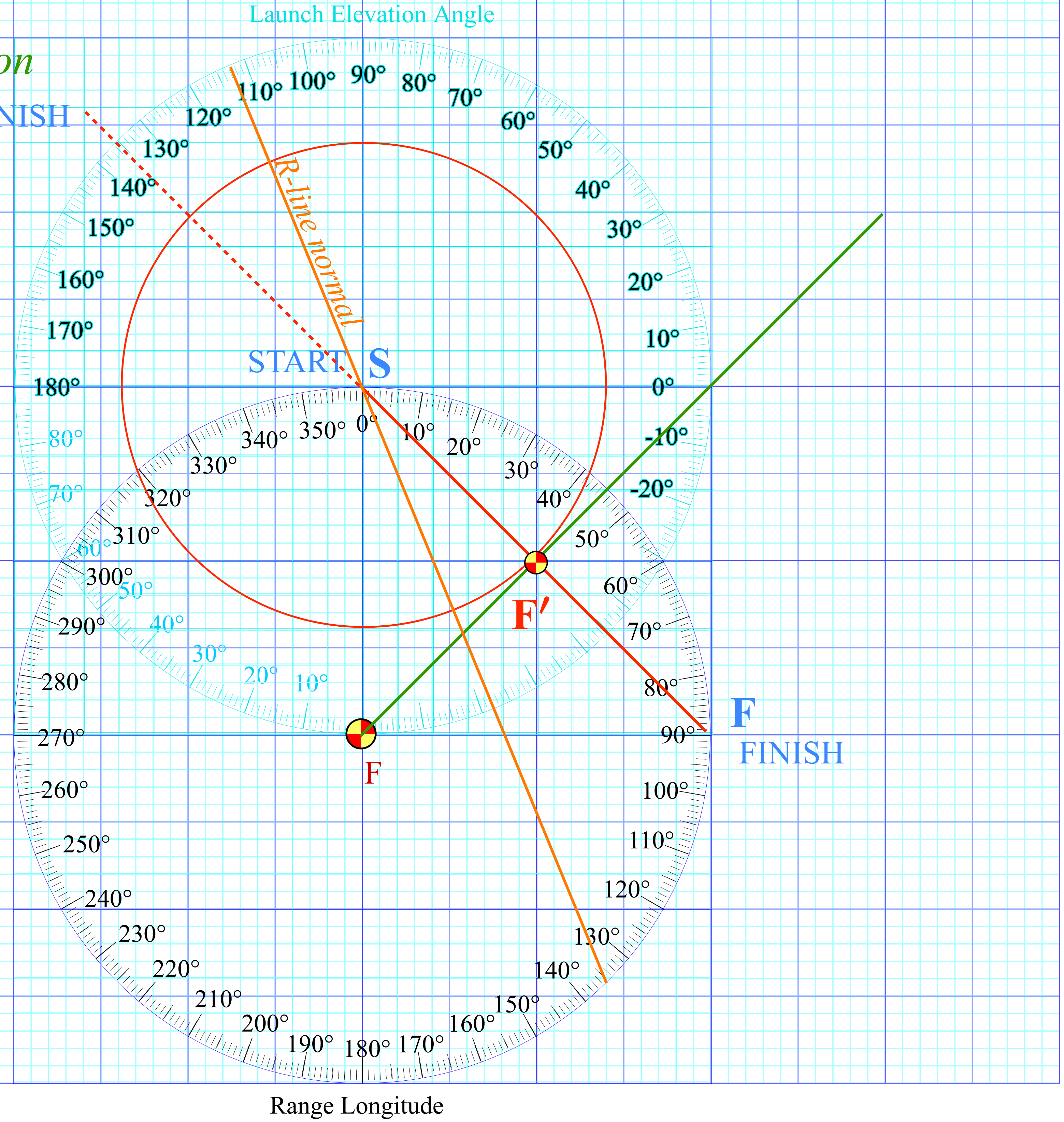
R-line normal must bisect angle **FSF′** *connecting foci* **F** *and* **F′** *and is normal to initial launch vector* $\mathbf{v}_0$



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Solution: Prime focus **F′** *lies on radial line that bisects longitude angle*

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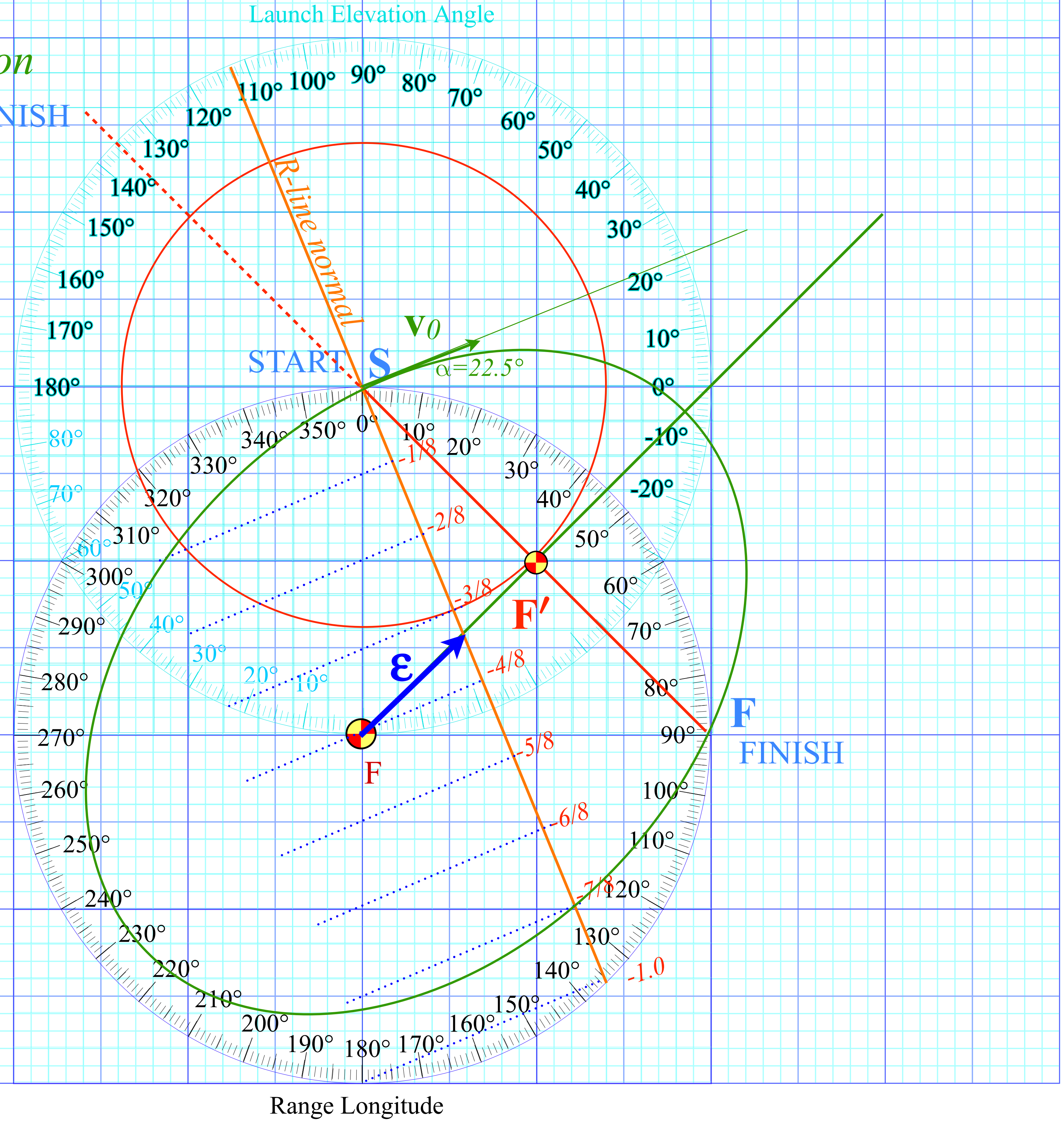
R-line normal must bisect angle **FSF′** *connecting foci* **F** *and* **F′** *and is normal to initial launch vector* $\mathbf{v}_0$ *with launch angle* α*=22.5°*

The **ε***-vector and R-value: slightly below R=-3/8...*



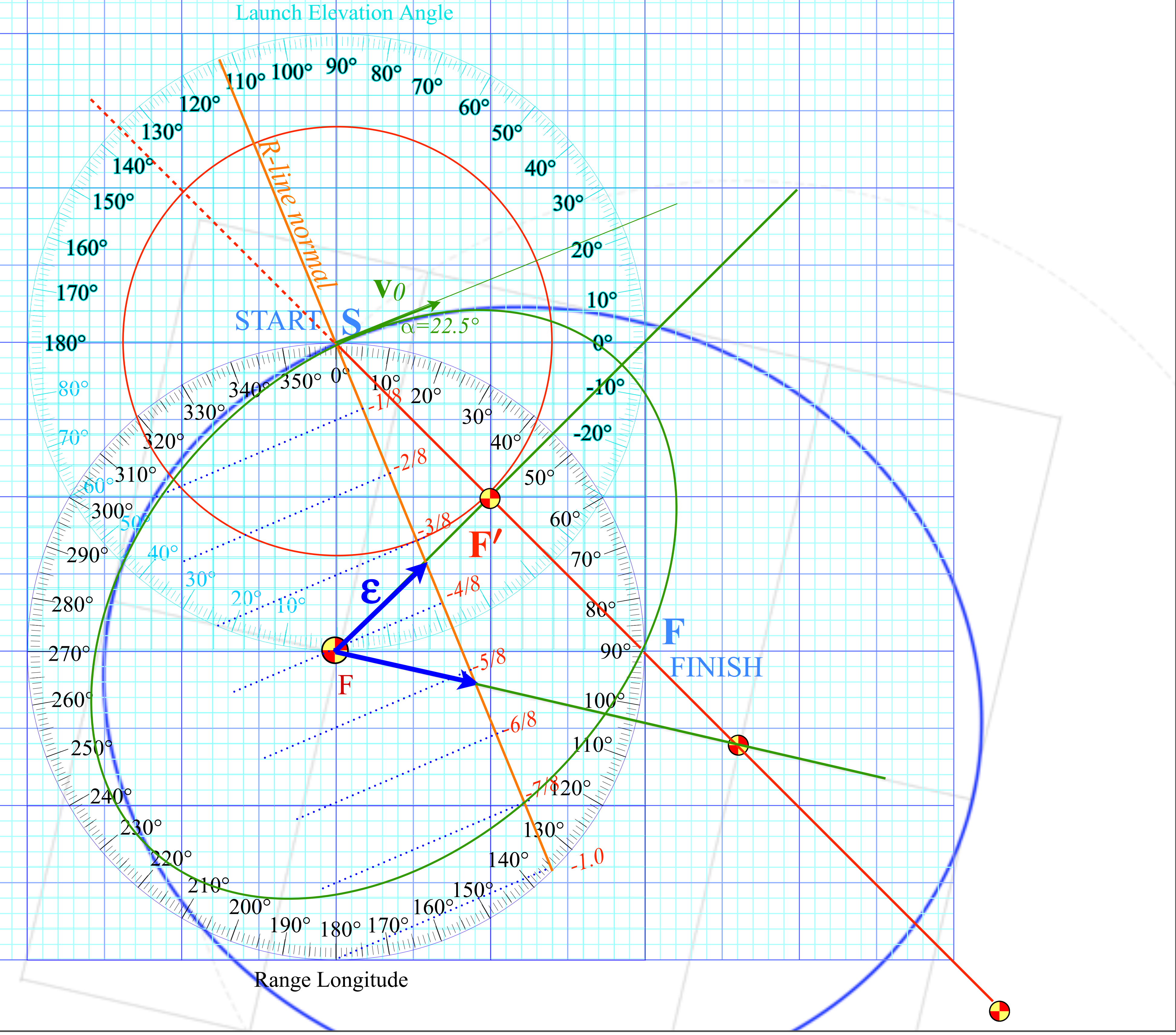
Graphs and protractors help Coulomb trajectory launch optimization and orbit family envelopes

Problem:

With launch angle α=22.5° find trajectory to fly 207° of longitude

*Solution: Prime focus **F′** lies on radial line at 103.5° that bisects longitude angle 207°*

The ε-vector and R-value: slightly below R=-5/8...



Problem:

With launch angle α=22.5° find maximum range of trajectory.

*Solution: Prime focus **F′** lies at infinity and gives parabola (ε_∞=1,R=-1) trajectory.*

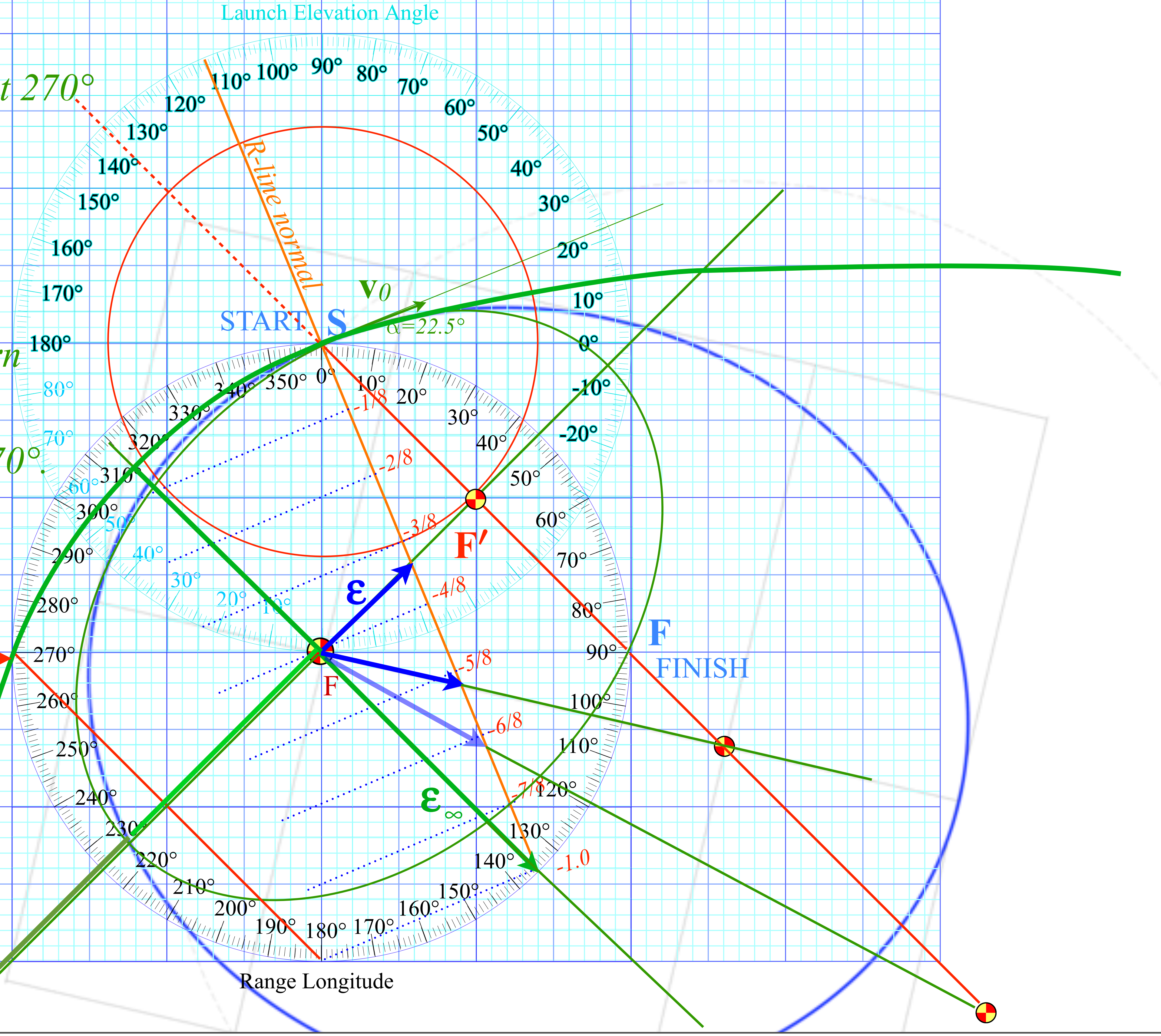
Trajectory axis is at 135°.

Trajectory would hit Earth at 270°

...if it actually returned...

...but a parabola cannot!

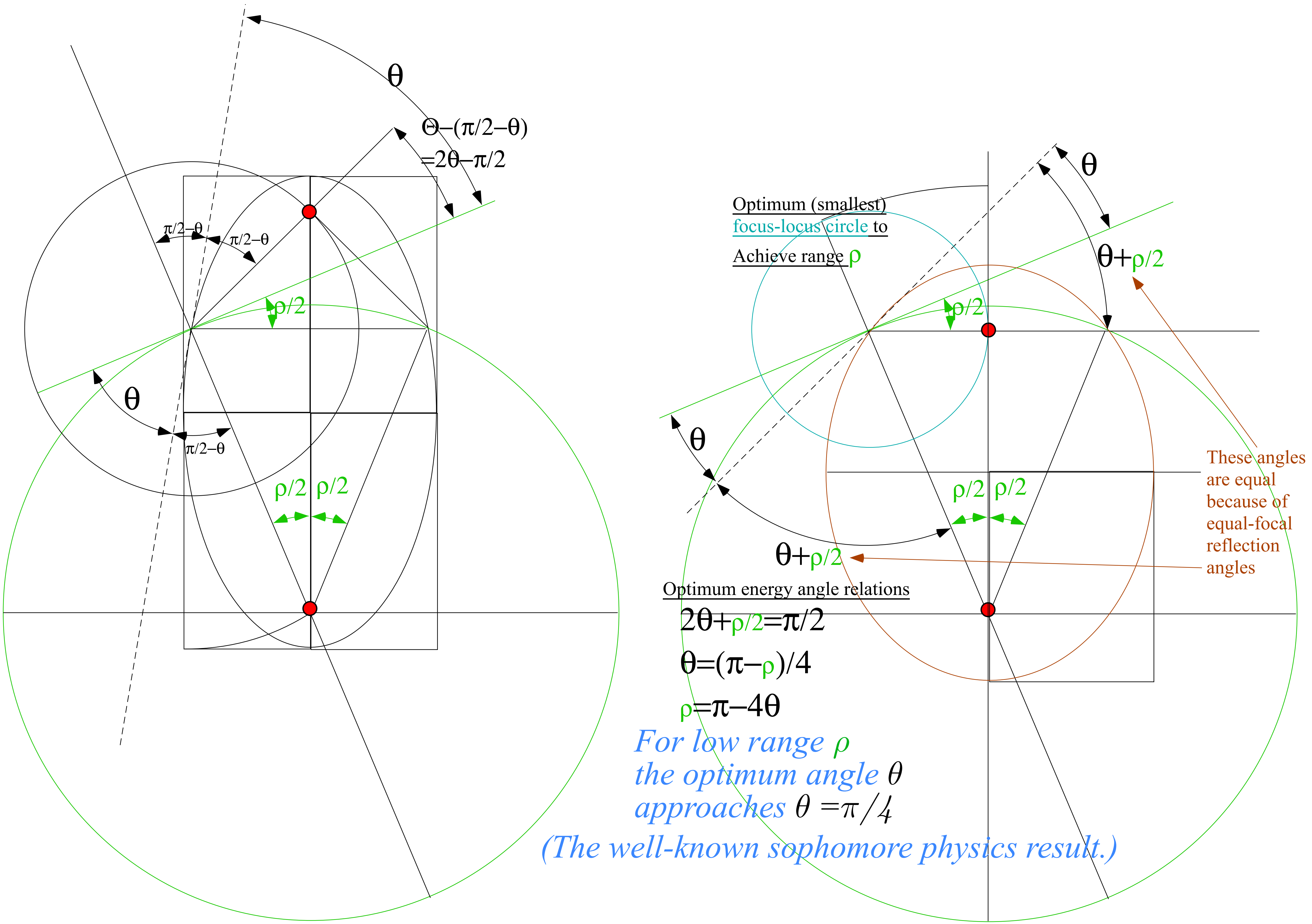
But at slightly less energy a very long ellipse would return after a very long time but at slightly less range than 270°.



Maximum range 269.999°:
(with launch angle α=22.5°)

Parabola escapes!...
...does not return...

Launch optimization



Optimum energy angle relations

$$2\theta+\rho/2=\pi/2$$

$$\theta=(\pi-\rho)/4$$

$$\rho=\pi-4\theta$$

For low range ρ
the optimum angle θ
approaches $\theta=\pi/4$

(The well-known sophomore physics result.)