Lecture 26
Tue. 12.03.2015

Geometry and Symmetry of Coulomb Orbital Dynamics

(Ch. 2-4 of Unit 5 12.03.15)

- *Rutherford scattering and hyperbolic orbit geometry*
  - *Backward vs forward scattering angles and orbit construction example*
  - *Parabolic "kite" and orbital envelope geometry*
  - *Differential and total scattering cross-sections*

- *Eccentricity vector* $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*
- *Projection* $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$
  - *Review and connection to usual orbital algebra (previous lecture)*
- *Projection* $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

- *General geometric orbit construction using* $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*
  - *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

- *Coulomb orbit algebra of* $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$
  - *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

- *Connection formulas for* (γ,R)*-parameters with* (a,b) *and* (ε,λ)

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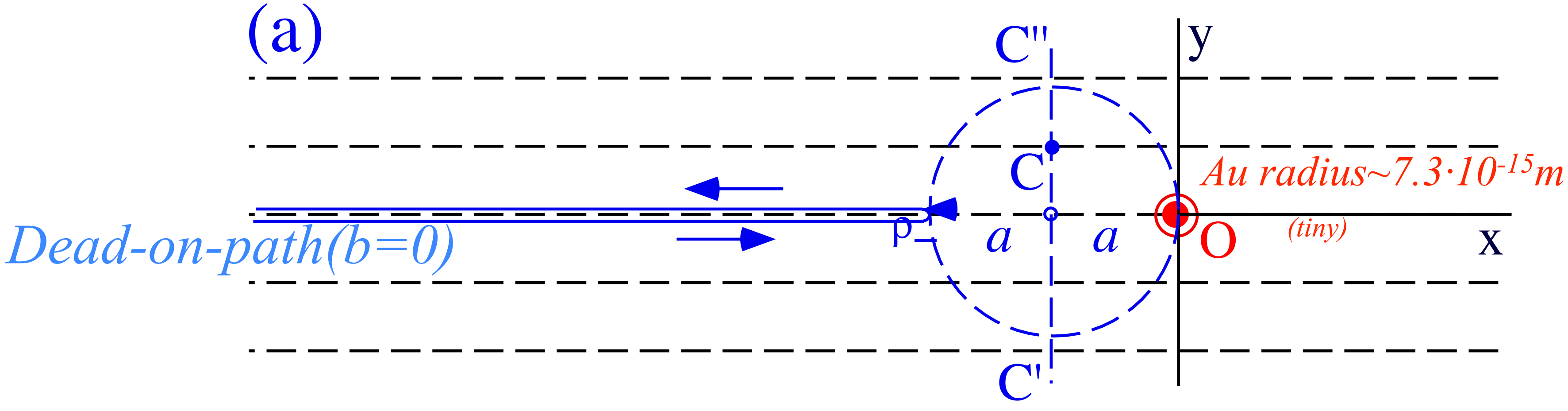
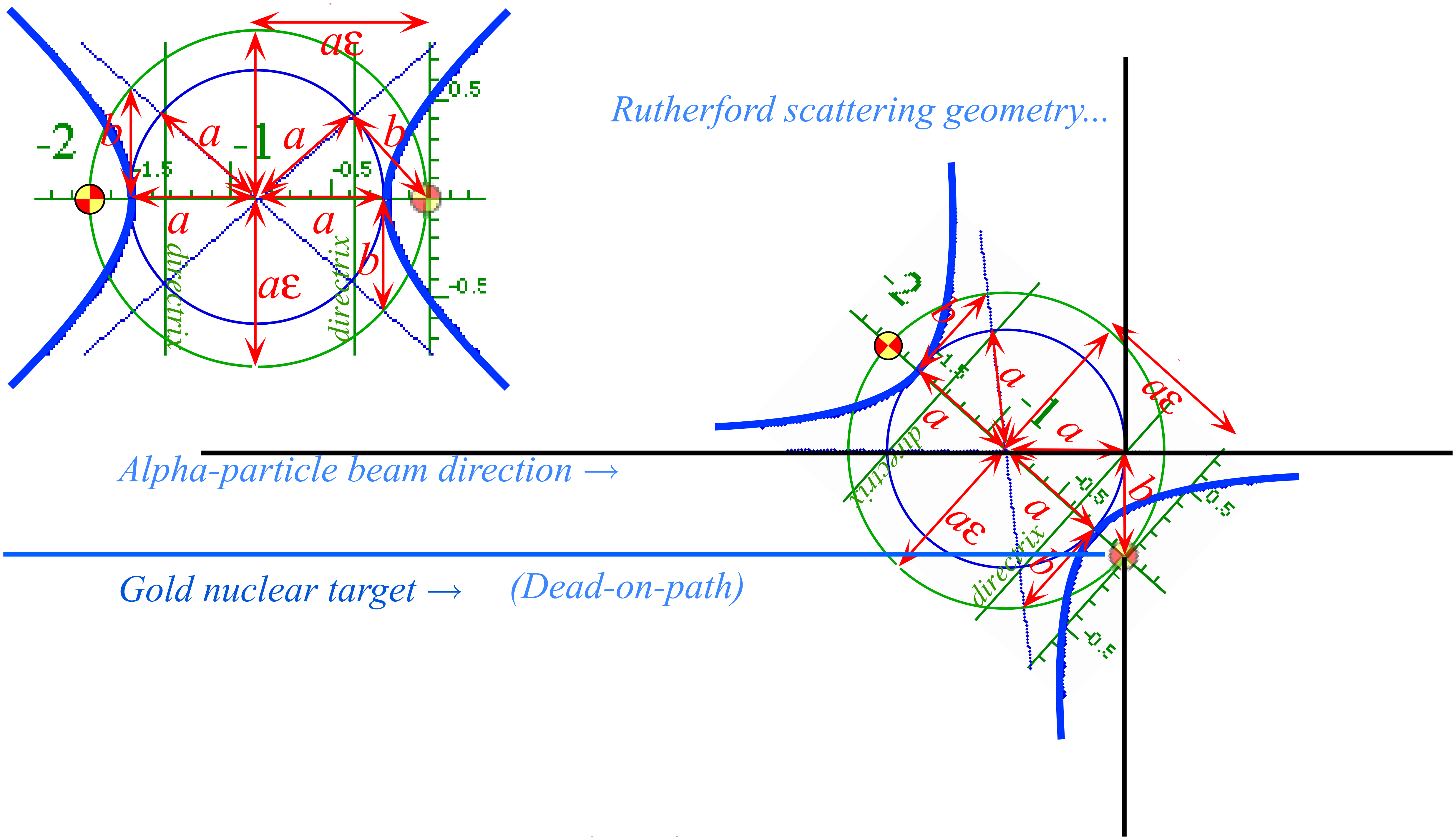


Rutherford scattering of α^{+2} particles from Au^{+79} nucleus at O

Assume "Dead-On" closest approach $2a$.

$(E=k/2a)$ $a \sim 10^{-11}m >> 7.3 \cdot 10^{-15}m$



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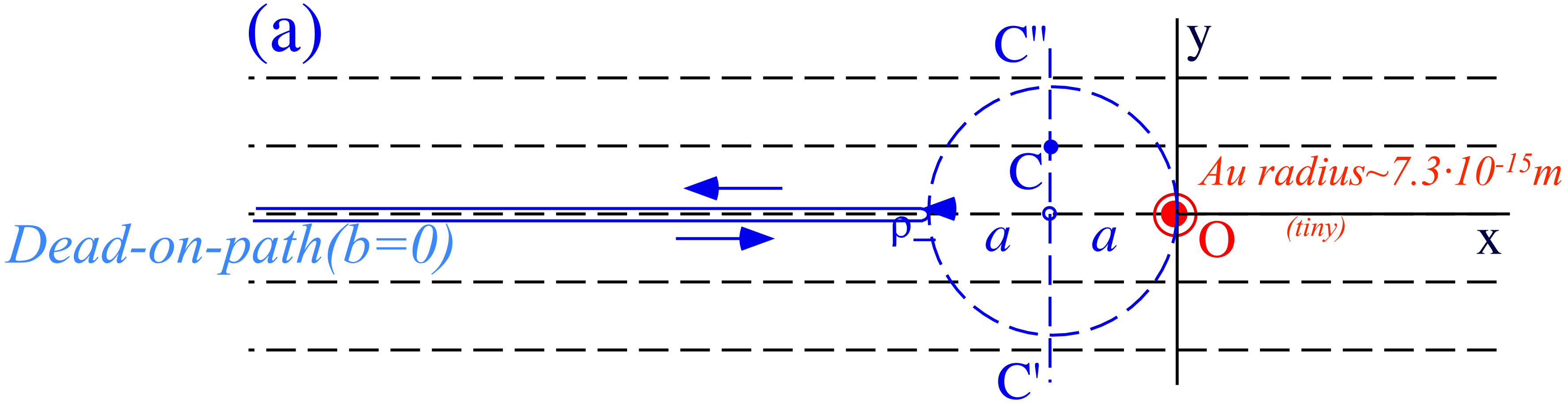

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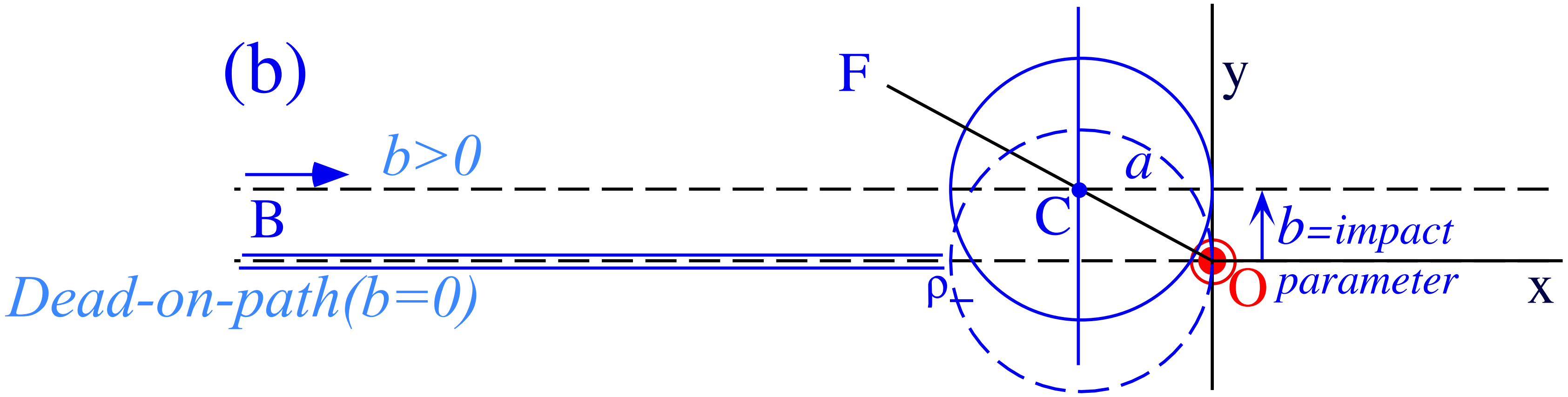


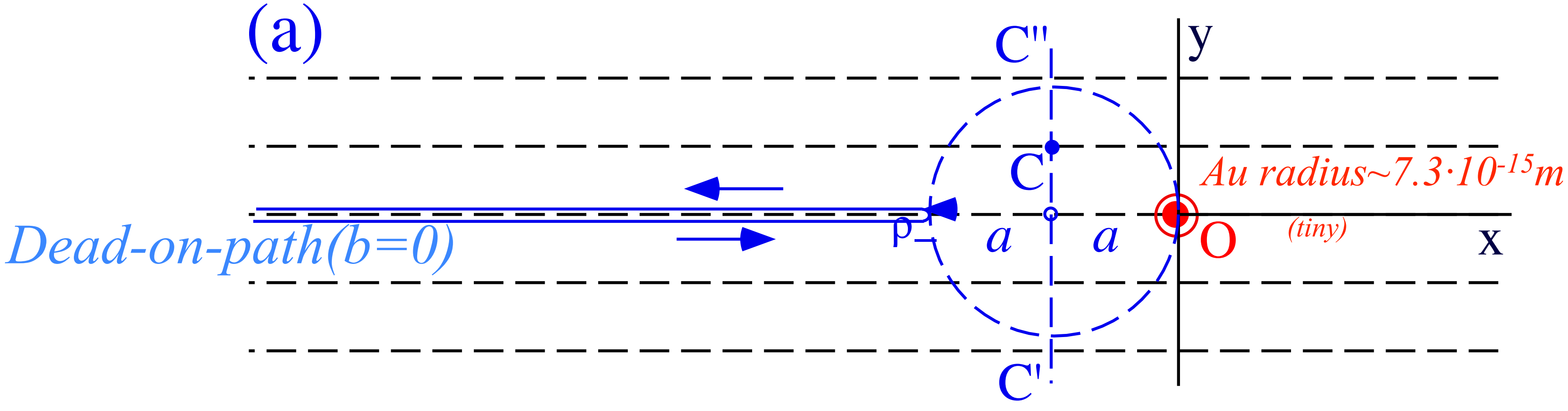
Pick an "impact parameter" line $y=b$.

Draw circle of radius a *around*

center point C$=(-a,b)$ *tangent to* y*-axis.*

Draw "focus-locus" line OCF.

Rutherford scattering of α^{+2} *particles from* Au^{+79} *nucleus at* O



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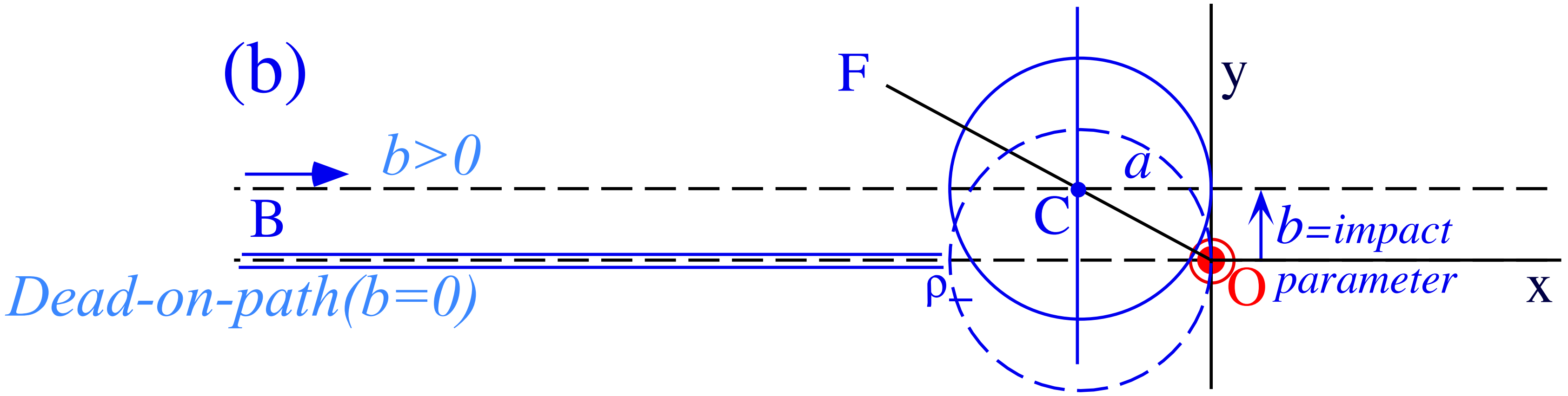

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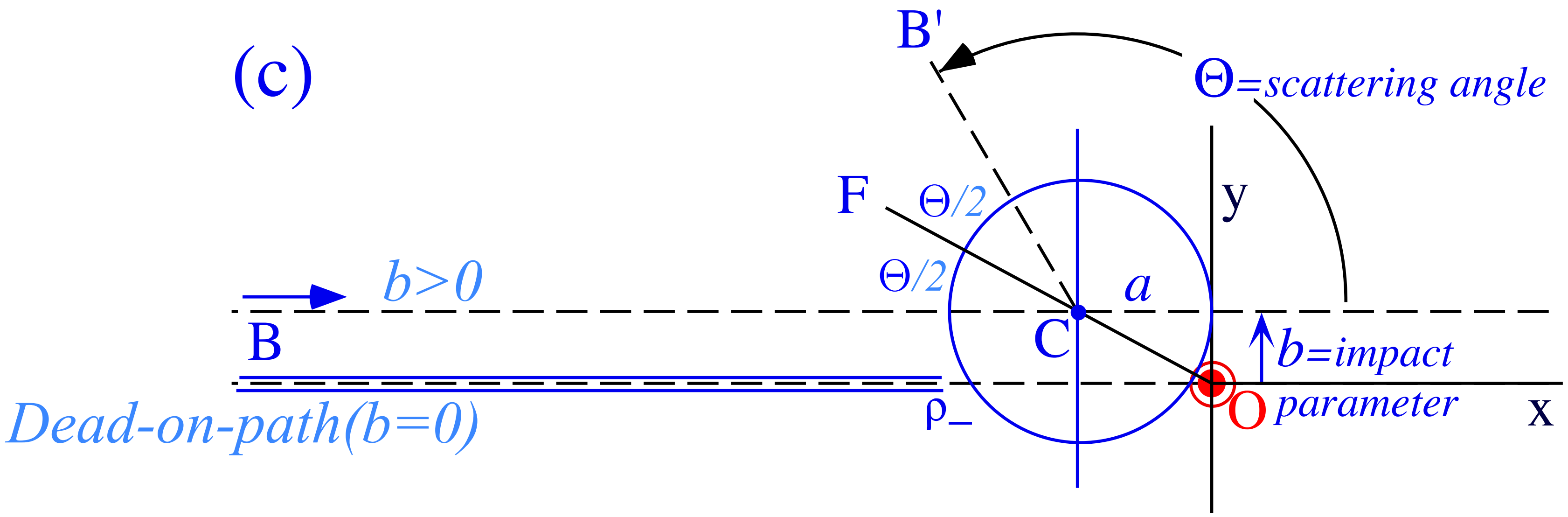

Copy angle ∠BCF *(equal to* Θ/2*)*

to make angle ∠FCB′ *(also equal to* Θ/2*)*

Resulting line CB′ *is outgoing asymptote*

at scattering angle Θ .

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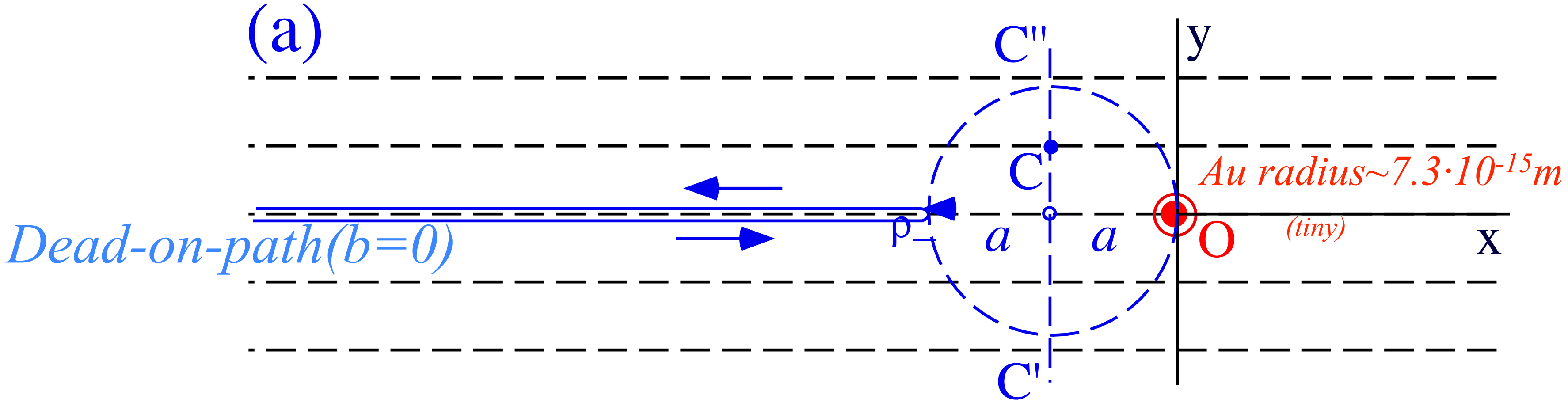

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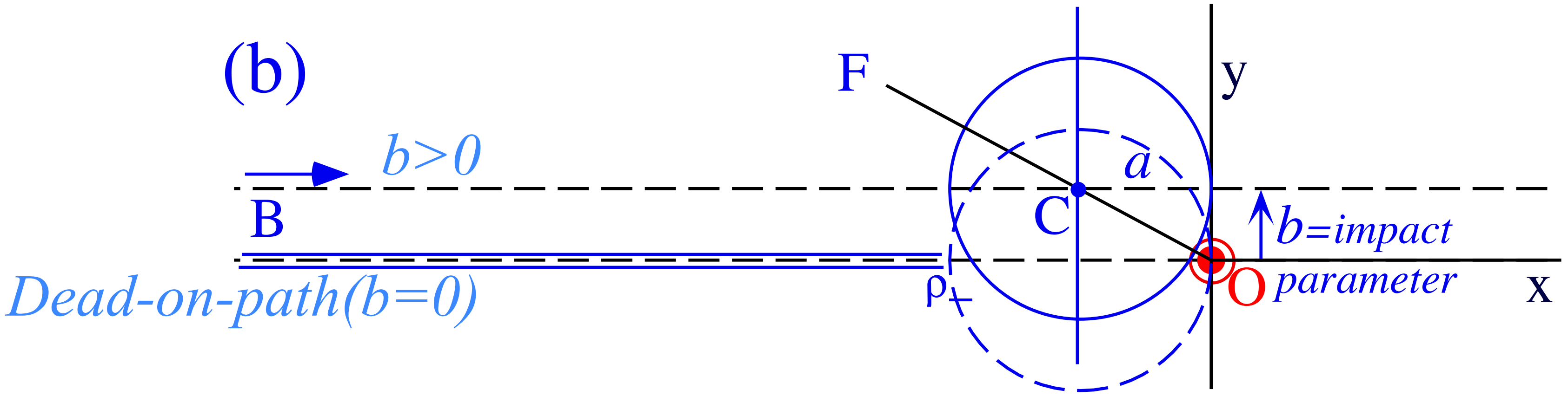

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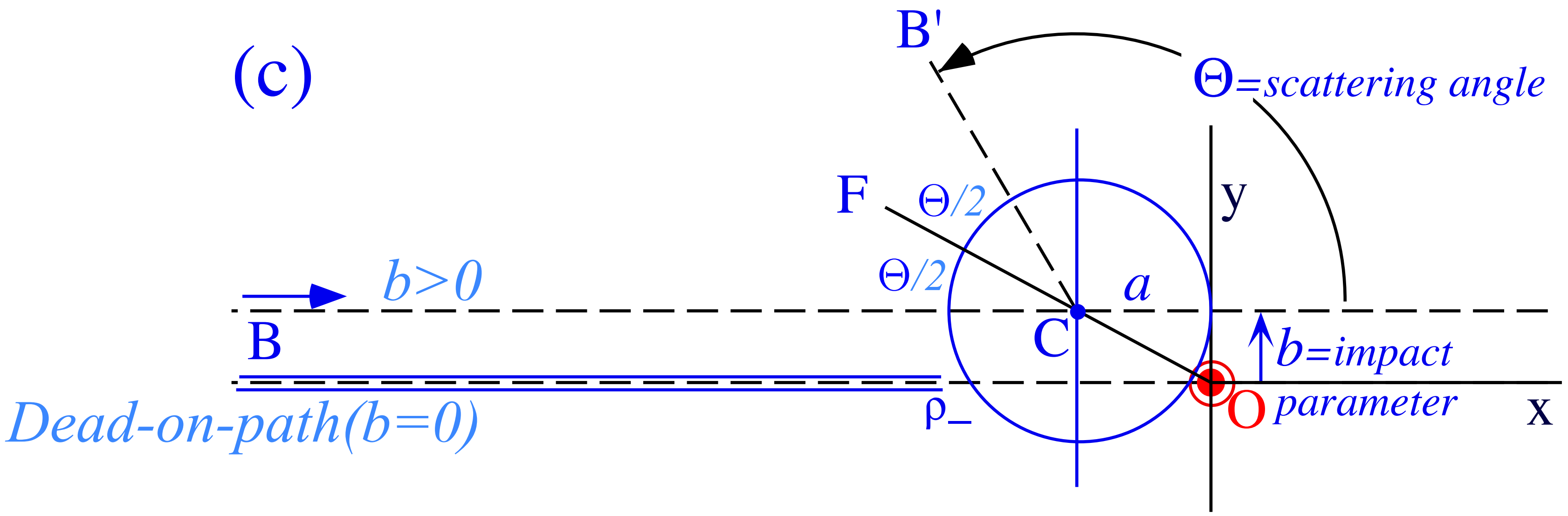
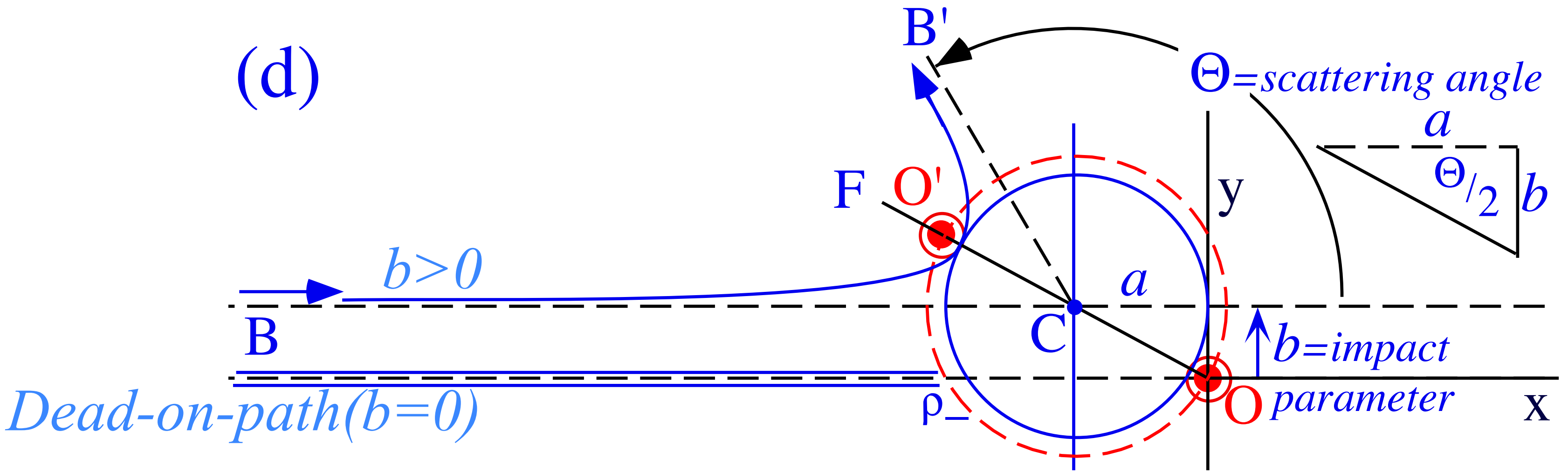


Copy angle ∠BCF (equal to Θ/2) to make angle ∠FCB′ (also equal to Θ/2)

Resulting line CB′ is outgoing asymptote at scattering angle Θ .



Locate secondary focus O′ by drawing circle around point C of diameter CO thru point O. Diameter O′CO is 2aε.

*Hyperbolic orbit points **P** now found using constant 2a=**P**O-**P**O′*

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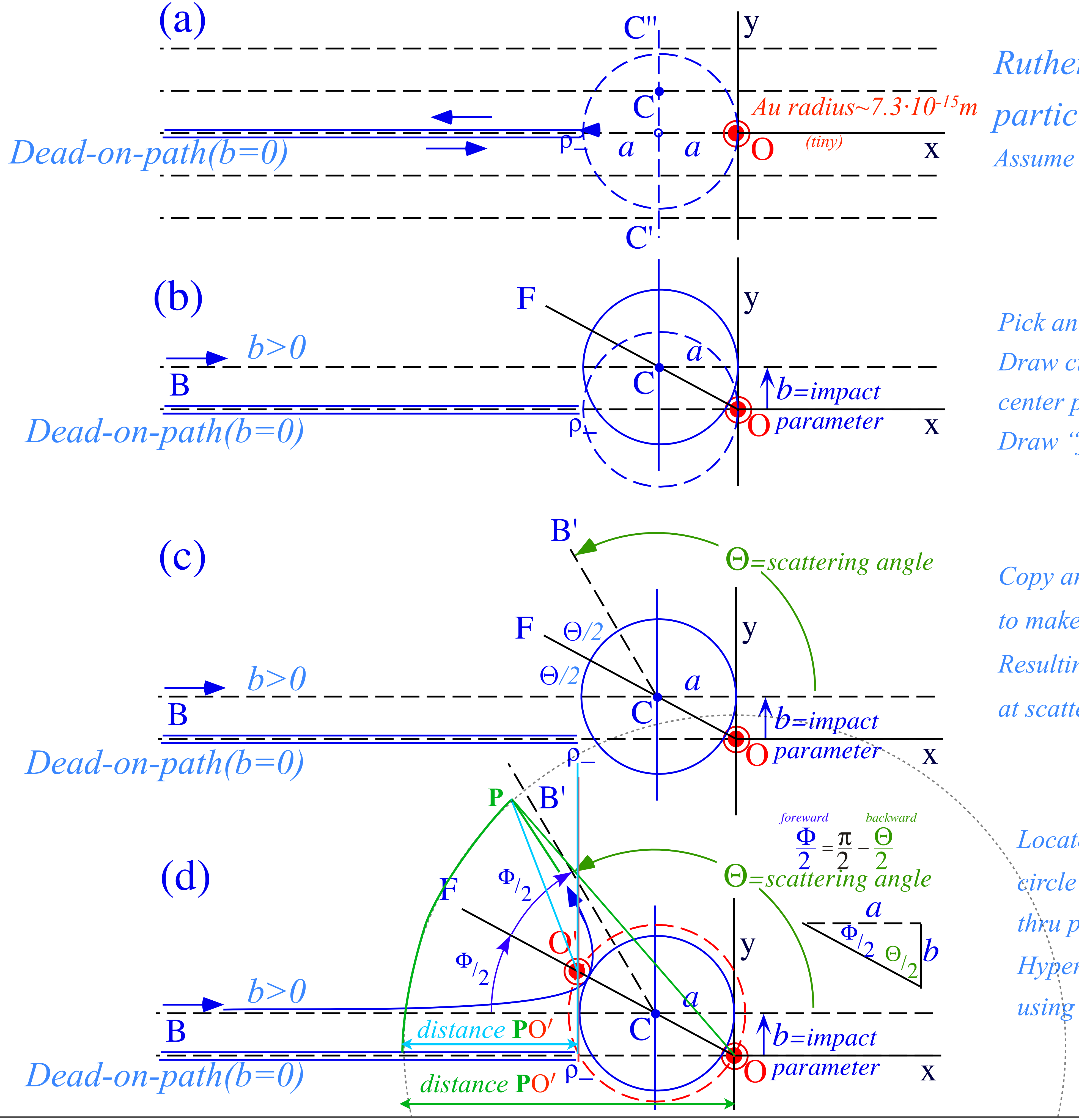

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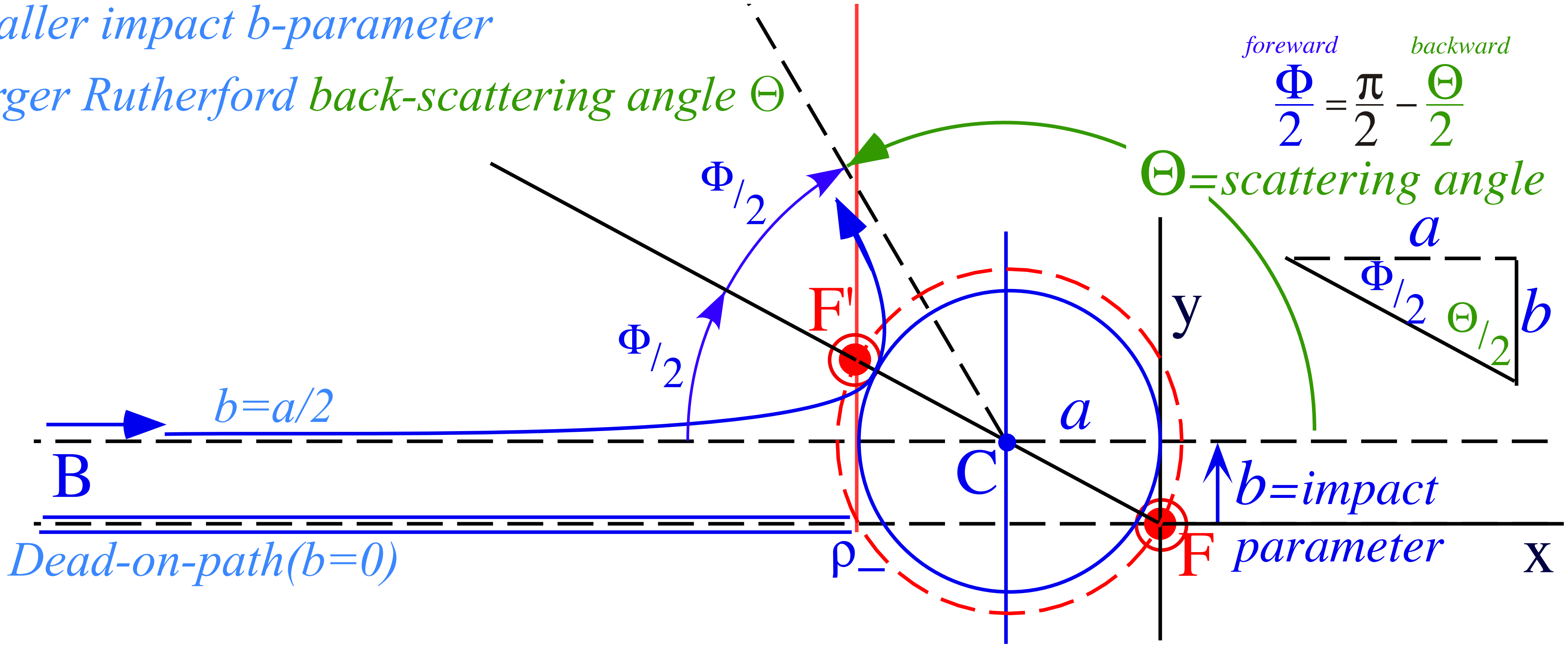
Connection formulas for (γ,R)*-parameters with* (a,b) *and* (ε,λ)

Smaller impact b-parameter

Larger Rutherford back-scattering angle Θ



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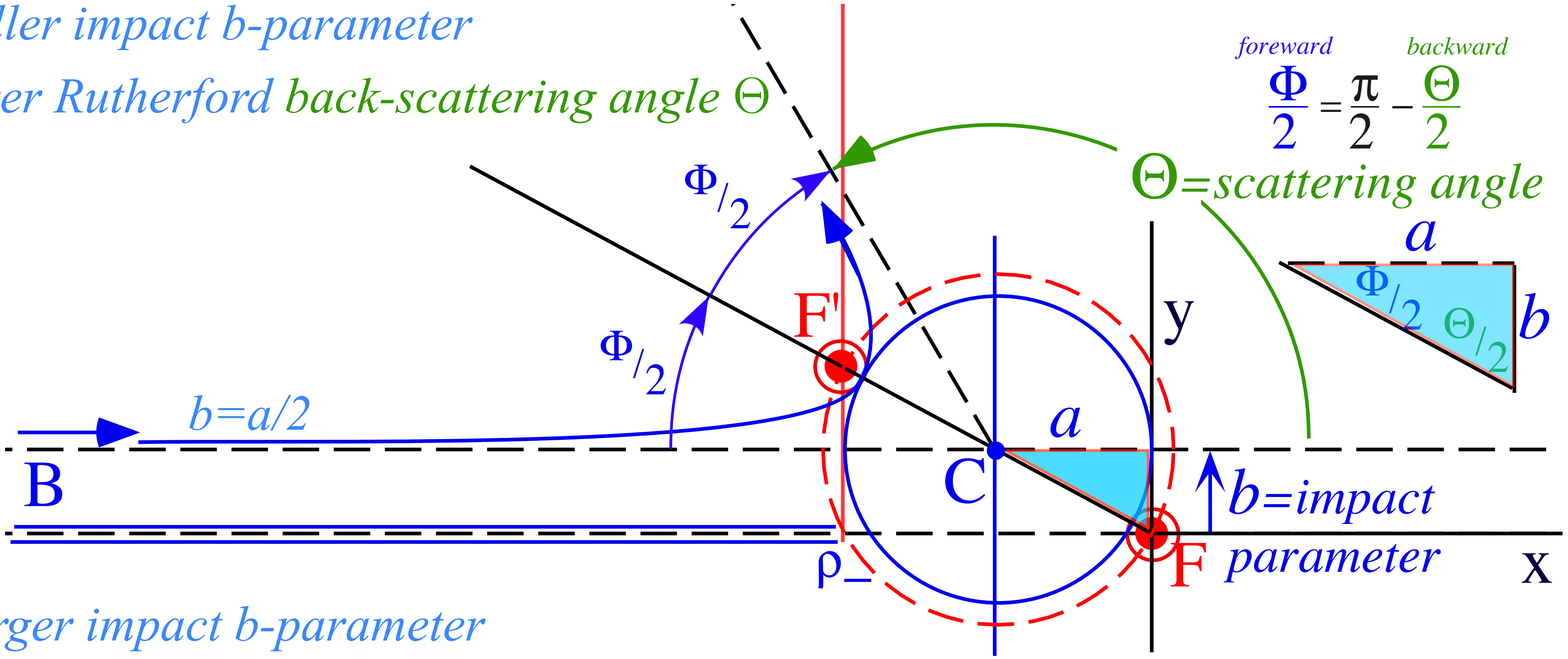
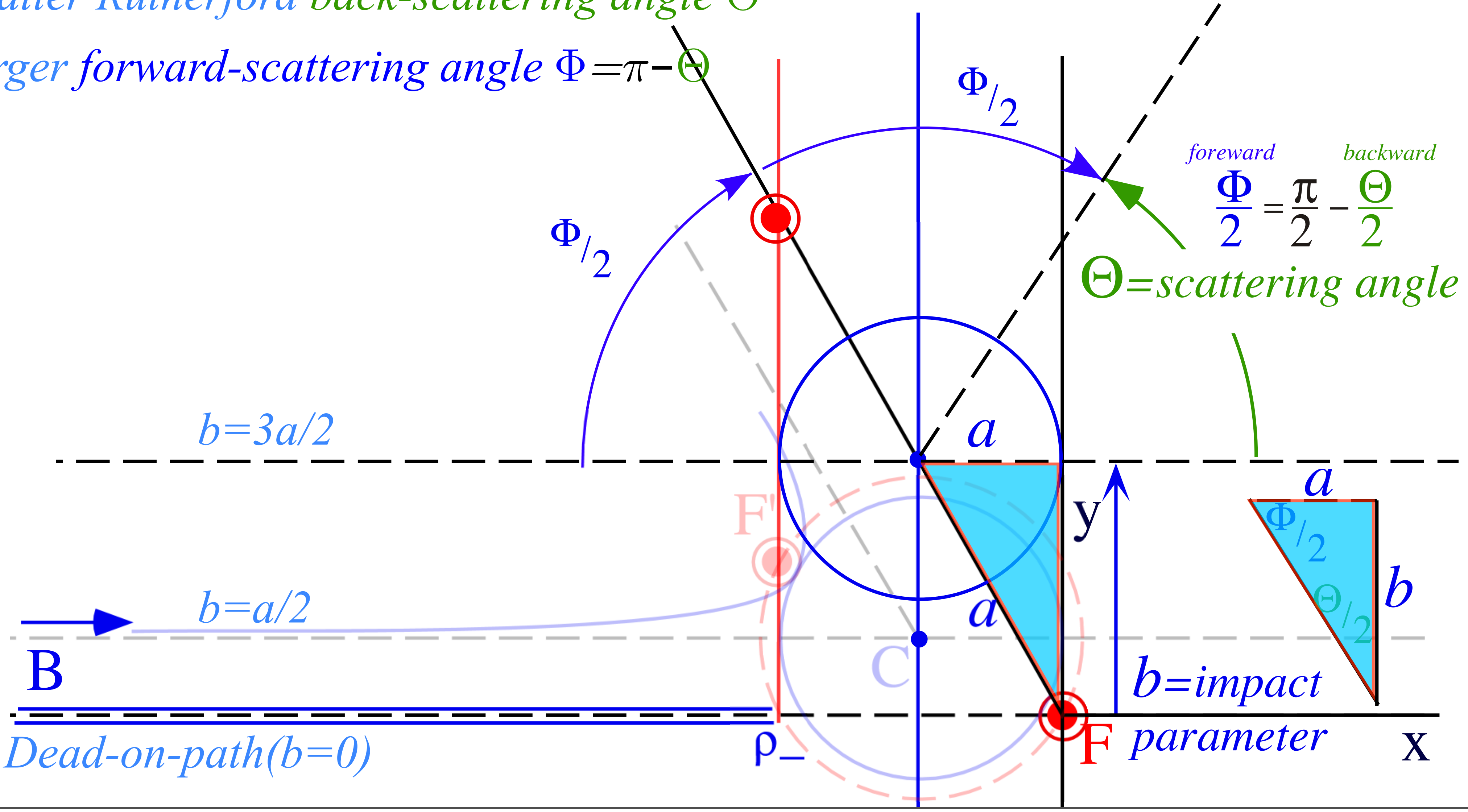
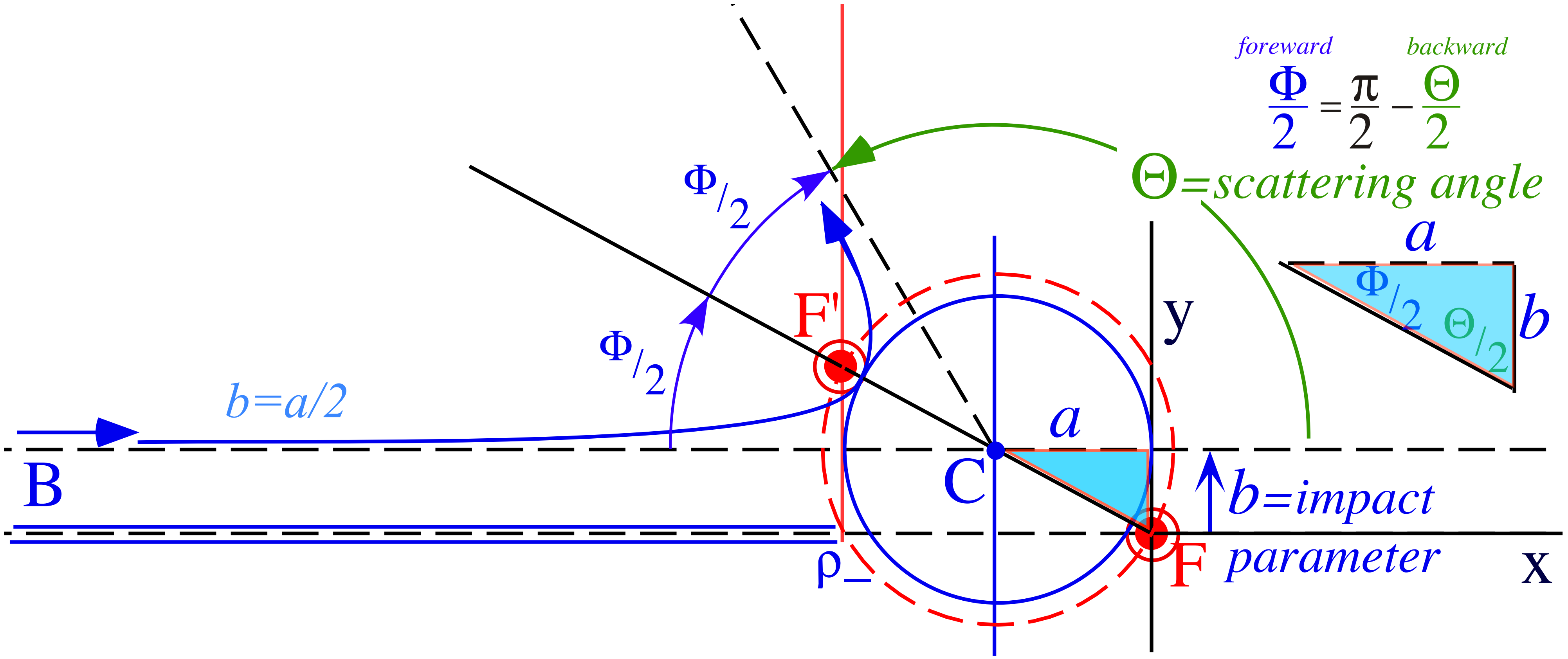


Larger impact b-parameter

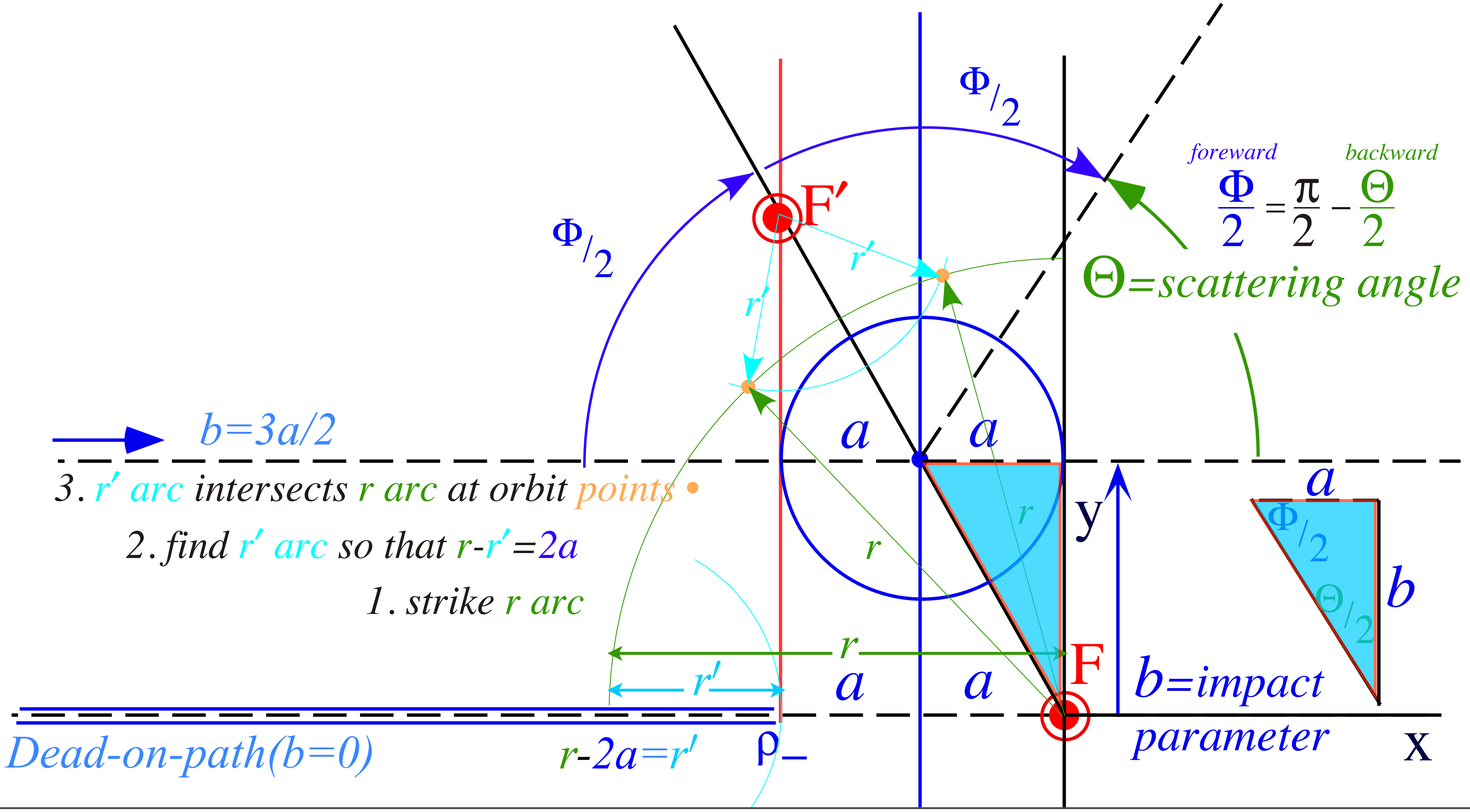
Smaller Rutherford back-scattering angle Θ

Larger forward-scattering angle Φ=π−Θ





$$\underset{foreward}{\frac{\Phi}{2}} = \frac{\pi}{2} - \underset{backward}{\frac{\Theta}{2}}$$

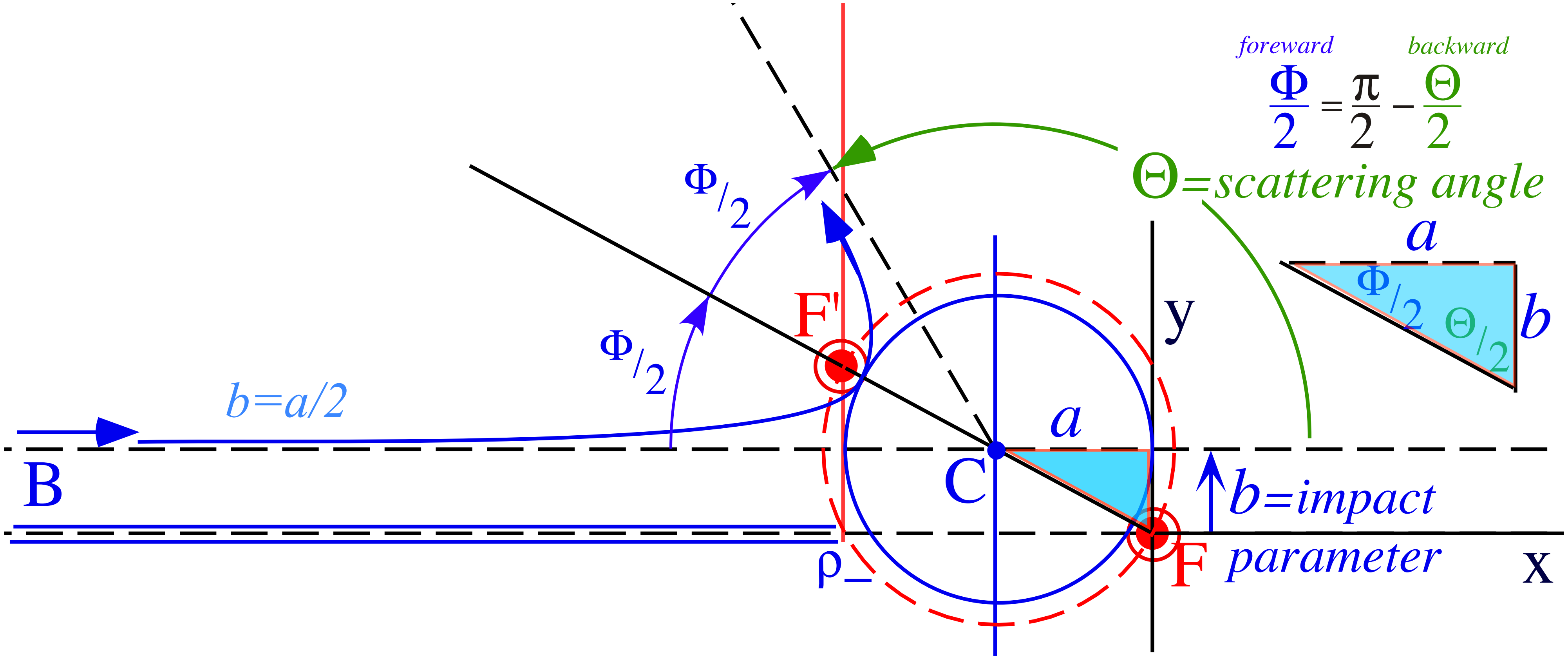


$$\underset{foreward}{\frac{\Phi}{2}} = \frac{\pi}{2} - \underset{backward}{\frac{\Theta}{2}}$$

3. *r′ arc intersects r arc at orbit points* •

2. *find r′ arc so that r-r′=2a*

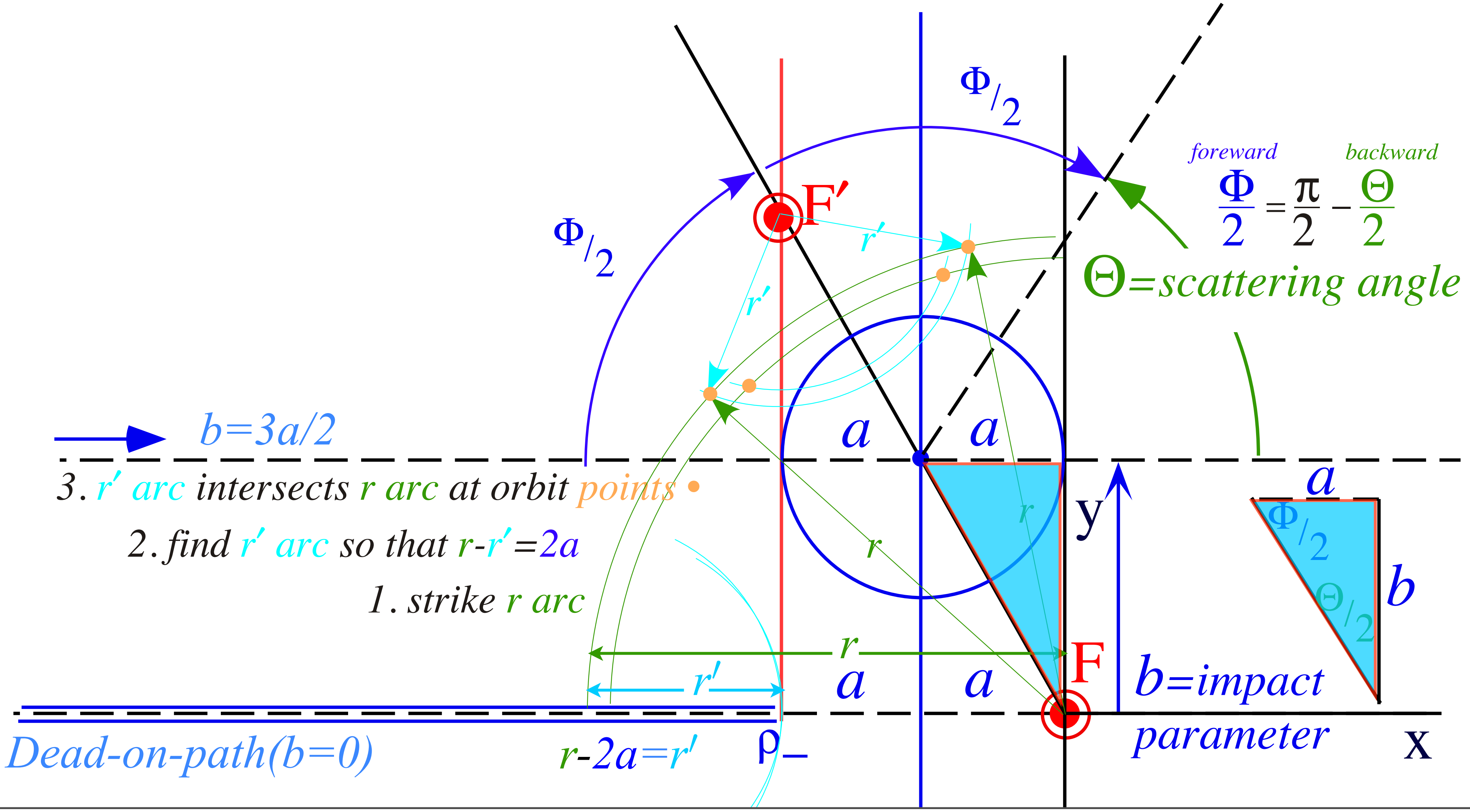
1. *strike r arc*



foreward *backward*

$$\frac{\Phi}{2}=\frac{\pi}{2}-\frac{\Theta}{2}$$

Θ=*scattering angle*



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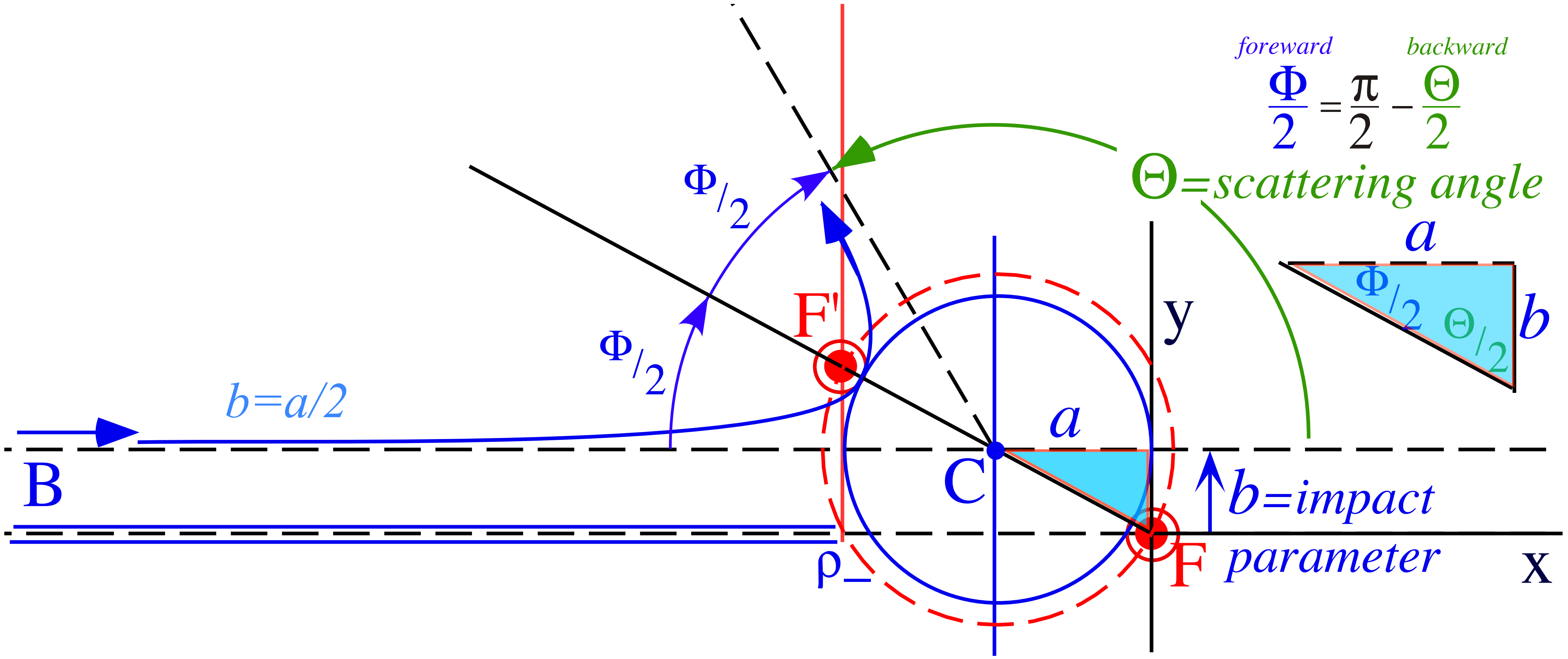
Dead-on-path(b=0)

r-2a=r′



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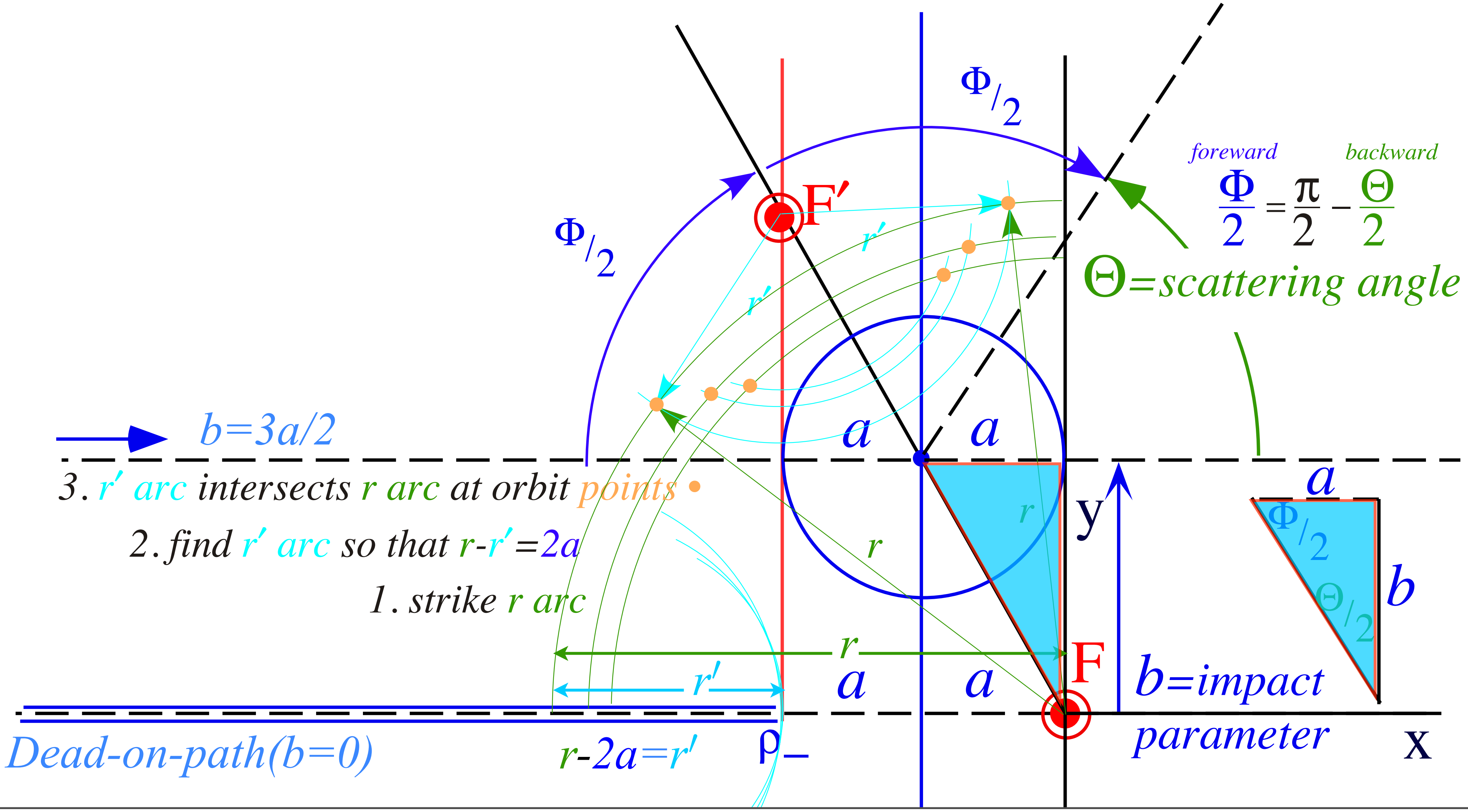

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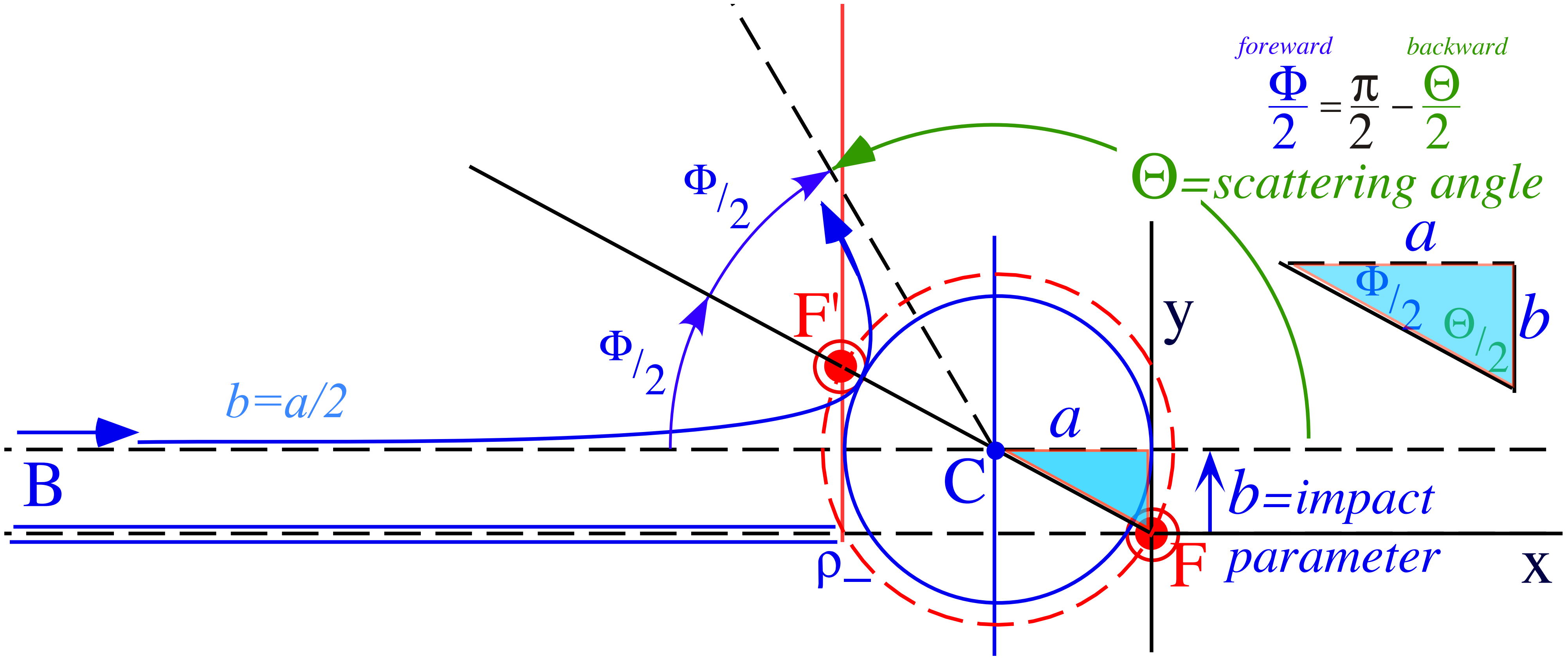

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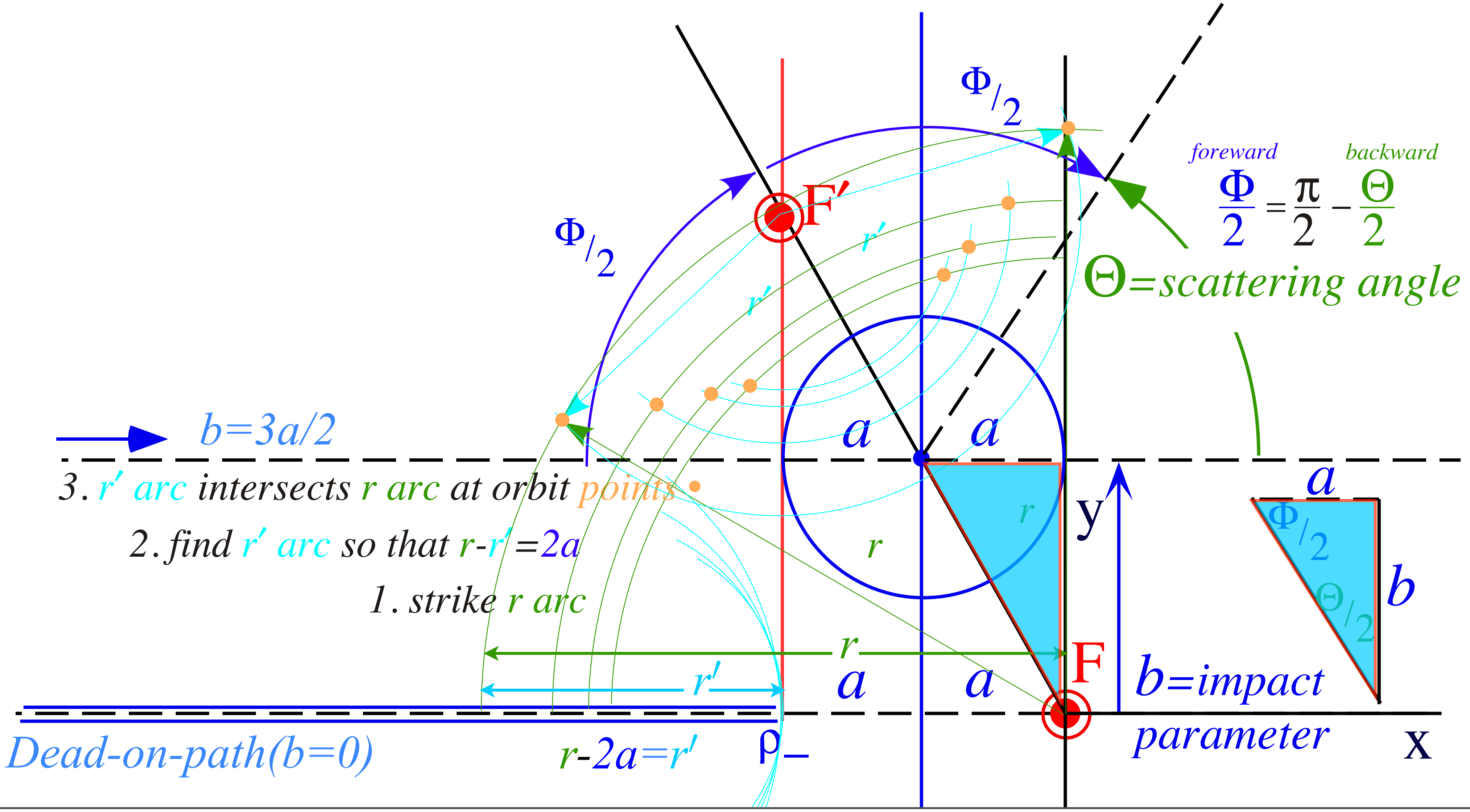

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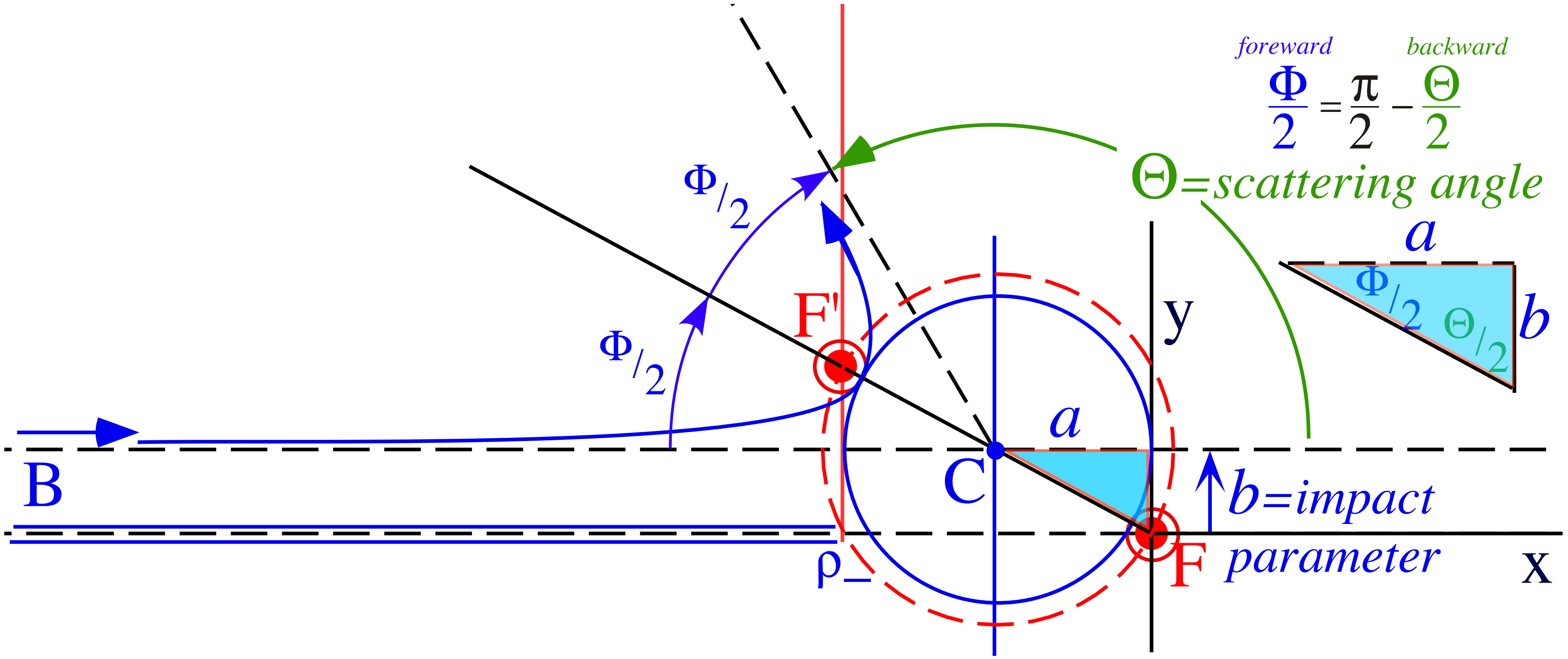
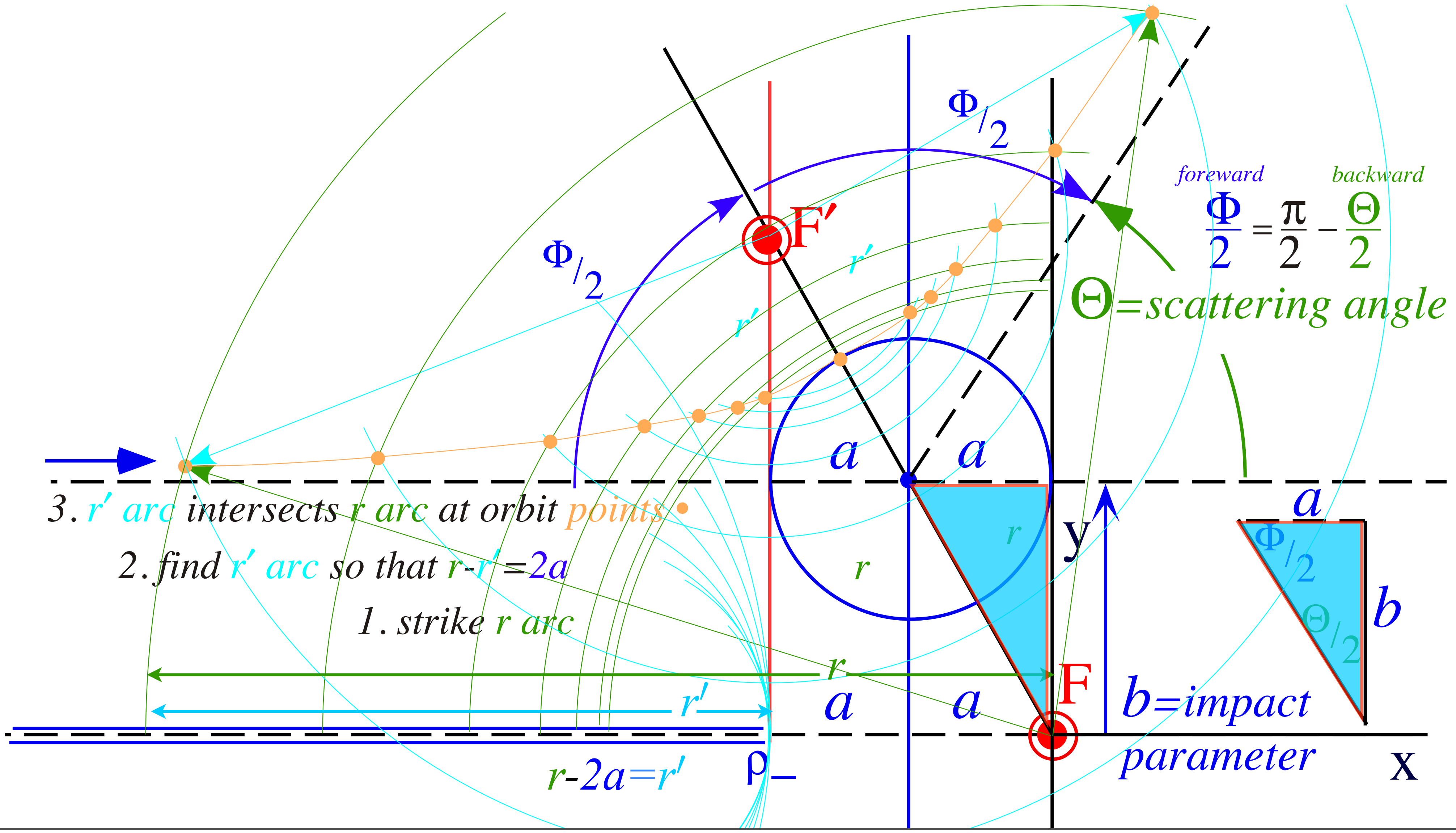


foreward $\frac{\Phi}{2} = \frac{\pi}{2} - \frac{\Theta}{2}$ *backward*

Θ=*scattering angle*

b=impact parameter



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r-2a=r′

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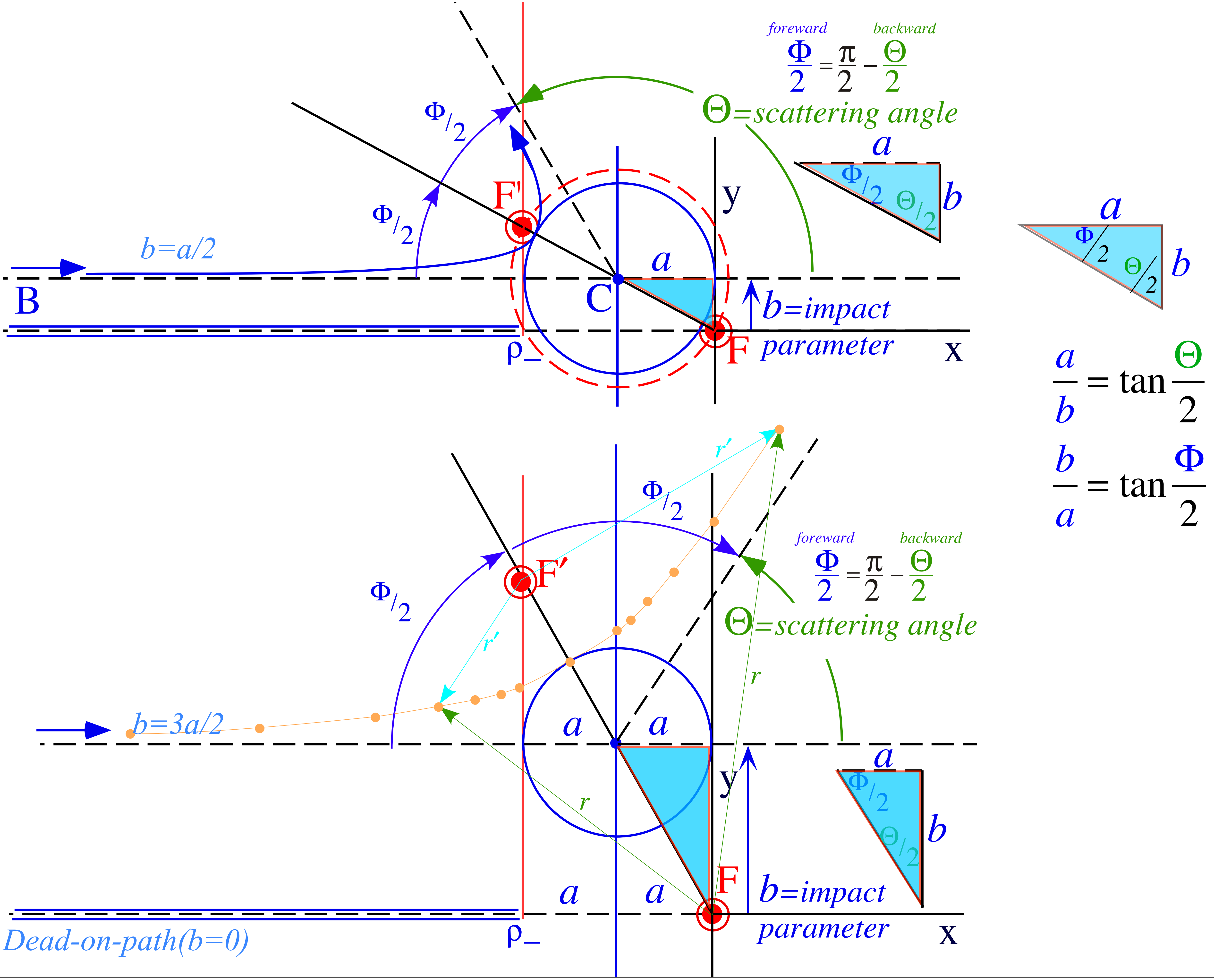

$$\frac{a}{b} = \tan\frac{\Theta}{2}$$

$$\frac{b}{a} = \tan\frac{\Phi}{2}$$

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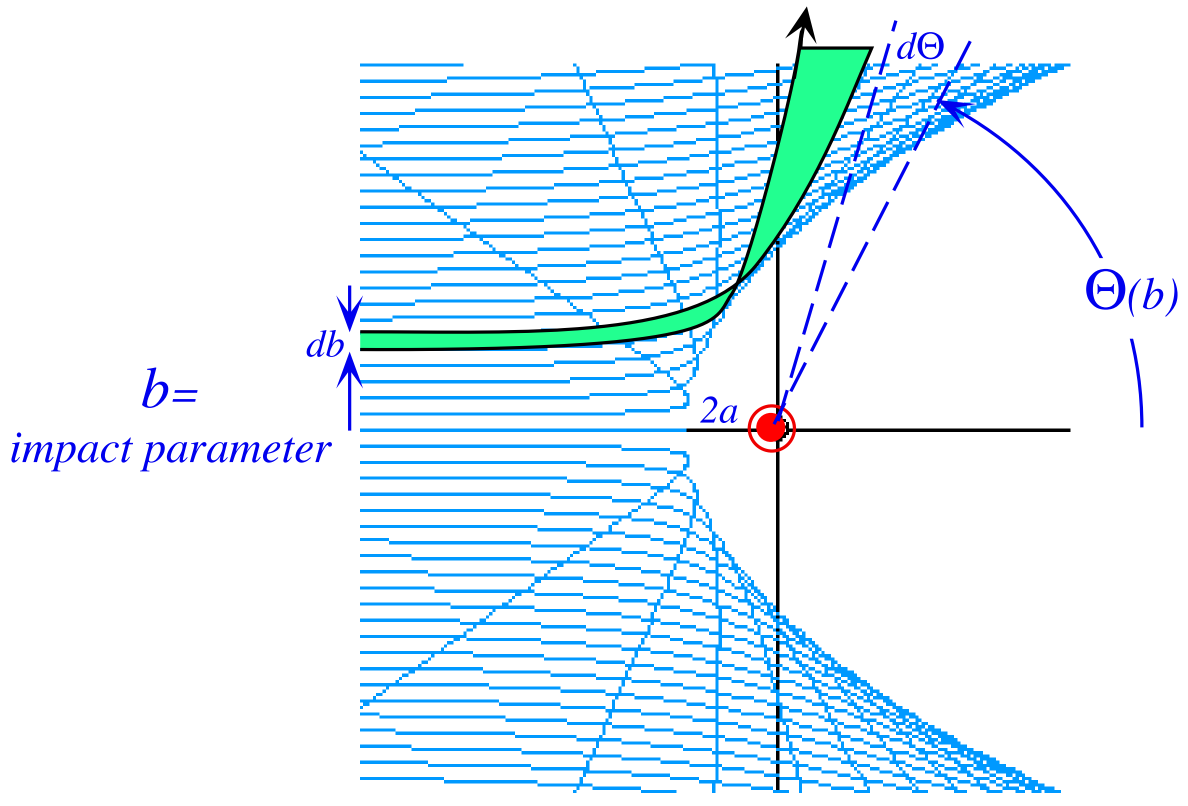

http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=Rutherford

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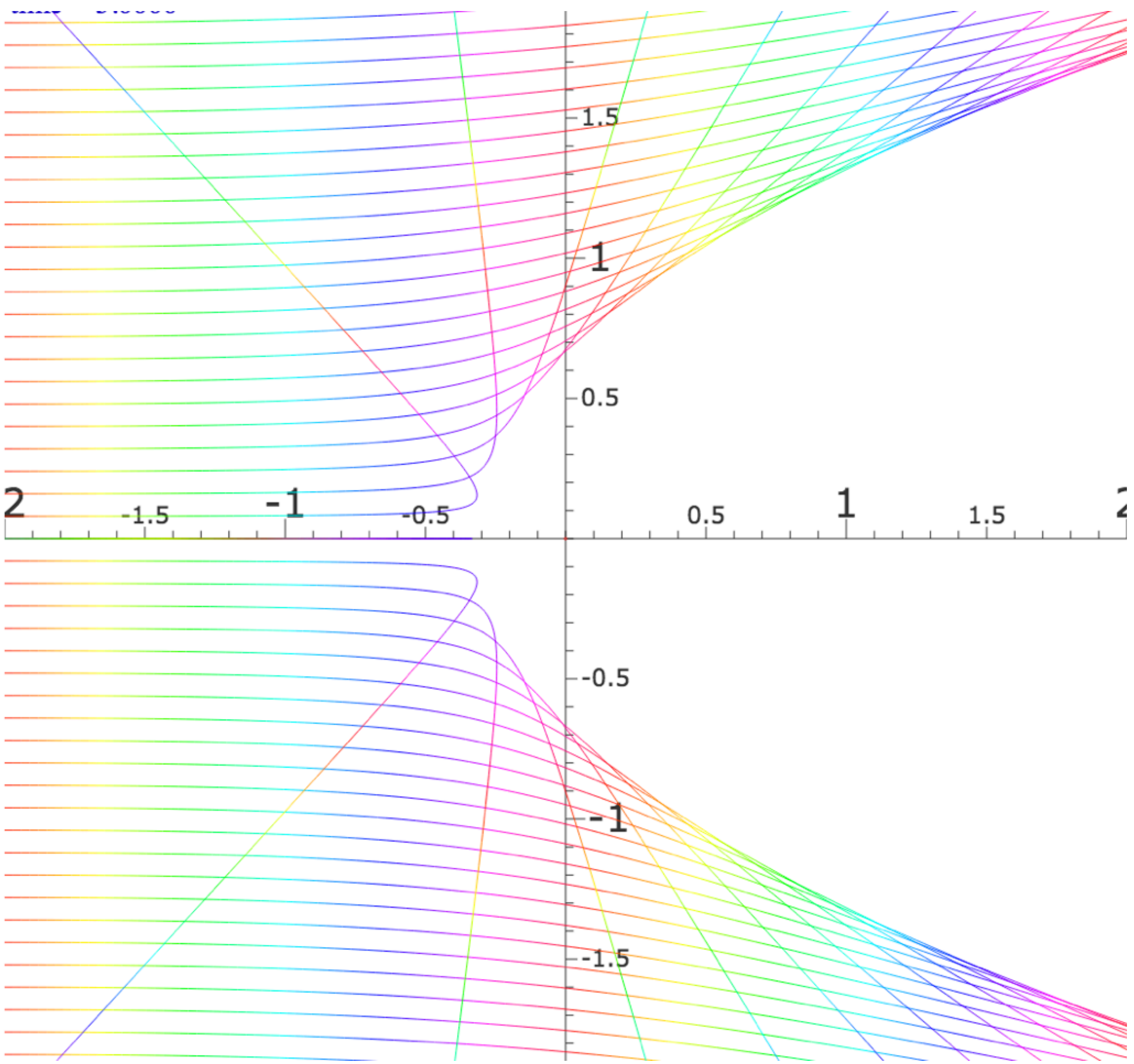
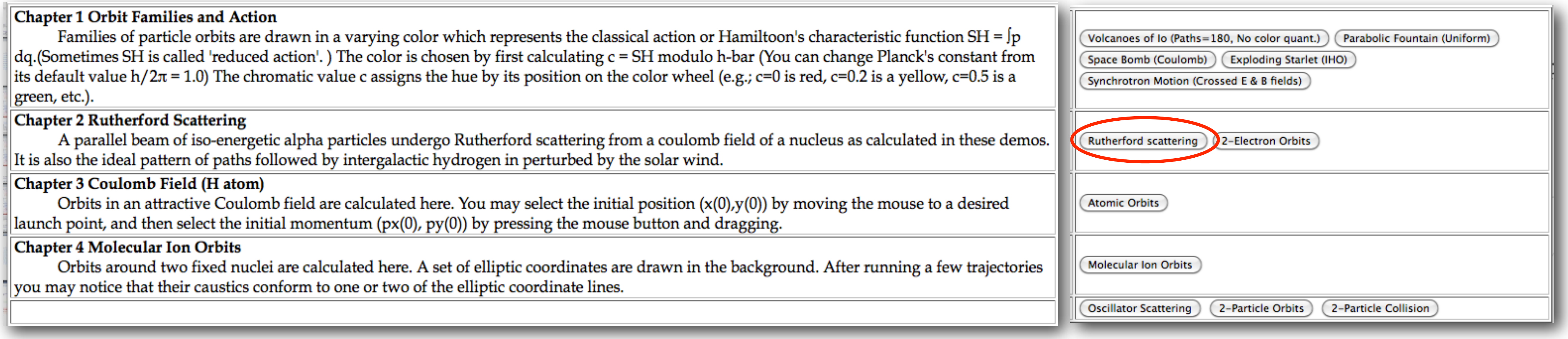




Chapter 1 Orbit Families and Action

Families of particle orbits are drawn in a varying color which represents the classical action or Hamilton's characteristic function SH = ∫p dq.(Sometimes SH is called 'reduced action'.) The color is chosen by first calculating c = SH modulo h-bar (You can change Planck's constant from its default value h/2π = 1.0) The chromatic value c assigns the hue by its position on the color wheel (e.g.; c=0 is red, c=0.2 is a yellow, c=0.5 is a green, etc.).

Chapter 2 Rutherford Scattering

A parallel beam of iso-energetic alpha particles undergo Rutherford scattering from a coulomb field of a nucleus as calculated in these demos. It is also the ideal pattern of paths followed by intergalactic hydrogen in perturbed by the solar wind.

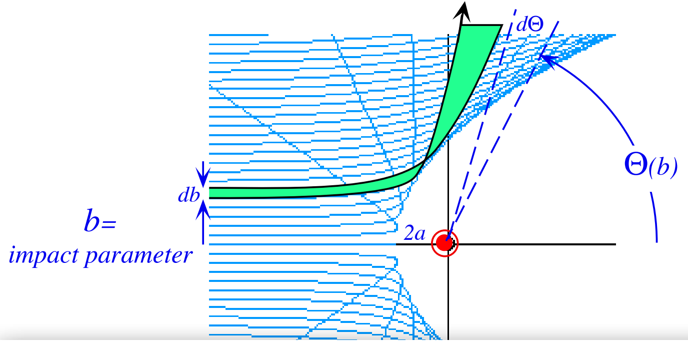
Chapter 3 Coulomb Field (H atom)

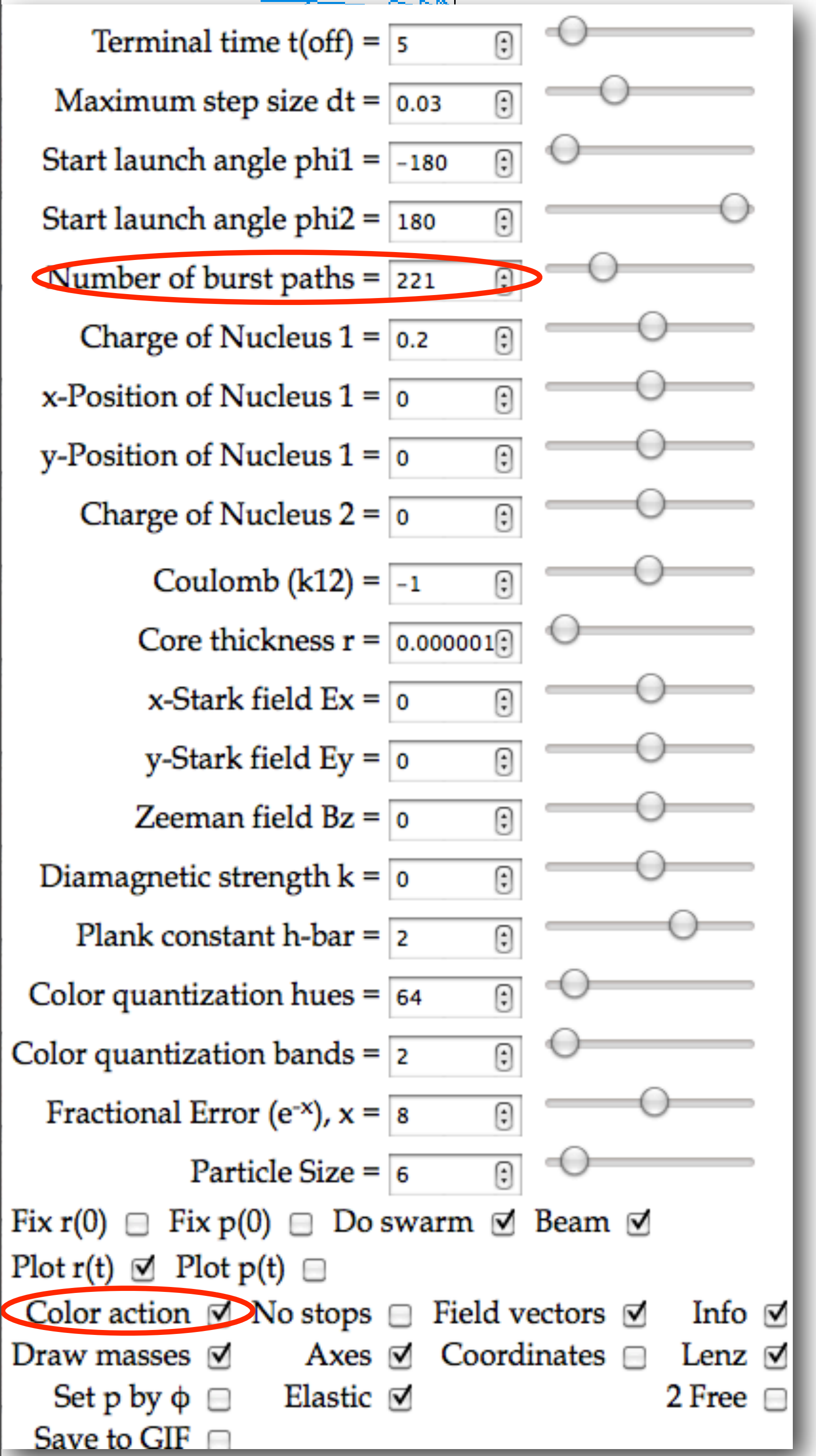
Orbits in an attractive Coulomb field are calculated here. You may select the initial position (x(0),y(0)) by moving the mouse to a desired launch point, and then select the initial momentum (px(0), py(0)) by pressing the mouse button and dragging.

Chapter 4 Molecular Ion Orbits

Orbits around two fixed nuclei are calculated here. A set of elliptic coordinates are drawn in the background. After running a few trajectories you may notice that their caustics conform to one or two of the elliptic coordinate lines.

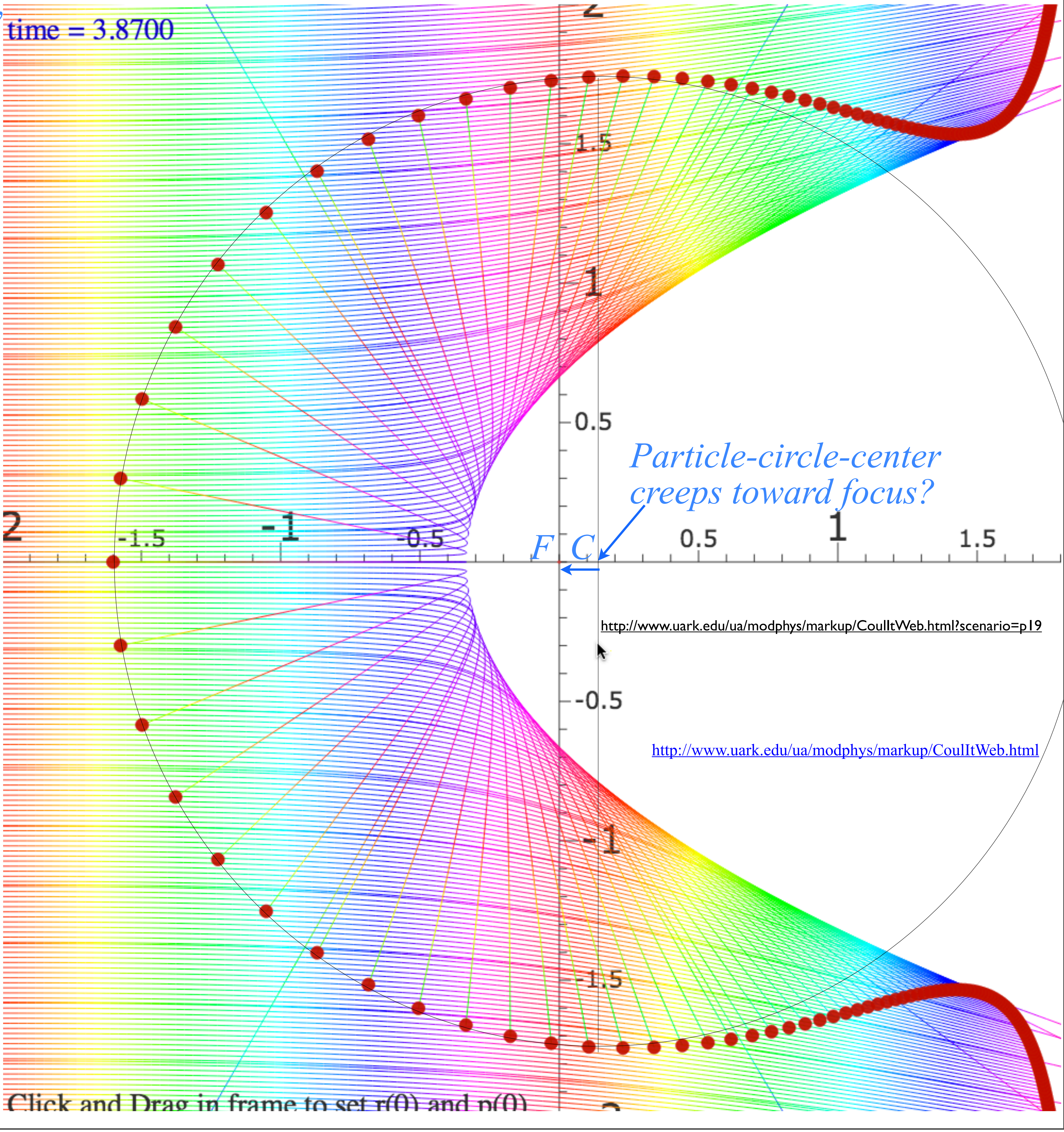
Rutherford scattering geometry





Terminal time t(off) = 5
Maximum step size dt = 0.03
Start launch angle phi1 = -180
Start launch angle phi2 = 180
Number of burst paths = 221
Charge of Nucleus 1 = 0.2
x-Position of Nucleus 1 = 0
y-Position of Nucleus 1 = 0
Charge of Nucleus 2 = 0
Coulomb (k12) = -1
Core thickness r = 0.000001
x-Stark field Ex = 0
y-Stark field Ey = 0
Zeeman field Bz = 0
Diamagnetic strength k = 0
Plank constant h-bar = 2
Color quantization hues = 64
Color quantization bands = 2
Fractional Error (e^{-x}), x = 8
Particle Size = 6

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☑ Beam ☑
Plot r(t) ☑ Plot p(t) ☐
Color action ☑ No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☑
Set p by ϕ ☐ Elastic ☑ 2 Free ☐
Save to GIF ☐

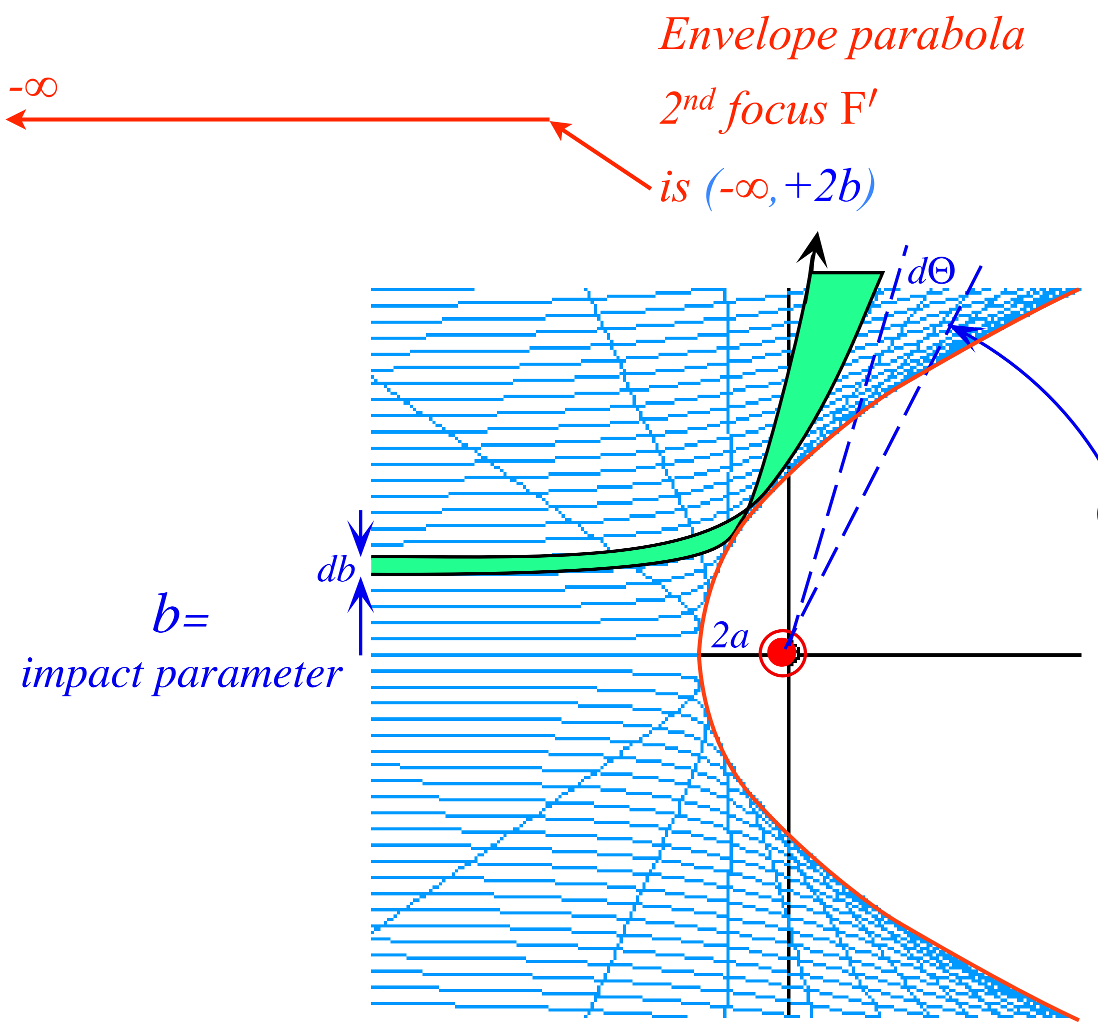


Particle-circle-center creeps toward focus?

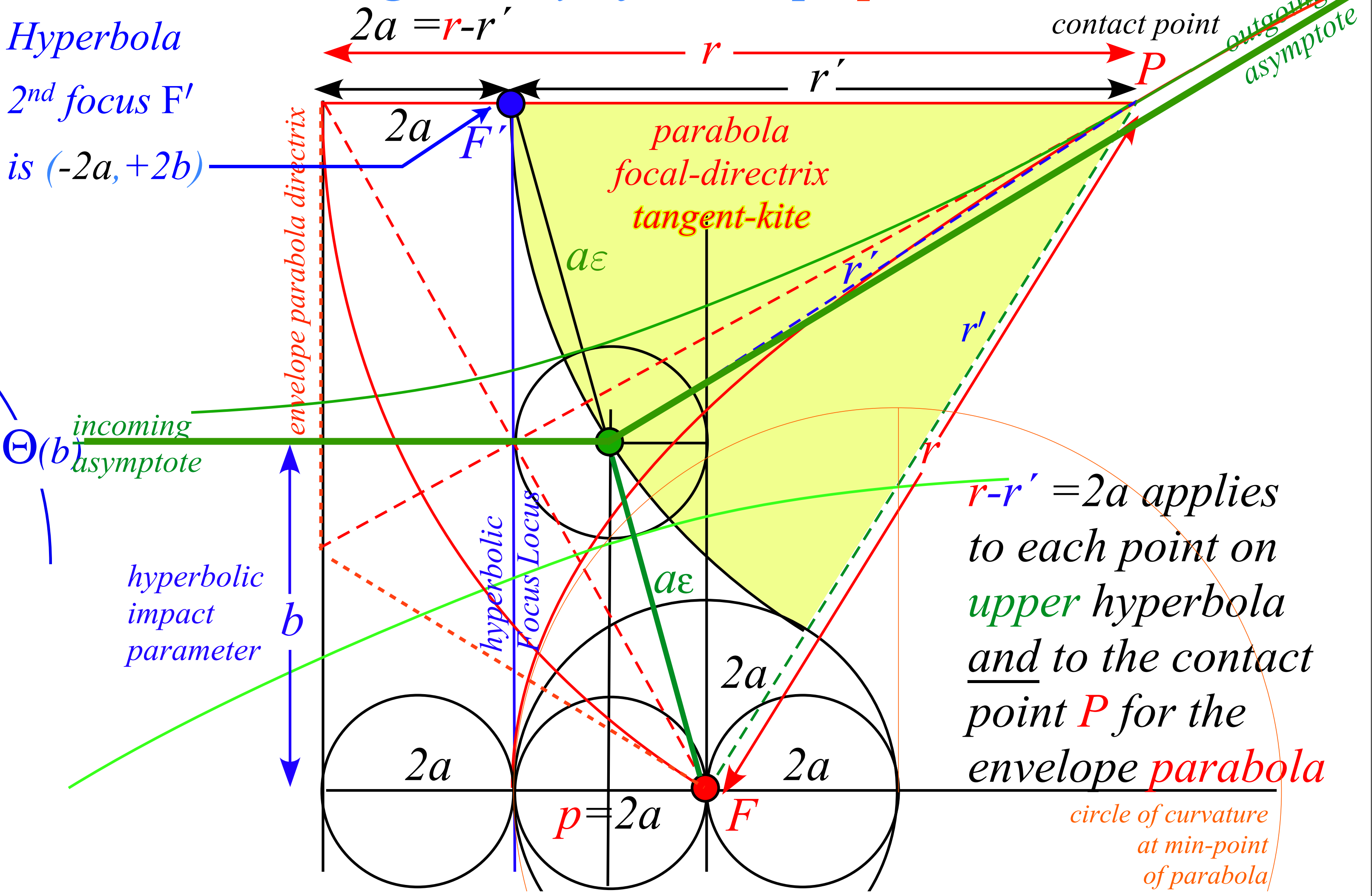
http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=p19

http://www.uark.edu/ua/modphys/markup/CoulItWeb.html

Rutherford scattering geometry

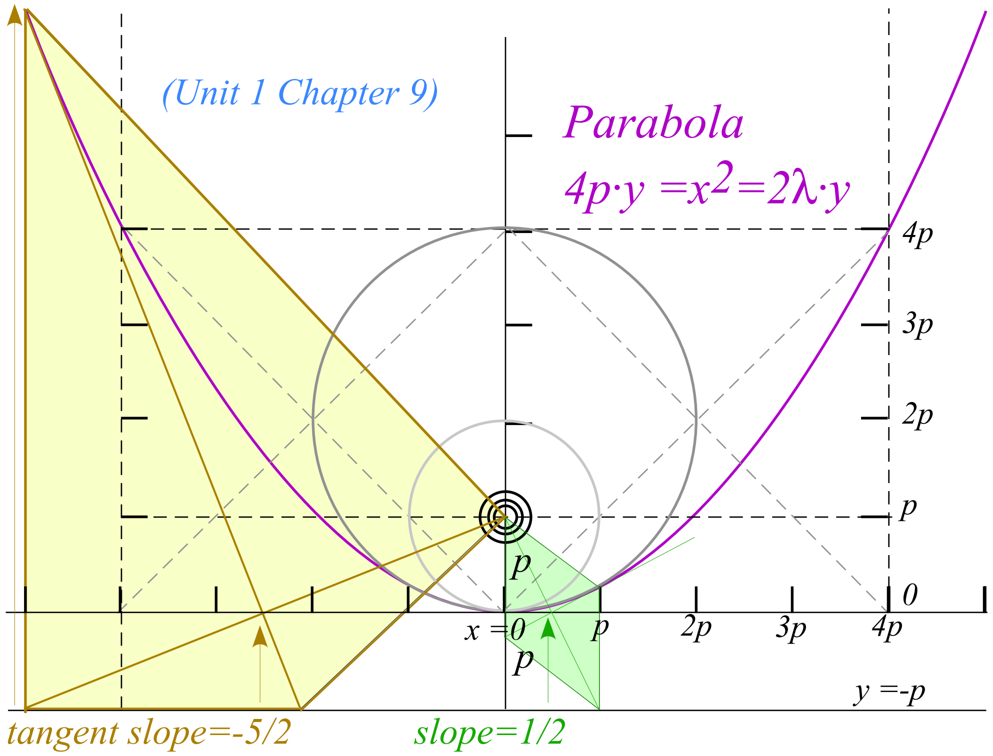


"Kite" geometry of envelope parabola



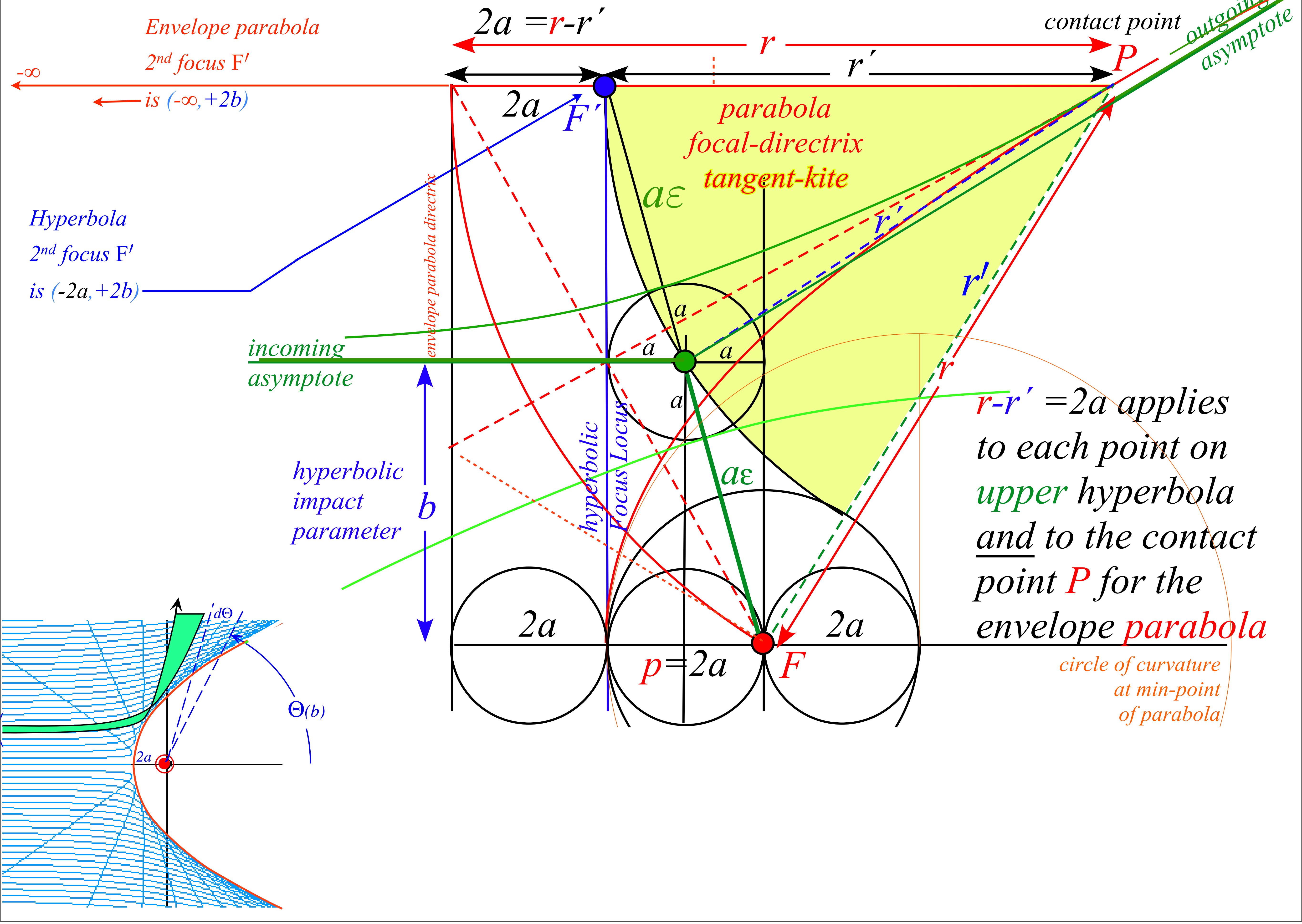
r-r' =2a applies to each point on upper hyperbola and to the contact point P for the envelope parabola

Recall parabolic " kite" geometry



Rutherford scattering geometry

“Kite” geometry of envelope parabola



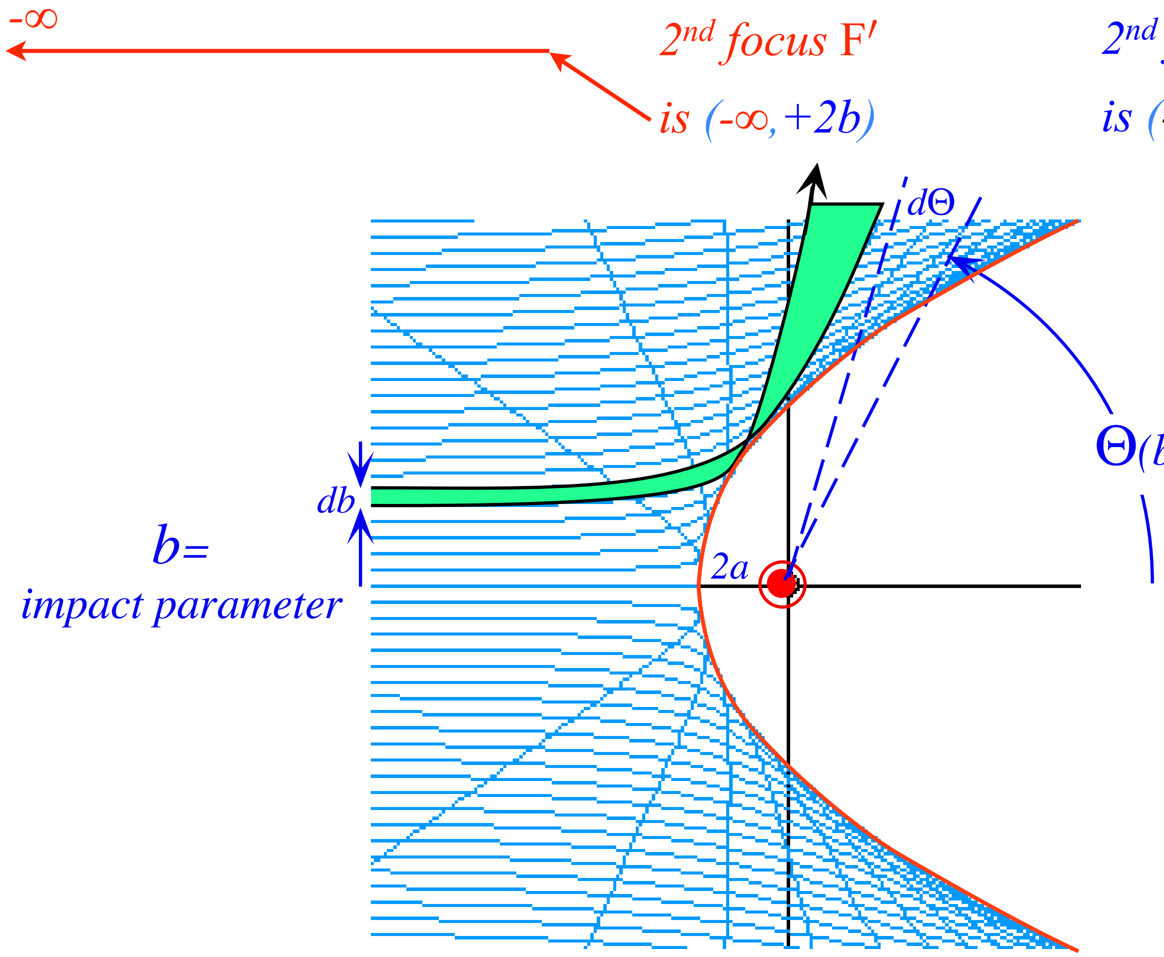
Envelope parabola
2^{nd} focus F′
is $(-\infty,+2b)$

Hyperbola
2^{nd} focus F′
is $(-2a,+2b)$

hyperbolic
impact
parameter

r-r' =2a applies
to each point on
upper hyperbola
and to the contact
point P for the
envelope parabola

Rutherford scattering geometry



Envelope parabola
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Hyperbola
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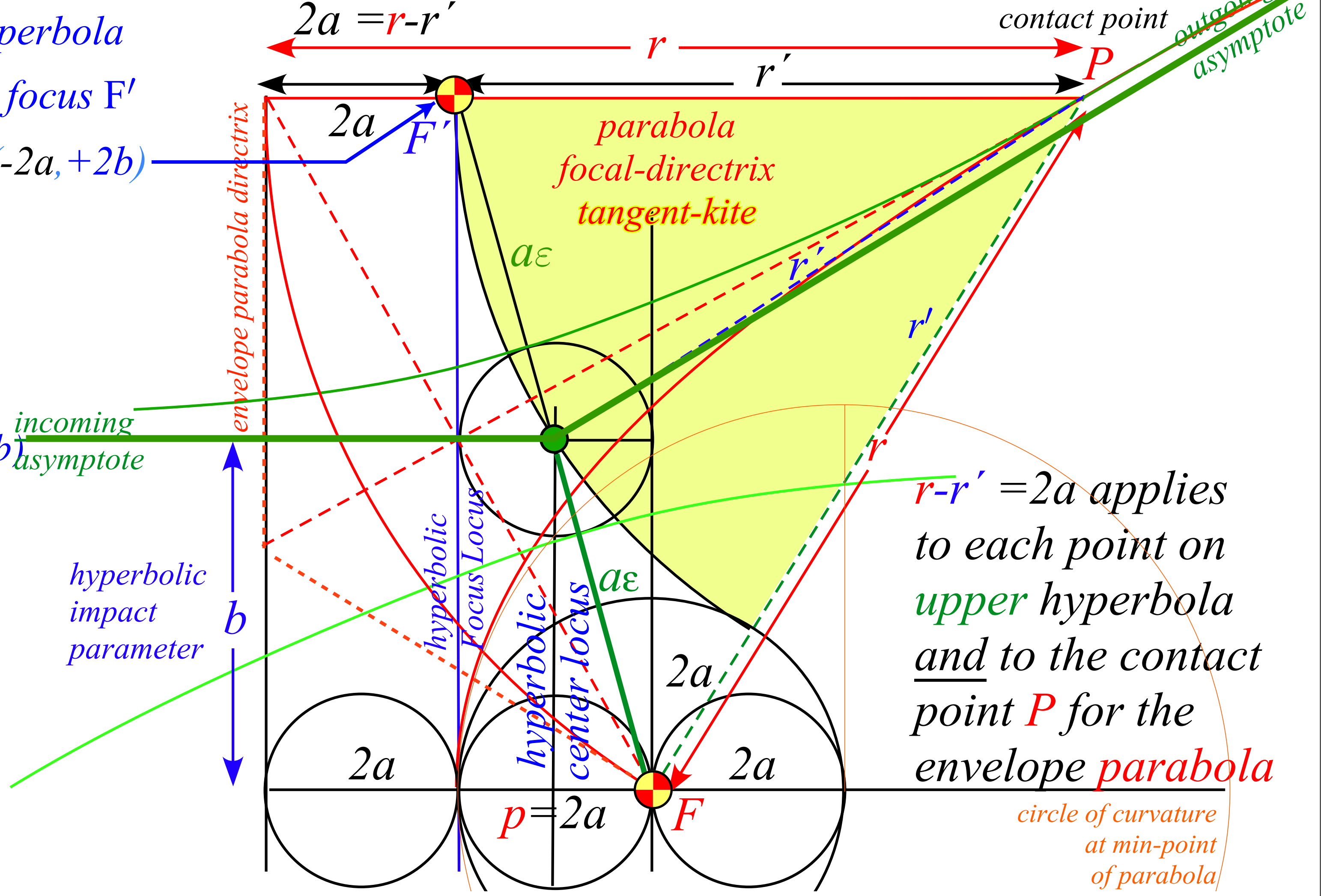
"Kite" geometry of envelope parabola



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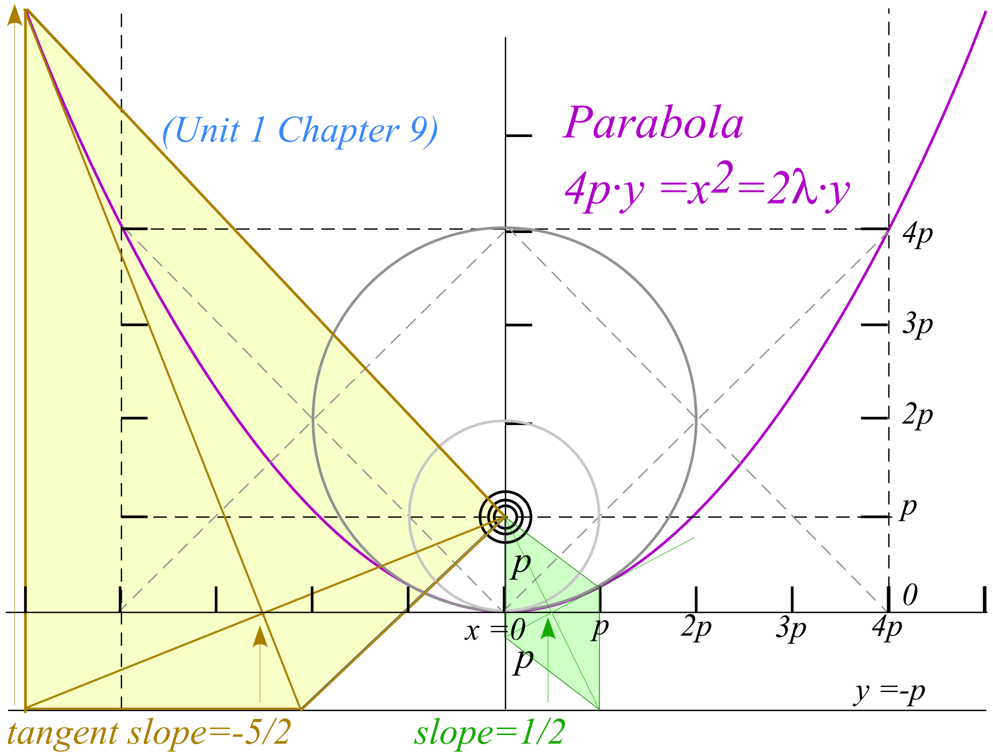
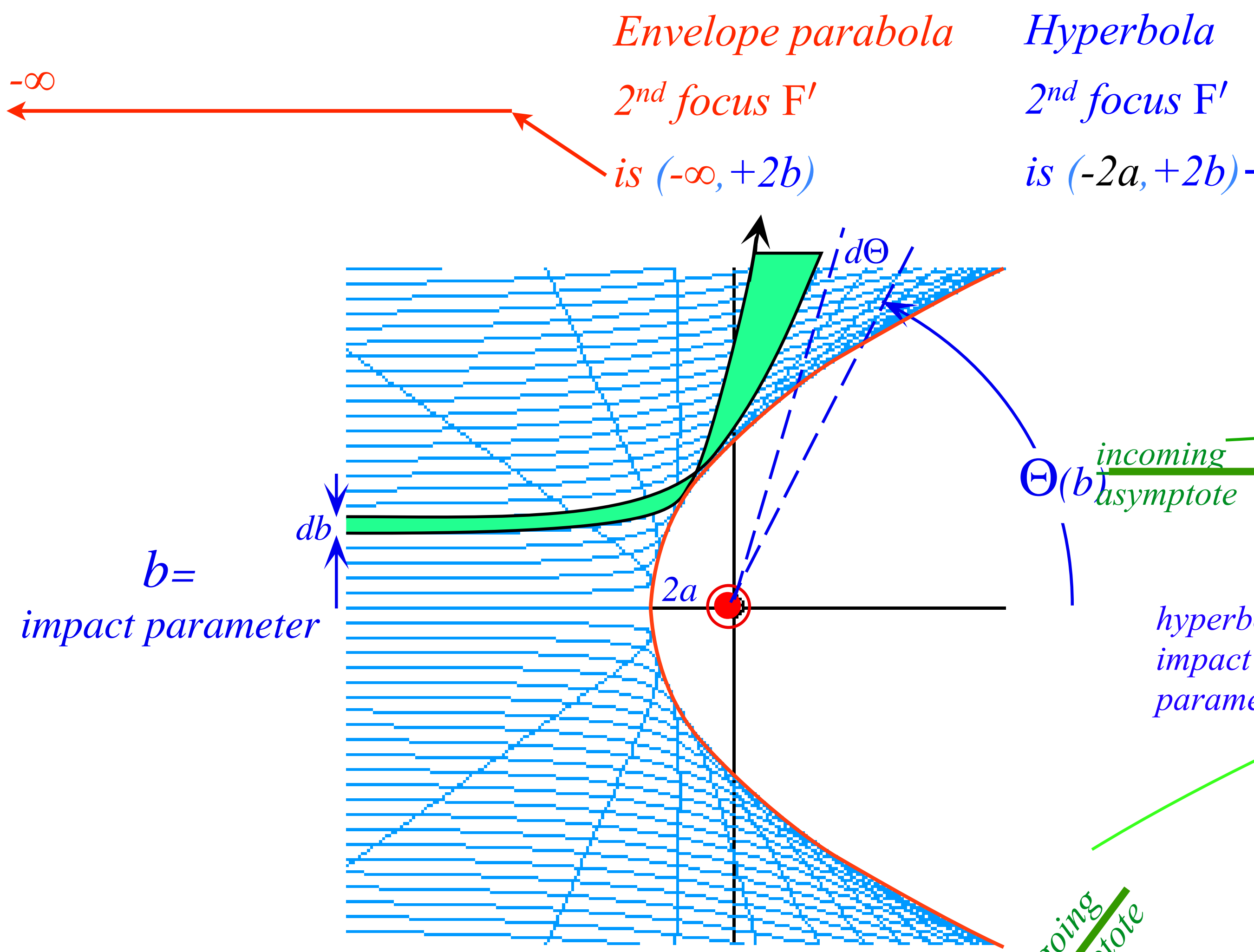
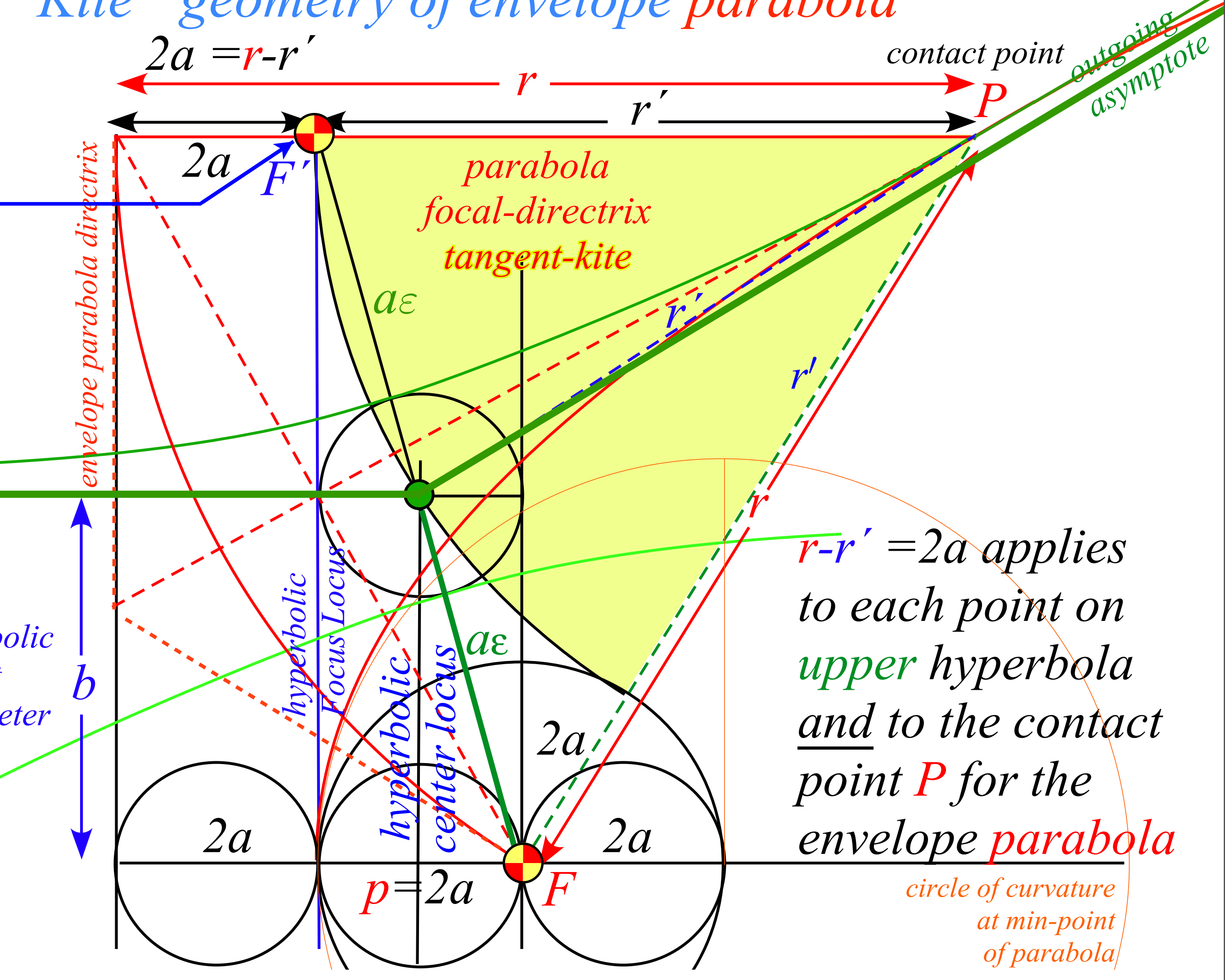


Rutherford scattering geometry



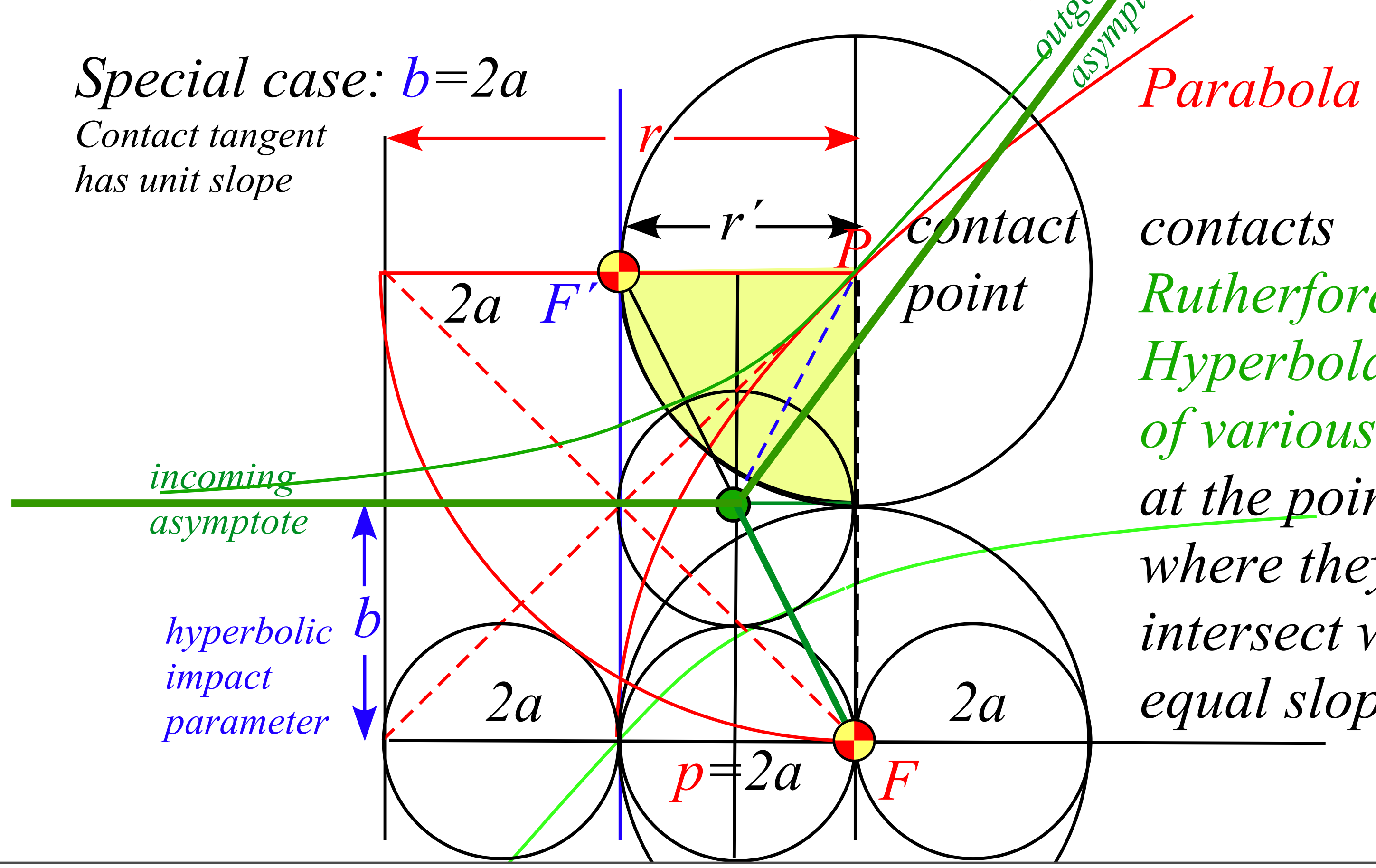
“Kite” geometry of envelope parabola



r-r′ =2a applies to each point on upper hyperbola and to the contact point P for the envelope parabola

Special case: b=2a

Contact tangent has unit slope



contacts Rutherford Hyperbolas of various b at the point where they intersect with equal slope

Rutherford scattering geometry

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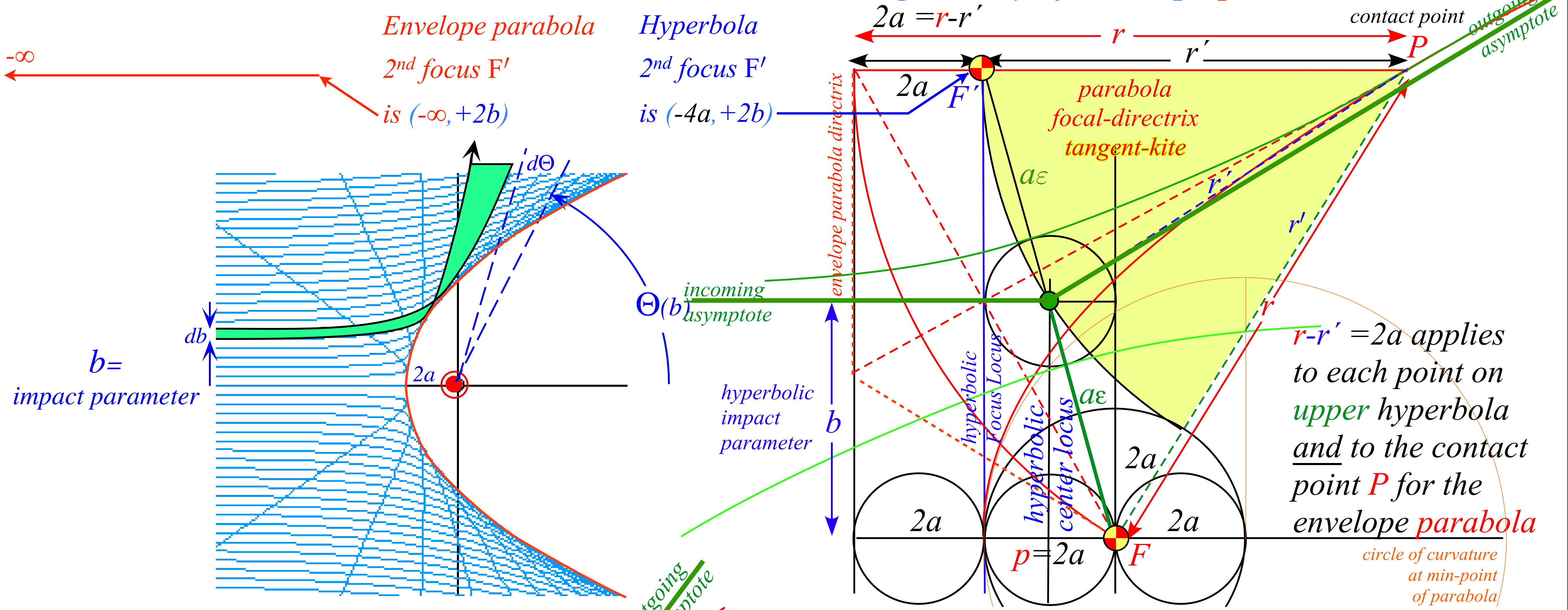

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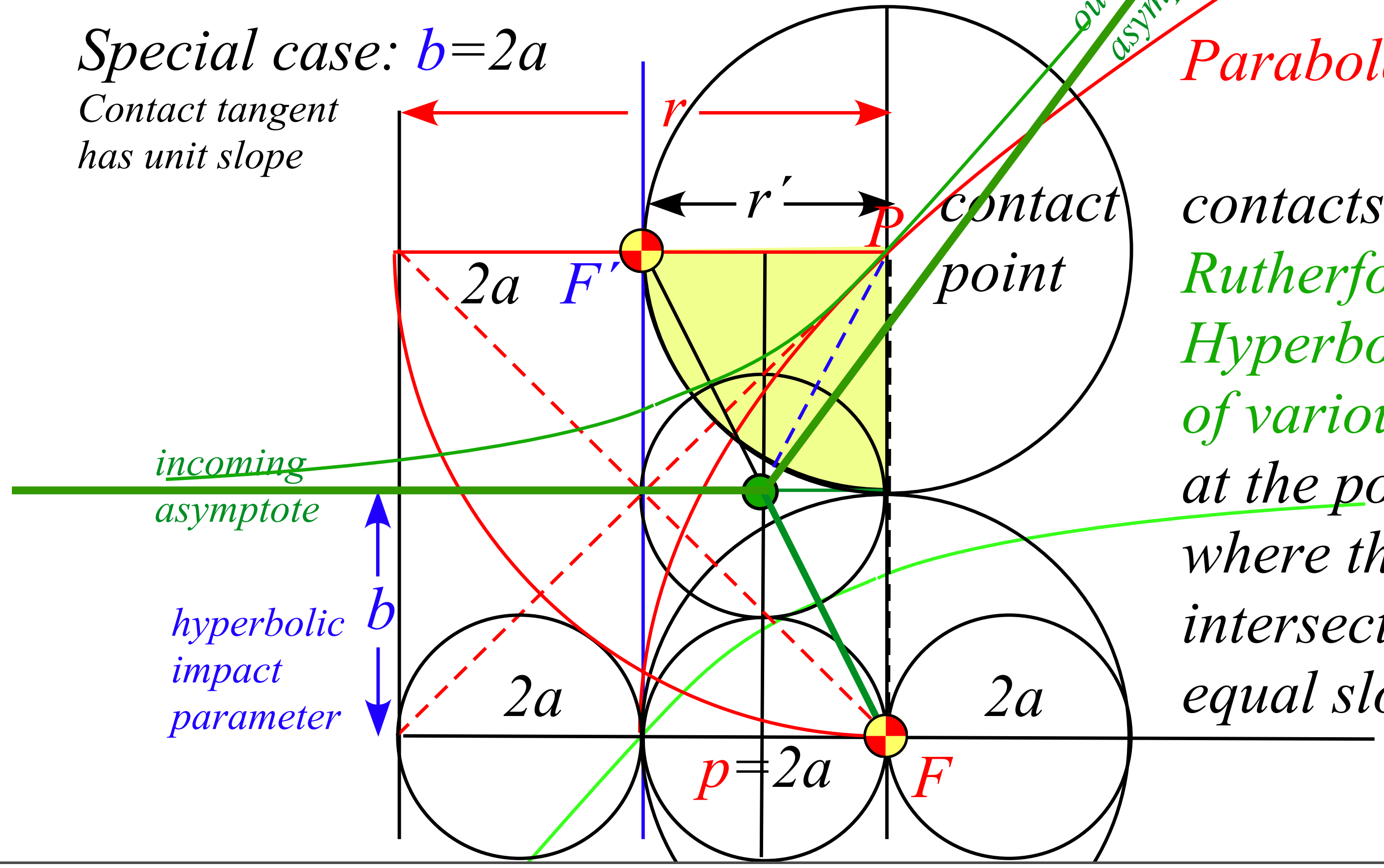
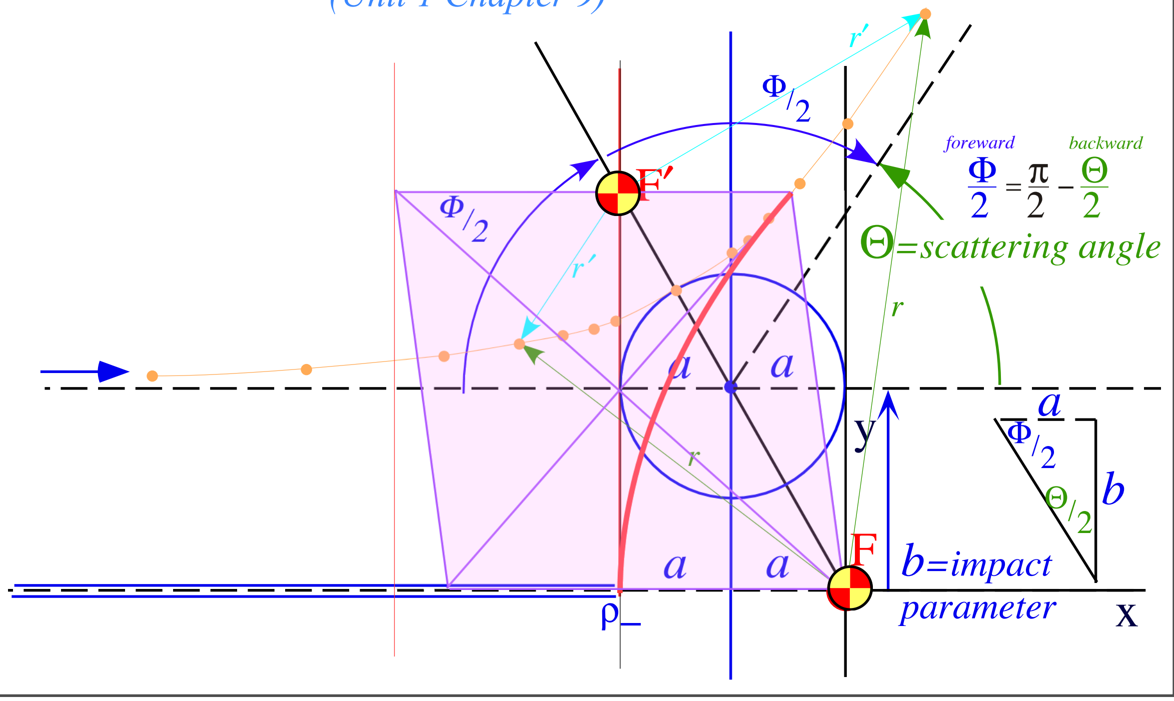


contacts Rutherford Hyperbolas of various b at the point where they intersect with equal slope

Recall parabolic " kite" geometry

(Unit 1 Chapter 9)



Rutherford scattering and hyperbolic orbit geometry

- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- ➡ *Differential and total scattering cross-sections*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

- *Review and connection to usual orbital algebra (previous lecture)*

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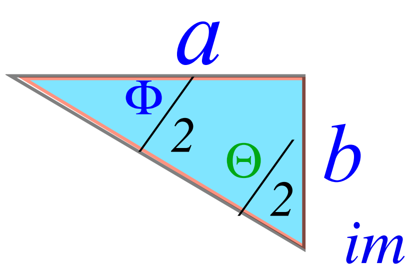
Rutherford scattering geometry



b= *impact parameter*

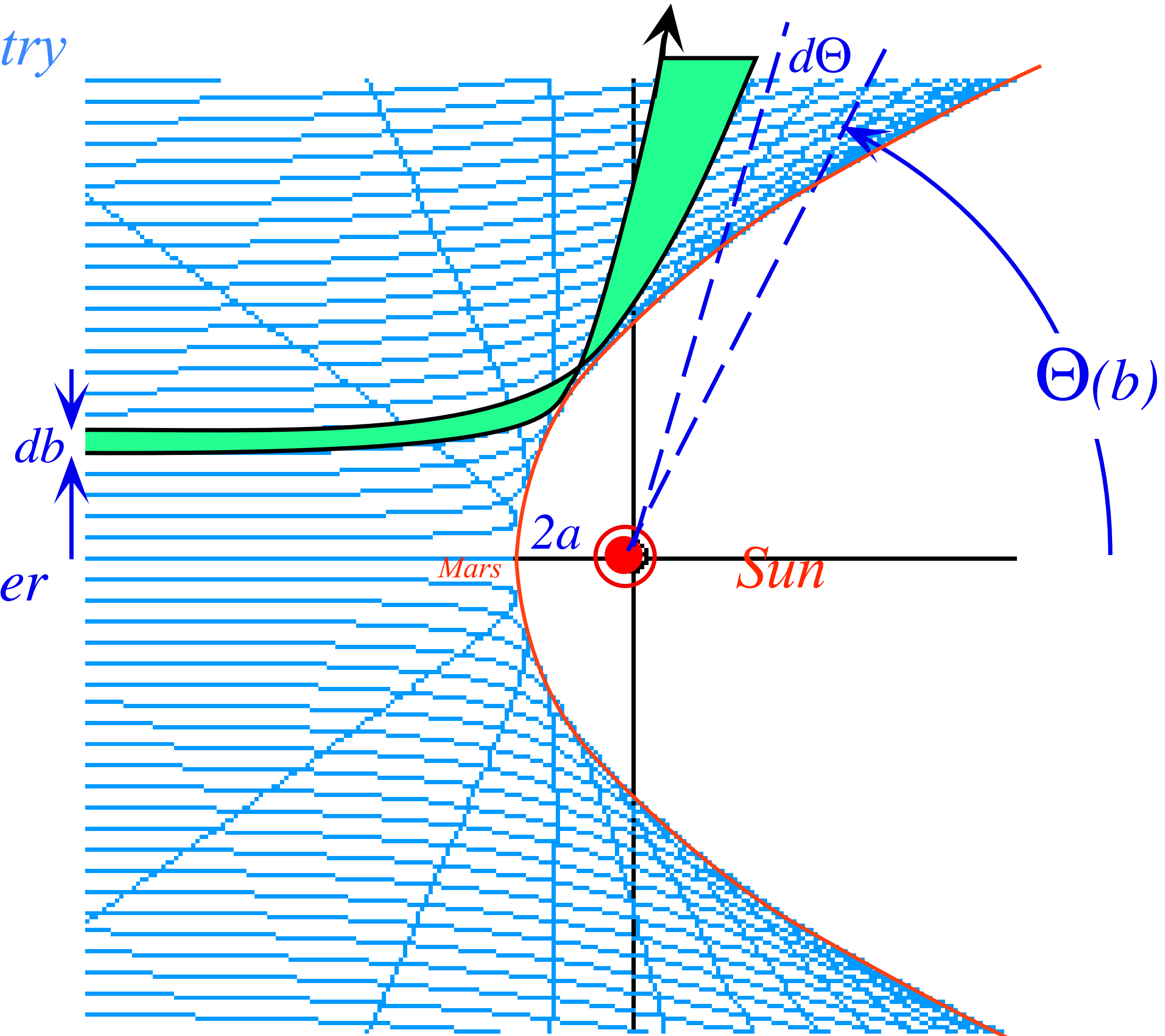
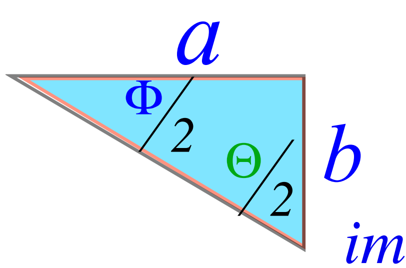


Fig. 5.3.2 Family of iso-energetic Rutherford scattering orbits with varying impact parameter.

Also: Approximate model of deep-space H-atom scattering from solar wind as our Sun travels around galaxy. Lyman-α shock wave found just inside Mars orbital radius 2a~1.2Au.

Incremental window dσ=*b*·*db* normal to beam axis at *x*=-∞ scatters to area $dA = R^2 \sin\Theta d\Theta d\varphi = R^2 d\Omega$ onto a sphere at *R*=+∞ where is called the *incremental solid angle* $d\Omega = \sin\Theta d\Theta d\varphi$

Rutherford scattering geometry



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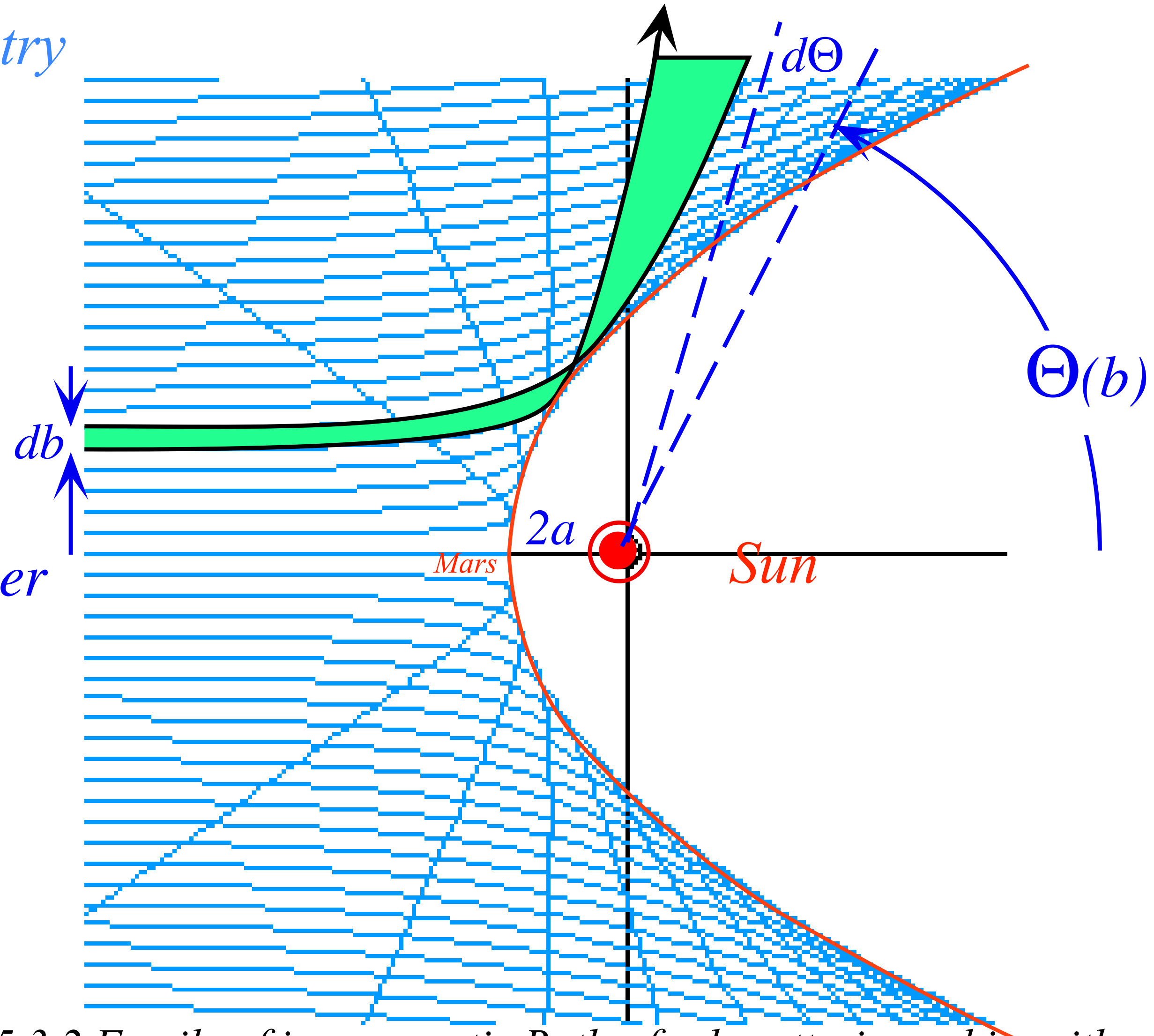

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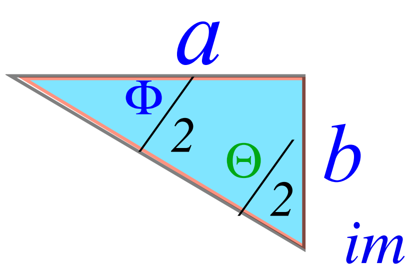
Rutherford scattering geometry



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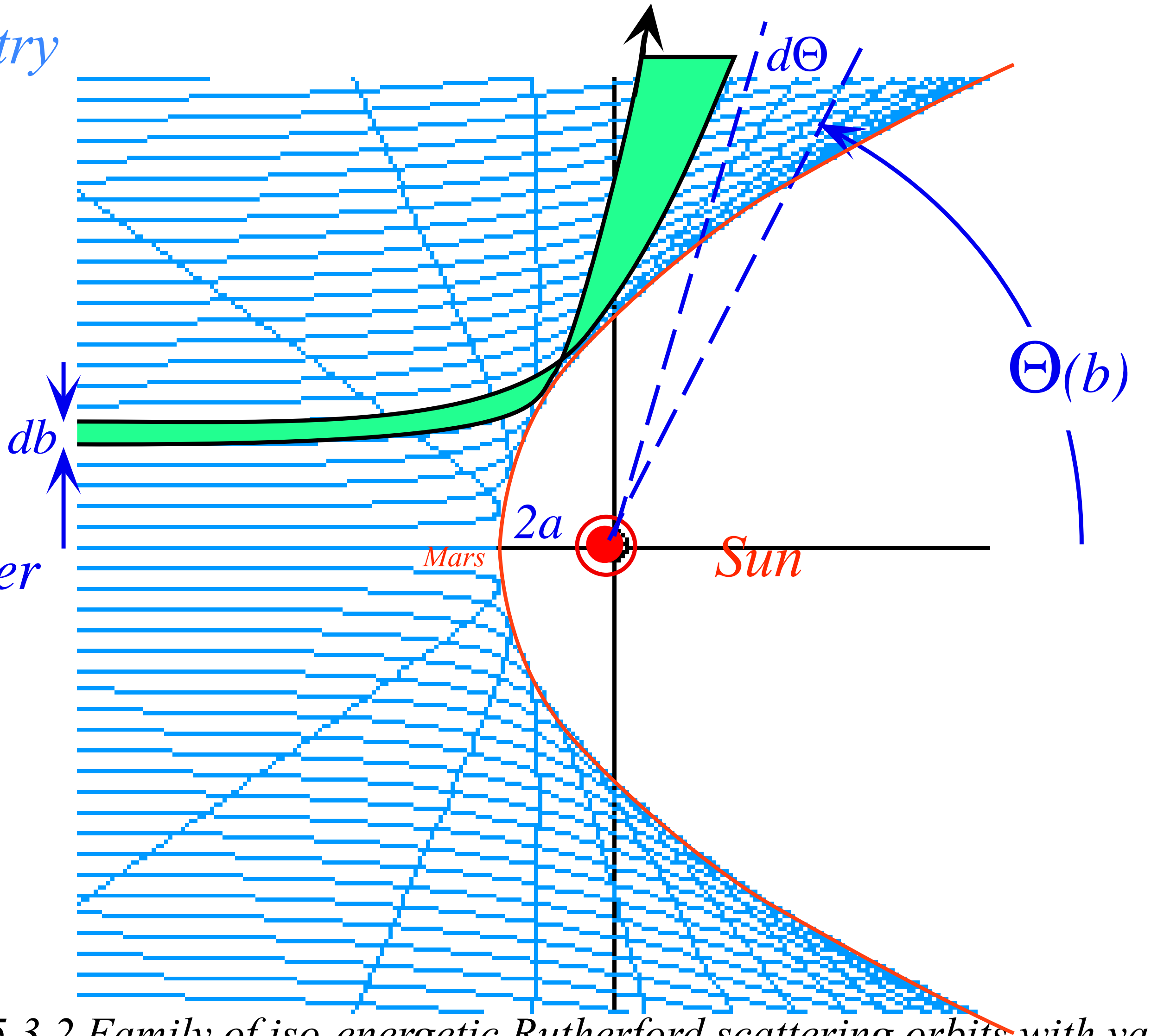
$$\frac{b}{a} = \cot\frac{\Theta}{2}$$



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with: $$\frac{db}{d\Theta} = \frac{-a}{2}\csc^2\frac{\Theta}{2}$$

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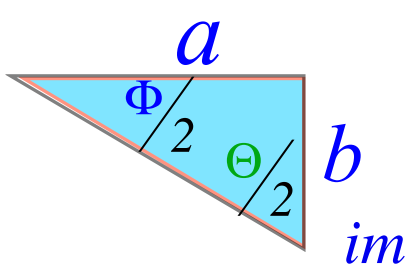

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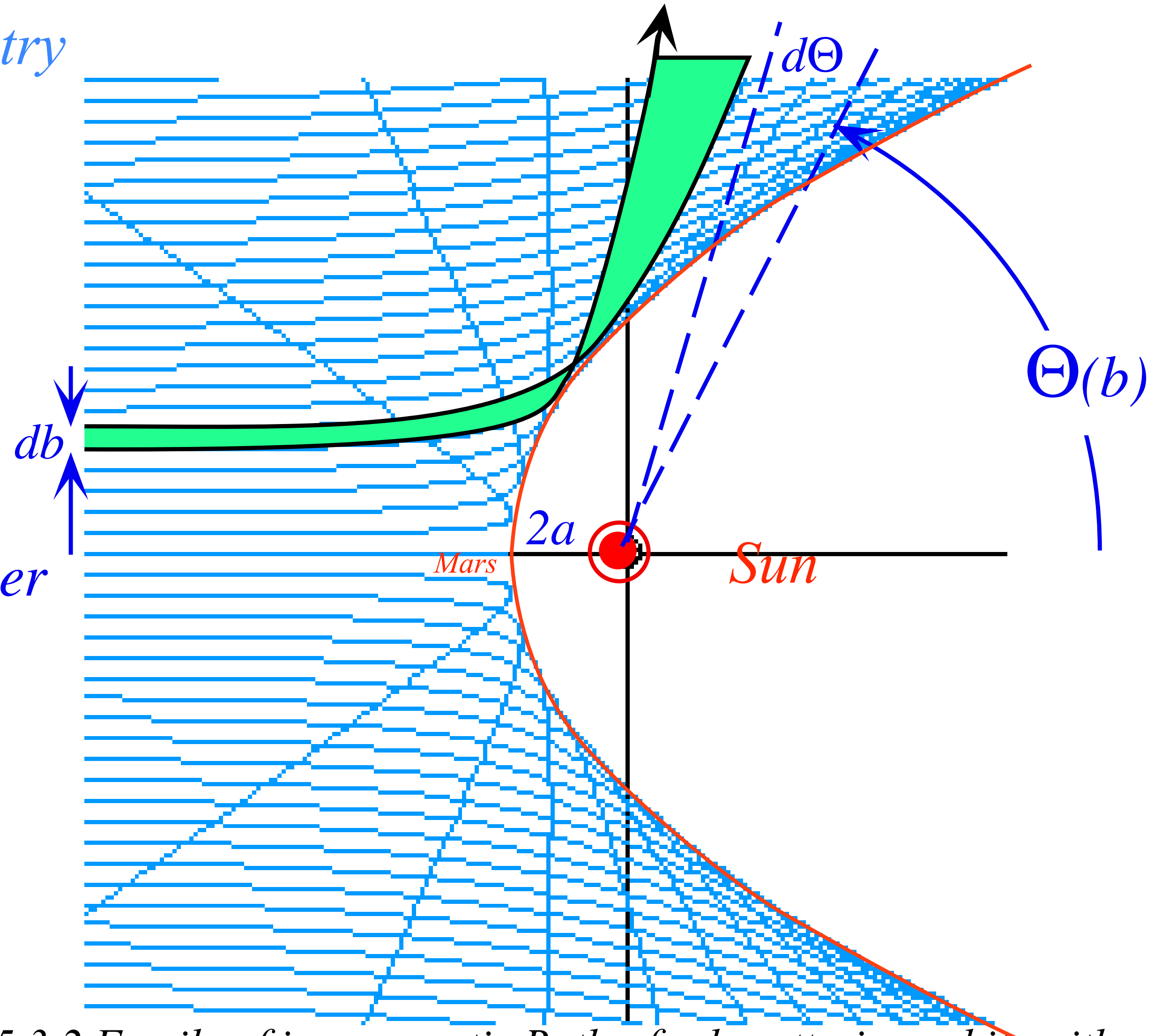

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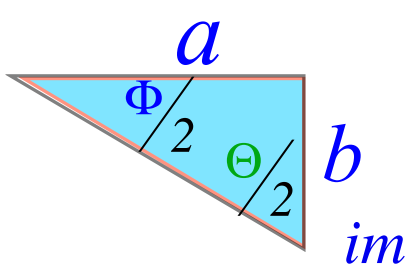
$(\textit{Never forget}!: a = \frac{-k}{2E})$

Rutherford scattering geometry



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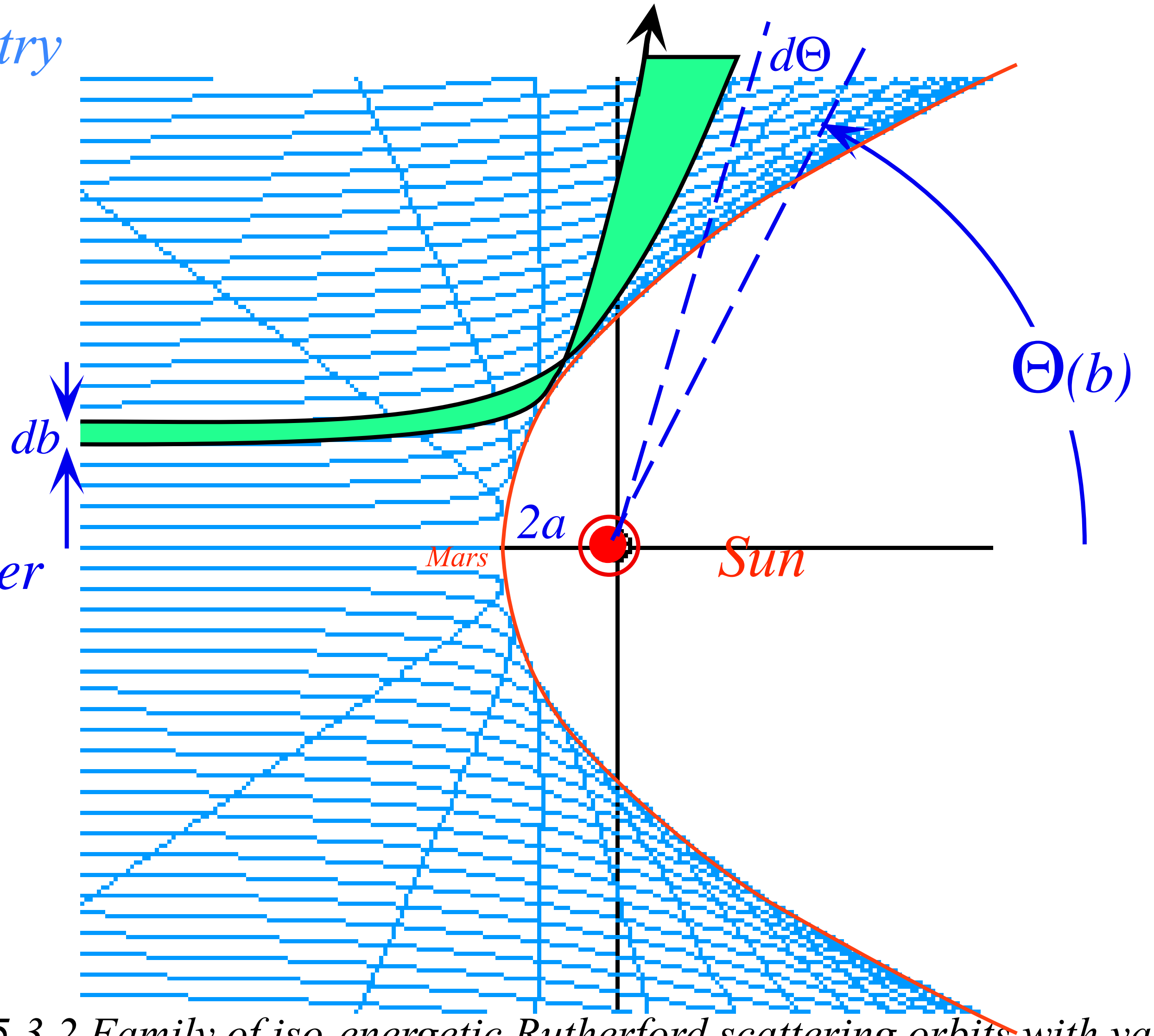

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Geometry: $b = a\cot\frac{\Theta}{2} = \frac{k}{2E}\cot\frac{\Theta}{2}$ gives the *Rutherford DSC*. $\frac{d\sigma}{d\Omega} = \frac{-a^2\cos\frac{\Theta}{2}}{2\sin\Theta\sin^3\frac{\Theta}{2}}$

with: $\frac{db}{d\Theta} = \frac{-a}{2}\csc^2\frac{\Theta}{2} = \frac{-a}{2\sin^2\frac{\Theta}{2}}$ and: $\sin\Theta = 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}$

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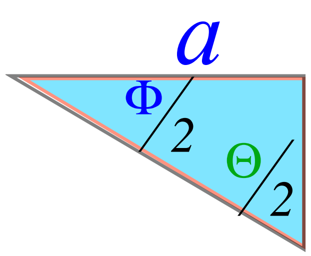

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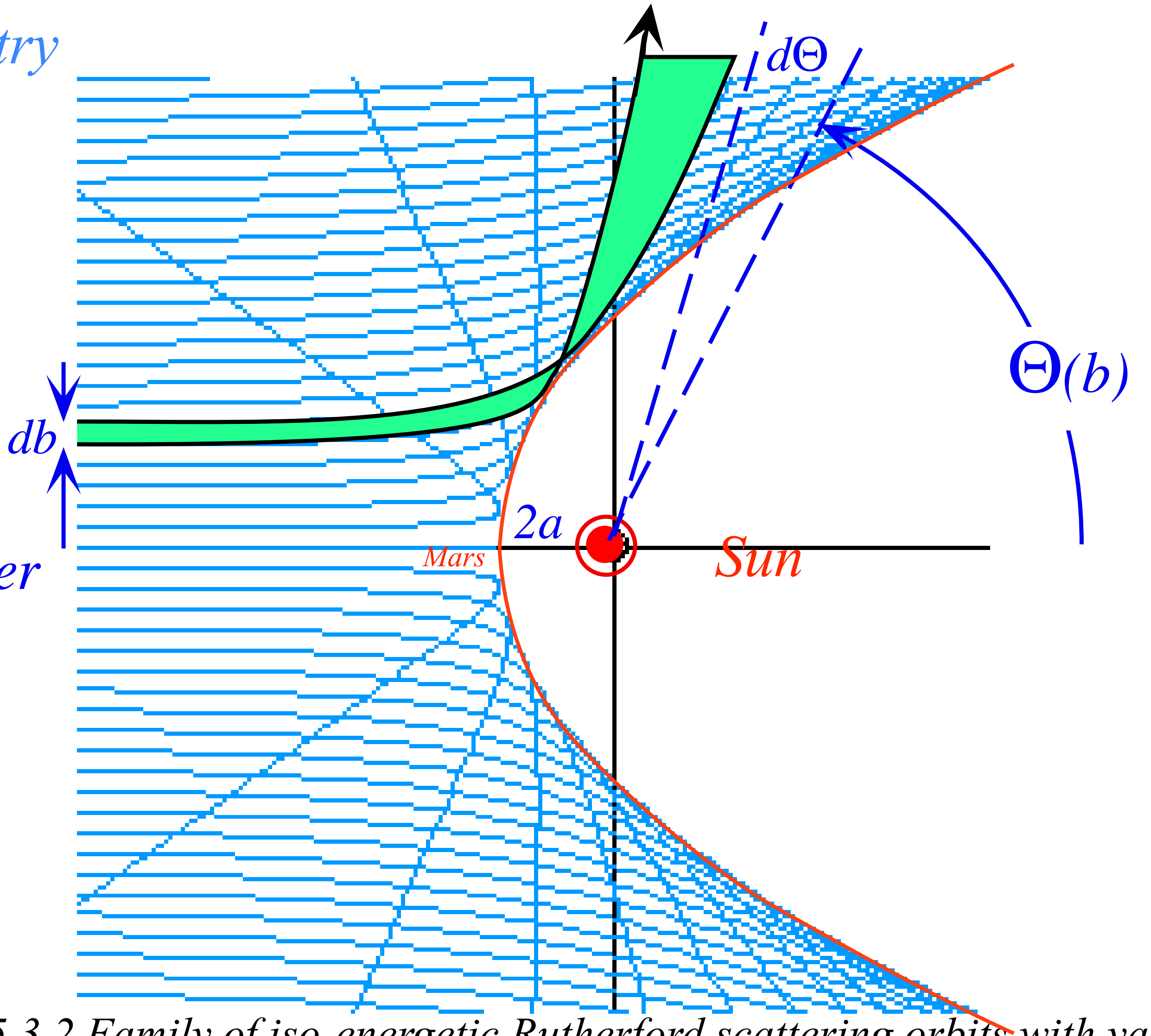

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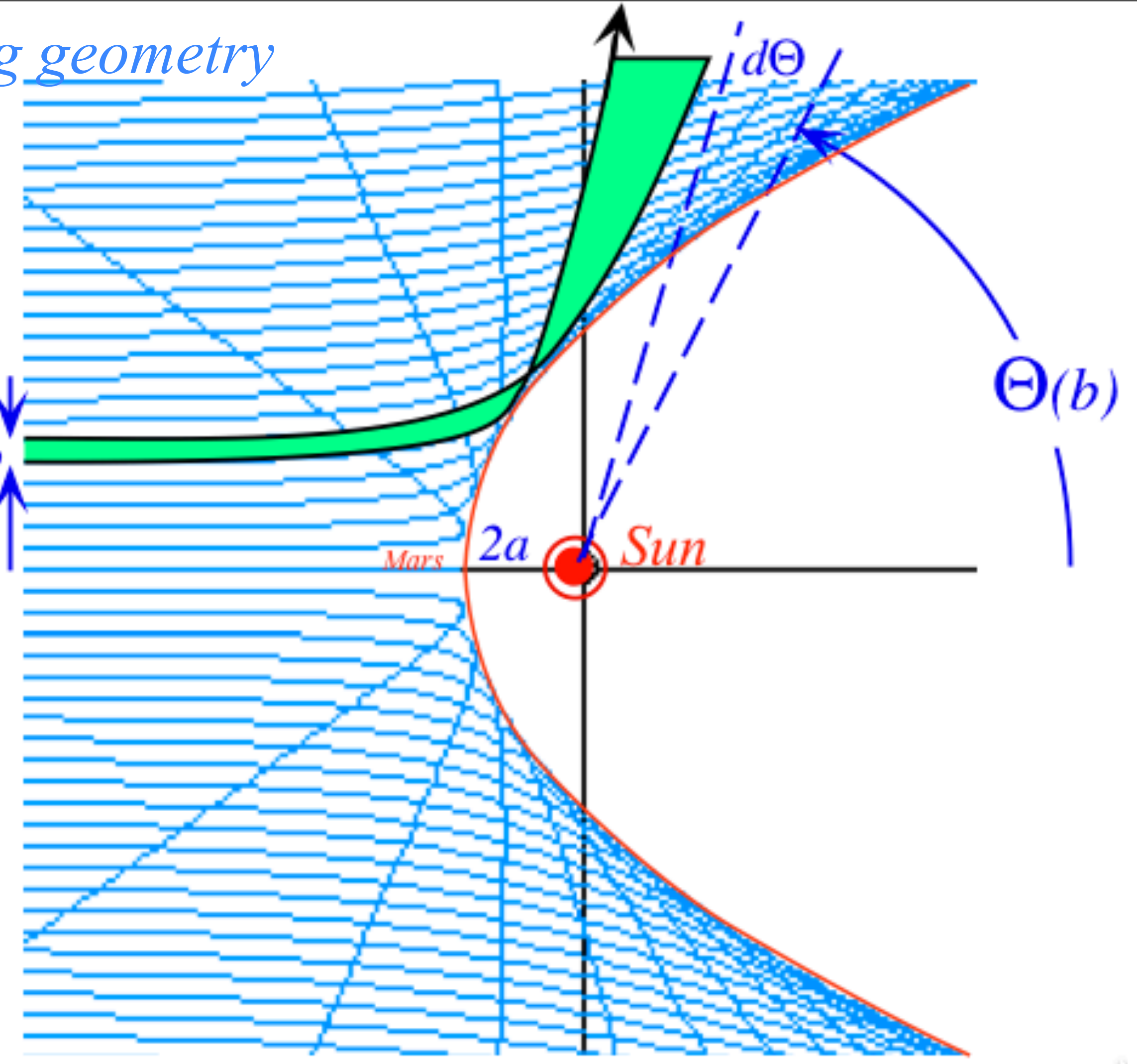
$$\frac{d\sigma}{d\Omega}=\frac{-a^2\cos\frac{\Theta}{2}}{2\sin\Theta\sin^3\frac{\Theta}{2}}=\frac{-k^4}{16E^2}\sin^{-4}\frac{\Theta}{2}$$

with: $\dfrac{db}{d\Theta}=\dfrac{-a}{2}\csc^2\dfrac{\Theta}{2}=\dfrac{-a}{2\sin^2\frac{\Theta}{2}}$ and: $\sin\Theta=2\sin\dfrac{\Theta}{2}\cos\dfrac{\Theta}{2}$

(*Never forget*!: $a=\dfrac{-k}{2E}$)

This classical result agrees exactly with 1st Born approximation to *quantum* Coulomb *DSC*!

Rutherford scattering geometry



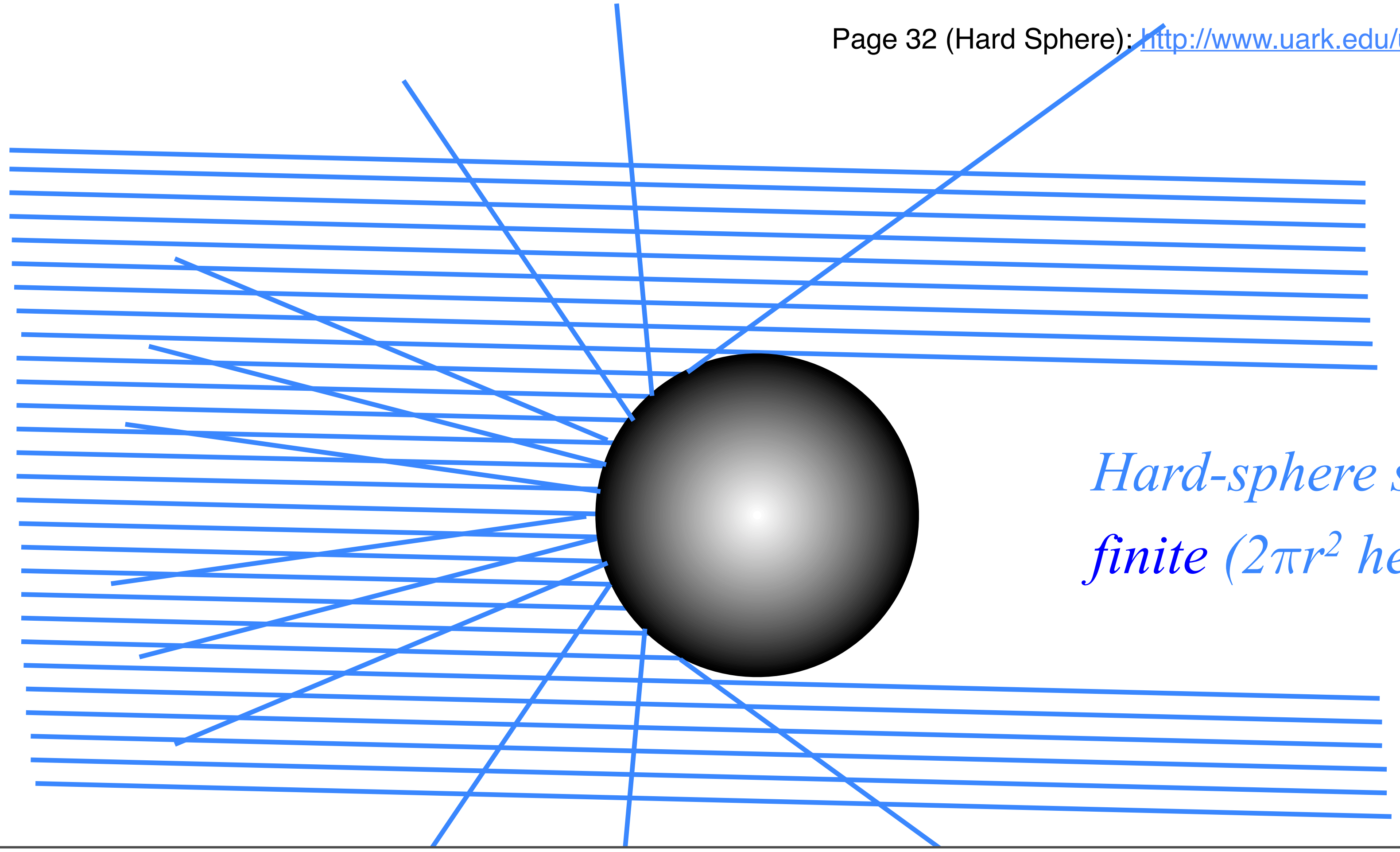
b= impact parameter

Two Extremes:

*Rutherford (Coulomb) scattering has **infinite** (∞) total cross section*

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\Omega \frac{k^4}{16E^2} \sin^{-4} \frac{\Theta}{2} = \infty$$

Page 32 (Hard Sphere): http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=p32



*Hard-sphere scattering has **finite** ($2\pi r^2$ here) total cross section*

Rutherford scattering and hyperbolic orbit geometry

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Isotropic field *V*=*V(r)* guarantees conservation *angular momentum vector* **L**

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \dot{\mathbf{r}}$$

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Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boxed{\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\frac{\mathbf{r}}{r}-\frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}}$$

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(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector* ***S***

$$S_A=\tfrac{1}{2}(x_1^2+p_1^2-x_2^2-p_2^2)$$

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Generates symmetry groups: $R(3)\subset R(3)\times R(3)\subset O(4)$

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Consider dot product of ε with a radial vector **r**:

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...or of $\boldsymbol{\varepsilon}$ with momentum vector **p**:

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$$\varepsilon\, r\cos\phi=r-\frac{L^2}{km} \qquad \text{or:} \qquad r=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

...or of ε with momentum vector **p**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r$$

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ) *geometry of orbital mechanics*

Isotropic field $V=V(r)$ guarantees conservation *angular momentum vector* **L**

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}=m\,\mathbf{r}\times\dot{\mathbf{r}}$$

Coulomb $V=-k/r$ also conserves *eccentricity vector* $\boldsymbol{\varepsilon}$

$$\boxed{\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{p}\times\mathbf{L}}{km}=\frac{\mathbf{r}}{r}-\frac{\mathbf{p}\times(\mathbf{r}\times\mathbf{p})}{km}}$$

(...for sake of comparison...)

IHO $V=(k/2)r^2$ also conserves *Stokes vector **S***

$$S_A=\tfrac{1}{2}(x_1{}^2+p_1{}^2-x_2{}^2-p_2{}^2)$$
$$S_B=x_1p_1+x_2p_2$$
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$\mathbf{A}=km\cdot\boldsymbol{\varepsilon}$ is known as the *Laplace-Hamilton-Gibbs-Runge-Lenz vector*.

Consider dot product of $\boldsymbol{\varepsilon}$ with a radial vector **r**:

$$\boldsymbol{\varepsilon}\bullet\mathbf{r}=\frac{\mathbf{r}\bullet\mathbf{r}}{r}-\frac{\mathbf{r}\bullet\mathbf{p}\times\mathbf{L}}{km}=r-\frac{\mathbf{r}\times\mathbf{p}\bullet\mathbf{L}}{km}=r-\frac{\mathbf{L}\bullet\mathbf{L}}{km}$$

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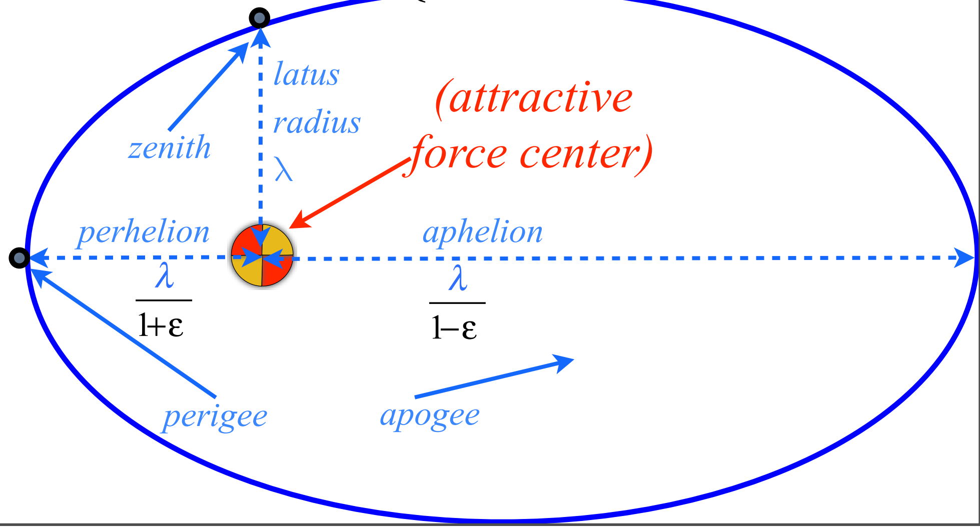

Eccentricity vector ε *and (ε,λ) geometry of orbital mechanics*

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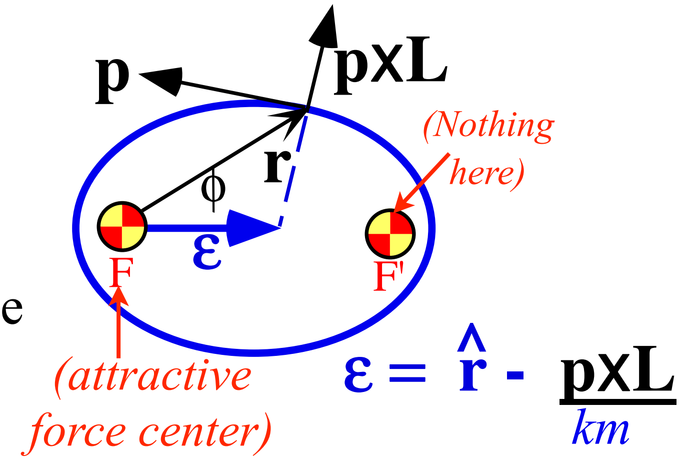
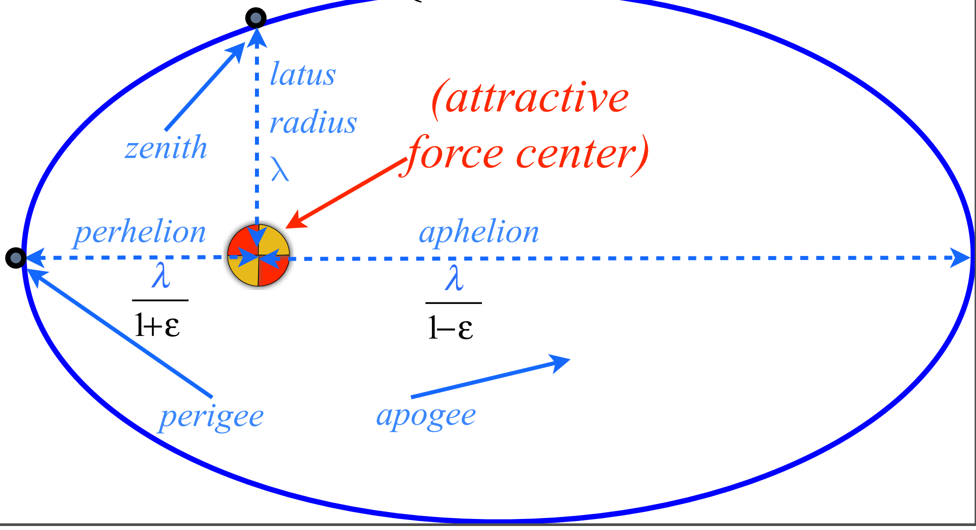
(a) Attractive (k>0)
Elliptic (E<0)

(Rotational momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is normal to the orbit plane.)





Eccentricity vector ε and (ε,λ) geometry of orbital mechanics

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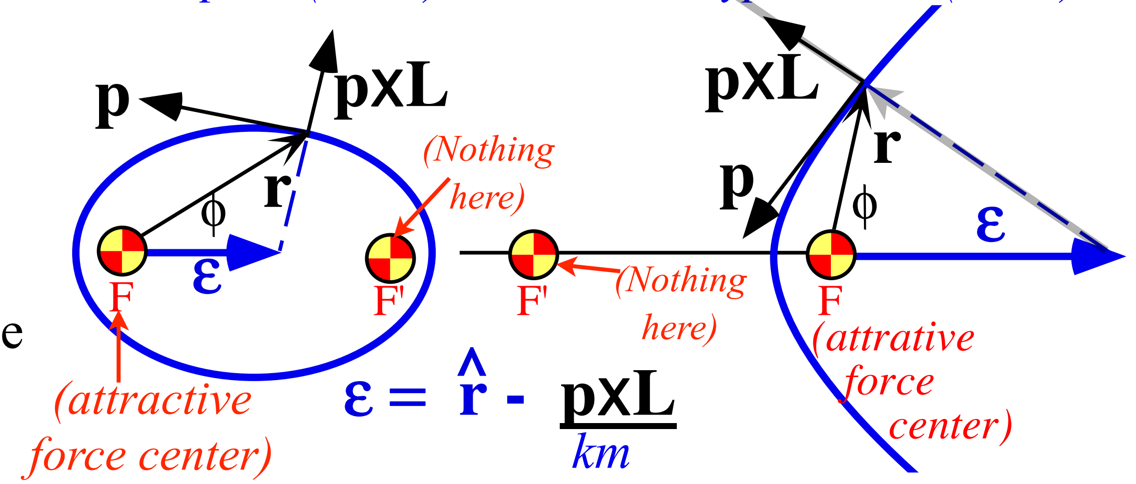
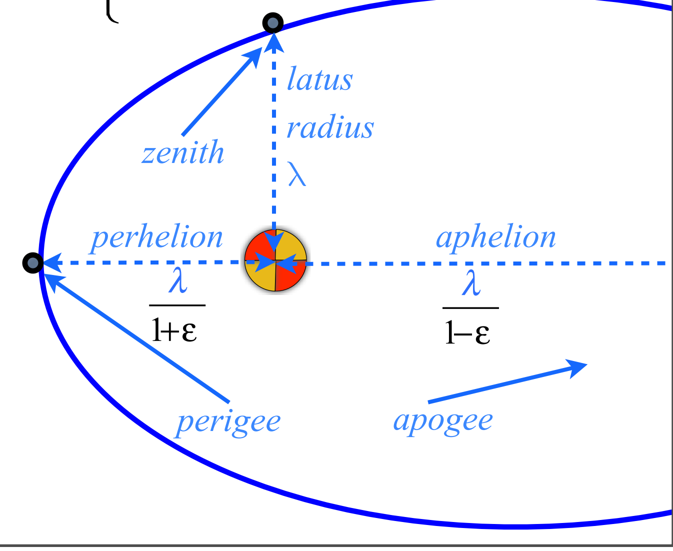
(b) Attractive (k>0) Hyperbolic (E>0)



$$\boldsymbol{\varepsilon}=\hat{\mathbf{r}}-\frac{\mathbf{pxL}}{km}$$



Eccentricity vector **ε** *and (ε,λ) geometry of orbital mechanics*

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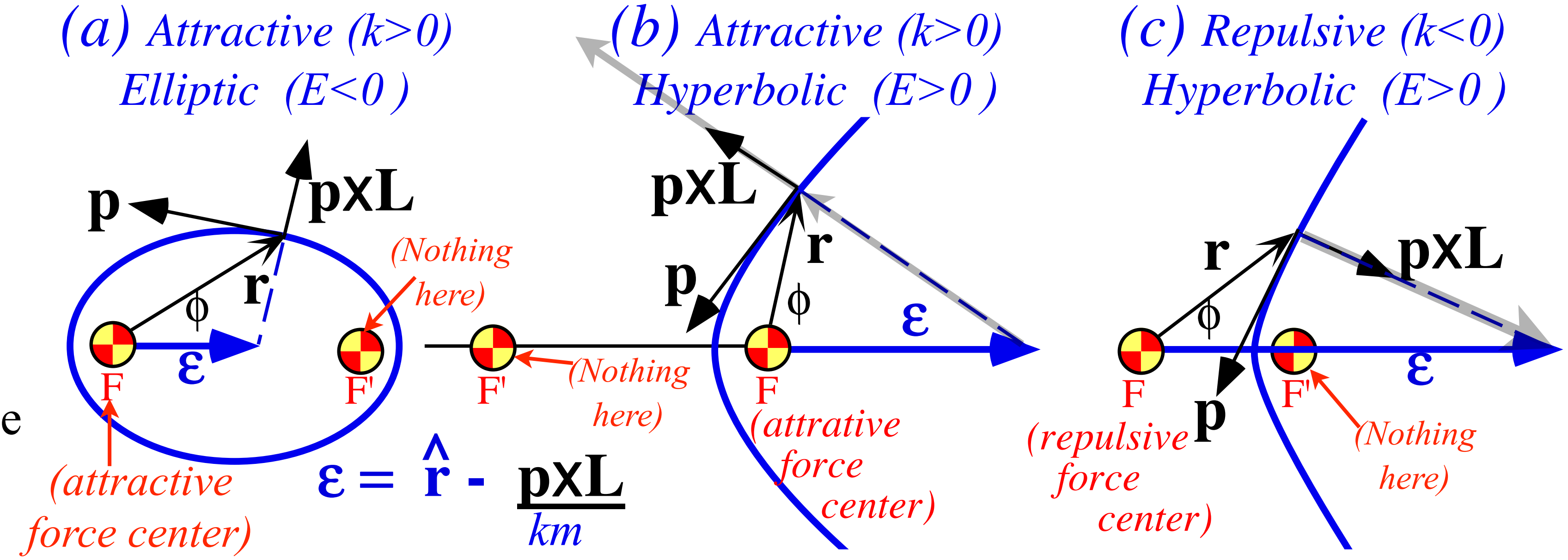
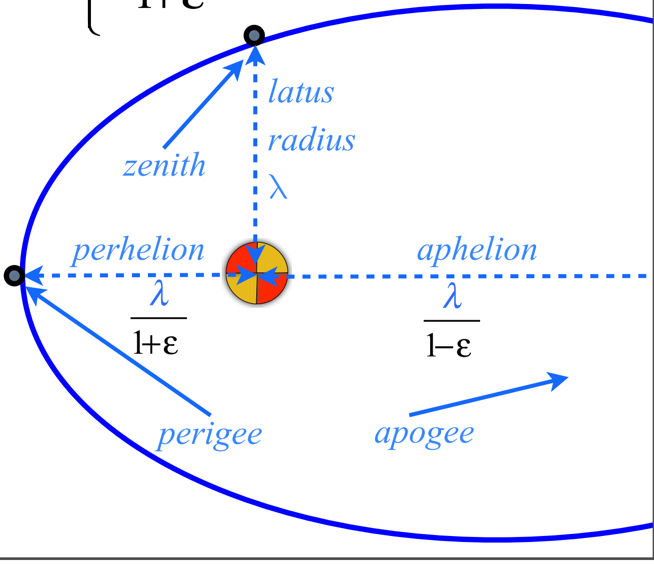


(a) Attractive (k>0) Elliptic (E<0)

(b) Attractive (k>0) Hyperbolic (E>0)

(c) Repulsive (k<0) Hyperbolic (E>0)



Rutherford scattering and hyperbolic orbit geometry

Backward vs forward scattering angles and orbit construction example

Parabolic "kite" and orbital envelope geometry

Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

➔ *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

Derivation of $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

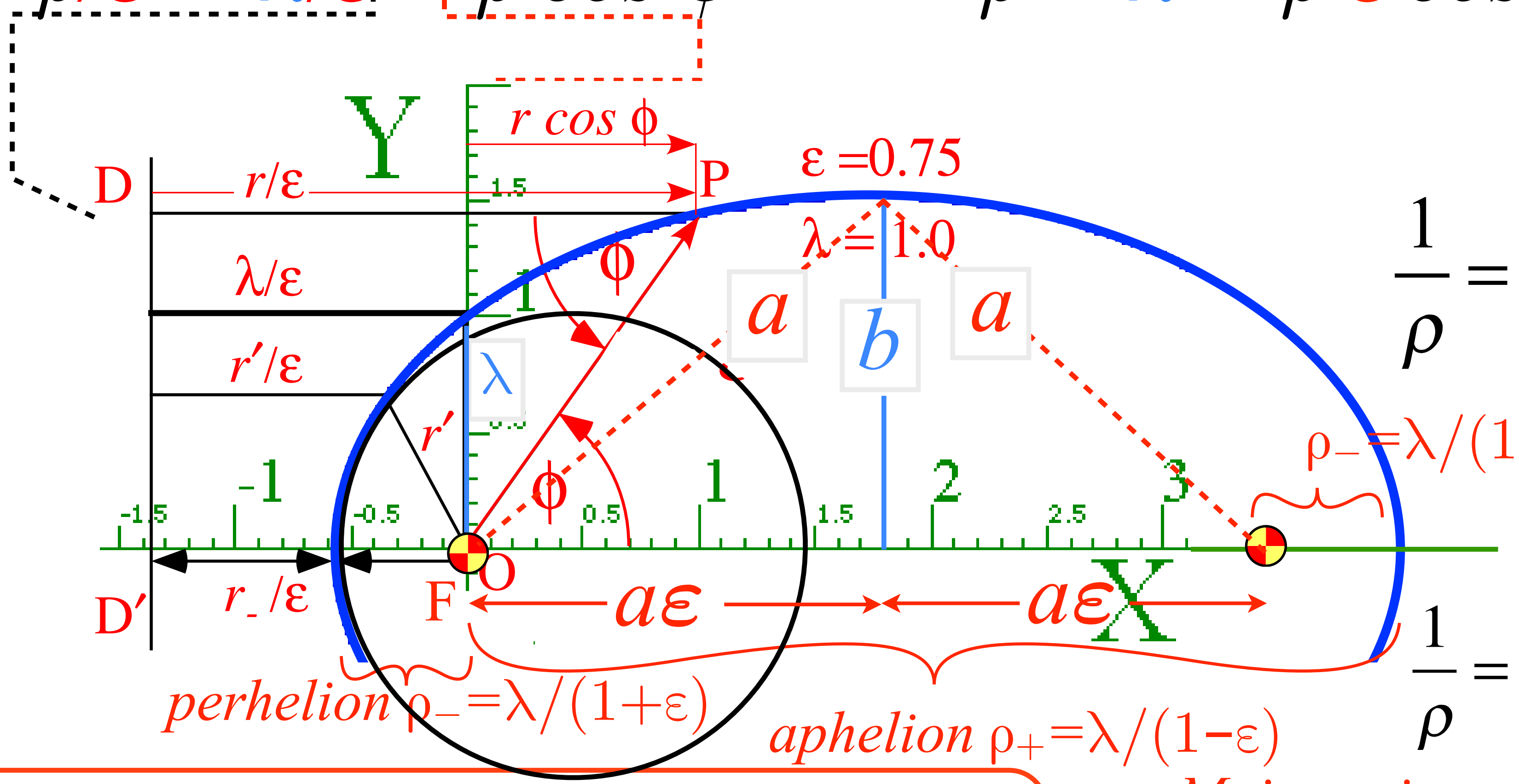
Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

Example of complete $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

(From Lecture 25 p. 64-74) *Geometry of Coulomb orbits (Let: r =ρ here)*

$$\rho/\varepsilon = \lambda/\varepsilon + \rho\cos\phi \qquad \rho = \lambda + \rho\,\varepsilon\cos\phi \qquad \rho = \frac{\lambda}{1-\varepsilon\cos\phi}$$



$$\frac{1}{\rho} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by:

Defining eccentricity ε

Distance to Focal-point $=\varepsilon\cdot$ Distance to Directrix-line

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon)+\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/|1-\varepsilon^2|$

Focal axis: $\rho_+ - \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon)-\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/|1-\varepsilon^2|$

Minor radius: $b=\sqrt{(a^2-a^2\varepsilon^2)}=\sqrt{(a\lambda)}$ *(ellipse:* $\varepsilon<1$*)*

Minor radius: $b=\sqrt{(a^2\varepsilon^2-a^2)}=\sqrt{(\lambda a)}$ *(hyperb:* $\varepsilon>1$*)*

(x,y) *parameters*	*physical constants*	(r,ϕ) *parameters*
major radius $a=\dfrac{k}{2E}$	*Energy* $E=\dfrac{k}{2a}$	$\varepsilon=\sqrt{\dfrac{k^2m+2\mu^2E}{k^2m}}=\sqrt{1\pm\dfrac{b^2}{a^2}}$ *eccentricity*
minor radius $b=\dfrac{\mu}{\sqrt{2m\lvert E\rvert}}$	*Orbital Momentum* $\mu=\sqrt{km\lambda}$	*latus radius* $\lambda=\dfrac{\mu^2}{km}=\dfrac{b^2}{a}$

$$\varepsilon^2 = 1-\frac{b^2}{a^2} \quad \text{(ellipse: } \varepsilon<1) \quad \frac{b}{a}=\sqrt{1-\varepsilon^2}$$

$$\varepsilon^2 = 1+\frac{b^2}{a^2} \quad \text{(hyperbola: } \varepsilon>1) \quad \frac{b}{a}=\sqrt{\varepsilon^2-1}$$

$\lambda = a(1-\varepsilon^2)$ *(ellipse:* $\varepsilon<1$*)*

$\lambda = a(\varepsilon^2-1)$ *(hyperb:* $\varepsilon>1$*)*

Rutherford scattering and hyperbolic orbit geometry

- Backward vs forward scattering angles and orbit construction example
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- Differential and total scattering cross-sections

Eccentricity vector $\boldsymbol{\varepsilon}$ and (ε,λ)-geometry of orbital mechanics

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ geometry of $\boldsymbol{\varepsilon}$-vector and orbital radius $\mathbf{r}$

- Review and connection to usual orbital algebra (previous lecture)

➡ Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ geometry of $\boldsymbol{\varepsilon}$-vector and momentum $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$-vector and (γ,R)-parameters

- Derivation of $\boldsymbol{\varepsilon}$-construction by analytic geometry

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

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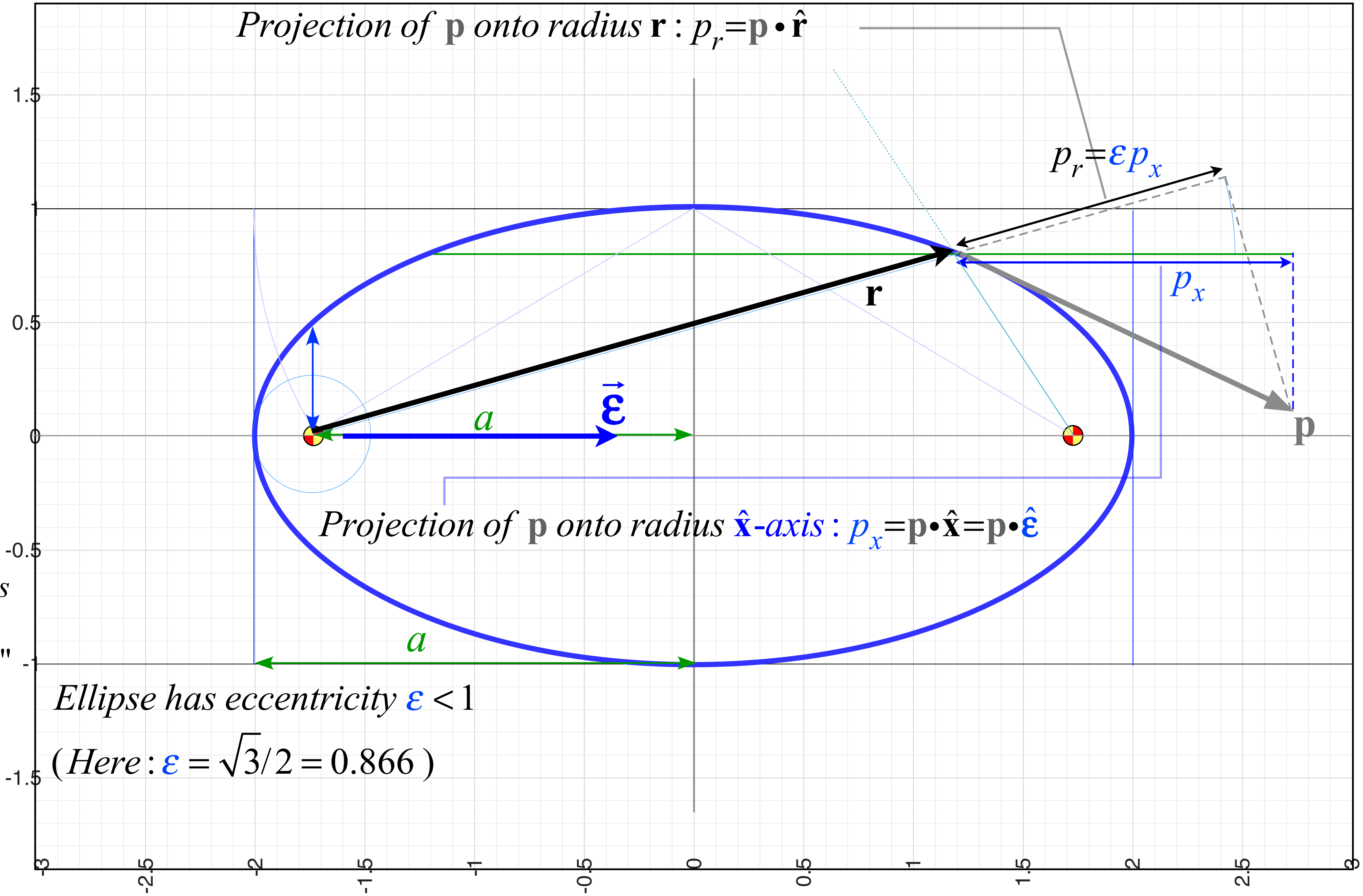
Dot product of ε with momentum vector $\mathbf{p}$:

$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}$$
$$=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r=\varepsilon p_x$$

This says:
"*Projection p_r of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis*: $(\hat{\mathbf{x}}=\hat{\boldsymbol{\varepsilon}})$ "



Projection of $\mathbf{p}$ *onto radius* $\mathbf{r}$: $p_r=\mathbf{p}\bullet\hat{\mathbf{r}}$

Projection of $\mathbf{p}$ *onto radius* $\hat{\mathbf{x}}$*-axis* : $p_x=\mathbf{p}\bullet\hat{\mathbf{x}}=\mathbf{p}\bullet\hat{\boldsymbol{\varepsilon}}$

Ellipse has eccentricity $\varepsilon<1$

$(Here: \varepsilon=\sqrt{3}/2=0.866\ )$

Dot product of ε with momentum vector $\mathbf{p}$:

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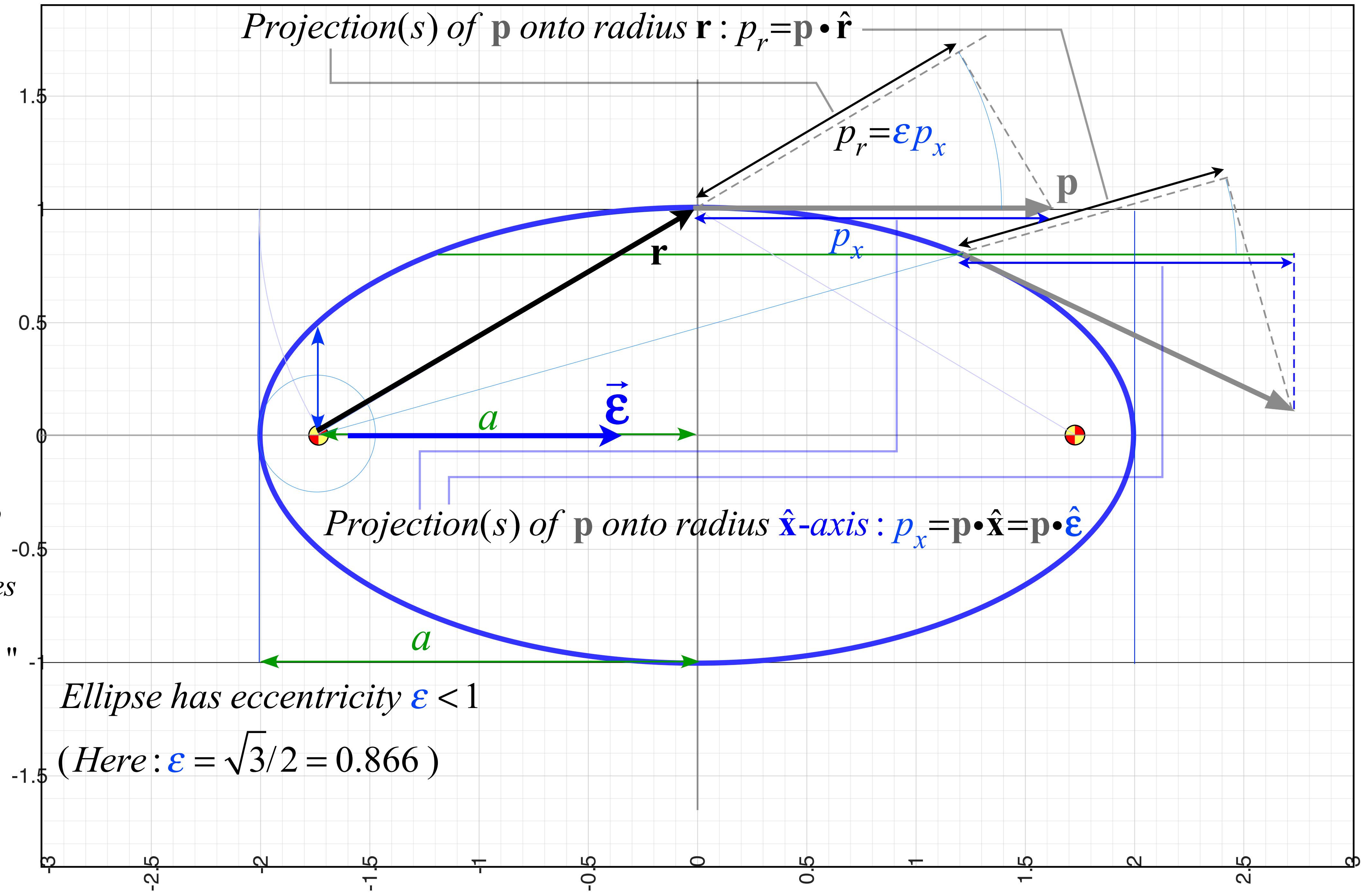

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NOTE: Lengths of vectors $\mathbf{p}$ and -$\mathbf{p}$ are not drawn to correctly show that momentum $\mathbf{p}=m\mathbf{v}$ grows as radial distance $r=|\mathbf{r}|$ falls. (To be shown on p. 85-90)

Dot product of ε with momentum vector $\mathbf{p}$:

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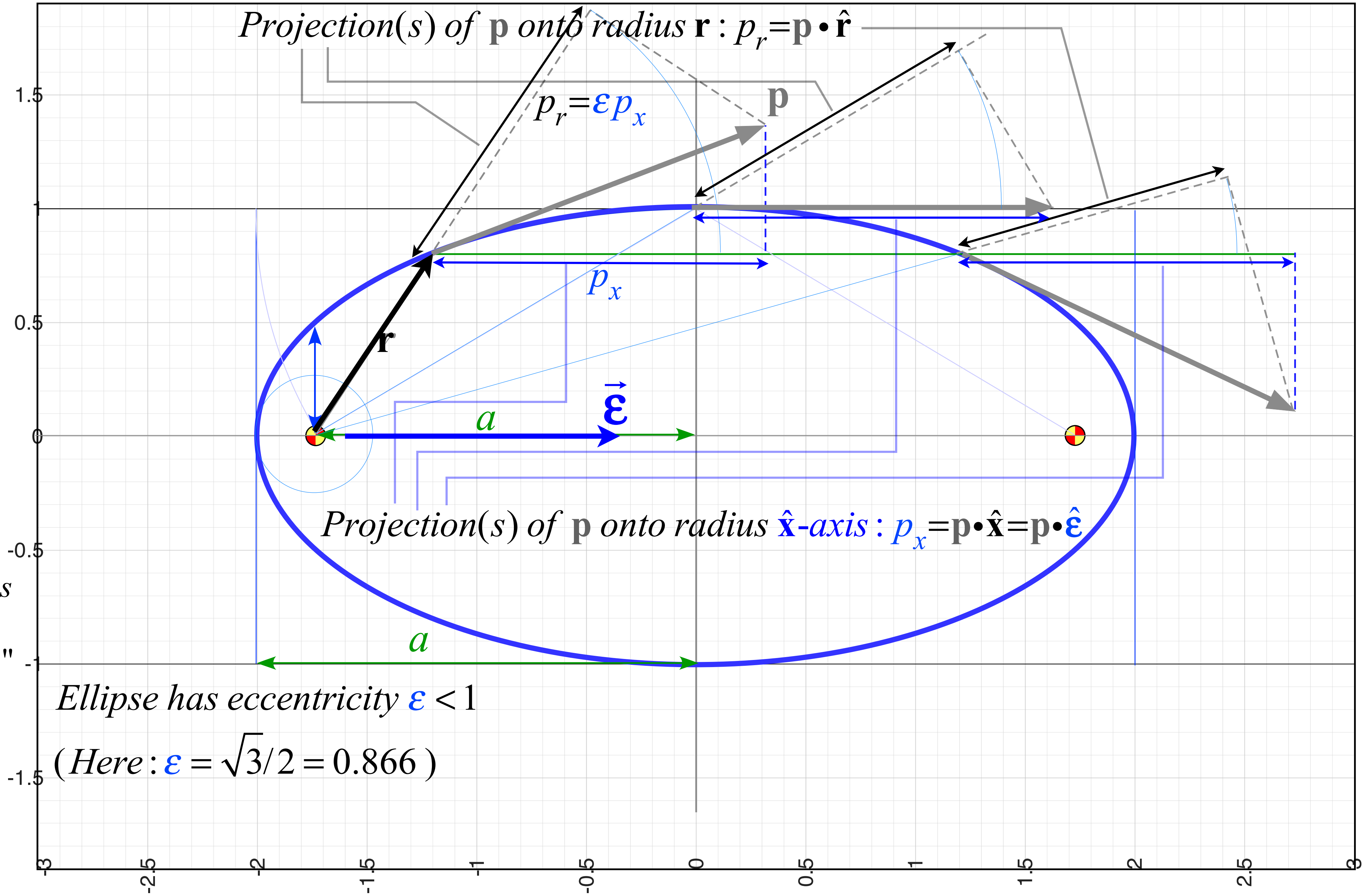

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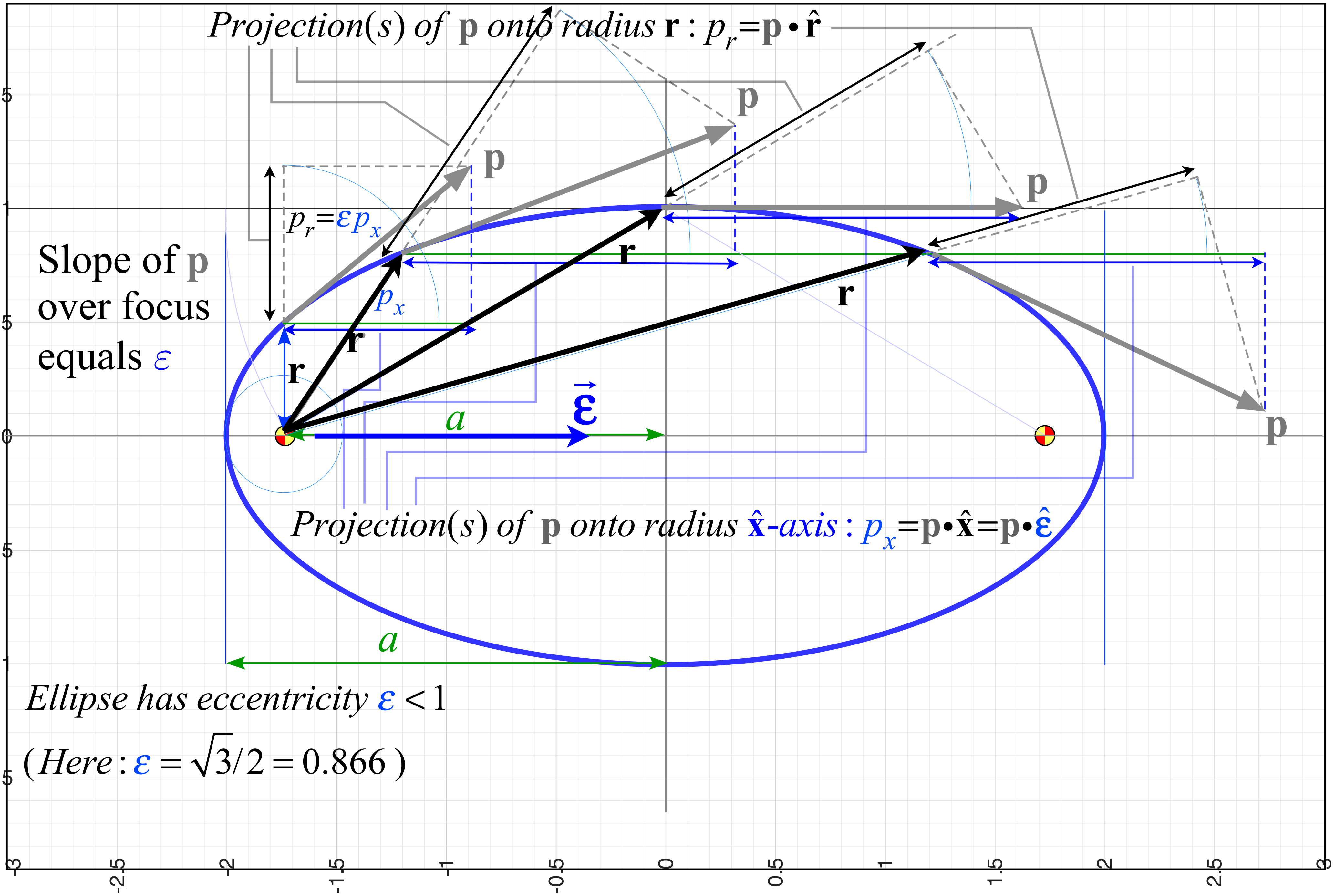

Projection(s) of $\mathbf{p}$ *onto radius* $\mathbf{r}$: $p_r=\mathbf{p}\bullet\hat{\mathbf{r}}$

Slope of $\mathbf{p}$ over focus equals ε

Projection(s) of $\mathbf{p}$ *onto radius* $\hat{\mathbf{x}}$*-axis* : $p_x=\mathbf{p}\bullet\hat{\mathbf{x}}=\mathbf{p}\bullet\hat{\boldsymbol{\varepsilon}}$

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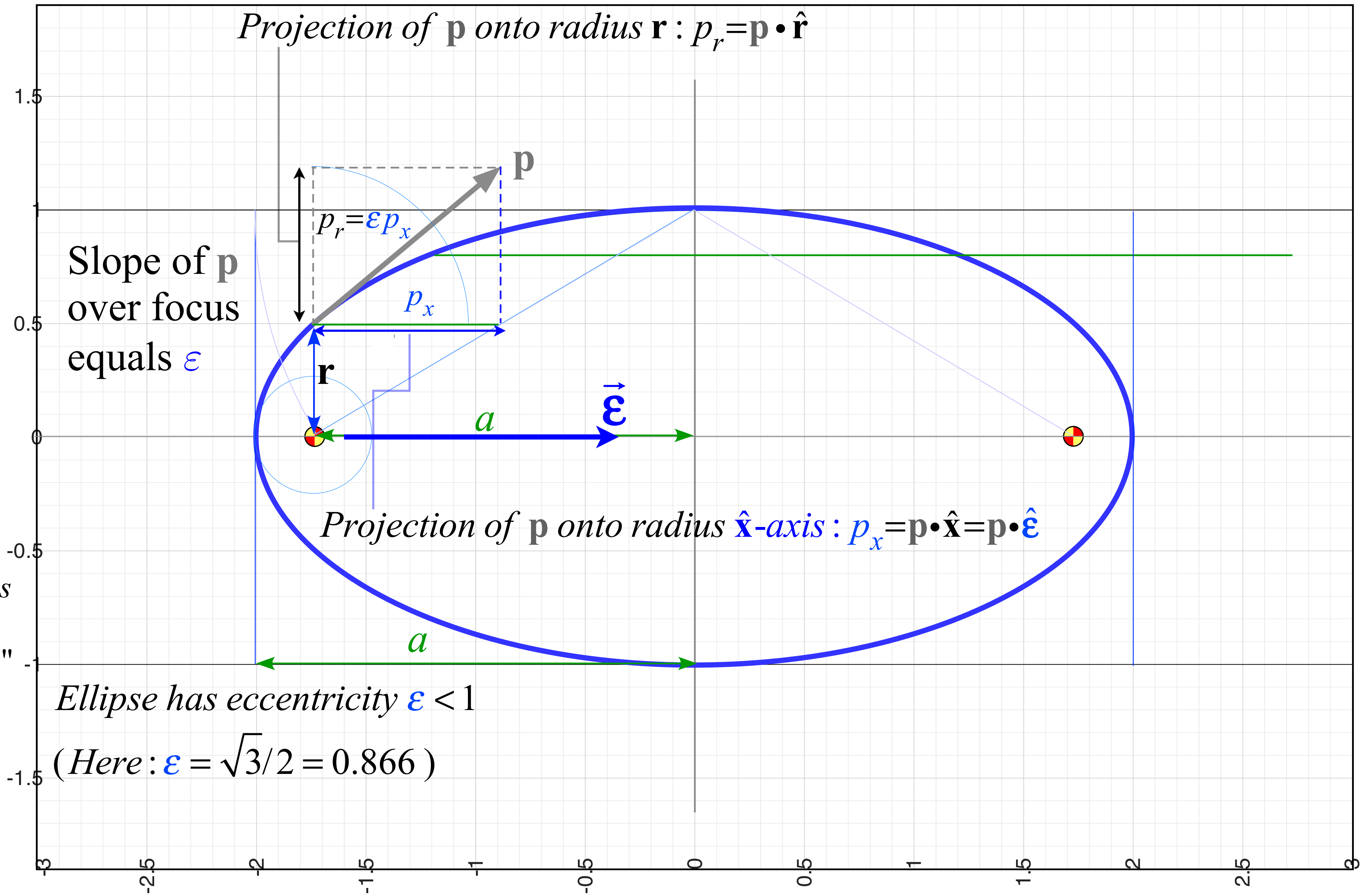

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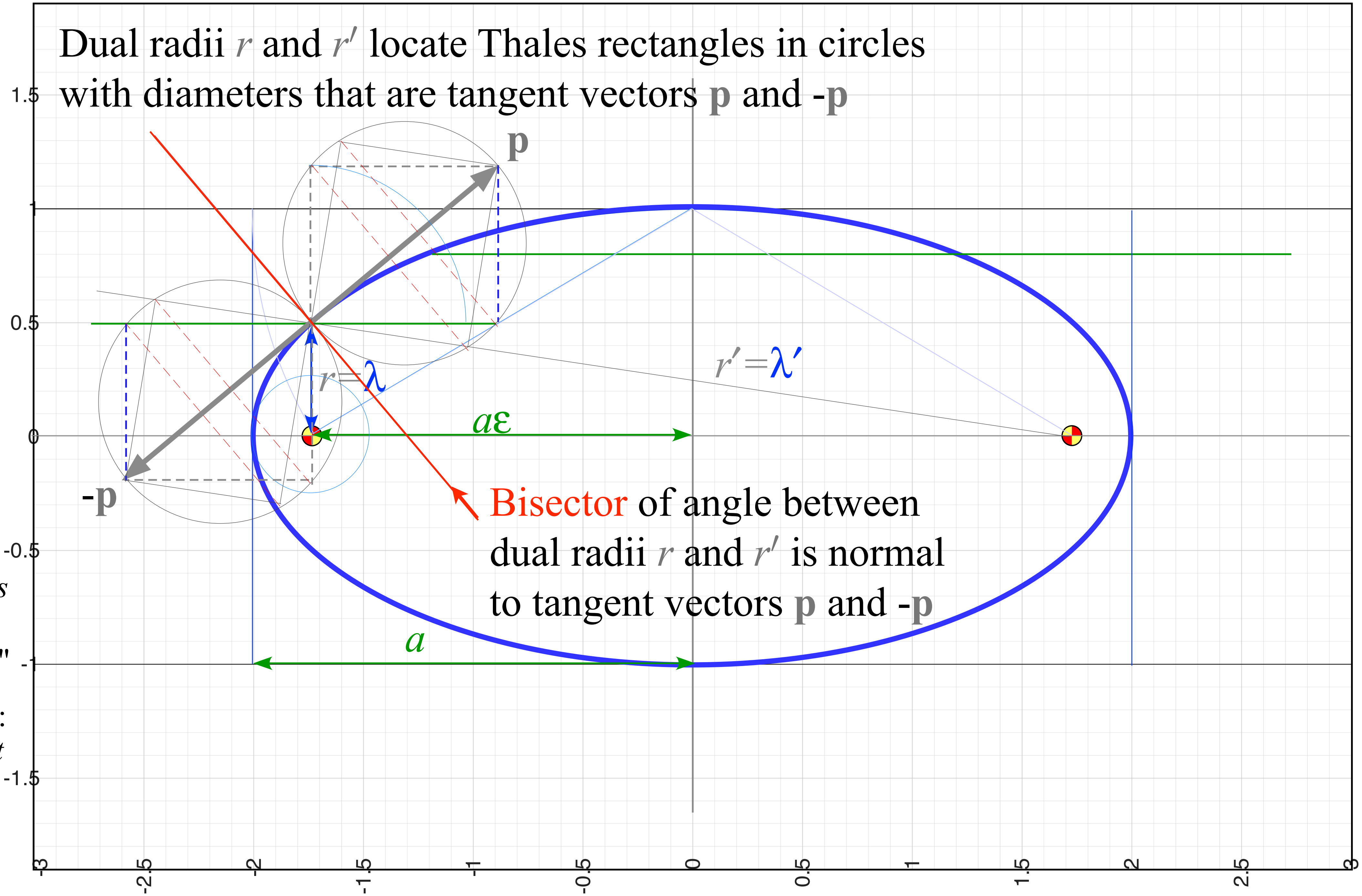
Focal geometry demands:
"*Momentum* $\mathbf{p}$ *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines.*"



Dual radii r and r' locate Thales rectangles in circles with diameters that are tangent vectors $\mathbf{p}$ and -$\mathbf{p}$

Bisector of angle between dual radii r and r' is normal to tangent vectors $\mathbf{p}$ and -$\mathbf{p}$

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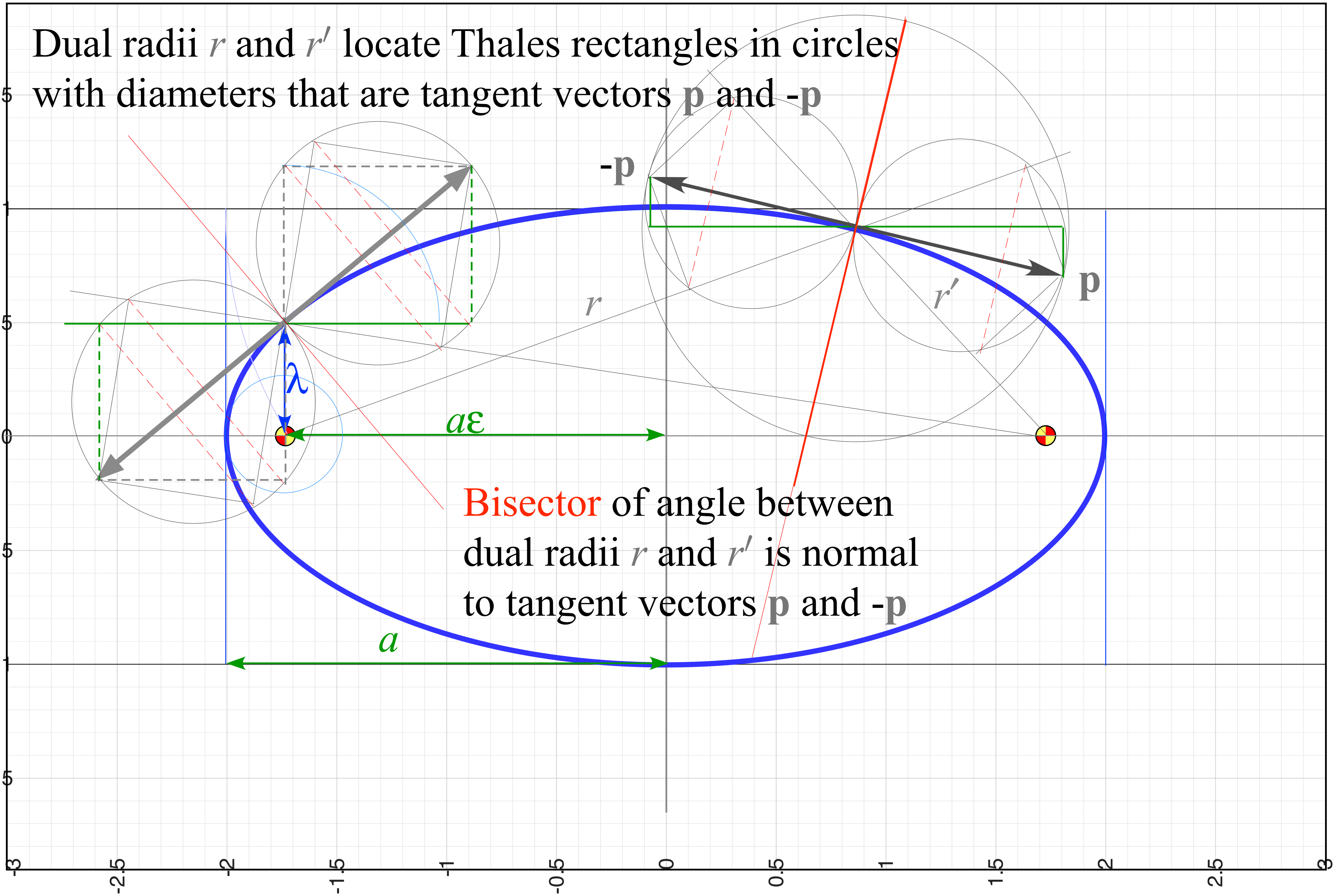

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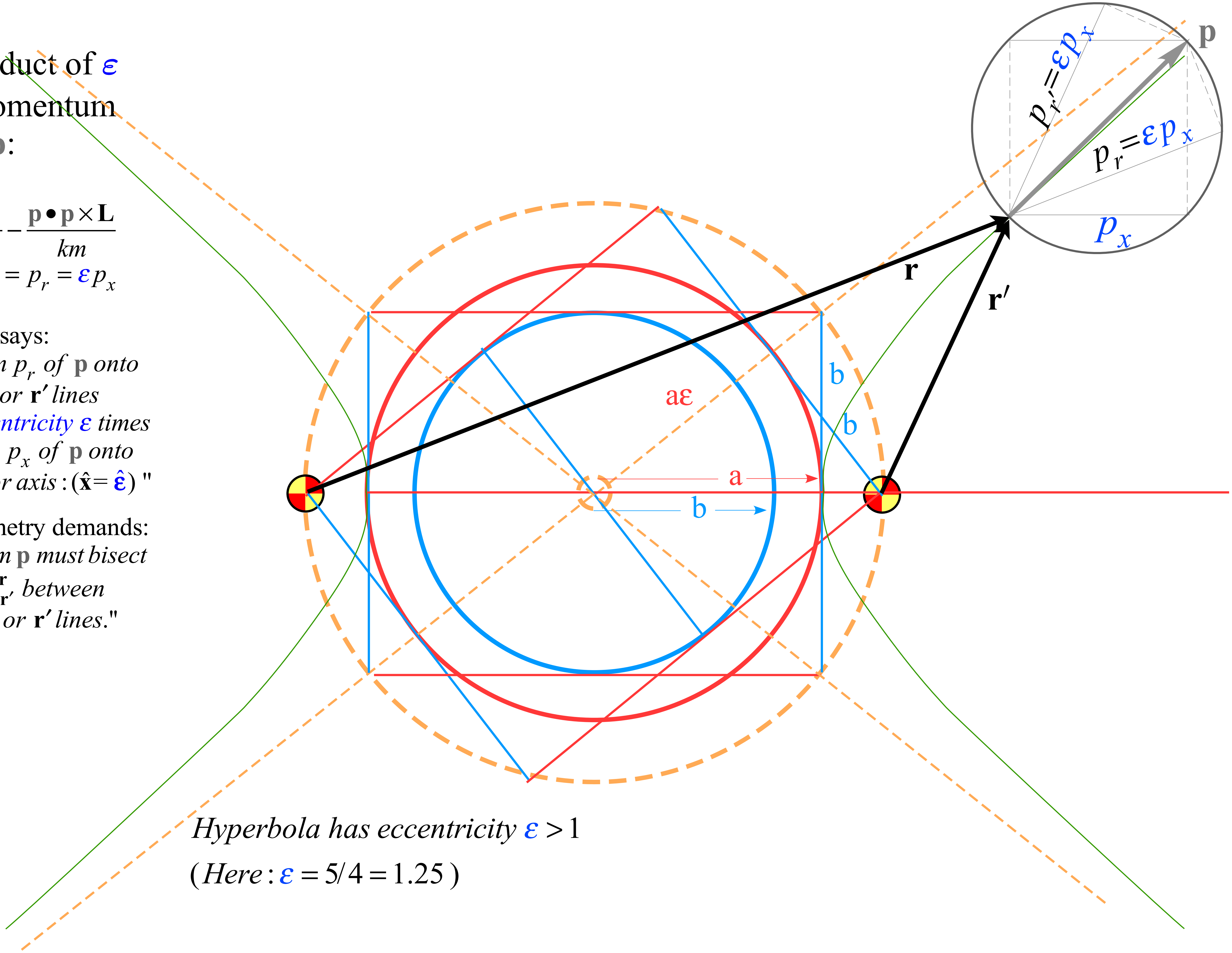
NOTE: Lengths of vectors **p** and -**p** are not drawn to correctly show that momentum **p**=m**v** grows as radial distance r=|**r**| falls. (To be shown on p. 85-90)

Dot product of ε with momentum vector $\mathbf{p}$:



$$\boldsymbol{\varepsilon}\bullet\mathbf{p}=\frac{\mathbf{p}\bullet\mathbf{r}}{r}-\frac{\mathbf{p}\bullet\mathbf{p}\times\mathbf{L}}{km}$$
$$=\mathbf{p}\bullet\hat{\mathbf{r}}=p_r=\varepsilon p_x$$

This says:
"*Projection* p_r *of* $\mathbf{p}$ *onto radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines equals eccentricity* ε *times projection* p_x *of* $\mathbf{p}$ *onto orbit major axis*: $(\hat{\mathbf{x}}=\hat{\boldsymbol{\varepsilon}})$ "

Focal geometry demands:
"*Momentum* $\mathbf{p}$ *must bisect angle* $\measuredangle^{\mathbf{r}}_{\mathbf{r}'}$ *between radial* $\mathbf{r}$ *or* $\mathbf{r}'$ *lines.*"

Hyperbola has eccentricity $\varepsilon>1$

(*Here*: $\varepsilon=5/4=1.25$)

Rutherford scattering and hyperbolic orbit geometry

- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

- *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

➔ *General geometric orbit construction using* $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

- *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

- *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

General geometric orbit construction using ε-vector and (γ,R)-parameters

Next several pages give step-by-step constructions of ε-vector and Coulomb orbit and trajectory physics

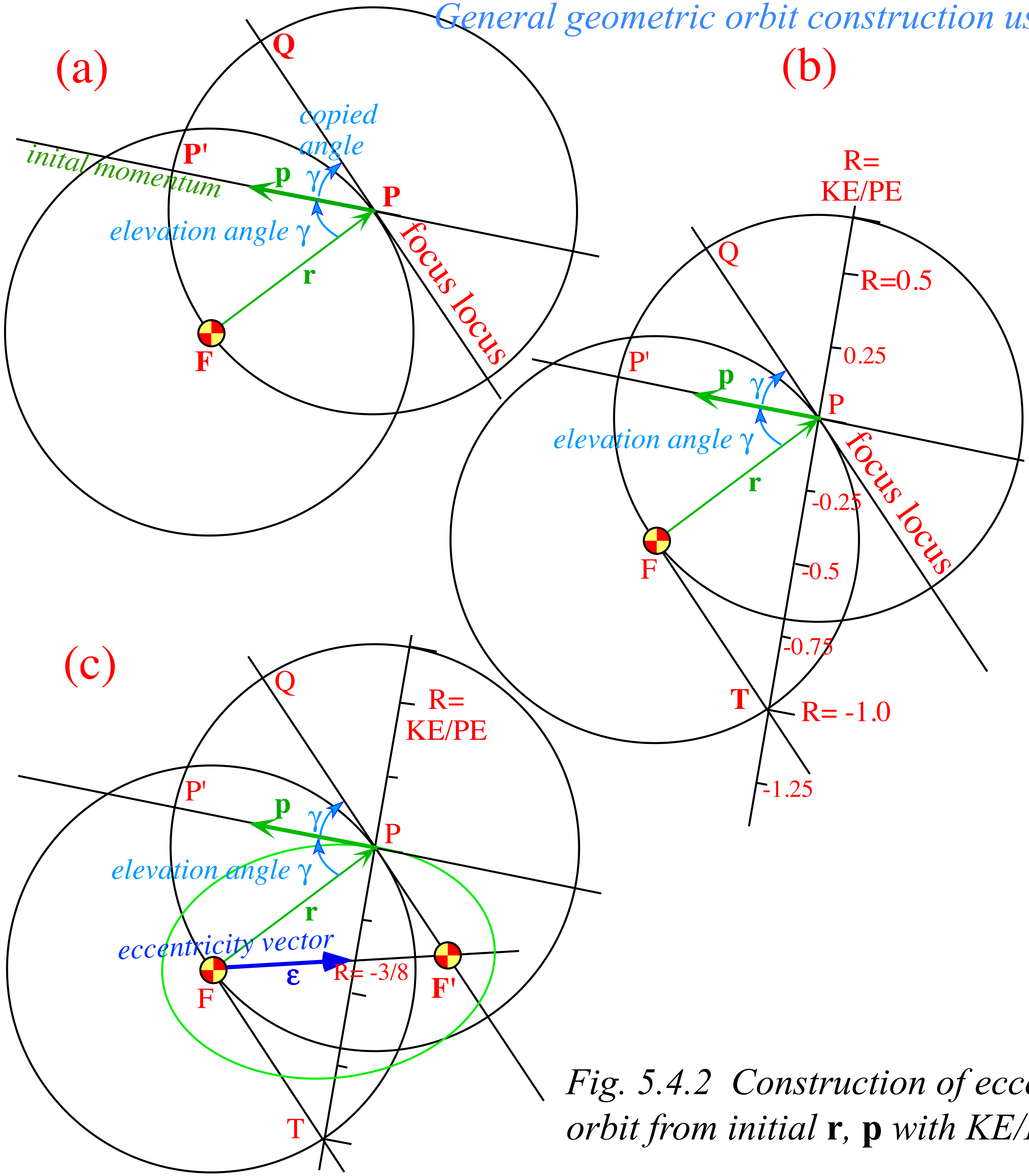


Fig. 5.4.2 Construction of eccentricity vector ε *and orbit from initial* **r**, **p** *with KE/PE=-3/8.*

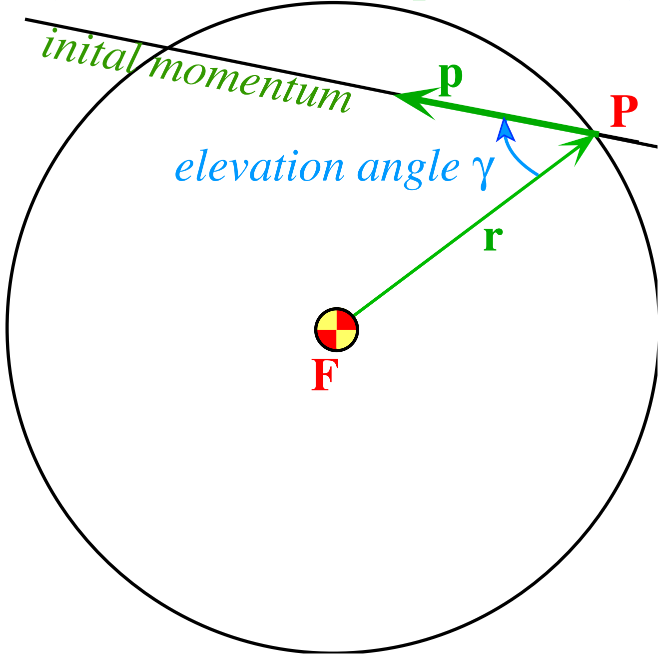
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

(momentum initial **p** *direction)*



General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

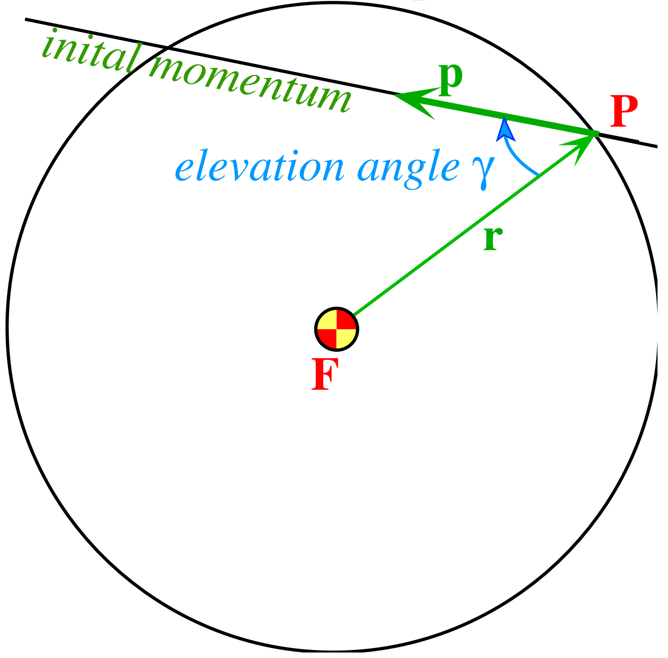
and elevation angle γ from radius

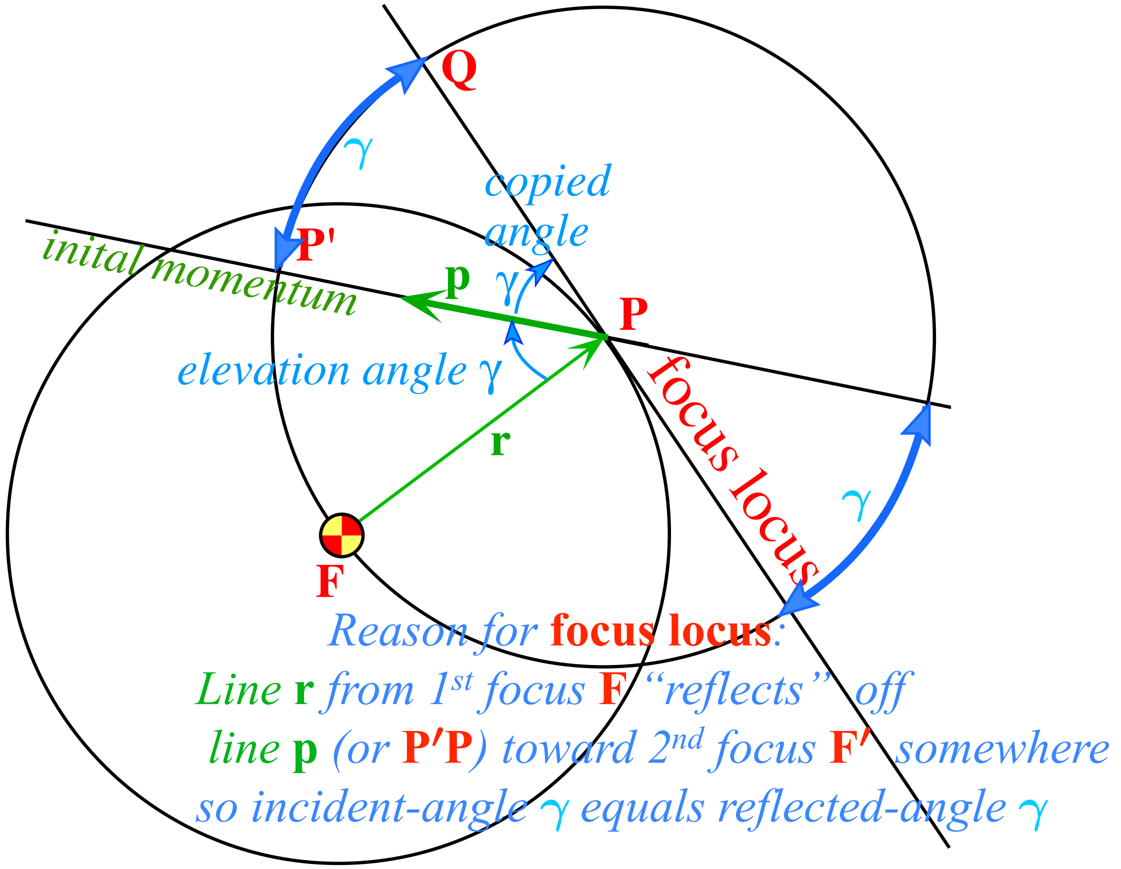
(momentum initial **p** *direction)*

Copy F-center circle around launch point P

Copy elevation angle γ (∠FPP′) onto ∠P′PQ

Extend resulting line QPQ′ *to make* **focus locus**





Reason for **focus locus**:

Line **r** *from* 1st *focus* **F** *“reflects” off*

line **p** *(or* **P′P***) toward* 2nd *focus* **F′** *somewhere*

so incident-angle γ equals reflected-angle γ

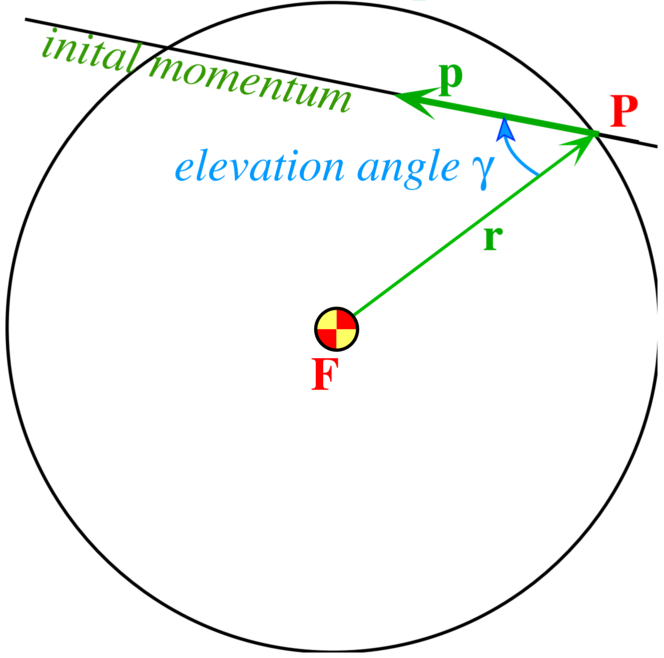
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

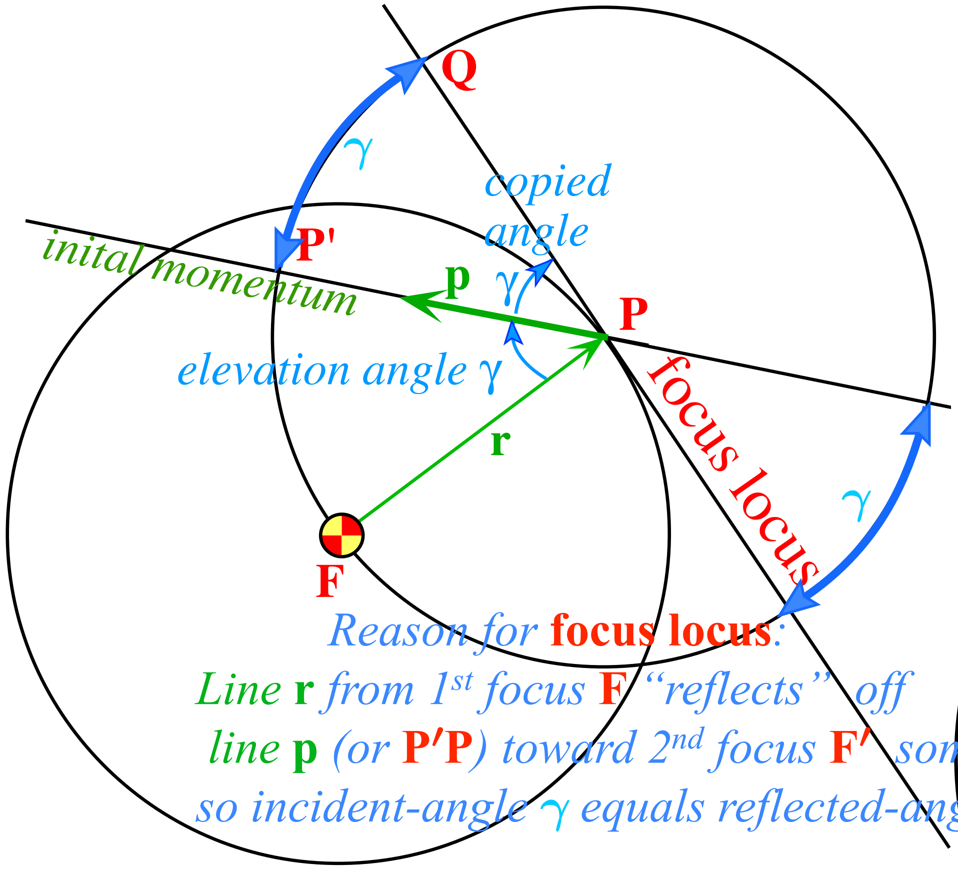
(momentum initial **p** *direction)*



Copy F-center circle around launch point P

Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*

Extend resulting line QPQ′ *to make* **focus locus**



Reason for **focus locus***:*

Line **r** *from 1st focus* **F** *“reflects” off*

line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*

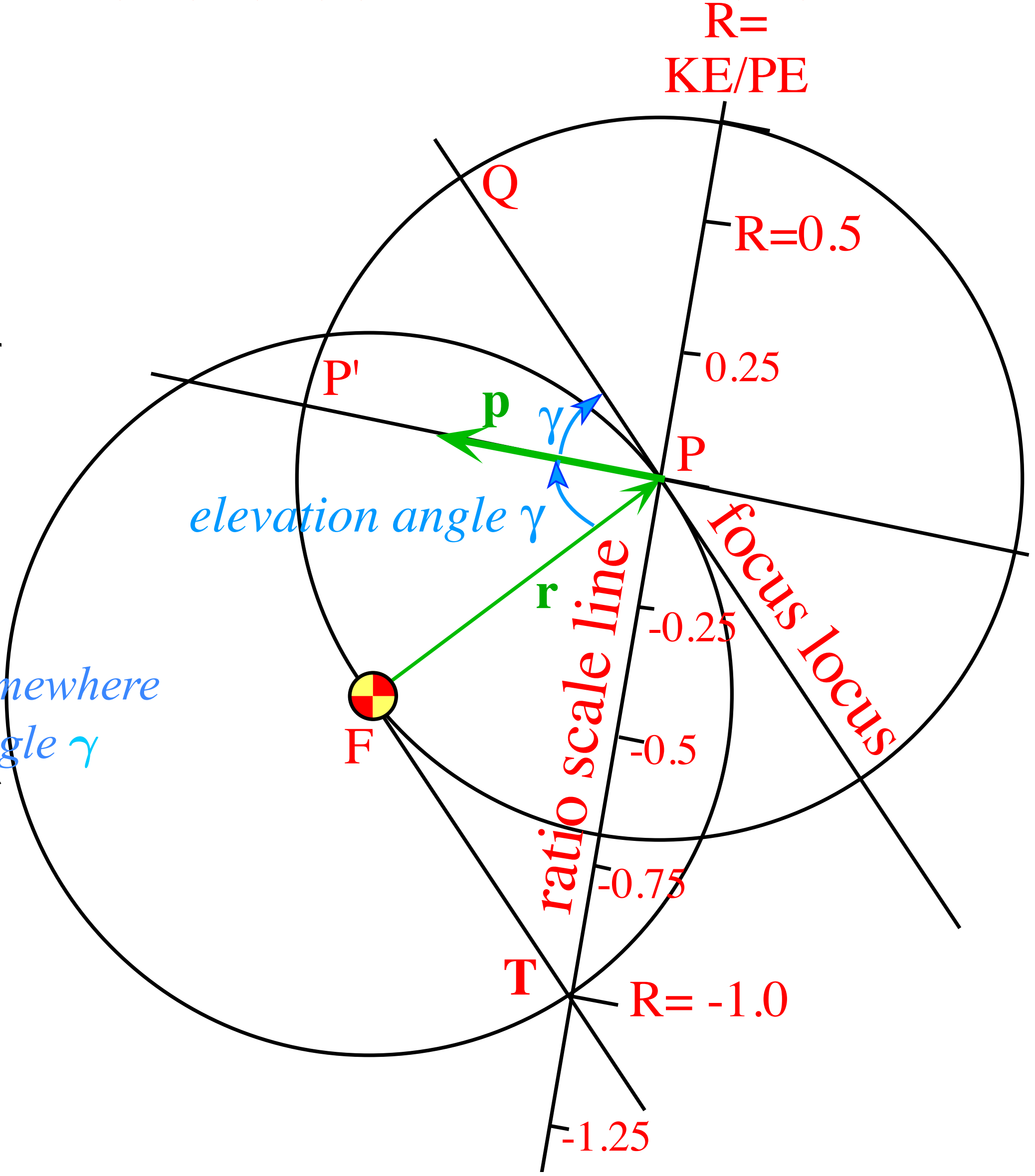
so incident-angle γ *equals reflected-angle* γ

Copy double angle 2γ*(∠FPQ) onto ∠PFT*

Extend ∠PFT chord PT *to make R-***ratio scale line**

Label chord PT *with* R*=0 at* P *and* R*=-1.0 at* T.

*Mark R-***line** *fractions* R*=0, +1/4, +1/2,... above* P *and*
R*=0, -1/8,-1/4,-1/2,...,-3/4 below* P *and -5/4,-3/2,... below* T.



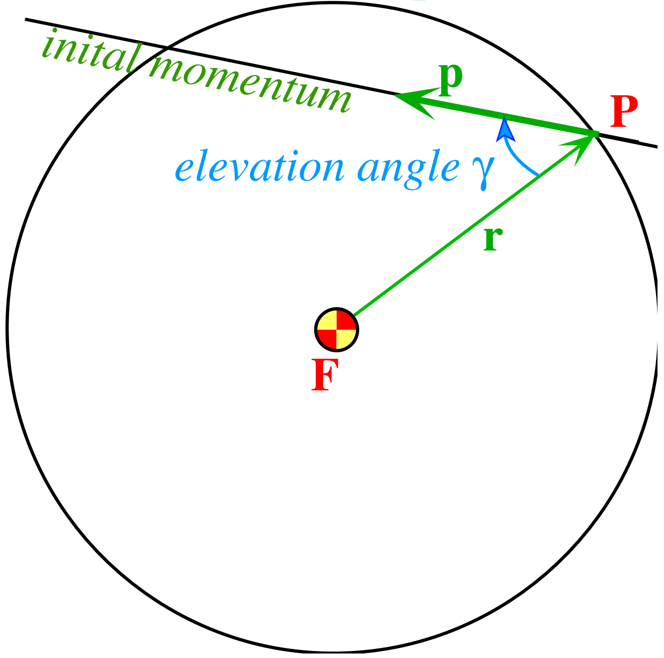
General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ from radius

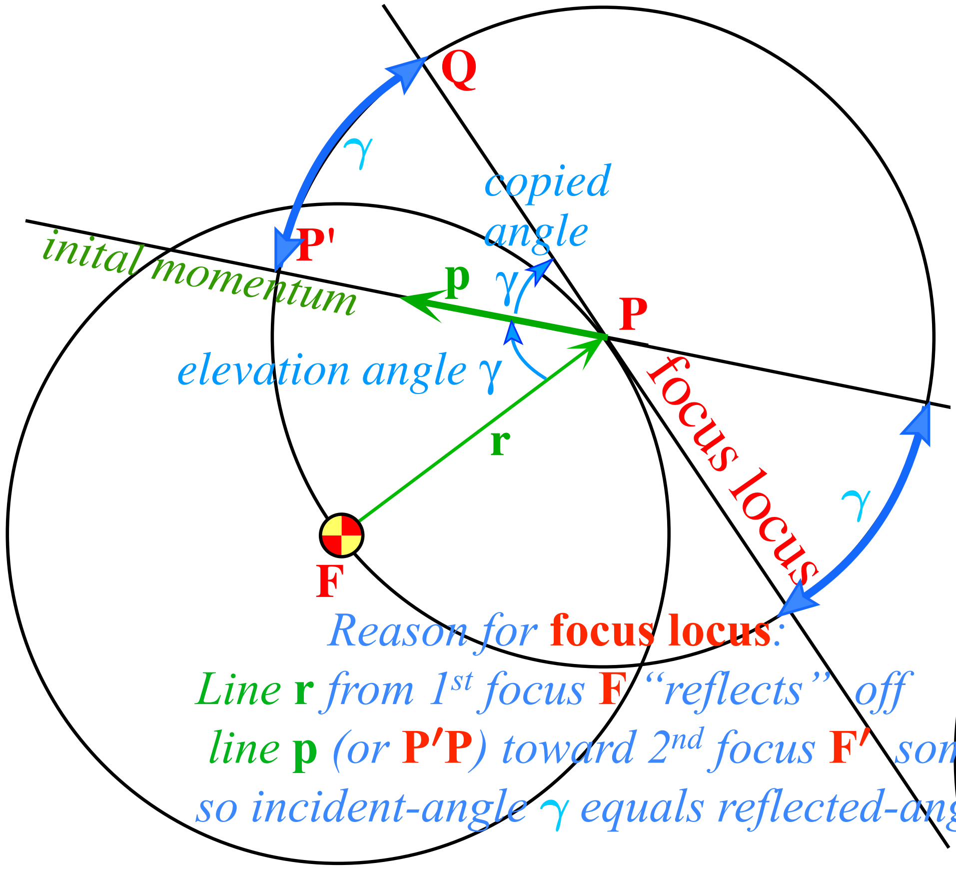
(momentum initial **p** *direction)*



Copy F-center circle around launch point P

Copy elevation angle γ (∠FPP′) onto ∠P′PQ

Extend resulting line QPQ′ *to make* **focus locus**



Reason for **focus locus***:*

Line **r** *from 1st focus* **F** *"reflects" off*

line **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere*

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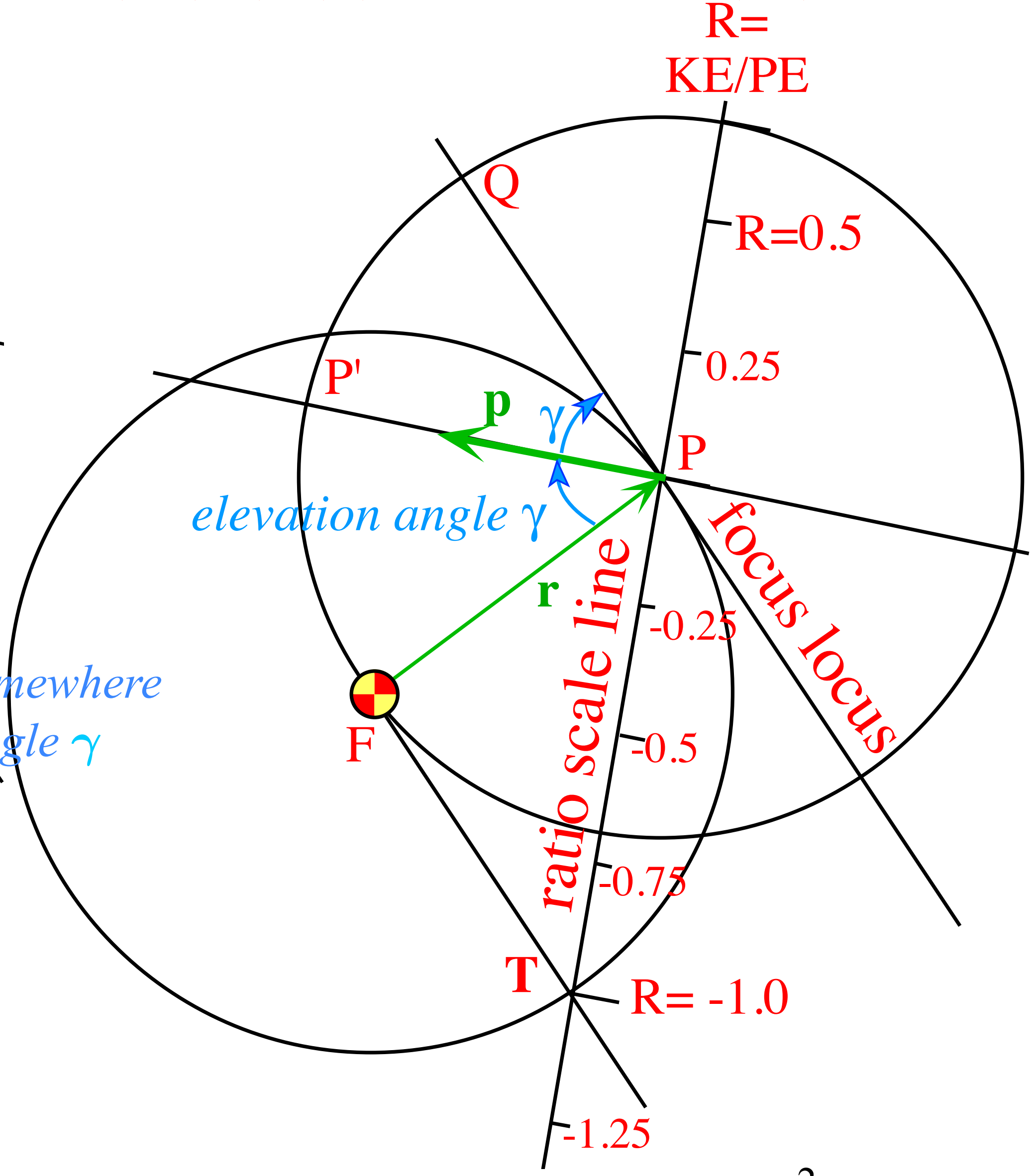
Label chord PT with R=0 at P and R=-1.0 at T.

*Mark R-***line** *fractions R=0, +1/4, +1/2,... above P and R=0, -1/8,-1/4,-1/2,...,-3/4 below P and -5/4,-3/2,... below T.*



$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

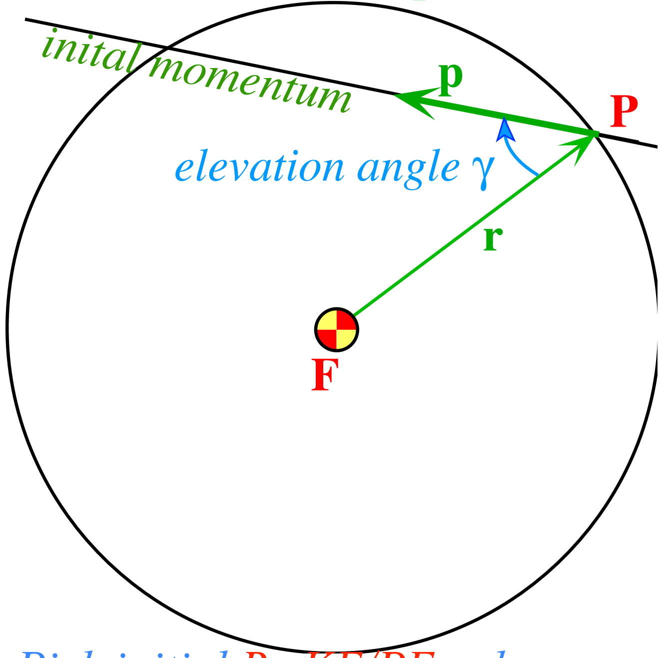
General geometric orbit construction using ***ε*****-vector** *and (γ,R)-parameters*

Pick launch point P

(radius vector **r** *)*

and elevation angle γ from radius

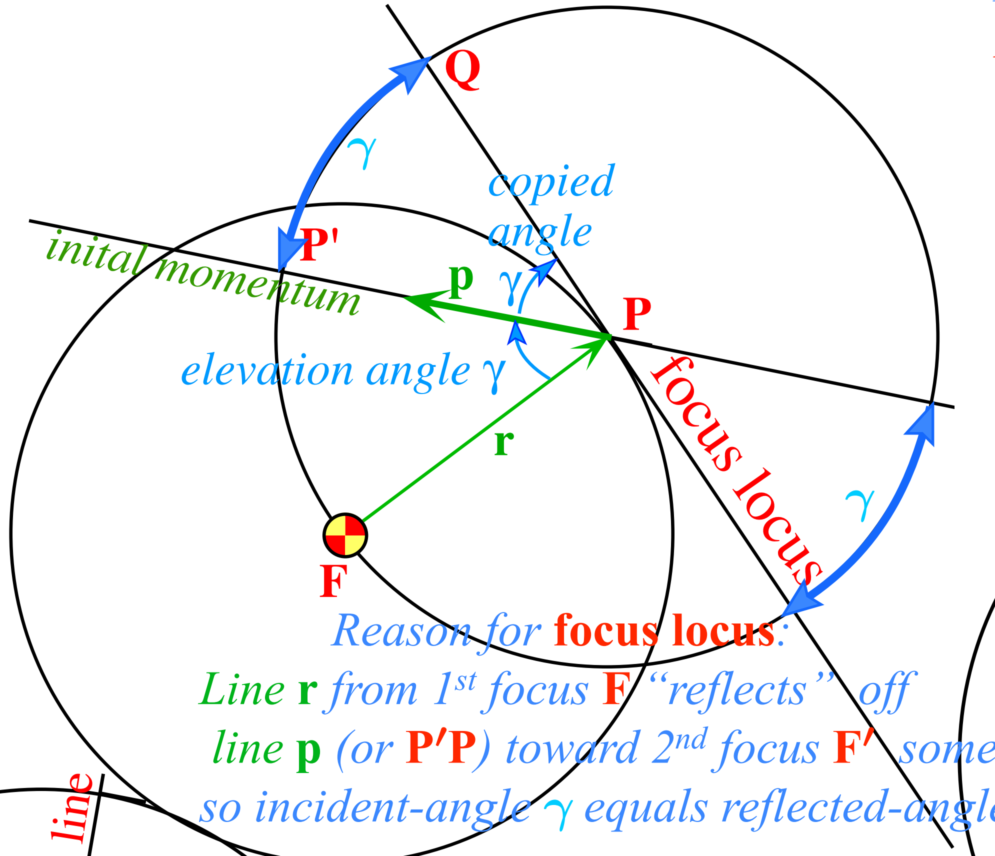
(momentum initial **p** *direction)*



Copy F-center circle around launch point P

Copy elevation angle γ (∠FPP′) onto ∠P′PQ

Extend resulting line QPQ′ to make **focus locus**



Reason for **focus locus***:*

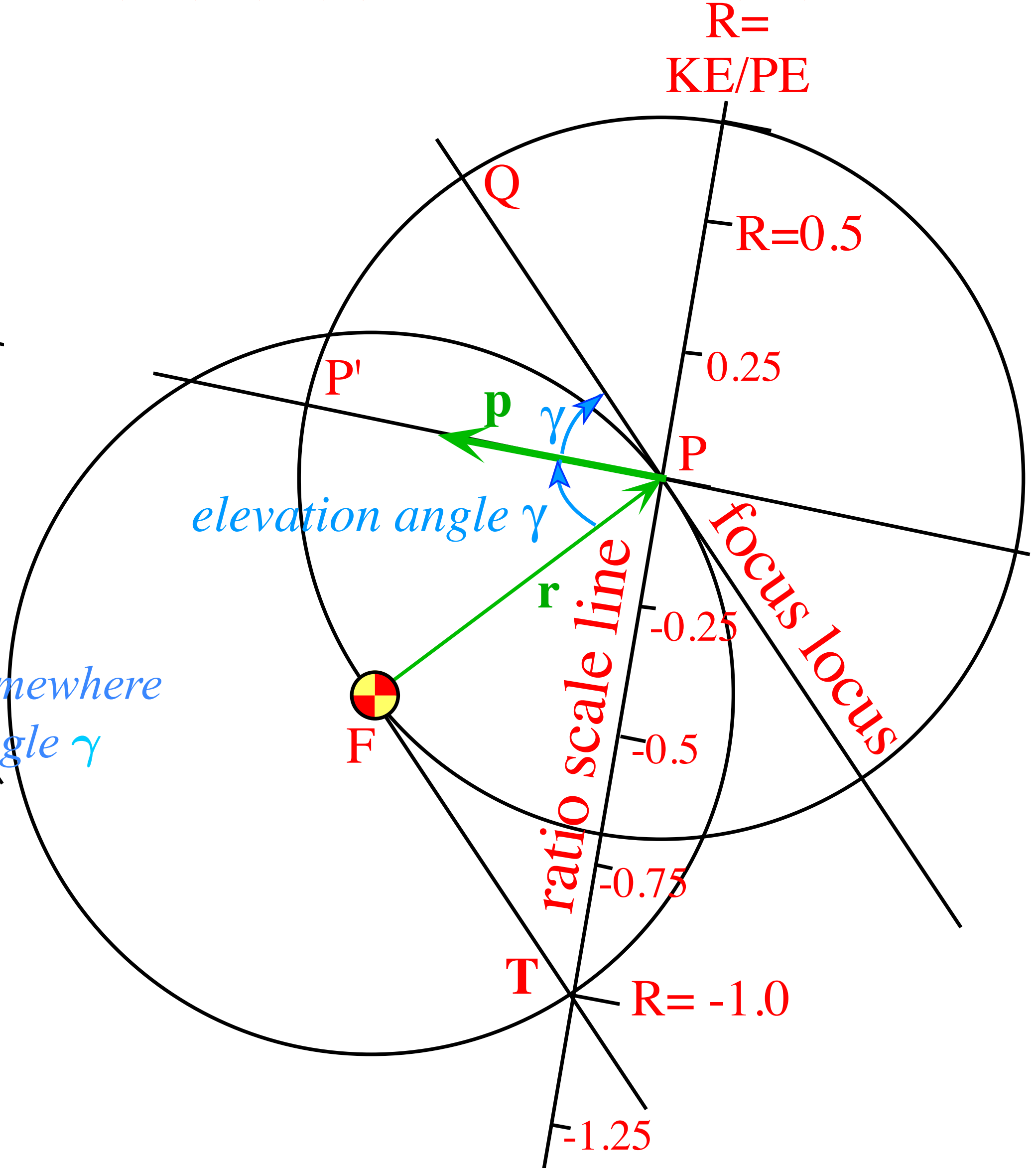
Line **r** *from 1st focus* **F** *"reflects" off line* **p** *(or* **P′P***) toward 2nd focus* **F′** *somewhere so incident-angle γ equals reflected-angle γ*

Copy double angle 2γ (∠FPQ) onto ∠PFT

*Extend ∠PFT chord PT to make R-***ratio scale line**

Label chord PT with R=0 at P and R=-1.0 at T.

*Mark R-***line** *fractions R=0, +1/4, +1/2,... above P and R=0, -1/8,-1/4,-1/2, ...,-3/4 below P and -5/4,-3/2,... below T.*

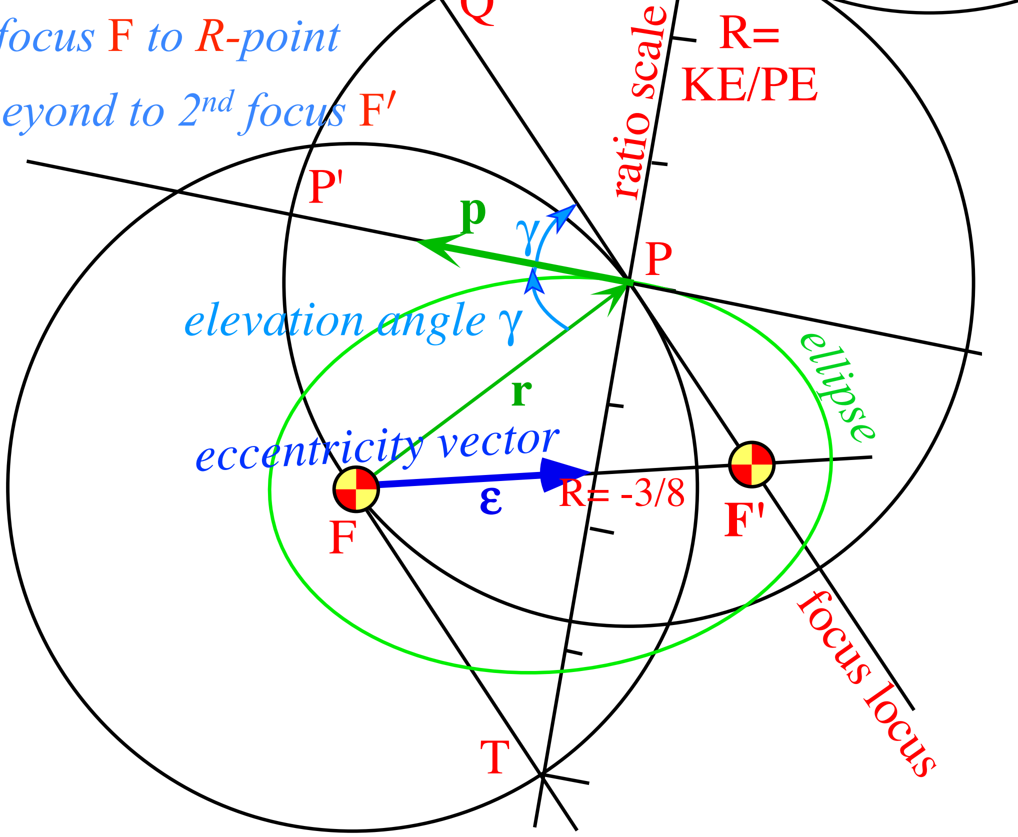


Pick initial R=KE/PE value

(here R=-3/8) Draw ***ε−vector***

from focus F to R-point

and beyond to 2nd focus F′



$$R=\frac{Initial\ KE}{Initial\ PE}=\frac{mv^2(0)/2}{-k/r(0)}$$

$$=\pm\left(\frac{Initial\ velocity}{Escape\ velocity}\right)^2=\pm\frac{v^2(0)}{v^2(\infty)}$$

focus F and 2nd focus F′ allow final construction of orbital trajectory. Here it is an R=-3/8 ellipse.

(Detailed Analytic geometry of ***ε*****-vector** *follows.)*

General geometric orbit construction using ε-vector and (γ,R)-parameters

Pick launch point P

(radius vector **r** *)*

and elevation angle γ *from radius*

(momentum initial **p** *direction)*

Copy F-center circle around launch point P

Copy elevation angle γ *(∠FPP′) onto ∠P′PQ*

Extend resulting line QPQ′ *to make* **focus locus**

*Copy double angle 2*γ *(∠FPQ) onto ∠PFT*

Extend ∠PFT chord PT *to make R-***ratio scale line**

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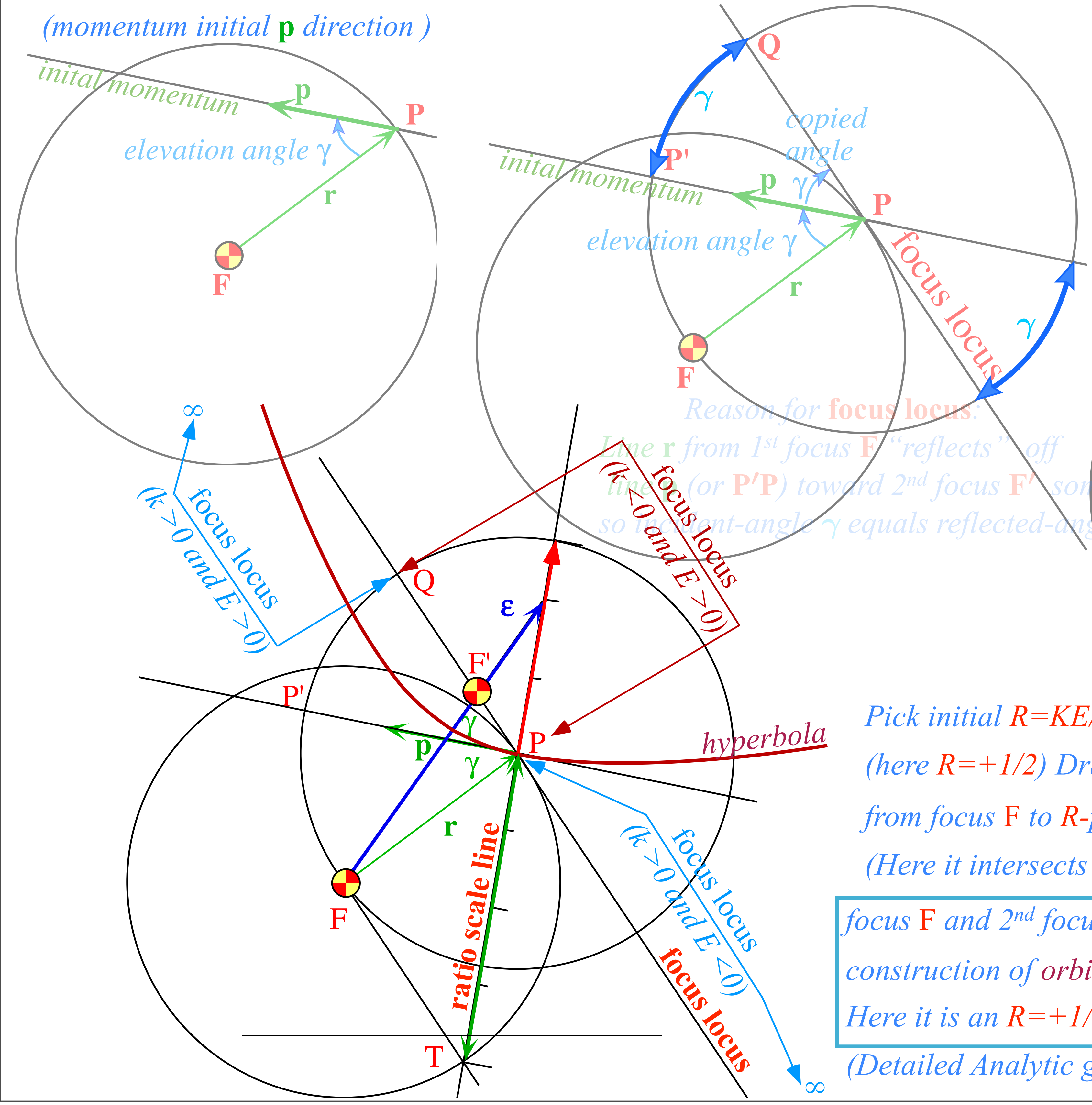
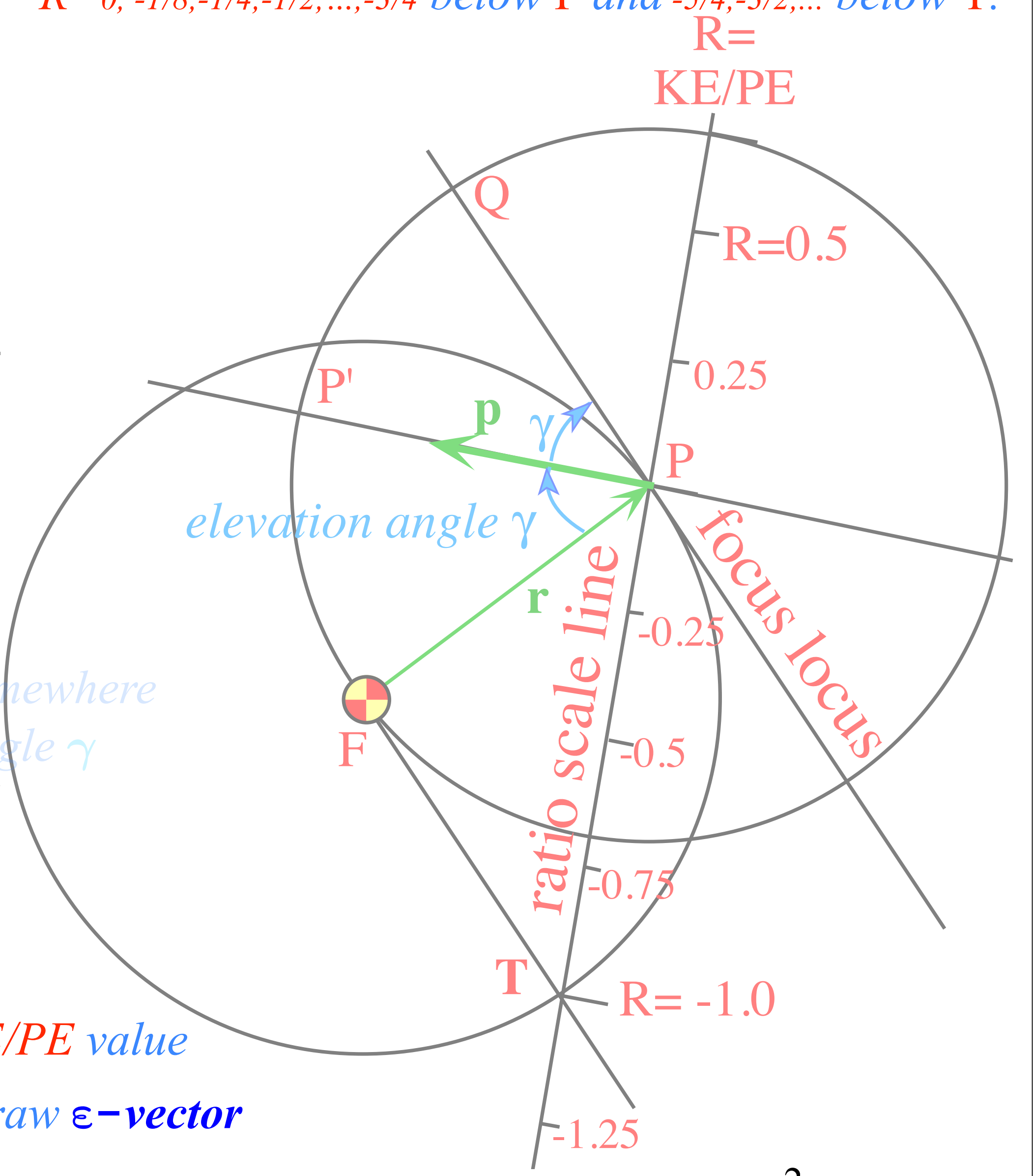
*Mark R-***line** *fractions R=0, +1/4, +1/2,... above* P *and R=0, -1/8,-1/4,-1/2,...,-3/4 below* P *and -5/4,-3/2,... below* T.



Reason for **focus locus**:
Line **r** *from 1st focus* F *"reflects" off line* **p** *(or* P′P*) toward 2nd focus* F′ *somewhere so incident-angle* γ *equals reflected-angle* γ



Pick initial R=KE/PE value

(here R=+1/2) Draw **ε−*vector***

from focus F *to R-point*

(Here it intersects 2nd focus F′

focus F *and 2nd focus* F′ *allow final construction of orbital trajectory. Here it is an R=+1/2 hyperbola.*

(Detailed Analytic geometry of ε-vector follows.)

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Rutherford scattering and hyperbolic orbit geometry

- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

- *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

- ➔ *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

- *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

Derivation of ε-construction by analytic geometry

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

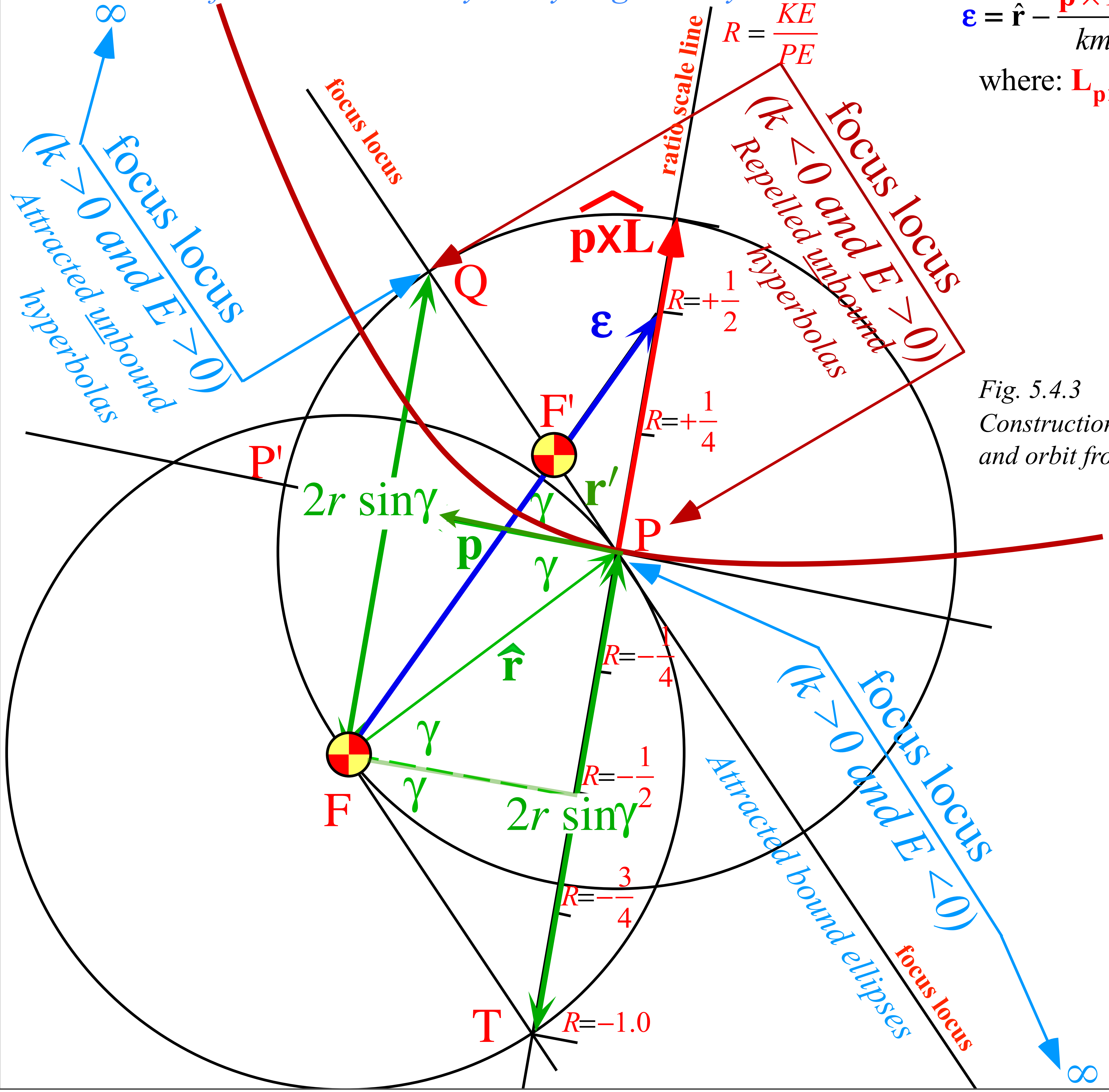
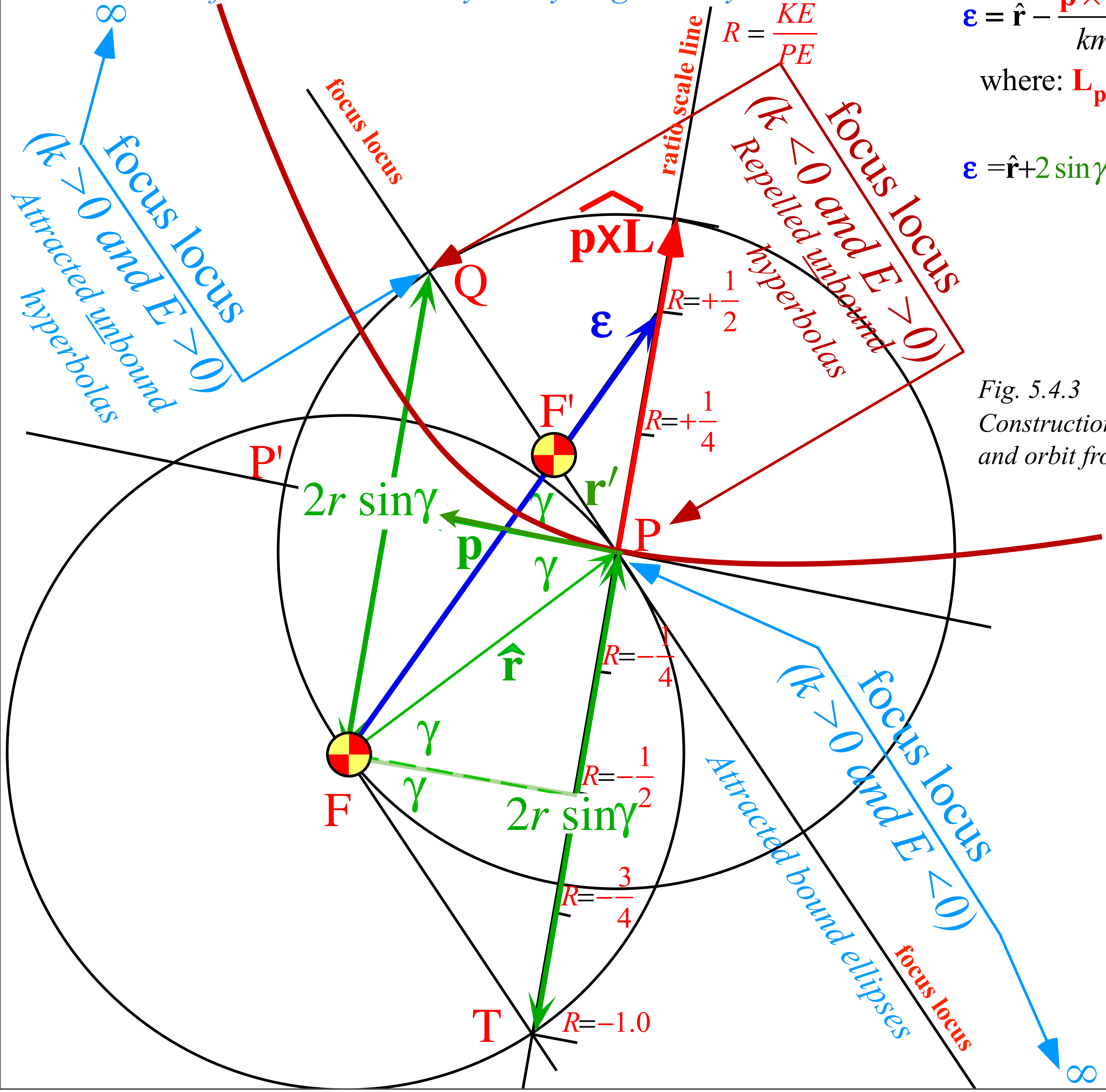


Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of ε-construction by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

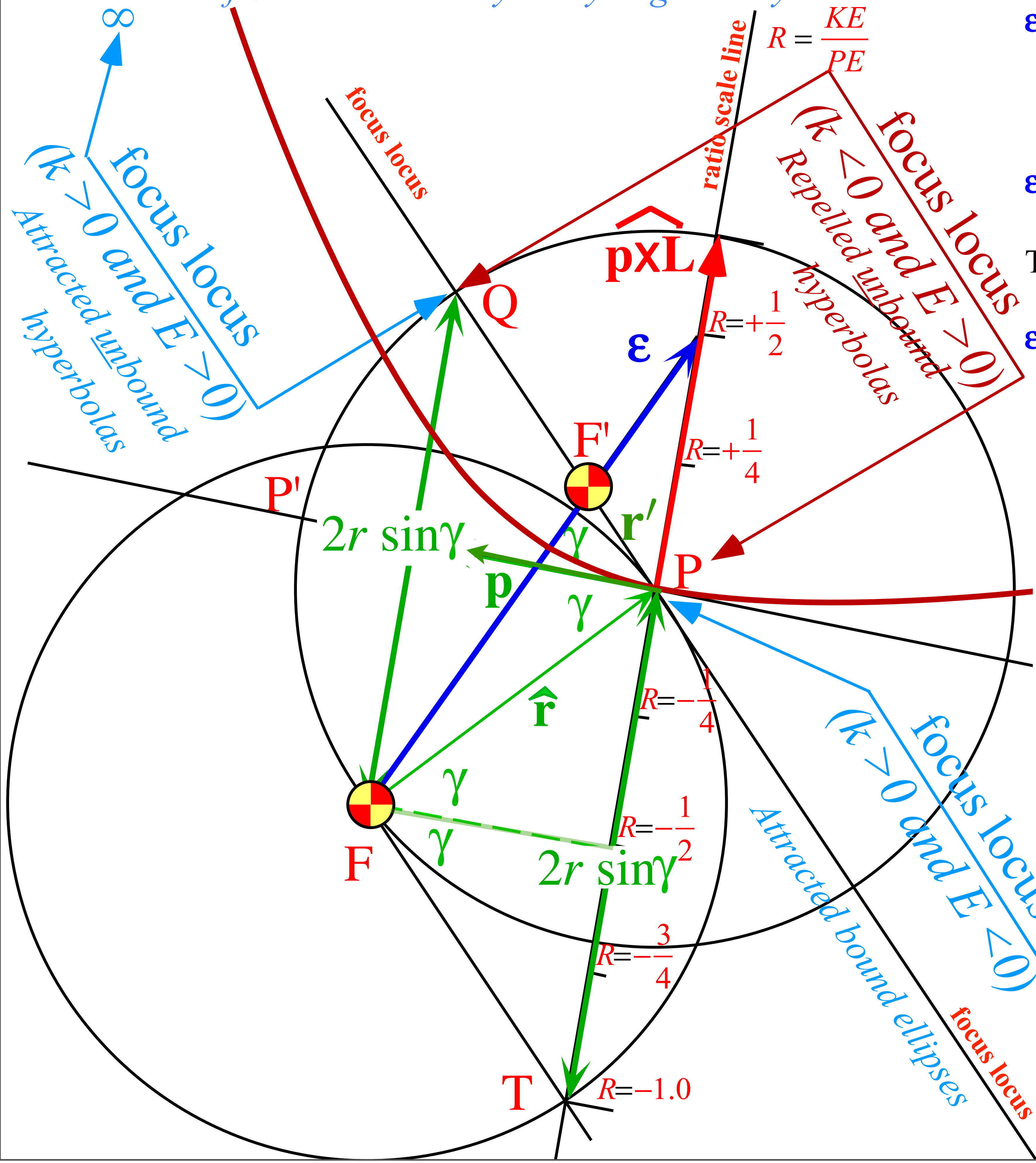
$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0{}^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

Fig. 5.4.3
Construction of eccentricity vector ε
and orbit from initial **r**, **p** *with KE/PE=+1/2.*

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of ε-construction by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{mv_0{}^2/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\,\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

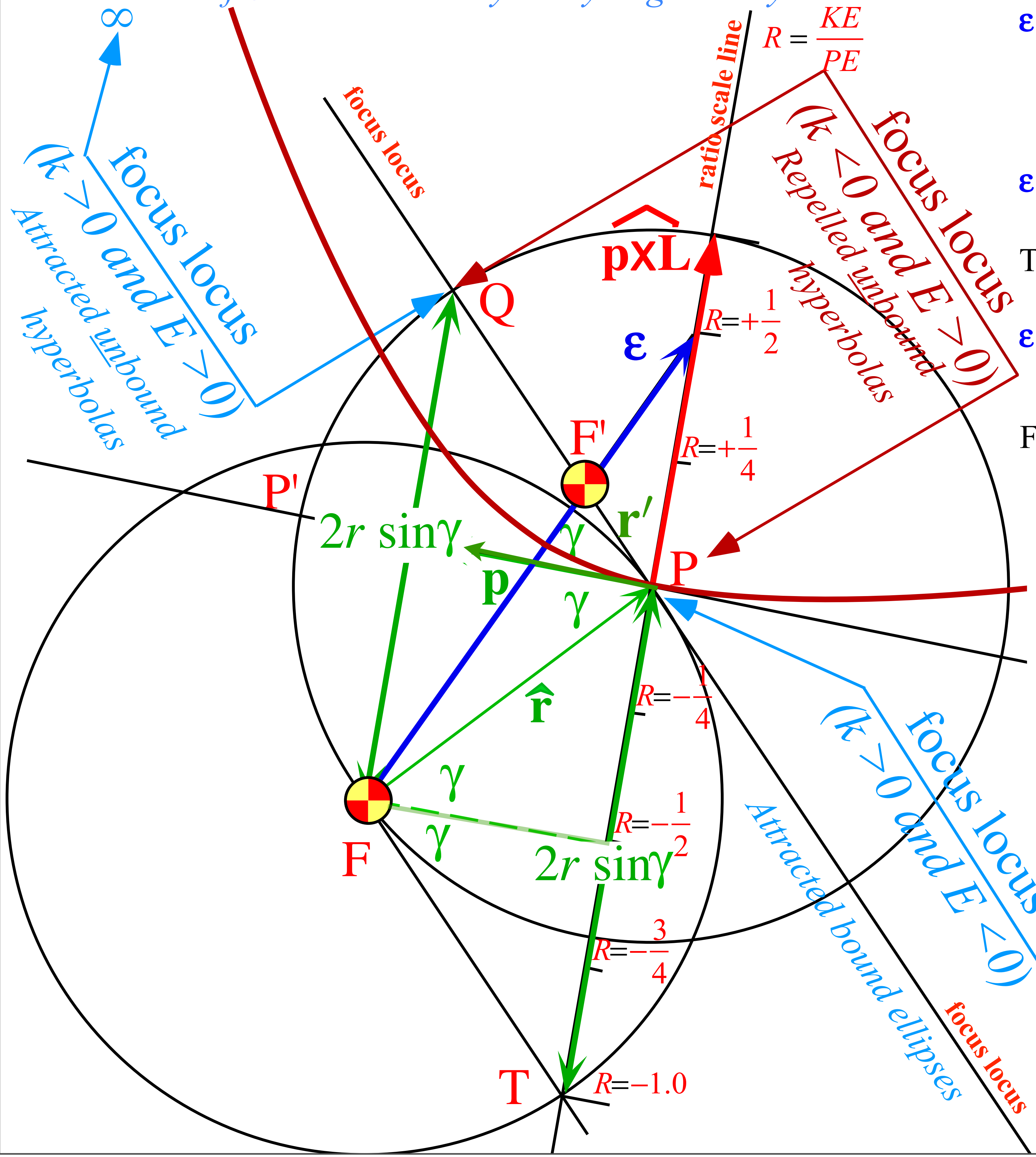
The *eccentricty* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\ 1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

$$R = \frac{Initial\ KE}{Initial\ PE} = \frac{m\,v^2(0)/2}{-k/r(0)}$$

$$= \pm\left(\frac{Initial\ velocity}{Escape\ velocity}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of ε-construction by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\frac{mv_0^{\ 2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricty* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

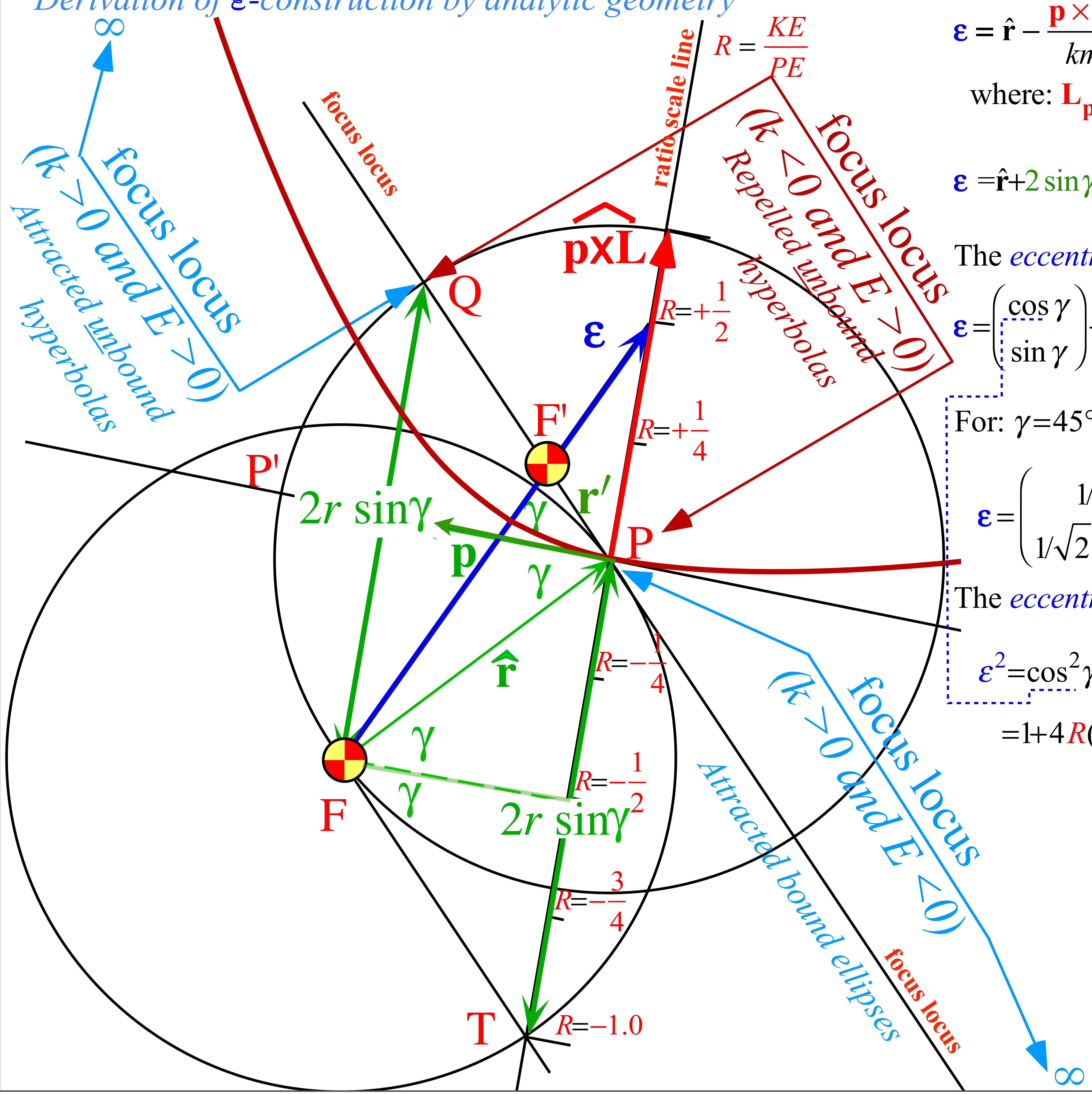
For: $\gamma=45°$ and: $R=+\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

$$R = \frac{Initial\ KE}{Initial\ PE} = \frac{m\,v^2(0)/2}{-k\,/\,r(0)}$$

$$= \pm\left(\frac{Initial\ velocity}{Escape\ velocity}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$

Derivation of ε-construction by analytic geometry



$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} - \frac{\mathbf{p}\times\mathbf{L}}{km} = \hat{\mathbf{r}} - \frac{(mv_0)(mv_0 r_0)\sin\gamma}{km}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

where: $\mathbf{L}_{\mathbf{p}\times} \equiv \mathbf{p}\times\mathbf{L}$

$$\boldsymbol{\varepsilon} = \hat{\mathbf{r}} + 2\sin\gamma\frac{mv_0^{\ 2}/2}{-k/r_0}\hat{\mathbf{L}}_{\mathbf{p}\times} = \hat{\mathbf{r}} + 2\sin\gamma\frac{KE}{PE}\hat{\mathbf{L}}_{\mathbf{p}\times}$$

The *eccentricity* vector is:

$$\boldsymbol{\varepsilon} = \begin{pmatrix}\cos\gamma\\ \sin\gamma\end{pmatrix} + 2\sin\gamma\begin{pmatrix}0\\1\end{pmatrix}R = \begin{pmatrix}\cos\gamma\\ (2R+1)\sin\gamma\end{pmatrix}$$

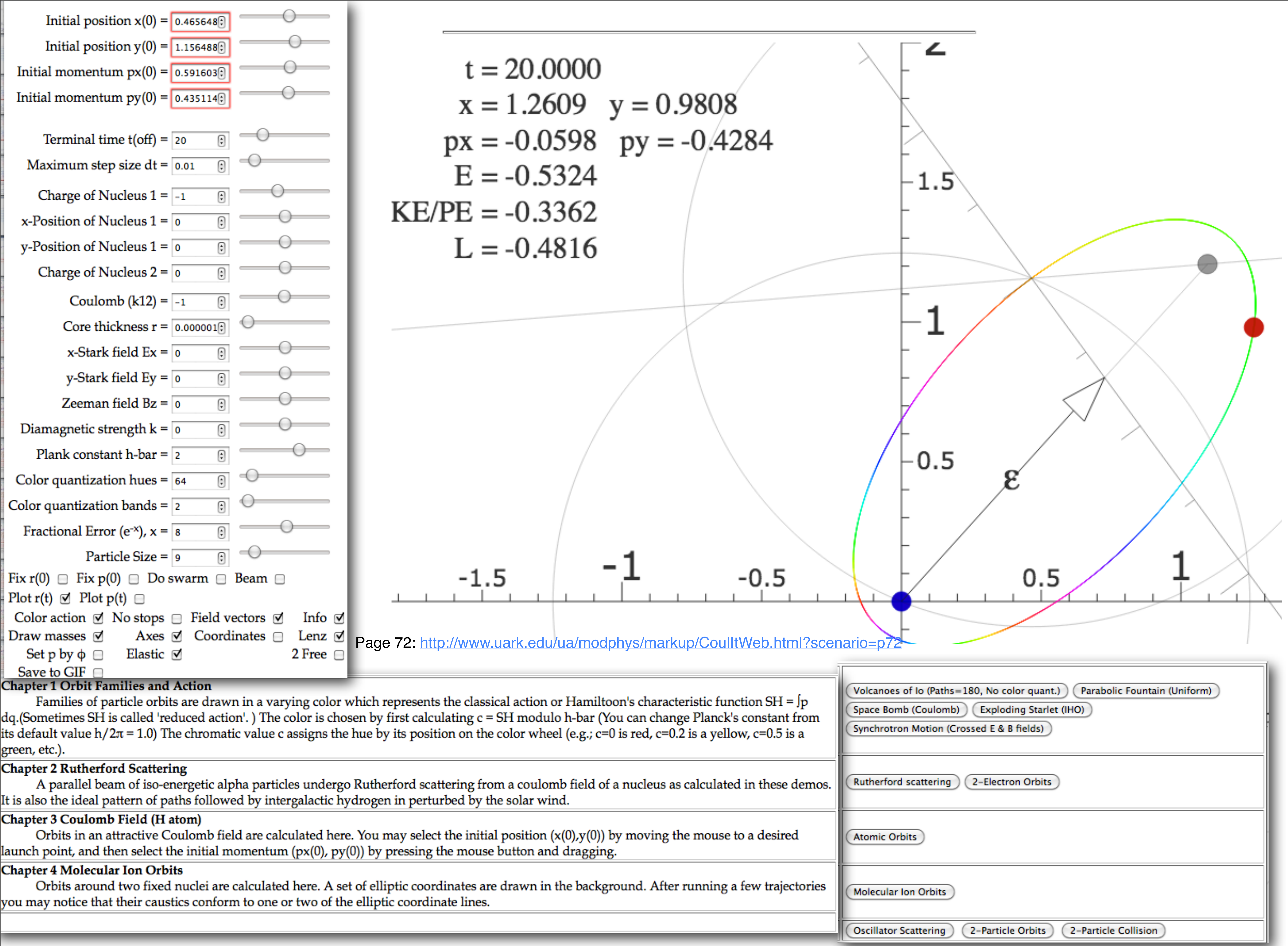
For: $\gamma=45°$ and: $R=+\frac{1}{2}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}(2R+1)\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 2/\sqrt{2}\end{pmatrix},$$

The *eccentricity* parameter defined by:

$$\varepsilon^2 = \cos^2\gamma + (2R+1)^2\sin^2\gamma = 1 \pm \frac{a^2}{b^2}$$
$$= 1 + 4R(R+1)\sin^2\gamma = \frac{5}{2}$$

$$R = \frac{\textit{Initial KE}}{\textit{Initial PE}} = \frac{m\,v^2(0)/2}{-k/r(0)}$$
$$= \pm\left(\frac{\textit{Initial velocity}}{\textit{Escape velocity}}\right)^2 = \pm\frac{v^2(0)}{v^2(\infty)}$$



Initial position x(0) = 0.465648
Initial position y(0) = 1.156488
Initial momentum px(0) = 0.591603
Initial momentum py(0) = 0.435114

Terminal time t(off) = 20
Maximum step size dt = 0.01
Charge of Nucleus 1 = -1
x-Position of Nucleus 1 = 0
y-Position of Nucleus 1 = 0
Charge of Nucleus 2 = 0
Coulomb (k12) = -1
Core thickness r = 0.000001
x-Stark field Ex = 0
y-Stark field Ey = 0
Zeeman field Bz = 0
Diamagnetic strength k = 0
Plank constant h-bar = 2
Color quantization hues = 64
Color quantization bands = 2
Fractional Error (e^{-x}), x = 8
Particle Size = 9

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
Color action ☑ No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☑
Set p by ϕ ☐ Elastic ☑ 2 Free ☐
Save to GIF ☐

```
        t = 20.0000
      x = 1.2609   y = 0.9808
   px = -0.0598   py = -0.4284
        E = -0.5324
KE/PE = -0.3362
        L = -0.4816
```

Page 72: http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=p72

Chapter 1 Orbit Families and Action

Families of particle orbits are drawn in a varying color which represents the classical action or Hamiltoon's characteristic function SH = ∫p dq.(Sometimes SH is called 'reduced action'.) The color is chosen by first calculating c = SH modulo h-bar (You can change Planck's constant from its default value h/2π = 1.0) The chromatic value c assigns the hue by its position on the color wheel (e.g.; c=0 is red, c=0.2 is a yellow, c=0.5 is a green, etc.).

Chapter 2 Rutherford Scattering

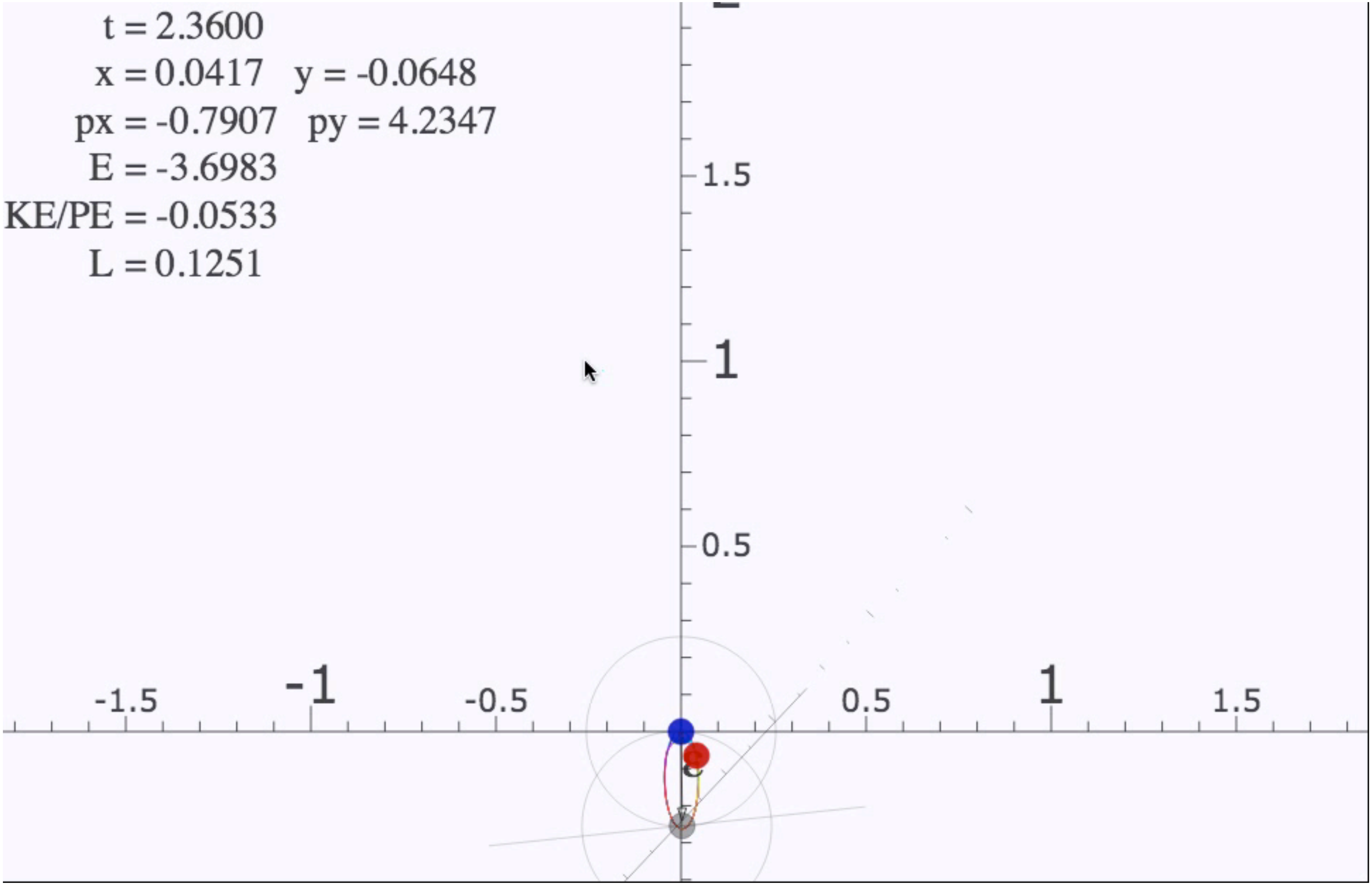
A parallel beam of iso-energetic alpha particles undergo Rutherford scattering from a coulomb field of a nucleus as calculated in these demos. It is also the ideal pattern of paths followed by intergalactic hydrogen in perturbed by the solar wind.

Chapter 3 Coulomb Field (H atom)

Orbits in an attractive Coulomb field are calculated here. You may select the initial position (x(0),y(0)) by moving the mouse to a desired launch point, and then select the initial momentum (px(0), py(0)) by pressing the mouse button and dragging.

Chapter 4 Molecular Ion Orbits

Orbits around two fixed nuclei are calculated here. A set of elliptic coordinates are drawn in the background. After running a few trajectories you may notice that their caustics conform to one or two of the elliptic coordinate lines.



```
        t = 2.3600
        x = 0.0417    y = -0.0648
       px = -0.7907   py = 4.2347
        E = -3.6983
    KE/PE = -0.0533
        L = 0.1251
```

Play this movie of ε-construction by CoulItWeb

Rutherford scattering and hyperbolic orbit geometry
- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*

Eccentricity vector ε *and (ε,λ)-geometry of orbital mechanics*

Projection ε•**r** *geometry of* ε*-vector and orbital radius* **r**
- *Review and connection to usual orbital algebra (previous lecture)*

Projection ε•**p** *geometry of* ε*-vector and momentum* **p**=*m***v**

General geometric orbit construction using ε*-vector and (γ,R)-parameters*
- *Derivation of* ε*-construction by analytic geometry*

➡ *Coulomb orbit algebra of* ε*-vector and Kepler dynamics of momentum* **p**=*m***v**
- *Example of complete (***r***,***p***)-geometry of elliptical orbit*

Connection formulas for (γ,R)-parameters with (a,b) and (ε,λ)

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=*m***v**

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

Radius r:

$$r=\frac{\lambda}{1-\varepsilon\cos\phi}=\frac{L^2/km}{1-\varepsilon\cos\phi}$$

Polar angle ϕ using: $L = mr^2\frac{d\phi}{dt} = mr^2\dot{\phi}$

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$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}$$

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using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

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Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum $\mathbf{p}=m\mathbf{v}$

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$$r\dot{\phi}=\frac{L}{mr}$$

using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=*m***v**

Finding time derivatives of orbital coordinates r, ϕ, x, y, and eventually velocity **v** *or momentum* **p**=*m***v**

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using: $\frac{1}{r}=\left(\frac{km}{L^2}\right)(1-\varepsilon\cos\phi)$

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Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=m**v**

Finding time derivatives of orbital coordinates r, φ, x, y, and eventually velocity **v** *or momentum* **p**=m**v**

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$$\dot{\phi}=\frac{L}{mr^2}=\frac{L}{m}\frac{1}{r^2}=\frac{L}{m}\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$$

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*Coulomb orbit algebra of ε-vector and Kepler dynamics of momentum **p**=m**v***

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$$\dot{r}=-\frac{k}{L^2}mr^2\dot{\phi}\,\varepsilon\sin\phi=-\frac{k}{L}\varepsilon\sin\phi$$

again using: $L=mr^2\dot{\phi}$

Polar angle ϕ using: $L=mr^2\frac{d\phi}{dt}=mr^2\dot{\phi}$

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using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

again using: $L=mr^2\dot{\phi}$

Cartesian x =r cos ϕ*:*

$$\dot{x}=\frac{dx}{dt}=\quad\dot{r}\cos\phi-\sin\phi\;r\dot{\phi}$$

Cartesian y =r sin ϕ*:*

$$\dot{y}=\frac{dy}{dt}=\quad\dot{r}\sin\phi+\cos\phi\;r\dot{\phi}$$

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using: $\frac{1}{r^2}=\left(\frac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

using: $\frac{1}{(1-\varepsilon\cos\phi)^2}=\left(\frac{km}{L^2}\right)^2 r^2$

again using: $L=mr^2\dot\phi$

Cartesian $x=r\cos\phi$*:*

$$\dot x=\frac{dx}{dt}=\quad\dot r\cos\phi-\sin\phi\; r\dot\phi$$

$$=-\frac{k}{L}\varepsilon\sin\phi\cos\phi-\sin\phi\;\frac{k}{L}(1-\varepsilon\cos\phi)$$

Cartesian $y=r\sin\phi$*:*

$$\dot y=\frac{dy}{dt}=\quad\dot r\sin\phi+\cos\phi\; r\dot\phi$$

$$=-\frac{k}{L}\varepsilon\sin\phi\sin\phi+\cos\phi\;\frac{k}{L}(1-\varepsilon\cos\phi)$$

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using: $\dfrac{1}{r^2}=\left(\dfrac{km}{L^2}\right)^2(1-\varepsilon\cos\phi)^2$

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$$\dot x=\frac{dx}{dt}=\quad\dot r\cos\phi-\sin\phi\;r\dot\phi$$

$$=-\frac{k}{L}\cancel{\varepsilon\sin\phi\cos\phi}-\sin\phi\;\frac{k}{L}(1-\cancel{\varepsilon\cos\phi})$$

$$=-\frac{k}{L}\sin\phi$$

Cartesian $y=r\sin\phi$*:*

$$\dot y=\frac{dy}{dt}=\quad\dot r\sin\phi+\cos\phi\;r\dot\phi$$

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$$=\frac{k}{L}(\cos\phi-\varepsilon)$$

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$$\dot{x}=\frac{dx}{dt}=\dot{r}\cos\phi-\sin\phi\,r\dot{\phi}$$

$$=-\frac{k}{L}\sin\phi$$

Cartesian $y=r\sin\phi$:

$$\dot{y}=\frac{dy}{dt}=\dot{r}\sin\phi+\cos\phi\,r\dot{\phi}$$

$$=\frac{k}{L}(\cos\phi-\varepsilon)$$

Velocity:

Momentum:

$$\boxed{p_x=m\dot{x}=-\frac{mk}{L}\sin\phi}$$

$$\boxed{p_y=m\dot{y}=\frac{mk}{L}(\cos\phi-\varepsilon)}$$

p *traces an off-center circle!*

Rutherford scattering and hyperbolic orbit geometry

- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

- *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

- *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

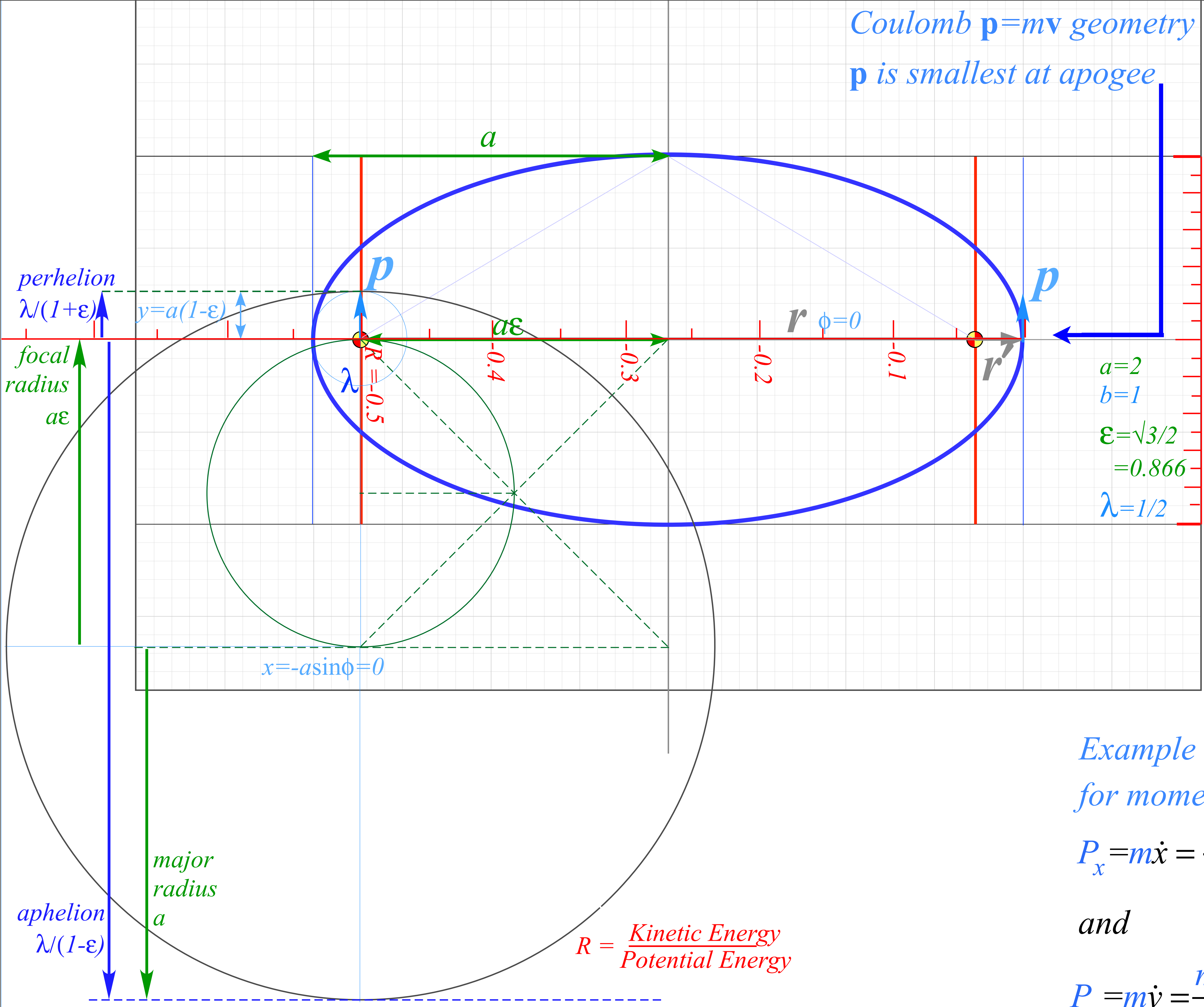
Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

➔ *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

Connection formulas for (γ,R)*-parameters with (a,b) and* (ε,λ)

Coulomb **p**=m**v** geometry (ϕ=0)

p is smallest at apogee



$$R = \frac{\text{Kinetic Energy}}{\text{Potential Energy}}$$

Example of geometry

for momentum functions:

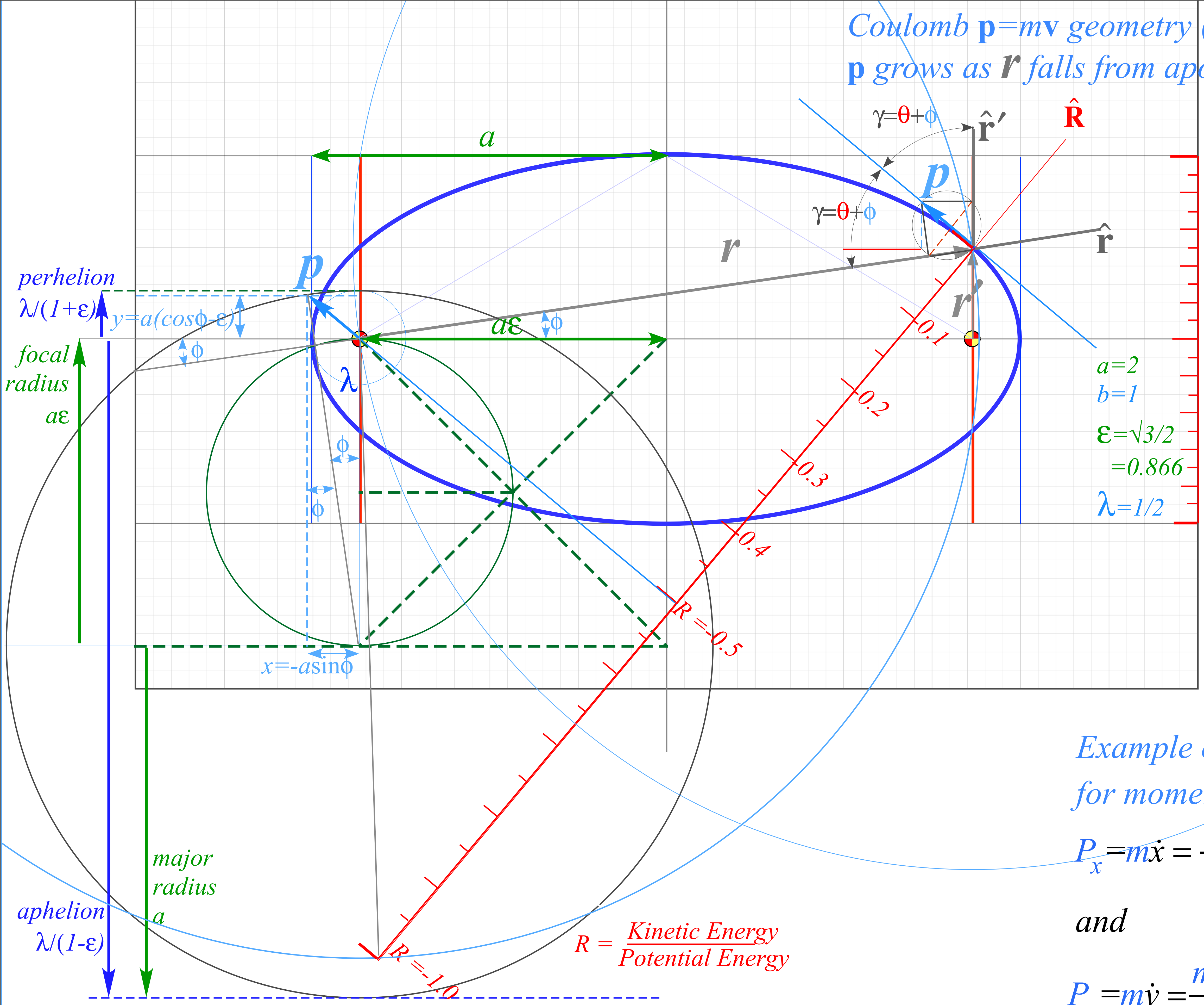
$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

Coulomb **p**=*mv geometry* (*φ* >0)

p *grows as* ***r*** *falls from apogee*



$R=\frac{KE}{PE}$ *scale line* bisects angle$\sphericalangle^{+\hat{\mathbf{r}}'}_{+\hat{\mathbf{r}}}=\pi-2\gamma$

$\hat{\mathbf{R}}$ is ⊥ to **p**

p *momentum line* bisects angle$\sphericalangle^{+\hat{\mathbf{r}}'}_{-\hat{\mathbf{r}}}=2\gamma$

Example of geometry for momentum functions:

$$P_x=m\dot{x}=-\frac{mk}{L}\sin\phi$$

and

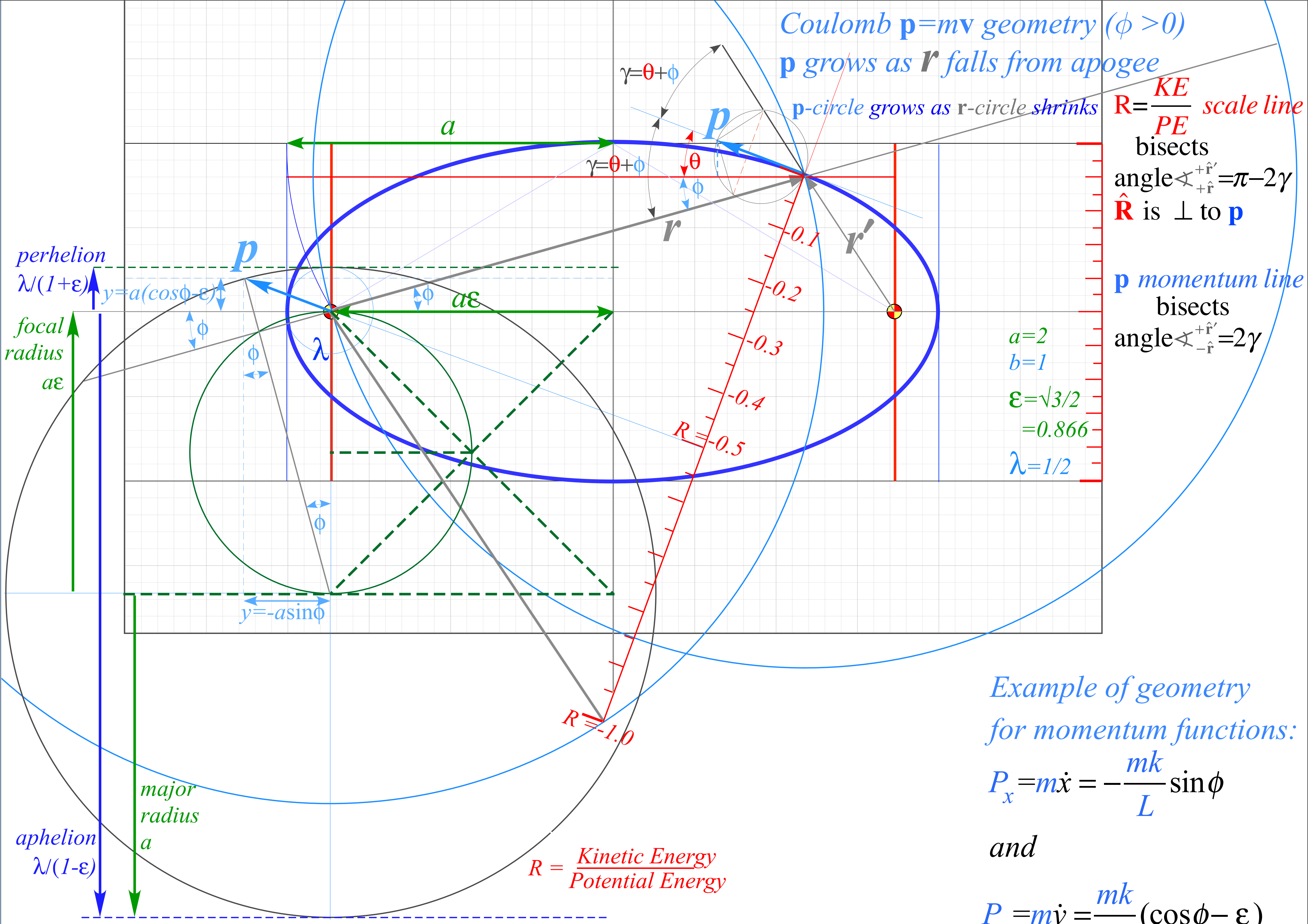
$$P_y=m\dot{y}=\frac{mk}{L}(\cos\phi-\varepsilon)$$

0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived before on p. 67-71.*

*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*

Coulomb **p**=*mv geometry* ($\phi>0$)

p *grows as* ***r*** *falls from apogee*



p-*circle grows as* **r**-*circle shrinks*

$R=\frac{KE}{PE}$ *scale line*
bisects
angle$\sphericalangle^{+\hat{\mathbf{r}}'}_{+\hat{\mathbf{r}}}=\pi-2\gamma$
$\hat{\mathbf{R}}$ is $\perp$ to **p**

p *momentum line*
bisects
angle$\sphericalangle^{+\hat{\mathbf{r}}'}_{-\hat{\mathbf{r}}}=2\gamma$

a=2
b=1
$\varepsilon=\sqrt{3}/2$
$=0.866$
$\lambda=1/2$

$R=\frac{\textit{Kinetic Energy}}{\textit{Potential Energy}}$

Example of geometry
for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

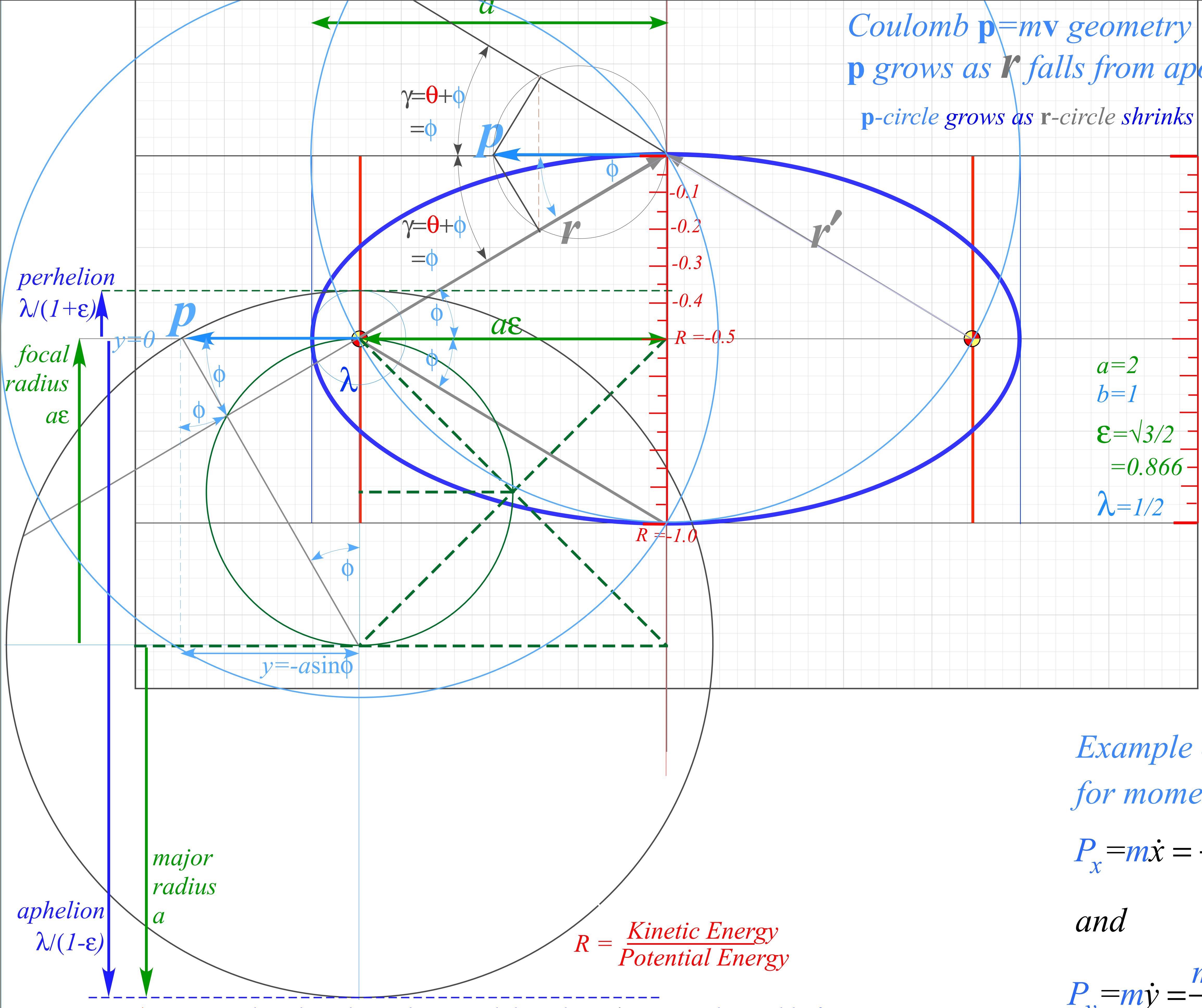
0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived before on p. 66-70.*

*Note similarity of (***R***,***r***)-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*

Coulomb **p**=m**v** geometry ($\phi > 0$)

p grows as **r** falls from apogee

p-circle grows as **r**-circle shrinks



$R=\frac{KE}{PE}$ scale line

bisects

angle$\sphericalangle^{+\hat{r}'}_{+\hat{r}}=\pi-2\gamma$

$\hat{\mathbf{R}}$ is $\perp$ to **p**

p momentum line

bisects

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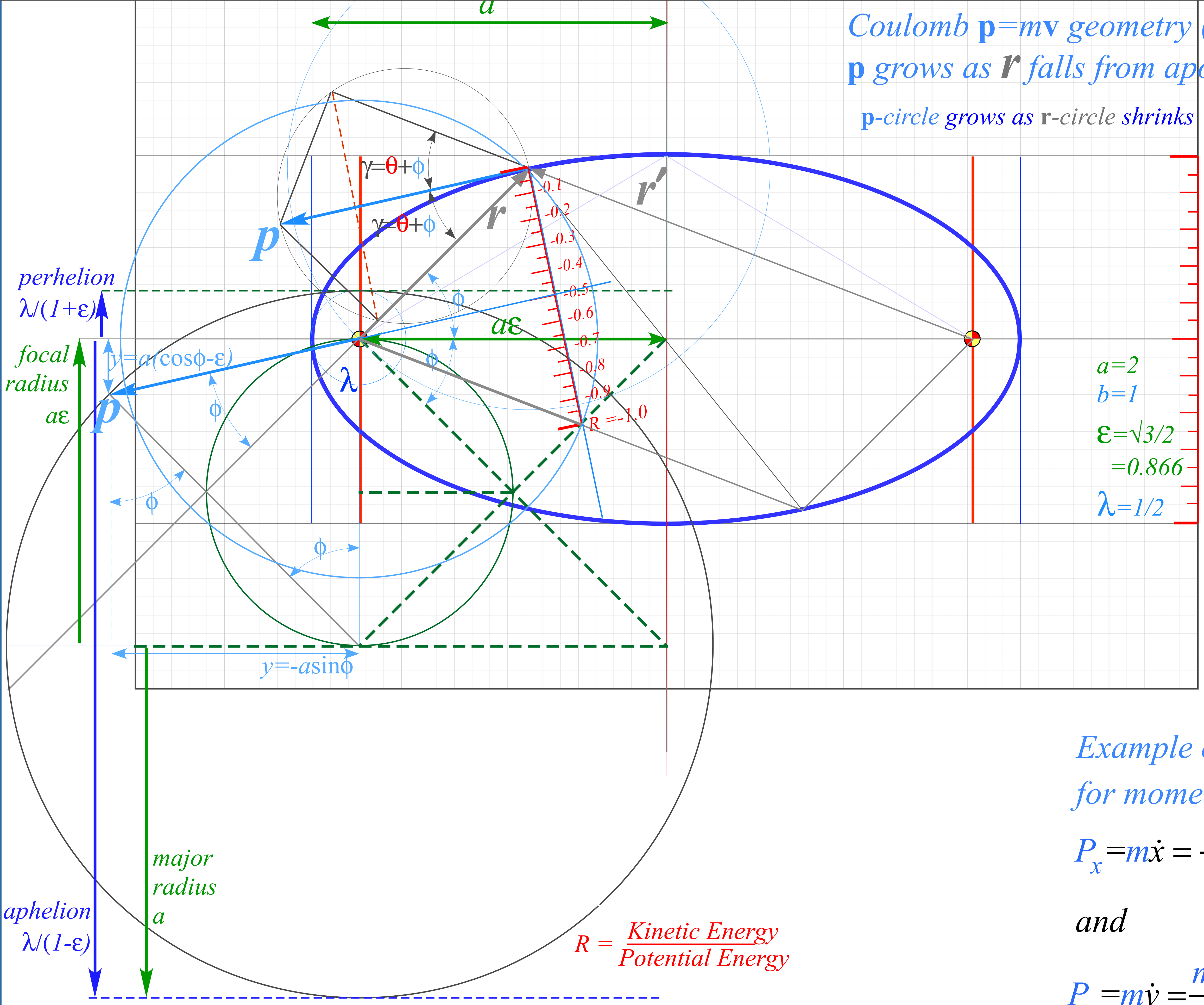
*Coulomb **p**=m**v** geometry (ϕ >0)*

***p** grows as **r** falls from apogee*

***p**-circle grows as **r**-circle shrinks*



$R=\frac{KE}{PE}$ scale line

bisects

angle $\sphericalangle^{+\hat{\mathbf{r}}'}_{+\hat{\mathbf{r}}}=\pi-2\gamma$

$\hat{\mathbf{R}}$ is ⊥ to **p**

p *momentum line*

bisects

angle $\sphericalangle^{+\hat{\mathbf{r}}'}_{-\hat{\mathbf{r}}}=2\gamma$

$R = \frac{\text{Kinetic Energy}}{\text{Potential Energy}}$

Example of geometry for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

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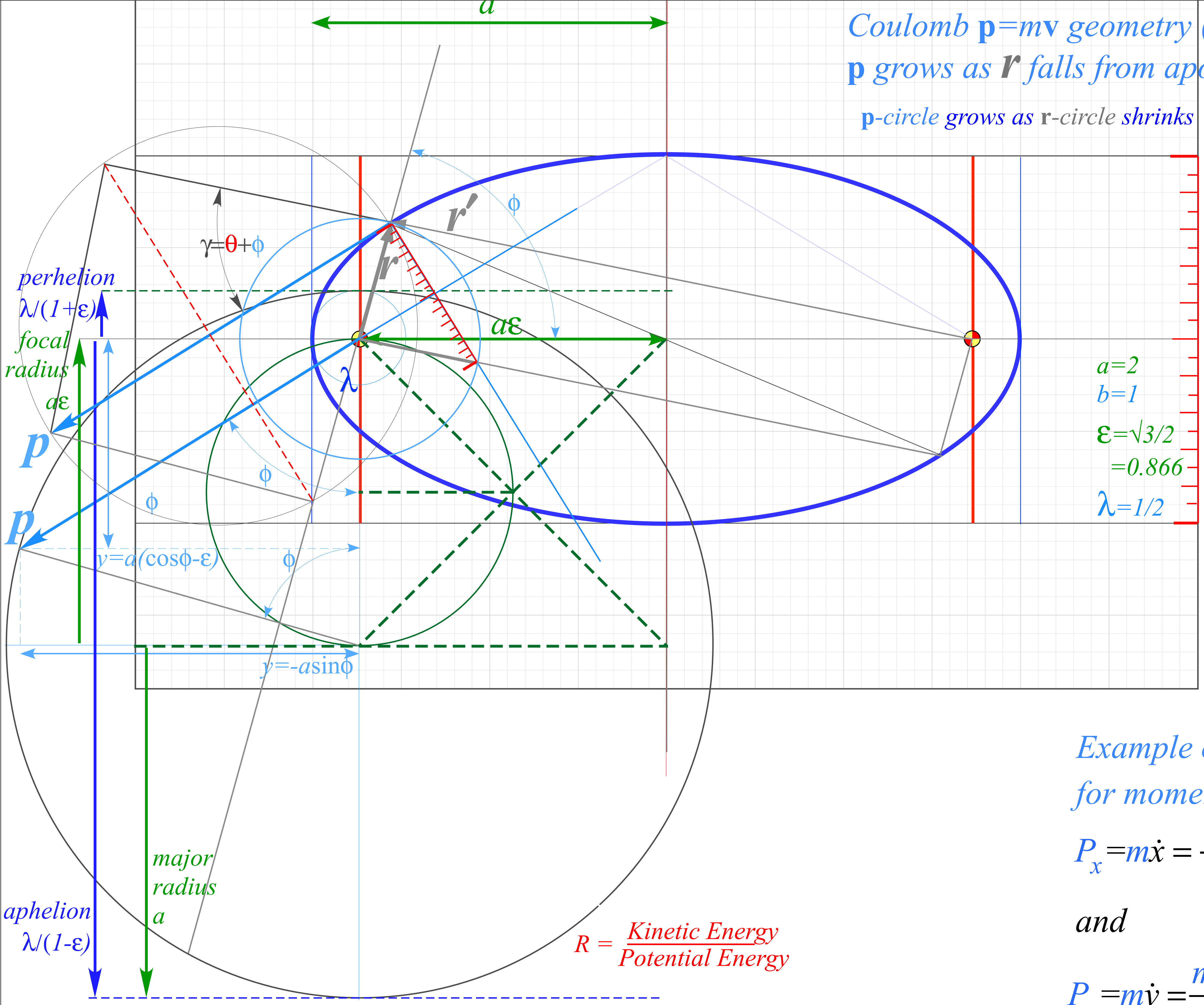
Coulomb **p**=*m***v** geometry ($\phi>0$)

p grows as *r* falls from apogee

p-circle grows as **r**-circle shrinks



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p *momentum line* bisects angle$\sphericalangle^{+\hat{r}'}_{-\hat{r}}=2\gamma$

$R=\frac{\text{Kinetic Energy}}{\text{Potential Energy}}$

0>*R*=*KE/PE*>-1 *scale subtends angle* 2γ *with length* $2r\sin\gamma$ *as is derived before on p. 66-70.*

*Note similarity of (***R***,***r***)-triangle in* **r***-circle of radius r to that in* **p***-circle of diameter p above.*

Example of geometry

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$$P_x=m\dot{x}=-\frac{mk}{L}\sin\phi$$

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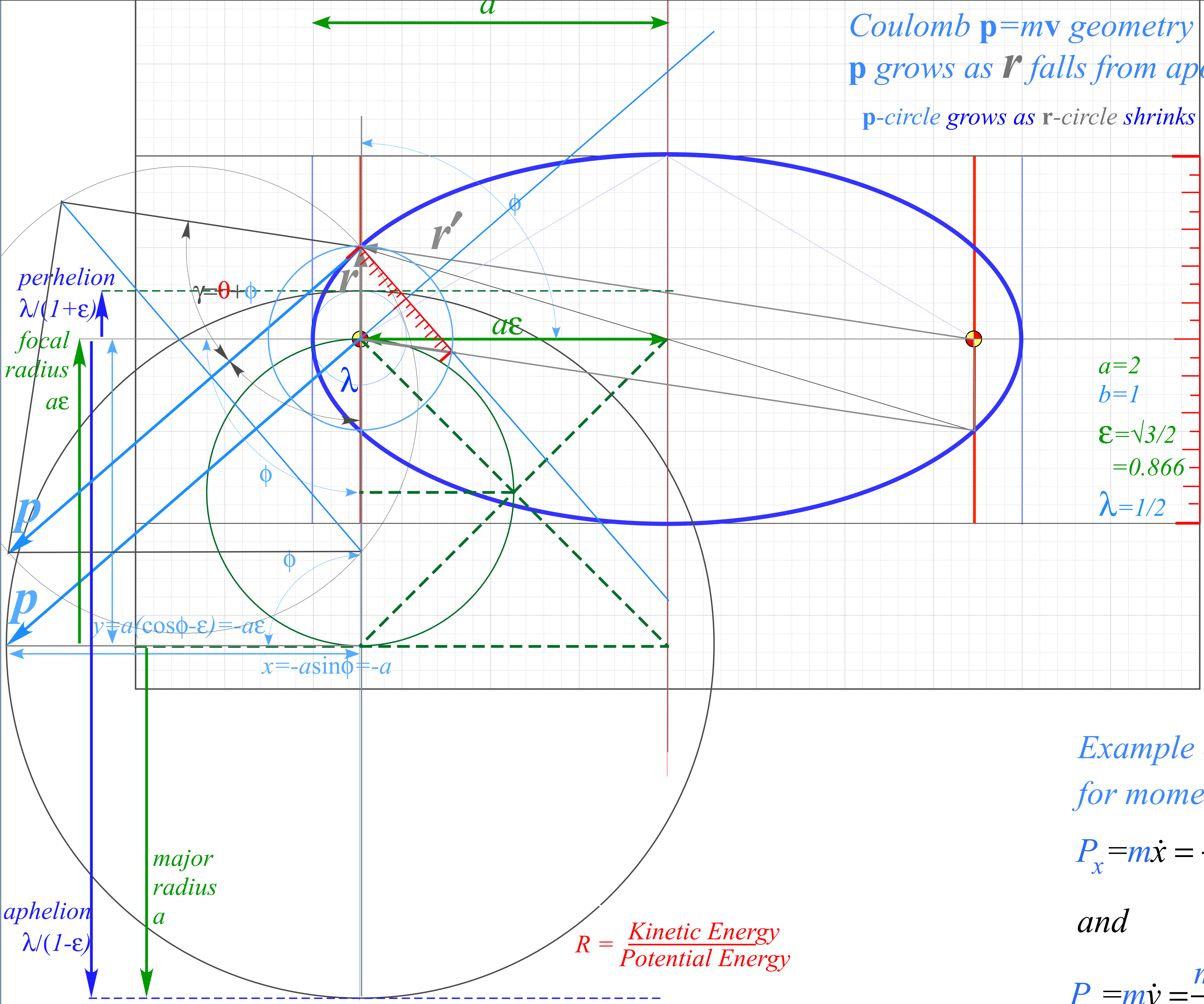
$$P_y=m\dot{y}=\frac{mk}{L}(\cos\phi-\varepsilon)$$

Coulomb **p**=*m***v** geometry (ϕ >0)

p grows as **r** falls from apogee

p-circle grows as **r**-circle shrinks



$\mathbf{R}=\dfrac{KE}{PE}$ scale line

bisects

angle$\sphericalangle^{+\hat{\mathbf{r}}'}_{+\hat{\mathbf{r}}}=\pi-2\gamma$

$\hat{\mathbf{R}}$ is $\perp$ to **p**

p *momentum line*

bisects

angle$\sphericalangle^{+\hat{\mathbf{r}}'}_{-\hat{\mathbf{r}}}=2\gamma$

$R = \dfrac{\text{Kinetic Energy}}{\text{Potential Energy}}$

Example of geometry

for momentum functions:

$$P_x = m\dot{x} = -\frac{mk}{L}\sin\phi$$

and

$$P_y = m\dot{y} = \frac{mk}{L}(\cos\phi - \varepsilon)$$

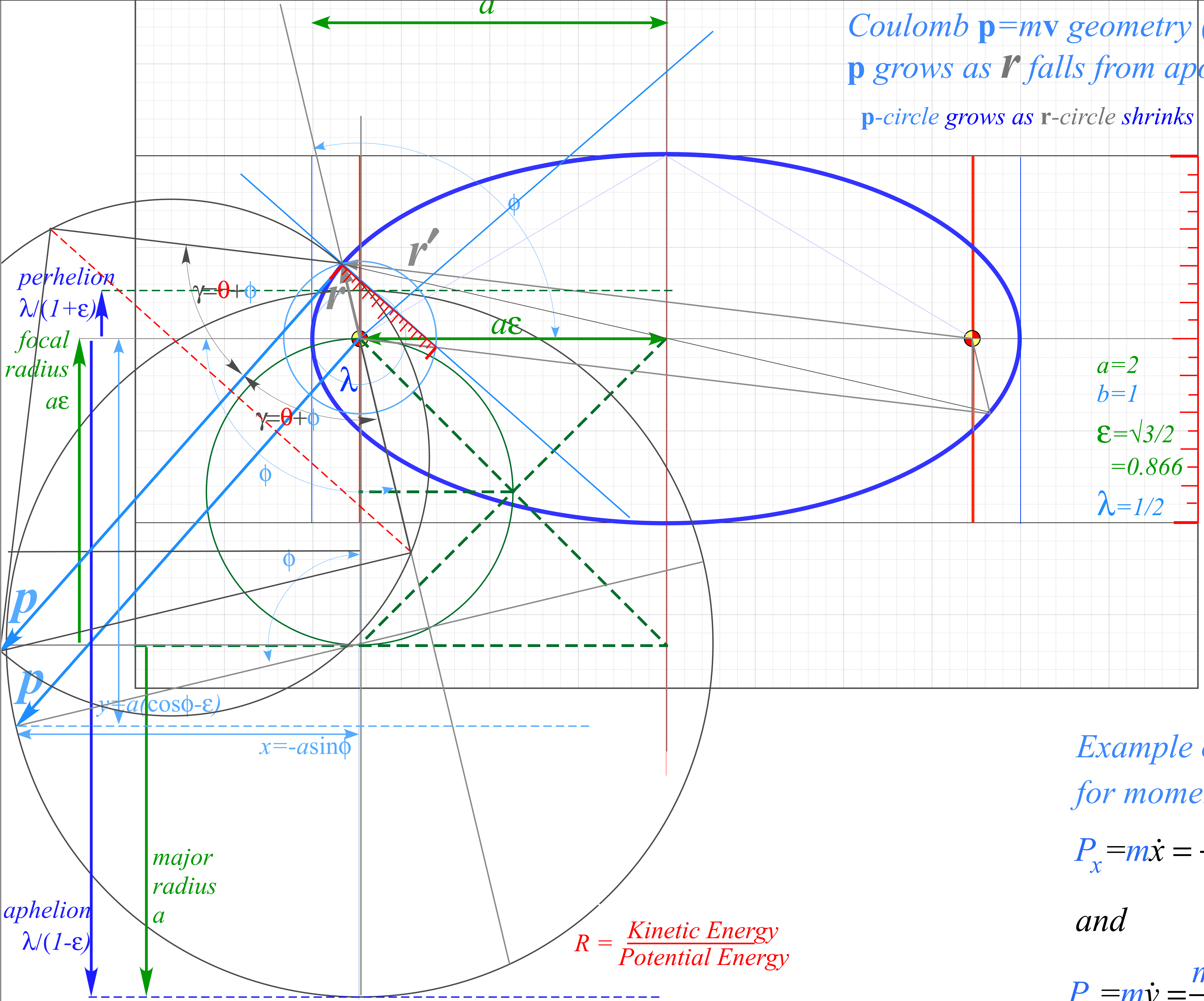
0>R=KE/PE>-1 scale subtends angle 2γ with length 2r sinγ *as is derived before on p. 66-70.*

*Note similarity of (**R**,**r**)-triangle in **r**-circle of radius r to that in **p**-circle of diameter p above.*

Coulomb **p**=*mv geometry* ($\phi>0$)

p *grows as* ***r*** *falls from apogee*

p-*circle grows as* **r**-*circle shrinks*



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bisects
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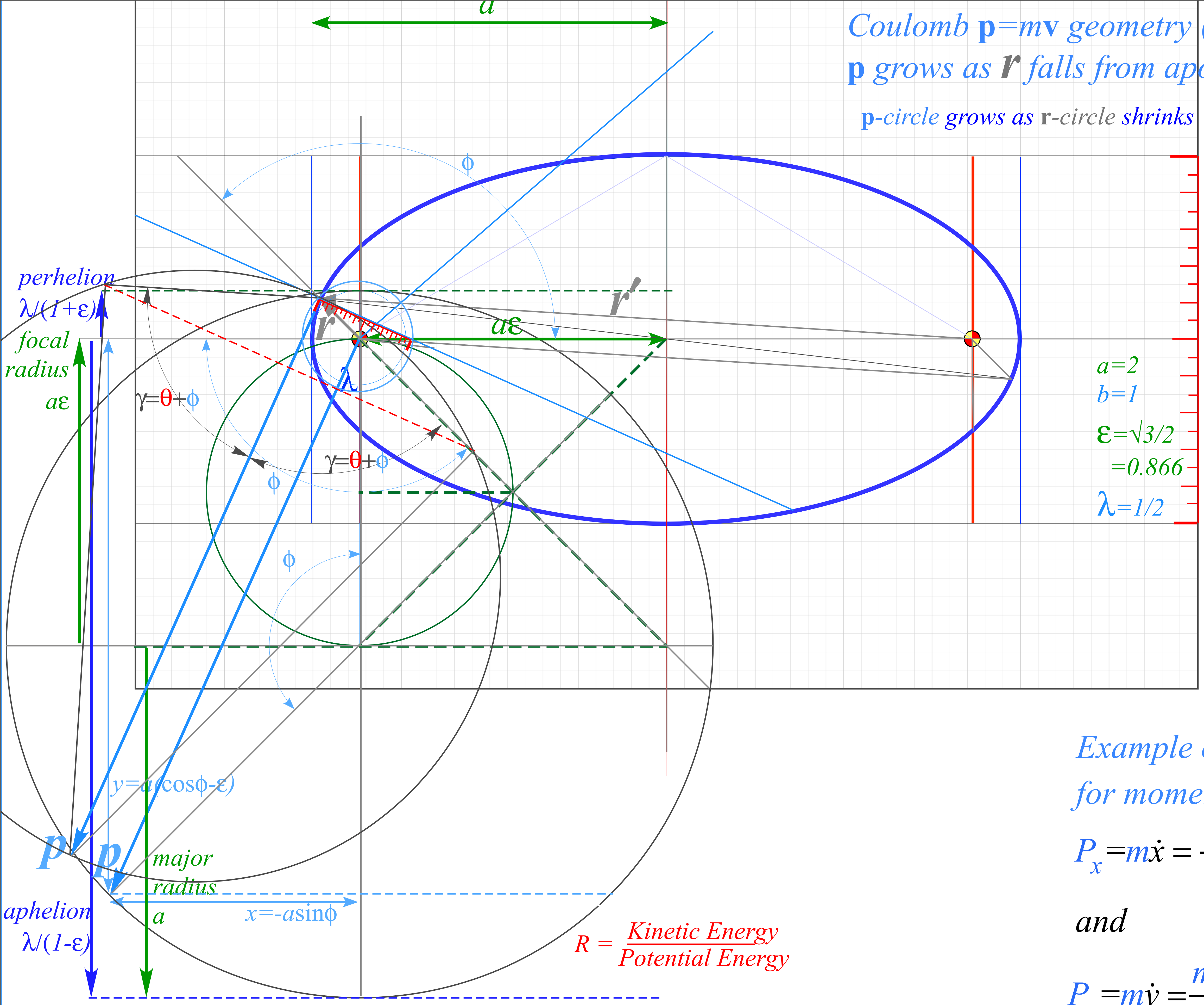

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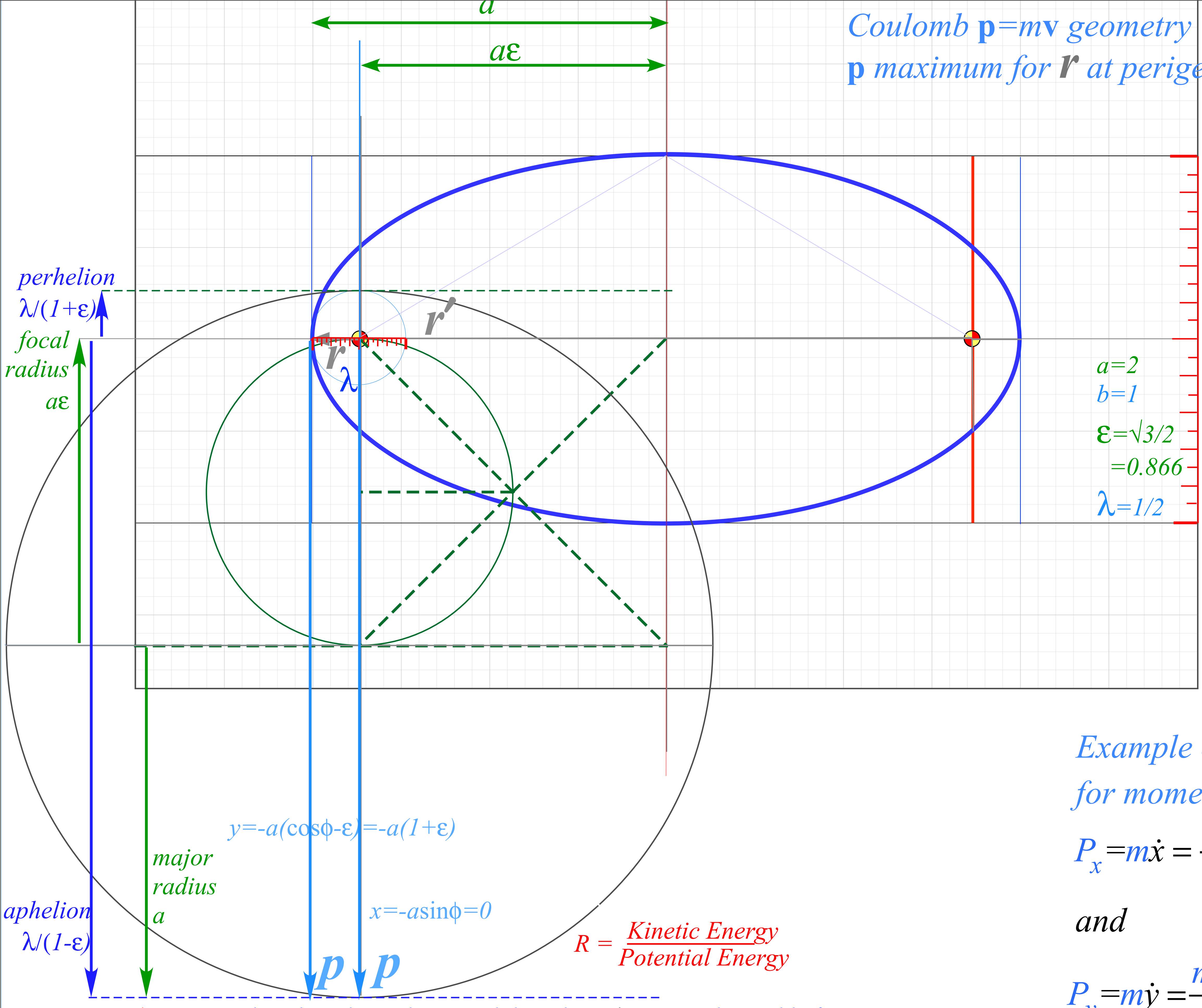
Coulomb **p**=*m***v** geometry ($\phi>0$)

p *maximum for* ***r*** *at perigee*



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Rutherford scattering and hyperbolic orbit geometry

- *Backward vs forward scattering angles and orbit construction example*
- *Parabolic "kite" and orbital envelope geometry*
- *Differential and total scattering cross-sections*

Eccentricity vector $\boldsymbol{\varepsilon}$ *and* (ε,λ)*-geometry of orbital mechanics*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{r}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and orbital radius* $\mathbf{r}$

- *Review and connection to usual orbital algebra (previous lecture)*

Projection $\boldsymbol{\varepsilon}\bullet\mathbf{p}$ *geometry of* $\boldsymbol{\varepsilon}$*-vector and momentum* $\mathbf{p}=m\mathbf{v}$

General geometric orbit construction using $\boldsymbol{\varepsilon}$*-vector and* (γ,R)*-parameters*

- *Derivation of* $\boldsymbol{\varepsilon}$*-construction by analytic geometry*

Coulomb orbit algebra of $\boldsymbol{\varepsilon}$*-vector and Kepler dynamics of momentum* $\mathbf{p}=m\mathbf{v}$

- *Example of complete* $(\mathbf{r},\mathbf{p})$*-geometry of elliptical orbit*

➔ *Connection formulas for* (γ,R)*-parameters with (a,b) and* (ε,λ)

Algebra of ε-construction geometry

Three pairs of parameters for Coulomb orbits:

1.Cartesian (a,b), 2.Physics (E,L), 3.Polar (ε,λ)

Now we relate a 4th pair: 4.Initial (γ,R)

The *eccentricty* parameter relates ratios $R = \frac{KE}{PE}$ and $\frac{b^2}{a^2}$

$$\varepsilon^2 = 1 + 4R(R+1)\sin^2\gamma$$

$$= 1 - \frac{b^2}{a^2} \quad \text{for ellipse} \quad (\varepsilon < 1)$$

$$= 1 + \frac{b^2}{a^2} \quad \text{for hyperbola } (\varepsilon > 1)$$

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Latus radius is similarly related:

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From ε^2 result (at top):

$$\frac{b}{a}=2\sqrt{\mp R(R+1)}\sin\gamma=\sqrt{\pm(1-\varepsilon^2)}$$