Lecture 25
Wed. 11.20.2019

Introduction to Orbital Dynamics

(Ch. 2-4 of Unit 5)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Review: "3 steps from Hell" (Lect. 7 Ch. 9 Unit 1)

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

(A mystery similarity appears)

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

Geometry and Symmetry of Coulomb orbits

Detailed elliptic geometry

Detailed hyperbolic geometry

This Lecture's Reference Link Listing

Web Resources - front page | Quantum Theory for the Computer Age | 2017 Group Theory for QM
UAF Physics UTube channel | Principles of Symmetry, Dynamics, and Spectroscopy | 2018 Adv CM
Classical Mechanics with a Bang! | 2018 AMOP
Modern Physics and its Classical Foundations | **2019 Advanced Mechanics**

Lecture #22-25

In reverse order

OscillatorPE Web App: IHO Scenario 2, Coulomb Scenario 3
RelaWavity Web App/Simulator/Calculator: Elliptical - IHO orbits

JerkIt Web App: 2-, 2+, Amp50Omega147-, Amp50Omega296, Amp50Omega602, Gap(1)
MolVibes Web App: C3vN3
WaveIt Web App:
  Dim = 3 w/Wave Components;
  Static Char Table: 6, 12, 12(b), 16, 36, 256
  Quantum Carpet with N=20: Gaussian, Boxcar
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-CPL-2015
QTCA Unit_5 Ch14 2013
Lester. R. Ford, Am. Math. Monthly 45,586(1938)
John Farey, Phil. Mag.(1816) Wolfram
Harter, J. Mol. Spec. 210, 166-182 (2001)
Harter, Li IMSS (2013)
Li, Harter, Chem.Phys.Letters (2015)

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
  Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
  Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
  Pure C-Type A,D=4.055, B=0, C=0.1
  Mixed AB-Type w/Cosine
  Mixed AB Type A=4.0, BU2=0.866…, CU2=0, & D=1.0 w/Stokes & Freq rats
  Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot
  Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot
  Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot

Select, exciting, and/or related Research

***This Indestructible NASA Camera Revealed Hidden Patterns on Jupiter* - seeker-yt-2019**
What did NASA's New Horizons discover around Pluto? - Astrum-yt-2018
Synthetic_Chiral_Light_for_Efficient_Control_of_Chiral_Light–Matter_Interaction_-_Ayuso-np-2019

Classical Mechanics with a Bang! 2018
  Lectures 8, 9, 23 page 93
  Text Unit 6, page=27
ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8,
  and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Spectral Decomposition with Repeated Eigenvalues - 2017 GTQM - Lecture 5

Web based 3D & XR (x∈{A,M,V}, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)

Recent In-House draft Articles:

Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 1 - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 2 - Reimer-www-2019

Quantum Computing (*QC*) and Geometric Algebra (*GA*) references:

Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
Quantum Computing for Computer Scientists - Helwer-mr-yt-2018, Slides
Quantum Computing and Workforce, Curriculum, and App Devel - Roetteler-MS-2019

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Excerpts (Page 44-47 in *Preliminary Draft*) for a GA take on the Complex Numbers
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
GA & QC references (Page 11-16 in Preliminary Draft)

In development, but close to role out.
More Advanced QM and classical references will *soon* be available through our: ***References Page***
Would be great to have our Apache SOLR *Search & Index system up for a bigger* Bang!*)*

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lectures #12 through #21

In reverse order

Wiki on Pafnuty Chebyshev
Nobelprize.org
2005 Physics Award

BoxIt Web Simulations:

A-Type w/Cosine, A-Type w/Freq ratios,
AB-Type w/Cosine, AB-Type 2:1 Freq ratio

OscillIt Web Simulations:

Default/Generic, Weakly Damped #18,
Forced : Way below resonance,On resonance
Way above resonance,Underdamped
Complex Response Plot

CoulIt Web Simulations:

Stark-Coulomb : Bound-state motion in parabolic coordinates
Molecular Ion : Bound-state motion in hyperbolic coordinates
Synchrotron Motion, Synchrotron Motion #2
Mechanical Analog to EM Motion (YouTube video)
iBall demo - Quasi-periodicity (YouTube video)

Trebuchet Web Simulations:

Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)
Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

Recent Articles of Interest:

A_Semi-Classical_Approach_to_the_Calculation_of_Highly_Excited_Rotational_Energies for …
Asymmetric-Top_Molecules_-_Schmiedt-pccp-2017
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*

Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10, pdf_page=135
Calculus of exponentials, logarithms, and complex fields,
RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Select, exciting, and related Research

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces -
Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_
Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

An assist from *Physics Girl* (YouTube Channel):

How to Make VORTEX RINGS in a Pool
Crazy pool vortex - pg-yt-2014
Fun with Vortex Rings in the Pool - pg-yt-2014

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74, Page=75, Page=79*
Pendulum Web Sim
Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted, FVPlot*

CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:
Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:
Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**
Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**
Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:
Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019
A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_* realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_, Nuclear_Collisions - Mishustin-PhysRevC-2007, APS Link & Abstract
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceItIt Web Animation - Scenarios:
49:1 y vs t, 49:1 V2 vs V1, 1:500:1 - 1D Gas Model w/ faux restorative force (Cool),
1:500:1 - 1D Gas (Warm), 1:500:1 - 1D Gas Model (Cool, Zoomed in),
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, 1000:1 with t vs x - Minkowski Plot
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, Quantum_Carpet_wMBars,
Quantum_Carpet_BCar, Quantum_Carpet_BCar_wMBars
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, (v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0)
y_2 vs y_1 plots: (v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0), (v_1, v_2)=(1, -1)
Estrangian plot *V_2 vs V_1: (v_1, v_2)=(0, 1), (v_1, v_2)=(1, -1)*
m_1:m_2 = 4:1
v2 vs v1, y2 vs y1
m_1:m_2 = 100:1, *(v_1, v_2)=(1, 0): V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<Under Construction>>

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- ➤ *Effective potentials for IHO and Coulomb orbits*
    - *Stable equilibrium radii and radial/angular frequency ratios*
    - *Classical turning radii and apogee/perigee parameters*
    - *Polar coordinate differential equations*
    - *Quadrature integration techniques*
        - *Detailed orbital functions*
    - *Relating orbital energy-momentum to conic-sectional orbital geometry*
        - *Kepler equation of time and phase geometry*

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum
μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

Angular momentum μ

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2}$$

where: $p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu$

For ALL central forces

$$\dot{\phi} = \frac{\mu}{m\rho^2}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu$$

$$\dot{\phi} = \frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

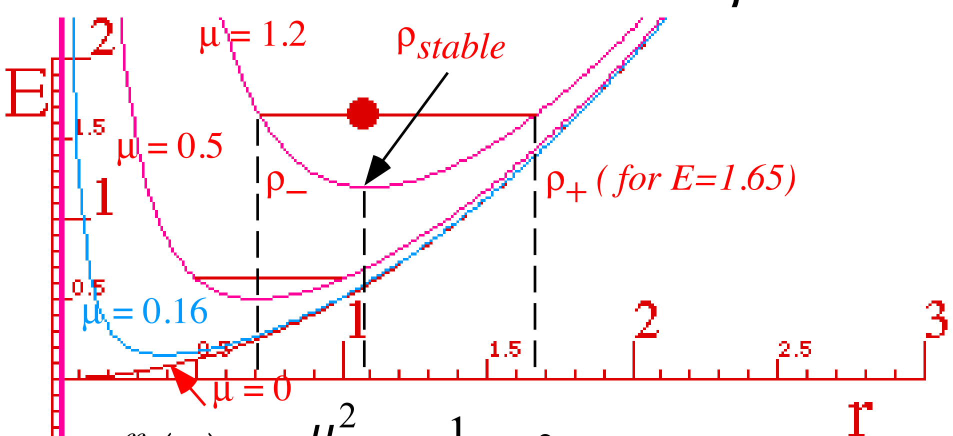
For ALL central forces

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for <u>constant</u> parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

Angular momentum μ

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

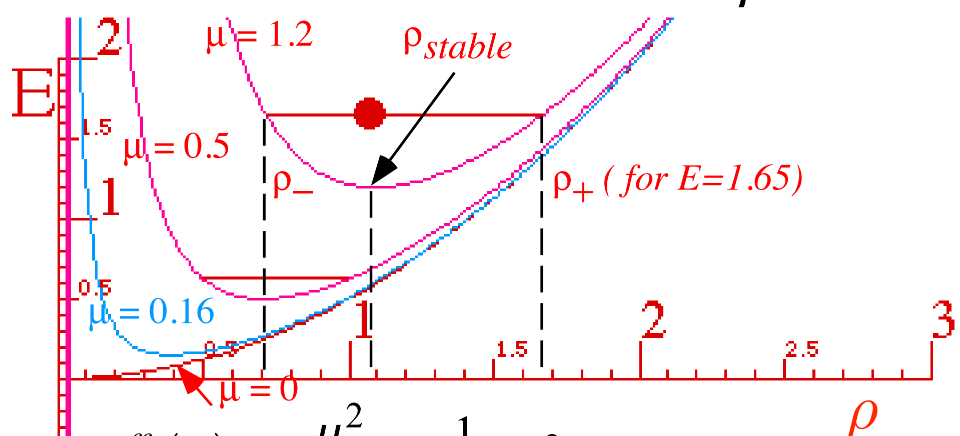
$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

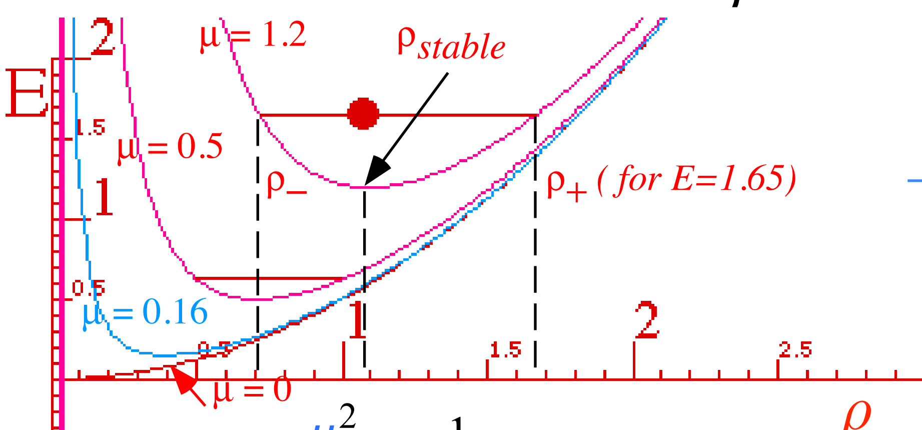
$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

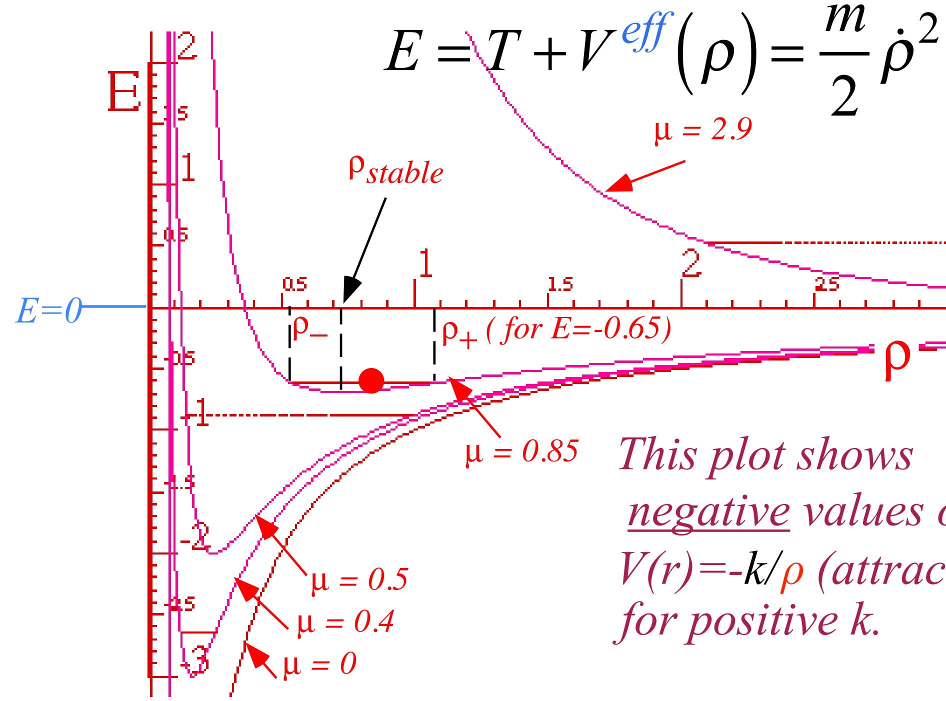
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows *negative values of* *$V(r)=-k/\rho$ (attractive)* *for positive k.*

(Web Simulation: OscillatorPE - IHO)

Web Simulation: OscillatorPE - Coulomb

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

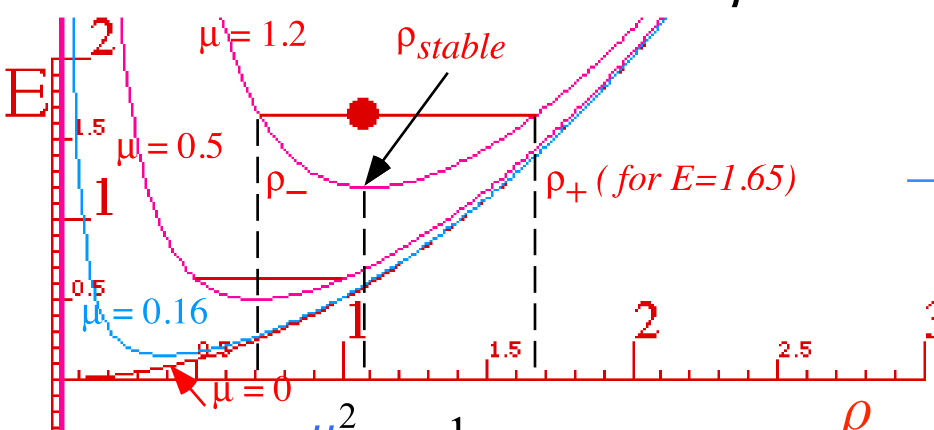
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

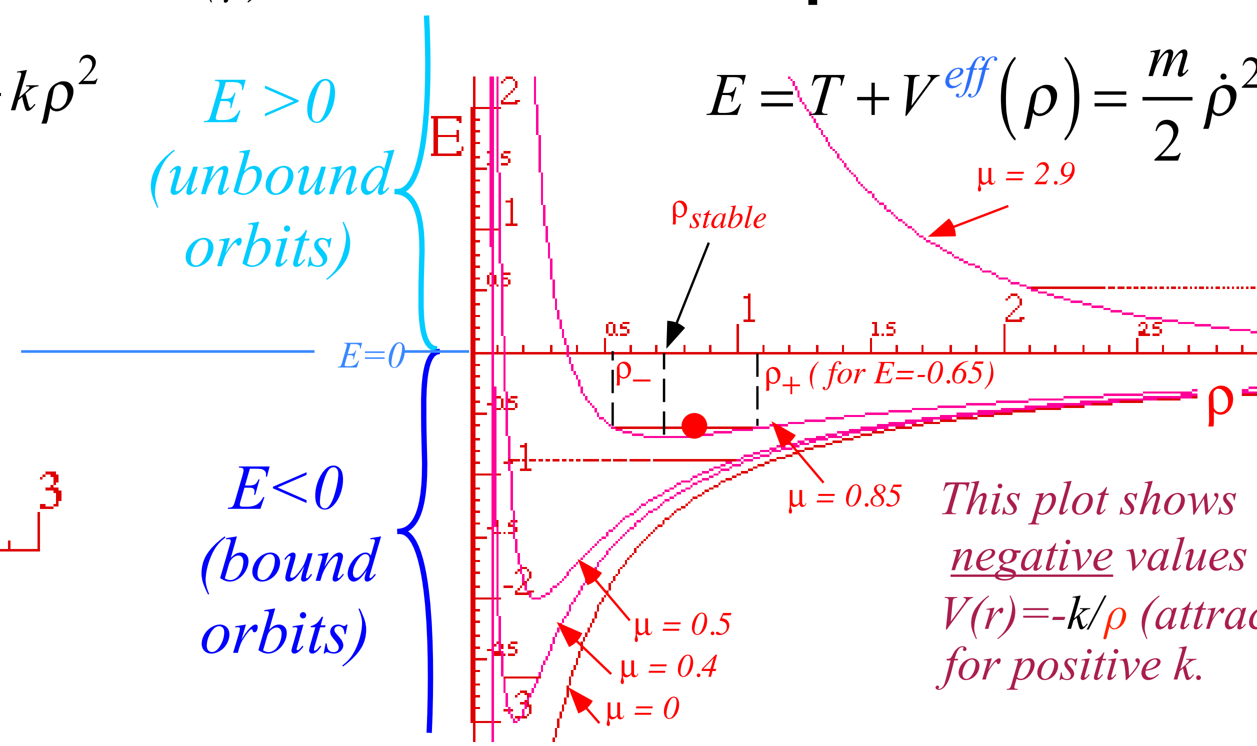
Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ | **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

(Web Simulation: OscillatorPE - IHO) | *Web Simulation: OscillatorPE - Coulomb*

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu$$

$$\dot{\phi} = \frac{\mu}{m\rho^2}$$

For ALL central forces

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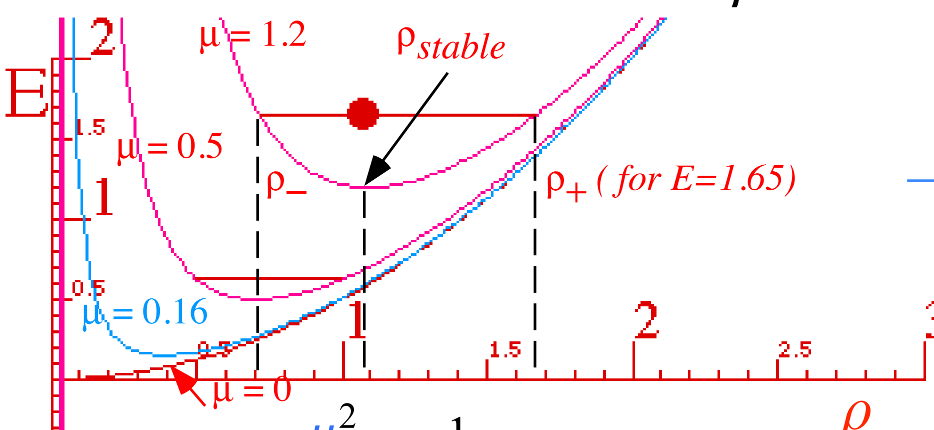


$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$

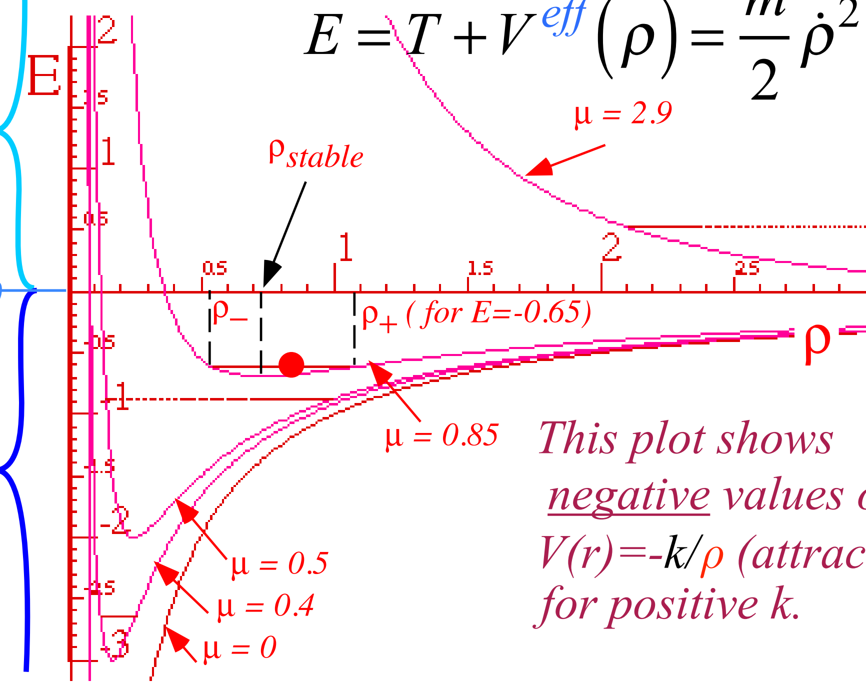
Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$

E >0 (unbound orbits)

E=0

E<0 (bound orbits)



$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

In either case: *IHO or Coulomb orbit blows up if k is negative.*

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

Angular momentum μ

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For ALL central forces

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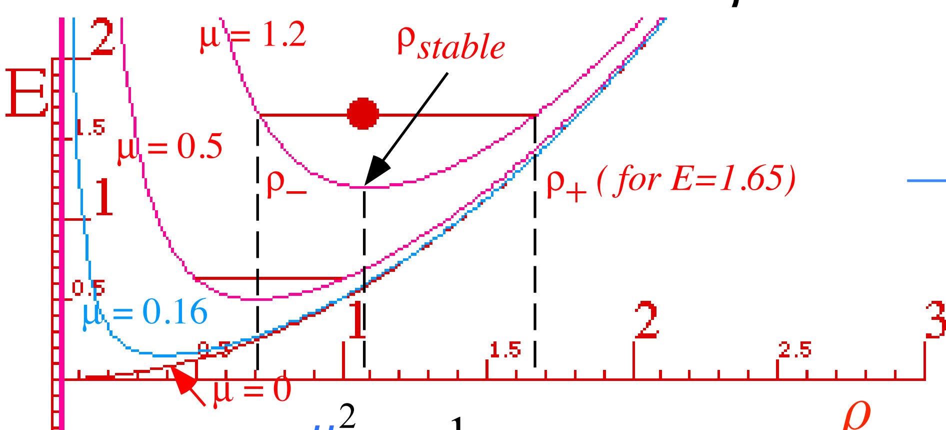

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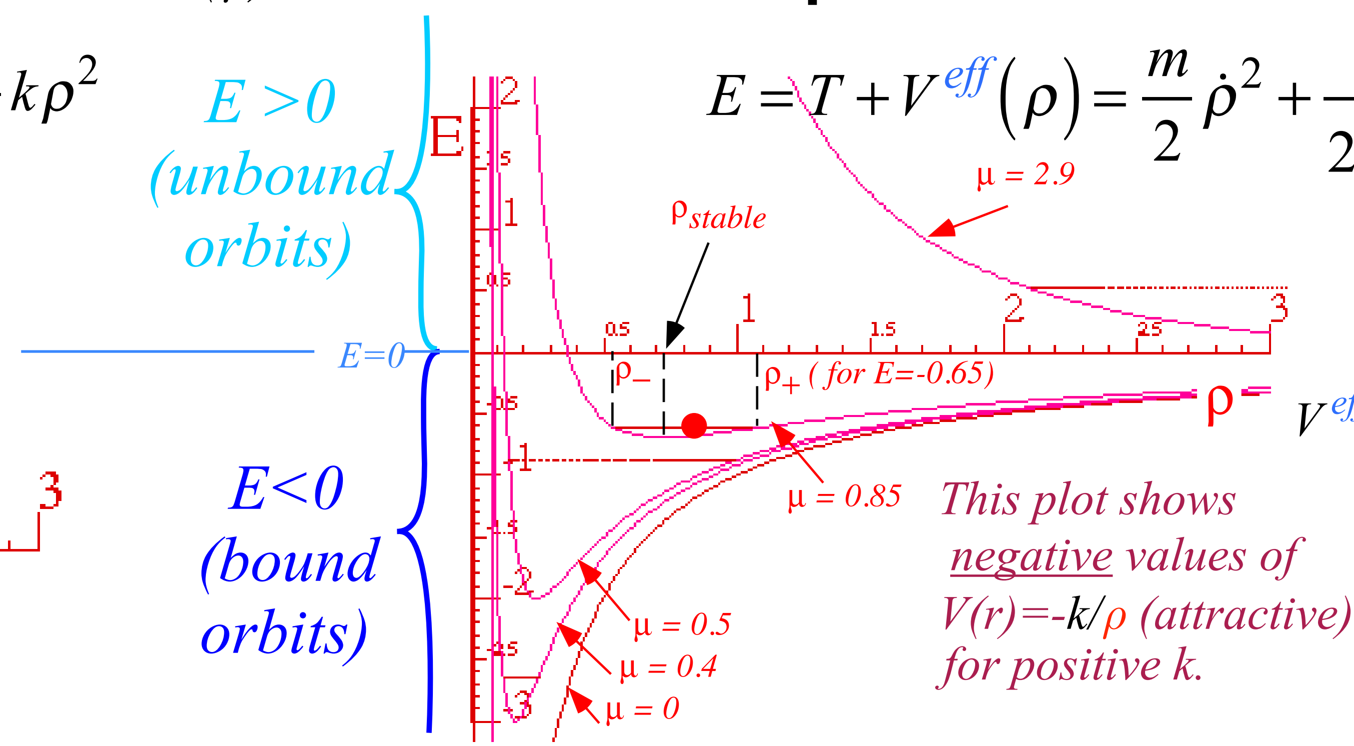


$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

In either case: *IHO or Coulomb orbit blows up if k is negative.*

NOTE: Our Coulomb field is attractive if k is positive

That is, if $-k/\rho$ is negative.

Coulomb $V(\rho)=-k/\rho$

(Explicit minus (-) convention)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits ➔ *Review: "3steps from Hell" (Lect. 7 Ch. 9 Unit 1)*

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

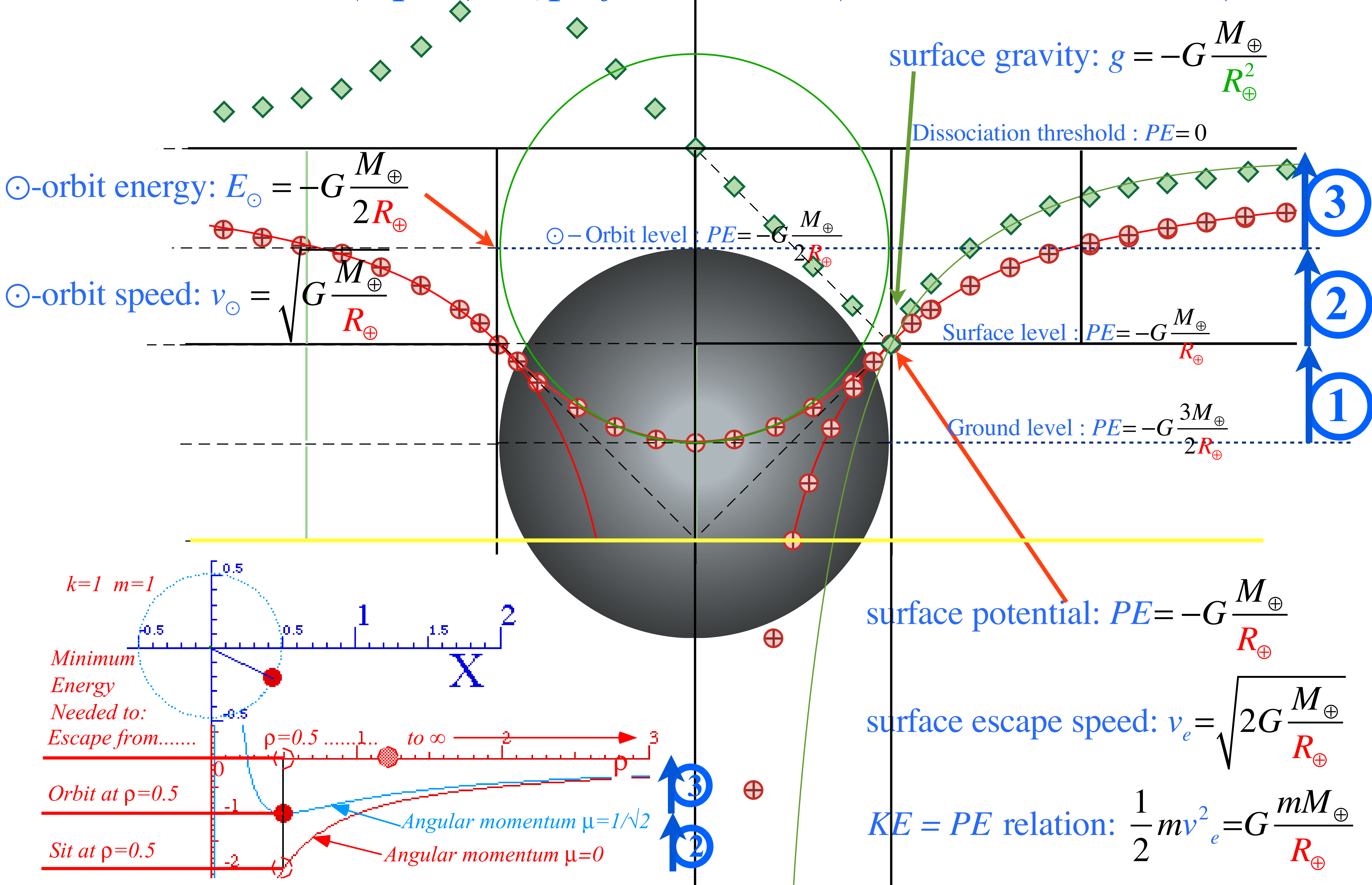
Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

Review: "Three (equal) steps from Hell" (Lect. 7 Ch. 9 Unit 1)



$G=0.67\cdot 10^{-10}$

Fig. 5.2.10 Three Coulomb threshold energies: to sit (E=-2), to orbit (E=-1), and to escape (E=0) from R.
A 4th energy (E=-3) is for sitting at the center (ρ=0) of a uniform mass planet of radius ρ=R.

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

➔ *Stable equilibrium radii and radial/angular frequency ratios*

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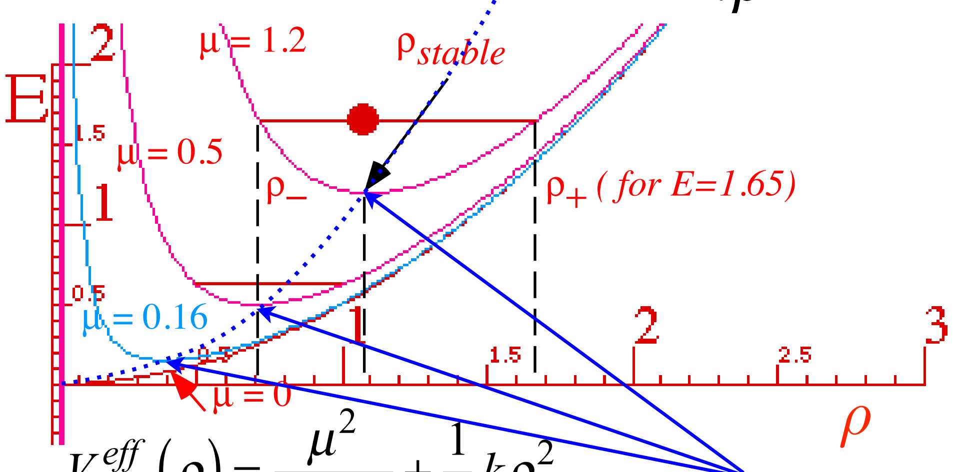

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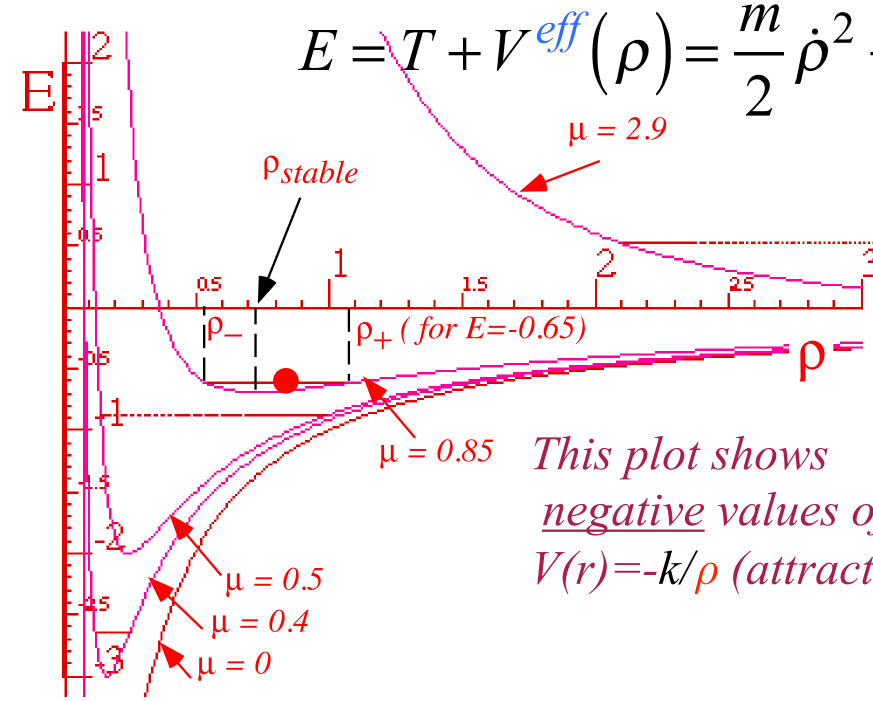

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Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\frac{\mu^2}{m}=+k\rho^4$$

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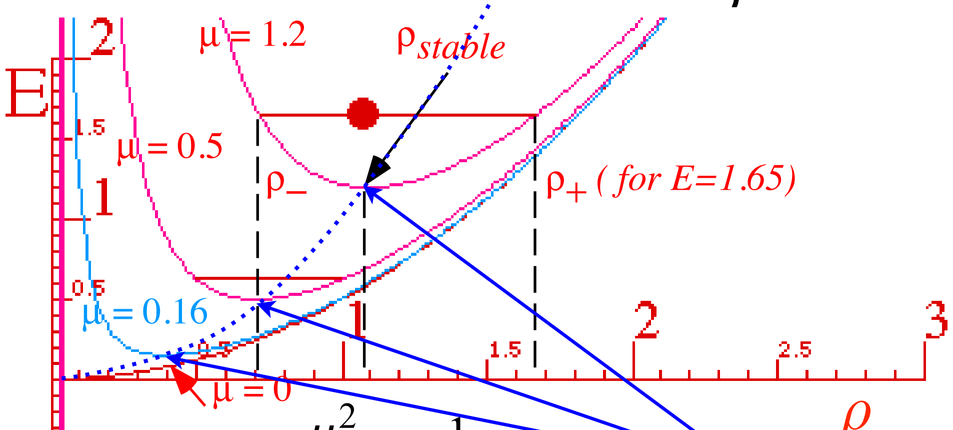

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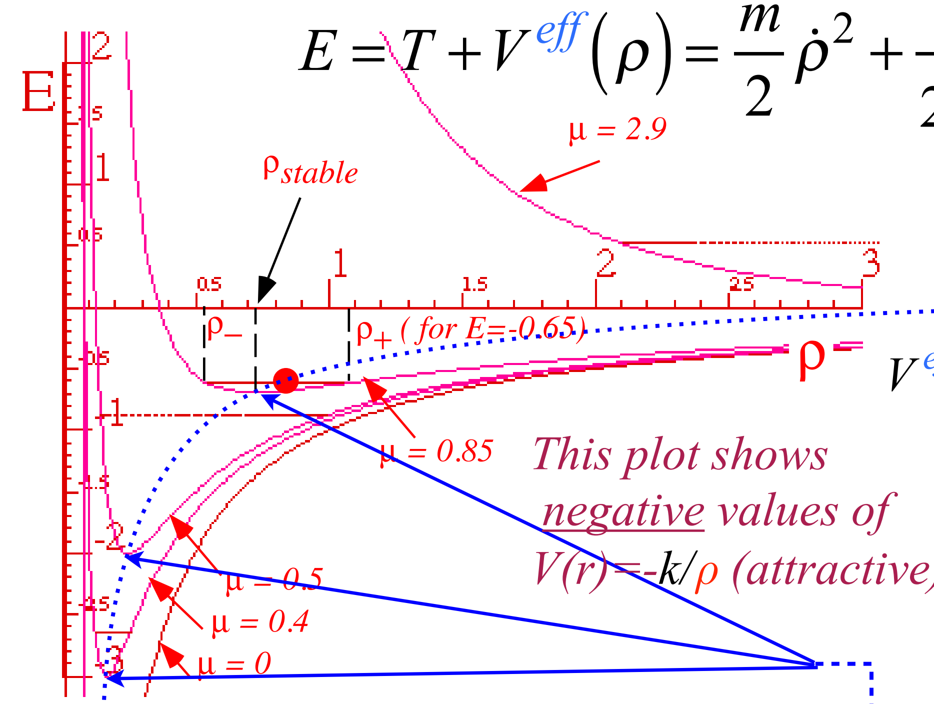

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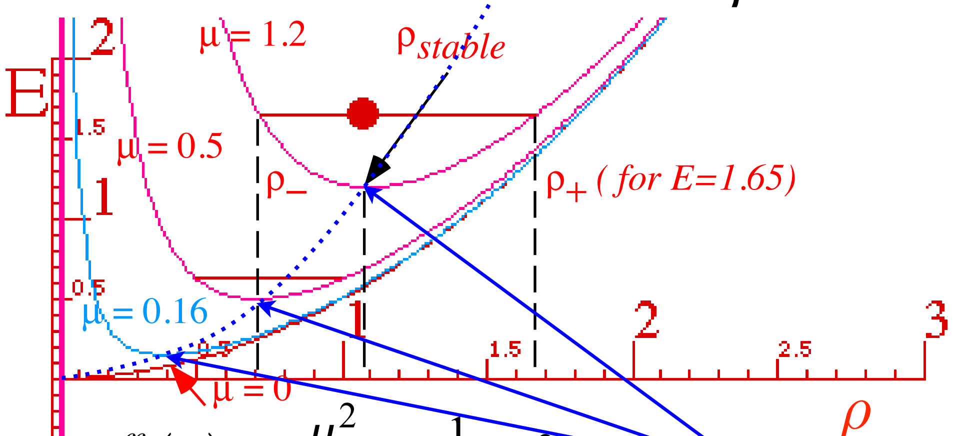

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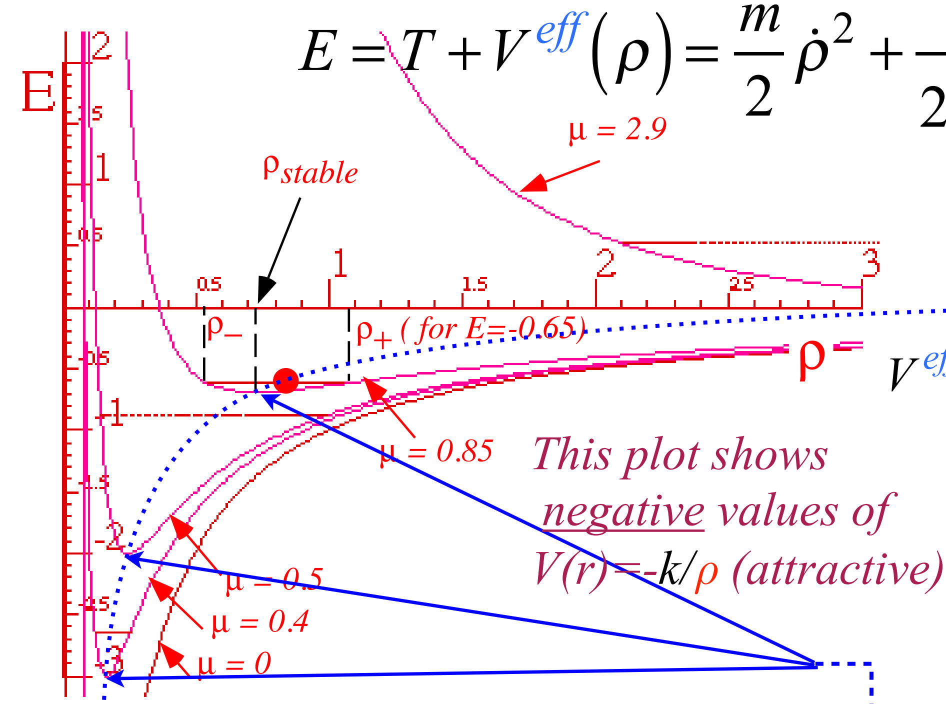

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Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

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For ALL central forces

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

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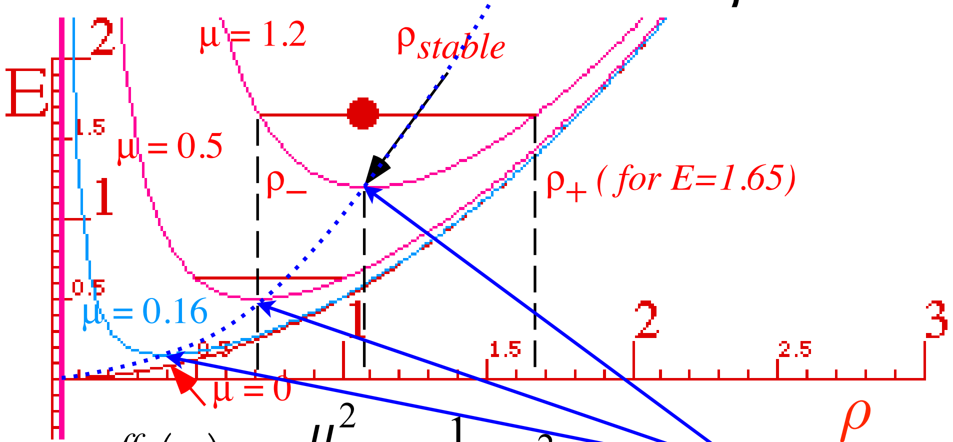

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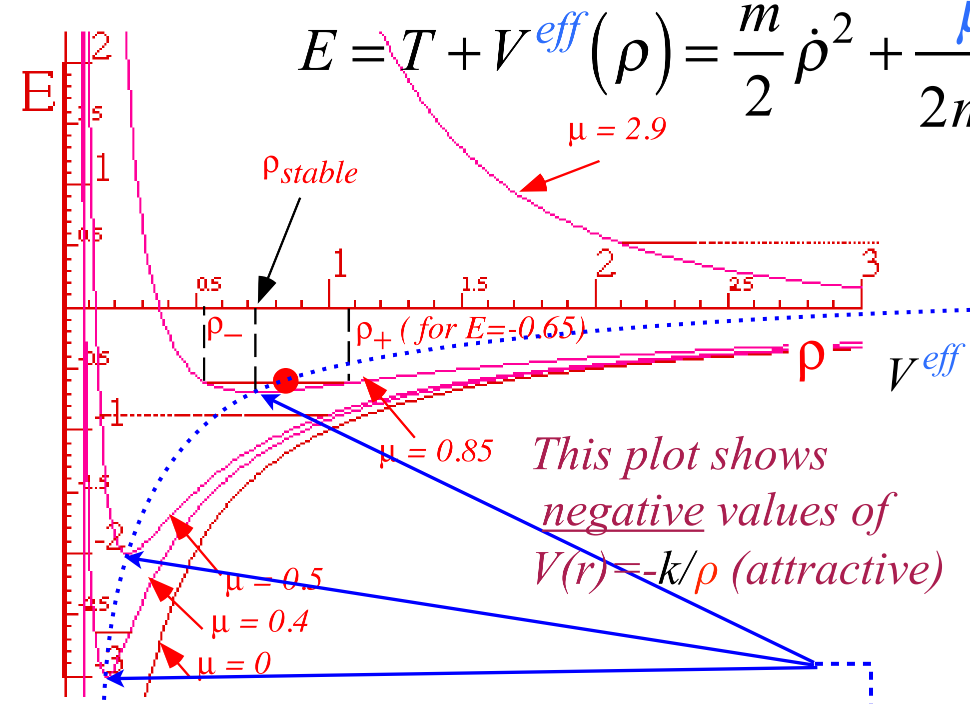

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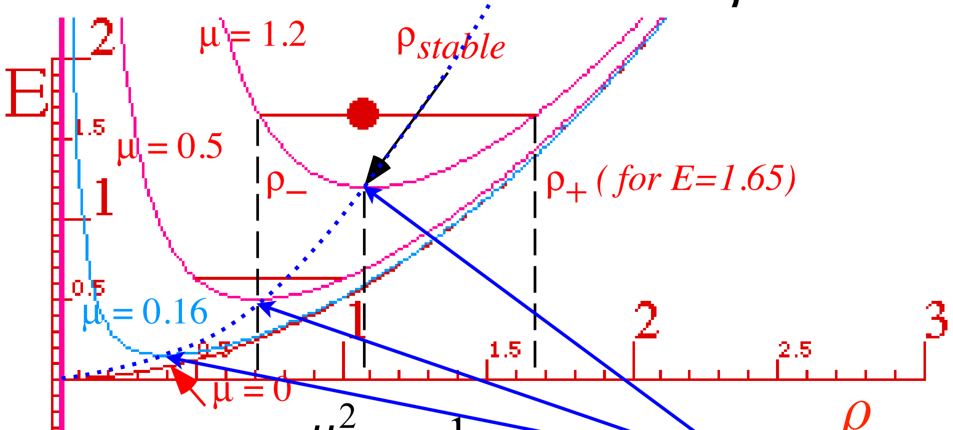

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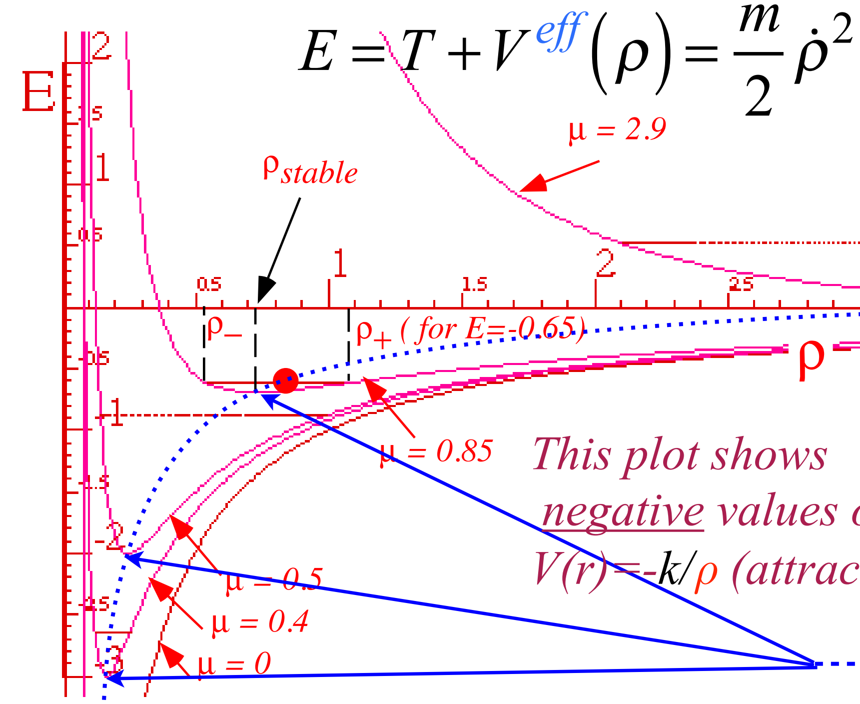

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Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}^2}=\sqrt{\dfrac{k}{m}}$

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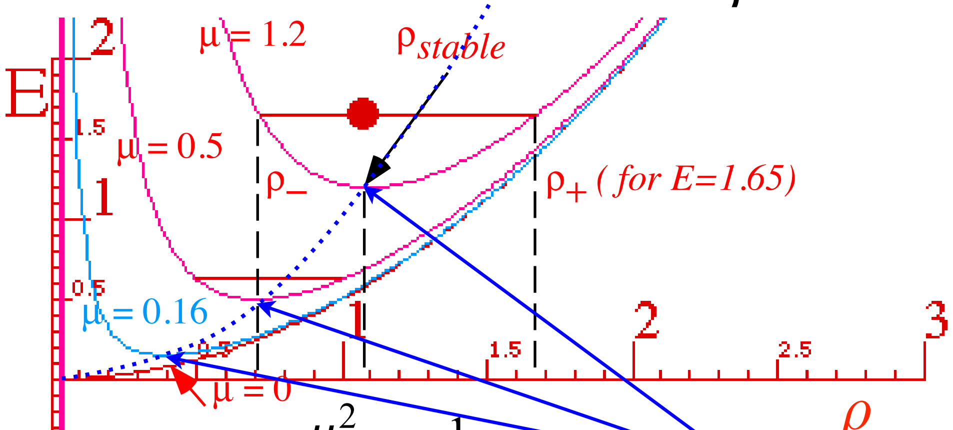

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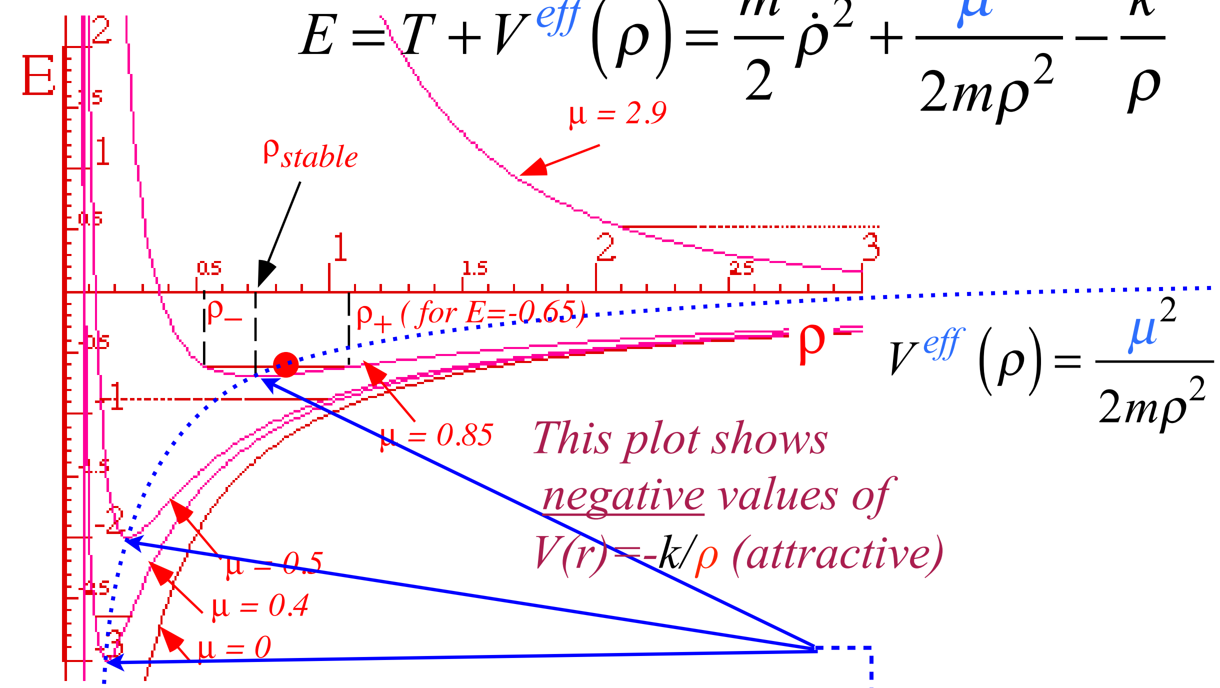

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Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4}-\frac{k}{\rho_{stable}^3}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}^2}=\sqrt{\dfrac{k}{m}}$

...angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}^2}=\dfrac{\mu}{m}\dfrac{m^2k^2}{\mu^4}=\dfrac{mk^2}{\mu^3}$

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

For ALL central forces

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

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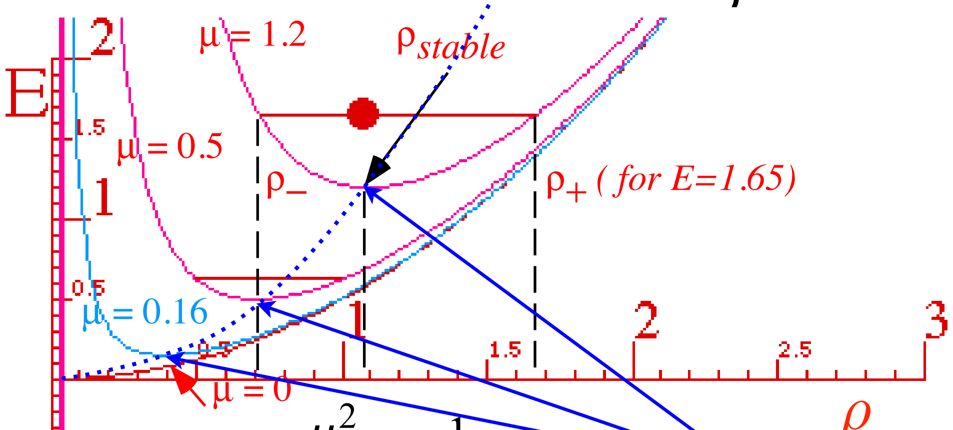
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



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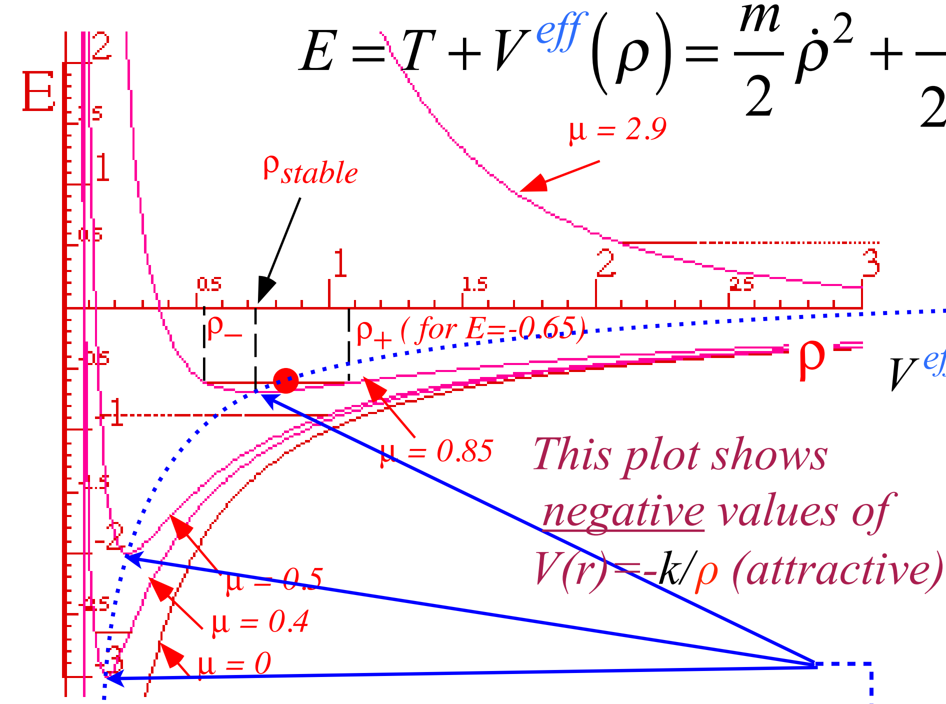

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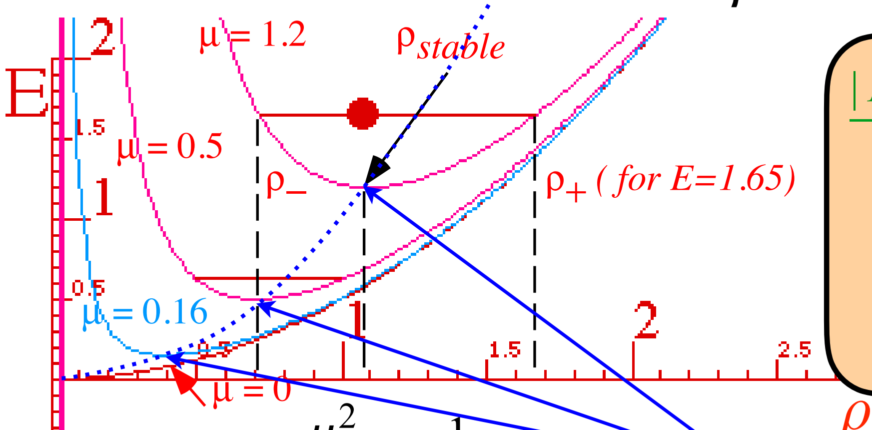

$$\frac{|F_{HO}|}{m}=\omega^2 r=\frac{GM_\oplus}{r_\oplus^3}r$$
$$=k_{HO}r\rightarrow\omega=\sqrt{k_{HO}}$$

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

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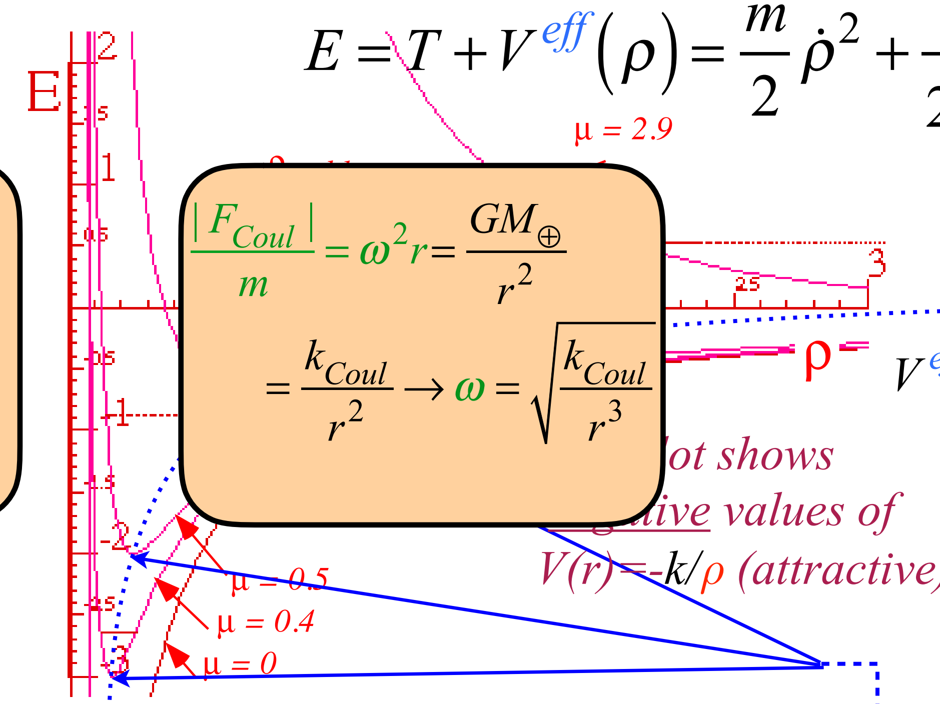

$$\frac{|F_{Coul}|}{m}=\omega^2 r=\frac{GM_\oplus}{r^2}$$
$$=\frac{k_{Coul}}{r^2}\rightarrow\omega=\sqrt{\frac{k_{Coul}}{r^3}}$$

$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

...ot shows ...ative values of $V(r)=-k/\rho$ (attractive)

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Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

- *Effective potentials for IHO and Coulomb orbits*
    - *Stable equilibrium radii and radial/angular frequency ratios*
    - ➤ *Classical turning radii and apogee/perigee parameters*
    - *Polar coordinate differential equations*
    - *Quadrature integration techniques*
        - *Detailed orbital functions*
    - *Relating orbital energy-momentum to conic-sectional orbital geometry*
        - *Kepler equation of time and phase geometry*

(A mystery similarity appears)

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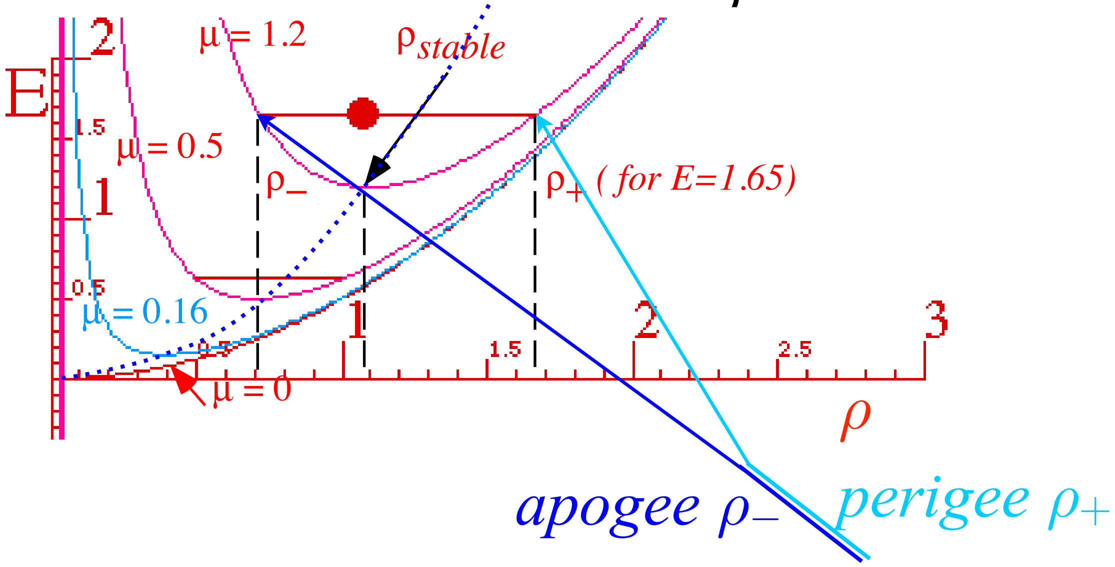

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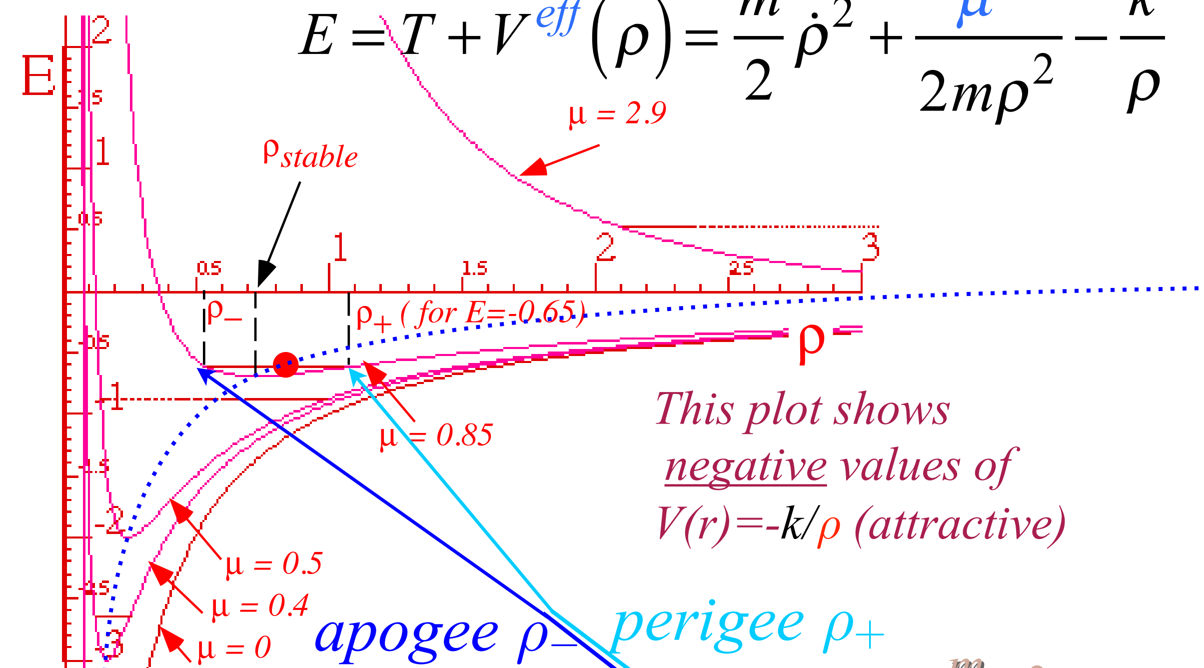

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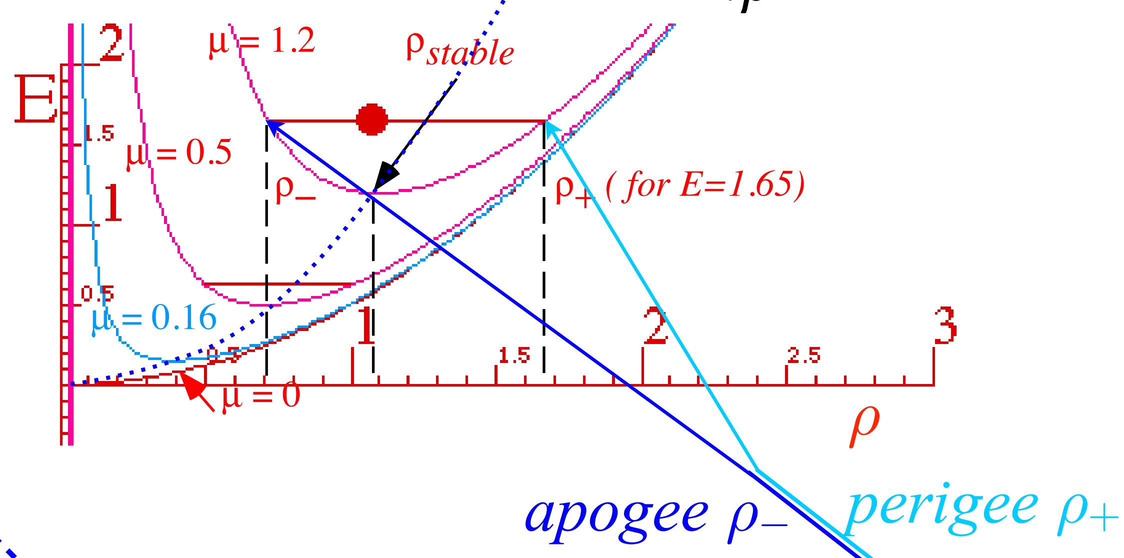
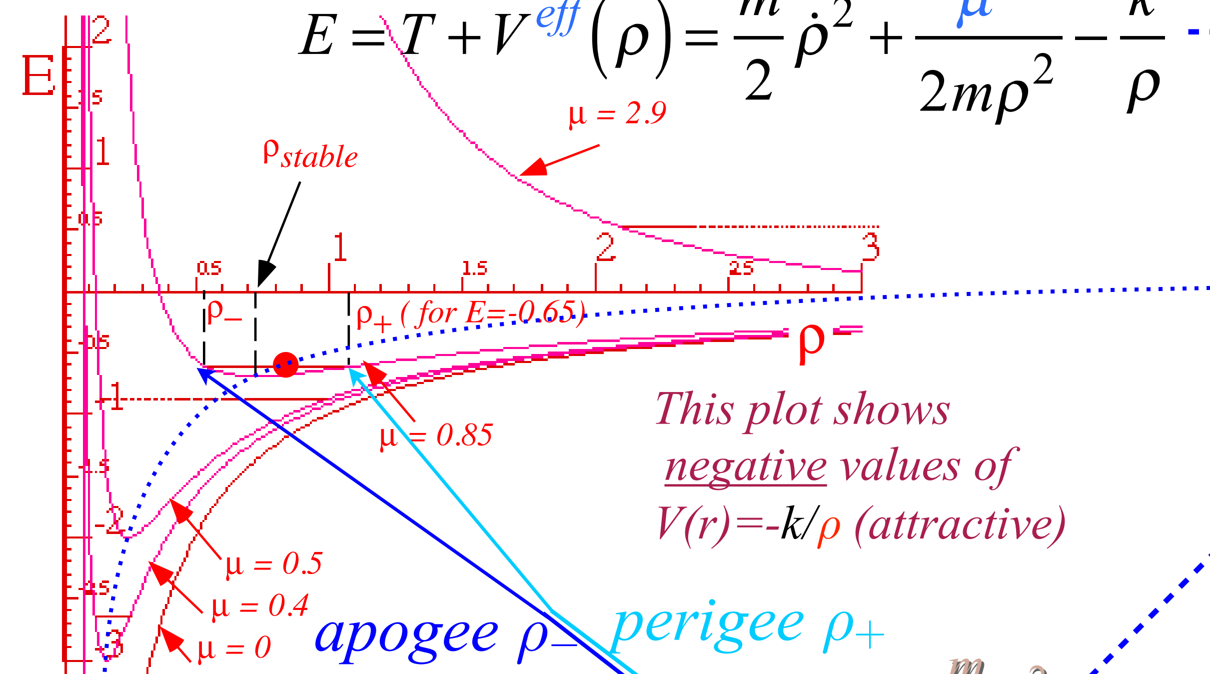


$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$



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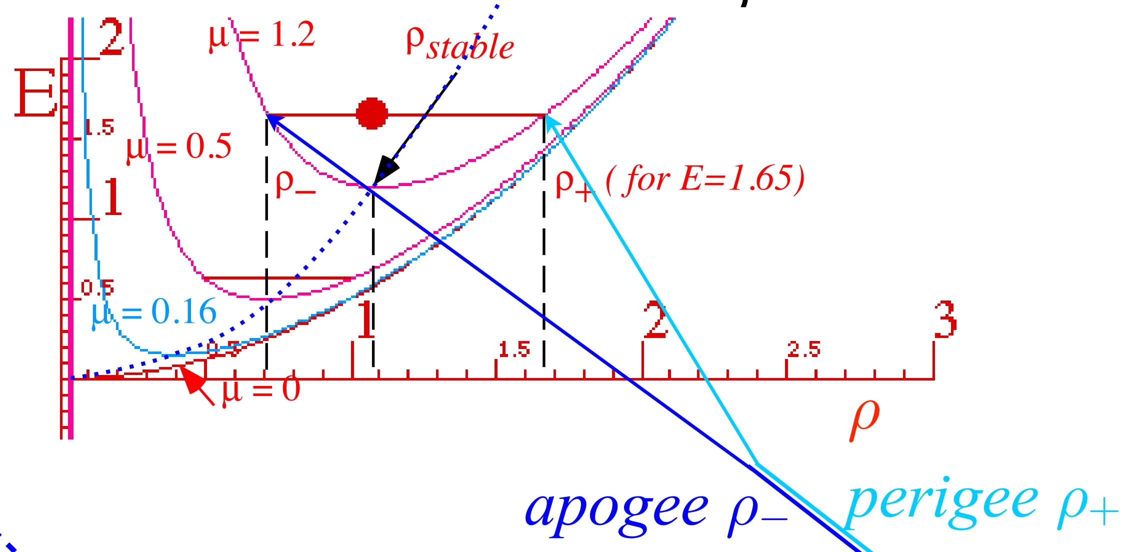

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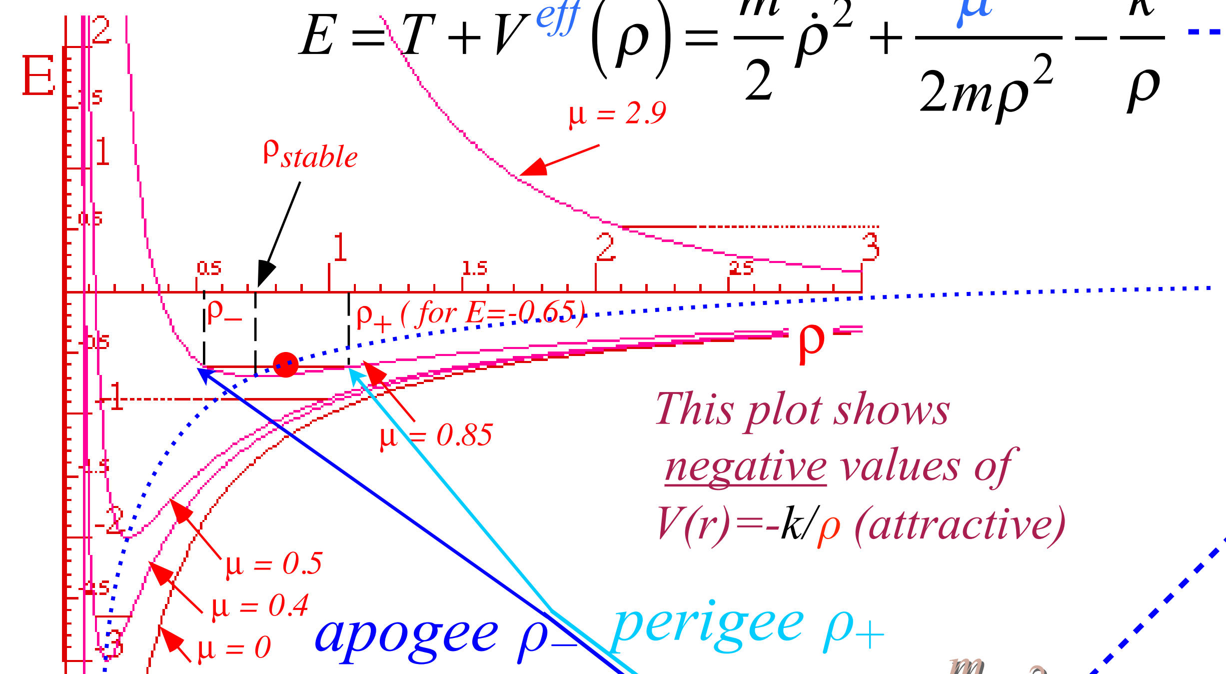

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$$0 = \frac{\mu^2}{2m} - E\rho^2 + \frac{k}{2}\rho^4 \quad \text{or else: } 0 = \frac{\mu^2}{2m}\frac{1}{\rho^4} - E\frac{1}{\rho^2} + \frac{k}{2}$$

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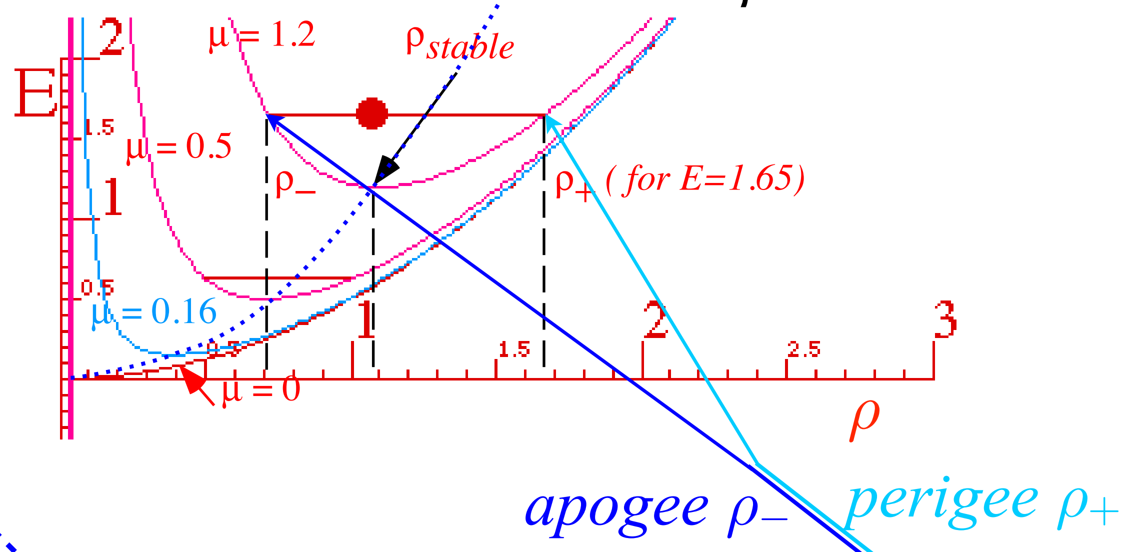

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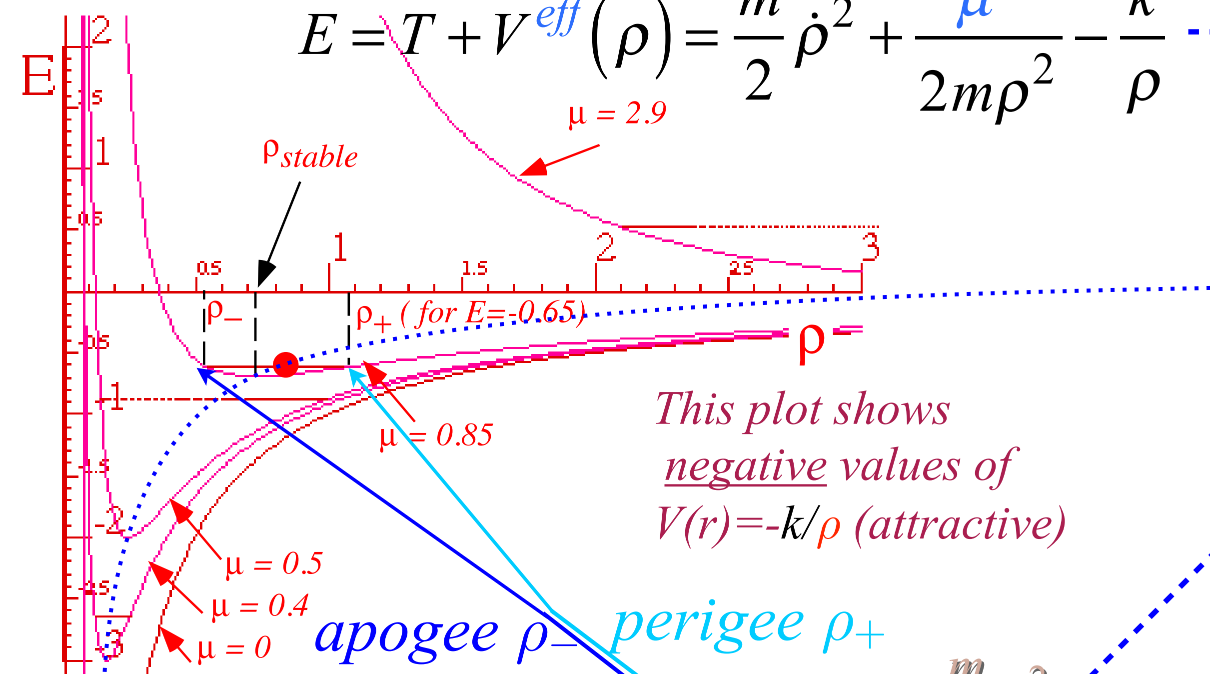

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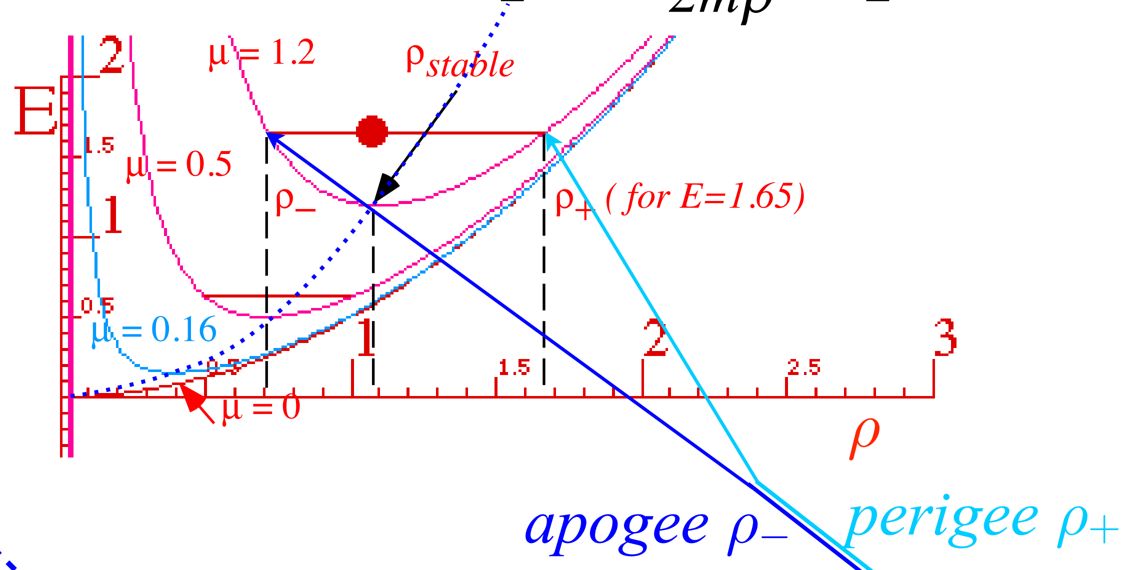

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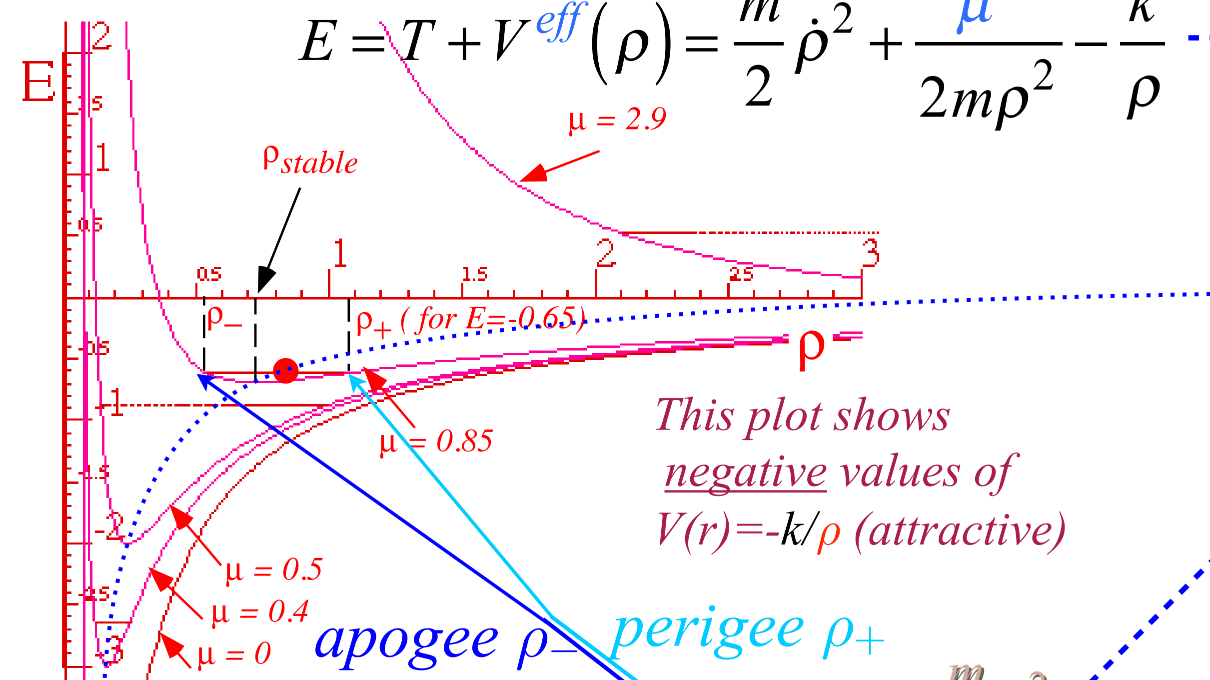

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$$\rho_\pm{}^2=\frac{E\pm\sqrt{E^2-k\mu^2/m}}{k}\quad\text{or else: }\frac{1}{\rho_\pm{}^2}=\frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

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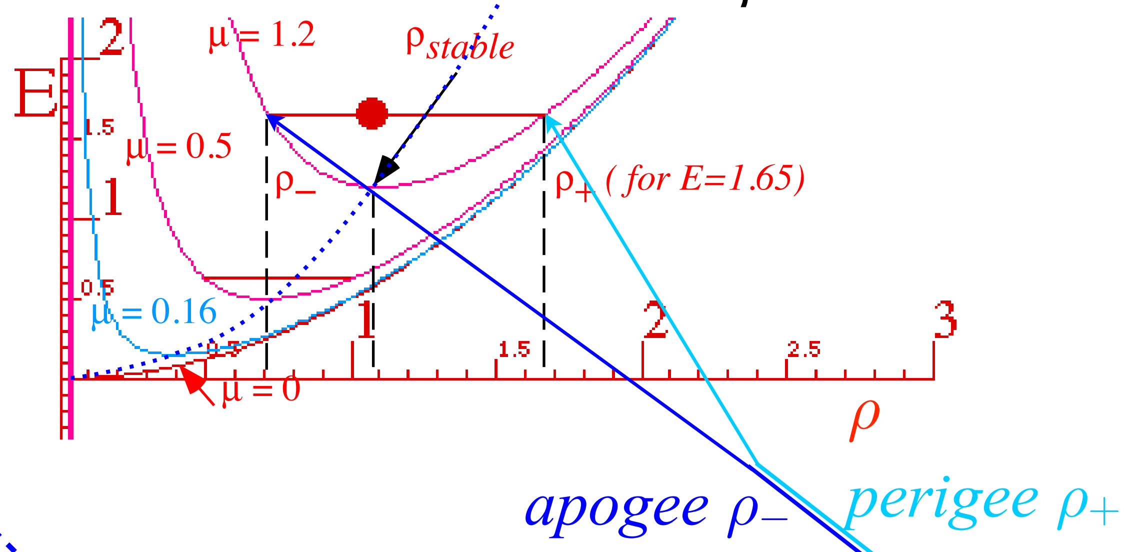

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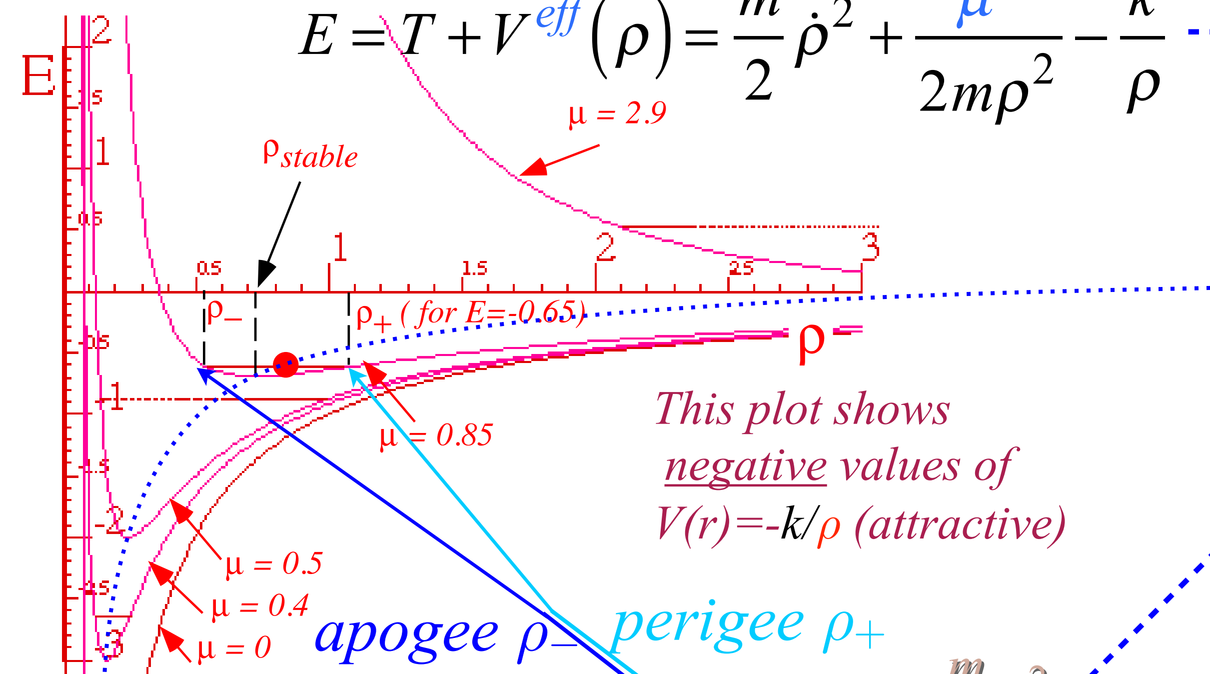

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$$\rho_\pm{}^2=\frac{E\pm\sqrt{E^2-k\mu^2/m}}{k} \quad \text{or else: } \frac{1}{\rho_\pm{}^2}=\frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m} \qquad \rho_\pm=\frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E} \quad \text{or else: } \frac{1}{\rho_\pm}=\frac{k\pm\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

Note: $\rho^2\to\rho$ similarity: $E\to k$ and $k\to 2E$

(See p.62)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

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➔ *Polar coordinate differential equations*

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Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

$$\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

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where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ *For ALL central forces*

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

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$$\frac{d\phi}{dt}\frac{dt}{d\rho}=\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.79

Orbits in Isotropic Oscillator and Coulomb Potentials

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$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}} \qquad \frac{d\phi}{dt}\frac{dt}{d\rho}=\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}} \qquad \dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

(Time solution begins: p. 81)
(Time solution ends: p. 90)

(Finding $\rho=\rho(\phi)$ trajectory equations)

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Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot\rho}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

(Finding $\rho=\rho(\phi)$ trajectory equations)

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For ALL central forces

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$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.77

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where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ *For ALL central forces*

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$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}} \qquad \frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}} \qquad \dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

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$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

Let: $x=u^2=\frac{1}{\rho^2}$ so: $dx=2udu$, $du=\frac{dx}{2\sqrt{x}}$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

(Finding $\rho=\rho(\phi)$ trajectory equations)

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$$\dot\phi=\frac{\mu}{m\rho^2}$$

For ALL central forces

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$$\dot\rho=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot\rho}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{k}{mx}}}$$

$$\dot\phi=\frac{\mu}{m\rho^2}\equiv\frac{d\phi}{dt}$$

$$\frac{d\phi}{d\rho}=\frac{\dot\phi}{\dot\rho}=\frac{\mu}{m\rho^2\dot\rho}$$

Let: $\frac{1}{\rho}=u$ so: $-\frac{d\rho}{\rho^2}=du$, $\quad d\rho=-\frac{du}{u^2}$

Let: $x=u^2=\frac{1}{\rho^2}$ so: $dx=2udu$, $\quad du=\frac{dx}{2\sqrt{x}}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot\rho^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot\rho=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot\rho}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

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(Finding $\rho=\rho(\phi)$ trajectory equations)

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For ALL central forces

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

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IHOscillator:

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

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$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Common:

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Let: $\dfrac{1}{\rho}=u$ so: $-\dfrac{d\rho}{\rho^2}=du$, $\quad d\rho=-\dfrac{du}{u^2}$

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(Finding $\rho=\rho(\phi)$ trajectory equations)

Parameter table on p.79

Coulomb:

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

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Note: $\rho^2\to\rho$ similarity: $E\to k$ and $k\to 2E$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

➡ *Quadrature integration techniques*

Detailed orbital functions

(A mystery similarity appears)

Relating orbital energy-momentum to conic-sectional orbital geometry

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Let: $x=u^2=\frac{1}{\rho^2}$

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Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}$$

(Finding $\rho=\rho(\phi)$ trajectory solutions)

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Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

(Finding $\rho=\rho(\phi)$ trajectory solutions)

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Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

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$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

Defining α and β:

$$z_\pm=\alpha\pm\beta$$

(Finding $\rho=\rho(\phi)$ trajectory solutions)

Orbits in Isotropic Oscillator and Coulomb Potentials

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$$\dot{\phi} = \frac{\mu}{m\rho^2}$$

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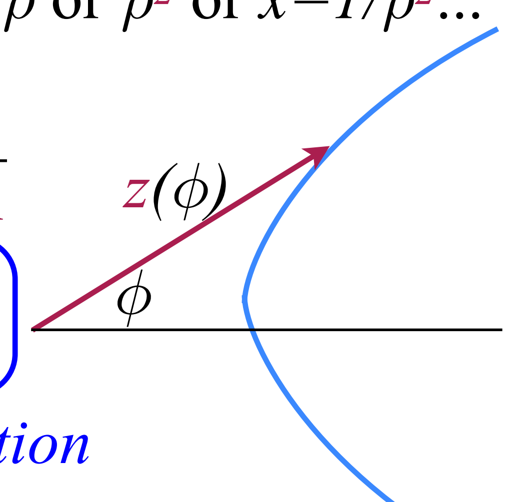
radial-polar-coordinate orbit function



Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

(A mystery similarity appears)

➡ *Quadrature integration techniques*

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

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Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

IHOscillator: $A=\frac{\mu^2}{m^2}$ $B=-\frac{2E}{m}$ $C=\frac{k}{m}$ $D=-\frac{\mu}{2m}$

$$\alpha=\frac{E}{\mu^2/m}$$

Coulomb: $A=\frac{\mu^2}{m^2}$ $B=\frac{2k}{m}$ $C=-\frac{2E}{m}$ $D=-\frac{\mu}{m}$

$$\alpha=\frac{-k}{\mu^2/m}\ ,$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

Angular momentum μ

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ $\quad\dot{\phi}=\frac{\mu}{m\rho^2}$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\frac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A},\quad\beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

Oscillator:

$A=\frac{\mu^2}{m^2}$ $\quad B=-\frac{2E}{m}$ $\quad C=\frac{k}{m}$ $\quad D=-\frac{\mu}{2m}$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

Coulomb:

$A=\frac{\mu^2}{m^2}$ $\quad B=\frac{2k}{m}$ $\quad C=-\frac{2E}{m}$ $\quad D=-\frac{\mu}{m}$

$$\alpha=\frac{-k}{\mu^2/m},$$

$$\beta=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

Algebra details on following pages

Orbits in Isotropic Oscillator and Coulomb Potentials

Angular momentum μ

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \dot{\phi}=\frac{\mu}{m\rho^2}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\frac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

Oscillator:

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Coulomb:

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

$$\alpha=\frac{-k}{\mu^2/m}\ ,$$

$$\beta=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Algebra details on following pages

Algebra details and checks

$$\alpha = \frac{-B}{2A}, \quad \beta = \frac{\sqrt{B^2 - 4AC}}{2A}$$

Angular momentum μ

(ρ,ϕ) orbits for IHOscillator $V(\rho) = k\rho^2/2$

$$A = \frac{\mu^2}{m^2} \quad B = -\frac{2E}{m} \quad C = \frac{k}{m} \quad D = -\frac{\mu}{2m}$$

$$\alpha = \frac{\frac{2E}{m}}{2\frac{\mu^2}{m^2}} = \frac{E}{\mu^2/m} = \frac{z_+ + z_-}{2}$$

$$\beta = \frac{\sqrt{\left(\frac{2E}{m}\right)^2 - 4\frac{\mu^2}{m^2}\frac{k}{m}}}{2\frac{\mu^2}{m^2}}$$

$$\beta = \frac{\sqrt{\left(\frac{2Em}{m\cdot m}\right)^2 - 4\frac{\mu^2}{m^2}\frac{km}{m^2}}}{2\frac{\mu^2}{m^2}} = \frac{\sqrt{E^2 - k\mu^2/m}}{\mu^2/m} = \frac{z_+ - z_-}{2}$$

(ρ,ϕ) equations for Coulomb $V(\rho) = -k/\rho$

$$A = \frac{\mu^2}{m^2} \quad B = \frac{2k}{m} \quad C = -\frac{2E}{m} \quad D = -\frac{\mu}{m}$$

$$\alpha = \frac{-\frac{2k}{m}}{2\frac{\mu^2}{m^2}} = \frac{-k}{\mu^2/m} = \frac{z_+ + z_-}{2}$$

$$\beta = \frac{\sqrt{\left(\frac{2k}{m}\right)^2 + 4\frac{\mu^2}{m^2}\frac{2E}{m}}}{2\frac{\mu^2}{m^2}} = \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m} = \frac{z_+ - z_-}{2}$$

Parameter table on p.79

Checking that roots $z_\pm$ are *classical turning points* (*perigee* $z_- = \alpha - \beta$, *apogee* $z_+ = \alpha + \beta$) (See p.34)

$$\rho_\pm^2 = \frac{E \pm \sqrt{E^2 - k\mu^2/m}}{k} \quad \text{or else:} \quad \frac{1}{\rho_\pm^2} = \frac{E \mp \sqrt{E^2 - k\mu^2/m}}{\mu^2/m}$$

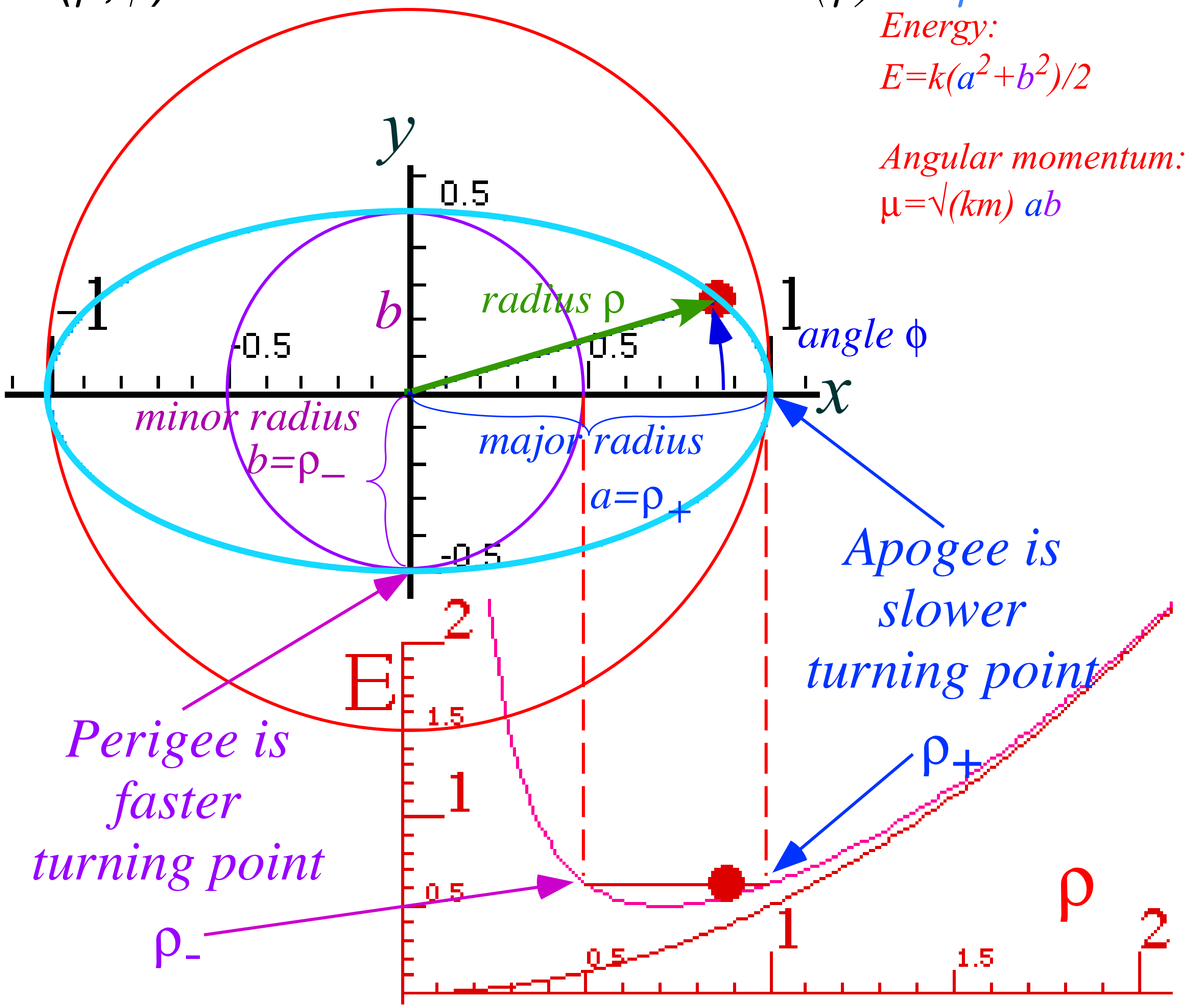
$$\rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E} \quad \text{or else:} \quad \frac{1}{\rho_\pm} = \frac{k \pm \sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

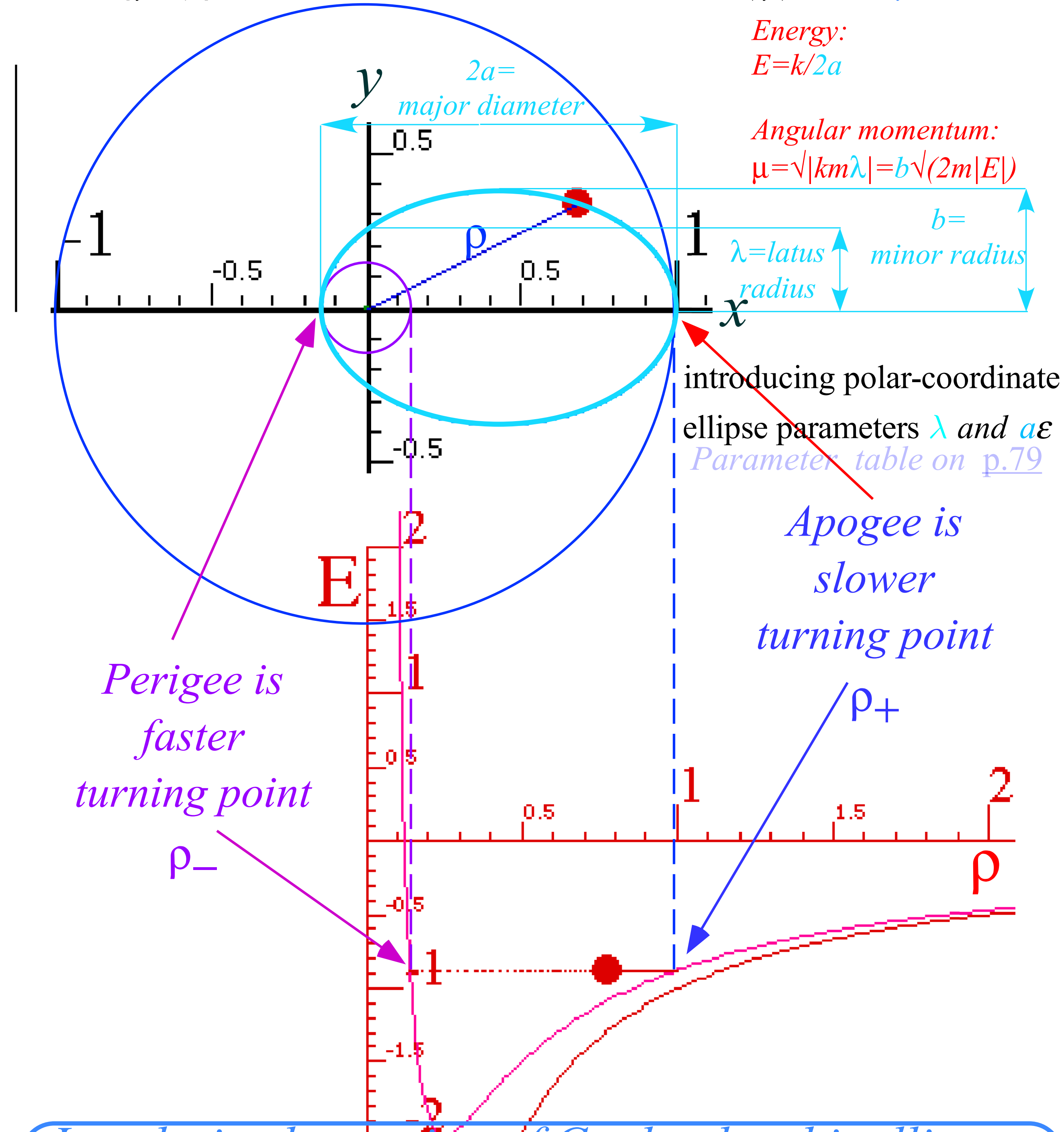
- *Effective potentials for IHO and Coulomb orbits*
  - *Stable equilibrium radii and radial/angular frequency ratios*
  - *Classical turning radii and apogee/perigee parameters*
  - *Polar coordinate differential equations*
  - *Quadrature integration techniques*
    - ➔ *Detailed orbital functions*
  - *Relating orbital energy-momentum to conic-sectional orbital geometry*
    - *Kepler equation of time and phase geometry*

Orbits in Isotropic Oscillator and Coulomb Potentials

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$



(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$



Just derived equation of IHO orbit ellipse

$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

$$\frac{1}{\rho^2}=+\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos(2\phi)$$

One of many equations of center-centered ellipse

Just derived equation of Coulomb orbit ellipse

$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

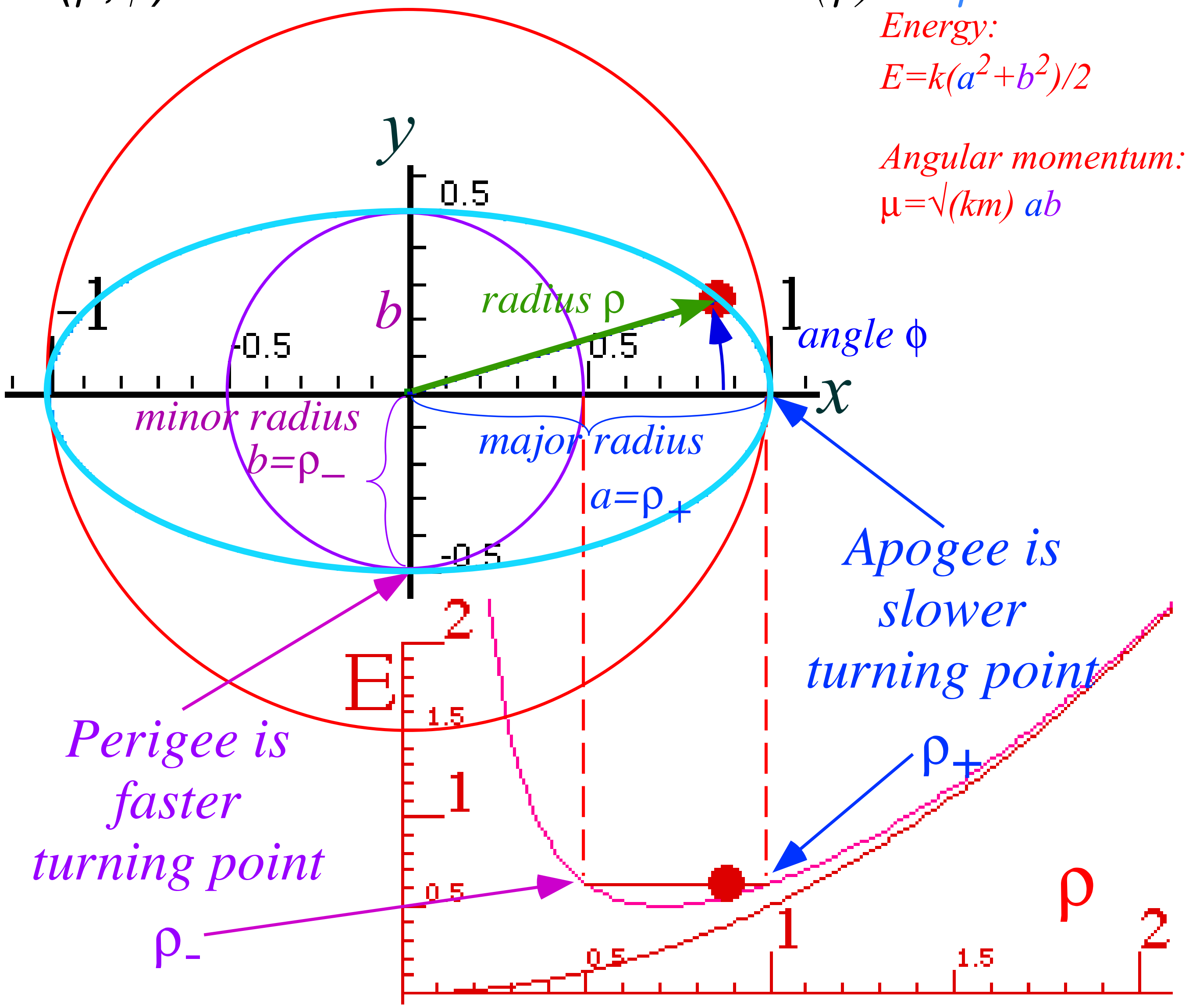
(to be discussed shortly)

$$\frac{1}{\rho}=\frac{a}{b^2}-\frac{\sqrt{a^2-b^2}}{b^2}\cos\phi$$

One of many equations of focus-centered ellipse

Orbits in Isotropic Oscillator and Coulomb Potentials

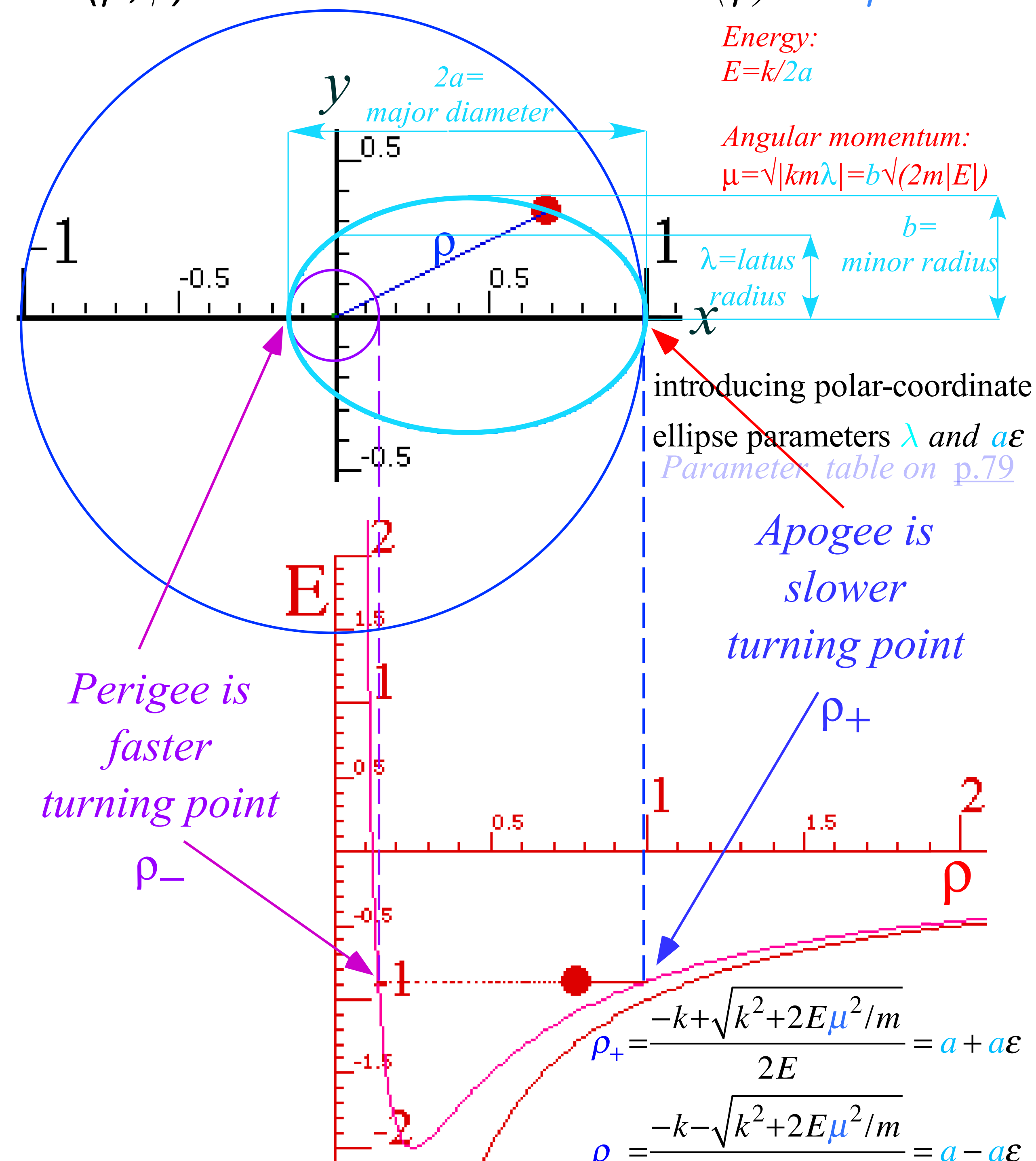
(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$



$$\rho_+{}^2=\frac{E+\sqrt{E^2-k\mu^2/m}}{k}=a^2$$

$$\rho_-{}^2=\frac{E-\sqrt{E^2-k\mu^2/m}}{k}=b^2$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$



$$\rho_+=\frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E}=a+a\varepsilon$$

$$\rho_-=\frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E}=a-a\varepsilon$$

(Turning points $\rho_\pm$ on p.62 or p.34)

(to be discussed first: turning point relations)

Just derived equation of IHO orbit ellipse

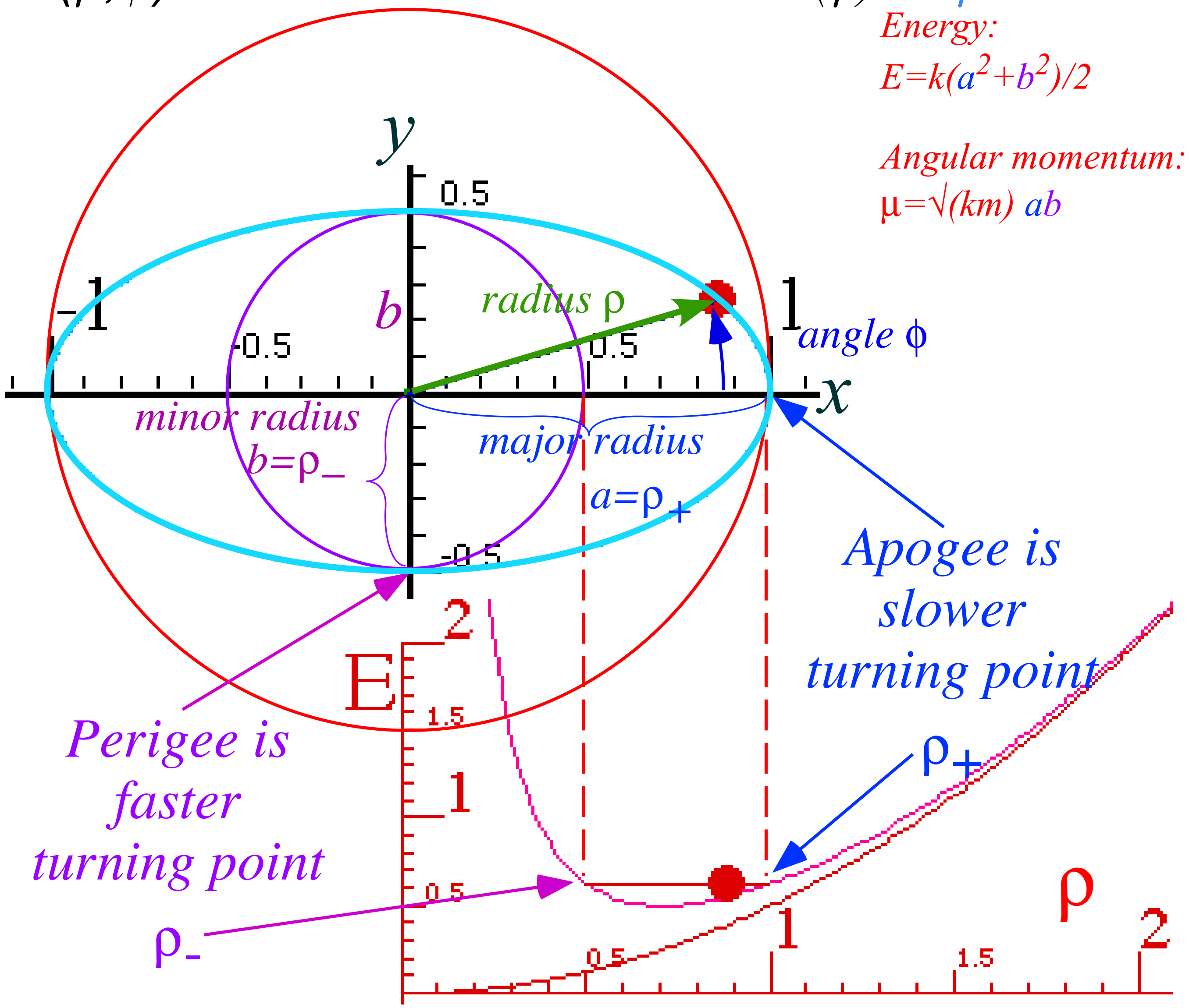
$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

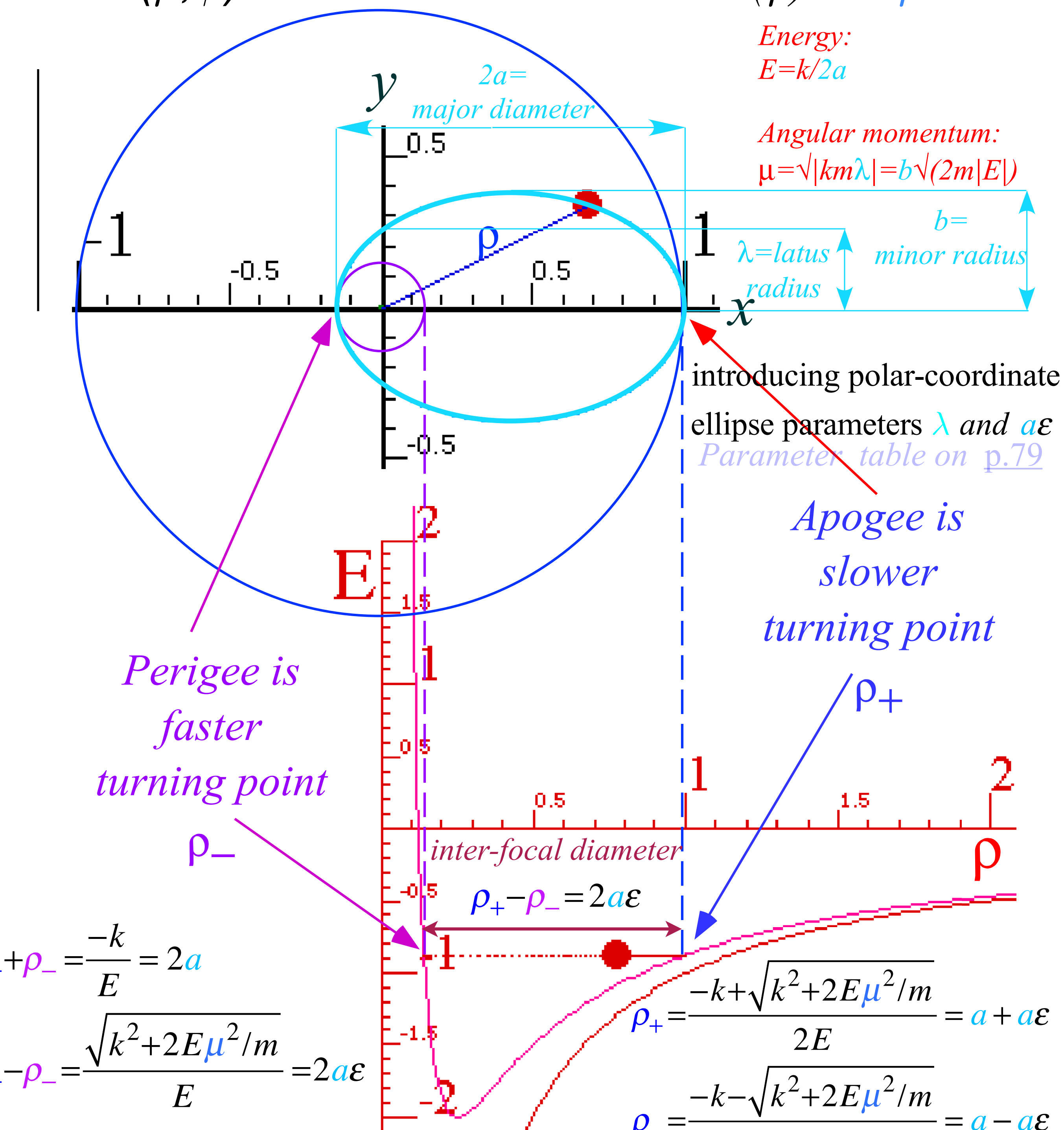
$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Orbits in Isotropic Oscillator and Coulomb Potentials

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$



(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$



$$\rho_+^{\,2}=\frac{E+\sqrt{E^2-k\mu^2/m}}{k}=a^2 \qquad \rho_+^{\,2}+\rho_-^{\,2}=\frac{2E}{k}=a^2+b^2$$

$$\rho_-^{\,2}=\frac{E-\sqrt{E^2-k\mu^2/m}}{k}=b^2 \qquad \rho_+^{\,2}-\rho_-^{\,2}=\frac{2\sqrt{E^2-k\mu^2/m}}{k}=a^2-b^2=a\varepsilon$$

$$\rho_+^{\,2}\rho_-^{\,2}=\frac{E^2-E^2-k\mu^2/m}{k^2}=a^2b^2=\frac{-\mu^2}{km}$$

$$\rho_++\rho_-=\frac{-k}{E}=2a$$

$$\rho_+-\rho_-=\frac{\sqrt{k^2+2E\mu^2/m}}{E}=2a\varepsilon$$

$$\rho_+=\frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E}=a+a\varepsilon$$

$$\rho_-=\frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E}=a-a\varepsilon$$

$$\rho_+\rho_-=\frac{k^2-k^2-2E\mu^2/m}{(2E)^2}=a^2-a^2\varepsilon^2=\frac{-\mu^2}{2Em}=b^2$$

(Turning points $\rho_\pm$ on p.62 or p.34)

(given above: turning point relations)

Just derived equation of IHO orbit ellipse

$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

(See p.69)

$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

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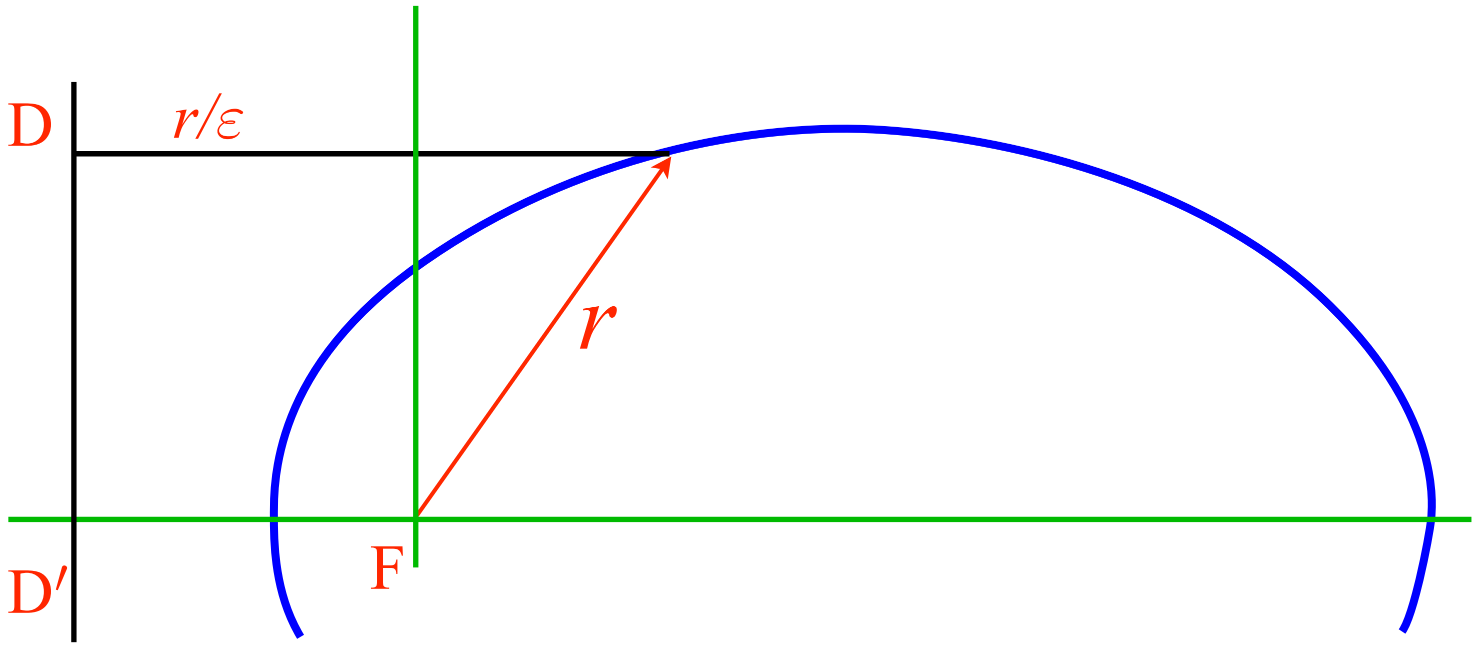
Detailed orbital functions

➤ *Relating orbital energy-momentum to conic-sectional orbital geometry*

Kepler equation of time and phase geometry

Geometry of ALL Coulomb conic section orbits (Let: $r = \rho$ here)



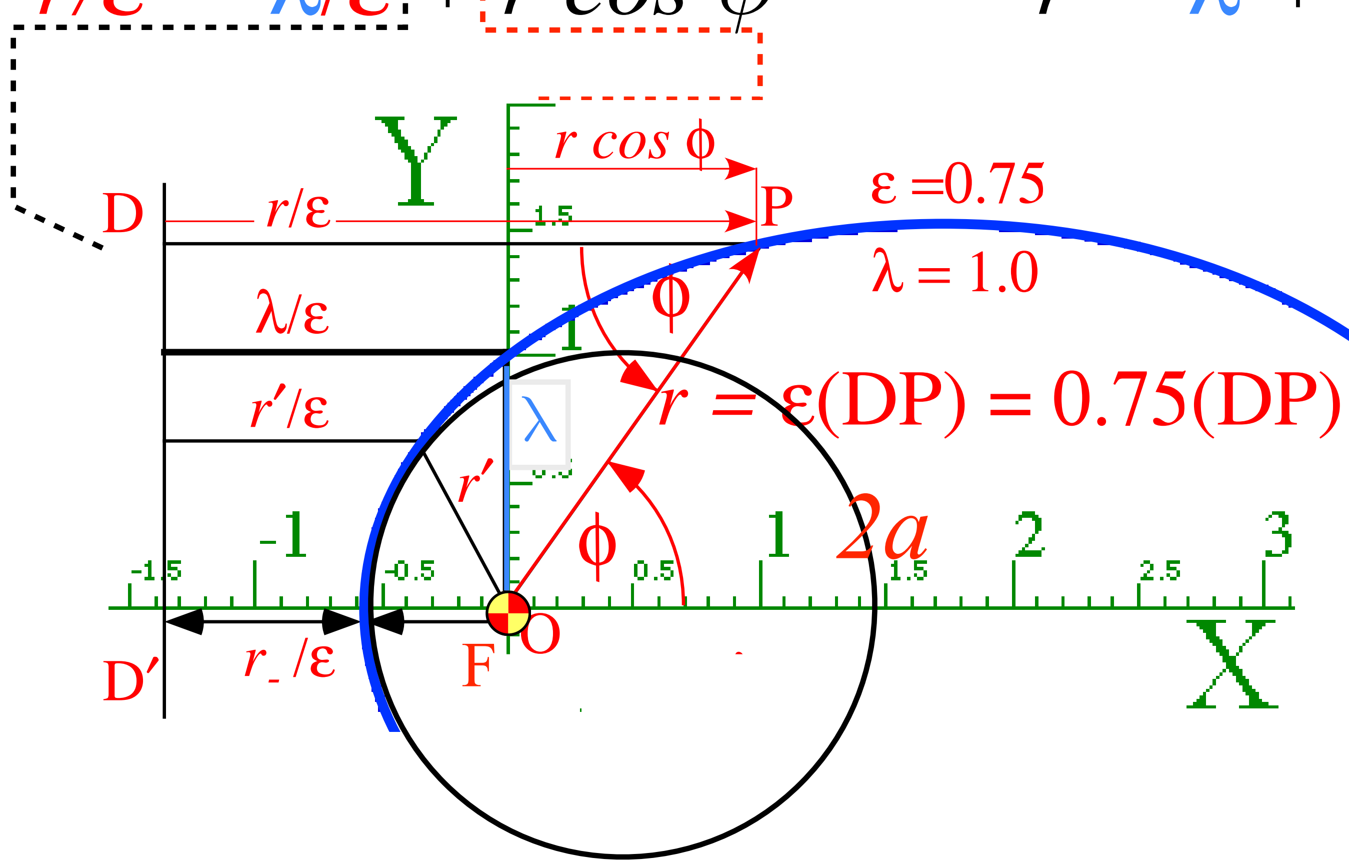
Parameter table on p.77

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Recall p.66 *formula:*

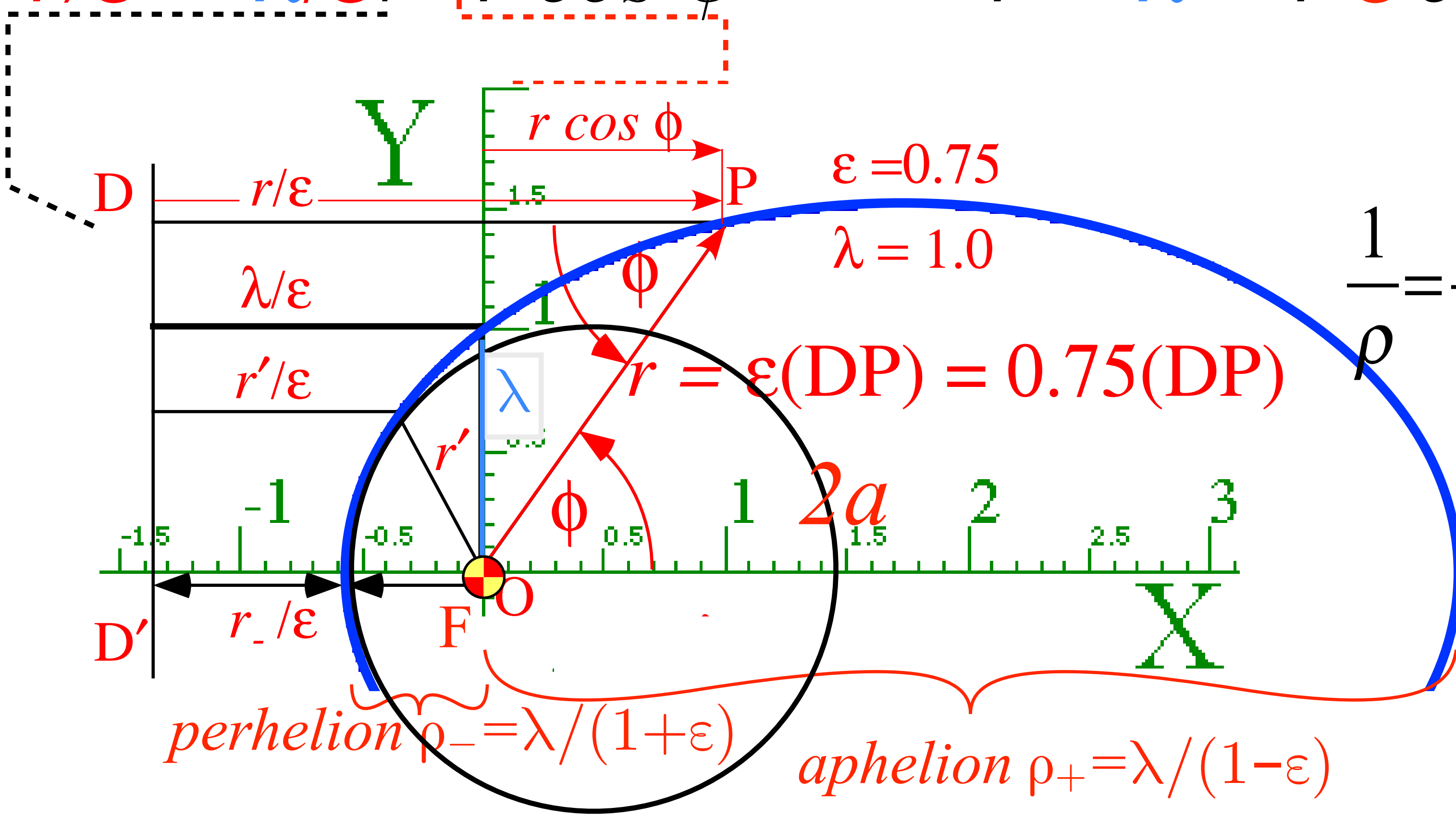
$$\frac{1}{r} = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = $\varepsilon\cdot$Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Parameter table on p.79

Recall p.66 formula:

$$r_{\pm} = \rho_{\pm} = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

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Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

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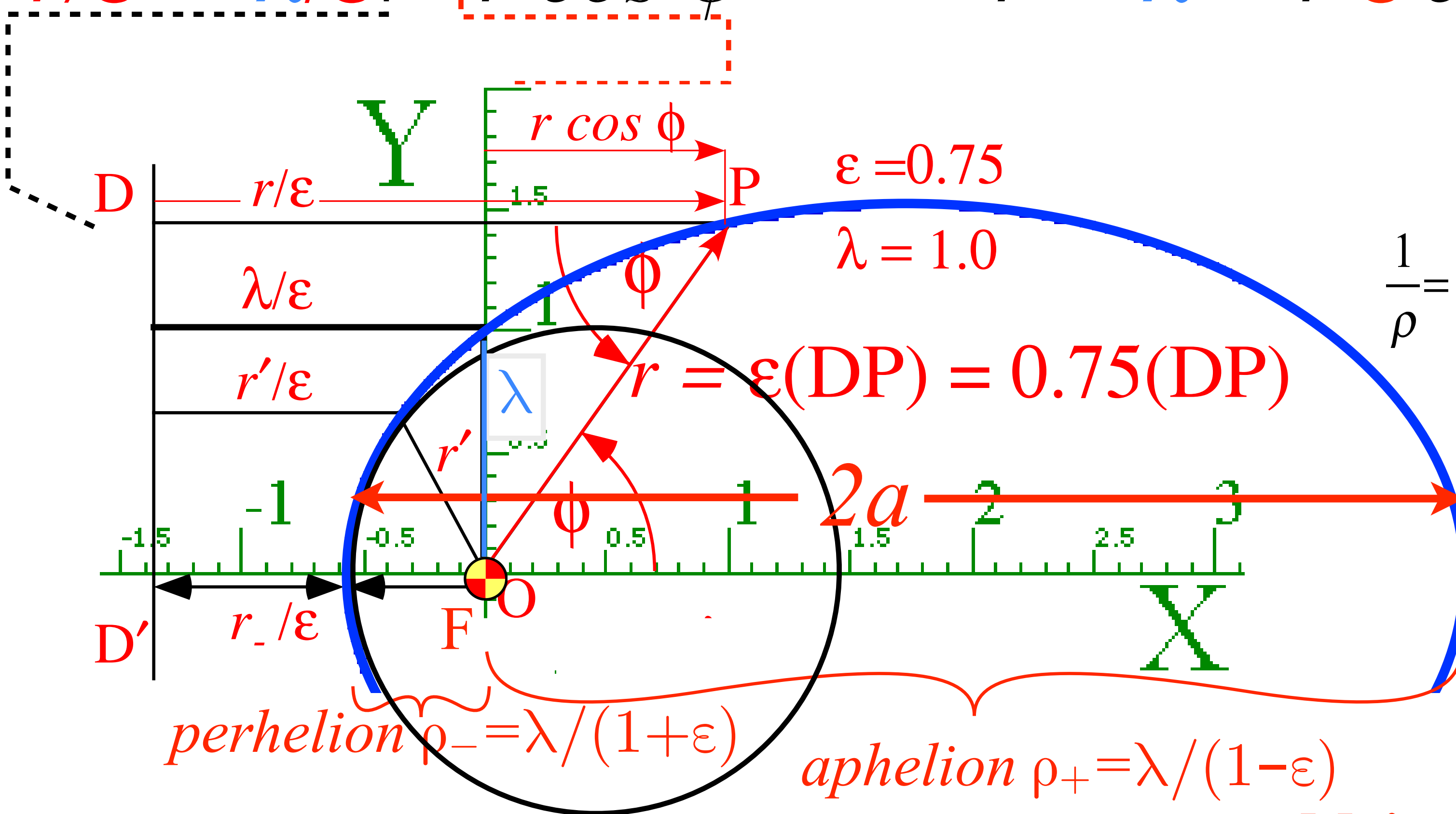


By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Parameter table on p.79

Recall p.66 formula:

$$r_\pm = \rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

Major axis: $\rho_+ + \rho_- = 2a$

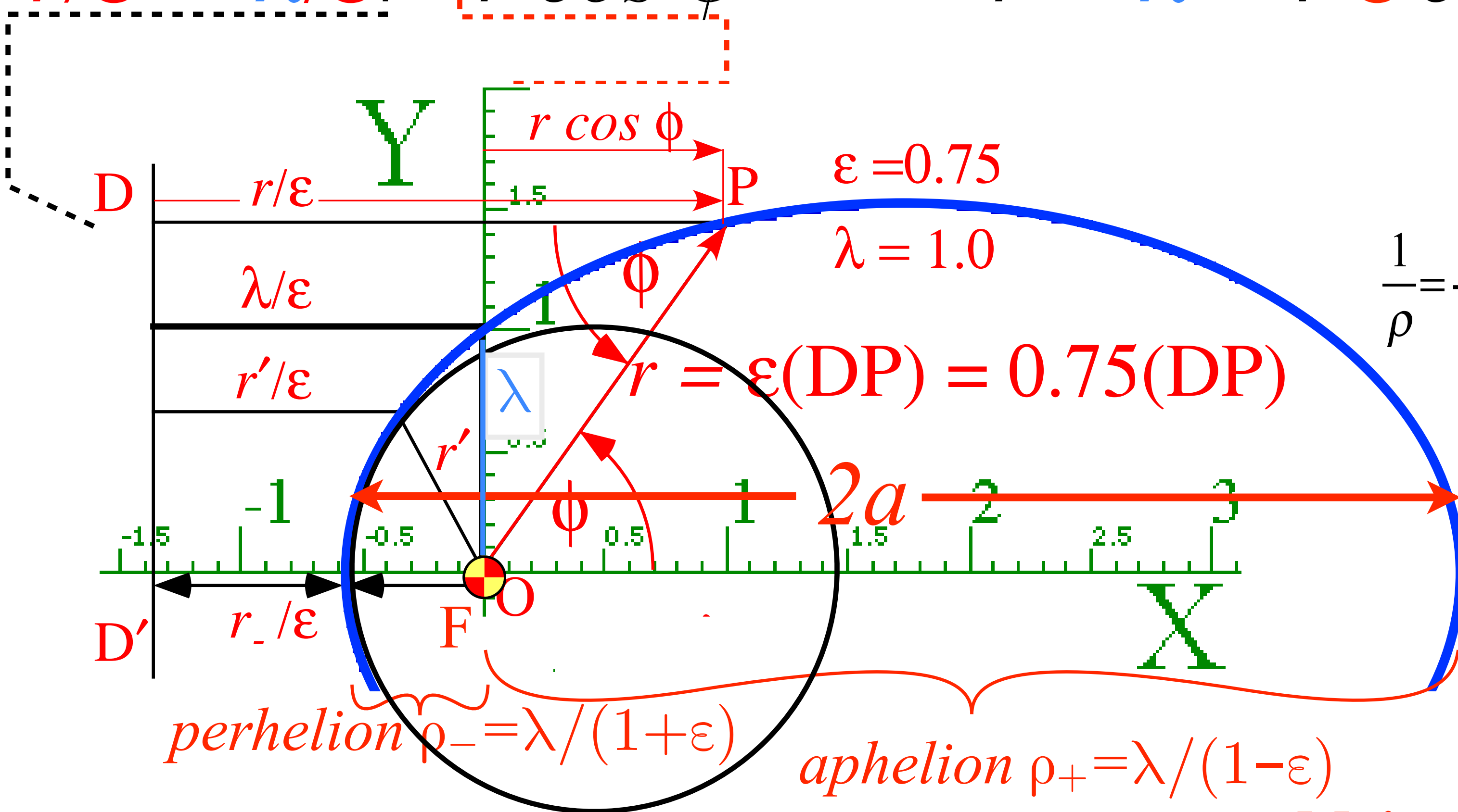
$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Parameter table on p.79

Recall p.66 formula:

$$r_{\pm} = \rho_{\pm} = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

All conics defined by: ***Eccentricity*** ε

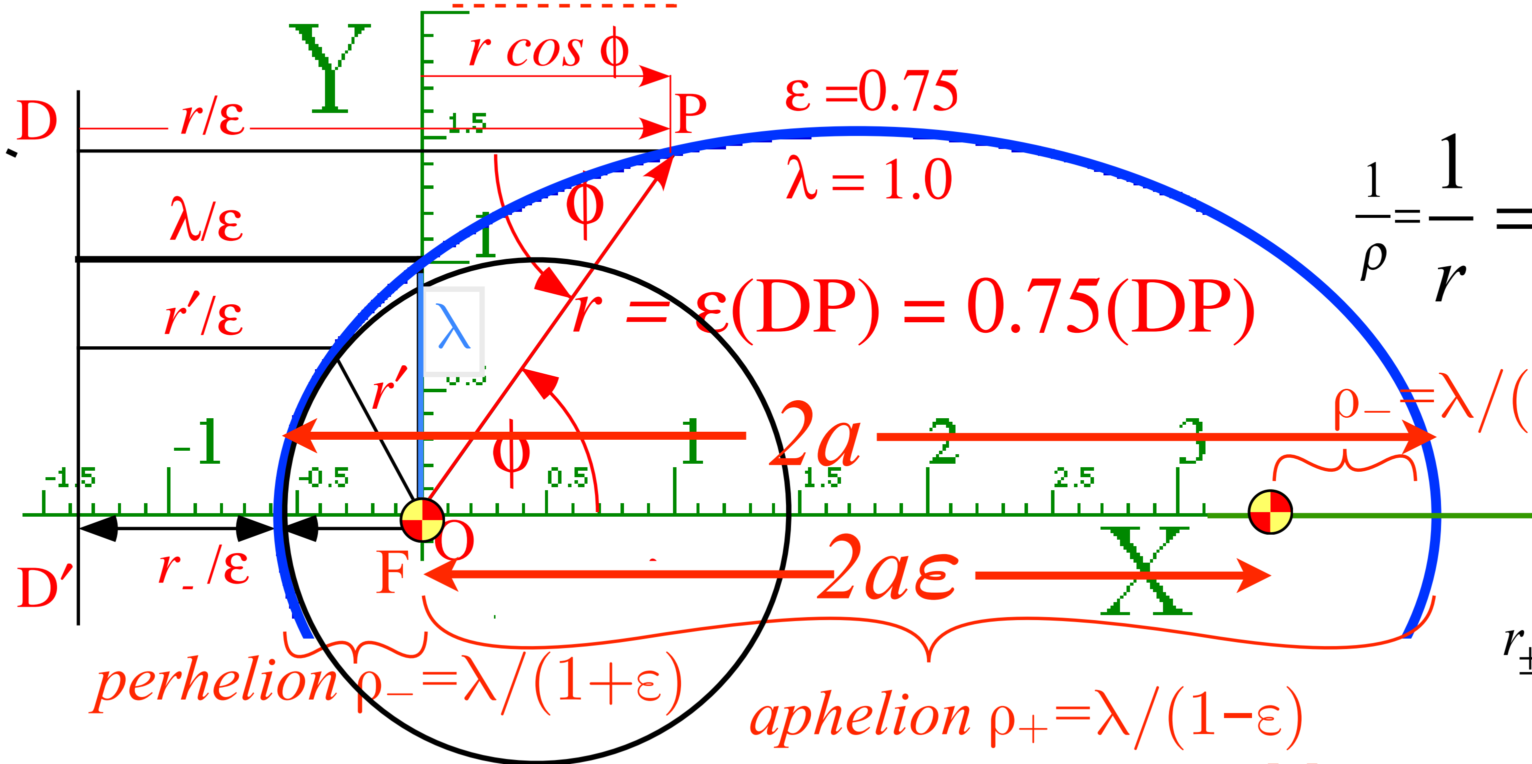
Distance to *Focus* **F** = $\varepsilon\cdot$Distance to *Directrix* **DD′**

Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Parameter table on p.79

Recall p.66 formula:

$$r_\pm = \rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

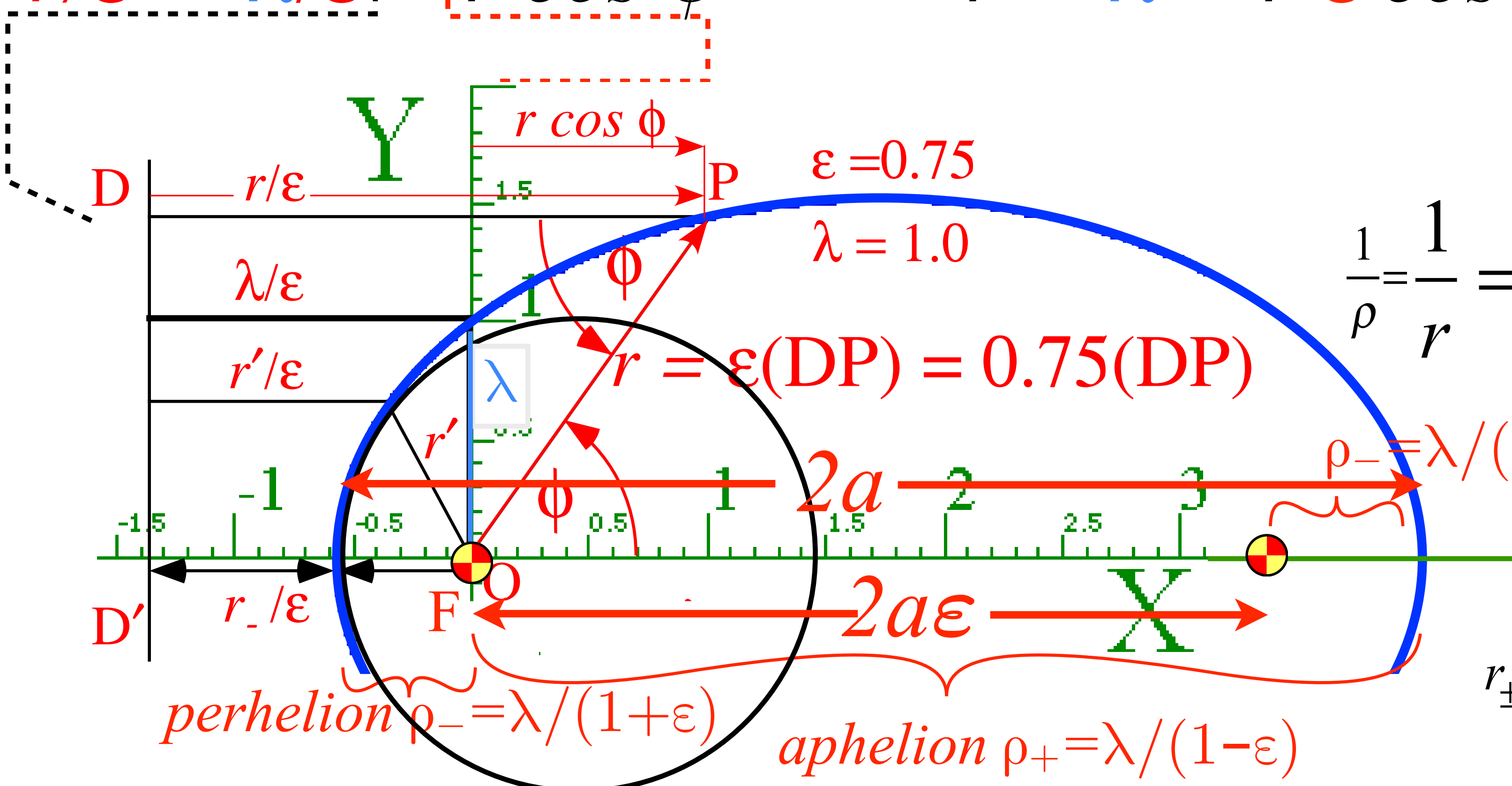
Very important result!

$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{E\,m}} = \left|\frac{k}{E}\right|\sqrt{1 + \frac{2\mu^2 E}{k^2\,m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{\left|1-\varepsilon^2\right|}$$

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$r/\varepsilon = \lambda/\varepsilon + r\cos\phi$ $\qquad$ $r = \lambda + r\,\varepsilon\cos\phi$ $\qquad$ $r = \dfrac{\lambda}{1-\varepsilon\cos\phi}$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Parameter table on p.79

Recall p.66 formula:

$$r_{\pm} = \rho_{\pm} = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

Focal axis: $\rho_+ + \rho_- = 2a\varepsilon$

$\rho_+ - \rho_- = [\lambda(1+\varepsilon) - \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda\varepsilon/(1-\varepsilon^2)$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Very important result!

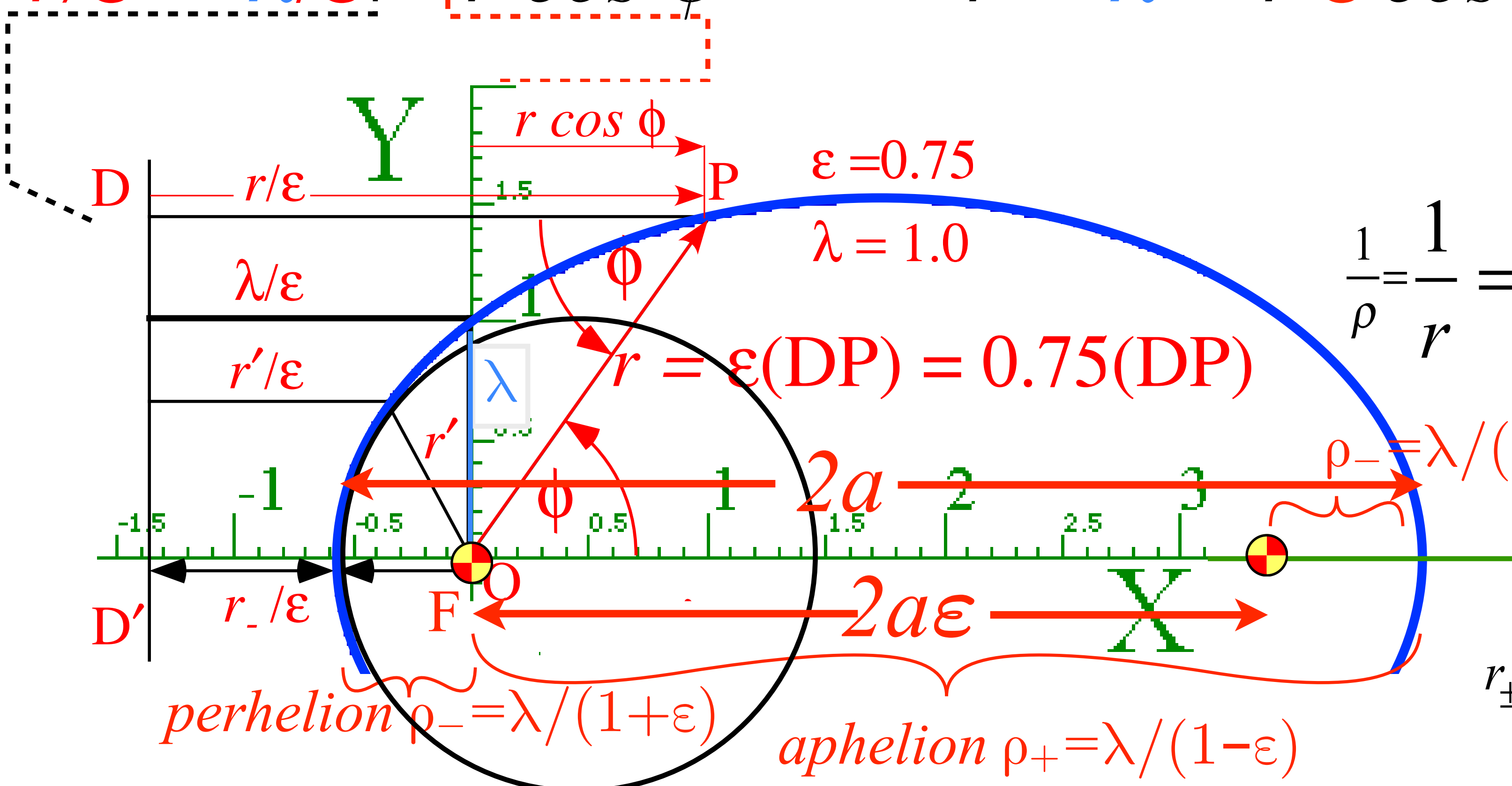
$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{Em}} = \left|\frac{k}{E}\right|\sqrt{1 + \frac{2\mu^2 E}{k^2 m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{|1-\varepsilon^2|}$$

$$\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} = \sqrt{1 + \frac{2\mu^2 E}{k^2 m}} = \varepsilon$$

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Parameter table on *p.79*

Recall *p.66* *formula:*

$$r_{\pm} = \rho_{\pm} = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E}$$

All conics defined by: ***Eccentricity*** ε

Distance to *Focus* **F** = ε·Distance to *Directrix* **DD′**

Major axis: $\rho_+ + \rho_- = 2a$

$\rho_+ + \rho_- = [\lambda(1+\varepsilon) + \lambda(1-\varepsilon)]/(1-\varepsilon^2) = 2\lambda/(1-\varepsilon^2)$

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Latus radius: $\lambda = a(1-\varepsilon^2)$

Very important result!

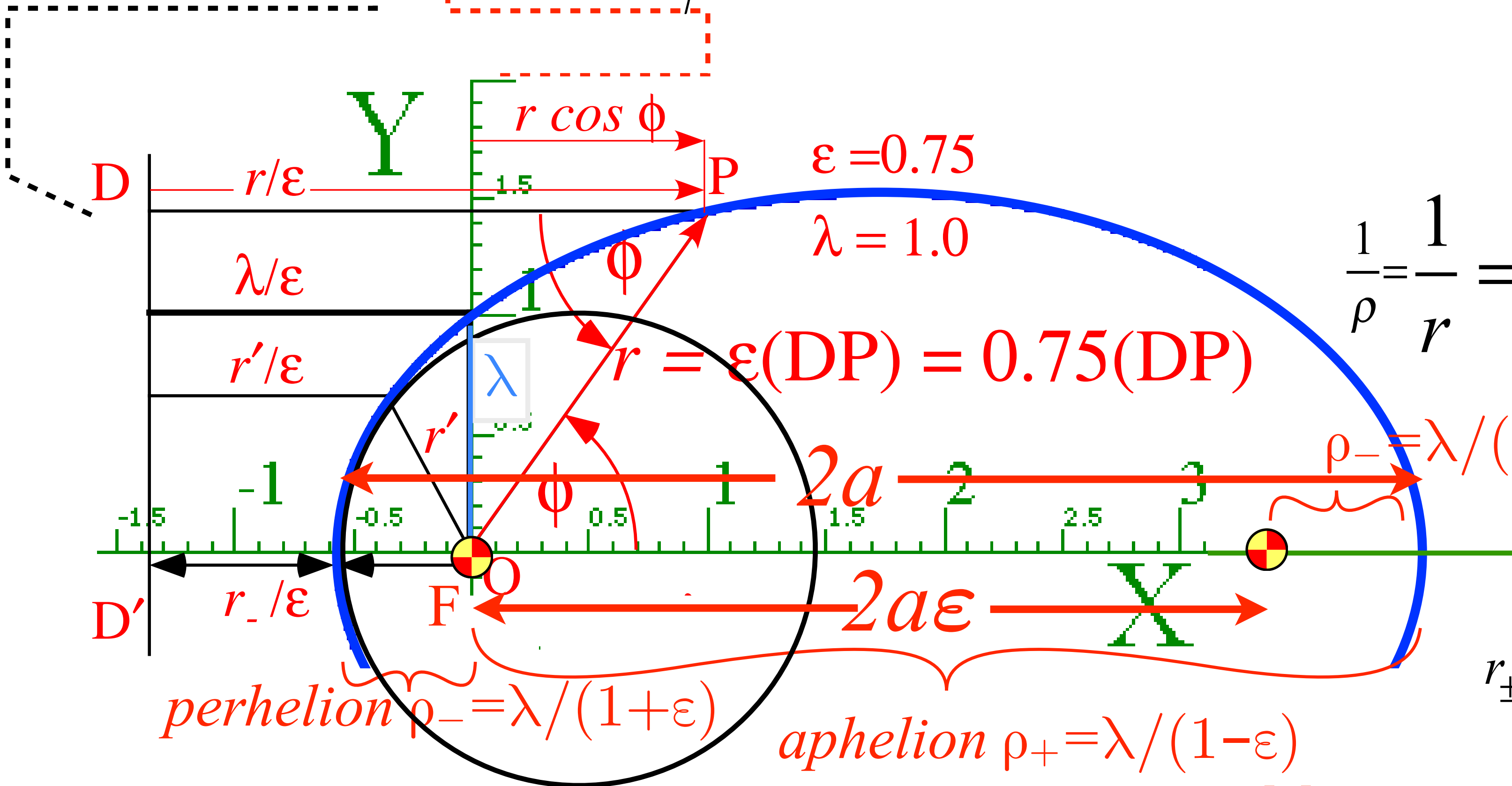
$$\rho_+ + \rho_- = \frac{-k}{E} = 2a \quad \text{implies:} \quad \boxed{E = \frac{-k}{2a}}$$

$$\rho_+ - \rho_- = \sqrt{\left(\frac{k}{E}\right)^2 + \frac{2\mu^2}{E\,m}} = \left|\frac{k}{E}\right|\sqrt{1 + \frac{2\mu^2 E}{k^2\,m}} = 2a\varepsilon = \frac{2\lambda\varepsilon}{|1-\varepsilon^2|}$$

$$\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-} = \sqrt{1 + \frac{2\mu^2 E}{k^2\,m}} = \varepsilon$$

Geometry of ALL Coulomb conic section orbits (Let: $r \equiv \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



By geometry:

$$\frac{1}{\rho} = \frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

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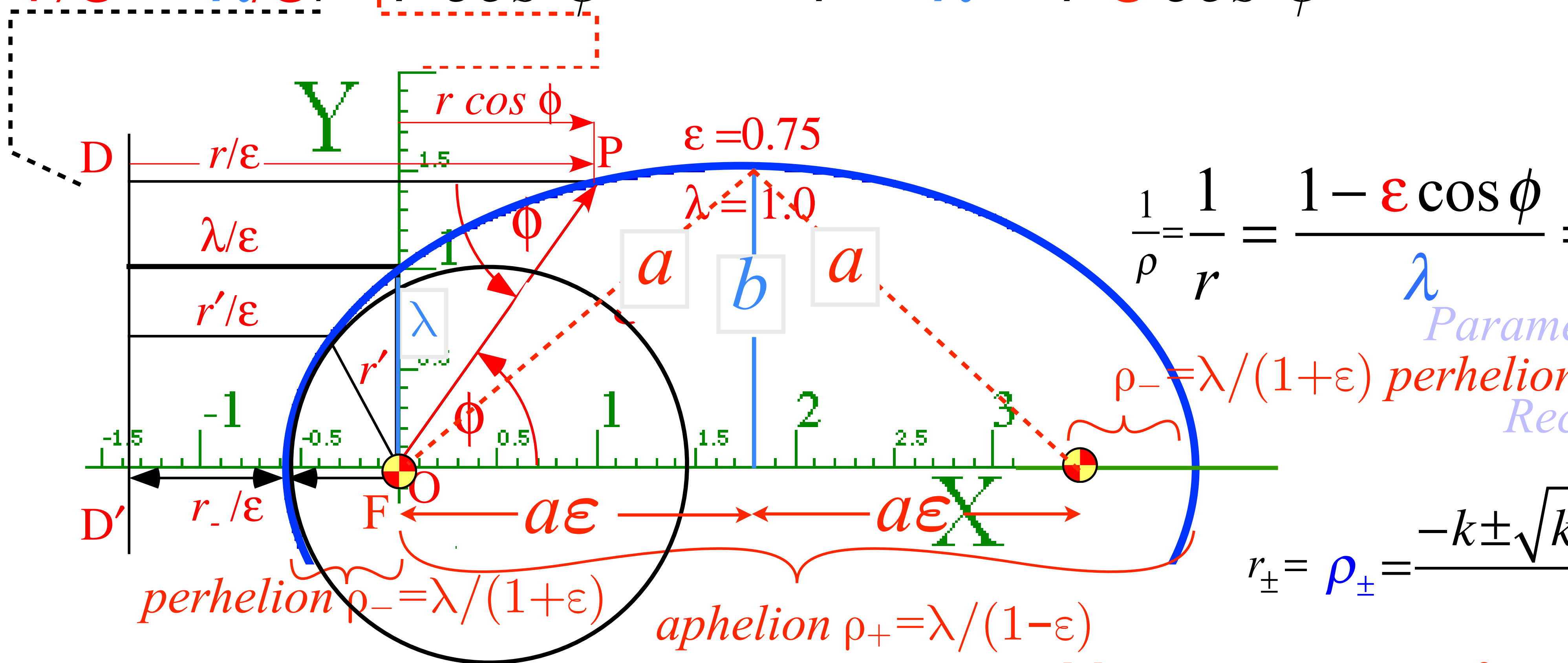

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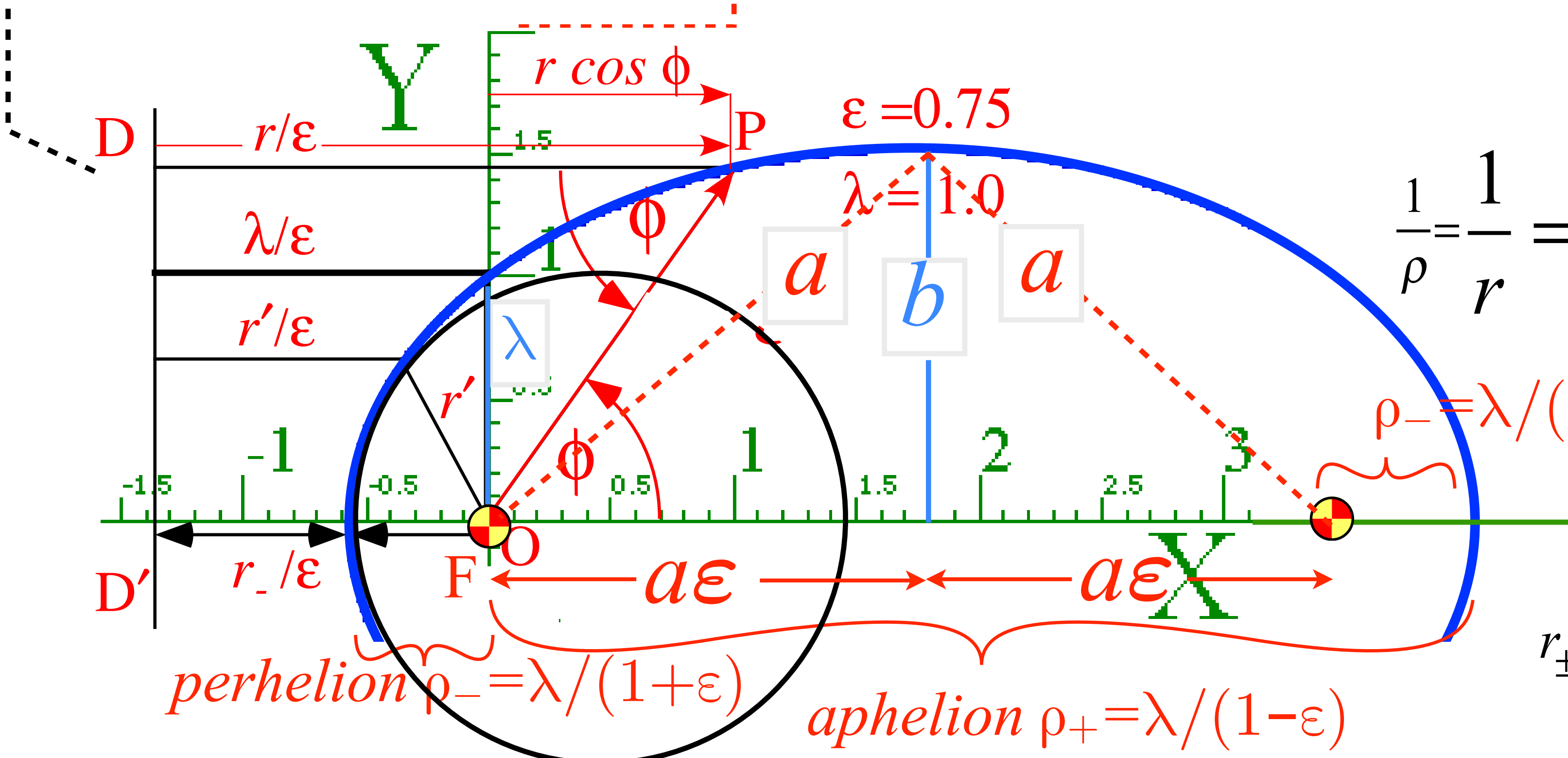

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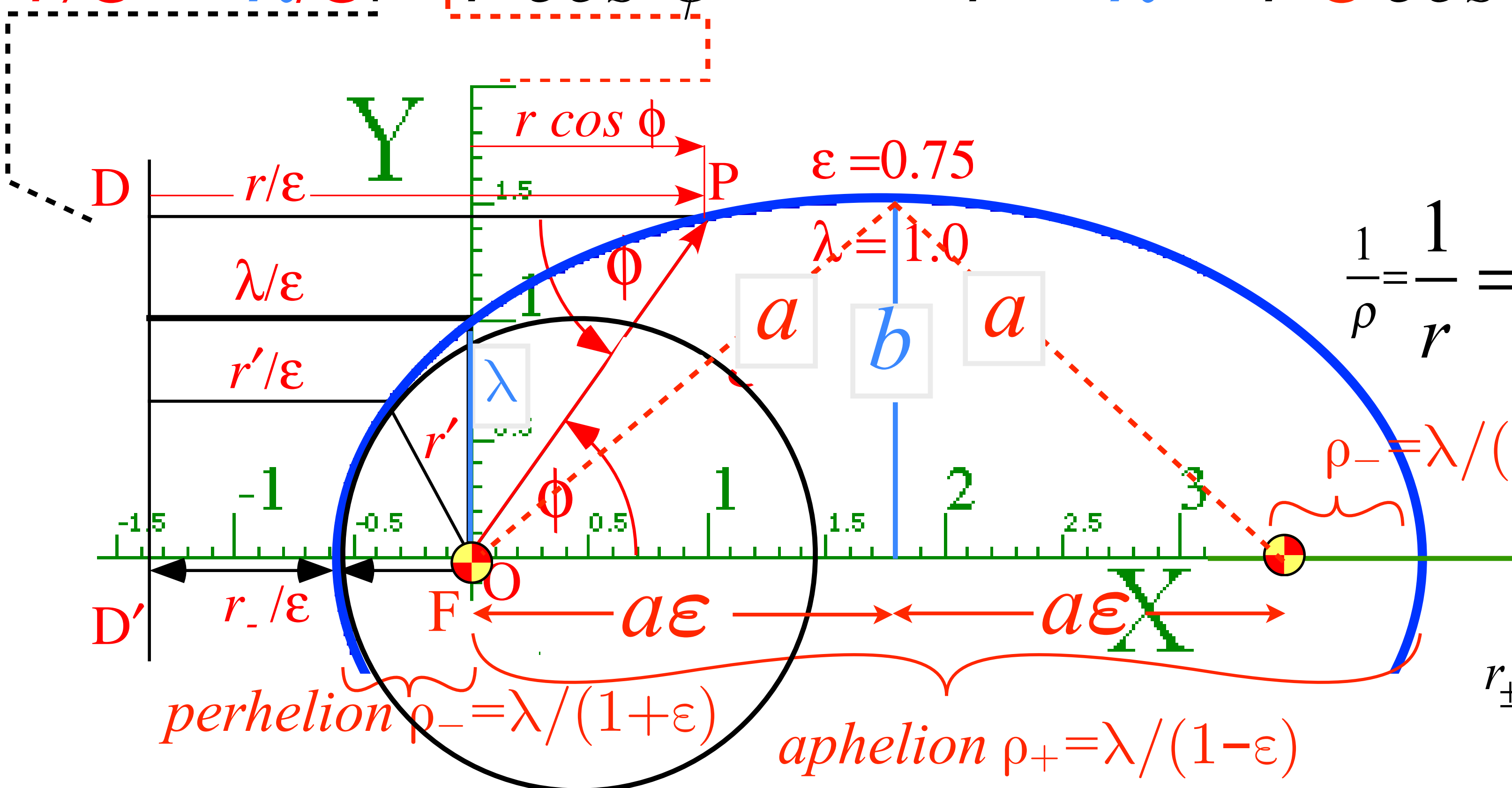

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(x,y) *parameters*	*physical parameters*	(r,ϕ) *parameters*
major radius $a = \frac{k}{2E}$	*Energy* $E = \frac{k}{2a}$	*eccentricity* $\varepsilon = \sqrt{\frac{k^2 m + 2L^2 E}{k^2 m}}$
minor radius $b = \frac{L}{\sqrt{2m\|E\|}}$	*∠-momentum* $L = \sqrt{km\lambda} \equiv \mu$	*latus radius* $\lambda = \frac{L^2}{km}$

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$\lambda = a(1-\varepsilon^2)$ (*ellipse:* $\varepsilon < 1$) $\quad a\varepsilon^2 = a - \lambda$

$\lambda = a(\varepsilon^2 - 1)$ (*hyperb:* $\varepsilon > 1$) $\quad a\varepsilon^2 = a + \lambda$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

➔ *Kepler equation of time and phase geometry*

Starting with KE-eff.-PE results: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or* p.38: $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

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Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

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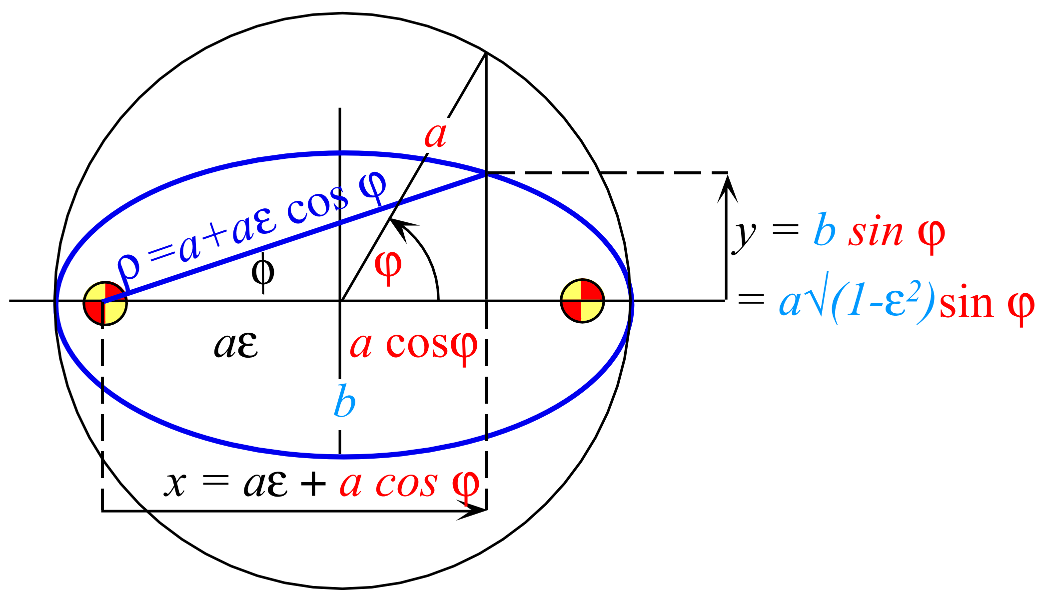
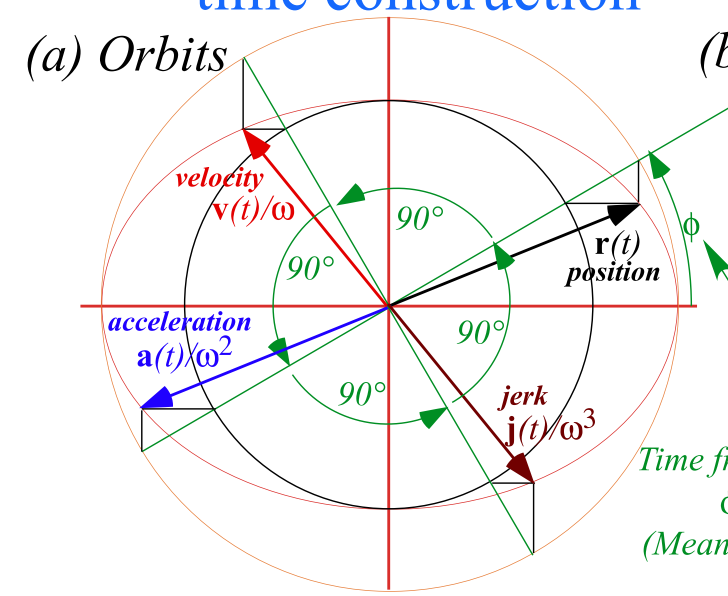
Unit 1 Ch. 9
Recall IHO orbit
time construction





RelaWavity Web Simulation:
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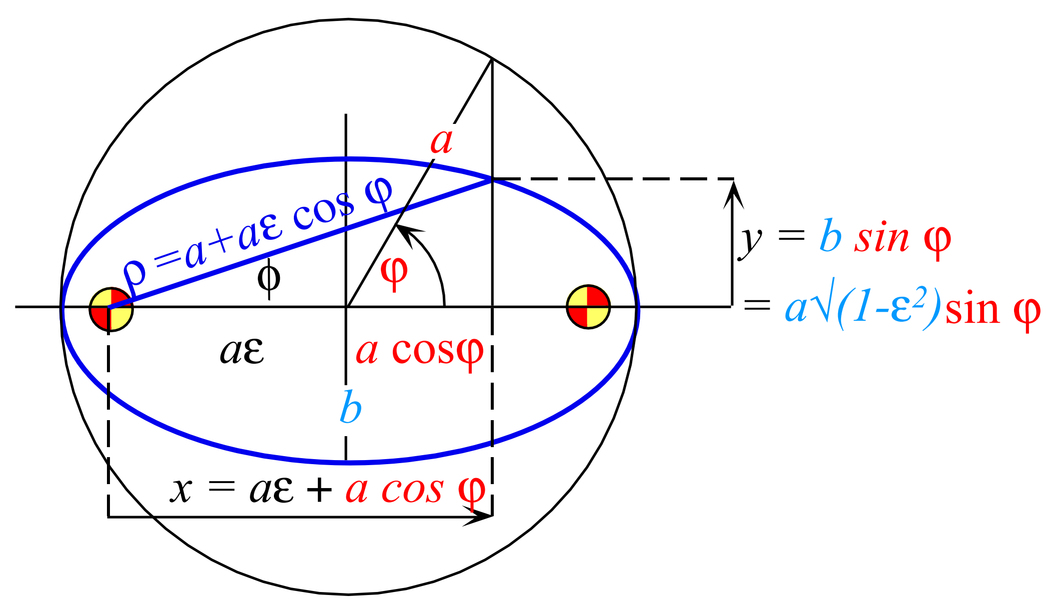

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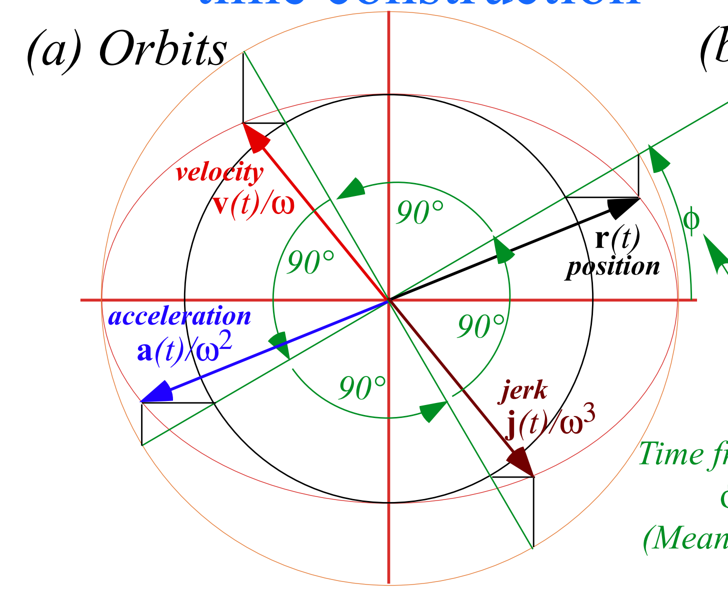

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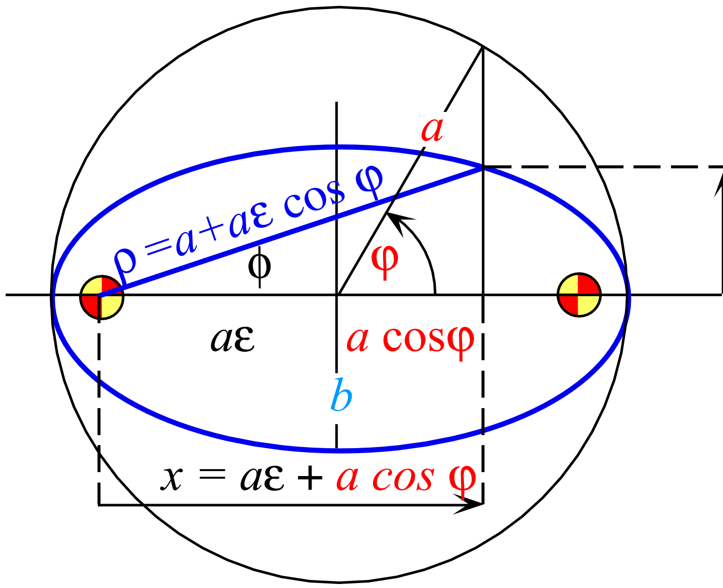

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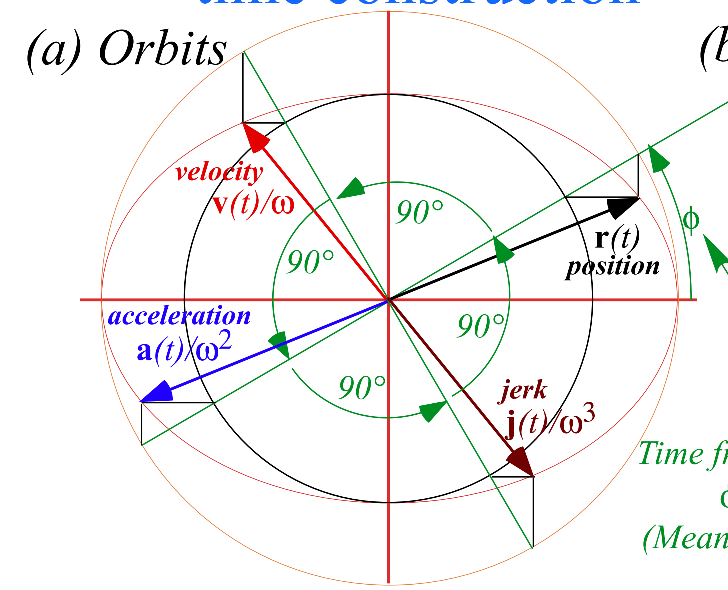

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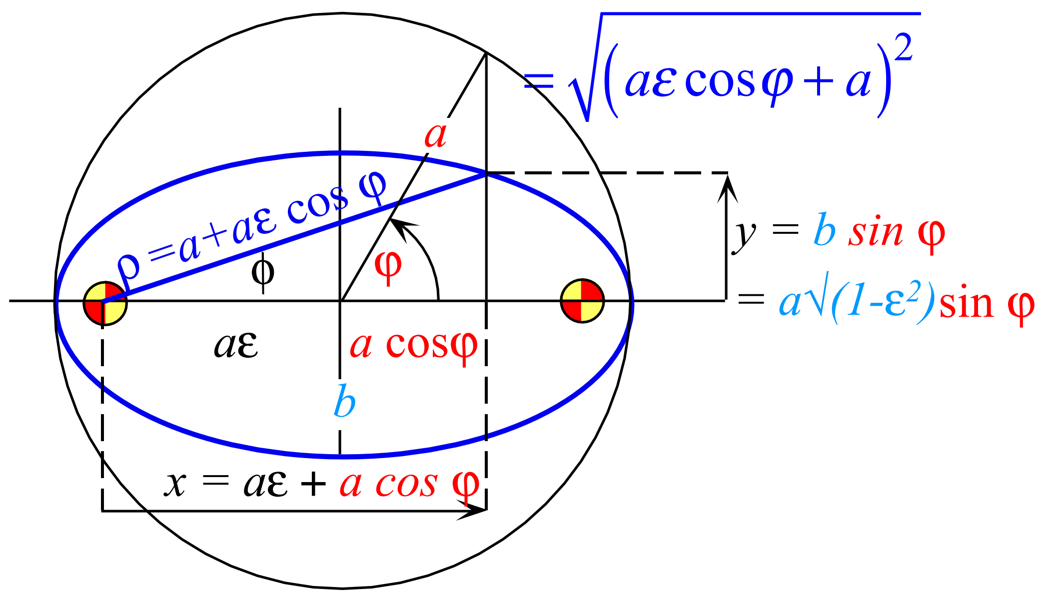

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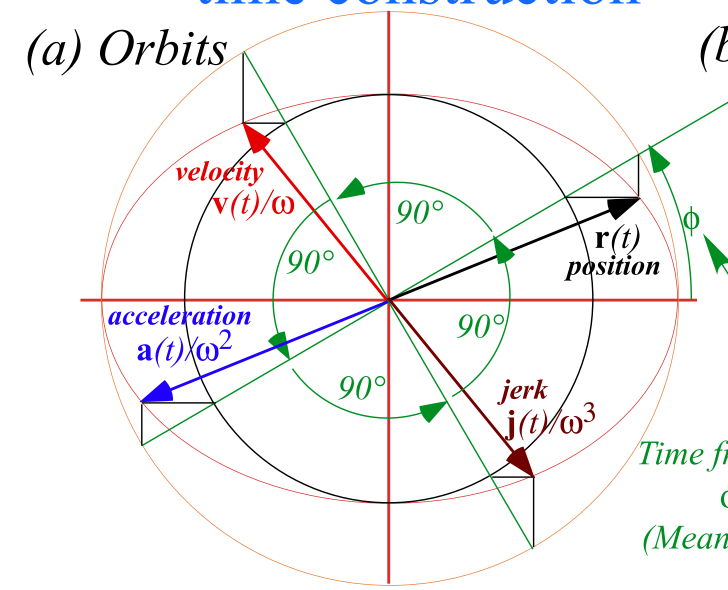

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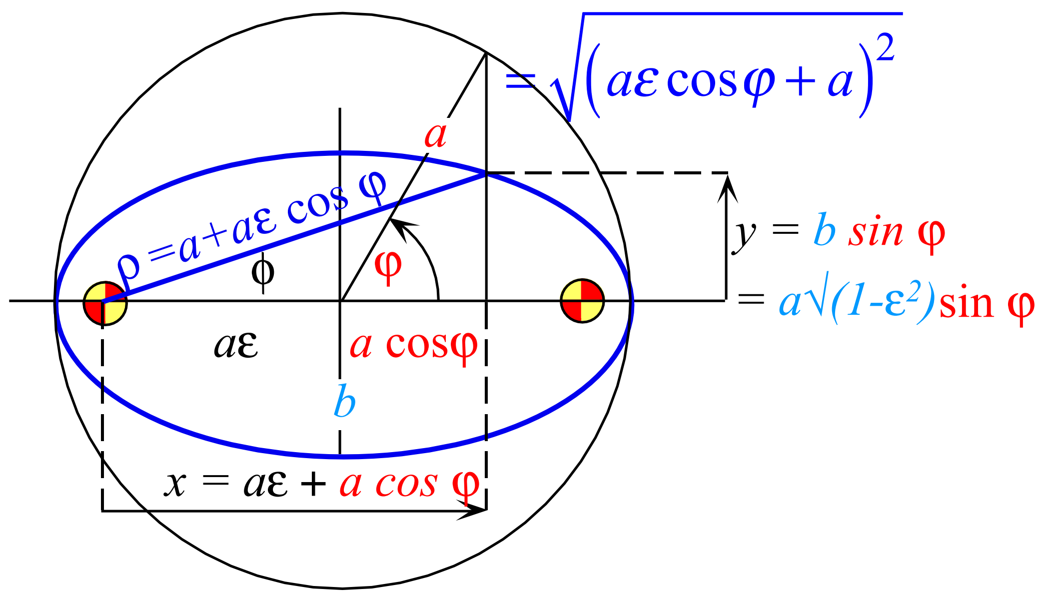

Unit 1 Ch. 9
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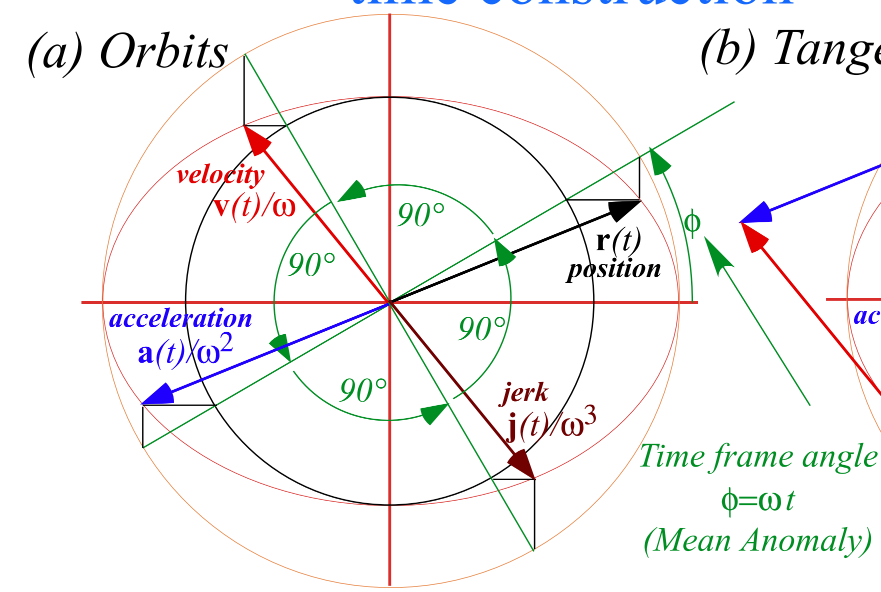

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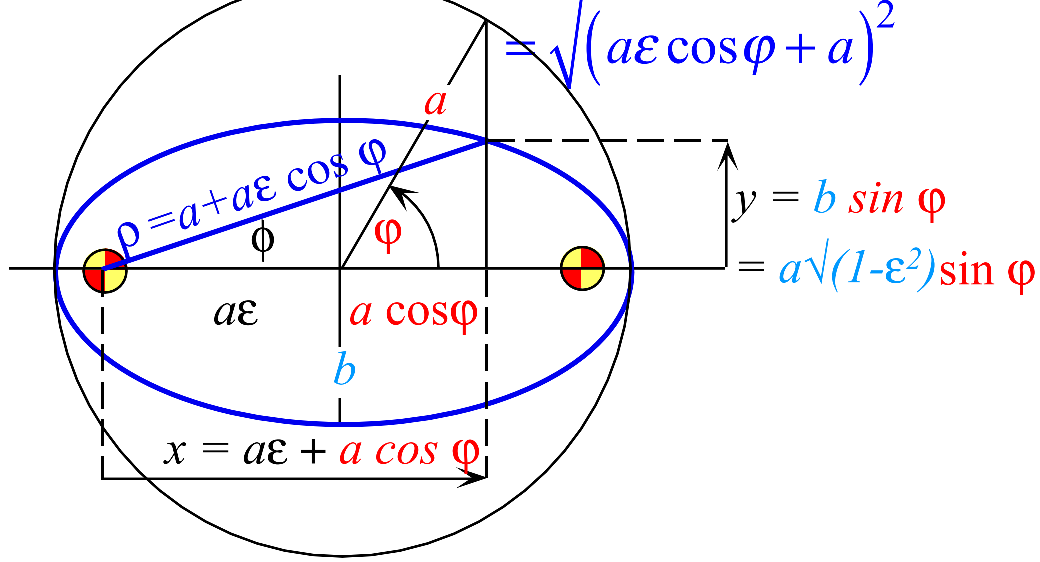
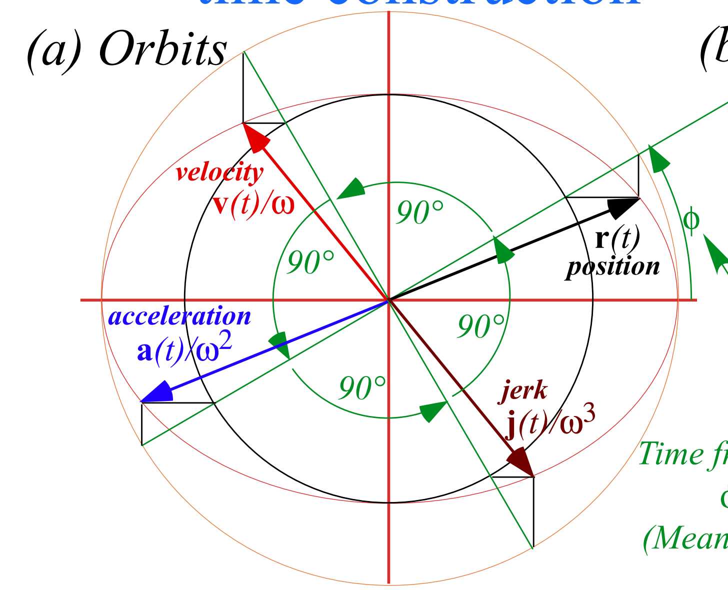
$$= \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$

$$= \sqrt{(a\varepsilon\cos\varphi + a)^2}$$

$$\rho = a(1 + \varepsilon\cos\varphi)$$



Unit 1 Ch. 9
Recall IHO orbit
time construction



$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-(a + a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}}$$

Starting with KE-eff.-PE results: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or* *p.38*: $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho\, d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho\, d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

$$x = a\varepsilon + a\cos\varphi\,, \qquad y = a\sqrt{1-\varepsilon^2}\,\sin\varphi\,,$$

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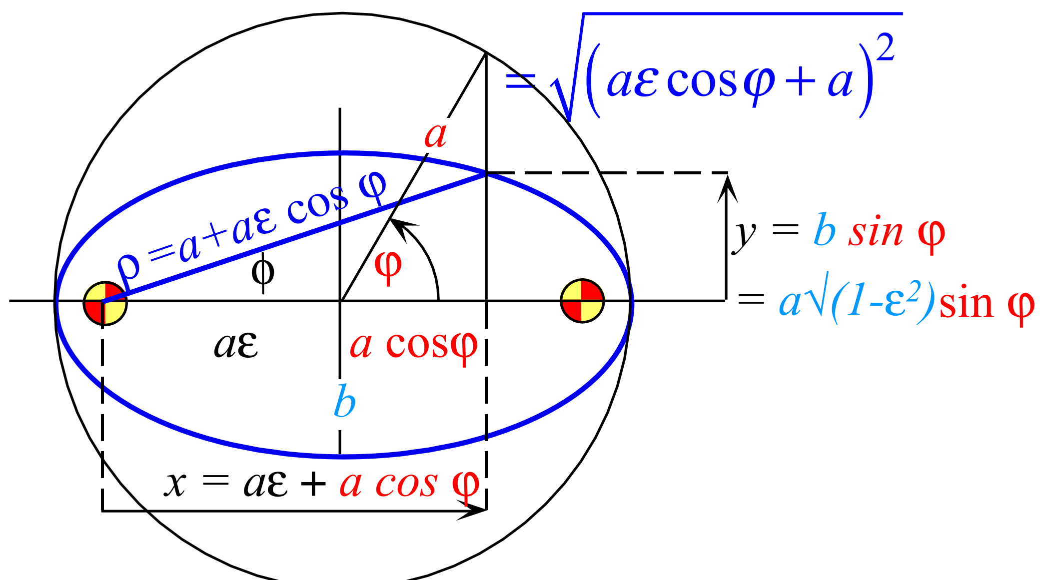
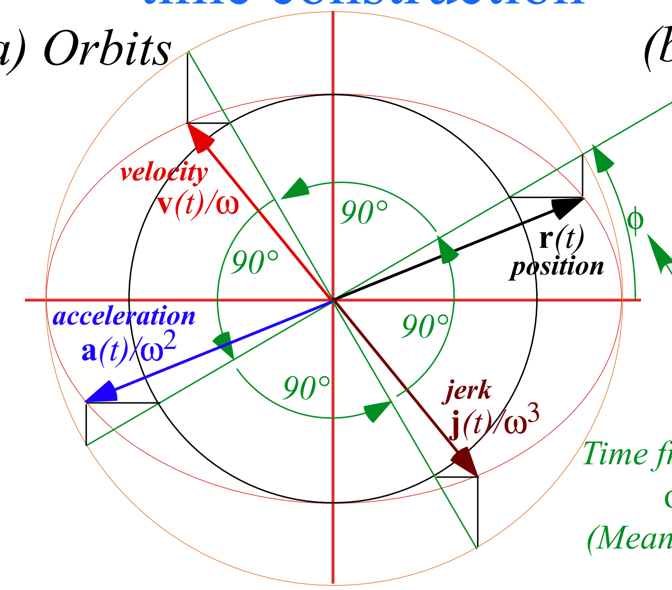
$$\rho = a(1+\varepsilon\cos\varphi) \qquad = \sqrt{(a\varepsilon\cos\varphi + a)^2}$$



Unit 1 Ch. 9
Recall IHO orbit
time construction

(a) Orbits



$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-(a+a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}} = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-a^2 - 2a^2\varepsilon\cos\varphi - a^2\varepsilon^2\cos^2\varphi + 2a^2 + 2a^2\varepsilon\cos\varphi - a^2 + a^2\varepsilon^2}{2a}\right)}}$$

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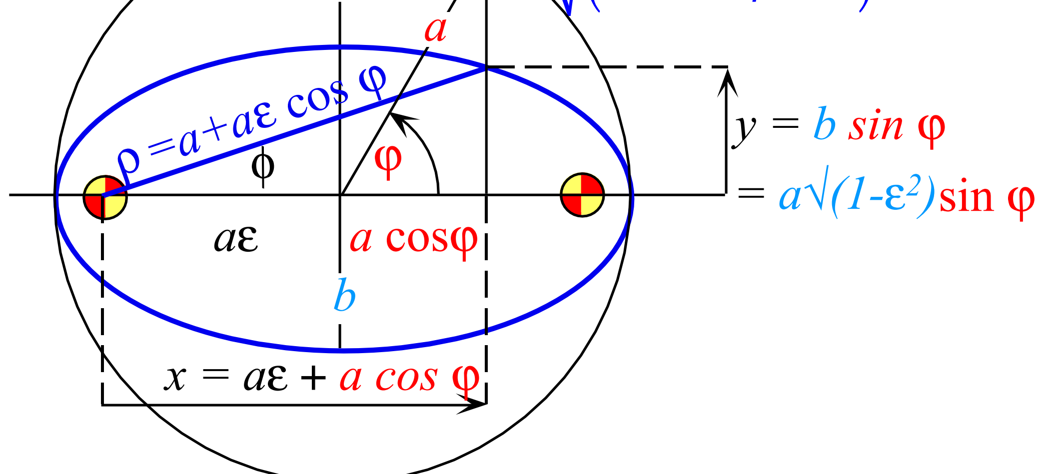
$$x = a\varepsilon + a\cos\varphi\,, \qquad y = a\sqrt{1-\varepsilon^2}\sin\varphi\,,$$

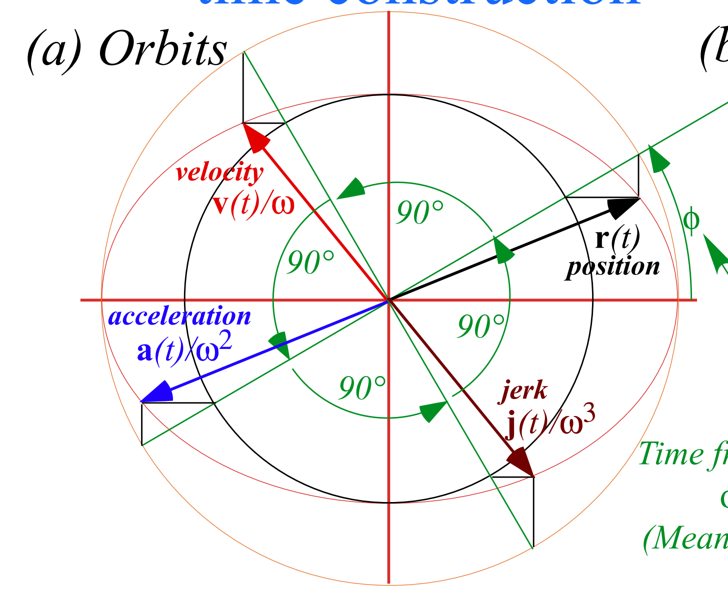
$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi - a^2\varepsilon^2\sin^2\varphi}$$

$$= \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$

Unit 1 Ch. 9
Recall IHO orbit time construction

$$\rho = a(1 + \varepsilon\cos\varphi) \qquad = \sqrt{(a\varepsilon\cos\varphi + a)^2}$$





$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-(a + a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}} = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-a^2 - 2a^2\varepsilon\cos\varphi - a^2\varepsilon^2\cos^2\varphi + 2a^2 + 2a^2\varepsilon\cos\varphi - a^2 + a^2\varepsilon^2}{2a}\right)}}$$

$$t = \sqrt{\frac{ma^3}{k}} \int (1 + \varepsilon\cos\varphi)\, d\varphi$$

Starting with KE-eff.-PE results: $\frac{m}{2}\dot{\rho}^2 = E - \frac{\mu^2}{2m\rho^2} - k/\rho$ *or* *p.38**:* $\dot{\rho} = \frac{d\rho}{dt} = \sqrt{\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} - \frac{2k}{m\rho}}$

Kepler equation of time for Coulomb orbits

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$$x = a\varepsilon + a\cos\varphi\ , \qquad y = a\sqrt{1-\varepsilon^2}\sin\varphi\ ,$$

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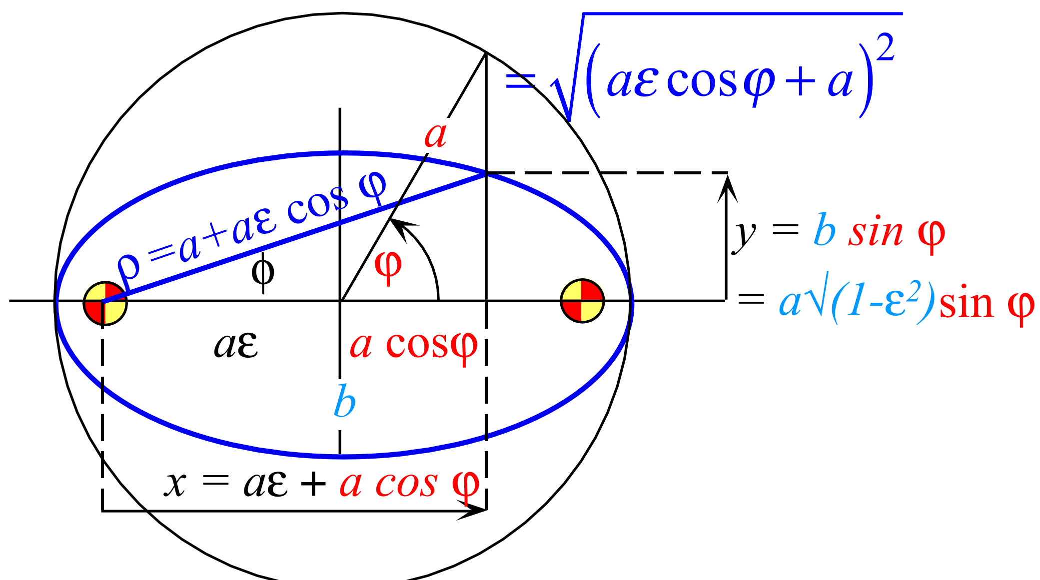
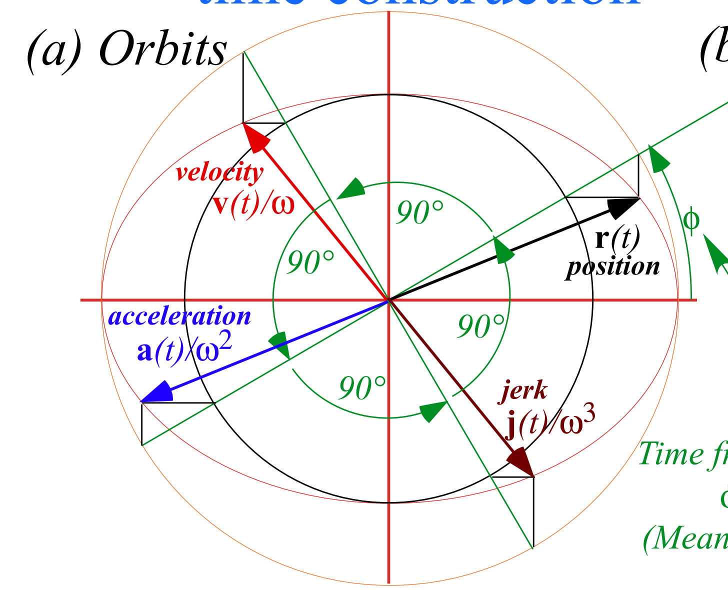
$$= \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon\cos\varphi + a^2}$$

$$= \sqrt{(a\varepsilon\cos\varphi + a)^2}$$

$$\rho = a(1 + \varepsilon\cos\varphi)$$



Unit 1 Ch. 9
Recall IHO orbit
time construction



$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-(a + a\varepsilon\cos\varphi)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2(1-\varepsilon^2)}{2a}\right)}} = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi\, d\varphi}{\sqrt{\left(\frac{-a^2 - 2a^2\varepsilon\cos\varphi - a^2\varepsilon^2\cos^2\varphi + 2a^2 + 2a^2\varepsilon\cos\varphi - a^2 + a^2\varepsilon^2}{2a}\right)}}$$

$$t = \sqrt{\frac{ma^3}{k}} \int (1 + \varepsilon\cos\varphi)\, d\varphi = \sqrt{\frac{ma^3}{k}}(\varphi + \varepsilon\sin\varphi)$$

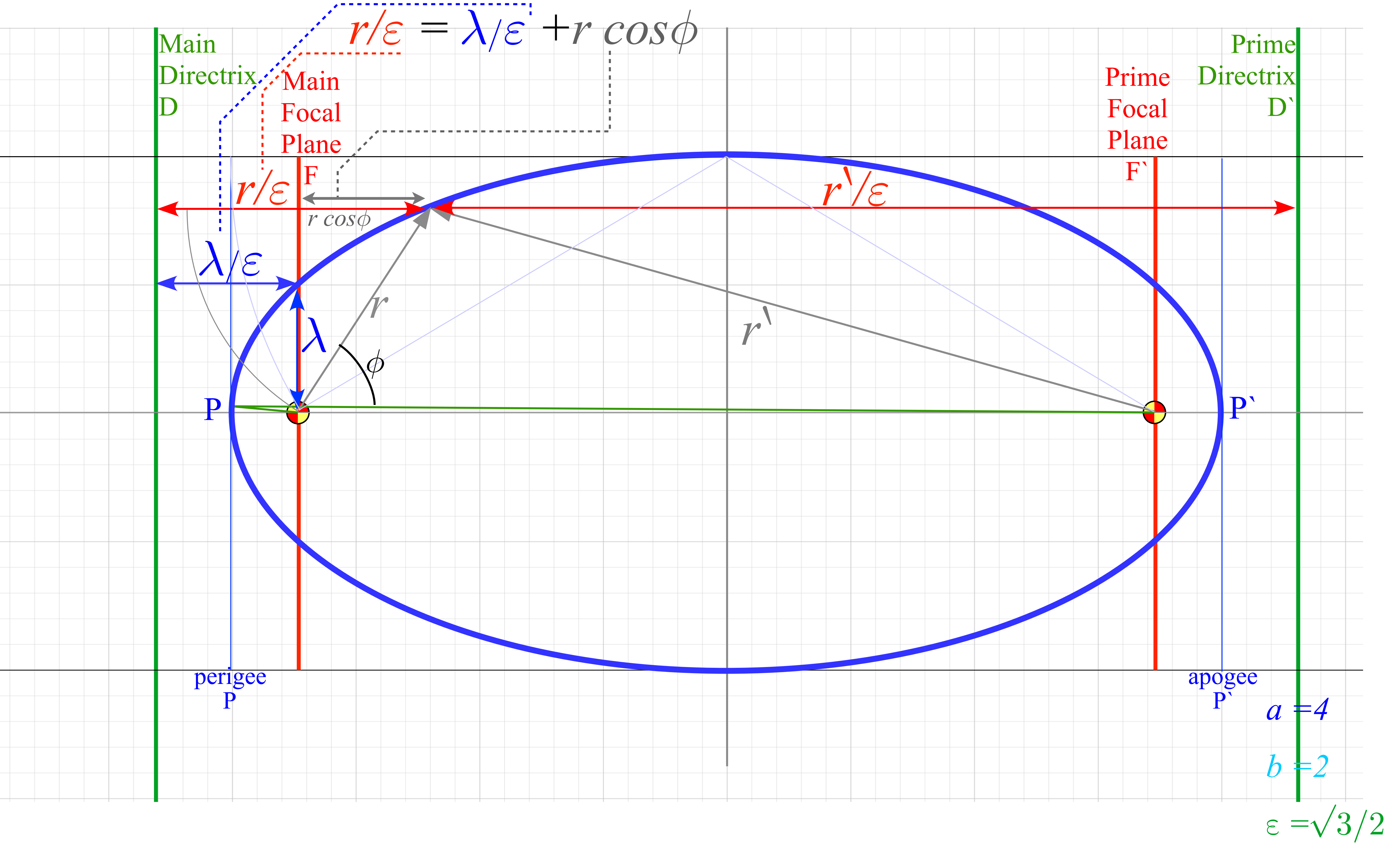
Kepler's equations of orbital time

Orbit Period $= T = \frac{2\pi}{\omega_\varphi} = 2\pi\sqrt{\frac{ma^3}{k}}$

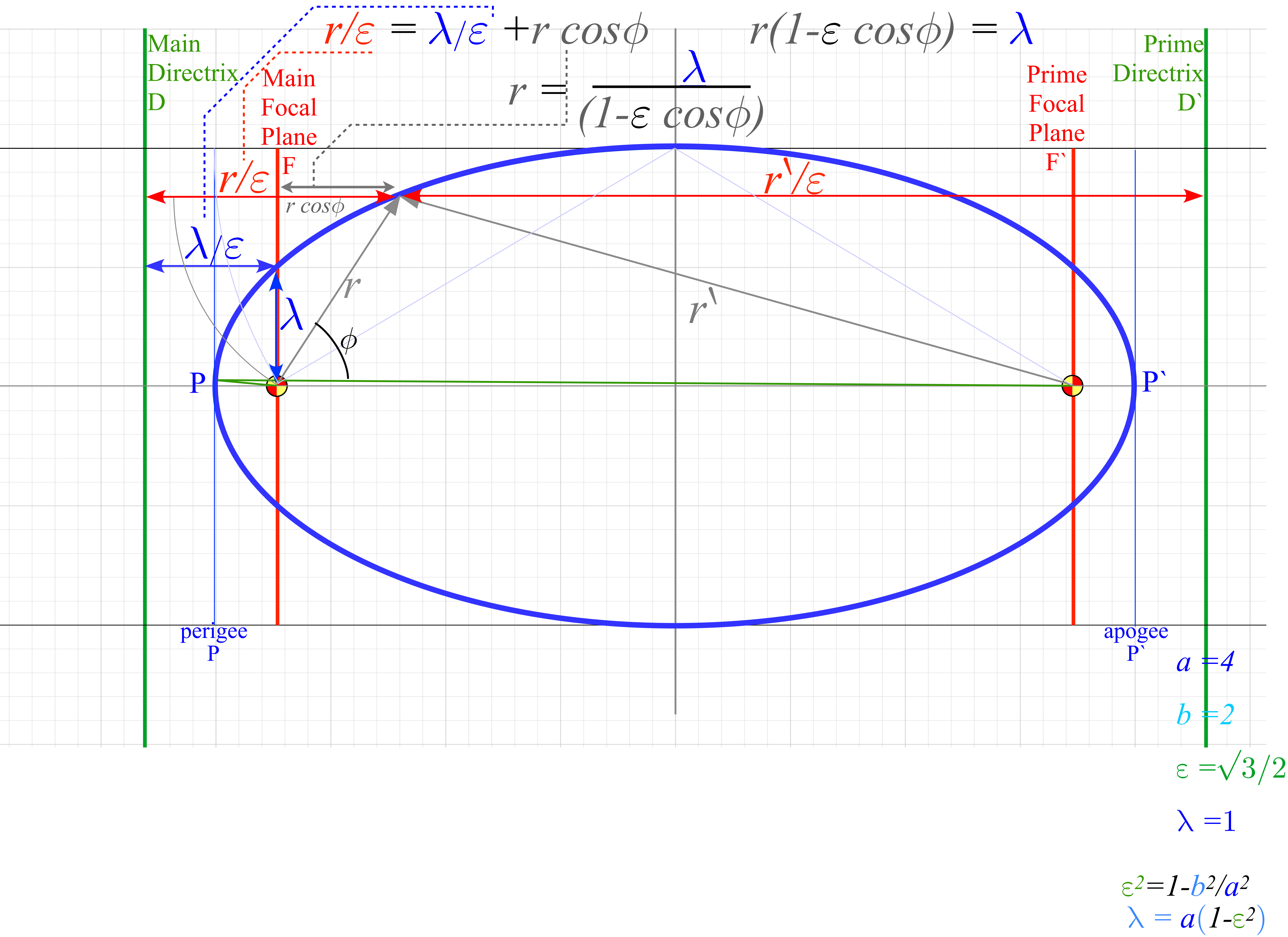
Geometry and Symmetry of Coulomb orbits

➤ *Detailed elliptic geometry*

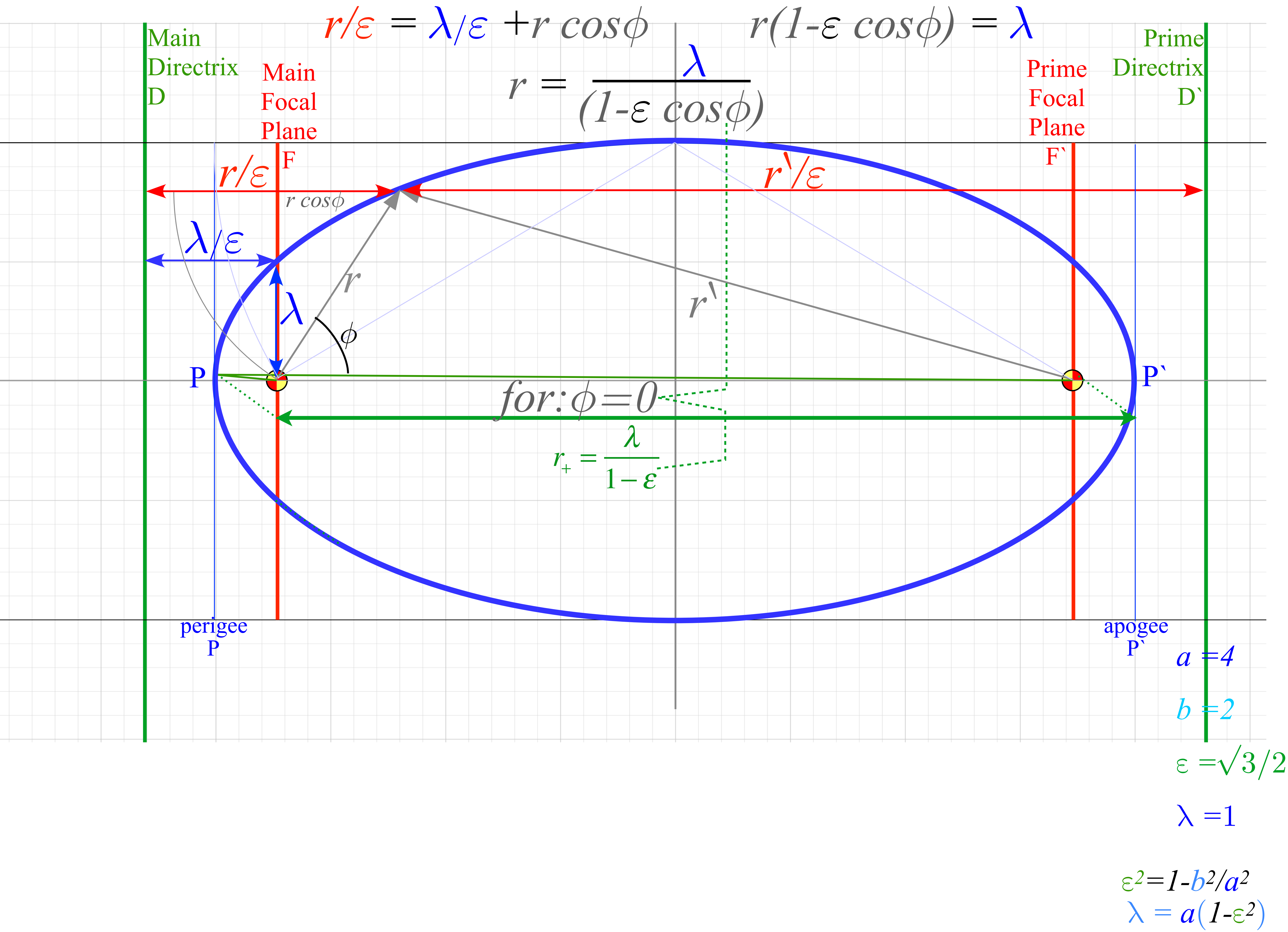
Detailed hyperbolic geometry



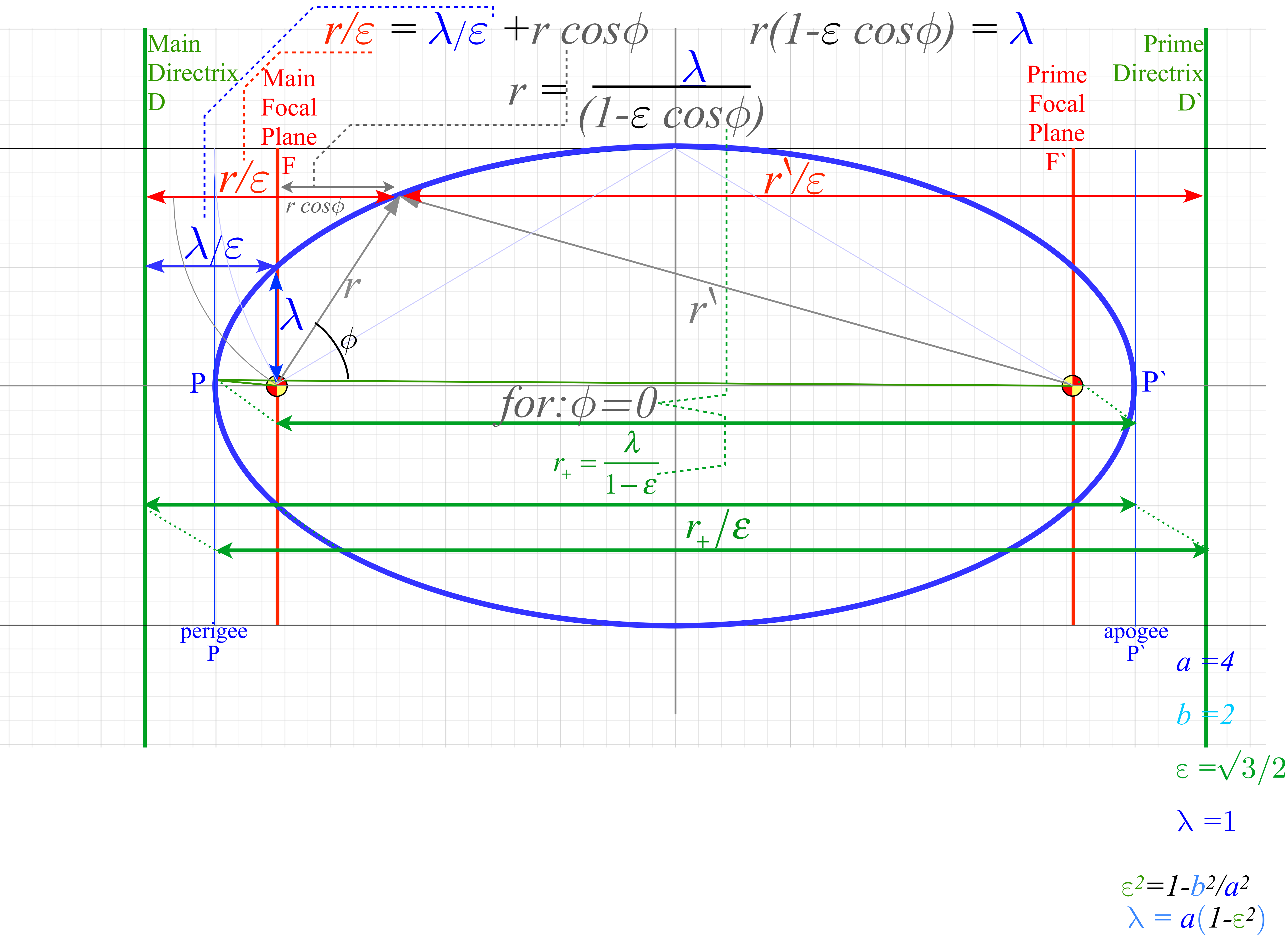
$$\varepsilon^2 = 1 - b^2/a^2$$
$$\lambda = a(1 - \varepsilon^2)$$

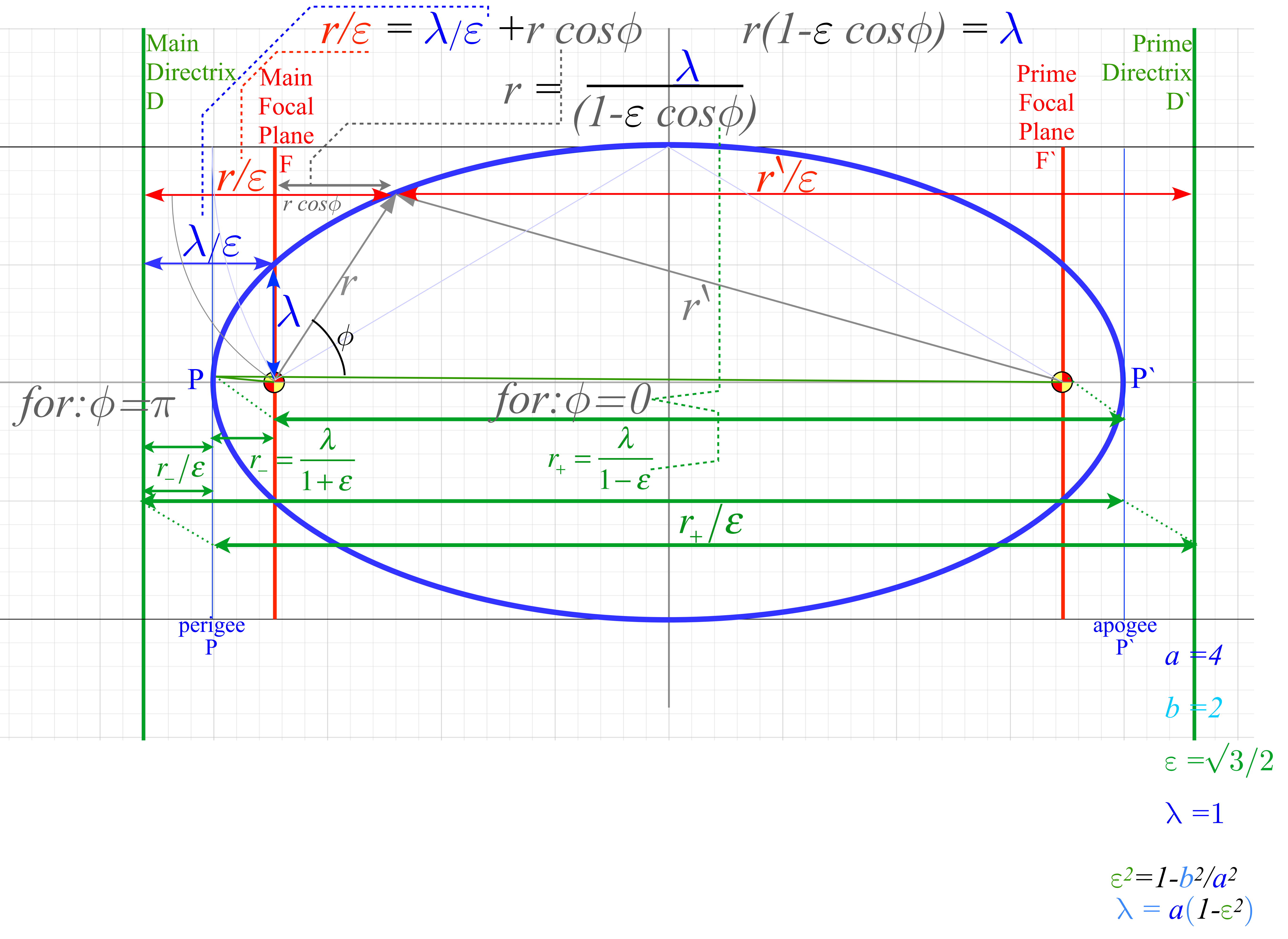


r/ε = λ/ε +r cosφ
r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main Directrix D
Main Focal Plane
Prime Focal Plane
Prime Directrix D`
r/ε
F
r cosφ
r`/ε
F`
λ/ε
λ
r
φ
r`
P
P`
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
ε2=1-b2/a2
λ = a(1-ε2)

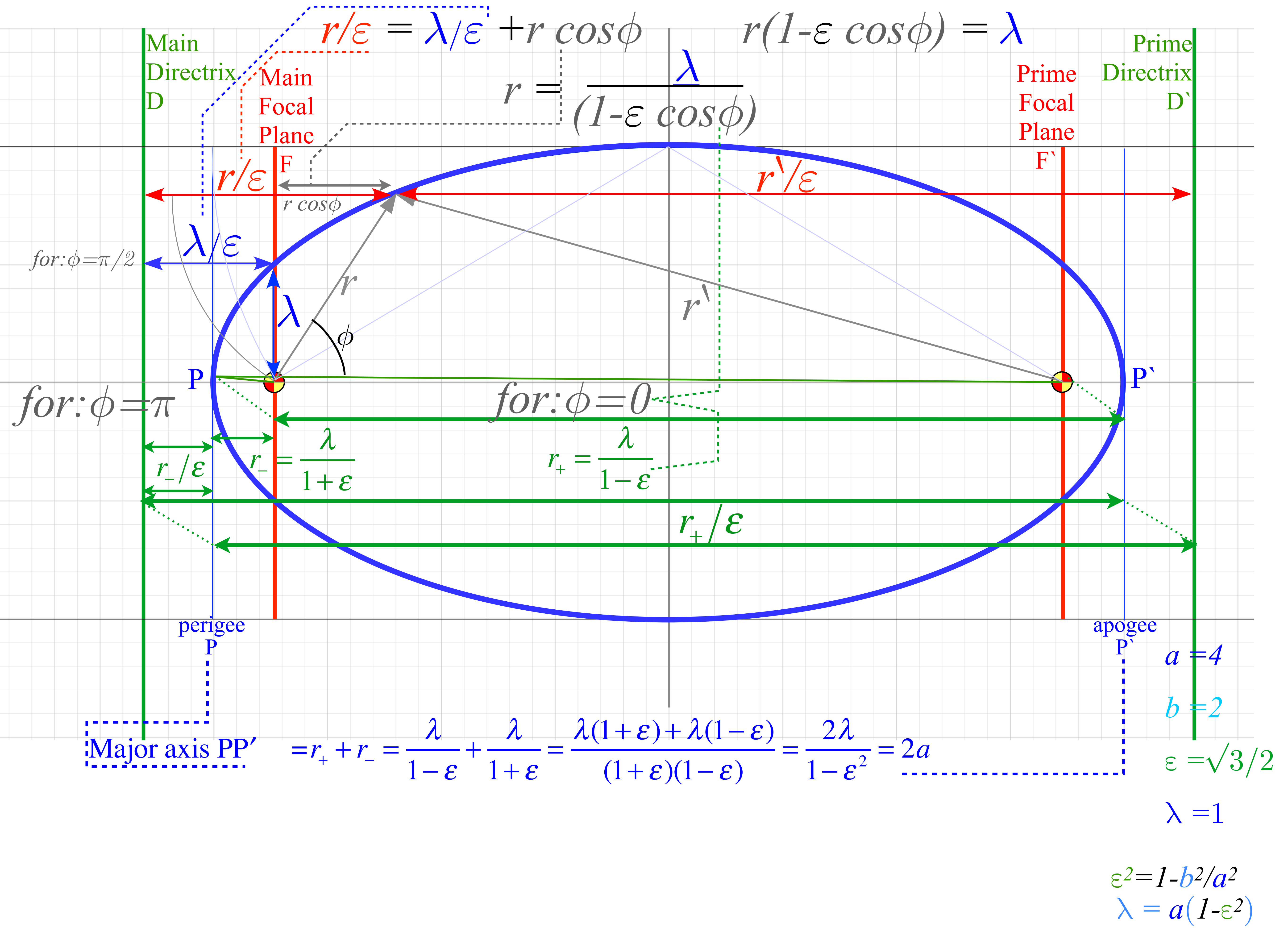


r/ε = λ/ε +r cosϕ
r(1-ε cosϕ) = λ
r = λ / (1-ε cosϕ)
Main Directrix D
Main Focal Plane
F
Prime Focal Plane
F`
Prime Directrix D`
r/ε
r cosϕ
r`/ε
λ/ε
λ
r
r`
ϕ
P
P`
for:ϕ=0
r+ = λ / (1−ε)
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
ε2=1-b2/a2
λ = a(1-ε2)

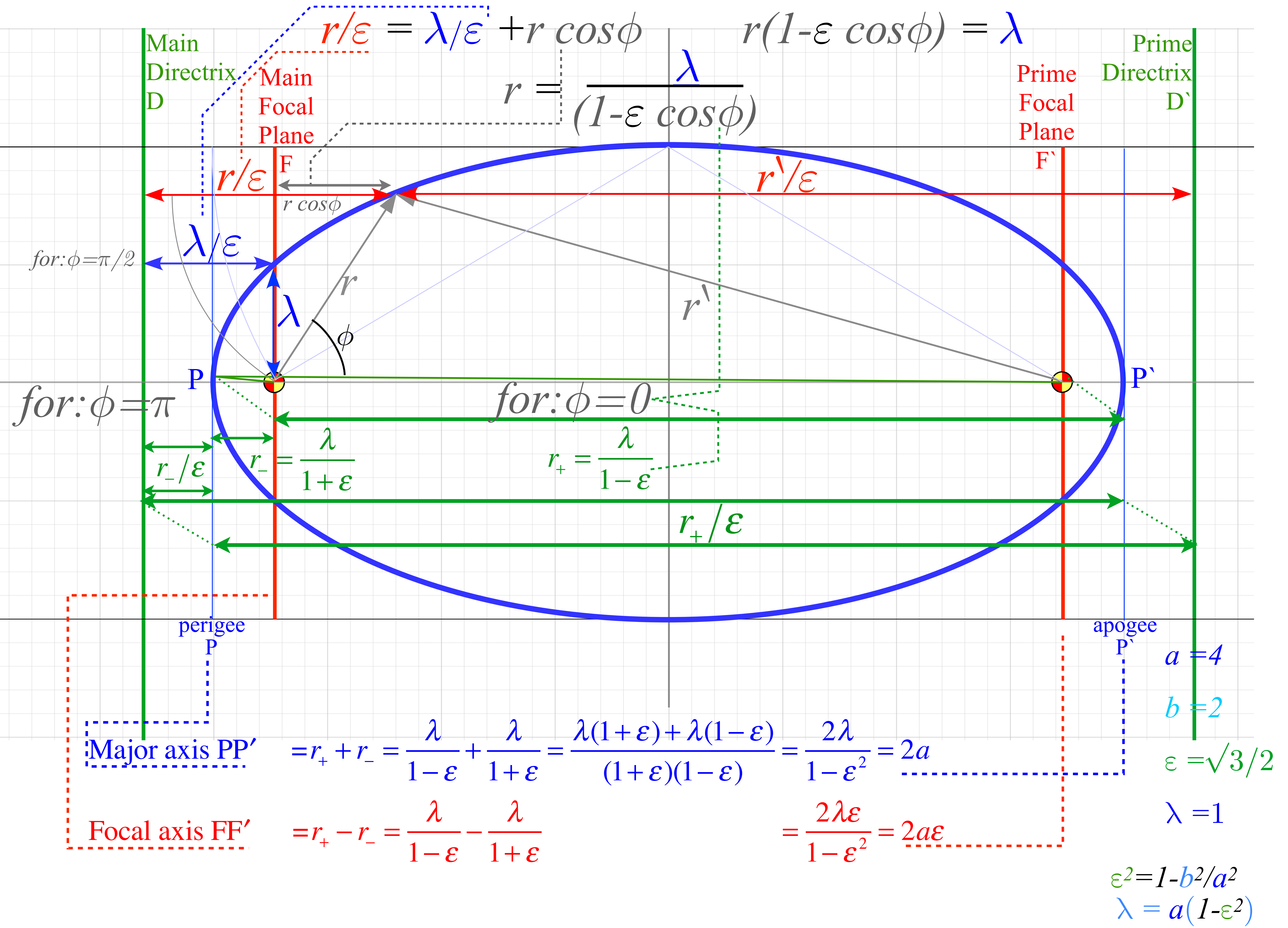

r/ε = λ/ε +r cosφ
r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main Directrix D
Main Focal Plane
F
r/ε
r cosφ
λ/ε
λ
r
φ
r`
r`/ε
Prime Focal Plane F`
Prime Directrix D`
P
P`
for:φ=0
r+ = λ / (1−ε)
r+/ε
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
ε²=1-b²/a²
λ = a(1-ε²)



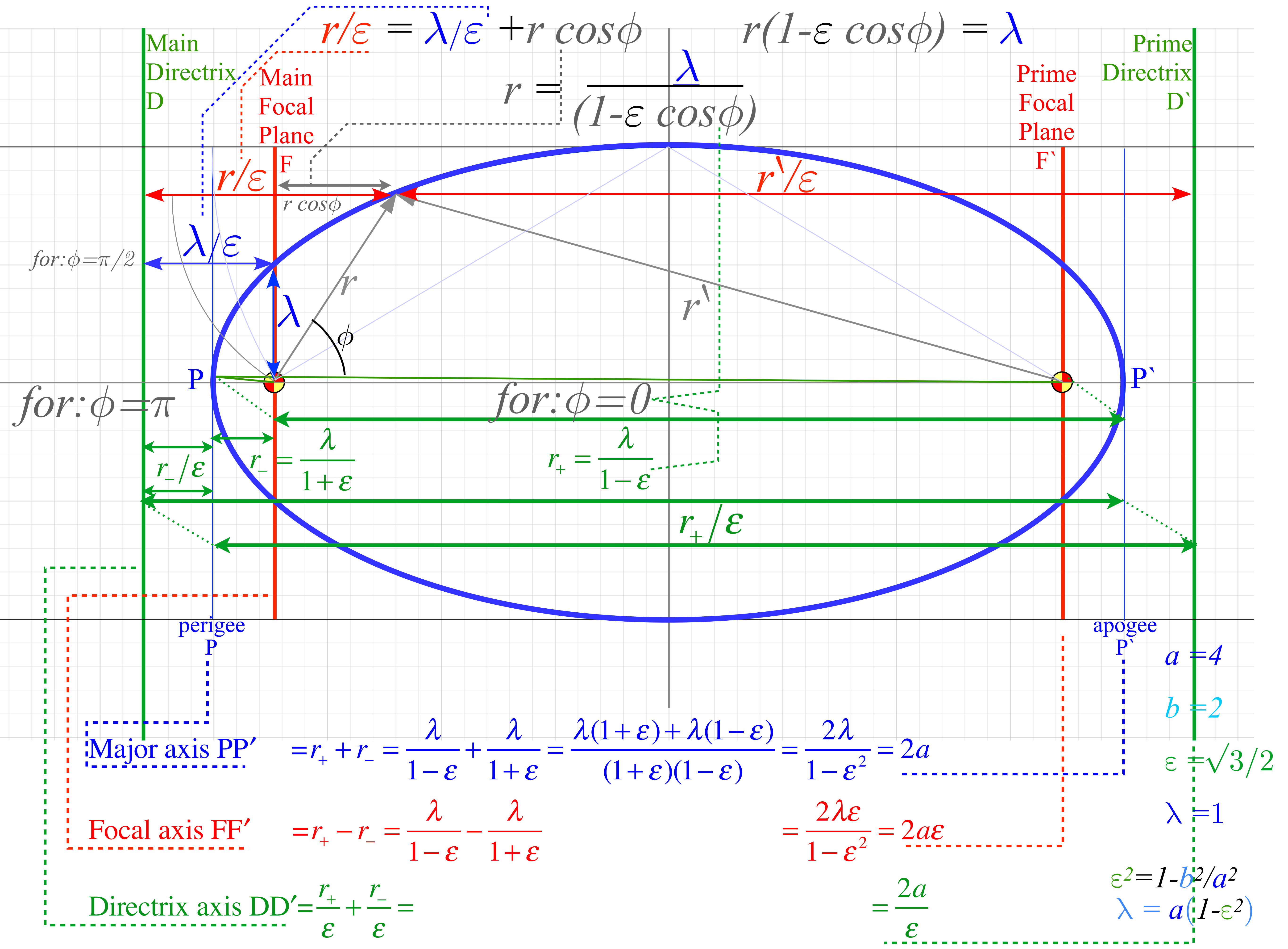
r/ε = λ/ε + r cosϕ
r(1-ε cosϕ) = λ
r = λ / (1-ε cosϕ)
Main Directrix D
Main Focal Plane F
Prime Focal Plane F`
Prime Directrix D`
r/ε
r'/ε
r cosϕ
λ/ε
λ
r
r'
ϕ
P
P`
for:ϕ=π
for:ϕ=0
r_ = λ / (1+ε)
r+ = λ / (1−ε)
r_/ε
r+/ε
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
ε²=1-b²/a²
λ = a(1-ε²)



r/ε = λ/ε +r cosϕ
r(1-ε cosϕ) = λ
r = λ / (1-ε cosϕ)
Main Directrix D
Main Focal Plane F
Prime Focal Plane F`
Prime Directrix D`
r/ε
r cosϕ
r`/ε
for:ϕ=π/2
λ/ε
λ
r
ϕ
r`
P
P`
for:ϕ=π
for:ϕ=0
r_/ε
r_ = λ/(1+ε)
r+ = λ/(1−ε)
r+/ε
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
Major axis PP′
=r+ + r_ = λ/(1−ε) + λ/(1+ε) = (λ(1+ε)+λ(1−ε))/((1+ε)(1−ε)) = 2λ/(1−ε²) = 2a
ε²=1-b²/a²
λ = a(1-ε²)

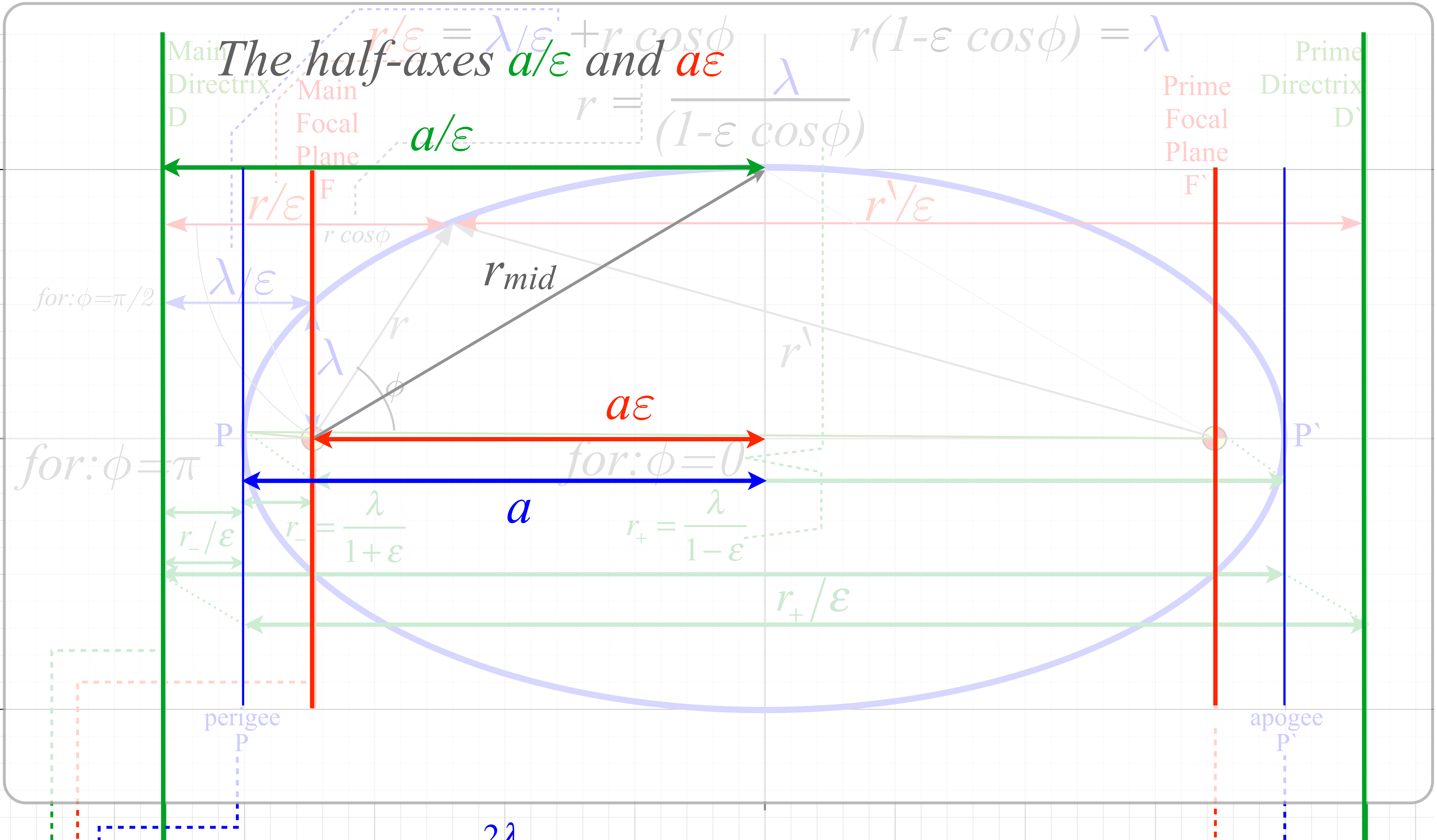


r/ε = λ/ε + r cosφ
r(1-ε cosφ) = λ
r = λ / (1-ε cosφ)
Main Directrix D
Main Focal Plane F
Prime Focal Plane F`
Prime Directrix D`
r/ε
r cosφ
r`/ε
λ/ε
for:φ=π/2
λ
r
φ
r`
P
P`
for:φ=π
for:φ=0
r_ = λ/(1+ε)
r_/ε
r+ = λ/(1-ε)
r+/ε
perigee P
apogee P`
a =4
b =2
ε =√3/2
λ =1
Major axis PP′ = r+ + r_ = λ/(1−ε) + λ/(1+ε) = (λ(1+ε)+λ(1−ε))/((1+ε)(1−ε)) = 2λ/(1−ε²) = 2a
Focal axis FF′ = r+ − r_ = λ/(1−ε) − λ/(1+ε) = 2λε/(1−ε²) = 2aε
ε²=1-b²/a²
λ = a(1-ε²)



$r/\varepsilon = \lambda/\varepsilon + r\cos\phi$
$r(1-\varepsilon\cos\phi) = \lambda$
$r = \dfrac{\lambda}{(1-\varepsilon\cos\phi)}$
Main Directrix D
Main Focal Plane F
Prime Focal Plane F`
Prime Directrix D`
r/ε
$r\cos\phi$
$r`/\varepsilon$
λ/ε
for: $\phi=\pi/2$
λ
r
$r`$
ϕ
P
P`
for: $\phi=\pi$
for: $\phi=0$
r_-/ε
$r_- = \dfrac{\lambda}{1+\varepsilon}$
$r_+ = \dfrac{\lambda}{1-\varepsilon}$
r_+/ε
perigee P
apogee P`
$a=4$
$b=2$
$\varepsilon=\sqrt{3}/2$
$\lambda=1$
Major axis PP′ $= r_+ + r_- = \dfrac{\lambda}{1-\varepsilon} + \dfrac{\lambda}{1+\varepsilon} = \dfrac{\lambda(1+\varepsilon)+\lambda(1-\varepsilon)}{(1+\varepsilon)(1-\varepsilon)} = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$
Focal axis FF′ $= r_+ - r_- = \dfrac{\lambda}{1-\varepsilon} - \dfrac{\lambda}{1+\varepsilon} = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$
Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$
$\varepsilon^2 = 1 - b^2/a^2$
$\lambda = a(1-\varepsilon^2)$

The half-axes a/ε and $a\varepsilon$

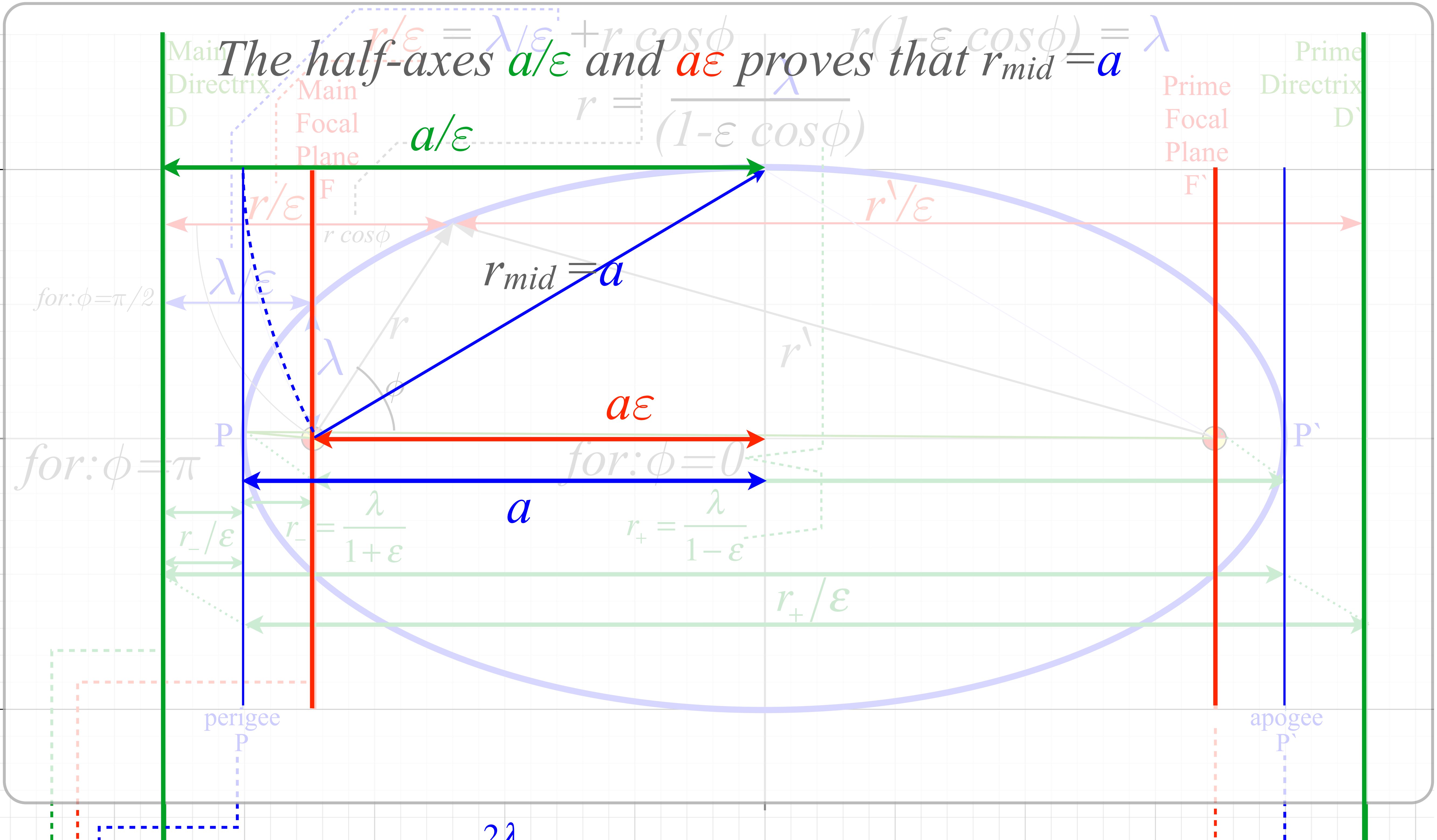


Major axis PP′ $= r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$

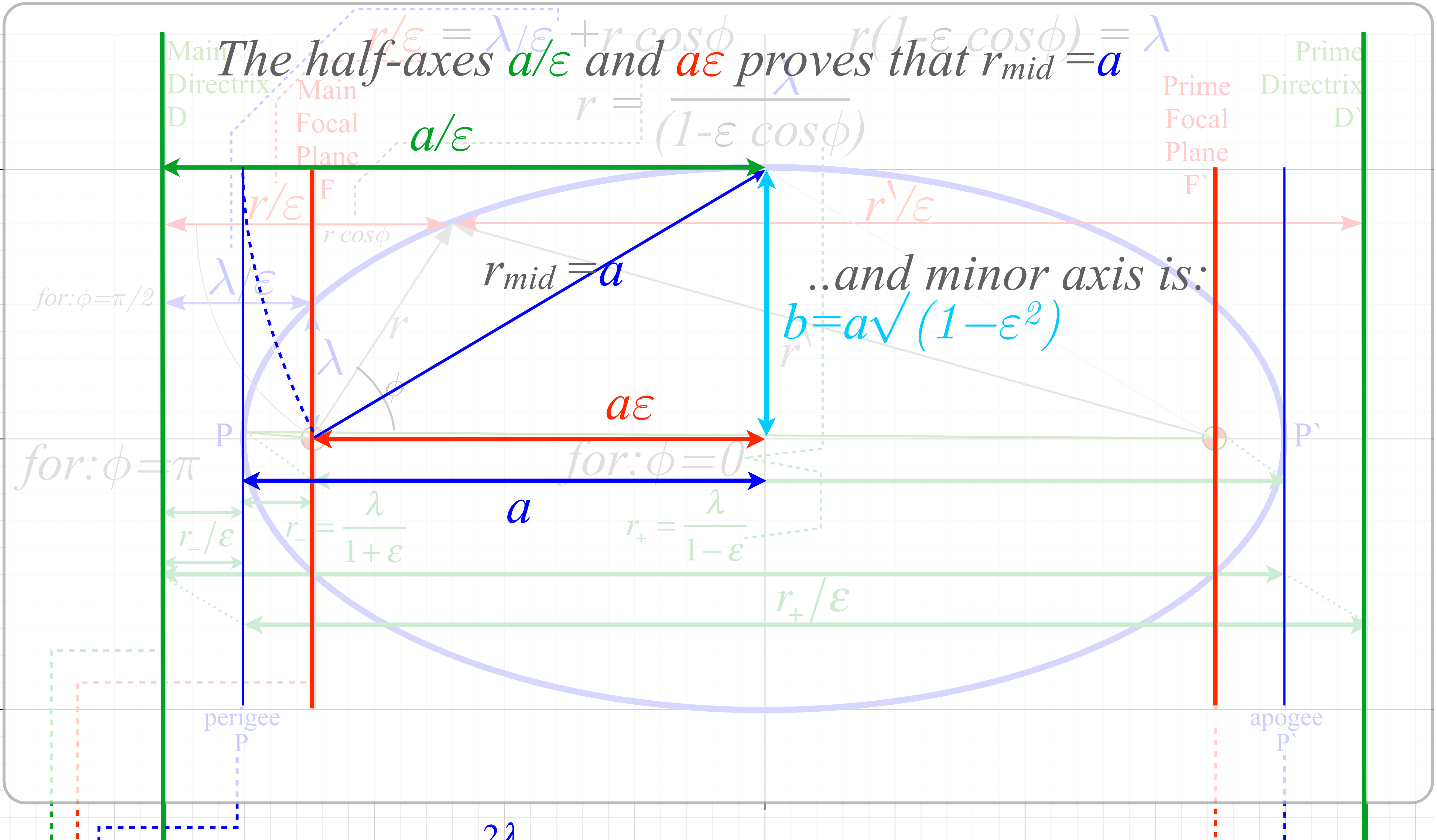
The half-axes a/ε and $a\varepsilon$ proves that $r_{mid} = a$



Major axis PP′ $=r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $=r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′ $=\dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$



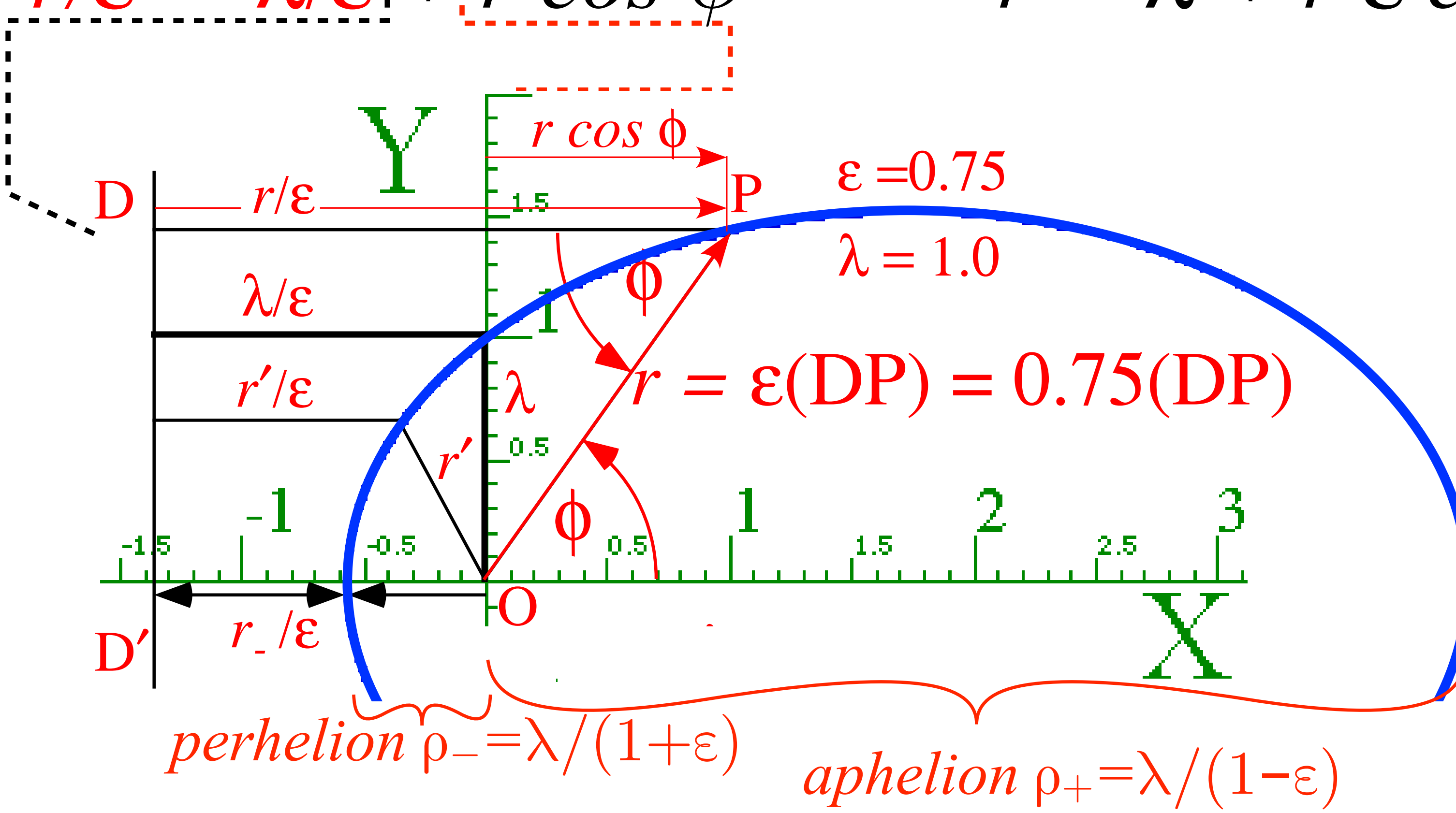
Major axis PP′ $= r_+ + r_- = \dfrac{2\lambda}{1-\varepsilon^2} = 2a$

Focal axis FF′ $= r_+ - r_- = \dfrac{2\lambda\varepsilon}{1-\varepsilon^2} = 2a\varepsilon$

Directrix axis DD′ $= \dfrac{r_+}{\varepsilon} + \dfrac{r_-}{\varepsilon} = \dfrac{2a}{\varepsilon}$

Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



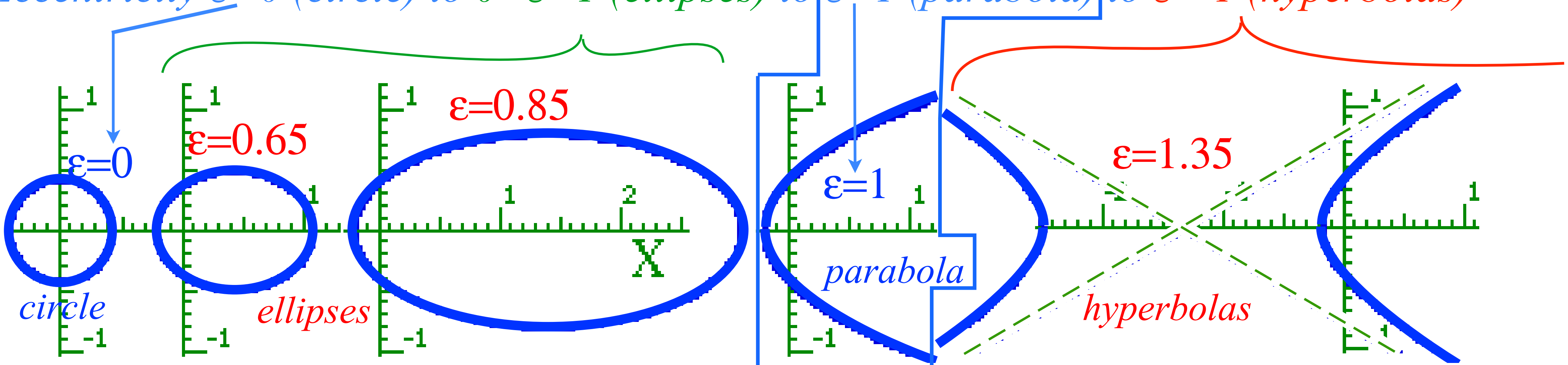
$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

Becoming more and more eccentric...

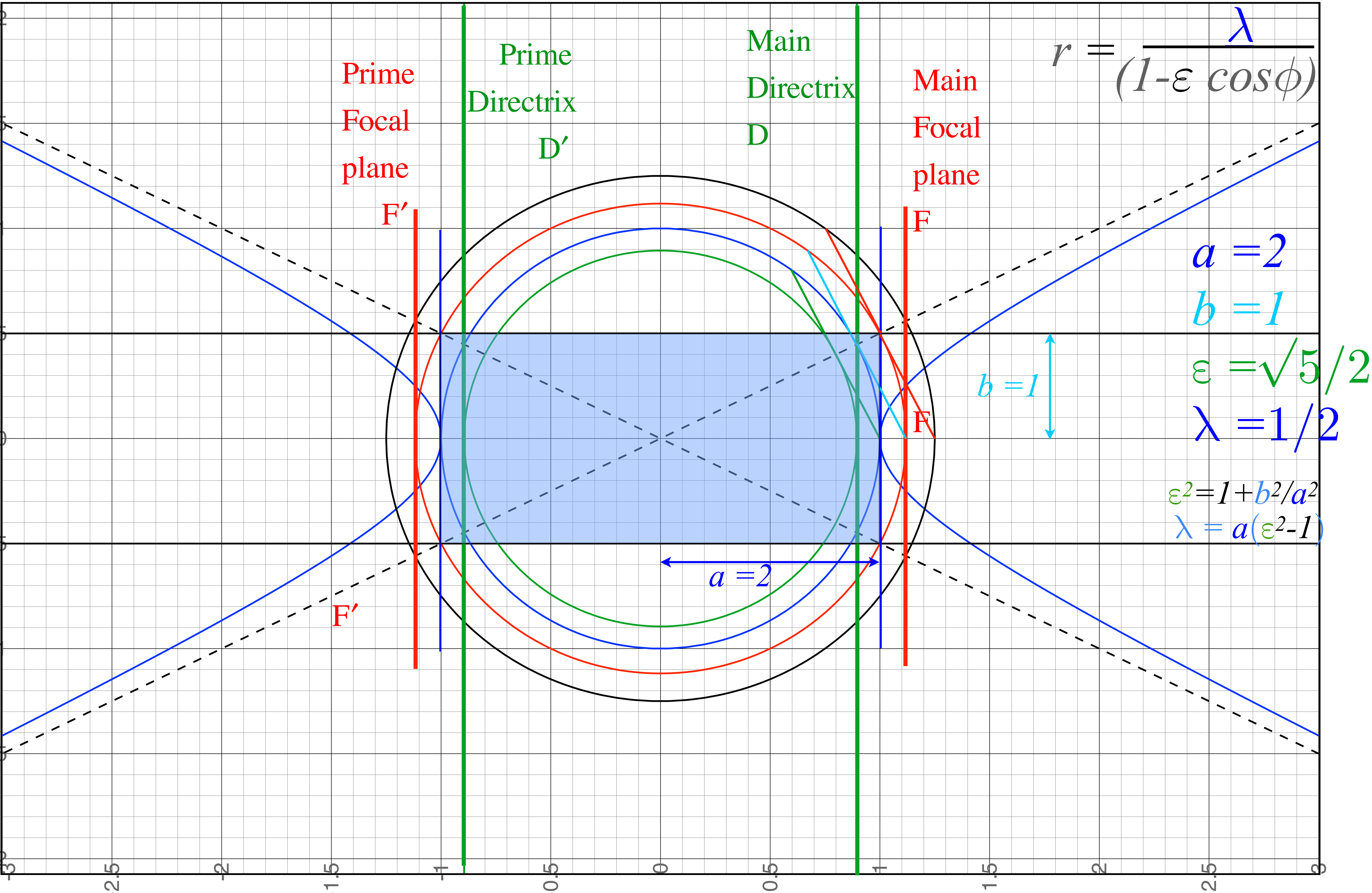
Singular Case!

Eccentricity $\varepsilon=0$ (circle) to $0<\varepsilon<1$ (ellipses) to $\varepsilon=1$ (parabola) to $\varepsilon >1$ (hyperbolas)

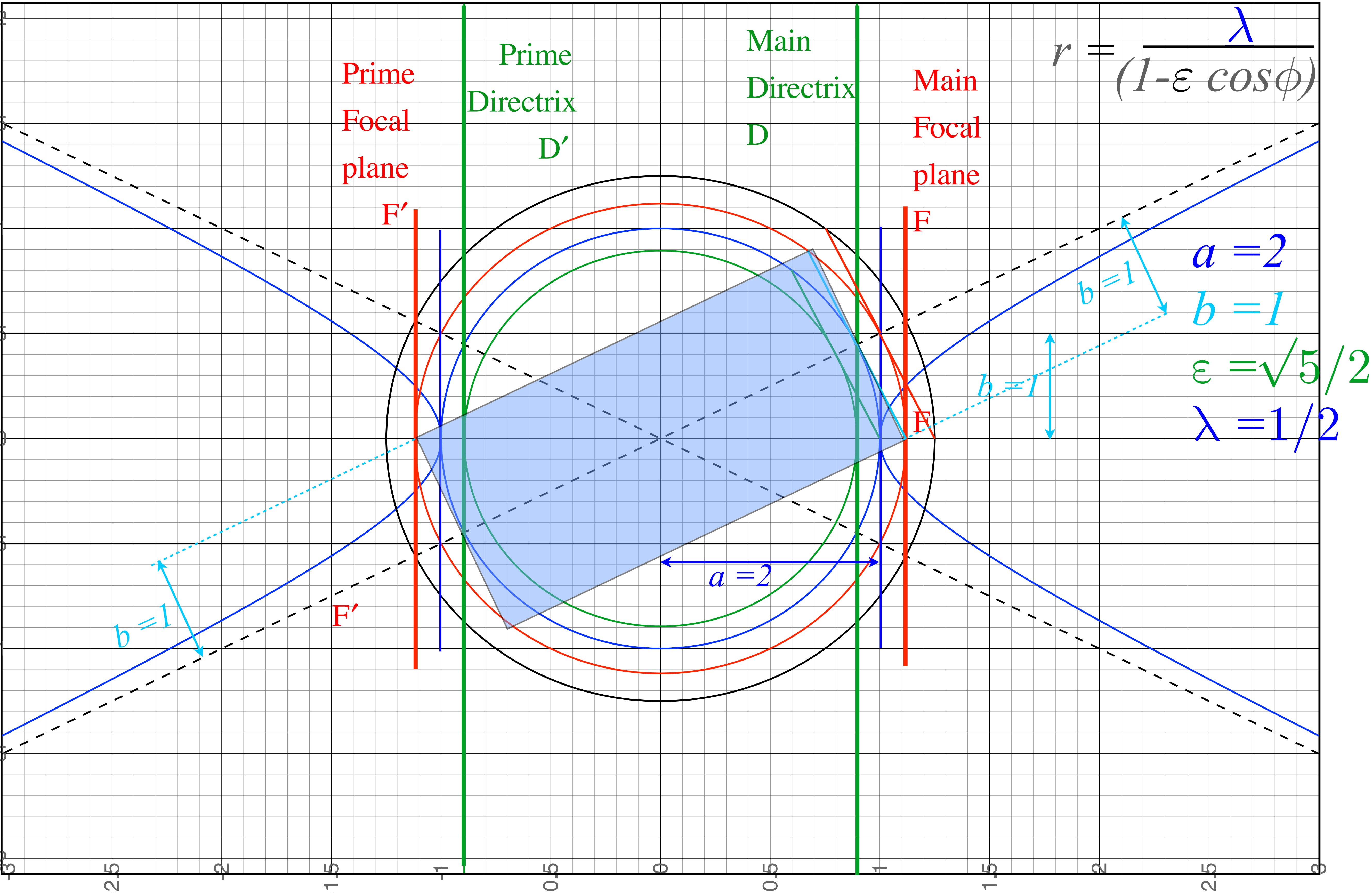


Geometry and Symmetry of Coulomb orbits

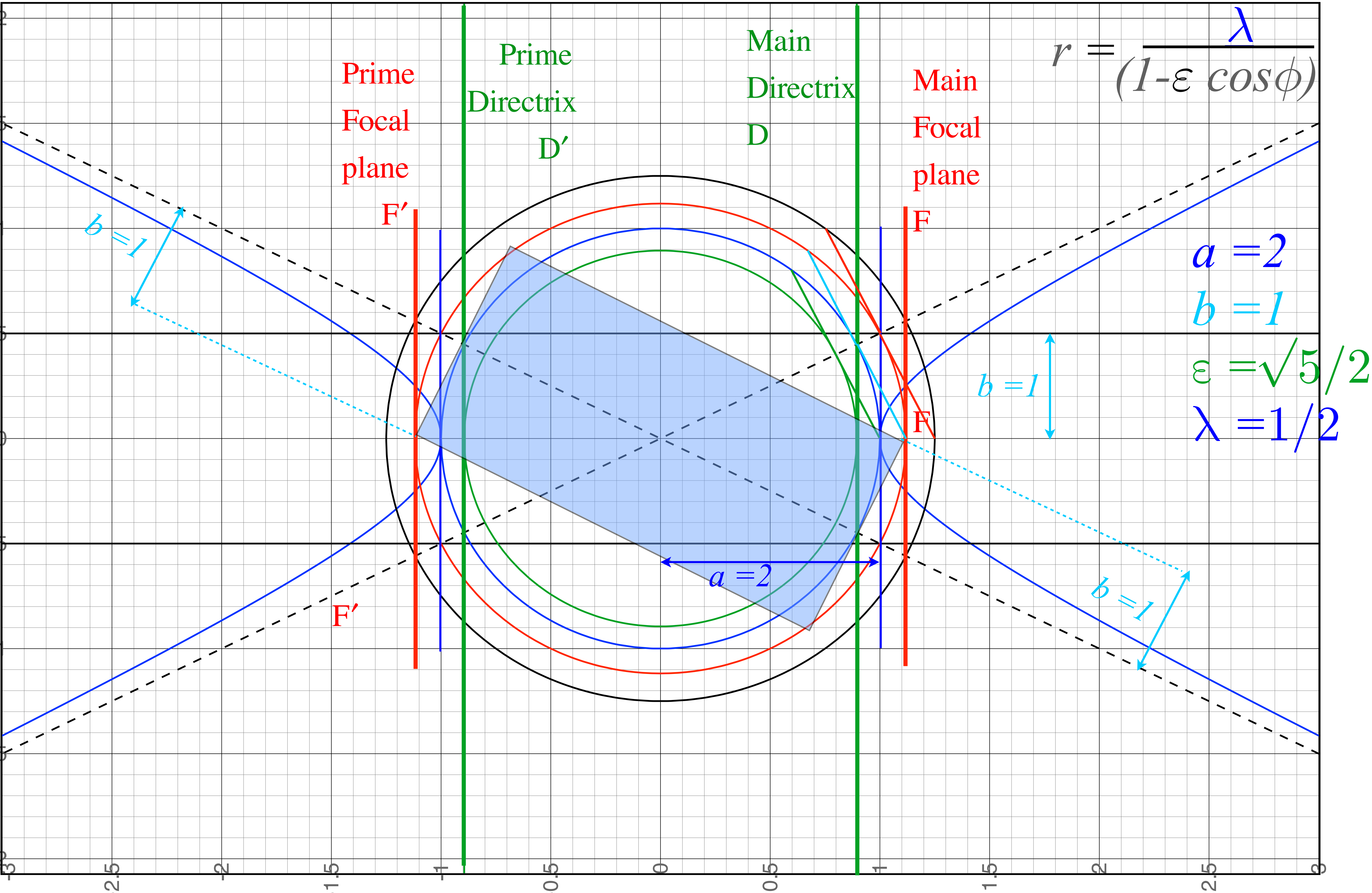
- *Detailed elliptic geometry*
- ➤ *Detailed hyperbolic geometry*



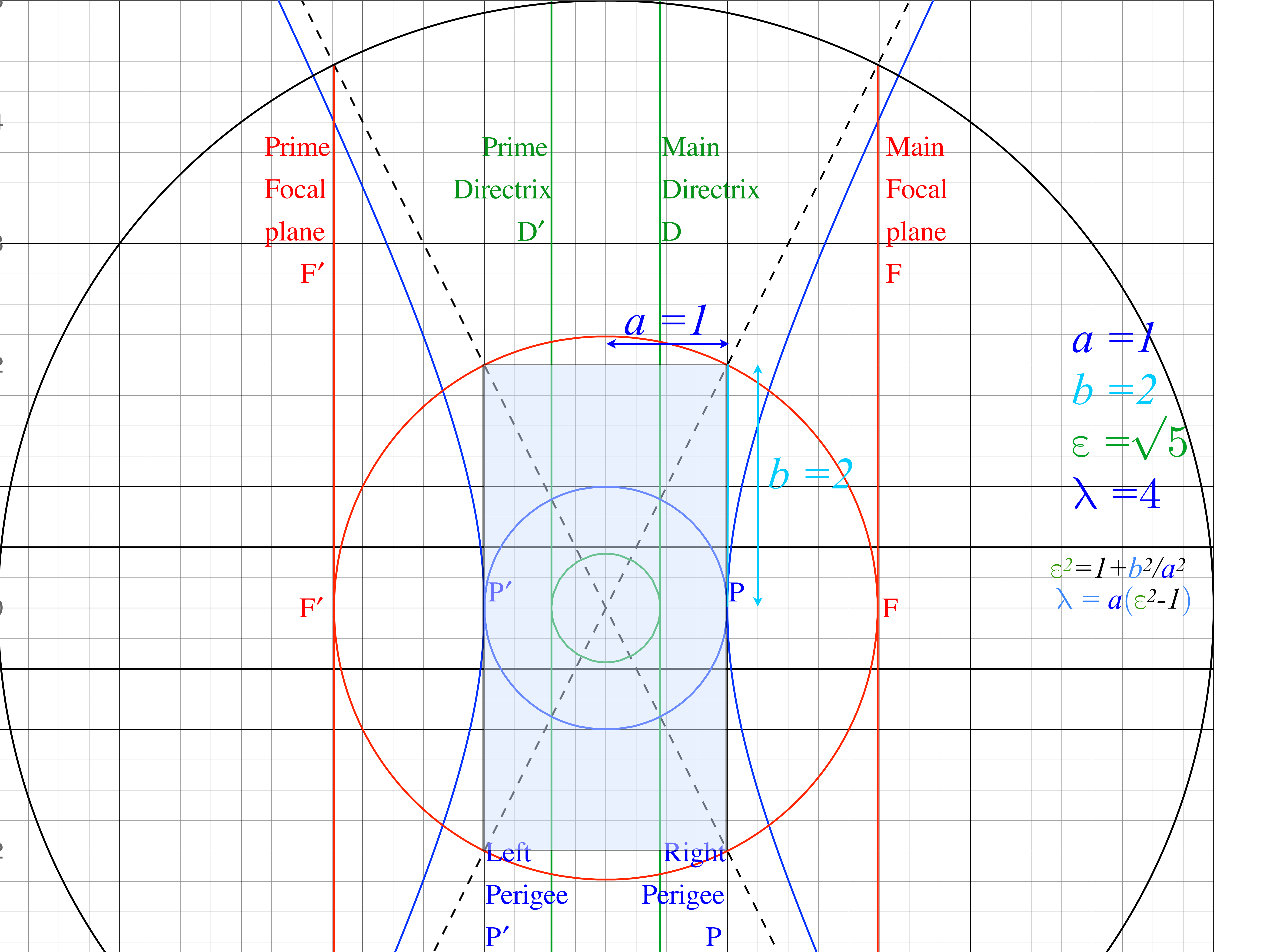
Prime Focal plane
F′
Prime Directrix D′
Main Directrix D
Main Focal plane
F
F
F′
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
$a = 2$
$b = 1$
$\varepsilon = \sqrt{5}/2$
$\lambda = 1/2$
$b = 1$
$a = 2$
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$



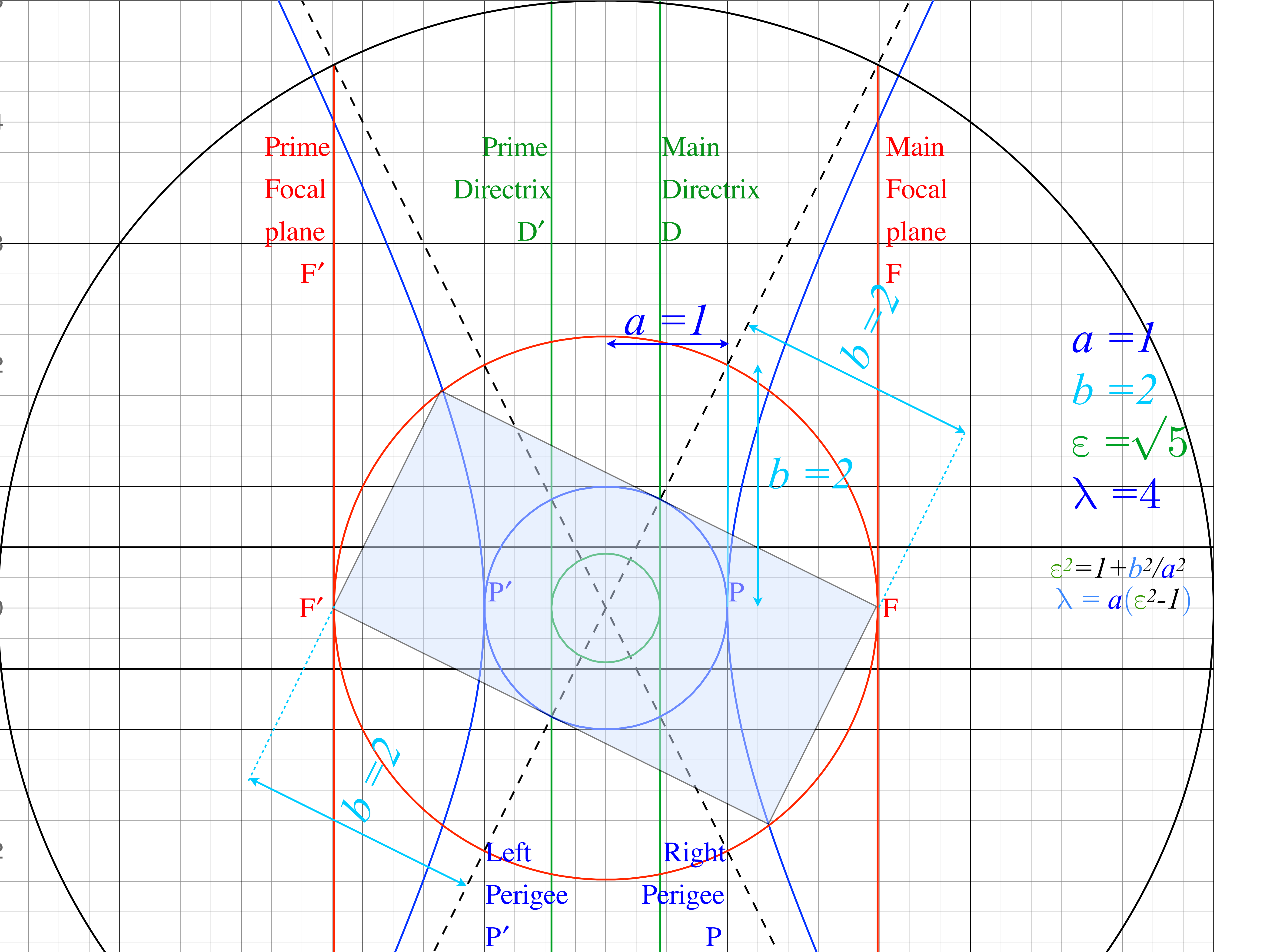
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r = λ / (1-ε cosφ)
a =2
b =1
ε =√5/2
λ =1/2
b =1
b =1
b =1
F
F′
a =2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3



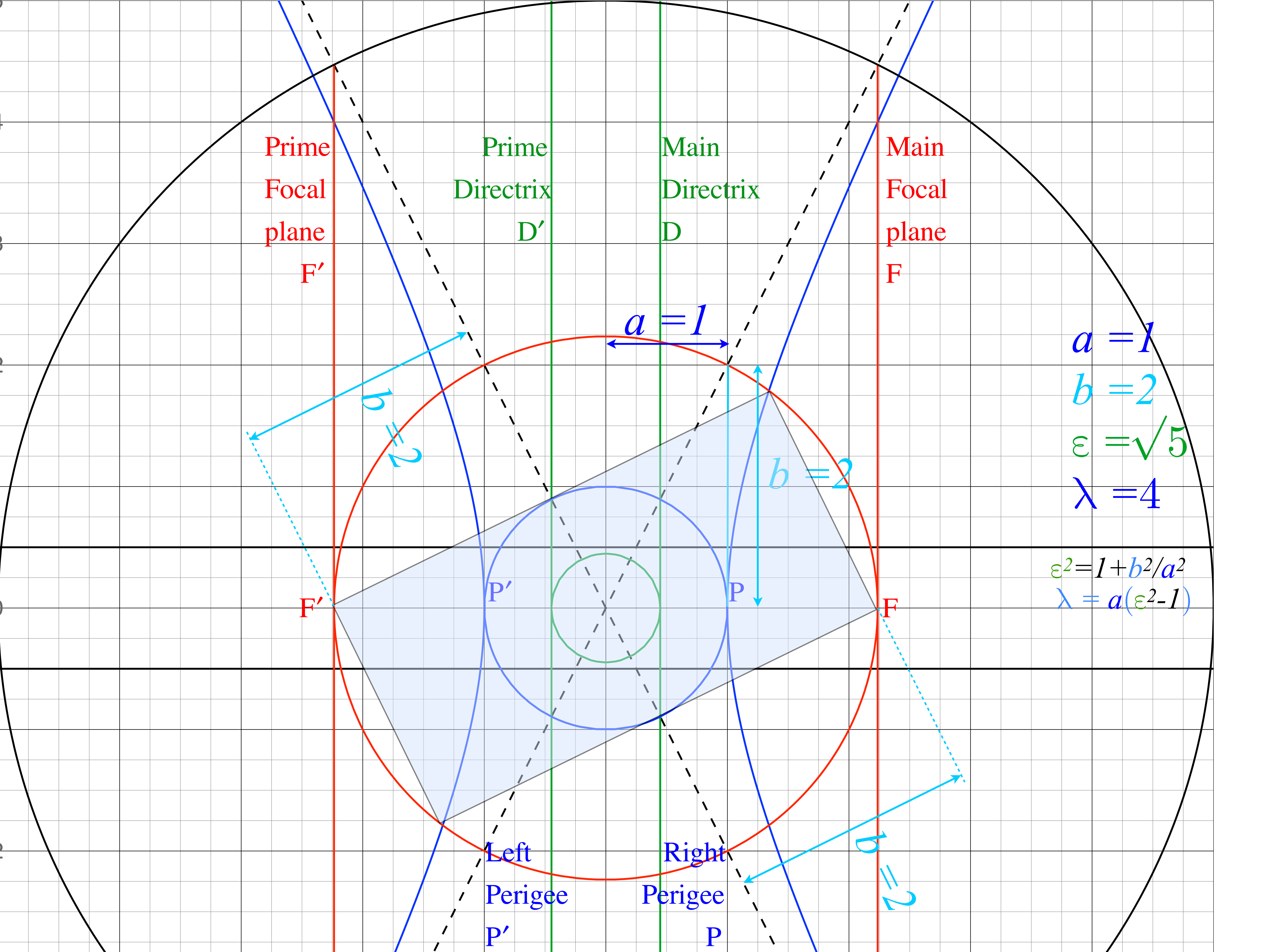
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
F
F′
$a = 2$
$b = 1$
$\varepsilon = \sqrt{5}/2$
$\lambda = 1/2$
$b = 1$
$b = 1$
$b = 1$
$a = 2$



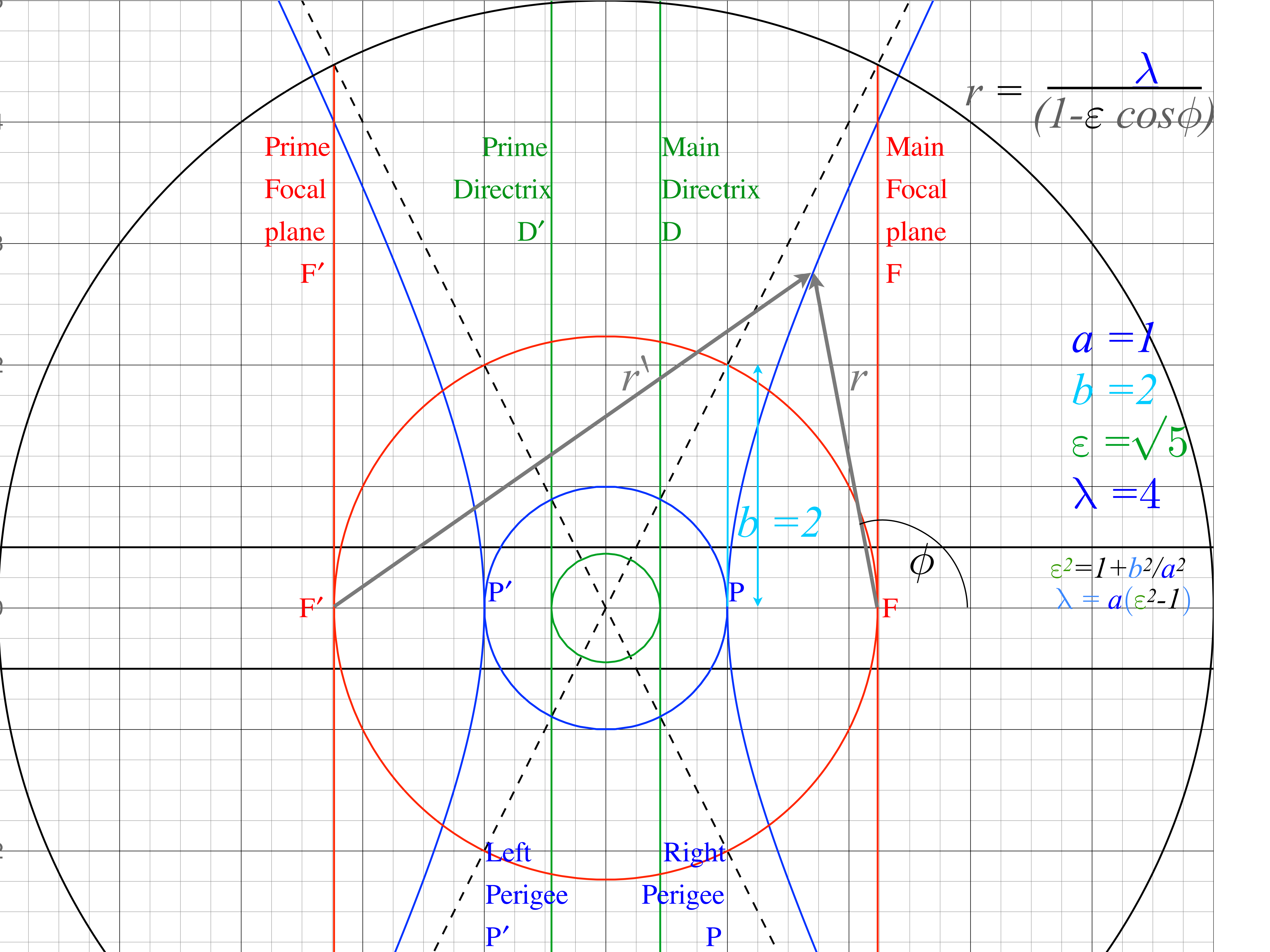
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
a =1
b =2
a =1
b =2
ε =√5
λ =4
ε2=1+b2/a2
λ = a(ε2-1)
P′
P
F′
F
Left
Perigee
P′
Right
Perigee
P



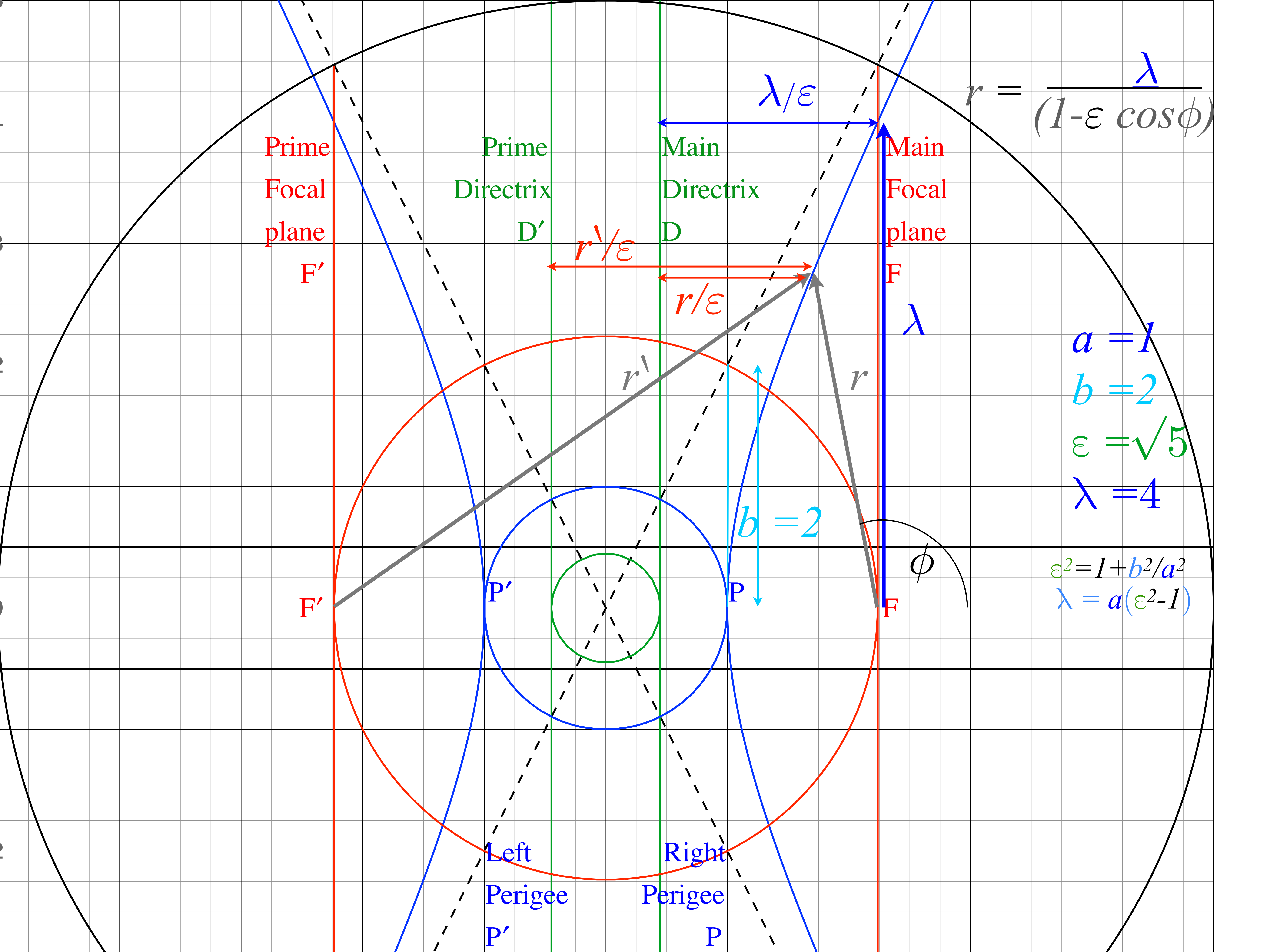
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
a = 1
b = 2
b = 2
b = 2
b = 2
a = 1
b = 2
ε = √5
λ = 4
ε2=1+b2/a2
λ = a(ε2-1)
F′
P′
P
F
Left
Perigee
P′
Right
Perigee
P



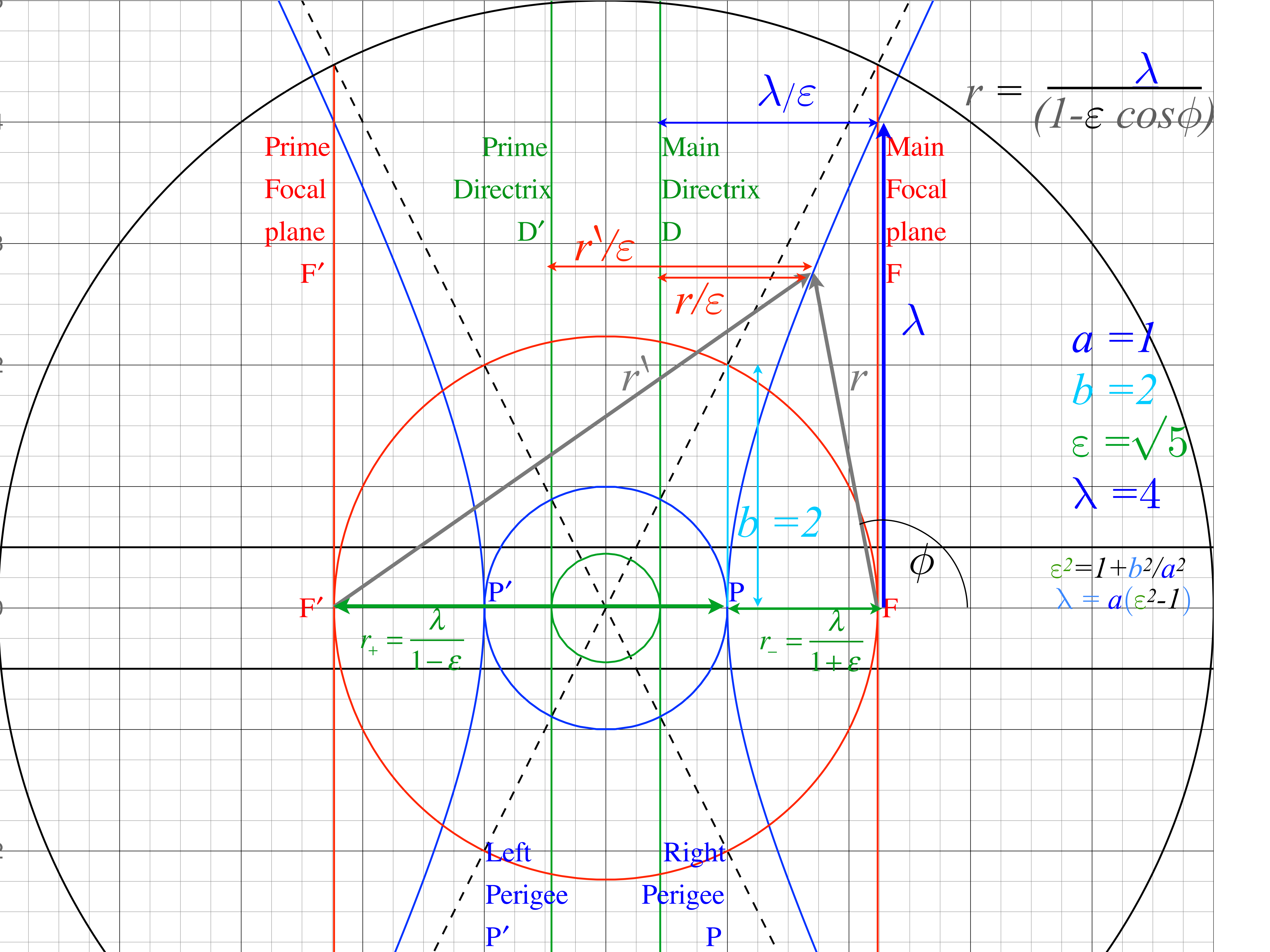
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
a = 1
b = 2
b = 2
b = 2
a = 1
b = 2
ε = √5
λ = 4
ε² = 1 + b²/a²
λ = a(ε² - 1)
F′
P′
P
F
Left Perigee P′
Right Perigee P



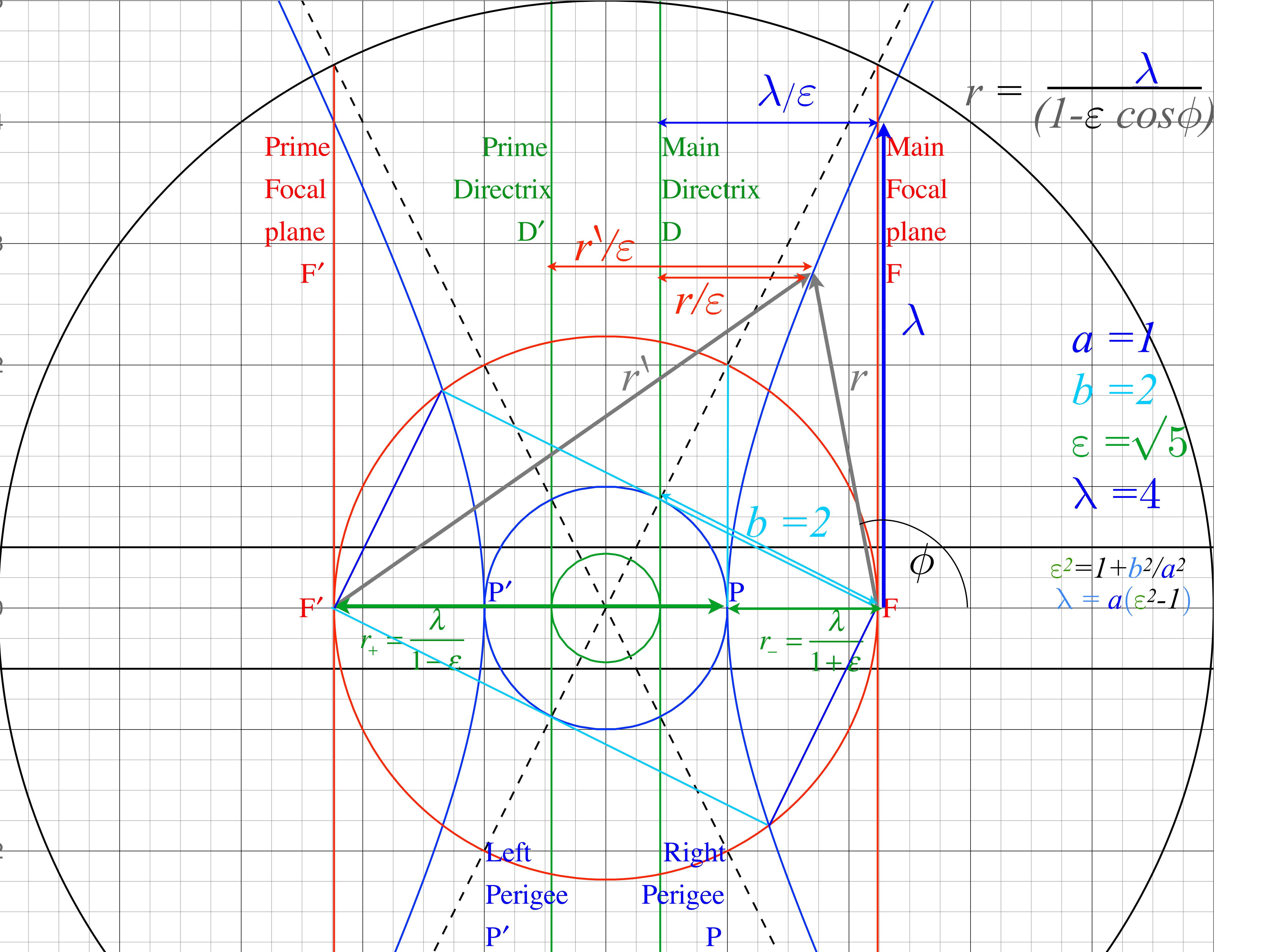
r = λ / (1-ε cosϕ)
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
a =1
b =2
ε =√5
λ =4
r′
r
b =2
ϕ
ε²=1+b²/a²
λ = a(ε²-1)
F′
P′
P
F
Left
Perigee
P′
Right
Perigee
P



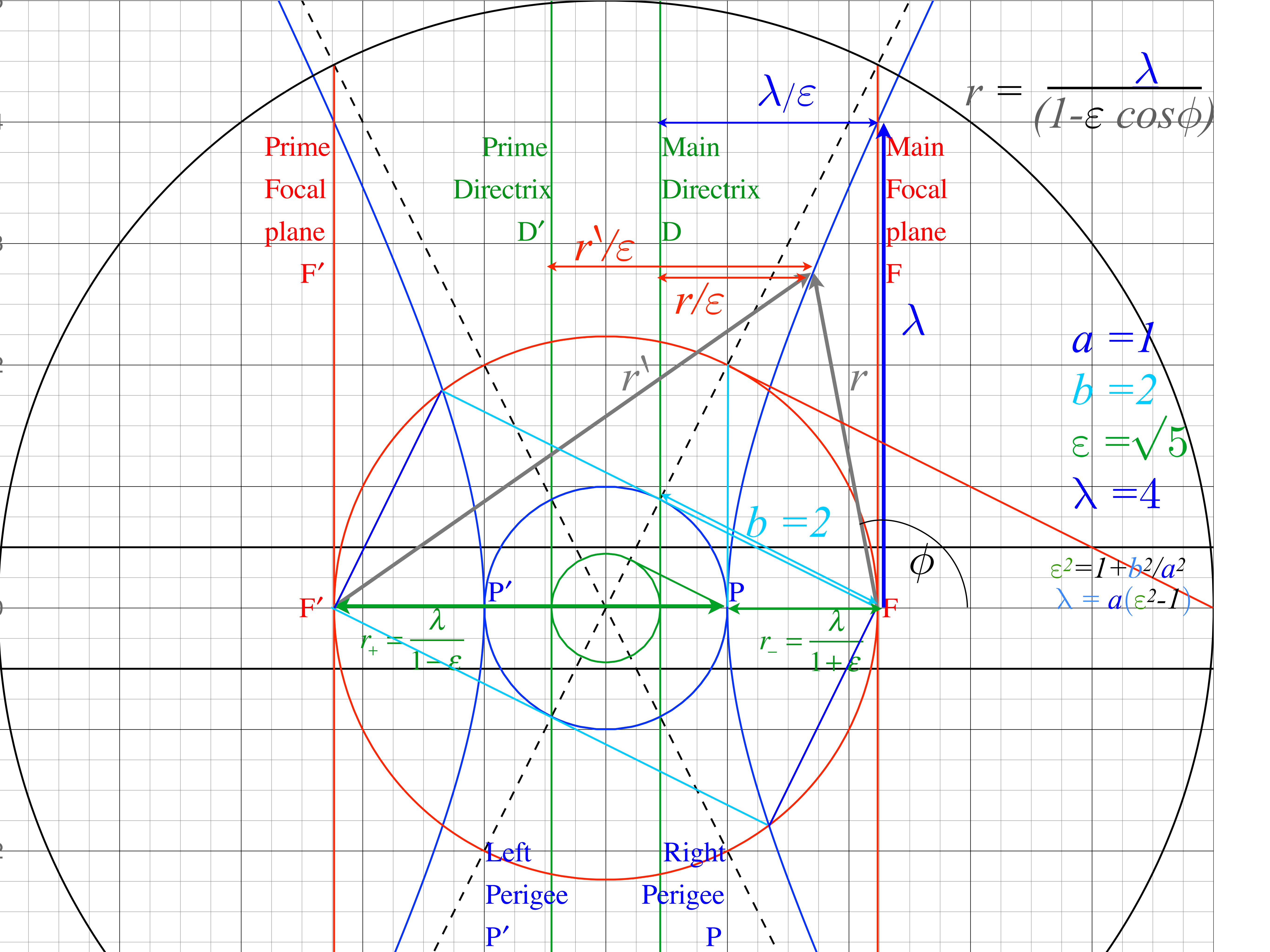
λ/ε
r = λ / (1-ε cosφ)
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
r′/ε
r/ε
λ
a =1
b =2
ε =√5
λ =4
r′
r
b =2
φ
ε²=1+b²/a²
λ = a(ε²-1)
F′
P′
P
F
Left
Perigee
P′
Right
Perigee
P



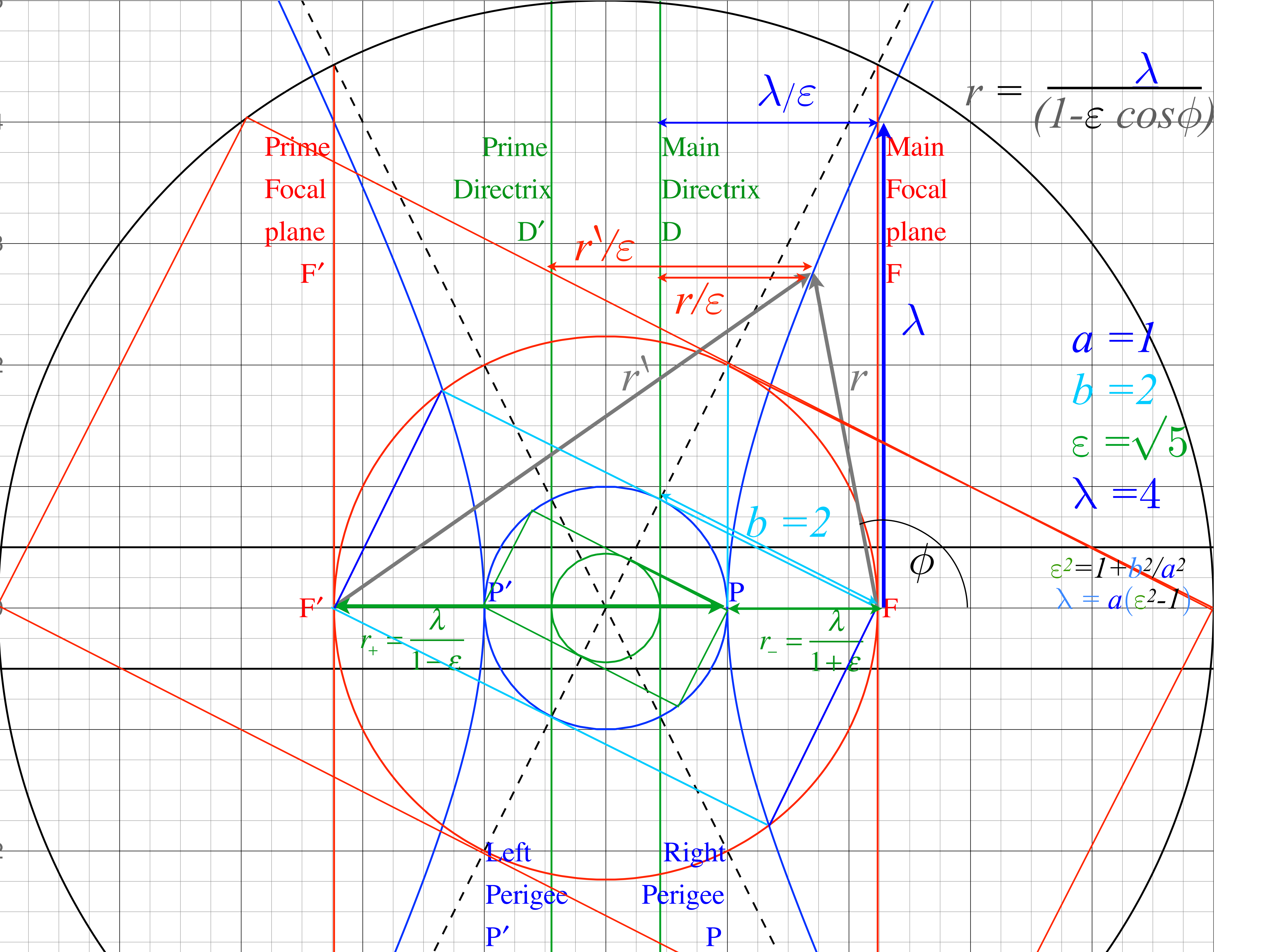
λ/ε
r = λ / (1-ε cosφ)
Prime
Focal
plane
F′
Prime
Directrix
D′
Main
Directrix
D
Main
Focal
plane
F
r′/ε
r/ε
λ
a =1
b =2
ε =√5
λ =4
r′
r
b =2
φ
ε²=1+b²/a²
λ = a(ε²-1)
F′
P′
P
F
r₊ = λ/(1−ε)
r₋ = λ/(1+ε)
Left
Perigee
P′
Right
Perigee
P



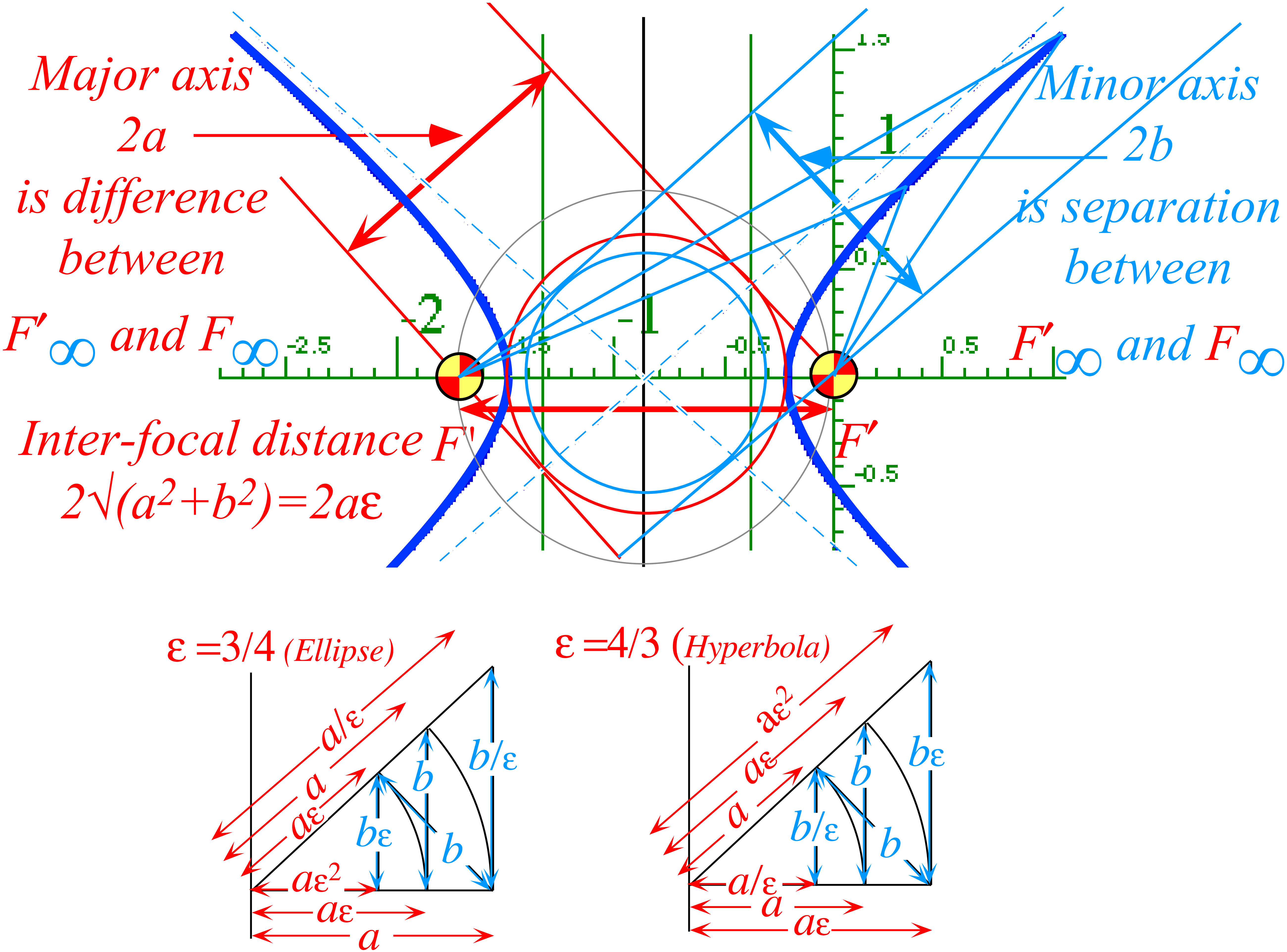
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r′/ε
r/ε
λ
r′
r
a =1
b =2
ε =√5
λ =4
b =2
φ
$\varepsilon^2=1+b^2/a^2$
$\lambda = a(\varepsilon^2-1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P

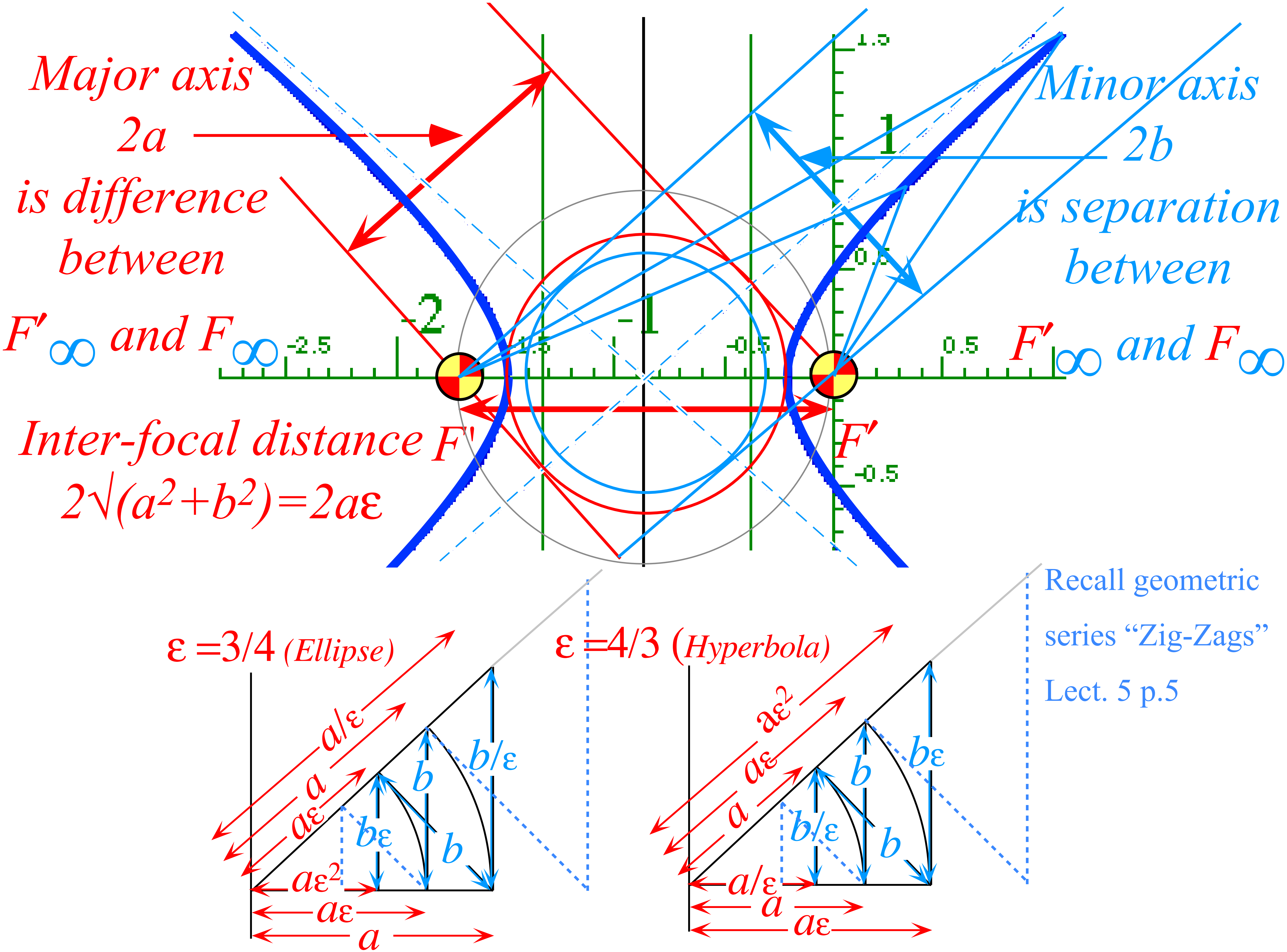


$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r′/ε
r/ε
λ
r′
r
a =1
b =2
ε =√5
λ =4
b =2
φ
ε²=1+b²/a²
λ = a(ε²-1)
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P



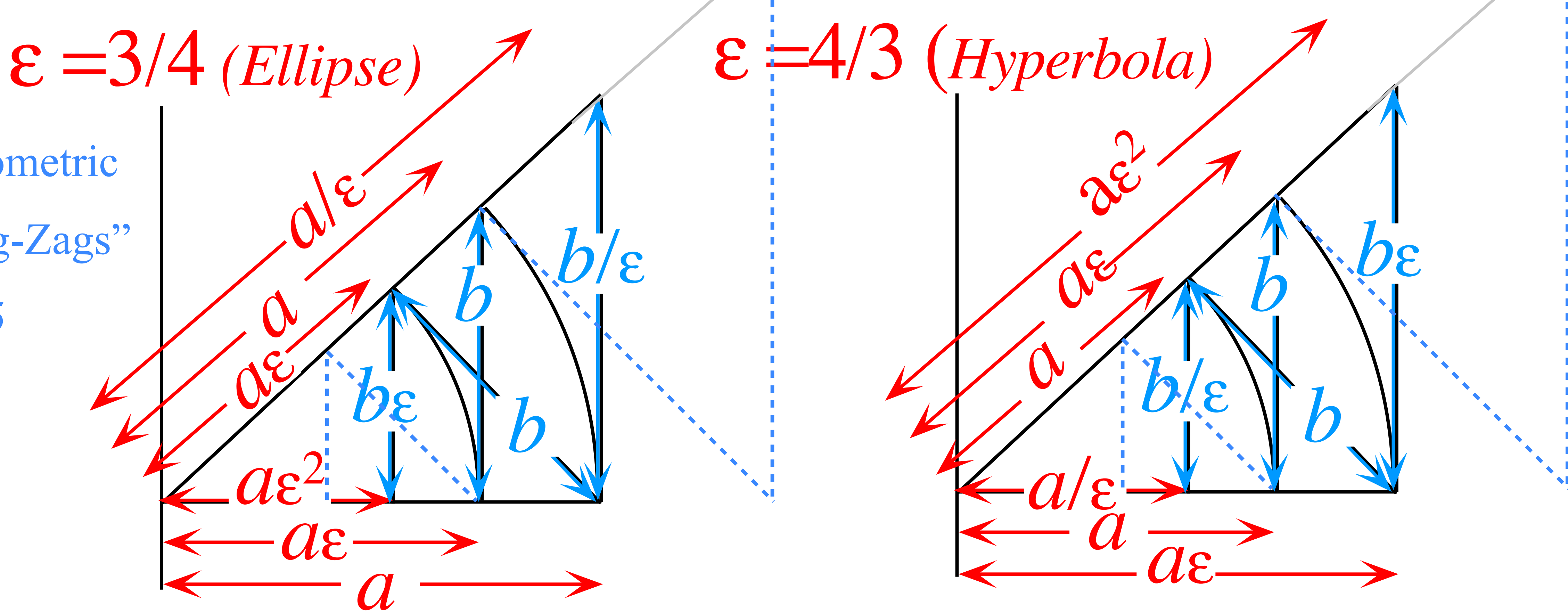
$r = \frac{\lambda}{(1-\varepsilon \cos\phi)}$
λ/ε
Prime Focal plane F′
Prime Directrix D′
Main Directrix D
Main Focal plane F
r'/ε
r/ε
λ
$a = 1$
$b = 2$
$\varepsilon = \sqrt{5}$
$\lambda = 4$
r'
r
$b = 2$
ϕ
$\varepsilon^2 = 1 + b^2/a^2$
$\lambda = a(\varepsilon^2 - 1)$
F′
P′
P
F
$r_+ = \frac{\lambda}{1-\varepsilon}$
$r_- = \frac{\lambda}{1+\varepsilon}$
Left Perigee P′
Right Perigee P


Major axis
2a
is difference
between
F′∞ and F∞
Minor axis
2b
is separation
between
F′∞ and F∞
Inter-focal distance F′
2√(a²+b²)=2aε
F′
-2
-1
1
-2.5
-1.5
-0.5
0.5
1.5
-0.5
ε =3/4 (Ellipse)
a/ε
a
aε
b/ε
b
bε
b
aε²
aε
a
ε =4/3 (Hyperbola)
aε²
aε
a
bε
b
b/ε
b
a/ε
a
aε



Major axis
2a
is difference
between
F′∞ and F∞
Minor axis
2b
is separation
between
F′∞ and F∞
Inter-focal distance F′
2√(a²+b²)=2aε
F′
-2.5
-2
-1.5
-1
-0.5
0.5
1.5
1
0.5
-0.5
ε =3/4 (Ellipse)
ε =4/3 (Hyperbola)
a/ε
a
aε
aε²
bε
b
b/ε
aε²
aε
a
b/ε
b
bε
a/ε
a
aε
Recall geometric
series "Zig-Zags"
Lect. 5 p.5

Recall geometric series "Zig-Zags" Lect. 5 p.5



For the elliptic geometry $(\varepsilon < 1)$:

$$b^2 = a^2 - a^2\varepsilon^2 = a\lambda\,,$$

$$b = a\sqrt{(1-\varepsilon^2)} = \sqrt{(a\lambda)},$$

For hyperbolic geometry $(\varepsilon > 1)$:

$$b^2 = a^2\varepsilon^2 - a^2 = a\lambda,$$

$$b = a\sqrt{(\varepsilon^2-1)} = \sqrt{(a\lambda)}.$$

(λ,ε)-(a,b) expressions and their inverses follow.

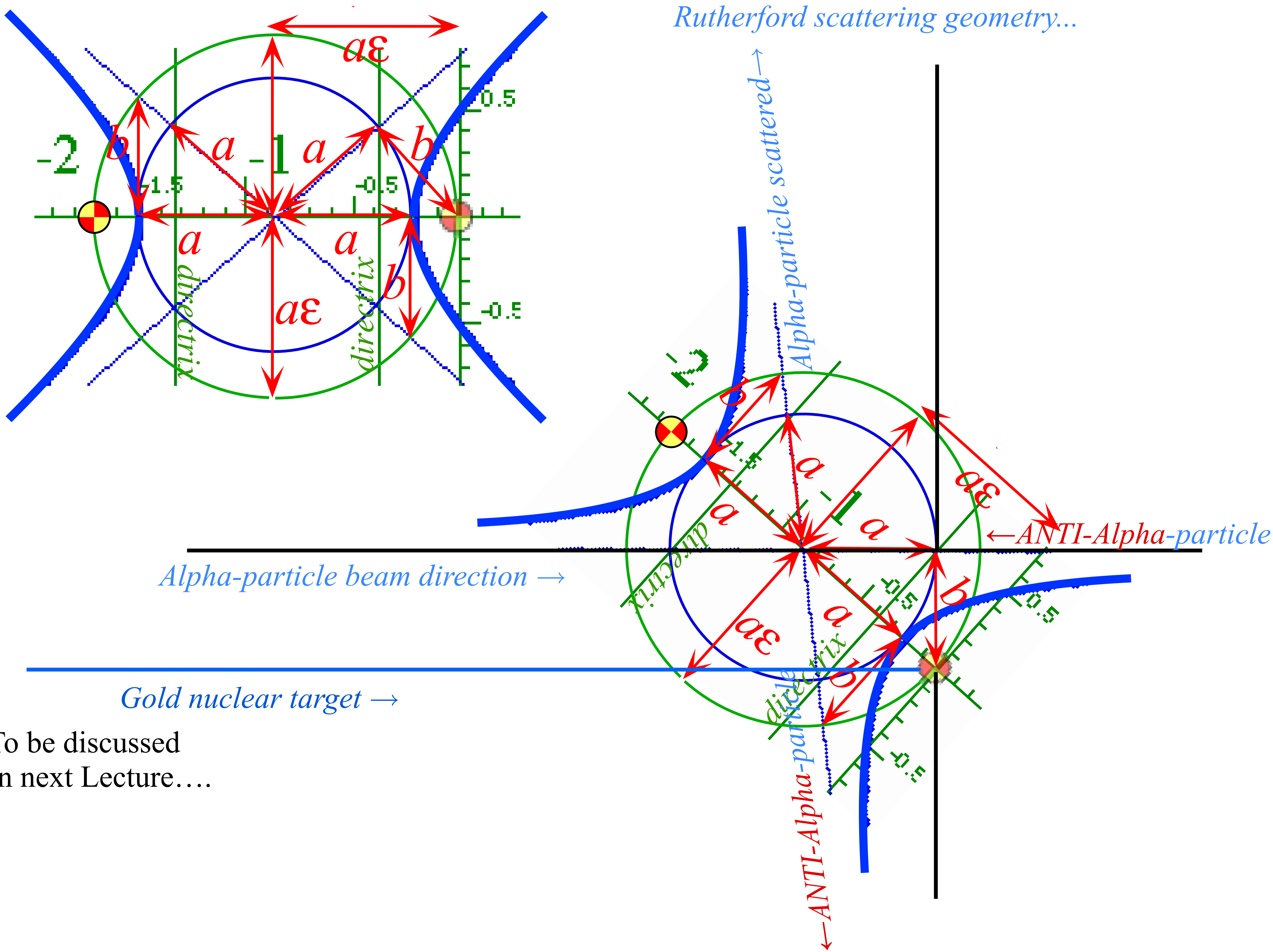
$a = \lambda/(1-\varepsilon^2)$	$a = \lambda/(\varepsilon^2-1)$
$b^2 = \lambda^2/(1-\varepsilon^2)$	$b^2 = \lambda^2/(\varepsilon^2-1)$
$\lambda = a(1-\varepsilon^2) = b^2/a$	$\lambda = a(\varepsilon^2-1) = b^2/a$
$\varepsilon^2 = 1- b^2/a^2$	$\varepsilon^2 = 1+b^2/a^2$

To be discussed
In next Lecture….

Cartesian Parameters	Physics	Polar Parameters
Semi-major axis $a=k/\|2E\|$	← Energy E →	Eccentricity $\varepsilon=\sqrt{(1+2\mu^2 E/(k^2 m))}$
Semi-minor axis $b=\mu/\sqrt{\|2mE\|}$	← Angular momentum $\mu=\ell$ →	Latus radius $\lambda=\mu^2/(km)$

Parameter table on p.79

Recall p.66 formula:



To be discussed
In next Lecture….