Lecture 24
Tue. 11.21.2017

Parametric Resonance and Multi-particle Wave Modes

(Ch. 7-8 of Unit 4-CMwBang 11.21.17)

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)
- *Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)*
- *Schrodinger wave equation related to Parametric resonance dynamics*
  - *Electronic band theory and analogous mechanics*

Wave resonance in cyclic symmetry
- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Resolving C_3 projectors and moving wave modes*
  - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
  - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ...)*
- *C_N symmetric mode models: Made-to order dispersion functions*
  - *Quadratic dispersion models: Super-beats and fractional revivals*
  - *Phase arithmetic*

Algebra and geometry of resonant revivals: Farey Sums and Ford Circles

Relating C_N symmetric H and K matrices to differential wave operators

Two Kinds of Resonance

Linear or *additive resonance*.
Example: oscillating electric **E**-field applied to a cyclotron orbit .

$$\ddot{x} + \omega_0^2 x = E_s \cos\left(\omega_s t\right)$$

Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0)$

Two Kinds of Resonance

Linear or *additive resonance*.
Example: oscillating electric **E**-field applied to a cyclotron orbit .

$$\ddot{x} + \omega_0^2 x = E_s \cos(\omega_s t)$$

Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0$)

Nonlinear or *multiplicative resonance.*
Example: oscillating magnetic **B**-field is applied to a cyclotron orbit.

$$\ddot{x} + \left(\omega_0^2 + B\cos(\omega_s t)\right)x = 0$$

Chapter 4.7

Also called *parametric resonance.*
Frequency parameter or spring constant $k=m\omega^2$ is being stimulated.

Two Kinds of Resonance

Linear or *additive resonance*.
Example: oscillating electric **E**-field applied to a cyclotron orbit .

$$\ddot{x} + \omega_0^2 x = E_s \cos(\omega_s t)$$

Chapter 4.2 study of FDHO
(Here damping $\Gamma \cong 0$)

Nonlinear or *multiplicative resonance.*
Example: oscillating magnetic **B**-field is applied to a cyclotron orbit.

$$\ddot{x} + \left(\omega_0^2 + B\cos(\omega_s t)\right)x = 0$$

Chapter 4.7

Also called *parametric resonance.*
Frequency parameter or spring constant $k=m\omega^2$ is being stimulated.
...Or pendulum accelerated up and down *(model to be used here)*

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)

Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)

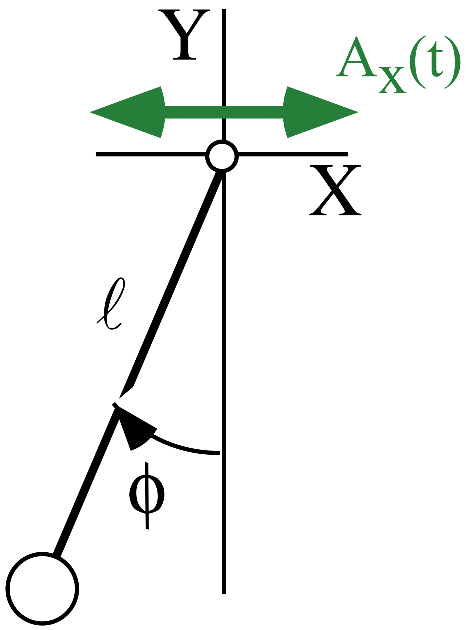
Schrodinger wave equation related to Parametric resonance dynamics

Electronic band theory and analogous mechanics

Coupled Rotation and Translation (Throwing)

Early non-human (or in-human) machines: trebuchets, whips..   (3000 BCE-1542 CE)

X-stimulated pendulum:
(Quasi-Linear Resonance)



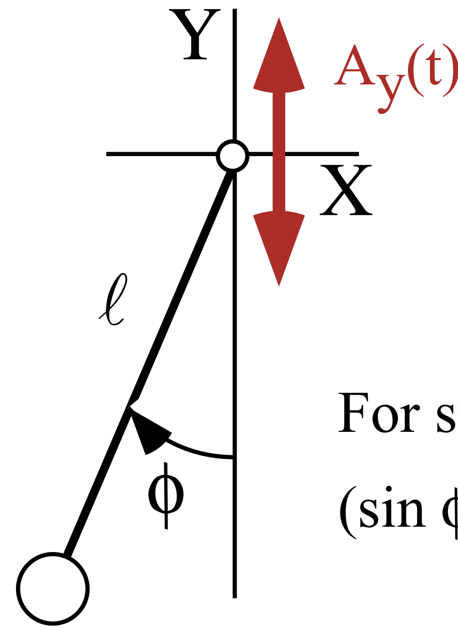
For small ϕ
($\cos\phi \sim 1$) :

Forced Harmonic Resonance

$$\frac{d^2\phi}{dt^2} + \frac{g}{\ell}\phi = \frac{A_x(t)}{\ell}$$

A Newtonian F=Ma equation
Lorentz equation (with Γ=0)

Y-stimulated pendulum:
(Non-Linear Resonance)

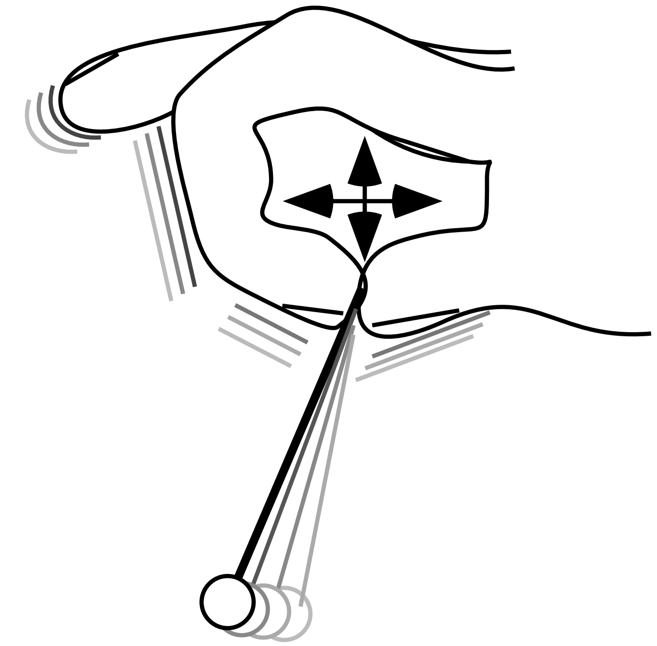


For small ϕ
($\sin\phi \sim \phi$) :

Parametric Resonance

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

A Schrodinger-like equation
(Time *t* replaces coord. *x*)



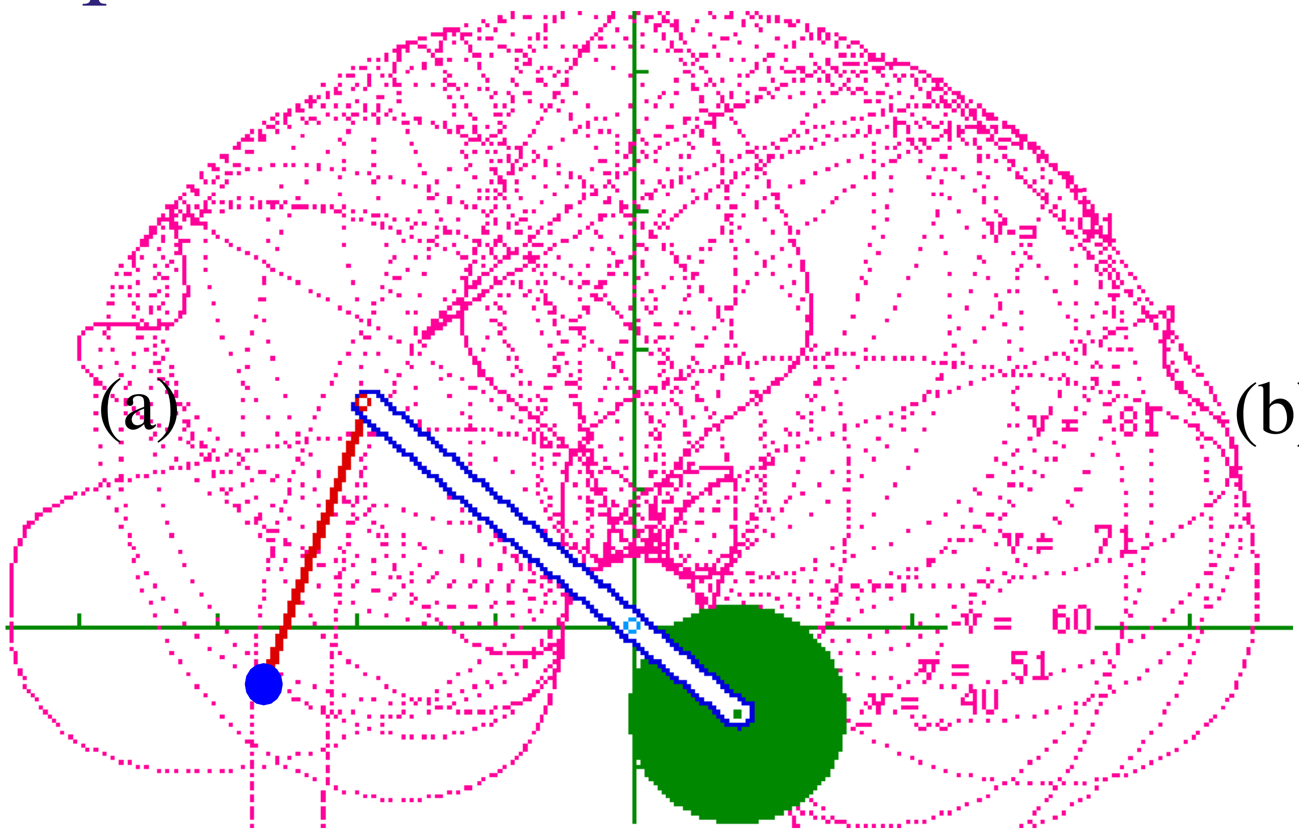
General ϕ :

(1542-2012 CE)

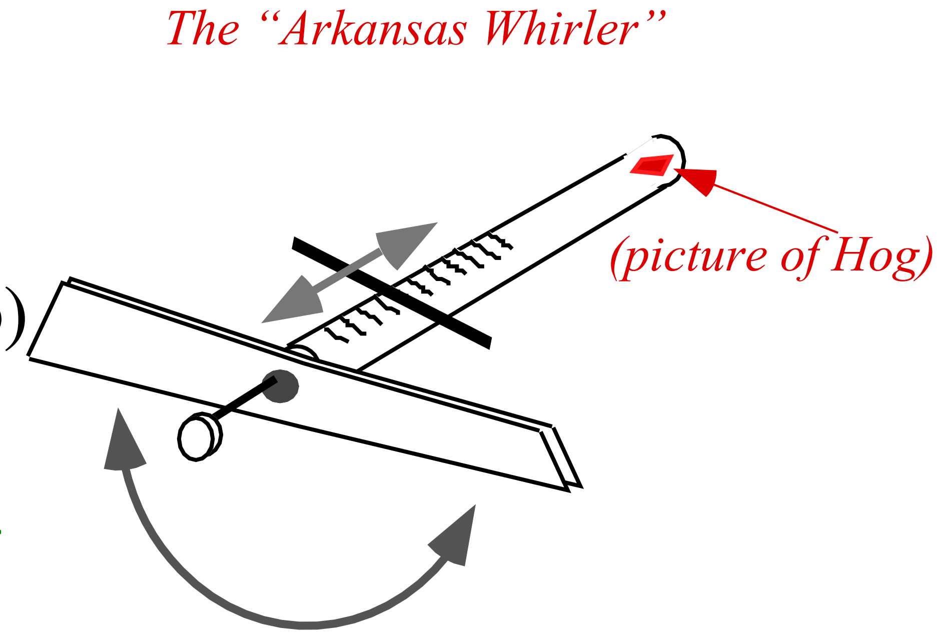
General case: A Nasty equation! $$\frac{d^2\phi}{dt^2} + \frac{g+A_y(t)}{\ell}\sin\phi + \frac{A_x(t)}{\ell}\cos\phi = 0$$

Coupled Rotation and Translation (Throwing)

(a)

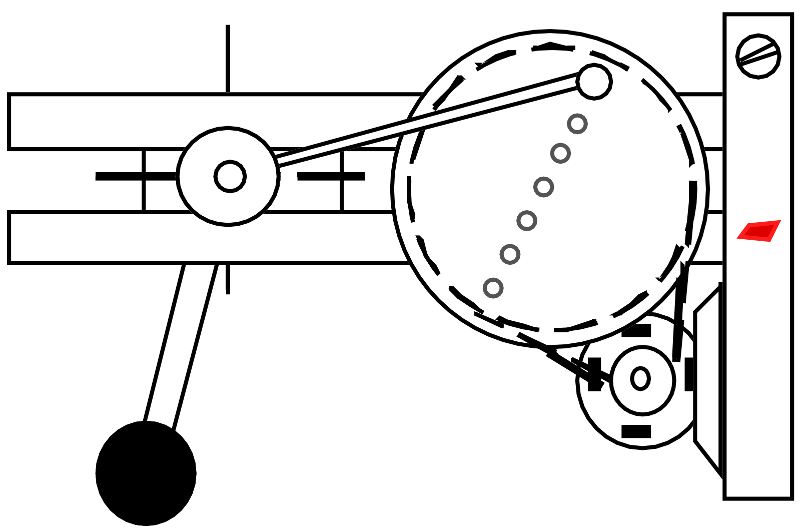


(b)



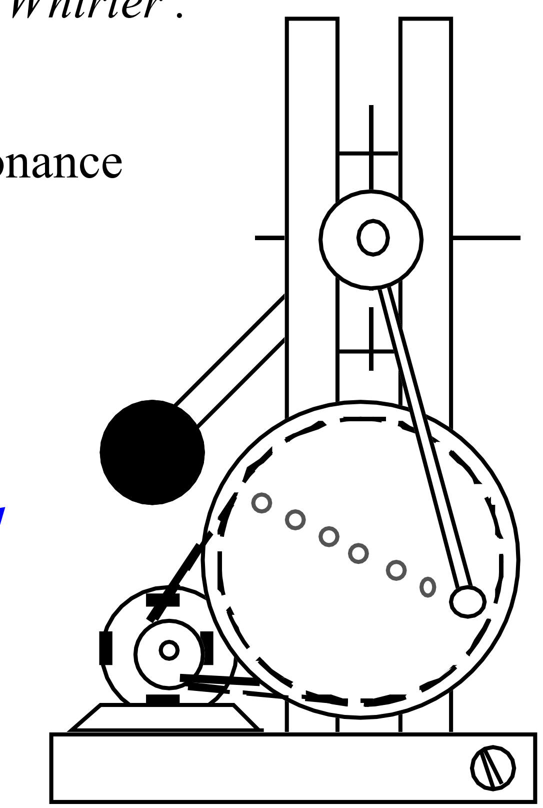
Chaotic motion from both linear and non-linear resonance (a) Trebuchet, (b) Whirler .

Positioned for linear resonance



Positioned for nonlinear resonance

device we hope to build (...someday)



Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)

Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)

Schrodinger wave equation related to Parametric resonance dynamics

Electronic band theory and analogous mechanics

Schrodinger Equation Parametric Resonance

Related to

Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation *(With m=1 and ħ=1)*

$$\frac{d^2\phi}{dx^2} + \left(E - V(x)\right)\phi = 0$$

With periodic potential

$$V(x) = -V_0 \cos(Nx)$$

main difference:
independent variable
space=x
becomes
time=t

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{\ddot{A}_y(t)}{\ell}\right)\phi = 0$$

On periodic roller coaster: $y=-A_y \cos w_y t$

$$\ddot{A}_y(t) = \omega_y^{\ 2} A_y \cos(\omega_y t)$$

Schrodinger Equation
Parametric Resonance

Related to

Jerked-Pendulum
Trebuchet Dynamics

Schrodinger Wave Equation *(With m=1 and ħ=1)*

$$\frac{d^2\phi}{dx^2}+\left(E-V(x)\right)\phi=0$$

With periodic potential

$$V(x)=-V_0\cos(Nx)$$

Mathieu Equation

$$\frac{d^2\phi}{dx^2}+\left(E+V_0\cos(Nx)\right)\phi=0$$

main difference:
independent variable
space=x
becomes
time=t

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{A_y(t)}{\ell}\right)\phi=0$$

On periodic roller coaster: $y=-A_y\cos w_y t$

$$A_y(t)=\omega_y{}^2A_y\cos(\omega_y t)$$

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{\omega_y{}^2A_y}{\ell}\cos(\omega_y t)\right)\phi=0$$

Schrodinger Equation Parametric Resonance

Related to

Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation (With m=1 and ħ=1)

$$\frac{d^2\phi}{dx^2} + \left(E - V(x)\right)\phi = 0$$

With periodic potential

$$V(x) = -V_0 \cos(Nx)$$

Mathieu Equation

$$\frac{d^2\phi}{dx^2} + \left(E + V_0 \cos(Nx)\right)\phi = 0$$

main difference:
independent variable
space=x
becomes
time=t

$$Nx = \omega_y t$$

Connection Relations

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

On periodic roller coaster: $y=-A_y \cos w_y t$

$$A_y(t) = \omega_y^2 A_y \cos(\omega_y t)$$

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{\omega_y^2 A_y}{\ell}\cos(\omega_y t)\right)\phi = 0$$

Schrodinger Equation Parametric Resonance — Related to — Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation *(With m=1 and ħ=1)*

$$\frac{d^2\phi}{dx^2}+\left(E-V(x)\right)\phi=0$$

With periodic potential

$$V(x)=-V_0\cos(Nx)$$

main difference:
independent variable
space=x
becomes
time=t

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{A_y(t)}{\ell}\right)\phi=0$$

On periodic roller coaster: $y=-A_y\cos w_y t$

$$A_y(t)=\omega_y^{\,2}A_y\cos(\omega_y t)$$

Mathieu Equation

$$Nx=\omega_y t$$

$$\frac{d^2\phi}{dx^2}+\left(E+V_0\cos(Nx)\right)\phi=0$$

$$\frac{d^2\phi}{dt^2}+\left(\frac{g}{\ell}+\frac{\omega_y^{\,2}A_y}{\ell}\cos(\omega_y t)\right)\phi=0$$

Connection Relations

$$\frac{N}{\omega_y}dx=dt \longrightarrow \frac{N^2}{\omega_y^{\,2}}dx^2=dt^2$$

Schrodinger Equation Parametric Resonance

Related to

Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation (With m=1 and ħ=1)

$$\frac{d^2\phi}{dx^2} + \left(E - V(x)\right)\phi = 0$$

With periodic potential

$$V(x) = -V_0 \cos(Nx)$$

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

On periodic roller coaster: $y=-A_y \cos w_y t$

$$A_y(t) = \omega_y^{\,2} A_y \cos(\omega_y t)$$

main difference: independent variable

space=x becomes time=t

Mathieu Equation

$$Nx = \omega_y t$$

$$\frac{d^2\phi}{dx^2} + \left(E + V_0 \cos(Nx)\right)\phi = 0$$

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{\omega_y^{\,2} A_y}{\ell}\cos(\omega_y t)\right)\phi = 0$$

Connection Relations

$$\frac{N}{\omega_y}dx = dt \qquad \frac{N^2}{\omega_y^{\,2}}dx^2 = dt^2$$

$$\frac{d^2\phi}{dx^2} + \frac{N^2}{\omega_y^{\,2}}\left(\frac{g}{\ell} + \frac{\omega_y^{\,2} A_y}{\ell}\cos(Nx)\right)\phi = 0$$

Let N=2 to get Band-edge modes

Schrodinger Equation Parametric Resonance

Related to

Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation (With m=1 and ħ=1)

$$\frac{d^2\phi}{dx^2} + \left(E - V(x)\right)\phi = 0$$

With periodic potential

$$V(x) = -V_0 \cos(Nx)$$

main difference:
independent variable
space=x
becomes
time=t

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

On periodic roller coaster: $y=-A_y \cos w_y t$

$$A_y(t) = \omega_y{}^2 A_y \cos(\omega_y t)$$

Mathieu Equation

$$Nx = \omega_y t$$

$$\frac{d^2\phi}{dx^2} + \left(E + V_0 \cos(Nx)\right)\phi = 0$$

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{\omega_y{}^2 A_y}{\ell}\cos(\omega_y t)\right)\phi = 0$$

Connection Relations

$$\frac{N}{\omega_y}dx = dt \longrightarrow \frac{N^2}{\omega_y{}^2}dx^2 = dt^2$$

$$\frac{d^2\phi}{dx^2} + \frac{N^2}{\omega_y{}^2}\left(\frac{g}{\ell} + \frac{\omega_y{}^2 A_y}{\ell}\cos(Nx)\right)\phi = 0$$

Let N=2 to get Band-edge modes

$$\boxed{E = \frac{N^2}{\omega_y{}^2}\frac{g}{\ell}}$$

QM Energy E-to-ω_y Jerk frequency Connection

Schrodinger Equation Parametric Resonance ⟺ Related to ⟺ Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation *(With m=1 and ħ=1)*

$$\frac{d^2\phi}{dx^2} + \left(E - V(x)\right)\phi = 0$$

With periodic potential

$$V(x) = -V_0 \cos(Nx)$$

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

On periodic roller coaster: $y=-A_y \cos w_y t$

$$A_y(t) = \omega_y^{\ 2} A_y \cos(\omega_y t)$$

main difference: independent variable

space=x ← *becomes* → *time=t*

Mathieu Equation

$$Nx = \omega_y t$$

$$\frac{d^2\phi}{dx^2} + \left(E + V_0 \cos(Nx)\right)\phi = 0$$

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{\omega_y^{\ 2} A_y}{\ell}\cos(\omega_y t)\right)\phi = 0$$

Connection Relations

$$\frac{N}{\omega_y} dx = dt \rightarrow \frac{N^2}{\omega_y^{\ 2}} dx^2 = dt^2$$

$$\frac{d^2\phi}{dx^2} + \frac{N^2}{\cancel{\omega_y^{\ 2}}}\left(\frac{g}{\ell} + \frac{\cancel{\omega_y^{\ 2}} A_y}{\ell}\cos(Nx)\right)\phi = 0$$

Let N=2 to get Band-edge modes

$$\boxed{E = \frac{N^2}{\omega_y^{\ 2}}\frac{g}{\ell}}$$

QM Energy E-to-ω_y Jerk frequency Connection

$$\boxed{V_0 = \frac{N^2 A_y}{\ell}}$$

QM Potential V_0-A_y Amplitude Connection

Schrodinger Equation Parametric Resonance ⟺ Related to ⟺ Jerked-Pendulum Trebuchet Dynamics

Schrodinger Wave Equation (With m=1 and ħ=1)

$$\frac{d^2\phi}{dx^2} + \left(E - V(x)\right)\phi = 0$$

With periodic potential

$$V(x) = -V_0 \cos(Nx)$$

main difference: independent variable
← *space=x*
becomes
time=t →

Jerked Pendulum Equation

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{A_y(t)}{\ell}\right)\phi = 0$$

On periodic roller coaster: y=-$A_y \cos w_y t$

$$A_y(t) = \omega_y{}^2 A_y \cos(\omega_y t)$$

Mathieu Equation

$$Nx = \omega_y t$$

$$\frac{d^2\phi}{dx^2} + \left(E + V_0 \cos(Nx)\right)\phi = 0$$

$$\frac{d^2\phi}{dt^2} + \left(\frac{g}{\ell} + \frac{\omega_y{}^2 A_y}{\ell}\cos(\omega_y t)\right)\phi = 0$$

Connection Relations

$$\frac{N}{\omega_y}dx = dt \longrightarrow \frac{N^2}{\omega_y{}^2}dx^2 = dt^2$$

$$\frac{d^2\phi}{dx^2} + \frac{N^2}{\cancel{\omega_y{}^2}}\left(\frac{g}{\ell} + \frac{\cancel{\omega_y{}^2} A_y}{\ell}\cos(Nx)\right)\phi = 0$$

Let N=2 to get Band-edge modes

$$\boxed{E = \frac{4}{\omega_y^2} g}$$

For N=2 and ℓ=1

$$\boxed{V_0 = 4A_y}$$

QM Energy E-to-ω_y Jerk frequency Connection

QM Potential V_0-A_y Amplitude Connection

Two Kinds of Resonance: Linear-additive vs. Nonlinear-multiplicative (Parametric resonance)

Coupled rotation and translation (Throwing revisited: trebuchet, atlatl, etc.)

Schrodinger wave equation related to Parametric resonance dynamics

➔ *Electronic band theory and analogous mechanics*

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi$$

independent variable
← *space=x*
becomes
time=t →

$$-\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi$$

independent variable
← *space=x*
becomes
time=t →

$$-\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x | k \rangle = \phi_k(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}, \text{ where: } E{=}k^2$$

$$\langle t | \omega \rangle = \phi_\omega(t) = \frac{e^{\pm i\omega_0 t}}{\sqrt{2\pi}}, \text{ where: } \omega_0{=}\sqrt{\frac{g}{\ell}}$$

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi$$

independent variable
space=x
becomes
time=t

$$-\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x|k\rangle = \phi_k(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}, \text{ where: } E = k^2$$

$$\langle t|\omega\rangle = \phi_\omega(t) = \frac{e^{\pm i\omega_0 t}}{\sqrt{2\pi}}, \text{ where: } \omega_0 = \sqrt{\frac{g}{\ell}}$$

Bohr has *periodic boundary conditions* x between 0 and L

$$\phi(0) = \phi(L) \Rightarrow e^{ikL} = 1, \text{ or: } k = \frac{2\pi m}{L}$$

Pendulum repeats perfectly after a time T.

$$\phi(0) = \phi(T) \Rightarrow e^{i\omega_0 T} = 1, \text{ or: } \omega_0 = \frac{2\pi m}{T}$$

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi$$

independent variable
space=x
becomes
time=t

$$-\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x|k\rangle = \phi_k(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}, \text{ where: } E=k^2$$

$$\langle t|\omega\rangle = \phi_\omega(t) = \frac{e^{\pm i\omega_0 t}}{\sqrt{2\pi}}, \text{ where: } \omega_0=\sqrt{\frac{g}{\ell}}$$

Bohr has *periodic boundary conditions* x between *0* and L

Pendulum repeats perfectly after a time T.

$$\phi(0)=\phi(L) \Rightarrow e^{ikL} = 1, \text{ or: } k=\frac{2\pi m}{L}$$

$$\phi(0)=\phi(T) \Rightarrow e^{i\omega_0 T} = 1, \text{ or: } \omega_0=\frac{2\pi m}{T}$$

Limit $L=2\pi=T$ for both analogies. Then the allowed energies and frequencies follow

$$E = k^2 = 0,1,4,9,16...$$

$$\omega_0 = m = 0,\pm1,\pm2,\pm3,\pm4,...$$

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi$$

independent variable
← *space=x*
becomes
time=t →

$$-\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x|k\rangle = \phi_k(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}, \text{ where: } E = k^2$$

$$\langle t|\omega\rangle = \phi_\omega(t) = \frac{e^{\pm i\omega_0 t}}{\sqrt{2\pi}}, \text{ where: } \omega_0 = \sqrt{\frac{g}{\ell}}$$

Bohr has *periodic boundary conditions* x between 0 and L

$$\phi(0) = \phi(L) \Rightarrow e^{ikL} = 1, \text{ or: } k = \frac{2\pi m}{L}$$

Pendulum repeats perfectly after a time T.

$$\phi(0) = \phi(T) \Rightarrow e^{i\omega_0 T} = 1, \text{ or: } \omega_0 = \frac{2\pi m}{T}$$

Limit $L=2\pi=T$ for both analogies. Then the allowed energies and frequencies follow

$$E = k^2 = 0,1,4,9,16...$$

$$\omega_0 = m = 0,\pm 1,\pm 2,\pm 3,\pm 4,...$$

Schrodinger equation with non-zero V solved in Fourier basis

$$-\frac{d^2\phi}{dx^2} + V_0\cos(Nx)\phi = E\phi\ , \qquad (\mathbf{D}+\mathbf{V})|\phi\rangle = E|\phi\rangle$$

Fourier representation: $\langle j|\mathbf{D}|k\rangle = j^2\delta_j^k$

$$\Sigma\langle j|(\mathbf{D}+\mathbf{V})|k\rangle\langle k|\phi\rangle = E\langle j|\phi\rangle$$

Matrix eigenvalue equation

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi$$

independent variable
space=x
becomes
time=t

$$-\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x|k\rangle = \phi_k(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}, \text{ where: } E=k^2$$

$$\langle t|\omega\rangle = \phi_\omega(t) = \frac{e^{\pm i\omega_0 t}}{\sqrt{2\pi}}, \text{ where: } \omega_0=\sqrt{\frac{g}{\ell}}$$

Bohr has *periodic boundary conditions* x between 0 and L Pendulum repeats perfectly after a time T.

$$\phi(0)=\phi(L) \Rightarrow e^{ikL}=1, \text{ or: } k=\frac{2\pi m}{L}$$

$$\phi(0)=\phi(T) \Rightarrow e^{i\omega_0 T}=1, \text{ or: } \omega_0=\frac{2\pi m}{T}$$

Limit $L=2\pi=T$ for both analogies. Then the allowed energies and frequencies follow

$$E = k^2 = 0,1,4,9,16...$$

$$\omega_0 = m = 0,\pm1,\pm2,\pm3,\pm4,...$$

Schrodinger equation with non-zero V solved in Fourier basis

$$-\frac{d^2\phi}{dx^2} + V_0\cos(Nx)\phi = E\phi \;, \qquad \left(\mathbf{D}+\mathbf{V}\right)|\phi\rangle = E|\phi\rangle$$

Fourier representation: $\langle j|\mathbf{D}|k\rangle = j^2\delta_j^k$ and $\langle j|\mathbf{V}|k\rangle = \int_0^{2\pi} dx \frac{e^{-ijx}}{\sqrt{2\pi}} V_0\cos(Nx)\frac{e^{+ikx}}{\sqrt{2\pi}} = \int_0^{2\pi} dx \frac{e^{-i(j-k)x}}{2\pi} V_0 \frac{e^{-iNx}+e^{iNx}}{2}$

$$= \frac{V_0}{2}\left(\delta_j^{k+N} + \delta_j^{k-N}\right)$$

$$\Sigma\langle j|\left(\mathbf{D}+\mathbf{V}\right)|k\rangle\langle k|\phi\rangle = E\langle j|\phi\rangle$$

Matrix eigenvalue equation

Electronic band theory and analogous mechanics

Suppose Schrodinger potential V is zero and, by analogy, the pendulum Y-stimulus A_y is zero

$$-\frac{d^2\phi}{dx^2} = E\phi \qquad\qquad -\frac{d^2\phi}{dt^2} = \omega_0^2\phi$$

Eigen-solutions are the familiar *Bohr orbitals* or, for the pendulum, the familiar phasor waves

$$\langle x|k\rangle = \phi_k(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}, \text{ where: } E = k^2 \qquad\qquad \langle t|\omega\rangle = \phi_\omega(t) = \frac{e^{\pm i\omega_0 t}}{\sqrt{2\pi}}, \text{ where: } \omega_0 = \sqrt{\frac{g}{\ell}}$$

Bohr has *periodic boundary conditions* x between 0 and L Pendulum repeats perfectly after a time T.

$$\phi(0) = \phi(L) \Rightarrow e^{ikL} = 1, \text{ or: } k = \frac{2\pi m}{L} \qquad\qquad \phi(0) = \phi(T) \Rightarrow e^{i\omega_0 T} = 1, \text{ or: } \omega_0 = \frac{2\pi m}{T}$$

Limit $L=2\pi=T$ for both analogies. Then the allowed energies and frequencies follow

$$E = k^2 = 0, 1, 4, 9, 16\ldots \qquad\qquad \omega_0 = m = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$$

Schrodinger equation with non-zero V solved in Fourier basis

$$-\frac{d^2\phi}{dx^2} + V_0\cos(Nx)\phi = E\phi\ , \qquad (\mathbf{D}+\mathbf{V})|\phi\rangle = E|\phi\rangle$$

Fourier representation: $\langle j|\mathbf{D}|k\rangle = j^2\delta_j^k$ and $\langle j|\mathbf{V}|k\rangle = \int_0^{2\pi} dx \frac{e^{-ijx}}{\sqrt{2\pi}} V_0\cos(Nx)\frac{e^{+ikx}}{\sqrt{2\pi}} = \int_0^{2\pi} dx \frac{e^{-i(j-k)x}}{2\pi} V_0 \frac{e^{-iNx}+e^{iNx}}{2}$

$$= \frac{V_0}{2}\left(\delta_j^{k+N} + \delta_j^{k-N}\right)$$

$$\Sigma\langle j|(\mathbf{D}+\mathbf{V})|k\rangle\langle k|\phi\rangle = E\langle j|\phi\rangle$$

Matrix eigenvalue equation

(Move Fourier reps. to top)

Electronic band theory and analogous mechanics

Schrodinger equation with non-zero V solved in Fourier basis

$$-\frac{d^2\phi}{dx^2} + V_0 \cos(Nx)\phi = E\phi \ , \qquad \left(\mathbf{D}+\mathbf{V}\right)\left|\phi\right\rangle = E\left|\phi\right\rangle$$

Fourier representation: $\left\langle j\middle|\mathbf{D}\middle|k\right\rangle = j^2\delta_j^k$ and $\left\langle j\middle|\mathbf{V}\middle|k\right\rangle = \int_0^{2\pi} dx \frac{e^{-i\,jx}}{\sqrt{2\pi}} V_0 \cos(Nx) \frac{e^{+i\,kx}}{\sqrt{2\pi}} = \int_0^{2\pi} dx \frac{e^{-i(j-k)x}}{2\pi} V_0 \frac{e^{-i\,Nx}+e^{i\,Nx}}{2}$

$$= \frac{V_0}{2}\left(\delta_j^{k+N} + \delta_j^{k-N}\right)$$

$$\Sigma \left\langle j\middle|\left(\mathbf{D}+\mathbf{V}\right)\middle|k\right\rangle\left\langle k\middle|\phi\right\rangle = E\left\langle j\middle|\phi\right\rangle$$

Matrix eigenvalue equation

Electronic band theory and analogous mechanics

Schrodinger equation with non-zero V solved in Fourier basis

$$-\frac{d^2\phi}{dx^2}+V_0\cos(Nx)\phi=E\phi\ , \qquad (\mathbf{D}+\mathbf{V})|\phi\rangle=E|\phi\rangle$$

Fourier representation: $\langle j|\mathbf{D}|k\rangle=j^2\delta_j^k$ and $\langle j|\mathbf{V}|k\rangle=\int_0^{2\pi}dx\frac{e^{-ijx}}{\sqrt{2\pi}}V_0\cos(Nx)\frac{e^{+ikx}}{\sqrt{2\pi}}=\int_0^{2\pi}dx\frac{e^{-i(j-k)x}}{2\pi}V_0\frac{e^{-iNx}+e^{iNx}}{2}$

$$=\frac{V_0}{2}\left(\delta_j^{k+N}+\delta_j^{k-N}\right)$$

$$\Sigma\langle j|(\mathbf{D}+\mathbf{V})|k\rangle\langle k|\phi\rangle=E\langle j|\phi\rangle$$

Matrix eigenvalue equation

$\langle j|(\mathbf{D}+\mathbf{V})|k\rangle=$ (for j and k *even*)

$$\begin{array}{c|ccccccccc} & \cdots & |-6\rangle & |-4\rangle & |-2\rangle & |0\rangle & |2\rangle & |4\rangle & |6\rangle & \cdots \\ \hline \vdots & \ddots & & & & & & & & \\ \langle -6| & & 6^2 & v & & & & & & \\ \langle -4| & & v & 4^2 & v & & & & & \\ \langle -2| & & & v & 2^2 & v & & & & \\ \langle -0| & & & & v & 0 & v & & & \\ \langle +2| & & & & & v & 2^2 & v & & \\ \langle +4| & & & & & & v & 4^2 & v & \\ \langle +6| & & & & & & & v & 6^2 & \\ \vdots & & & & & & & & & \ddots \end{array}$$

,

$\langle j|(\mathbf{D}+\mathbf{V})|k\rangle=$ (for j and k *odd*)

$$\begin{array}{c|ccccccccc} & \cdots & |-7\rangle & |-5\rangle & |-3\rangle & |-1\rangle & |1\rangle & |3\rangle & |5\rangle & \cdots \\ \hline \vdots & \ddots & & & & & & & & \\ \langle -7| & & 7^2 & v & & & & & & \\ \langle -5| & & v & 5^2 & v & & & & & \\ \langle -3| & & & v & 3^2 & v & & & & \\ \langle -1| & & & & v & 1^2 & v & & & \\ \langle +1| & & & & & v & 1^2 & v & & \\ \langle +3| & & & & & & v & 3^2 & v & \\ \langle +5| & & & & & & & v & 5^2 & \\ \vdots & & & & & & & & & \ddots \end{array}$$

Connection relations from p. 15-16

Here: $v=\frac{V_0}{2}=\frac{4A_y}{2\ell}=\frac{2A_y}{\ell}=2A_y$

For N=2 and ℓ=1

E_m-values vary with amplitude V_0 or wiggle amplitude $A_y = V_0\ell/N^2 = 2v/N^2 = v/2$. *($N=2$ and $\ell=1$ here)*

Eigenvalues for $V_0=0.2$ or $v=0.1$ and $V_0=2.0$ or $v=1.0$.

Connection relations from p. 15-16

$E_0 =$	-0.0050	← *inverted*
$E_{1-} =$	0.8988	
$E_{1+} =$	1.0987	
$E_{2-} =$	3.9992	
$E_{2+} =$	4.0042	
$E_{3-} =$	9.0006	
$E_{3+} =$	9.0006	

$E_0 =$	-0.4551	← *inverted*
$E_{1-} =$	-0.1102	← *inverted*
$E_{1+} =$	1.8591	
$E_{2-} =$	3.9170	
$E_{2+} =$	4.3713	
$E_{3-} =$	9.0477	
$E_{3+} =$	9.0784	

When pendulum is "normal" and near its lowest point ($\phi\sim 0$) then $\cos\phi\sim 1$ and $\sin\phi\sim\phi$

$$\frac{d^2\phi}{dx^2}+\frac{N^2}{\omega_y^2}\left(\frac{g}{\ell}-\frac{\omega_y^2 A_y}{\ell}\cos(Nx)\right)\phi=0=\frac{d^2\phi}{dx^2}+\left(\frac{N^2}{\omega_y^2}\frac{g}{\ell}-\frac{N^2 A_y}{\ell}\cos(Nx)\right)\phi,\quad \text{(where: } \phi\sim 0)$$

When pendulum is "inverted" near highest point ($\phi\sim\pi$) then $\cos\phi\sim -1$ and $\sin\phi\sim\pi-\phi$.

$$\frac{d^2\phi}{dt^2}-\left(\frac{g}{\ell}-\frac{\omega_y^2 A_y}{\ell}\cos(\omega_y t)\right)(\phi-\pi)=0\,,\qquad \text{(where: } \phi\sim\pi)$$

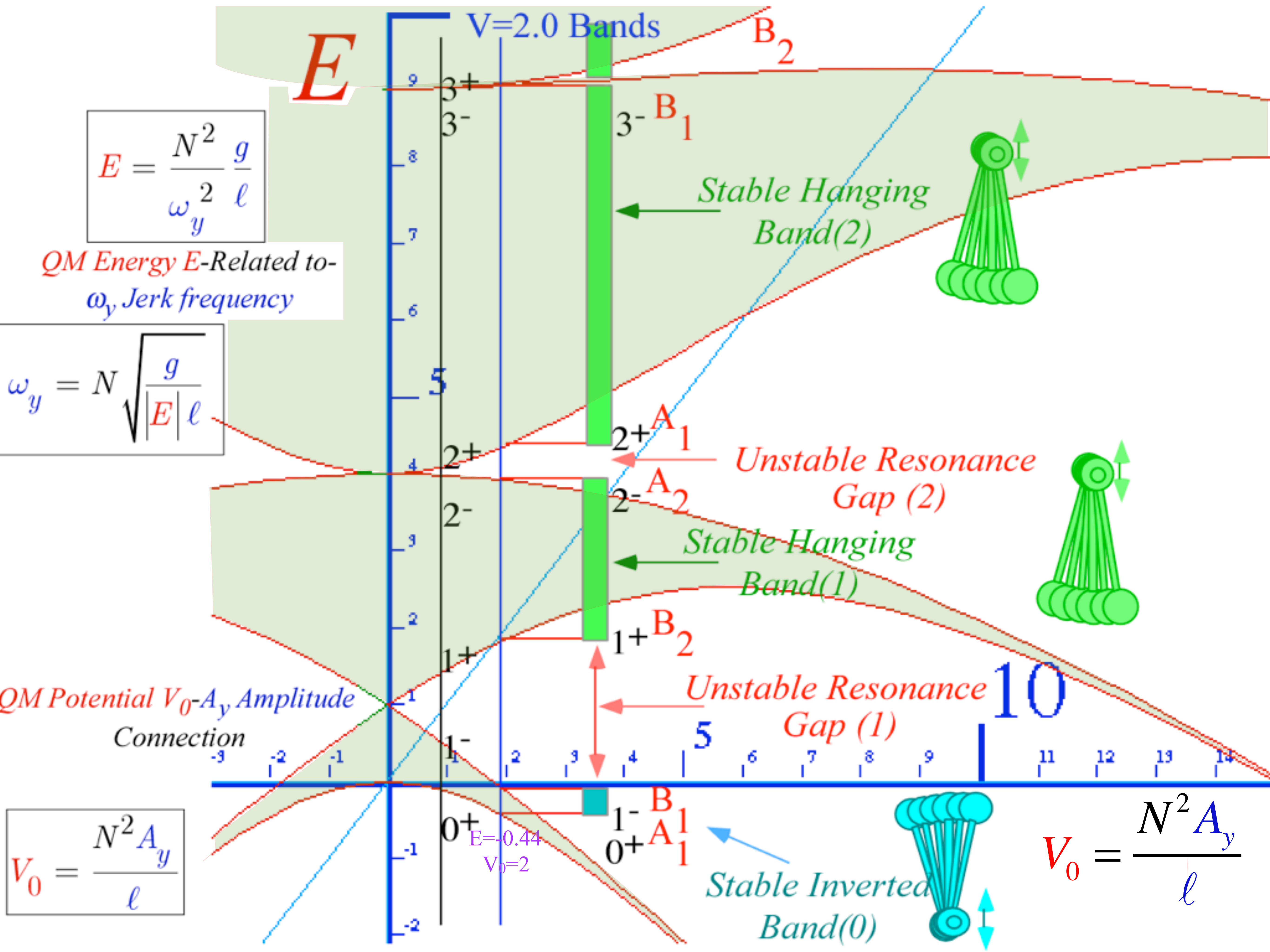
E_m-eigenvalue determines pendulum Y-wiggle frequency $\omega_{y(m)}$.

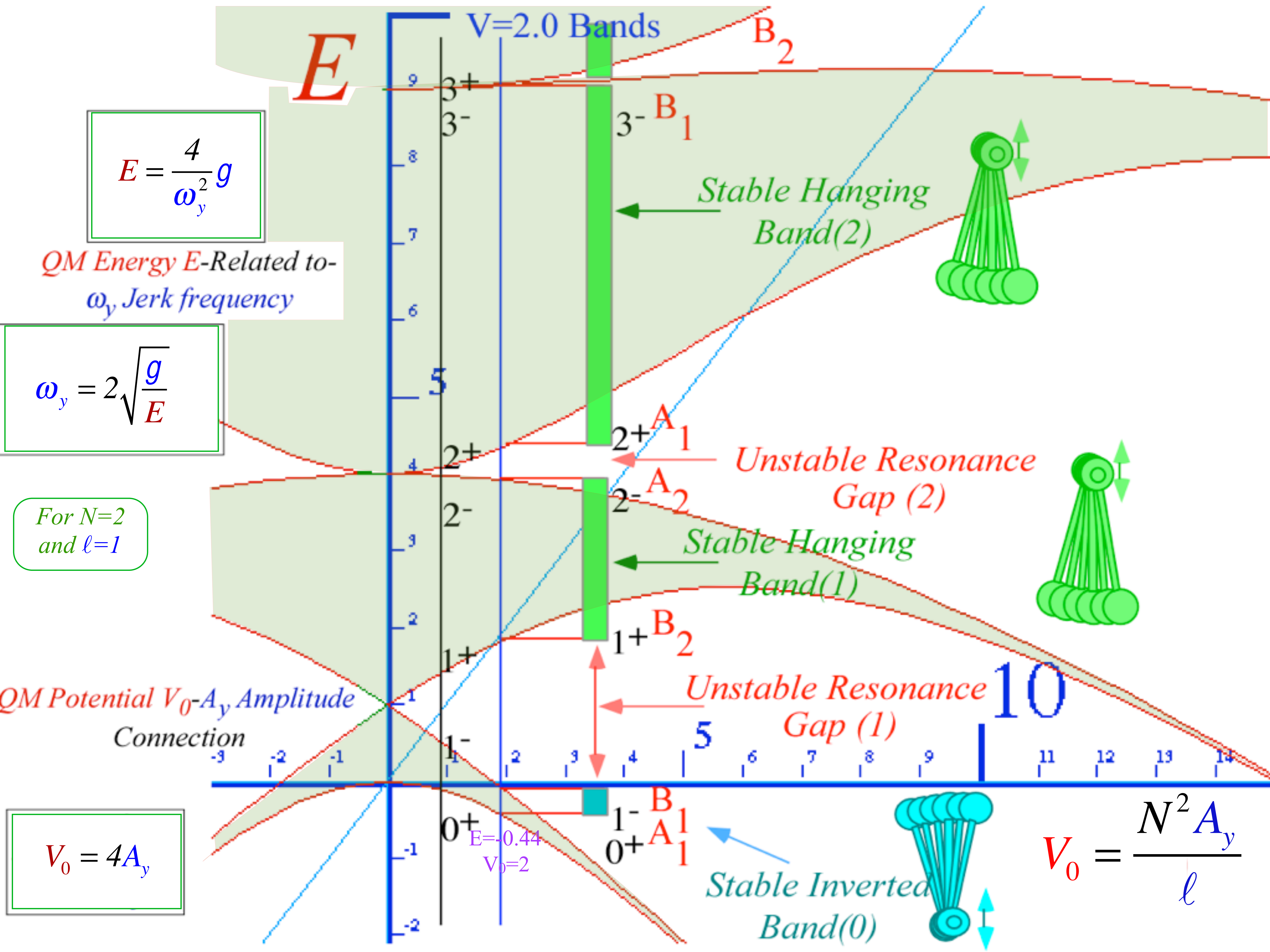
$$E_m=\frac{N^2}{\omega_{y(m)}^2}\frac{g}{\ell}\qquad \text{implies:}\qquad \omega_{y(m)}=\frac{N}{\sqrt{E_m}}\sqrt{\frac{g}{\ell}}=\frac{2}{\sqrt{E_m}}\qquad (g=1\text{, too})$$

Pendulum Y-wiggle frequency $\omega_{y(m)}$ for $V_0=0.2$ and for $V_0=2.0$.

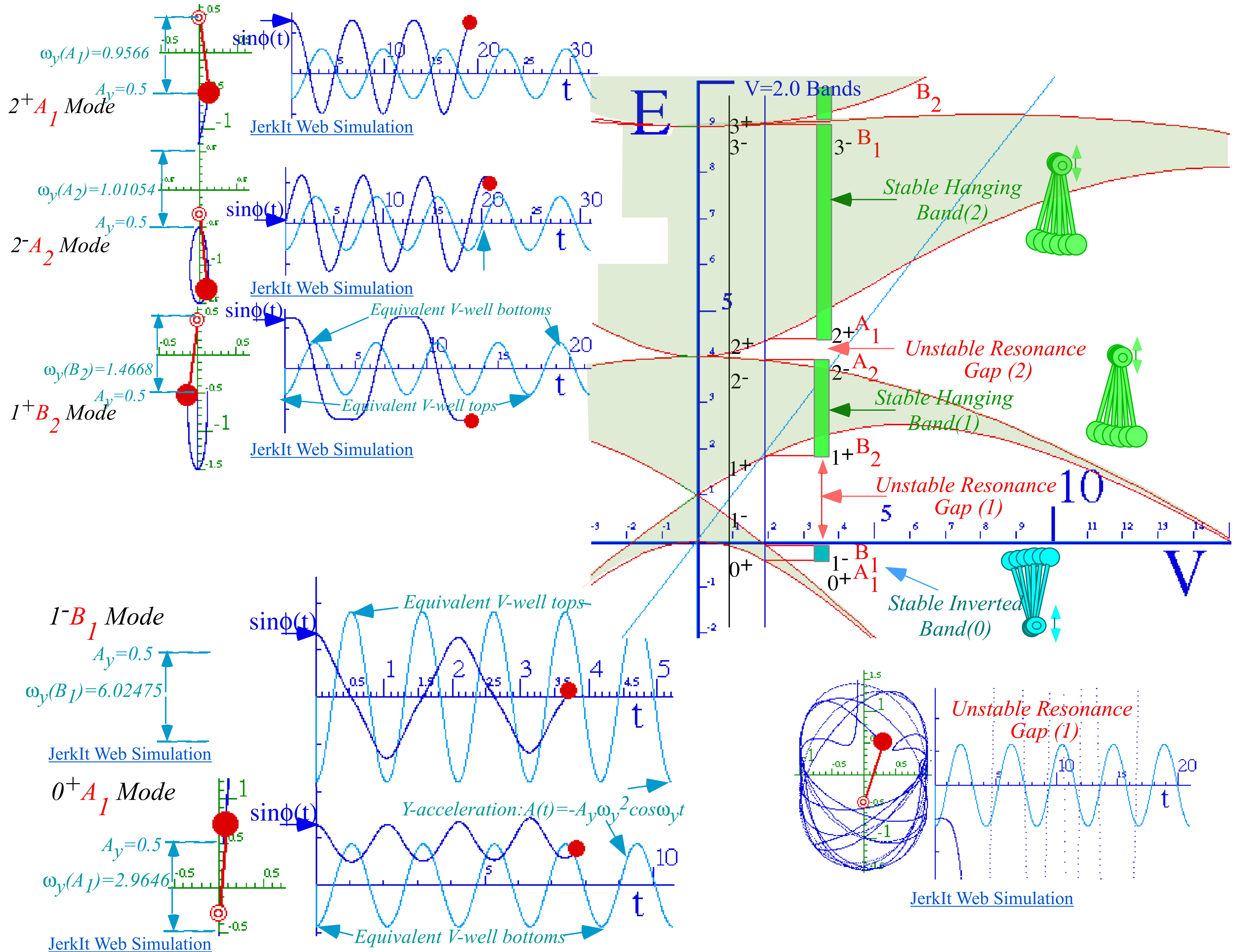
$\omega_{y(0)}=2/\sqrt{.0050}$	$=28.2843$	← *inverted*
$\omega_{y(1^-)}=2/\sqrt{.8988}$	$=2.10959$	
$\omega_{y(1^+)}=2/\sqrt{1.0987}$	$=1.90805$	
$\omega_{y(2^-)}=2/\sqrt{3.9992}$	$=1.00010$	
$\omega_{y(2^+)}=2/\sqrt{4.0042}$	$=0.99948$	

$\omega_{y(0)}=2/\sqrt{.4551}$	$=2.9646$	← *inverted*
$\omega_{y(1^-)}=2/\sqrt{.1102}$	$=6.02475$	← *inverted*
$\omega_{y(1^+)}=2/\sqrt{1.8591}$	$=1.4668$	
$\omega_{y(2^-)}=2/\sqrt{3.9170}$	$=1.0105$	
$\omega_{y(2^+)}=2/\sqrt{4.3713}$	$=0.9566$	


V=2.0 Bands
E
B_2
3+
3-
B_1
$E = \frac{N^2}{\omega_y^2} \frac{g}{\ell}$
QM Energy E-Related to-
ω_y Jerk frequency
$\omega_y = N\sqrt{\frac{g}{|E|\ell}}$
Stable Hanging Band(2)
2+
A_1
Unstable Resonance Gap (2)
A_2
2-
Stable Hanging Band(1)
1+
B_2
Unstable Resonance Gap (1)
QM Potential V_0-A_y Amplitude
Connection
1-
0+
E=-0.44
V_0=2
1-
B_1
0+
A_1
Stable Inverted Band(0)
$V_0 = \frac{N^2 A_y}{\ell}$

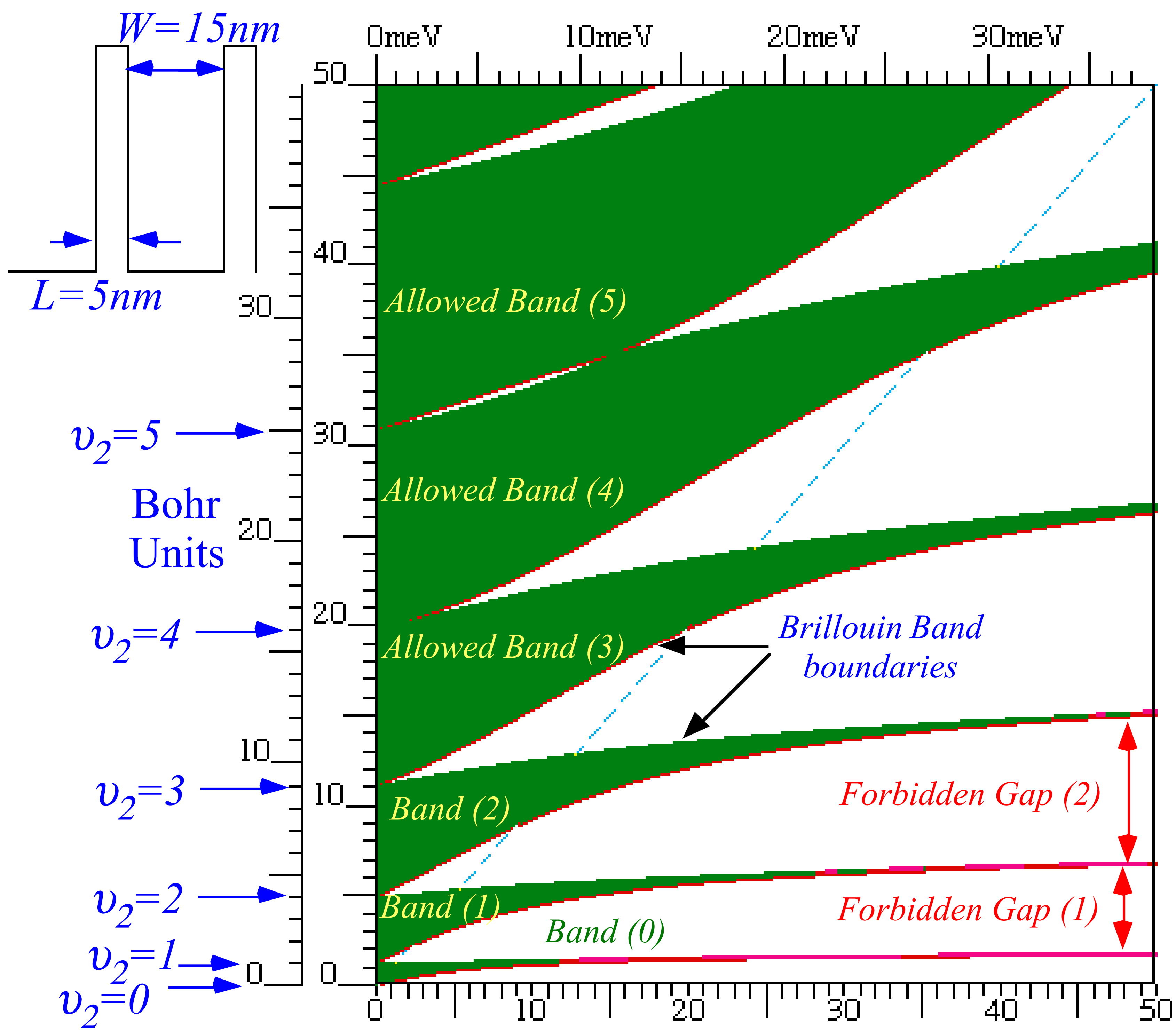


V=2.0 Bands
E
$E = \frac{4}{\omega_y^2} g$
QM Energy E-Related to-
ω_y Jerk frequency
$\omega_y = 2\sqrt{\frac{g}{E}}$
For N=2
and ℓ=1
QM Potential V_0-A_y Amplitude
Connection
$V_0 = 4A_y$
B_2
3^+
3^-
3^- B_1
Stable Hanging
Band(2)
2^+ A_1
2^+
Unstable Resonance
Gap (2)
2^- A_2
2^-
Stable Hanging
Band(1)
1^+ B_2
1^+
Unstable Resonance
Gap (1)
1^-
1^- B_1
0^+
0^+ A_1
E=-0.44
V_0=2
Stable Inverted
Band(0)
$V_0 = \frac{N^2 A_y}{\ell}$



2+A1 Mode
ωy(A1)=0.9566
Ay=0.5
sinϕ(t)
JerkIt Web Simulation
2-A2 Mode
ωy(A2)=1.01054
Ay=0.5
sinϕ(t)
JerkIt Web Simulation
1+B2 Mode
ωy(B2)=1.4668
Ay=0.5
sinϕ(t)
Equivalent V-well bottoms
Equivalent V-well tops
JerkIt Web Simulation
E
V=2.0 Bands
B2
3+
3-
3- B1
Stable Hanging Band(2)
2+A1
Unstable Resonance Gap (2)
2-A2
Stable Hanging Band(1)
1+B2
Unstable Resonance Gap (1)
1-
1- B1
0+
0+A1
Stable Inverted Band(0)
V
1-B1 Mode
Ay=0.5
ωy(B1)=6.02475
sinϕ(t)
Equivalent V-well tops
JerkIt Web Simulation
0+A1 Mode
Ay=0.5
ωy(A1)=2.9646
sinϕ(t)
Y-acceleration:A(t)=-Ayωy2cosωyt
Equivalent V-well bottoms
JerkIt Web Simulation
Unstable Resonance Gap (1)
JerkIt Web Simulation

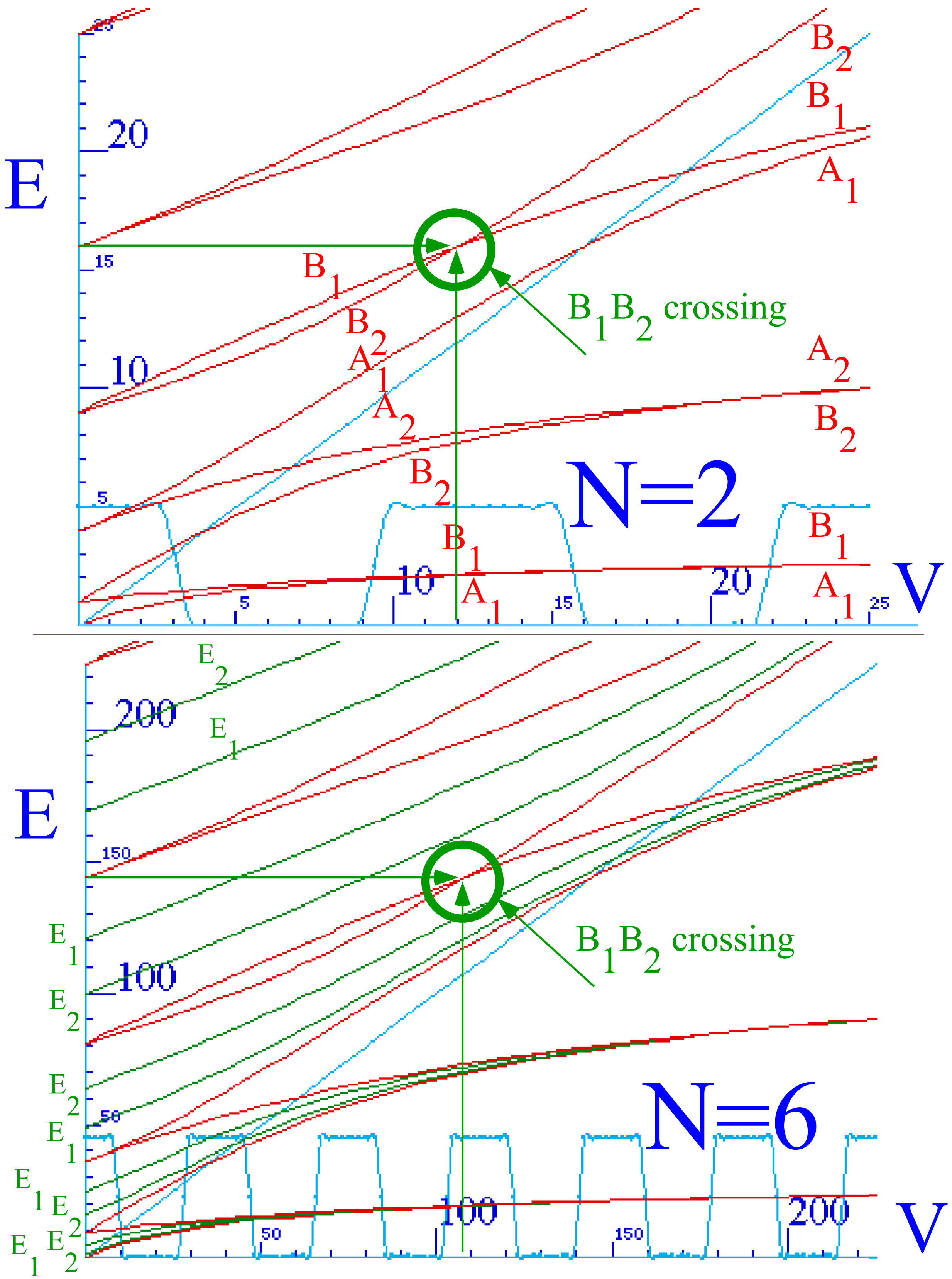
A quick look at band splitting for a square periodic potential (Kronig-Penney Model)



(From Ch. 14 Unit 5 Quantum Theory for the Computer Age (QTftCA)

Fig. 14.2.7 Bands vs. V.(W=15nm well ,L=5nm barrier) showing Bohr splitting for (N=2)-ring.

A quick look at band splitting for a square periodic potential (Kronig-Penney Model)



(From Ch. 14 Unit 5 Quantum Theory for the Computer Age (QTftCA)

Fig. 14.2.13 (B_1, B_2) crossing for:(N=2) at V=12 and E=16, and (N=6) at V=144 and E=108.

Wave resonance in cyclic symmetry

- Harmonic oscillator with cyclic C_2 symmetry
  - C_2 symmetric (B-type) modes
- Harmonic oscillator with cyclic C_3 symmetry
  - C_3 symmetric spectral decomposition by 3rd roots of unity
  - Resolving C_3 projectors and moving wave modes
  - Dispersion functions and standing waves
- C_6 symmetric mode model:Distant neighbor coupling
  - C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, .
- C_N symmetric mode models: Made-to order dispersion functions
  - Quadratic dispersion models: Super-beats and fractional revivals
  - Phase arithmetic

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} + B\cdot\boldsymbol{\sigma}_B$$

$$\mathbf{K} = \mathbf{H}^2 = \begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} + B\cdot\boldsymbol{\sigma}_B$$

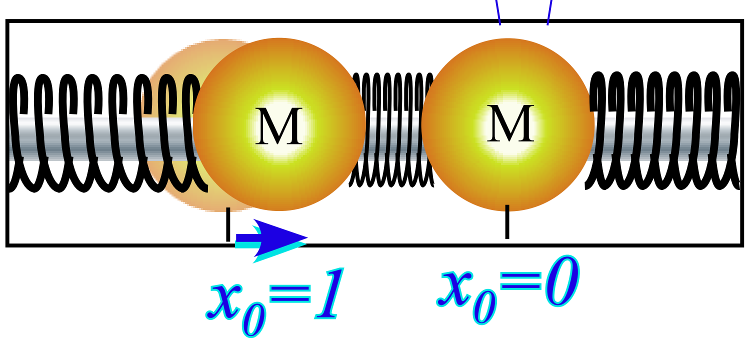
$$\mathbf{K} = \mathbf{H}^2 = \begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

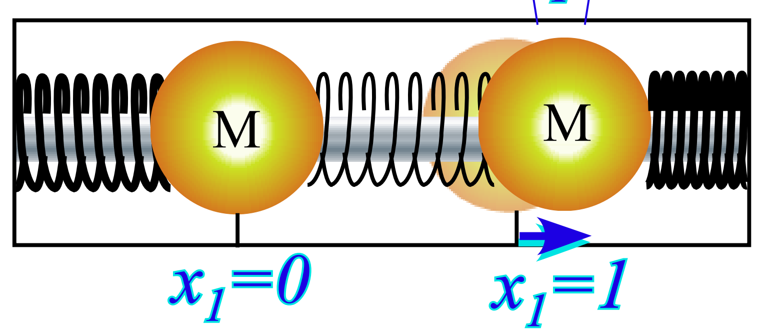
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

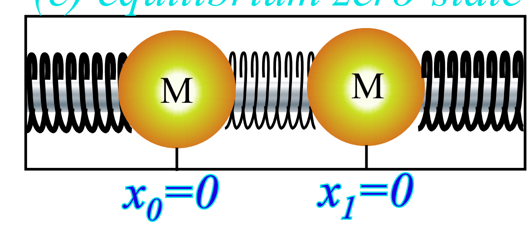


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} + B\cdot\boldsymbol{\sigma}_B$$

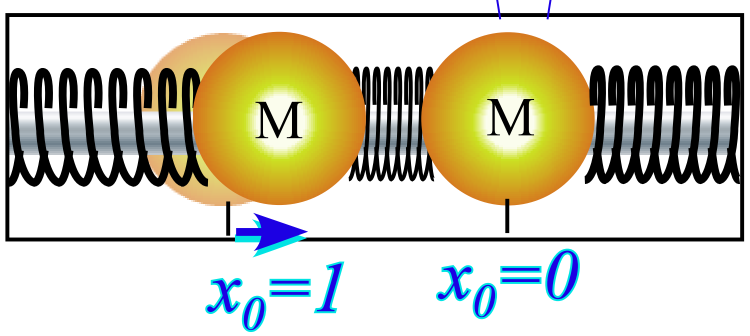
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

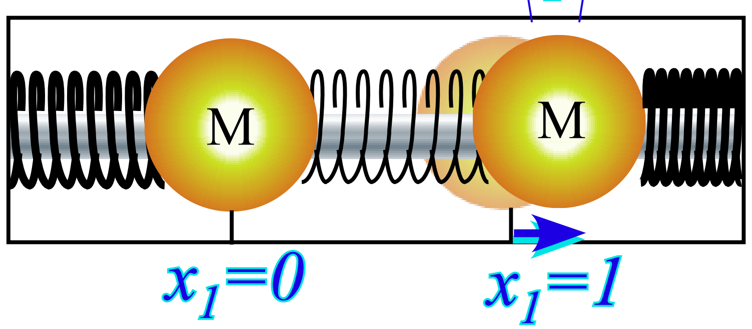
(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

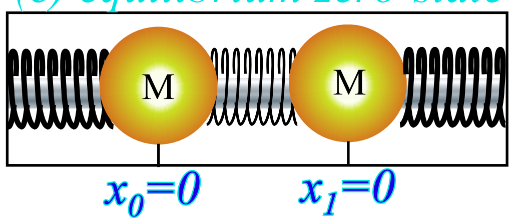


(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1=0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\,\mathbf{p}^{(-1)}$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix $\mathbf{K}=\mathbf{H}^2$ with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$=A\cdot\mathbf{1}\qquad +B\cdot\boldsymbol{\sigma}_B$$

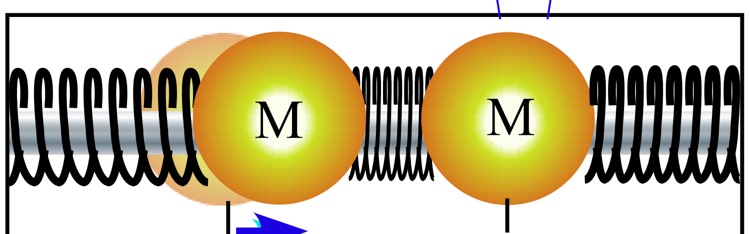
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$
$$=(A^2+B^2)\cdot\mathbf{1}\qquad +2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

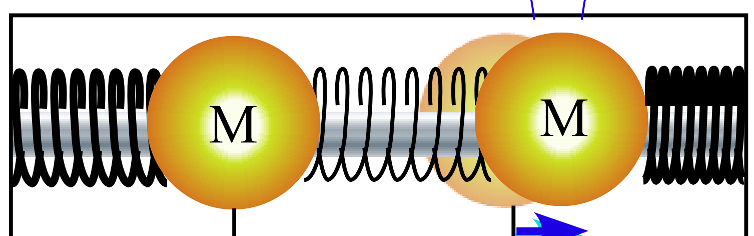
$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$x_0=1$ $\quad x_0=0$

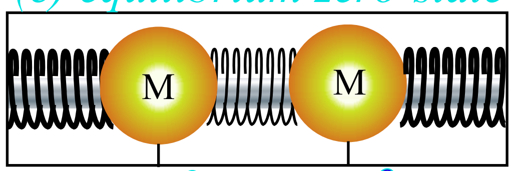
(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x_1=0$ $\quad x_1=1$

(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x_0=0$ $\quad x_1=0$

$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix $\mathbf{K}=\mathbf{H}^2$ with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}=A\cdot\mathbf{1}+B\cdot\boldsymbol{\sigma}_B$$

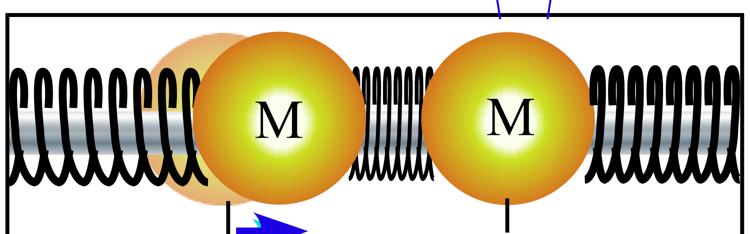
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}=(A^2+B^2)\cdot\mathbf{1}+2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

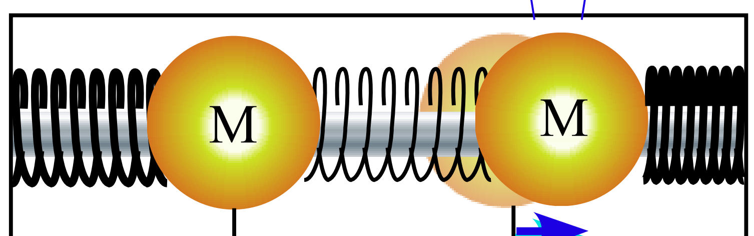
$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$x_0=1$ $\quad x_0=0$

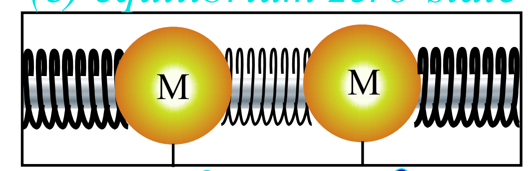
(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x_1=0$ $\quad x_1=1$

(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x_0=0$ $\quad x_1=0$

$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

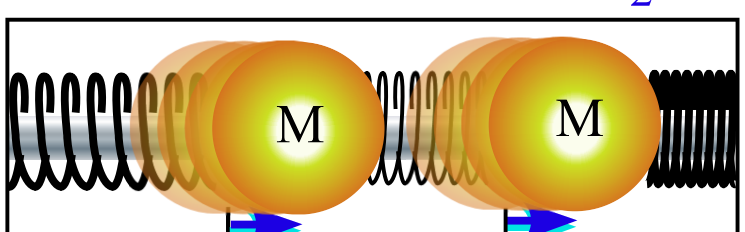
$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

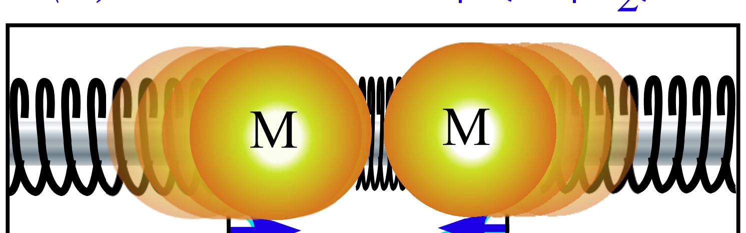
C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad x_1=1/\sqrt{2}$

(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad x_1=-1/\sqrt{2}$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A\cdot\mathbf{1} \quad + B\cdot\boldsymbol{\sigma}_B$$

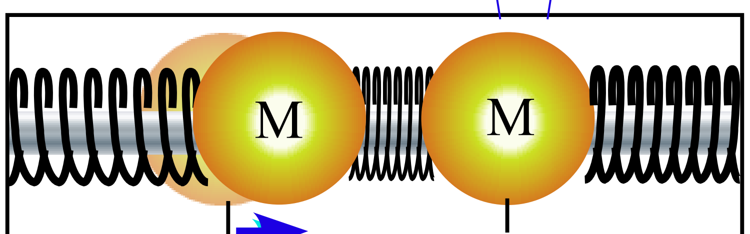
$$\mathbf{K} = \mathbf{H}^2 = \begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix} = (A^2+B^2)\cdot\mathbf{1} \quad + 2AB\cdot\boldsymbol{\sigma}_B$$

C_2	**1**	σ_B
1	**1**	σ_B
σ_B	σ_B	**1**

Reflection symmetry σ_B defined by $(\sigma_B)^2$=**1** in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

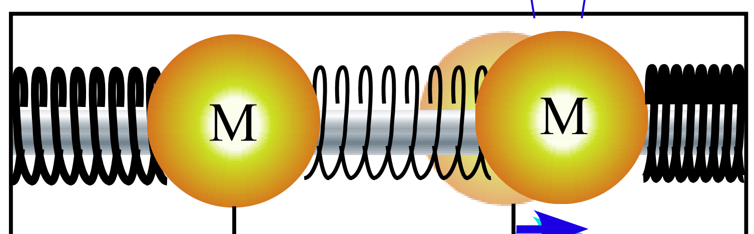
$|0\rangle=|x\rangle=|2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$x_0=1$ $\quad$ $x_0=0$

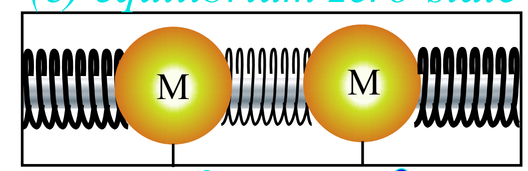
(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x_1=0$ $\quad$ $x_1=1$

(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x_0=0$ $\quad$ $x_1=0$

$(\sigma_B)^2$=**1** or: $(\sigma_B)^2$-**1**=**0** gives projectors:

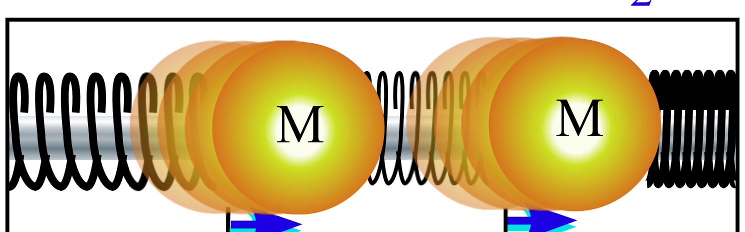
$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

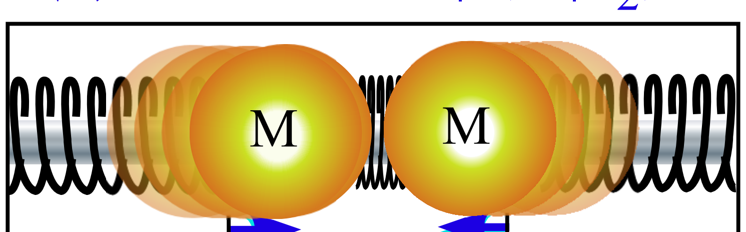
C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad$ $x_1=1/\sqrt{2}$

(b) Odd mode $|-\rangle=|1_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad$ $x_1=-1/\sqrt{2}$

Mode state projection:

$$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$$
$$=(|0\rangle+|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle+|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

$$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2}$$
$$=(|0\rangle-|2\rangle)/\sqrt{2}$$
$$=(|\mathbf{1}\rangle-|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry (B-type)

Hamiltonian matrix **H** or spring-constant matrix **K=H**^{2} with B-type or *bilateral-balanced* symmetry

$$\mathbf{H}=\begin{pmatrix} A & B \\ B & A \end{pmatrix}=A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$=A\cdot\mathbf{1}\qquad+B\cdot\boldsymbol{\sigma}_B$$

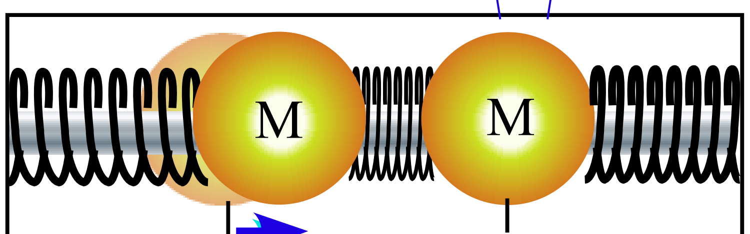
$$\mathbf{K}=\mathbf{H}^2=\begin{pmatrix} A^2+B^2 & 2AB \\ 2AB & A^2+B^2 \end{pmatrix}$$
$$=(A^2+B^2)\cdot\mathbf{1}\qquad+2AB\cdot\boldsymbol{\sigma}_B$$

C_2	$\mathbf{1}$	σ_B
$\mathbf{1}$	$\mathbf{1}$	σ_B
σ_B	σ_B	$\mathbf{1}$

Reflection symmetry σ_B defined by $(\sigma_B)^2=\mathbf{1}$ in C_2 group product table.

(a) unit base state $|\mathbf{1}\rangle=\mathbf{1}|\mathbf{1}\rangle$

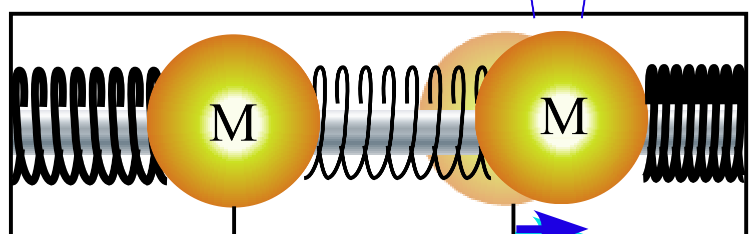
$|0\rangle=|x\rangle=|2\rangle=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$x_0=1$ $\quad x_0=0$

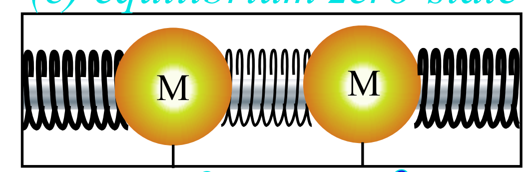
(b) unit base state $|\boldsymbol{\sigma}_B\rangle=\boldsymbol{\sigma}_B|\mathbf{1}\rangle$

$|1\rangle=|y\rangle=|-1\rangle=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$x_1=0$ $\quad x_1=1$

(c) equilibrium zero-state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$x_0=0$ $\quad x_1=0$

$(\sigma_B)^2=\mathbf{1}$ or: $(\sigma_B)^2-\mathbf{1}=\mathbf{0}$ gives projectors:

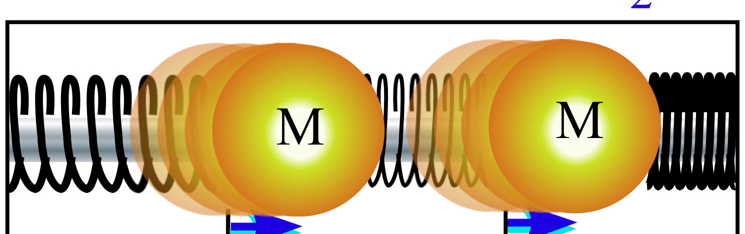
$(\sigma_B+\mathbf{1})\cdot(\sigma_B-\mathbf{1})=\mathbf{0}=\mathbf{p}^{(+1)}\cdot\mathbf{p}^{(-1)}$

$\mathbf{P}^{(+)}=(\sigma_B+\mathbf{1})/2$ and $\mathbf{P}^{(-)}=(\sigma_B-\mathbf{1})/2$

(Normed so: $\mathbf{P}^{(+)}+\mathbf{P}^{(-)}=\mathbf{1}$ and: $\mathbf{P}^{(m)}\cdot\mathbf{P}^{(m)}=\mathbf{P}^{(m)}$)

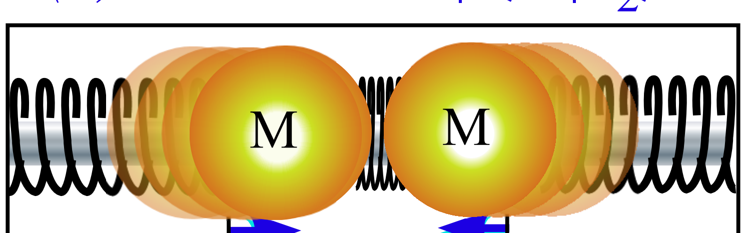
C_2 symmetry (B-type) modes

(a) Even mode $|+\rangle=|0_2\rangle=\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad x_1=1/\sqrt{2}$

(b) Odd mode $|-\rangle=|1_2\rangle=\begin{pmatrix} 1 \\ -1 \end{pmatrix}/\sqrt{2}$



$x_0=1/\sqrt{2}$ $\quad x_1=-1/\sqrt{2}$

Mode state projection:

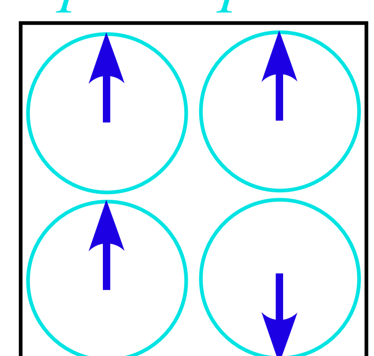
$|+\rangle=|0_2\rangle=\mathbf{P}^{(+)}|0\rangle\sqrt{2}$
$=(|0\rangle+|2\rangle)/\sqrt{2}$
$=(|\mathbf{1}\rangle+|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$

$|-\rangle=|0_2\rangle=\mathbf{P}^{(-)}|0\rangle\sqrt{2}$
$=(|0\rangle-|2\rangle)/\sqrt{2}$
$=(|\mathbf{1}\rangle-|\boldsymbol{\sigma}_B\rangle)/\sqrt{2}$

C_2 mode phase & character tables

p= position point (modulo-2)

p=0 p=1



	$p=0$	$p=1$
$m=0_2$	1	1
$m=1_2$	1	-1

State norm: $1/\sqrt{2}$

m=wave-number or "momentum" (modulo-2)

Operator norm: 1/2

Wave resonance in cyclic symmetry

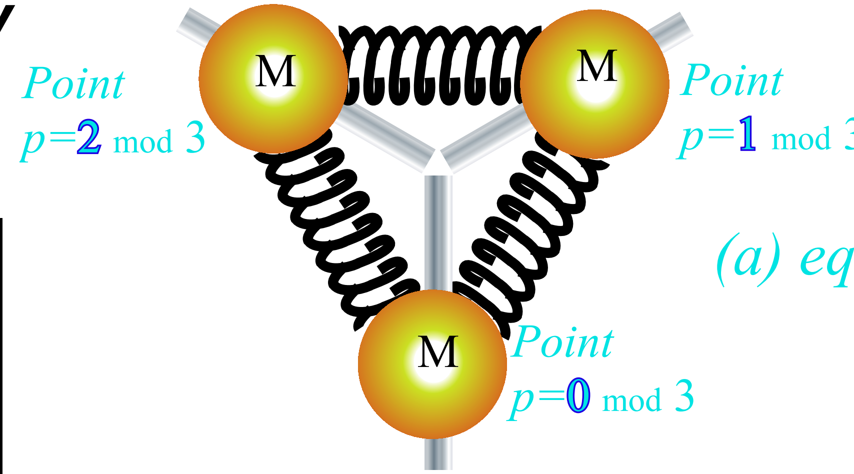
- Harmonic oscillator with cyclic C_2 symmetry
    - C_2 symmetric (B-type) modes
- Harmonic oscillator with cyclic C_3 symmetry
    - C_3 symmetric spectral decomposition by 3rd roots of unity
    - Resolving C_3 projectors and moving wave modes
    - Dispersion functions and standing waves
- C_6 symmetric mode model:Distant neighbor coupling
    - C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, .
- C_N symmetric mode models: Made-to order dispersion functions
    - Quadratic dispersion models: Super-beats and fractional revivals
    - Phase arithmetic

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



$x_0=x_1=x_2=0 \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} = r_0\cdot\mathbf{1} + r_1\cdot\mathbf{r}^1 + r_2\cdot\mathbf{r}^2$$

$\mathbf{r}^0=\mathbf{1}$

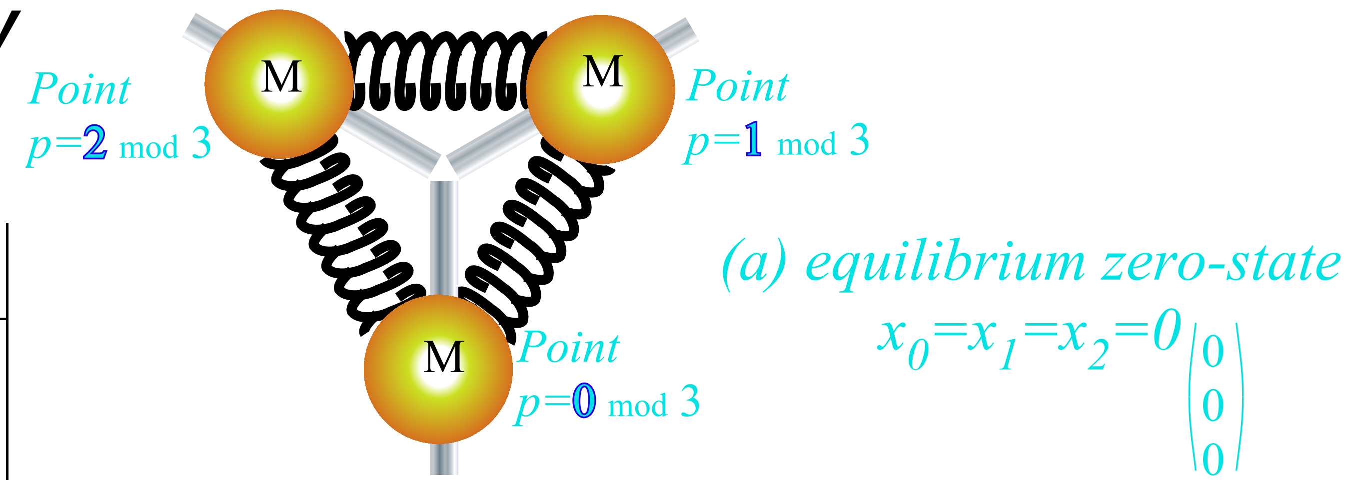
Fig. 4.8.1
Unit 4
CMwBang

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} = \underbrace{r_0\cdot\mathbf{1}}_{\mathbf{r}^0=\mathbf{1}} + r_1\cdot\mathbf{r}^1 + r_2\cdot\mathbf{r}^2$$

C_3 unit base states

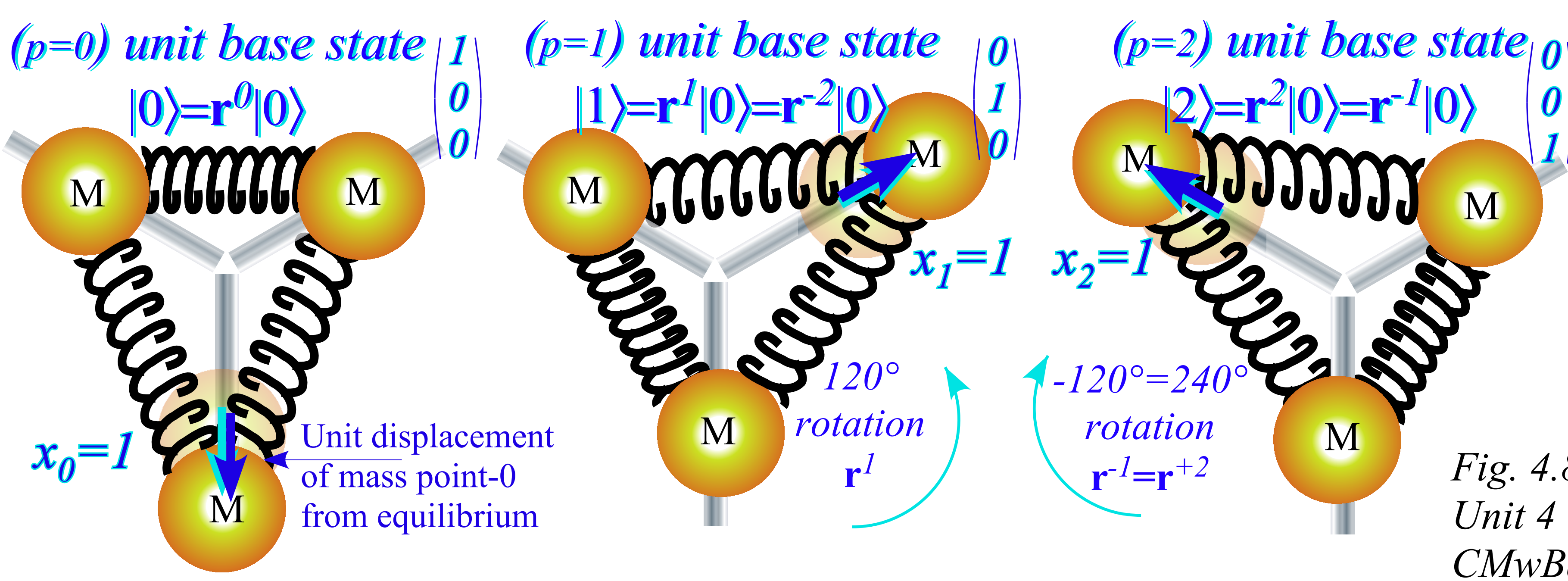


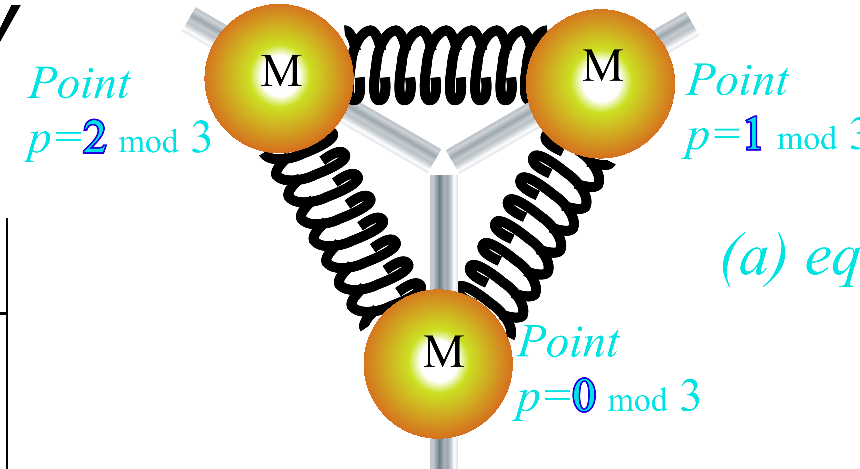
Fig. 4.8.1
Unit 4
CMwBang

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_3 symmetry

3-fold ±120° rotations $\mathbf{r}=\mathbf{r}^1$ and $(\mathbf{r})^2=\mathbf{r}^2=\mathbf{r}^{-1}$
obey: $(\mathbf{r})^3=\mathbf{r}^3=\mathbf{1}=\mathbf{r}^0$ and a C_3 **g†g**-*product-table*

C_3	$\mathbf{r}^0=\mathbf{1}$	$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^2=\mathbf{r}^{-1}$
$\mathbf{r}^0=\mathbf{1}$	$\mathbf{1}$	$\mathbf{r}^1$	$\mathbf{r}^2$
$\mathbf{r}^2=\mathbf{r}^{-1}$	$\mathbf{r}^2$	$\mathbf{1}$	$\mathbf{r}^1$
$\mathbf{r}^1=\mathbf{r}^{-2}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{1}$



(a) equilibrium zero-state

$x_0=x_1=x_2=0 \begin{pmatrix}0\\0\\0\end{pmatrix}$

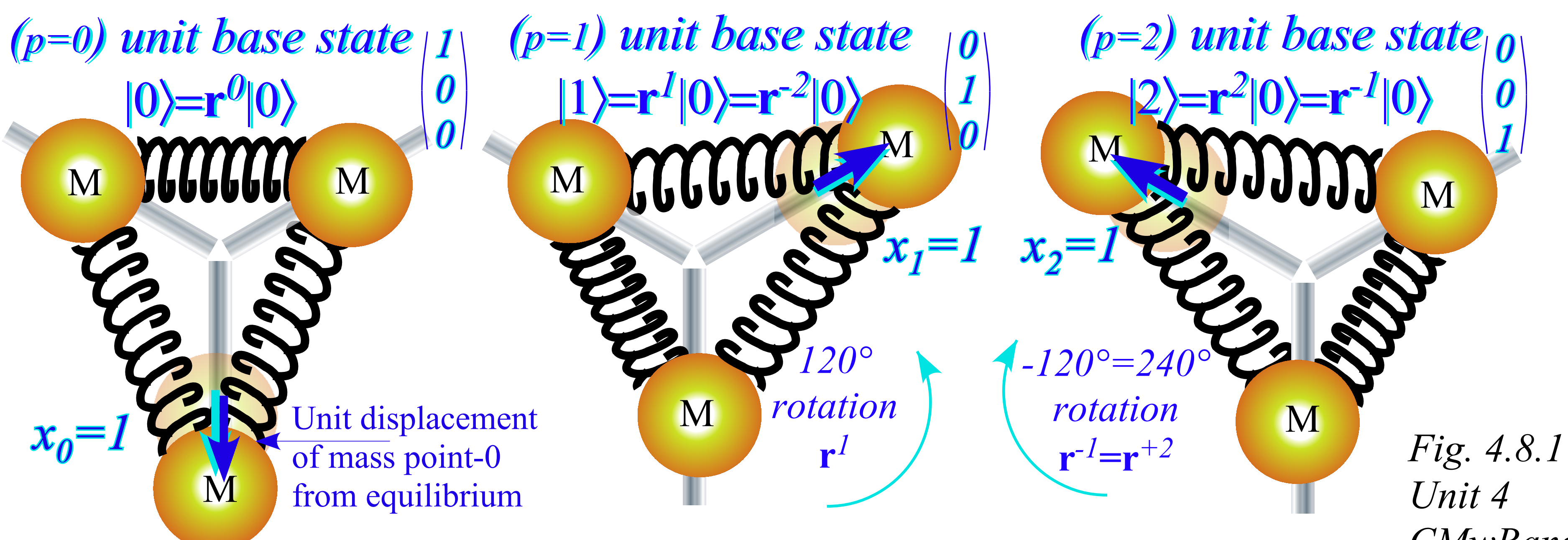
$\mathbf{g}=\mathbf{r}^p$ heads p^{th}-column. Inverse $\mathbf{g}^\dagger=\mathbf{g}^{-1}$ heads p^{th}-row then unit $\mathbf{g}^\dagger\mathbf{g}=\mathbf{1}=\mathbf{g}^{-1}\mathbf{g}$ occupies p^{th}-diagonal.

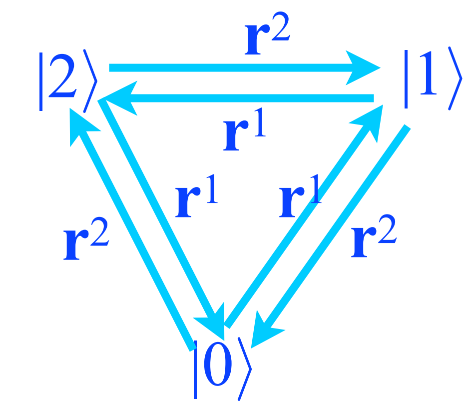
H-matrix and each $\mathbf{r}^p$-matrix based on **g†g**-table.

$$\begin{pmatrix} r_0 & r_1 & r_2 \\ r_2 & r_0 & r_1 \\ r_1 & r_2 & r_0 \end{pmatrix} = r_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + r_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{H} = r_0 \cdot \underbrace{\mathbf{1}}_{\mathbf{r}^0=\mathbf{1}} + r_1 \cdot \mathbf{r}^1 + r_2 \cdot \mathbf{r}^2$$

C_3 unit base states





$\mathbf{r}^0=\mathbf{1}$
$=\mathbf{r}^3$

Fig. 4.8.1
Unit 4
CMwBang

Each **H**-matrix coupling constant $r_p=\{r_0, r_1, r_2\}$ is amplitude of its operator power $\mathbf{r}^p=\{\mathbf{r}^0, \mathbf{r}^1, \mathbf{r}^2\}$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Resolving C_3 projectors and moving wave modes

Dispersion functions and standing waves

C_6 symmetric mode model:Distant neighbor coupling

C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

Phase arithmetic

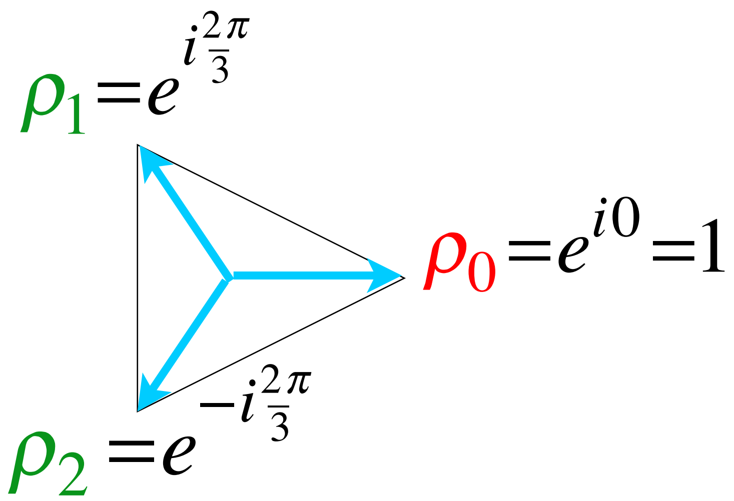
C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.



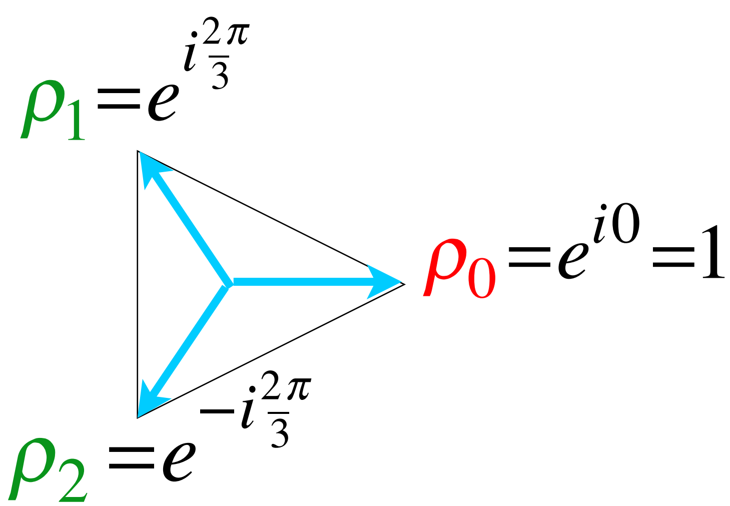
C_3 Spectral resolution: 3rd roots of unity

We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.

$$\mathbf{1}=\mathbf{r}^3 \;\; \textit{implies}: \;\; \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \;\; \textit{where}: \;\; \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.



C_3 Spectral resolution: 3^{rd} roots of unity

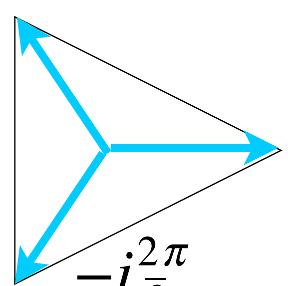
We can spectrally resolve **H** if we resolve **r** since is **H** a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

r-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.

$$\mathbf{1}=\mathbf{r}^3 \textit{ implies: } \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \textit{ where: } \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of **r**, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\mathbf{P}^{(n)}=\delta_{mn}\mathbf{P}^{(m)}$) and *complete* (sum to unit **1**).

$\rho_1=e^{i\frac{2\pi}{3}}$



$\rho_0=e^{i0}=1$

$\rho_2=e^{-i\frac{2\pi}{3}}$

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

C_3 Spectral resolution: 3rd roots of unity

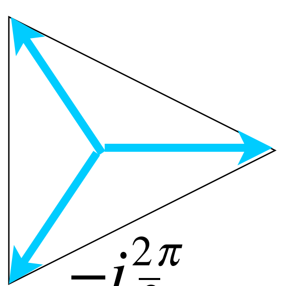
We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

$\mathbf{r}$-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.

$$\mathbf{1}=\mathbf{r}^3 \; implies: \; \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \; where: \; \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of $\mathbf{r}$, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\mathbf{P}^{(n)}=\delta_{mn}\mathbf{P}^{(m)}$) and *complete* (sum to unit $\mathbf{1}$).

$$\rho_1=e^{i\frac{2\pi}{3}}$$



$$\rho_0=e^{i0}=1$$

$$\rho_2=e^{-i\frac{2\pi}{3}}$$

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\mathbf{P}^{(0)} + \rho_1\mathbf{P}^{(1)} + \rho_2\mathbf{P}^{(2)}$$

C_3 Spectral resolution: 3rd roots of unity

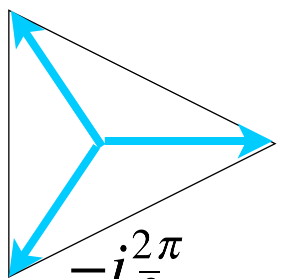
We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

$\mathbf{r}$-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.

$$\mathbf{1}=\mathbf{r}^3 \textit{ implies: } \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \textit{ where: } \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of $\mathbf{r}$, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\mathbf{P}^{(n)}=\delta_{mn}\mathbf{P}^{(m)}$) and *complete* (sum to unit $\mathbf{1}$).

$\rho_1=e^{i\frac{2\pi}{3}}$



$\rho_0=e^{i0}=1$

$\rho_2=e^{-i\frac{2\pi}{3}}$

$$\mathbf{1}=\mathbf{P}^{(0)}+\mathbf{P}^{(1)}+\mathbf{P}^{(2)}$$

$$\mathbf{r}=\rho_0\mathbf{P}^{(0)}+\rho_1\mathbf{P}^{(1)}+\rho_2\mathbf{P}^{(2)}$$

$$\mathbf{r}^2=(\rho_0)^2\mathbf{P}^{(0)}+(\rho_1)^2\mathbf{P}^{(1)}+(\rho_2)^2\mathbf{P}^{(2)}$$

C_3 Spectral resolution: 3rd roots of unity

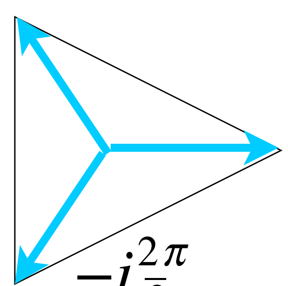
We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

$\mathbf{r}$-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.

$$\mathbf{1}=\mathbf{r}^3 \; implies: \; \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \; where: \; \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of $\mathbf{r}$, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\mathbf{P}^{(n)}=\delta_{mn}\mathbf{P}^{(m)}$) and *complete* (sum to unit $\mathbf{1}$).

$$\rho_1=e^{i\frac{2\pi}{3}}$$



$$\rho_0=e^{i0}=1$$

$$\rho_2=e^{-i\frac{2\pi}{3}}$$

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\,\mathbf{P}^{(0)} + \rho_1\,\mathbf{P}^{(1)} + \rho_2\,\mathbf{P}^{(2)}$$

$$\mathbf{r}^2=(\rho_0)^2\mathbf{P}^{(0)}+(\rho_1)^2\mathbf{P}^{(1)}+(\rho_2)^2\mathbf{P}^{(2)}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

C_3 Spectral resolution: 3rd roots of unity

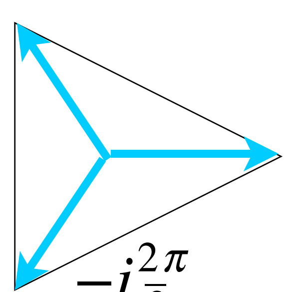
We can spectrally resolve $\mathbf{H}$ if we resolve $\mathbf{r}$ since is $\mathbf{H}$ a combination $r_p\mathbf{r}^p$ of powers $\mathbf{r}^p$.

$\mathbf{r}$-symmetry is cubic $\mathbf{r}^3=\mathbf{1}$, or $\mathbf{r}^3-\mathbf{1}=\mathbf{0}$ and resolves to factors of *3rd roots of unity* $\rho_m=e^{im2\pi/3}$.

$$\mathbf{1}=\mathbf{r}^3 \;\textit{implies:}\; \mathbf{0}=\mathbf{r}^3-\mathbf{1}=(\mathbf{r}-\rho_0\mathbf{1})(\mathbf{r}-\rho_1\mathbf{1})(\mathbf{r}-\rho_2\mathbf{1}) \;\textit{where:}\; \rho_m=e^{im\frac{2\pi}{3}}$$

Each eigenvalue ρ_m of $\mathbf{r}$, has idempotent projector $\mathbf{P}^{(m)}$ such that $\mathbf{r}\cdot\mathbf{P}^{(m)}=\rho_m\mathbf{P}^{(m)}$.
All three $\mathbf{P}^{(m)}$ are *orthonormal* ($\mathbf{P}^{(m)}\mathbf{P}^{(n)}=\delta_{mn}\mathbf{P}^{(m)}$) and *complete* (sum to unit $\mathbf{1}$).

$\rho_1=e^{i\frac{2\pi}{3}}$



$\rho_0=e^{i0}=1$

$\rho_2=e^{-i\frac{2\pi}{3}}$

$$\mathbf{1} = \mathbf{P}^{(0)} + \mathbf{P}^{(1)} + \mathbf{P}^{(2)}$$

$$\mathbf{r} = \rho_0\mathbf{P}^{(0)} + \rho_1\mathbf{P}^{(1)} + \rho_2\mathbf{P}^{(2)}$$

$$\mathbf{r}^2 = (\rho_0)^2\mathbf{P}^{(0)} + (\rho_1)^2\mathbf{P}^{(1)} + (\rho_2)^2\mathbf{P}^{(2)}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle(m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

$$\langle(0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad 1\quad 1\;)$$

$$\langle(1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{-i2\pi/3}\quad e^{+i2\pi/3})$$

$$\langle(2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(1\quad e^{+i2\pi/3}\quad e^{-i2\pi/3})$$

(m_3) means: *m-modulo-3* (Details follow)

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

➤ *Resolving C_3 projectors and moving wave modes*

Dispersion functions and standing waves

C_6 symmetric mode model:Distant neighbor coupling

C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

Phase arithmetic

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle (m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

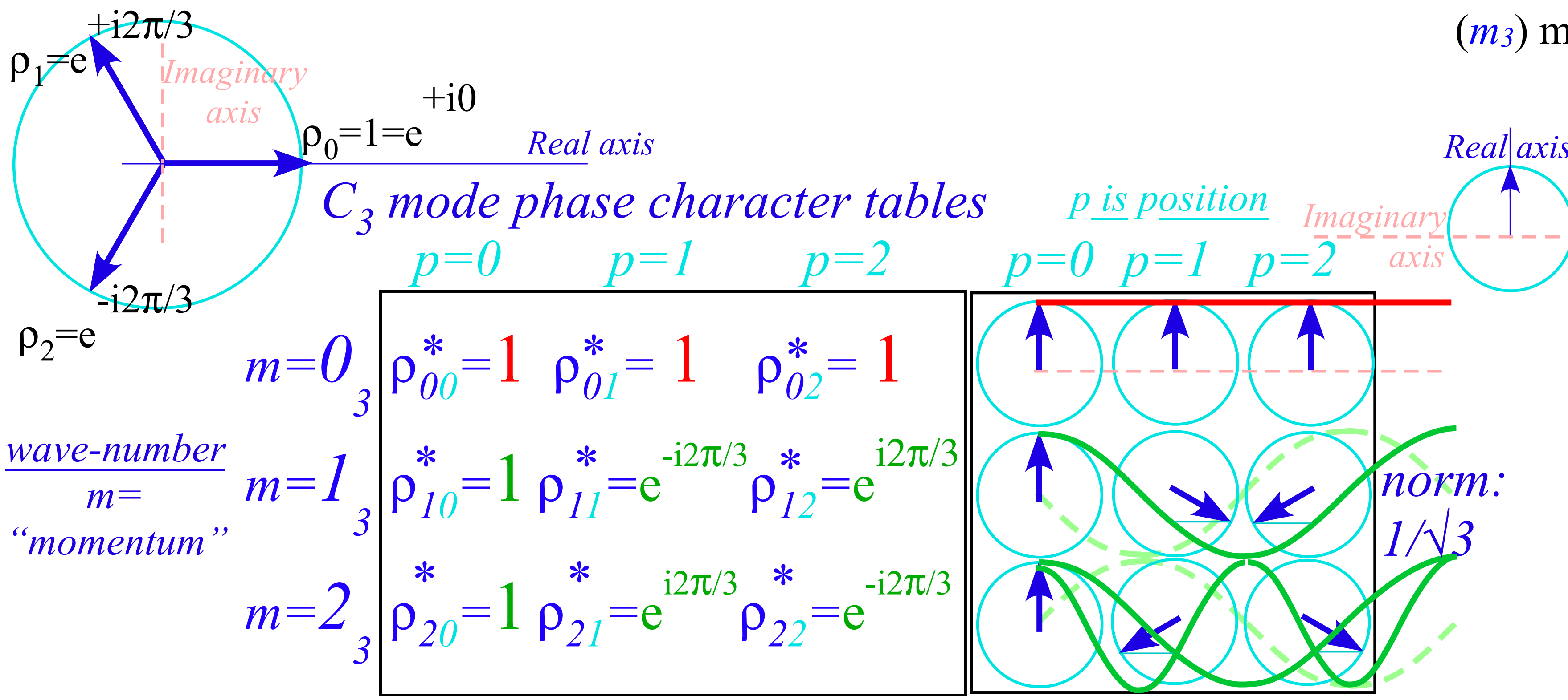
$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

$$\langle (0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad 1\quad 1\;)$$

$$\langle (1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{-i2\pi/3}\quad e^{+i2\pi/3}\;)$$

$$\langle (2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{+i2\pi/3}\quad e^{-i2\pi/3}\;)$$

(m_3) means: *m-modulo-3* (Details follow)



Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle (m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

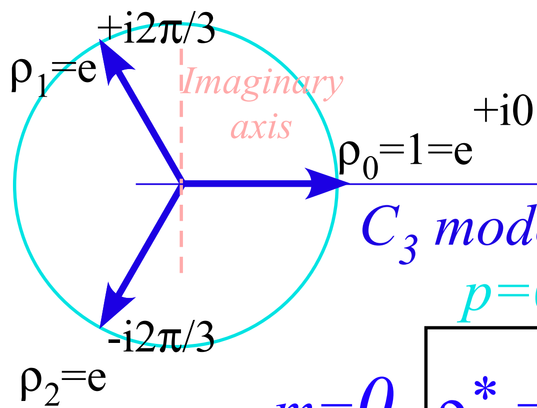
$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

$$\langle (0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad 1\quad 1\;)$$

$$\langle (1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{-i2\pi/3}\quad e^{+i2\pi/3}\;)$$

$$\langle (2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{+i2\pi/3}\quad e^{-i2\pi/3}\;)$$

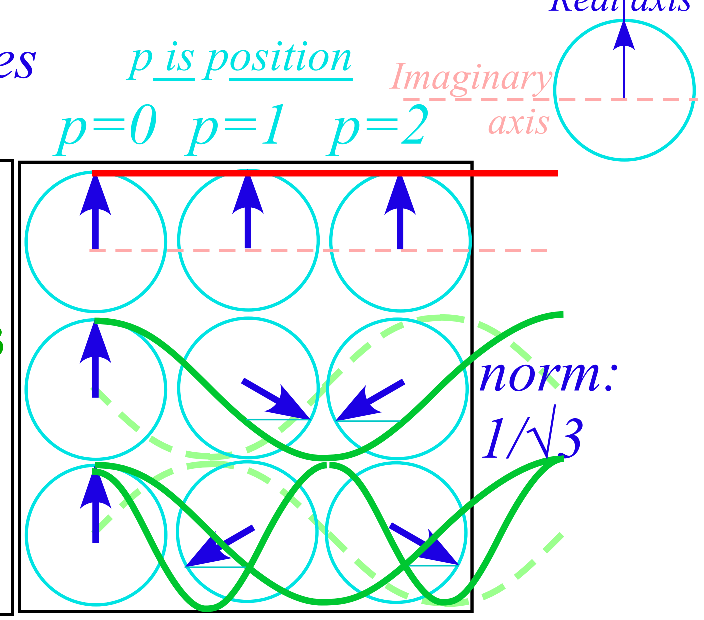
(m_3) means: *m-modulo-3* (Details follow)

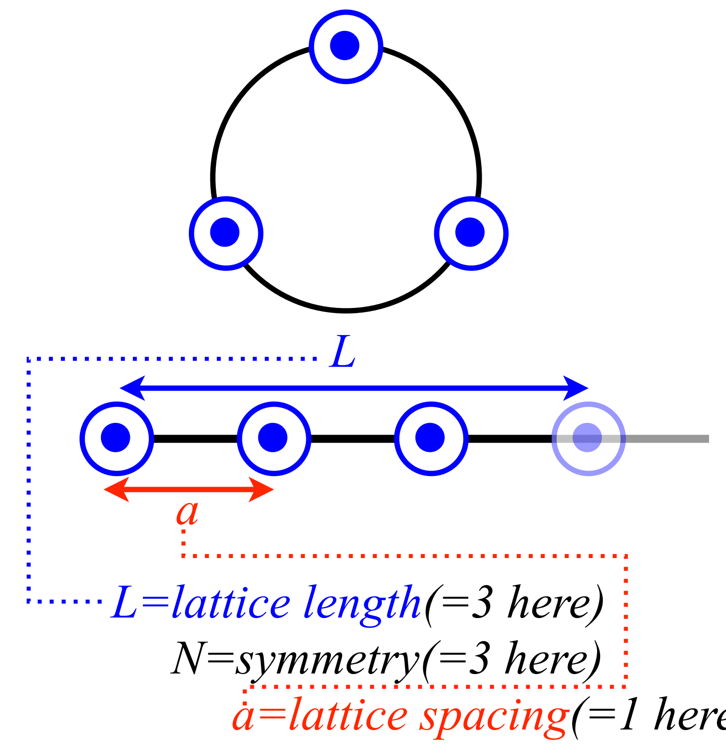


C_3 mode phase character tables

wave-number $m=$ "momentum"

	p=0	*p=1*	*p=2*
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
$m=1_3$	$\rho^*_{10}=1$	$\rho^*_{11}=e^{-i2\pi/3}$	$\rho^*_{12}=e^{i2\pi/3}$
$m=2_3$	$\rho^*_{20}=1$	$\rho^*_{21}=e^{i2\pi/3}$	$\rho^*_{22}=e^{-i2\pi/3}$





Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle (m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

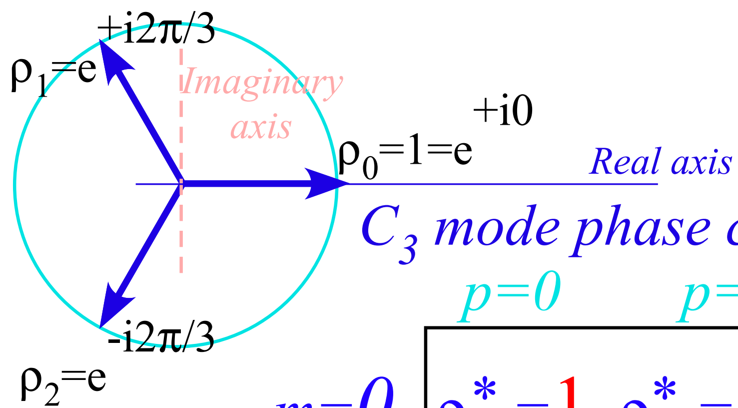
$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

$$\langle(0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad 1\quad 1\;)$$

$$\langle(1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{-i2\pi/3}\quad e^{+i2\pi/3})$$

$$\langle(2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{+i2\pi/3}\quad e^{-i2\pi/3})$$

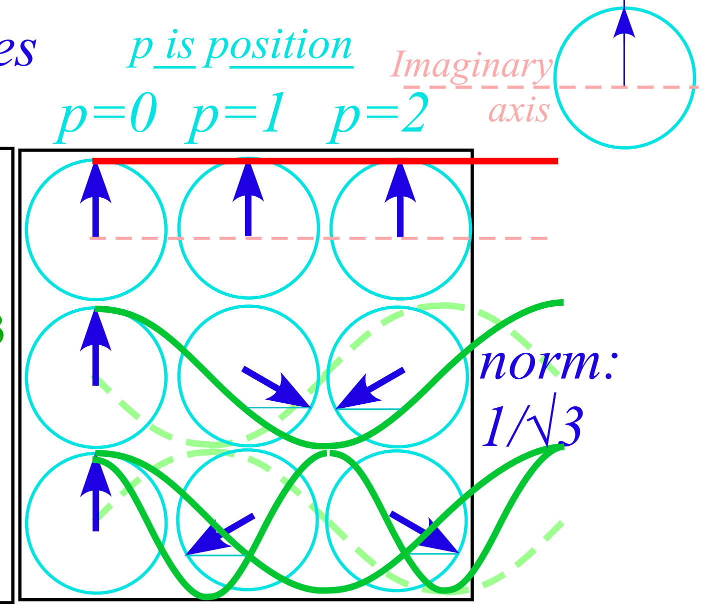
(m_3) means: *m-modulo-3* (Details follow)

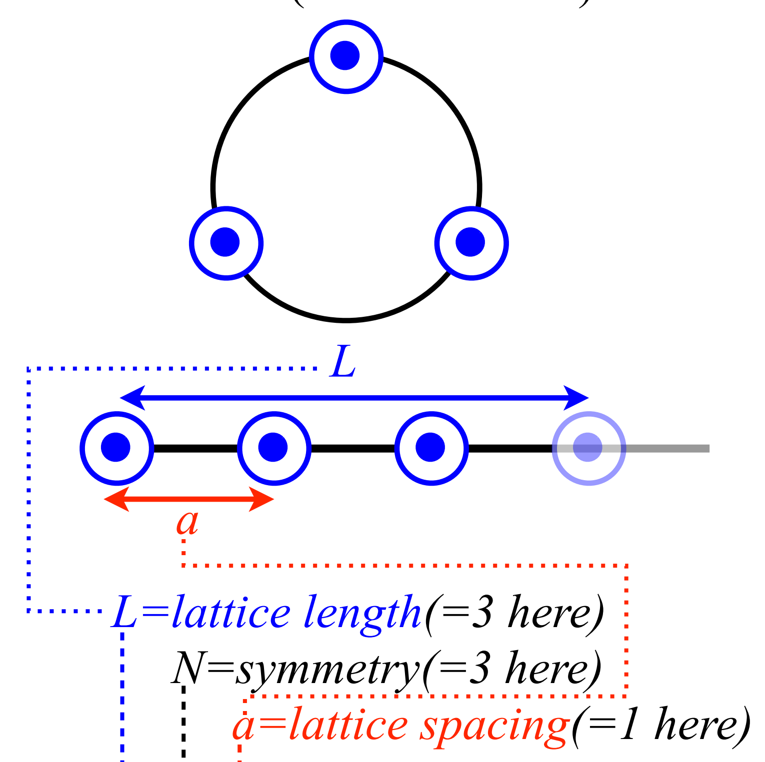


C_3 mode phase character tables

wave-number $m=$ "momentum"

	$p=0$	$p=1$	$p=2$
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
$m=1_3$	$\rho^*_{10}=1$	$\rho^*_{11}=e^{-i2\pi/3}$	$\rho^*_{12}=e^{i2\pi/3}$
$m=2_3$	$\rho^*_{20}=1$	$\rho^*_{21}=e^{i2\pi/3}$	$\rho^*_{22}=e^{-i2\pi/3}$





Two distinct types of "quantum" numbers.

$p=0,1,$or 2 is *power p* of operator $\mathbf{r}^p$ and defines each oscillator's *position point p*.

$m=0,1,$or 2 is *mode momentum m* of the waves or wavevector $k_m=2\pi/\lambda_m=2\pi m/L$. $(L=Na=3)$

wavelength $\lambda_m=2\pi/k_m=L/m$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle (m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

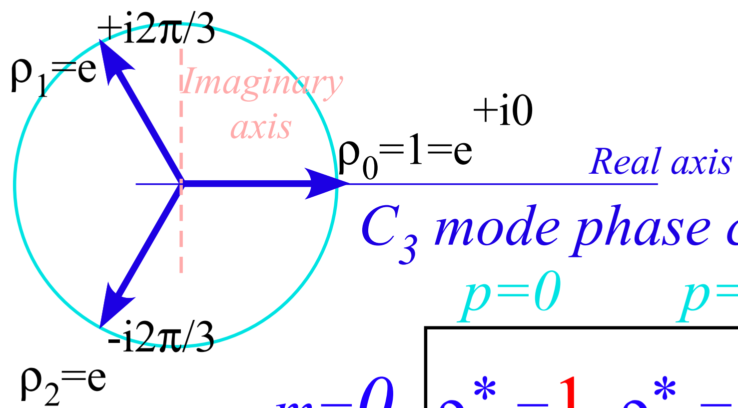
$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

$$\langle (0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad 1\quad 1\;)$$

$$\langle (1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{-i2\pi/3}\quad e^{+i2\pi/3}\;)$$

$$\langle (2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}(\;1\quad e^{+i2\pi/3}\quad e^{-i2\pi/3}\;)$$

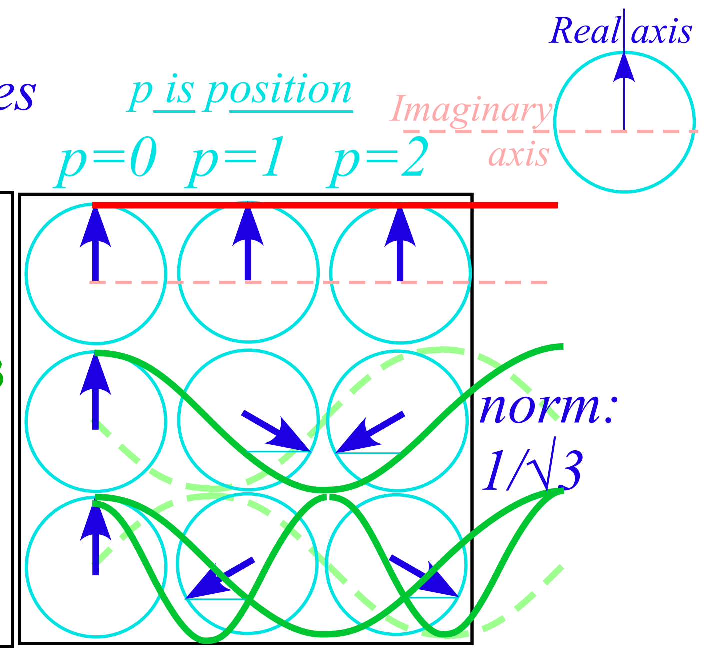
(m_3) means: *m-modulo-3* (Details follow)

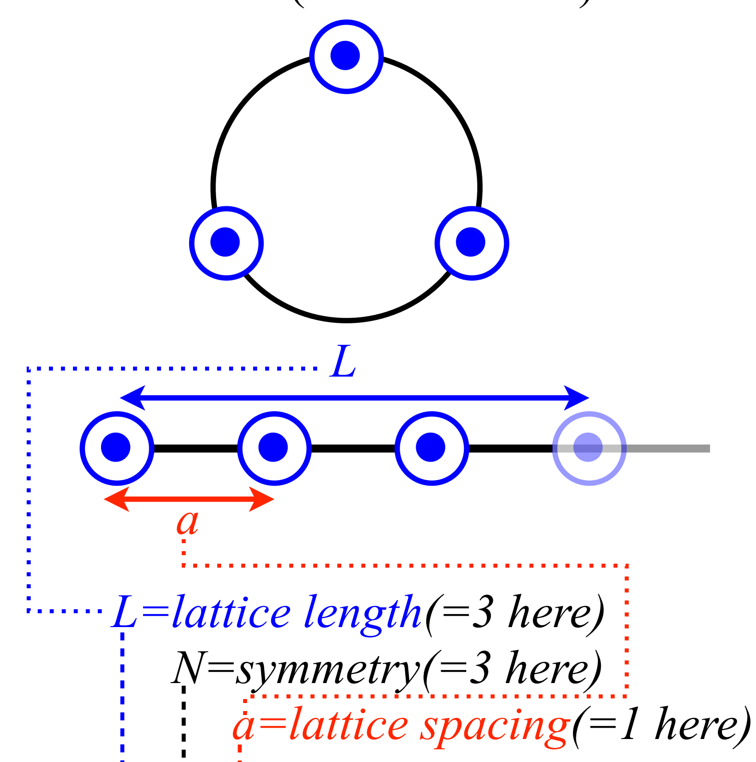


C_3 mode phase character tables

wave-number m= "momentum"

	$p=0$	$p=1$	$p=2$
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
$m=1_3$	$\rho^*_{10}=1$	$\rho^*_{11}=e^{-i2\pi/3}$	$\rho^*_{12}=e^{i2\pi/3}$
$m=2_3$	$\rho^*_{20}=1$	$\rho^*_{21}=e^{i2\pi/3}$	$\rho^*_{22}=e^{-i2\pi/3}$





Two distinct types of "quantum" numbers.

p=0,1,or 2 is *power p* of operator $\mathbf{r}^p$ and defines each oscillator's *position point p*.

m=0,1,or 2 is *mode momentum m* of the waves or wavevector $k_m=2\pi/\lambda_m=2\pi m/L$. ($L=Na=3$)

wavelength $\lambda_m=2\pi/k_m=L/m$

Each quantum number follows *modular arithmetic:* sums or products are an *integer-modulo-3*, that is, always *0*,*1*,or *2,* or else *-1*,*0*,or *1,* or else *-2*,*-1*,or *0, etc.,* depending on choice of origin.

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigen-bra-vectors $\langle (m)|$

$$\mathbf{P}^{(0)}=\tfrac{1}{3}(\mathbf{r}^0+\mathbf{r}^1+\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+\mathbf{r}^1+\mathbf{r}^2)$$

$$\mathbf{P}^{(1)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_1^*\mathbf{r}^1+\rho_2^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{-i2\pi/3}\mathbf{r}^1+e^{+i2\pi/3}\mathbf{r}^2)$$

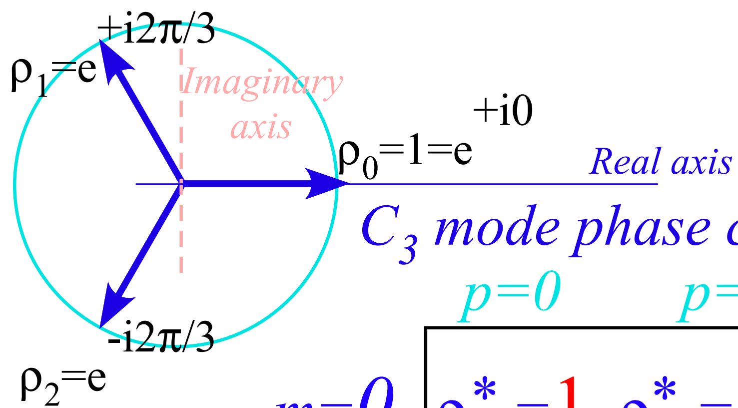
$$\mathbf{P}^{(2)}=\tfrac{1}{3}(\mathbf{r}^0+\rho_2^*\mathbf{r}^1+\rho_1^*\mathbf{r}^2)=\tfrac{1}{3}(\mathbf{1}+e^{+i2\pi/3}\mathbf{r}^1+e^{-i2\pi/3}\mathbf{r}^2)$$

$$\langle (0_3)|=\langle 0|\mathbf{P}^{(0)}\sqrt{3}=\sqrt{\tfrac{1}{3}}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$$

$$\langle (1_3)|=\langle 0|\mathbf{P}^{(1)}\sqrt{3}=\sqrt{\tfrac{1}{3}}\begin{pmatrix}1 & e^{-i2\pi/3} & e^{+i2\pi/3}\end{pmatrix}$$

$$\langle (2_3)|=\langle 0|\mathbf{P}^{(2)}\sqrt{3}=\sqrt{\tfrac{1}{3}}\begin{pmatrix}1 & e^{+i2\pi/3} & e^{-i2\pi/3}\end{pmatrix}$$

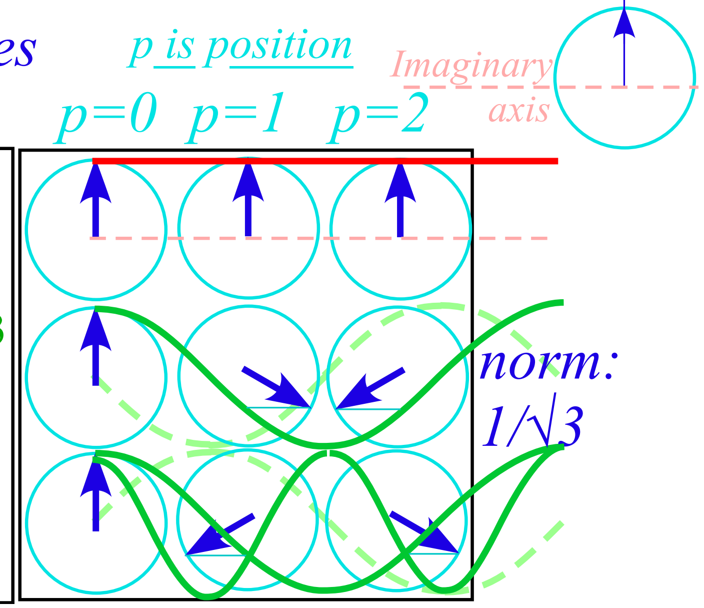
(m_3) means: *m-modulo-3* (Details follow)

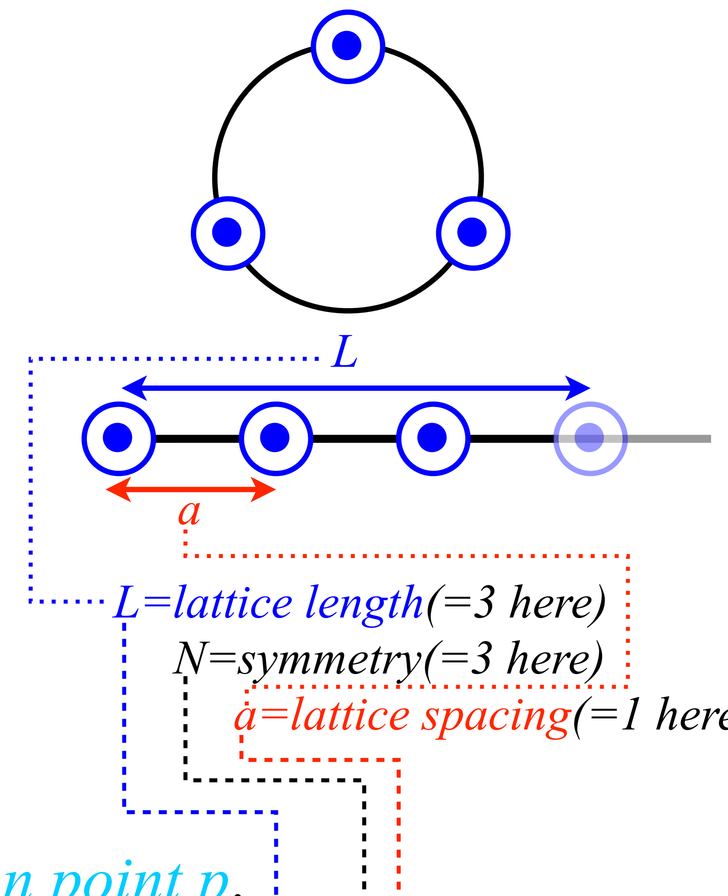


C_3 mode phase character tables

	p=0	*p=1*	*p=2*
$m=0_3$	$\rho^*_{00}=1$	$\rho^*_{01}=1$	$\rho^*_{02}=1$
$m=1_3$	$\rho^*_{10}=1$	$\rho^*_{11}=e^{-i2\pi/3}$	$\rho^*_{12}=e^{i2\pi/3}$
$m=2_3$	$\rho^*_{20}=1$	$\rho^*_{21}=e^{i2\pi/3}$	$\rho^*_{22}=e^{-i2\pi/3}$

wave-number m= "momentum"





Two distinct types of "quantum" numbers.

*p=0,1,*or *2* is *power p* of operator $\mathbf{r}^p$ and defines each oscillator's *position point p*.

*m=0,1,*or *2* is *mode momentum m* of the waves or wavevector $k_m=2\pi/\lambda_m=2\pi m/L$. $(L=Na=3)$

wavelength $\lambda_m=2\pi/k_m=L/m$

Each quantum number follows *modular arithmetic:* sums or products are an *integer-modulo-3*, that is, always *0,1,*or *2,* or else *-1,0,*or *1,* or else *-2,-1,*or *0, etc.,* depending on choice of origin.

For example, for $m=2$ and $p=2$ the number $(\rho_m)^p=(e^{im2\pi/3})^p$ is $e^{imp\cdot 2\pi/3}=e^{i4\cdot 2\pi/3}=e^{i1\cdot 2\pi/3}\,e^{i2\pi}=e^{i2\pi/3}=\rho_1$.

That is, (*2-times-2*) mod *3* is not *4* but *1* (*4* mod *3=1*, the remainder of *4* divided by *3*.)

Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Resolving C_3 projectors and moving wave modes*
  - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
  - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
- *C_N symmetric mode models: Made-to order dispersion functions*
  - *Quadratic dispersion models: Super-beats and fractional revivals*
  - *Phase arithmetic*

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{i\,m\cdot p\,2\pi/3}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} Eigenvalue of $\mathbf{r}^p$

$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{i\,m\cdot p\,2\pi/3}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{i\,m\cdot p\,2\pi/3}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

$$\boxed{\begin{array}{l} m^{th} \textit{ Eigenvalue of } \mathbf{r}^p \\ \langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3} \end{array}}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\frac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\, m\cdot 0\, \frac{2\pi}{3}} + r_1 e^{i\, m\cdot 1\, \frac{2\pi}{3}} + r_2 e^{i\, m\cdot 2\, \frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\ i\, m\cdot p\ 2\pi/3}$

$$= r_0 e^{i\, m\cdot 0\, \frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\frac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

$$\boxed{\begin{array}{l} m^{th}\ \textit{Eigenvalue of } \mathbf{r}^p \\ \langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3} \end{array}}$$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\frac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\frac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}$ – *eigenfrequencies*	$\mathbf{K}$ – *eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{2m\pi}{3}\right) = r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)} = \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos\left(\frac{-2m\pi}{3}\right) = r_0 - r$	$\sqrt{k_0 - 2k\cos\left(\frac{2m\pi}{3}\right)} = \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\frac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m=0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

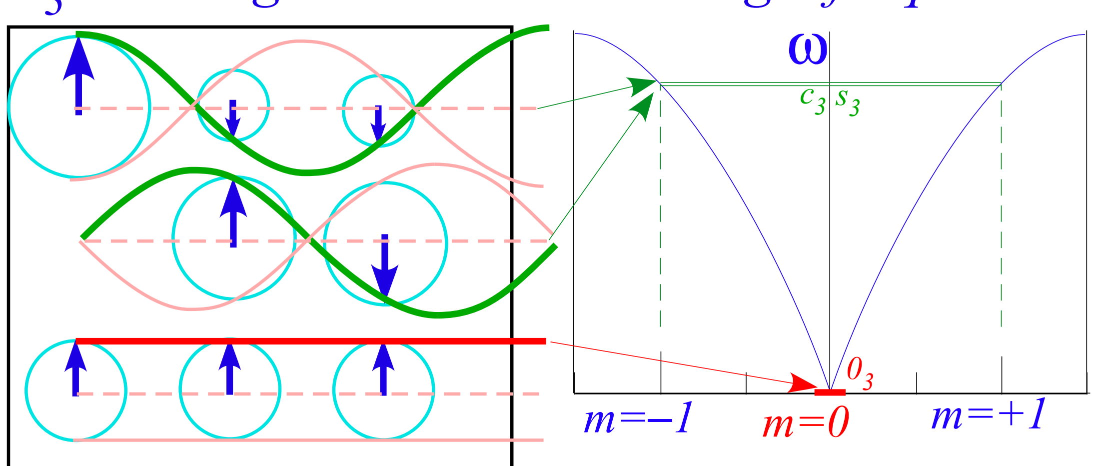
K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

Moving eigenwave	*Standing eigenwaves*	**H** – *eigenfrequencies*	**K** – *eigenfrequencies*
$\|(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\|c_3\rangle = \frac{\|(+1)_3\rangle + \|(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\|(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\|s_3\rangle = \frac{\|(+1)_3\rangle - \|(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\|(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

	$p=0$	$p=1$	$p=2$
c_3	$2/\sqrt{6}$	$-1/\sqrt{6}$	$-1/\sqrt{6}$
s_3	0	$1/\sqrt{2}$	$-1/\sqrt{2}$
$m=0_3$	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$

C_3 standing wave modes and eigenfrequencies of **K**



Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\,m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\,m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$\langle m|\ \mathbf{r}^p\ |m\rangle = e^{\,i\,m\cdot p\,2\pi/3}$

$$= r_0 e^{i\,m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\frac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

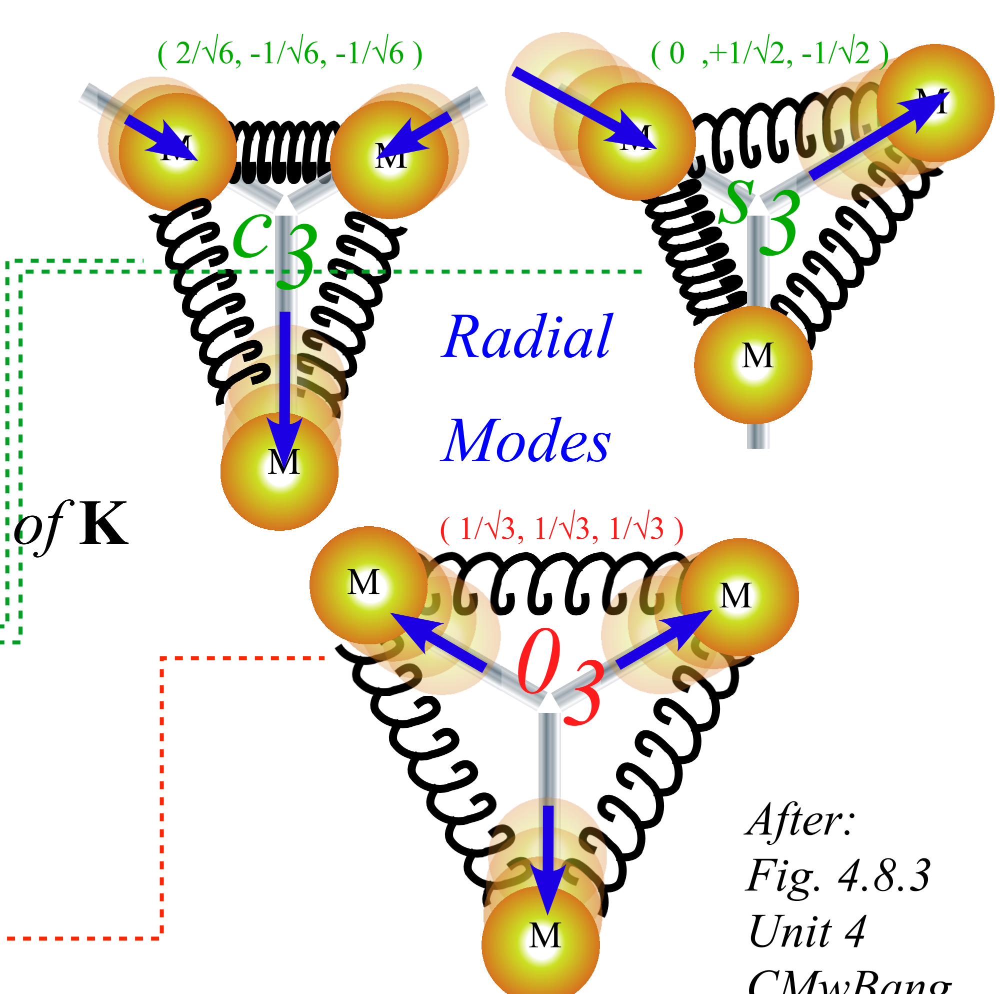
H-eigenvalues:

$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\frac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

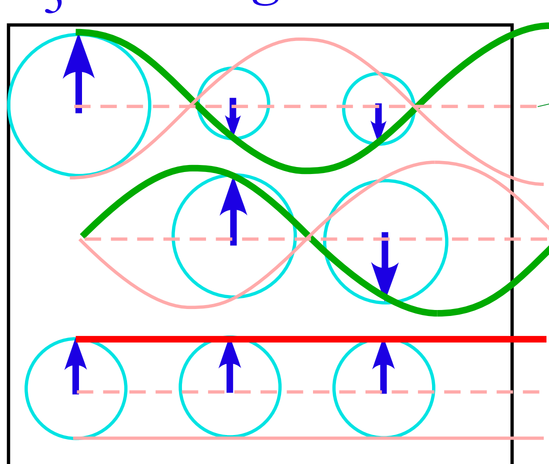
$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\frac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

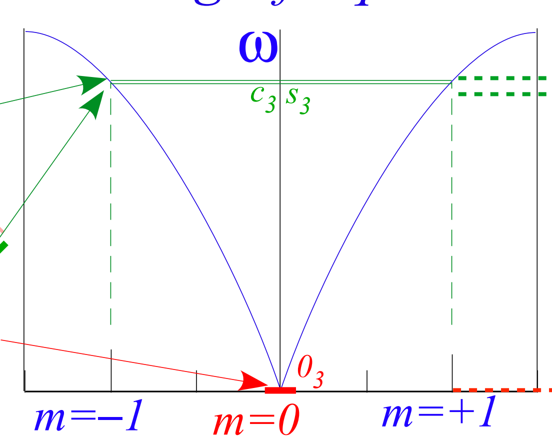
Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}-$*eigenfrequencies*	$\mathbf{K}-$*eigenfrequencies*
$\vert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\vert c_3\rangle = \frac{\vert(+1)_3\rangle + \vert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
$\vert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\vert s_3\rangle = \frac{\vert(+1)_3\rangle - \vert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3})$ $= r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})}$ $= \sqrt{k_0 + k}$
	$\vert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$



C_3 standing wave modes and eigenfrequencies of **K**

	$p=0$	$p=1$	$p=2$
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3





After: Fig. 4.8.3 Unit 4 CMwBang

Easy to resolve spectral projectors $\mathbf{P}^{(m)}$ and eigenvalues ω_m or dispersion functions $\omega(k_m)$

$$\langle m|\mathbf{H}|m\rangle = \langle m|r_0\mathbf{r}^0 + r_1\mathbf{r}^1 + r_2\mathbf{r}^2|m\rangle = r_0 e^{i\, m\cdot 0\,\frac{2\pi}{3}} + r_1 e^{i\, m\cdot 1\,\frac{2\pi}{3}} + r_2 e^{i\, m\cdot 2\,\frac{2\pi}{3}}$$

m^{th} *Eigenvalue of* $\mathbf{r}^p$

$$\langle m|\,\mathbf{r}^p\,|m\rangle = e^{\,i\, m\cdot p\; 2\pi/3}$$

$$= r_0 e^{i\, m\cdot 0\,\frac{2\pi}{3}} + r\left(e^{i\frac{2\pi m}{3}} + e^{-i\frac{2\pi m}{3}}\right) = r_0 + 2r\cos\left(\tfrac{2\pi m}{3}\right) = \begin{cases} r_0 + 2r & (\text{for } m = 0) \\ r_0 - r & (\text{for } m = \pm 1) \end{cases}$$

H-eigenvalues:

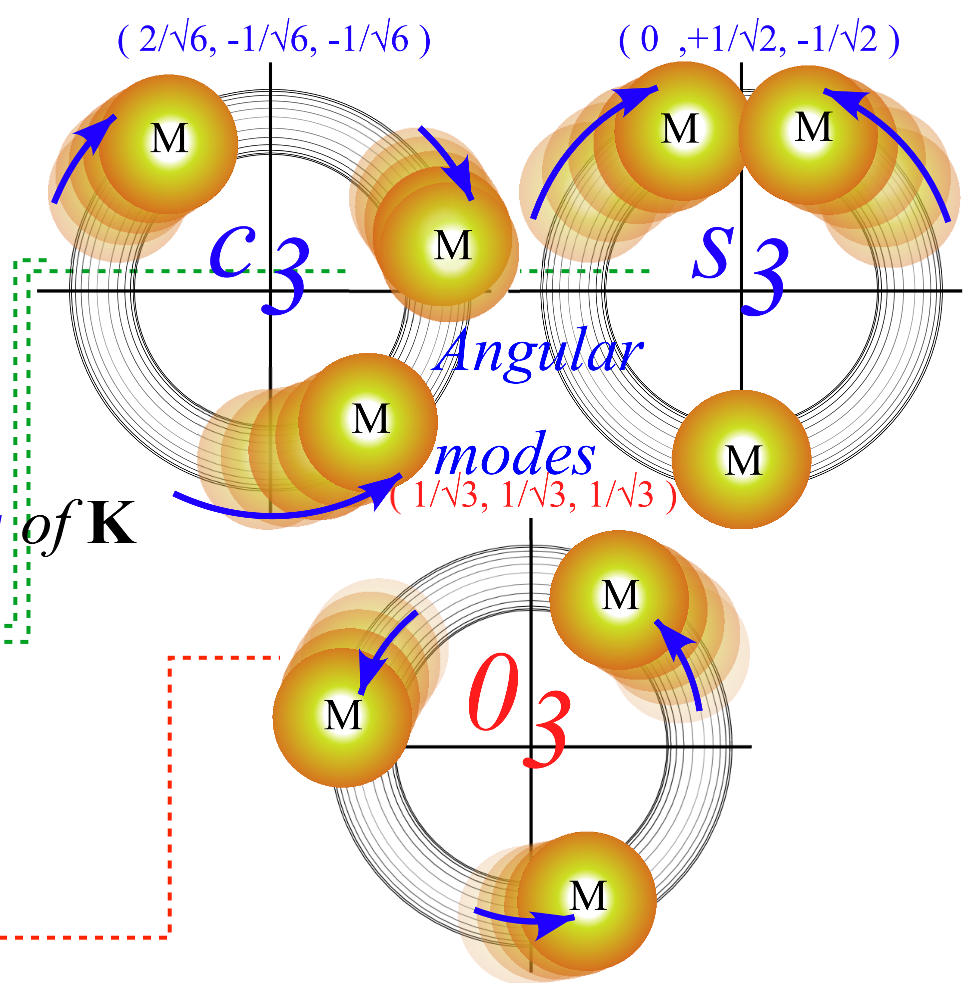
$$\begin{pmatrix} r_0 & r & r \\ r & r_0 & r \\ r & r & r_0 \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(r_0 + 2r\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

K-eigenvalues:

$$\begin{pmatrix} K & -k & -k \\ -k & K & -k \\ -k & -k & K \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix} = \left(K - 2k\cos\left(\tfrac{2m\pi}{3}\right)\right)\begin{pmatrix} 1 \\ e^{i\frac{2m\pi}{3}} \\ e^{-i\frac{2m\pi}{3}} \end{pmatrix}$$

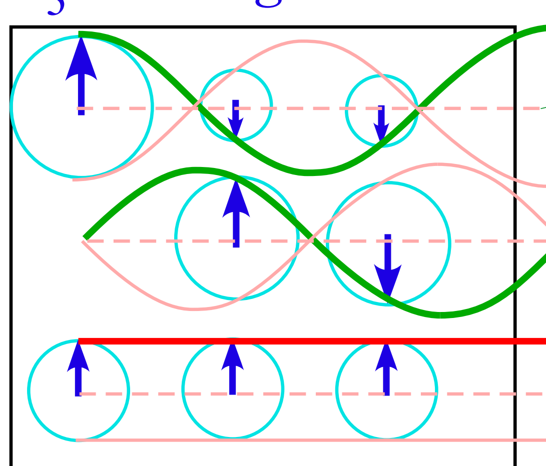
Moving eigenwave	*Standing eigenwaves*	$\mathbf{H}-$*eigenfrequencies*	$\mathbf{K}-$*eigenfrequencies*
$\lvert(+1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i2\pi/3} \\ e^{-i2\pi/3} \end{pmatrix}$	$\lvert c_3\rangle = \frac{\lvert(+1)_3\rangle + \lvert(-1)_3\rangle}{\sqrt{2}} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{2m\pi}{3}) = r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})} = \sqrt{k_0 + k}$
$\lvert(-1)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i2\pi/3} \\ e^{+i2\pi/3} \end{pmatrix}$	$\lvert s_3\rangle = \frac{\lvert(+1)_3\rangle - \lvert(-1)_3\rangle}{i\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$	$r_0 + 2r\cos(\frac{-2m\pi}{3}) = r_0 - r$	$\sqrt{k_0 - 2k\cos(\frac{2m\pi}{3})} = \sqrt{k_0 + k}$
	$\lvert(0)_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$r_0 + 2r$	$\sqrt{k_0 - 2k}$

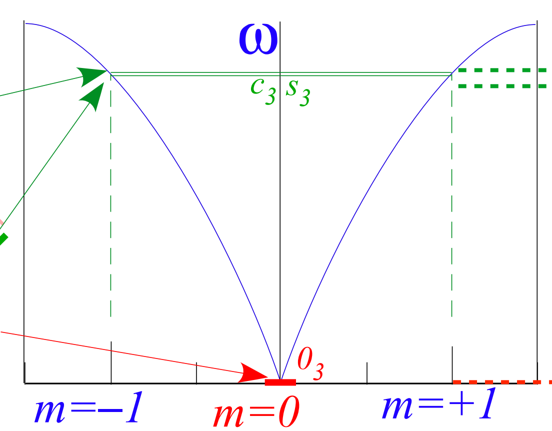
Longitudinal (to k) Waves



C_3 standing wave modes and eigenfrequencies of **K**

	p=0	*p=1*	*p=2*
c_3	2/√6	-1/√6	-1/√6
s_3	0	1/√2	-1/√2
$m=0_3$	1/√3	1/√3	1/√3





Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Resolving C_3 projectors and moving wave modes

Dispersion functions and standing waves

➔ *C_6 symmetric mode model:Distant neighbor coupling*

C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

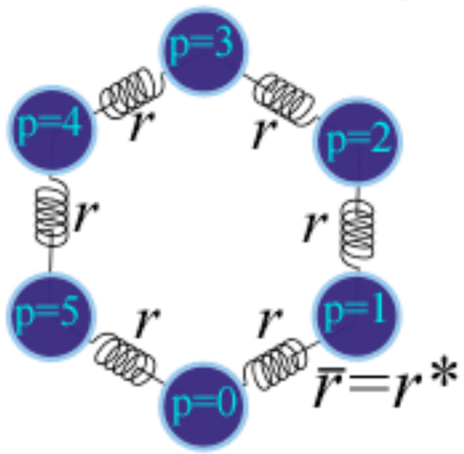
C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

Phase arithmetic

C_6 Symmetric Mode Model: Distant neighbor coupling

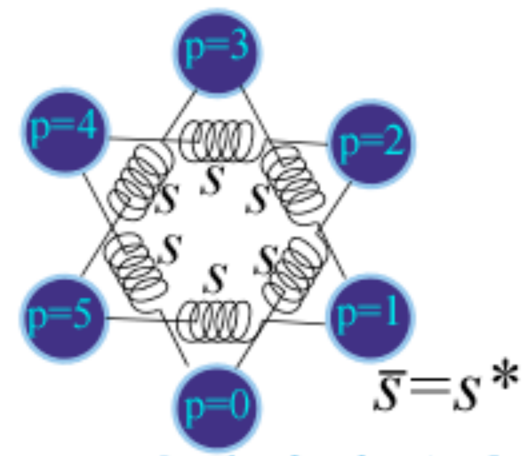
(a) 1^{st} *Neighbor* C_6



$$\mathbf{H}^{B1(6)}=\begin{array}{c}\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5\end{array}\\ \left(\begin{array}{cccccc} H_1 & -r & \cdot & \cdot & \cdot & -\bar{r}\\ -\bar{r} & H_1 & -r & \cdot & \cdot & \cdot\\ \cdot & -\bar{r} & H_1 & -r & \cdot & \cdot\\ \cdot & \cdot & -\bar{r} & H_1 & -r & \cdot\\ \cdot & \cdot & \cdot & -\bar{r} & H_1 & -r\\ -r & \cdot & \cdot & \cdot & -\bar{r} & H_1 \end{array}\right)\end{array}\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\end{array}$$

$$= H_1\mathbf{1} - r\mathbf{r} - \bar{r}\mathbf{r}^{-1}$$

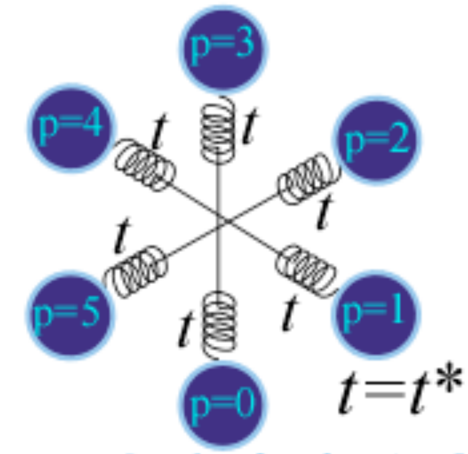
(b) 2^{nd} *Neighbor* C_6



$$\mathbf{H}^{B2(6)}=\begin{array}{c}\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5\end{array}\\ \left(\begin{array}{cccccc} H_2 & \cdot & -s & \cdot & -\bar{s} & \cdot\\ \cdot & H_2 & \cdot & -s & \cdot & -\bar{s}\\ -\bar{s} & \cdot & H_2 & \cdot & -s & \cdot\\ \cdot & -\bar{s} & \cdot & H_2 & \cdot & -s\\ -s & \cdot & -\bar{s} & \cdot & H_2 & \cdot\\ \cdot & -s & \cdot & -\bar{s} & \cdot & H_2 \end{array}\right)\end{array}\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\end{array}$$

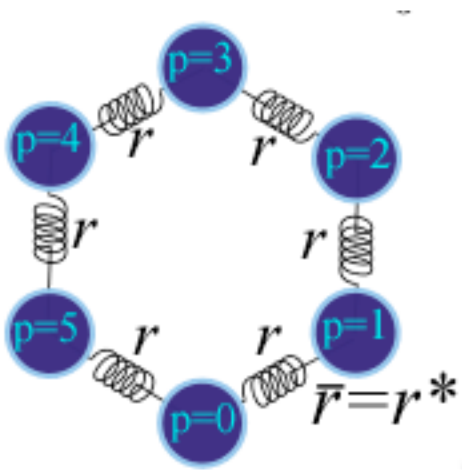
$$= H_2\mathbf{1} - s\mathbf{r}^2 - \bar{s}\mathbf{r}^{-2}$$

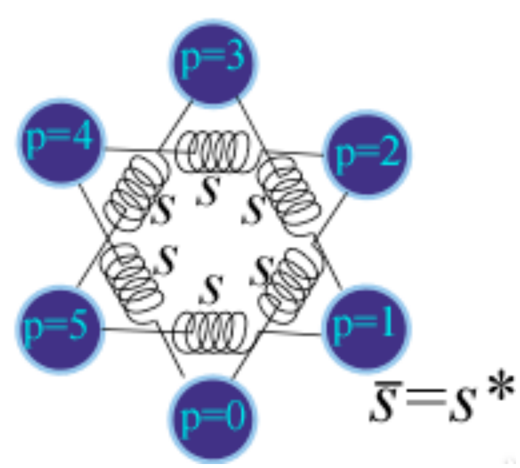
(c) 3^{rd} *Neighbor* C_6



$$\mathbf{H}^{B3(6)}=\begin{array}{c}\begin{array}{cccccc} p=0 & 1 & 2 & 3 & 4 & 5\end{array}\\ \left(\begin{array}{cccccc} H_3 & \cdot & \cdot & -t & \cdot & \cdot\\ \cdot & H_3 & \cdot & \cdot & -t & \cdot\\ \cdot & \cdot & H_3 & \cdot & \cdot & -t\\ -t & \cdot & \cdot & H_3 & \cdot & \cdot\\ \cdot & -t & \cdot & \cdot & H_3 & \cdot\\ \cdot & \cdot & -t & \cdot & \cdot & H_3 \end{array}\right)\end{array}\begin{array}{c} \\ 0\\1\\2\\3\\4\\5\end{array}$$

$$= H_3\mathbf{1} - t\mathbf{r}^3 - t\mathbf{r}^{-3}$$





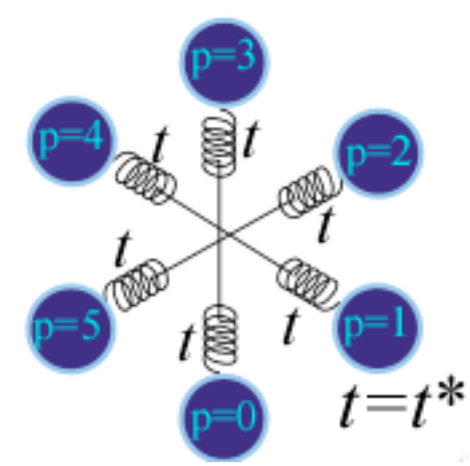
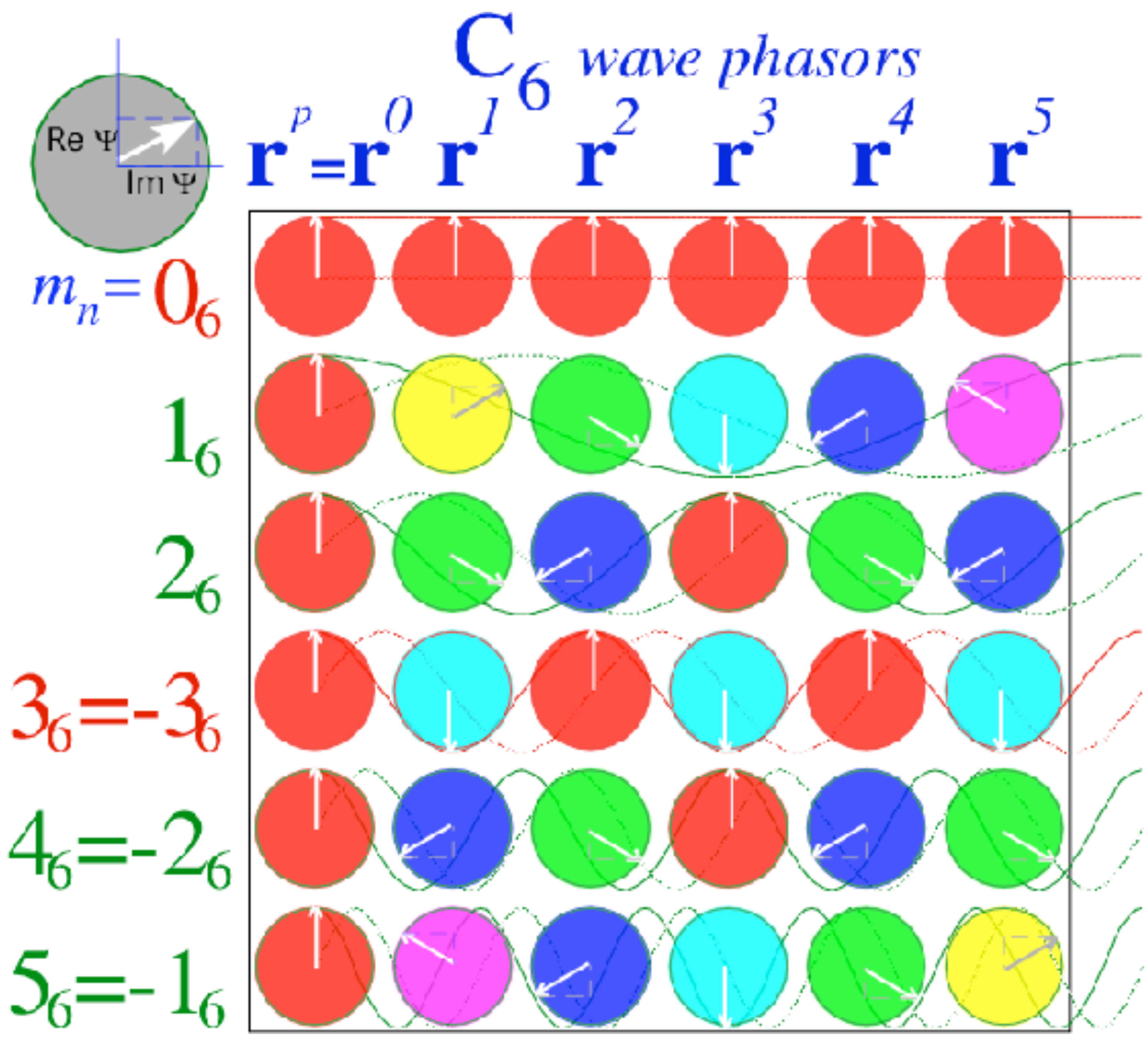


Fig. 12 International Journal of Molecular Science 14, 749 (2013)

C_6 Spectral resolution: 6^{th} roots of unity

$\chi_p^{m*}(C_6)$	$\mathbf{r}^{p=0}$	$\mathbf{r}^1$	$\mathbf{r}^2$	$\mathbf{r}^3$	$\mathbf{r}^4$	$\mathbf{r}^5$
$m=0_6$	1	1	1	1	1	1
1_6	1	ε^*	ε^{2*}	1	ε^2	ε
2_6	1	ε^{2*}	ε^2	1	ε^{2*}	ε^2
$3_6=-3_6$	1	-1	1	-1	1	-1
$4_6=\ 2_6$	1	ε^2	ε^{2*}	1	ε^2	ε^{2*}
$5_6=-1_6$	1	ε	ε^2	-1	ε^{2*}	ε^*

$$\textit{Wavefunction}: \ \Psi^m(x_p)=\chi_p^{m*}=D^{m*}(\mathbf{r}^p)$$



$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{6}}$$

Fig. 13 International Journal of Molecular Science 14, 752 (2013)

C_6 Spectral resolution of n^{th} Neighbor H: Same modes but different dispersion

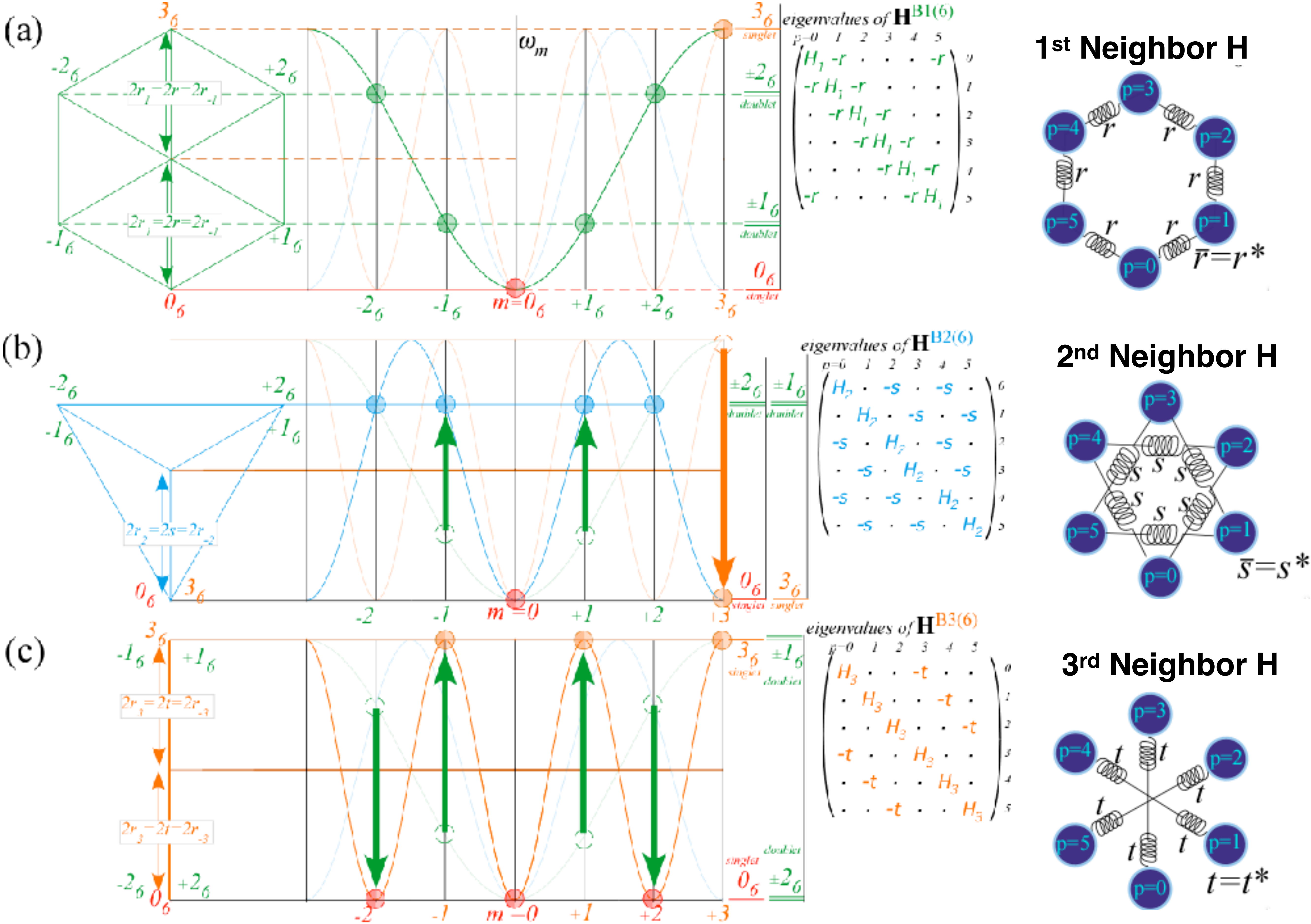
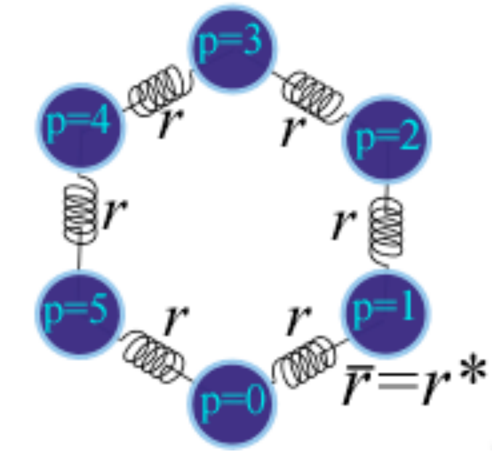


Fig. 14 International Journal of Molecular Science 14, 754 (2013)

Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
  - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
  - *C_3 symmetric spectral decomposition by 3rd roots of unity*
  - *Resolving C_3 projectors and moving wave modes*
  - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
  - ➔ *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
- *C_N symmetric mode models: Made-to order dispersion functions*
  - *Quadratic dispersion models: Super-beats and fractional revivals*
  - *Phase arithmetic*

C_6 Spectra of 1st neighbor gauge splitting by C-type (Chiral, Coriolis,…,



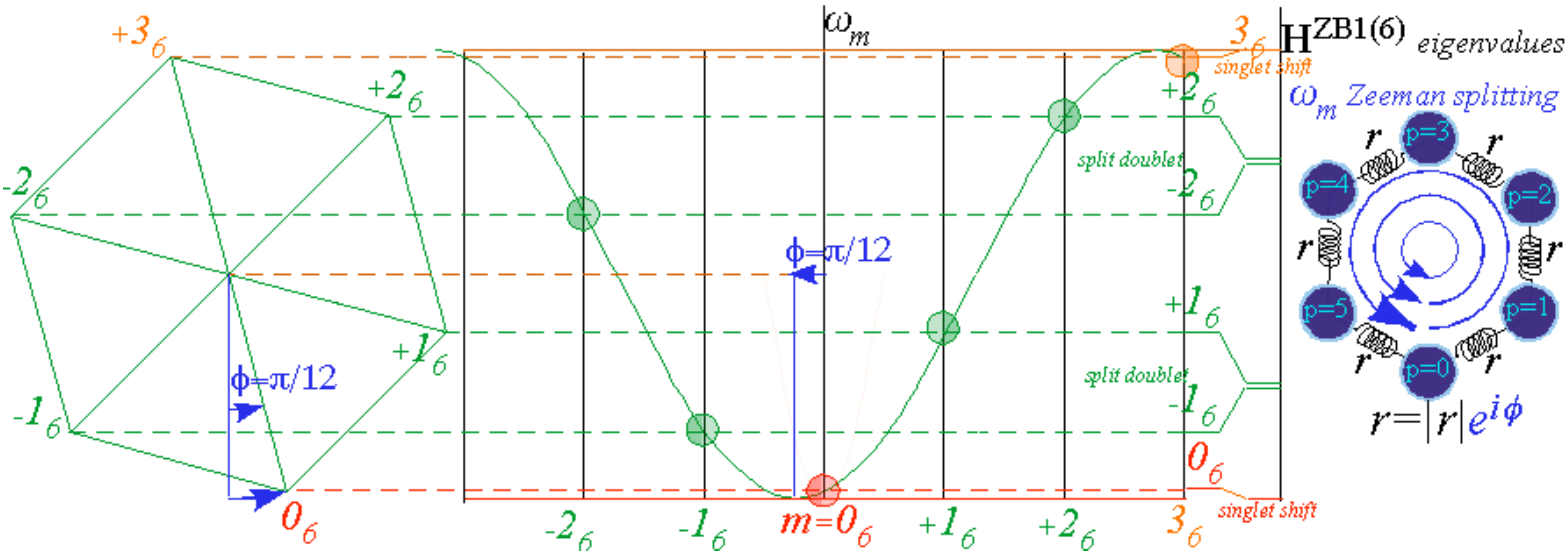


Fig. 15 International Journal of Molecular Science 14, 755 (2013)

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Resolving C_3 projectors and moving wave modes

Dispersion functions and standing waves

C_6 symmetric mode model:Distant neighbor coupling

C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

C_N symmetric mode models: Made-to order dispersion functions

Quadratic dispersion models: Super-beats and fractional revivals

Phase arithmetic

C_N Symmetric Mode Models:

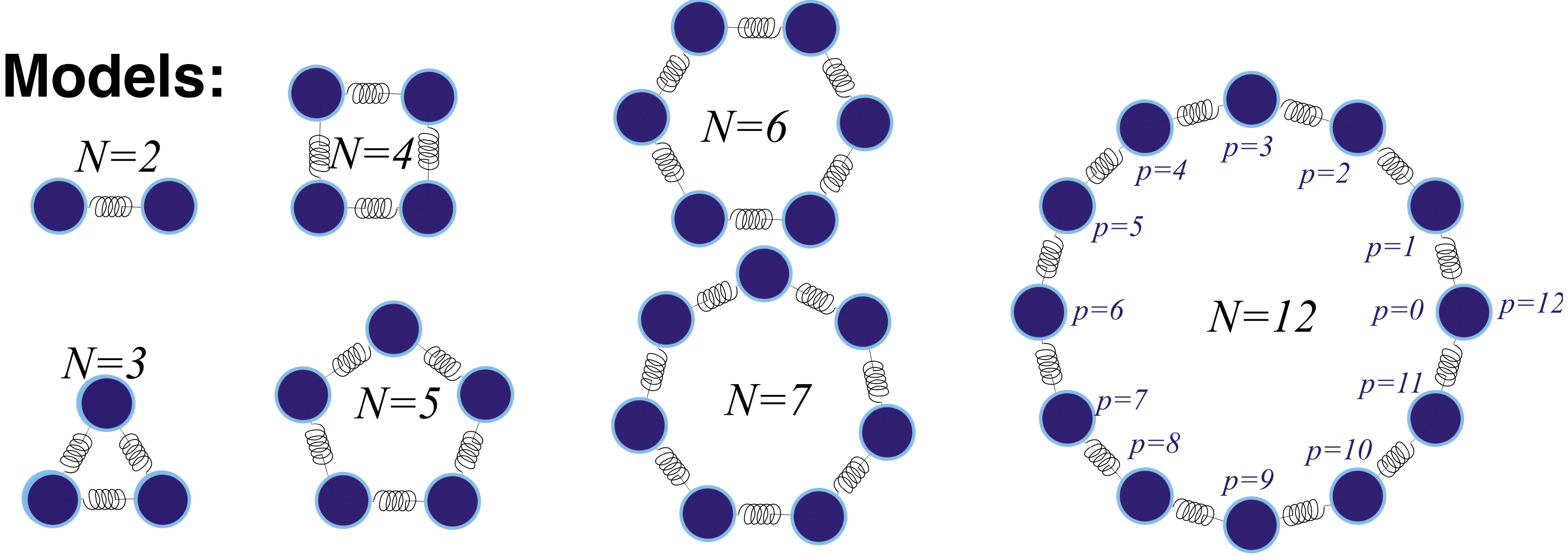


Fig. 4.8.4
Unit 4
CMwBang

C_N Symmetric Mode Models:

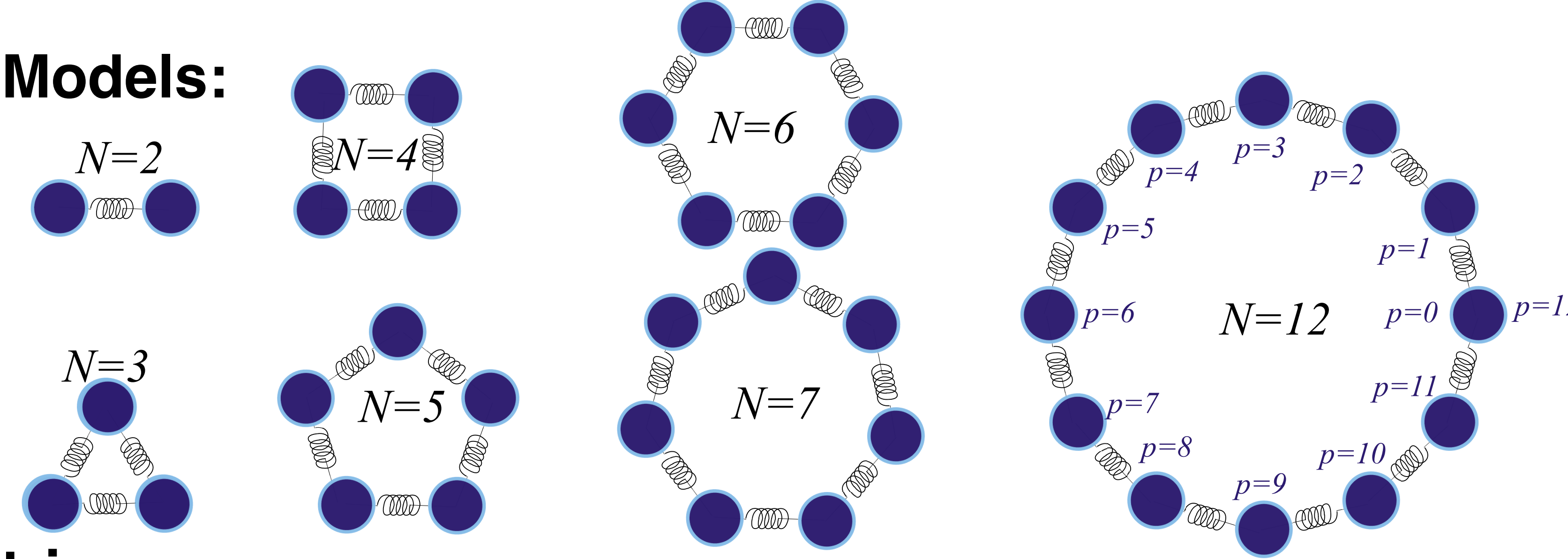


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix} = \begin{pmatrix} K & -k_{12} & . & . & . & \cdots & -k_{12} \\ -k_{12} & K & -k_{12} & . & . & \cdots & . \\ . & -k_{12} & K & -k_{12} & . & \cdots & . \\ . & . & -k_{12} & K & -k_{12} & \cdots & . \\ . & . & . & -k_{12} & K & \cdots & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\ -k_{12} & . & . & . & . & -k_{12} & K \end{pmatrix} \bullet \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix}$$

where: $K = k + 2k_{12}$

$$k = \frac{Mg}{\ell}$$

$$(\cdot) = 0$$

C_N Symmetric Mode Models:

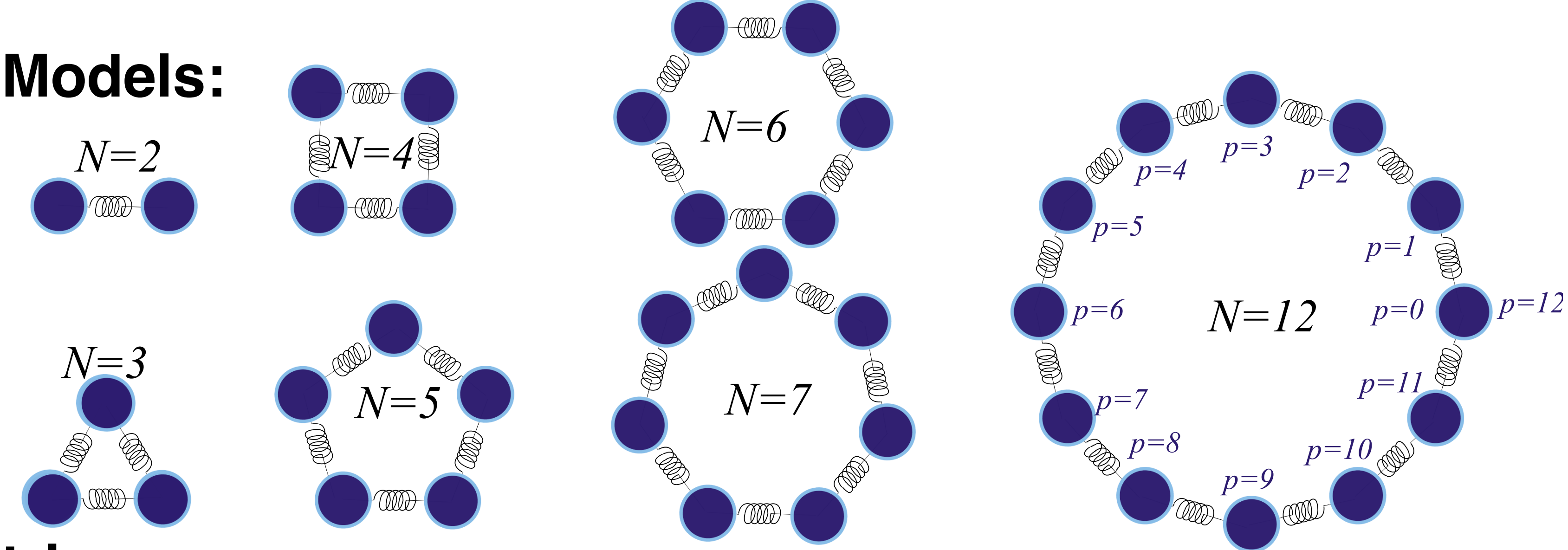


Fig. 4.8.4
Unit 4
CMwBang

1st Neighbor K-matrix

$$
-\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_{N-1} \end{pmatrix}
=
\begin{pmatrix}
K & -k_{12} & \cdot & \cdot & \cdot & \cdots & -k_{12} \\
-k_{12} & K & -k_{12} & \cdot & \cdot & \cdots & \cdot \\
\cdot & -k_{12} & K & -k_{12} & \cdot & \cdots & \cdot \\
\cdot & \cdot & -k_{12} & K & -k_{12} & \cdots & \cdot \\
\cdot & \cdot & \cdot & -k_{12} & K & \cdots & \cdot \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -k_{12} \\
-k_{12} & \cdot & \cdot & \cdot & \cdot & -k_{12} & K
\end{pmatrix}
\bullet
\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-1} \end{pmatrix}
$$

where: $K = k + 2k_{12}$, $k = \dfrac{Mg}{\ell}$, $(\cdot) = 0$

N^{th} roots of 1 *$e^{i\,m\cdot p\,2\pi/N} = \langle m|\,\mathbf{r}^p\,|m\rangle$ serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*

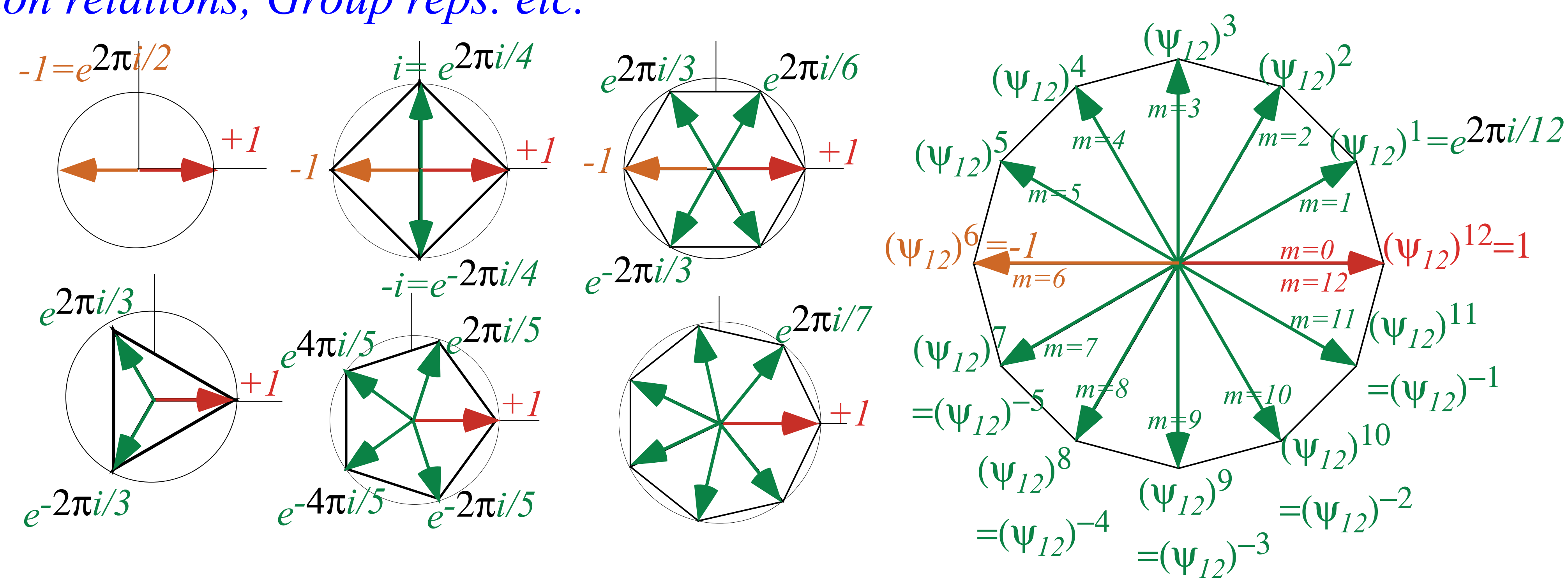
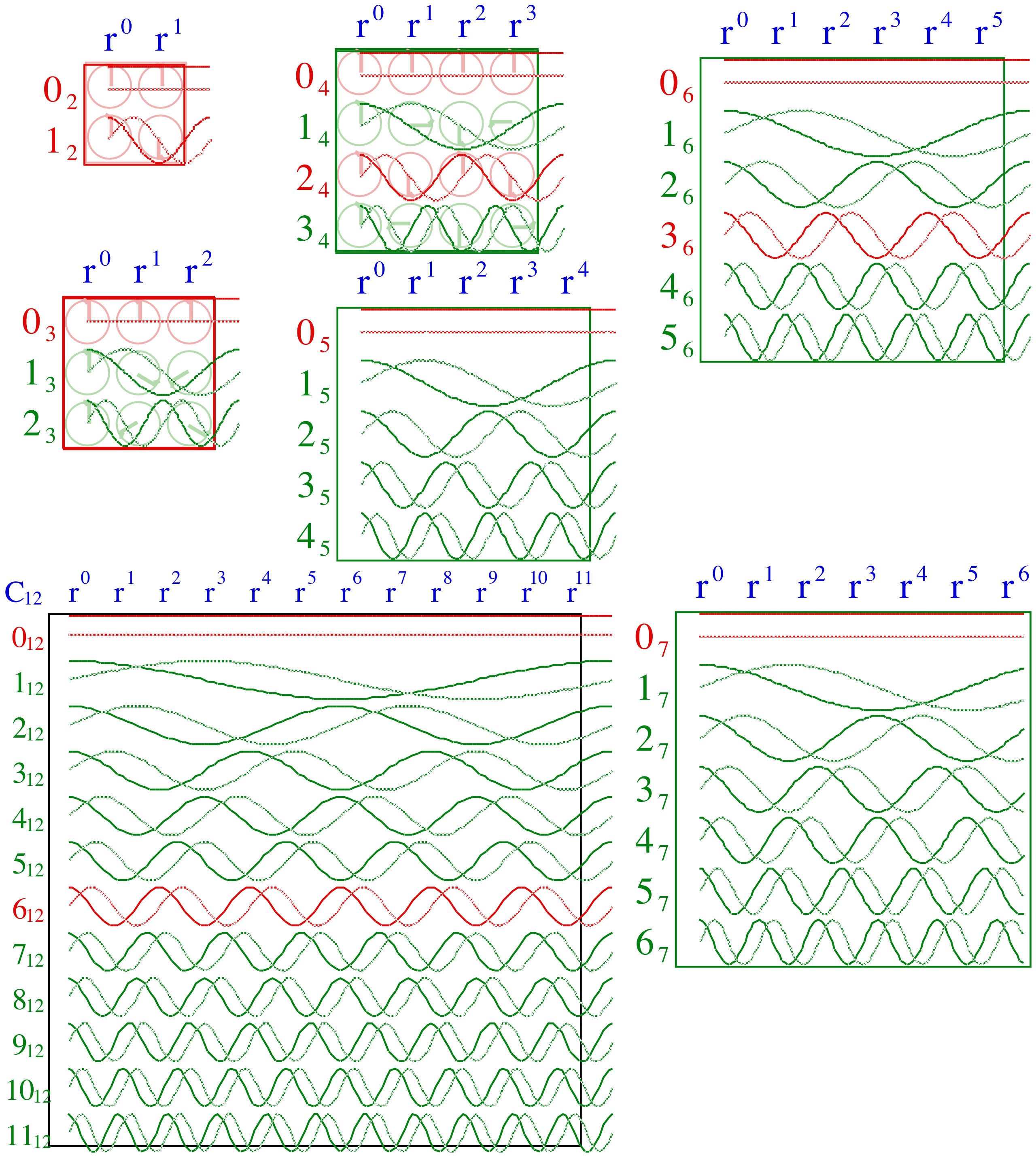


Fig. 4.8.5
Unit 4
CMwBang

C_N Symmetric Mode Models:

N^{th} roots of 1 *$e^{i\, m\cdot p\, 2\pi/N} = \langle m|\, \mathbf{r}^p\, |m\rangle$ serving as e-values, eigenfunctions, transformation matrices, dispersion relations, Group reps. etc.*



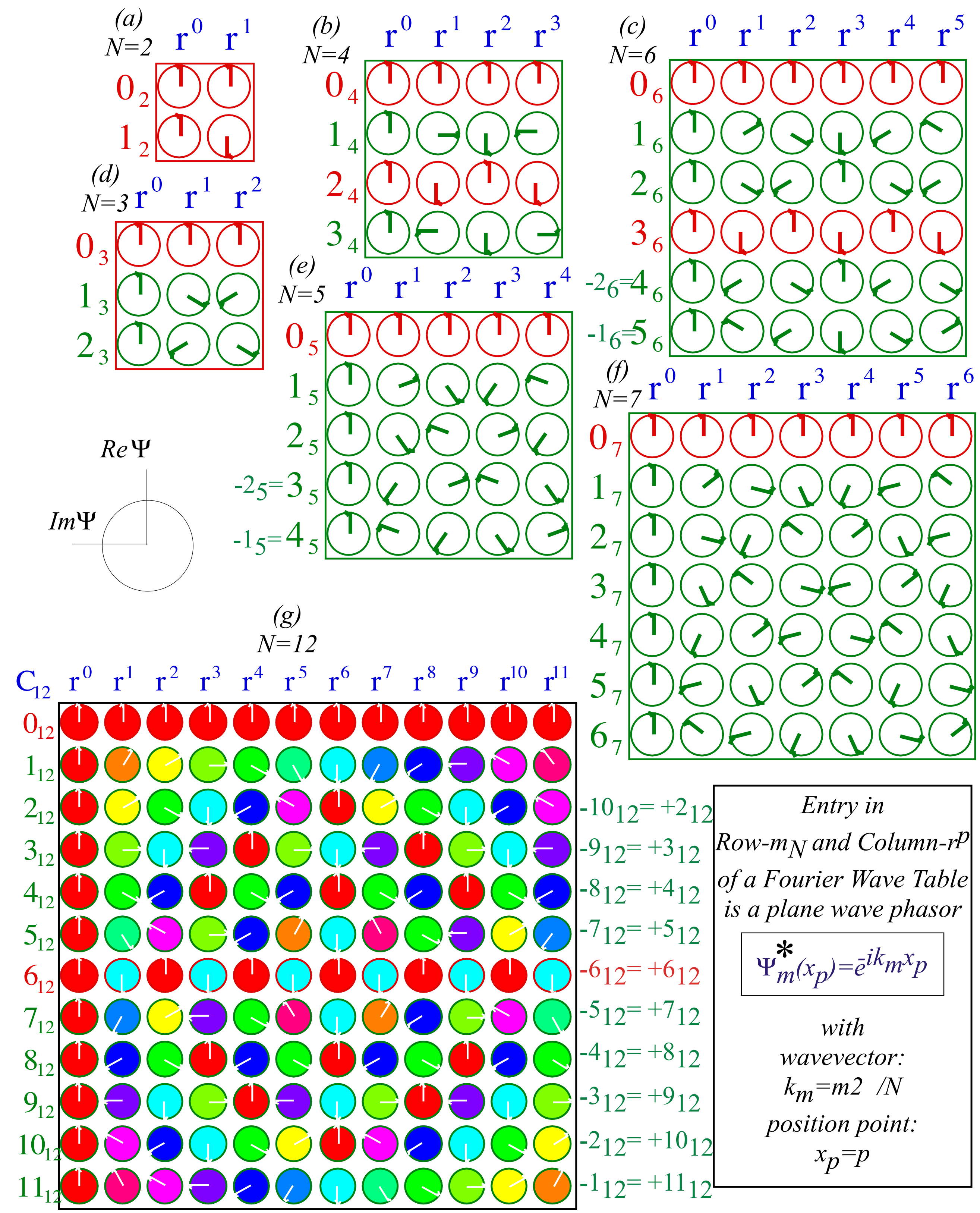
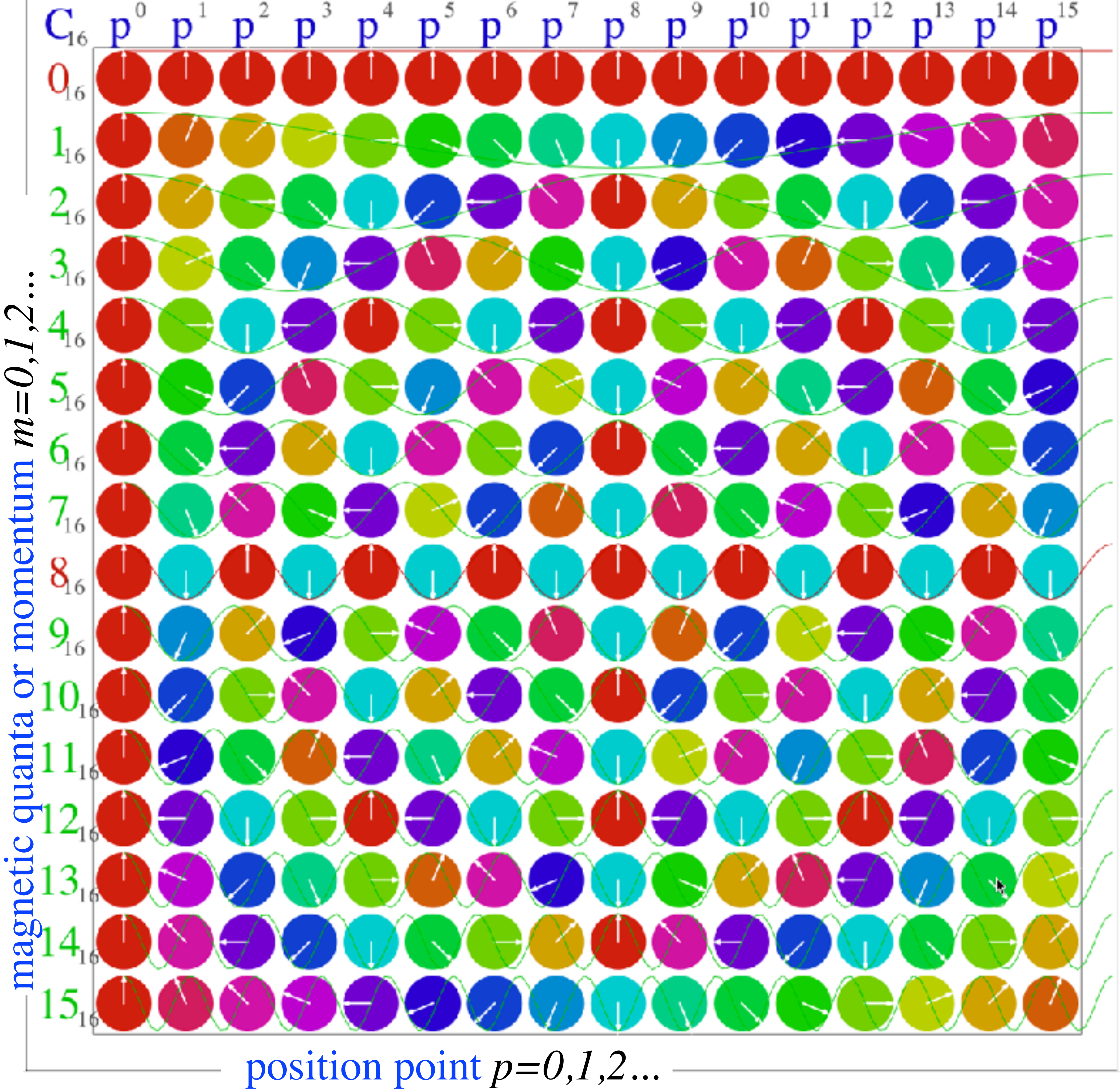


Fig. 4.8.6-7
Unit 4
CMwBang

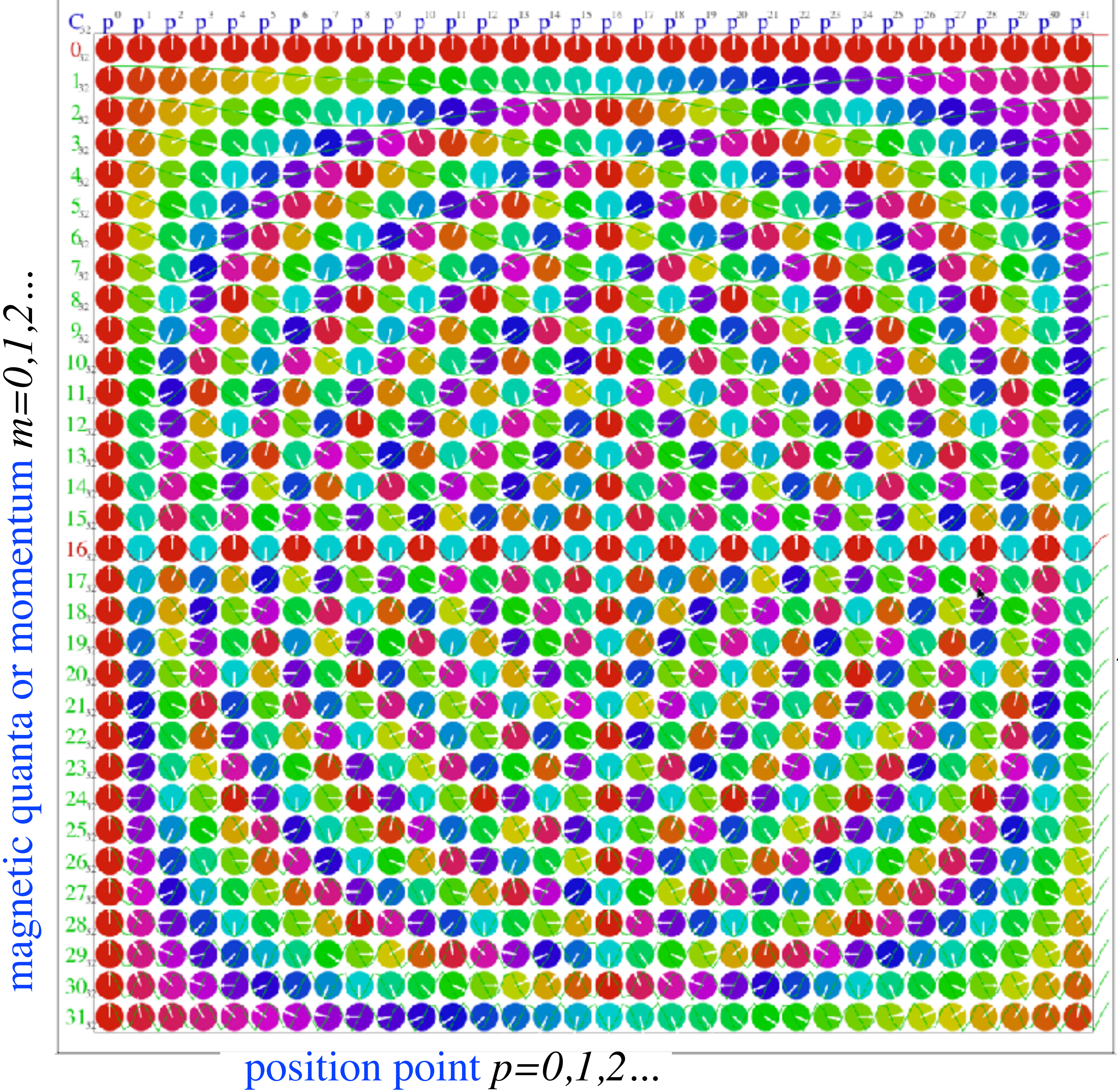
Fourier transformation matrices



C_{16}

phasor character table

$$\chi_p^m = e^{ik_m r^p} = e^{\frac{2\pi imp}{16}}$$

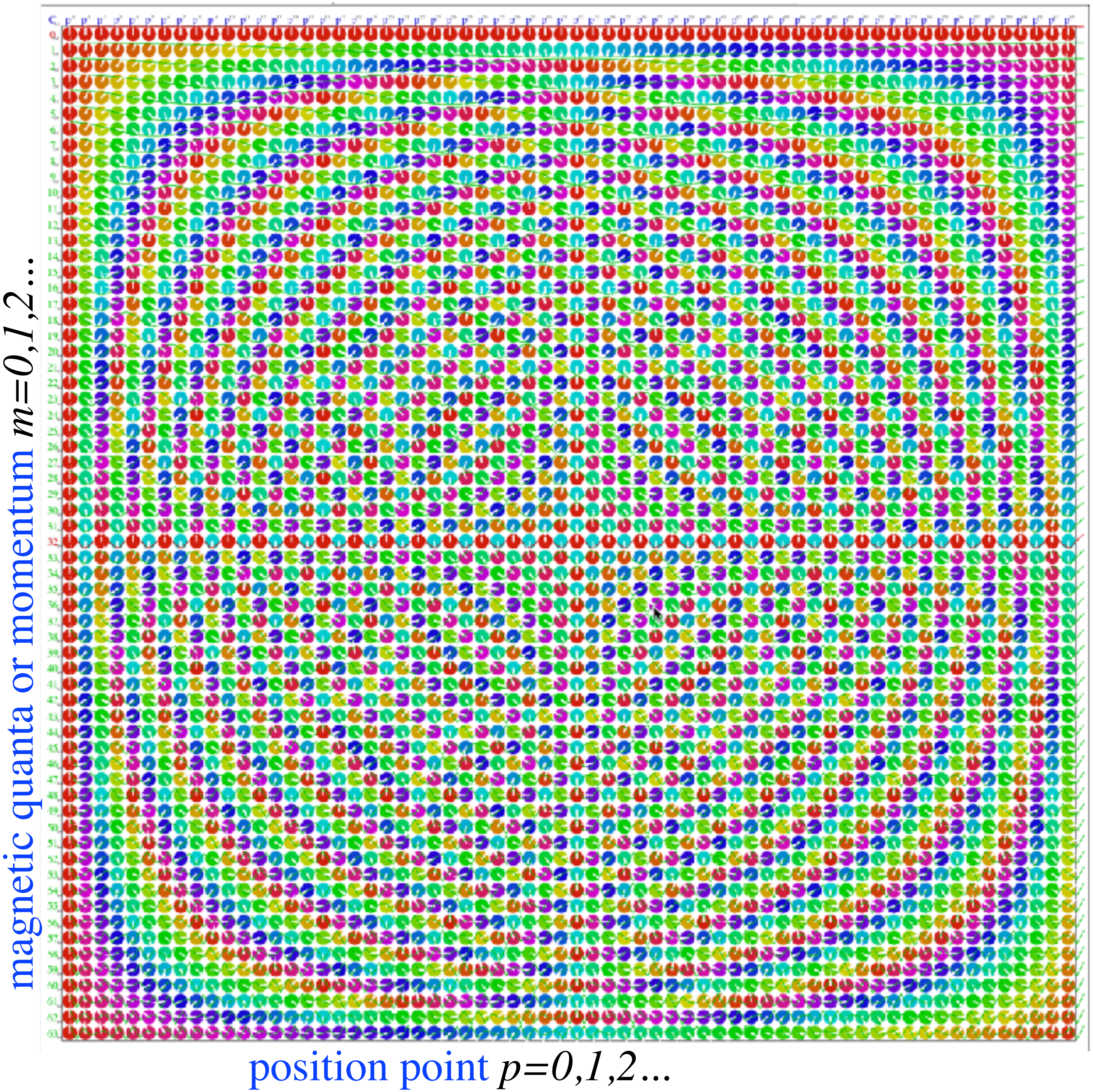


C_{32}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi i m p}{32}}$$



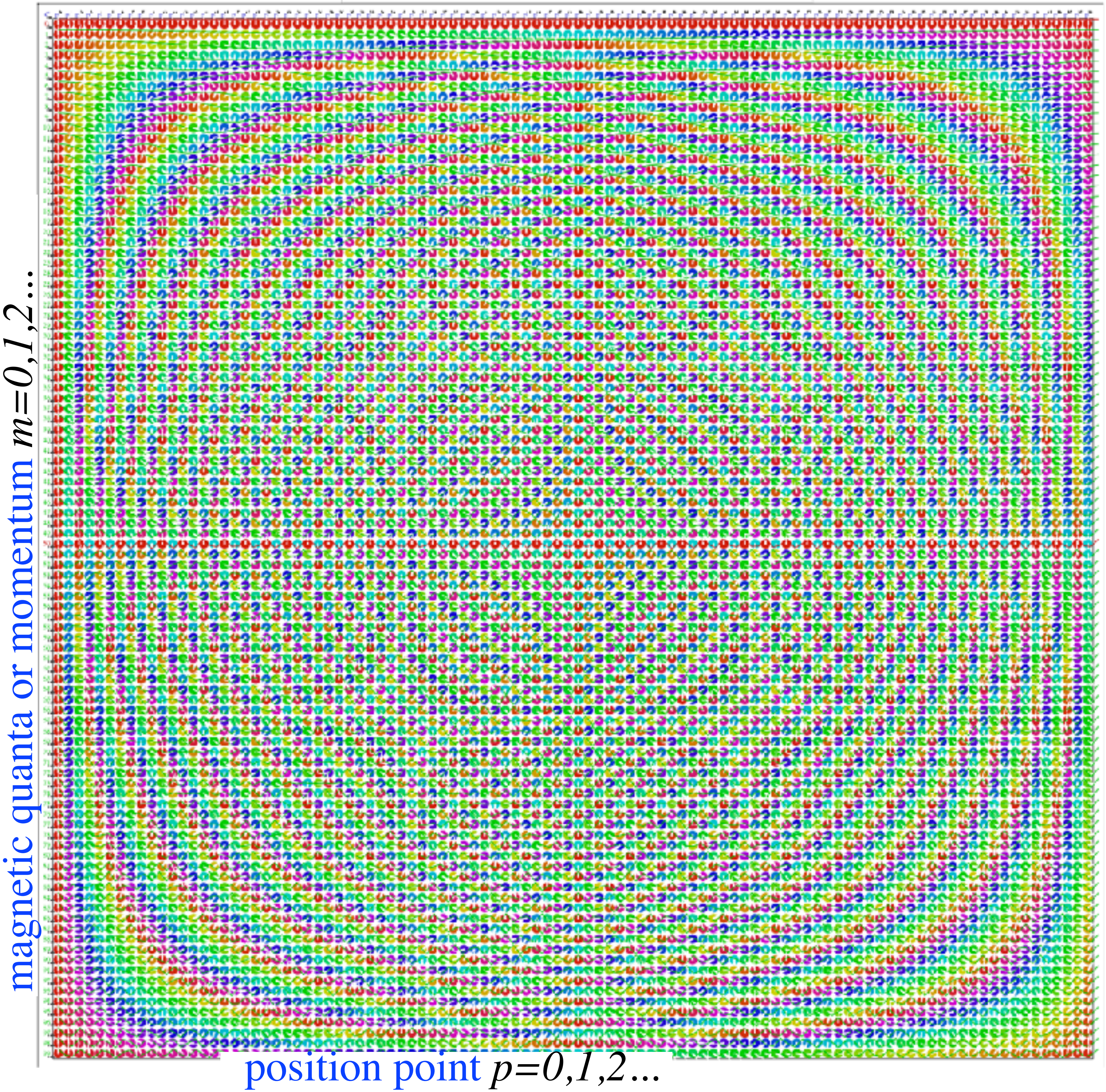
C_{64}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{64}}$$

Invariant phase "Uncertainty" hyperbolas: $m\cdot p = const.$



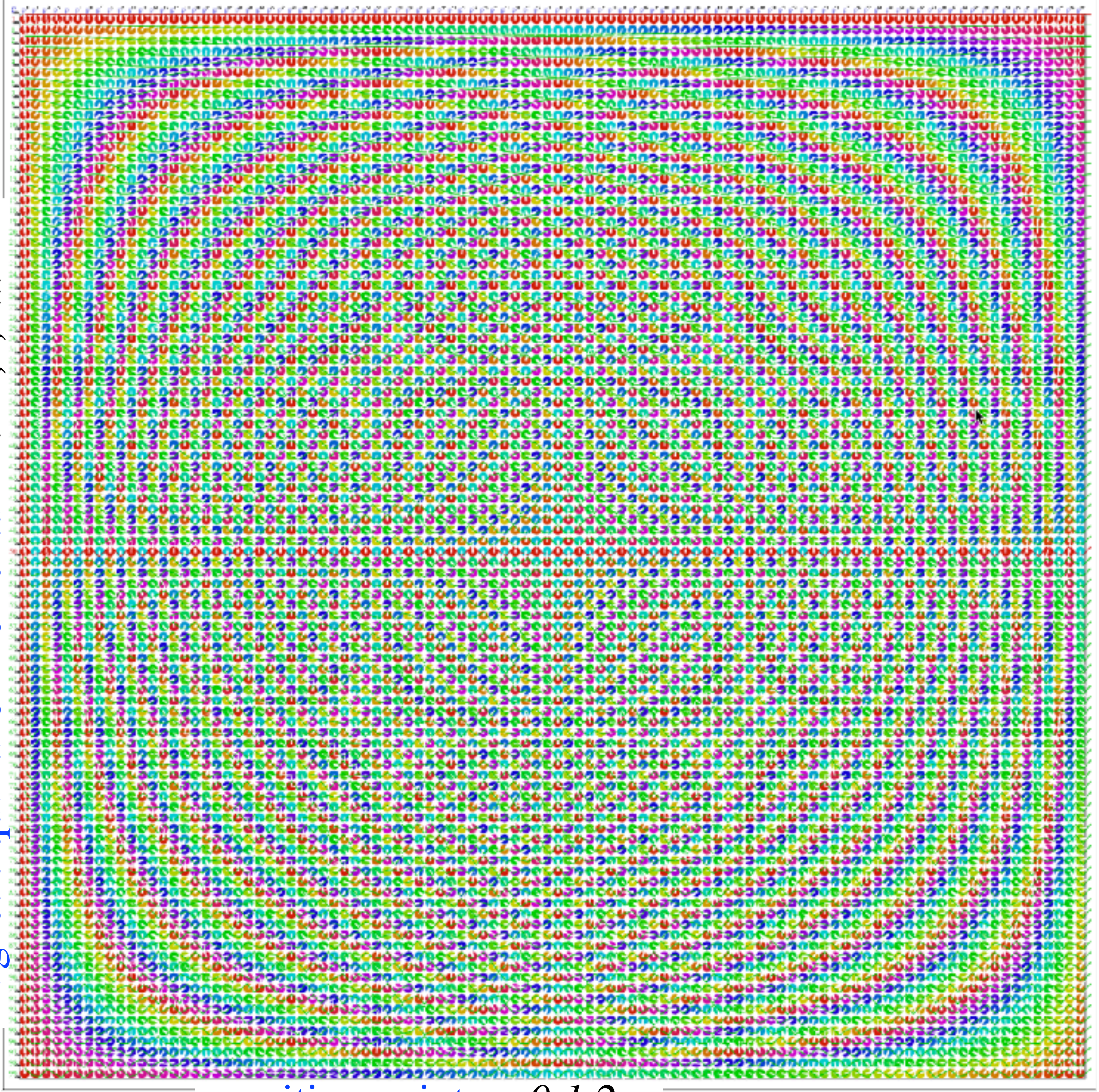
C_{100}

phasor
character
table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi imp}{100}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m \cdot p = const.$



C_{256}

phasor character table

$$\chi_p^m = e^{ik_m r^p}$$

$$= e^{\frac{2\pi i m p}{256}}$$

Invariant phase
"Uncertainty"
hyperbolas:
$m \cdot p = const.$

Wave resonance in cyclic symmetry

Harmonic oscillator with cyclic C_2 symmetry

C_2 symmetric (B-type) modes

Harmonic oscillator with cyclic C_3 symmetry

C_3 symmetric spectral decomposition by 3rd roots of unity

Resolving C_3 projectors and moving wave modes

Dispersion functions and standing waves

C_6 symmetric mode model:Distant neighbor coupling

C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..

C_N symmetric mode models: Made-to order dispersion functions

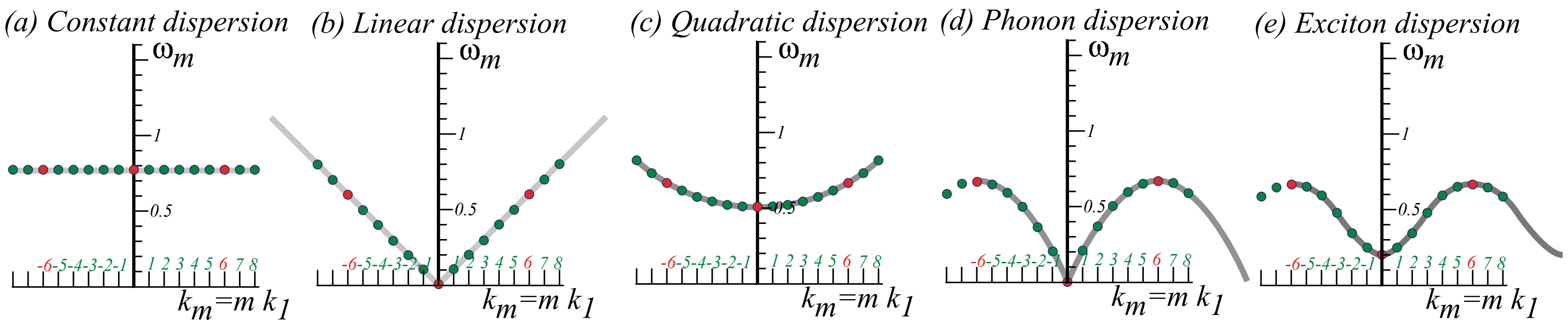
Quadratic dispersion models: Super-beats and fractional revivals

Phase arithmetic

C_N Symmetric Mode Models: Made-to-Order Dispersion *(and wave dynamics)*

(Making pure linear $\omega=ck$, quadratic $\omega=ck^2$, etc. ?)

Archetypical Examples of Dispersion Functions



Applications:

Uncoupled pendulums	*Weakly coupled pendulums (No gravity)*	*Weakly coupled pendulums (With gravity)*	*Strongly coupled pendulums (No gravity)*	*Strongly coupled pendulums (With gravity)*
Movie marquis Xmas lights	*Light in vacuum (Exactly) Sound (Approximately)*	*Light in fiber (Approx) Non-relativistic Schrodinger matter wave*	*Acoustic mode in solids*	*Optical mode in solids Relativistic matter (If exact hyperbola)*

Reading Wave Velocity From Dispersion Function by (k,ω) Vectors

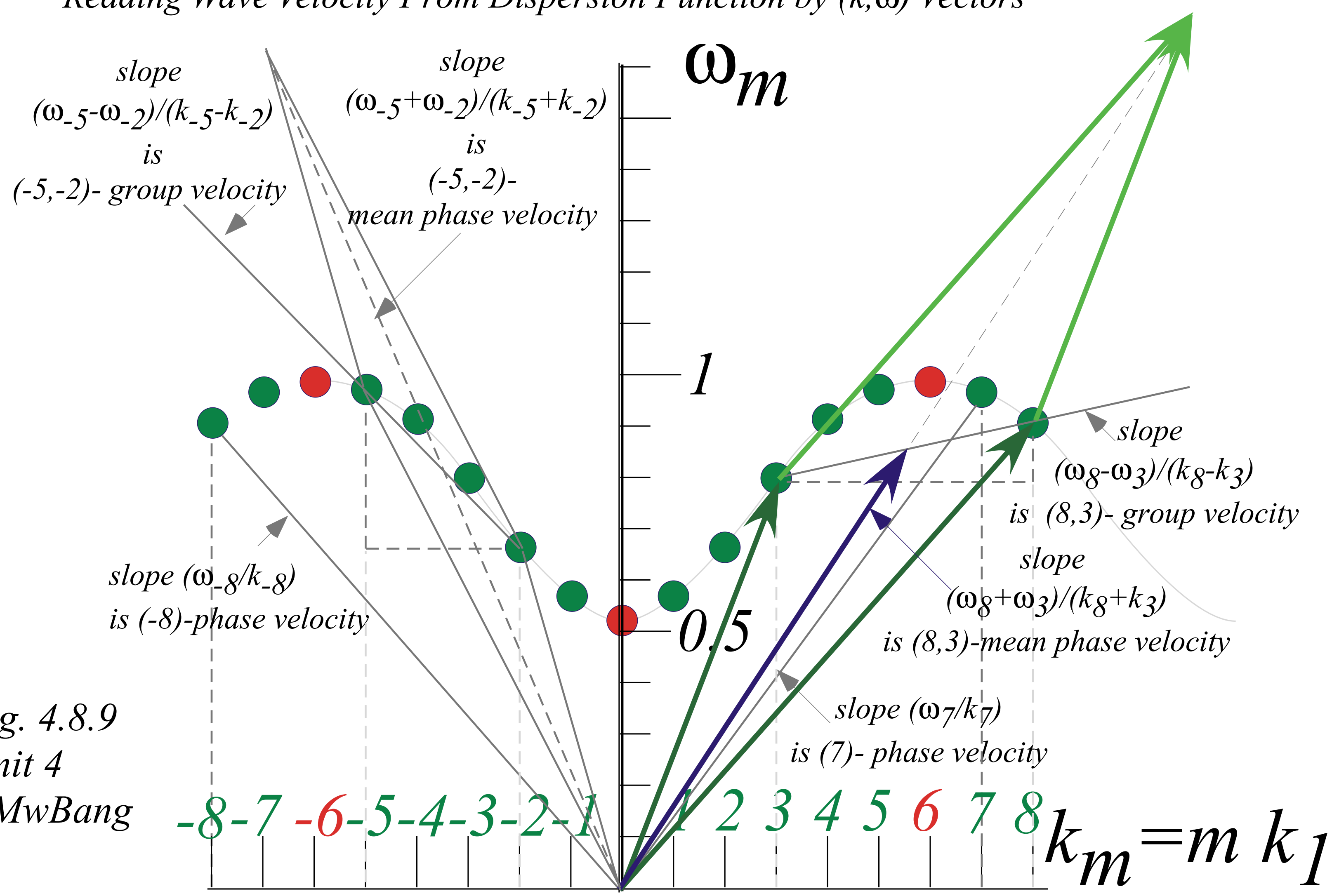


Fig. 4.8.9
Unit 4
CMwBang

$$a = k_a \cdot x - \omega_a \cdot t$$

$$b = k_b \cdot x - \omega_b \cdot t$$

$$\frac{e^{ia} + e^{ib}}{2} = e^{i\frac{a+b}{2}} \left(\frac{e^{i\frac{a-b}{2}} + e^{-i\frac{a-b}{2}}}{2} \right)$$

$$= e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2}\right)$$

Things determined by

Dispersion $\omega = \omega(k)$

Individual phase velocity:

$$V_{phase-1} = \frac{\omega(k)}{k}$$

Pairwise phase velocity:

$$V_{phase-2} = \frac{\omega(k_a) + \omega(k_b)}{k_a + k_b}$$

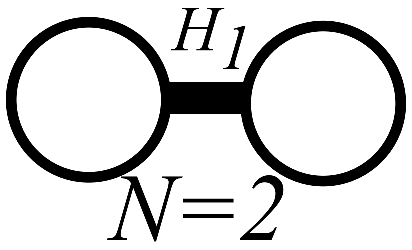
Pairwise group velocity:

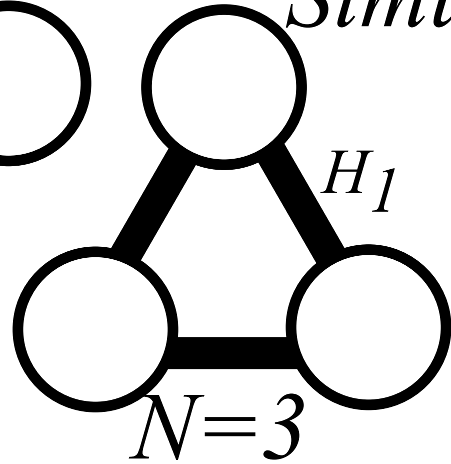
$$V_{group-2} = \frac{\omega(k_a) - \omega(k_b)}{k_a - k_b}$$

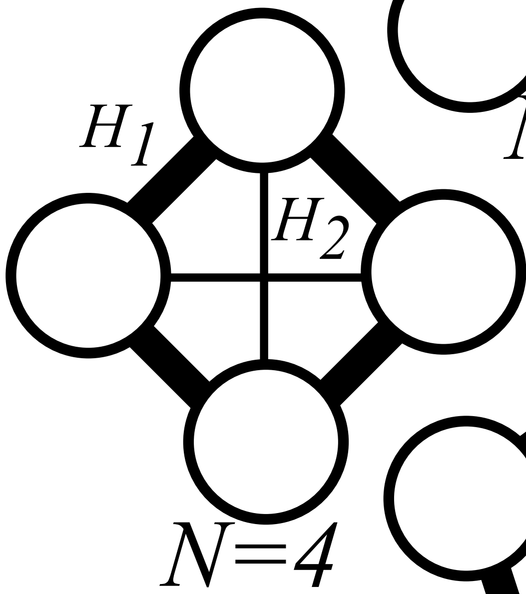
Simulating Complex Systems With Simpler Ones

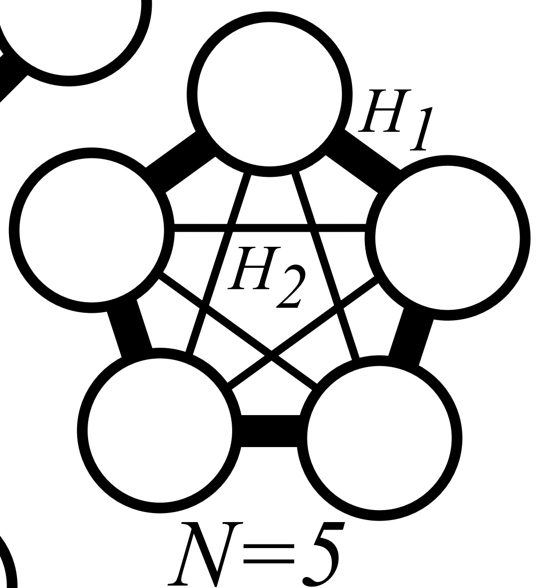
Made of Quantum Dots

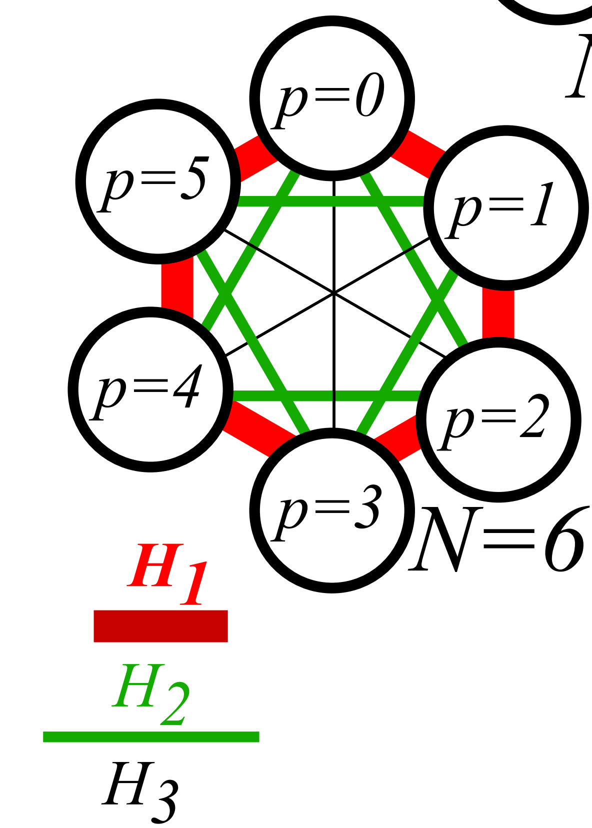
[Harter, J. Mol. Spec. 210, 166-182 (2001)]











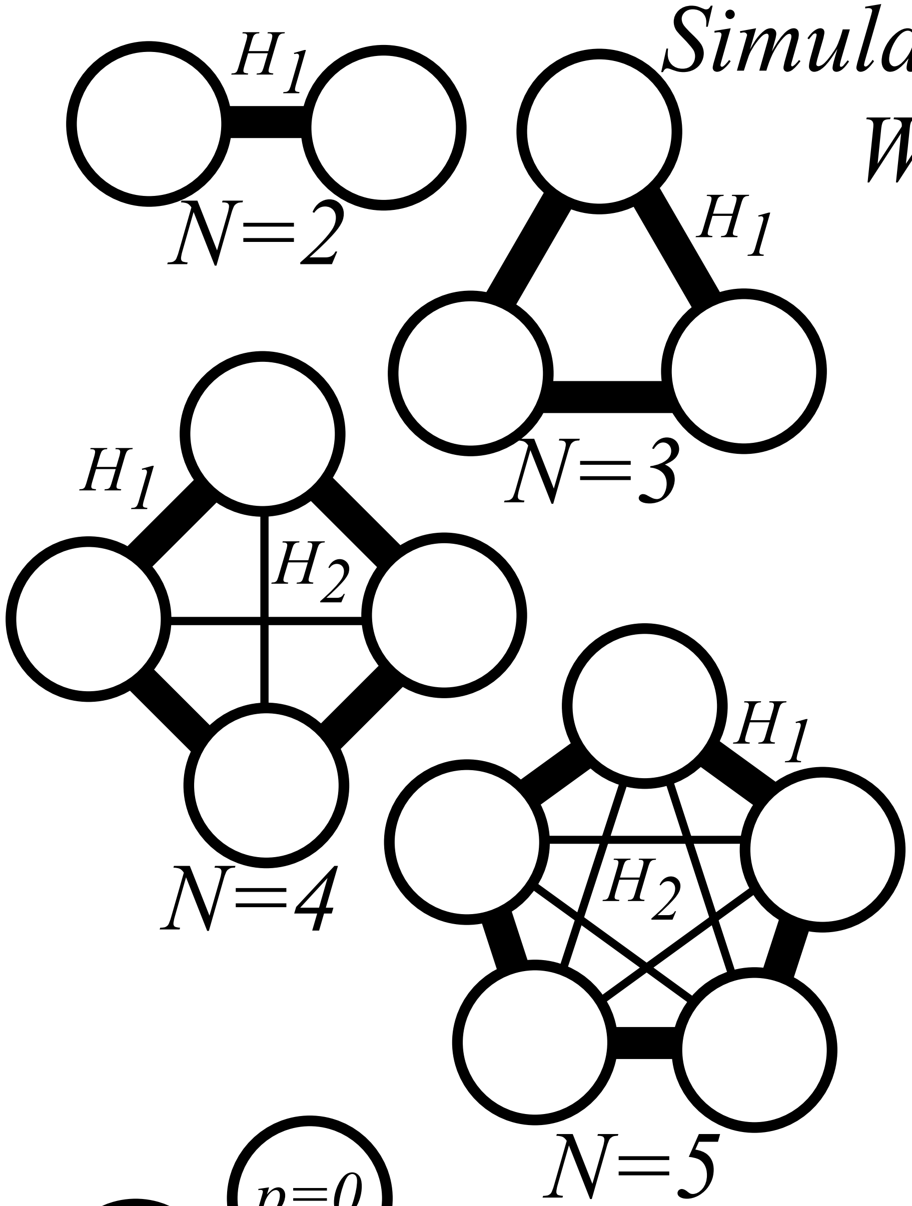
Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H_1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

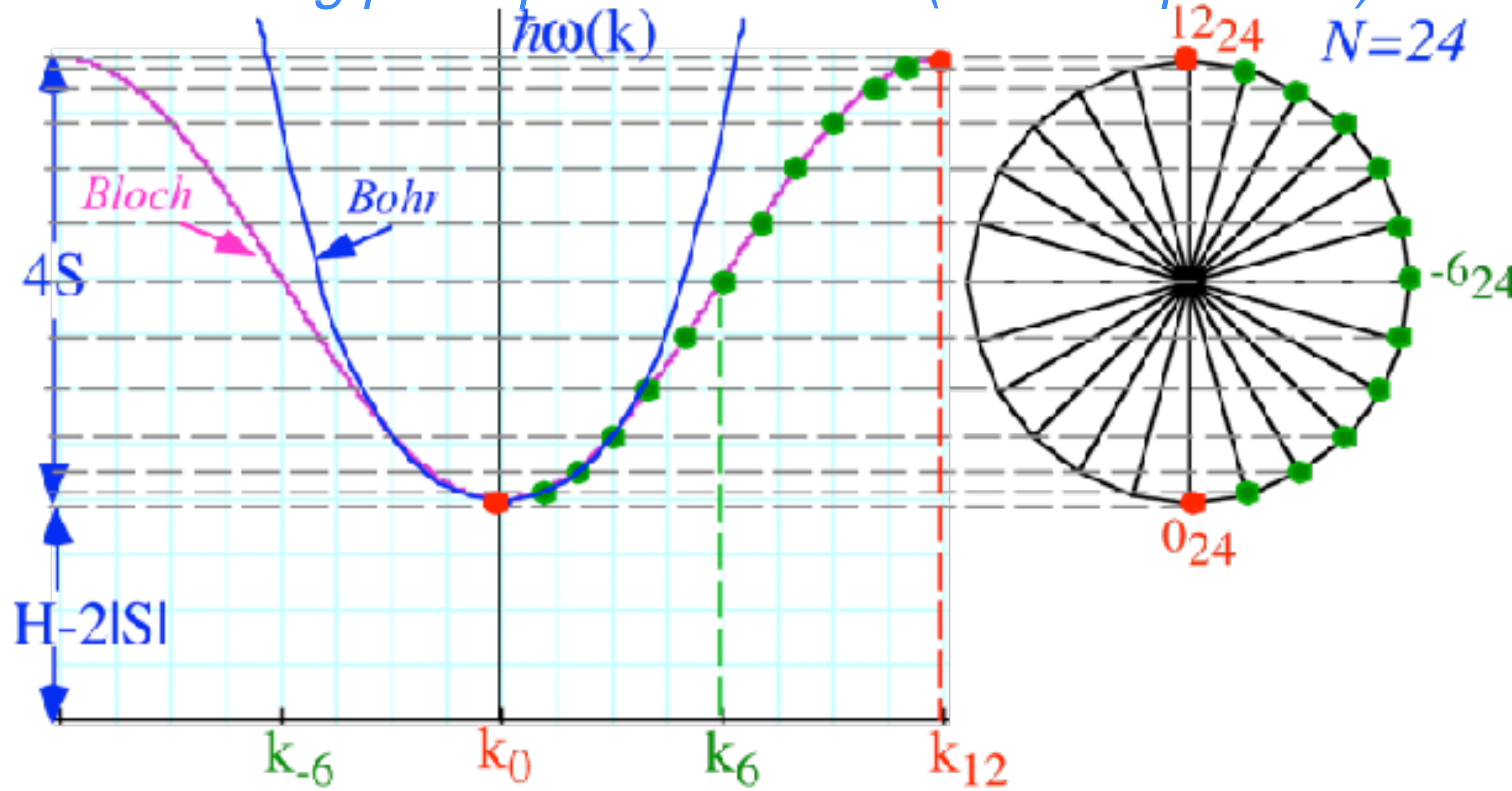
Simulating Complex Systems With Simpler Ones

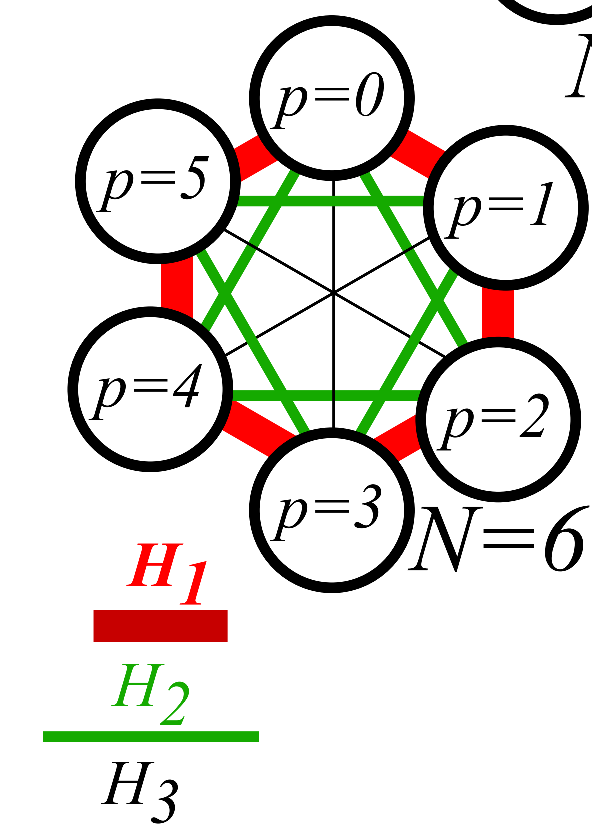
Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Making pure quadratic $\omega=ck^2$ (Bohr dispersion)



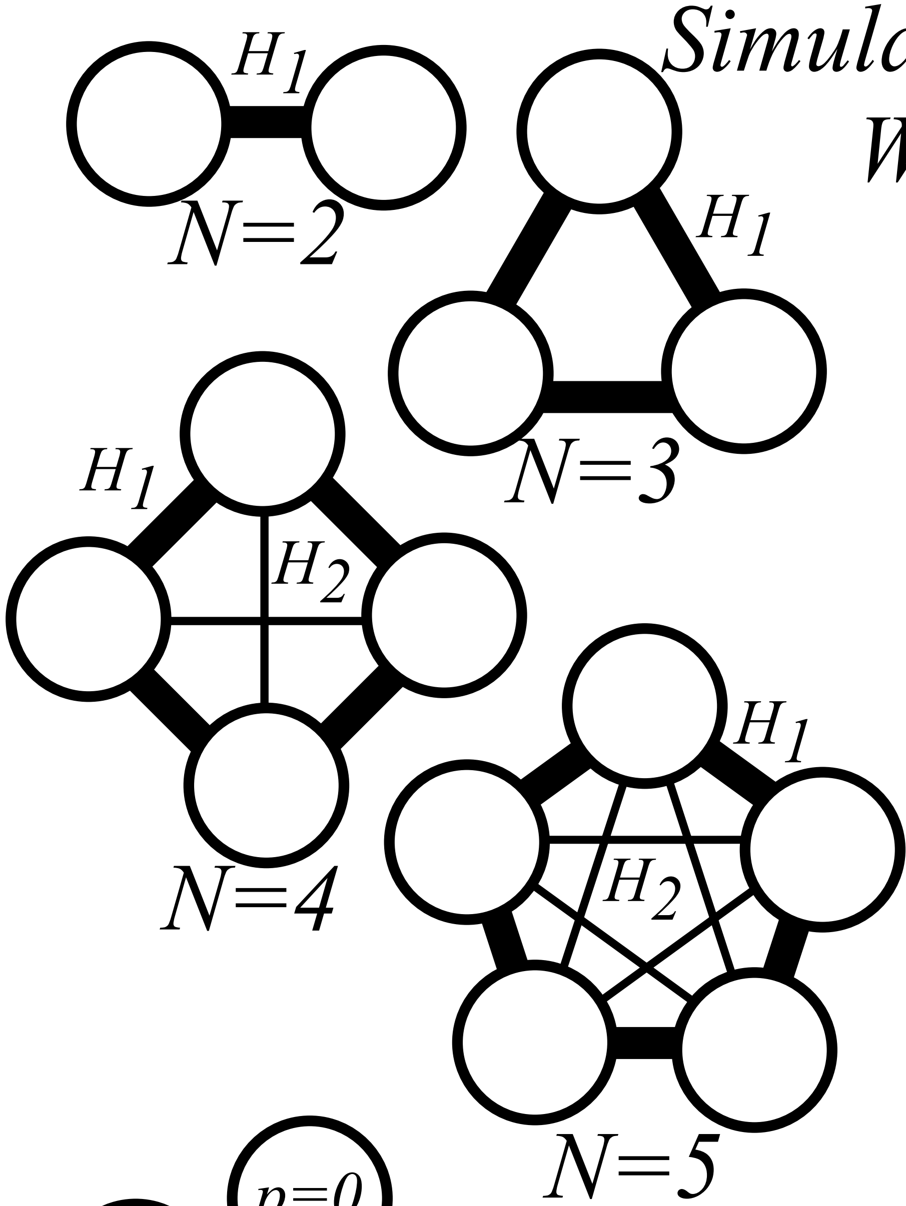


Hexagonal 2D Rotor

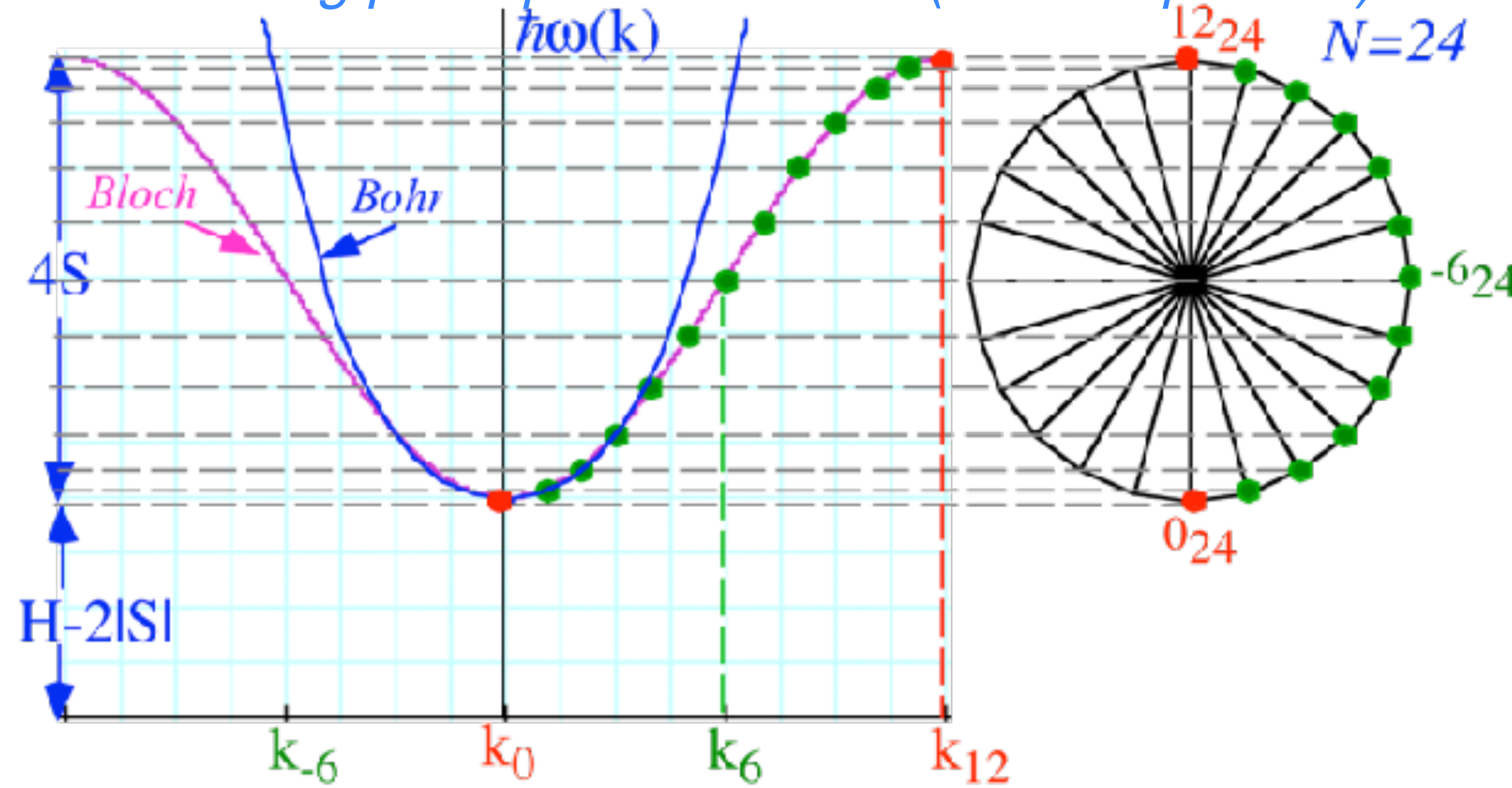
$$
\begin{pmatrix}
H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\
H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\
H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\
H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\
H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\
H_1 & H_2 & H_3 & H_2 & H_1 & H_0
\end{pmatrix}
$$

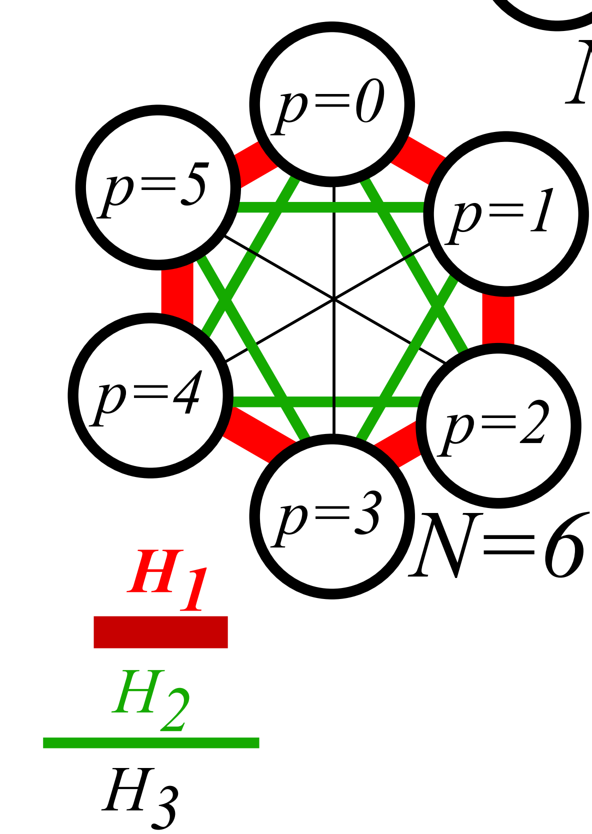
Simulating Complex Systems
With Simpler Ones
Made of Quantum Dots

[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Making pure quadratic $\omega=ck^2$ (Bohr dispersion)





Hexagonal 2D Rotor

$$\begin{pmatrix} H_0 & H_1 & H_2 & H_3 & H_2 & H_1 \\ H1 & H_0 & H_1 & H_2 & H_3 & H_2 \\ H_2 & H_1 & H_0 & H_1 & H_2 & H_3 \\ H_3 & H_2 & H_1 & H_0 & H_1 & H_2 \\ H_2 & H_3 & H_2 & H_1 & H_0 & H_1 \\ H_1 & H_2 & H_3 & H_2 & H_1 & H_0 \end{pmatrix}$$

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$N=2$	1/2	-1/2							
$N=3$	2/3	-1/3							
$N=4$	3/2	-1	1/2						
$N=5$	2	-1.1708	0.1708						
$N=6$	19/6	-2	2/3	-1/2					
$N=7$	4	-2.393	0.51	-0.1171					
$N=8$	11/2	-3.4142	1	-0.5858	1/2				
$N=9$	20/3	-4.0165	0.9270	-1/3	0.0895				
$N=10$	17/2	-5.2361	1.4472	-0.7639	0.5528	-1/2			
$N=11$	10	-6.0442	1.4391	-0.5733	0.2510	-0.0726			
$N=12$	73/6	-7.4641	2	-1	2/3	-0.5359	1/2		
$N=13$	14	-8.4766	2.0500	-0.8511	0.4194	-0.2028	0.06116		
$N=14$	33/2	-10.098	2.6560	-1.2862	0.8180	-0.6160	0.5260	-1/2	
$N=15$	57/3	-11.314	2.7611	-1.1708	0.6058	-1/3	0.1708	-0.0528	
$N=16$	43/2	-13.137	3.4142	-1.6199	1	-0.7232	0.5858	-0.5198	1/2
$N=17$	24	-14.557	3.5728	-1.5340	0.81413	-0.4732	0.2781	-0.1479	0.0465

Wave resonance in cyclic symmetry

- *Harmonic oscillator with cyclic C_2 symmetry*
    - *C_2 symmetric (B-type) modes*
- *Harmonic oscillator with cyclic C_3 symmetry*
    - *C_3 symmetric spectral decomposition by 3rd roots of unity*
    - *Resolving C_3 projectors and moving wave modes*
    - *Dispersion functions and standing waves*
- *C_6 symmetric mode model:Distant neighbor coupling*
    - *C_6 spectra of gauge splitting by C-type symmetry(complex, chiral, coriolis, current, ..*
- *C_N symmetric mode models: Made-to order dispersion functions*
    - *Quadratic dispersion models: Super-beats and fractional revivals*
    - ➔ *Phase arithmetic*

2-level-system and C_2 symmetry phase dynamics

C_2 Character Table describes eigenstates

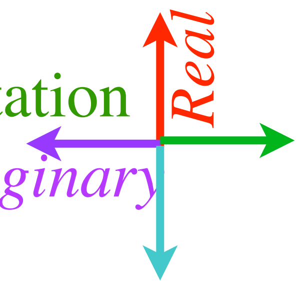
symmetric A_1

vs.

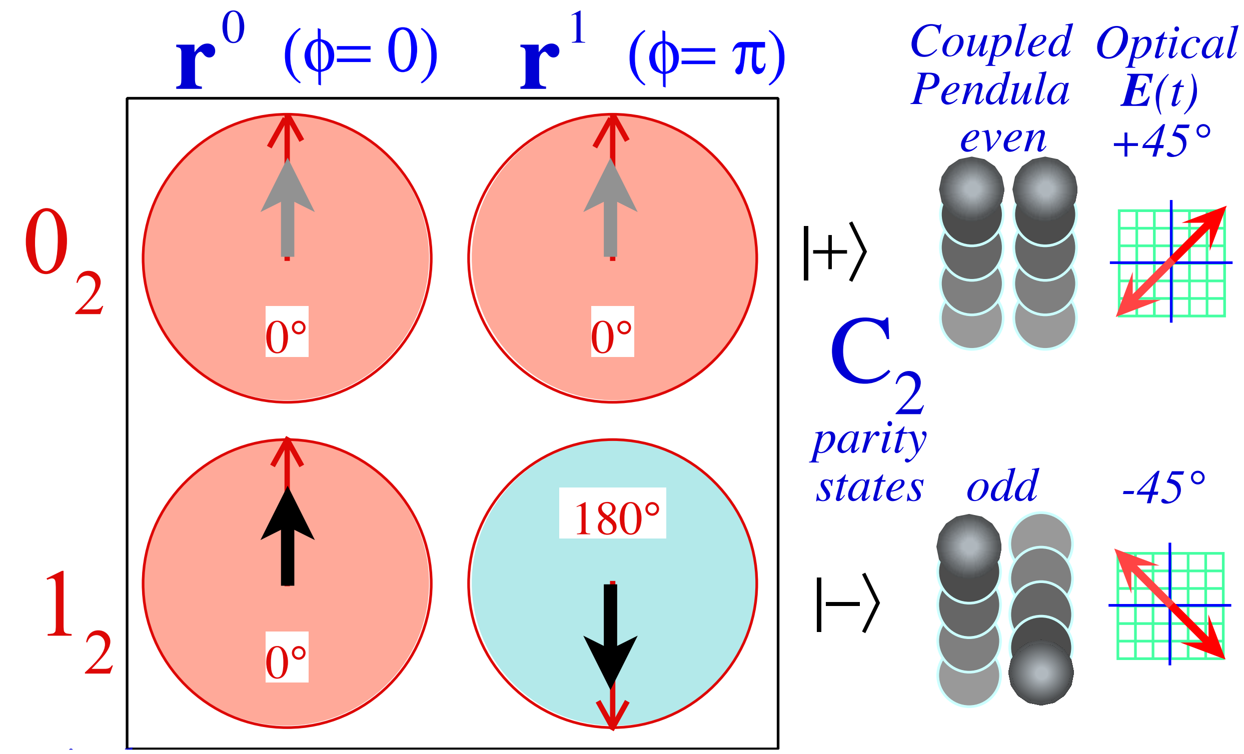
antisymmetric A_2

	$1=r^0$	$r=r^1$
$0 \bmod 2$	1	1
$\pm 1 \bmod 2$	1	-1

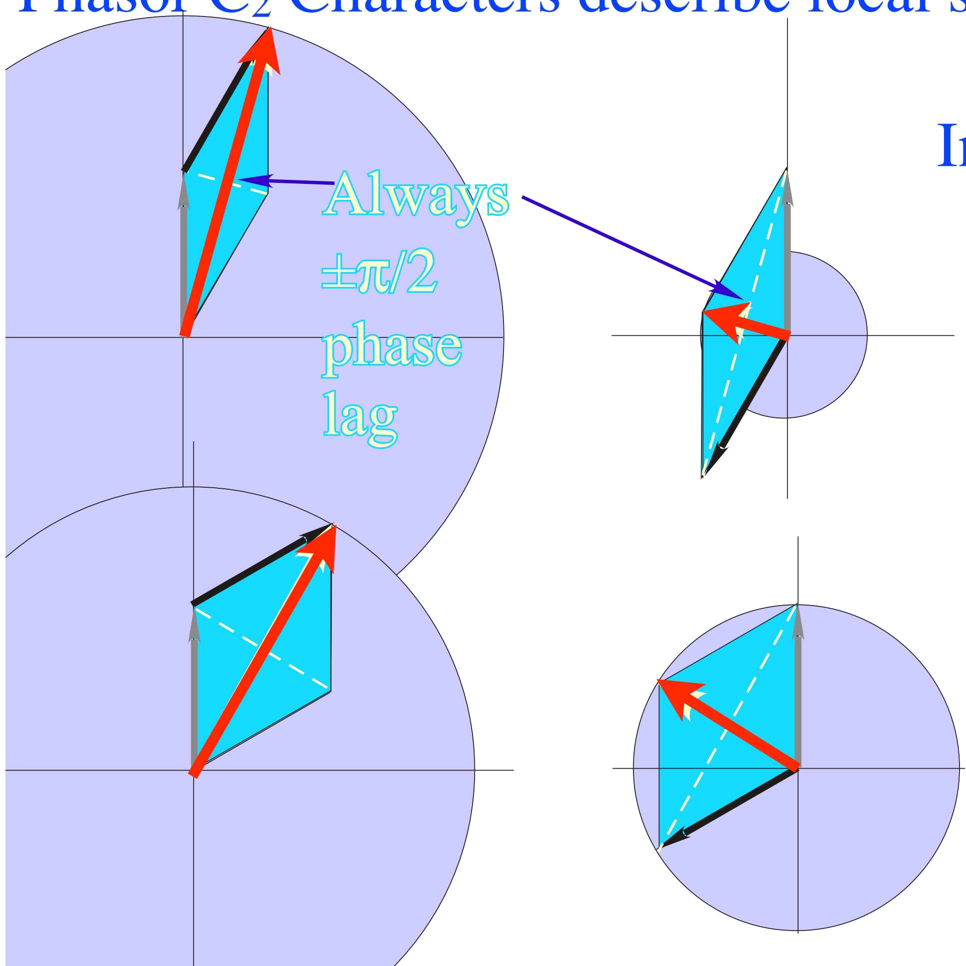
Phasor notation

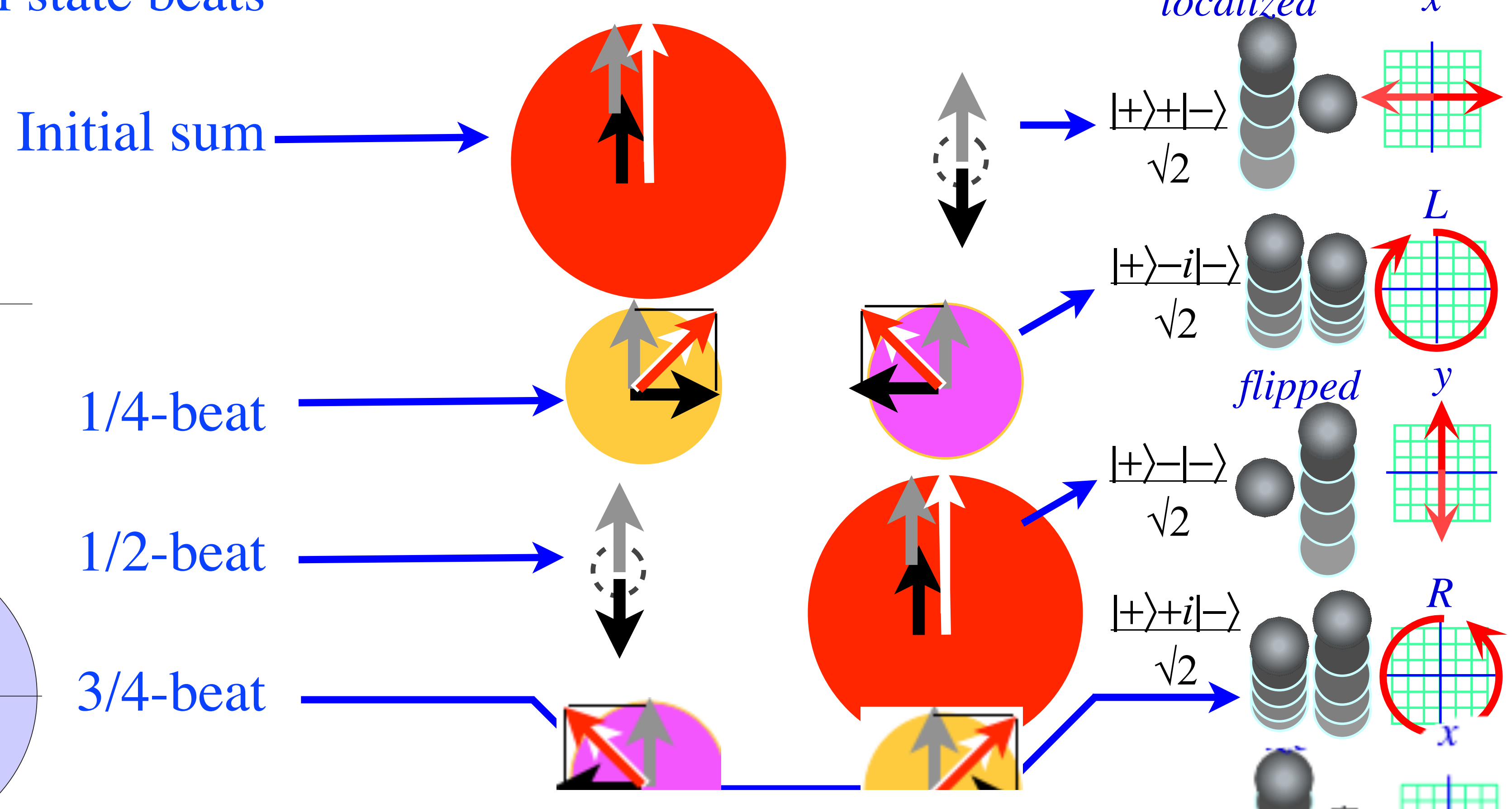


C_2 *Phasor*-Character Table



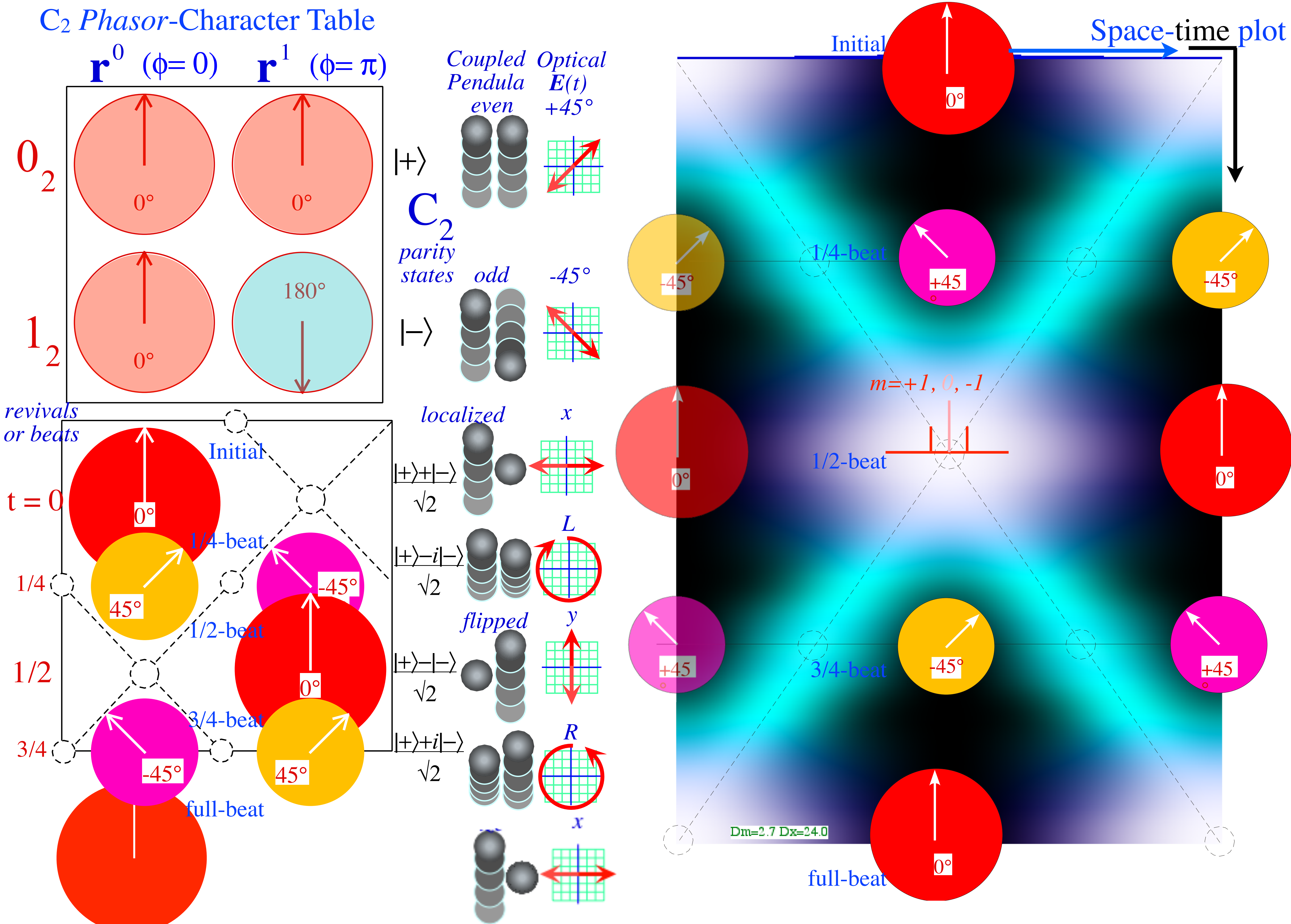
Phasor C_2 Characters describe local state beats



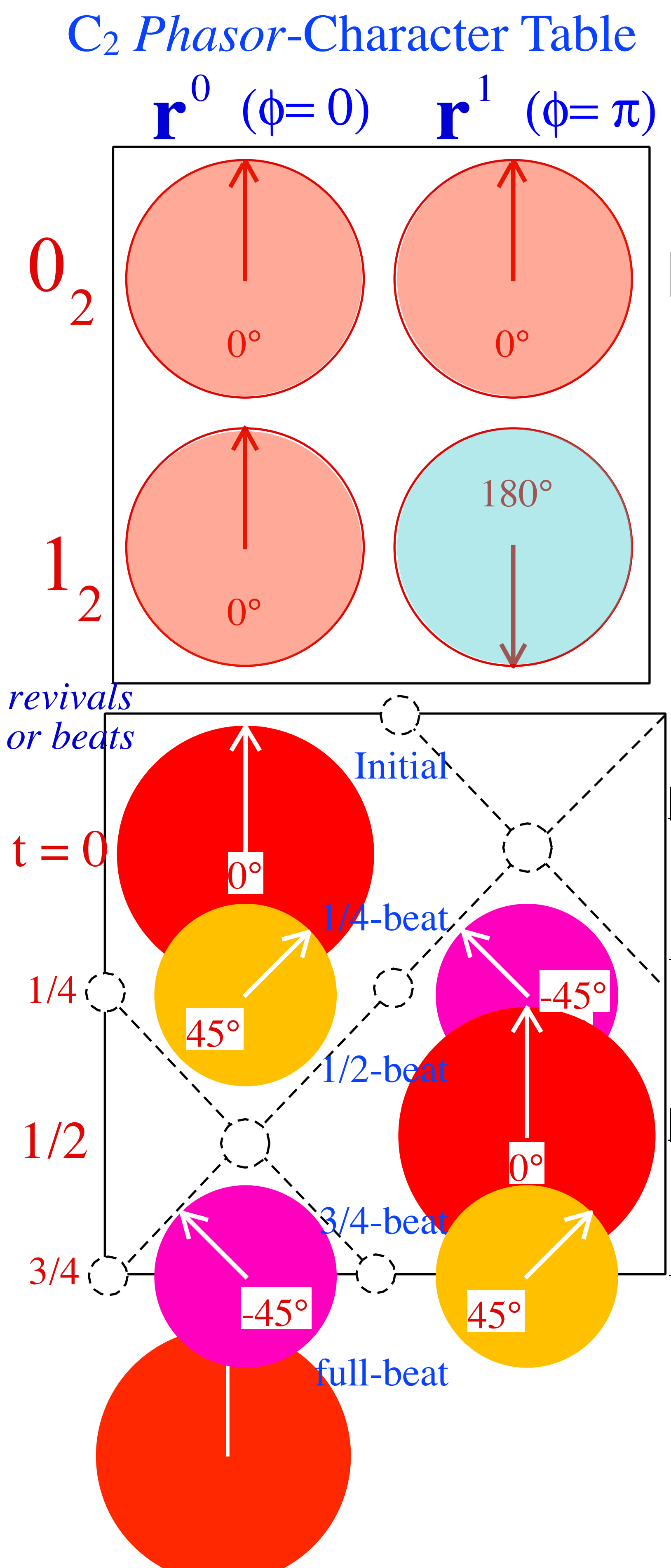


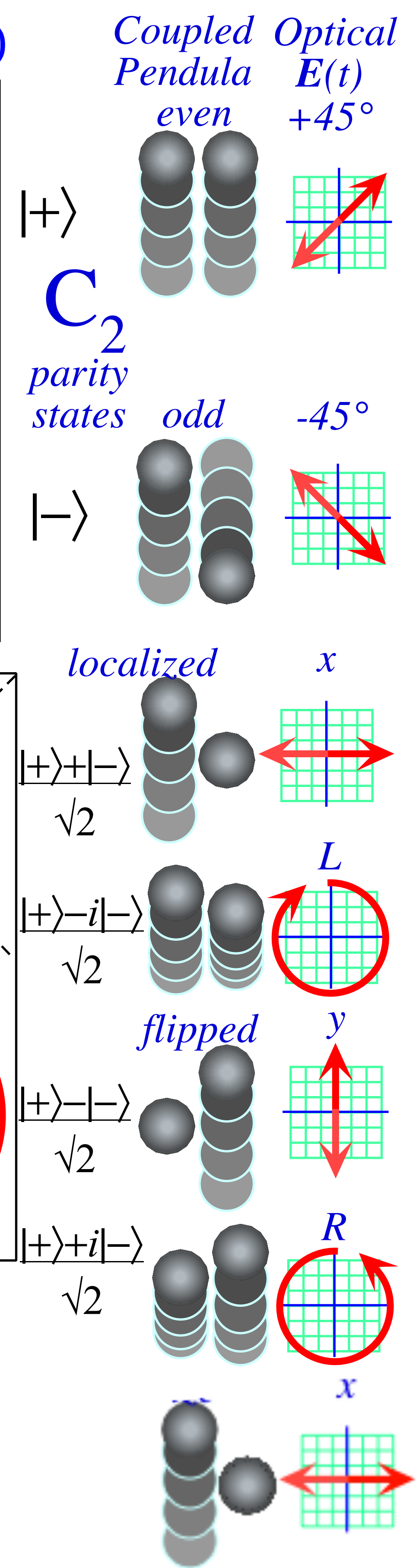
is simplest example of a *revival*

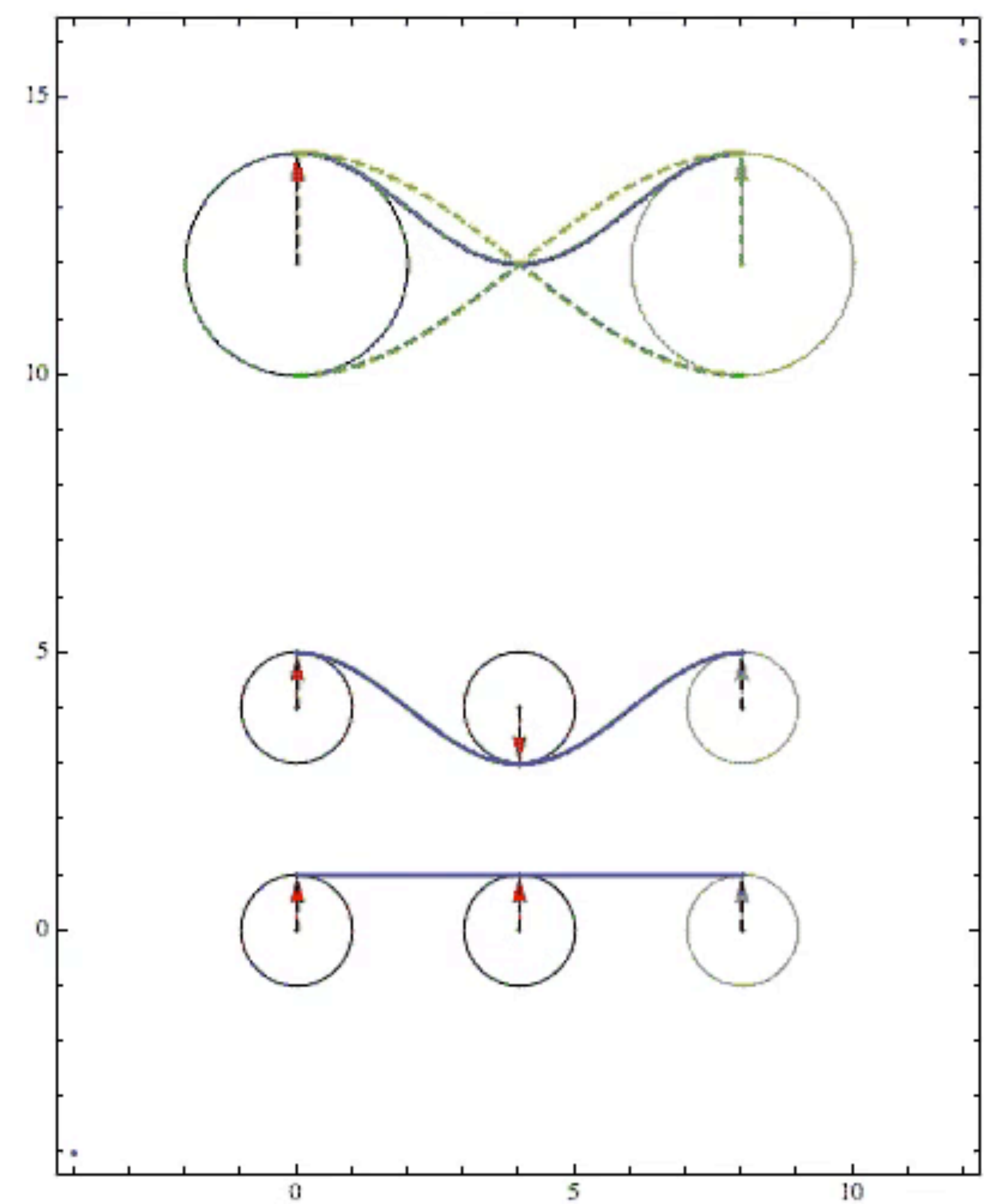
2-level-system and C_2 symmetry phase dynamics



2-level-system and C_2 symmetry phase dynamics



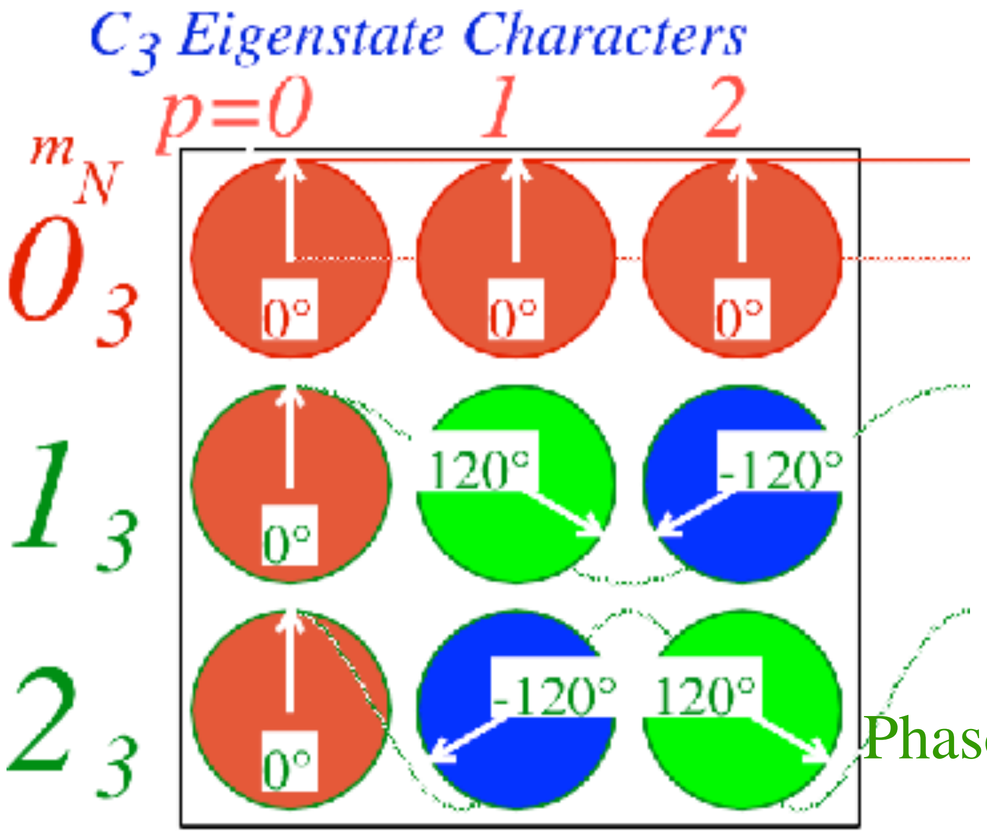


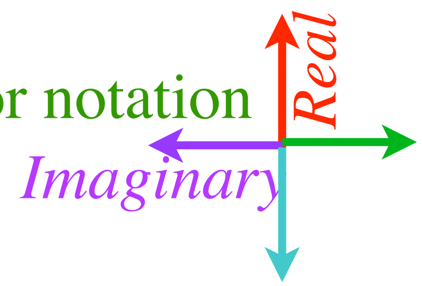


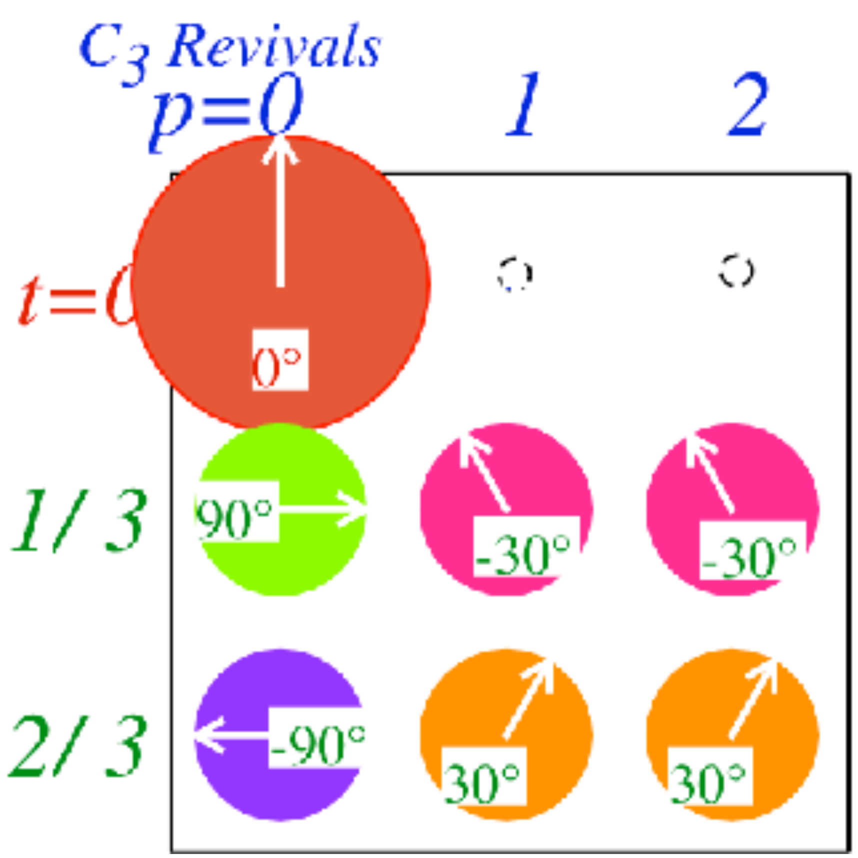
C_3 symmetry phase in *1*, *2*, *or 3*-level-systems

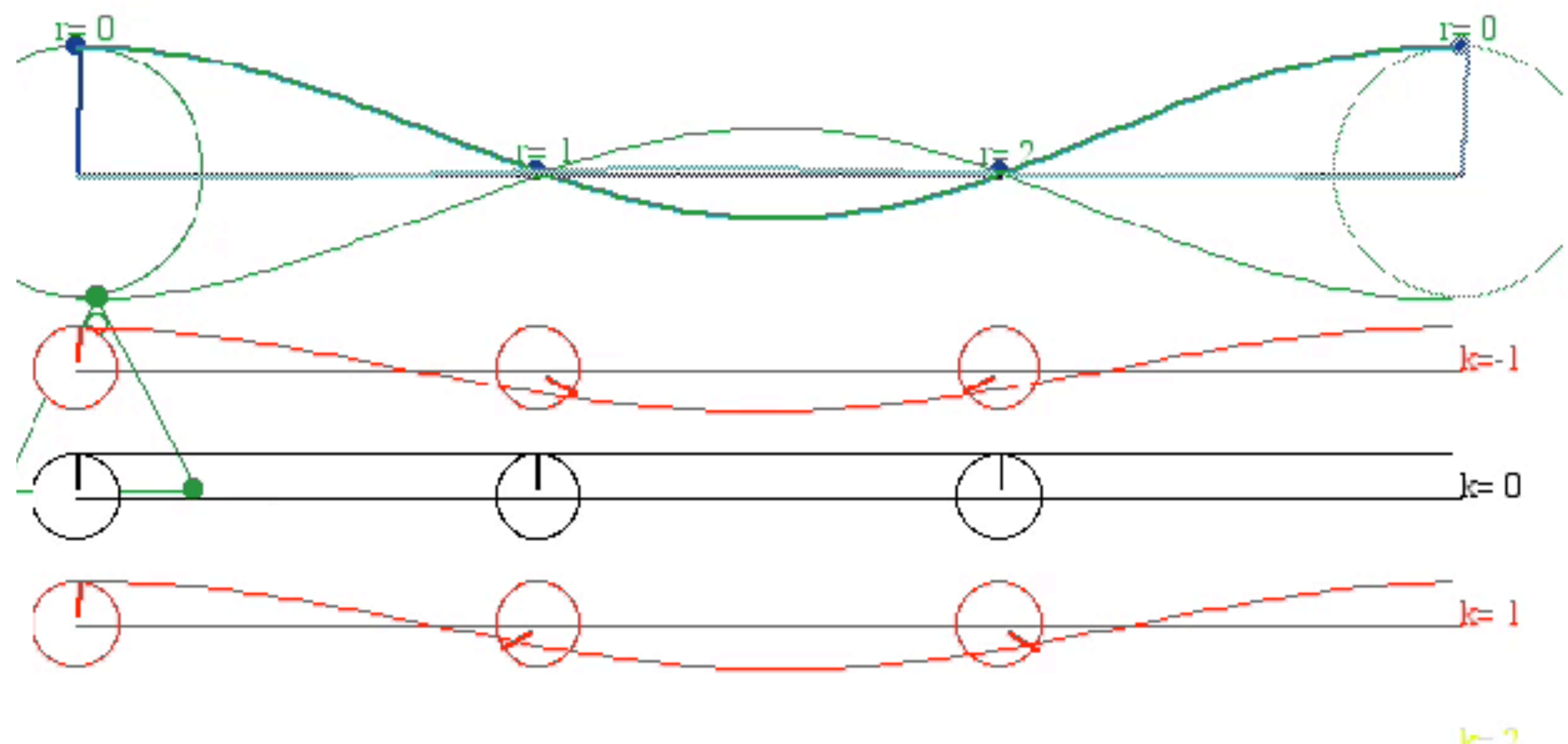
Non - chiral
C_{3v} system

Chiral
"quantum-Hall-like"
systems
deserve special treatment



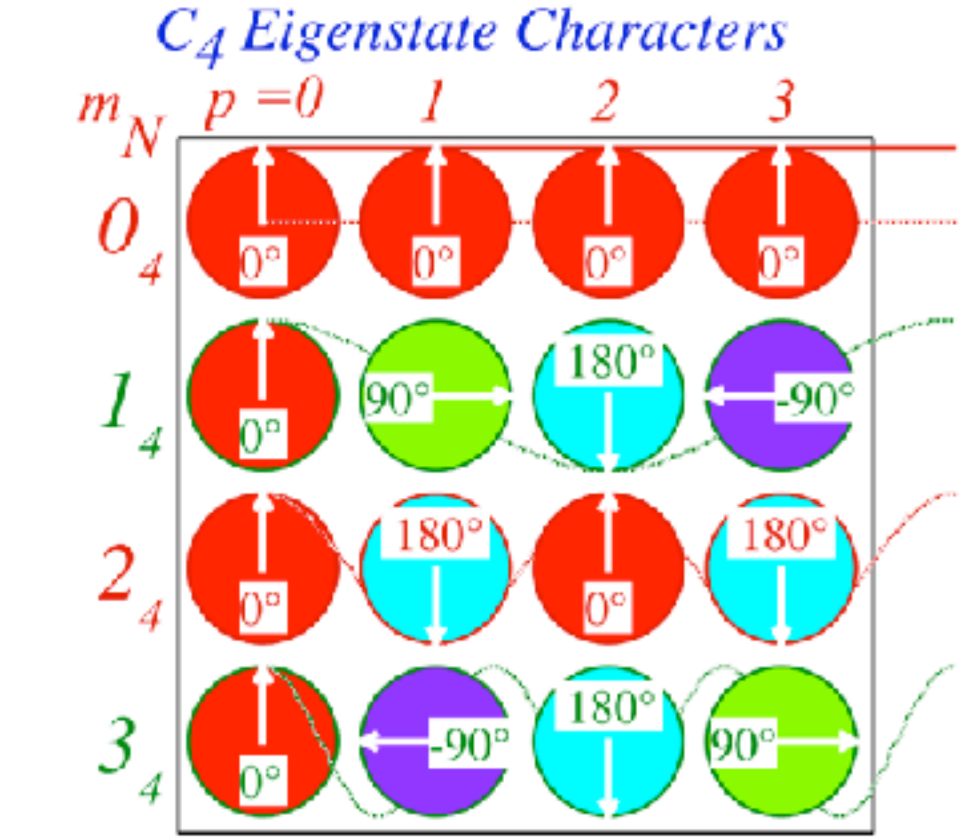


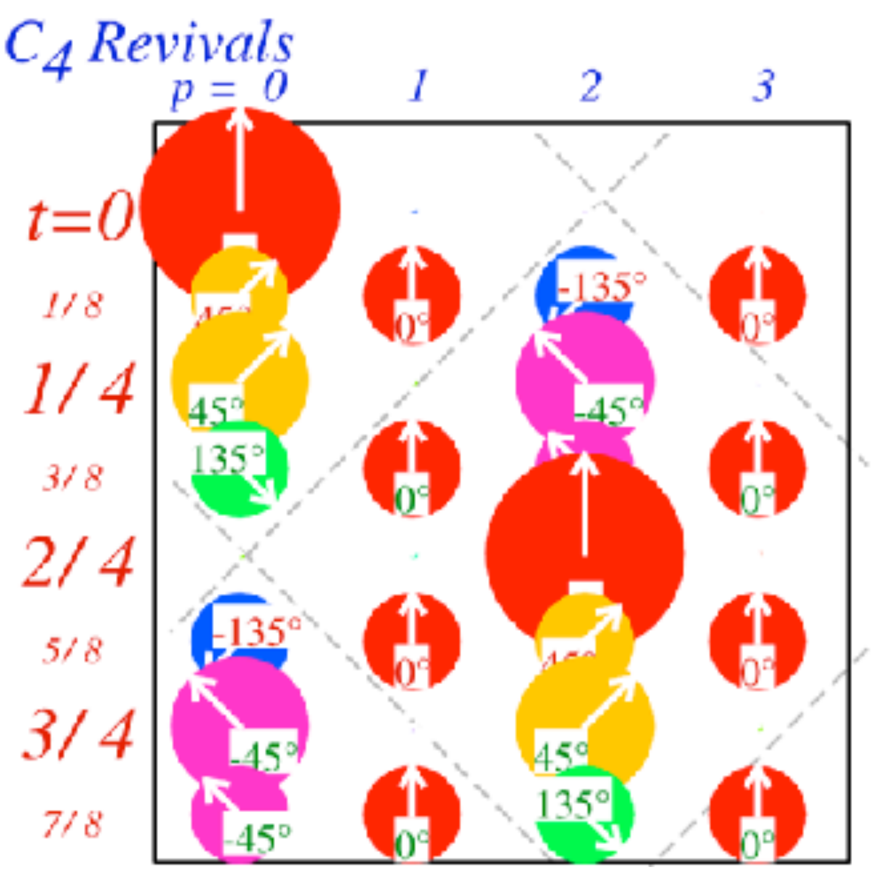


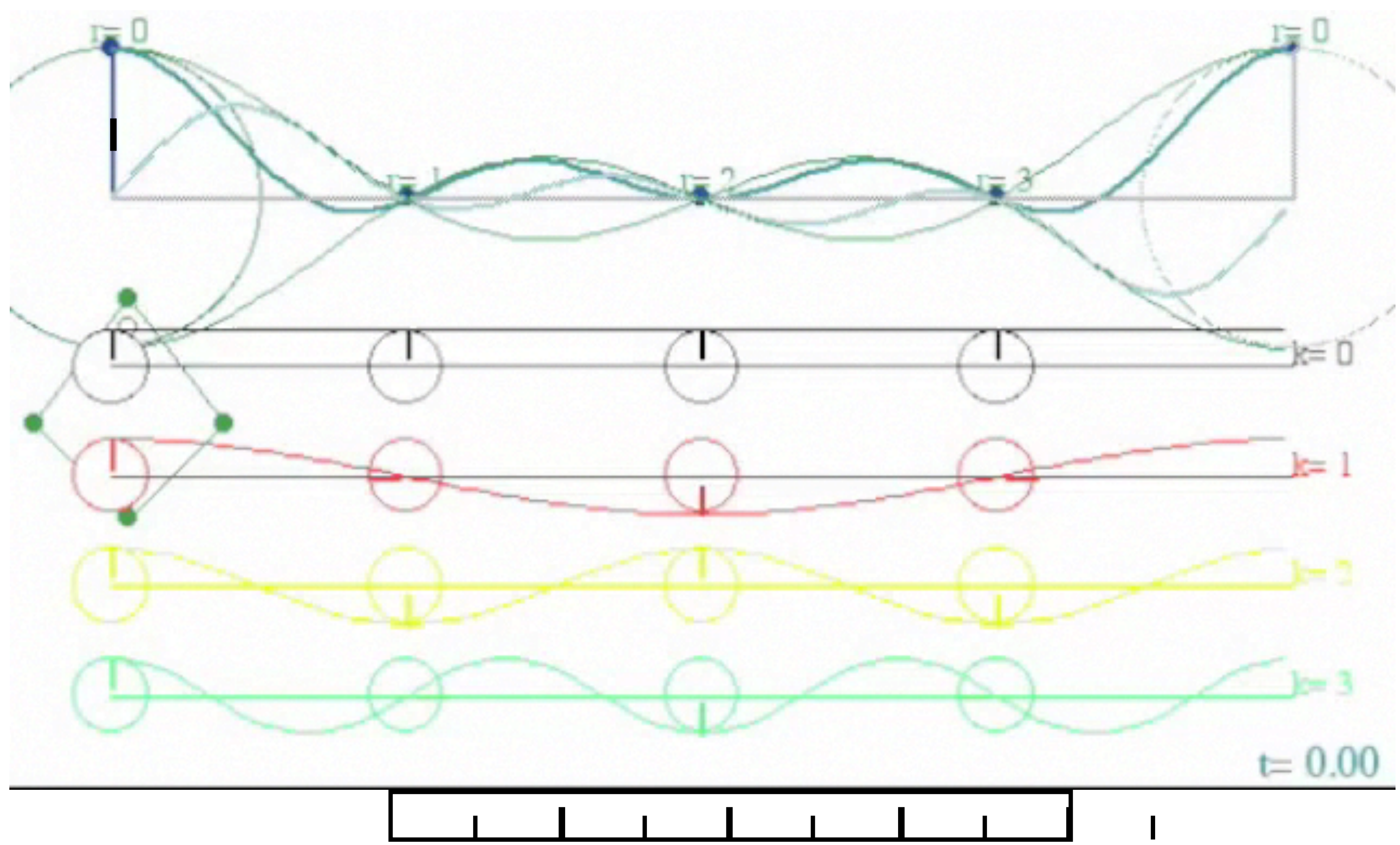


C_4 symmetry phase in *1*, 2, 3 , *or 4* level-systems

Non - chiral
C_{4v} system

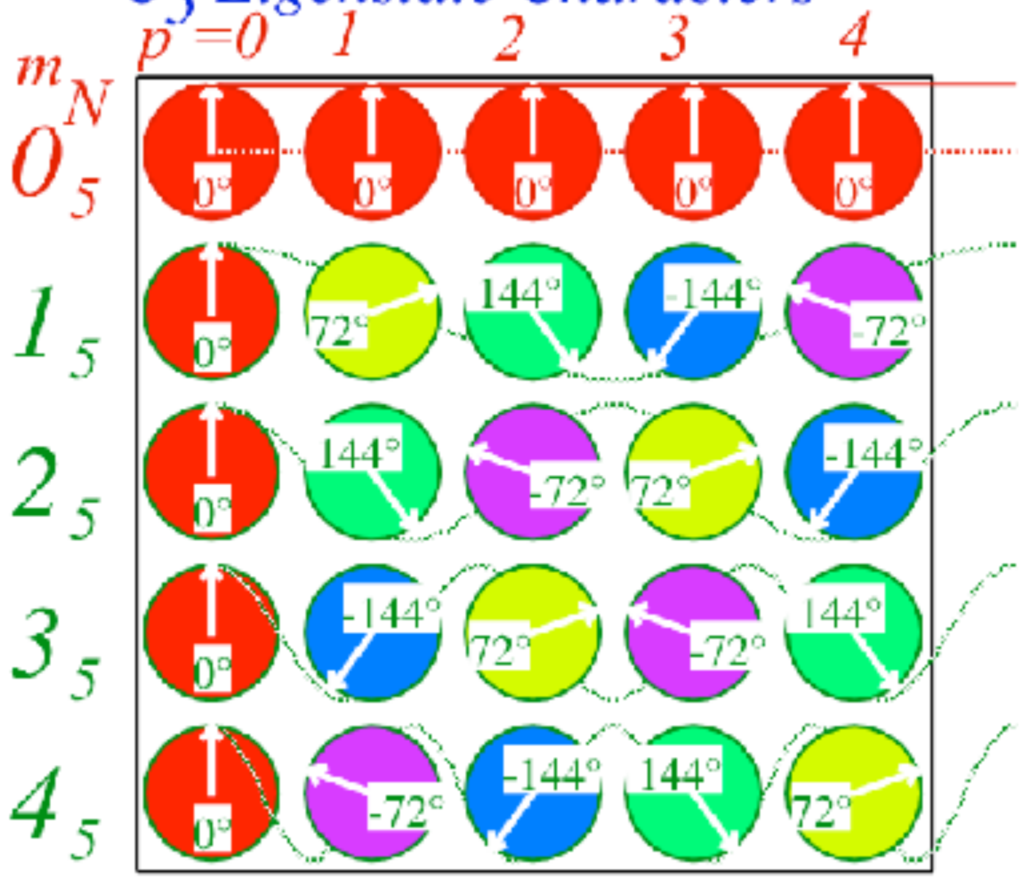




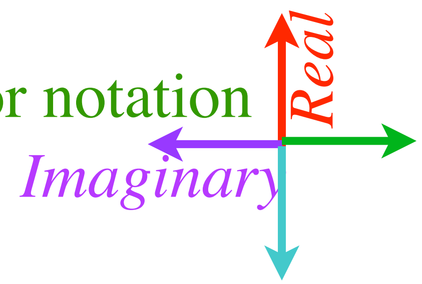


C_5 symmetry phase in *1*, *2*,...*5* level-systems

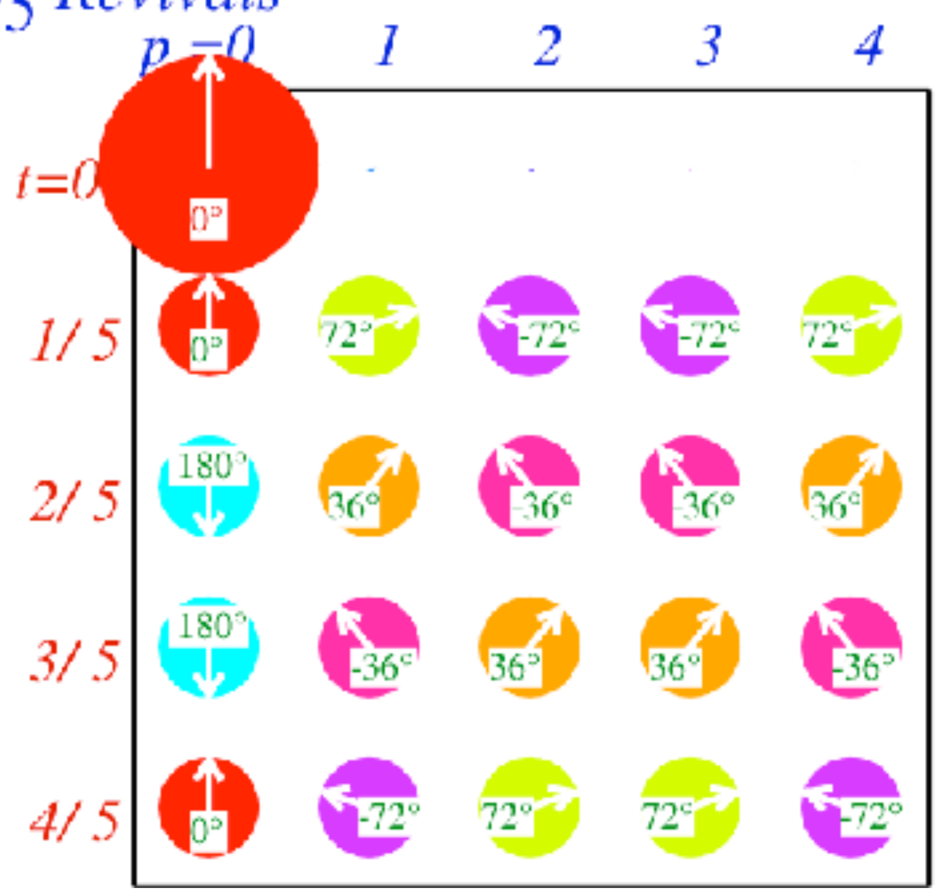
C_5 Eigenstate Characters

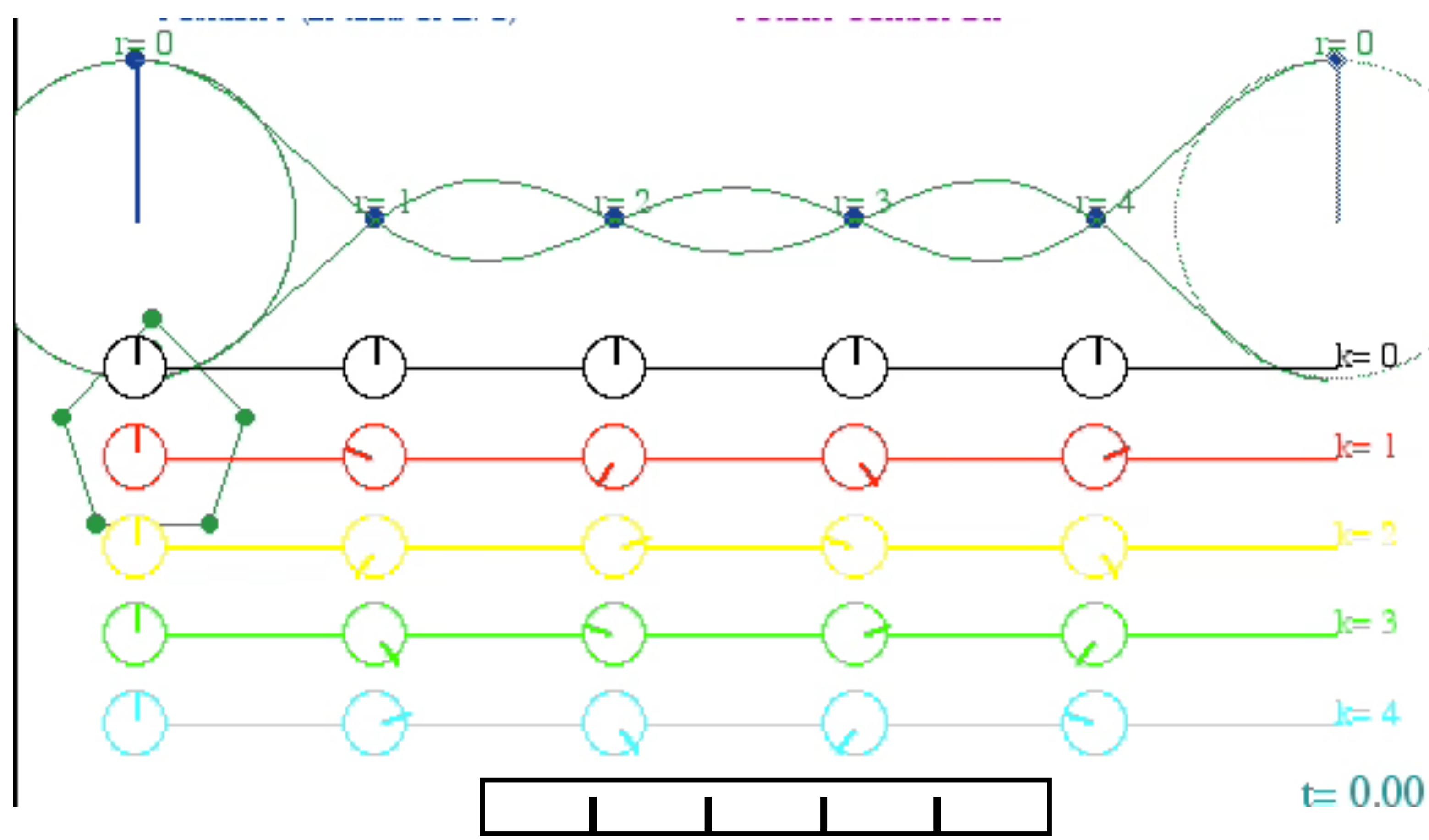


Phasor notation

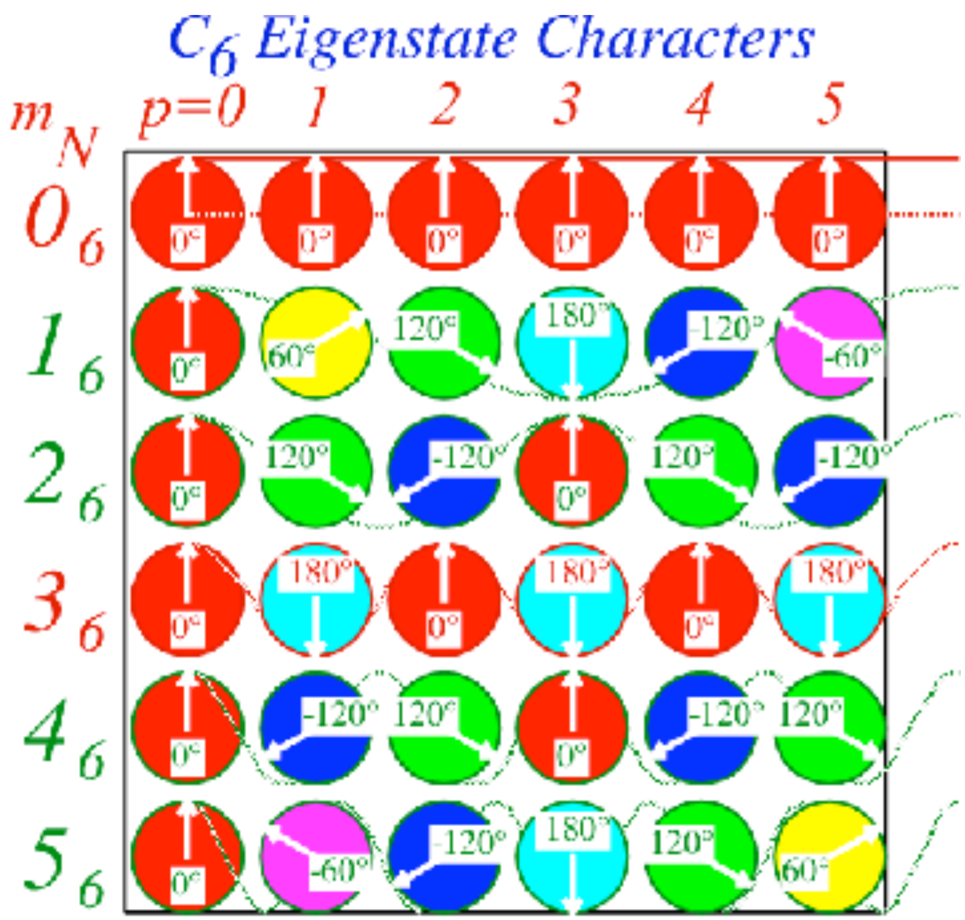


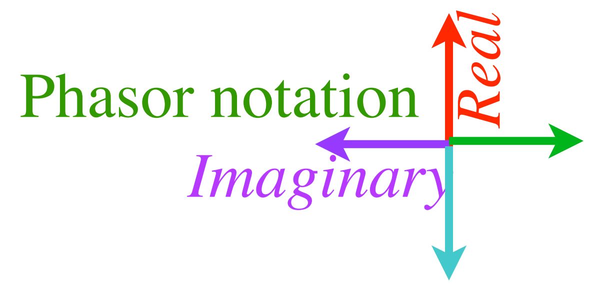
C_5 Revivals

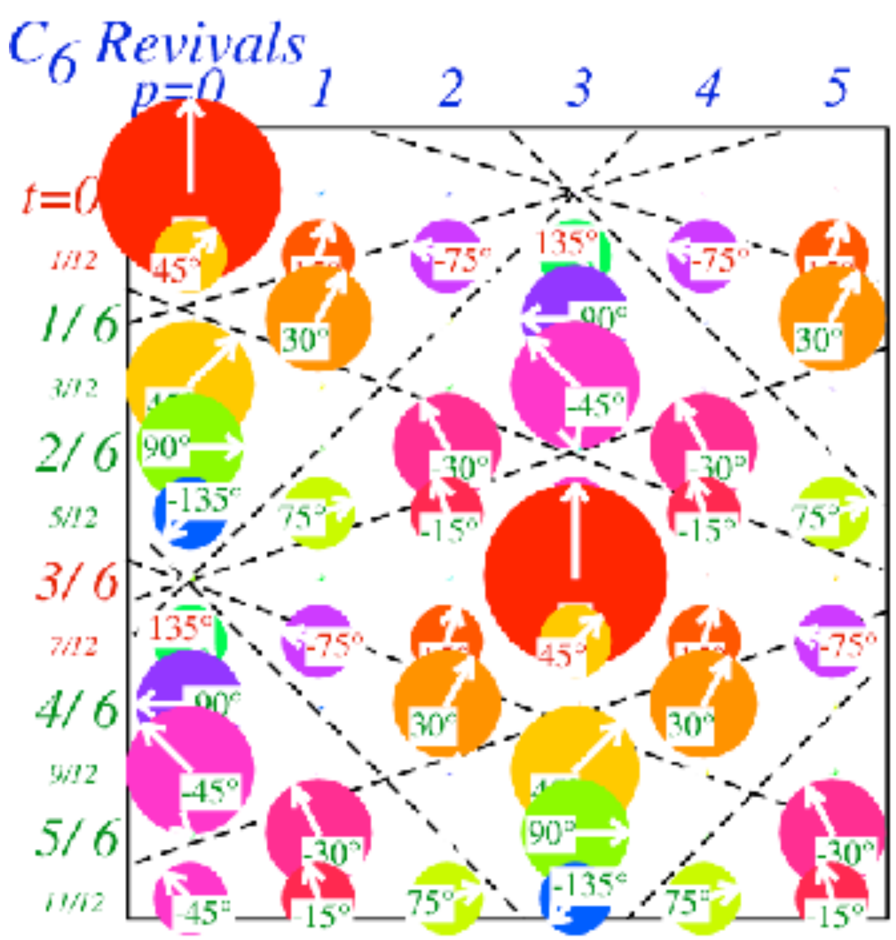


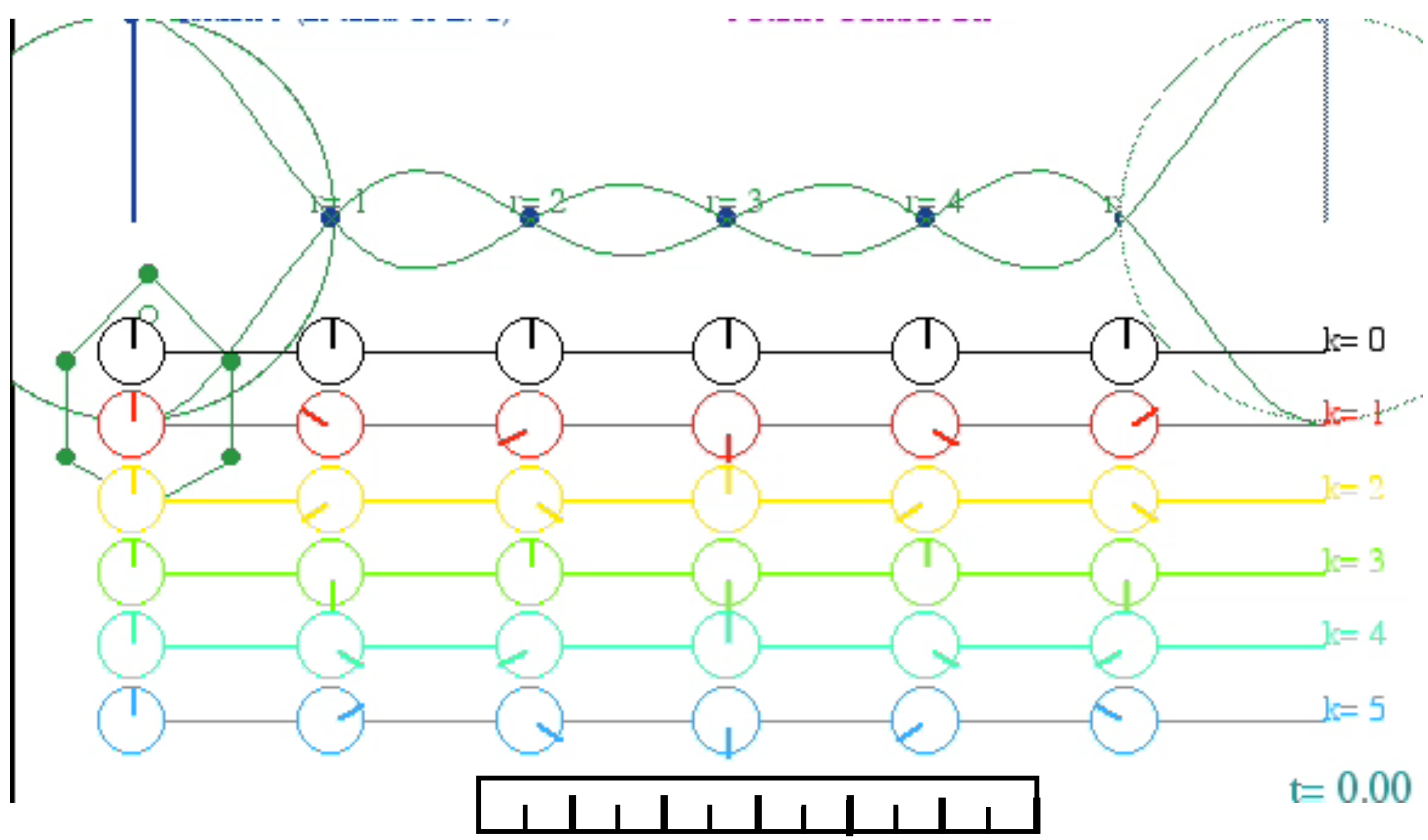


C_6 symmetry phase in *1*, ...*6* level-systems





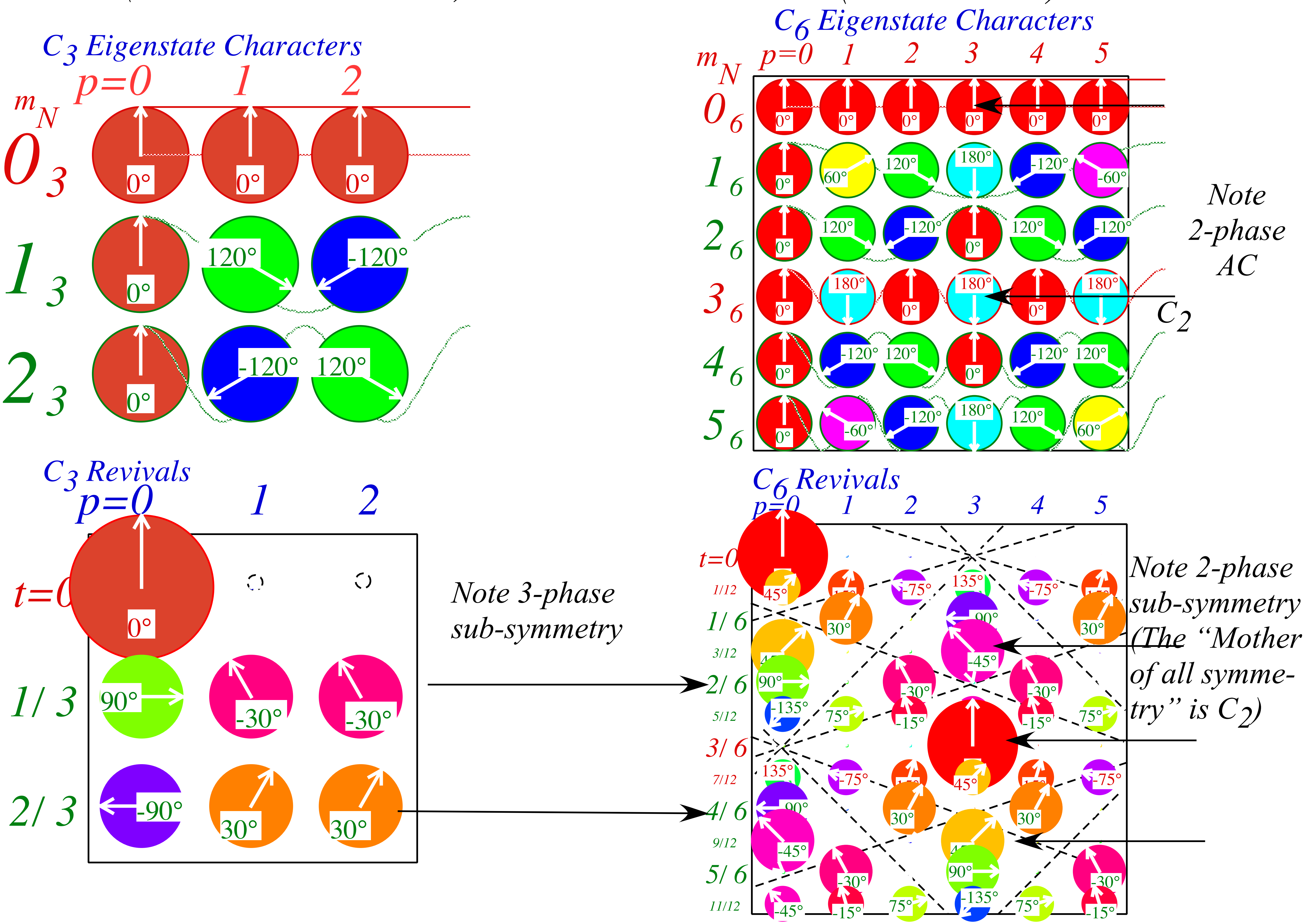




C_m algebra of revival-phase dynamics

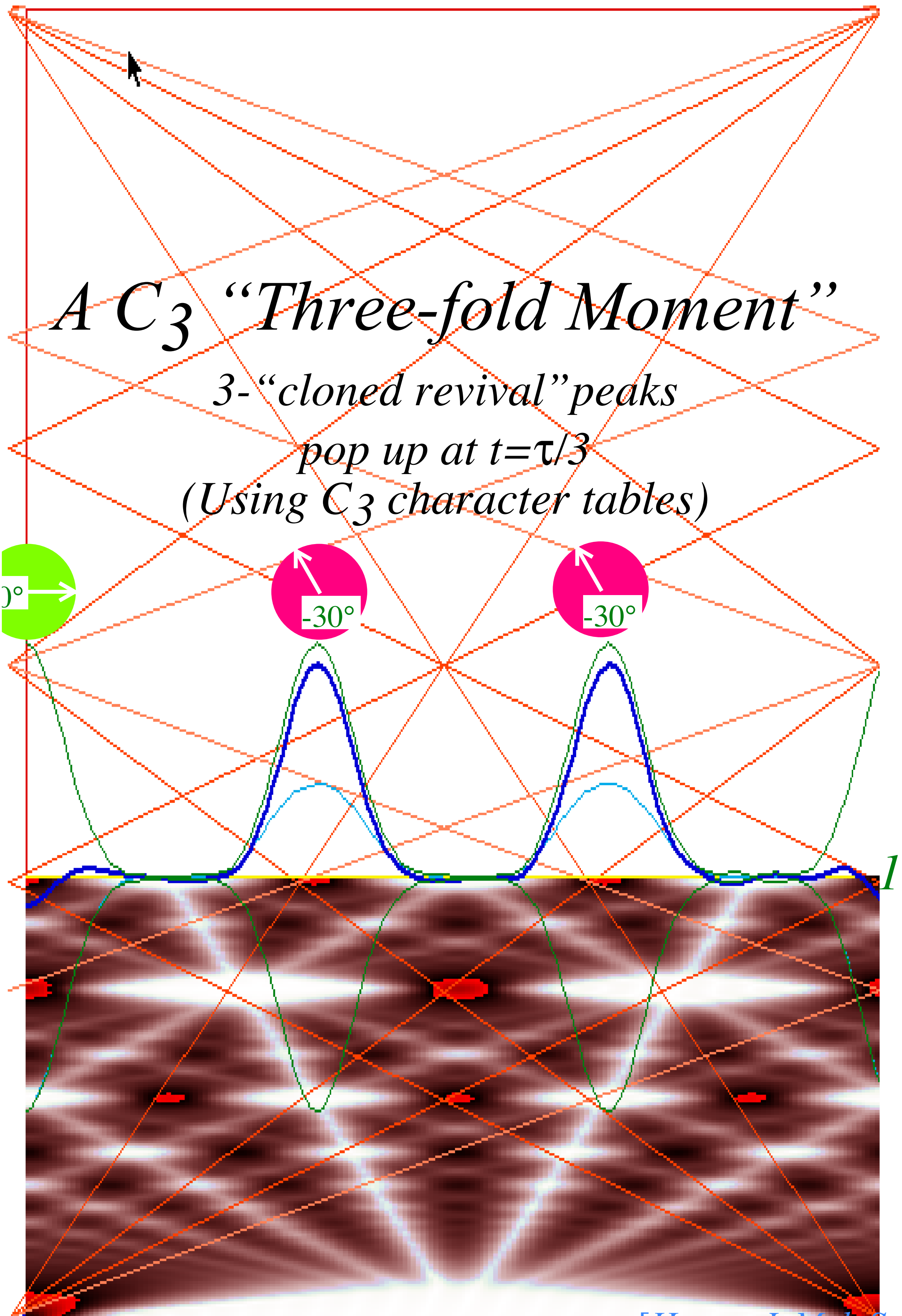
Discrete 3-State or Trigonal System (Tesla's 3-Phase AC)

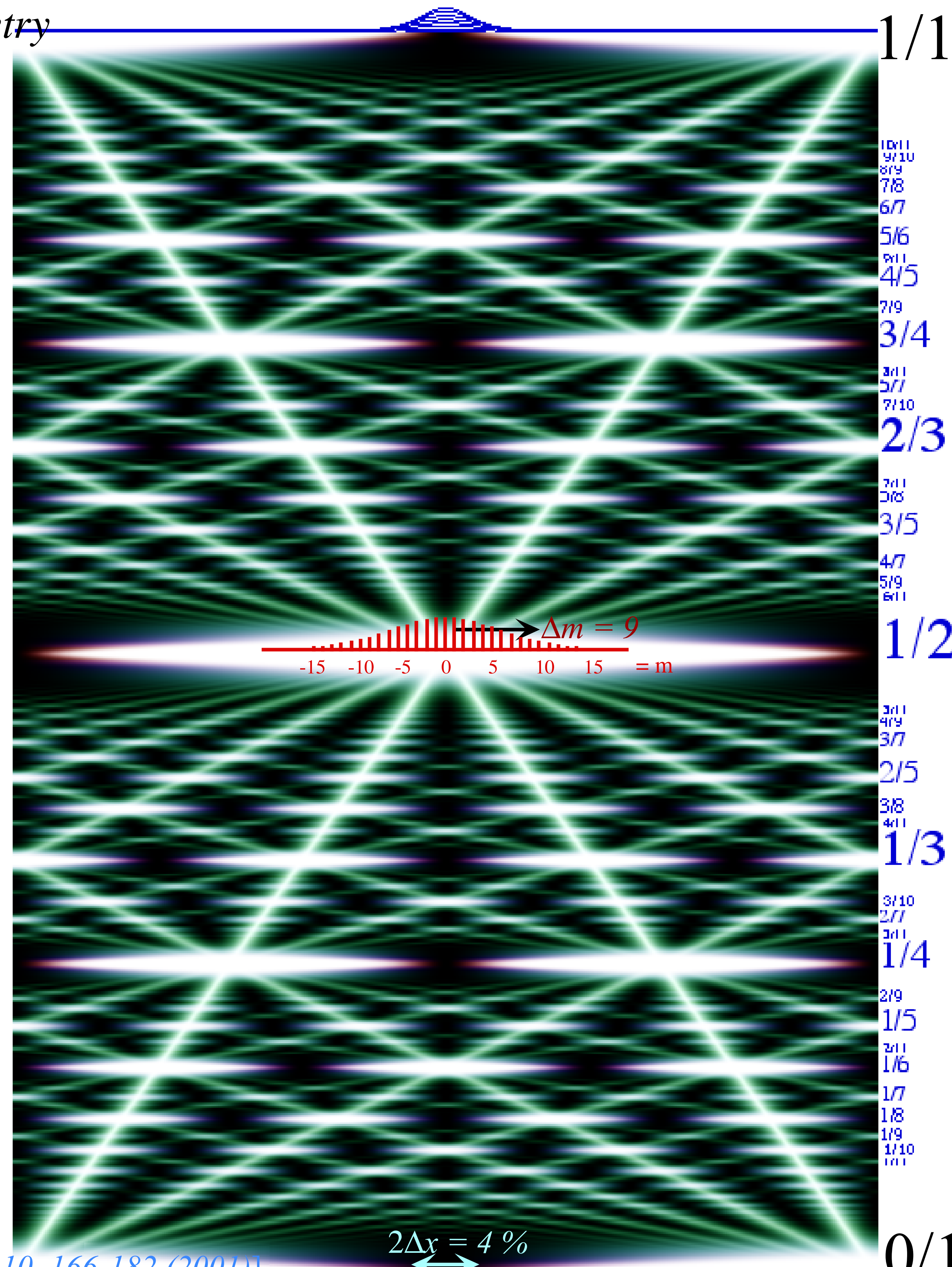
Discrete 6-State or Hexagonal System (6-Phase AC)



C_m algebra of revival-phase dynamics

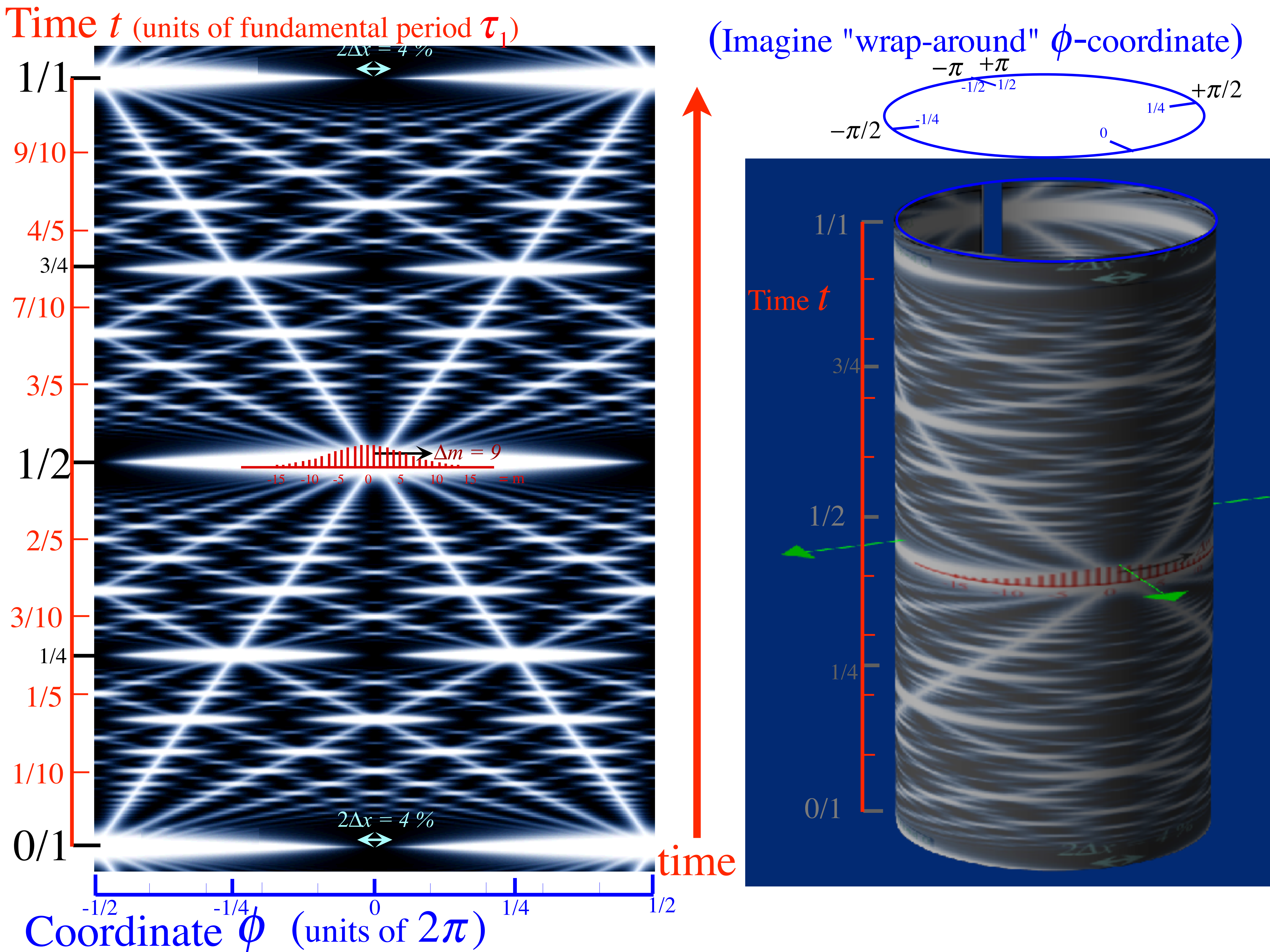
Quantum rotor fractional take turns at Cn symmetry





[Harter, J. Mol. Spec. 210, 166-182 (2001)]

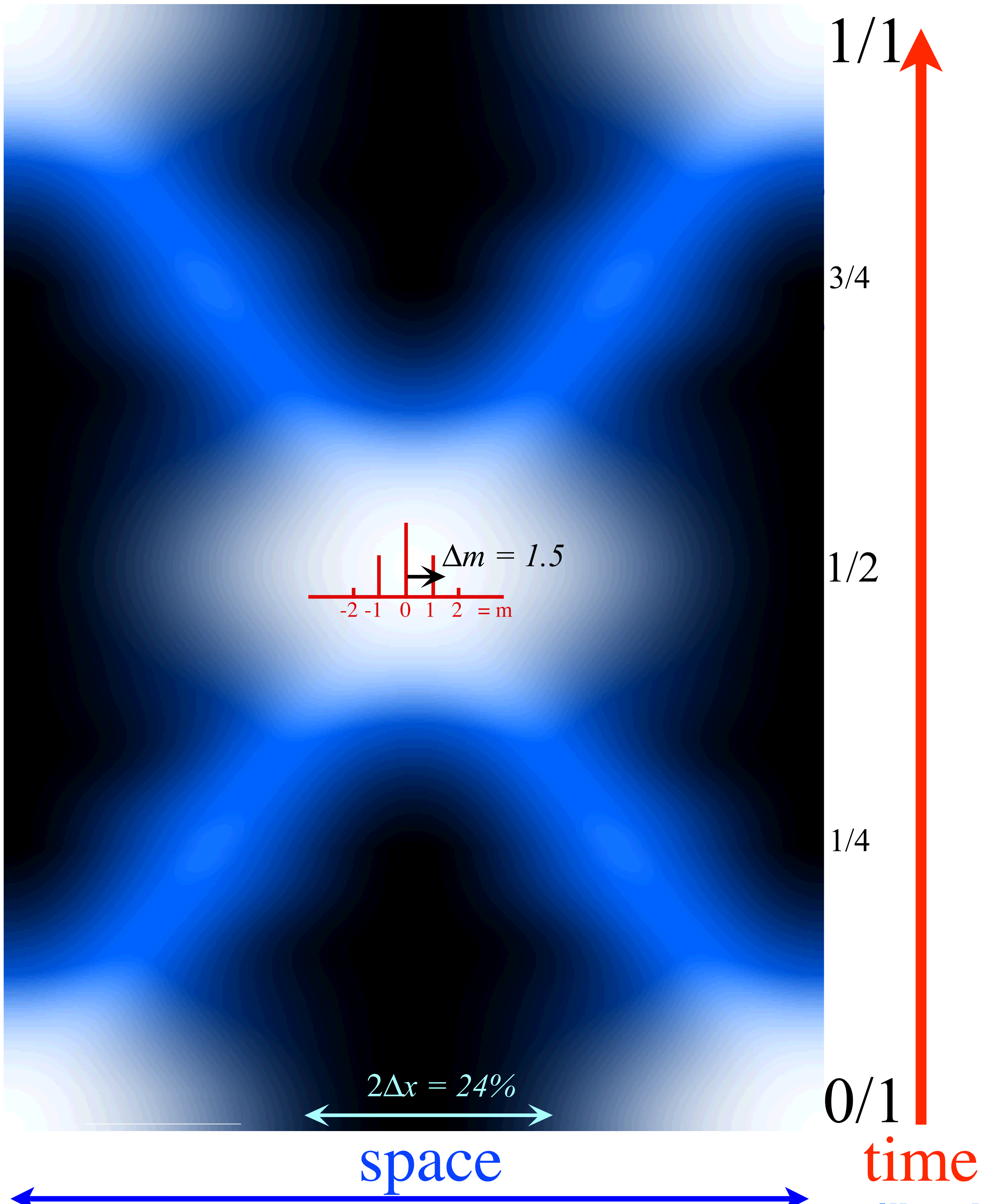
Algebra and geometry of resonant revivals: Farey Sums and Ford Circles



Time t (units of fundamental period τ_1)
(Imagine "wrap-around" ϕ-coordinate)
$2\Delta x = 4$ %
1/1
9/10
4/5
3/4
7/10
3/5
1/2
$\Delta m = 9$
-15 -10 -5 0 5 10 15 = m
2/5
3/10
1/4
1/5
1/10
0/1
time
-1/2 -1/4 0 1/4 1/2
Coordinate ϕ (units of 2π)
$-\pi$ $+\pi$
-1/2 1/2
$+\pi/2$
1/4
$-\pi/2$
-1/4
0
Time t
1/1
3/4
1/2
1/4
0/1

N-level-rotor system revival-beat wave dynamics

(Just 2-levels (0, ±1) (and some ±2) excited)



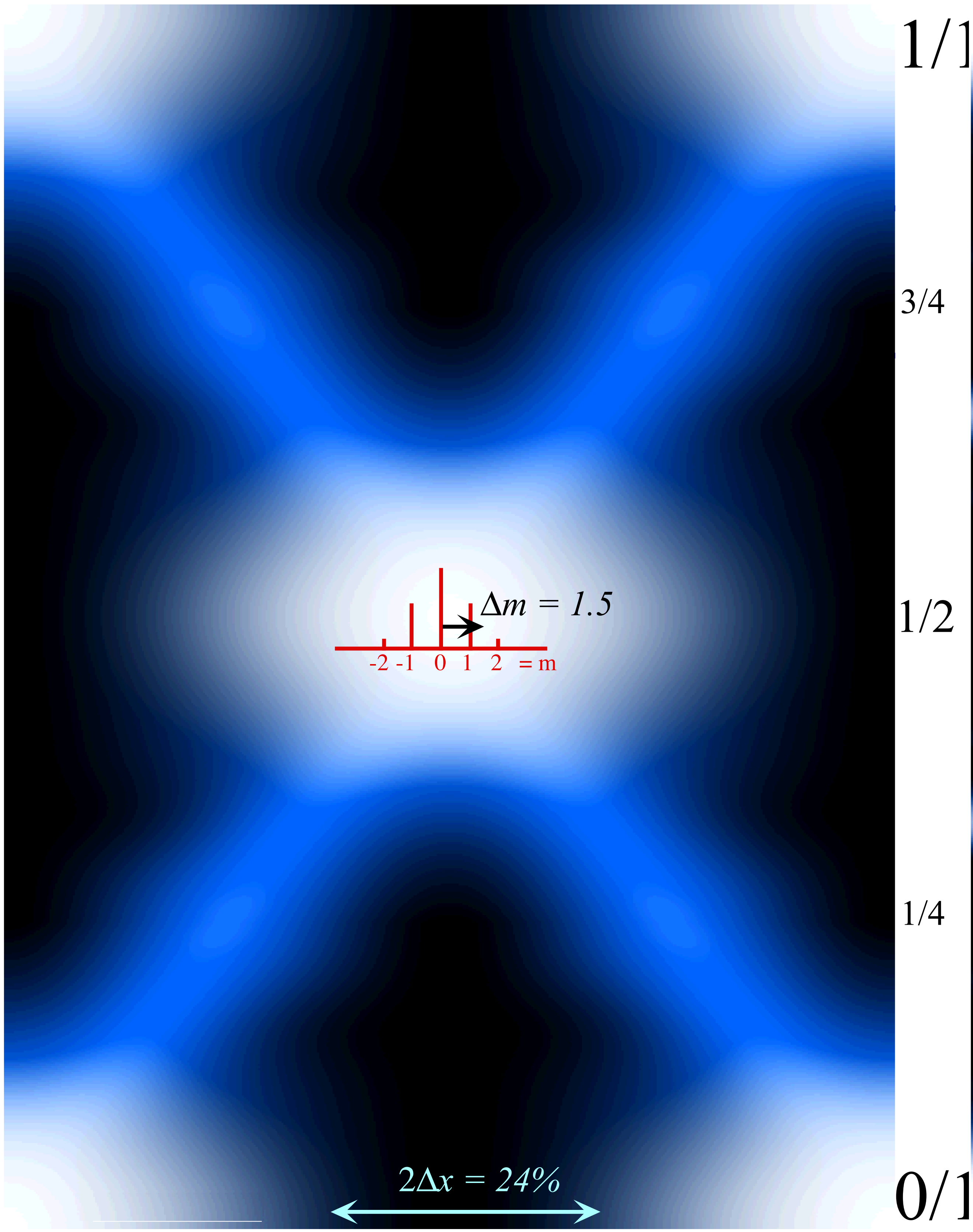
|Ψ(x,t)| in space-time

Simplest quantum revival:
Exciting first two levels
(**l**=0 and **l**=±1)
is like a
2-level system quantum beat
in space-time

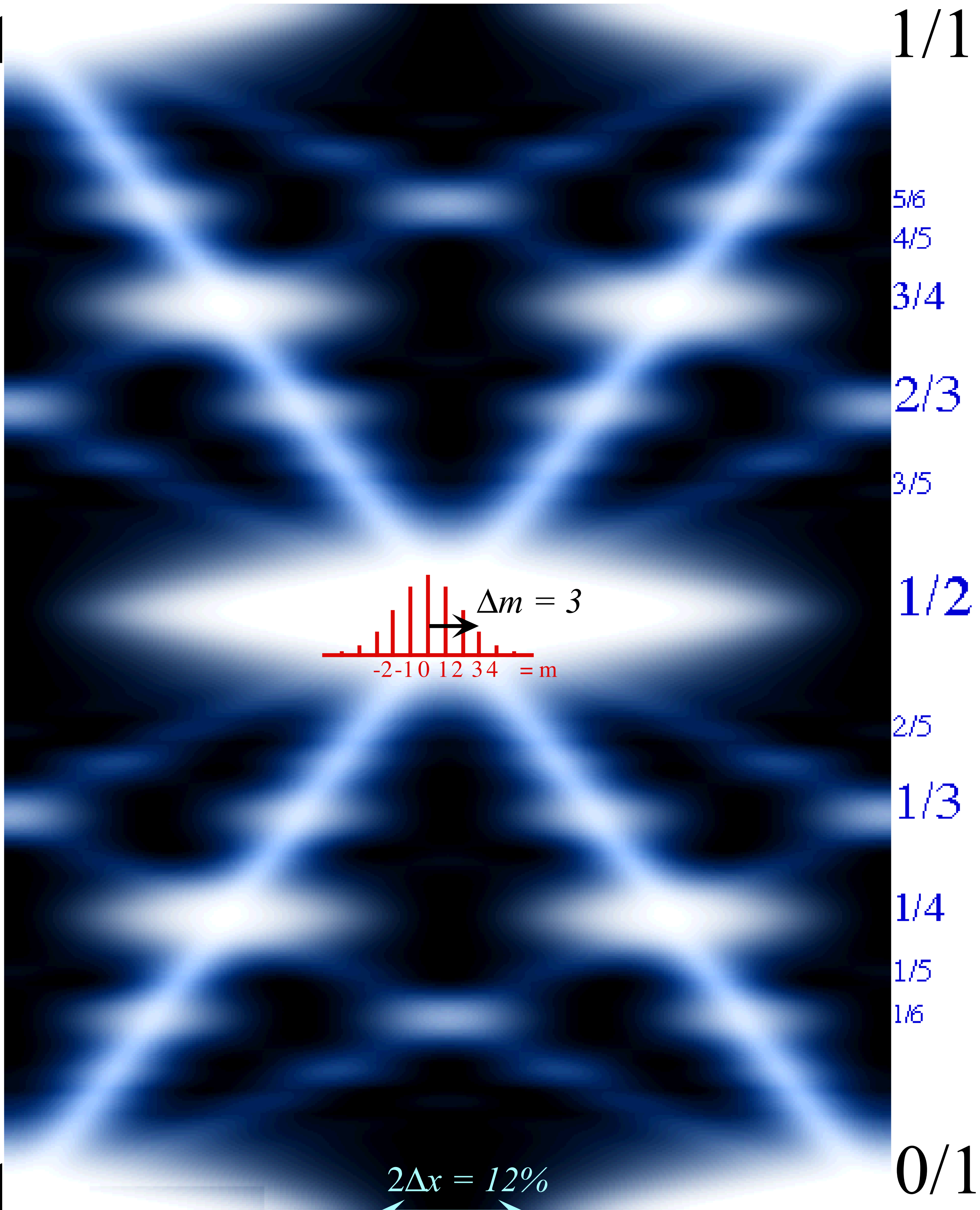
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

N-level-rotor system revival-beat wave dynamics

(Just 2-levels (0, ±1) (and some ±2) excited)



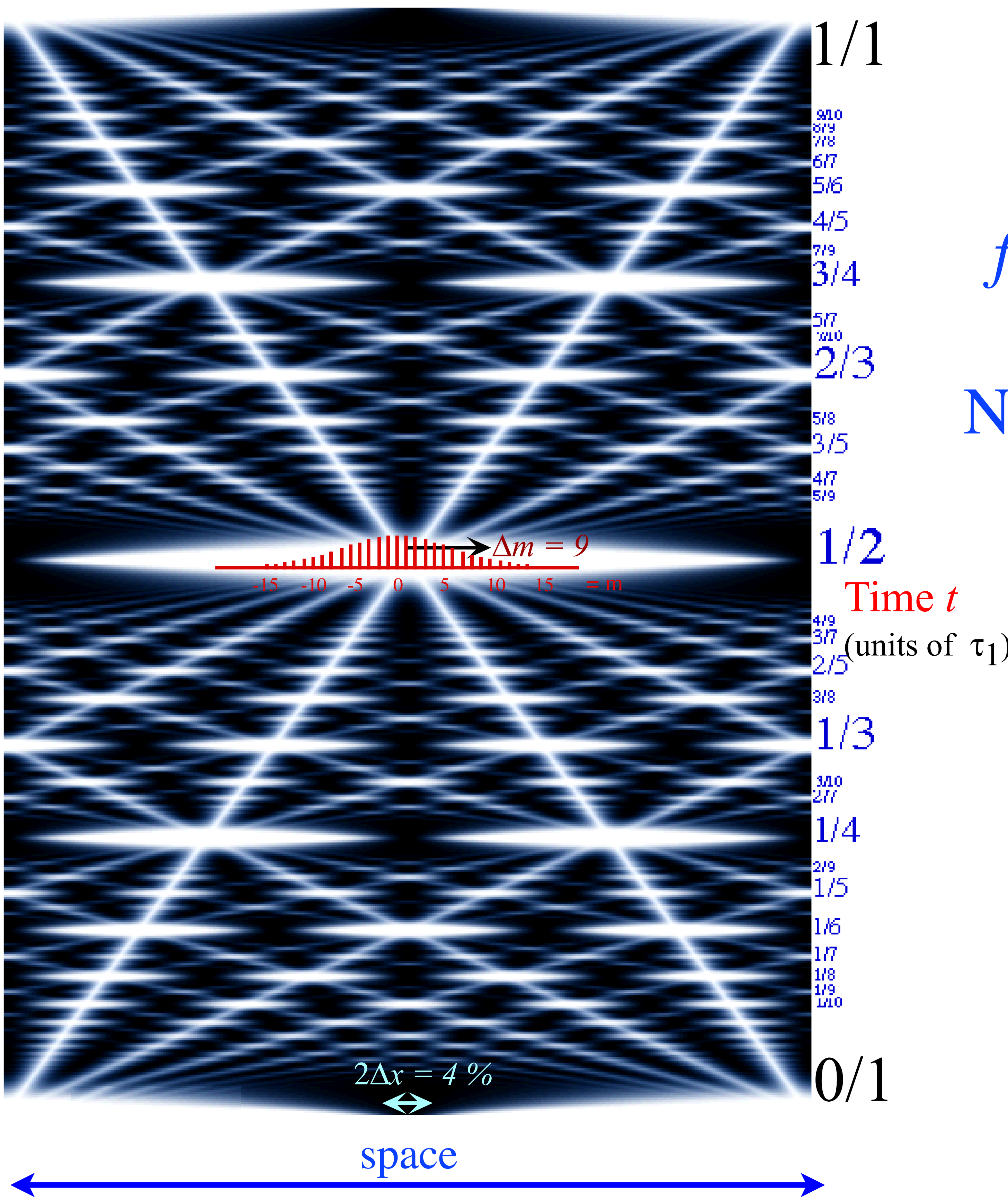
(4-levels (0, ±1,±2,±3) (and some ±4) excited)



Simplest *fractional* quantum revivals: 3,4,5-level systems

N-level-rotor system revival-beat wave dynamics

(9 or10-levels (0, ±1, ±2, ±3, ±4,..., ±9, ±10, ±11...) excited)



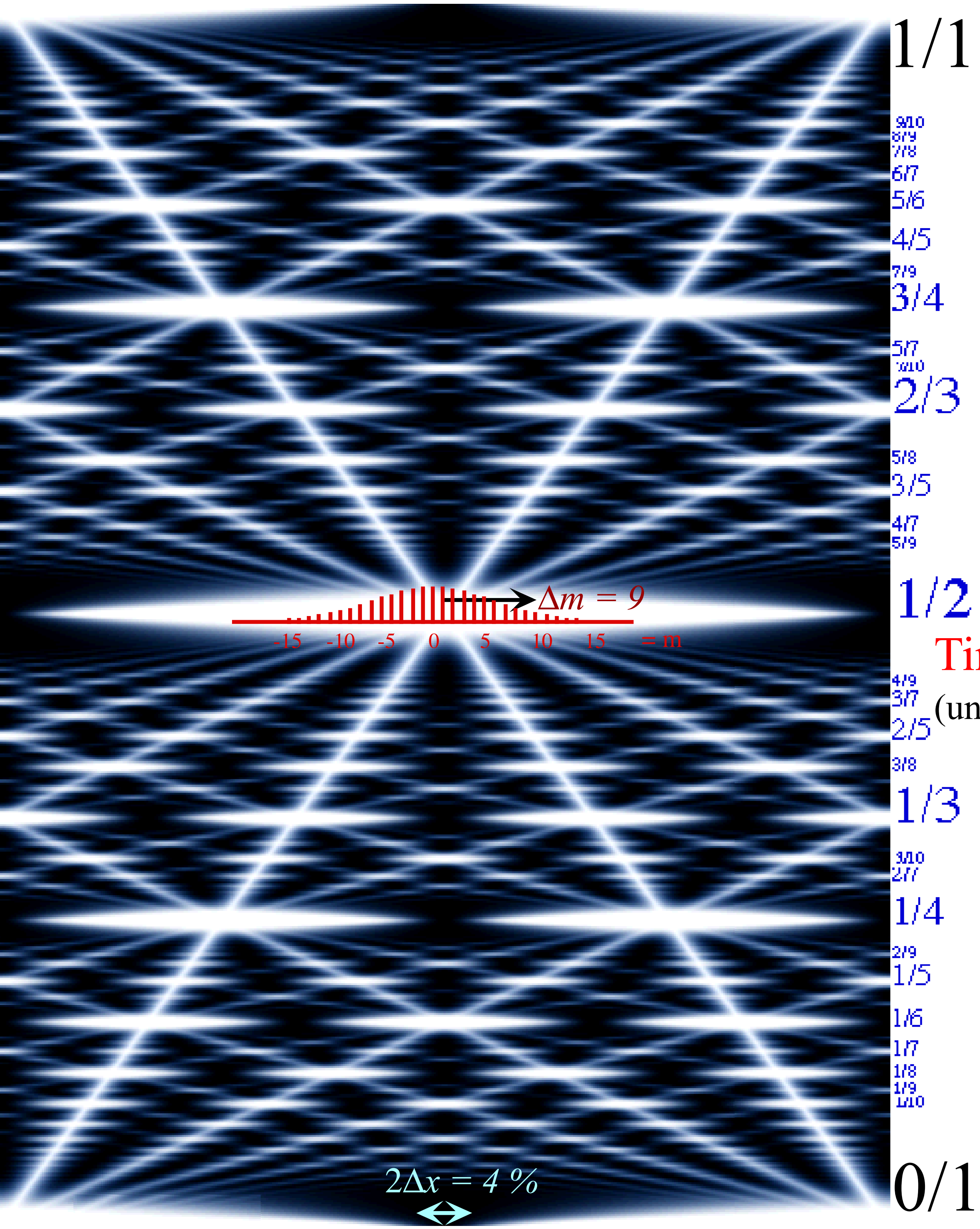
fractional quantum revivals:
in 3,4,..., N-level systems
Number increases rapidly with
number of levels
and/or bandwidth
of excitation

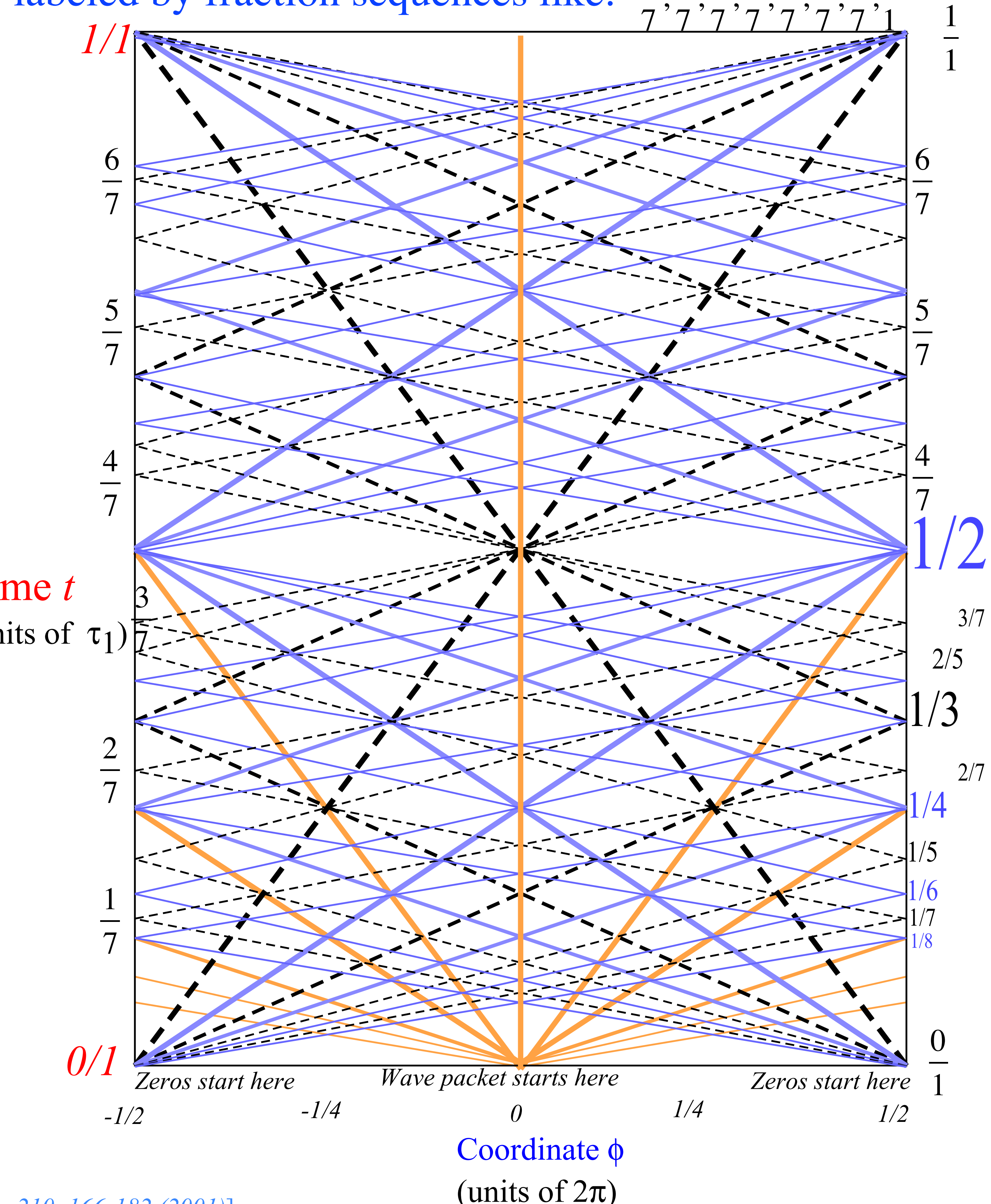
[Harter, J. Mol. Spec. 210, 166-182 (2001)]

N-level-rotor system revival-beat wave dynamics

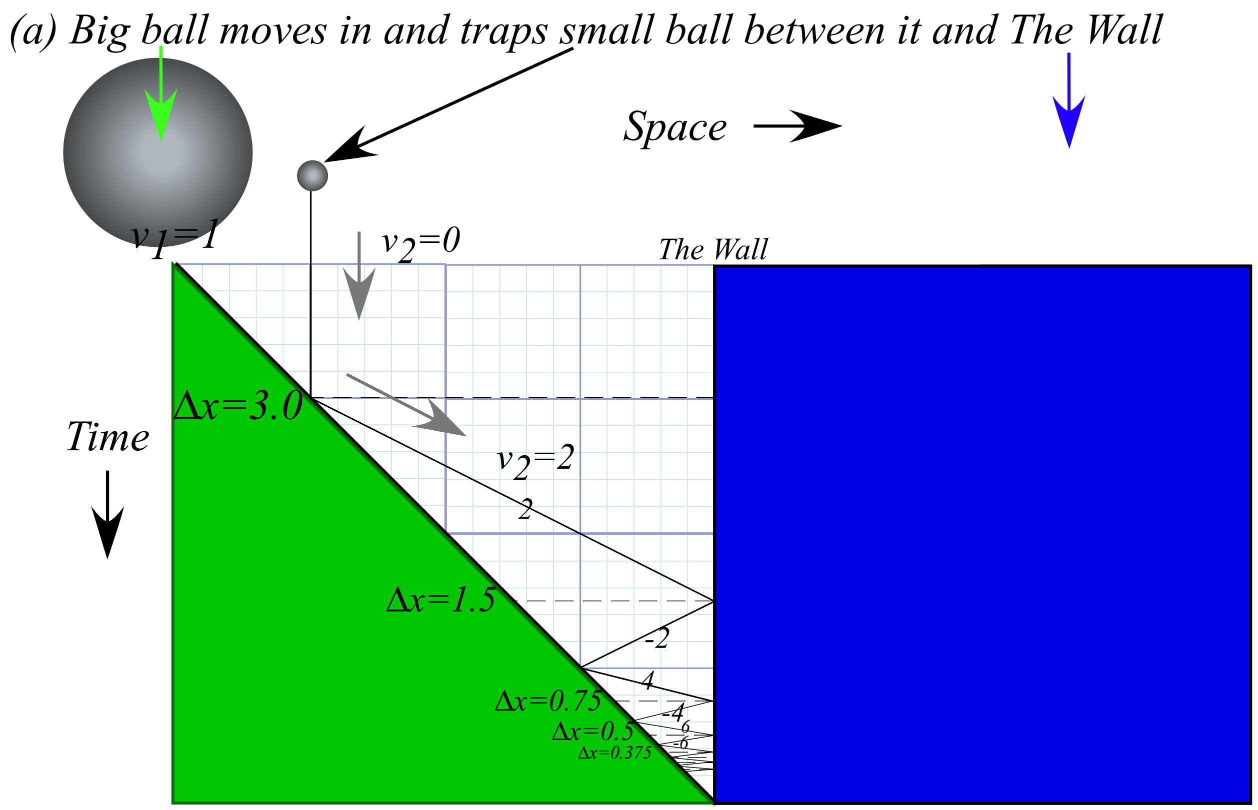
(9 or10-levels $(0, \pm1, \pm2, \pm3, \pm4, ..., \pm9, \pm10, \pm11...)$ excited)

Zeros (clearly) and "particle-packets" (faintly) have paths labeled by fraction sequences like: $\frac{0}{7}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{1}$





[Harter, J. Mol. Spec. 210, 166-182 (2001)]



Lect. 5 (9.11.14)

The Classical "Monster Mash"

Classical introduction to

Heisenberg "Uncertainty" Relations

$$v_2 = \frac{const.}{Y} \quad \text{or:} \quad Y \cdot v_2 = const.$$

is analogous to: $\Delta x \cdot \Delta p = N \cdot \hbar$

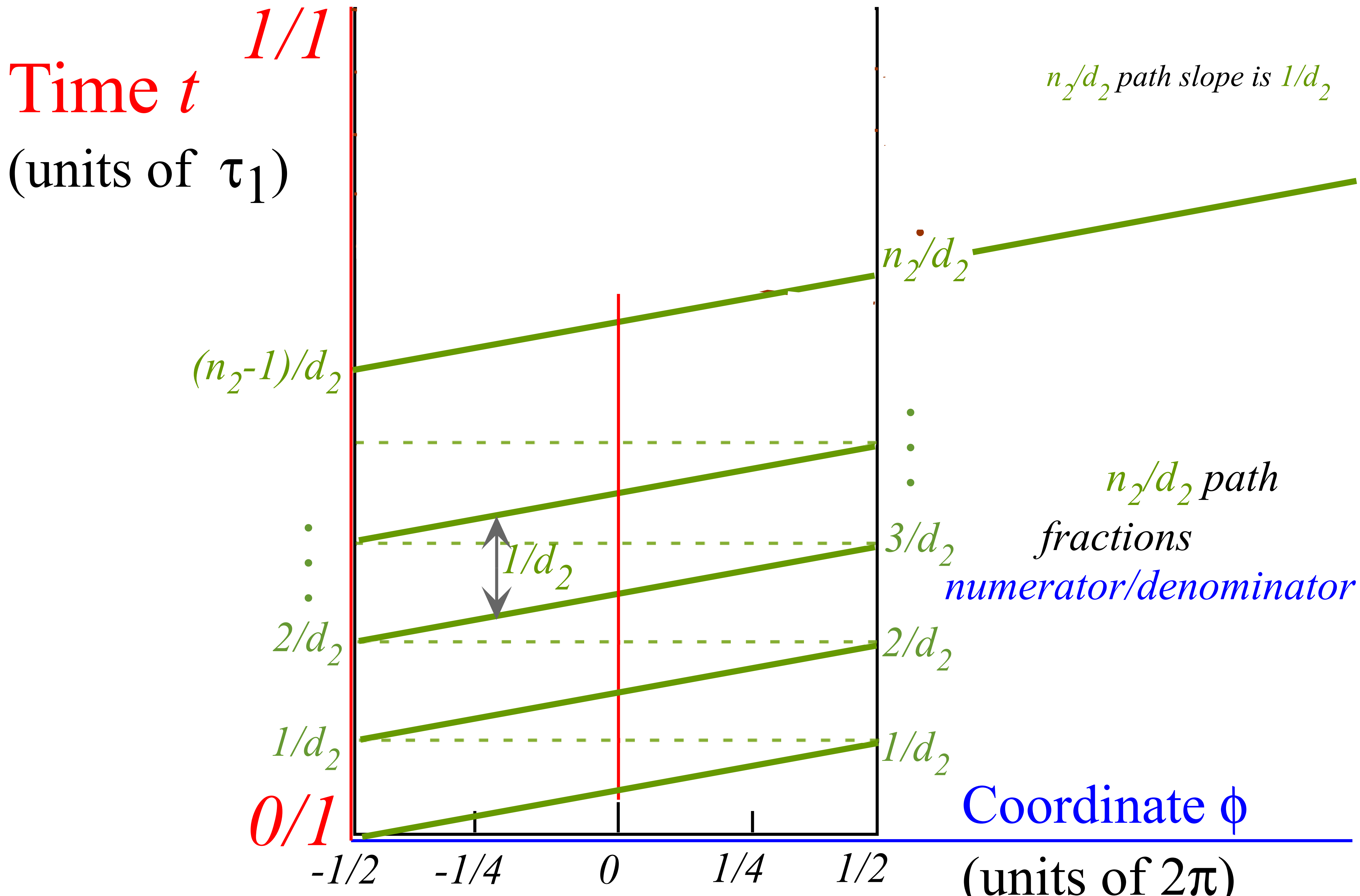
Recall classical "Monster Mash" in Lecture 5

with small-ball trajectory paths having same geometry
as revival beat wave-zero paths

Farey-Sum arithmetic of revival wave-zero paths
(How *Rational Fractions N/D* occupy real space-time)

Farey Sum algebra of revival-beat wave dynamics

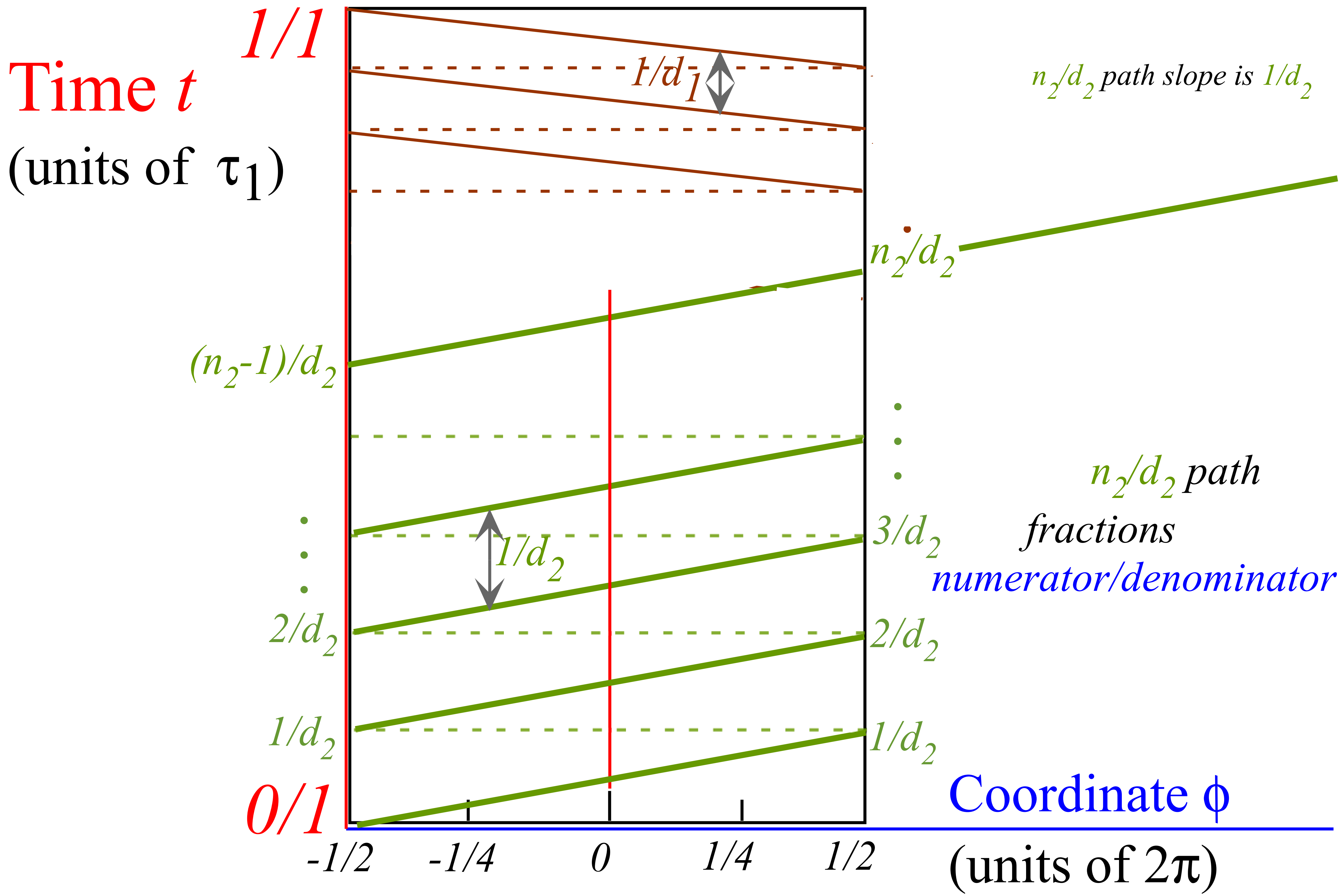
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

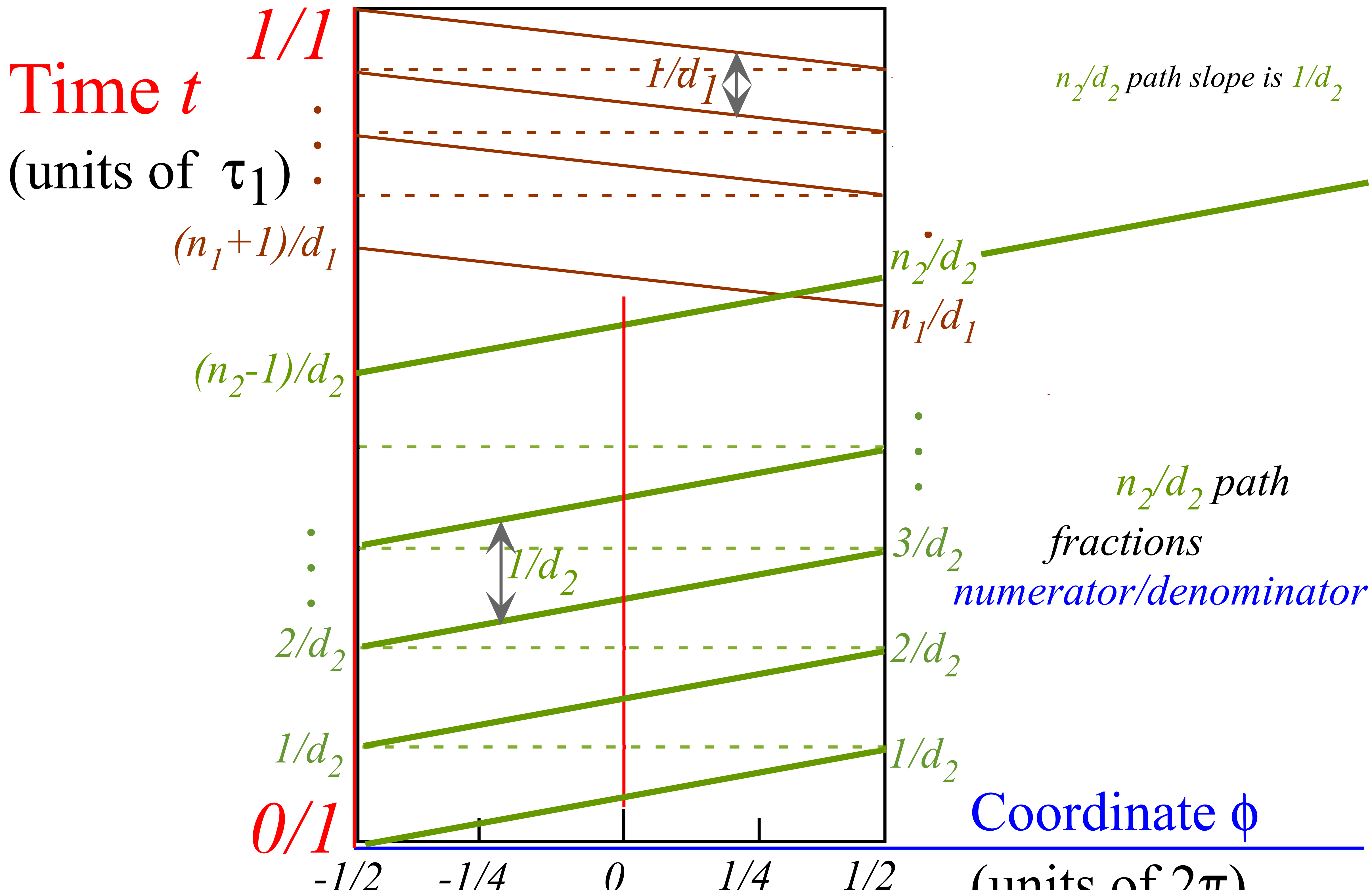
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

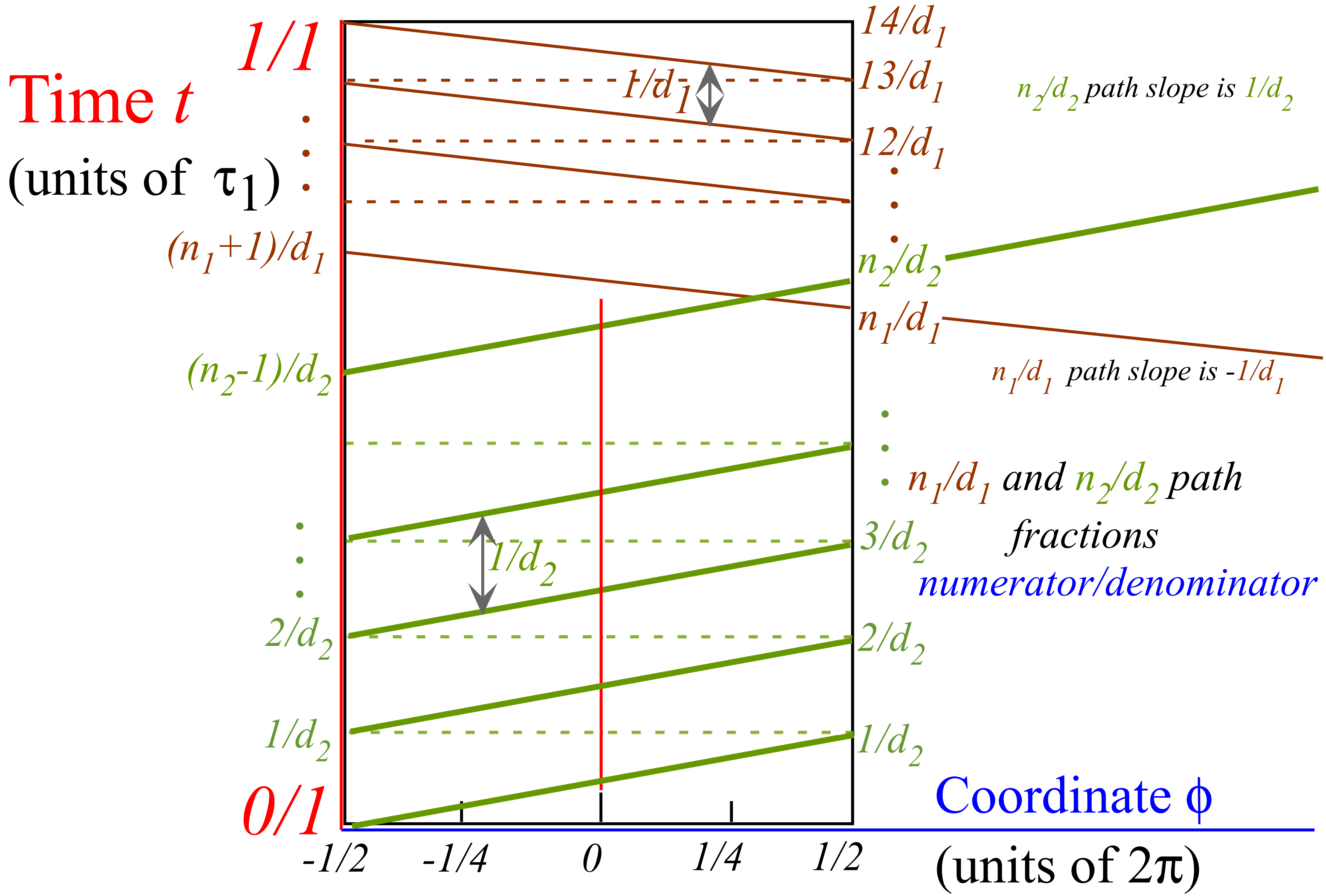
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

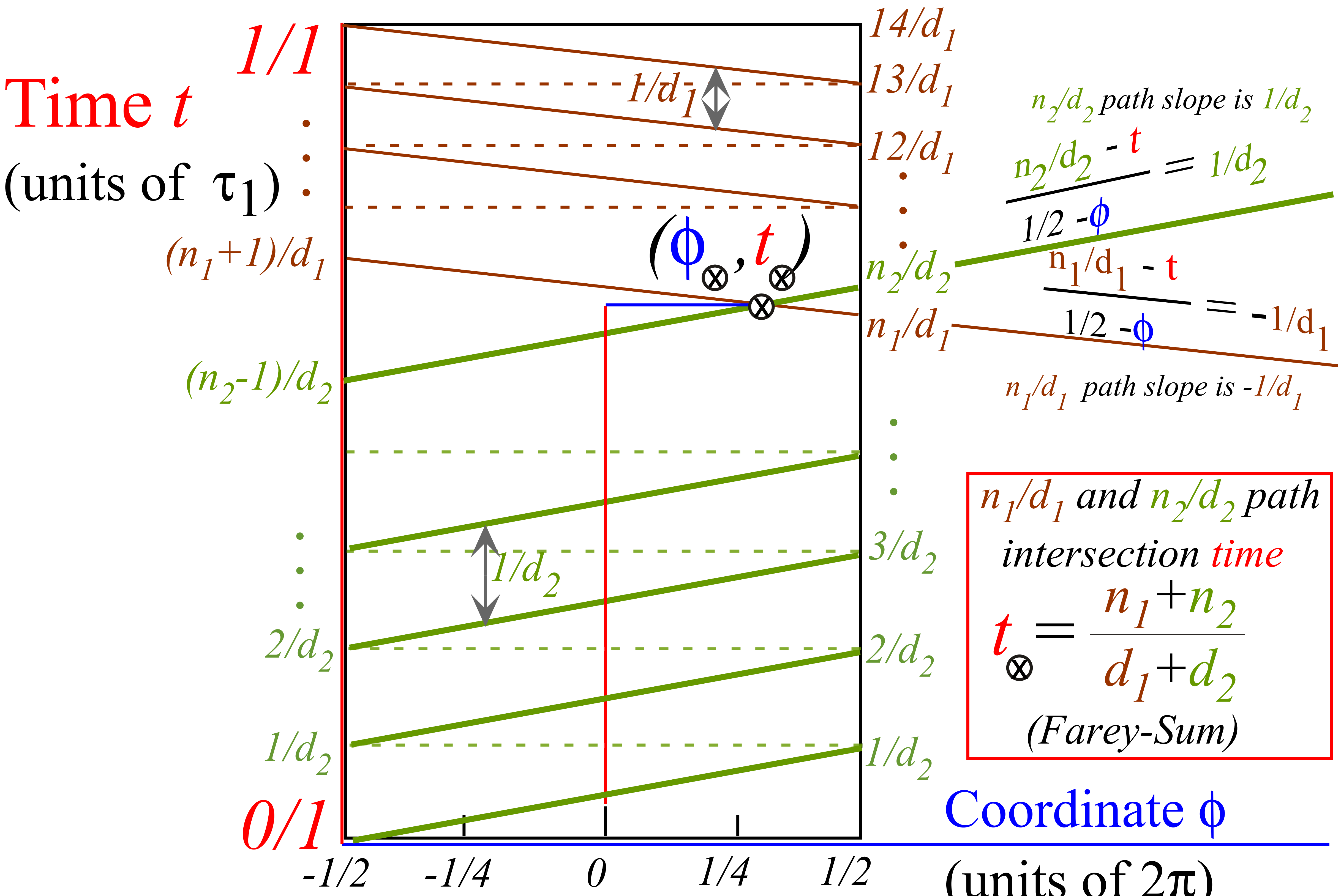
Label by *numerators N* and *denominators D* of rational fractions *N/D*



Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1+n_2}{d_1+d_2}$$

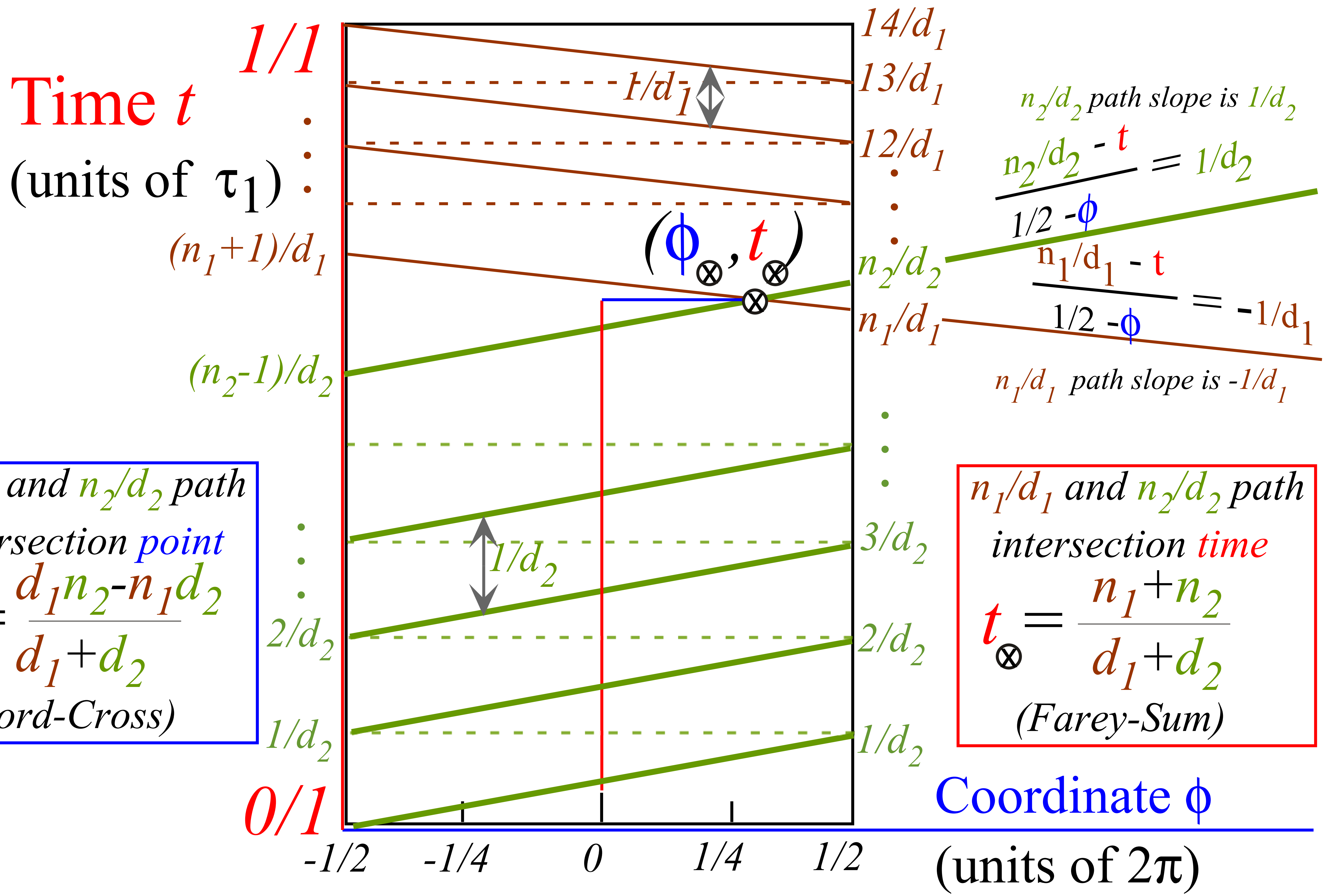
(Farey-Sum)

[*John Farey, Phil. Mag.(1816)*]

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

Farey Sum algebra of revival-beat wave dynamics

Label by *numerators N* and *denominators D* of rational fractions *N/D*



n_1/d_1 and n_2/d_2 path intersection point

$$\phi_\otimes = \frac{d_1 n_2 - n_1 d_2}{d_1 + d_2}$$

(Ford-Cross)

n_1/d_1 and n_2/d_2 path intersection time

$$t_\otimes = \frac{n_1 + n_2}{d_1 + d_2}$$

(Farey-Sum)

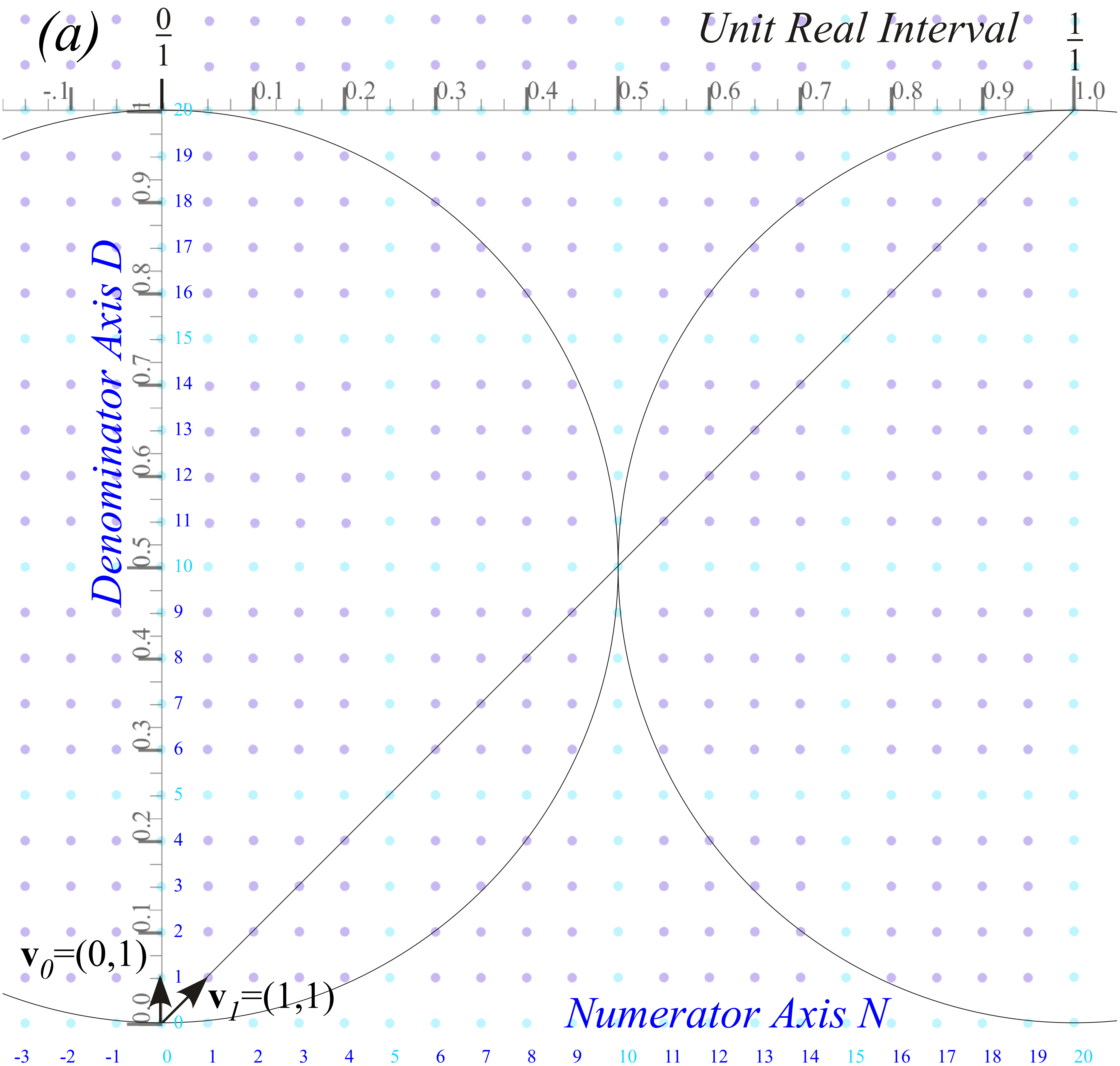
[*Lester. R. Ford, Am. Math. Monthly 45,586(1938)*]

Harter, J. Mol. Spec. 210, 166-182 (2001) and ISMS (2013)

[*John Farey, Phil. Mag.(1816)*]

Ford-Circle geometry of revival paths

(How *Rational Fractions N/D* occupy real space-time)

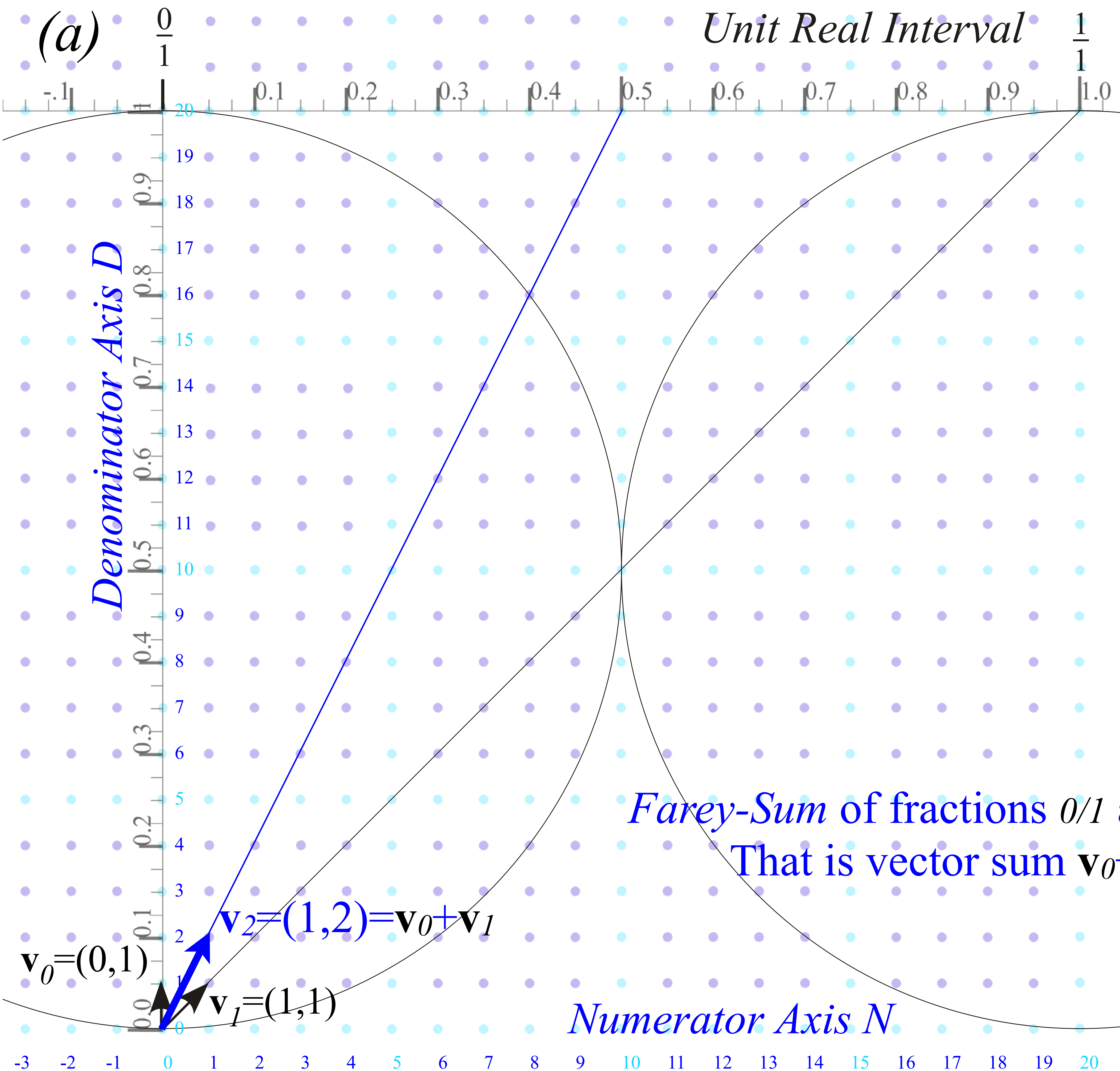


Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)

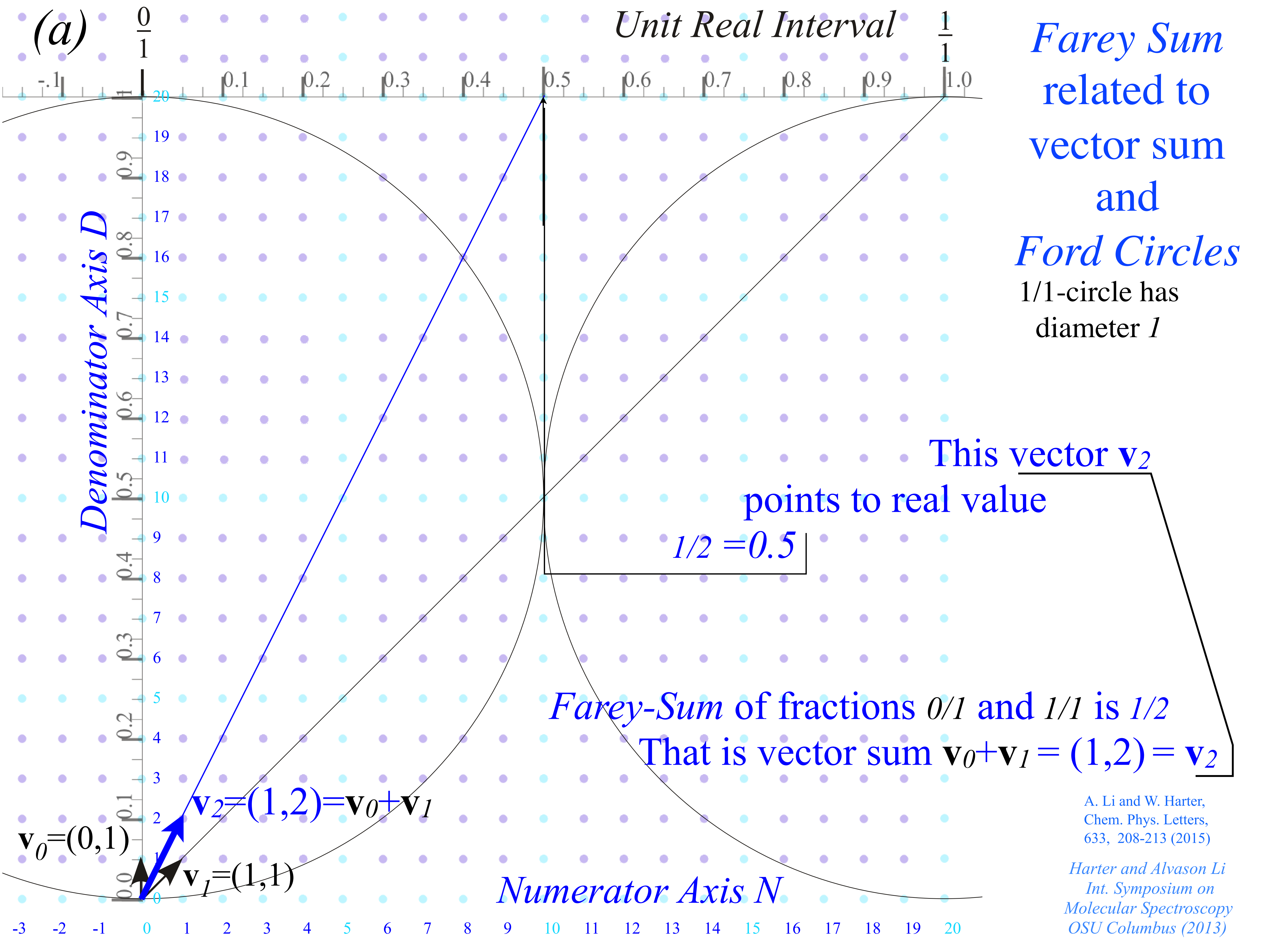


Farey Sum related to vector sum and *Ford Circles*

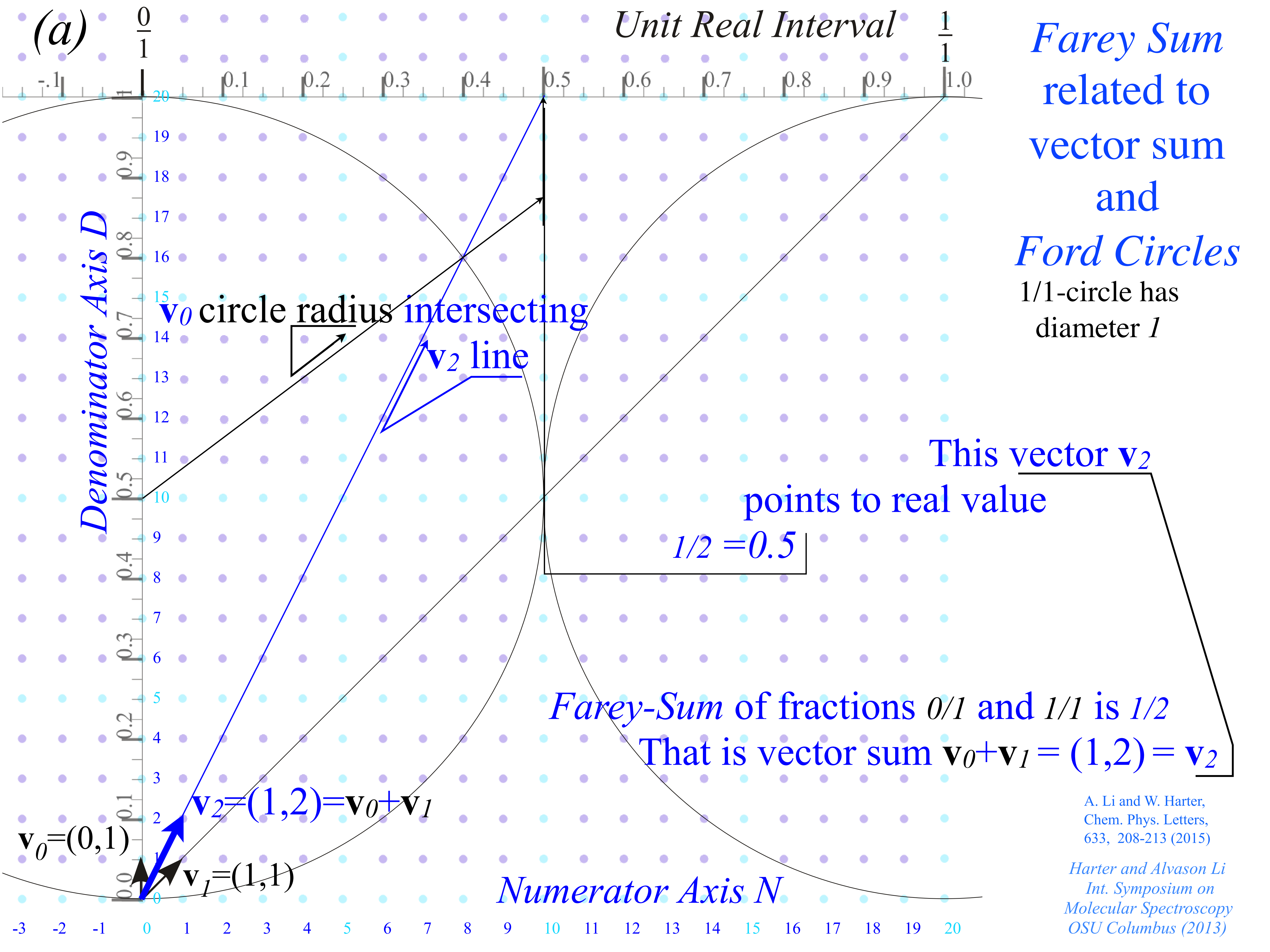
1/1-circle has diameter *1*

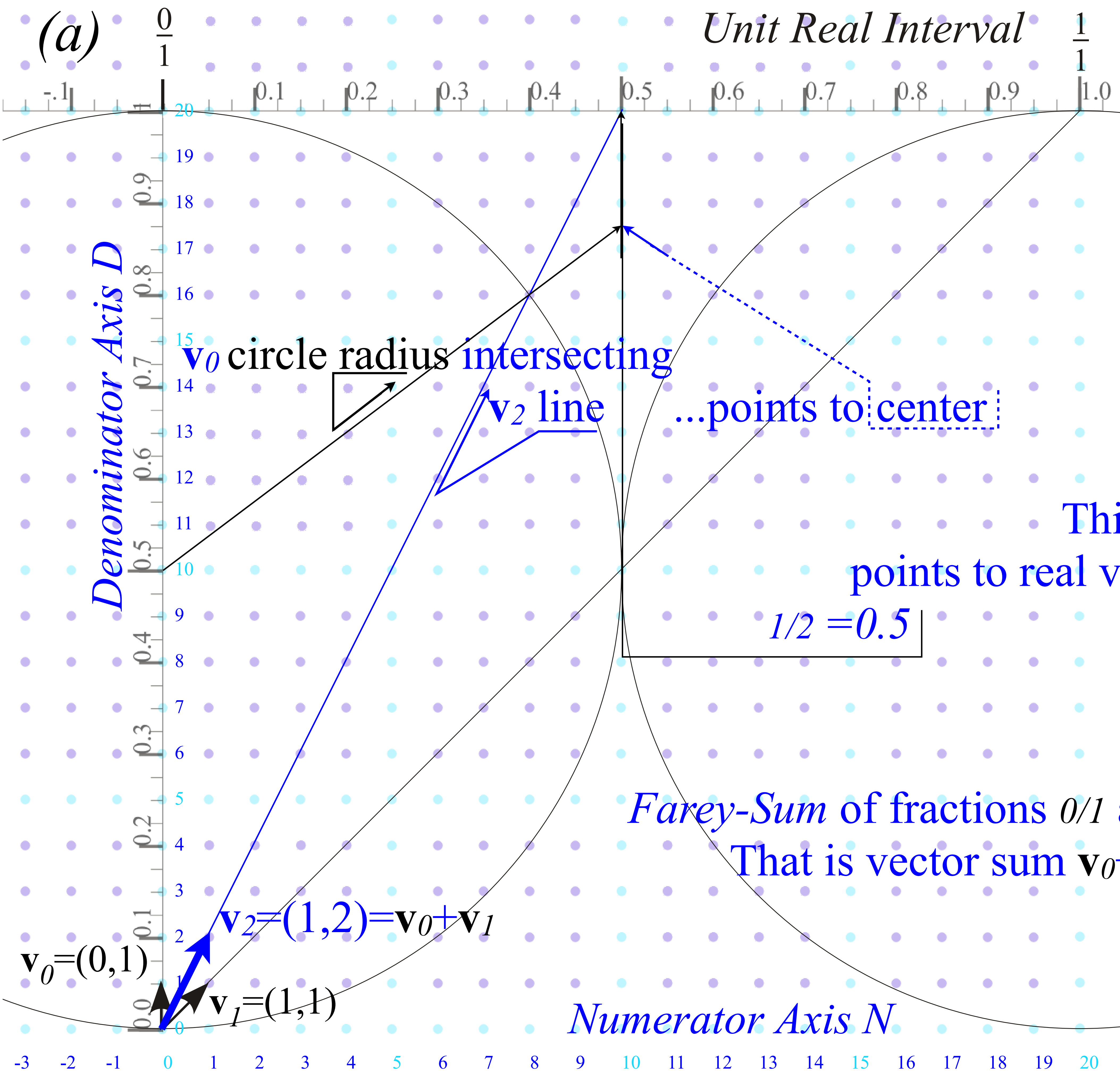
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)

Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



(a)
0/1
Unit Real Interval
1/1
Farey Sum
related to
vector sum
and
Ford Circles
1/1-circle has
diameter 1
Denominator Axis D
This vector v2
points to real value
1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
Numerator Axis N
A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)
Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)


(a)
Unit Real Interval
Farey Sum
related to
vector sum
and
Ford Circles
1/1-circle has
diameter 1
Denominator Axis D
$\mathbf{v}_0$ circle radius intersecting
$\mathbf{v}_2$ line
This vector $\mathbf{v}_2$
points to real value
$1/2 = 0.5$
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum $\mathbf{v}_0+\mathbf{v}_1 = (1,2) = \mathbf{v}_2$
$\mathbf{v}_2=(1,2)=\mathbf{v}_0+\mathbf{v}_1$
$\mathbf{v}_0=(0,1)$
$\mathbf{v}_1=(1,1)$
Numerator Axis N
A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)
Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)

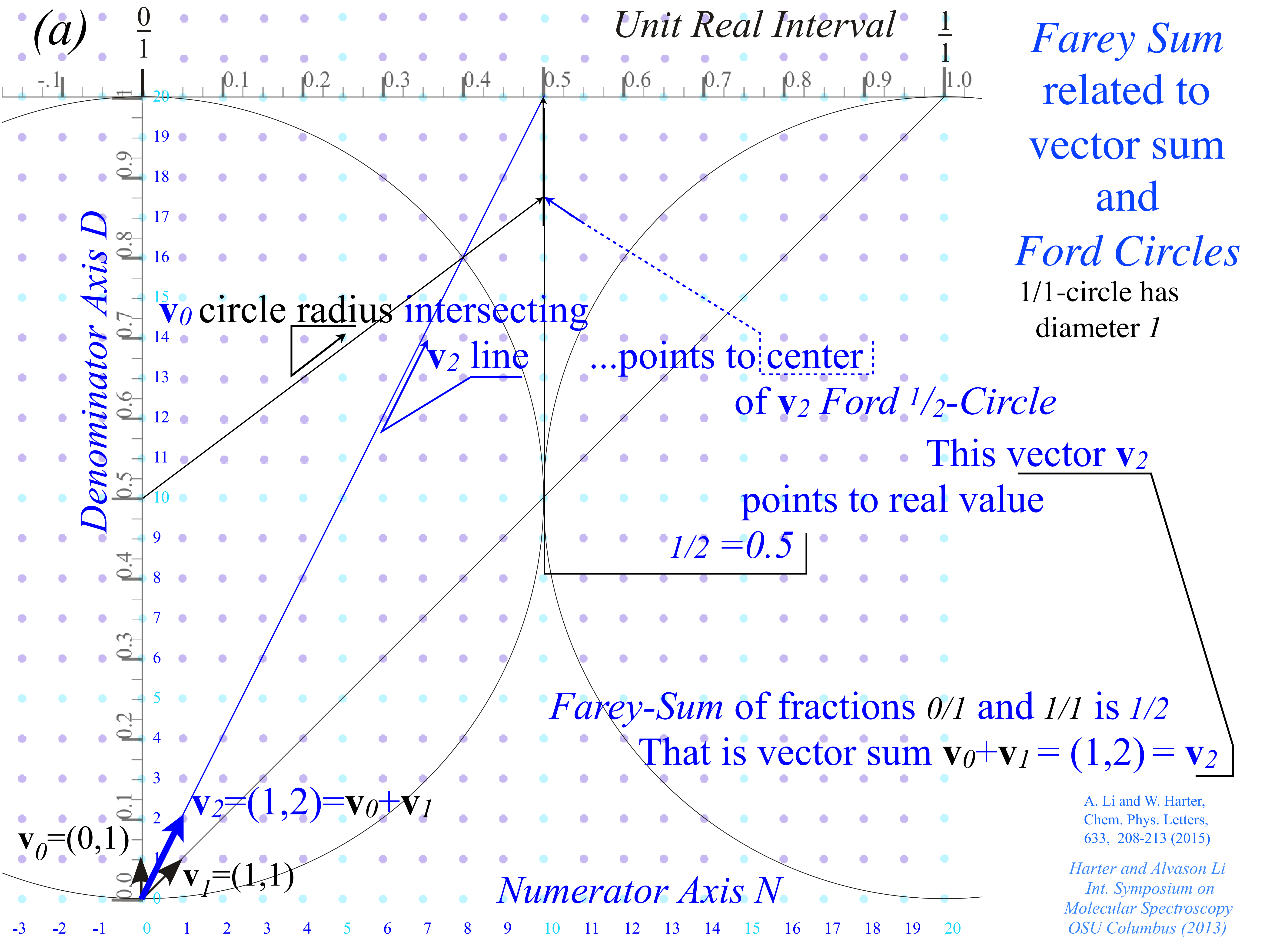


Farey Sum related to vector sum and *Ford Circles*

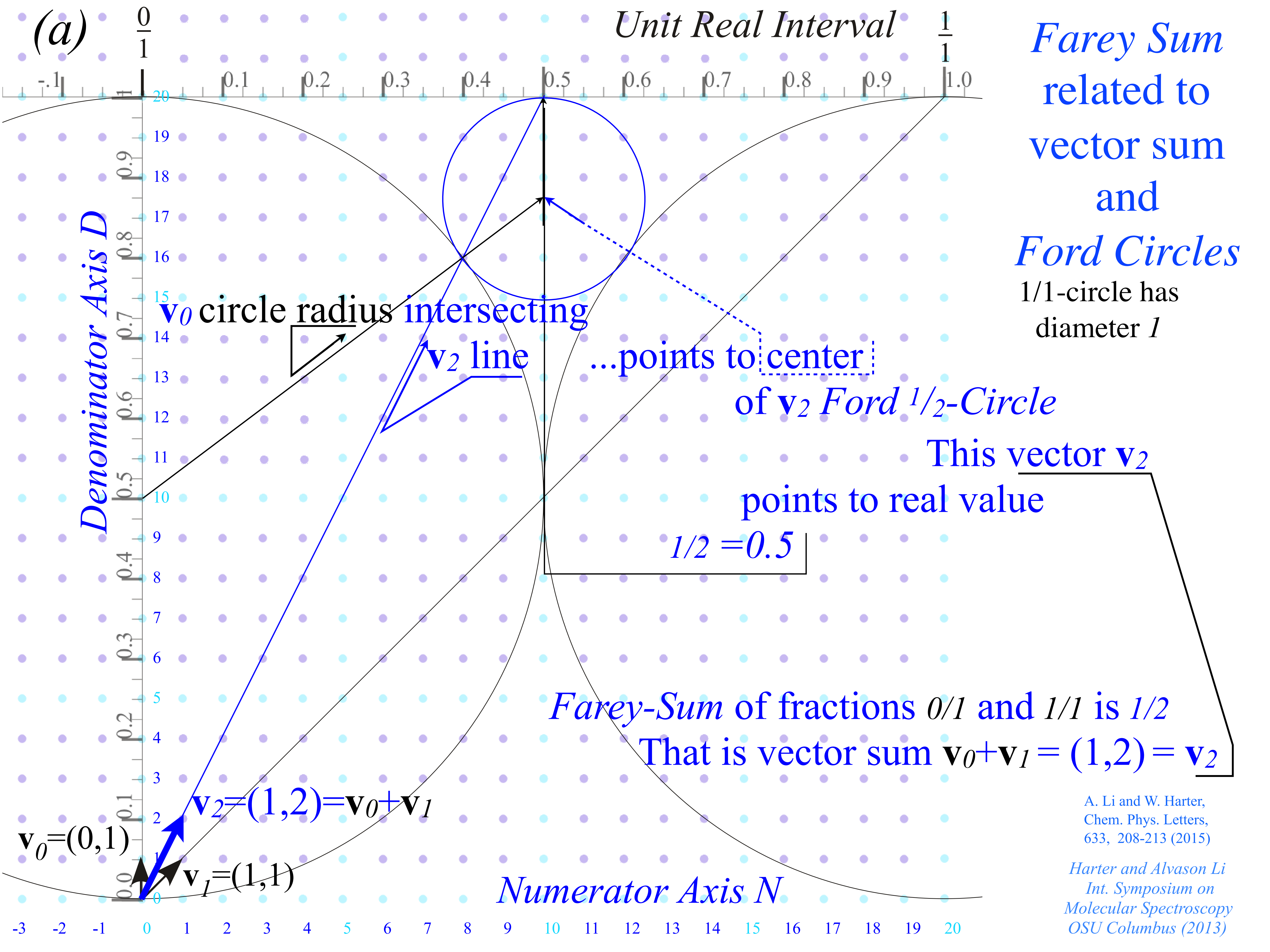
1/1-circle has diameter *1*

A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

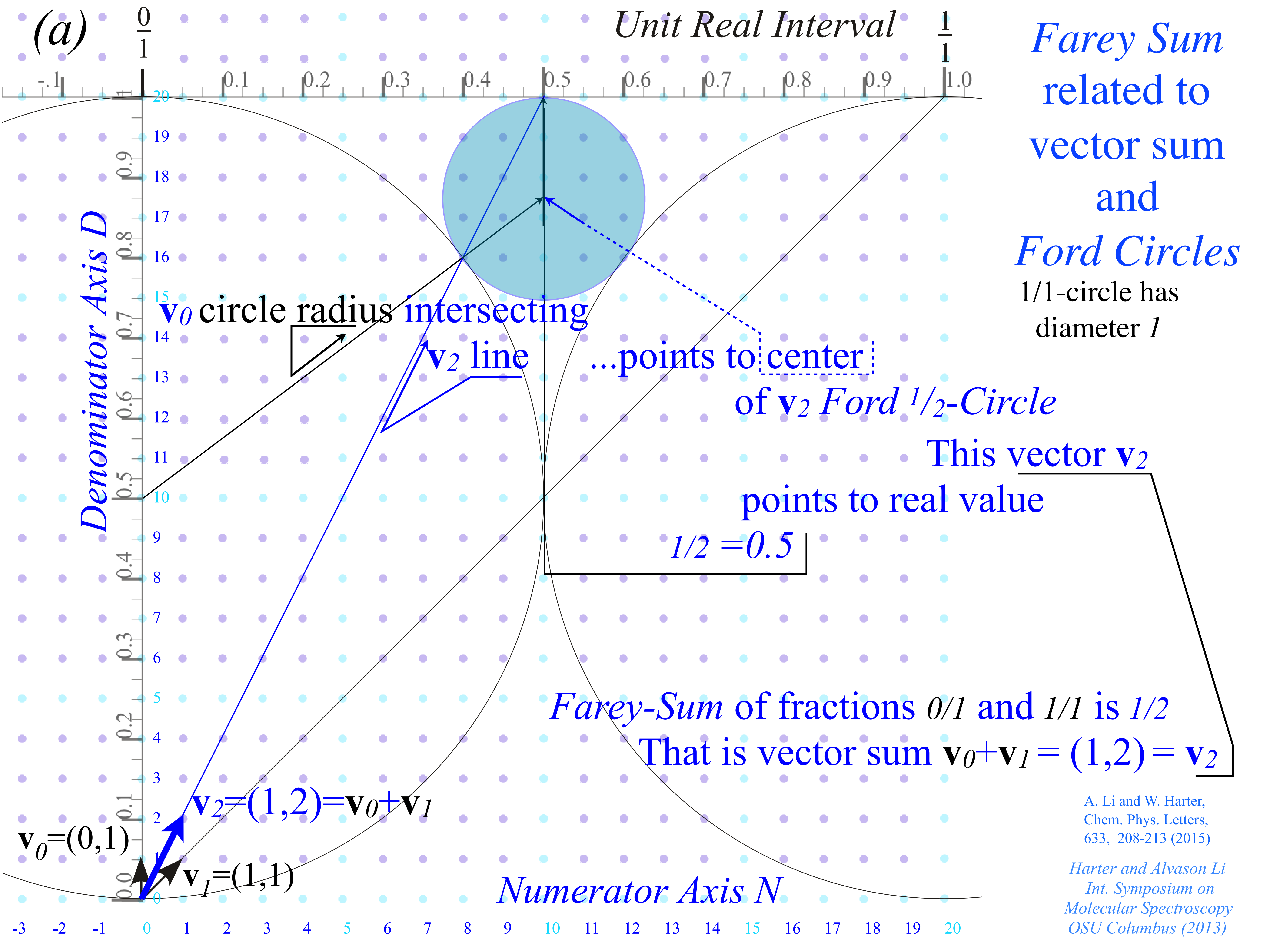
Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)



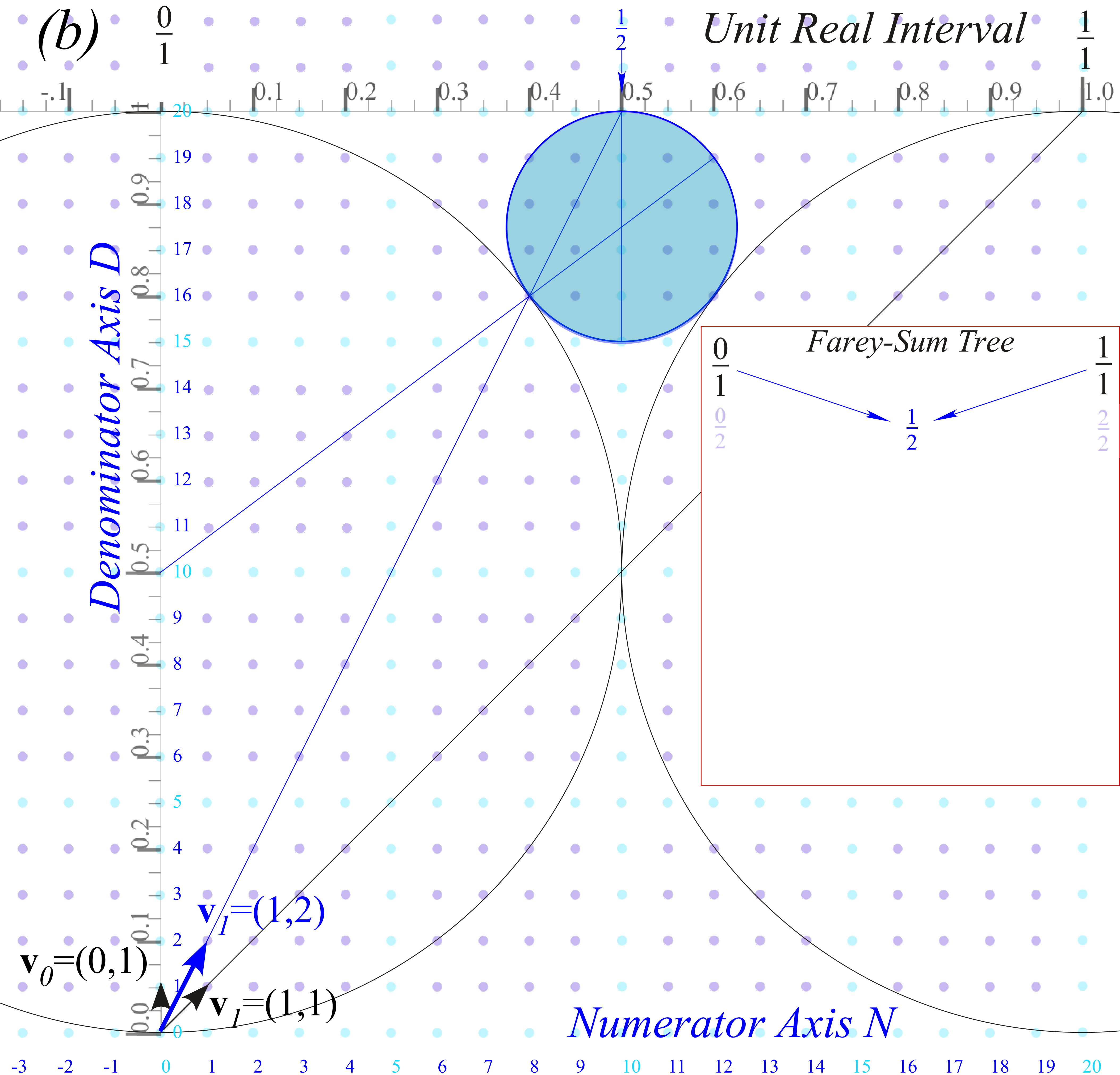
(a)
0/1
Unit Real Interval
1/1
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
v0 circle radius intersecting
v2 line
...points to center
of v2 Ford 1/2-Circle
This vector v2
points to real value
1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
Numerator Axis N
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)
Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



(a)
Unit Real Interval
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
Numerator Axis N
v0 circle radius intersecting v2 line
...points to center of v2 Ford 1/2-Circle
This vector v2 points to real value 1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)
Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



(a)
Unit Real Interval
0/1
1/1
Farey Sum related to vector sum and Ford Circles
1/1-circle has diameter 1
Denominator Axis D
Numerator Axis N
v0 circle radius intersecting
v2 line
...points to center of v2 Ford 1/2-Circle
This vector v2 points to real value 1/2 =0.5
Farey-Sum of fractions 0/1 and 1/1 is 1/2
That is vector sum v0+v1 = (1,2) = v2
v2=(1,2)=v0+v1
v0=(0,1)
v1=(1,1)
A. Li and W. Harter, Chem. Phys. Letters, 633, 208-213 (2015)
Harter and Alvason Li Int. Symposium on Molecular Spectroscopy OSU Columbus (2013)



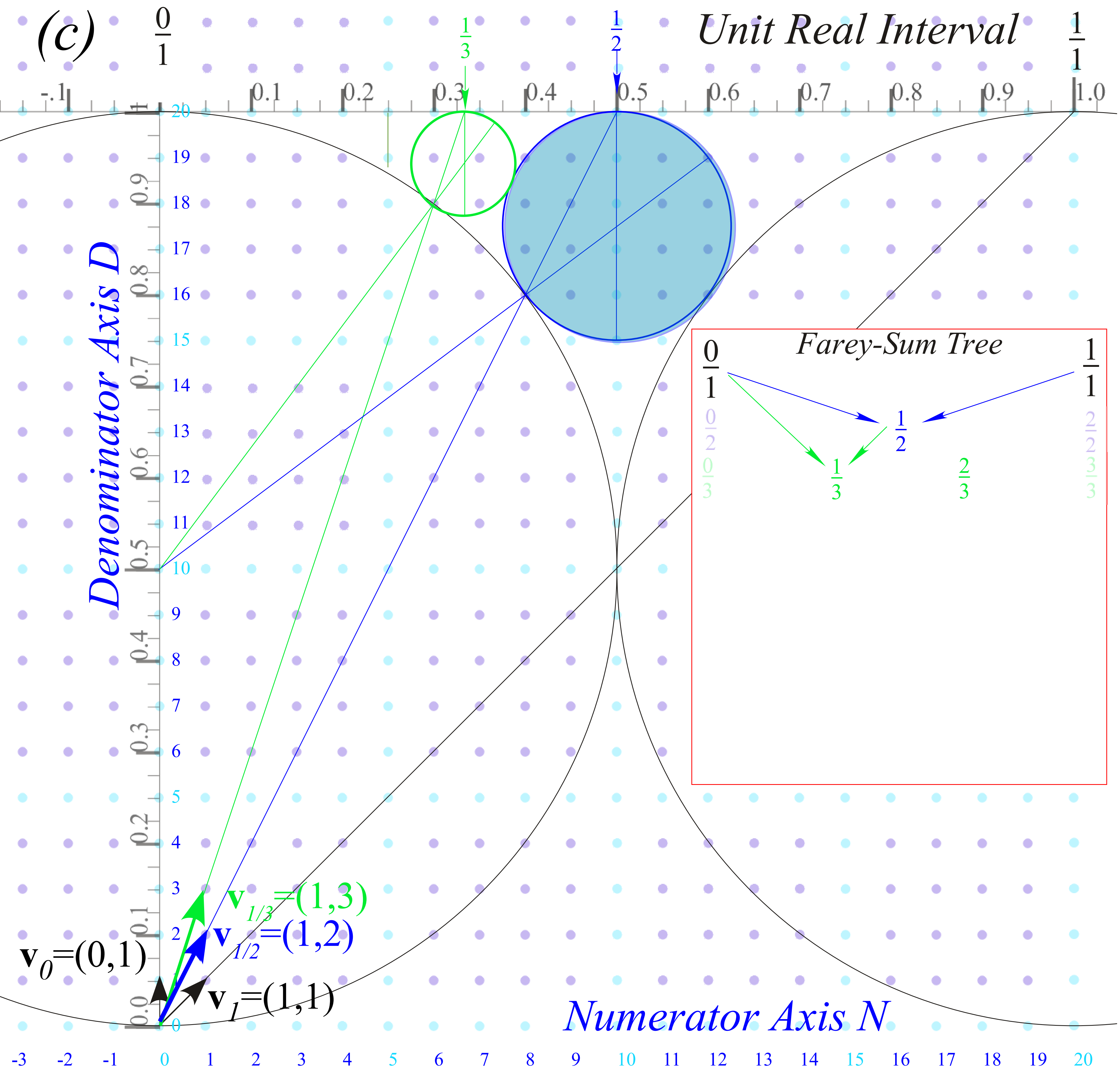
Farey Sum related to vector sum and *Ford Circles*

1/1-circle has diameter *1*

1/2-circle has diameter $1/2^2=1/4$

A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)



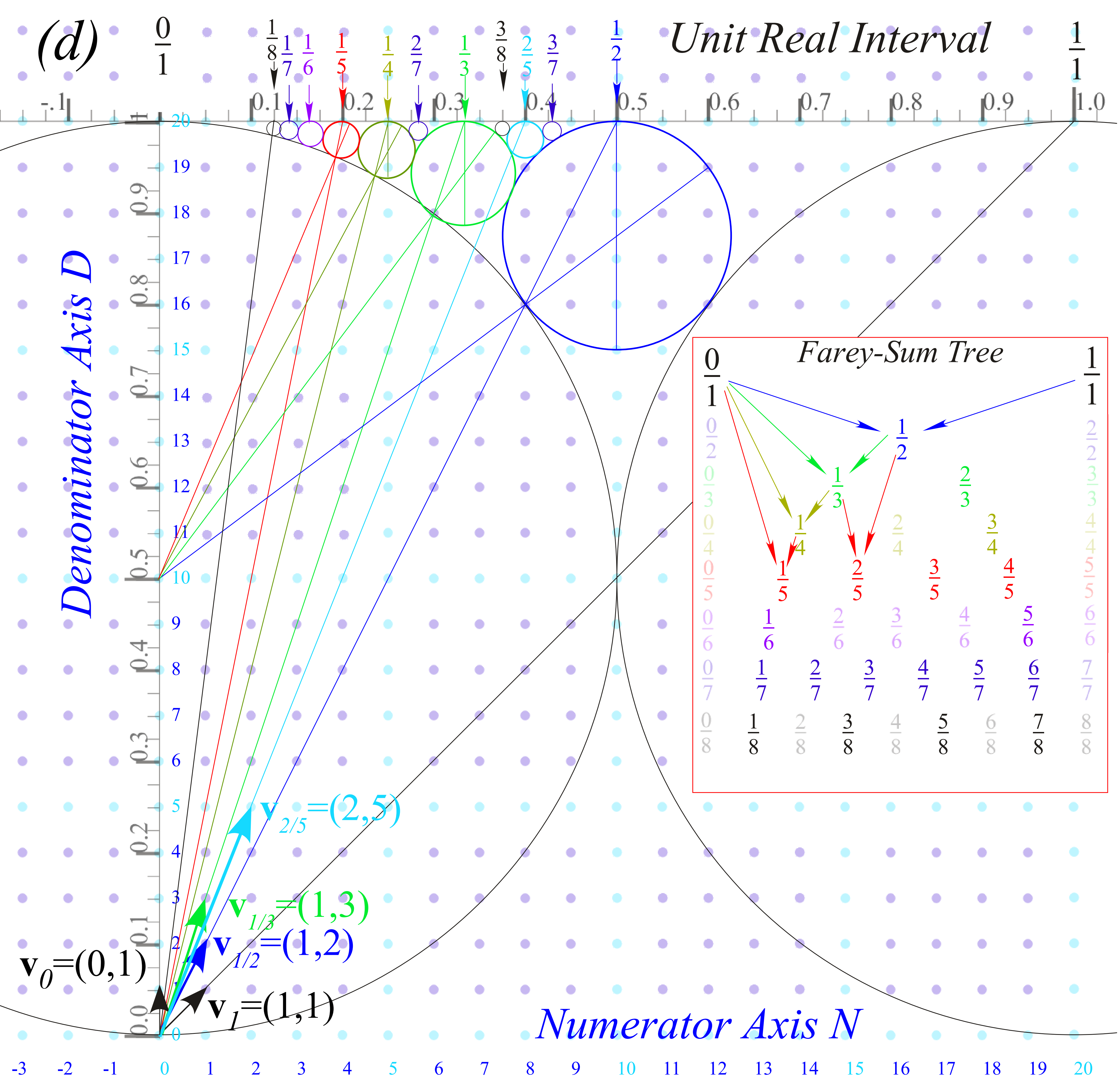
Farey Sum related to vector sum and *Ford Circles*

1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)



Farey Sum related to vector sum and *Ford Circles*

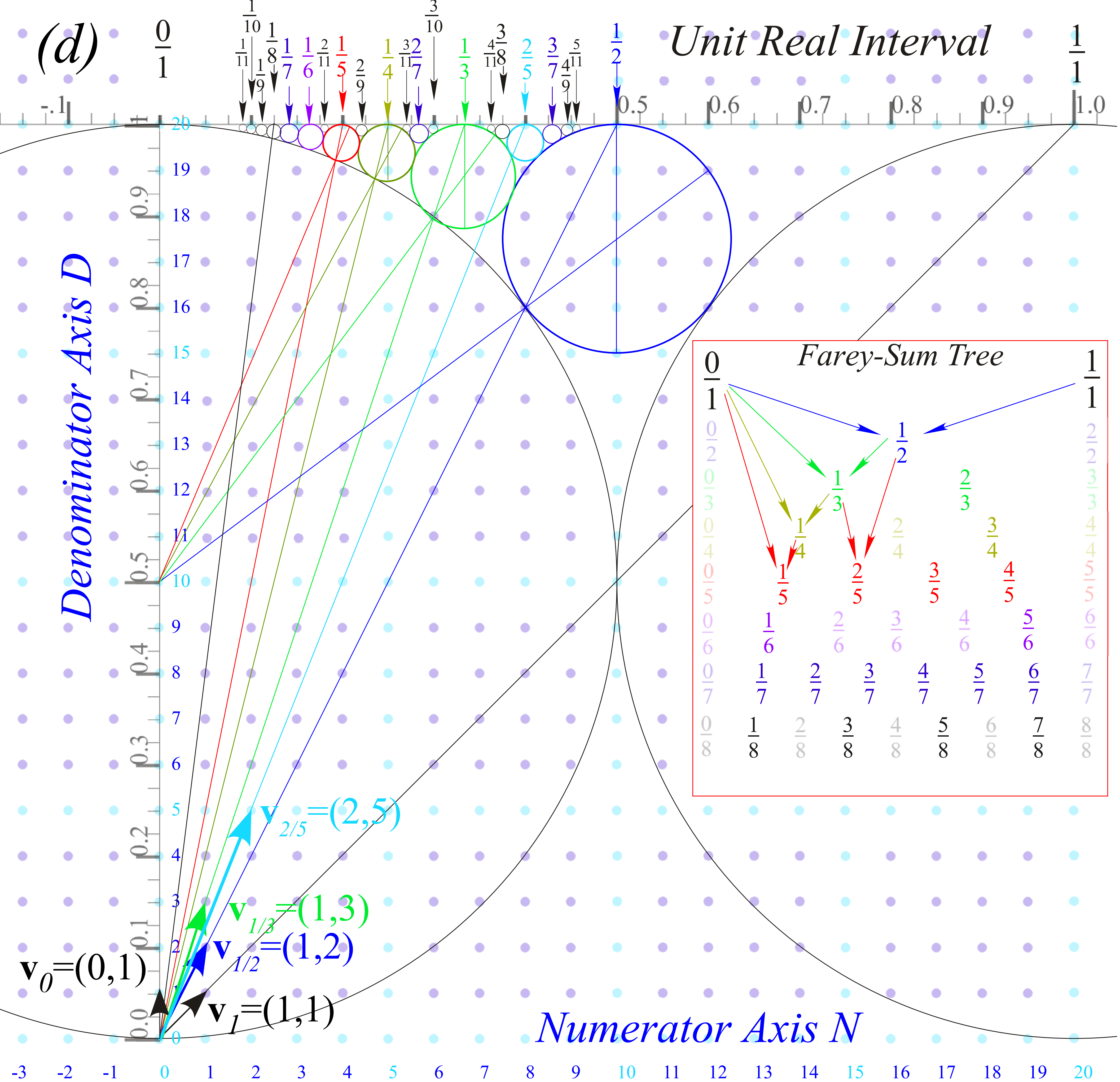
1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on Molecular Spectroscopy
OSU Columbus (2013)



Farey Sum related to vector sum and *Ford Circles*

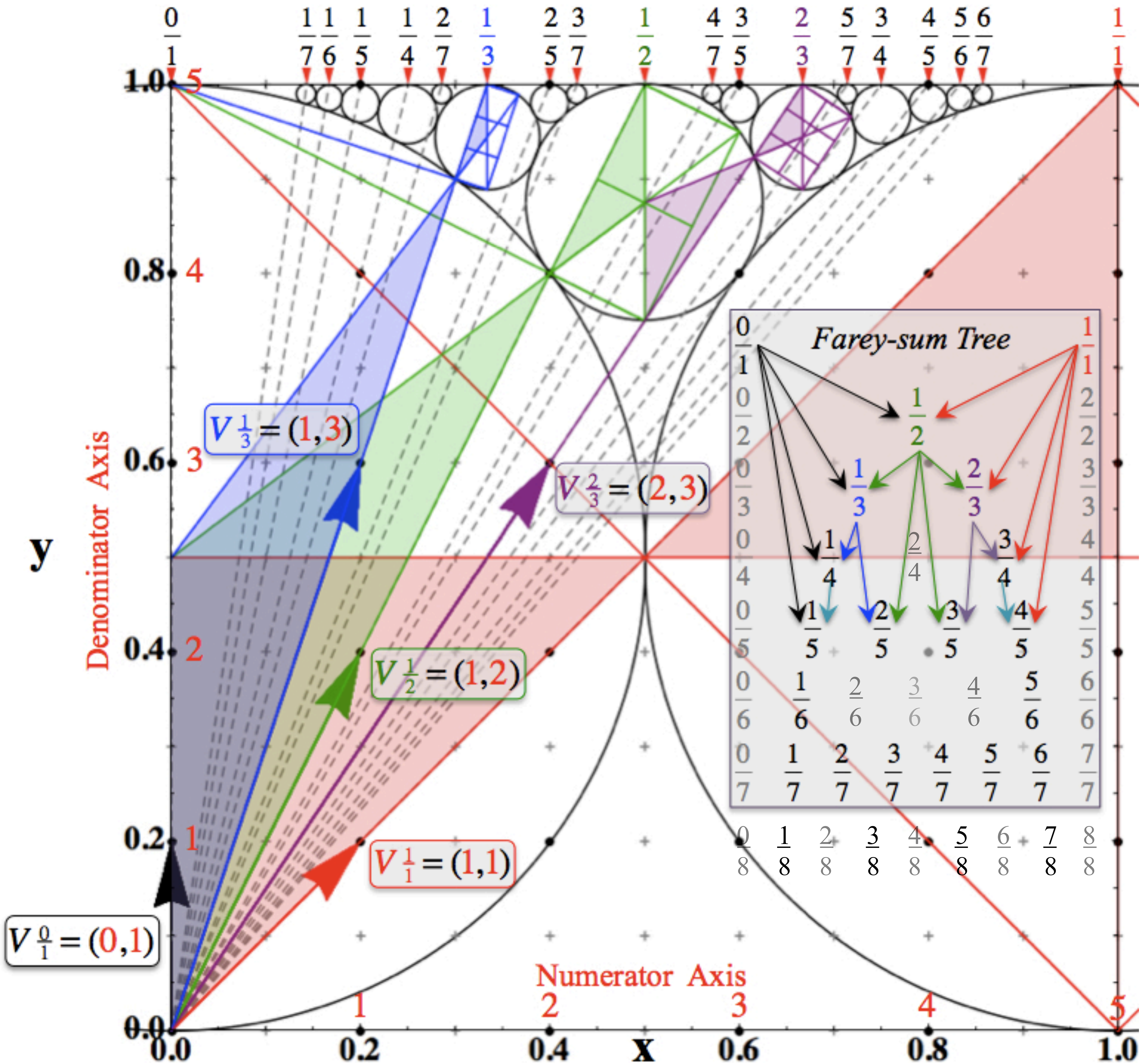
1/2-circle has diameter $1/2^2=1/4$

1/3-circles have diameter $1/3^2=1/9$

n/d-circles have diameter $1/d^2$

A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)

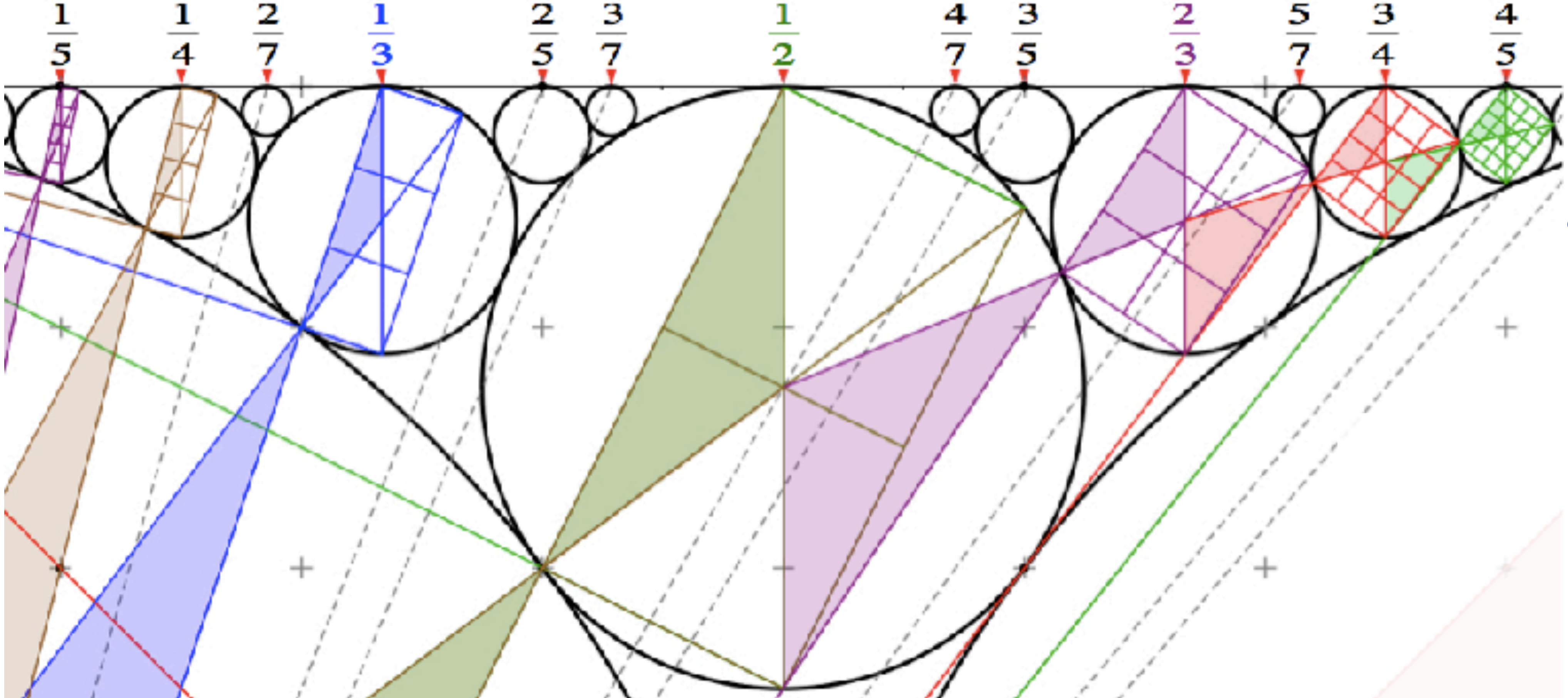


Thales Rectangles

provide
analytic geometry
of
fractal structure

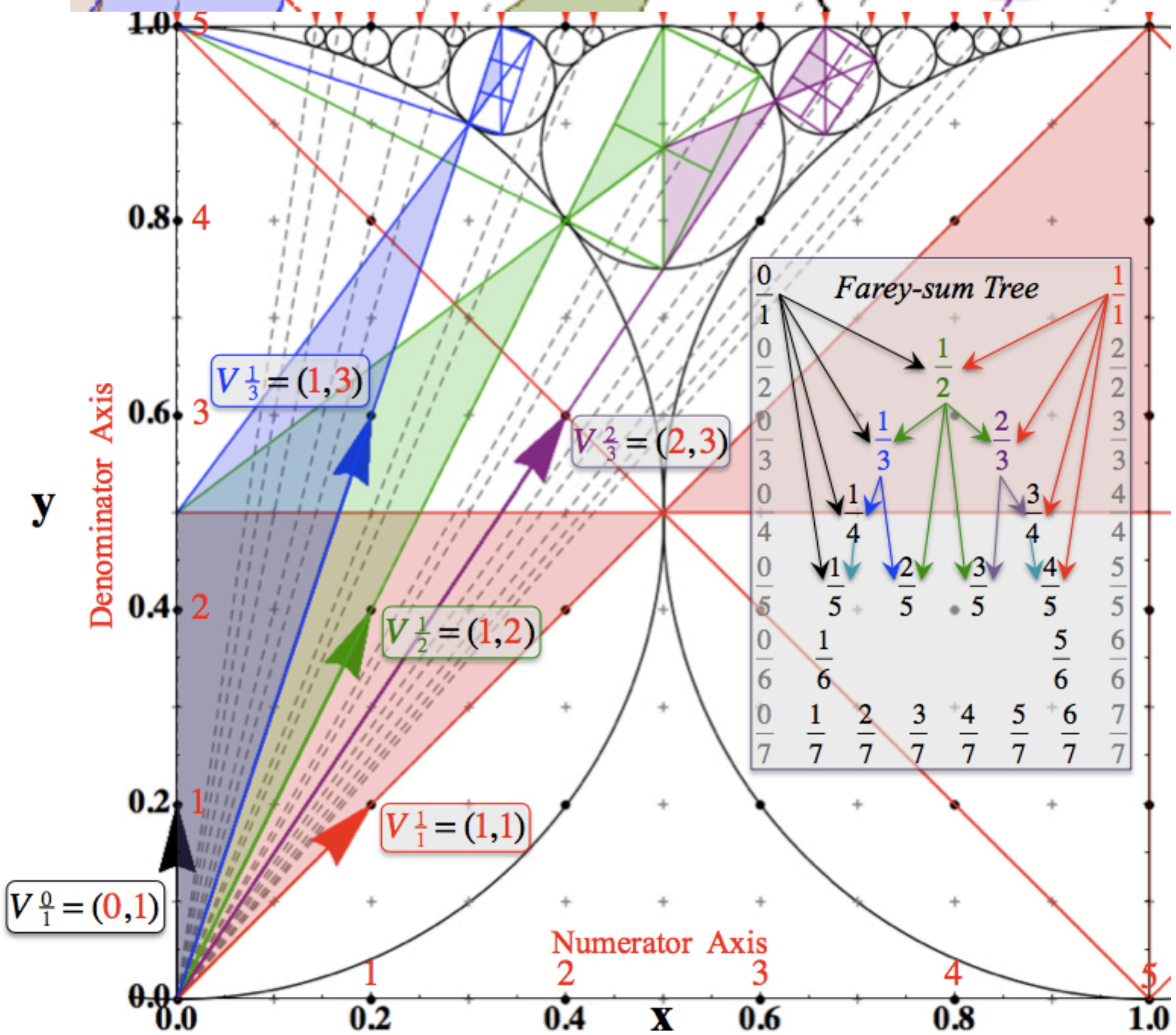
A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)



"Quantized" Thales Rectangles

provide
analytic geometry
of
fractal structure



A. Li and W. Harter,
Chem. Phys. Letters,
633, 208-213 (2015)

Harter and Alvason Li
Int. Symposium on
Molecular Spectroscopy
OSU Columbus (2013)

Relating C_N symmetric H and K matrices to differential wave operators

Relating C_N symmetric H and K matrices to wave differential operators

The 1st neighbor **K** matrix relates to a 2nd *finite-difference* matrix of 2nd x-derivative for high C_N.

$$\mathbf{K} = k(2\mathbf{1} - \mathbf{r} - \mathbf{r}^{-1}) \text{ analogous to:} -k\frac{\partial^2}{\partial x^2}$$

1st derivative momentum: $p = \frac{\hbar}{i}\frac{\partial y}{\partial x} \approx \frac{\hbar}{i}\frac{y(x+\Delta x) - y(x)}{(\Delta x)}$

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \frac{\hbar}{i}\begin{pmatrix} \cdot \\ y_1 - y_0 \\ y_2 - y_1 \\ y_3 - y_2 \\ y_4 - y_3 \\ \cdot \end{pmatrix}$$

2nd derivative KE: $2mE = \text{-}\hbar^2\frac{\partial^2 y}{\partial x^2} \approx \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{(\Delta x)^2}$

$$\text{-}\hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} \cdot \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot \\ y_0 - 2y_1 + y_2 \\ y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \cdot \end{pmatrix}$$

H and **K** matrix equations are finite-difference versions of quantum and classical wave equations.

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \mathbf{H}\left|\psi\right\rangle \quad (\mathbf{H}\text{-}\textit{matrix equation})$$

$$-\frac{\partial^2}{\partial t^2}\left|y\right\rangle = \mathbf{K}\left|y\right\rangle \quad (\mathbf{K}\text{-}\textit{matrix equation})$$

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right)\left|\psi\right\rangle \quad (\textit{Scrodinger equation})$$

$$-\frac{\partial^2}{\partial t^2}\left|y\right\rangle = -k\frac{\partial^2}{\partial x^2}\left|y\right\rangle \quad (\textit{Classical wave equation})$$

Square p^2 gives 1st neighbor **K** matrix. Higher order p^3, p^4,.. involve 2nd, 3rd, 4th..neighbor **H**

$$\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\frac{\hbar}{i}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \hbar^2\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$p^4 \cong \begin{pmatrix} \ddots & \vdots & 1 & \cdot & \cdot & \cdot \\ \cdots & 6 & -4 & 1 & \cdot & \cdot \\ 1 & -4 & 6 & -4 & 1 & \cdot \\ \cdot & 1 & -4 & 6 & -4 & 1 \\ \cdot & \cdot & 1 & -4 & 6 & -4 \\ \cdot & \cdot & \cdot & 1 & -4 & 6 \end{pmatrix}$$

Symmetrized finite-difference operators

$$\overline{\Delta} = \frac{1}{2}\begin{pmatrix} \ddots & \vdots & & & & \\ \cdots & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}, \quad \overline{\Delta}^3 = \frac{1}{2^3}\begin{pmatrix} \ddots & \vdots & 0 & -1 & & \\ \cdots & 0 & 3 & 0 & -1 & \\ 0 & -3 & 0 & 3 & 0 & -1 \\ 1 & 0 & -3 & 0 & 3 & 0 \\ & 1 & 0 & -3 & 0 & 3 \\ & & 1 & 0 & -3 & 0 \end{pmatrix}$$

$$\overline{\Delta}^2 = \frac{1}{2^2}\begin{pmatrix} \ddots & \vdots & 1 & & & \\ \cdots & -2 & 0 & 1 & & \\ 1 & 0 & -2 & 0 & 1 & \\ & 1 & 0 & -2 & 0 & 1 \\ & & 1 & 0 & -2 & 0 \\ & & & 1 & 0 & -2 \end{pmatrix}, \quad \overline{\Delta}^4 = \frac{1}{2^4}\begin{pmatrix} \ddots & \vdots & -4 & 0 & 1 & \\ \cdots & 6 & 0 & -4 & 0 & 1 \\ -4 & 0 & 6 & 0 & -4 & 0 \\ 0 & -4 & 0 & 6 & 0 & -4 \\ 1 & 0 & -4 & 0 & 6 & 0 \\ & 1 & 0 & -4 & 0 & 6 \end{pmatrix}$$