Lecture 25
Tue. 12.01.2015

Introduction to Orbital Dynamics

(Ch. 2-4 of Unit 5 12.01.15)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

Review: "3steps from Hell"
(Lect. 7 Ch. 9 Unit 1)

(A mystery similarity appears)

Geometry and Symmetry of Coulomb orbits

- *Detailed elliptic geometry*
- *Detailed hyperbolic geometry*

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

➔ Effective potentials for IHO and Coulomb orbits
- Stable equilibrium radii and radial/angular frequency ratios
- Classical turning radii and apogee/perigee parameters
- Polar coordinate differential equations
- Quadrature integration techniques
  - Detailed orbital functions
- Relating orbital energy-momentum to conic-sectional orbital geometry
  - Kepler equation of time and phase geometry

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

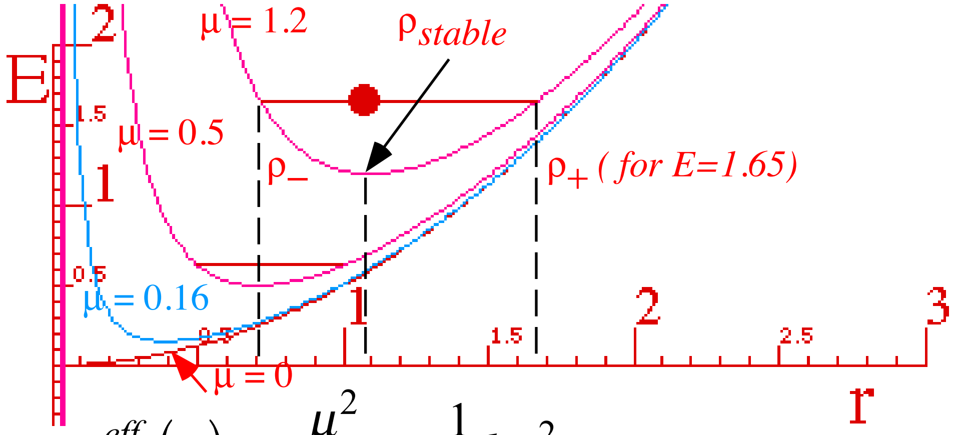
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For <u>ALL</u> central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for <u>constant</u> parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

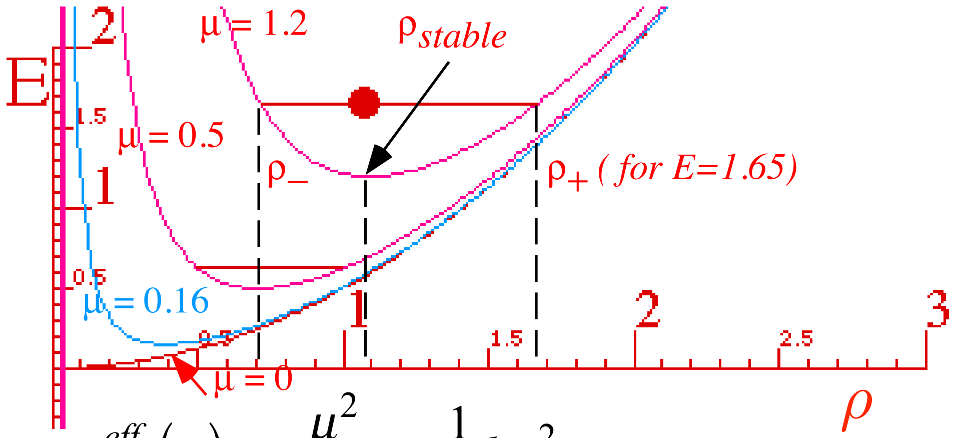
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

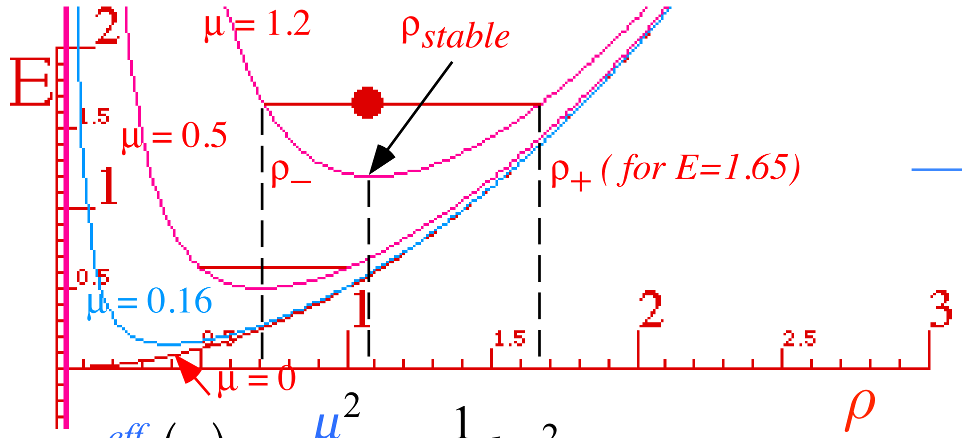
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

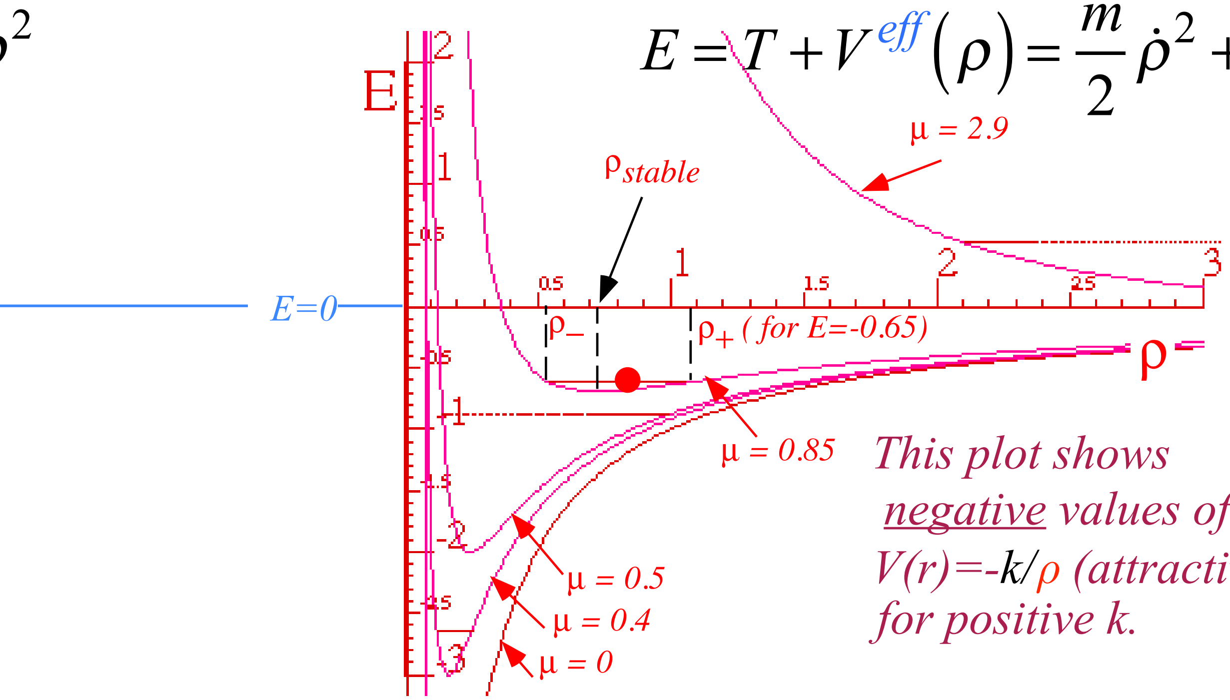
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

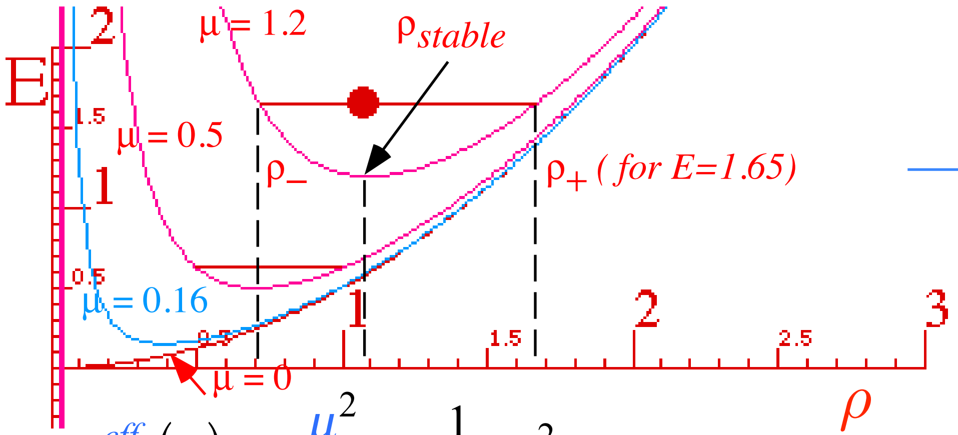
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



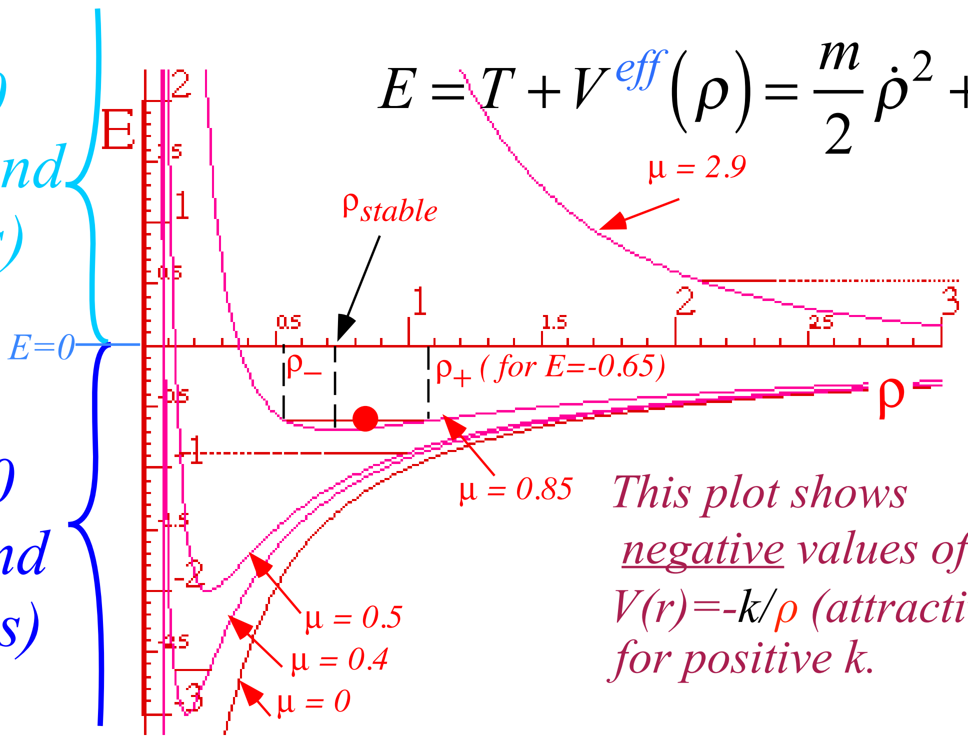
$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

E >0 (unbound orbits)

E<0 (bound orbits)

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

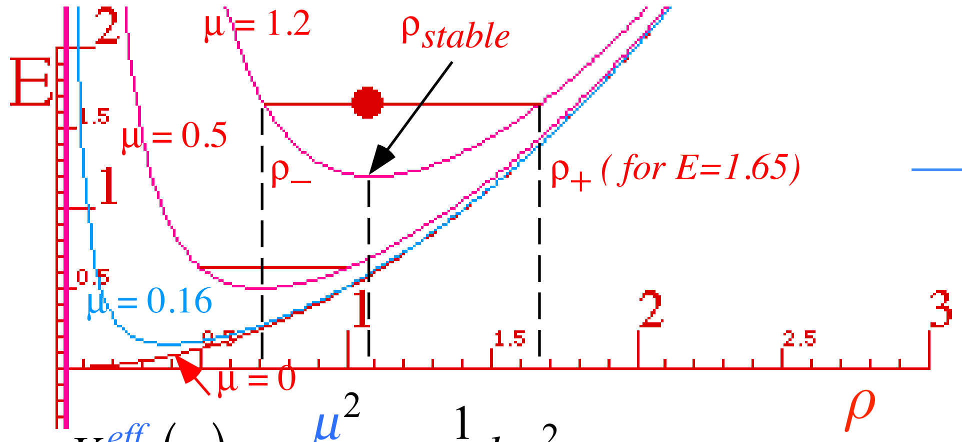
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



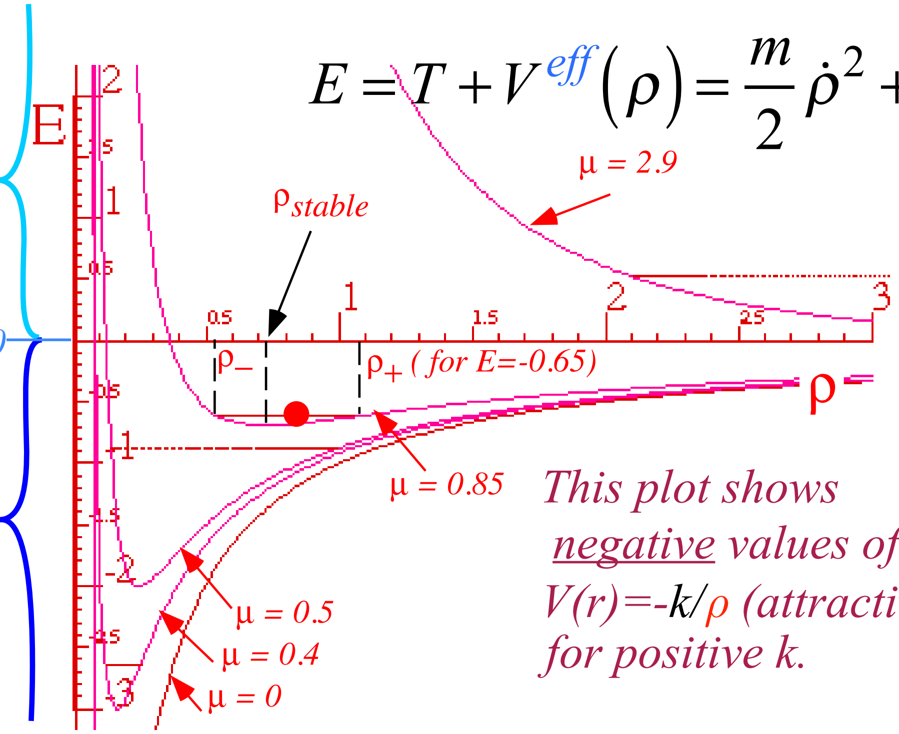
$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

E >0 (unbound orbits)

E<0 (bound orbits)



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of V(r)=-k/ρ (attractive) for positive k.

In either case: *IHO or Coulomb orbit blows up if* k *is negative.*

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

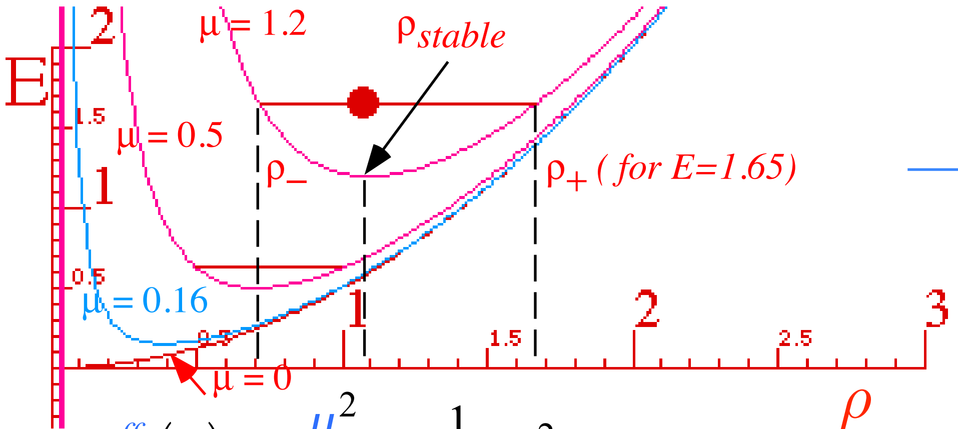
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

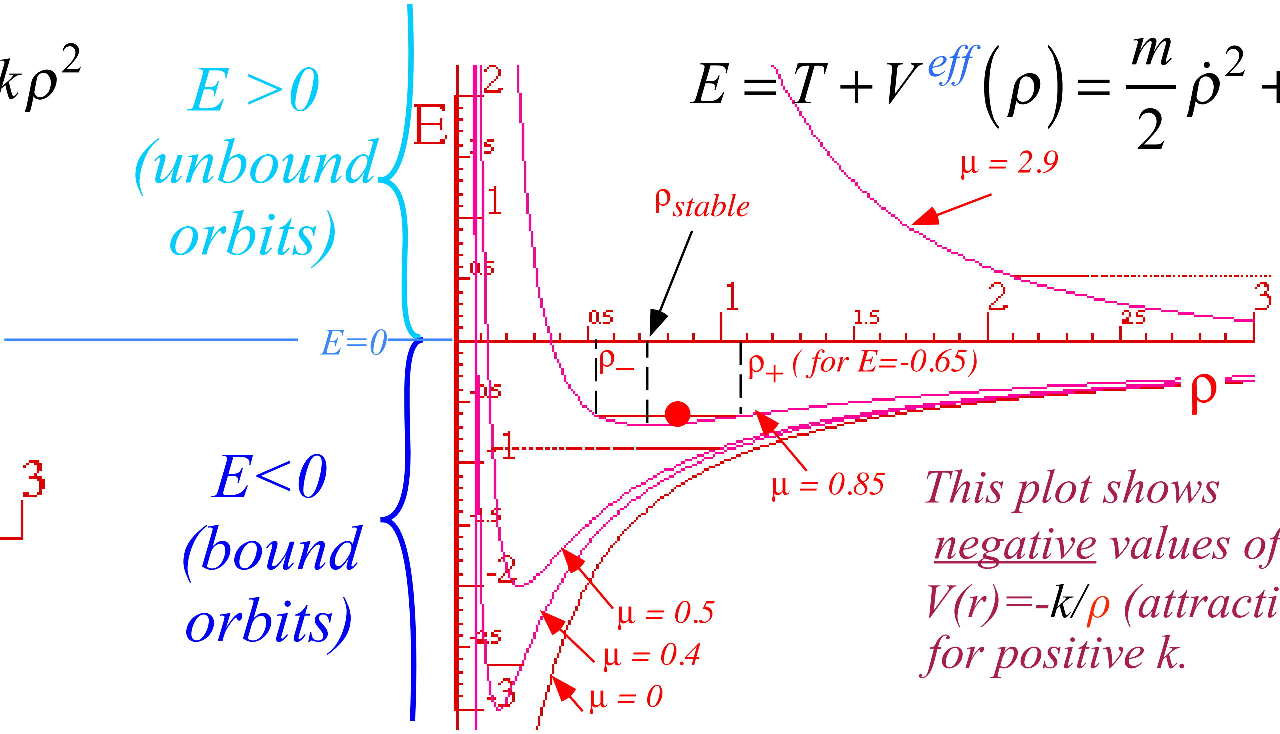
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows
negative values of
V(r)=-k/ρ (attractive)
for positive k.

In either case: *IHO or Coulomb orbit blows up if k is negative.*

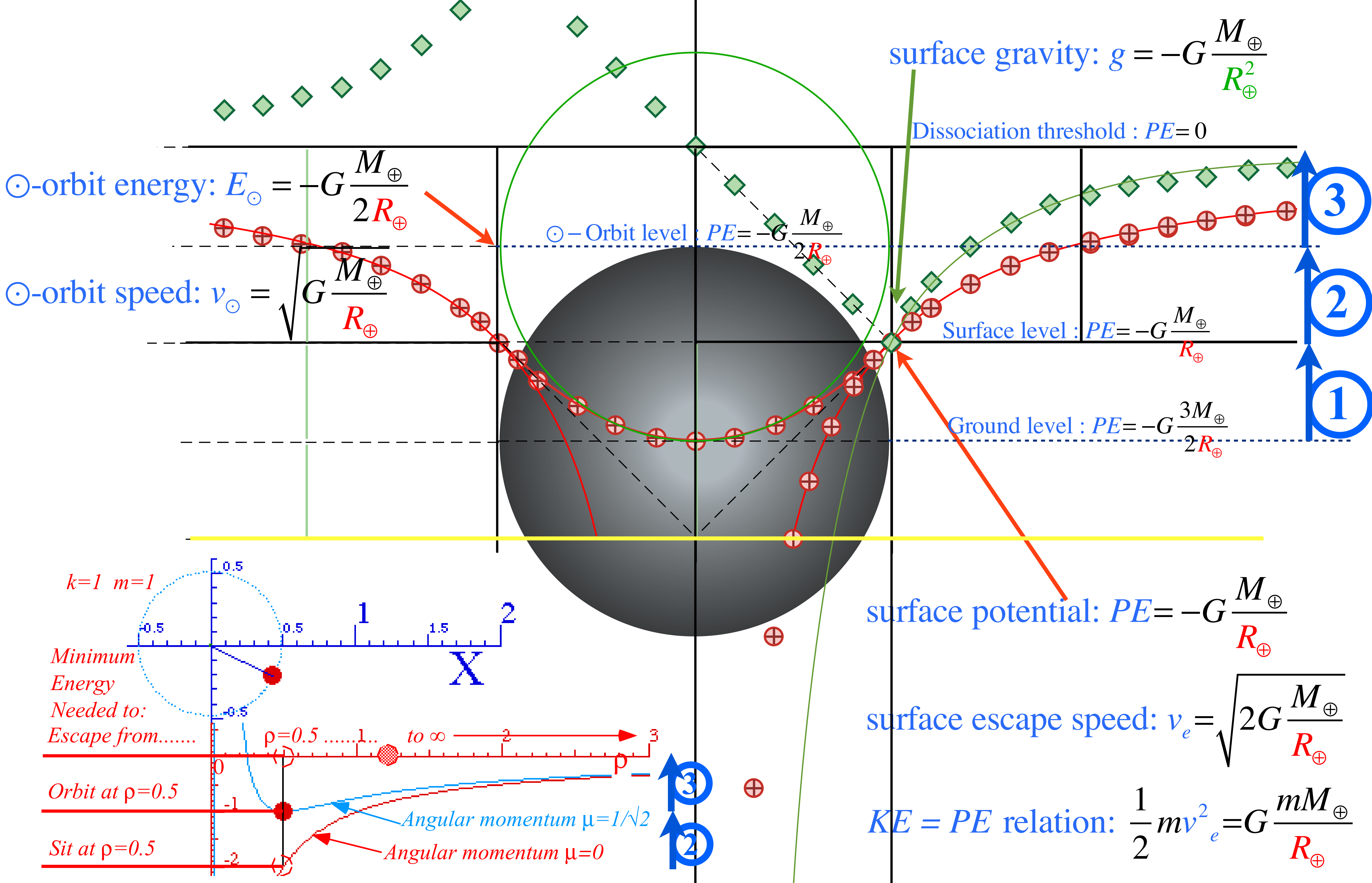
NOTE: Our Coulomb field is attractive if k is positive
That is,if -k/ρ is negative.

Coulomb $V(\rho)=-k/\rho$
(Explicit minus (-) convention)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits* ➔ *Review: "3steps from Hell" (Lect. 7 Ch. 9 Unit 1)*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

Review: "Three (equal) steps from Hell" (Lect. 7 Ch. 9 Unit 1)



surface gravity: $g = -G\frac{M_\oplus}{R_\oplus^2}$

Dissociation threshold : $PE = 0$

⊙-orbit energy: $E_\odot = -G\frac{M_\oplus}{2R_\oplus}$

⊙ – Orbit level : $PE = -G\frac{M_\oplus}{2R_\oplus}$

⊙-orbit speed: $v_\odot = \sqrt{G\frac{M_\oplus}{R_\oplus}}$

Surface level : $PE = -G\frac{M_\oplus}{R_\oplus}$

Ground level : $PE = -G\frac{3M_\oplus}{2R_\oplus}$

surface potential: $PE = -G\frac{M_\oplus}{R_\oplus}$

surface escape speed: $v_e = \sqrt{2G\frac{M_\oplus}{R_\oplus}}$

KE = PE relation: $\frac{1}{2}mv^2_e = G\frac{mM_\oplus}{R_\oplus}$

$G = 0.67\cdot 10^{-10}$

Fig. 5.2.10 Three Coulomb threshold energies: to sit (E=-2), to orbit (E=-1), and to escape (E=0) from R. A 4th energy (E=-3) is for sitting at the center (ρ=0) of a uniform mass planet of radius ρ=R.

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - ➡ *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

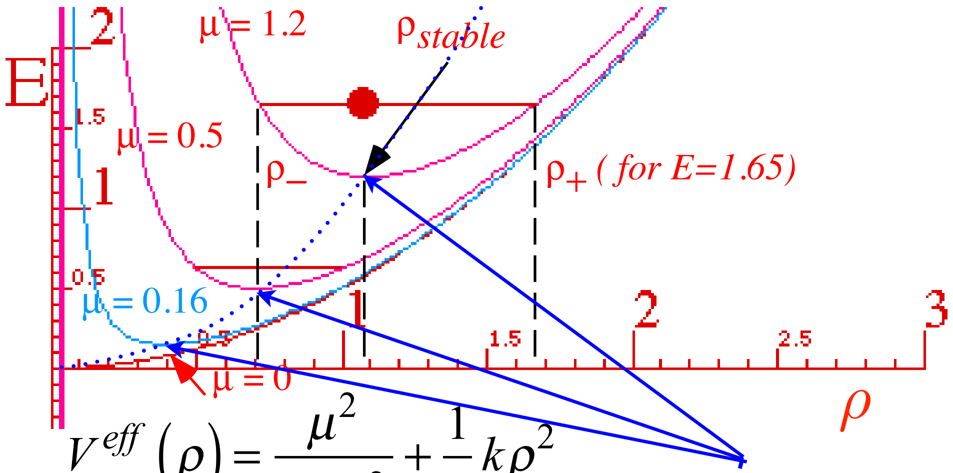
For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

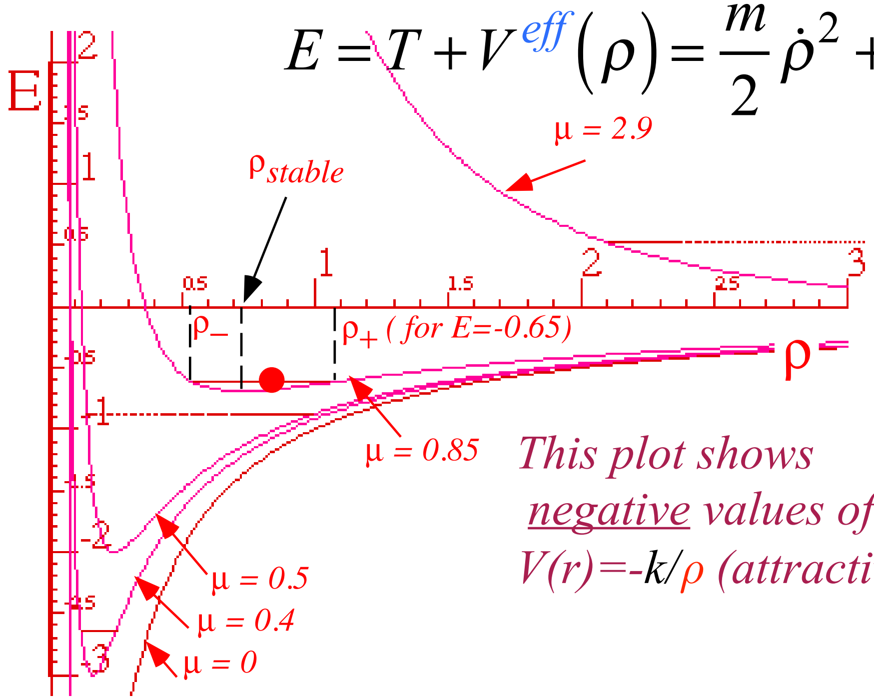
Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \boxed{\rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}}$$

$$\frac{\mu^2}{m}=+k\rho^4$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

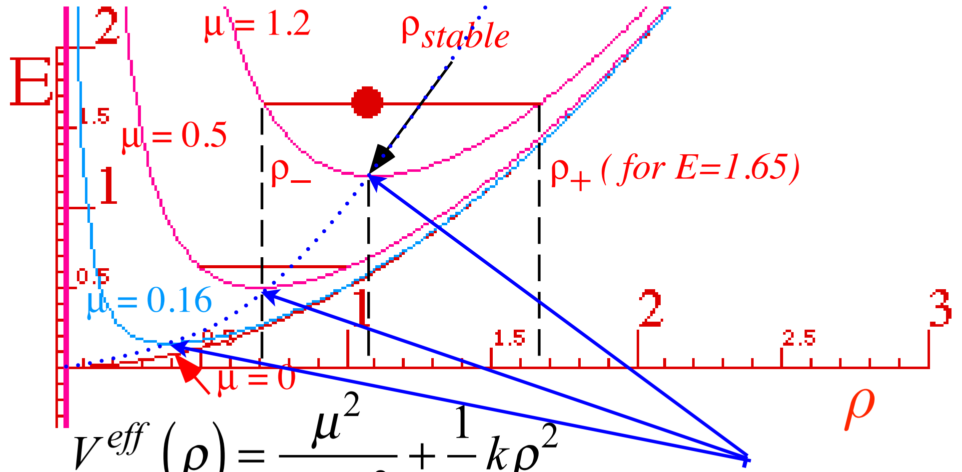
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu\qquad\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

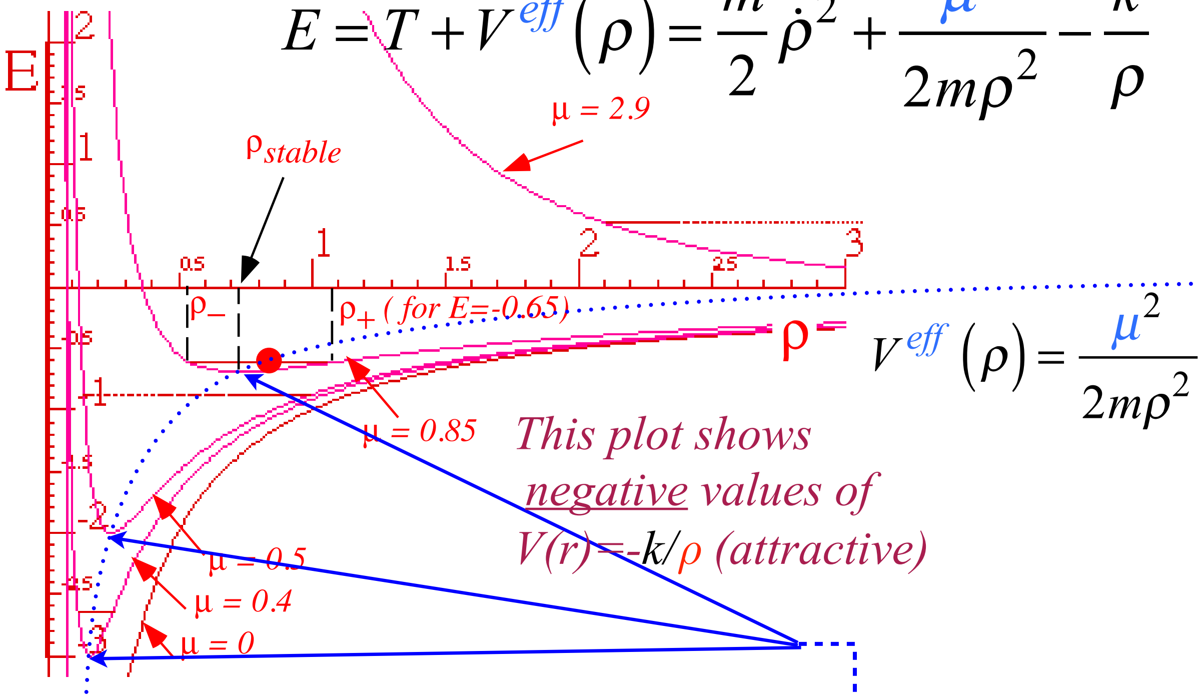
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho\qquad\text{or: }\rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\frac{\mu^2}{m}=+k\rho^4$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2}\qquad\text{or: }\rho_{stable}=\frac{\mu^2}{mk}$$

$$\frac{\mu^2}{m}=+k\rho$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

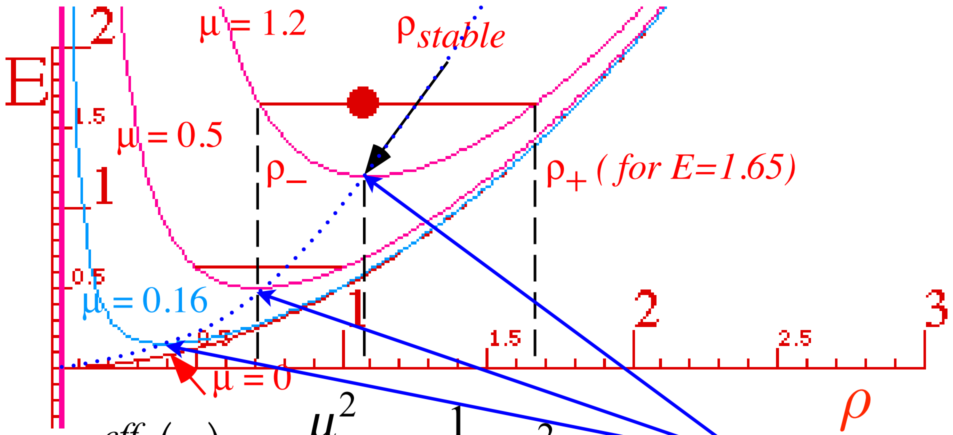
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu\qquad\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

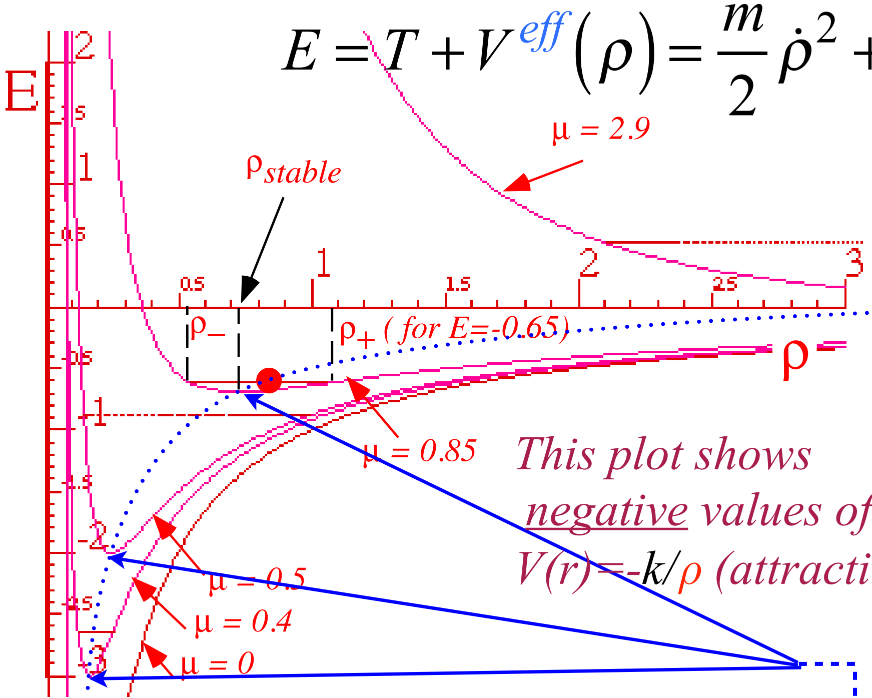
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho\quad\text{or: }\rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2}\quad\text{or: }\rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

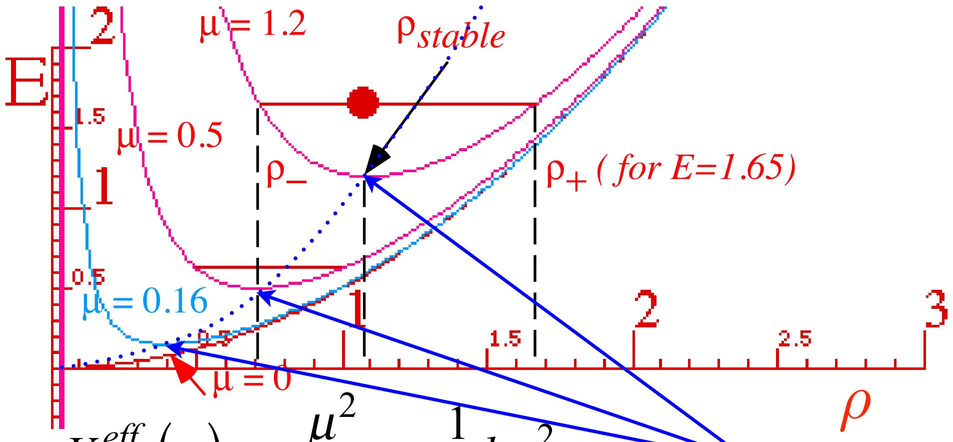
For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

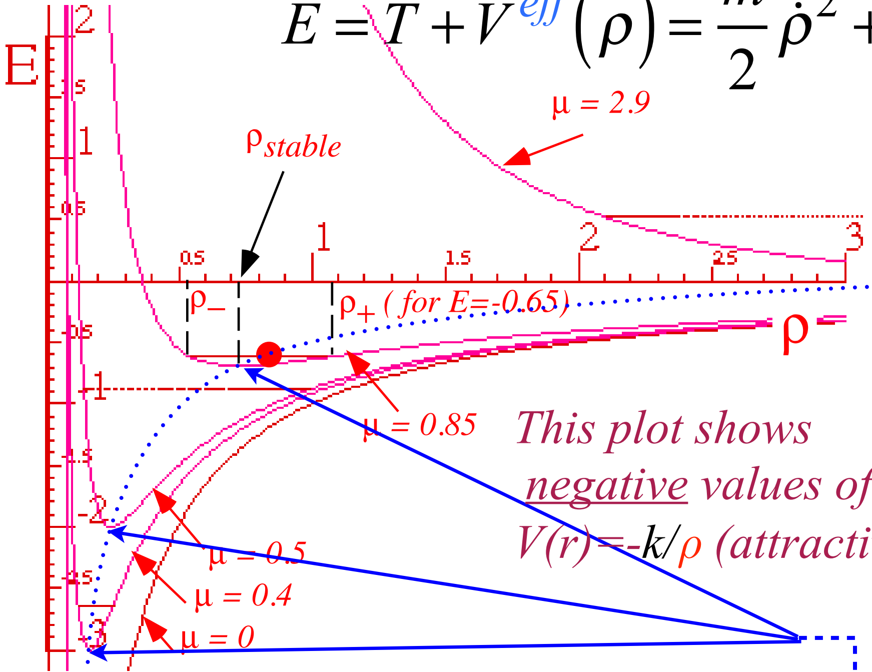
Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



This plot shows negative values of $V(r)=-k/\rho$ (attractive)

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \quad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}-\frac{k}{\rho_{stable}{}^3}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

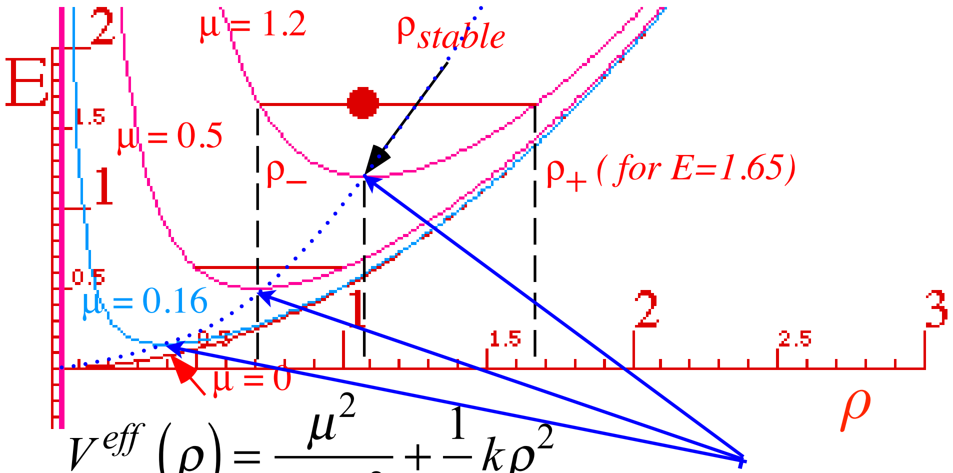
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

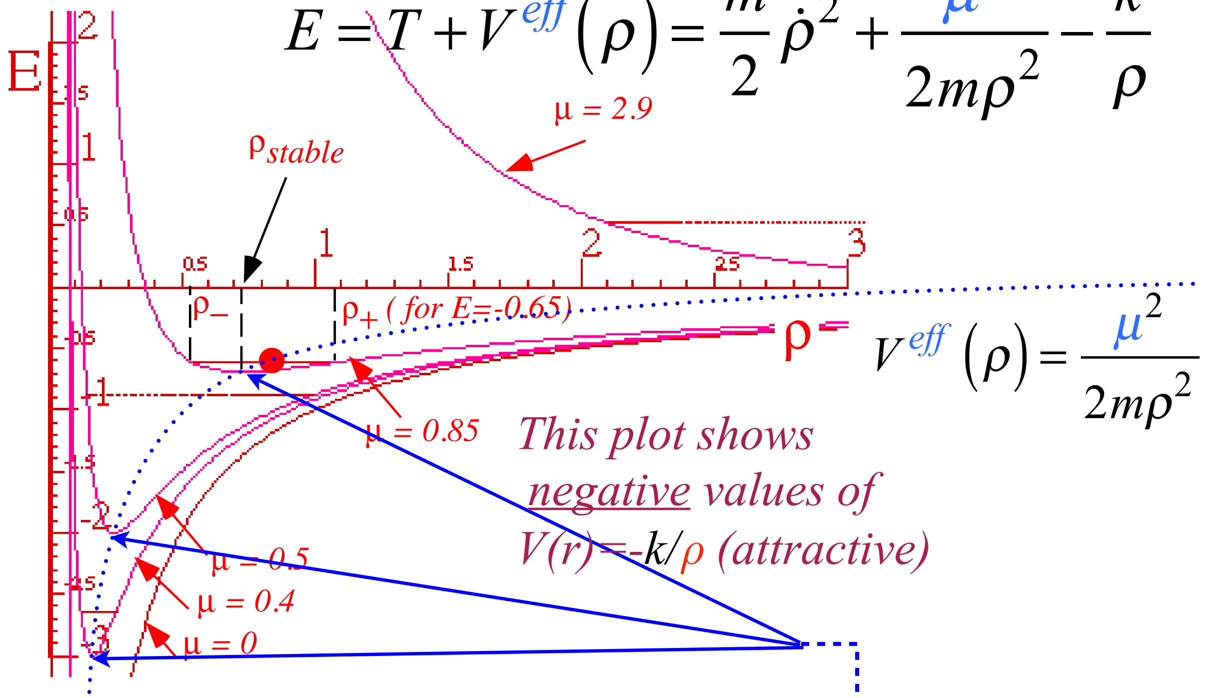
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \quad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}-\frac{k}{\rho_{stable}{}^3}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\frac{\mu}{m\rho_{stable}{}^2}=\sqrt{\frac{k}{m}}$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

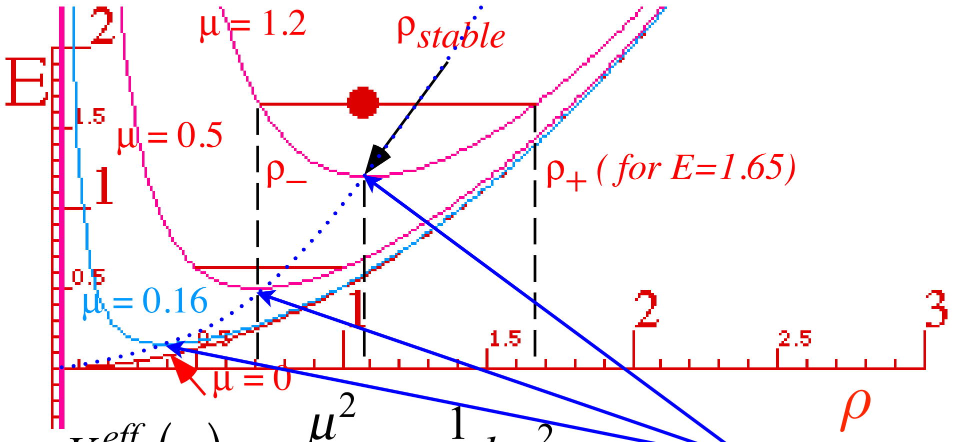
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

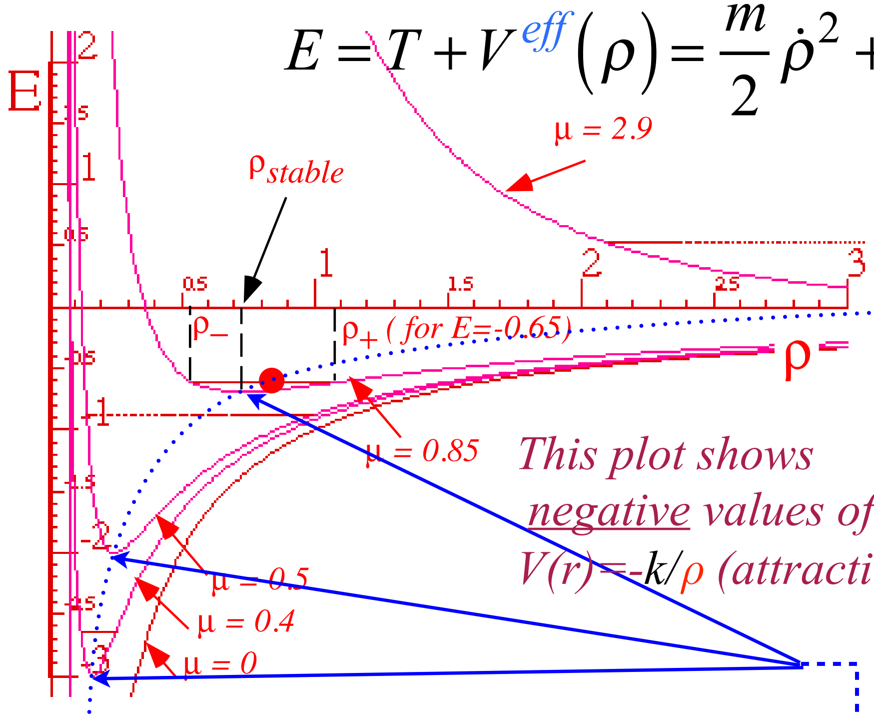
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows *negative values of* *$V(r)=-k/\rho$ (attractive)*

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \quad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}-\frac{k}{\rho_{stable}{}^3}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}{}^2}=\sqrt{\dfrac{k}{m}}$

...angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}{}^2}=\dfrac{\mu}{m}\dfrac{m^2k^2}{\mu^4}=\dfrac{mk^2}{\mu^3}$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

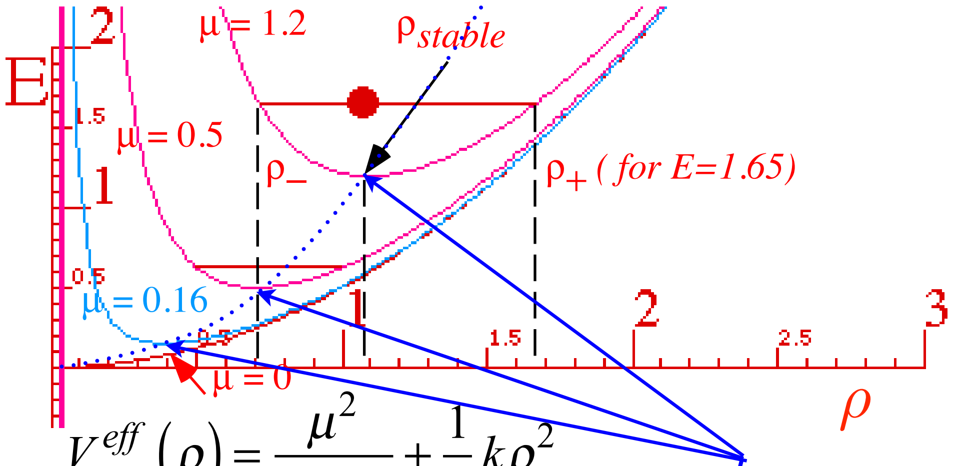
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

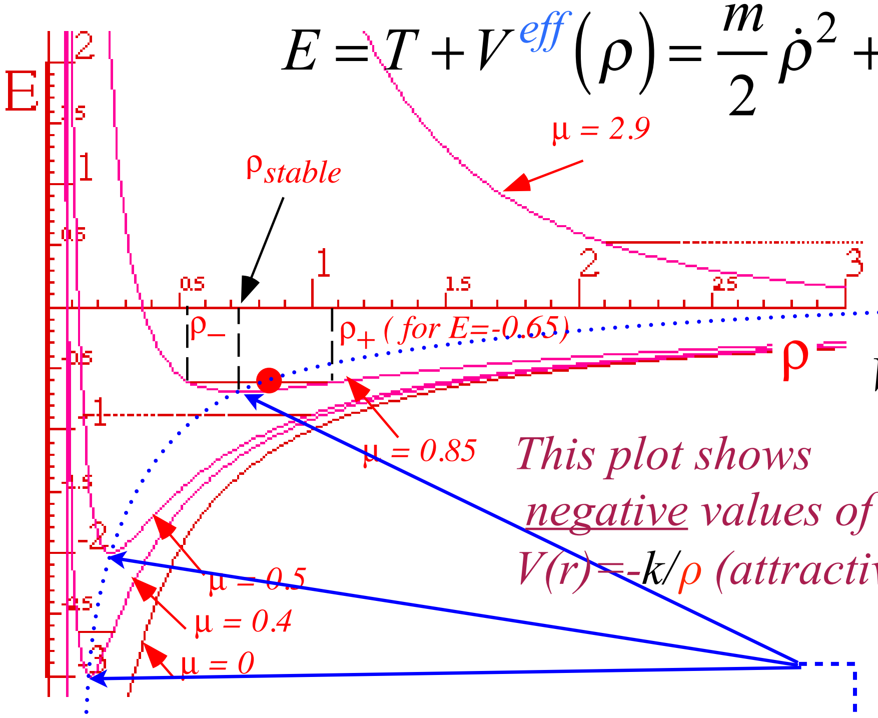
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \quad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}-\frac{k}{\rho_{stable}{}^3}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}{}^2}=\sqrt{\dfrac{k}{m}}$

$\omega_{\rho_{stable}}:\omega_\phi=2:1$

...angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}{}^2}=\dfrac{\mu}{m}\dfrac{m^2k^2}{\mu^4}=\dfrac{mk^2}{\mu^3}$

$\omega_{\rho_{stable}}:\omega_\phi=1:1$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - ➡ *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

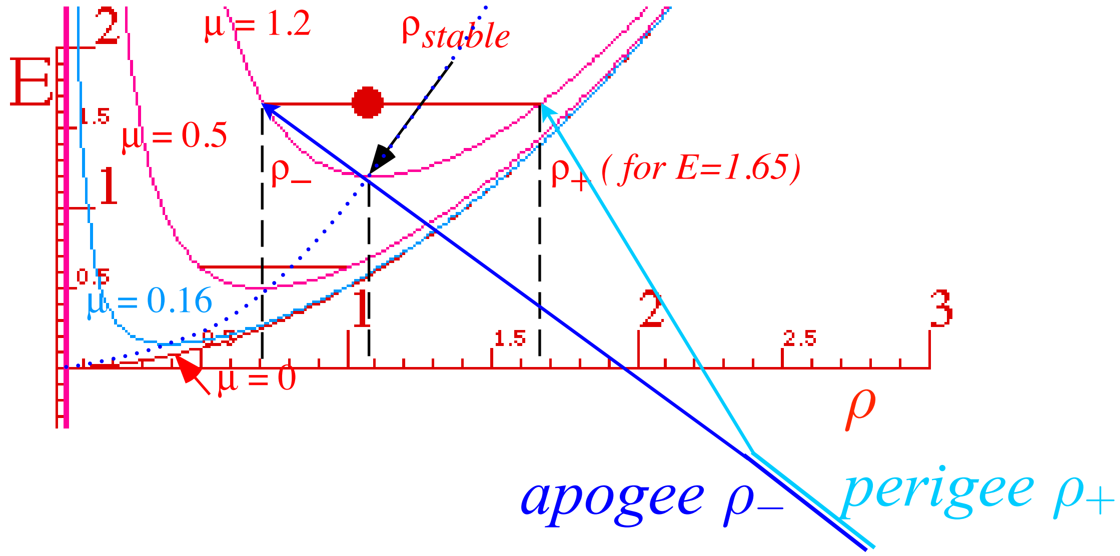
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

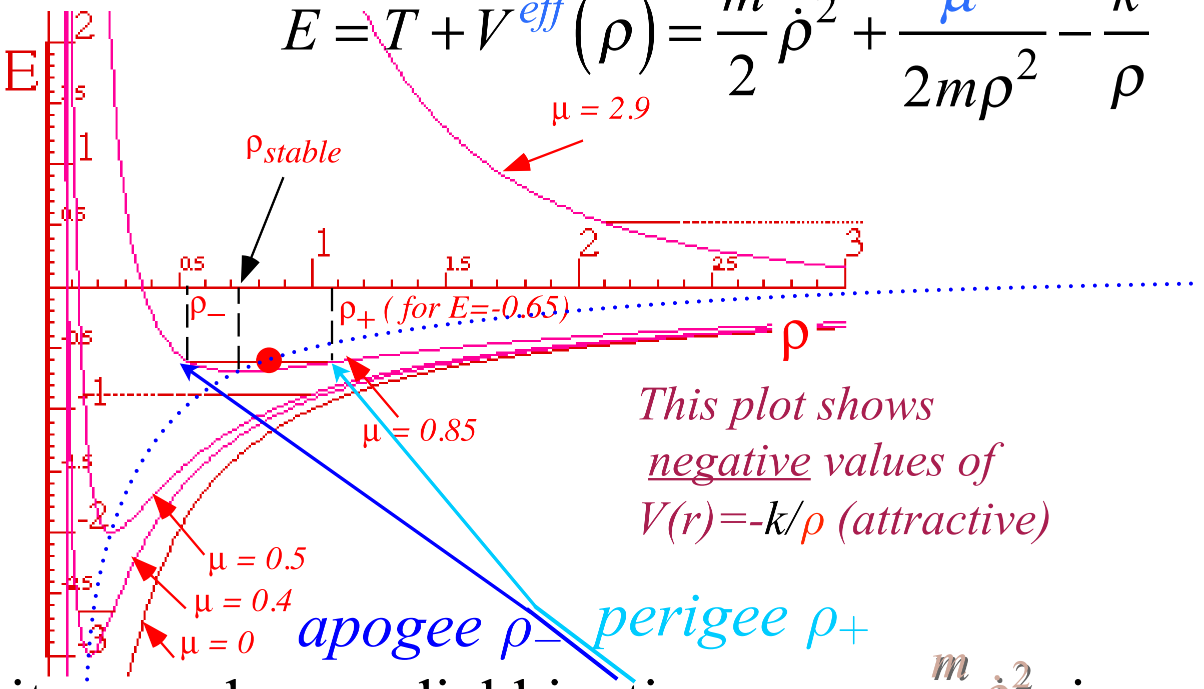
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



apogee ρ_- *perigee* ρ_+

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



This plot shows negative values of $V(r)=-k/\rho$ (attractive)

apogee ρ_- *perigee* ρ_+

Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: }p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu\qquad\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

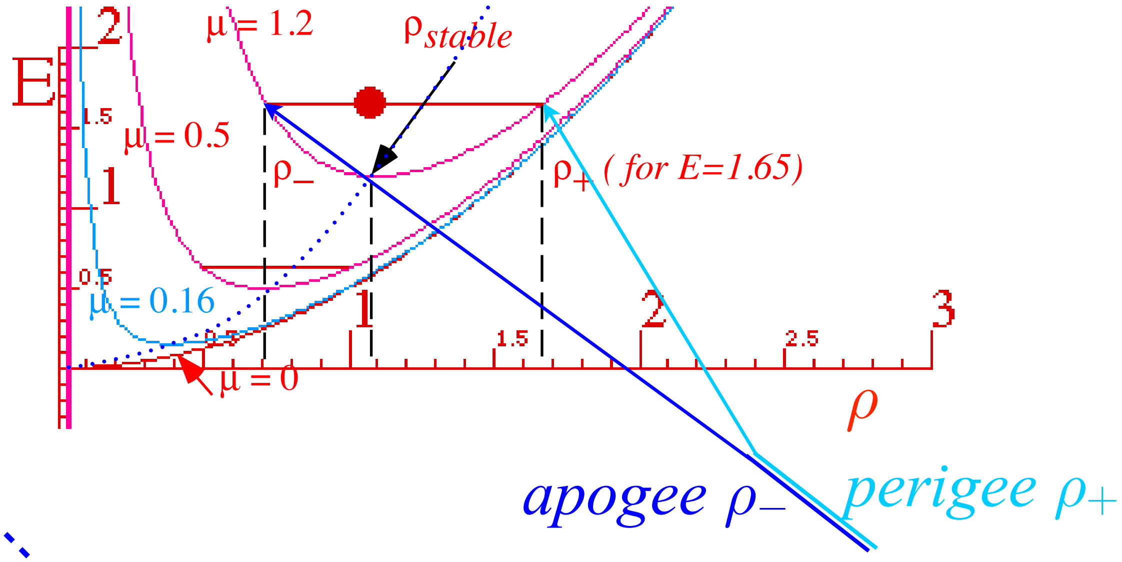
Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

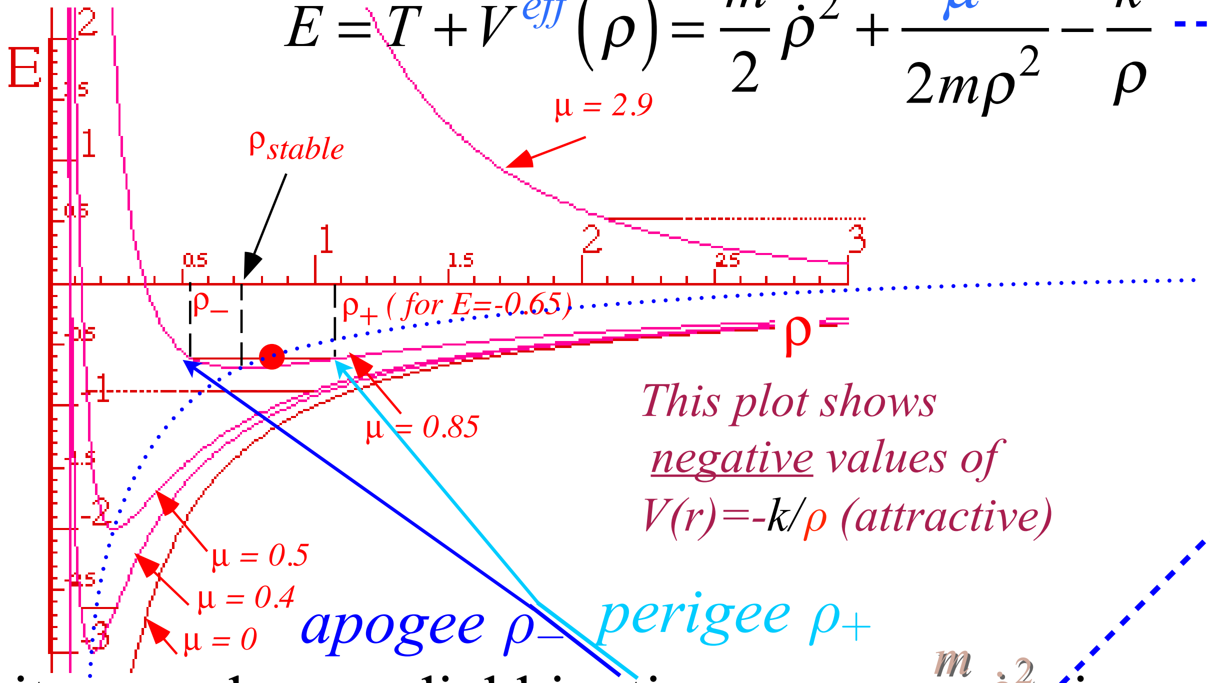
Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$





This plot shows negative values of V(r)=-k/ρ (attractive)

Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

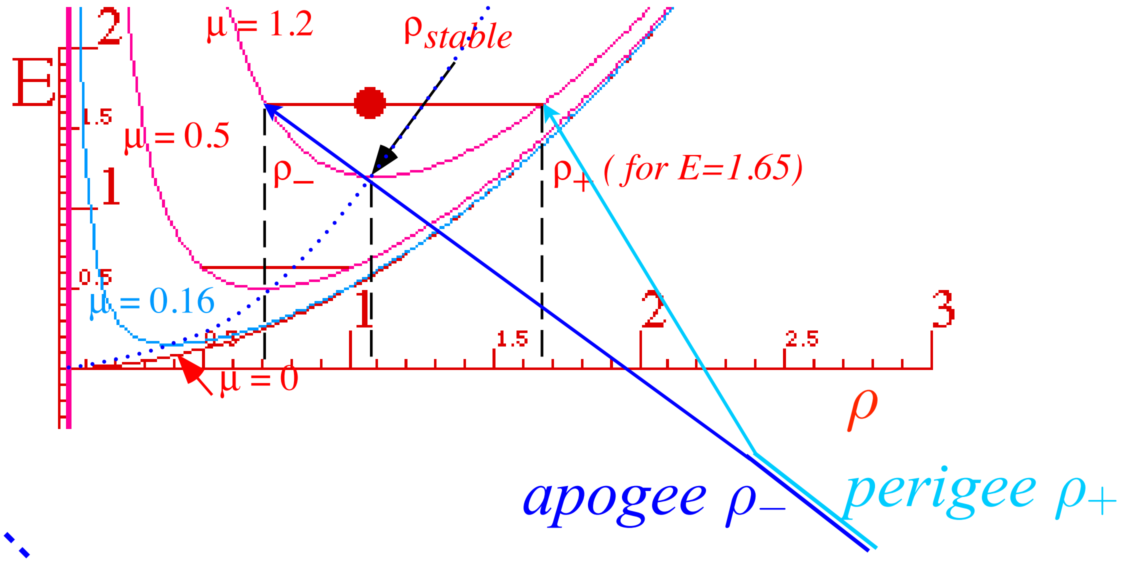
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

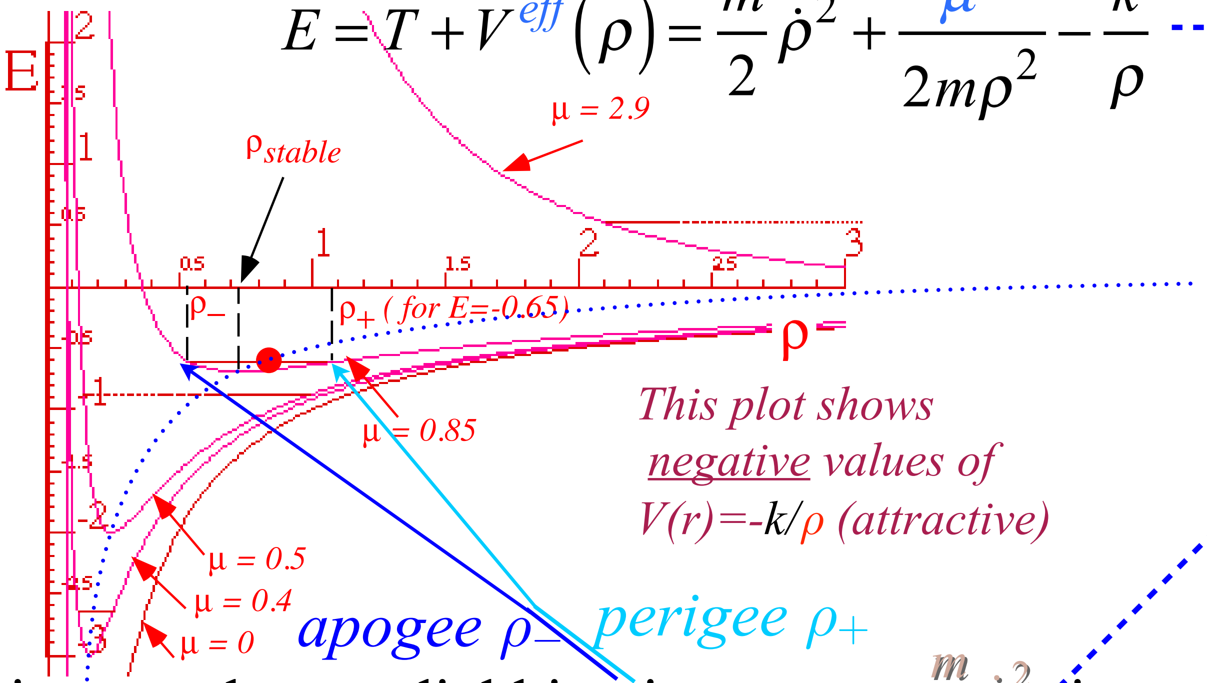
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



apogee ρ_- perigee ρ_+

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



This plot shows negative values of V(r)=-k/ρ (attractive)

apogee ρ_- perigee ρ_+

Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \quad \text{or else: } 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

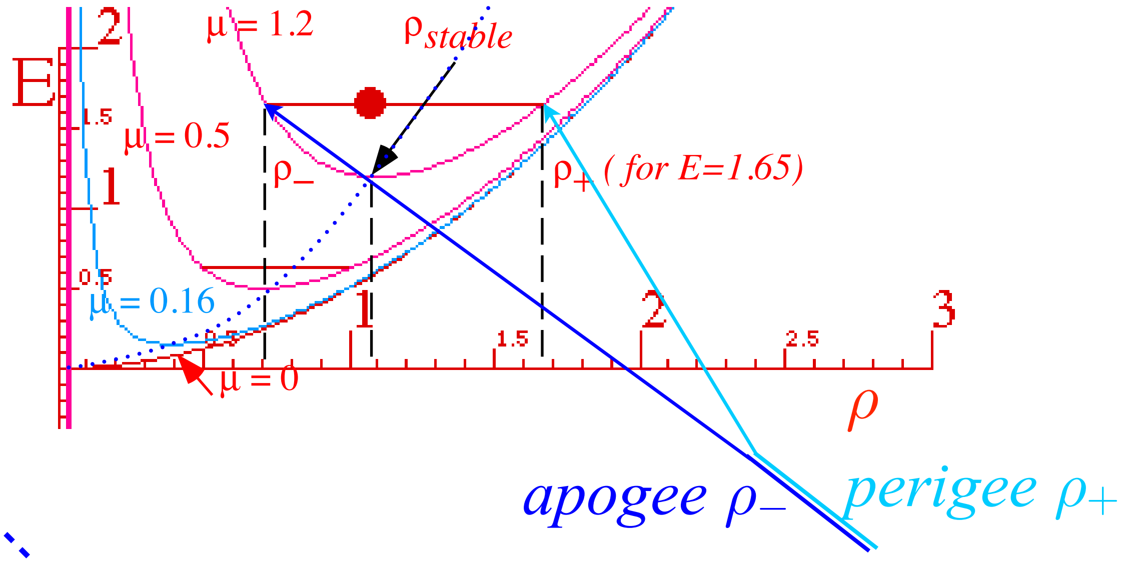
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

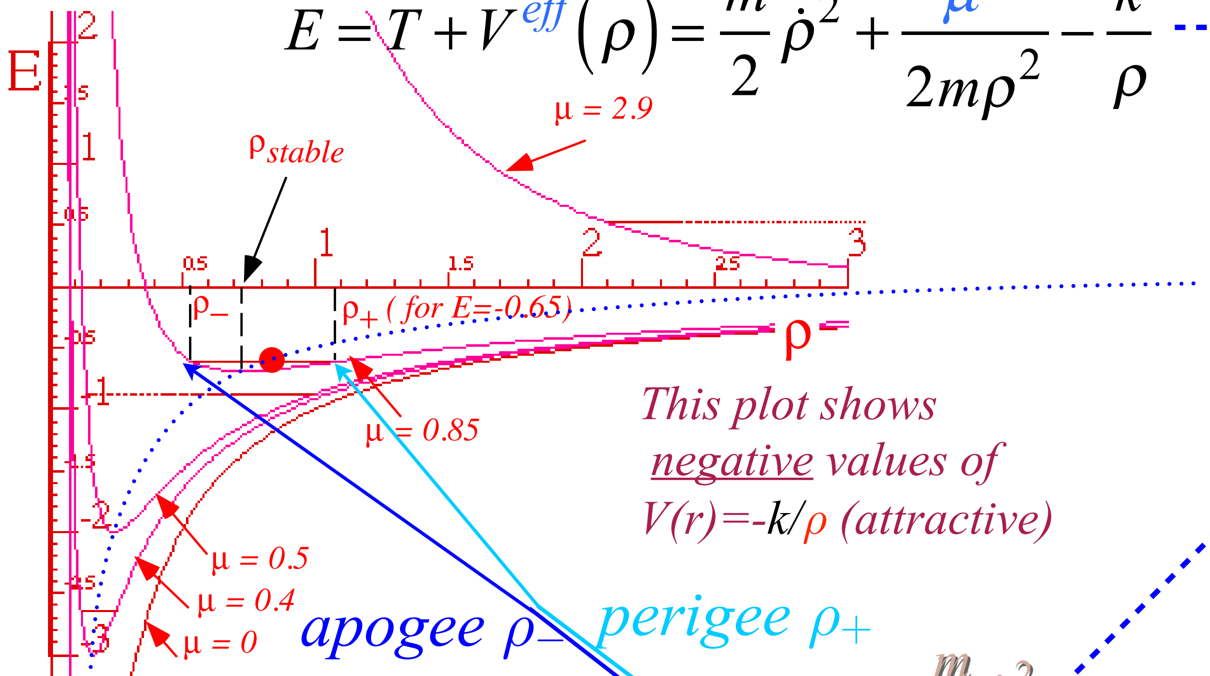
Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



This plot shows negative values of V(r)=-k/ρ (attractive)

Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad\qquad 0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \quad \text{or else: } 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2} \qquad 0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0 \quad \text{or else: } 0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu \qquad \boxed{\dot{\phi} = \frac{\mu}{m\rho^2}}$$

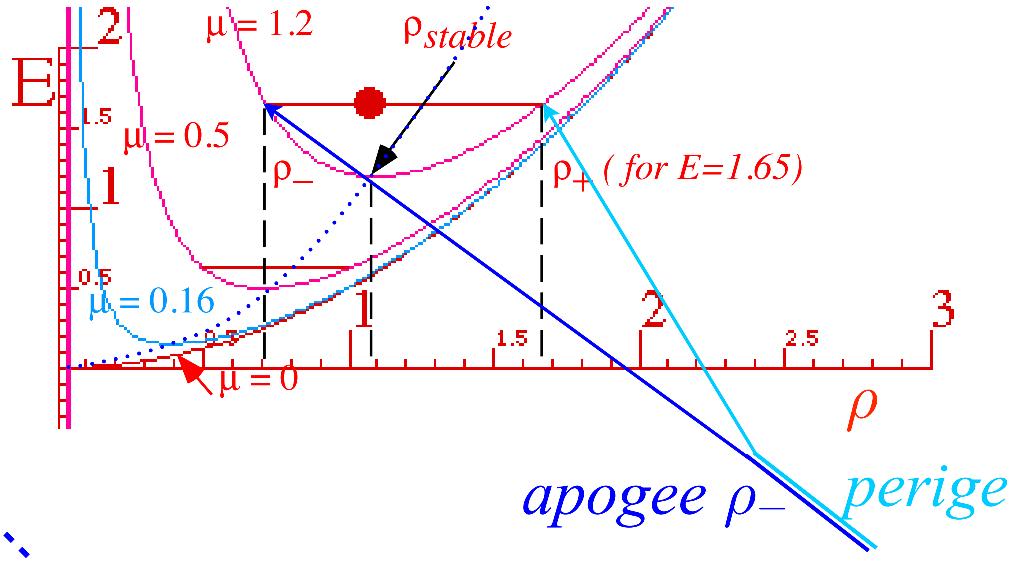
For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

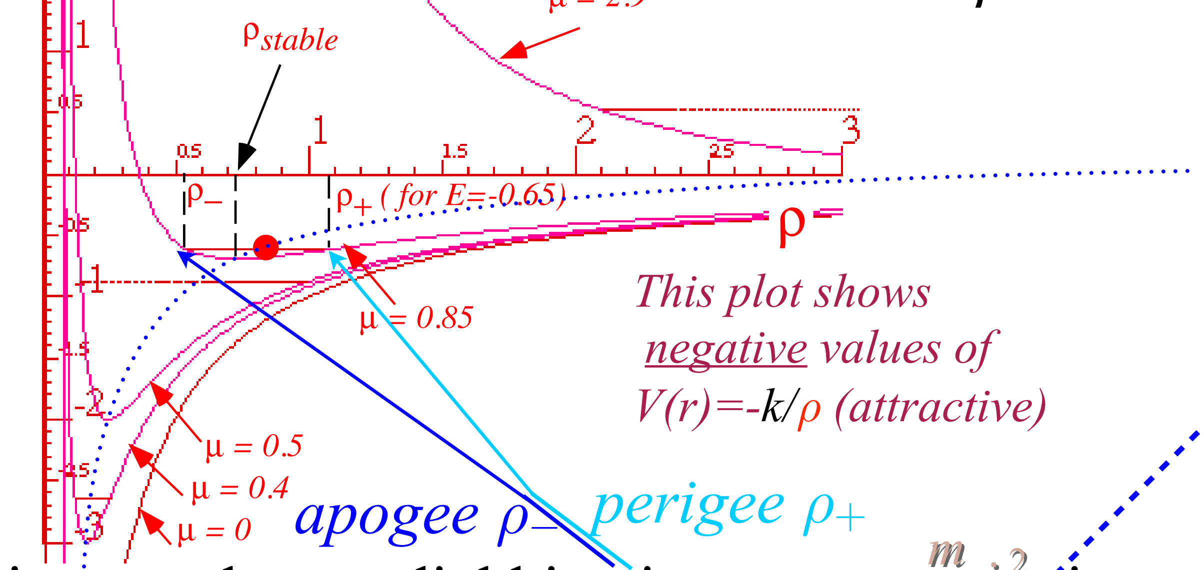
Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$



$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$



Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0 = -E + \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$

$$0 = \frac{\mu^2}{2m} - E\rho^2 + \frac{k}{2}\rho^4 \quad \text{or else: } 0 = \frac{\mu^2}{2m}\frac{1}{\rho^4} - E\frac{1}{\rho^2} + \frac{k}{2}$$

$${\rho_\pm}^2 = \frac{E \pm \sqrt{E^2 - k\mu^2/m}}{k} \quad \text{or else: } \frac{1}{{\rho_\pm}^2} = \frac{E \mp \sqrt{E^2 - k\mu^2/m}}{\mu^2/m}$$

$$0 = -E + \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$

$$0 = \frac{-\mu^2}{2m} + k\rho + E\rho^2 0 \quad \text{or else: } 0 = \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho} - E$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

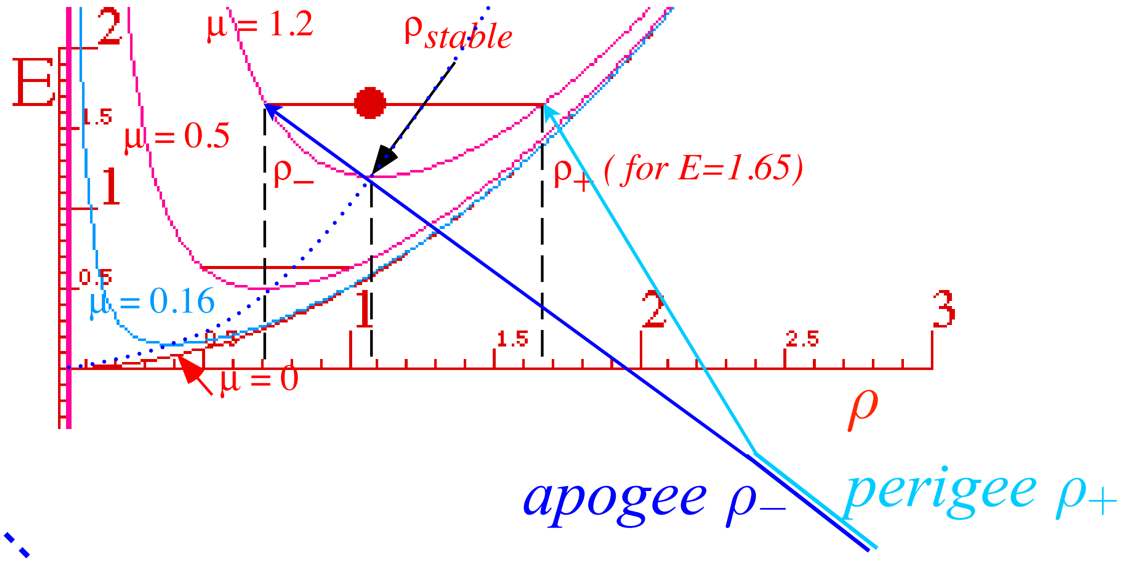
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: }\; p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

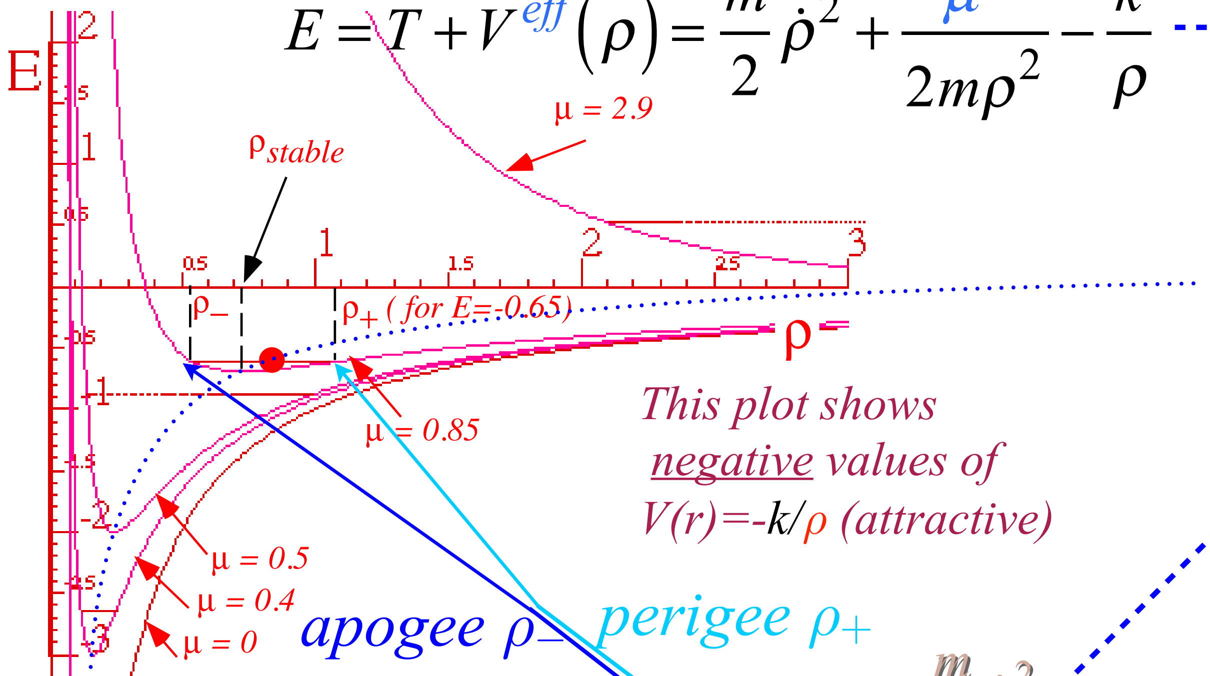
Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



This plot shows negative values of V(r)=-k/ρ (attractive)

Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad\qquad 0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4\quad\text{or else: }\; 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2} \qquad 0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0\quad\text{or else: }\; 0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

$$\rho_\pm{}^2=\frac{E\pm\sqrt{E^2-k\mu^2/m}}{k}\quad\text{or else: }\;\frac{1}{\rho_\pm{}^2}=\frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m} \qquad \rho_\pm=\frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}\quad\text{or else: }\;\frac{1}{\rho_\pm}=\frac{k\pm\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

Notice mysterious similarity: E→k and k→2E

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - ➔ *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\frac{d\phi}{dt}\frac{dt}{d\rho}=\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$

For ALL central forces

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$\frac{d\phi}{dt}\frac{dt}{d\rho}=\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu\qquad\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

$$d\phi=\frac{\mu d\rho}{m\rho^2\dot{\rho}}$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

For ALL central forces

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$\text{Let: }\frac{1}{\rho}=u\quad\text{so: }\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$

For ALL central forces

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu\qquad\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

$$\text{Let: }\frac{1}{\rho}=u\quad\text{so: }\begin{cases}-\dfrac{d\rho}{\rho^2}=du\\ d\rho=-\dfrac{du}{u^2}\end{cases}$$

$$\text{Let: }x=u^2=\frac{1}{\rho^2}\quad\text{so: }\begin{cases}dx=2udu\\ du=\dfrac{dx}{2\sqrt{x}}\end{cases}$$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$

For ALL central forces

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{k}{mx}}}$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

Let: $x=u^2=\frac{1}{\rho^2}$ so: $\begin{cases}dx=2udu\\ du=\frac{dx}{2\sqrt{x}}\end{cases}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{2k\sqrt{x}}{m}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ *For ALL central forces*

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{k}{mx}}}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

Let: $x=u^2=\frac{1}{\rho^2}$ so: $\begin{cases}dx=2u\,du\\ du=\frac{dx}{2\sqrt{x}}\end{cases}$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{2k\sqrt{x}}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - ➡ *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

Defining α and β:

$$z_\pm=\alpha\pm\beta$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ $\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

Defining α and β:

$z_\pm=\alpha\pm\beta$, where: $\alpha=\frac{z_++z_-}{2}=\frac{-B}{2A}$, and: $\beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$

Solution based on quadratic roots of $Az^2+Bz+C=0$.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: }p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu\qquad\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

Defining α and β:

$$z_\pm=\alpha\pm\beta\text{ , where: }\alpha=\frac{z_++z_-}{2}=\frac{-B}{2A}\text{ , and: }\beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

Solution based on quadratic roots of $Az^2+Bz+C=0$.

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ $\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-\left(z-z_+\right)\left(x-x_-\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\left(z_+-z\right)\left(z-z_-\right)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

Defining α and β:

$z_\pm=\alpha\pm\beta$, where: $\alpha=\frac{z_++z_-}{2}=\frac{-B}{2A}$, and: $\beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$

Solution based on quadratic roots of $Az^2+Bz+C=0$. Variable z may be ρ or $u=1/\rho$ or ρ^2 or $x=1/\rho^2$...

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

$$z(\phi)=\frac{\sqrt{B^2-4AC}}{2A}\sin\frac{\sqrt{A}}{D}\phi-\frac{B}{2A}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For <u>ALL</u> central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for <u>constant</u> parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

Defining α *and* β*:*

$$z_\pm=\alpha\pm\beta \text{ , where: } \alpha=\frac{z_++z_-}{2}=\frac{-B}{2A} \text{ , and: } \beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

Solution based on quadratic roots of $Az^2+Bz+C=0$. Variable z may be ρ or $u=1/\rho$ or ρ^2 or $x=1/\rho^2$...

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

$$z(\phi)=\frac{\sqrt{B^2-4AC}}{2A}\sin\frac{\sqrt{A}}{D}\phi-\frac{B}{2A}$$

$$z(\phi)=\beta\cdot\sin\frac{\sqrt{A}}{D}\phi-\alpha$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

$$z_\pm=\alpha\pm\beta \text{ , where: } \alpha=\frac{z_++z_-}{2}=\frac{-B}{2A} \text{ , and: } \beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

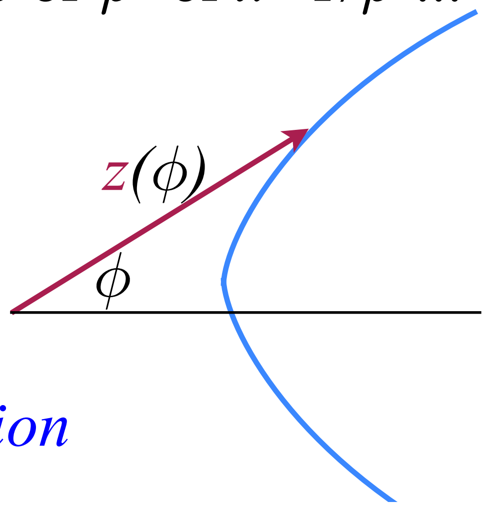
Solution based on quadratic roots of $Az^2+Bz+C=0$. Variable z may be ρ or $u=1/\rho$ or ρ^2 or $x=1/\rho^2$...

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

$$z(\phi)=\frac{\sqrt{B^2-4AC}}{2A}\sin\frac{\sqrt{A}}{D}\phi-\frac{B}{2A}$$

$$\boxed{z(\phi)=\beta\cdot\sin\frac{\sqrt{A}}{D}\phi-\alpha}$$

radial-polar-coordinate orbit function



Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - ➤ *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$

For ALL central forces

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\frac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-\left(z-z_+\right)\left(z-z_-\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\left(z_+-z\right)\left(z-z_-\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-\left(z-\alpha\right)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\frac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A} \qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$

Let: $u=\frac{1}{\rho}$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A} \qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

$$A=\frac{\mu^2}{m^2} \quad B=-\frac{2E}{m} \quad C=\frac{k}{m} \quad D=-\frac{\mu}{2m}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$

For ALL central forces

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\frac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

$A=\frac{\mu^2}{m^2}$ $B=-\frac{2E}{m}$ $C=\frac{k}{m}$ $D=-\frac{\mu}{2m}$

$A=\frac{\mu^2}{m^2}$ $B=\frac{2k}{m}$ $C=-\frac{2E}{m}$ $D=-\frac{\mu}{m}$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$

For ALL central forces

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\frac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\;,\;\beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

$A=\frac{\mu^2}{m^2}$ $B=-\frac{2E}{m}$ $C=\frac{k}{m}$ $D=-\frac{\mu}{2m}$

$$\alpha=\frac{E}{\mu^2/m}$$

$A=\frac{\mu^2}{m^2}$ $B=\frac{2k}{m}$ $C=-\frac{2E}{m}$ $D=-\frac{\mu}{m}$

$$\alpha=\frac{-k}{\mu^2/m}\;,$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}\qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}\qquad \text{Let: } u=\frac{1}{\rho}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

$$\alpha=\frac{-k}{\mu^2/m}\ ,$$

$$\beta=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

Algebra details on following pages

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}\qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$

$$\text{Let: } u=\frac{1}{\rho}\qquad d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$\boxed{x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)}$$

Algebra details on following pages

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

$$\alpha=\frac{-k}{\mu^2/m}\ ,$$

$$\beta=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

$$\boxed{u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)}$$

Algebra details and checks

$$\alpha = \frac{-B}{2A}\ ,\ \beta = \frac{\sqrt{B^2 - 4AC}}{2A}$$

(ρ,ϕ) orbits for IHOscillator $V(\rho) = k\rho^2/2$

$$A = \frac{\mu^2}{m^2} \quad B = -\frac{2E}{m} \quad C = \frac{k}{m} \quad D = -\frac{\mu}{2m}$$

$$\alpha = \frac{\frac{2E}{m}}{2\frac{\mu^2}{m^2}} = \frac{E}{\mu^2/m} = \frac{z_+ + z_-}{2}$$

$$\beta = \frac{\sqrt{\left(\frac{2E}{m}\right)^2 - 4\frac{\mu^2}{m^2}\frac{k}{m}}}{2\frac{\mu^2}{m^2}}$$

$$\beta = \frac{\sqrt{\left(\frac{2Em}{m\cdot m}\right)^2 - 4\frac{\mu^2}{m^2}\frac{km}{m^2}}}{2\frac{\mu^2}{m^2}} = \frac{\sqrt{E^2 - k\mu^2/m}}{\mu^2/m} = \frac{z_+ - z_-}{2}$$

(ρ,ϕ) equations for Coulomb $V(\rho) = -k/\rho$

$$A = \frac{\mu^2}{m^2} \quad B = \frac{2k}{m} \quad C = -\frac{2E}{m} \quad D = -\frac{\mu}{m}$$

$$\alpha = \frac{-\frac{2k}{m}}{2\frac{\mu^2}{m^2}} = \frac{-k}{\mu^2/m} = \frac{z_+ + z_-}{2}$$

$$\beta = \frac{\sqrt{\left(\frac{2k}{m}\right)^2 + 4\frac{\mu^2}{m^2}\frac{2E}{m}}}{2\frac{\mu^2}{m^2}} = \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m} = \frac{z_+ - z_-}{2}$$

Checking that roots $z_\pm$ are *classical turning points* (*perigee* $z_- = \alpha - \beta$, *apogee* $z_+ = \alpha + \beta$) from p.27-29.

$$\rho_\pm{}^2 = \frac{E \pm \sqrt{E^2 - k\mu^2/m}}{k} \quad \text{or else:} \quad \frac{1}{\rho_\pm{}^2} = \frac{E \mp \sqrt{E^2 - k\mu^2/m}}{\mu^2/m}$$

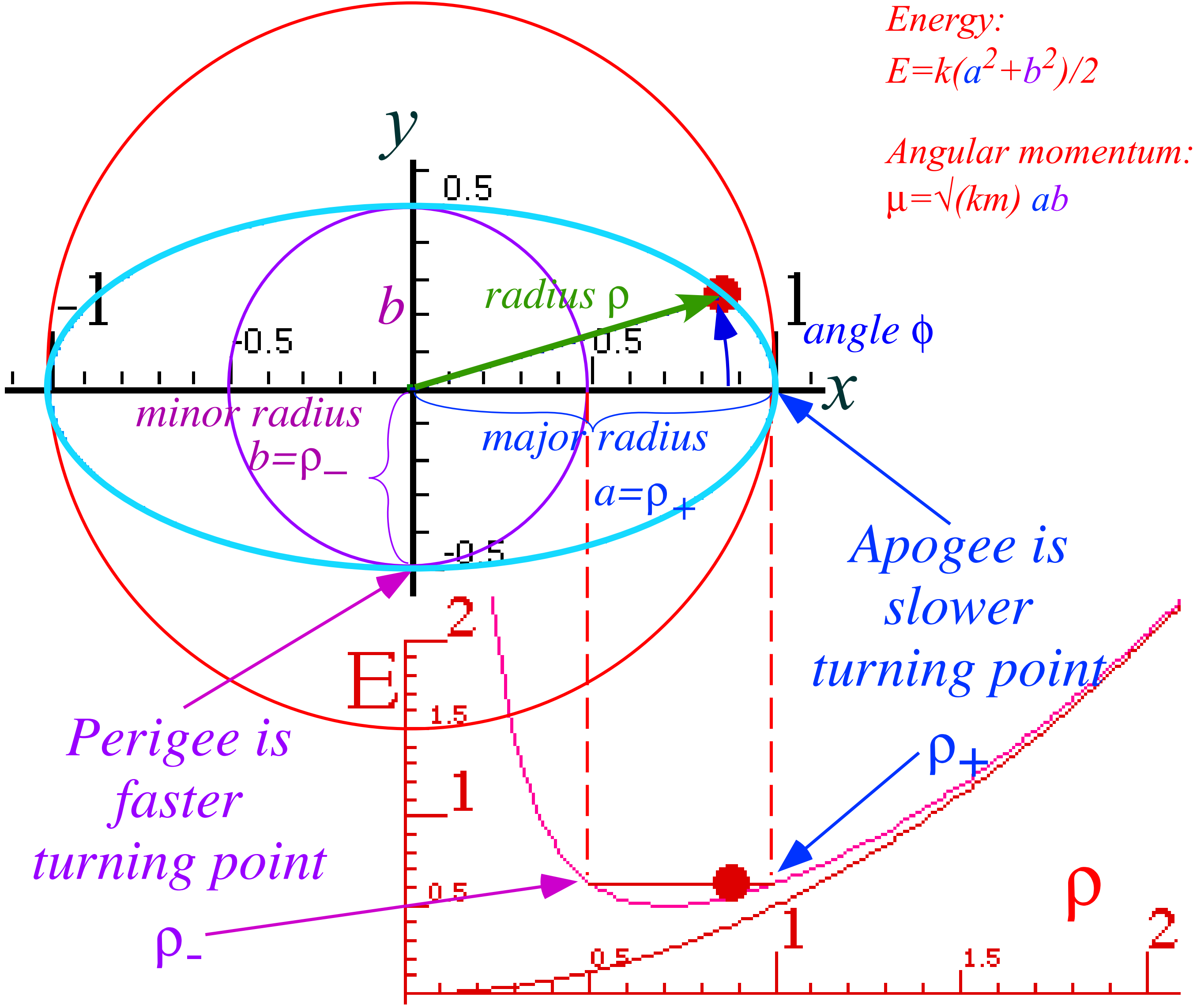
$$\rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E} \quad \text{or else:} \quad \frac{1}{\rho_\pm} = \frac{k \pm \sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - ➔ *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

Orbits in Isotropic Oscillator and Coulomb Potentials

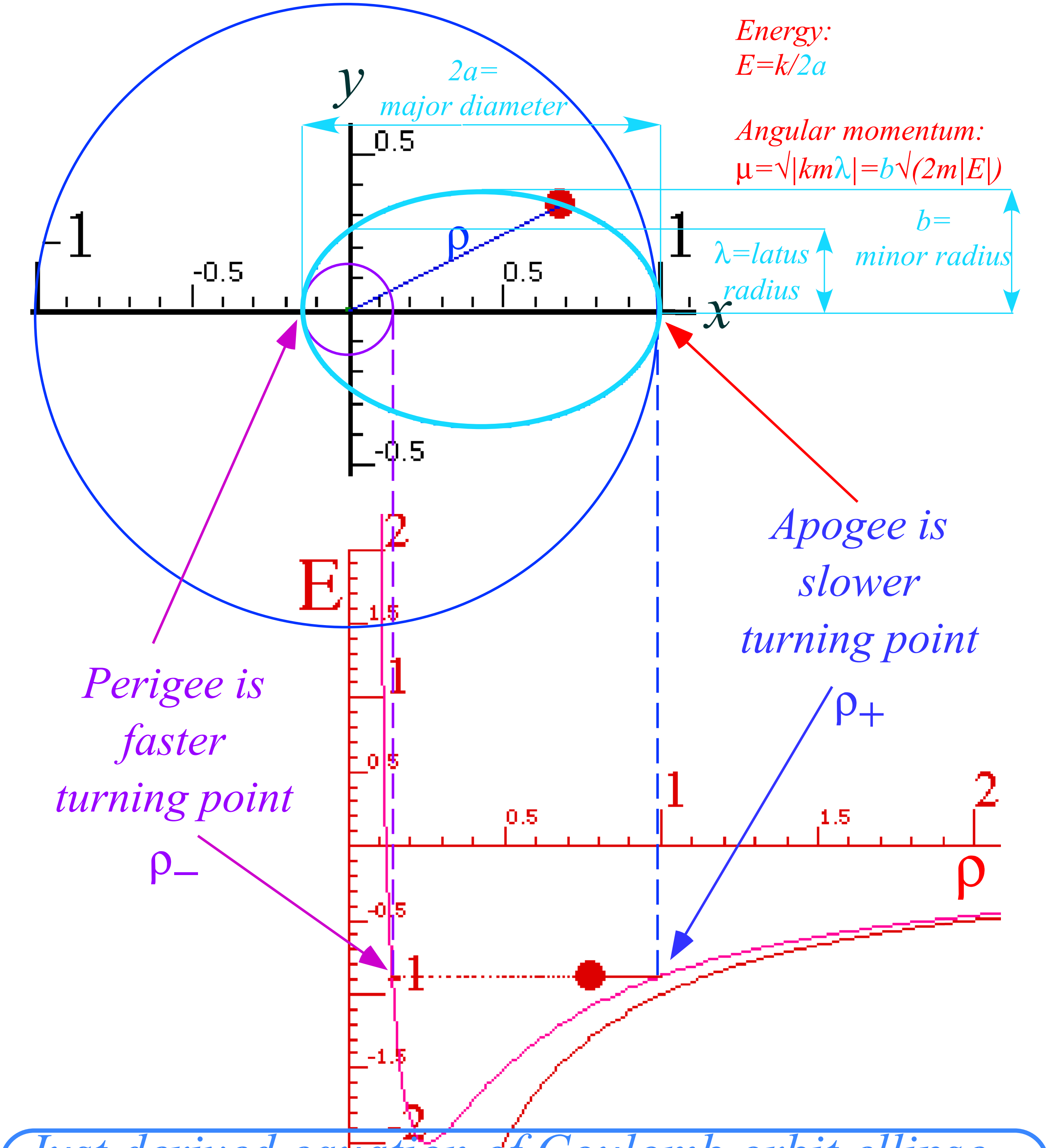
(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$



Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\,ab$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$



Energy:
$E=k/2a$

Angular momentum:
$\mu=\sqrt{|km\lambda|}=b\sqrt{(2m|E|)}$

Just derived equation of IHO orbit ellipse

$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

$$\frac{1}{\rho^2}=+\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos(2\phi)$$

One of many equations of center-centered ellipse

Just derived equation of Coulomb orbit ellipse

$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

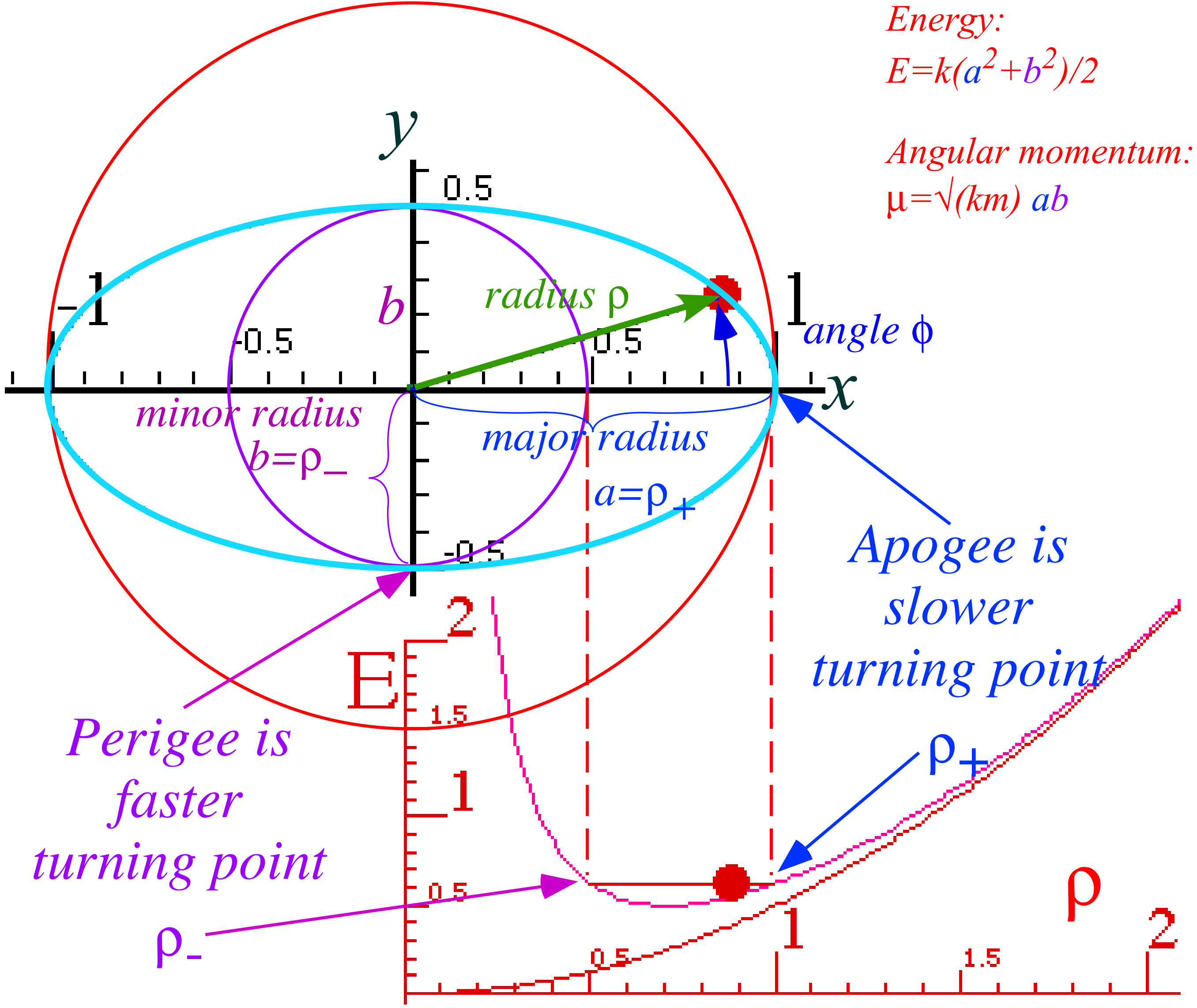
(to be discussed shortly)

$$\frac{1}{\rho}=\frac{a}{b^2}-\frac{\sqrt{a^2-b^2}}{b^2}\cos\phi$$

One of many equations of focus-centered ellipse

Orbits in Isotropic Oscillator and Coulomb Potentials

(ρ,ϕ) **orbits for IHOscillator** $V(\rho) = k\rho^2/2$



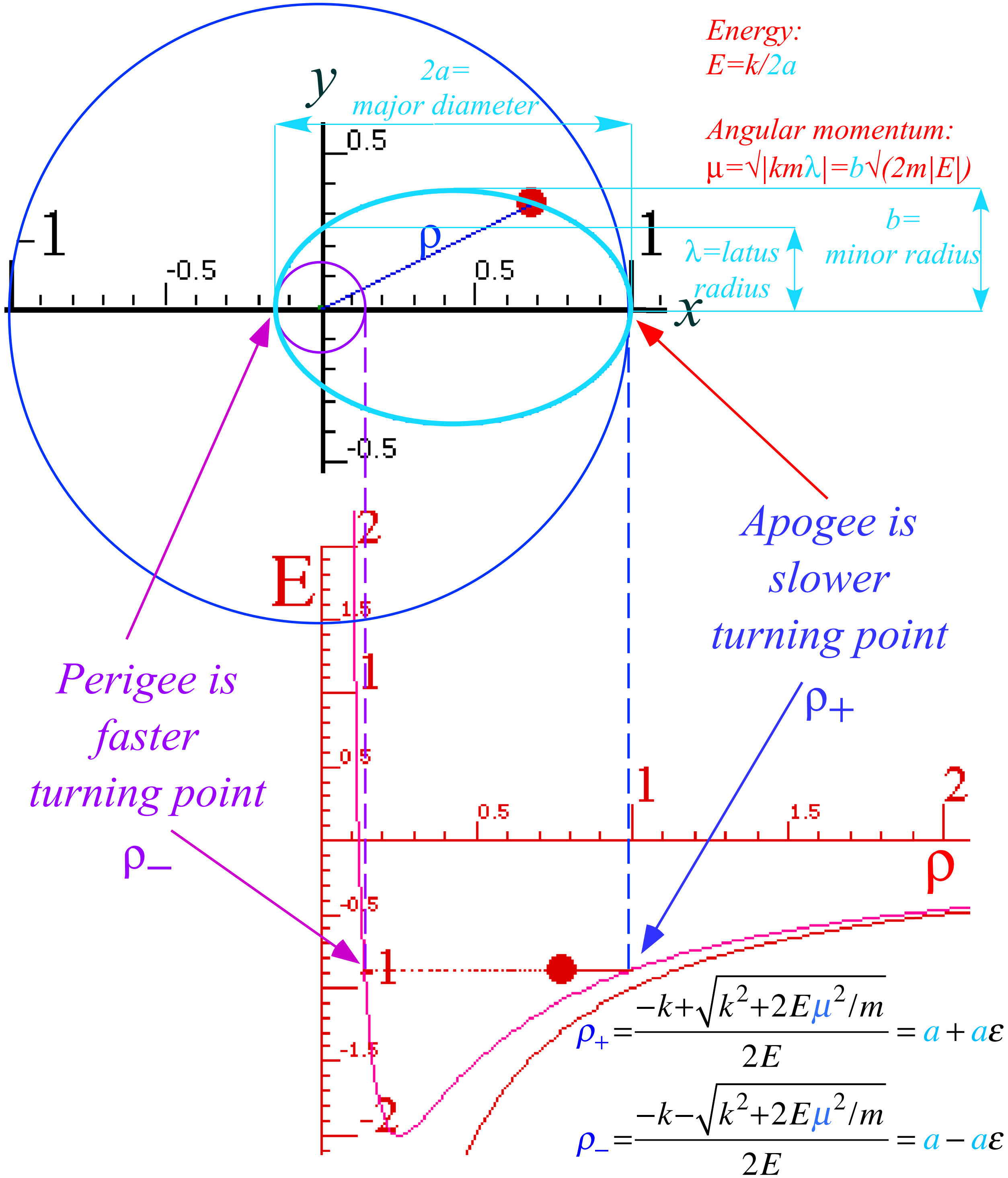
Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\,ab$

$$\rho_+{}^2 = \frac{E+\sqrt{E^2-k\mu^2/m}}{k} = a^2$$

$$\rho_-{}^2 = \frac{E-\sqrt{E^2-k\mu^2/m}}{k} = b^2$$

(ρ,ϕ) **orbits for Coulomb** $V(\rho) = -k/\rho$



Energy:
$E=k/2a$

Angular momentum:
$\mu=\sqrt{|km\lambda|}=b\sqrt{(2m|E|)}$

$$\rho_+ = \frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E} = a+a\varepsilon$$

$$\rho_- = \frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E} = a-a\varepsilon$$

(from p.29 or p.57)

(to be discussed first: turning point relations)

Just derived equation of IHO orbit ellipse

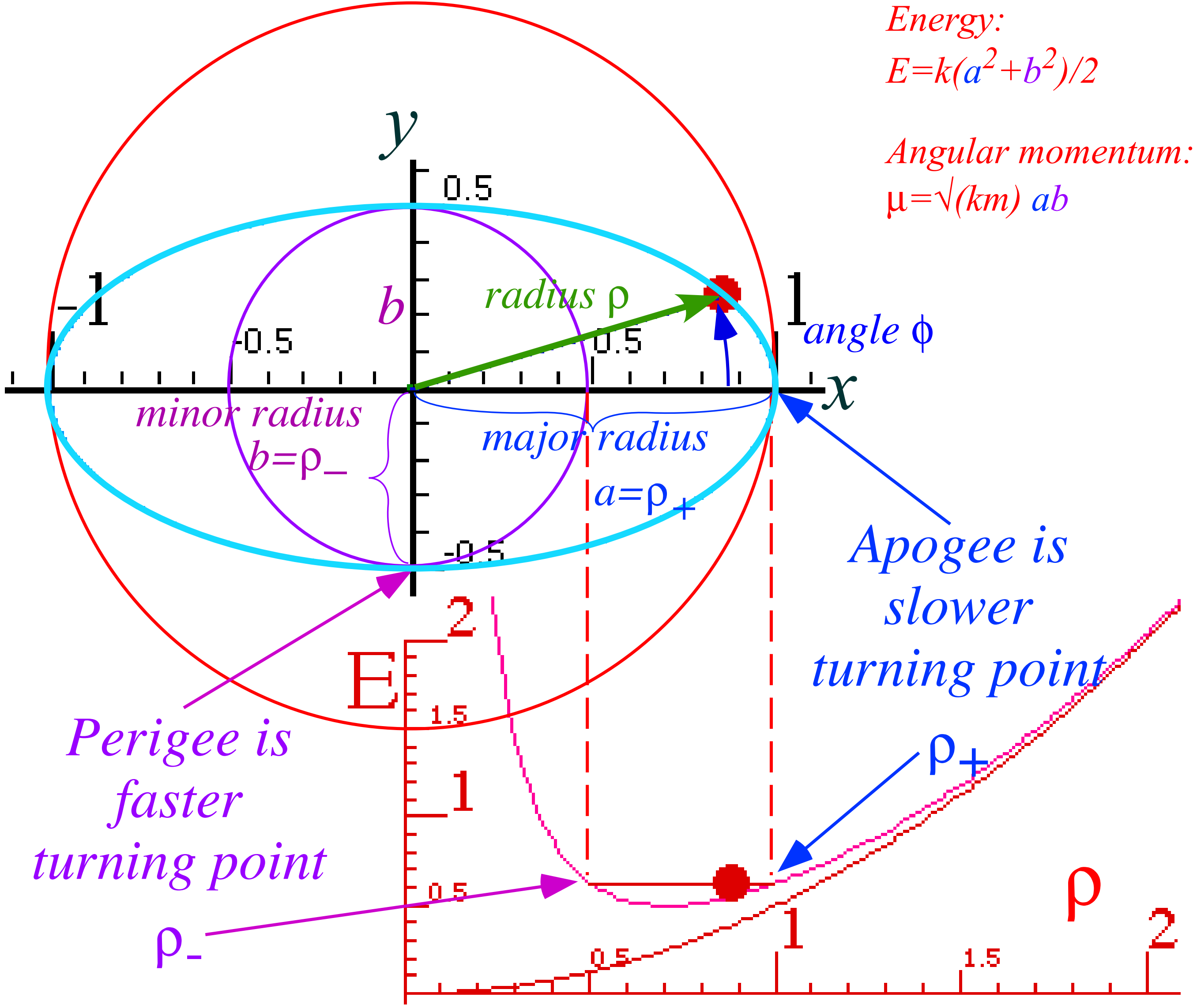
$$x = \frac{1}{\rho^2} = \frac{E}{\mu^2/m} + \frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

$$u = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Orbits in Isotropic Oscillator and Coulomb Potentials

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$



Energy:
$E=k(a^2+b^2)/2$

Angular momentum:
$\mu=\sqrt{(km)}\,ab$

Apogee is slower turning point

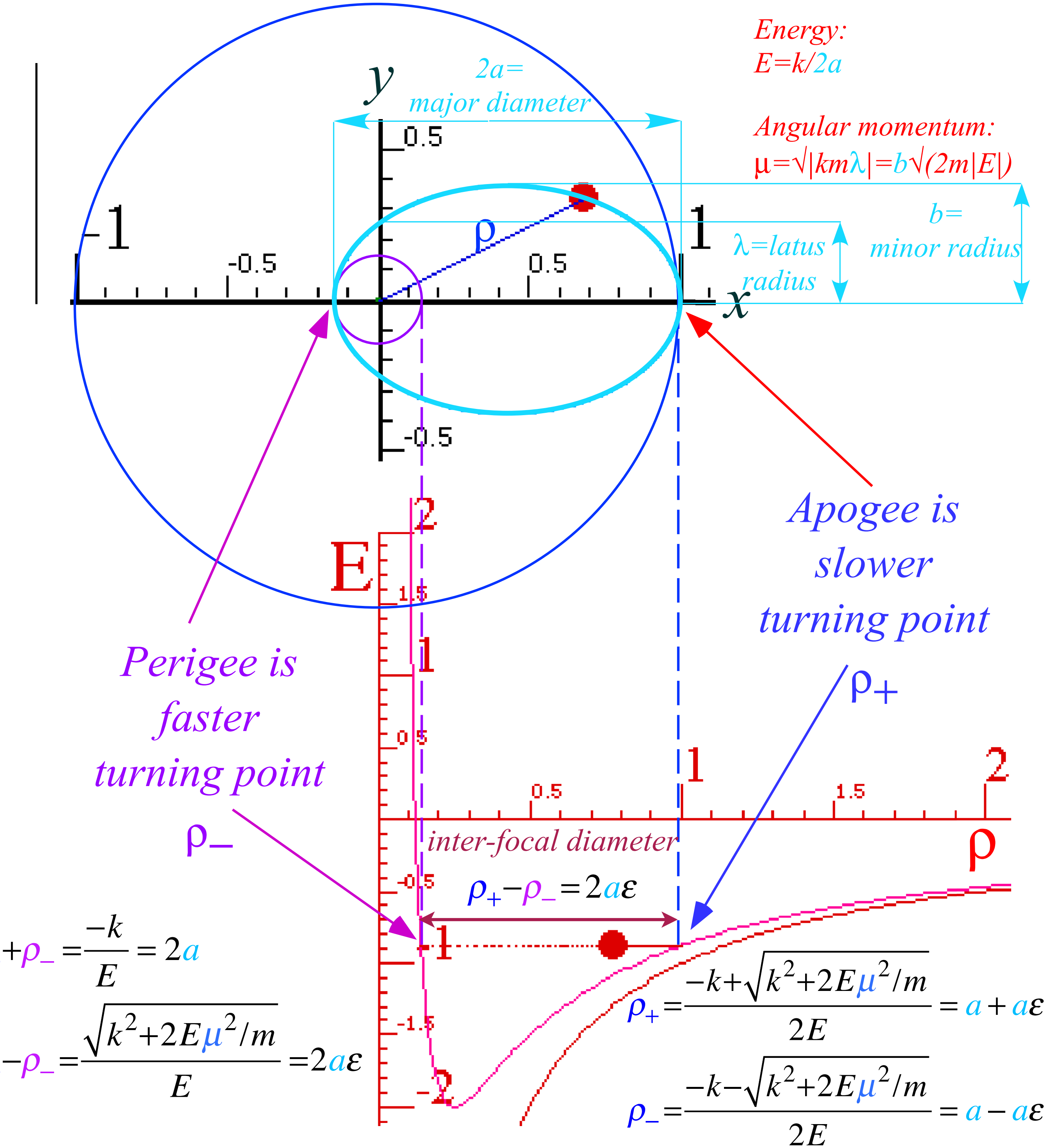
Perigee is faster turning point

$$\rho_+{}^2 = \frac{E+\sqrt{E^2-k\mu^2/m}}{k} = a^2 \qquad \rho_+{}^2+\rho_-{}^2 = \frac{2E}{k} = a^2+b^2$$

$$\rho_-{}^2 = \frac{E-\sqrt{E^2-k\mu^2/m}}{k} = b^2 \qquad \rho_+{}^2-\rho_-{}^2 = \frac{2\sqrt{E^2-k\mu^2/m}}{k} = a^2-b^2 = a\varepsilon$$

$$\rho_+{}^2\rho_-{}^2 = \frac{E^2-E^2-k\mu^2/m}{k^2} = a^2b^2 = \frac{-\mu^2}{km}$$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$



Energy:
$E=k/2a$

Angular momentum:
$\mu=\sqrt{|km\lambda|}=b\sqrt{(2m|E|)}$

Apogee is slower turning point

Perigee is faster turning point

$$\rho_+ = \frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E} = a+a\varepsilon$$

$$\rho_- = \frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E} = a-a\varepsilon$$

$$\rho_++\rho_- = \frac{-k}{E} = 2a$$

$$\rho_+-\rho_- = \frac{\sqrt{k^2+2E\mu^2/m}}{E} = 2a\varepsilon$$

$$\rho_+\rho_- = \frac{k^2-k^2-2E\mu^2/m}{(2E)^2} = a^2-a^2\varepsilon^2 = \frac{-\mu^2}{2Em} = b^2$$ (from p.29 or p.57)

(to be discussed first: turning point relations)

Just derived equation of IHO orbit ellipse

$$x = \frac{1}{\rho^2} = \frac{E}{\mu^2/m} + \frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

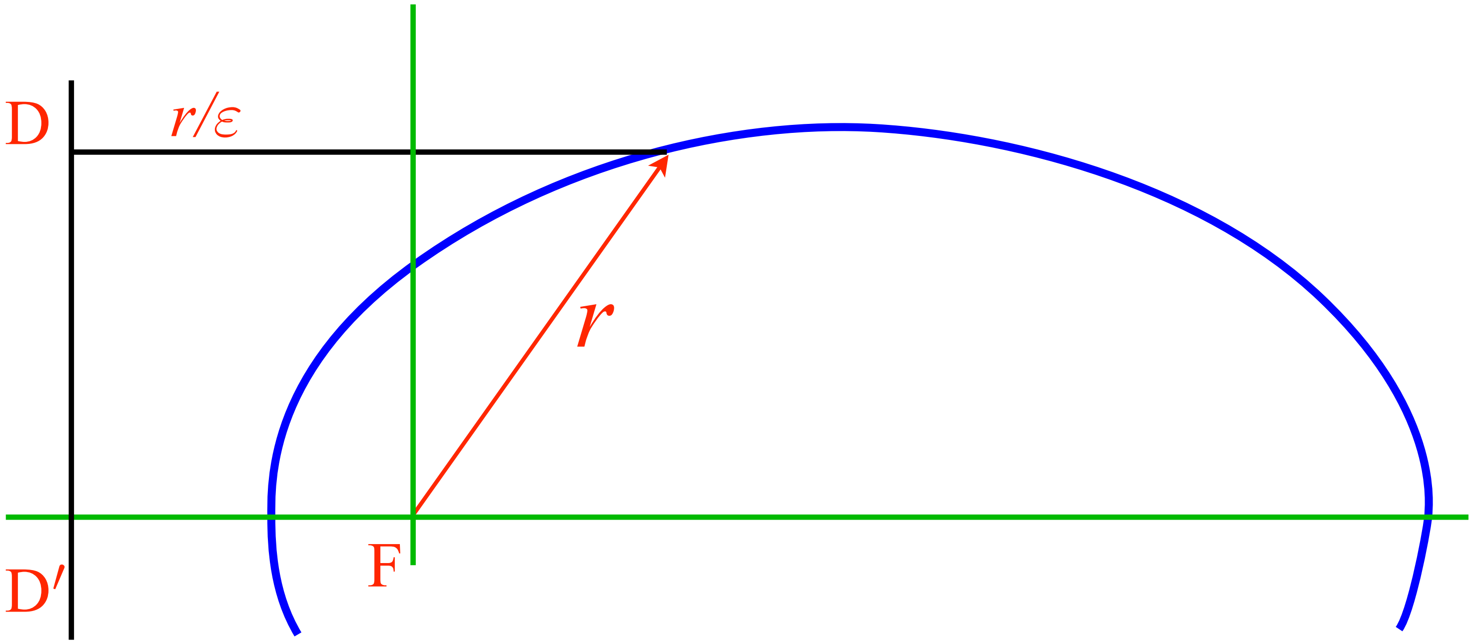
Just derived equation of Coulomb orbit ellipse

$$u = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - ➔ *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

Geometry of ALL Coulomb conic section orbits (Let:r =ρ here)

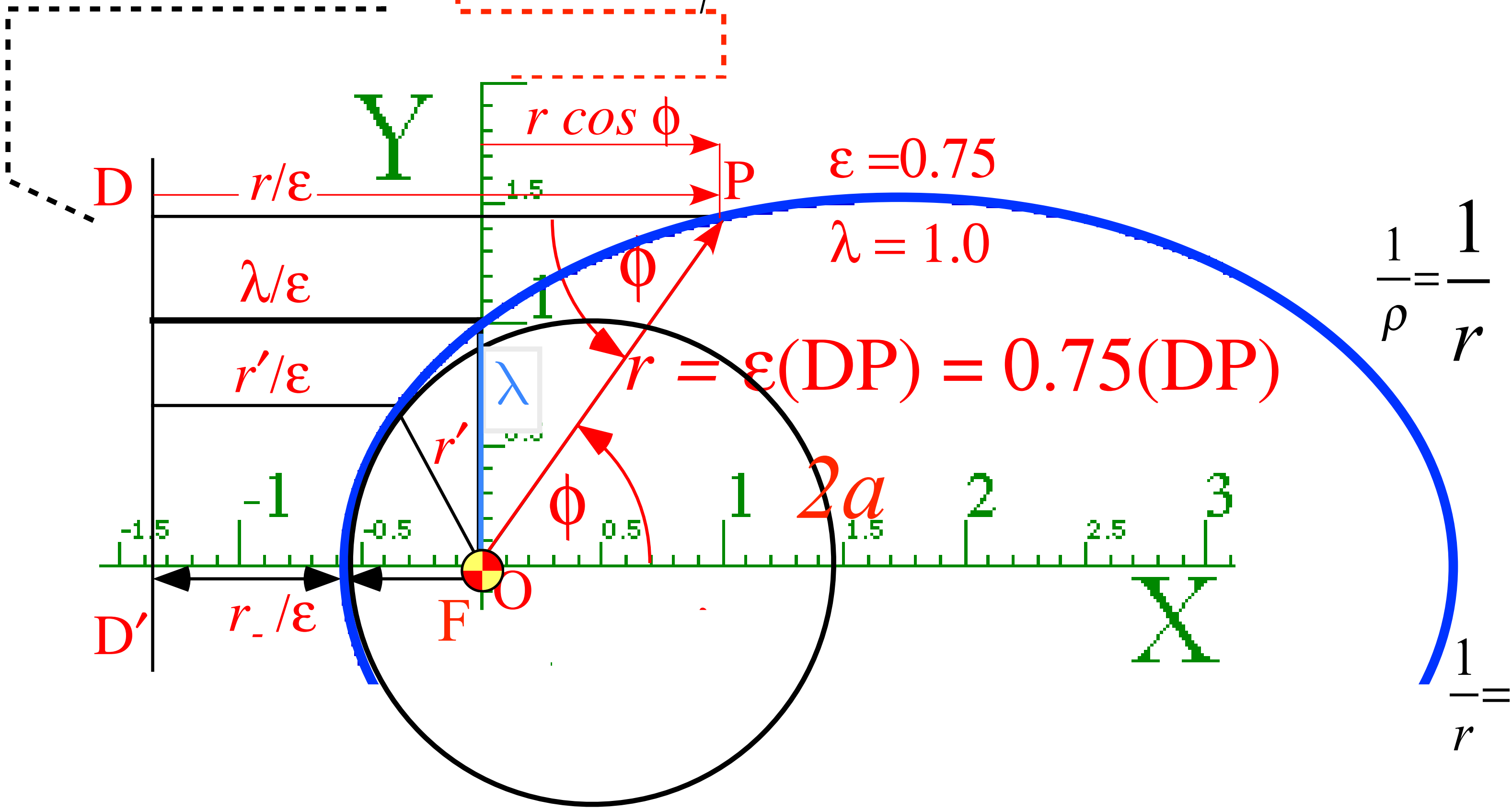


All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.59 physics:

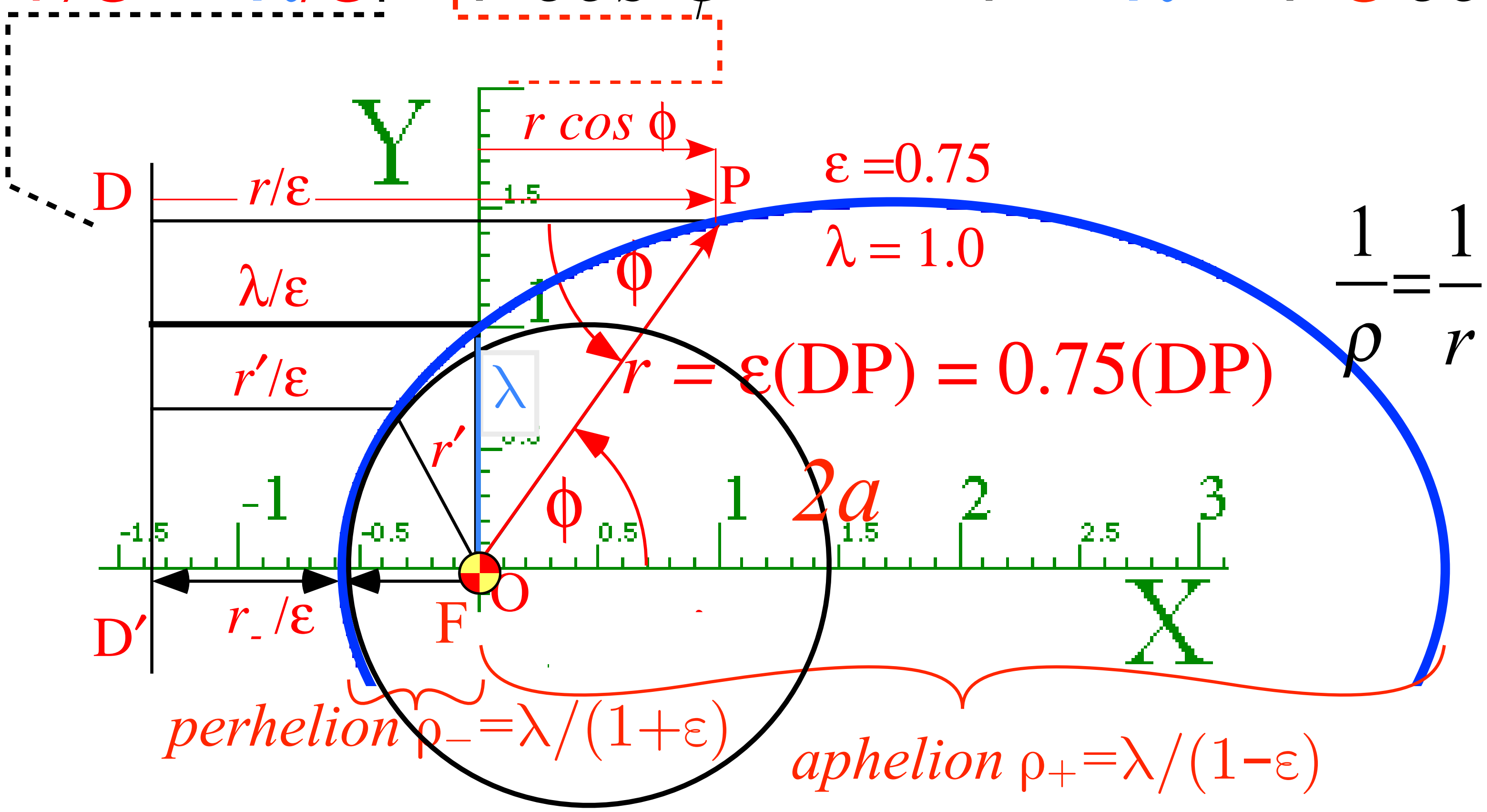
$$\frac{1}{r} = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let:r≡ρ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho}=\frac{1}{r}=\frac{1-\varepsilon\cos\phi}{\lambda}=\frac{1}{\lambda}-\frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

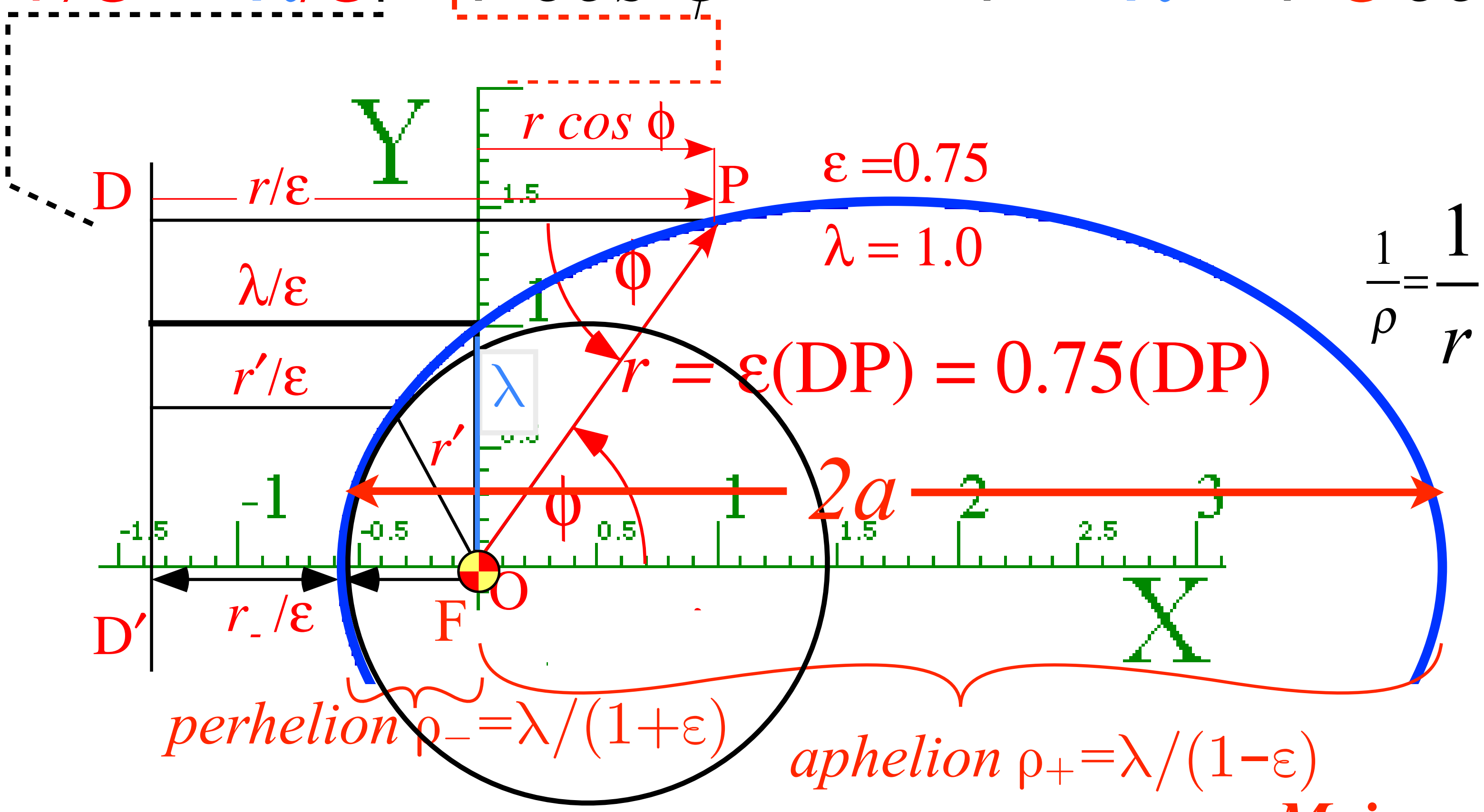
$$r_\pm = \rho_\pm = \frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** *ε*

Distance to *Focus* **F** = *ε*·Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let:$r\equiv\rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



$r = \varepsilon(DP) = 0.75(DP)$

perhelion $\rho_- = \lambda/(1+\varepsilon)$

aphelion $\rho_+ = \lambda/(1-\varepsilon)$

By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_\pm = \rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

Major axis: $\rho_+ + \rho_- = 2a$

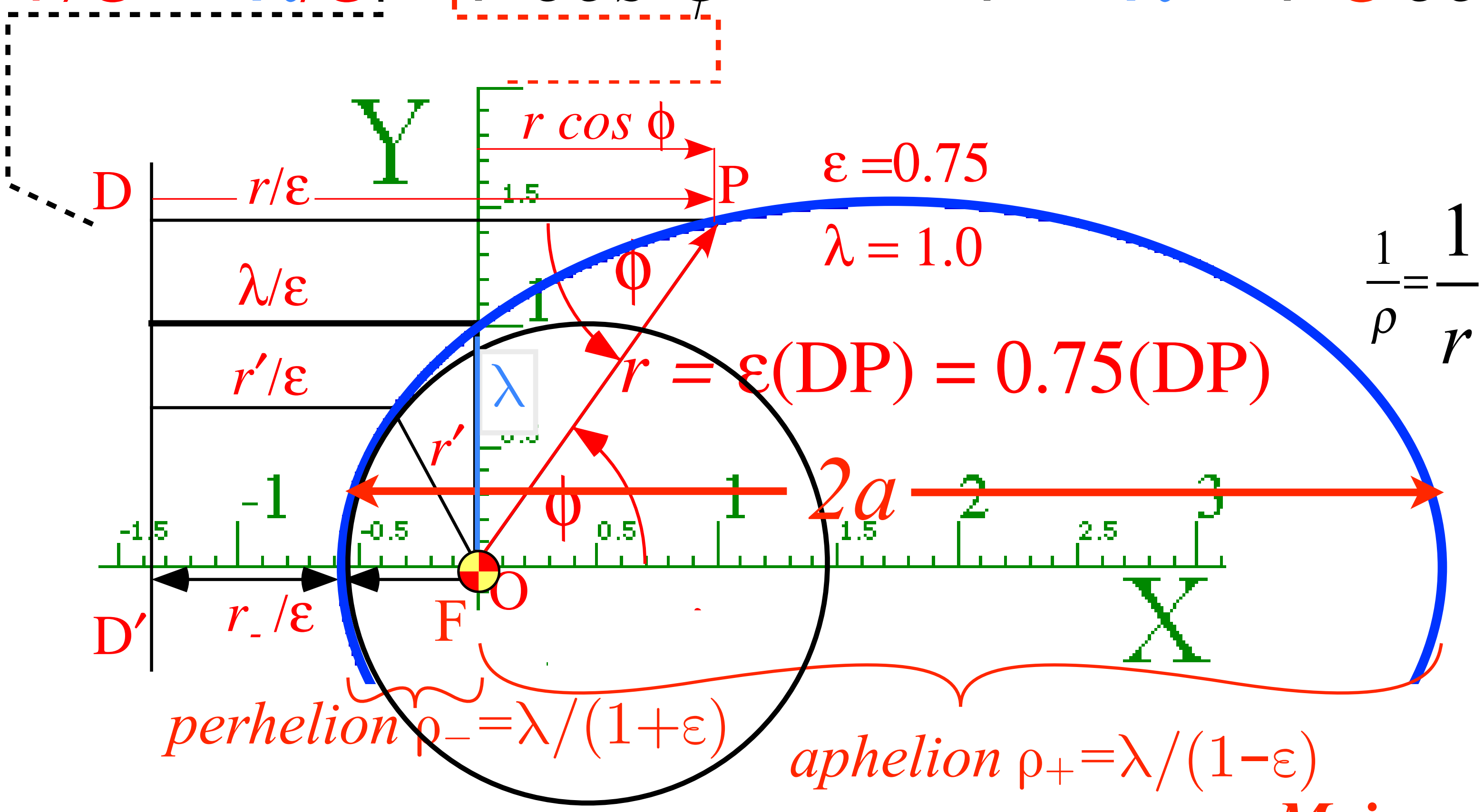
$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let:$r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_\pm = \rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

***Major axis**:* $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

All conics defined by: ***Eccentricity*** ε

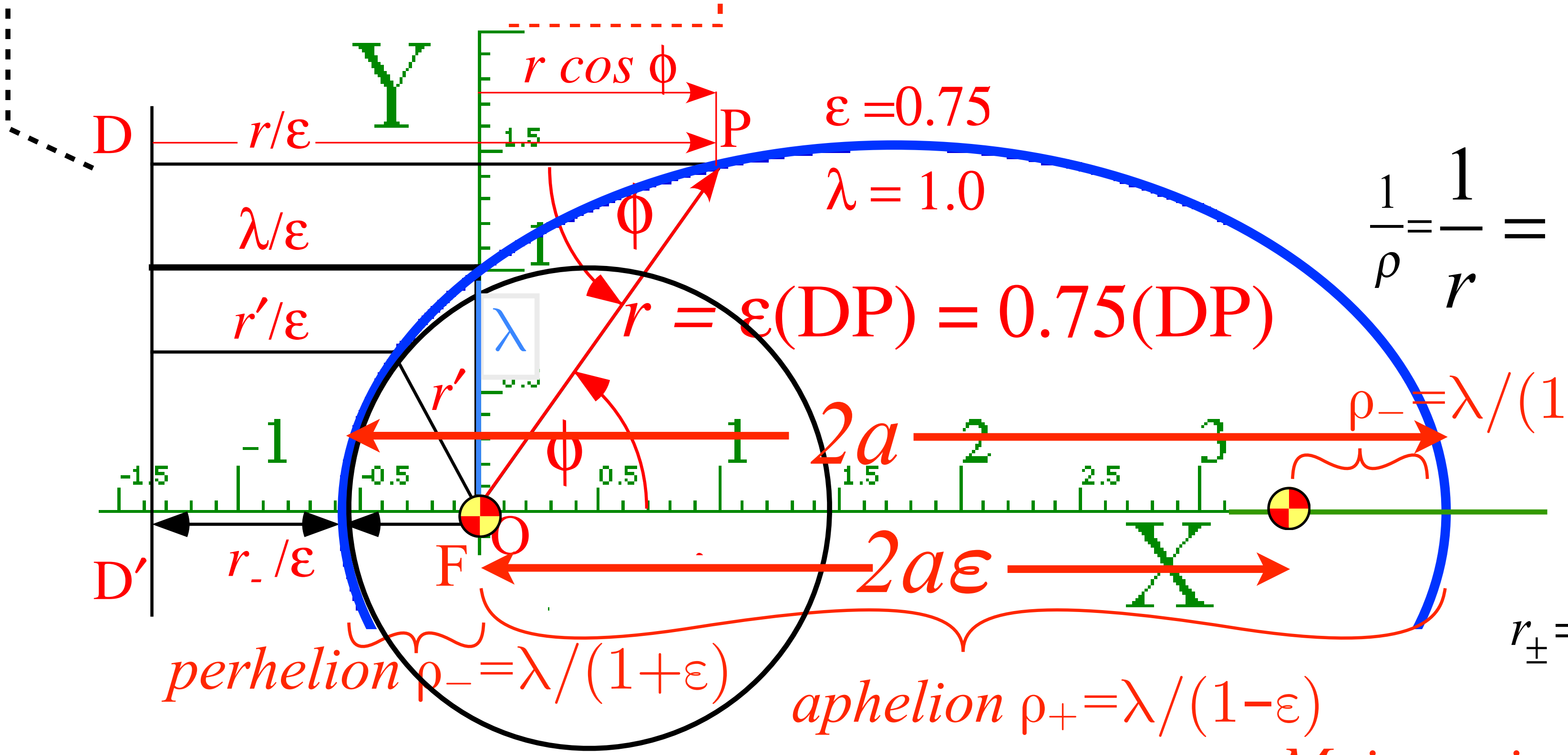
Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho}=\frac{1}{r}=\frac{1-\varepsilon\cos\phi}{\lambda}=\frac{1}{\lambda}-\frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_{\pm}=\rho_{\pm}=\frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}$$

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon)+\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon)-\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

All conics defined by: ***Eccentricity*** *ε*

Distance to *Focus* **F** = *ε*·Distance to *Directrix* **DD′**

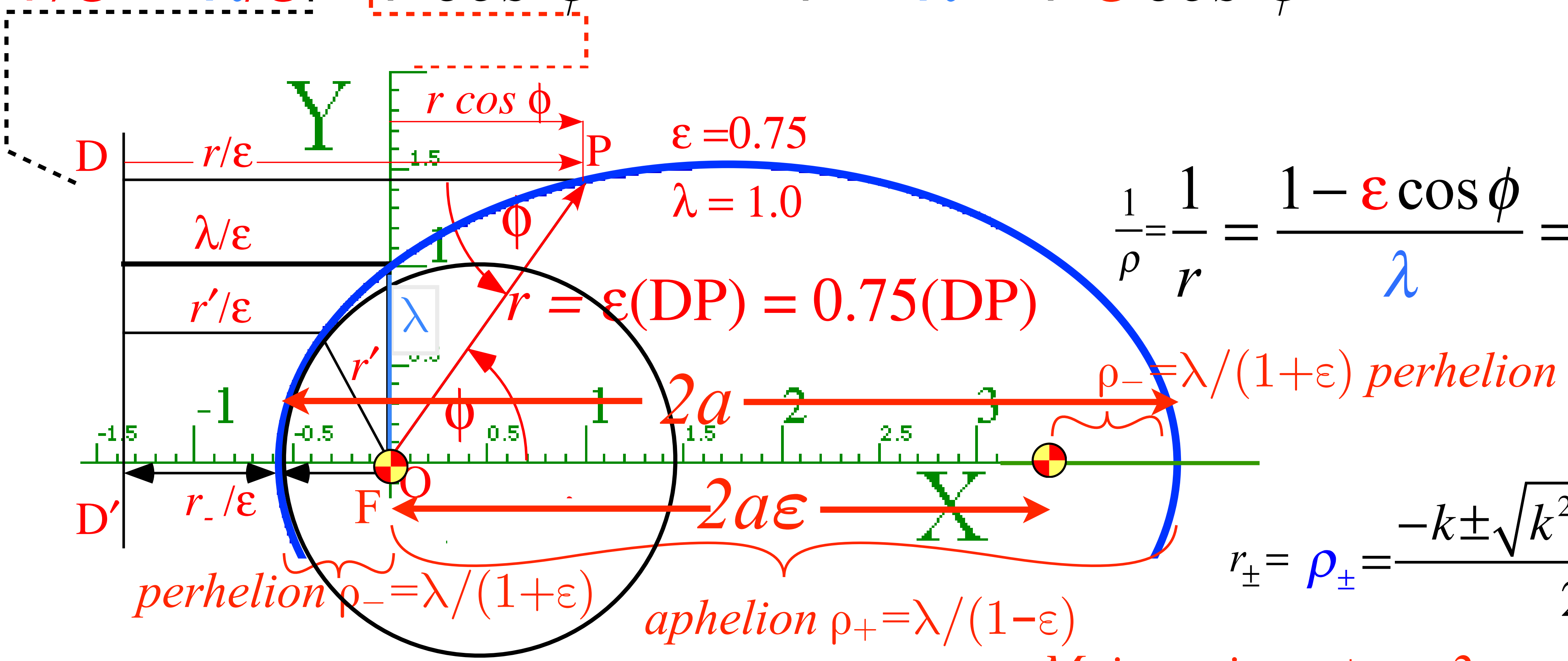
Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{E\,m}} = \left|\frac{k}{E}\right|\sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{\left|1-\varepsilon^2\right|}$$

Geometry of ALL Coulomb conic section orbits (Let: $r\equiv\rho$ here)

$$\frac{r}{\varepsilon} = \frac{\lambda}{\varepsilon} + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho}=\frac{1}{r}=\frac{1-\varepsilon\cos\phi}{\lambda}=\frac{1}{\lambda}-\frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_\pm = \rho_\pm = \frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon)+\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon)-\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

Very important result!

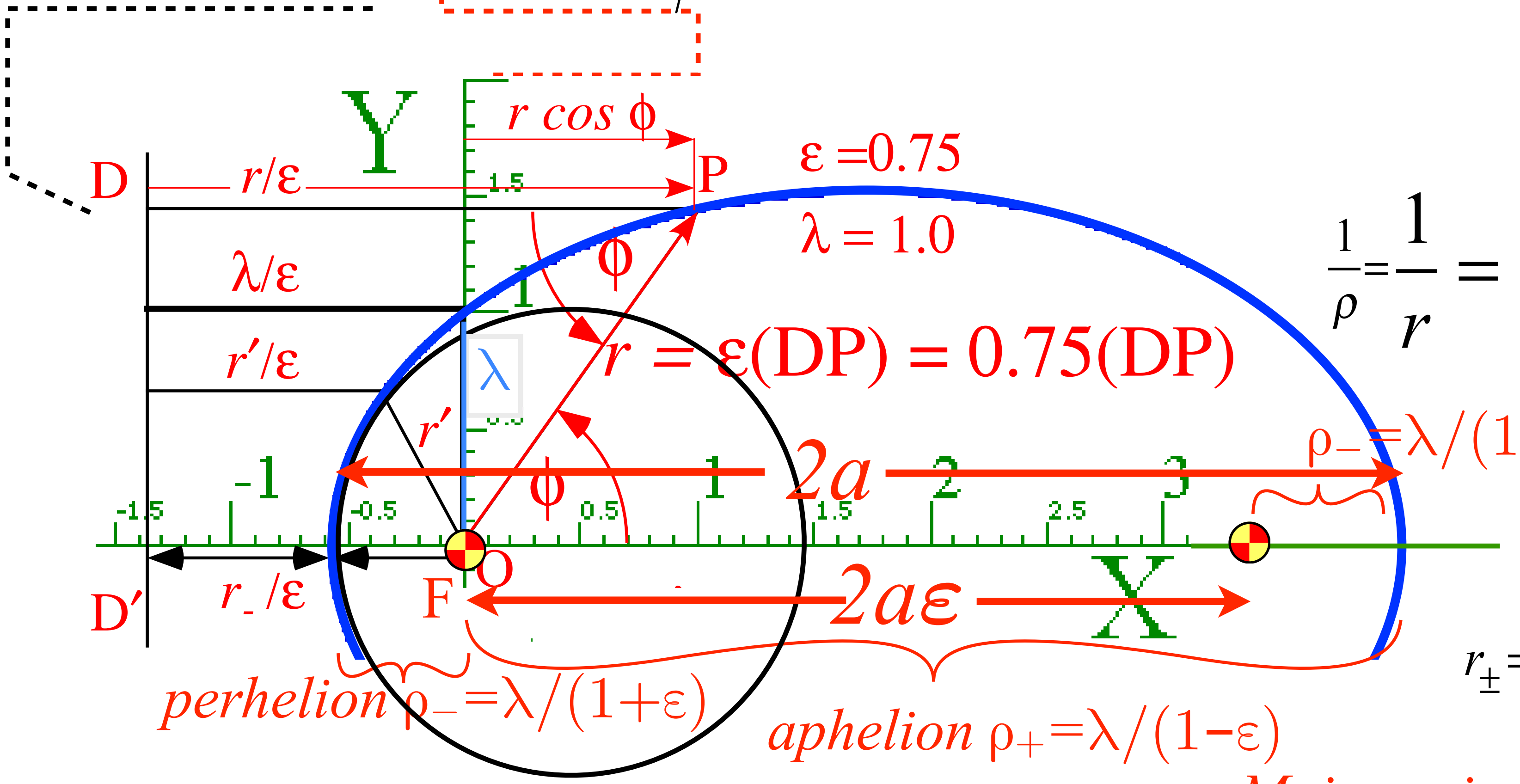
$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{E\,m}} = \left|\frac{k}{E}\right|\sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{\left|1-\varepsilon^2\right|}$$

$$\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} = \sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = \varepsilon$$

Geometry of ALL Coulomb conic section orbits (Let:r≡ρ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_\pm = \rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** *ε*

Distance to *Focus* **F** = *ε*·Distance to *Directrix* **DD′**

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

Latus radius: $\lambda = a(1-\varepsilon^2)$

Very important result!

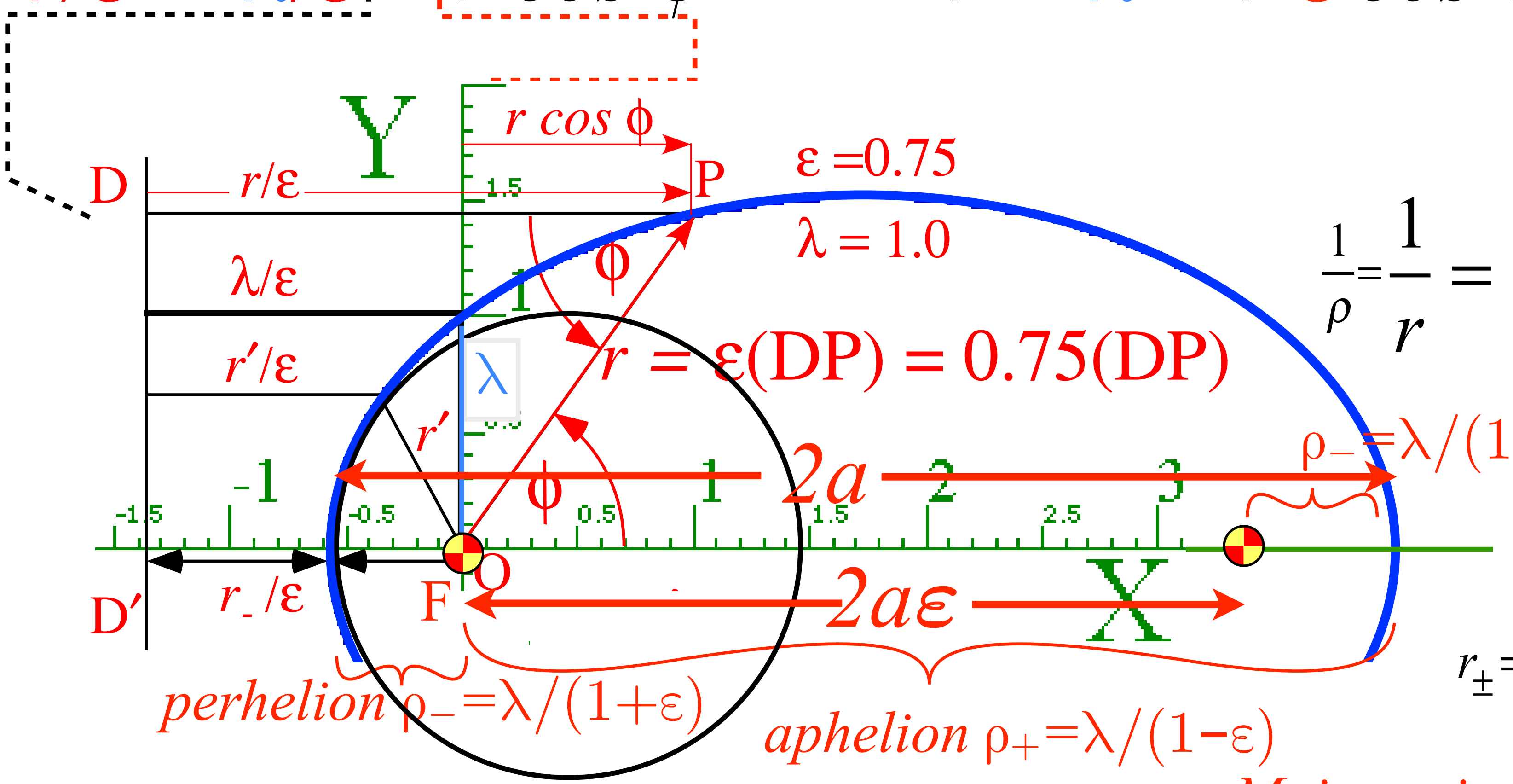
$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{E\,m}} = \left|\frac{k}{E}\right|\sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{|1-\varepsilon^2|}$$

$$\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} = \sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = \varepsilon$$

Geometry of ALL Coulomb conic section orbits (Let: $r\equiv\rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho}=\frac{1}{r}=\frac{1-\varepsilon\cos\phi}{\lambda}=\frac{1}{\lambda}-\frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_\pm = \rho_\pm = \frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}$$

All conics defined by: *Eccentricity* ε

Distance to *Focus* F = ε·Distance to *Directrix* DD′

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon)+\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon)-\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

Latus radius: $\lambda = a(1-\varepsilon^2)$

Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

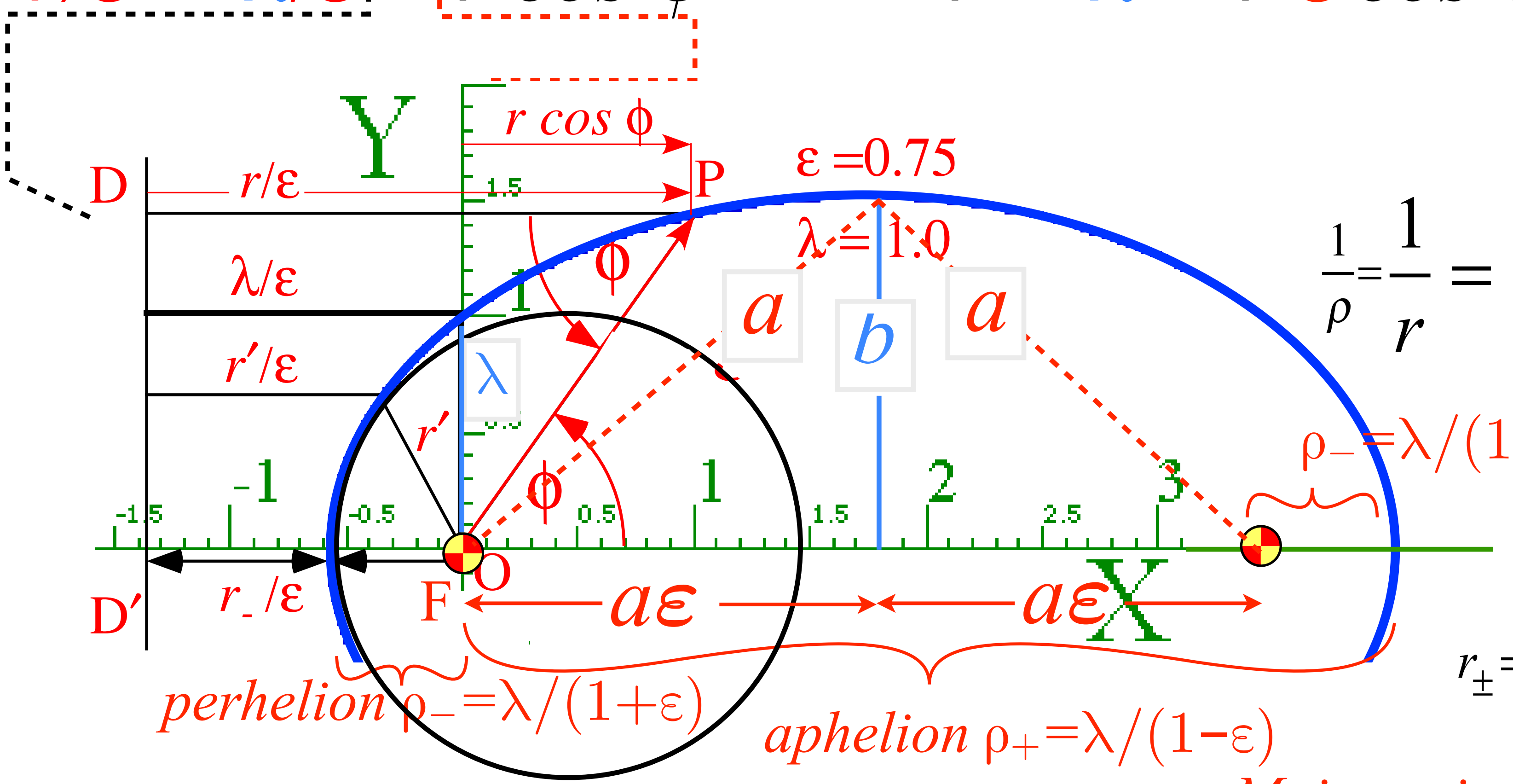
$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{E\,m}} = \left|\frac{k}{E}\right|\sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{\left|1-\varepsilon^2\right|}$$

$$\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} = \sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = \varepsilon \quad \text{implies: } \lambda = a(1-\varepsilon^2) = a\frac{2\mu^2 E}{k^2\,m} = \frac{\mu^2}{km}$$

Also important! $\boxed{\mu = \sqrt{km\lambda}}$

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_{\pm} = \rho_{\pm} = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{E\,m}} = \left|\frac{k}{E}\right|\sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{\left|1-\varepsilon^2\right|}$$

$$\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} = \sqrt{1+\frac{2\mu^2 E}{k^2\,m}} = \varepsilon \quad \text{implies: } \lambda = a(1-\varepsilon^2) = a\frac{2\mu^2 E}{k^2\,m} = \frac{\mu^2}{km}$$

Also important! $\boxed{\mu = \sqrt{km\lambda}}$

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

Latus radius: $\lambda = a(1-\varepsilon^2)$

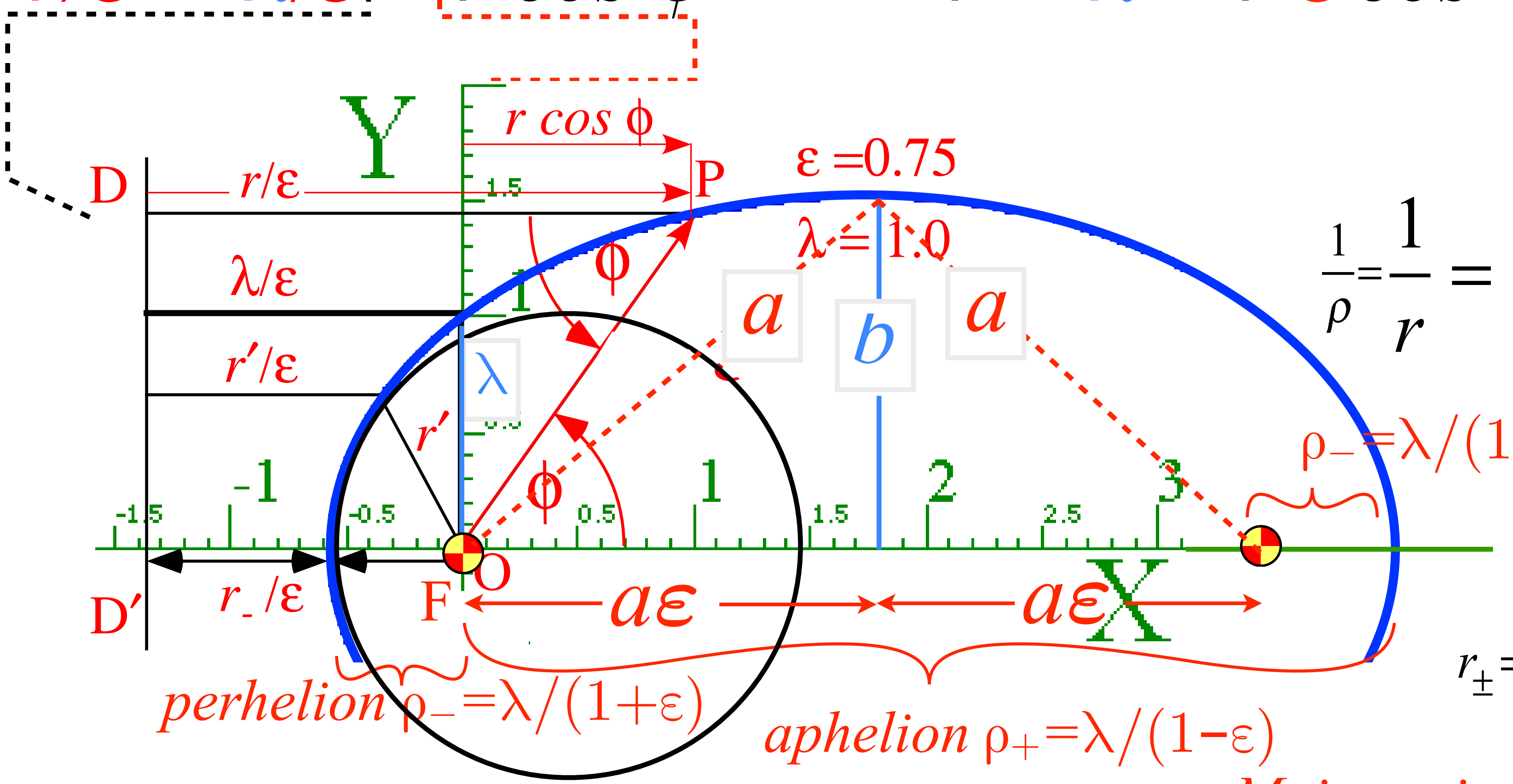
Minor radius:

$b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon < 1$)*

$b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon > 1$)*

Geometry of ALL Coulomb conic section orbits (Let:r≡ρ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r=\frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho}=\frac{1}{r}=\frac{1-\varepsilon\cos\phi}{\lambda}=\frac{1}{\lambda}-\frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_\pm = \rho_\pm = \frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E=\frac{-k}{2a}}$$

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2+\frac{2\mu^2}{E\,m}} = \left|\frac{k}{E}\right|\sqrt{1+\frac{2\mu^2E}{k^2\,m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{\left|1-\varepsilon^2\right|}$$

$$\frac{\rho_+-\rho_-}{\rho_++\rho_-} = \sqrt{1+\frac{2\mu^2E}{k^2\,m}} = \varepsilon \quad \text{implies: } \lambda = a(1-\varepsilon^2) = a\frac{2\mu^2E}{k^2\,m} = \frac{\mu^2}{km}$$

Also important! $\boxed{\mu=\sqrt{km\lambda}}$

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon)+\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon)-\lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

Latus radius: $\lambda = a(1-\varepsilon^2)$

***Minor radius**:*

$b=\sqrt{(a^2-a^2\varepsilon^2)}=\sqrt{(a\lambda)}$ (ellipse: $\varepsilon<1$)

$b=\sqrt{(a^2\varepsilon^2-a^2)}=\sqrt{(\lambda a)}$ (hyperb: $\varepsilon>1$)

$b/a=\sqrt{(1-\varepsilon^2)}$ (ellipse: $\varepsilon<1$)

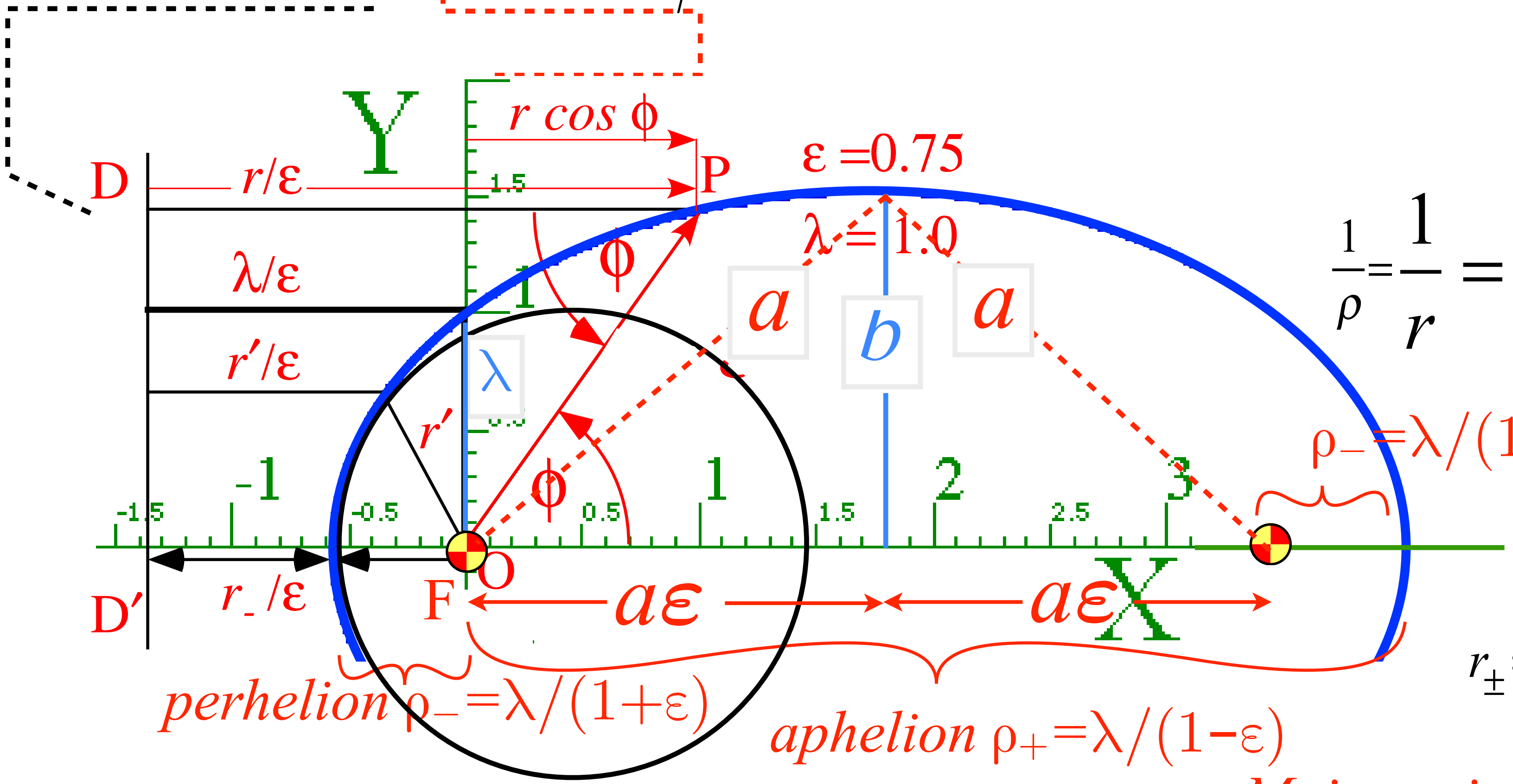
$b/a=\sqrt{(\varepsilon^2-1)}$ (hyperb: $\varepsilon>1$)

$\lambda = a(1-\varepsilon^2)$ (ellipse: $\varepsilon<1$)

$\lambda = a(\varepsilon^2-1)$ (hyperb: $\varepsilon>1$)

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad \rho \equiv r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

By p.29 physics:

$$r_{\pm} = \rho_{\pm} = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

(x,y) parameters	*physical parameters*	*(r,ϕ) parameters*
major radius	*Energy*	*eccentricity*
$a = \frac{k}{2E}$	$E = \frac{k}{2a}$	$\varepsilon = \sqrt{\frac{k^2 m + 2L^2 E}{k^2 m}}$
minor radius	*∠-momentum*	*latus radius*
$b = \frac{L}{\sqrt{2m\|E\|}}$	$L = \sqrt{km\lambda} \equiv \mu$	$\lambda = \frac{L^2}{km}$

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

Minor radius: $b = \sqrt{(a^2 - a^2\varepsilon^2)} = \sqrt{(a\lambda)}$ *(ellipse: $\varepsilon < 1$)*

Minor radius: $b = \sqrt{(a^2\varepsilon^2 - a^2)} = \sqrt{(\lambda a)}$ *(hyperb: $\varepsilon > 1$)*

$b/a = \sqrt{(1-\varepsilon^2)}$ *(ellipse: $\varepsilon < 1$)* $\quad \varepsilon^2 = 1 - b^2/a^2$

$b/a = \sqrt{(\varepsilon^2 - 1)}$ *(hyperb: $\varepsilon > 1$)* $\quad \varepsilon^2 = 1 + b^2/a^2$

$\lambda = a(1-\varepsilon^2)$ *(ellipse: $\varepsilon < 1$)* $\quad a\varepsilon^2 = a - \lambda$

$\lambda = a(\varepsilon^2 - 1)$ *(hyperb: $\varepsilon > 1$)* $\quad a\varepsilon^2 = a + \lambda$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - ➔ *Kepler equation of time and phase geometry*

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

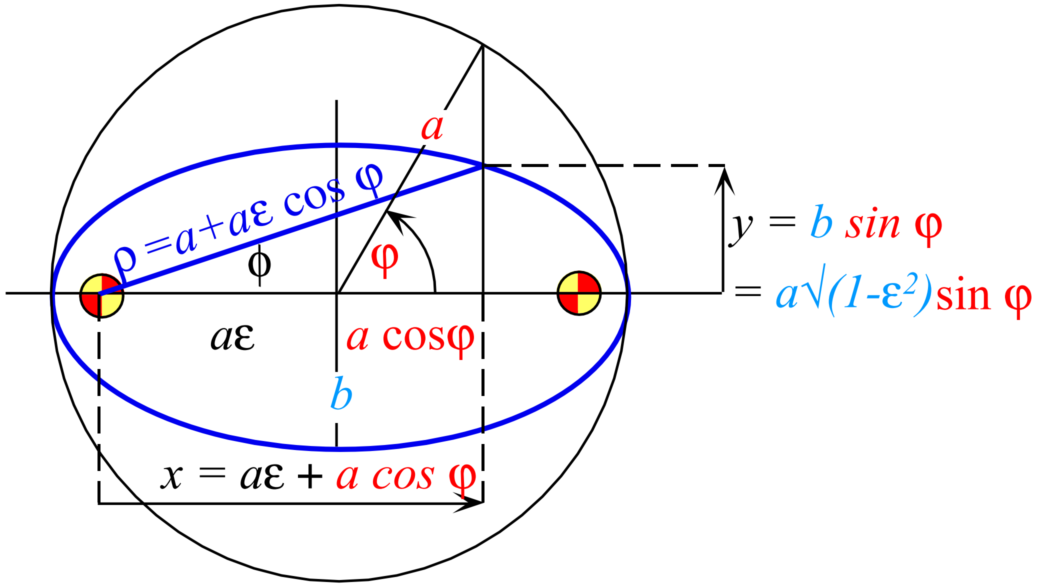
Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

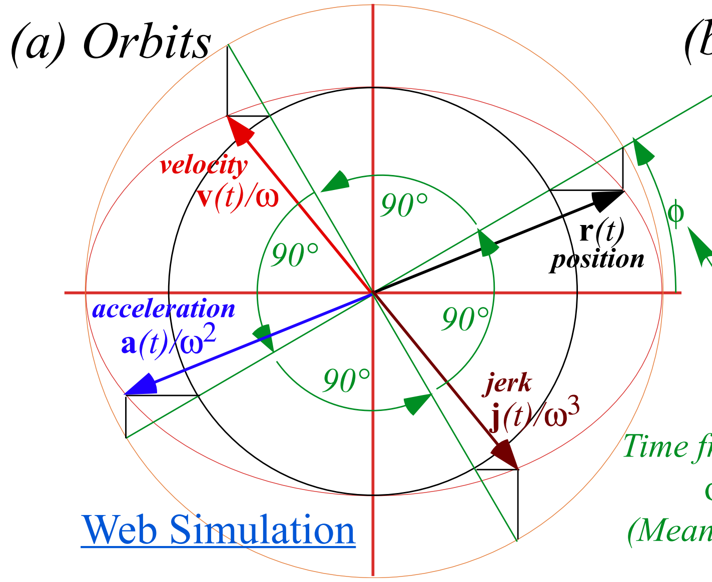
Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

$$x = a\varepsilon + a\cos\varphi \ , \qquad y = a\sqrt{1-\varepsilon^2}\,\sin\varphi \ ,$$



Unit 1 Ch. 9
Recall IHO orbit
time construction



Web Simulation

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

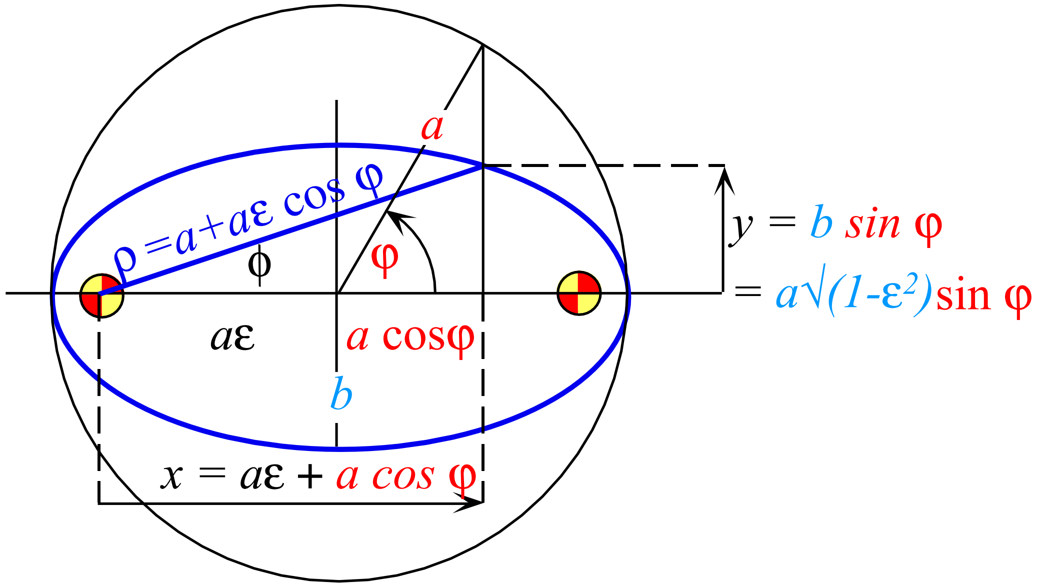
Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

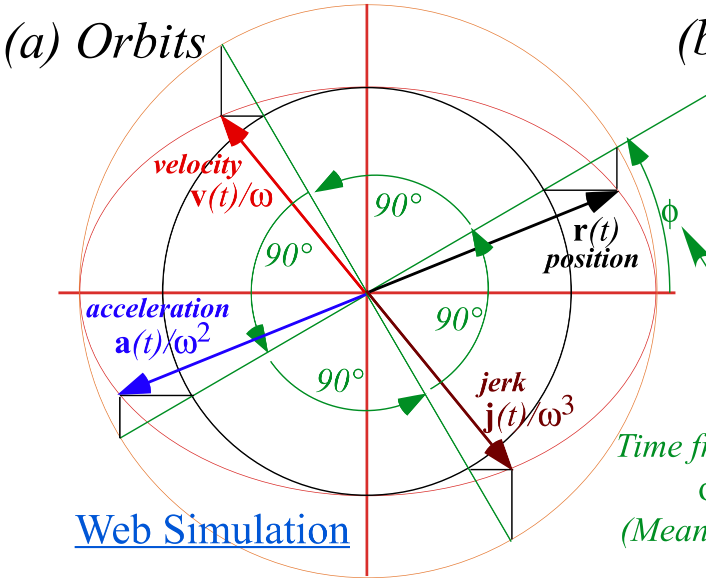
$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

$$x = a\varepsilon + a\cos\varphi \ , \qquad y = a\sqrt{1\text{-}\varepsilon^2}\,\sin\varphi \ ,$$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi \text{ - } a^2\varepsilon^2\sin^2\varphi}$$



Unit 1 Ch. 9
Recall IHO orbit
time construction



Web Simulation

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

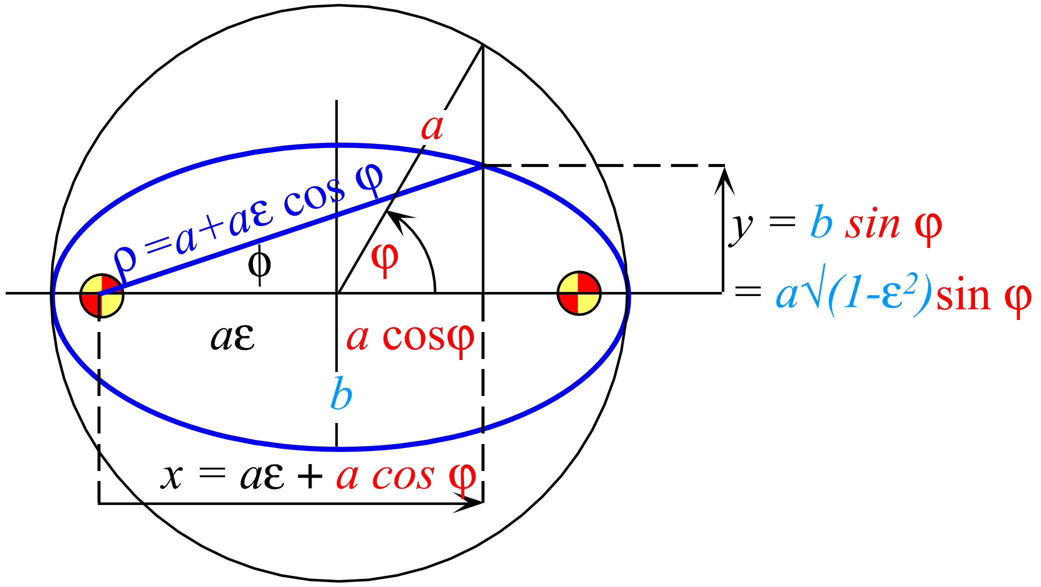
Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

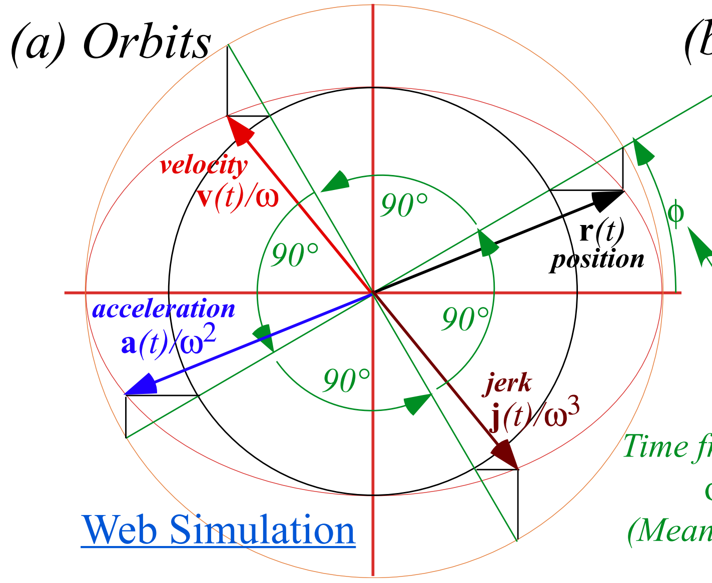
$$x = a\varepsilon + a\cos\varphi \; , \qquad y = a\sqrt{1\text{-}\varepsilon^2}\,\sin\varphi \; ,$$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi \text{-} a^2\varepsilon^2\sin^2\varphi}$$

$$= \sqrt{a^2\varepsilon^2 \text{-} a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$



Unit 1 Ch. 9
Recall IHO orbit
time construction



Web Simulation

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

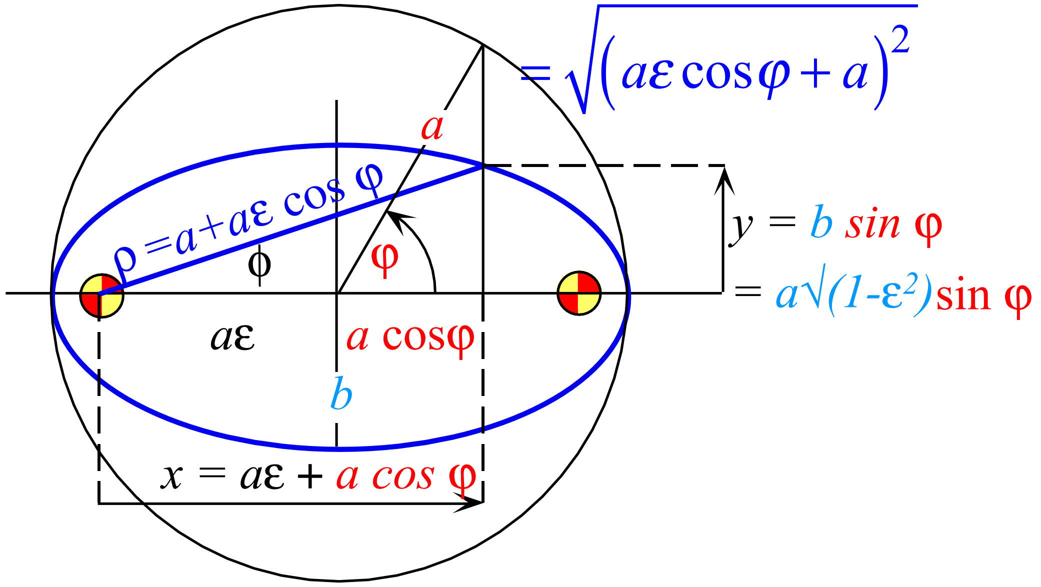
$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

$$x = a\varepsilon + a\cos\varphi \,, \qquad y = a\sqrt{1\text{-}\varepsilon^2}\,\sin\varphi \,,$$

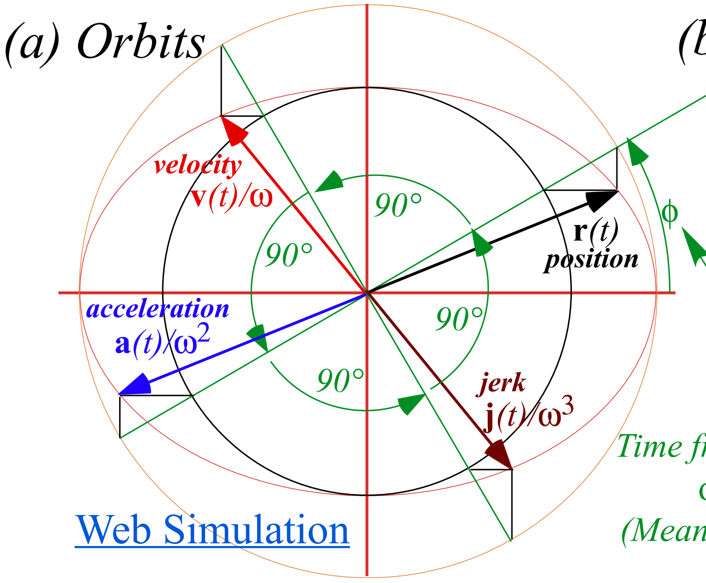
$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi - a^2\varepsilon^2\sin^2\varphi}$$

$$= \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$

$$= \sqrt{\left(a\varepsilon\cos\varphi + a\right)^2}$$



Unit 1 Ch. 9
Recall IHO orbit
time construction



Web Simulation

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

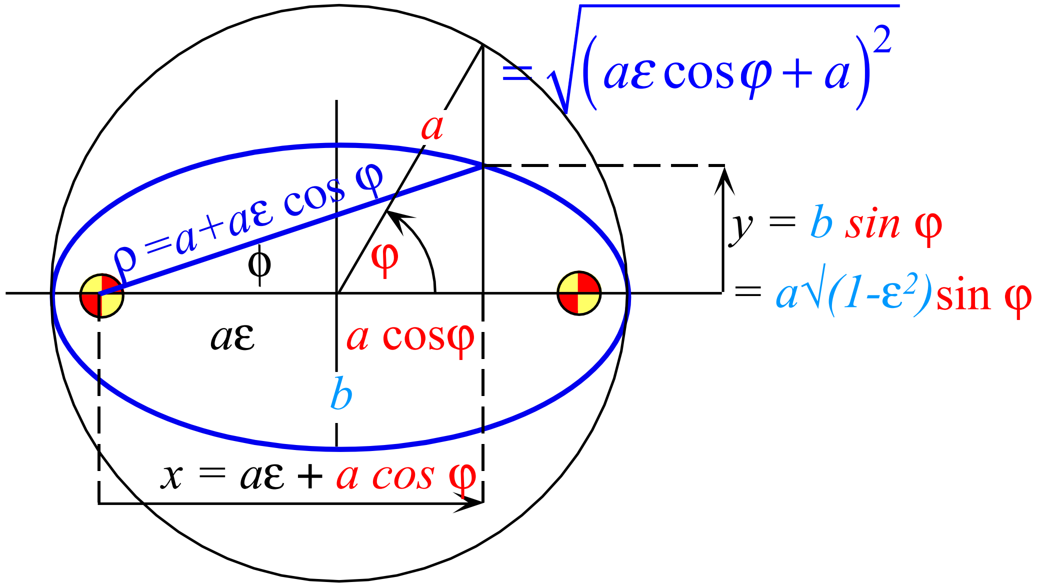
$$x = a\varepsilon + a\cos\varphi\ , \qquad y=a\sqrt{1\text{-}\varepsilon^2}\,\sin\varphi\ ,$$

$$\rho=\sqrt{x^2+y^2}=\sqrt{a^2\varepsilon^2+2a^2\varepsilon\cos\varphi+a^2\cos^2\varphi+a^2\sin^2\varphi\text{ - }a^2\varepsilon^2\sin^2\varphi}$$

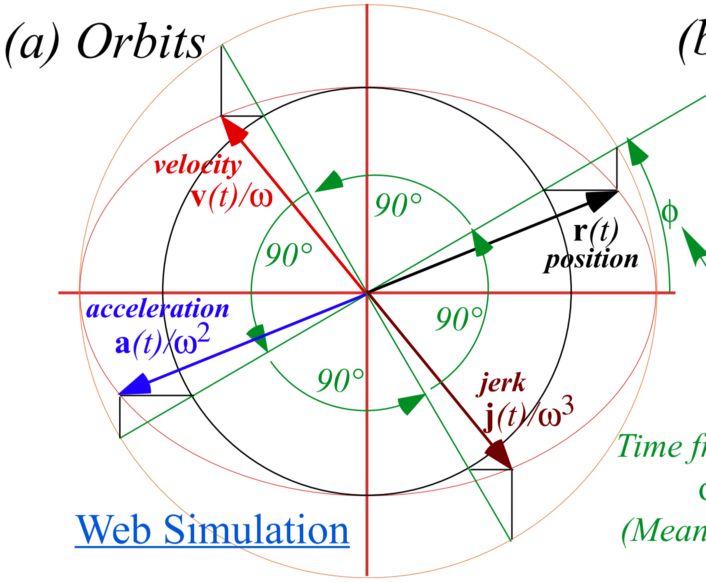
$$=\sqrt{a^2\varepsilon^2\text{ - }a^2\varepsilon^2\sin^2\varphi+2a^2\varepsilon\cos\varphi+a^2}=\sqrt{a^2\varepsilon^2\cos^2\varphi+2a^2\varepsilon\cos\varphi+a^2}$$

$$=\sqrt{(a\varepsilon\cos\varphi+a)^2}$$

$$\rho= a(1+\varepsilon\cos\varphi)$$



Unit 1 Ch. 9
Recall IHO orbit
time construction



Web Simulation

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}}\int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}}\int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

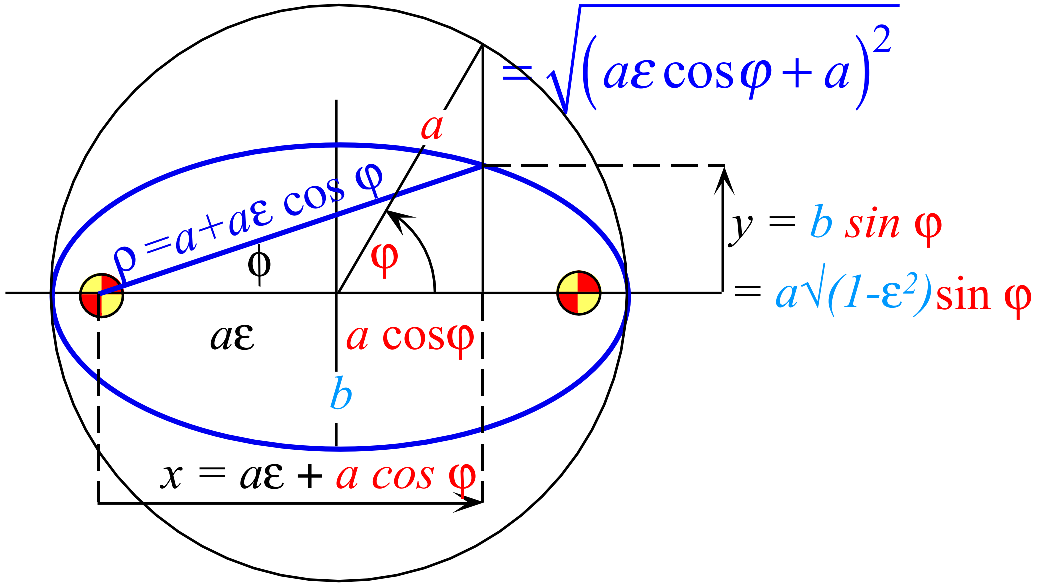
$$x = a\varepsilon + a\cos\varphi\ , \qquad y = a\sqrt{1-\varepsilon^2}\sin\varphi\ ,$$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi - a^2\varepsilon^2\sin^2\varphi}$$

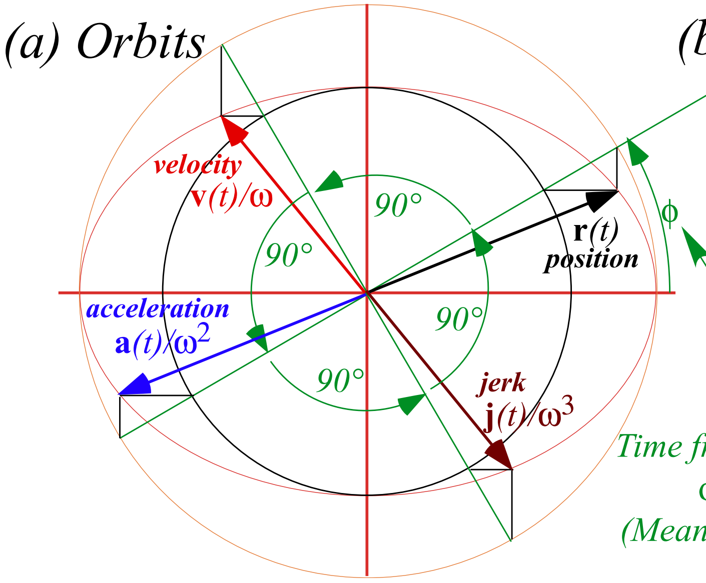
$$= \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$

$$= \sqrt{(a\varepsilon\cos\varphi + a)^2}$$

$$\rho = a(1 + \varepsilon\cos\varphi)$$



Unit 1 Ch. 9
Recall IHO orbit
time construction



$$t = \sqrt{\frac{m}{2k}}\int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi d\varphi}{\sqrt{\left(\frac{-(a + a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}}$$

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

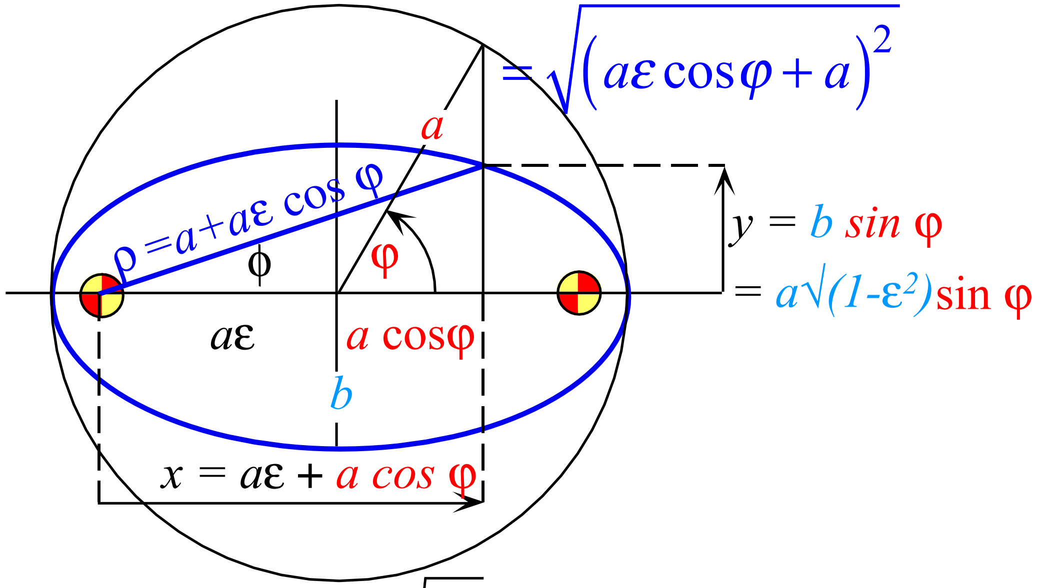
$$x = a\varepsilon + a\cos\varphi\ , \qquad y = a\sqrt{1\text{-}\varepsilon^2}\sin\varphi\ ,$$

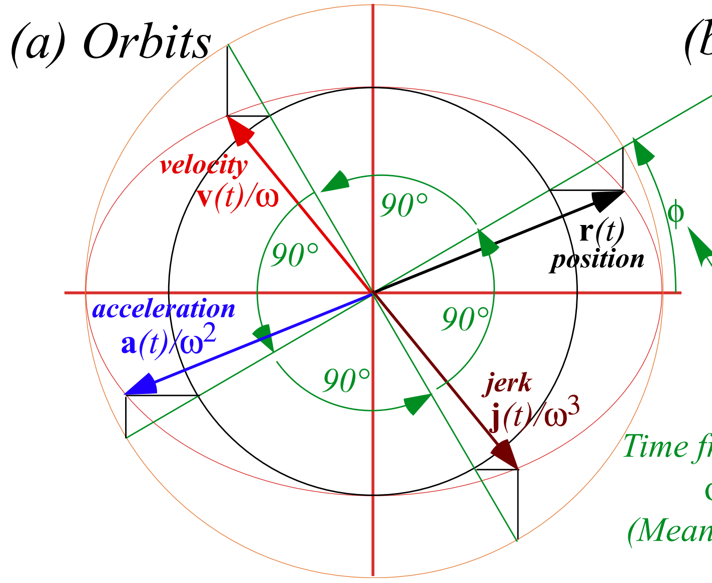
$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi\ \text{-}\ a^2\varepsilon^2\sin^2\varphi}$$

$$= \sqrt{a^2\varepsilon^2\ \text{-}\ a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$

$$\rho = a\left(1 + \varepsilon\cos\varphi\right) \qquad = \sqrt{\left(a\varepsilon\cos\varphi + a\right)^2}$$

Unit 1 Ch. 9
Recall IHO orbit
time construction





$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi d\varphi}{\sqrt{\left(\frac{-(a + a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}} = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi d\varphi}{\sqrt{\left(\frac{\cancel{-a^2} - 2a^2\varepsilon\cos\varphi \boxed{- a^2\varepsilon^2\cos^2\varphi} + \cancel{2a^2} + 2a^2\varepsilon\cos\varphi \cancel{- a^2} + \boxed{a^2\varepsilon^2}}{2a}\right)}}$$

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho\, d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho\, d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

$$x = a\varepsilon + a\cos\varphi \ , \qquad y = a\sqrt{1\text{-}\varepsilon^2}\,\sin\varphi \ ,$$

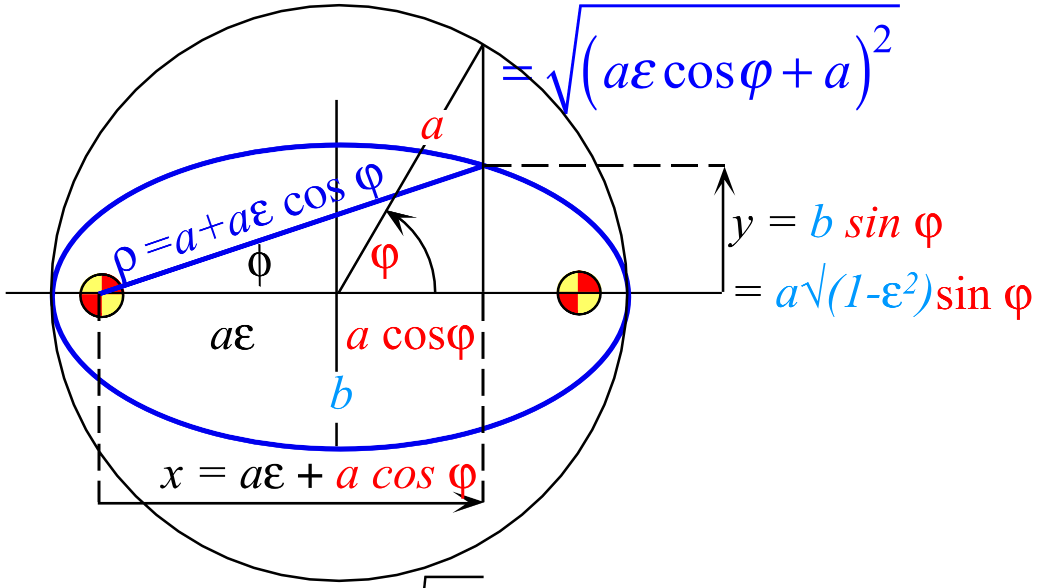
$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi \text{ - } a^2\varepsilon^2\sin^2\varphi}$$

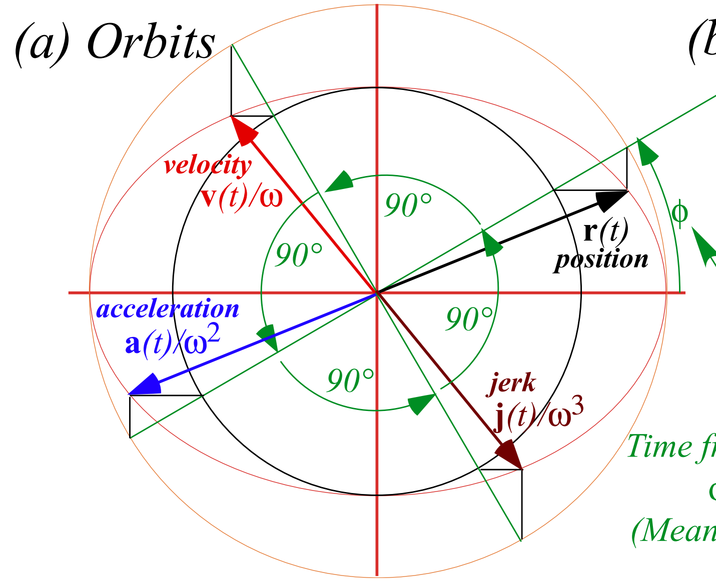
$$= \sqrt{a^2\varepsilon^2 \text{ - } a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$

$$\rho = a(1 + \varepsilon\cos\varphi)$$

$$= \sqrt{(a\varepsilon\cos\varphi + a)^2}$$

Unit 1 Ch. 9
Recall IHO orbit
time construction





$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-(a + a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}} = \sqrt{\frac{m}{\cancel{2}k}} \int \frac{(a + a\varepsilon\cos\varphi)\cancel{a\varepsilon\sin\varphi}\, d\varphi}{\sqrt{\left(\frac{\cancel{-a^2} \cancel{- 2a^2\varepsilon\cos\varphi} \cancel{- a^2\varepsilon^2\cos^2\varphi} \cancel{+ 2a^2} \cancel{+ 2a^2\varepsilon\cos\varphi} \cancel{- a^2} \cancel{+ a^2\varepsilon^2}}{\cancel{2}a}\right)}}$$

$$t = \sqrt{\frac{ma^3}{k}} \int (1 + \varepsilon\cos\varphi)\, d\varphi$$

Starting with KE-eff.-PE results on p.31: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or p.33:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

$$x = a\varepsilon + a\cos\varphi \ , \qquad y = a\sqrt{1\text{-}\varepsilon^2}\,\sin\varphi \ ,$$

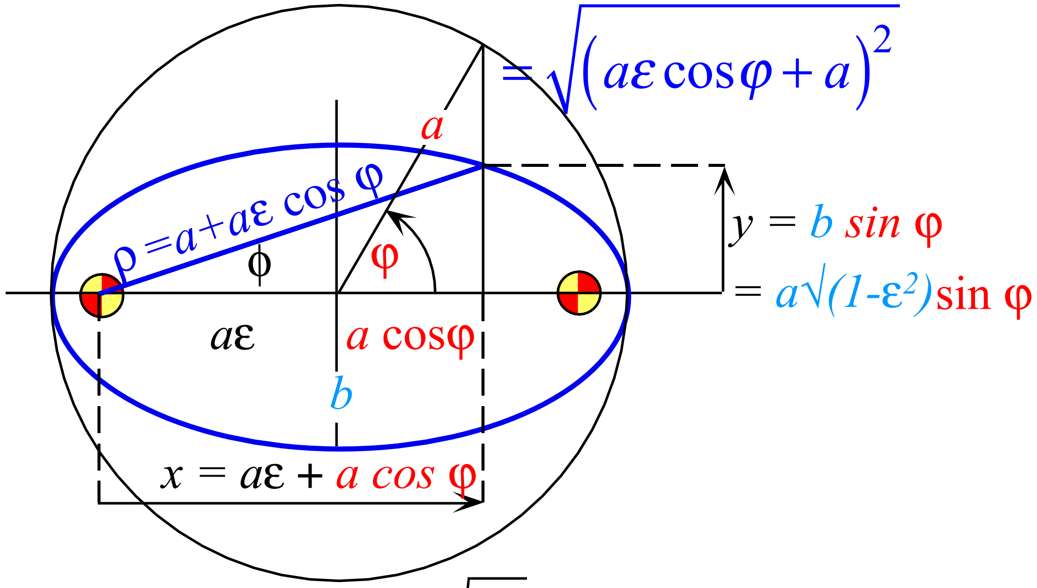
$$\rho = \sqrt{x^2+y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi - a^2\varepsilon^2\sin^2\varphi}$$

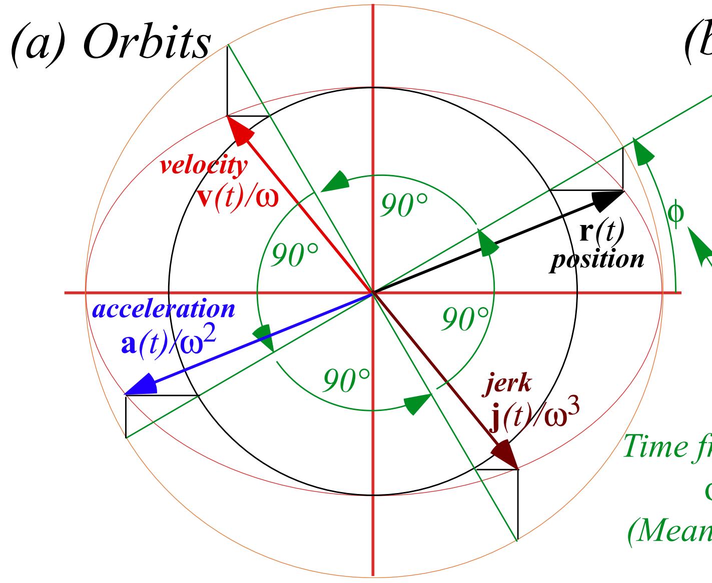
$$= \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$

Unit 1 Ch. 9
Recall IHO orbit
time construction

$$\rho = a(1+\varepsilon\cos\varphi)$$

$$= \sqrt{(a\varepsilon\cos\varphi + a)^2}$$





$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi d\varphi}{\sqrt{\left(\frac{-(a+a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}} = \sqrt{\frac{m}{\cancel{2}k}} \int \frac{(a + a\varepsilon\cos\varphi)\cancel{a\varepsilon\sin\varphi} d\varphi}{\sqrt{\left(\frac{\cancel{-a^2} - \cancel{2a^2\varepsilon\cos\varphi} \cancel{- a^2\varepsilon^2\cos^2\varphi} + \cancel{2a^2} + \cancel{2a^2\varepsilon\cos\varphi} - \cancel{a^2} \cancel{+ a^2\varepsilon^2}}{\cancel{2}a}\right)}}$$

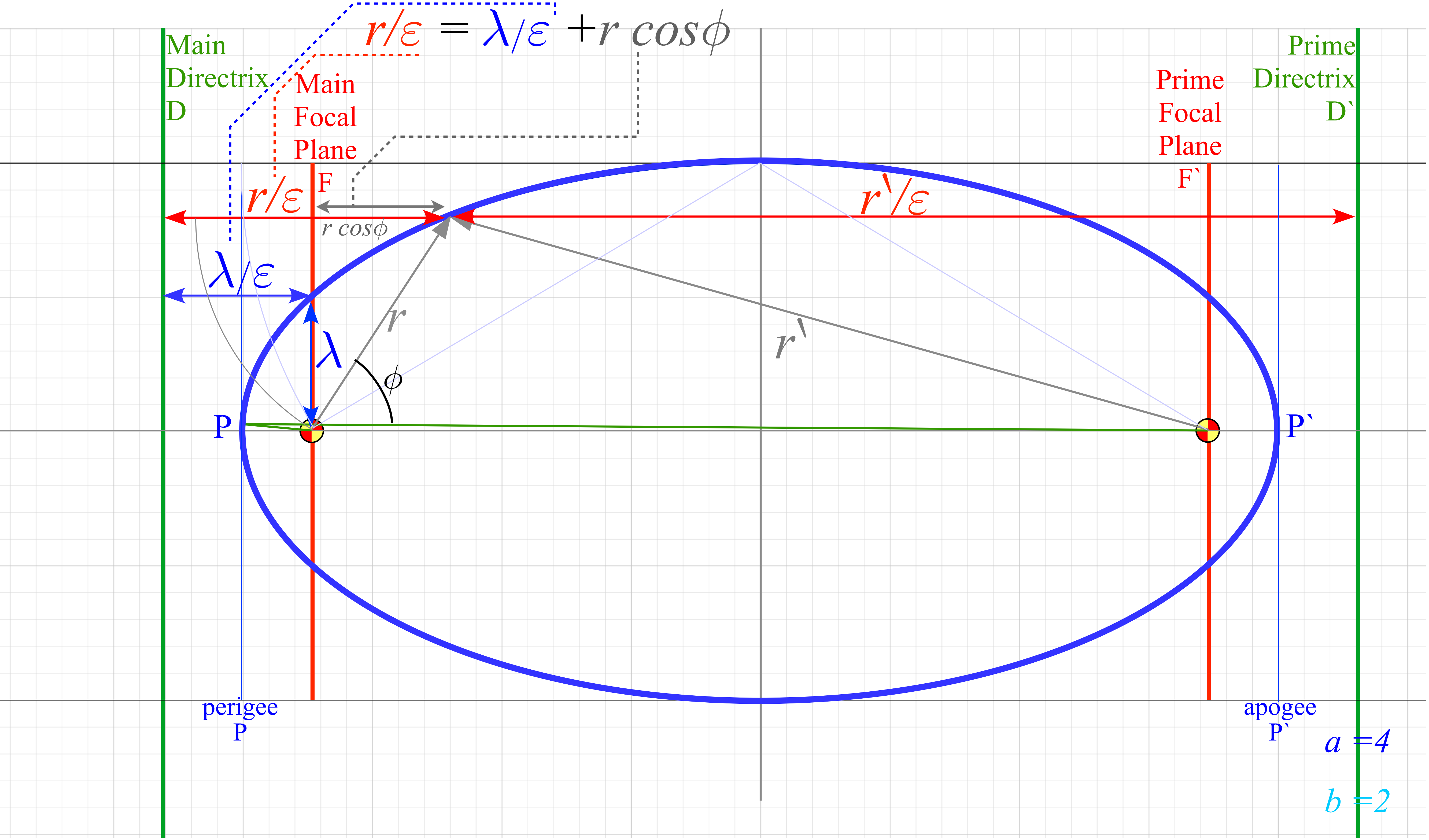
$$t = \sqrt{\frac{ma^3}{k}} \int (1 + \varepsilon\cos\varphi)d\varphi = \sqrt{\frac{ma^3}{k}}(\varphi + \varepsilon\sin\varphi)$$

Kepler's equations
of orbital time

Orbit Period $= T = \frac{2\pi}{\omega_\varphi} = 2\pi\sqrt{\frac{ma^3}{k}}$

Geometry and Symmetry of Coulomb orbits

- ➔ *Detailed elliptic geometry*
- *Detailed hyperbolic geometry*



$a = 4$

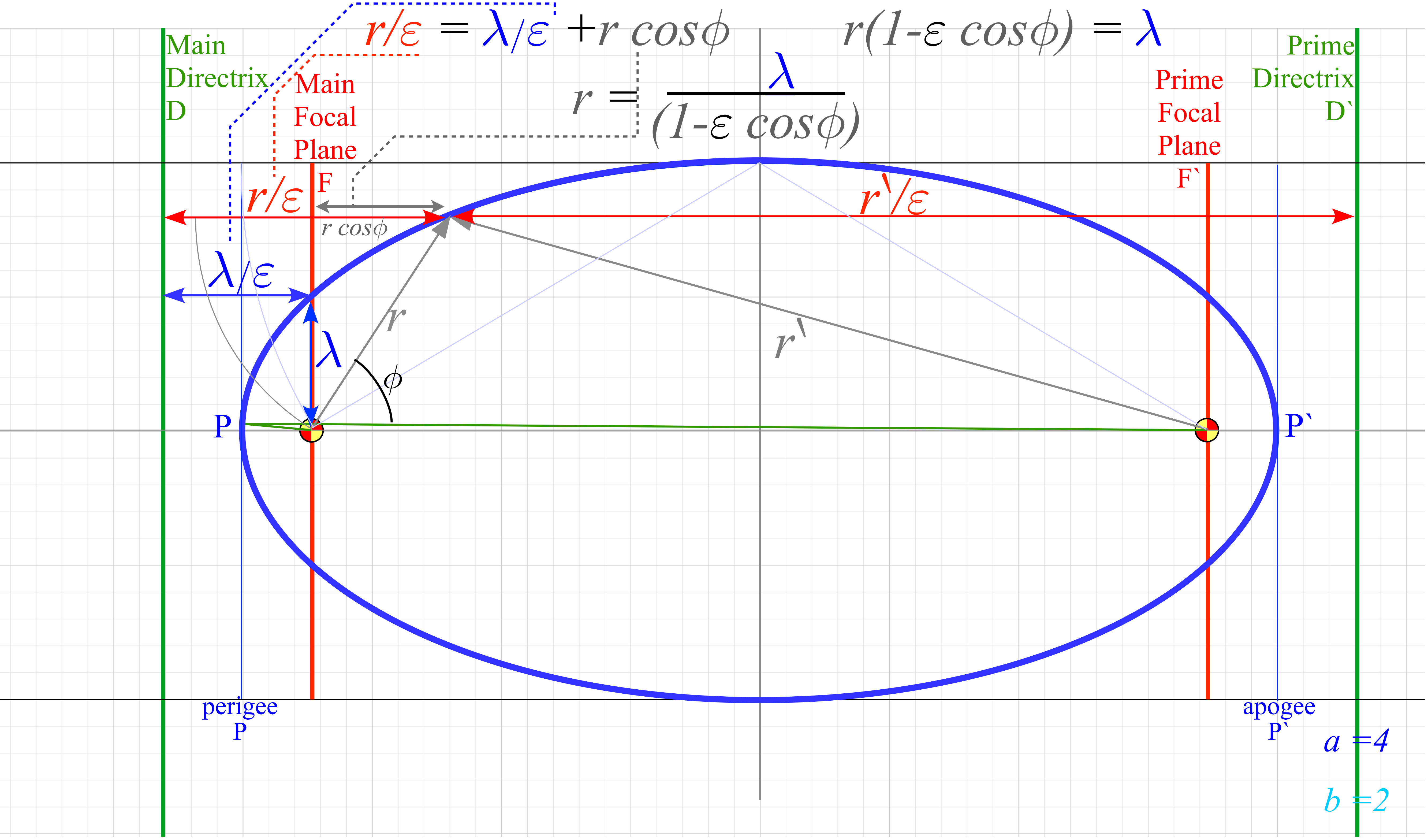
$b = 2$

$\varepsilon = \sqrt{3}/2$

$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1 - \varepsilon^2)$



$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r(1-\varepsilon\cos\phi) = \lambda$$

$$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$$

$a = 4$

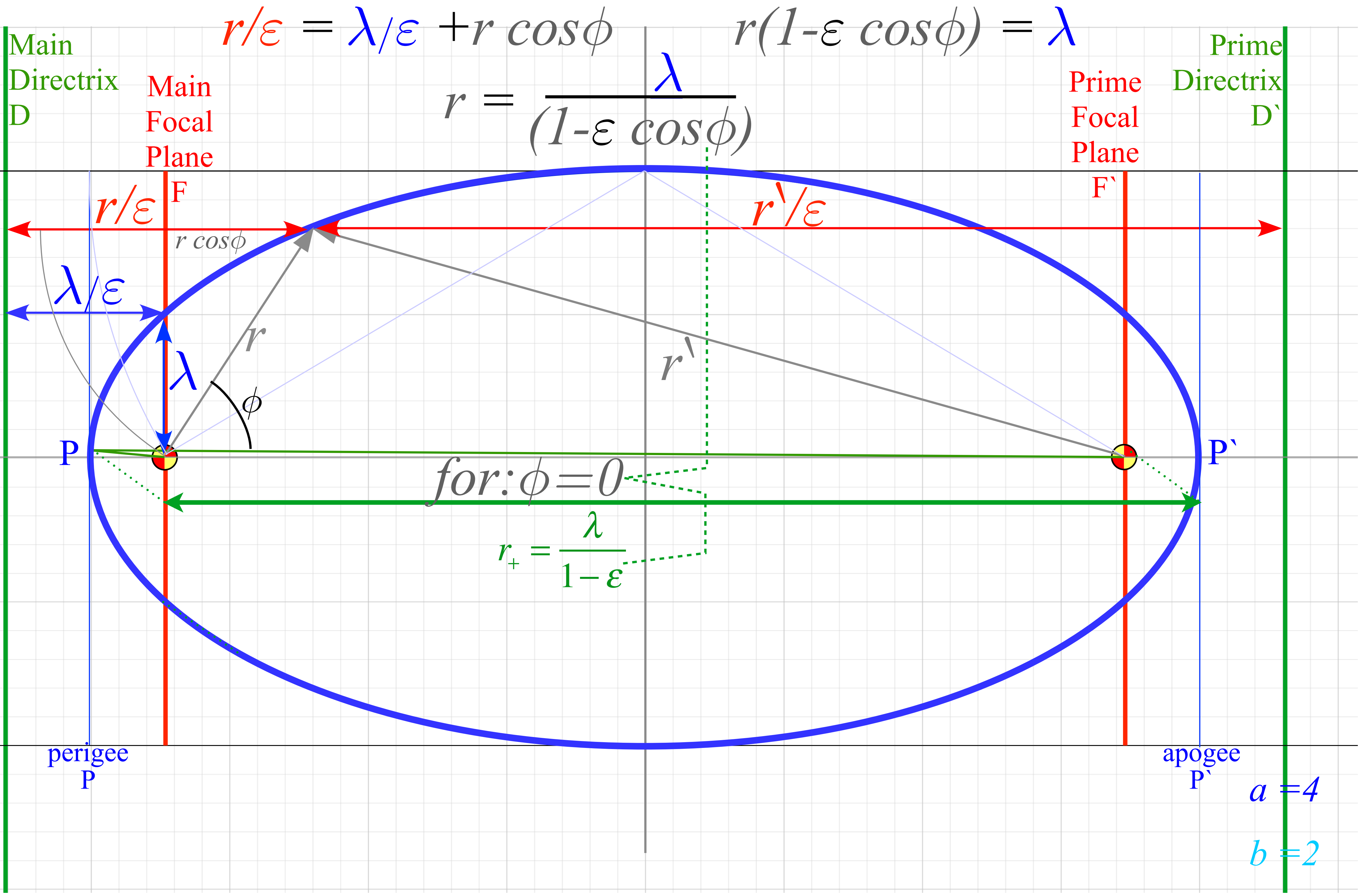
$b = 2$

$\varepsilon = \sqrt{3}/2$

$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1-\varepsilon^2)$



$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r(1-\varepsilon\cos\phi) = \lambda$$

$$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$$

for: $\phi=0$

$$r_+ = \frac{\lambda}{1-\varepsilon}$$

a =4

b =2

$\varepsilon = \sqrt{3}/2$

$\lambda = 1$

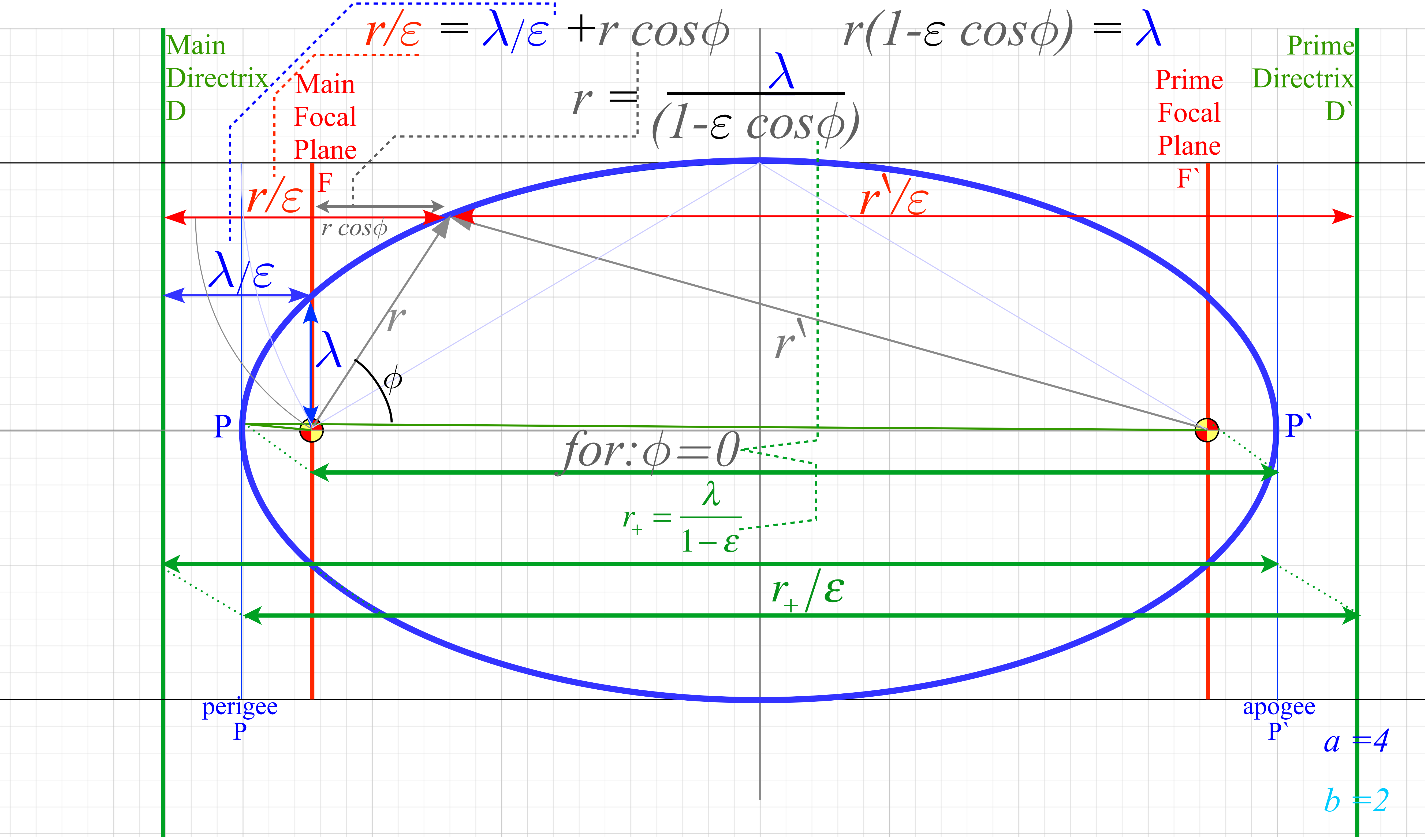
$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1-\varepsilon^2)$

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi$$

$$r(1-\varepsilon\cos\phi) = \lambda$$

$$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$$



$a = 4$

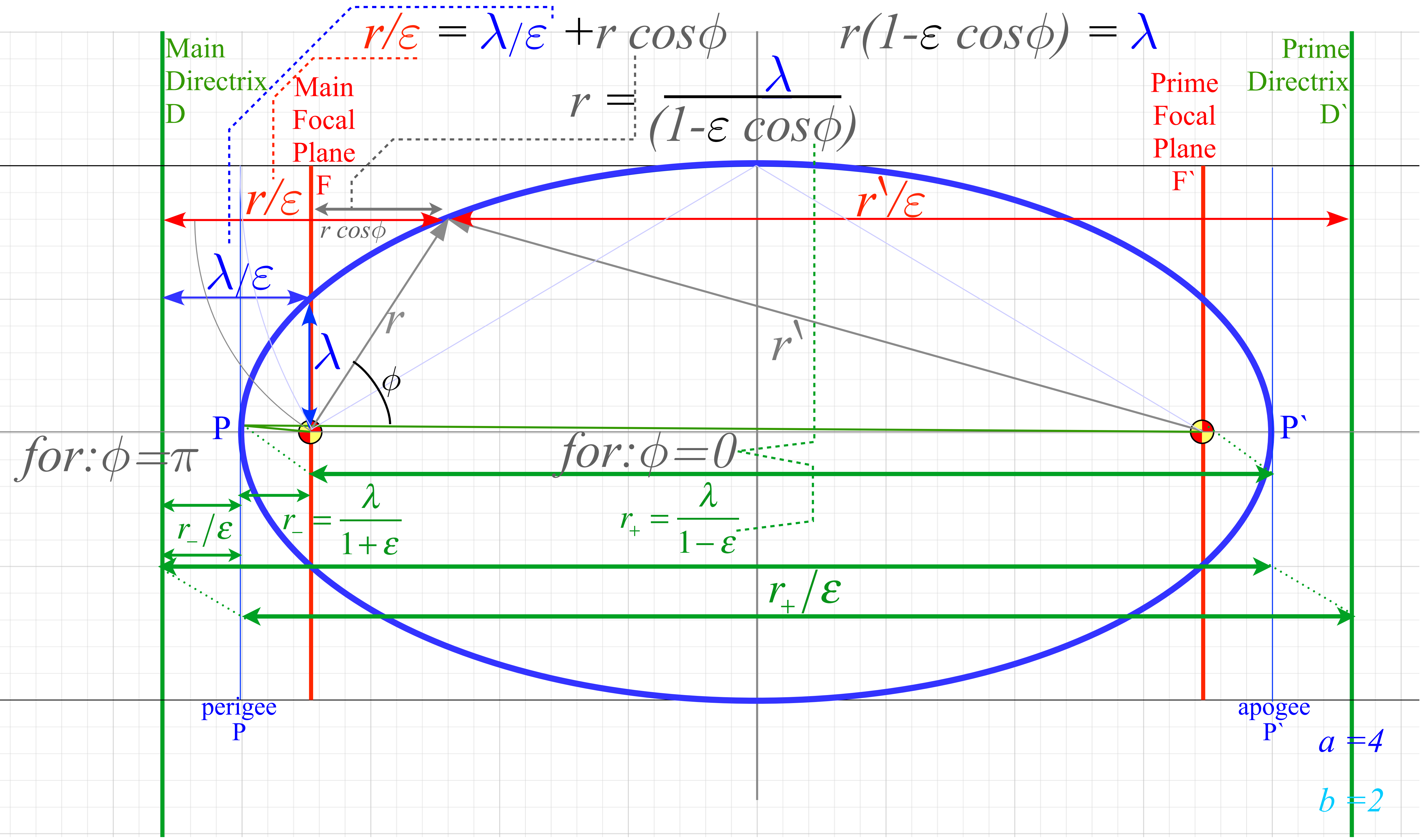
$b = 2$

$\varepsilon = \sqrt{3}/2$

$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1-\varepsilon^2)$



$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r(1-\varepsilon\cos\phi) = \lambda$$

$$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$$

$a = 4$

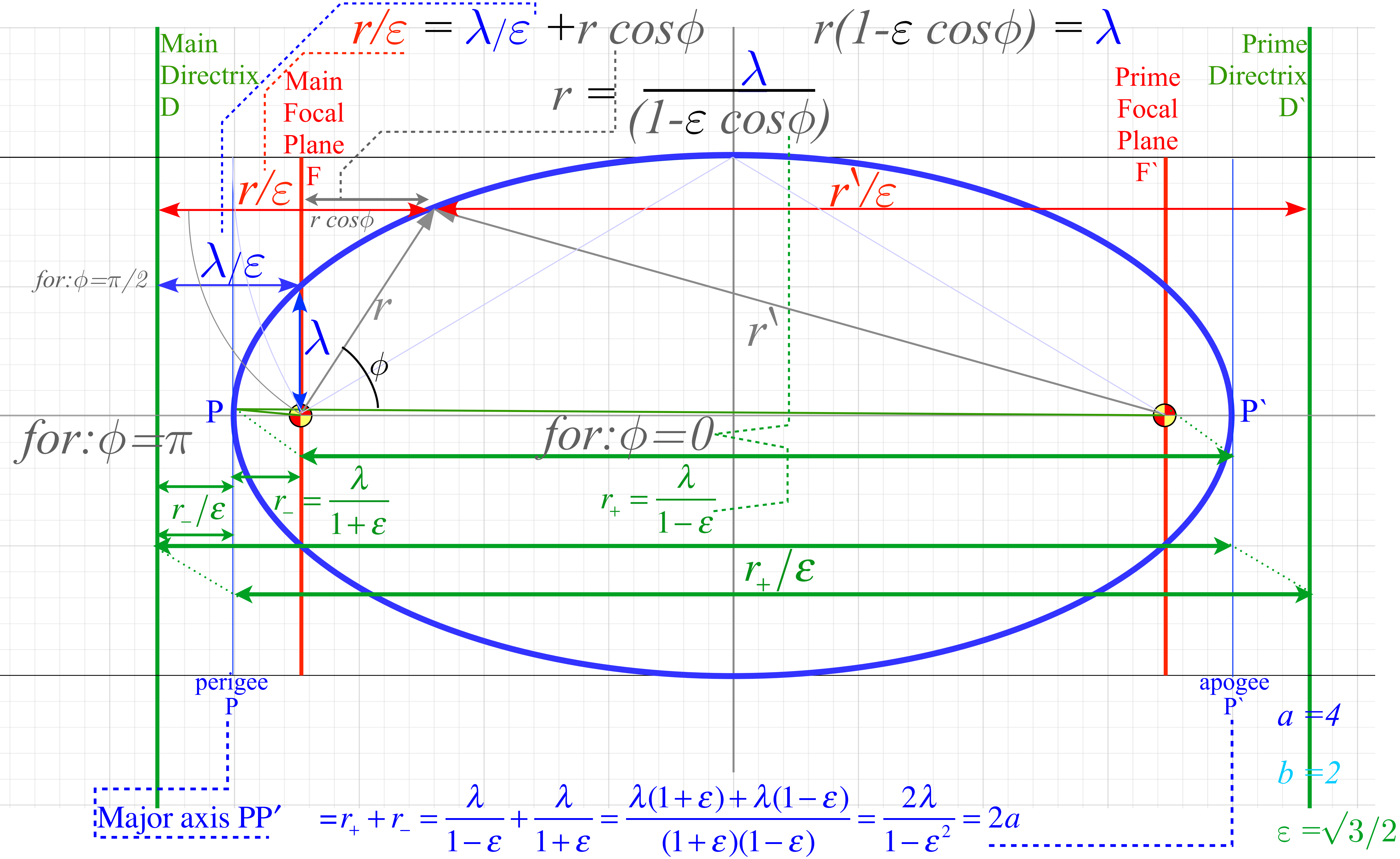
$b = 2$

$\varepsilon = \sqrt{3}/2$

$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1-\varepsilon^2)$



$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi$$

$$r(1-\varepsilon\cos\phi) = \lambda$$

$$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$$

$$r_- = \frac{\lambda}{1+\varepsilon}$$

$$r_+ = \frac{\lambda}{1-\varepsilon}$$

Major axis PP′ $= r_+ + r_- = \frac{\lambda}{1-\varepsilon} + \frac{\lambda}{1+\varepsilon} = \frac{\lambda(1+\varepsilon)+\lambda(1-\varepsilon)}{(1+\varepsilon)(1-\varepsilon)} = \frac{2\lambda}{1-\varepsilon^2} = 2a$

$a = 4$

$b = 2$

$\varepsilon = \sqrt{3}/2$

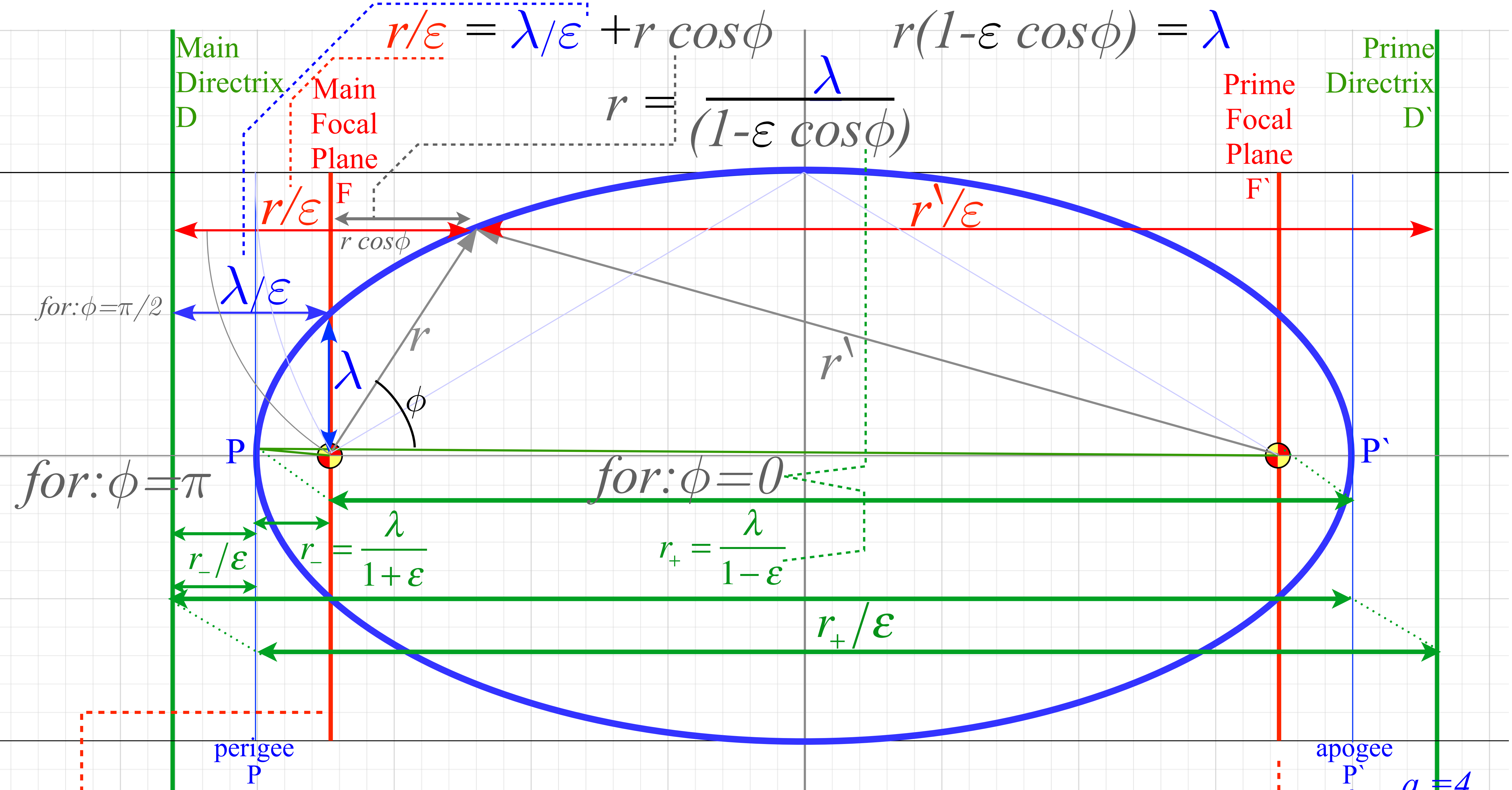
$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1-\varepsilon^2)$

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r(1-\varepsilon\cos\phi) = \lambda$$

$$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$$



Major axis PP′ $= r_+ + r_- = \frac{\lambda}{1-\varepsilon} + \frac{\lambda}{1+\varepsilon} = \frac{\lambda(1+\varepsilon)+\lambda(1-\varepsilon)}{(1+\varepsilon)(1-\varepsilon)} = \frac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \frac{\lambda}{1-\varepsilon} - \frac{\lambda}{1+\varepsilon} = \frac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

$a = 4$

$b = 2$

$\varepsilon = \sqrt{3}/2$

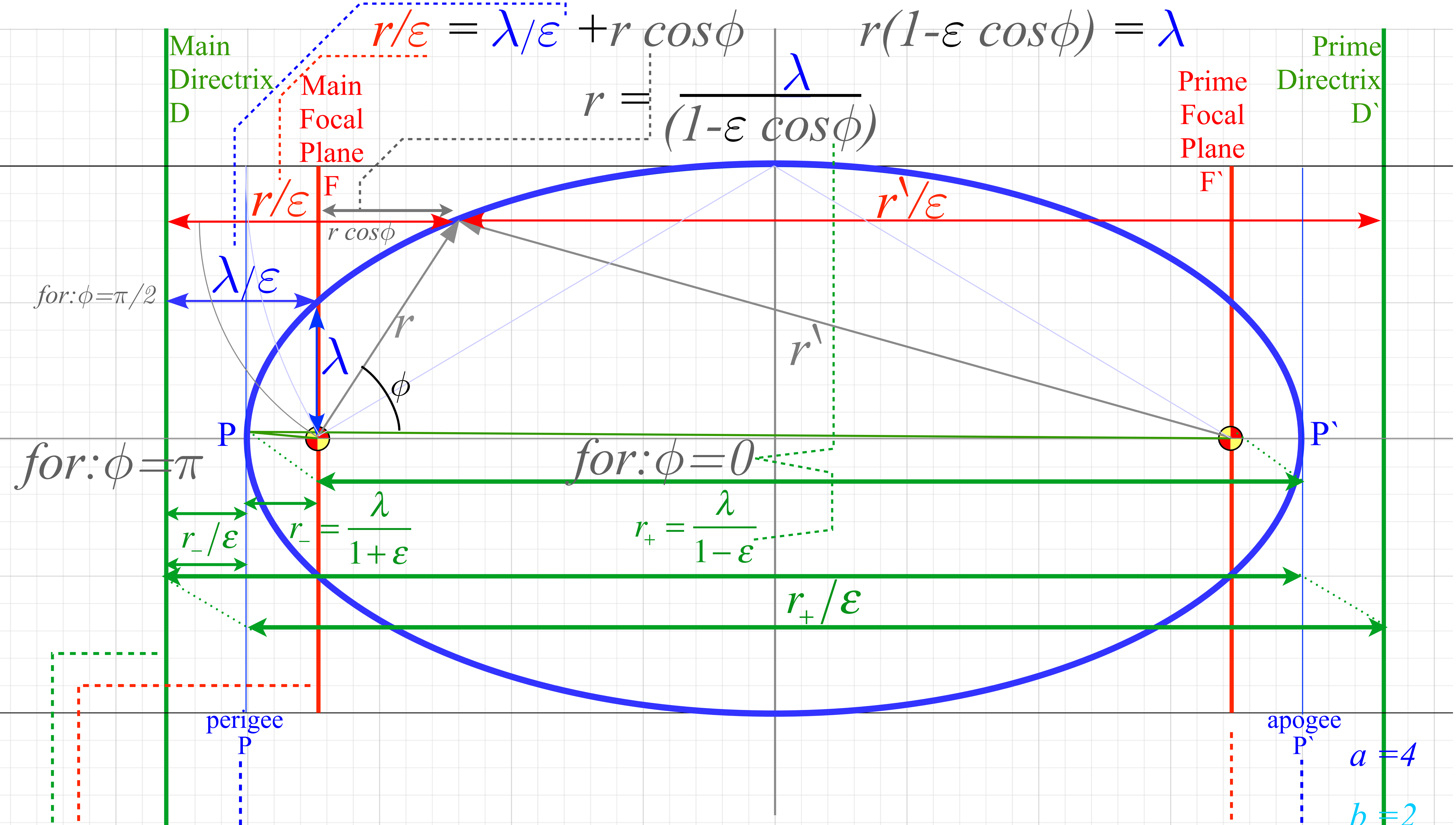
$\lambda = 1$

$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1-\varepsilon^2)$

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r(1-\varepsilon\cos\phi) = \lambda$$

$$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$$



$a = 4$

$b = 2$

$\varepsilon = \sqrt{3}/2$

$\lambda = 1$

Major axis PP′ $= r_+ + r_- = \frac{\lambda}{1-\varepsilon} + \frac{\lambda}{1+\varepsilon} = \frac{\lambda(1+\varepsilon)+\lambda(1-\varepsilon)}{(1+\varepsilon)(1-\varepsilon)} = \frac{2\lambda}{1-\varepsilon^2} = 2a$

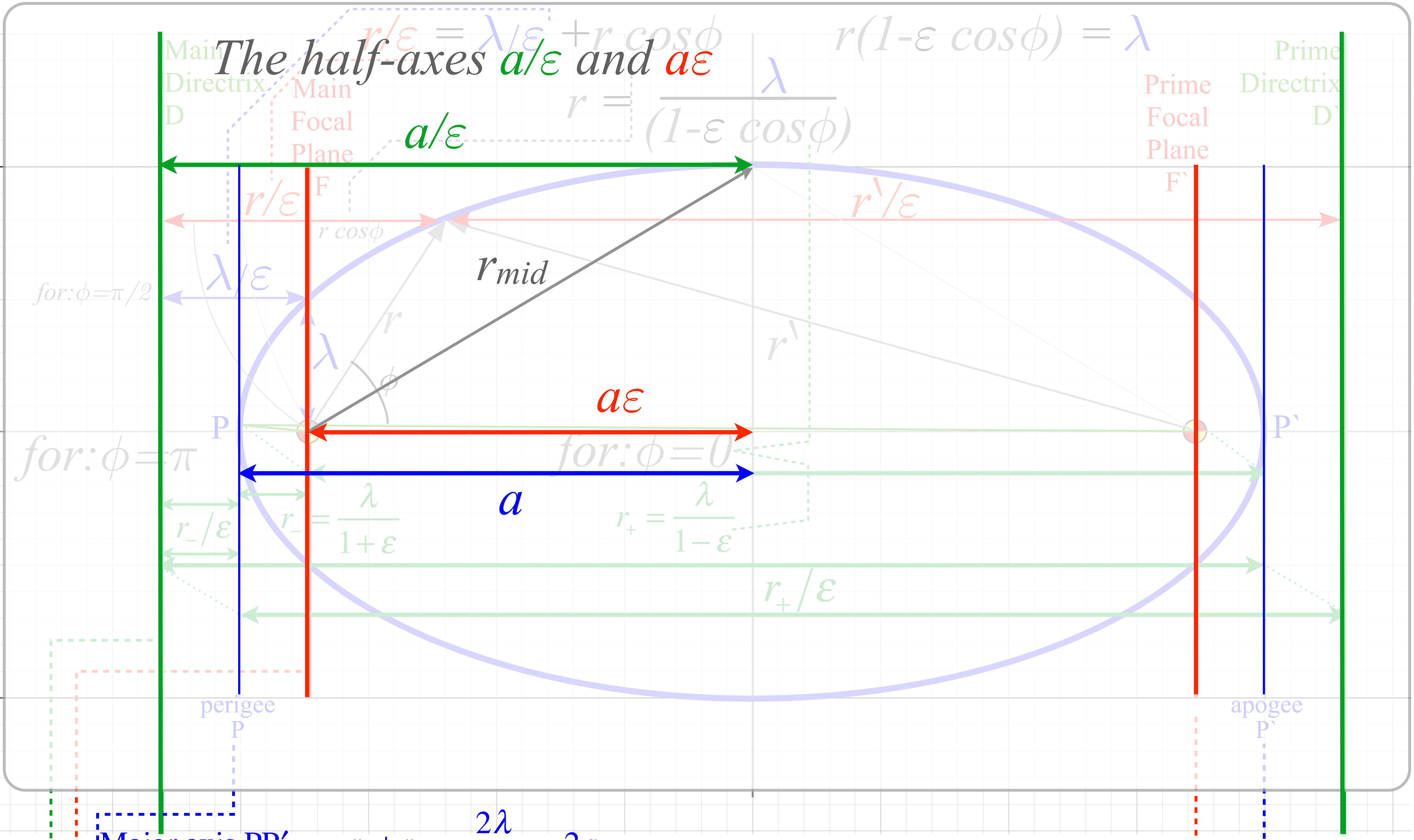
Focal axis FF′ $= r_+ - r_- = \frac{\lambda}{1-\varepsilon} - \frac{\lambda}{1+\varepsilon} = \frac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′ $= \frac{r_+}{\varepsilon} + \frac{r_-}{\varepsilon} = \qquad = \frac{2a}{\varepsilon}$

$\varepsilon^2 = 1 - b^2/a^2$

$\lambda = a(1-\varepsilon^2)$

The half-axes a/ε and $a\varepsilon$

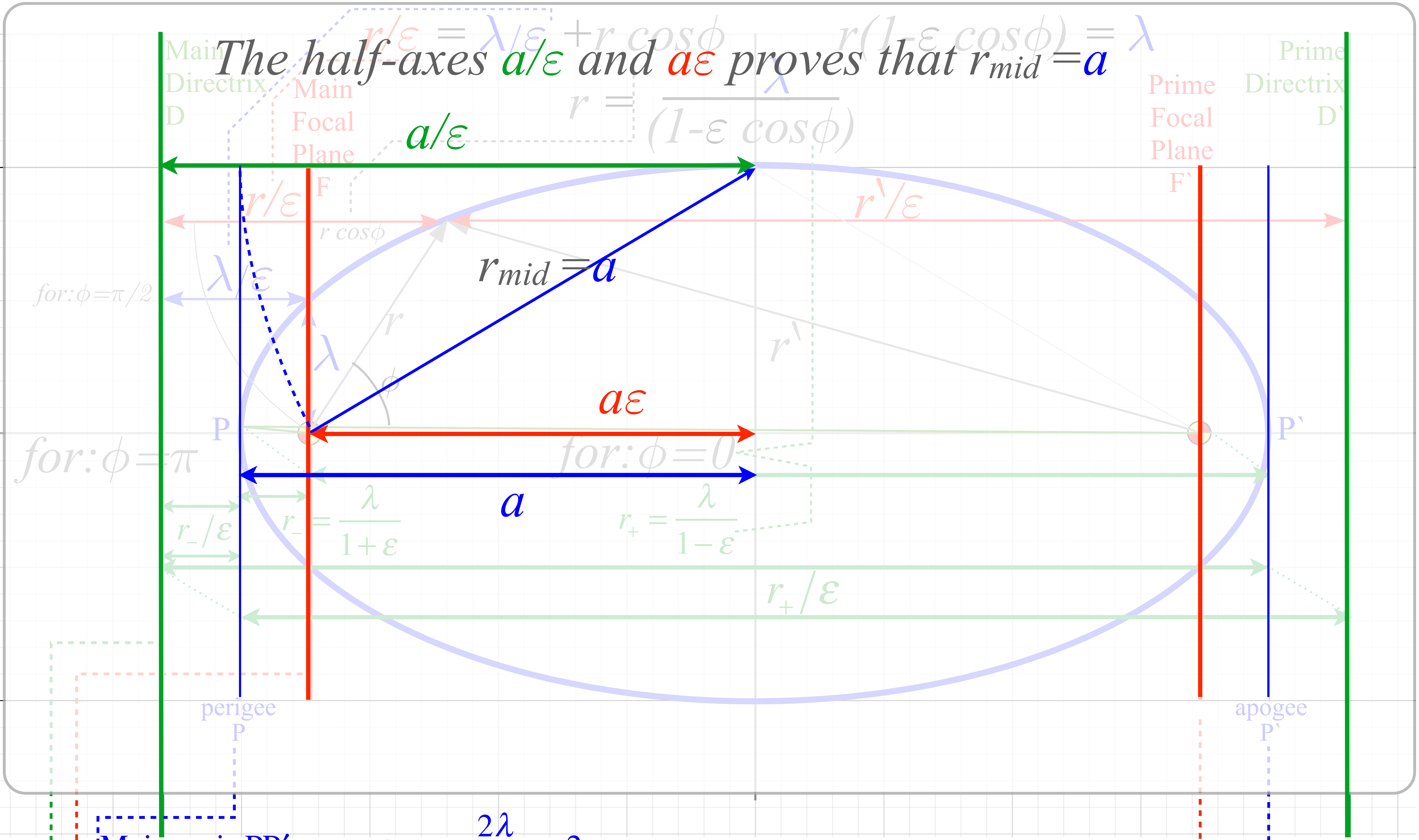


Major axis PP′ $= r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$

The half-axes a/ε and $a\varepsilon$ proves that $r_{mid} = a$

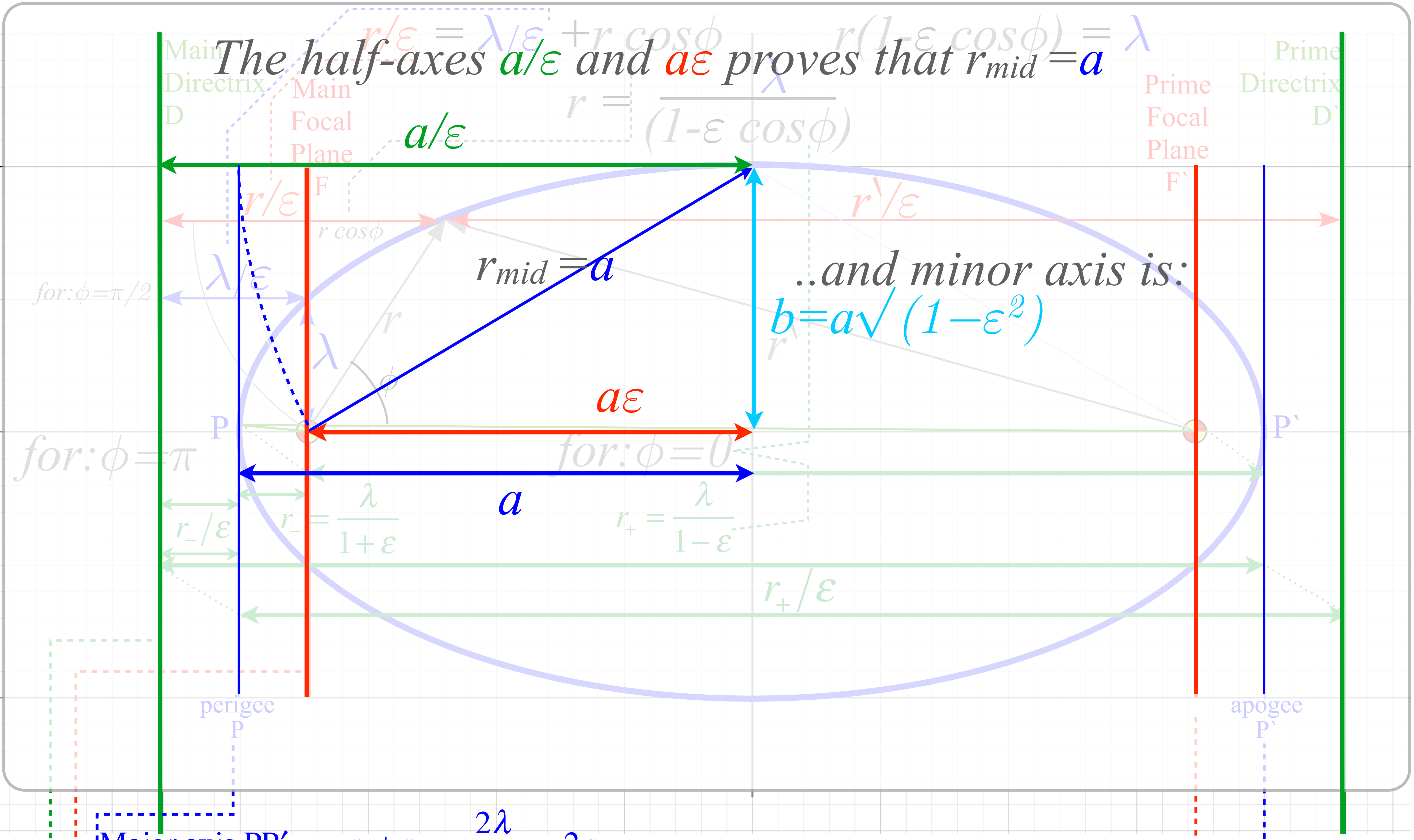


Major axis PP′ $= r_+ + r_- = \frac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \frac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′$= \frac{r_+}{\varepsilon} + \frac{r_-}{\varepsilon} = \frac{2a}{\varepsilon}$

The half-axes a/ε and $a\varepsilon$ proves that $r_{mid} = a$



..and minor axis is:

$b = a\sqrt{(1-\varepsilon^2)}$

Major axis PP′ $= r_+ + r_- = \frac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \frac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

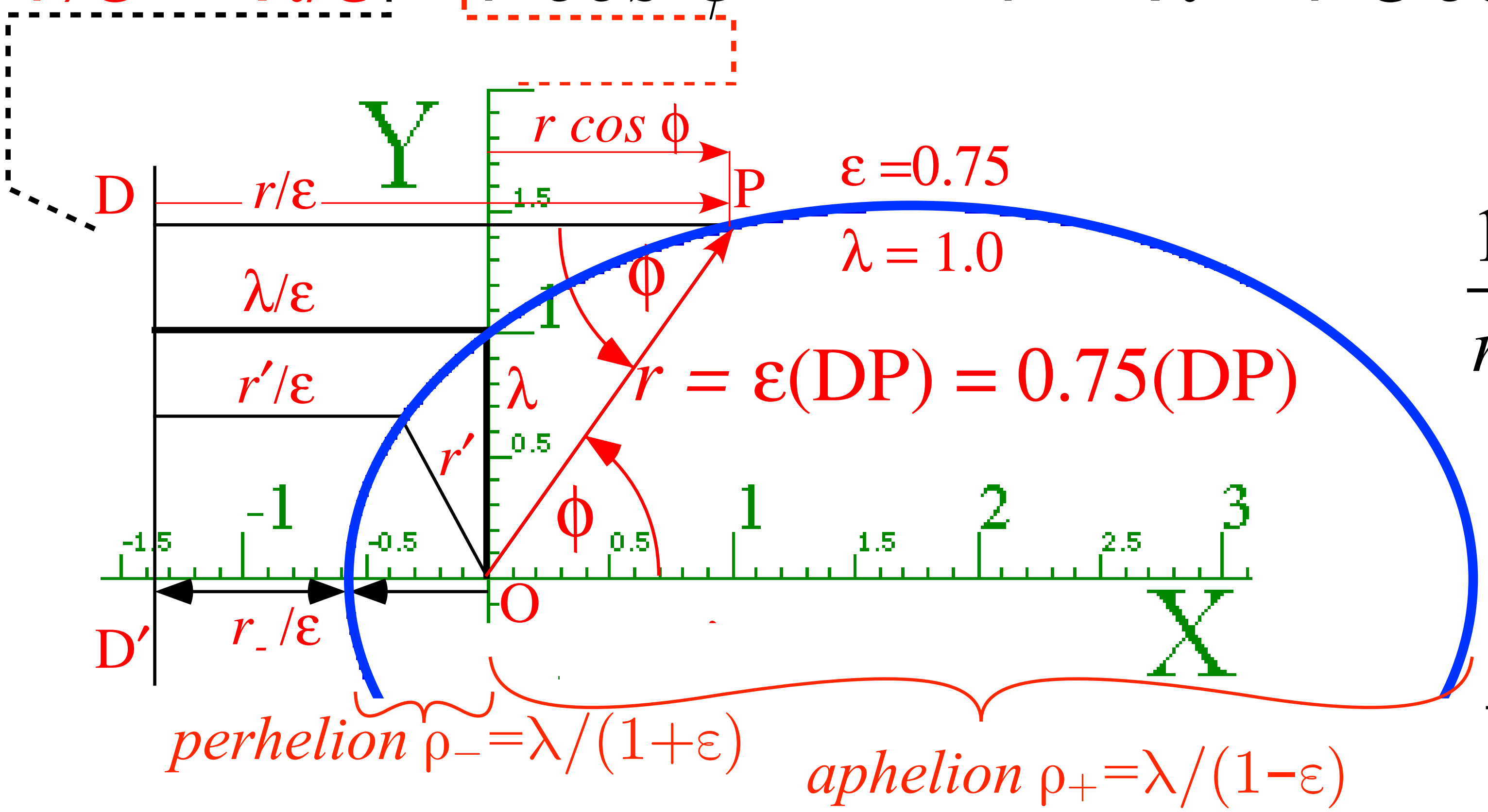
Directrix axis DD′ $= \frac{r_+}{\varepsilon} + \frac{r_-}{\varepsilon} = \frac{2a}{\varepsilon}$

Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi$$

$$r = \lambda + r\,\varepsilon\cos\phi$$

$$r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



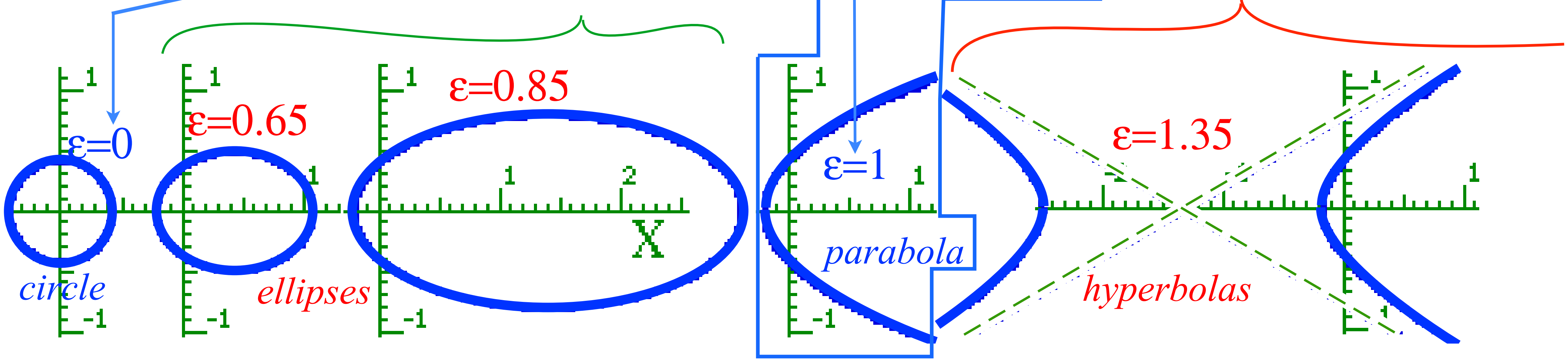
$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\cos\phi$$

Becoming more and more eccentric...

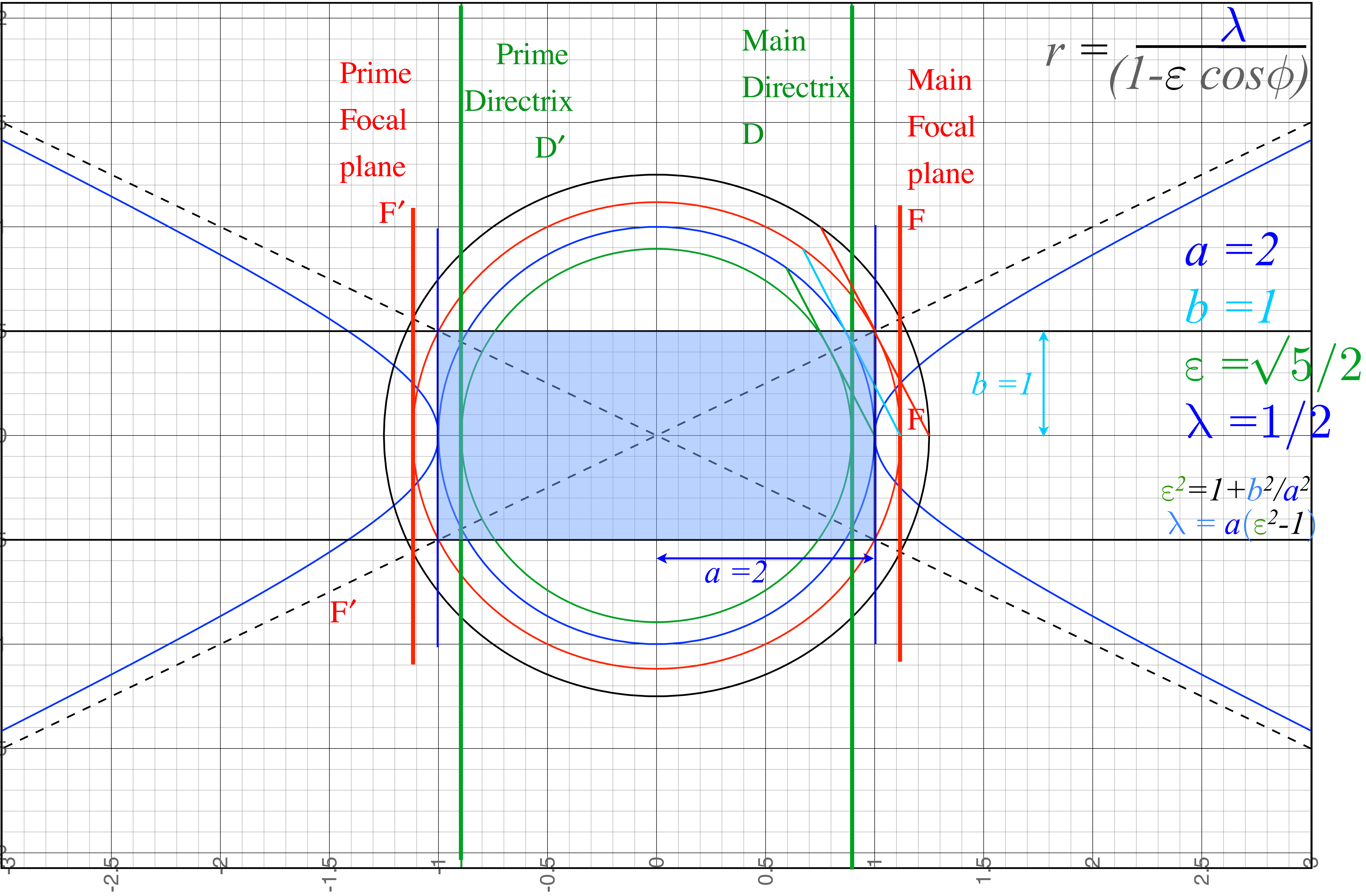
Singular Case!

Eccentricity $\varepsilon=0$ (circle) to $0<\varepsilon<1$ (ellipses) to $\varepsilon=1$ (parabola) to $\varepsilon>1$ (hyperbolas)



Geometry and Symmetry of Coulomb orbits

- *Detailed elliptic geometry*
- ➤ *Detailed hyperbolic geometry*



$$r = \frac{\lambda}{(1-\varepsilon\ \cos\phi)}$$

$a = 2$

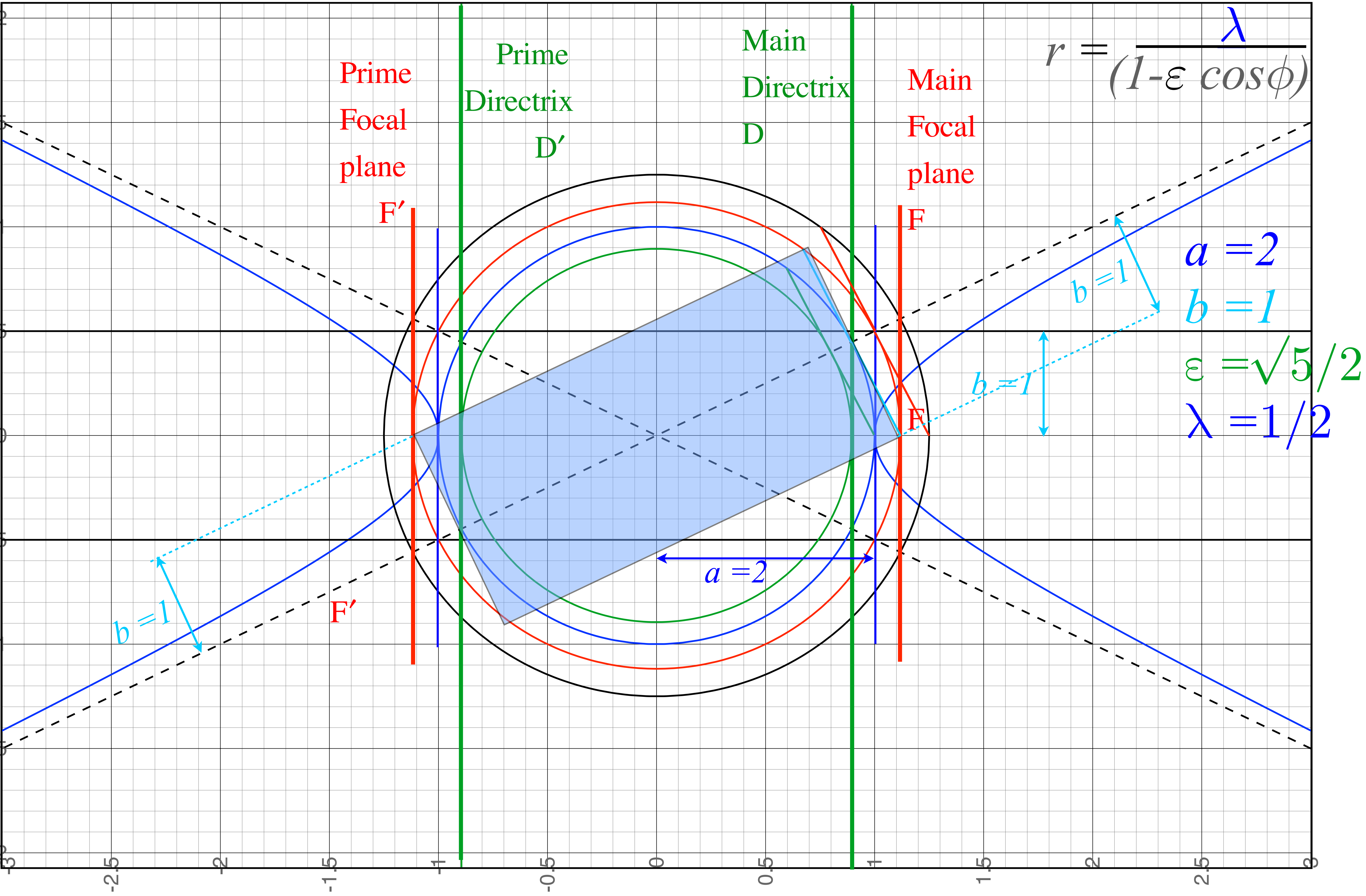
$b = 1$

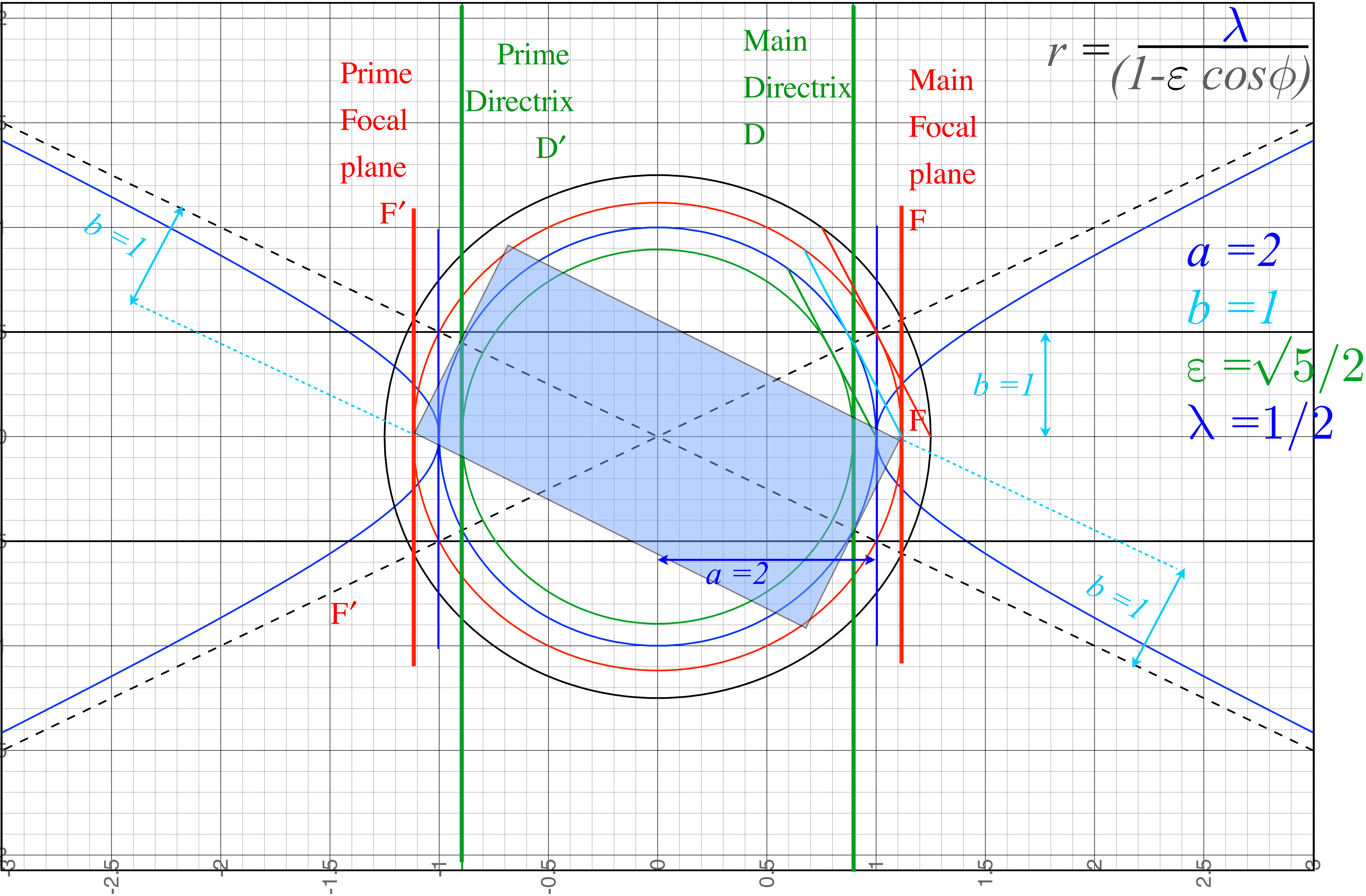
$\varepsilon = \sqrt{5}/2$

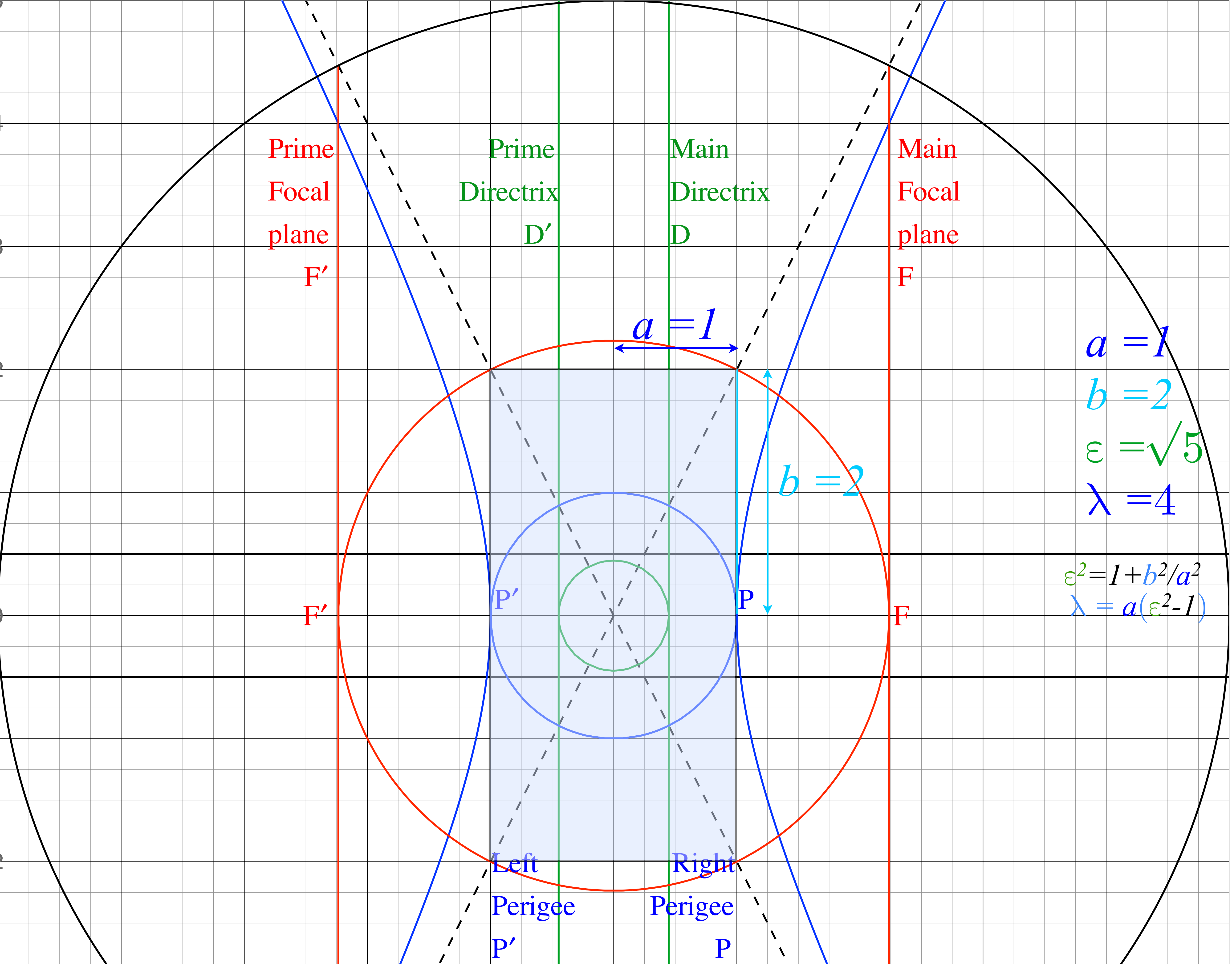
$\lambda = 1/2$

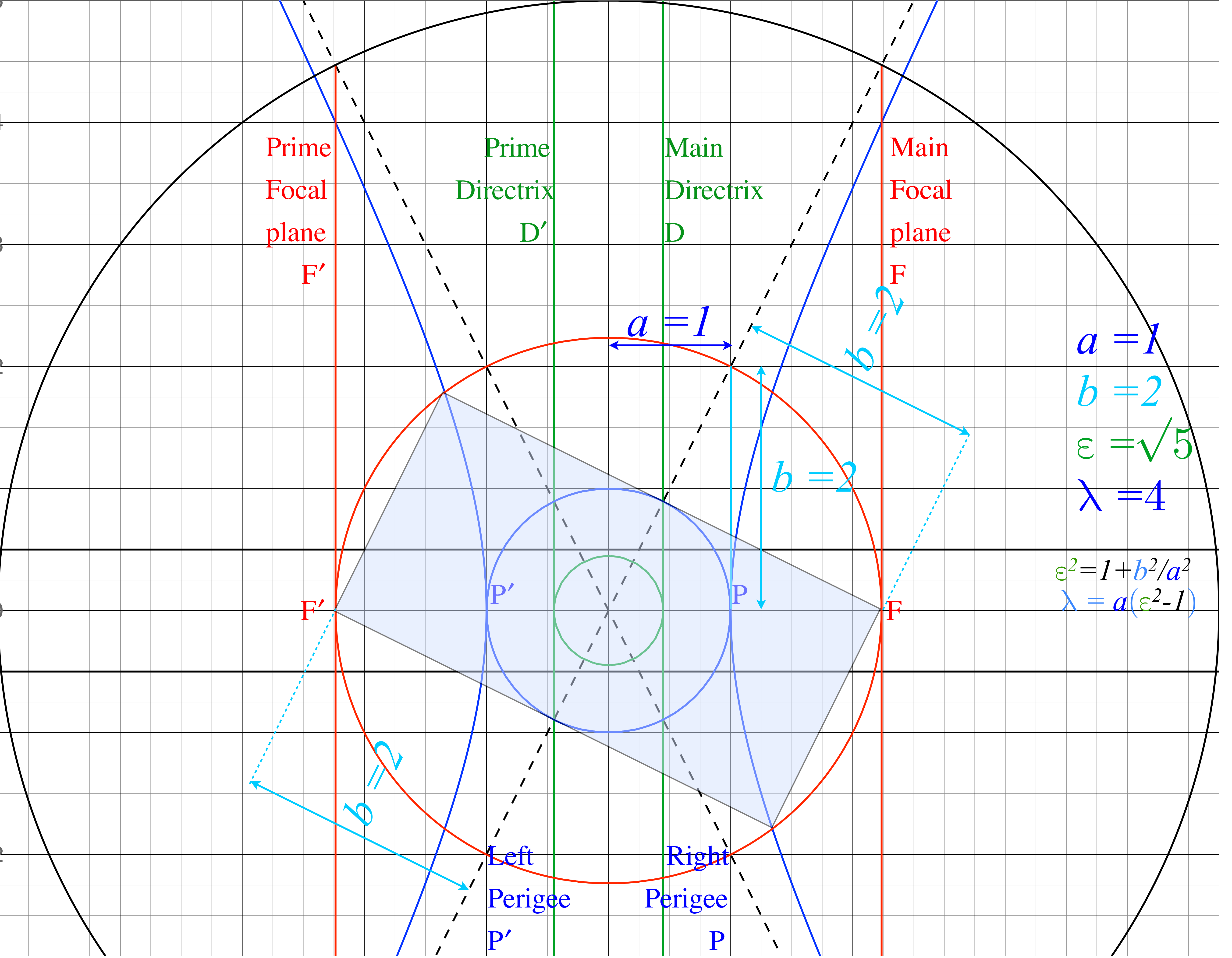
$\varepsilon^2 = 1 + b^2/a^2$

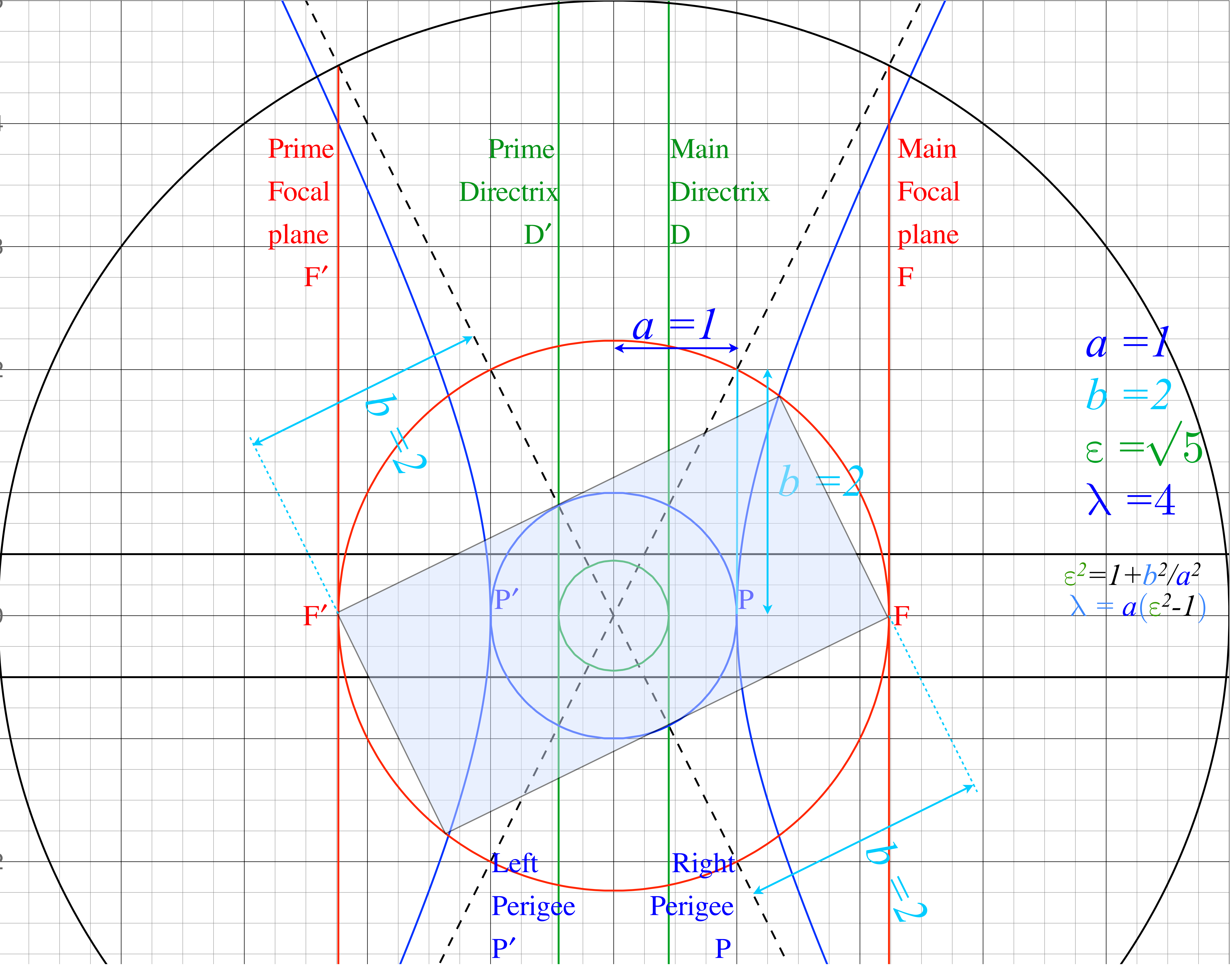
$\lambda = a(\varepsilon^2 - 1)$

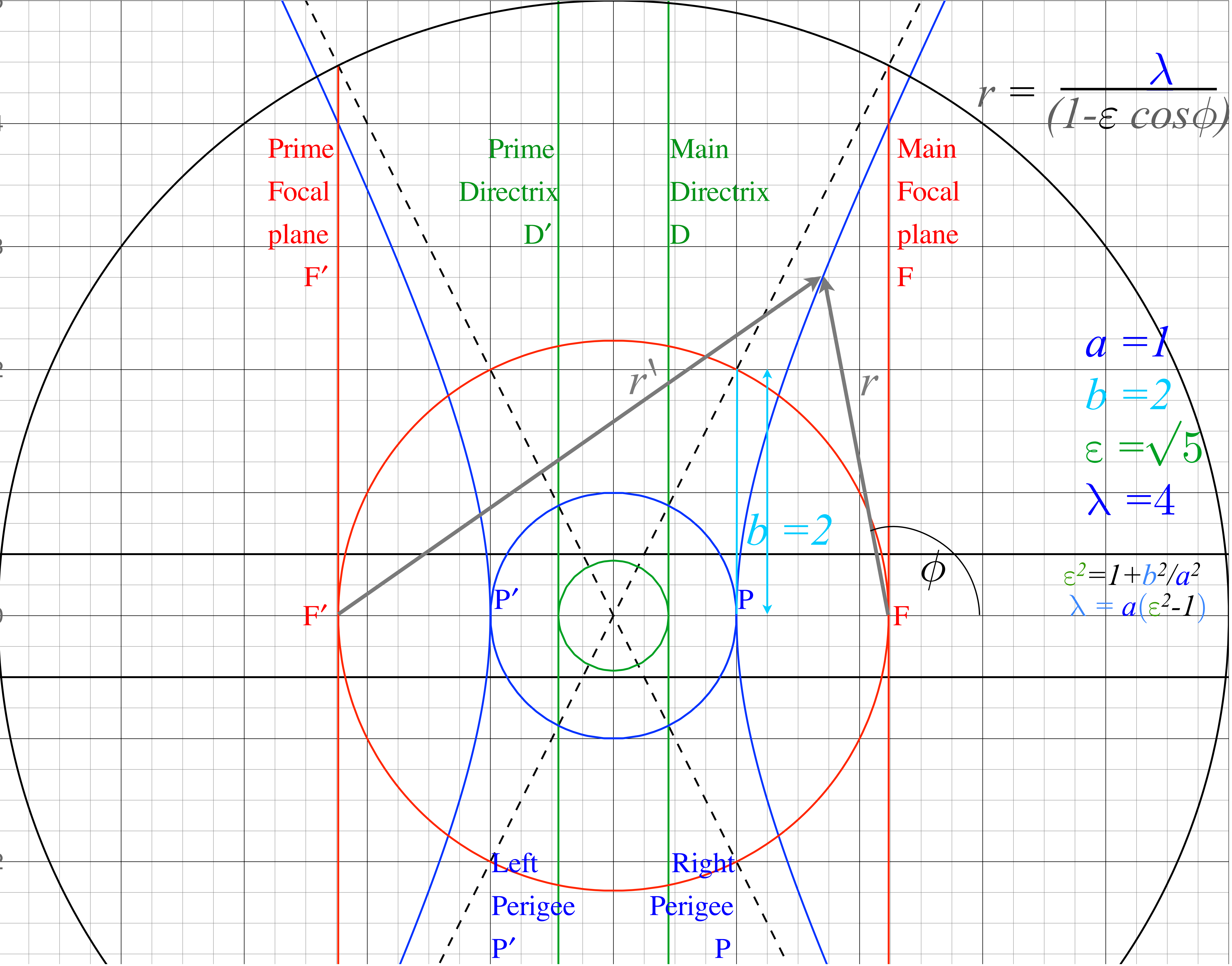

Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
$r = \frac{\lambda}{(1-\varepsilon\ cos\phi)}$
a =2
b =1
ε =√5/2
λ =1/2
b =1
b =1
F
b =1
F′
a =2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3

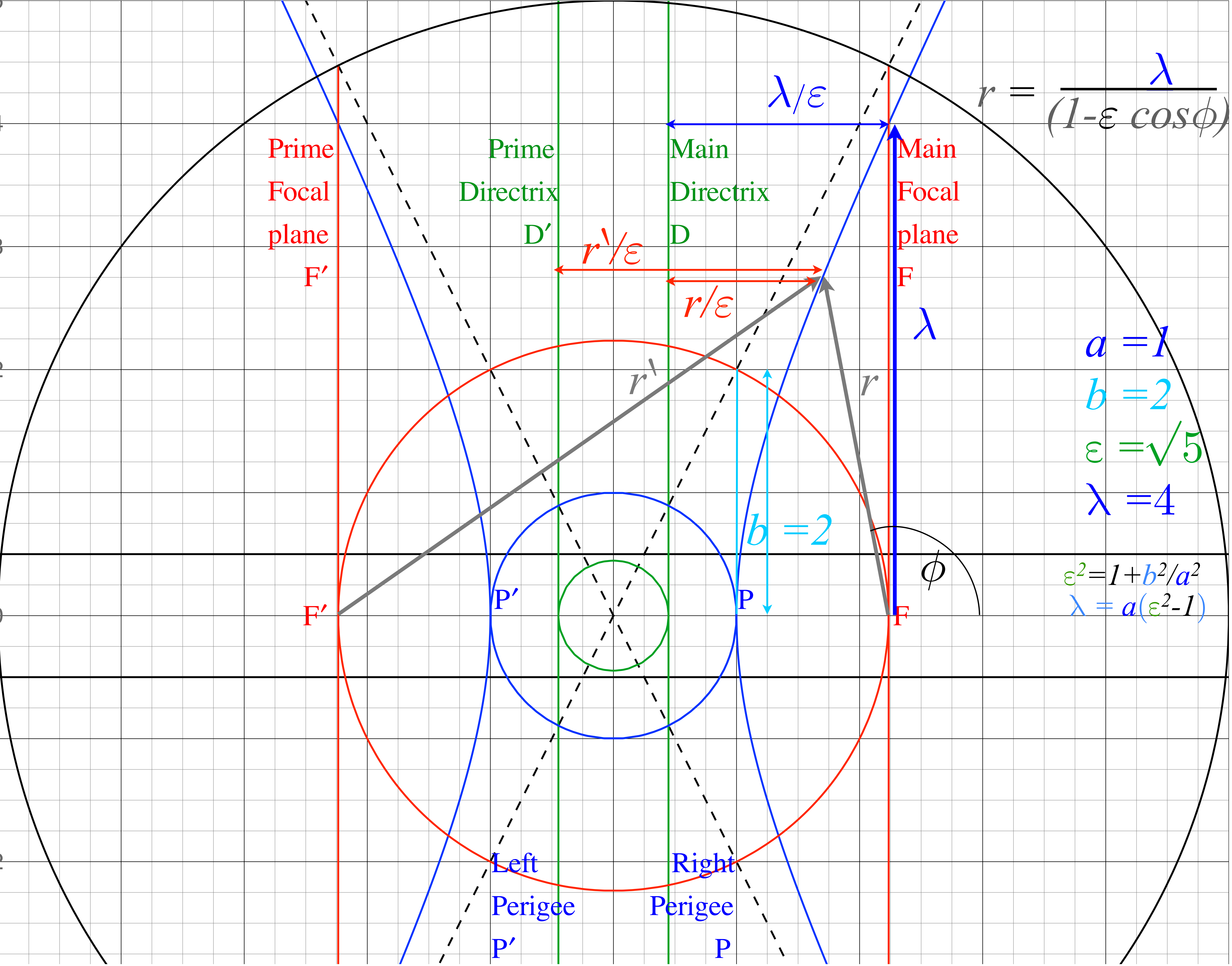

$r = \frac{\lambda}{(1\text{-}\varepsilon \cos\phi)}$
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
F
F′
a =2
b =1
ε =√5/2
λ =1/2
b =1
b =1
b =1
a =2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3

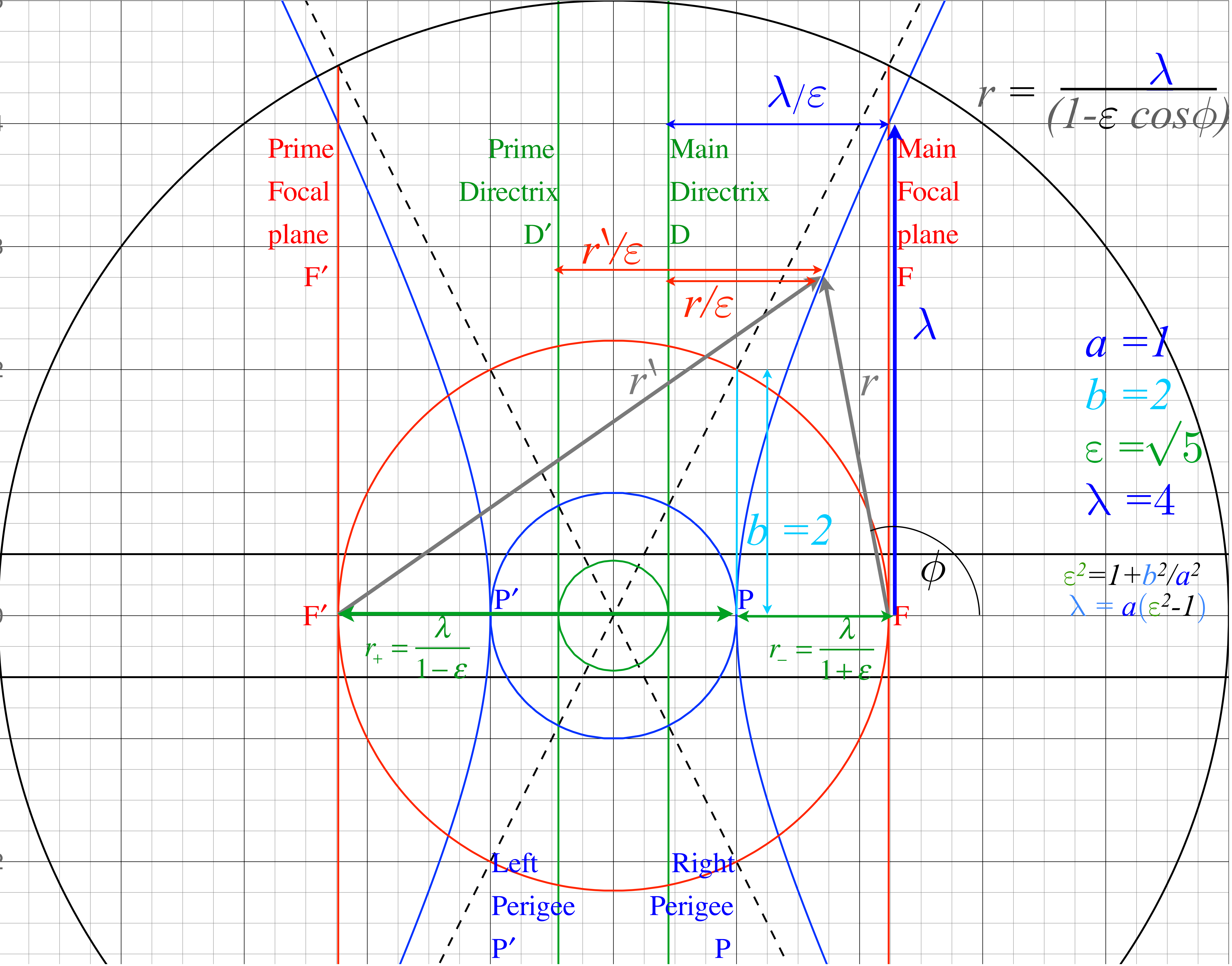

Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
$a = 1$
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
$b = 2$
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
Left Perigee P′
Right Perigee P

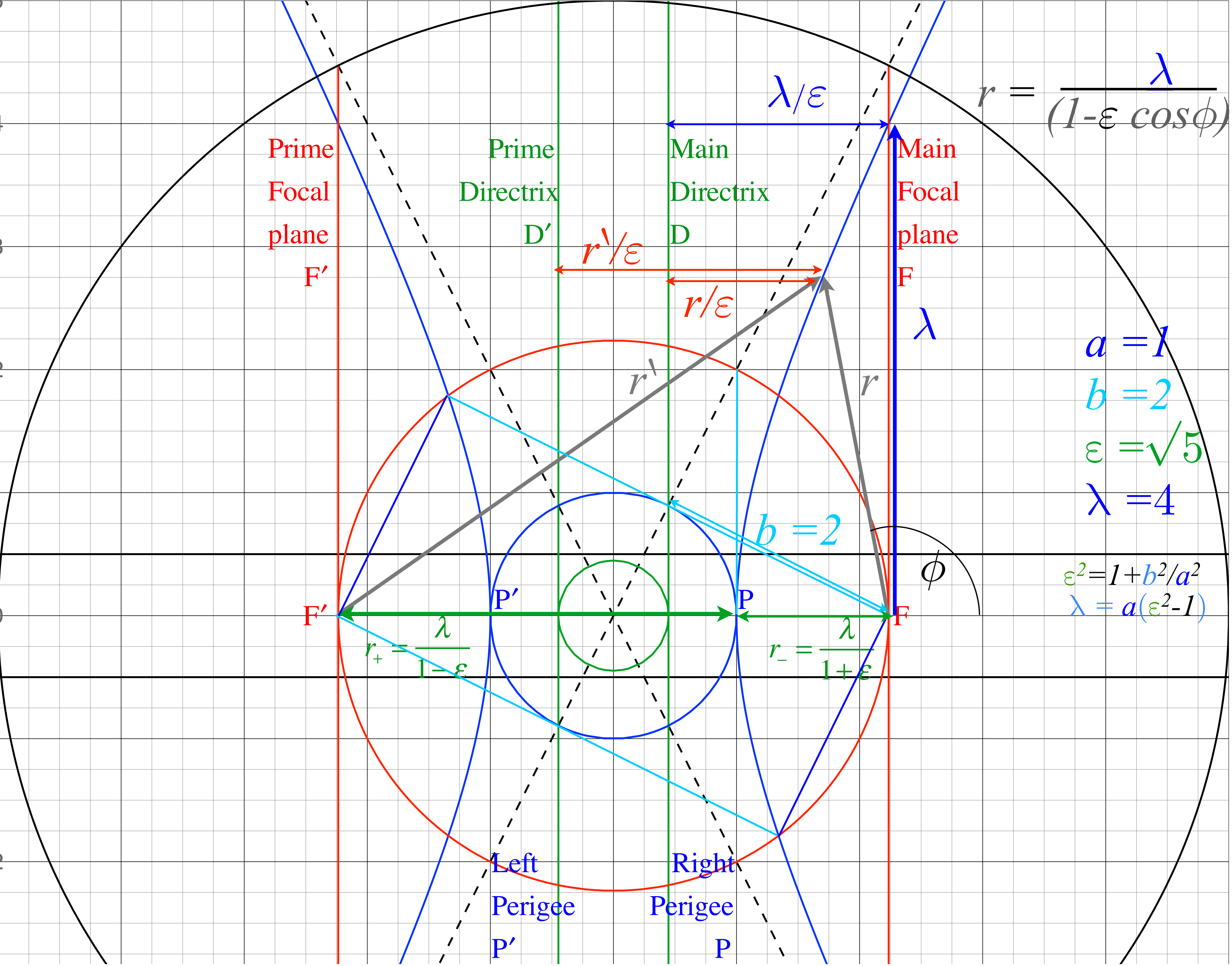

Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
$a = 1$
$b = 2$
$b = 2$
$b = 2$
$b = 2$
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
Left
Perigee
P′
Right
Perigee
P

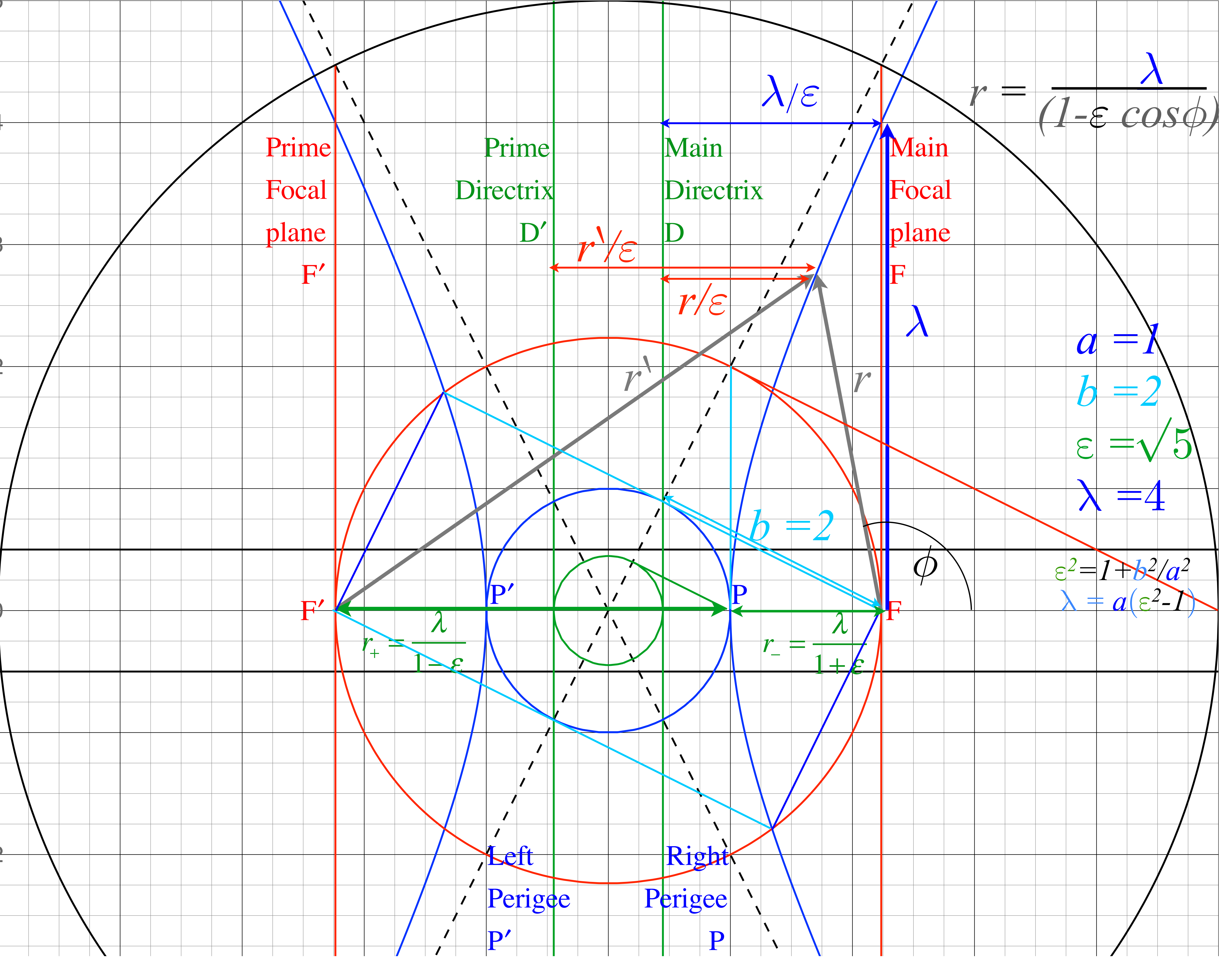

Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
a =1
b =2
b =2
b =2
a =1
b =2
ε =√5
λ =4
ε²=1+b²/a²
λ = a(ε²-1)
P′
P
F′
F
Left
Perigee
P′
Right
Perigee
P

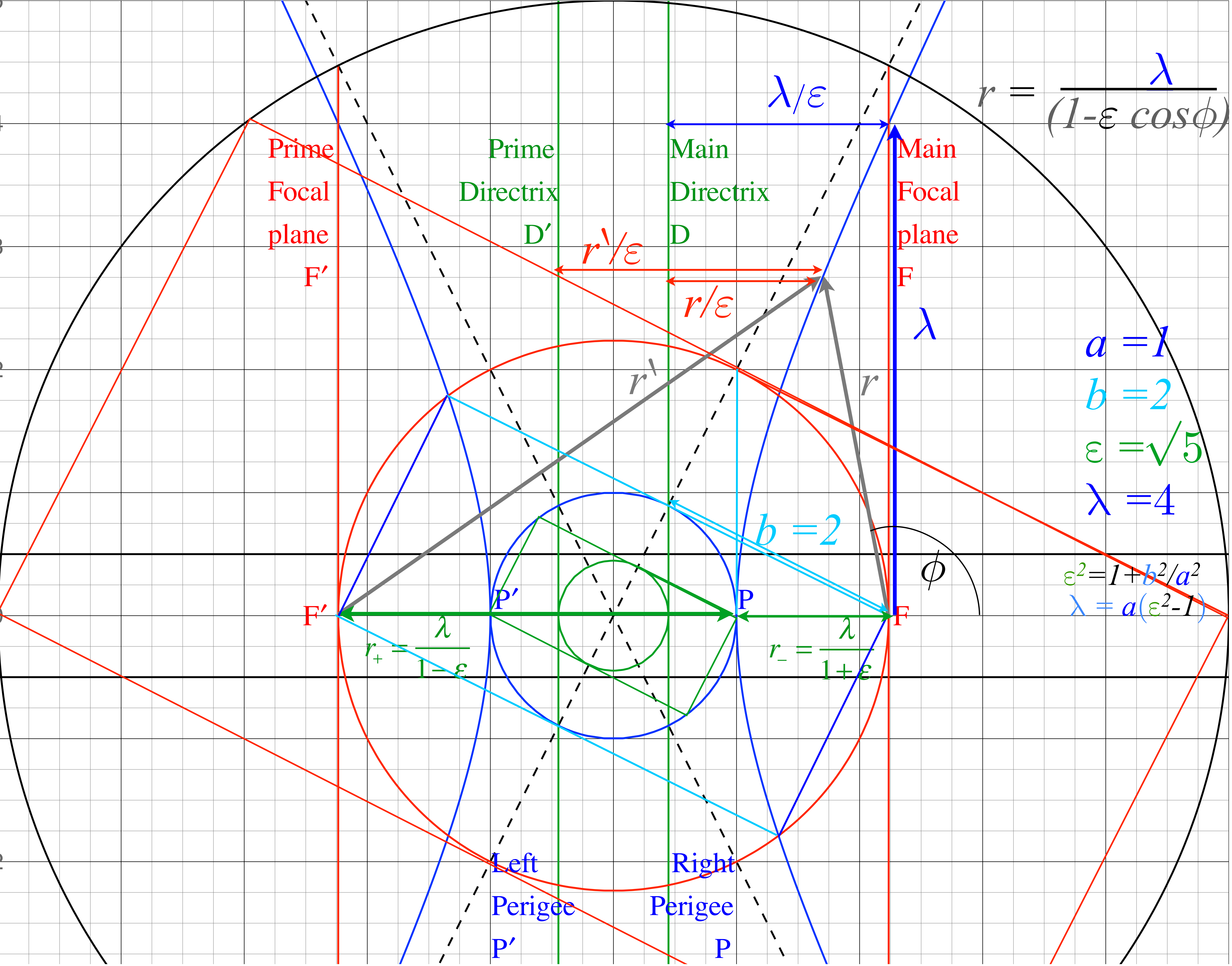

$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
Left Perigee P′
Right Perigee P


$r = \frac{\lambda}{(1-\varepsilon\cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
Left Perigee P′
Right Perigee P


$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P


$r = \frac{\lambda}{(1\text{-}\varepsilon\, cos\phi)}$
λ/ε
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
r'/ε
r/ε
λ
r'
r
$a =1$
$b =2$
$\varepsilon =\sqrt{5}$
$\lambda =4$
$b =2$
ϕ
$\varepsilon^2=1+b^2/a^2$
$\lambda = a(\varepsilon^2\text{-}1)$
P′
P
F′
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left
Perigee
P′
Right
Perigee
P


$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
r'
r
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P



$$r = \frac{\lambda}{(1\text{-}\varepsilon\ cos\phi)}$$

$a = 1$

$b = 2$

$\varepsilon = \sqrt{5}$

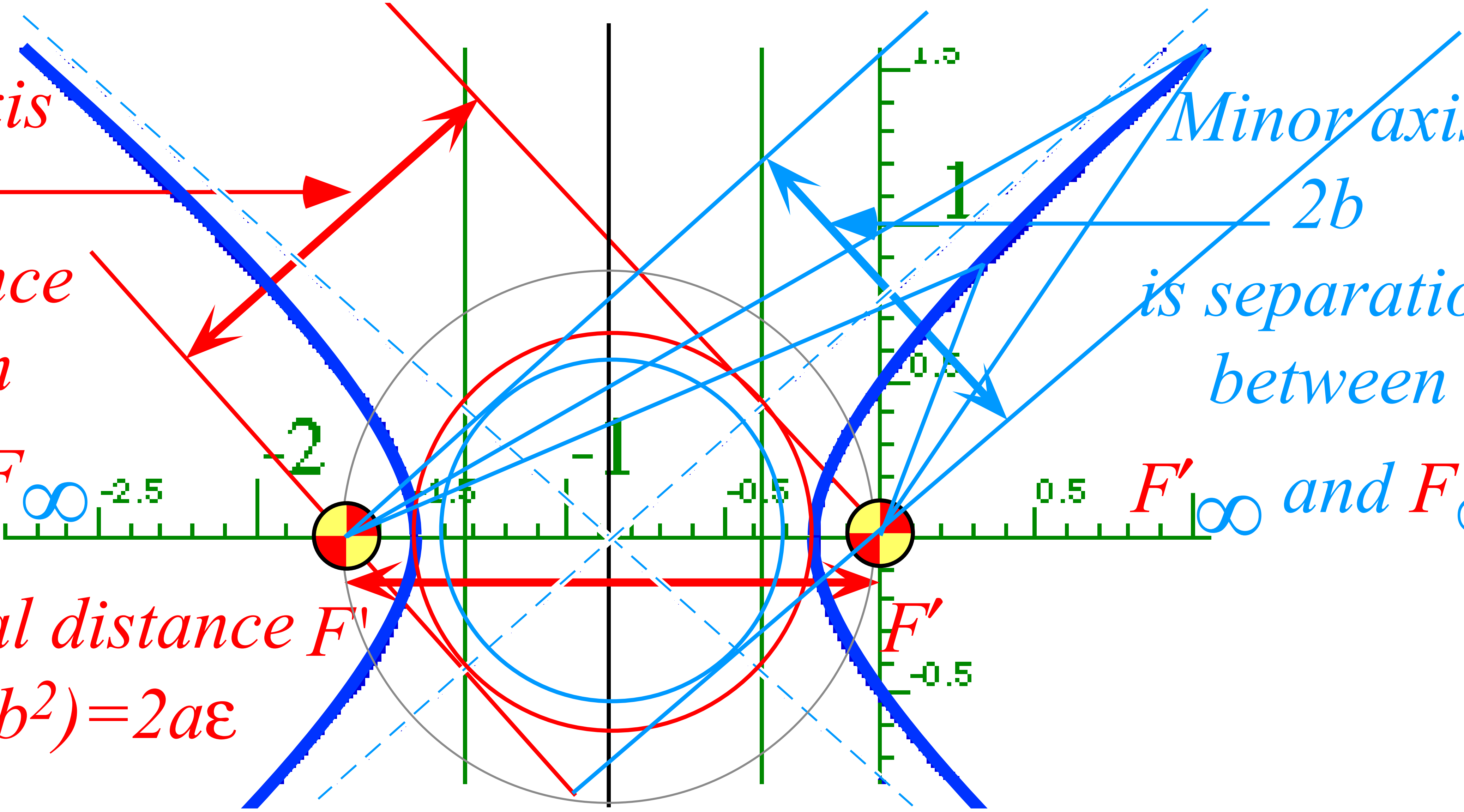
$\lambda = 4$

$\varepsilon^2 = 1 + b^2/a^2$

$\lambda = a(\varepsilon^2 - 1)$

$$r_+ = \frac{\lambda}{1-\varepsilon}$$

$$r_- = \frac{\lambda}{1+\varepsilon}$$



Major axis
2a
is difference
between

F′∞ and F∞

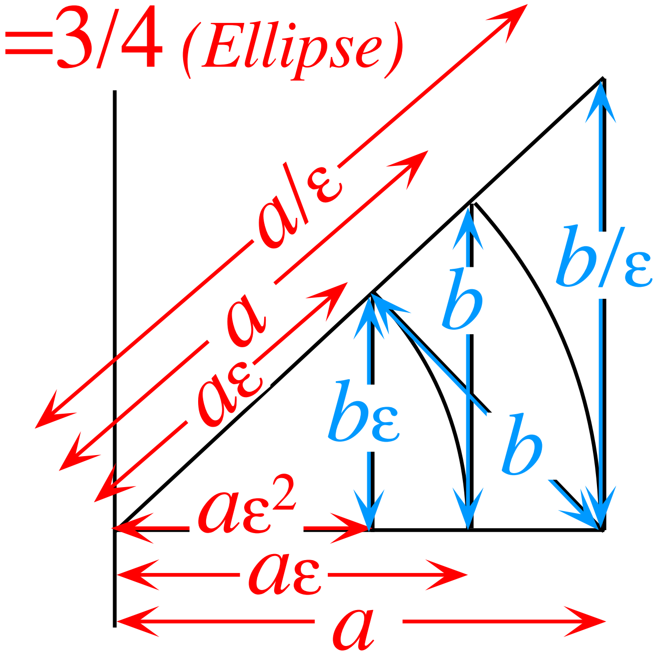
Minor axis
2b
is separation
between

F′∞ and F∞

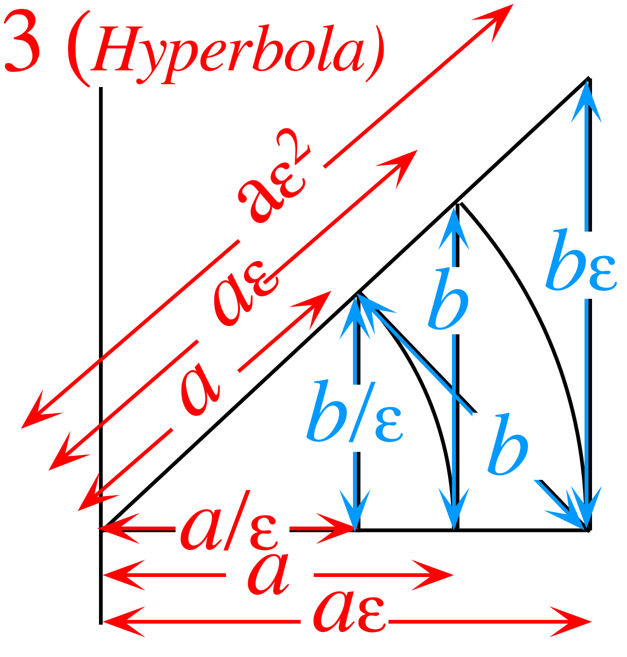
Inter-focal distance F′

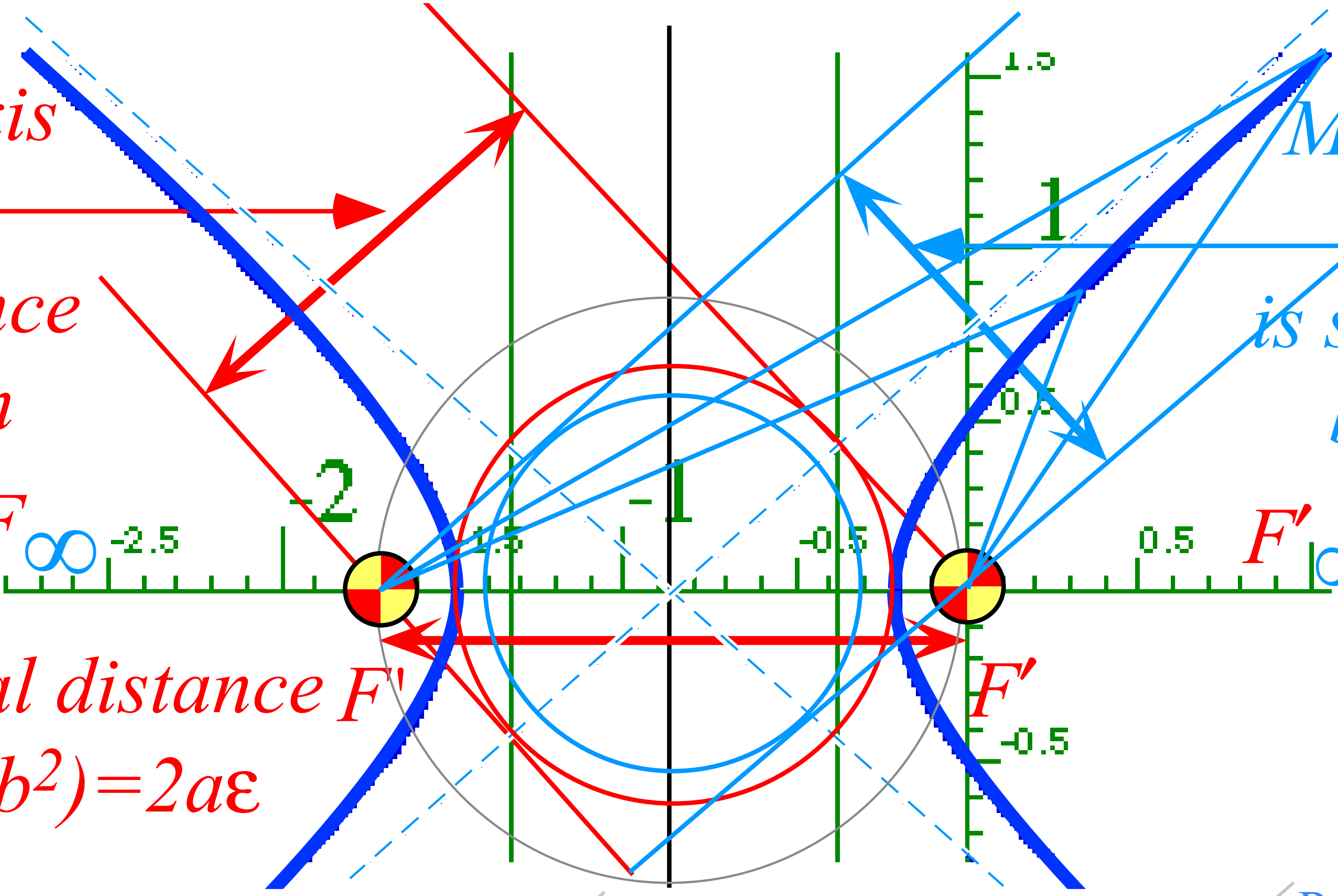
$2\sqrt{(a^2+b^2)}=2a\varepsilon$

$\varepsilon=3/4$ *(Ellipse)*



$\varepsilon=4/3$ *(Hyperbola)*



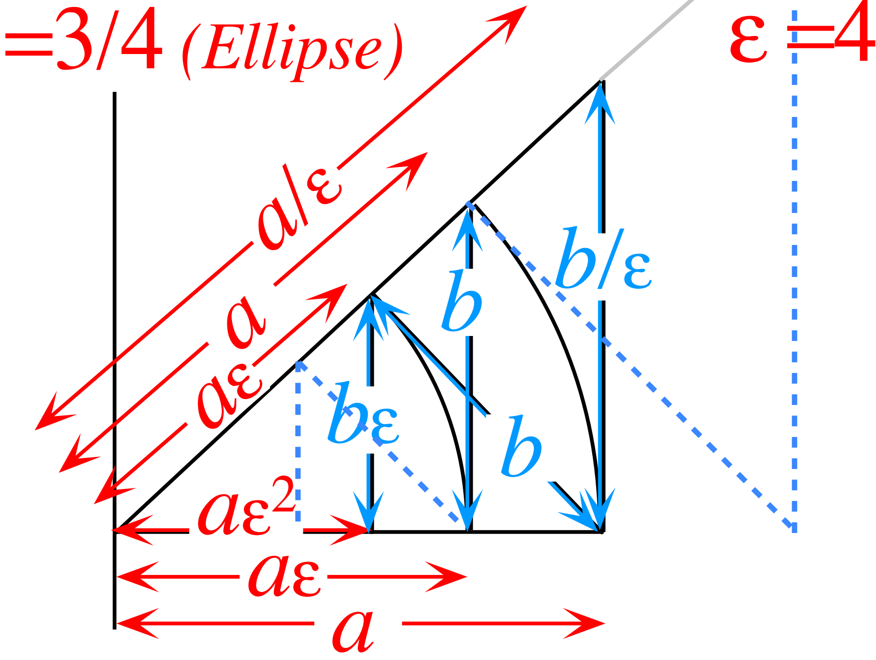


Major axis
2a
is difference
between
F'_∞ and F_∞

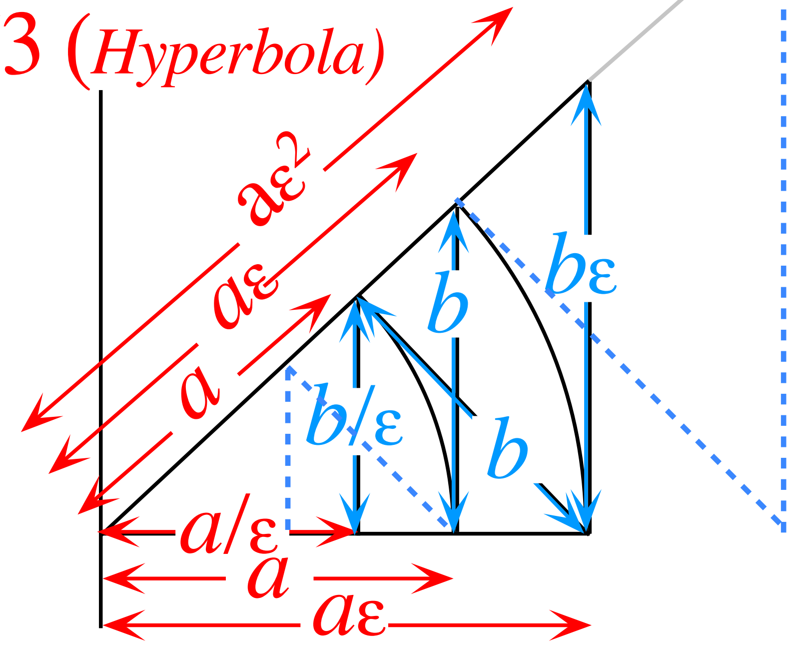
Minor axis
2b
is separation
between
F'_∞ and F_∞

Inter-focal distance F' F'
$2\sqrt{(a^2+b^2)}=2a\varepsilon$

$\varepsilon=3/4$ *(Ellipse)*



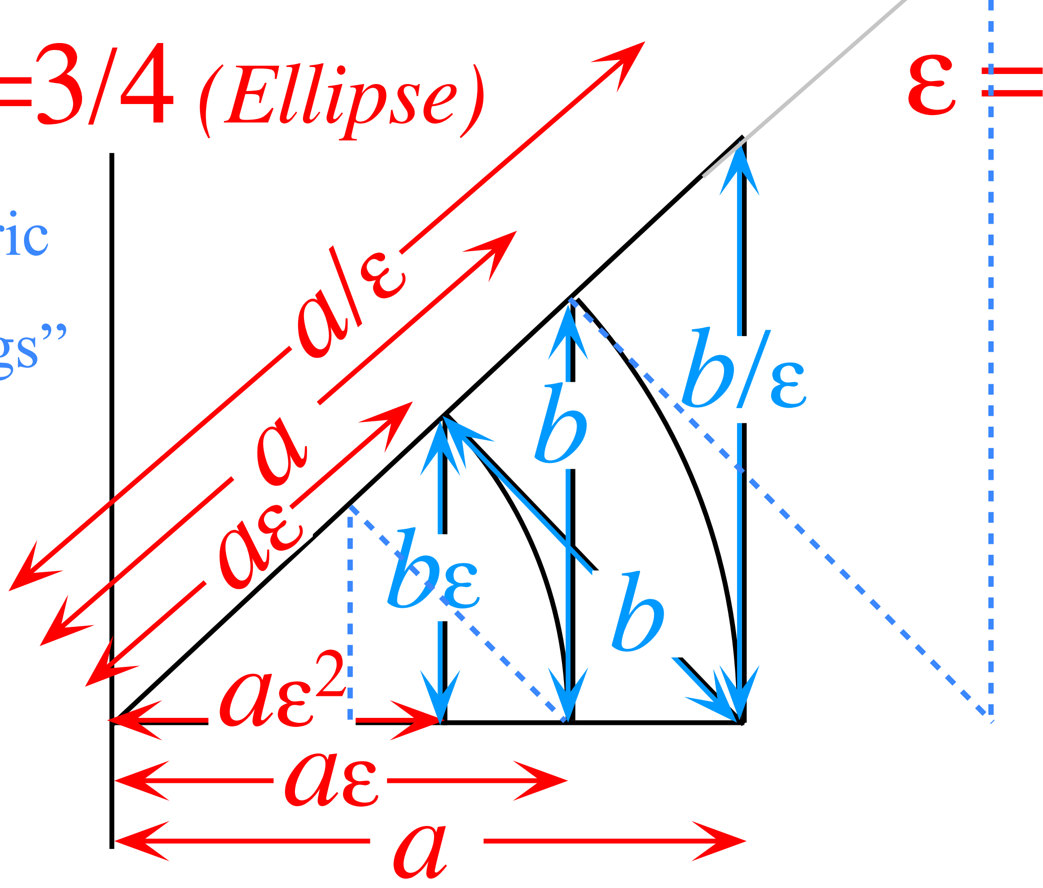
$\varepsilon=4/3$ (*Hyperbola*)



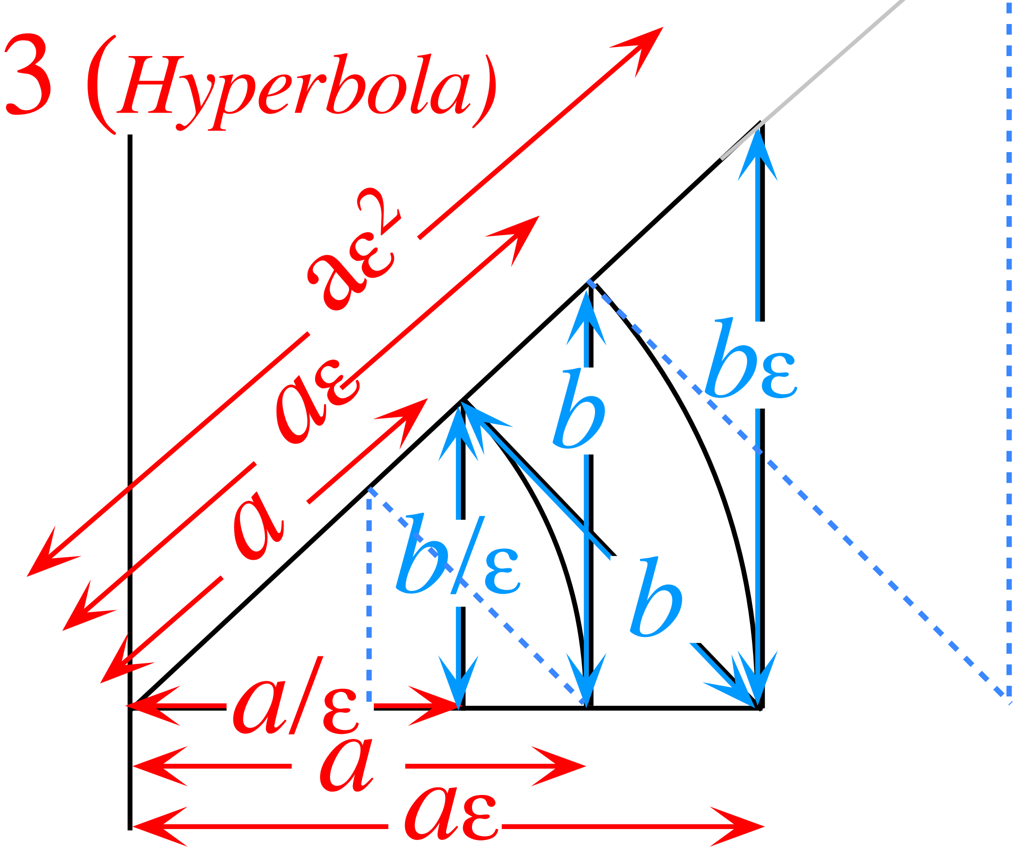
Recall geometric
series "Zig-Zags"
Lect. 5 p.5

Recall geometric series "Zig-Zags" Lect. 5 p.5

$\varepsilon = 3/4$ (*Ellipse*)



$\varepsilon = 4/3$ (*Hyperbola*)



For the elliptic geometry ($\varepsilon < 1$):

$$b^2 = a^2 - a^2\varepsilon^2 = a\lambda\ ,$$
$$b = a\sqrt{(1-\varepsilon^2)} = \sqrt{(a\lambda)},$$

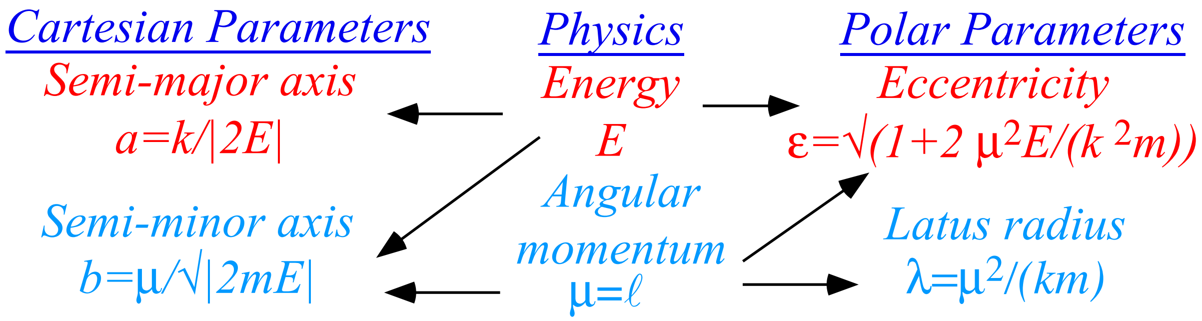
For hyperbolic geometry($\varepsilon > 1$):

$$b^2 = a^2\varepsilon^2 - a^2 = a\lambda,$$
$$b = a\sqrt{(\varepsilon^2-1)} = \sqrt{(a\lambda)}.$$

(λ,ε)-(a,b) expressions and their inverses follow.

$$a = \lambda/(1-\varepsilon^2)$$
$$b^2 = \lambda^2/(1-\varepsilon^2)$$
$$\lambda = a(1-\varepsilon^2) = b^2/a$$
$$\varepsilon^2 = 1 - b^2/a^2$$

$$a = \lambda/(\varepsilon^2-1)$$
$$b^2 = \lambda^2/(\varepsilon^2-1)$$
$$\lambda = a(\varepsilon^2-1) = b^2/a$$
$$\varepsilon^2 = 1 + b^2/a^2$$



Cartesian Parameters

Semi-major axis
$a = k/|2E|$

Semi-minor axis
$b = \mu/\sqrt{|2mE|}$

Physics

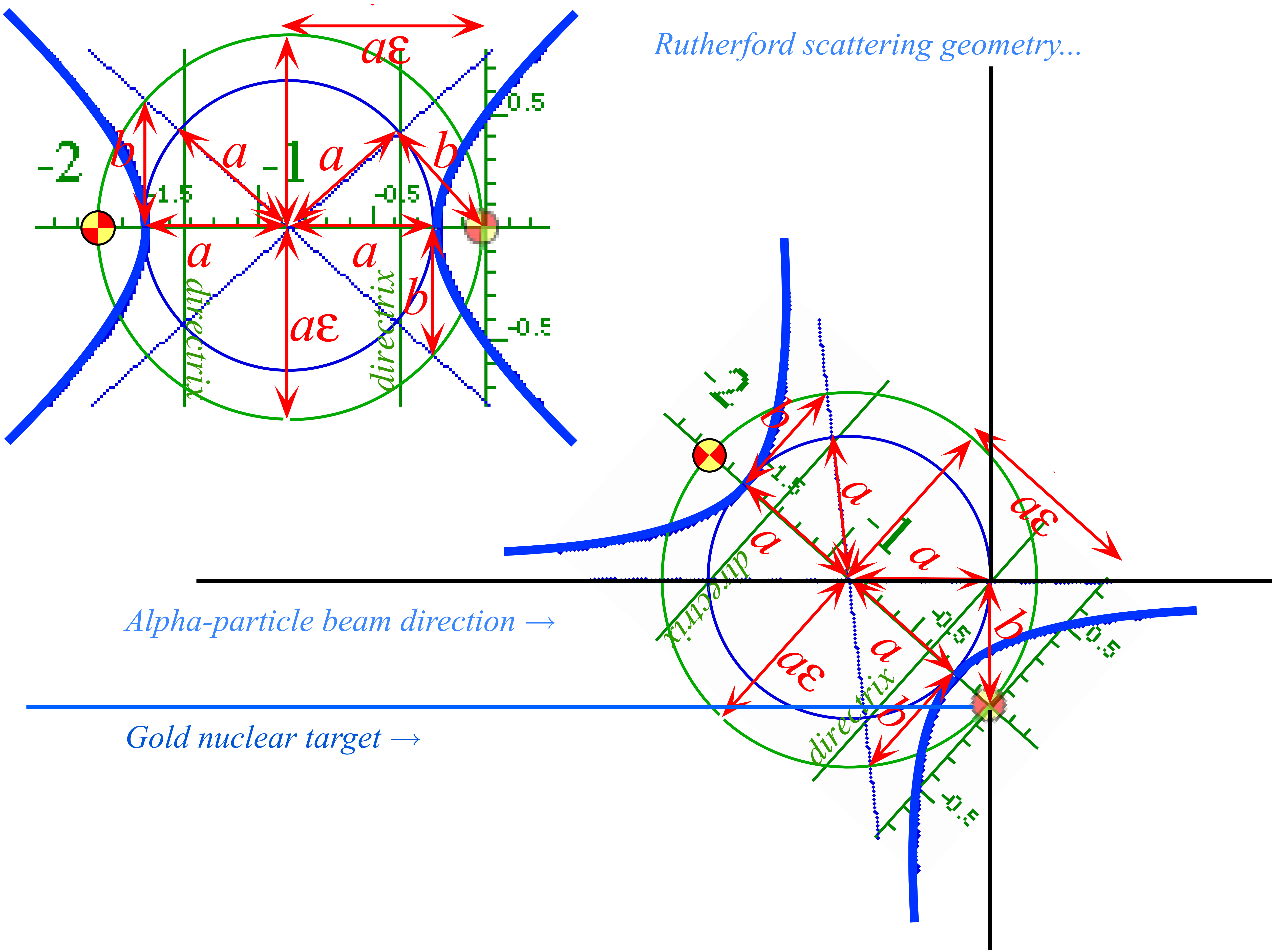
Energy
E

Angular momentum
$\mu = \ell$

Polar Parameters

Eccentricity
$\varepsilon = \sqrt{(1+2\mu^2 E/(k^2 m))}$

Latus radius
$\lambda = \mu^2/(km)$



Rutherford scattering geometry...

Alpha-particle beam direction →

Gold nuclear target →