Lecture 23
Wed. 11.13.2019

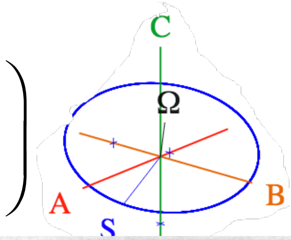
U(2)~R(3) algebra/geometry in classical or quantum theory

(Classical Mechanics with a BANG! Units 4-6, Quantum Theory for Computer Age - Ch. 10A-B of Unit 3)
(Principles of Symmetry, Dynamics, and Spectroscopy - Sec. 1-3 of Ch. 5 and Ch. 7)

Reviewing fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

- *Euler-defined state |αβγ⟩ described by Stoke's* **S***-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* **H**[φϑΘ]=*exp(-i***Ω•S***)·t and angular velocity* **Ω**(φϑ)·*t*=**Θ***-vector*
- *Euler-defined operator* **R***(αβγ) derived from Darboux-defined* **R**[φϑΘ] *and vice versa*
- *Euler* **R***(αβγ) rotation* **Θ**=0−4π*-sequence* [φϑ] *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

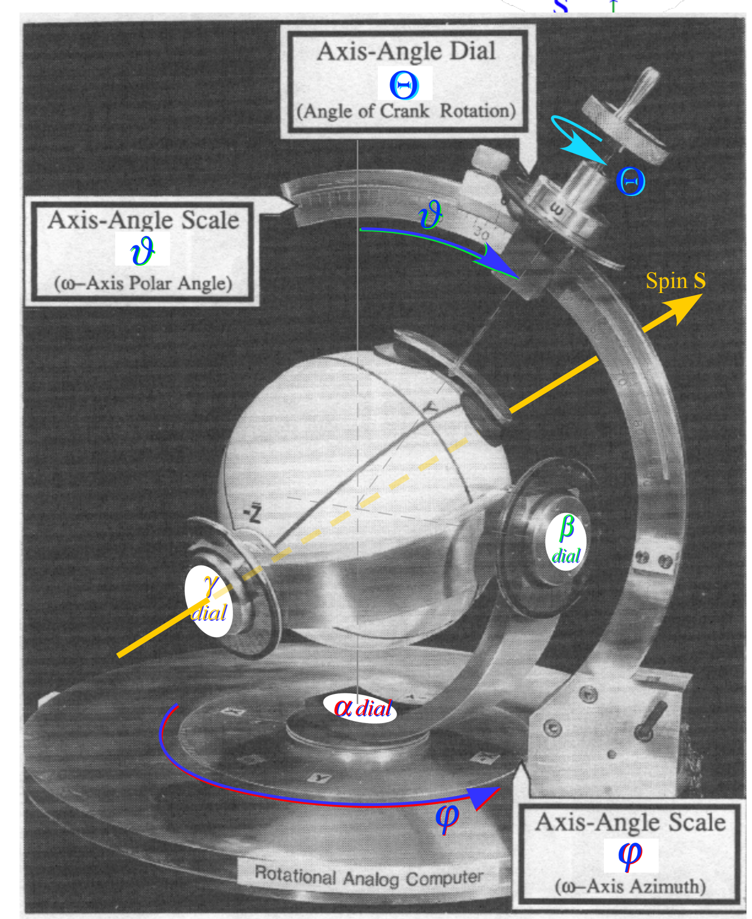


The ABC's of U(2) dynamics-Archetypes
- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes
- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates
- *Conventional amp-phase ellipse coordinates*
- *Euler Angle (αβγ) ellipse coordinates*

Addenda: U(2) density matrix formalism
- *Bloch equation for density operator*



This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lecture #22-23

In reverse order

Advanced Atomic and Molecular Optical Physics 2018 Class #9, pages: 5, 61
BoxIt Web Simulations
- Pure A-Type A=4.9, B=0 ,C=0, & D=4.0
- Pure B-Type: A=4.0, B=-0.2, C=0, & D=4.0
- Pure C-Type A,D=4.055, B=0, C=0.1
- Mixed AB-Type w/Cosine
- Mixed AB Type A=4.0, BU2=0.866…, CU2=0, & D=1.0 w/Stokes & Freq rats
- Mixed AB Type A=5.086 B=-0.27 C=0 D=2.024 w/Stokes plot
- Mixed ABC Type A=4.833 B=0.2403 C=0.4162 D=4.277 w/Stokes plot
- Recent mixed ABC Type A=0.325 B=0.375 C=0.825 D=0.05 w/Stokes plot

Classical Mechanics with a Bang! 2018
Lectures 8, 9, 23 page 93
Text Unit 6, page=27
ColorU2 for the Web - in development
Group Theory for Quantum Mechanics - 2017 Lectures: 6, 7, 8,
and the combined 9-10
Quantum Theory for the Computer Age Unit 3 Ch.7-10, page=90
Spectral Decomposition with Repeated Eigenvalues - 2017 GTQM - Lecture 5

Web based 3D & XR (x∈{A,M,V}, R=Reality) https://www.babylonjs.com/
Web based 3D graphics WebGL API (Graphics Layer modeled after OpenGL)

Select, exciting, and/or related Research

This Indestructible NASA Camera Revealed Hidden Patterns on Jupiter - seeker-yt-2019
What did NASA's New Horizons discover around Pluto? - Astrum-yt-2018
Synthetic_Chiral_Light_for_Efficient_Control_of_Chiral_Light–Matter_Interaction_-_Ayuso-np-2019

Quantum Computing (*QC*) and Geometric Algebra (*GA*) references:

Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
Quantum Computing for Computer Scientists - Helwer-mr-yt-2018, Slides
Quantum Computing and Workforce, Curriculum, and App Devel - Roetteler-MS-2019

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Excerpts (Page 44-47 in *Preliminary Draft*) for a GA take on the Complex Numbers
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
GA & QC references (Page 11-16 in Preliminary Draft)

Recent In-House draft Articles:

Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Auxiliary slides, June 19, 2018

Quantum_Computing - (Current) State of the Art - Reimer-www-2019
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 1 - Reimer-www-2019
Wildlife Monitoring Identification and Behavioral Study - Section 2 - Reimer-www-2019

In development, but close to role out.
More Advanced QM and classical references will *soon* be available through our: ***References Page***
Would be great to have our Apache SOLR *Search & Index system up for a bigger* Bang!*)*

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lectures #12 through #21

In reverse order

Wiki on Pafnuty Chebyshev
Nobelprize.org
    2005 Physics Award

BoxIt Web Simulations:

A-Type w/Cosine, A-Type w/Freq ratios,
AB-Type w/Cosine, AB-Type 2:1 Freq ratio

OscillIt Web Simulations:

Default/Generic, Weakly Damped #18,
Forced : Way below resonance,On resonance
    Way above resonance,Underdamped
    Complex Response Plot

CoulIt Web Simulations:

Stark-Coulomb : Bound-state motion in parabolic coordinates
Molecular Ion : Bound-state motion in hyperbolic coordinates
Synchrotron Motion, Synchrotron Motion #2
Mechanical Analog to EM Motion (YouTube video)
iBall demo - Quasi-periodicity (YouTube video)

Trebuchet Web Simulations:

Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

Recent Articles of Interest:

A_Semi-Classical_Approach_to_the_Calculation_of_Highly_Excited_Rotational_Energies for ...
    Asymmetric-Top_Molecules_-_Schmiedt-pccp-2017
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*

    Using Earth as a clock,
    Tesla's AC Phasors ,
    Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10, pdf_page=135
    Calculus of exponentials, logarithms, and complex fields,
RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Select, exciting, and related Research

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces -
    Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_
    Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

An assist from *Physics Girl* (YouTube Channel):

How to Make VORTEX RINGS in a Pool
Crazy pool vortex - pg-yt-2014
Fun with Vortex Rings in the Pool - pg-yt-2014

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74, Page=75, Page=79*

Pendulum Web Sim

Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted, FVPlot*

CMwithBang Lecture 8, page=20

WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:
Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:
Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**
Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**
Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**
Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:
Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019
A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_* realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_, Nuclear_Collisions - Mishustin-PhysRevC-2007, APS Link & Abstract
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceItIt Web Animation - Scenarios:
49:1 y vs t, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
1:500:1 - 1D Gas (Warm), *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, *1000:1 with t vs x - Minkowski Plot*
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, *Quantum_Carpet_wMBars*,
Quantum_Carpet_BCar, *Quantum_Carpet_BCar_wMBars*
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceIt Dual plots
m_1:m_2 = 3:1
v_2 vs v_1 and V_2 vs V_1, (v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0)
y_2 vs y_1 plots: (v_1, v_2)=(1, 0.1), (v_1, v_2)=(1, 0), (v_1, v_2)=(1, -1)
Estrangian plot *V_2 vs V_1: (v_1, v_2)=(0, 1), (v_1, v_2)=(1, -1)*
m_1:m_2 = 4:1
v2 vs v1, *y2 vs y1*
m_1:m_2 = 100:1, *(v_1, v_2)=(1, 0): V2 vs V1 Estrangian plot, y2 vs y1 plot*

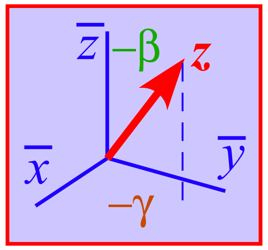
With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

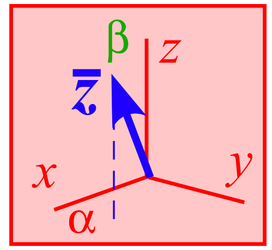
X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<Under Construction>>

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

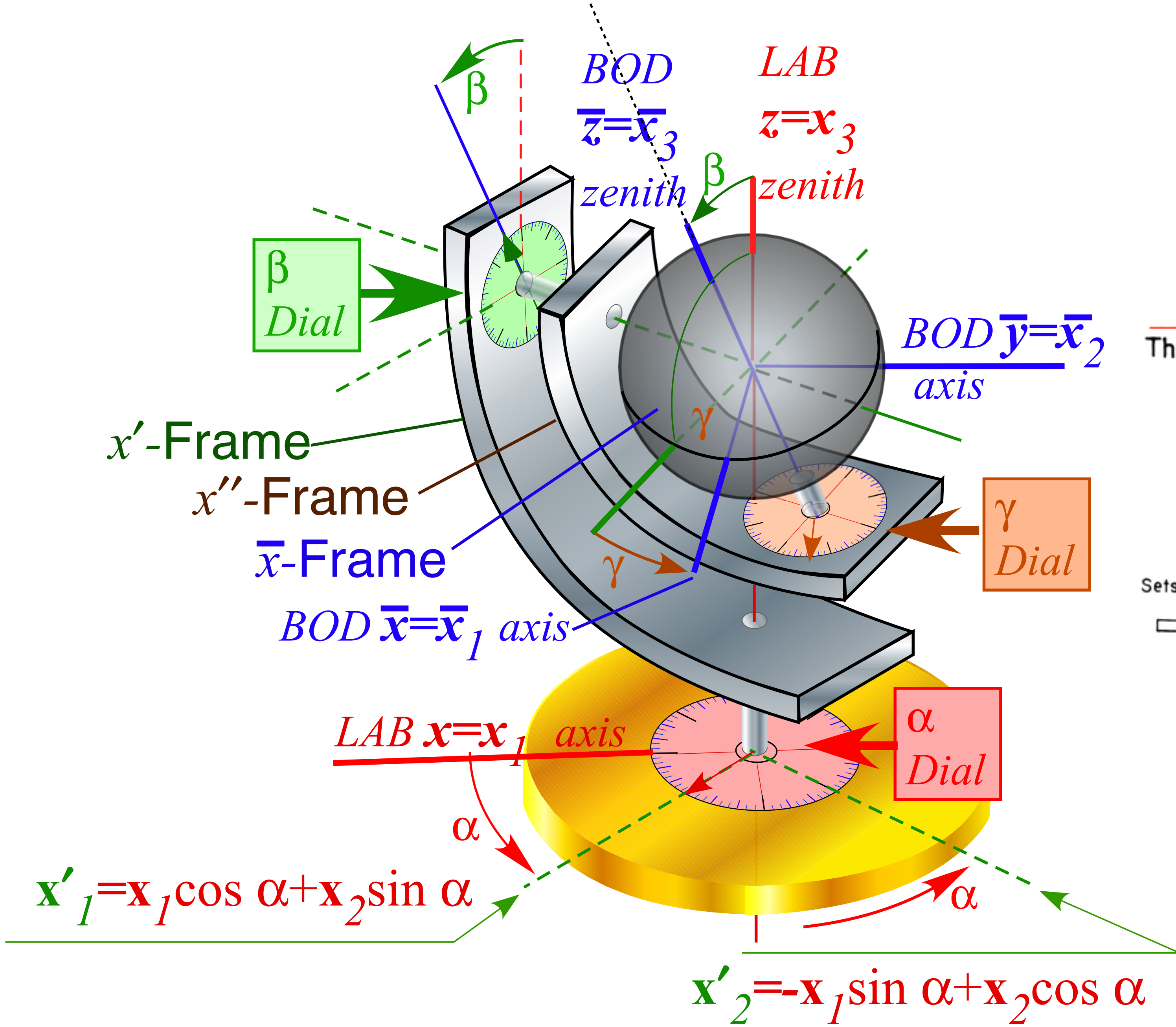
3D-real **S**-*vector represents state* $|\alpha,\beta,\gamma\rangle$ *of U(2) oscillator*

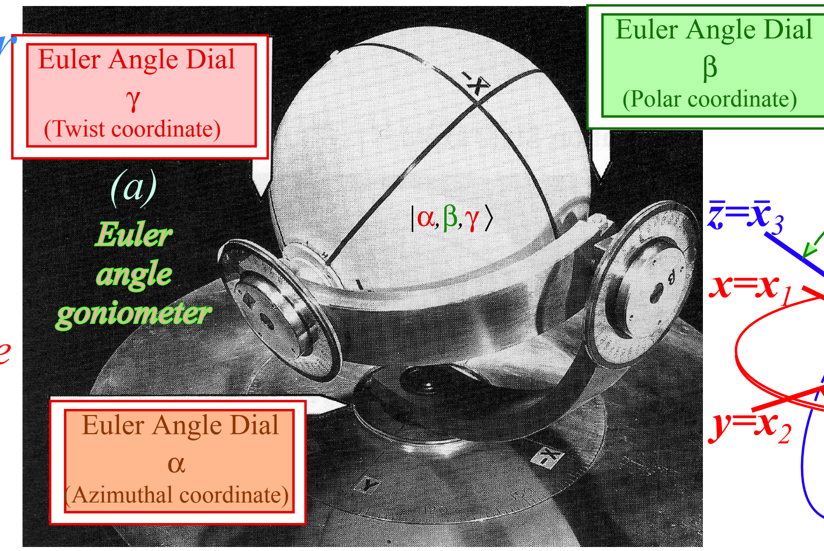


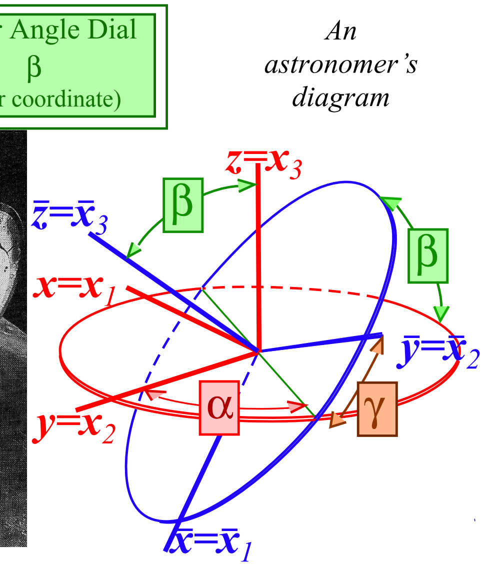
BOD frame view
Polar angles of
LAB zenith $z=x_3$ *are*
(azimuth angle$=-\gamma$,
polar angle$=-\beta$ *)*



LAB frame view
Polar angles of
BOD zenith $\bar{z}=\bar{x}_3$ *are*
(azimuth angle$=\alpha$,
polar angle$=\beta$ *)*

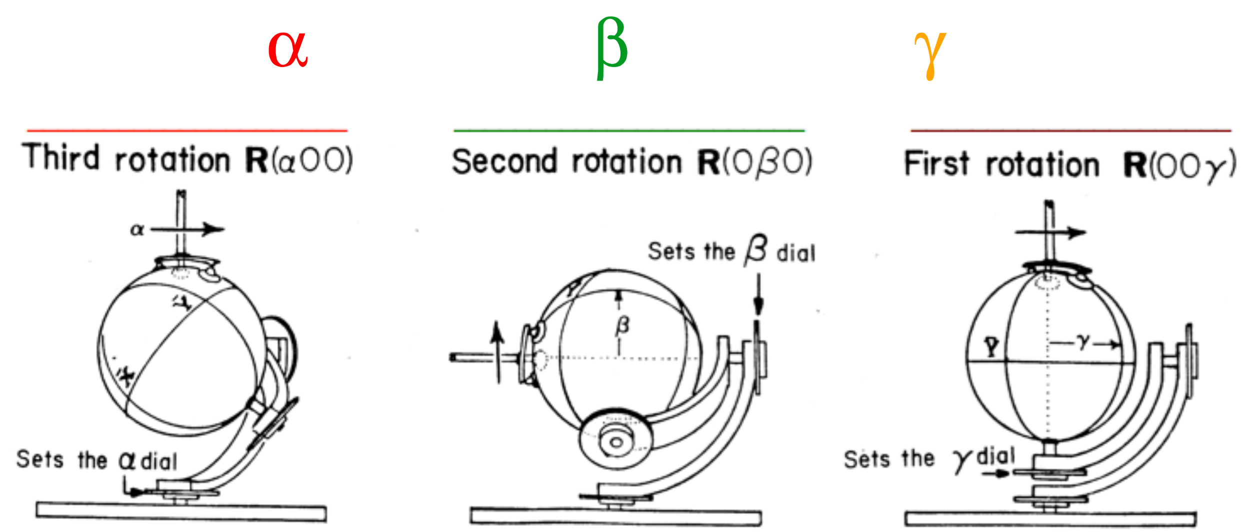






From Lecture 22
page 67

Euler angles



Development has begun on a web based version of this tool,
but much of the App is at present (10/7/2018), in an '*indeterminate state*'.
The App's 3D will in future be handled by Babylon.JS, *to act as a* shim
to buttress the WebGL (web graphics layer) that is already in place.

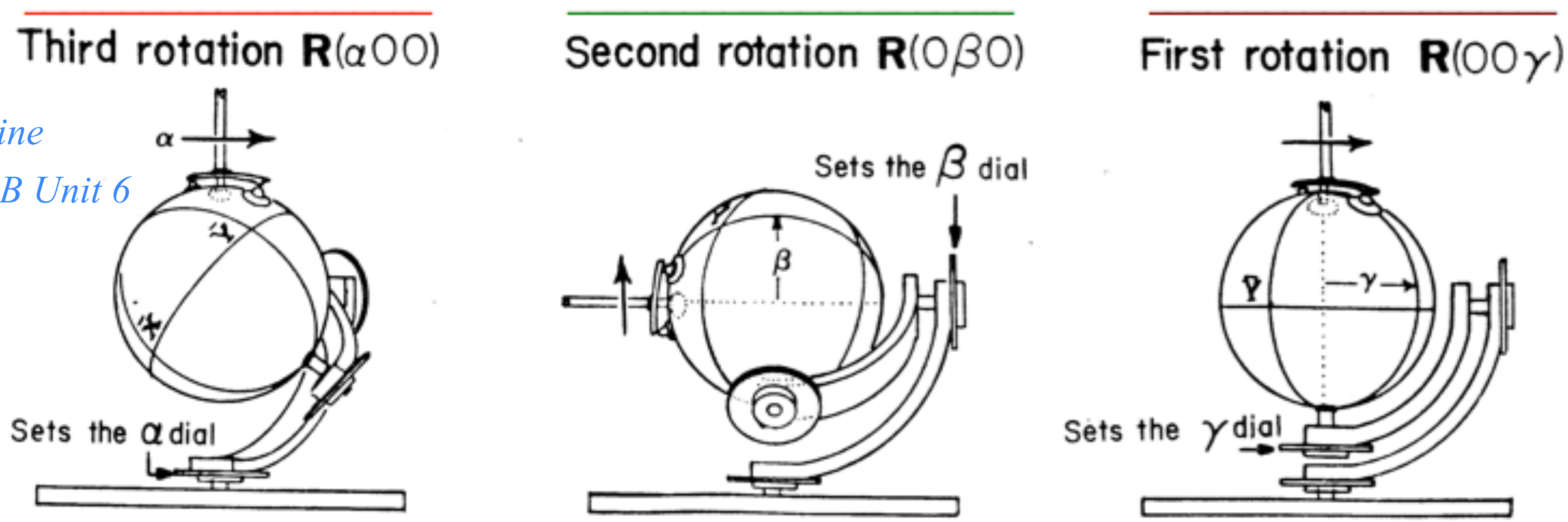
Web based U(2) Calculator - Euler State

Fig. 10.A.3-4 Mechanical device demonstrating Euler angles (α,β,γ)

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{cccc} & |\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle & |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle & |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle \\ \langle \mathbf{e}_X| & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \langle \mathbf{e}_Y| & \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ \langle \mathbf{e}_Z| & -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{array}$$

$$(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B \rangle) = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

Note lab-frame polar coordinates of Z-body vector $|\mathbf{e}_{\bar{Z}}\rangle$

...and body-frame polar coordinates of Z-lab $|\mathbf{e}_Z\rangle$

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$
\begin{aligned}
\left|a\right\rangle &= \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle \\
&= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle \\
&= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}
\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}
\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}
\begin{pmatrix} A \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}
\begin{pmatrix} A \\ 0 \end{pmatrix} \\
&= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}
\end{aligned}
$$

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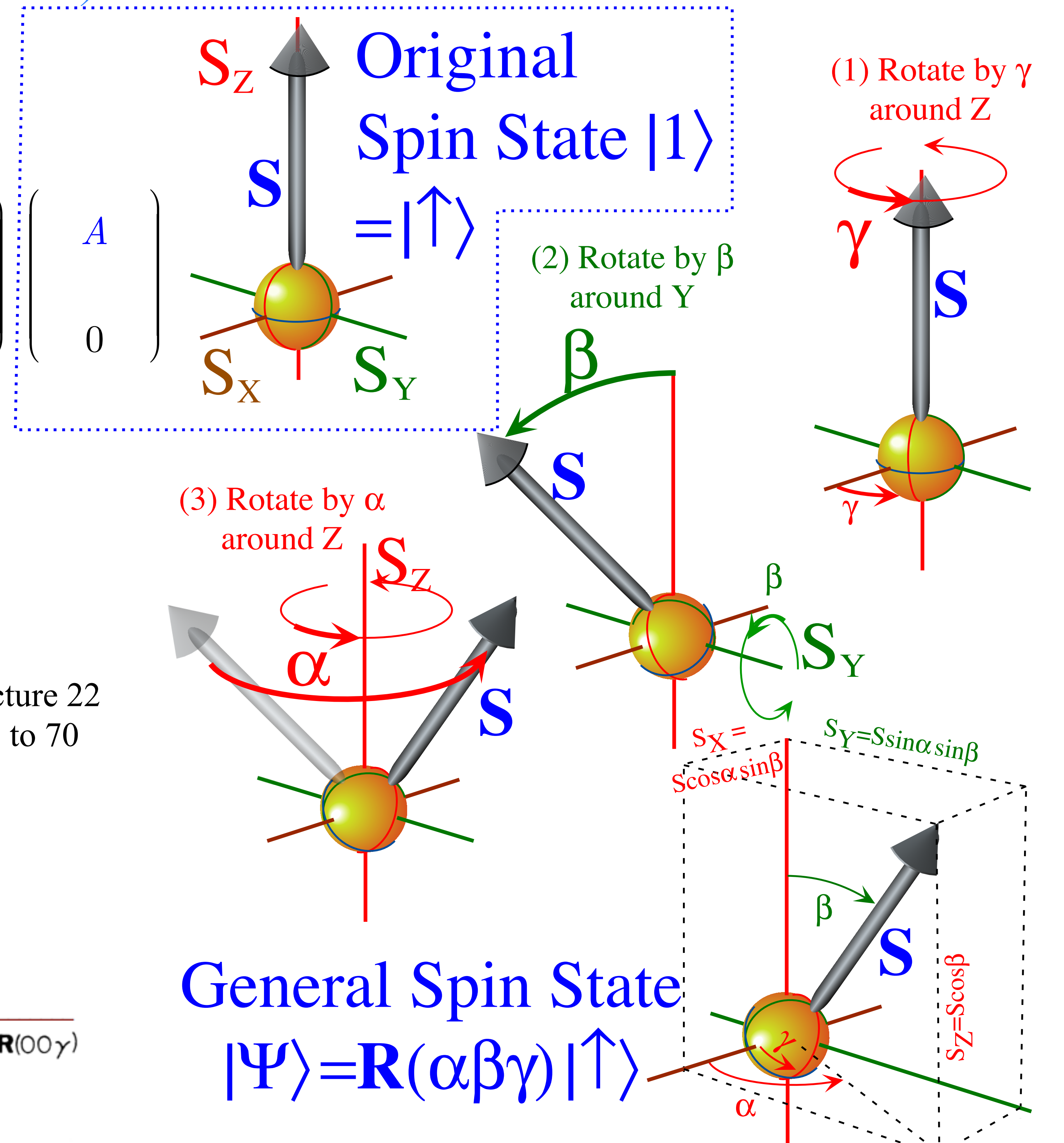
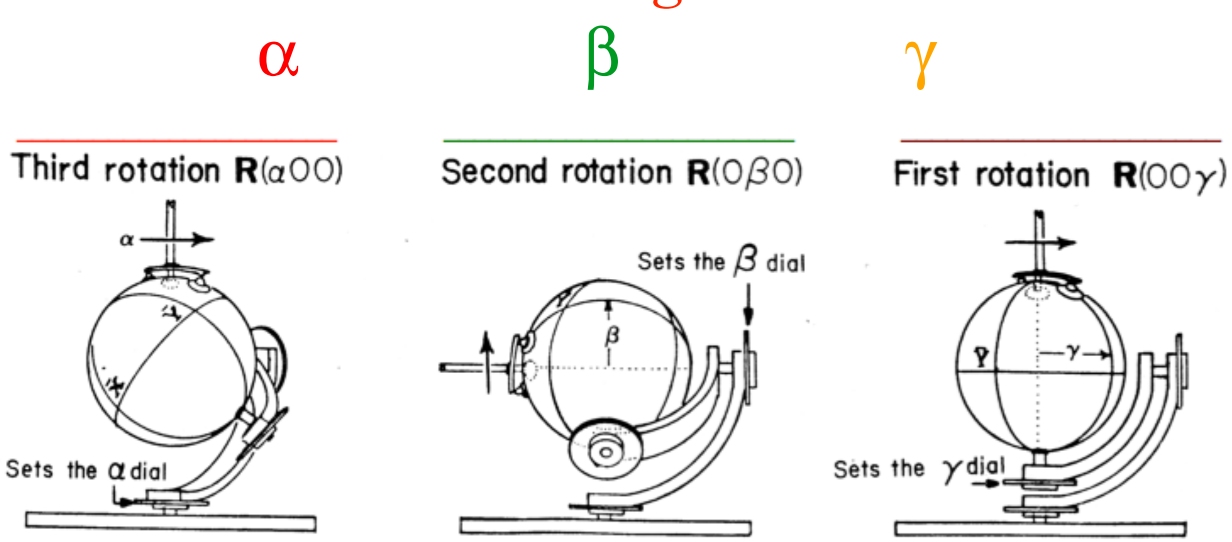


Euler angles



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

From Lecture 22
page 72 to 74

Each point $\{E_1,E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$

This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\textit{Asymmetry}\; S_A = \frac{1}{2}(a|\sigma_A|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \frac{I}{2}\cos\beta$$

$$\textit{Balance}\; S_B = \frac{1}{2}(a|\sigma_B|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$$

$$\textit{Chirality}\; S_C = \frac{1}{2}(a|\sigma_C|a) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\sin\alpha\sin\beta$$

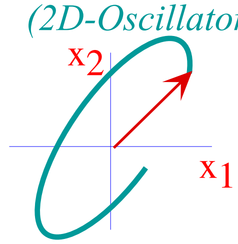
azimuth angle α

polar angle β

Three ways to picture U(2) spin or pseudo-spin states

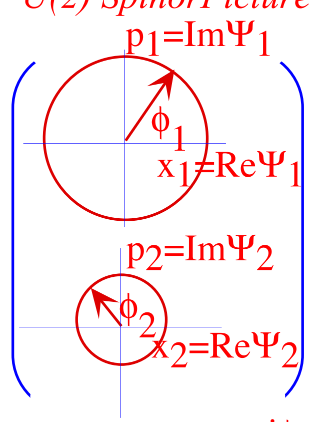
From Lecture 22
page 74 to 76

(a) Real Spinor Space Picture (2D-Oscillator Orbit)



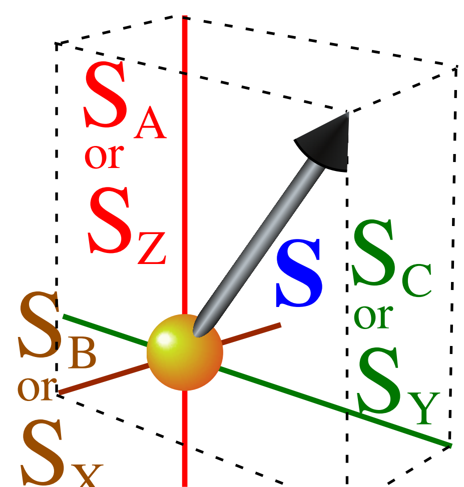
$x_1 = \mathrm{Re}\Psi_1$
$x_2 = \mathrm{Re}\Psi_2$

(b) 2-Phasor U(2) SpinorPicture



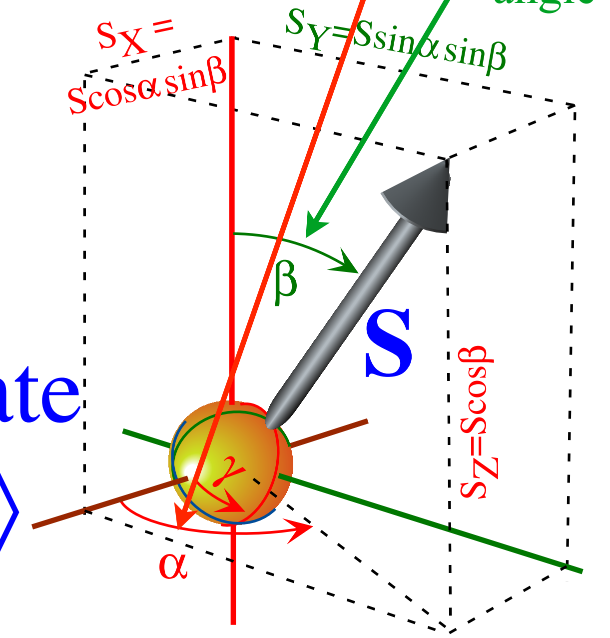
$\Psi_1 = x_1 + ip_1 = |\Psi_1| e^{i\phi_1}$
$\Psi_2 = x_2 + ip_2 = |\Psi_2| e^{i\phi_2}$

(c) 3-Dimensional Real R(3)-SU(2)Vector Picture



$S_A = (\Psi_1^* \Psi_1 - \Psi_2^* \Psi_2)/2$
$S_B = (\Psi_1^* \Psi_2 + \Psi_2^* \Psi_1)/2$
$S_C = (\Psi_1^* \Psi_2 - \Psi_2^* \Psi_1)/2i$

General Spin State
$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\uparrow\rangle$



From Lecture 22
page 70 to 76

(a)	*(b)*	*(c)*
Ellipsometry	*U(2) phasors*	*3D real R(3) vectors*

Fig. 10.5.2 Spinor, phasor, and vector descriptions of 2-state systems .

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\frac{\alpha}{2}} \cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}} \sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}}$

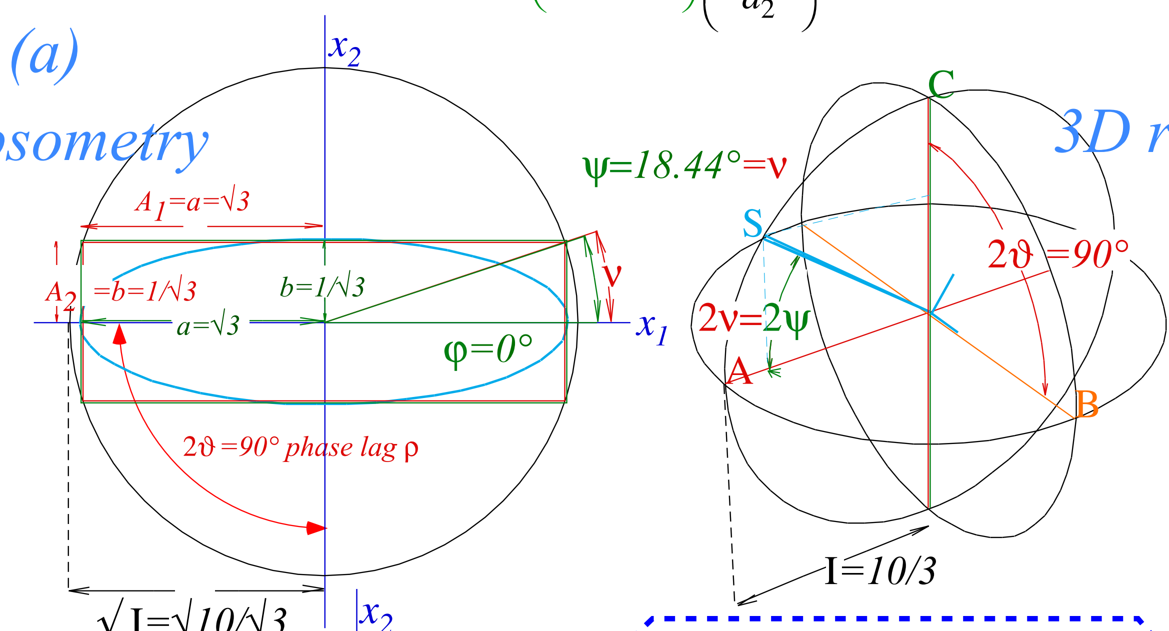
This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\text{Asymmetry } S_A = \frac{1}{2}\langle a|\sigma_A|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right] = \frac{I}{2}\left[\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right] = \boxed{\frac{I}{2}\cos\beta}$$

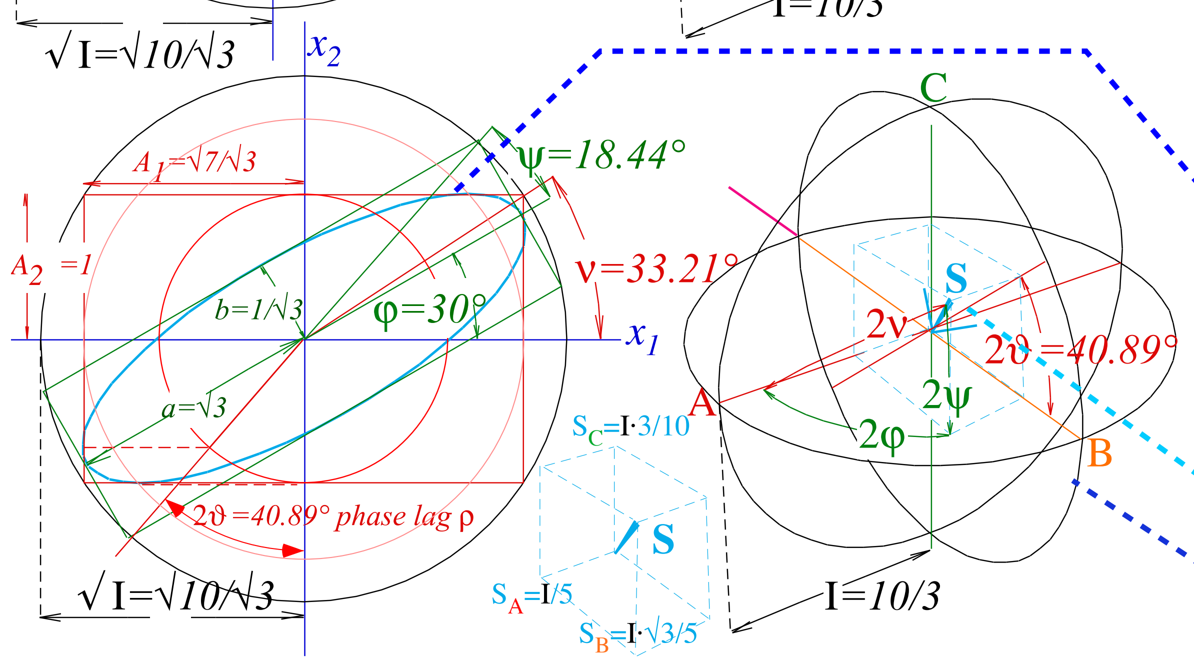
$$\text{Balance } S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \boxed{\frac{I}{2}\cos\alpha\sin\beta}$$

$$\text{Chirality } S_C = \frac{1}{2}\langle a|\sigma_C|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right] = I\left[\cos\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} - \cos\frac{\alpha-\gamma}{2}\cdot -\sin\frac{\alpha+\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \boxed{\frac{I}{2}\sin\alpha\sin\beta}$$

(a) Ellipsometry



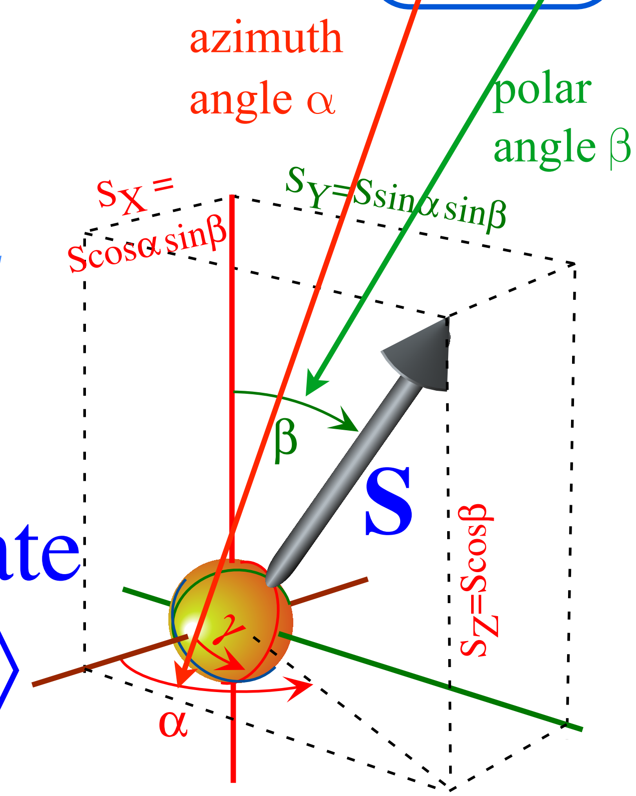
(c) 3D real R(3) **S**-vectors



Ellipsometry of U(2) states detailed at end of this Lecture

General Spin State $|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$

Complex U(2) ellipse of any state corresponds to a single point **S** in R(3) on the Stoke's sphere



Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).

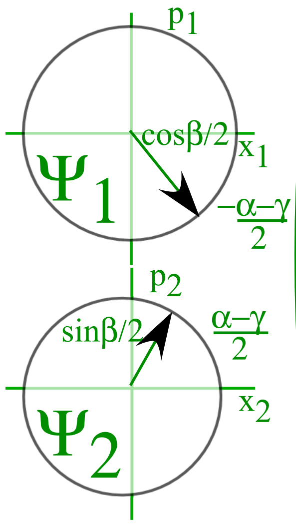
U(2) World : Complex 2D Spinors

Ellipsometry of U(2) states described by Two "Worlds" U(2) or R(3)

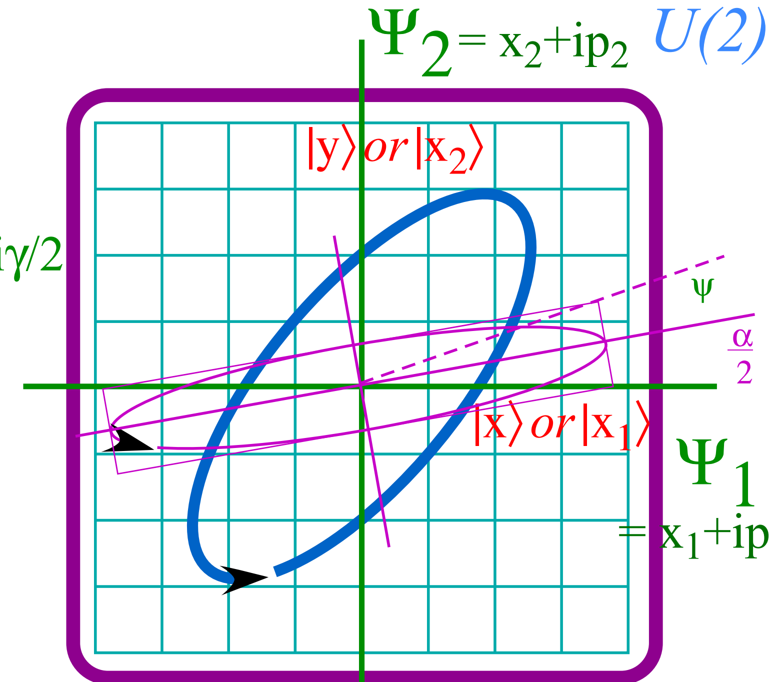
U(2) World labeled by two complex phasors and driven by complex operator

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

2-State ket $|\Psi\rangle=$



$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{N}e^{-i\alpha/2}\cos\beta/2 \\ \sqrt{N}e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S

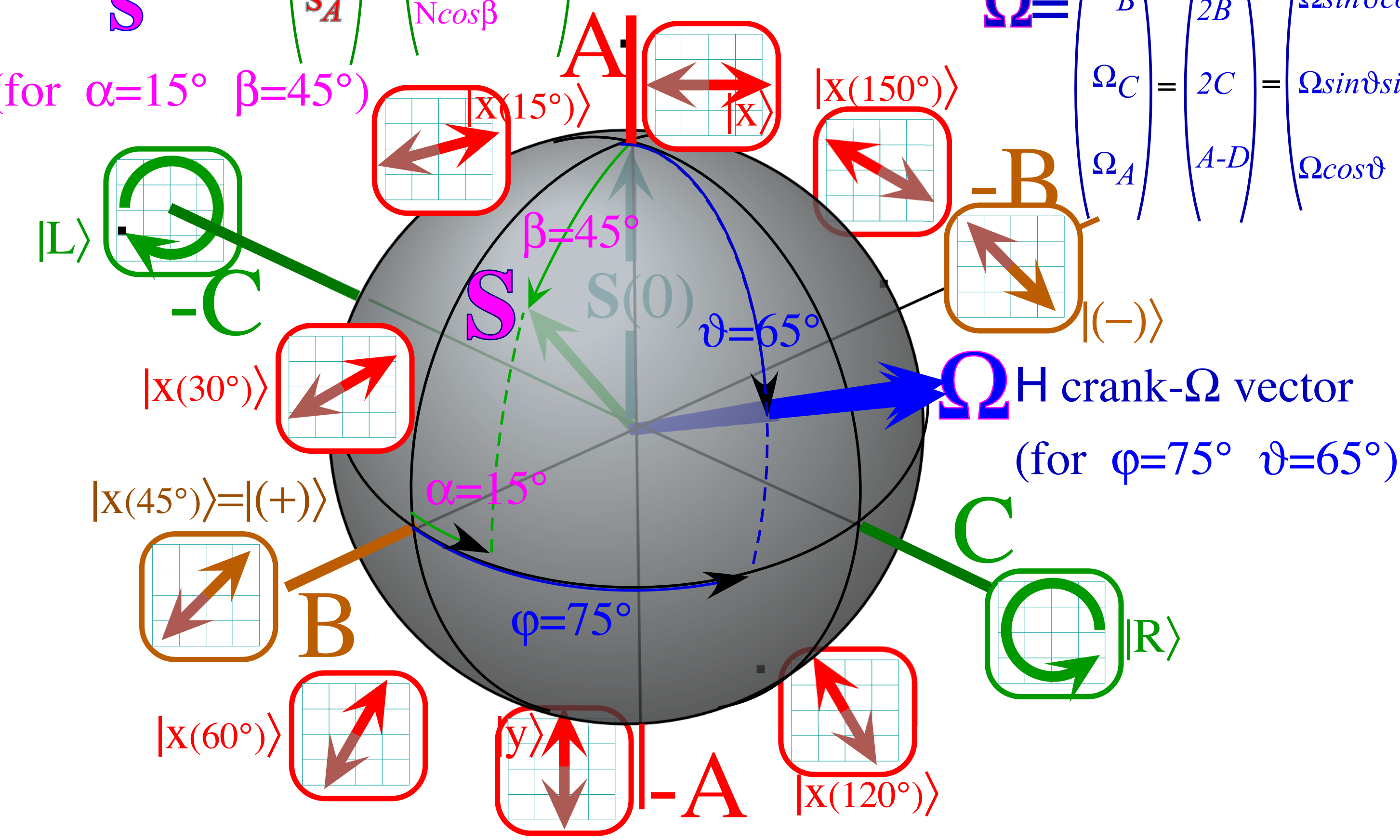
$$\begin{pmatrix} S_B \\ S_C \\ S_A \end{pmatrix} = \begin{pmatrix} N\sin\beta\cos\alpha \\ N\sin\beta\sin\alpha \\ N\cos\beta \end{pmatrix} \frac{1}{2}$$

H-Operator $\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

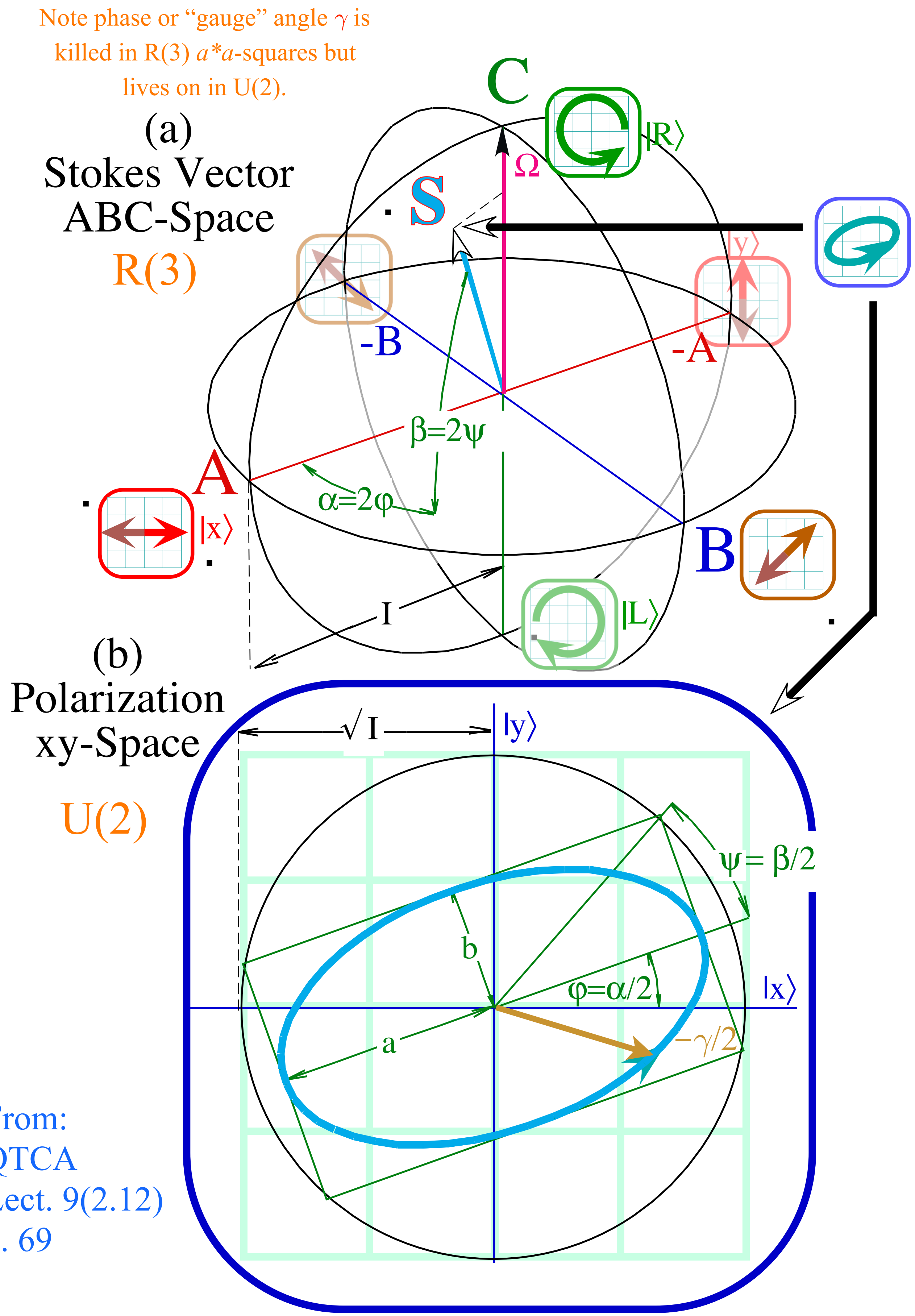
Angular velocity

$$\boldsymbol{\Omega} = \begin{pmatrix} \Omega_B \\ \Omega_C \\ \Omega_A \end{pmatrix} = \begin{pmatrix} 2B \\ 2C \\ A-D \end{pmatrix} = \begin{pmatrix} \Omega\sin\vartheta\cos\varphi \\ \Omega\sin\vartheta\sin\varphi \\ \Omega\cos\vartheta \end{pmatrix}$$

*R(3) World labeled by real 3-D "spin" vector **S** of angular momentum and driven by real 3-D "spin" vector **Ω** of angular velocity*



Polarization ellipse and spinor state dynamics



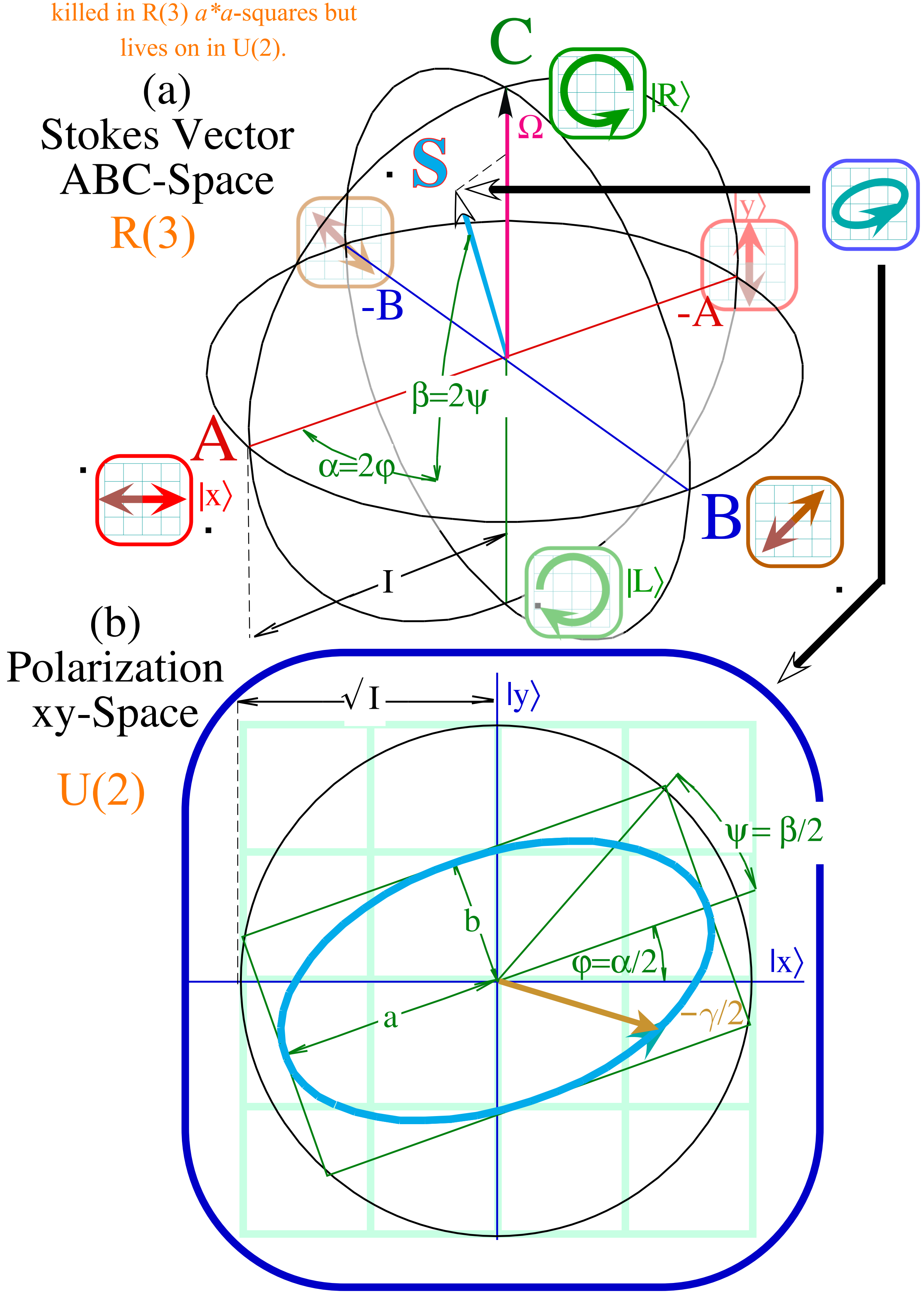
From:
QTCA
Lect. 9(2.12)
p. 69

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

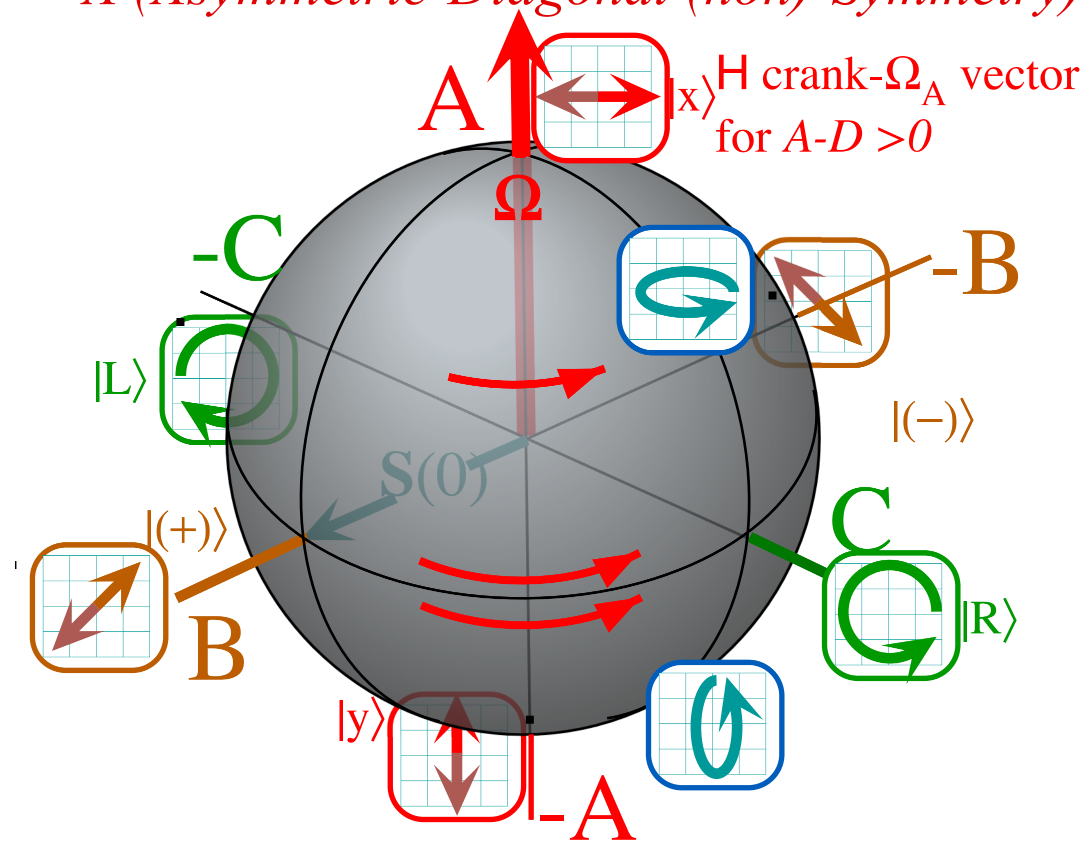
Further explanation of polarization geometry given in *Lecture 23 p. 93 to 125*

Polarization ellipse and spinor state dynamics (A-Type motion)

Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).



A (Asymmetric-Diagonal (non)-Symmetry)

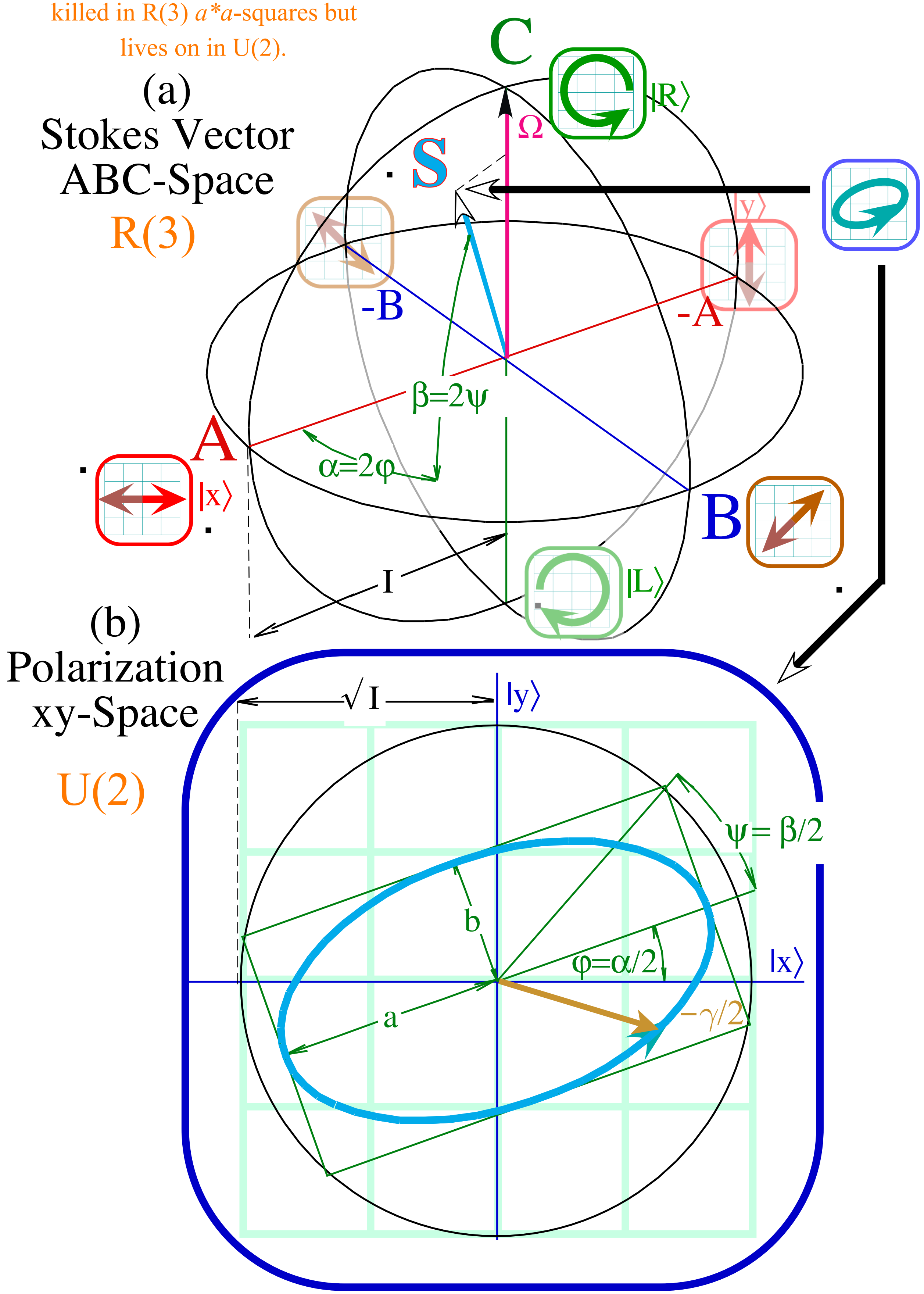


BoxIt (*A-Type*) Web Simulation:
A=4.9, B=C=0, D=4.0

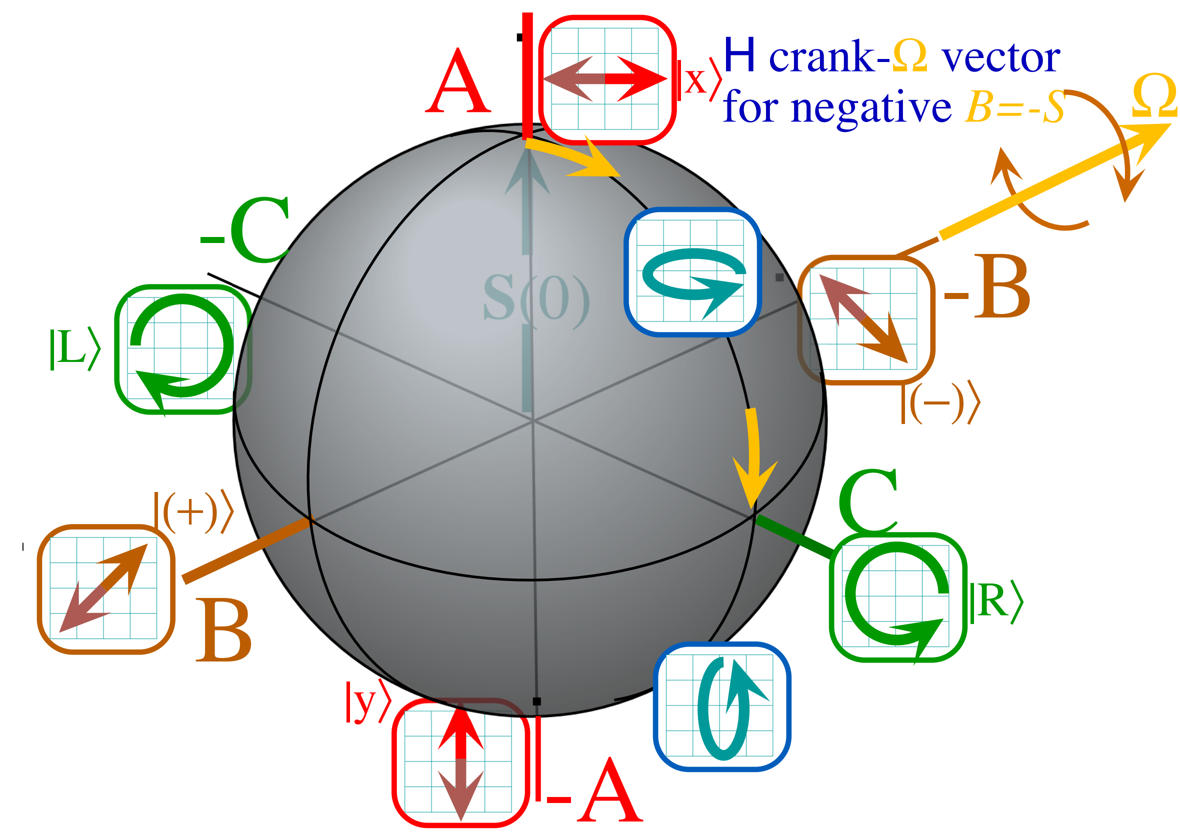
Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

Polarization ellipse and spinor state dynamics (B-Type motion)

Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).



B (Bilateral Balanced Symmetry)

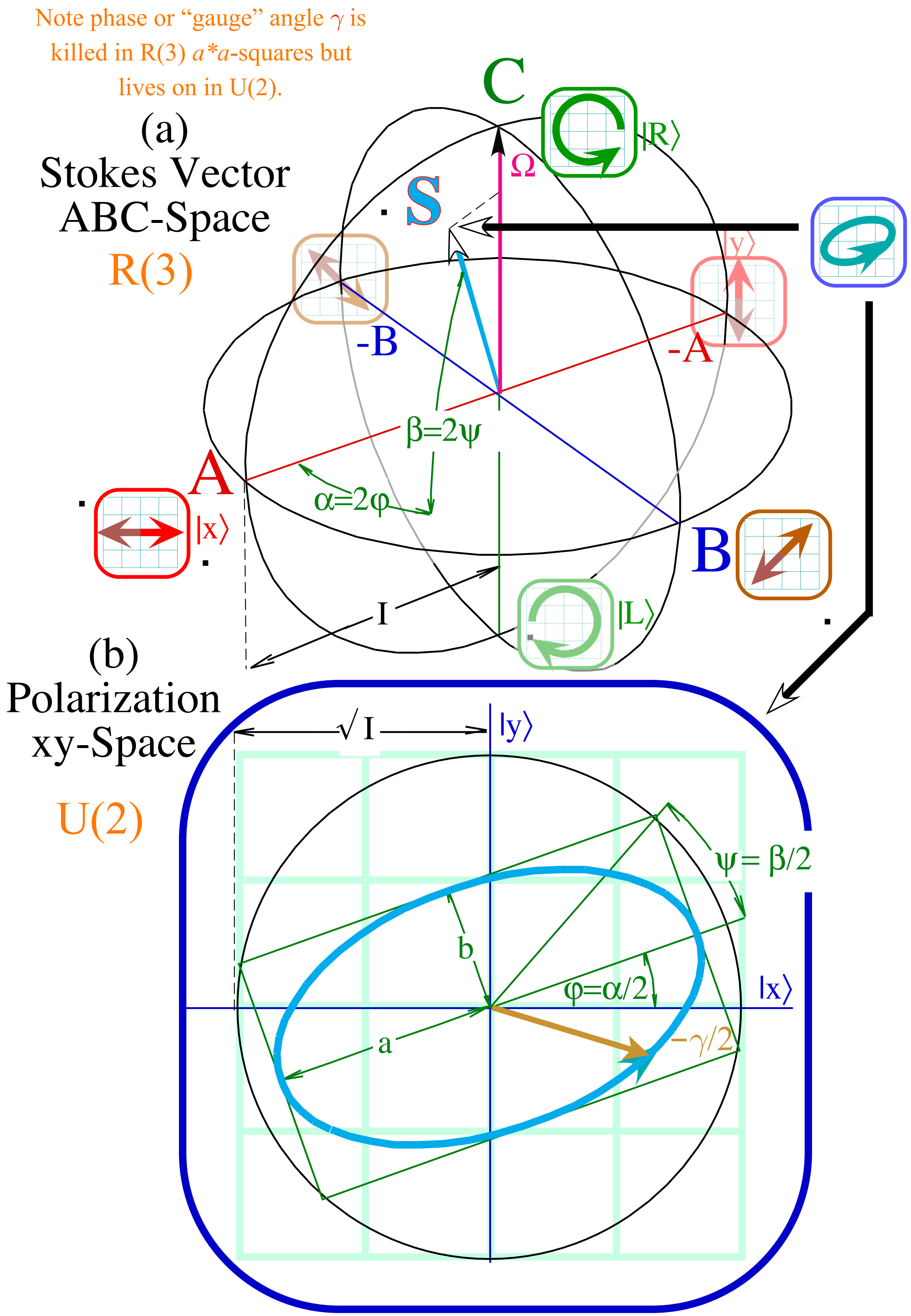


BoxIt (*A-Type*) Web Simulation:
A=4.9, B=C=0, D=4.0

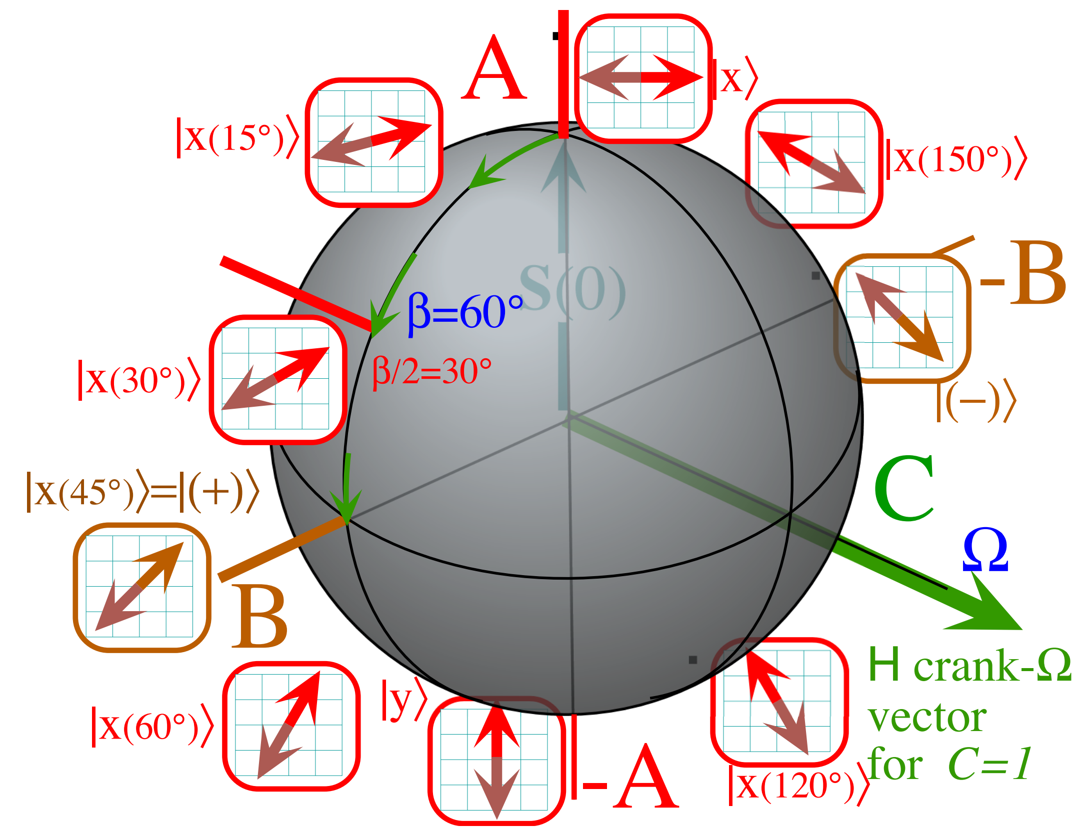
BoxIt (*B-Type*) *Simulation*
A=4.0, B=-0.2, C=0, D=4.0

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

Polarization ellipse and spinor state dynamics (A-Type motion)



C (Chiral-circular-complex-Coriolis-cyclotron-curly…current-carrier..symmetry)



BoxIt (*A-Type*) Web Simulation:
A=4.9, B=C=0, D=4.0

BoxIt (*B-Type*) *Simulation*
A=4.0, B=-0.2, C=0, D=4.0

BoxIt (*C-Type*) *Simulation*
A=4.055, B=0, C=0.1, D=4.055

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\boldsymbol{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\boldsymbol{\Omega}(\varphi\vartheta)\cdot t=\boldsymbol{\Theta}$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\boldsymbol{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

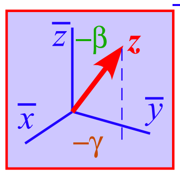
- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

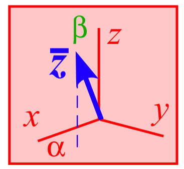
- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Here spin-rotor S-polar coordinates are Euler angles

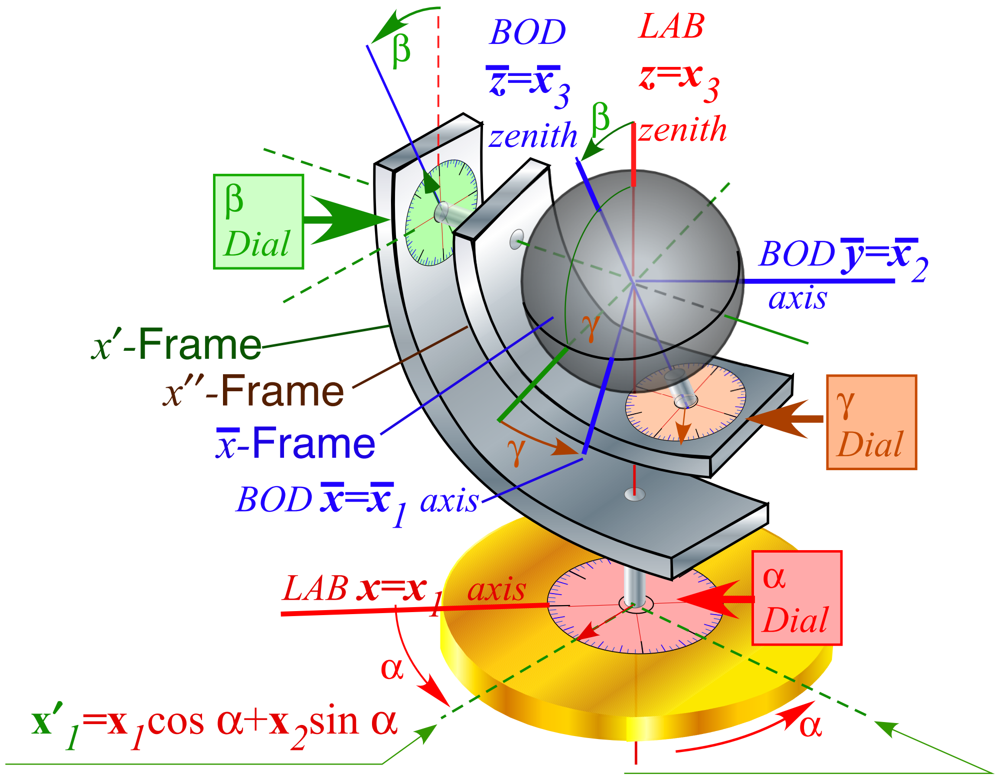
From Lecture 7 page 86



BOD frame view
Polar angles of LAB zenith $z=x_3$ are (azimuth angle=−γ, polar angle=−β)



LAB frame view
Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ are (azimuth angle=α, polar angle=β)



Polar coordinates for unit Spin vector $\hat{S}$

$$\hat{S}_X = \cos\alpha \ \sin\beta$$
$$\hat{S}_Y = \sin\alpha \ \sin\beta$$
$$\hat{S}_Z = \cos\beta$$

Spin State&Operator

$$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$$

by Euler angles

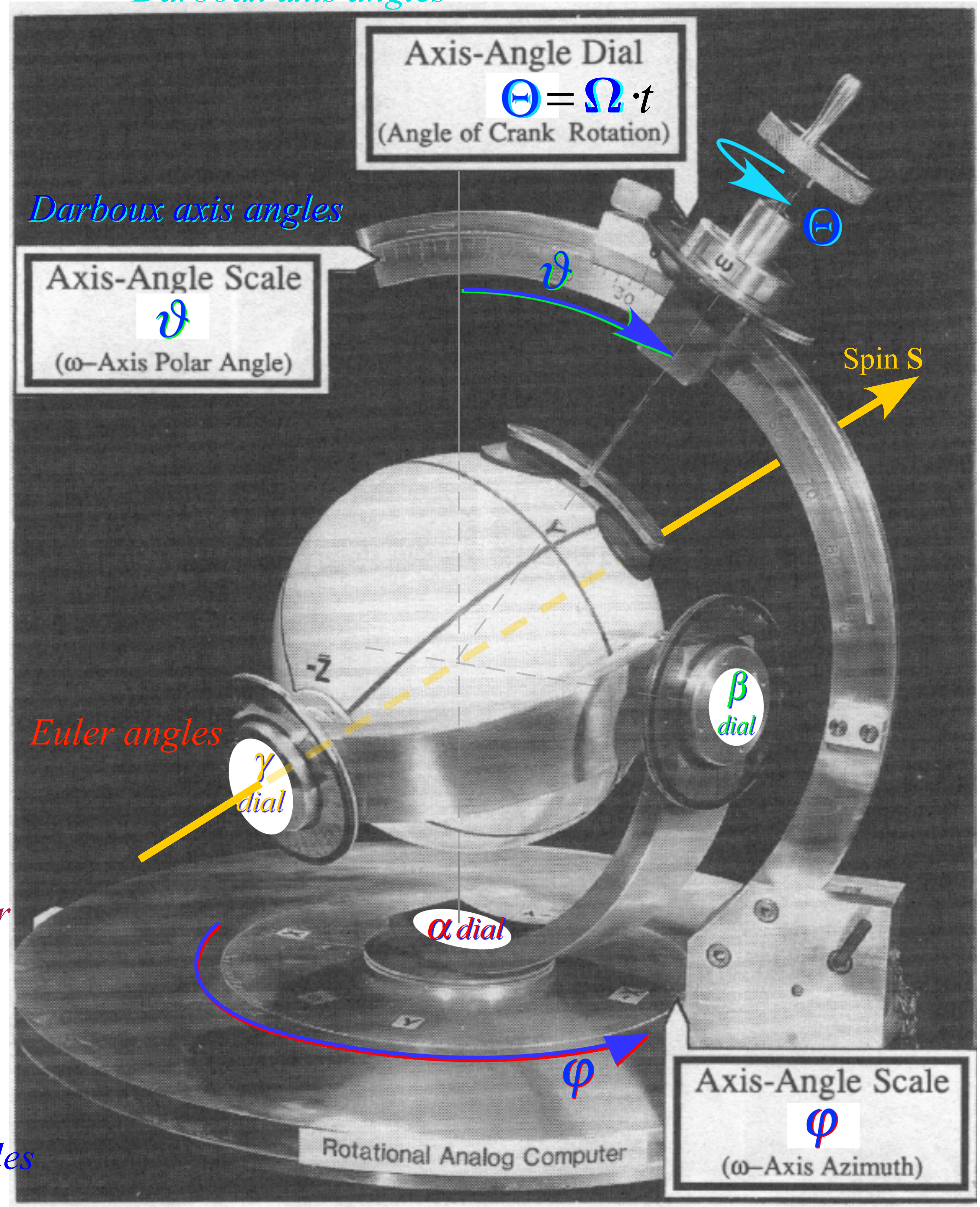
Polar coordinates for unit axis vector $\hat{\Theta}$

$$\hat{\Theta}_X = \cos\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Y = \sin\varphi \ \sin\vartheta$$
$$\hat{\Theta}_Z = \cos\vartheta$$

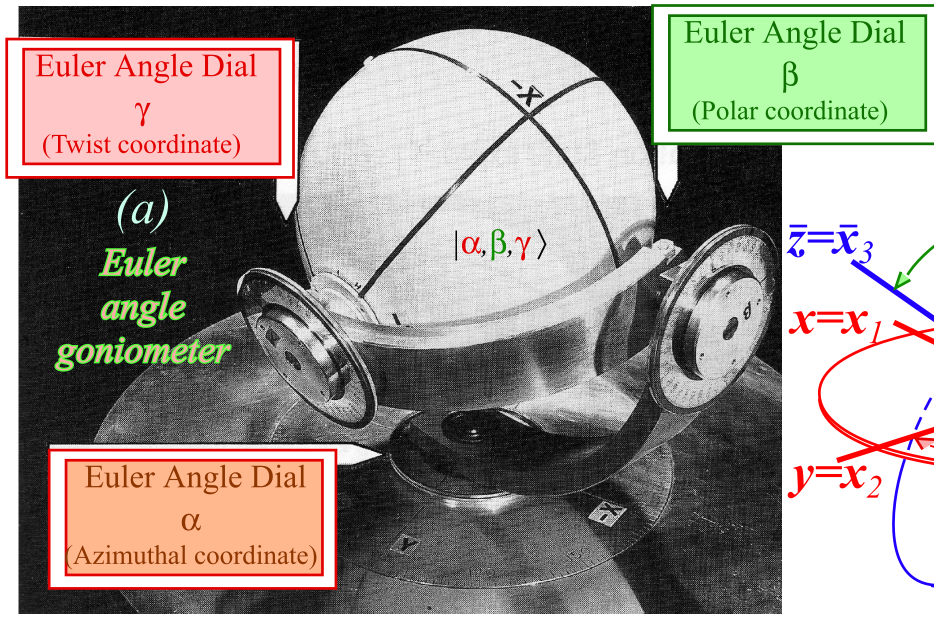
State&Operator

$$|[\varphi\vartheta\Theta]\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\uparrow\rangle$$

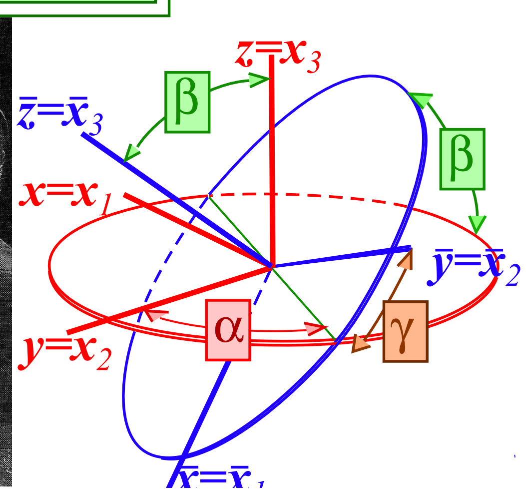
by Darboux axis angles

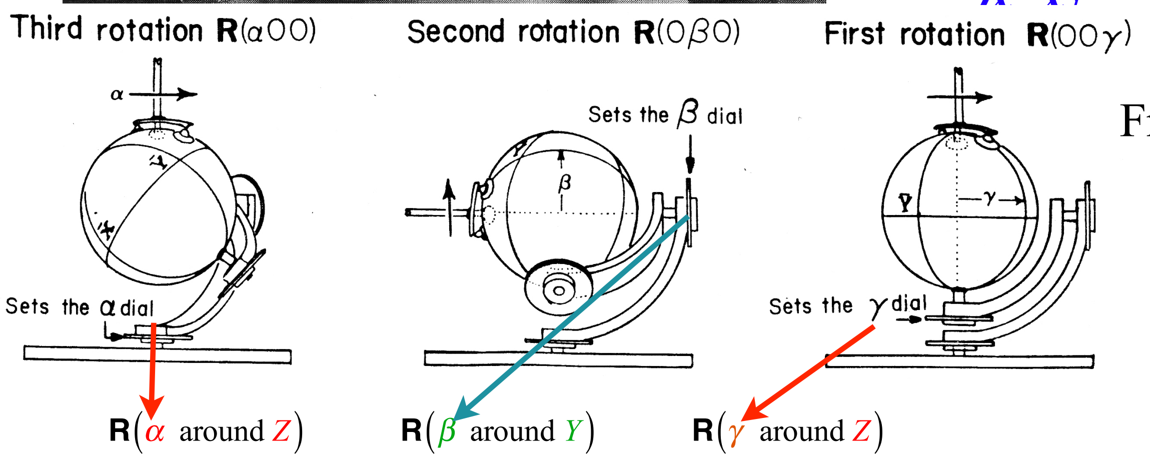


Euler R(αβγ) versus *Darboux* R[φϑΘ]



An astronomer's diagram



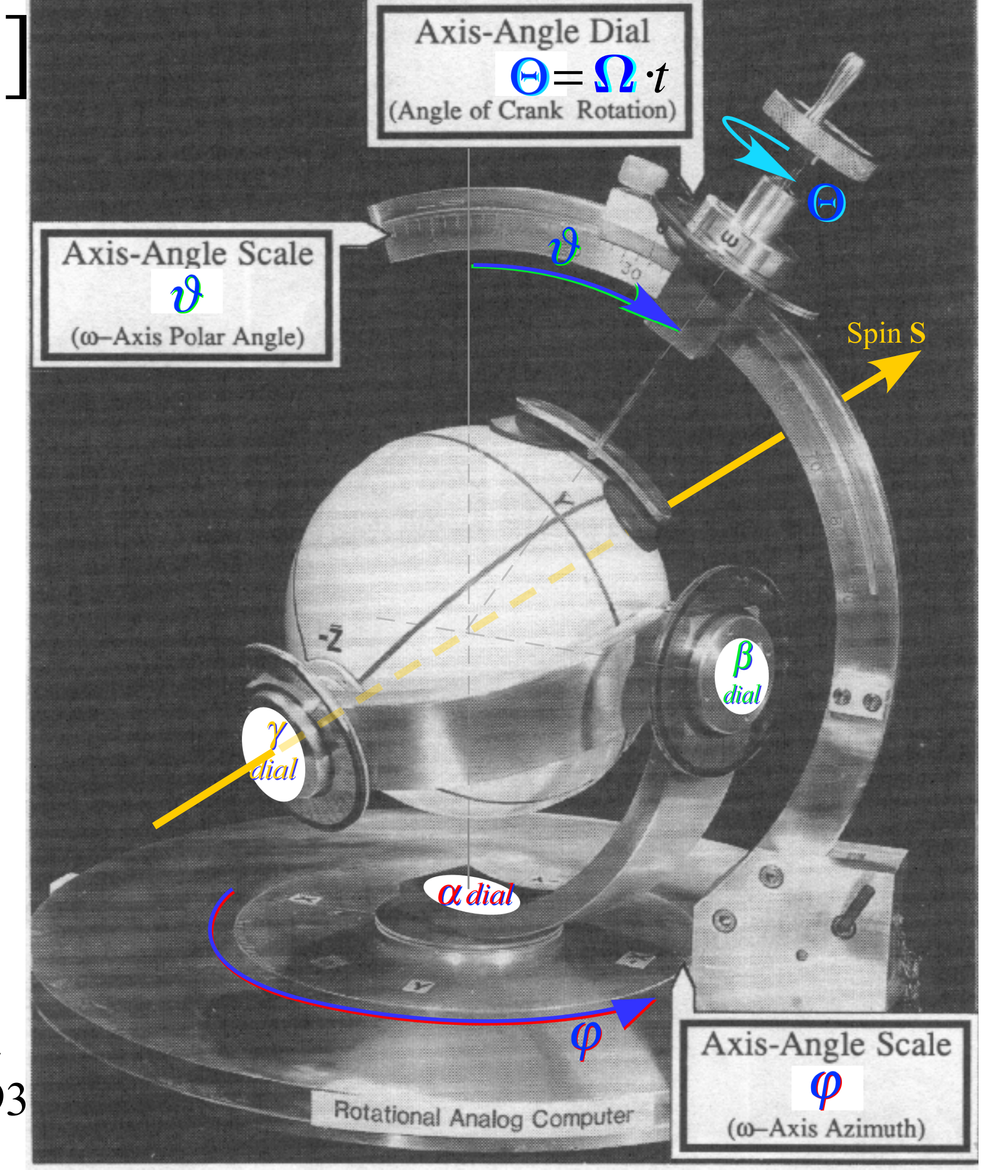


From Lecture 22 page 62 to 70

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].



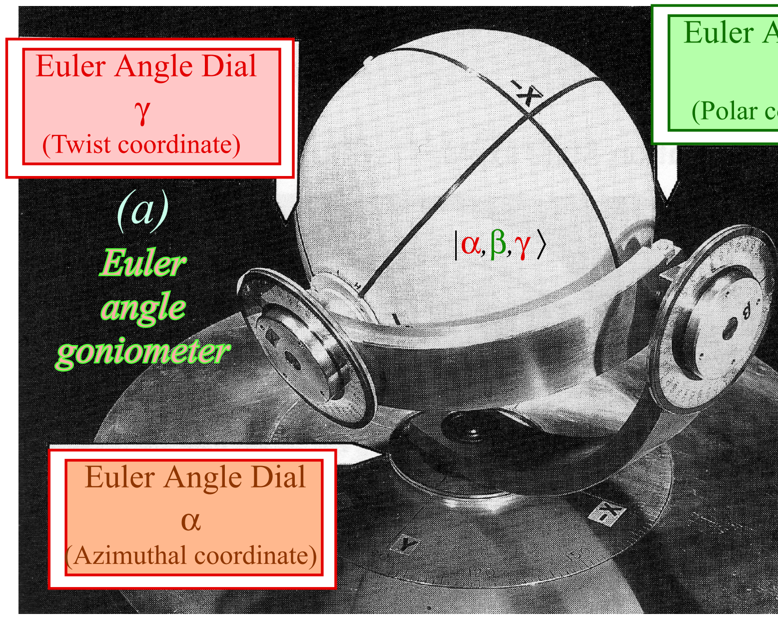
Lecture 22 page 92 to 93

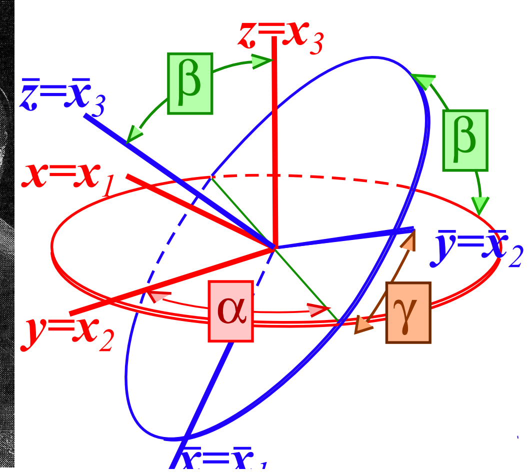
$$\mathbf{R}\left[\vec{\Theta}\right]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}\left[\varphi\vartheta\Theta\right]=e^{-i\mathbf{H}t}$$

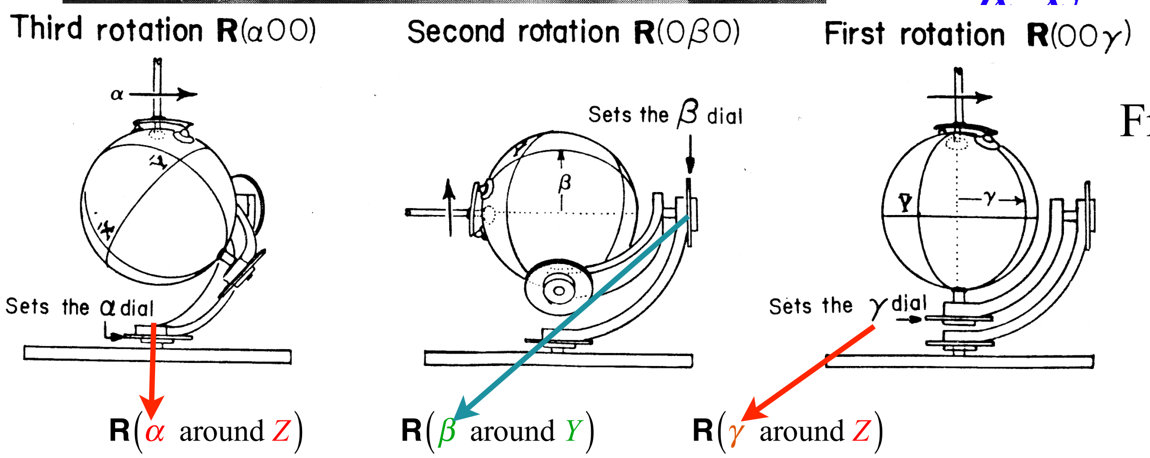
$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$







From Lecture 22
page 62 to 70

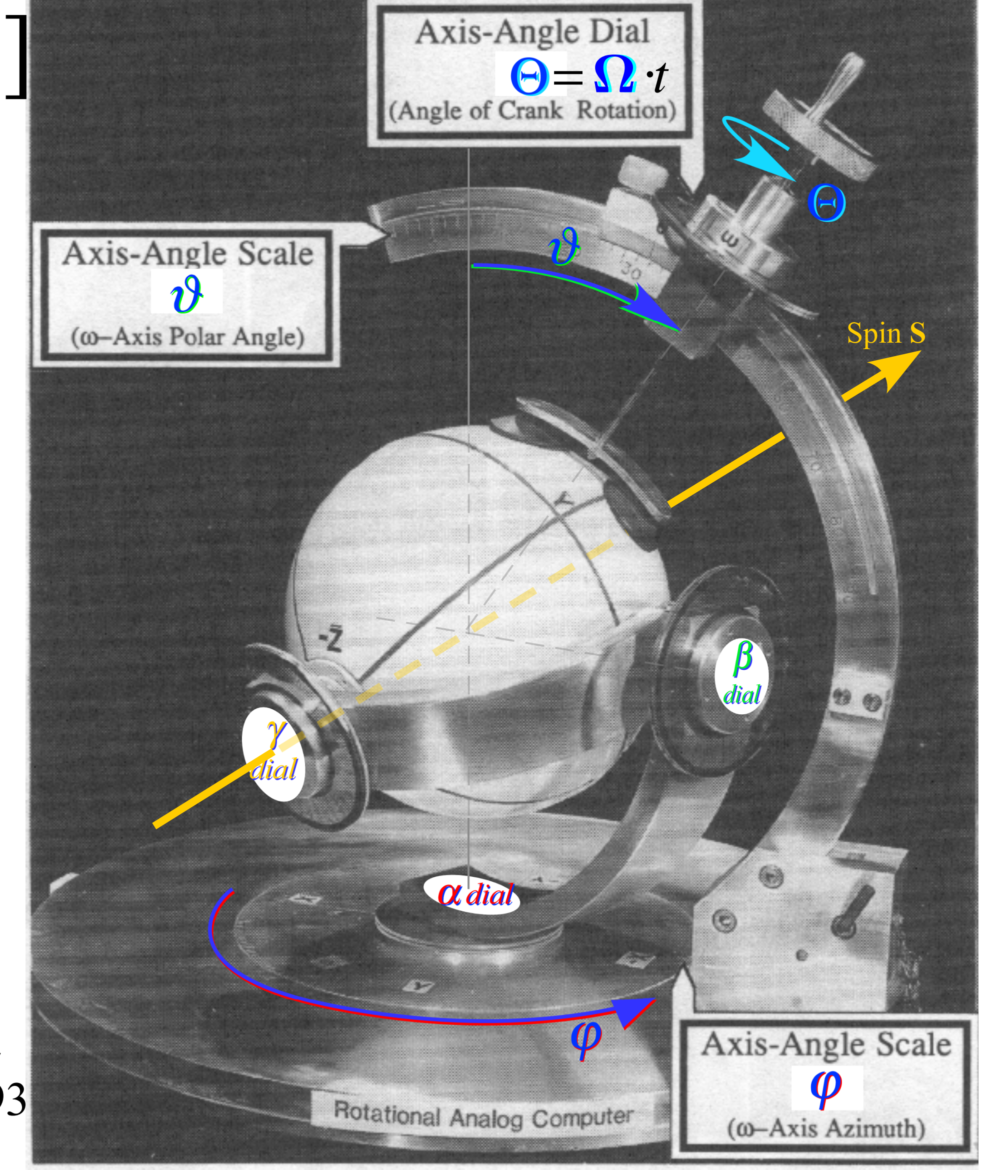
$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition*:

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*



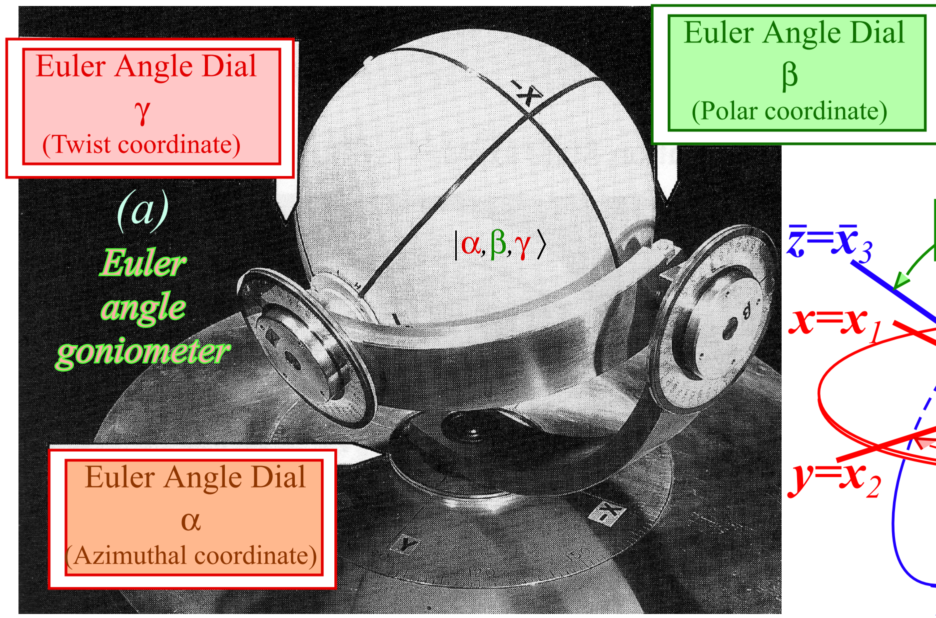
Lecture 22
page 92 to 93

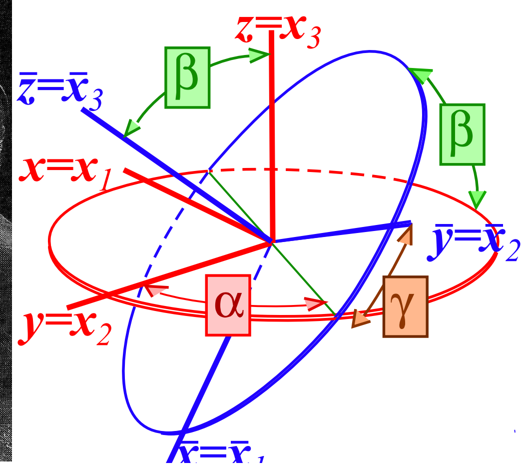
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

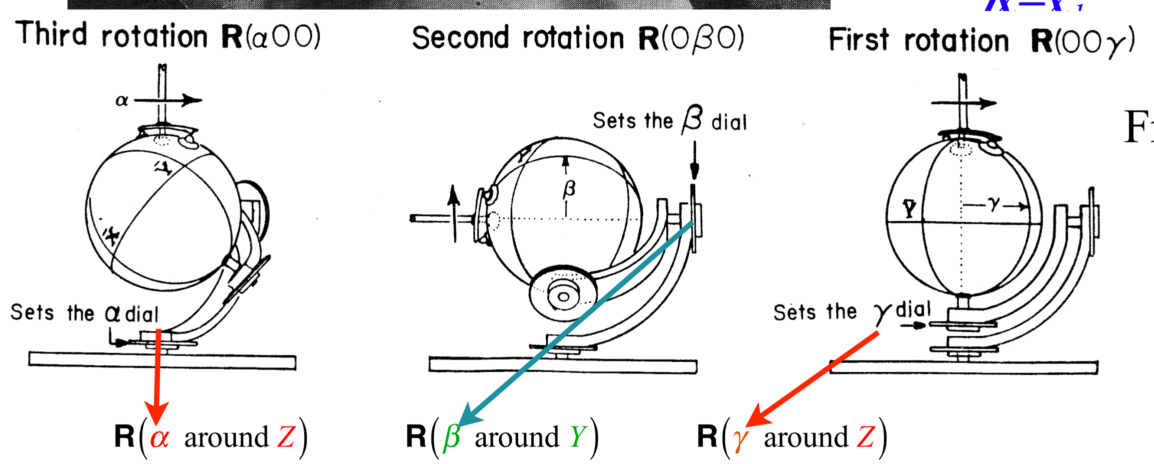
$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$







From Lecture 22
page 62 to 70

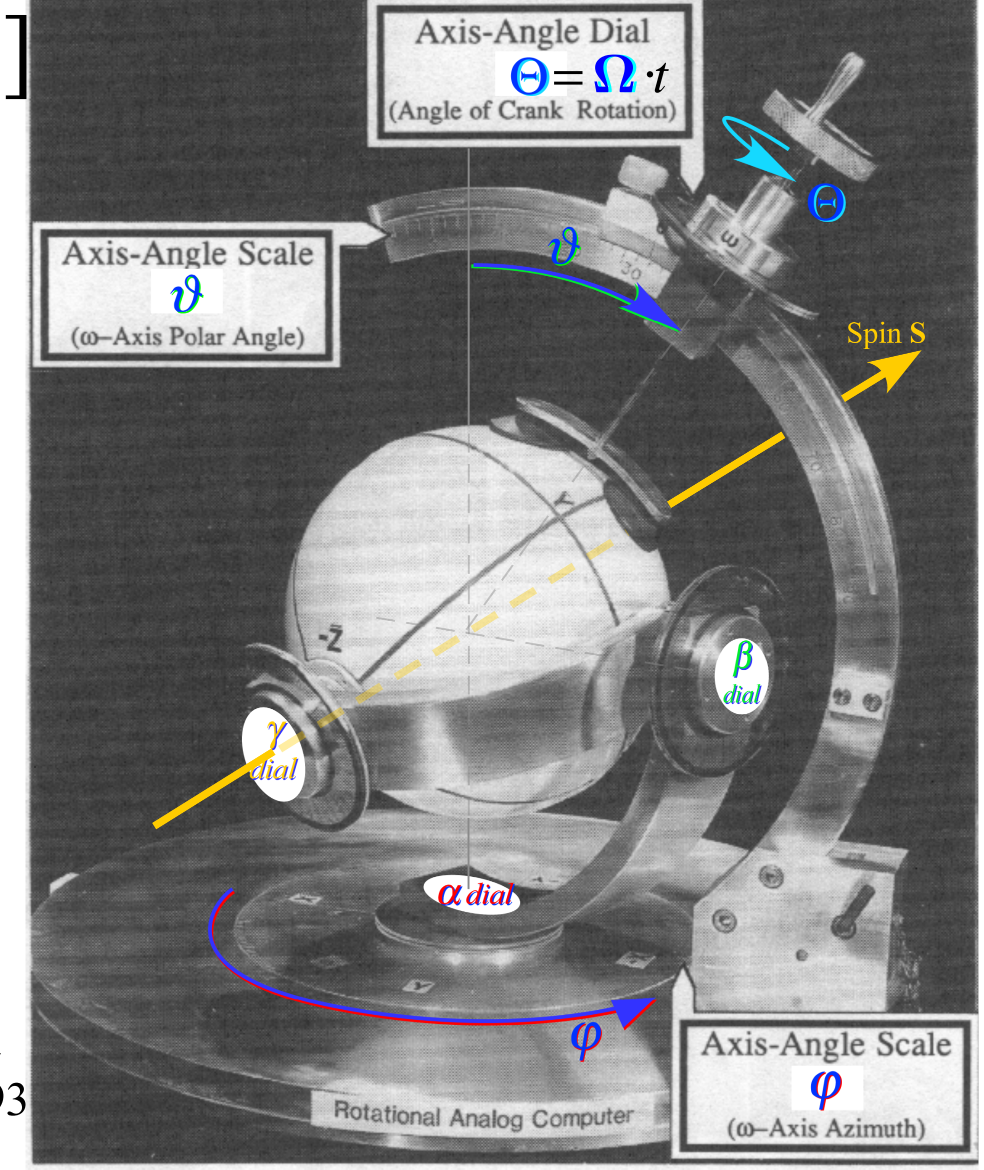
$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*



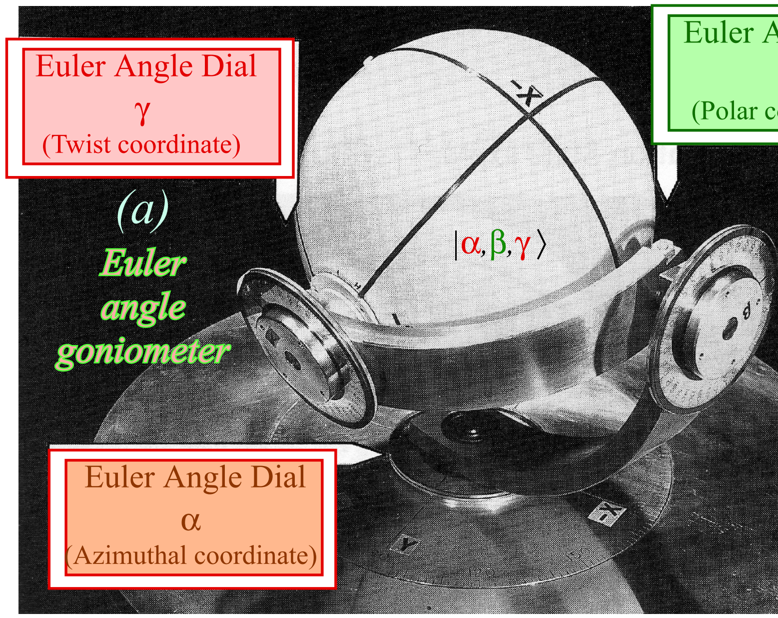
Lecture 22
page 92 to 93

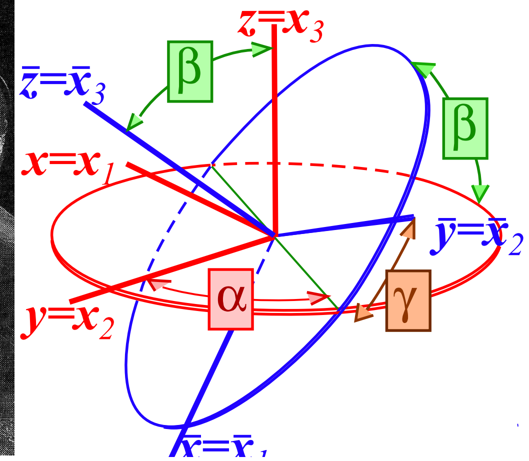
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

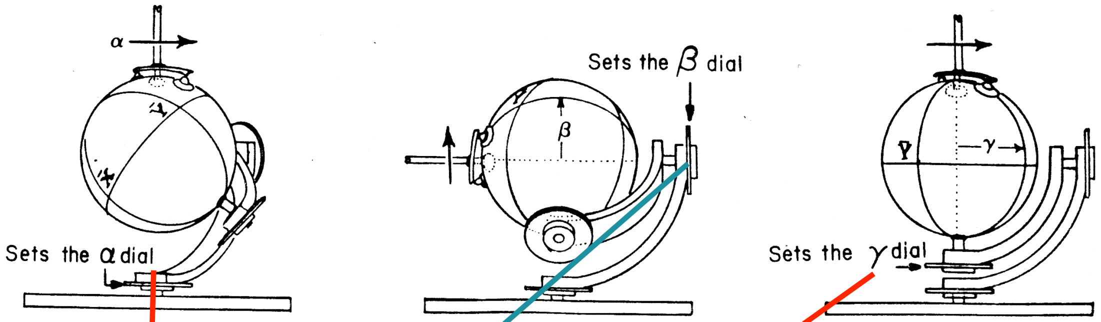
$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$





Third rotation $\mathbf{R}(\alpha 0 0)$ Second rotation $\mathbf{R}(0\beta 0)$ First rotation $\mathbf{R}(0 0\gamma)$



$\mathbf{R}(\alpha \text{ around } Z)$ $\mathbf{R}(\beta \text{ around } Y)$ $\mathbf{R}(\gamma \text{ around } Z)$

From Lecture 22 page 62 to 70

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

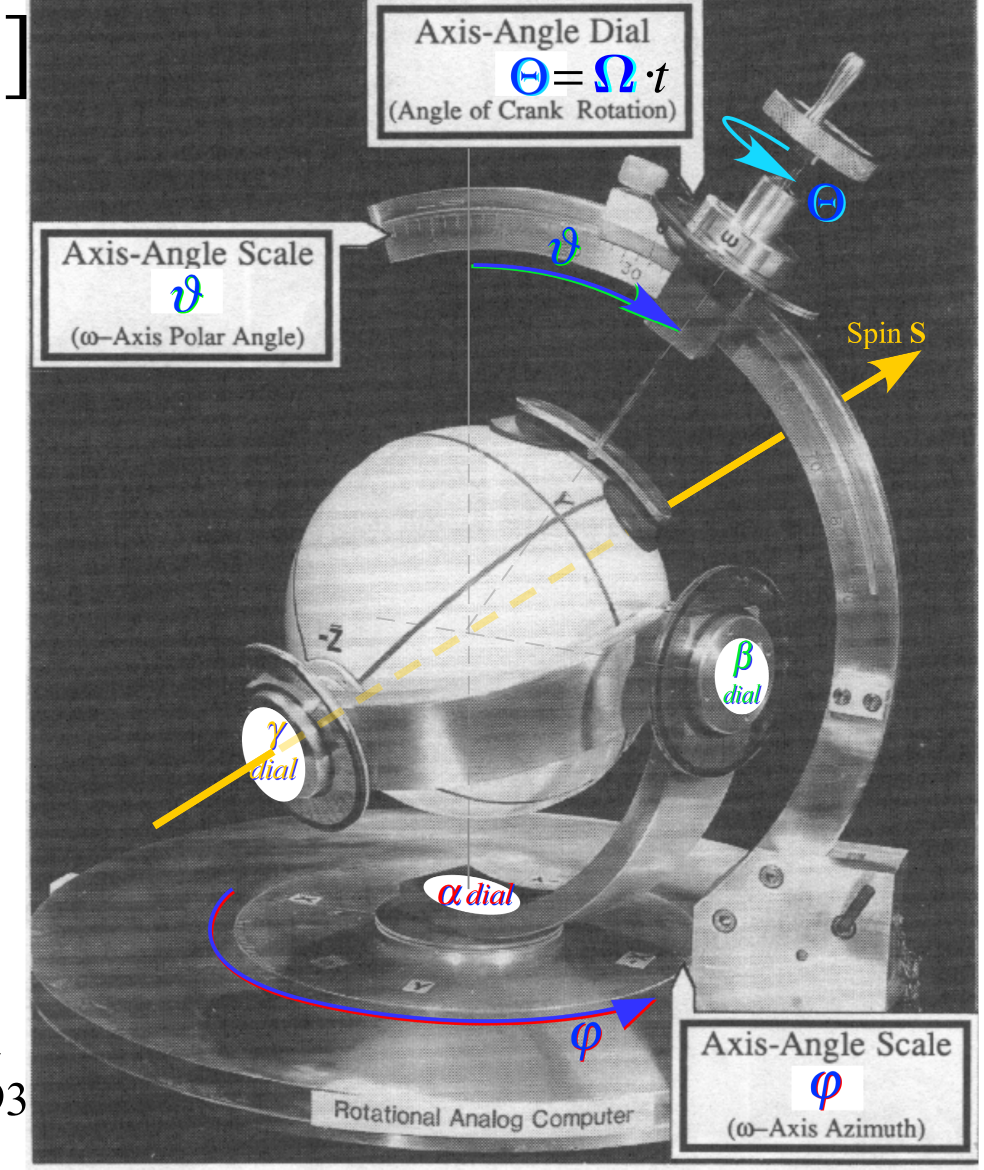
$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma) = \mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$



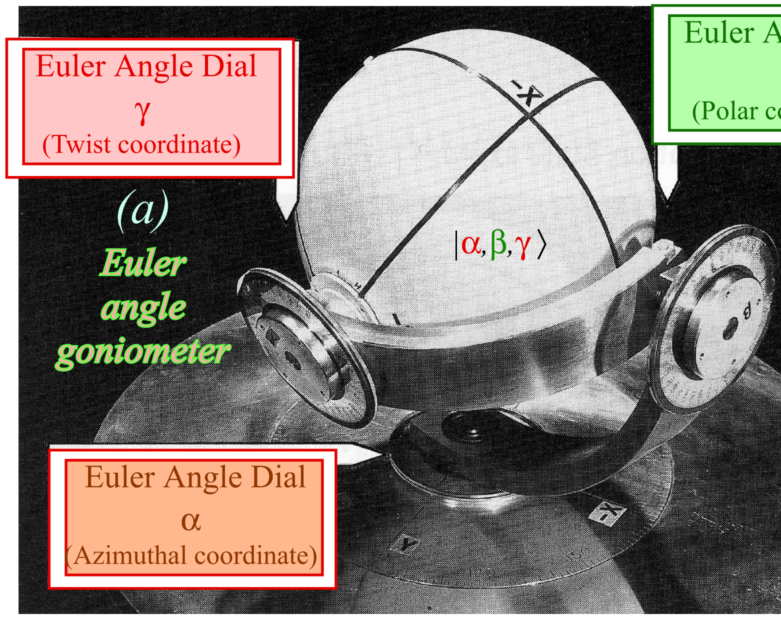
Lecture 22 page 92 to 93

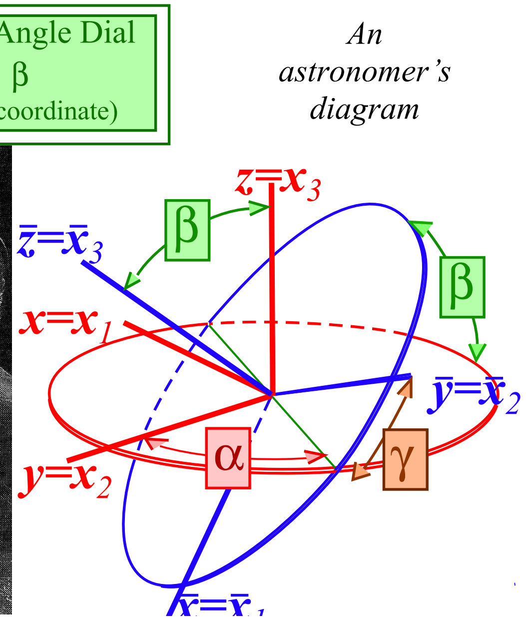
$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

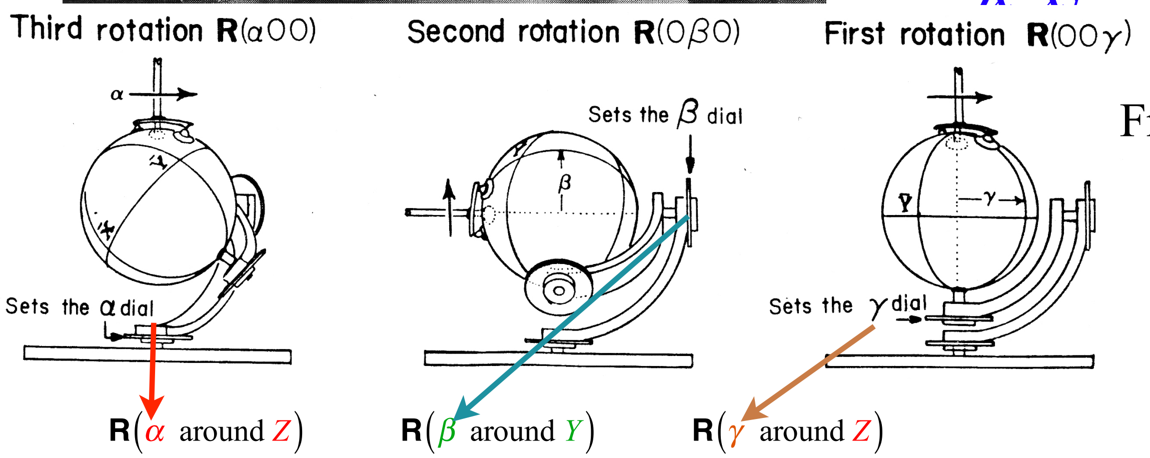
$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right)\\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

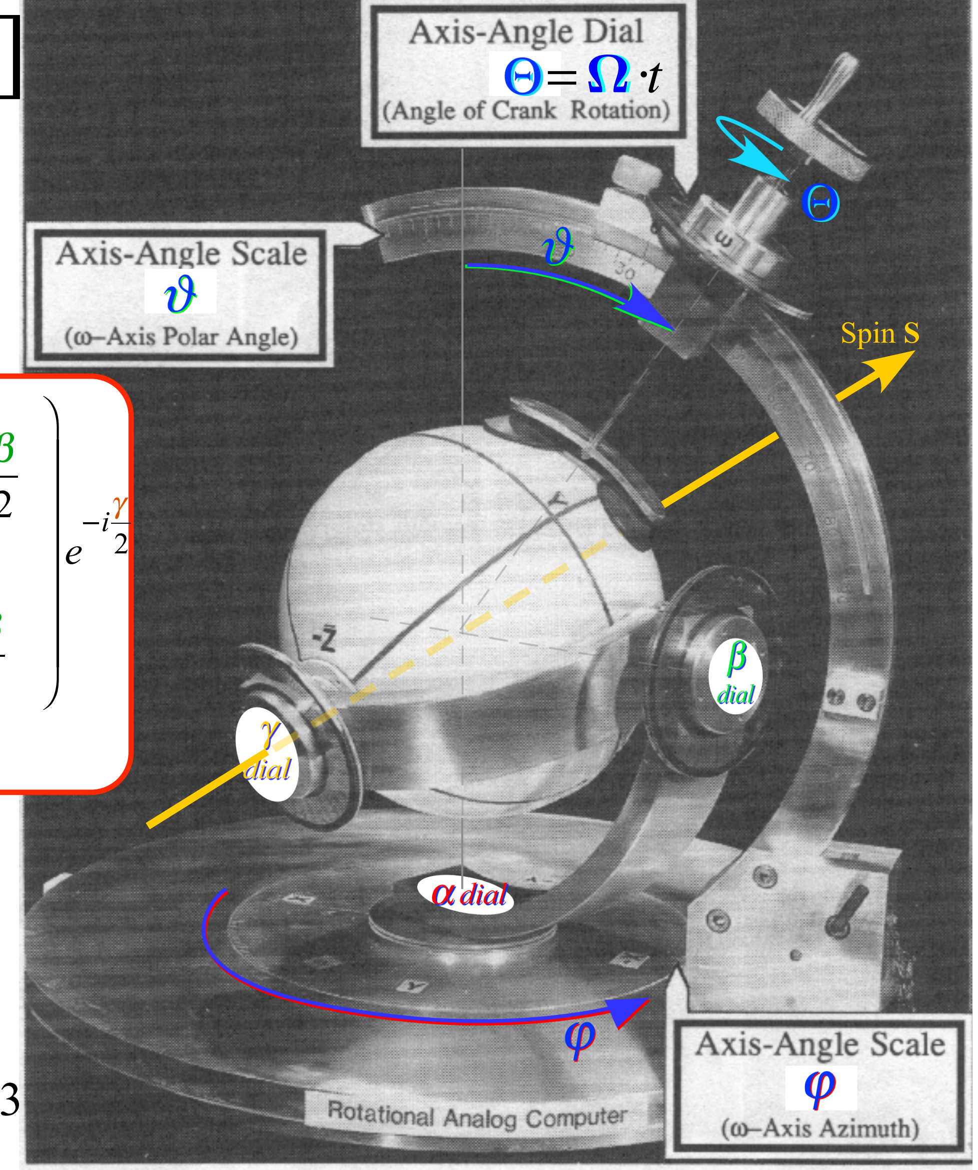




$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



From Lecture 22 page 62 to 70



Lecture 22 page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

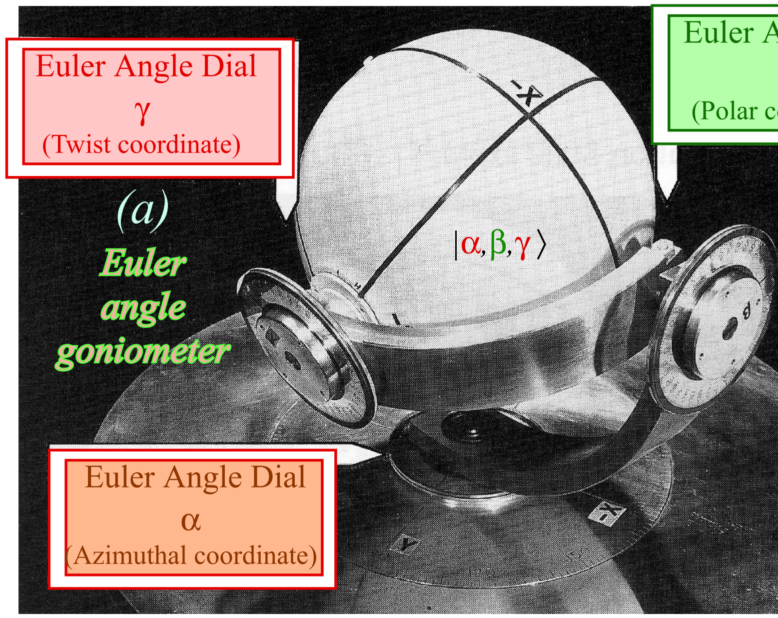
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} 1 \\ 0\end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}}$$

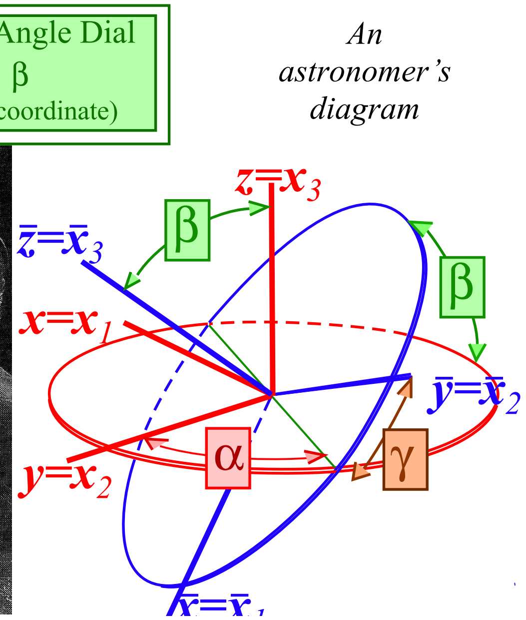
Phase coherence angle

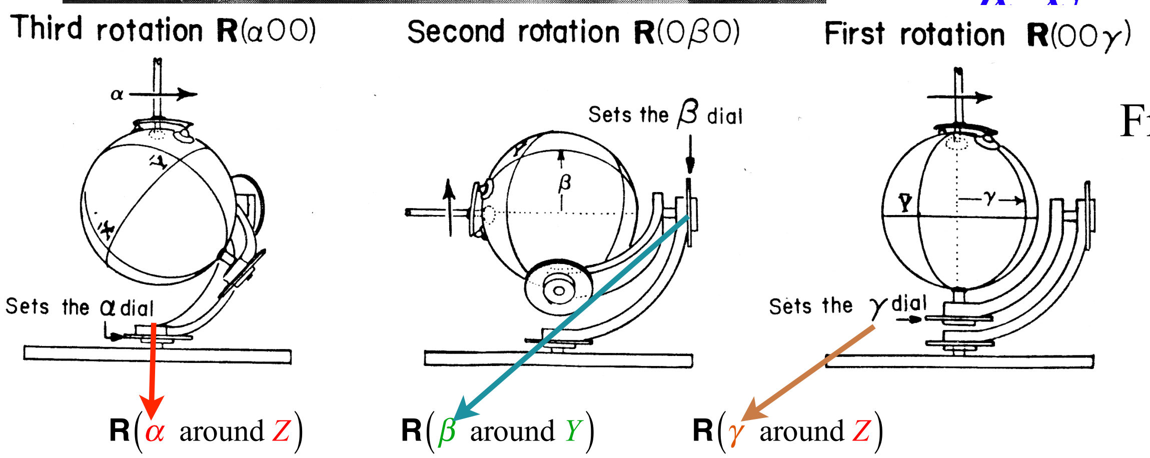
Population inversion angle

Overall phase angle

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$







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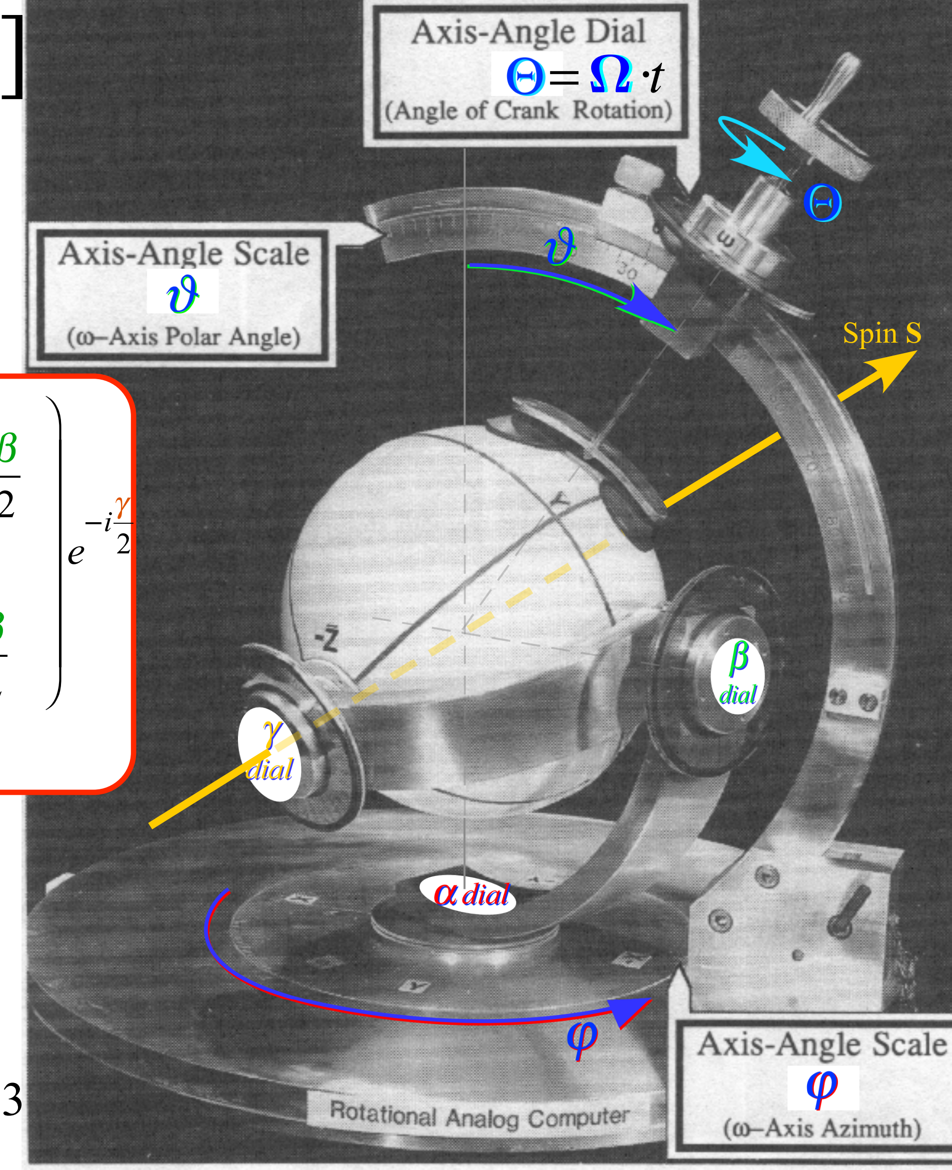
$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Lecture 22 page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix} =$$

$$= \cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}] = \begin{pmatrix} \cos\frac{\Theta}{2} - i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X - i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X + i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2} + i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix} = \mathbf{R}[\varphi\vartheta\Theta] = e^{-i\mathbf{H}t}$$

$$= \cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2} - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\Theta}{2} - i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta + i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta - i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2} + i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

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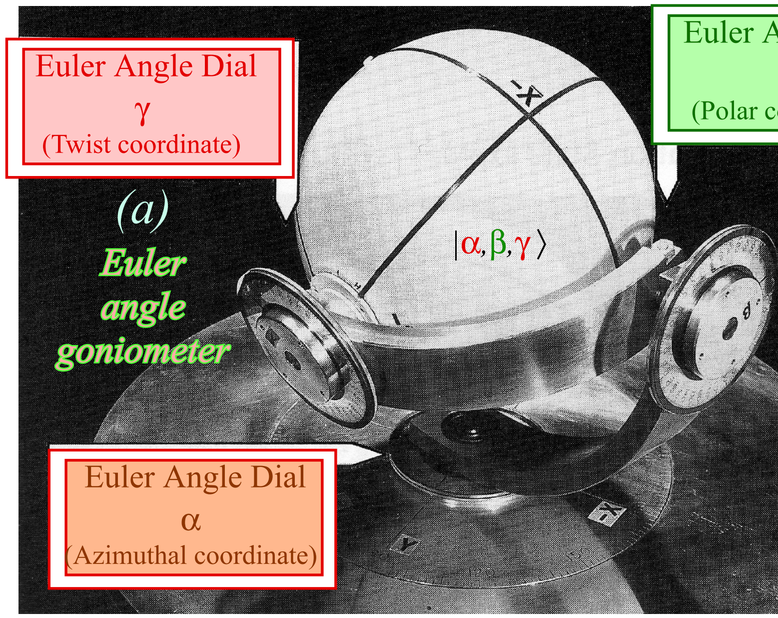
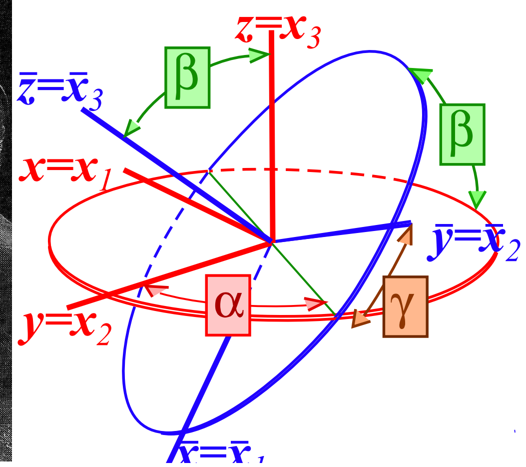
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

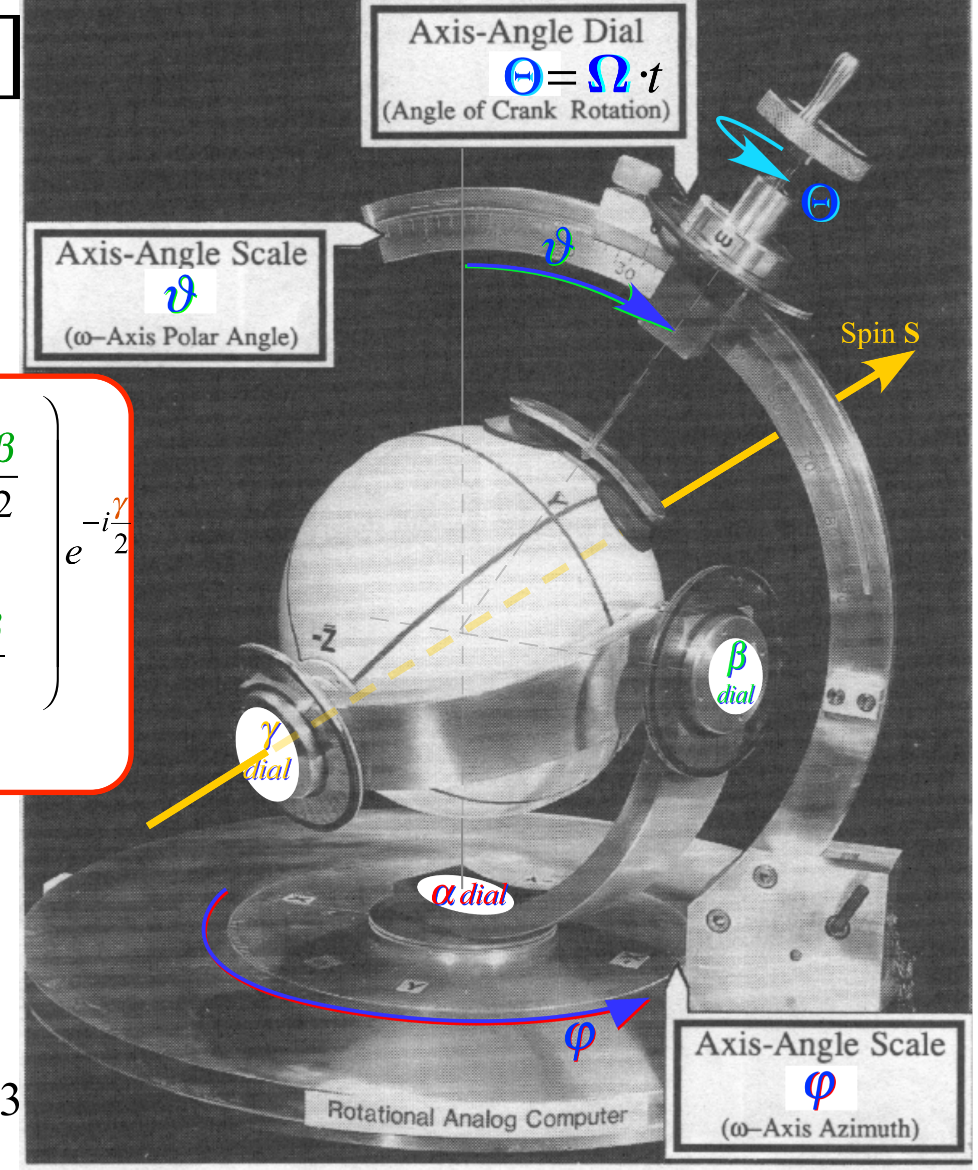
Euler R(αβγ) versus *Darboux* R[φϑΘ]

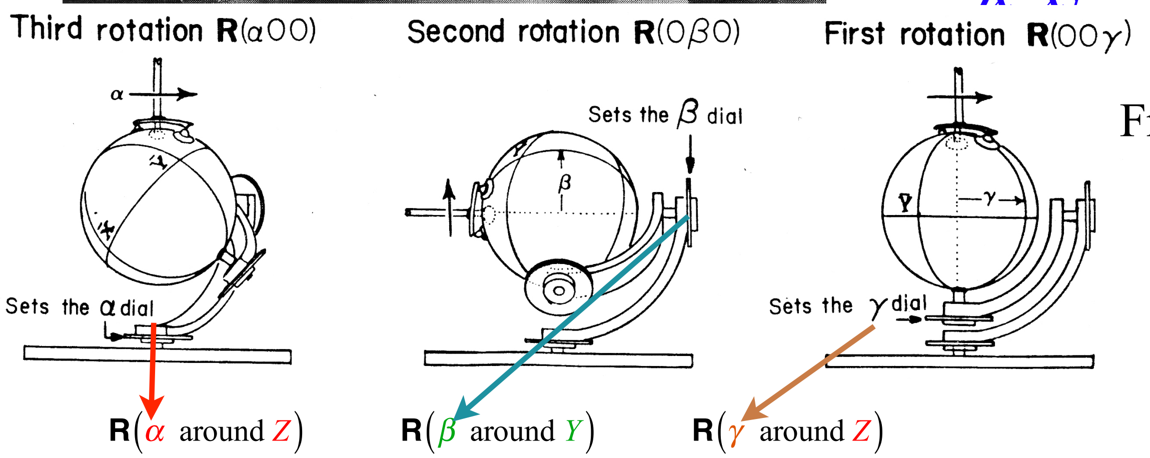


An astronomer's diagram



$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$





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Lecture 22 page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler R(αβγ) is simpler to form than Θ-axis *Darboux* R[φϑΘ].

Euler *state definition* lets us relate R(αβγ) = R[φϑΘ] ...

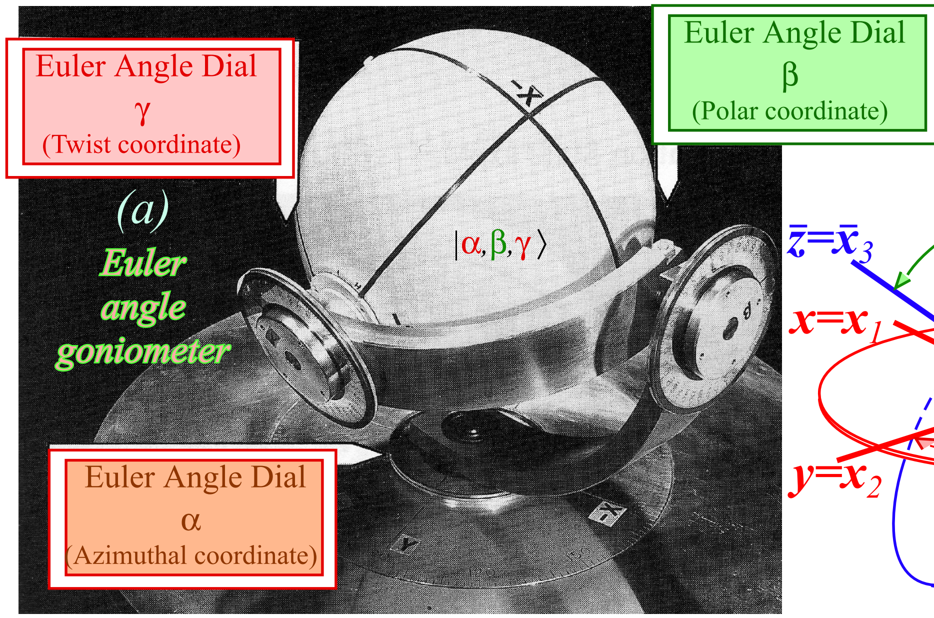
$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

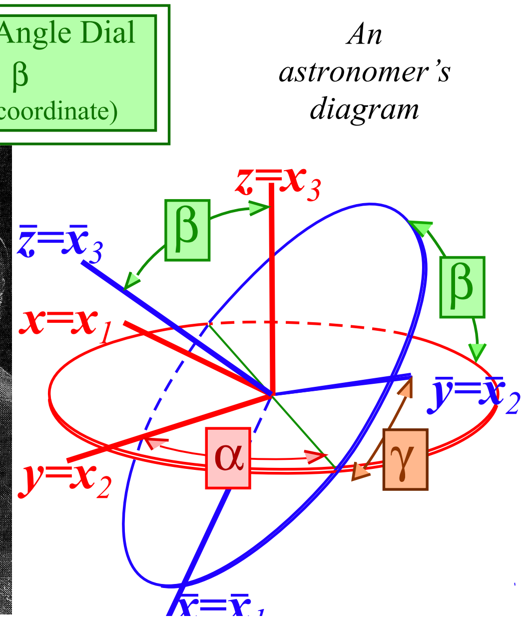
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

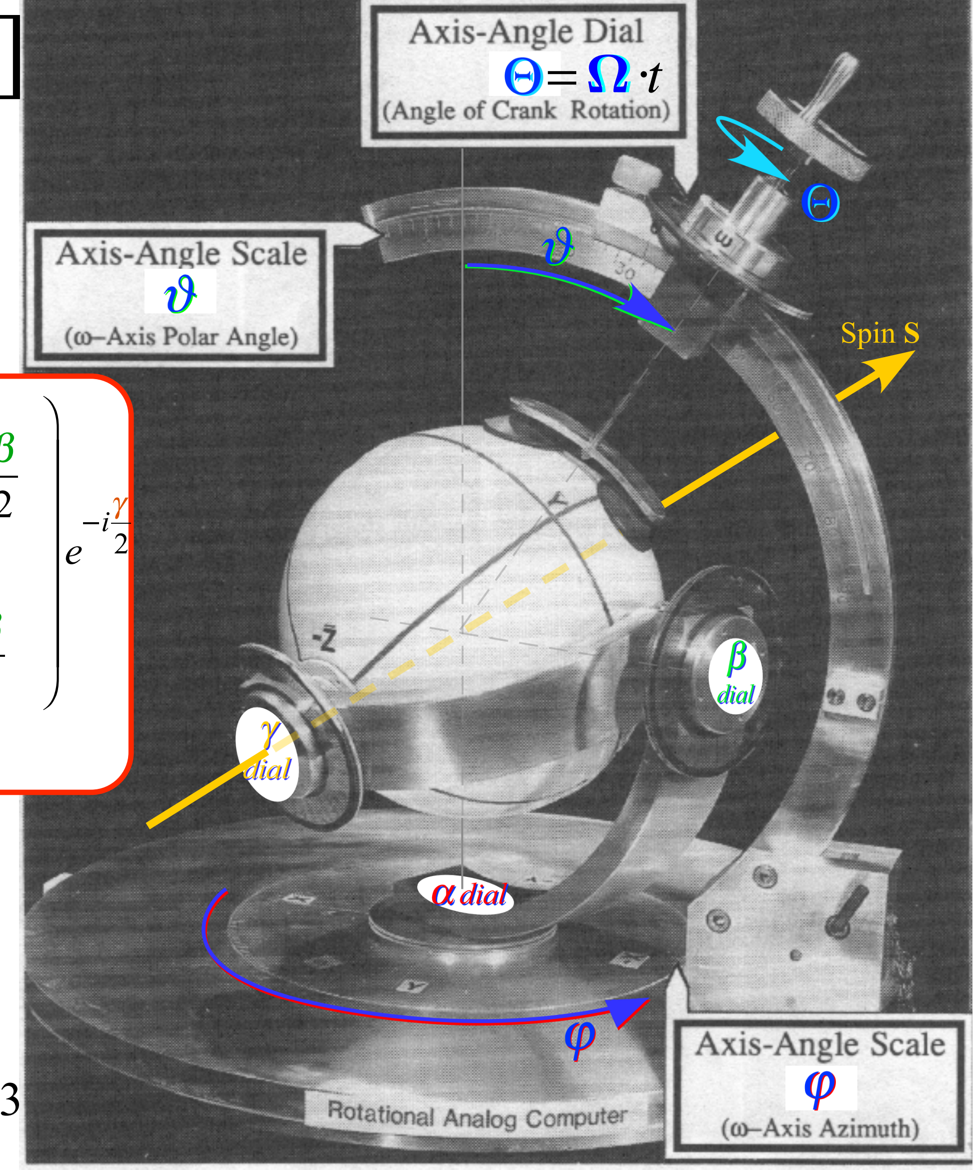
$$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$

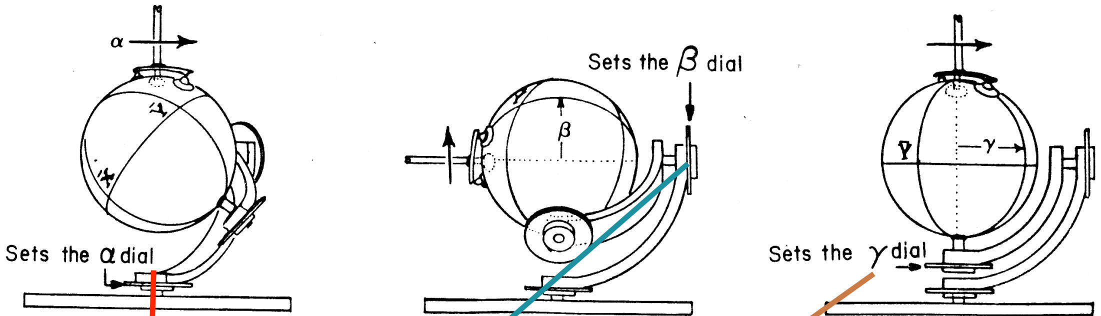




$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



Third rotation $\mathbf{R}(\alpha 00)$ Second rotation $\mathbf{R}(0\beta 0)$ First rotation $\mathbf{R}(00\gamma)$



$\mathbf{R}(\alpha$ around $Z)$ $\mathbf{R}(\beta$ around $Y)$ $\mathbf{R}(\gamma$ around $Z)$

From Lecture 22 page 62 to 70

Lecture 22 page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}\left[\vec{\Theta}\right]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right) \\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}\left[\varphi\vartheta\Theta\right]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right) \\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma) = \mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

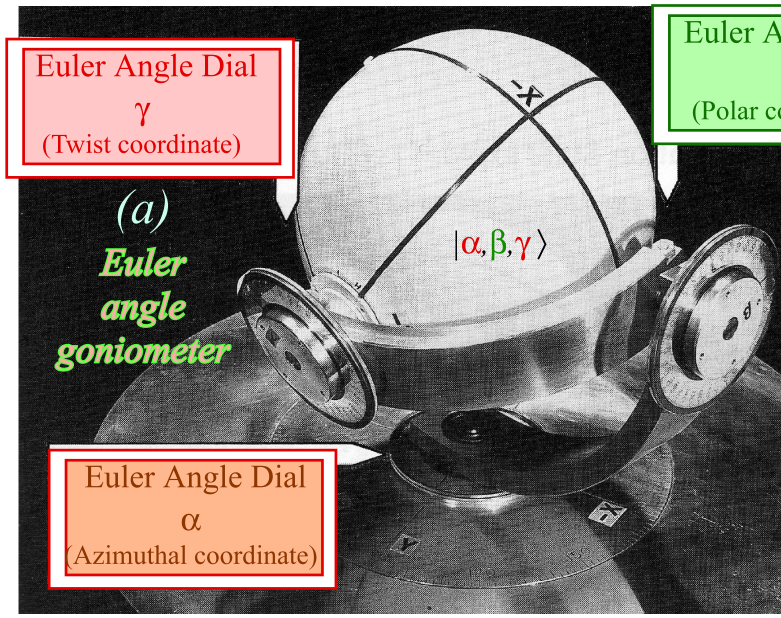
$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

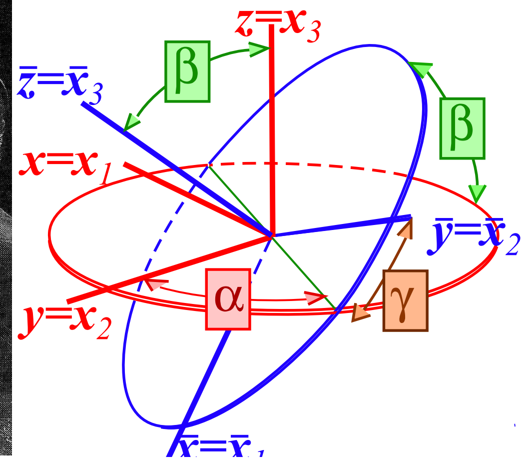
$$-p_2 = \sin[(\gamma-\alpha)/2]\ \sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$$

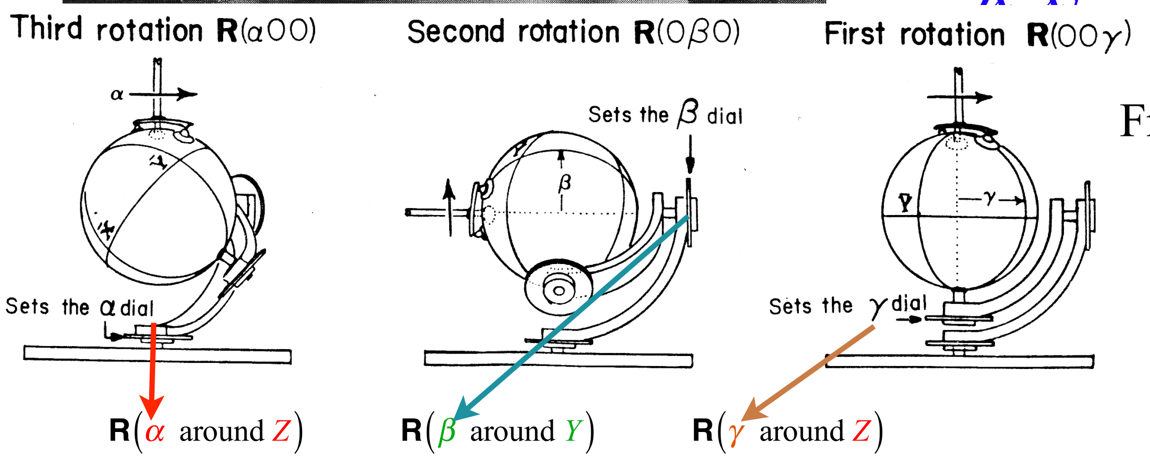
Euler $\mathbf{R}(\alpha\beta\gamma)$ versus *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$



An astronomer's diagram



$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle=\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2}\\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2}\end{pmatrix}e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



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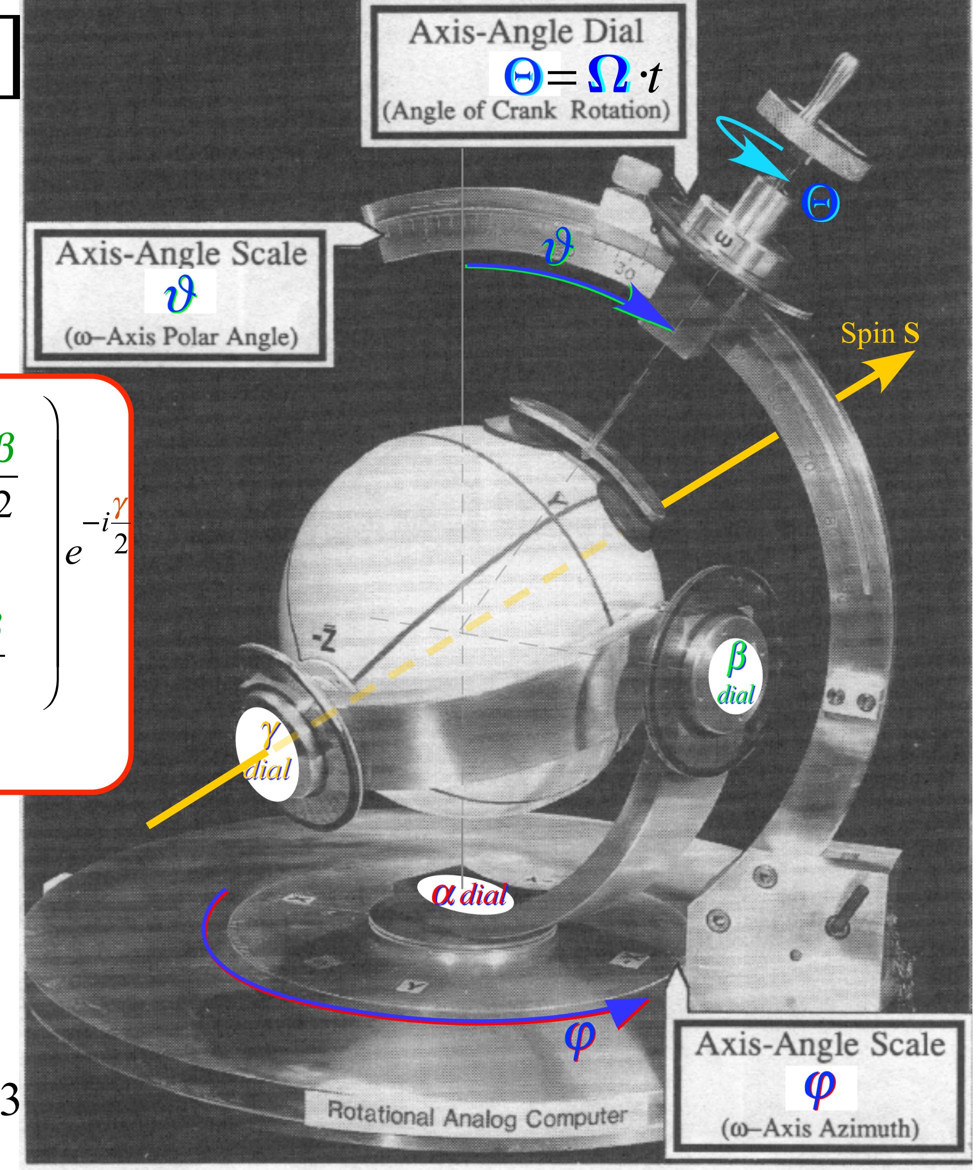

Lecture 22 page 92 to 93

$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0\\ 0 & e^{i\frac{\alpha}{2}}\end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0\\ 0 & e^{i\frac{\gamma}{2}}\end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}\left[\vec{\Theta}\right]=\begin{pmatrix}\cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X-i\hat{\Theta}_Y\right)\\ -i\sin\frac{\Theta}{2}\left(\hat{\Theta}_X+i\hat{\Theta}_Y\right) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2}\end{pmatrix}=\mathbf{R}\left[\varphi\vartheta\Theta\right]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}-i\begin{pmatrix}0&1\\1&0\end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix}\cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta\right)\\ \sin\frac{\Theta}{2}\left(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta\right) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2}\end{pmatrix}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ is simpler to form than Θ-axis *Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$.

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *($\alpha\beta\gamma$ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2=\cos\Theta/2$

$-p_2=\sin[(\gamma-\alpha)/2]\ \sin\beta/2=\hat{\Theta}_X\sin\Theta/2=\cos\varphi\ \sin\vartheta\ \ \sin\Theta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2=\hat{\Theta}_Y\sin\Theta/2=\sin\varphi\ \sin\vartheta\ \ \sin\Theta/2$

$-p_1=\sin[(\gamma+\alpha)/2]\cos\beta/2=\hat{\Theta}_Z\sin\Theta/2=\cos\vartheta\ \sin\Theta/2$

$U(2)$ *operator vs* (x,p)

$$\begin{pmatrix}x_1+ip_1 & -x_2+ip_2\\ x_2+ip_2 & x_1-ip_1\end{pmatrix}$$

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*

Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$\begin{aligned} x_1 &= \cos[(\gamma+\alpha)/2]\cos\beta/2 = && \cos\Theta/2 \\ -p_2 &= \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X \sin\Theta/2 = \cos\varphi\sin\vartheta && \sin\Theta/2 \\ x_2 &= \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y \sin\Theta/2 = \sin\varphi\sin\vartheta && \sin\Theta/2 \\ -p_1 &= \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \cos\vartheta && \sin\Theta/2 \end{aligned}$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\;\sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\;\sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

Euler **R**(αβγ) *related to Darboux* **R**[φϑΘ] *(So:* **R**(αβγ)=**R**[φϑΘ]*)*

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ αβγ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2$$

$$\tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2]$$

$$(\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

This gives *Euler angles* (αβγ) in terms of *Darboux angles* [φϑΘ]

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

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$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

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$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

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$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

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This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

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$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z \sin\Theta/2 = \qquad \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\ \sin(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ (So: $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$)

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2\end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\sin\vartheta\ \sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\sin\vartheta\ \sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\frac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\frac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\ \sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\ \sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2 \qquad \cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ *related to Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *(So:* $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$*)*

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$$

$$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\,\sin\Theta/2 = \cos\varphi\sin\vartheta\,\sin\Theta/2$$

$$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\,\sin\Theta/2 = \sin\varphi\sin\vartheta\,\sin\Theta/2$$

$$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\,\sin\Theta/2 = \qquad \cos\vartheta\,\sin\Theta/2$$

$$\tan[(\gamma+\alpha)/2] = \cos\vartheta\tan\Theta/2 \qquad \tan[(\gamma-\alpha)/2] = \cot\varphi = \tan[\tfrac{\pi}{2}-\varphi]$$

$$(\gamma+\alpha)/2 = \tan^{-1}[\cos\vartheta\tan\Theta/2] \qquad (\gamma-\alpha)/2 = \frac{\pi}{2}-\varphi$$

$$\sin[(\gamma-\alpha)/2] = \sin[\tfrac{\pi}{2}-\varphi] = \cos\varphi$$

$$\sin\beta/2 = \sin\vartheta\,\sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

$$\beta = 2\sin^{-1}(\sin\Theta/2\,\sin\vartheta)$$

$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2 \qquad \cos[(\gamma-\alpha)/2] = \cos[\tfrac{\pi}{2}-\varphi] = \sin\varphi$$

$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\,\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

$$\Theta = 2\cos^{-1}[\cos\beta/2\,\cos(\alpha+\gamma)/2]$$

$$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \qquad \cos\Theta/2$$

Example: *Euler angles* $(\alpha=50^\circ\ \beta=60^\circ\ \gamma=70^\circ)$

$$\varphi = (50^\circ - 70^\circ + 180^\circ)/2 = 80^\circ$$

$$\vartheta = \tan^{-1}[\tan 60^\circ/2/\sin(50^\circ+\gamma)/2] = 33.7^\circ$$

$$\Theta = 2\cos^{-1}[\cos 60^\circ/2\,\cos(50^\circ+\gamma)/2] = 128.7^\circ$$

Euler $\mathbf{R}(\alpha\beta\gamma)$ related to Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ (So: $\mathbf{R}(\alpha\beta\gamma)=\mathbf{R}[\varphi\vartheta\Theta]$)

Euler *state definition* lets us relate $\mathbf{R}(\alpha\beta\gamma)$ to $\mathbf{R}[\varphi\vartheta\Theta]$ ...

$|\alpha\beta\gamma\rangle = \mathbf{R}(\alpha\beta\gamma)|000\rangle$ $\alpha\beta\gamma$ *make better coordinates but:* $\mathbf{R}(\alpha\beta\gamma)|000\rangle = \mathbf{R}(\alpha\beta\gamma)\,|\mathbf{1}\rangle = \mathbf{R}[\varphi\vartheta\Theta]|\mathbf{1}\rangle$

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}$$

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$$\sin\beta/2 = \sin\vartheta\;\sin\Theta/2$$

This gives *Euler angles* $(\alpha\beta\gamma)$ in terms of *Darboux angles* $[\varphi\vartheta\Theta]$

$$\alpha = \varphi - \pi/2 + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$
$$\beta = 2\sin^{-1}(\sin\Theta/2\;\sin\vartheta)$$
$$\gamma = \pi/2 - \varphi + \tan^{-1}(\cos\vartheta\tan\Theta/2)$$

Inverse relations have *Darboux axis angles* $[\varphi\vartheta\Theta]$ in terms of *Euler angles* $(\alpha\beta\gamma)$

$$\varphi = (\alpha-\gamma+\pi)/2$$
$$\vartheta = \tan^{-1}[\tan\beta/2/\sin(\alpha+\gamma)/2]$$
$$\Theta = 2\cos^{-1}[\cos\beta/2\cos(\alpha+\gamma)/2]$$

$$\cos[(\gamma-\alpha)/2] = \cos[\frac{\pi}{2}-\varphi] = \sin\varphi$$

$$\frac{\cos[(\gamma-\alpha)/2]\sin\beta/2}{\sin[(\gamma+\alpha)/2]\cos\beta/2} = \sin\varphi\tan\vartheta \Rightarrow \frac{\tan\beta/2}{\sin[(\gamma+\alpha)/2]} = \tan\vartheta$$

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$$\varphi = (50° - 70° + 180°)/2 = 80°$$
$$\vartheta = \tan^{-1}[\tan 60°/2/\sin(50°+\gamma)/2] = 33.7°$$
$$\Theta = 2\cos^{-1}[\cos 60°/2\cos(50°+\gamma)/2] = 128.7°$$

Reverse check: $(\alpha\beta\gamma)$ in terms of $[\varphi\vartheta\Theta]$

$$\alpha = 80° - 90° + \tan^{-1}(\tan(128.7°/2)\cos 33.7°) = 50.007°$$
$$\beta = 2\sin^{-1}(\sin 128.7°/2\;\sin 33.7°) = 60.022°$$
$$\gamma = \pi/2 - 128.7° + \tan^{-1}(\tan(128.7°/2)) = 70.007°$$

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

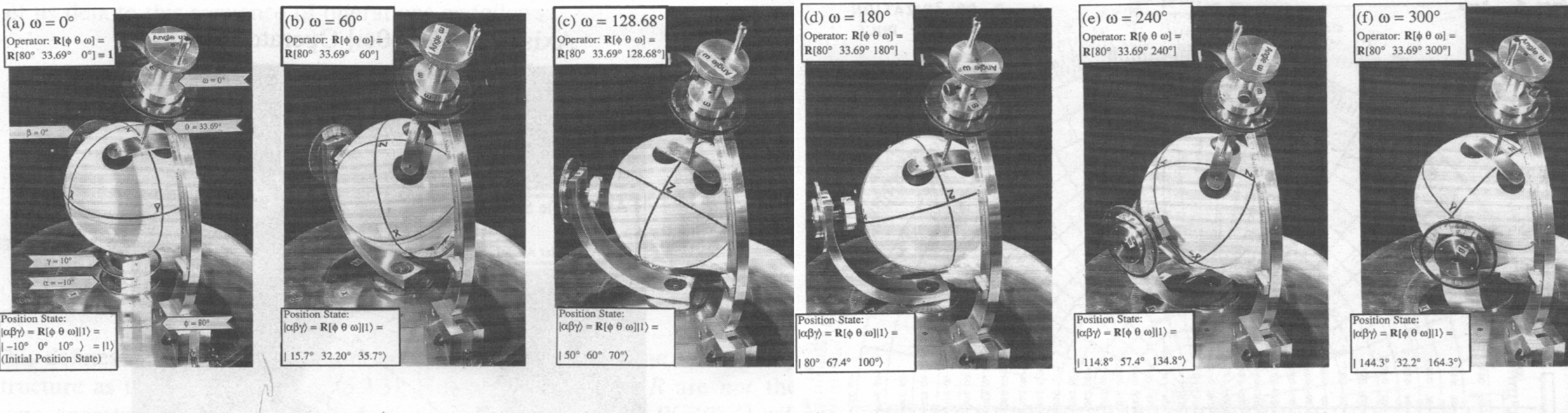
- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Euler **R**$(\alpha\beta\gamma)$ *rotation* Θ=0−4π-*sequence* [φϑ] *fixed*

Θ=0°　Θ=60°　Θ=128.7°　Θ=180°　Θ=240°　Θ=300°



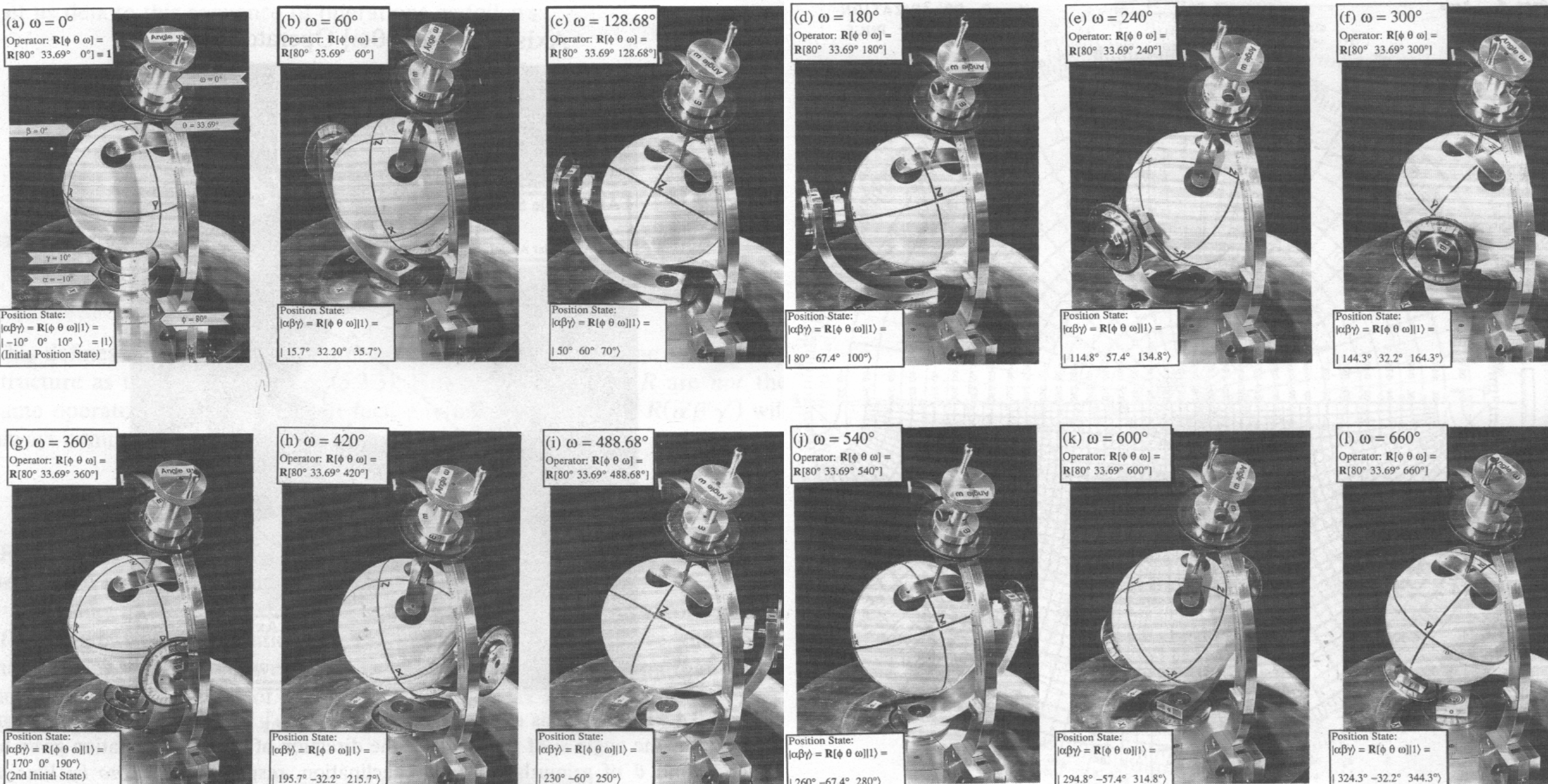
Principles of Symmetry, Dynamics, and Spectroscopy - W. G. Harter - Wiley (1994)

Development has begun on a web based version of this tool,
but much of the App is at present (10/7/2018), in an '*indeterminate state*'.
The App's 3D will in future be handled by *Babylon.JS, to act as a* shim
to buttress the WebGL (web graphics layer) that is already in place.

Web based U(2) Calculator - Euler State

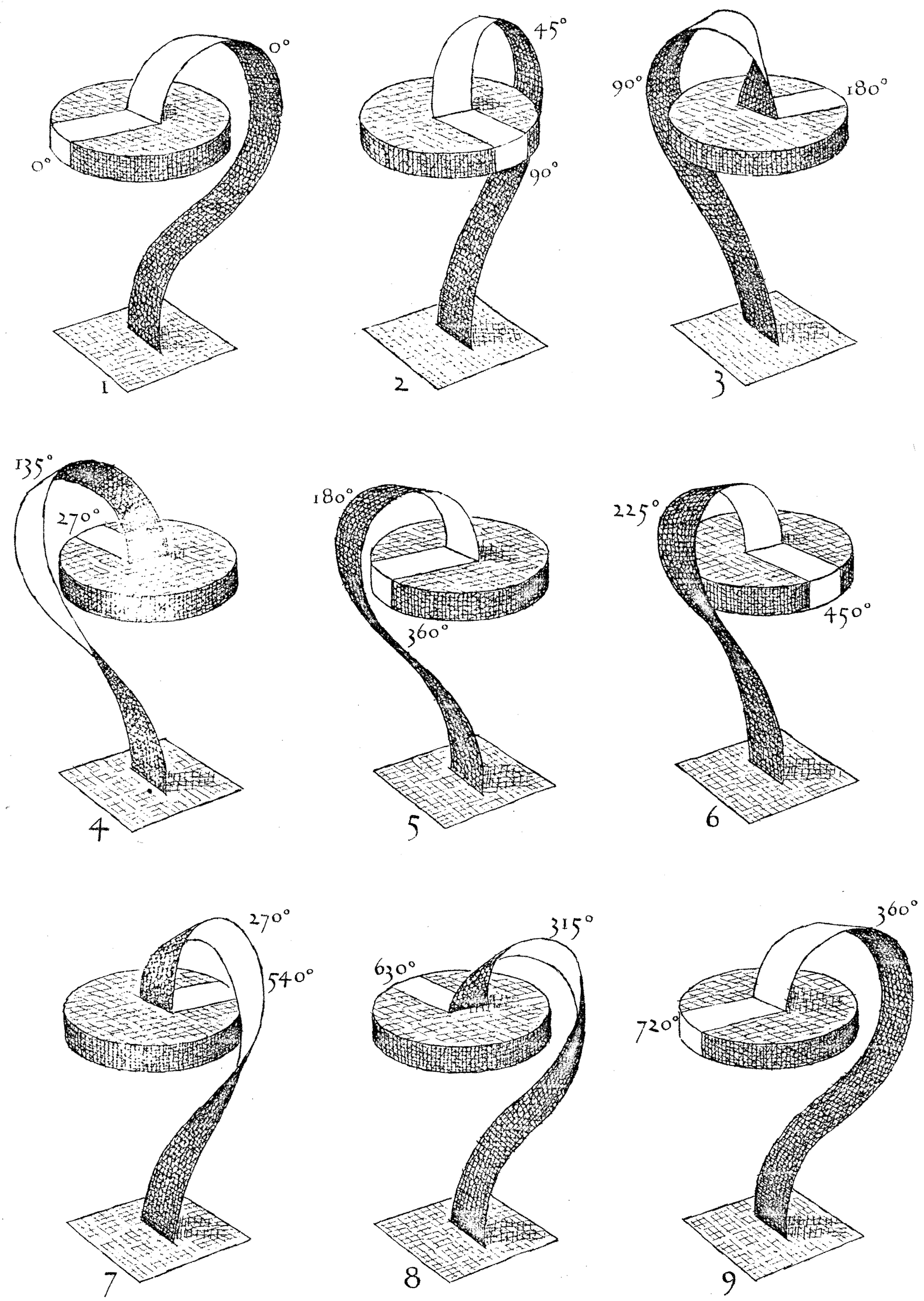
Euler **R**$(\alpha\beta\gamma)$ *rotation* Θ=0−4π-*sequence* [φϑ] *fixed*

Θ=0° Θ=60° Θ=128.7° Θ=180° Θ=240° Θ=300°



Θ=360° Θ=420° Θ=488.7° Θ=540° Θ=600° Θ=660°

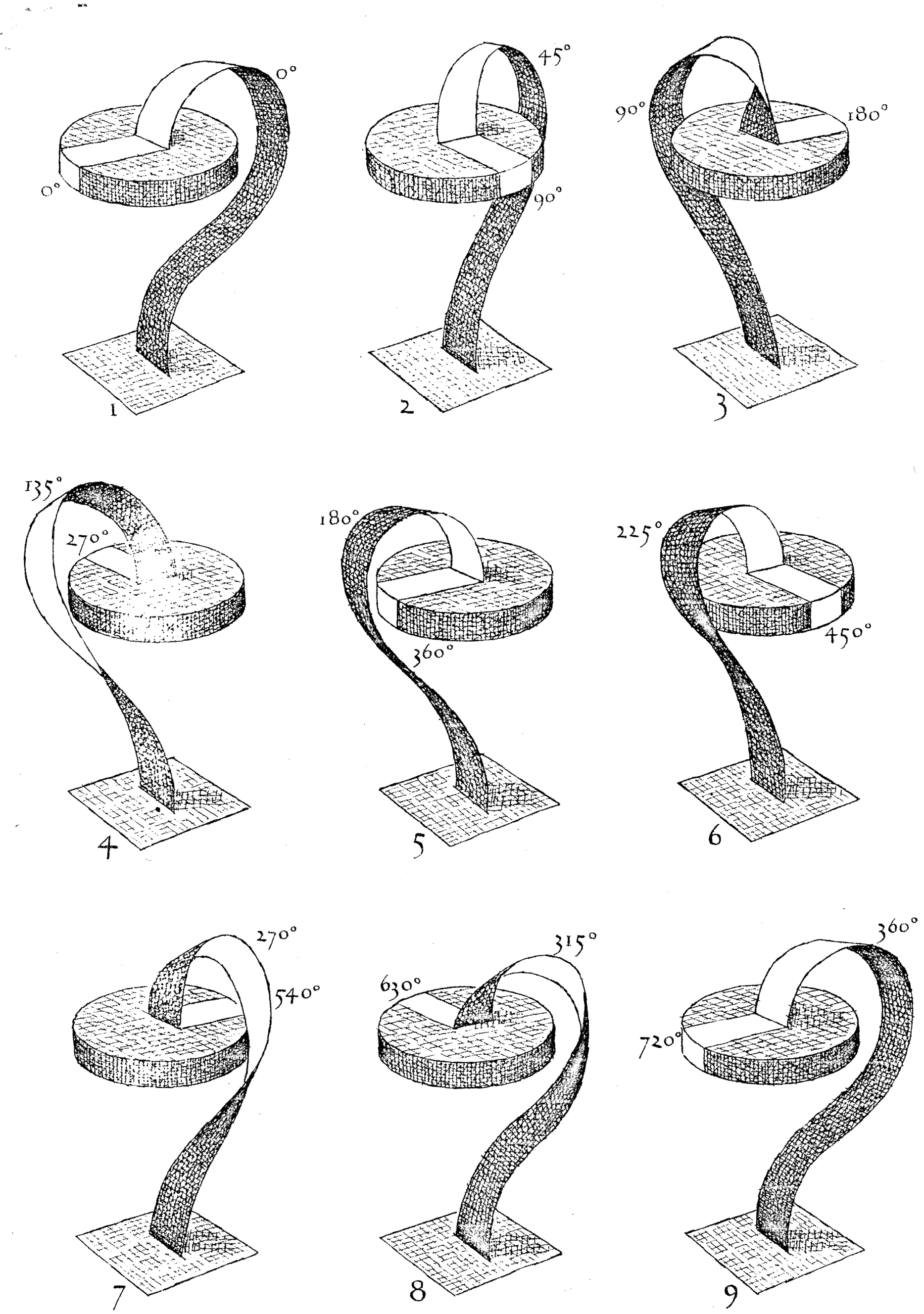
Some "real-world" applications of the U(2)-R(3) spinor-vector topology



Sequential models of D. A. Adams' antitwister mechanism

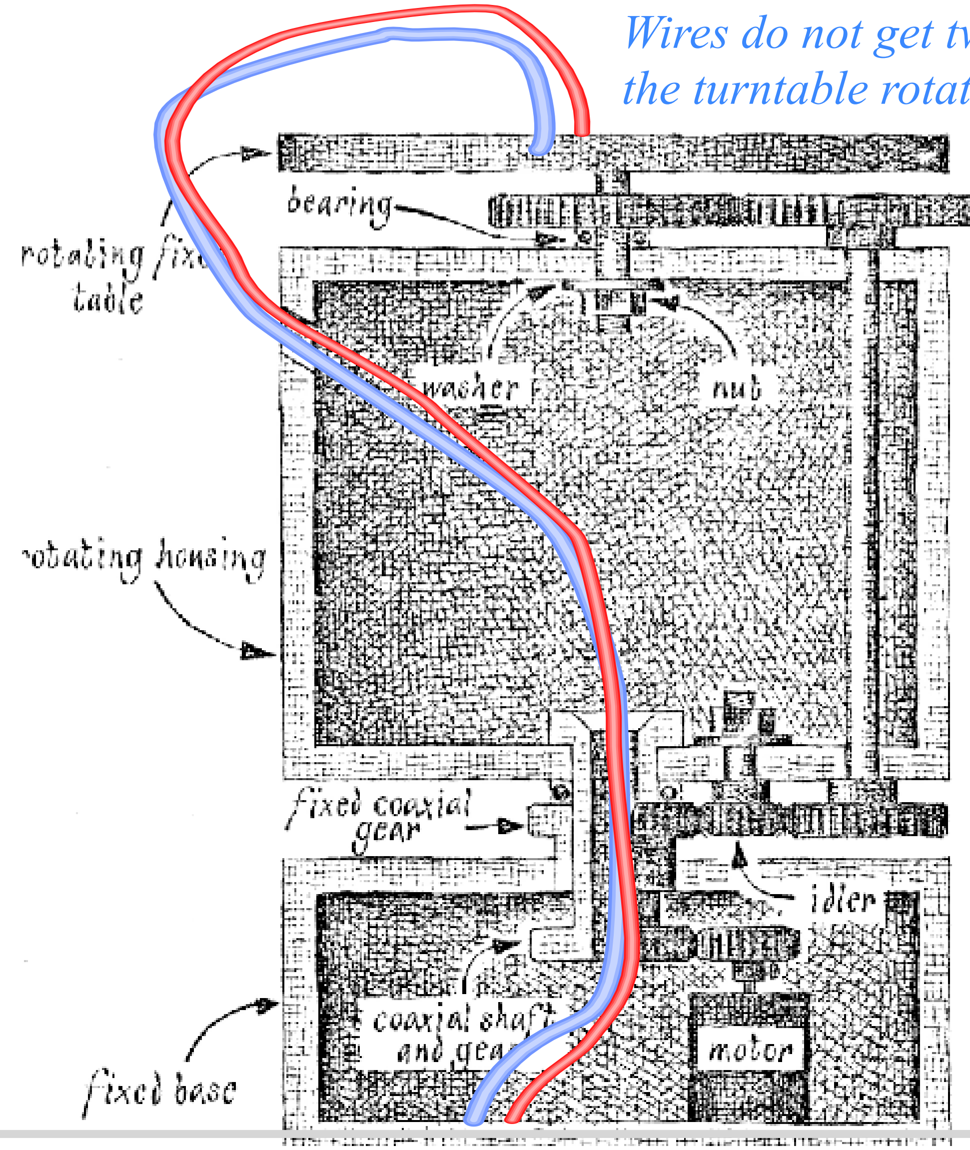
From Scientific American December 1975-p.120-125

Some "real-world" applications of the U(2)-R(3) spinor-vector topology



Sequential models of D. A. Adams' antitwister mechanism

Wires do not get twisted up as the turntable rotates



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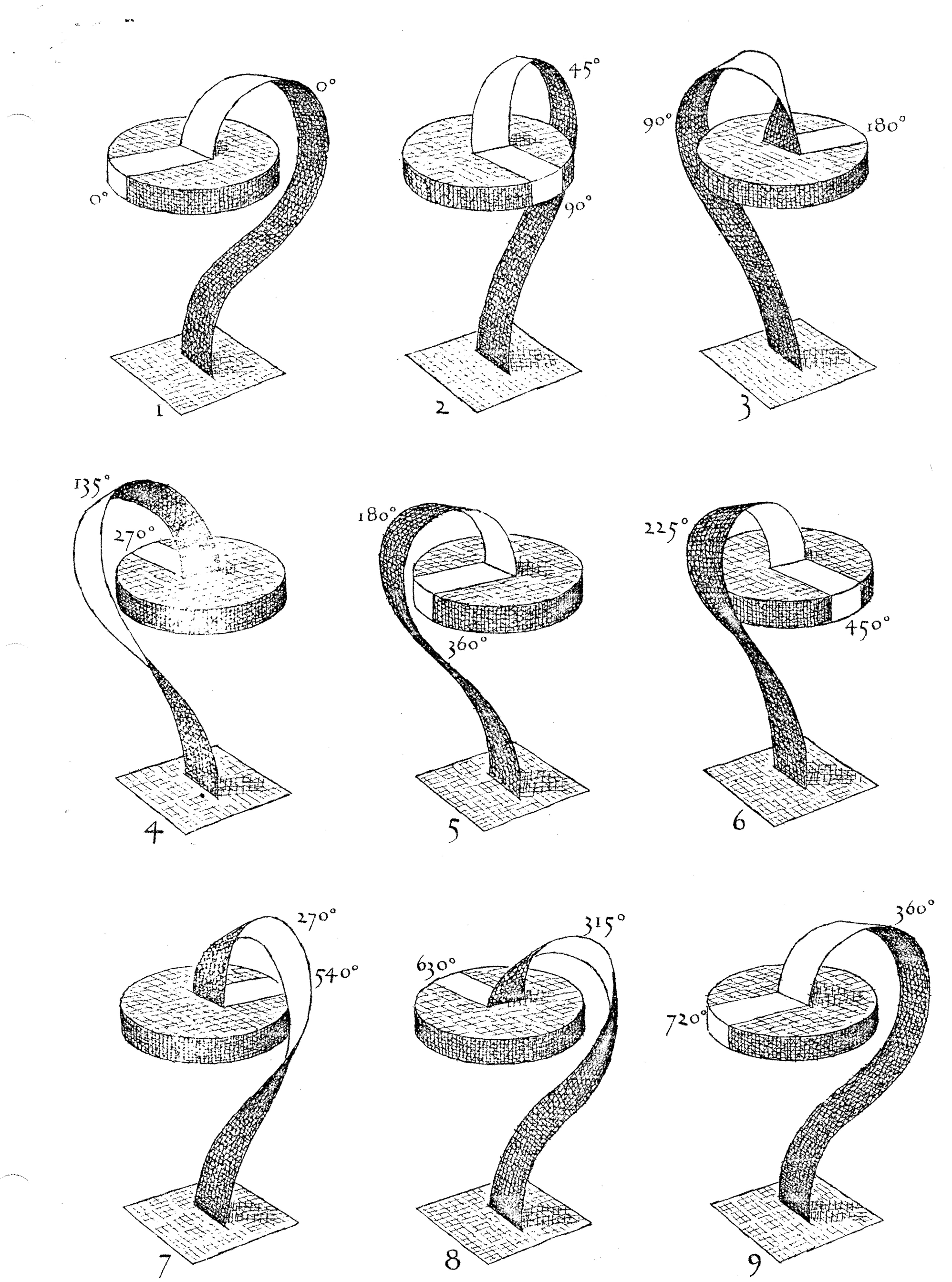
Some "real-world" applications of the U(2)-R(3) spinor-vector topology



Sequential models of D. A. Adams' antitwister mechanism

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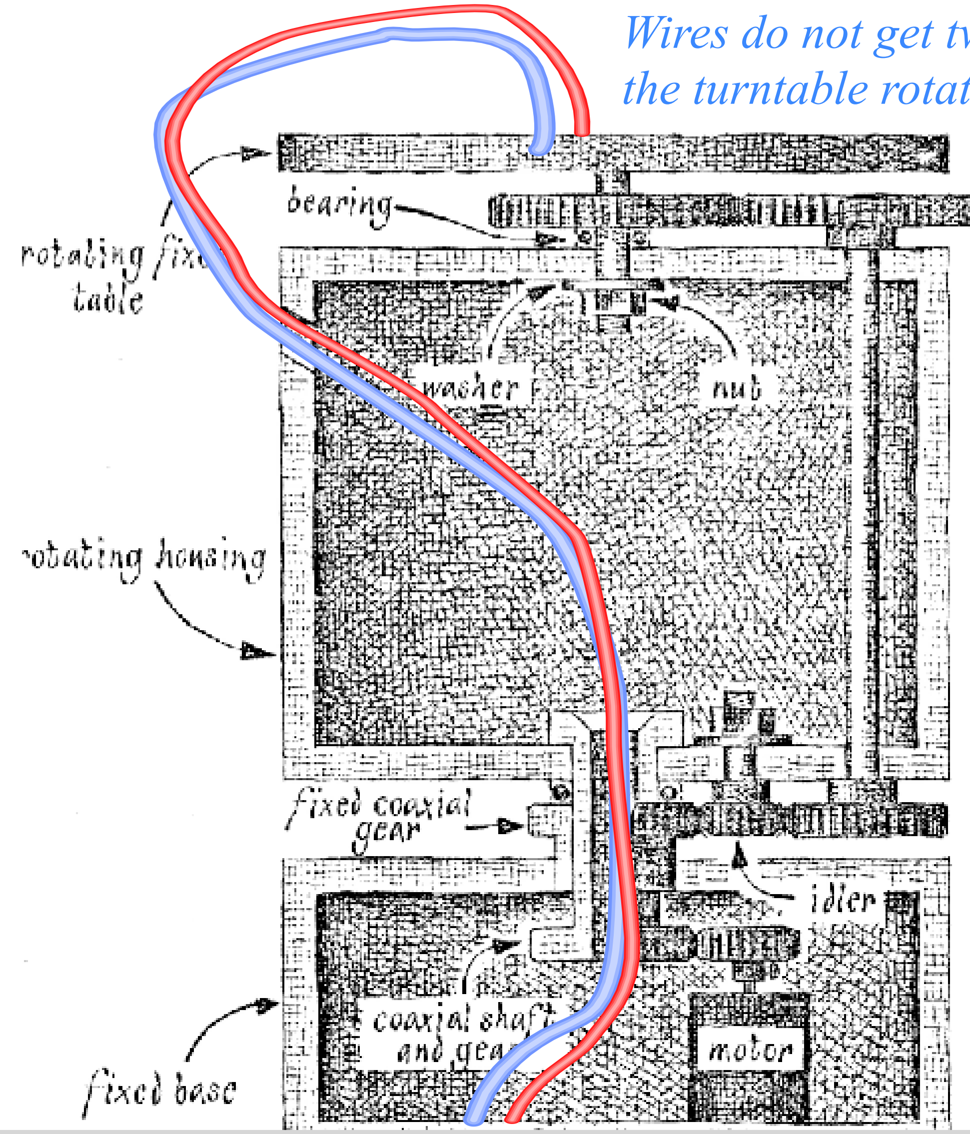

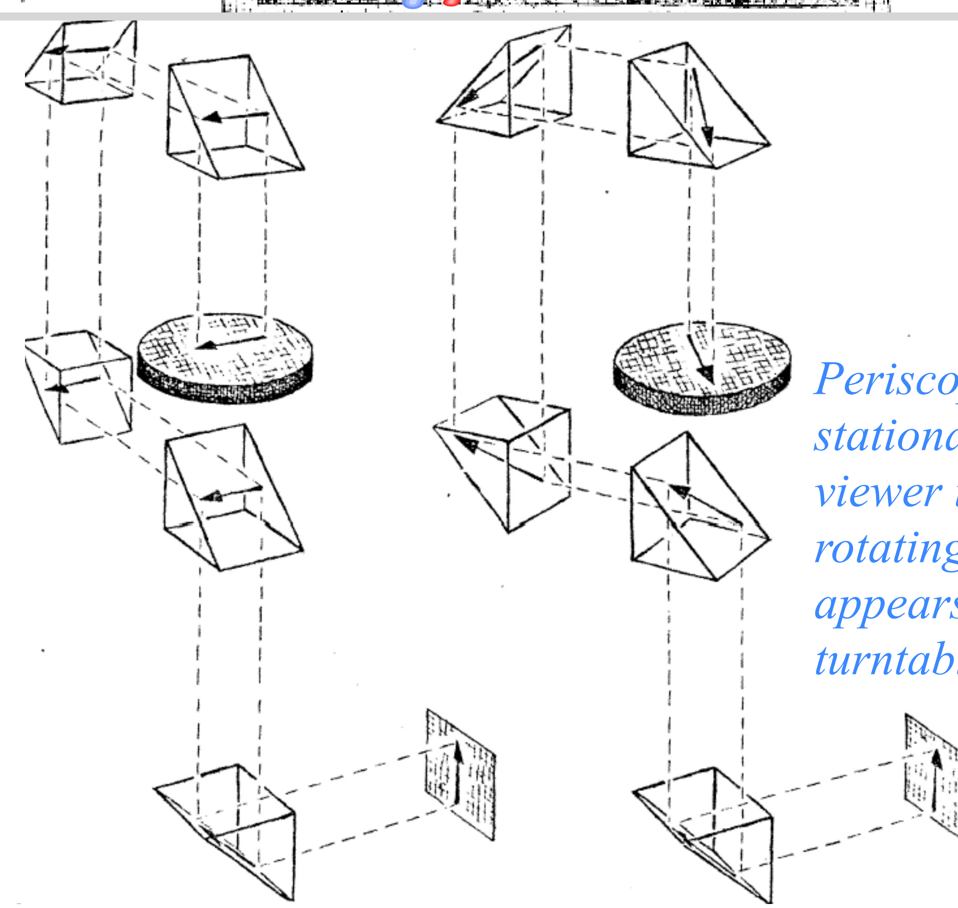

Periscope allows stationary outside viewer to see into a rotating frame that appears fixed as the turntable rotates

Reviewing fundamental Euler **R***(αβγ) and Darboux* **R**[φϑΘ] *representations of U(2) and R(3)*

Euler-defined state |αβγ⟩ *described by Stoke's* **S***-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian **H**[φϑΘ]=*exp(-i***Ω•S***)·t and angular velocity* **Ω**(φϑ)·*t*=**Θ**-*vector*

Euler-defined operator **R***(αβγ) derived from Darboux-defined* **R**[φϑΘ] *and vice versa*

Euler **R***(αβγ) rotation* **Θ**=0−4π*-sequence* [φϑ] *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle (αβγ) ellipse coordinates

Quick U(2) way to find eigen-solutions for 2-by-2 $\mathbf{H}$

Steps to find eigen-solutions for 2-by-2 $\mathbf{H}$ *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian $\mathbf{H}$

$$\begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix} = \mathbf{H}$$

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\boldsymbol{\Omega} = \boldsymbol{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

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$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

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Quick U(2) way to find eigen-solutions for 2-by-2 **H**

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Eigenvalues: $\Omega_{\pm} = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

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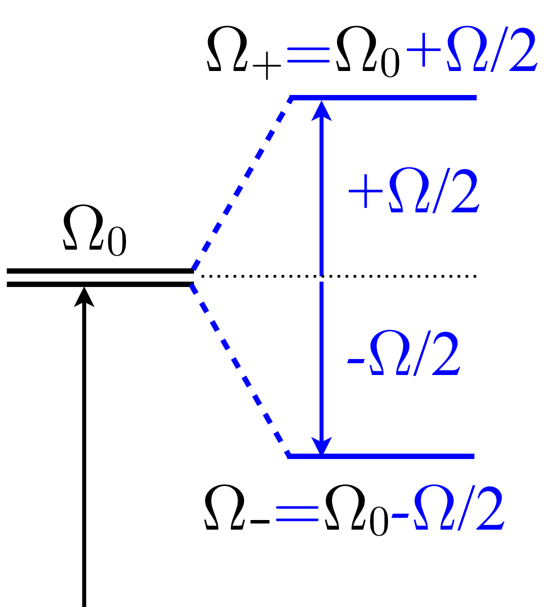
Frequency level diagram



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

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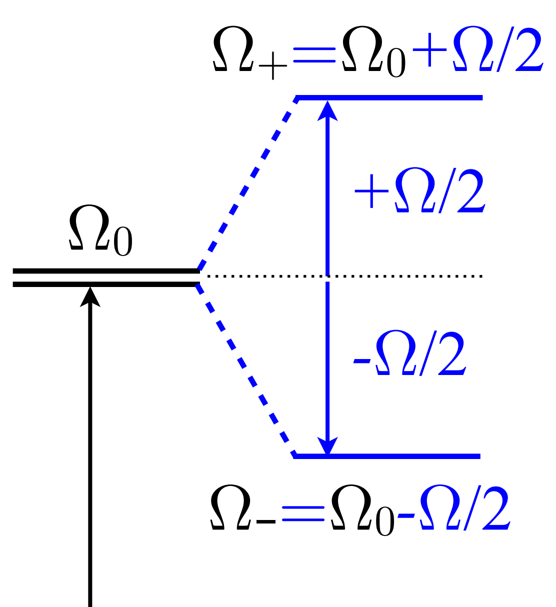

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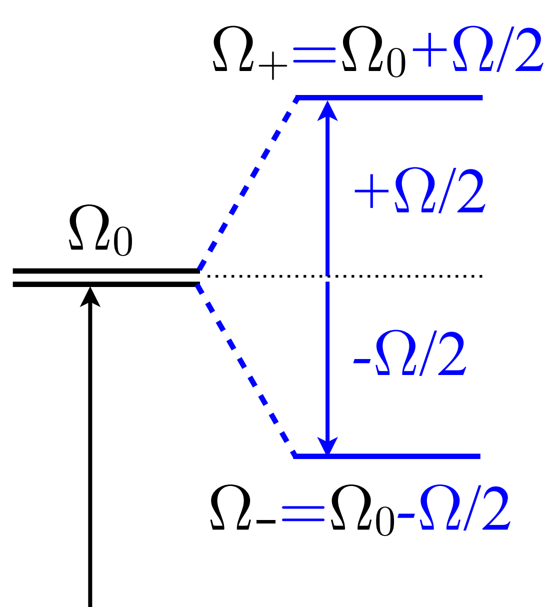

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Step 3.To find eigenvectors replace Euler angles (azimuth α, *polar* β *) of Euler-state*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$

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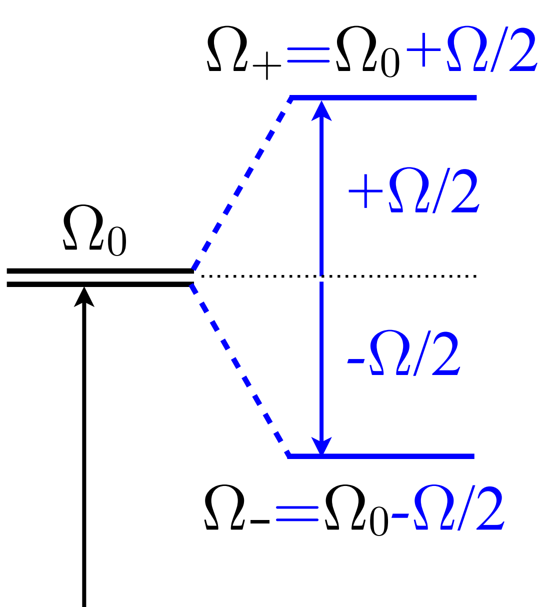

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

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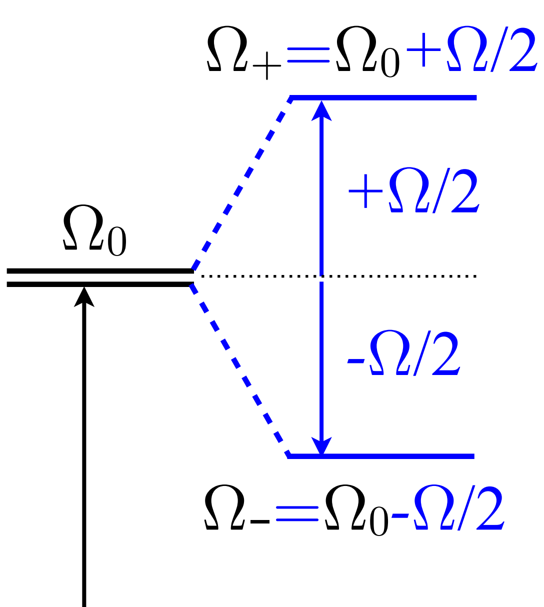

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

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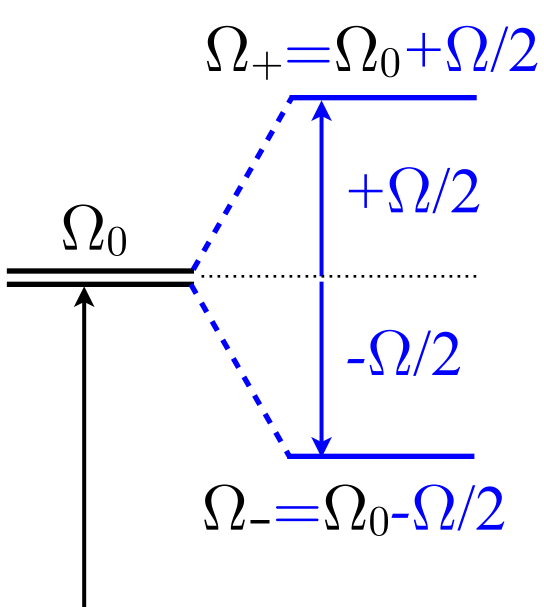
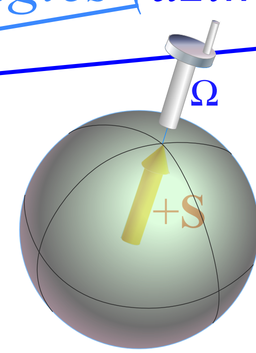


$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$



Spin +S
Up-Crank

Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

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where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A^{\,2} + \Omega_B^{\,2} + \Omega_C^{\,2}} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B^{\,2}+\Omega_C^{\,2}}\,]$

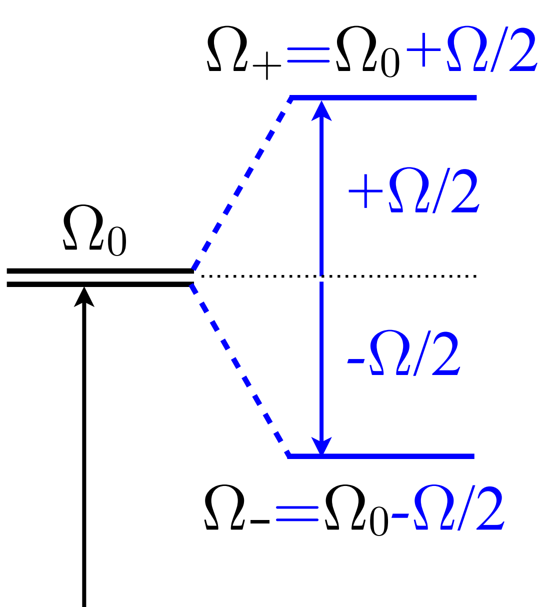
or: $\vartheta = \cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2+4B^2+4C^2}\,]$, $\quad\varphi = \cos^{-1}[B/\sqrt{B^2+C^2}\,]$

$$\boxed{\textit{Eigenvalues: } \Omega_\pm = \Omega_0 \pm \Omega/2 = \frac{A+D \pm\sqrt{(A-D)^2+4B^2+4C^2}}{2}}$$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

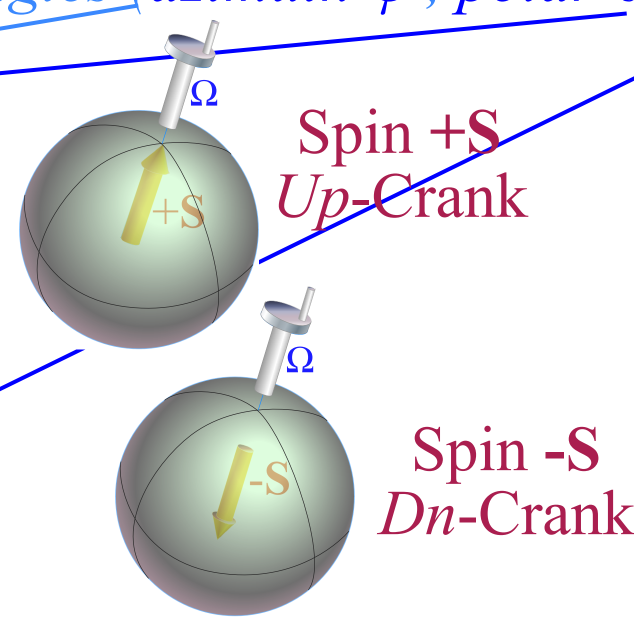
with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta\pm\pi$*) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$$|\Omega_+\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$

$$|\Omega_-\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta\pm\pi}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta\pm\pi}{2} \end{pmatrix}$$



Quick U(2) way to find eigen-solutions for 2-by-2 **H**

Steps to find eigen-solutions for 2-by-2 **H** *matrix:*

Step 1 Find components $(\Omega_A, \Omega_B, \Omega_C)$ *of crank vector* $\mathbf{\Omega} = \mathbf{\Theta}/t$

Hamiltonian **H**

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \mathbf{H} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (A-D)\tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2B\tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2C\tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\mathbf{H} = \Omega_0 \quad \mathbf{1} \quad + \quad \Omega_A \quad \mathbf{S}_A \quad + \quad \Omega_B \quad \mathbf{S}_B \quad + \quad \Omega_C \quad \mathbf{S}_C \quad = \Omega_0 \mathbf{1} + \vec{\Omega} \bullet \mathbf{S}$$

Step 2.Convert Cartesian to polar form: ($\Omega_A = \Omega\cos\vartheta$, $\Omega_B = \Omega\cos\varphi\,\sin\vartheta$, $\Omega_C = \Omega\sin\varphi\,\sin\vartheta$)

where: $\Omega_0 = \dfrac{A+D}{2}$ *and:* $\Omega = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2}$

and: $\vartheta = \cos^{-1}(\Omega_A/\Omega)$, *and:* $\varphi = \cos^{-1}(\Omega_B/\Omega\,\sin\vartheta) = \cos^{-1}[\Omega_B/\sqrt{\Omega_B{}^2 + \Omega_C{}^2}\,]$

or: $\vartheta = \cos^{-1}[(A\text{-}D)\,/\,\sqrt{(A-D)^2 + 4B^2 + 4C^2}\,]$, $\quad \varphi = \cos^{-1}[B/\sqrt{B^2 + C^2}\,]$

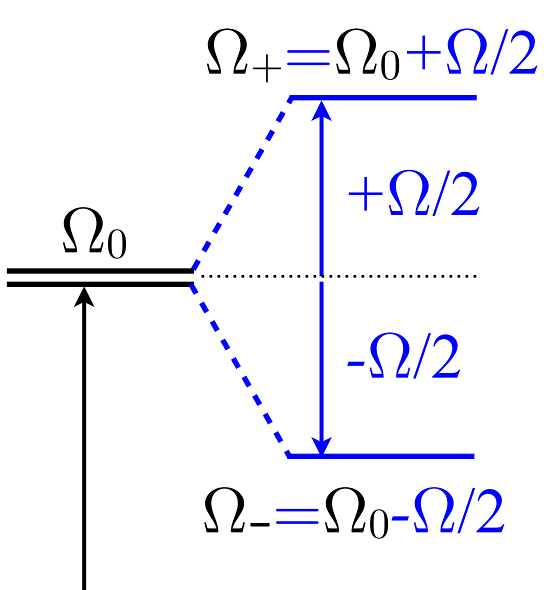
Eigenvalues: $\Omega_\pm = \Omega_0 \pm \Omega/2$

$$= \frac{A+D \pm \sqrt{(A-D)^2 + 4B^2 + 4C^2}}{2}$$

Step 3.To find eigenvectors replace Euler angles (azimuth α*, polar* β *) of Euler-state*

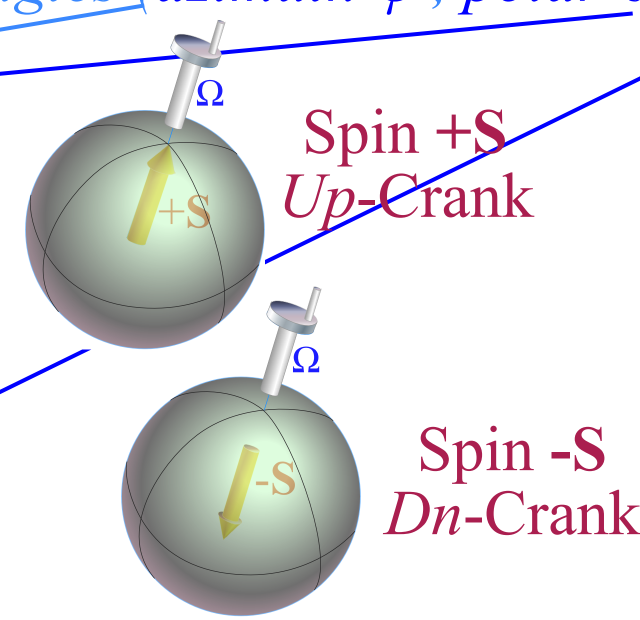
with the Darboux axis polar angles (azimuth φ *, polar* ϑ *or* $\vartheta \pm \pi$*) of* **H***-matrix*

$$\left|\uparrow_{\alpha\beta\gamma}\right\rangle = \begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \mathbf{R}(\alpha\beta\gamma)\left|\uparrow_{000}\right\rangle$$



$$\left|\Omega_+\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix}$$

$$\left|\Omega_-\right\rangle = \begin{pmatrix} e^{-i\frac{\varphi}{2}}\cos\frac{\vartheta \pm \pi}{2} \\ e^{i\frac{\varphi}{2}}\sin\frac{\vartheta \pm \pi}{2} \end{pmatrix}$$



More reliable computation:

$\varphi = atan2(C, B)$

[$\tan^{-1}(C\,/\,B)$ *is unreliable*]

$\vartheta = atan2(2\sqrt{B^2 + C^2}, A-D)$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H} = \begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

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$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$

Quick U(2) way example for 2-by-2 **H**

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$A = 12, \quad B = \sqrt{6}, \quad C = \sqrt{6}, \quad D = 8,$

Step 2.Convert Cartesian to polar form: ($\Omega_A = \Omega\cos\vartheta, \quad \Omega_B = \Omega\cos\varphi\,\sin\vartheta, \quad \Omega_C = \Omega\sin\varphi\,\sin\vartheta$)

$$\Omega_0 = \frac{A+D}{2} = 10$$

and: $\Omega \quad = \sqrt{\Omega_A{}^2 + \Omega_B{}^2 + \Omega_C{}^2} = \sqrt{(A-D)^2 + 4B^2 + 4C^2} = \sqrt{(4)^2 + 4\sqrt{6}^2 + 4\sqrt{6}^2} = \sqrt{16 + 24 + 24} = \sqrt{64} = 8$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

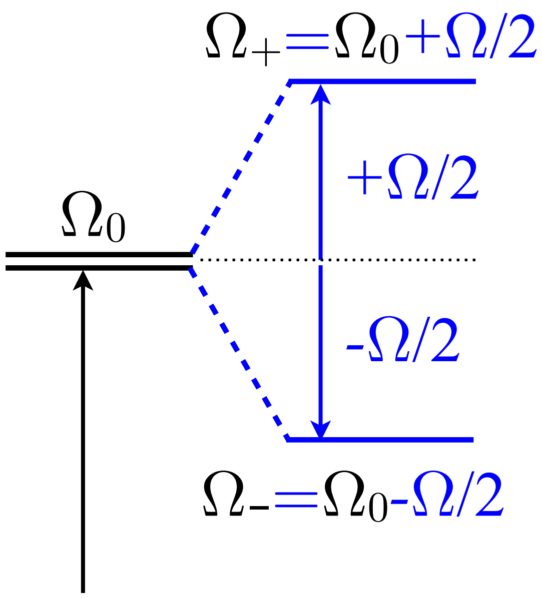
$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$$

$$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$$

Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta$, $\Omega_B=\Omega\cos\varphi\sin\vartheta$, $\Omega_C=\Omega\sin\varphi\sin\vartheta$)

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega\ =\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$



$\boxed{eigenvalue-1}$

$$\omega_{\uparrow}=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10+4=14$$

$\boxed{eigenvalue-2}$

$$\omega_{\downarrow}=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10-4=6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$

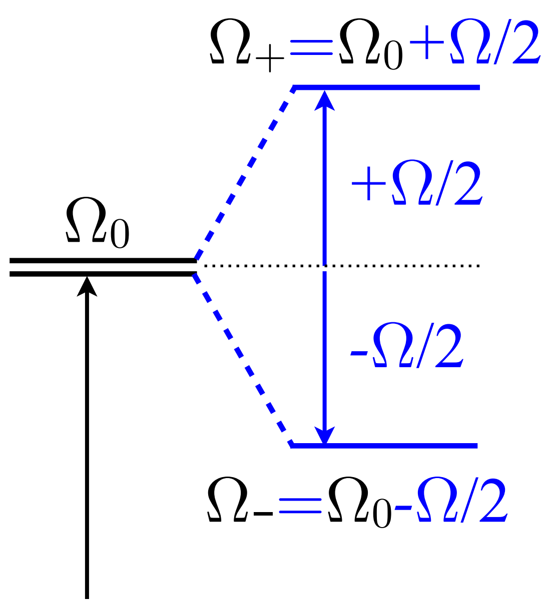
Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta$, $\Omega_B=\Omega\cos\varphi\sin\vartheta$, $\Omega_C=\Omega\sin\varphi\sin\vartheta$)

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega=\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$

or: $\vartheta=\cos^{-1}[(A\text{-}D)/\sqrt{(A-D)^2+4B^2+4C^2}\,]=\cos^{-1}[(4)/8]=\pi/3,$

$\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}\,]=\cos^{-1}[\sqrt{6}/\sqrt{12}]=\pi/4$



$\boxed{eigenvalue-1}$

$$\omega_\uparrow=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10+4=14$$

$\boxed{eigenvalue-2}$

$$\omega_\downarrow=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}$$
$$=10-4=6$$

Quick U(2) way example for 2-by-2 **H**

Can you write down all eigensolutions to the following **H** -matrix in 60 seconds?

$$\mathbf{H}=\begin{pmatrix} 12 & \sqrt{6}(1-i) \\ \sqrt{6}(1+i) & 8 \end{pmatrix}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\begin{pmatrix} 10+4\cos\frac{\pi}{3} & 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}-i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} \\ 4\cos\frac{\pi}{4}\sin\frac{\pi}{3}+i4\sin\frac{\pi}{4}\sin\frac{\pi}{3} & 10-4\cos\frac{\pi}{3} \end{pmatrix}$$

$A=12,\quad B=\sqrt{6},\quad C=\sqrt{6},\quad D=8,$

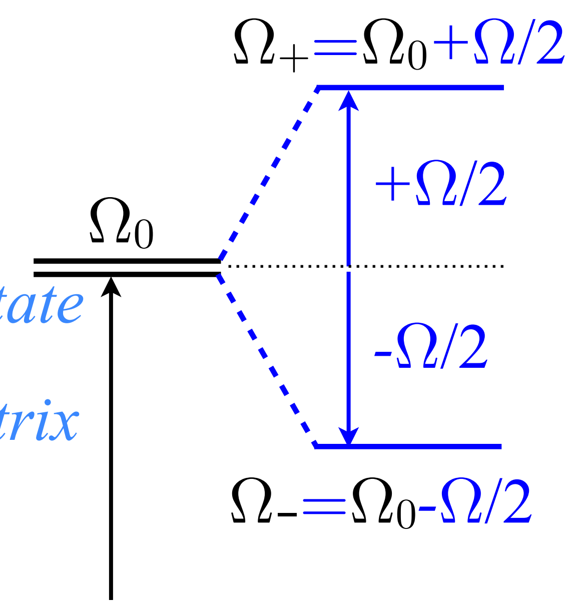
Step 2.Convert Cartesian to polar form: ($\Omega_A=\Omega\cos\vartheta,\quad \Omega_B=\Omega\cos\varphi\sin\vartheta,\quad \Omega_C=\Omega\sin\varphi\sin\vartheta$)

$$\Omega_0=\frac{A+D}{2}=10$$

and: $\Omega=\sqrt{\Omega_A{}^2+\Omega_B{}^2+\Omega_C{}^2}=\sqrt{(A-D)^2+4B^2+4C^2}=\sqrt{(4)^2+4\sqrt{6}^2+4\sqrt{6}^2}=\sqrt{16+24+24}=\sqrt{64}=8$

or: $\vartheta=\cos^{-1}[(A-D)/\sqrt{(A-D)^2+4B^2+4C^2}\,]=\cos^{-1}[(4)/8]=\pi/3,$

$\varphi=\cos^{-1}[B/\sqrt{B^2+C^2}\,]=\cos^{-1}[\sqrt{6}/\sqrt{12}]=\pi/4$



Step 3.To find eigenvectors replace Euler angles (azimuth α, polar β) of Euler-state

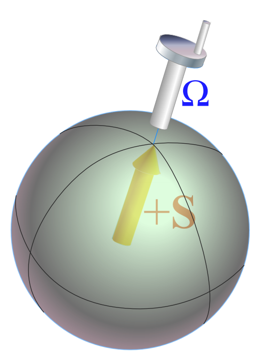
with the Darboux axis polar angles (azimuth φ , polar ϑ or $\vartheta\pm\pi$) of **H***-matrix*

eigenvalue − 1

$$\omega_\uparrow=10+\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}=10+4=14$$

eigenvector − 1

$$|\uparrow\rangle=\begin{pmatrix} e^{-i\frac{\pi}{8}}\cos\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\sin\frac{\pi}{6} \end{pmatrix}=\begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$

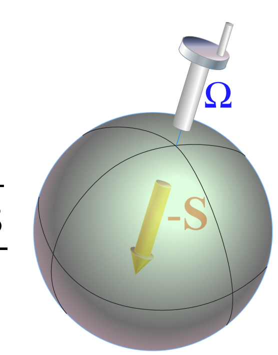


eigenvalue − 2

$$\omega_\downarrow=10-\sqrt{\left(\frac{12-8}{2}\right)^2+\left(\sqrt{6}\right)^2+\left(\sqrt{6}\right)^2}=10-4=6$$

eigenvector − 2

$$|\downarrow\rangle=\begin{pmatrix} -e^{-i\frac{\pi}{8}}\sin\frac{\pi}{6} \\ e^{+i\frac{\pi}{8}}\cos\frac{\pi}{6} \end{pmatrix}=\begin{pmatrix} -e^{-i\frac{\pi}{4}}\frac{\sqrt{3}}{3} \\ 1 \end{pmatrix}\frac{e^{-i\frac{\pi}{8}}\sqrt{3}}{2}$$



Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S**-*vector, phasors, or ellipsometry*
Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$-*vector*
Euler-defined operator **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*
Euler **R**$(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$-*sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes
Asymmetric-Diagonal A-Type motion
Bilateral-Balanced B-Type motion
Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes
AB-Type motion and Wigner's Avoided-Symmetry-Crossings
ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates
Conventional amp-phase ellipse coordinates
Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\rightarrow$ $\boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{array}}$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147) $\rightarrow$ $\boxed{\begin{gathered}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}\end{gathered}}$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\boldsymbol{\Omega}_B}{2}\boldsymbol{\sigma}_B + \frac{\boldsymbol{\Omega}_C}{2}\boldsymbol{\sigma}_C + \frac{\boldsymbol{\Omega}_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\boldsymbol{\Omega}} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\boldsymbol{\Omega}_A}{2}\boldsymbol{\sigma}_A$$

Crank : $\vec{\boldsymbol{\Omega}} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix}$ *Eigen−Spin* : $\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$

The *ABC's* of U(2) dynamics

Density operator $\boldsymbol{\rho}$ *(see p.128-147)*

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

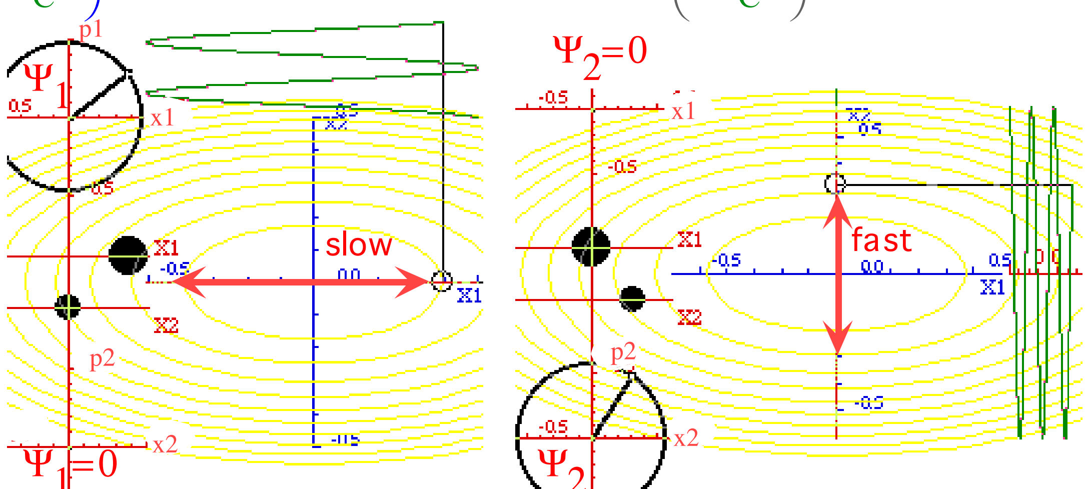
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Asymmetric Diagonal *A*-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin:\ \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$



The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H}=\frac{A+D}{2}\mathbf{1}+B\,\boldsymbol{\sigma}_B+C\,\boldsymbol{\sigma}_C+\frac{A-D}{2}\boldsymbol{\sigma}_A$$

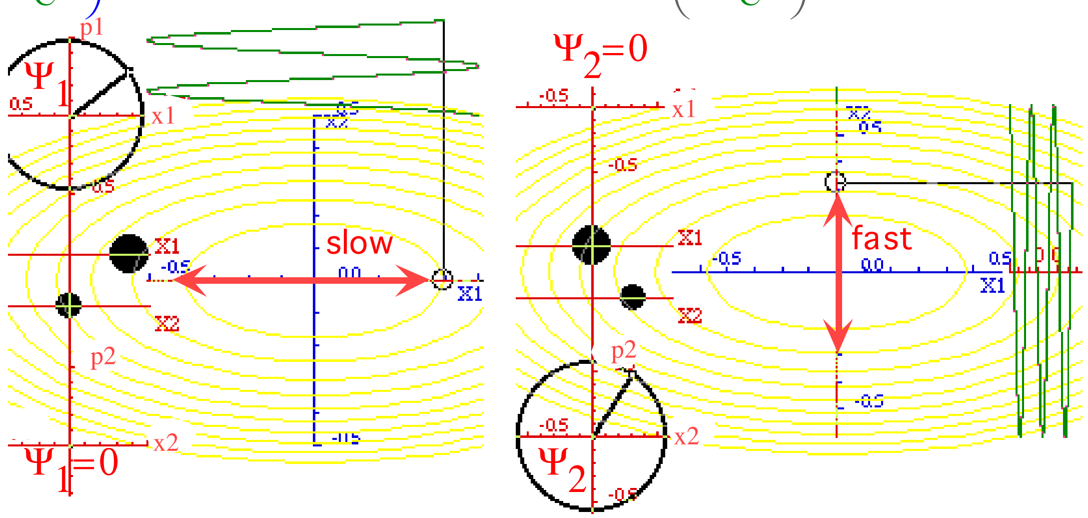
$$\mathbf{H}=\frac{A+D}{2}\boldsymbol{\sigma}_0+\frac{\Omega_B}{2}\boldsymbol{\sigma}_B+\frac{\Omega_C}{2}\boldsymbol{\sigma}_C+\frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

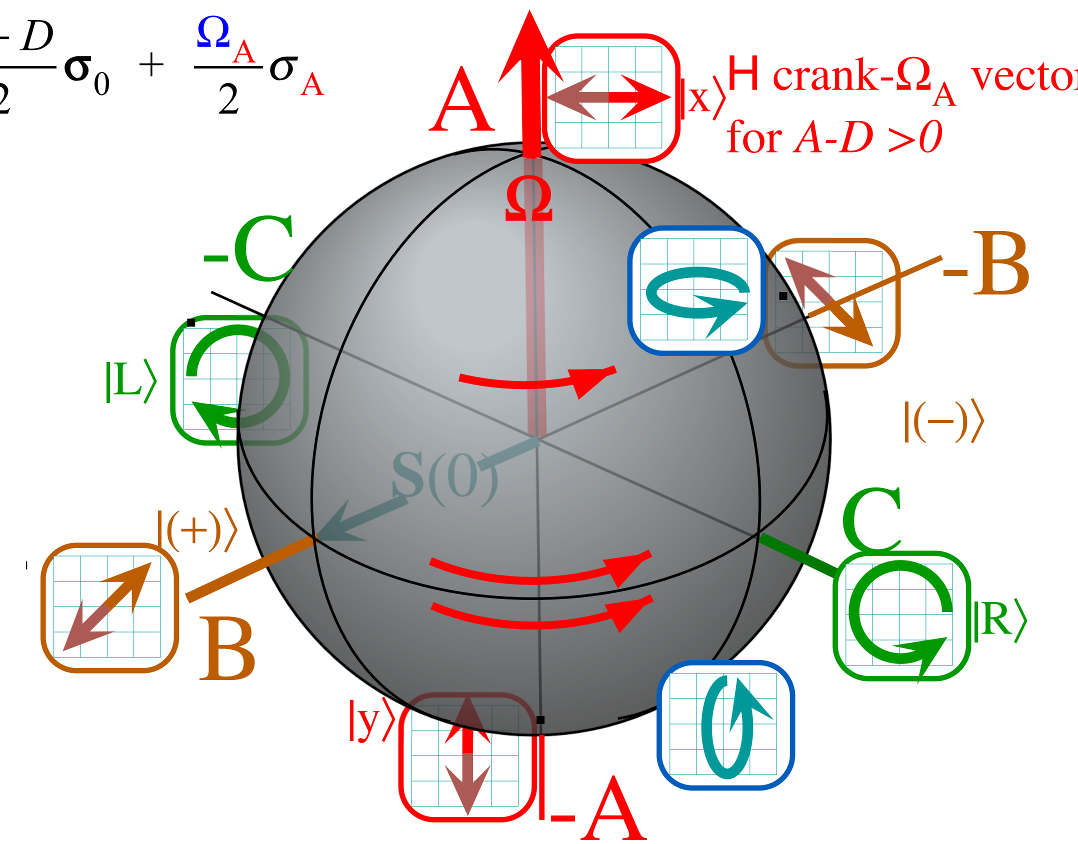
$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Asymmetric Diagonal A-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{A+D}{2}\boldsymbol{\sigma}_0+\frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\text{Crank}:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix}\qquad \text{Eigen-Spin}:\ \vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$





The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

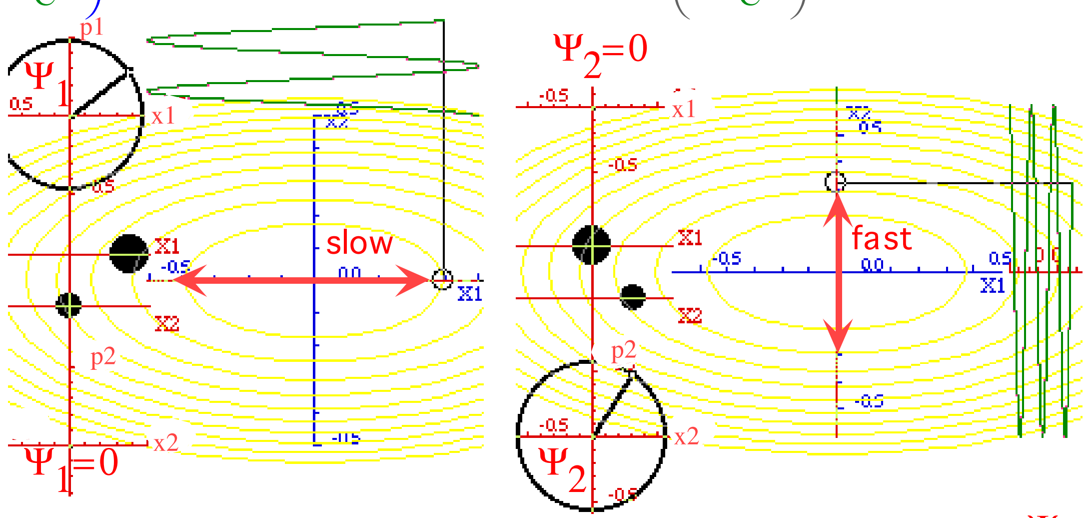
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

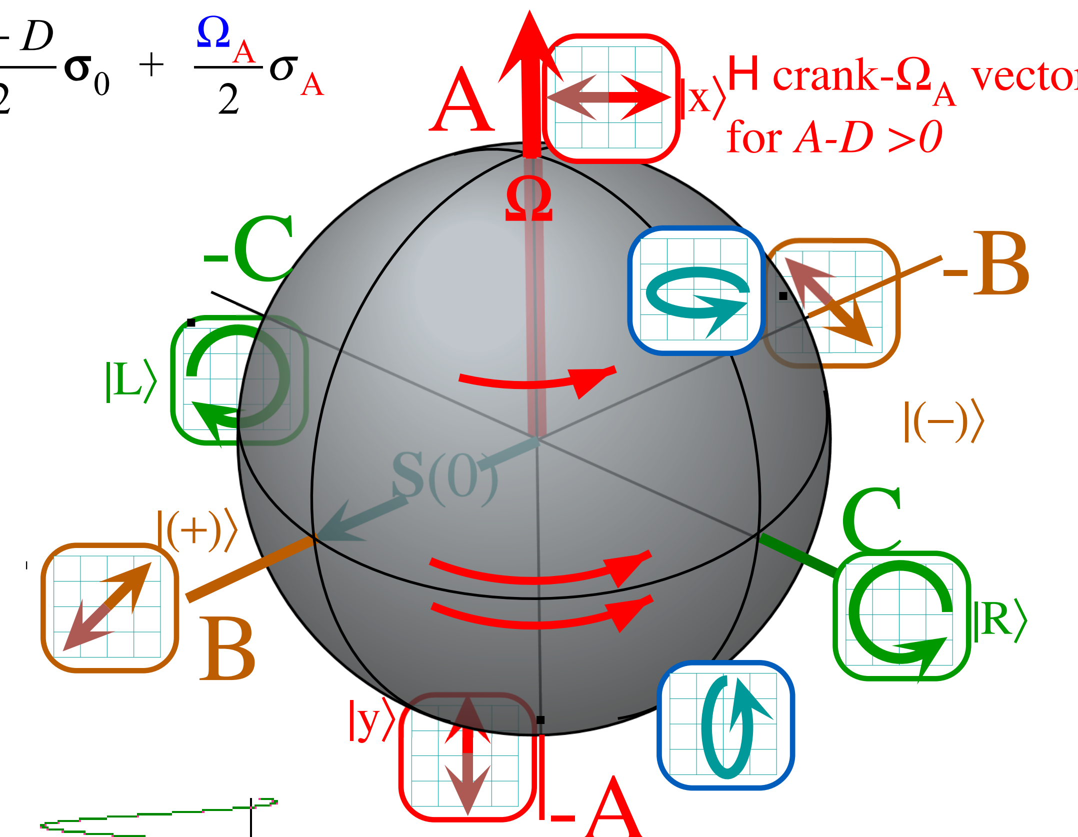
$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

Asymmetric Diagonal A-Type motion

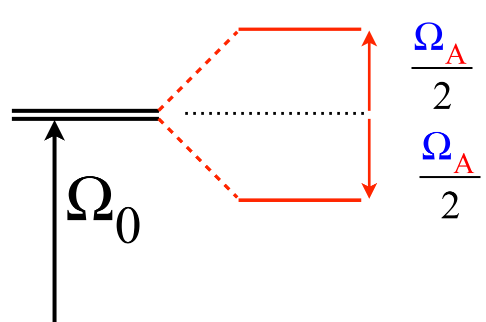
$$\begin{pmatrix}\langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle\\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle\end{pmatrix}=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\frac{A+D}{2}\boldsymbol{\sigma}_0+\frac{\Omega_A}{2}\sigma_A$$

$$\textit{Crank}:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 0\\ 0\end{pmatrix}\qquad \textit{Eigen-Spin}:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}\pm S\\ 0\\ 0\end{pmatrix}$$

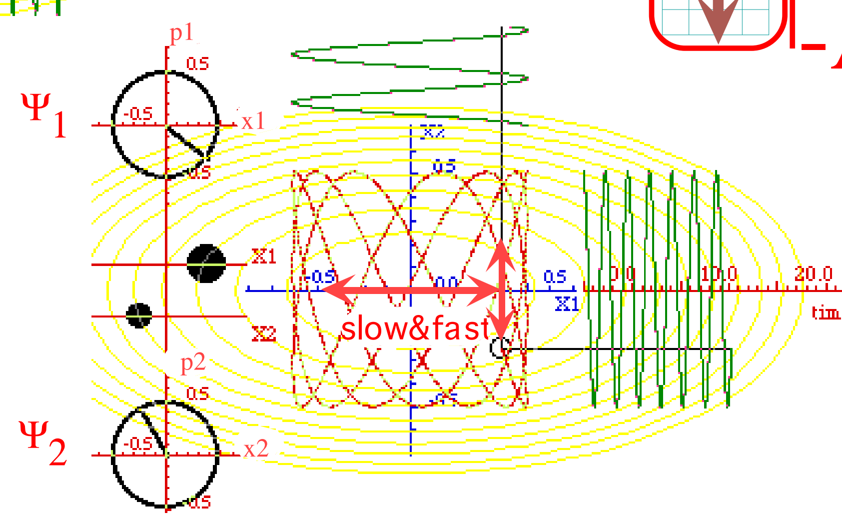




Beat dynamics:

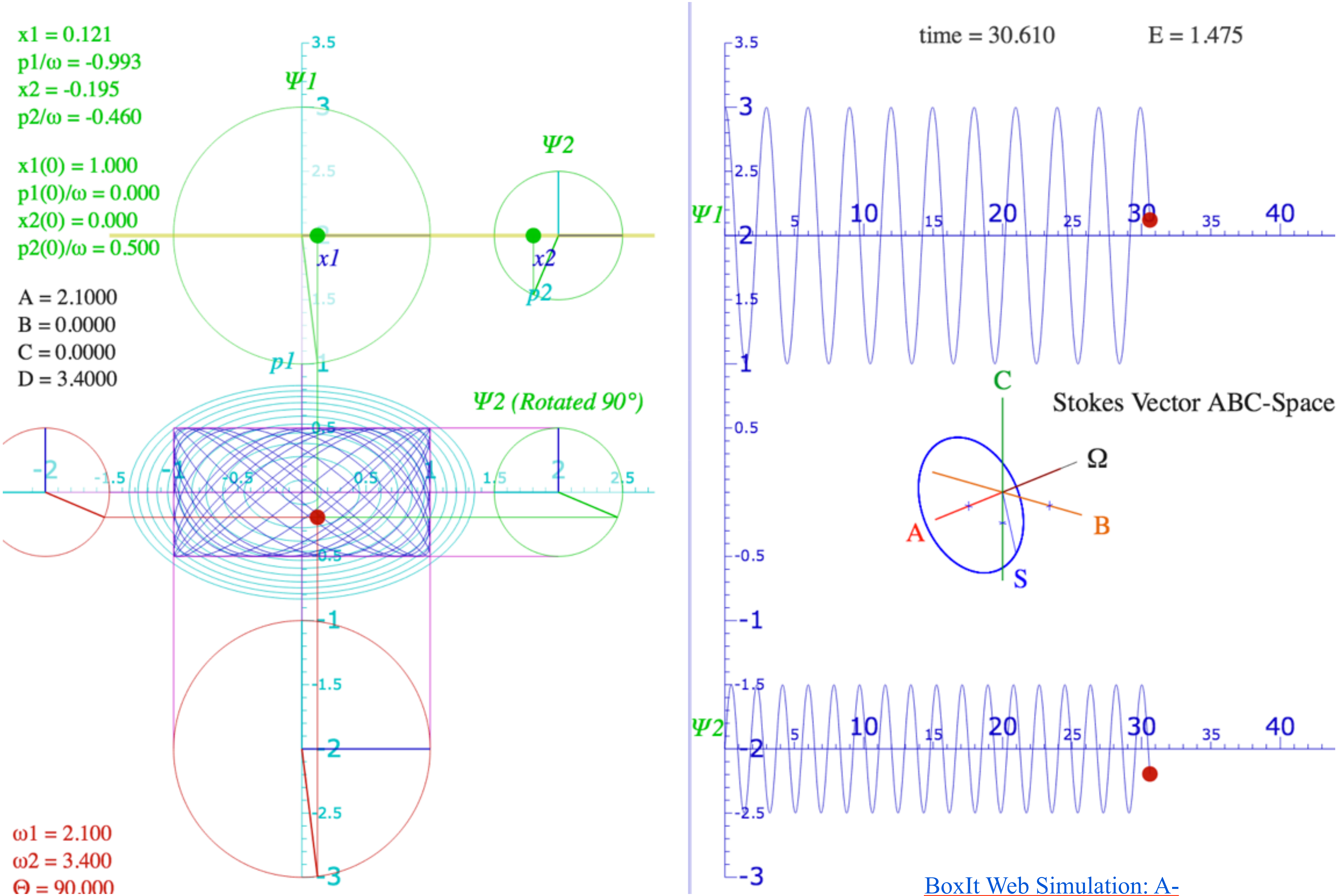


Total beat frequency Ω_A



BoxIt Web Simulation
Pure A-Type

A-Type elliptical polarized motion



[BoxIt Web Simulation: A-Type with A=2.1, D=3.4]

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* **S***-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian **H**$[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*

Euler-defined operator **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

Euler **R**$(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

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$$\boxed{\begin{array}{l}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{array}}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B$$

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Density operator $\boldsymbol{\rho}$ (see p.128-147)

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In general:

$$\mathbf{H} = \frac{A+D}{2}\,\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

$$\mathbf{H} = \frac{A+D}{2}\,\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

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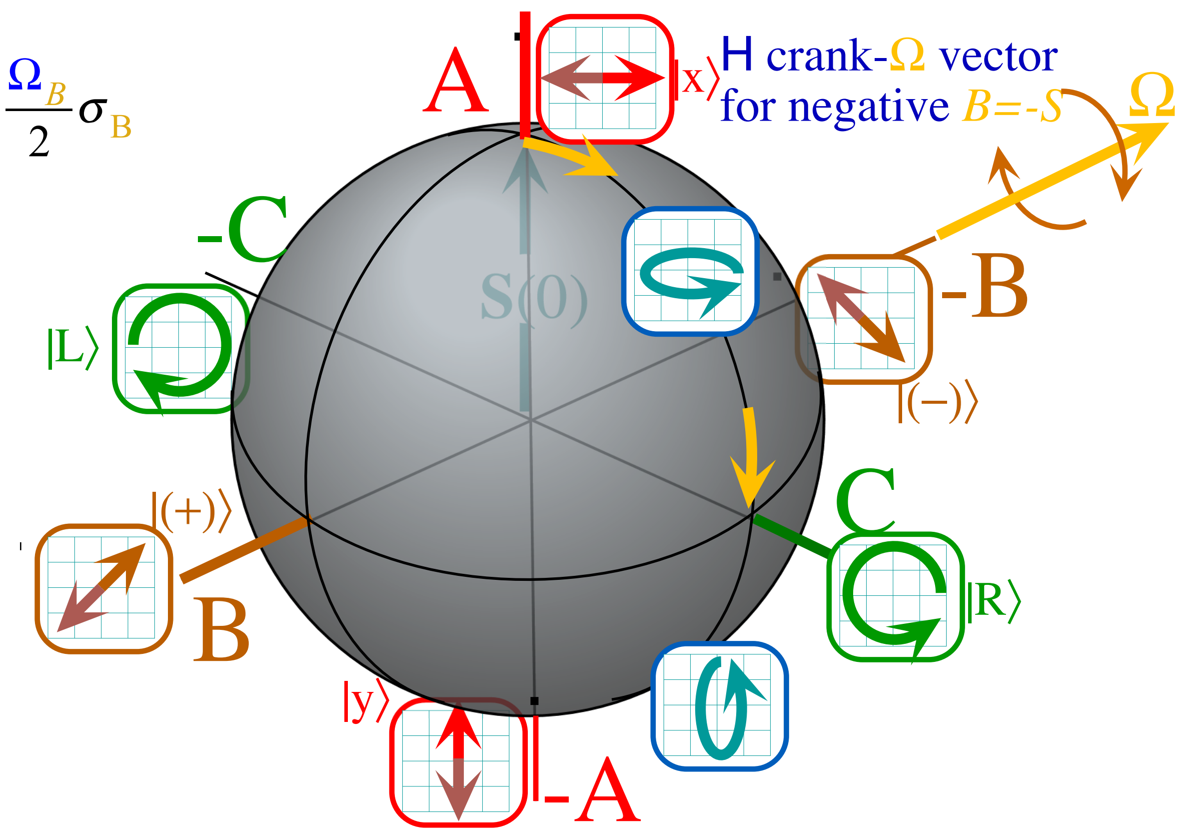
$$\begin{pmatrix}\langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle\\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle\end{pmatrix}=\begin{pmatrix}\Omega_0 & B\\ B & \Omega_0\end{pmatrix}=\Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}=\Omega_0\boldsymbol{\sigma}_0+\frac{\Omega_B}{2}\boldsymbol{\sigma}_B$$

$$\textit{Crank}:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}0\\ 2B\\ 0\end{pmatrix}\qquad \textit{Eigen-Spin}:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}0\\ \pm S\\ 0\end{pmatrix}$$



The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ (see p.128-147)

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

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$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

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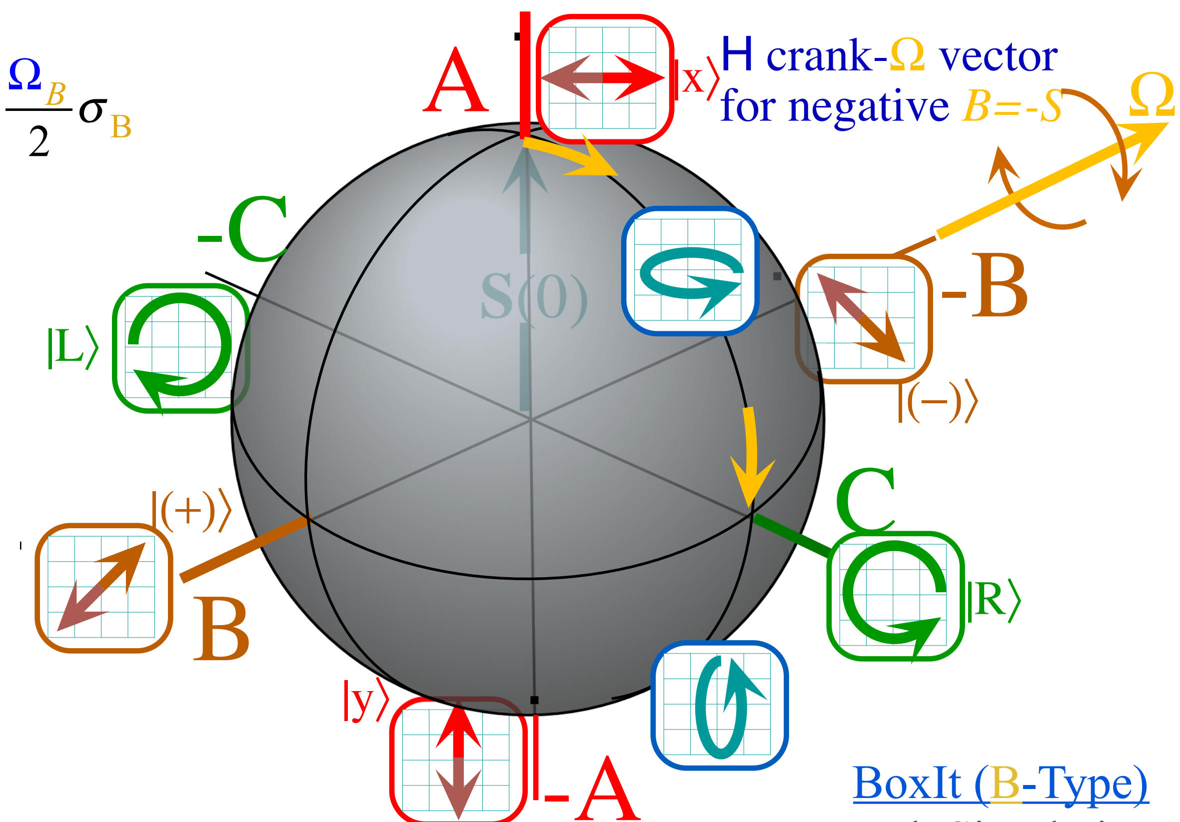
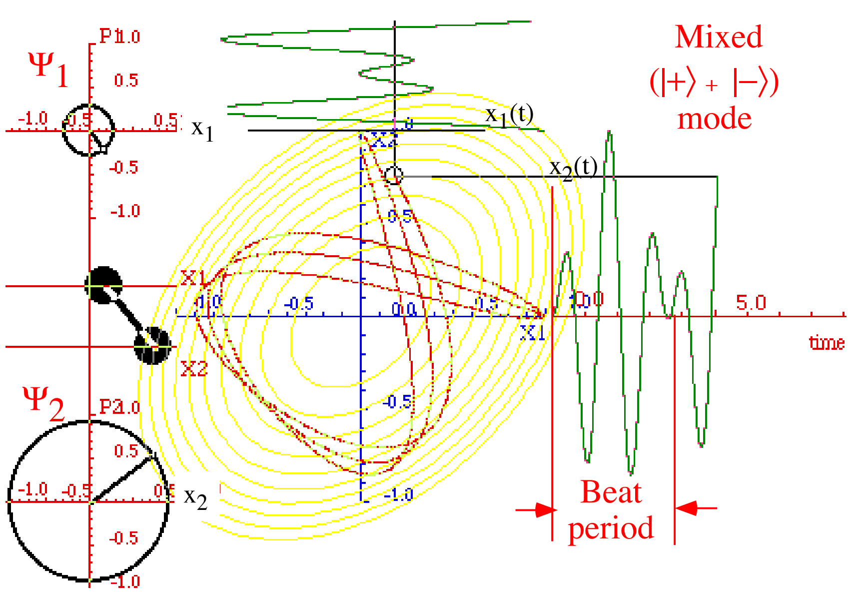
Crank : $\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix}$ *Eigen−Spin :* $\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$



[BoxIt (B-Type) Web Simulation]

Beat dynamics:



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$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\,\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\,\sigma_A$$

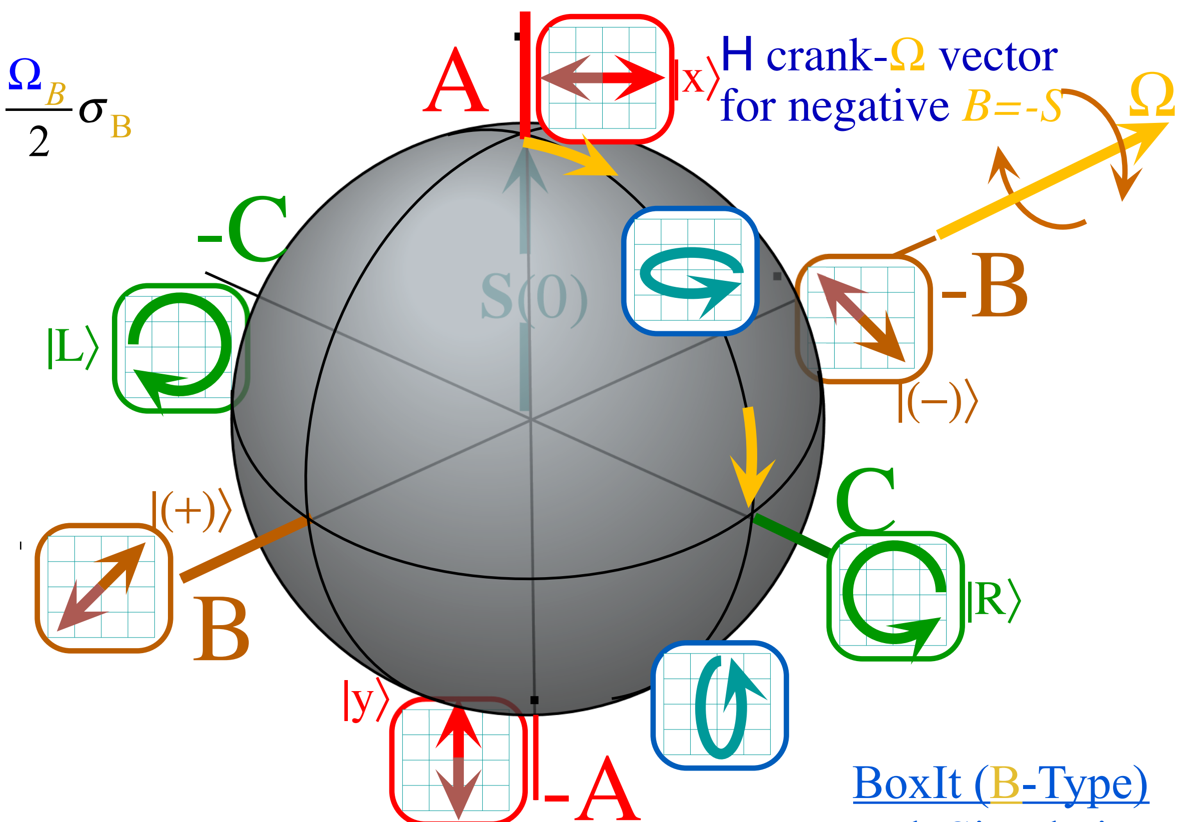
$$\mathbf{H} = \frac{A+D}{2}\,\sigma_0 + \frac{\Omega_B}{2}\,\sigma_B + \frac{\Omega_C}{2}\,\sigma_C + \frac{\Omega_A}{2}\,\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

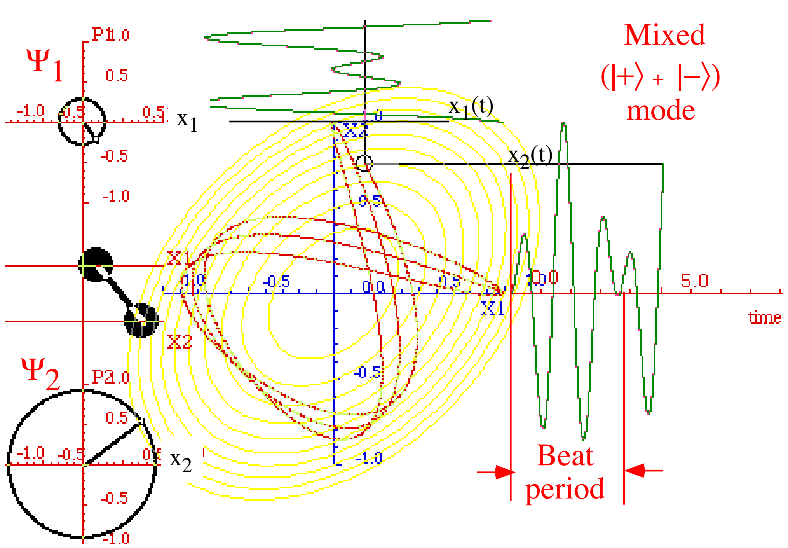
$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\sigma_0 + \frac{\Omega_B}{2}\sigma_B$$

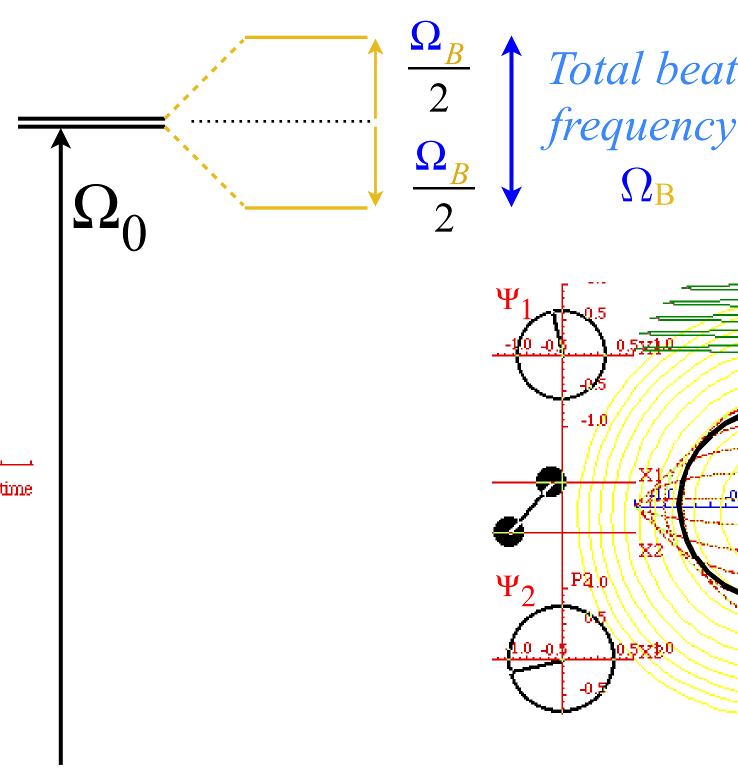
Crank : $\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix}$ *Eigen − Spin* : $\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$

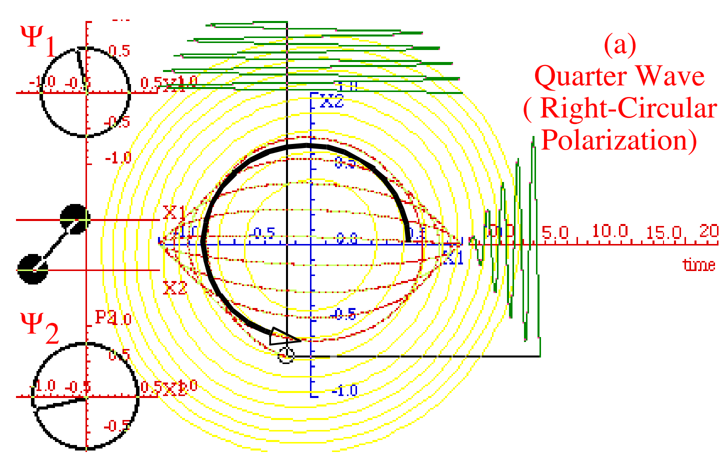


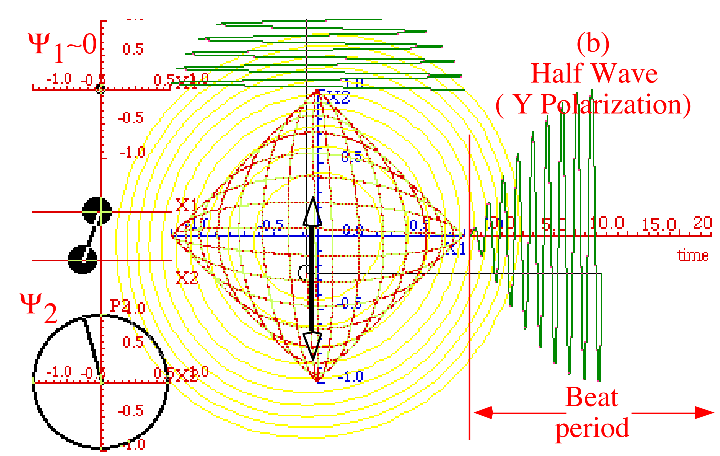
BoxIt (B-Type) Web Simulation

Beat dynamics:

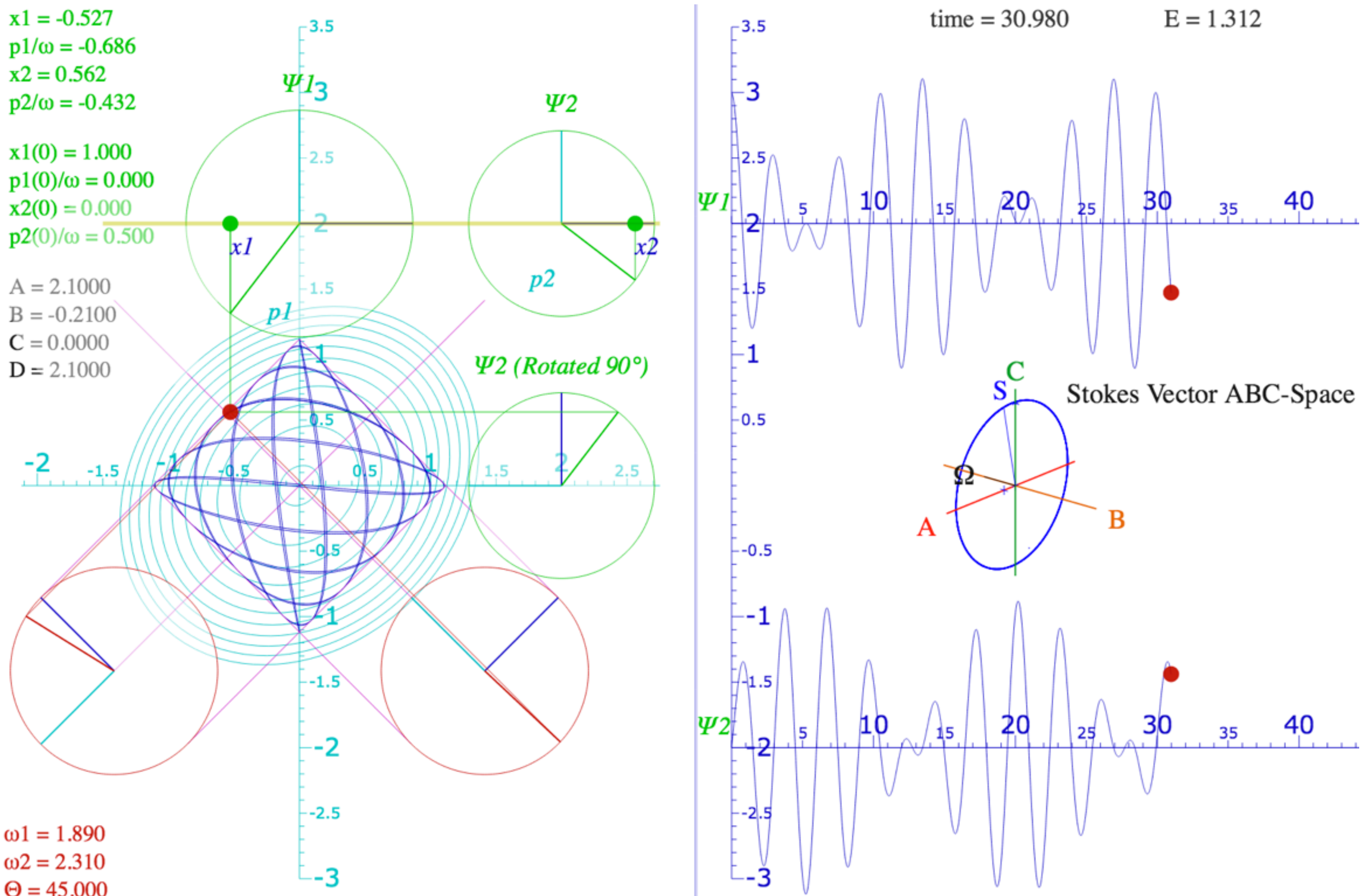






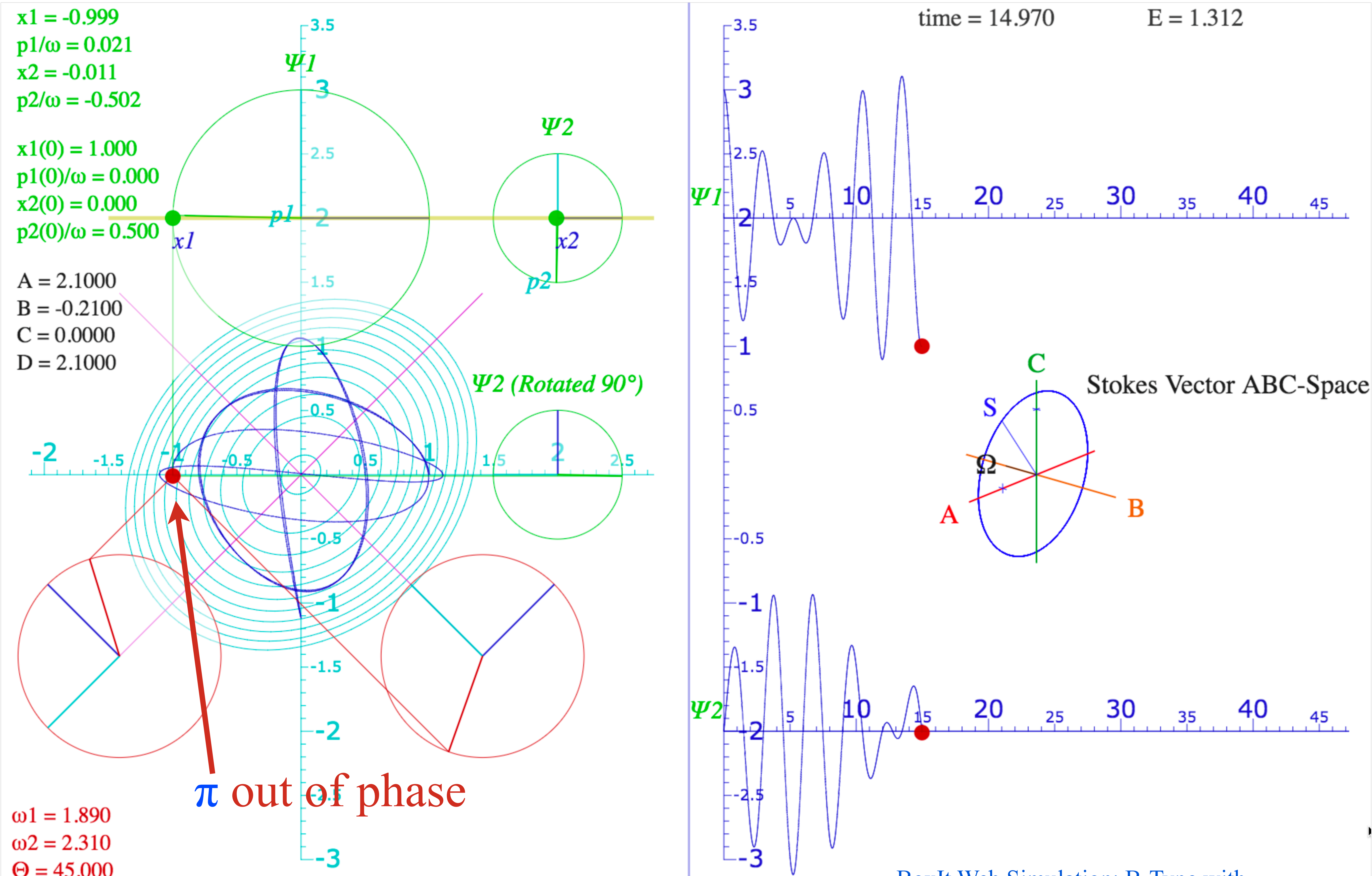


B-Type elliptical polarized motion



[BoxIt Web Simulation: B-Type with A, D=2.1; B=-0.21]

B-Type elliptical polarized motion Note that one 360°=2π rotation of **S** leaves (x_1,x_2) at $-(x_1,x_2)$



BoxIt Web Simulation: B-Type with A, D=2.1; B=-0.21

To assess the rationality of any number we approximate it using successive levels of *continued fractions*.

$$\alpha = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \ddots}}}}$$

Example 1: the number π = 3.1415926..., and recipe for getting n_k

$$A_0 = \alpha = 3.14159265\ldots \qquad n_0 = INT(A_0) = 3 \qquad \pi \cong \quad = 3.000..$$

$$A_1 = \frac{1}{A_0 - n_0} = 7.06\ldots \qquad n_1 = INT(A_1) = 7 \qquad \pi \cong 3 + \frac{1}{7} = \frac{22}{7} = 3.1428$$

$$A_2 = \frac{1}{A_1 - n_1} = 15.99\ldots \qquad n_2 = INT(A_2) = 15 \qquad \pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106} = 3.141509$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.003\ldots \qquad n_3 = INT(A_3) = 1 \qquad \pi \cong 3 + \cfrac{1}{7 + \cfrac{1}{15 + 1}} = \frac{355}{113} = 3.14159292$$

Example 2: the *Golden Mean G*=(1+√5)/2=1.618033989…

$$A_0 = G = 1.618033989\ldots \qquad n_0 = INT(A_0) = 1 \qquad G \cong \quad = 1.000..$$

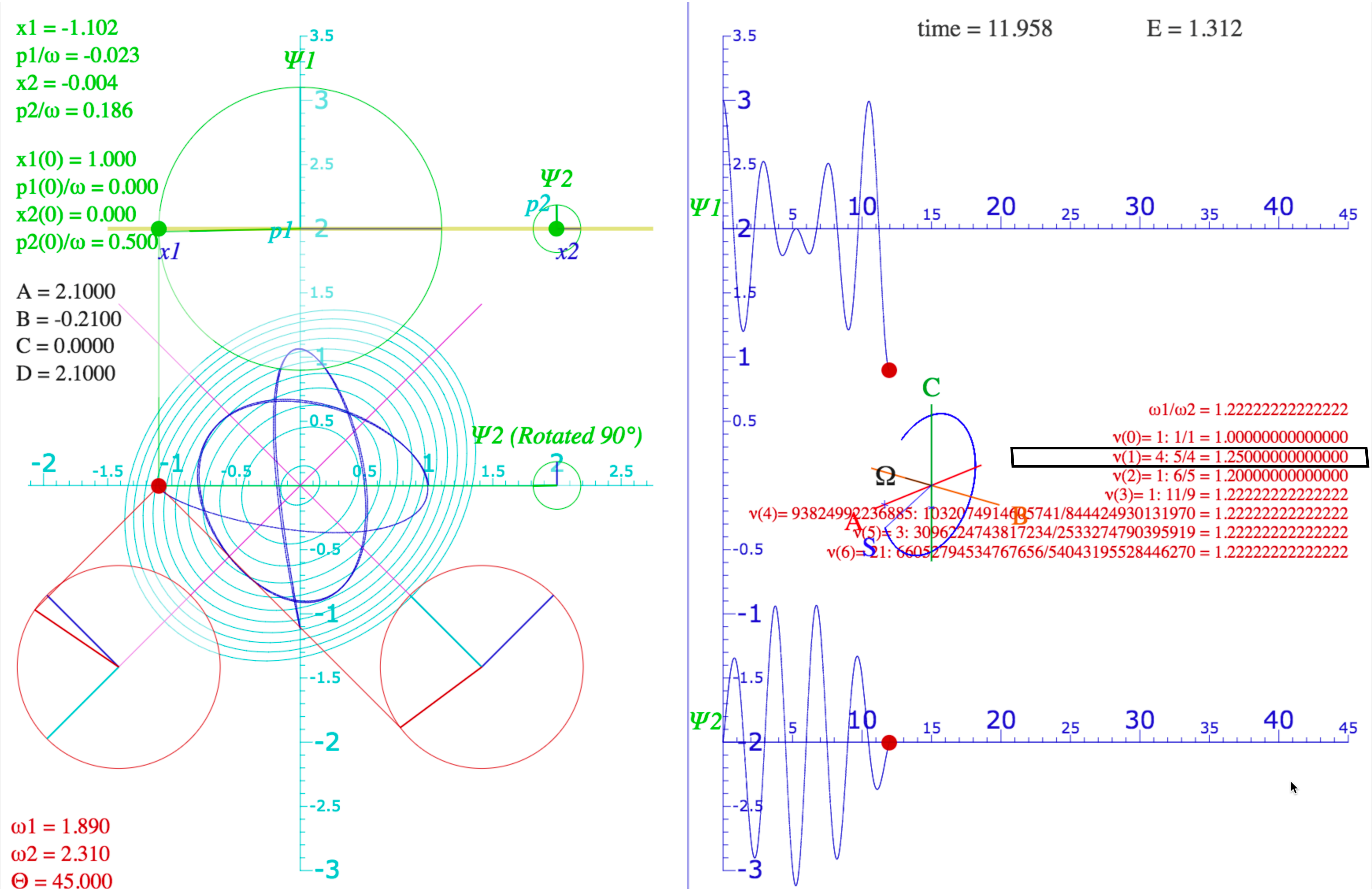
$$A_1 = \frac{1}{A_0 - n_0} = 1.6180\ldots \qquad n_1 = INT(A_1) = 1 \qquad G \cong 1 + \frac{1}{1} = \frac{2}{1} = 2.000$$

$$A_2 = \frac{1}{A_1 - n_1} = 1.6180\ldots \qquad n_2 = INT(A_2) = 1 \qquad G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2} = 1.500$$

$$A_3 = \frac{1}{A_2 - n_2} = 1.6180\ldots \qquad n_3 = INT(A_3) = 1 \qquad G \cong 1 + \cfrac{1}{1 + \cfrac{1}{1 + 1}} = \frac{5}{3} = 1.666\ldots$$

Note: *Fibonacci* numbers: 1,2,3,5,8,…

The most irrational number is closest to being rational!



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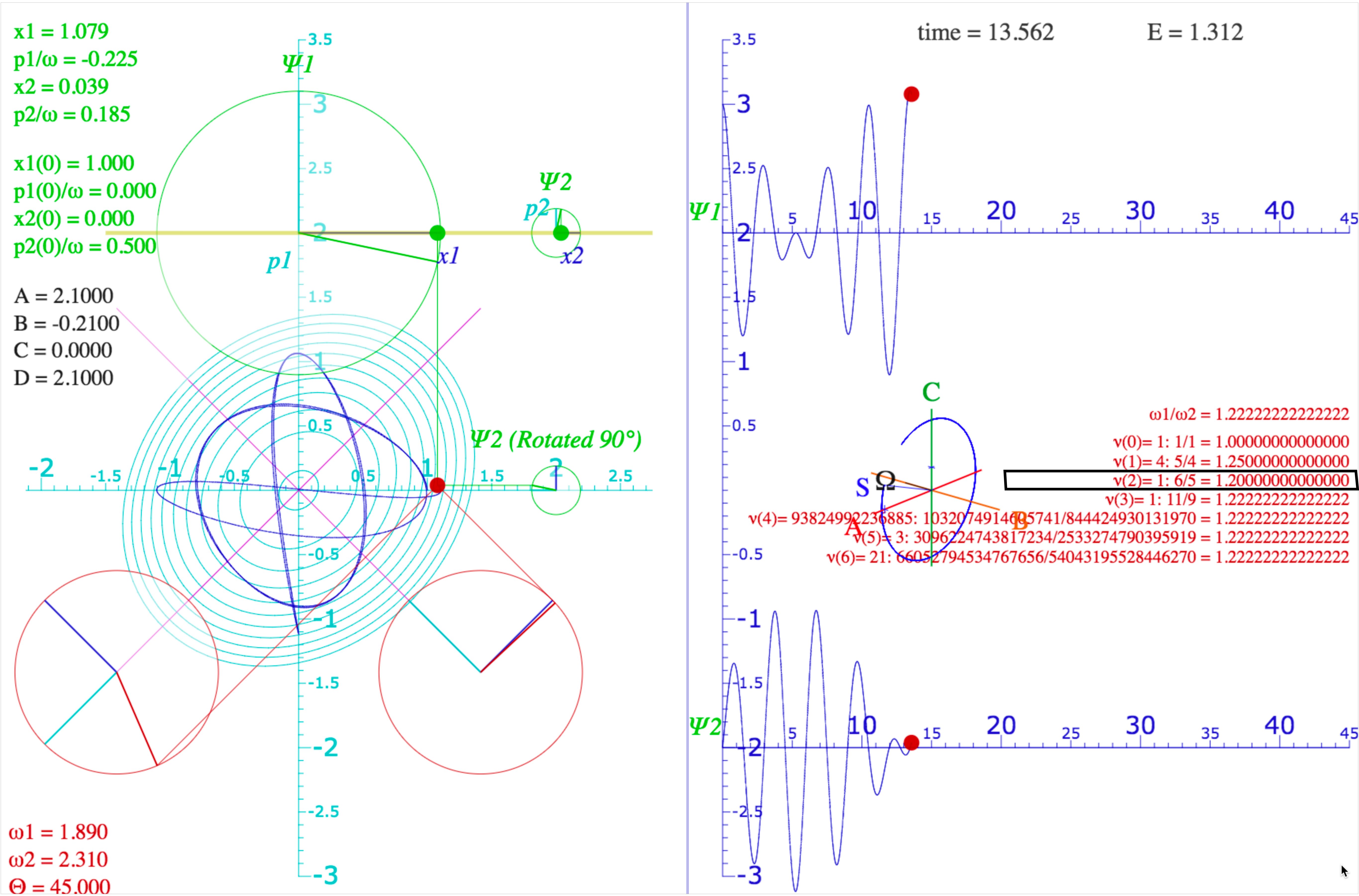

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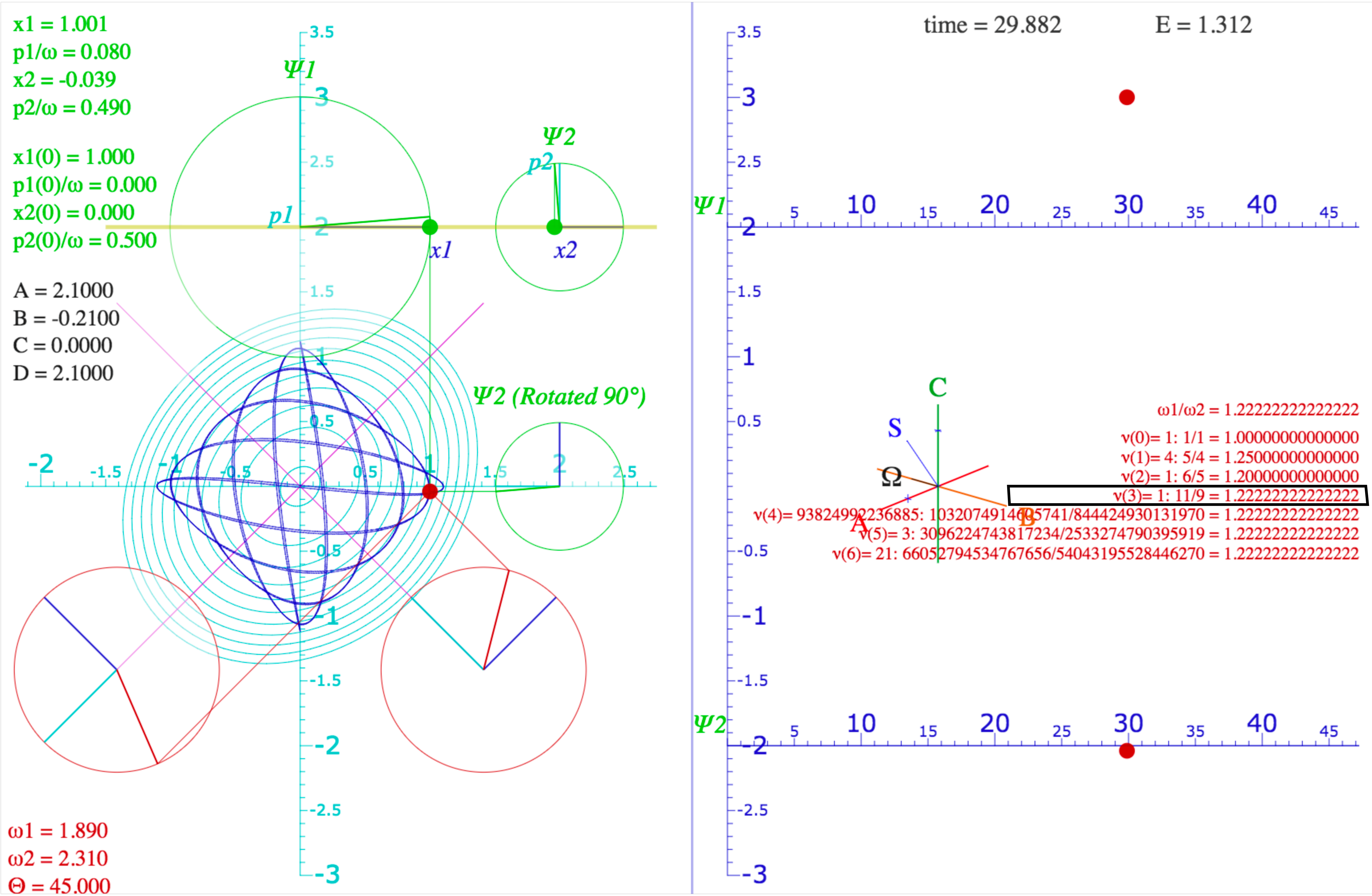

x1 = 1.001
p1/ω = 0.080
x2 = -0.039
p2/ω = 0.490
x1(0) = 1.000
p1(0)/ω = 0.000
x2(0) = 0.000
p2(0)/ω = 0.500
A = 2.1000
B = -0.2100
C = 0.0000
D = 2.1000
ω1 = 1.890
ω2 = 2.310
Θ = 45.000
Ψ1
Ψ2
p1
p2
x1
x2
Ψ2 (Rotated 90°)
time = 29.882
E = 1.312
C
S
Ω
A
B
ω1/ω2 = 1.2222222222222
ν(0)= 1: 1/1 = 1.0000000000000
ν(1)= 4: 5/4 = 1.2500000000000
ν(2)= 1: 6/5 = 1.2000000000000
ν(3)= 1: 11/9 = 1.2222222222222
ν(4)= 938249922236885: 1032074914605741/844424930131970 = 1.2222222222222
ν(5)= 3: 3096224743817234/2533274790395919 = 1.2222222222222
ν(6)= 21: 66052794534767656/54043195528446270 = 1.2222222222222

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Euler-defined operator **R**$(\alpha\beta\gamma)$ *derived from Darboux-defined* **R**$[\varphi\vartheta\Theta]$ *and vice versa*

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Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

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$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A-D}{2}\boldsymbol{\sigma}_A$$

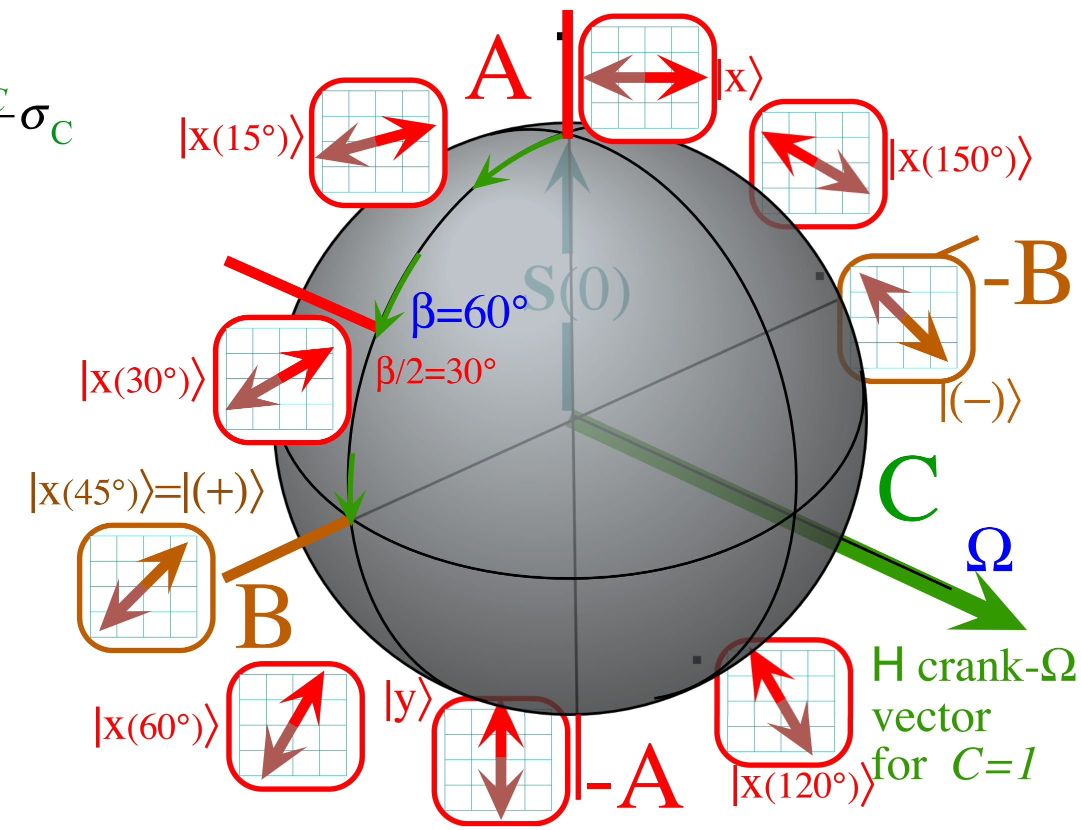
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\boldsymbol{\sigma}_B + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C + \frac{\Omega_A}{2}\boldsymbol{\sigma}_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\boldsymbol{\sigma}_C$$

$$\text{Crank}: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad \text{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$



The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ *(see p.128-147)* $\rightarrow$ $\boxed{\begin{array}{c}\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma} \\ \mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{array}}$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

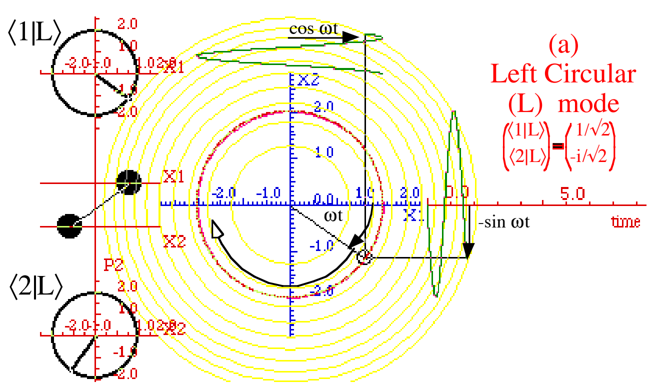
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

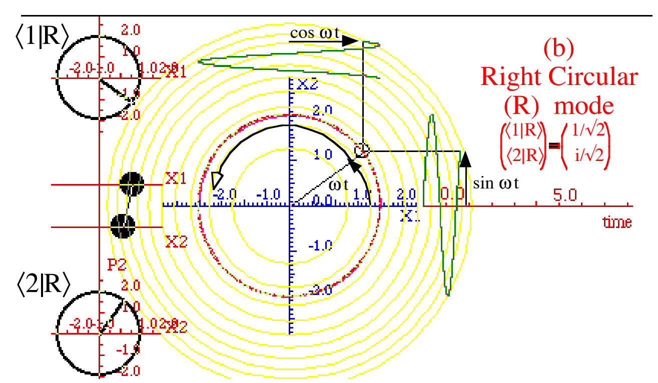
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

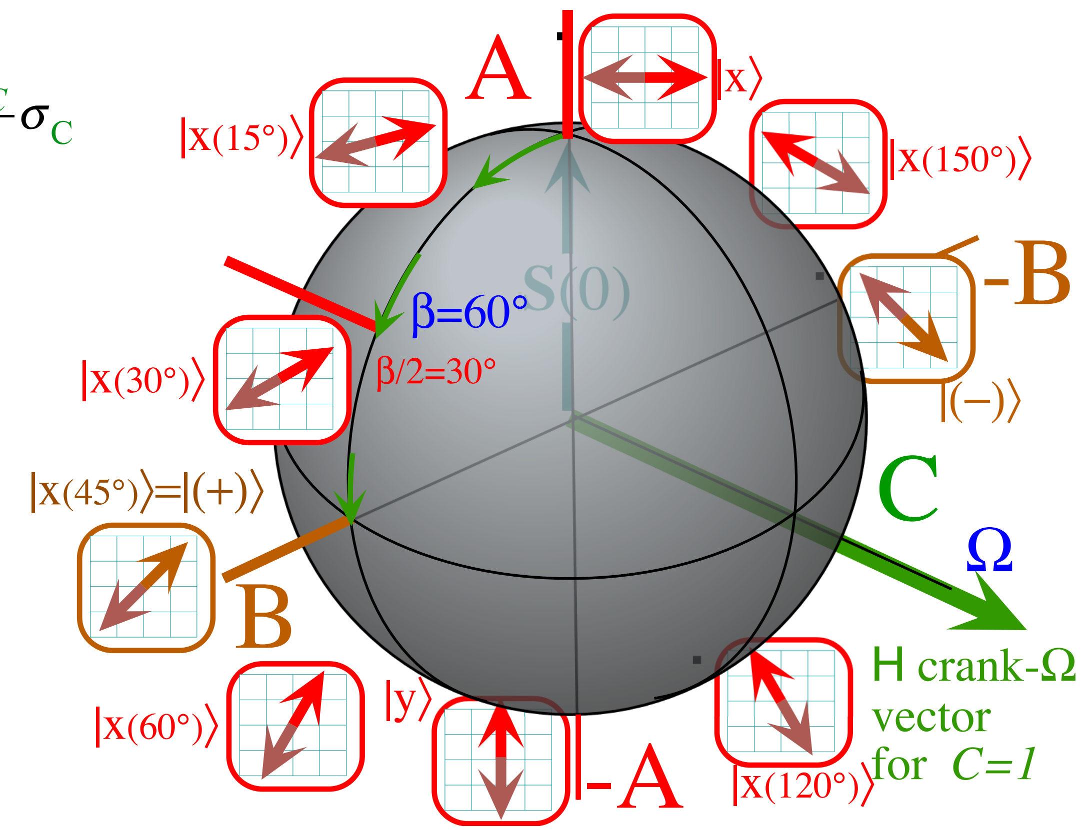
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

$$\textit{Crank}: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad \textit{Eigen-Spin}: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$







The ABC's of U(2) dynamics

Density operator $\boldsymbol{\rho}$ *(see p.128-147)*

$$\boxed{\begin{aligned}\boldsymbol{\rho} &= \tfrac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H} &= \Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

$$\begin{pmatrix}\langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle\\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle\end{pmatrix}=\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+B\begin{pmatrix}0&1\\1&0\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

In general:

$$\mathbf{H} = \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

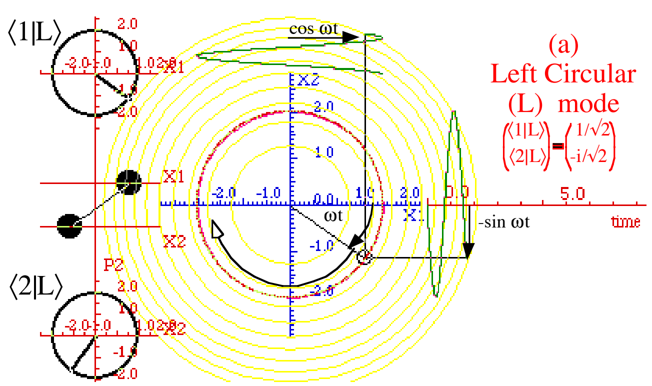
$$\mathbf{H} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}A-D\\ 2B\\ 2C\end{pmatrix}$$

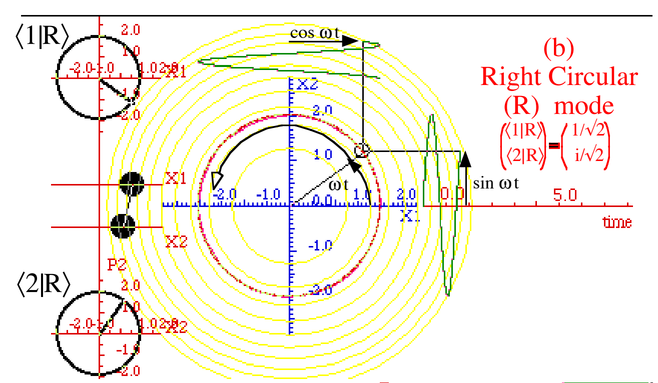
Circular-Coriolis... C-Type motion

$$\begin{pmatrix}\langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle\\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle\end{pmatrix}=\begin{pmatrix}\Omega_0 & -iC\\ iC & \Omega_0\end{pmatrix}=\Omega_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+C\begin{pmatrix}0&-i\\i&0\end{pmatrix}=\Omega_0\boldsymbol{\sigma}_0+\frac{\Omega_C}{2}\sigma_C$$

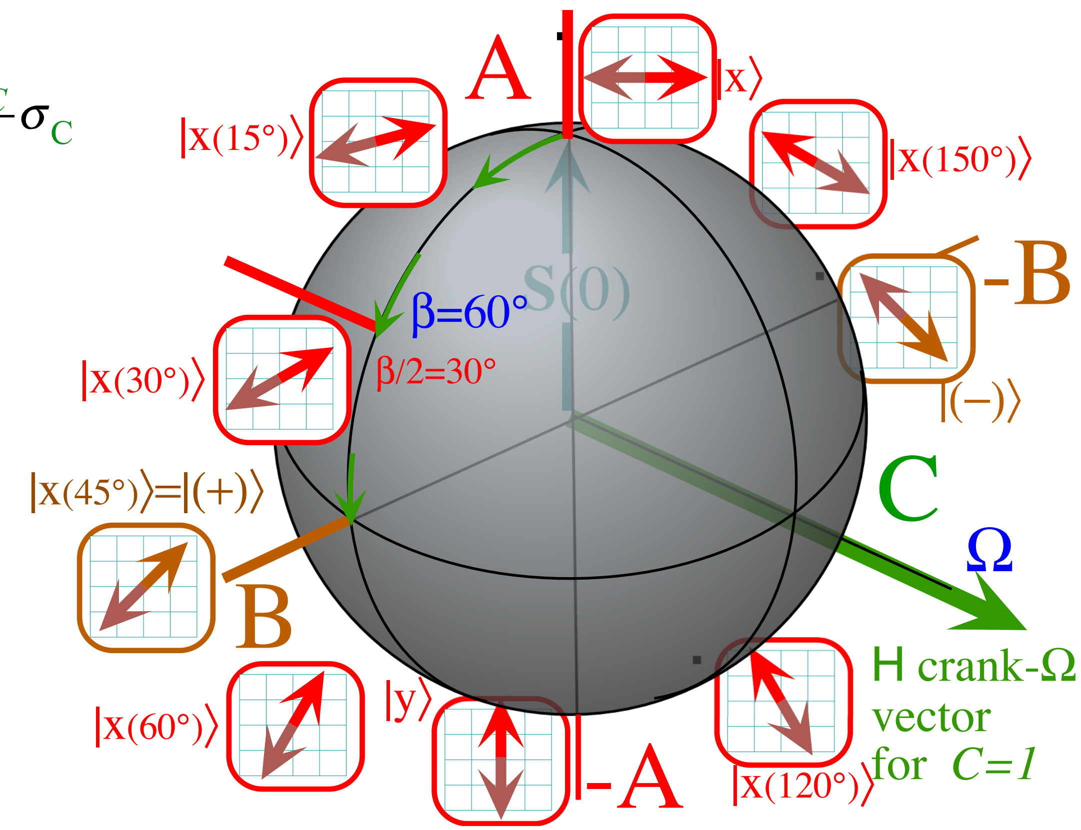
$$\textit{Crank}:\ \vec{\Omega}=\begin{pmatrix}\Omega_A\\ \Omega_B\\ \Omega_C\end{pmatrix}=\begin{pmatrix}0\\0\\2C\end{pmatrix}\qquad \textit{Eigen-Spin}:\vec{\mathbf{S}}=\begin{pmatrix}S_A\\ S_B\\ S_C\end{pmatrix}=\begin{pmatrix}0\\0\\ \pm S\end{pmatrix}$$



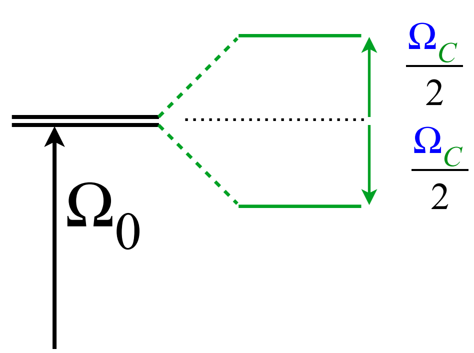
$\begin{pmatrix}\langle 1|L\rangle\\ \langle 2|L\rangle\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ -i/\sqrt{2}\end{pmatrix}$



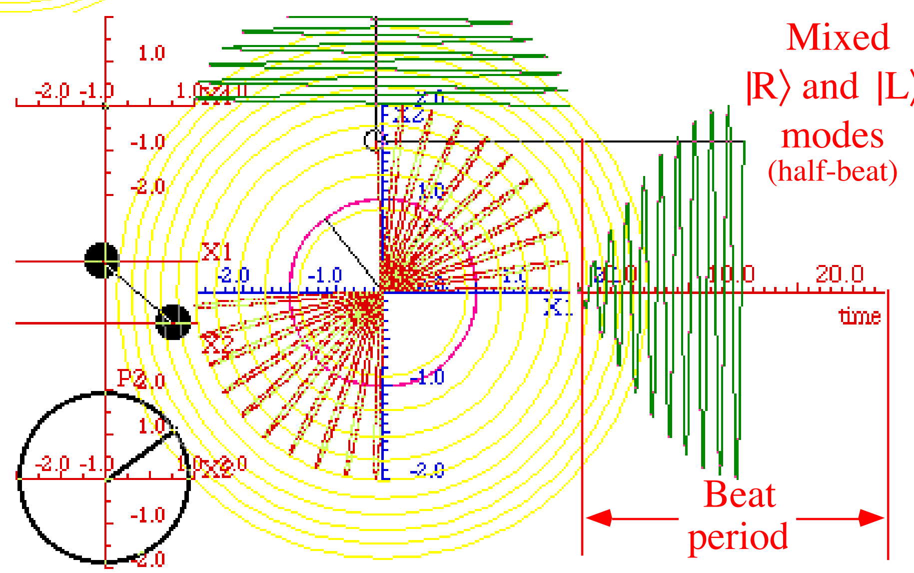
$\begin{pmatrix}\langle 1|R\rangle\\ \langle 2|R\rangle\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ i/\sqrt{2}\end{pmatrix}$



Beat dynamics:

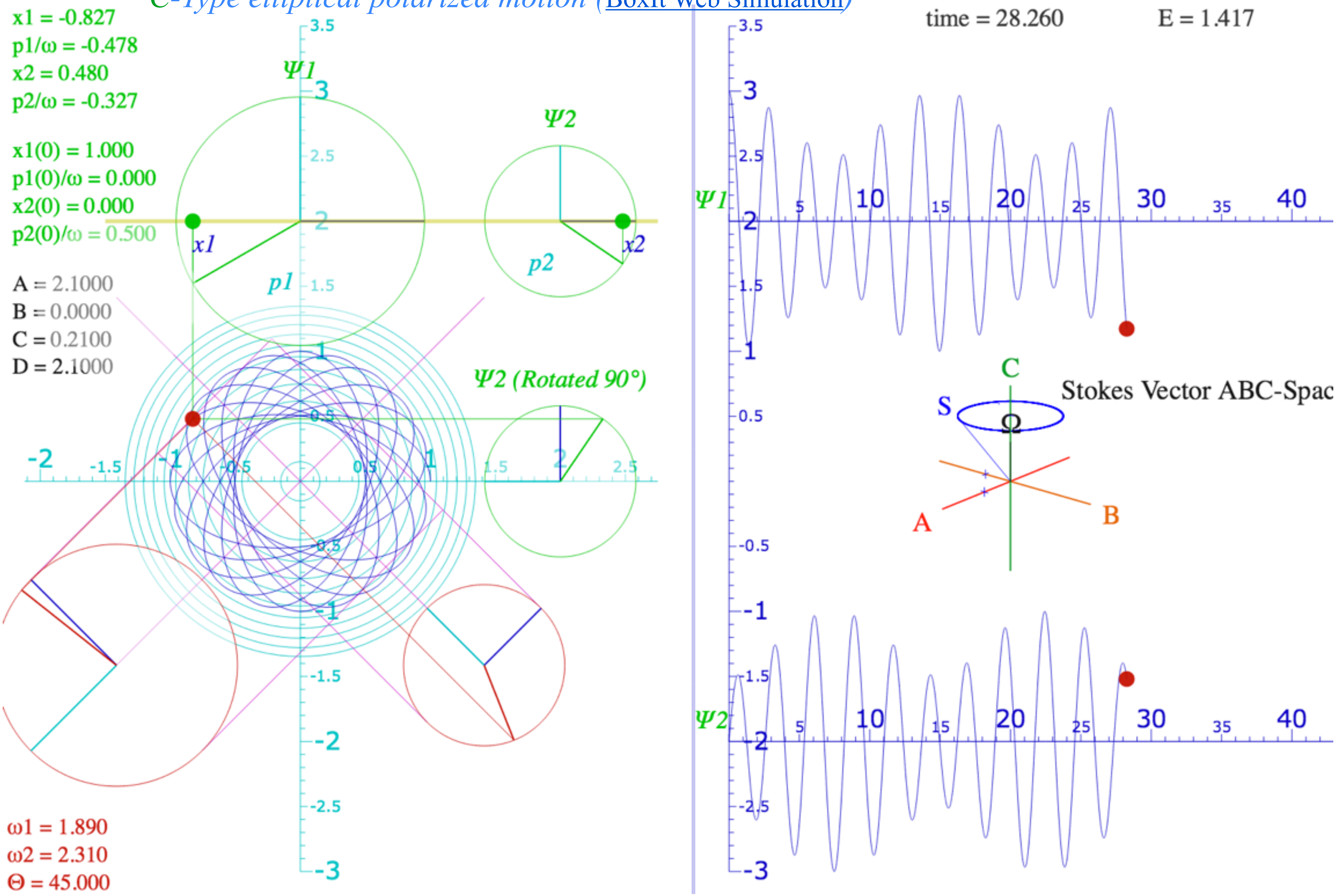


Total beat frequency Ω_C



[BoxIt (C-Type) Web Simulation]()

C-Type elliptical polarized motion ([BoxIt Web Simulation]())



[BoxIt Web Simulation:
C-Type with A, D=2.1; C=-0.21]()

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

The ABC's of U(2) dynamics-Mixed modes (AB-Type motion)

$$\rho = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0 \mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From:
QTCA
Lect. 9(2.12)
p.60

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

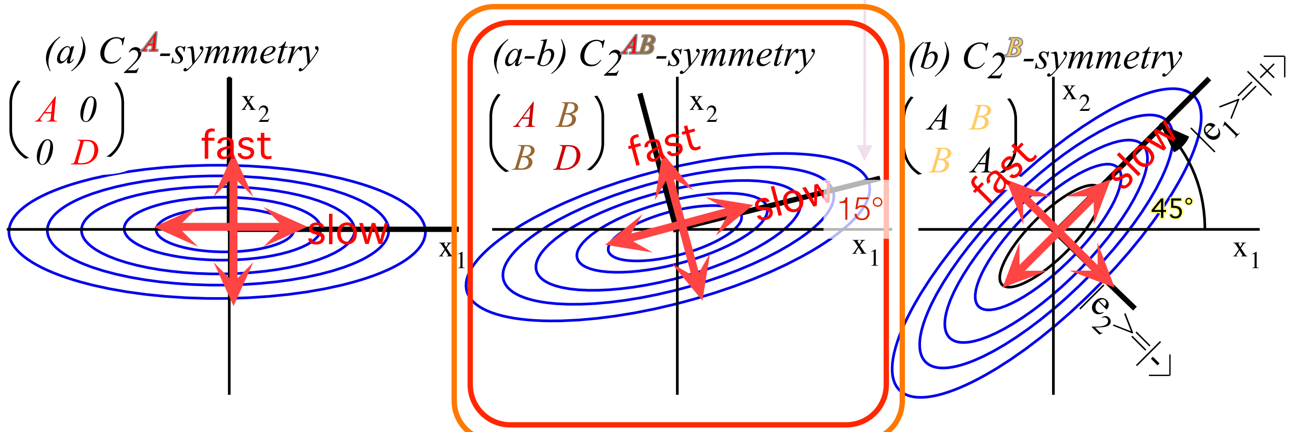
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Tilted-plane polarization AB-Type motion

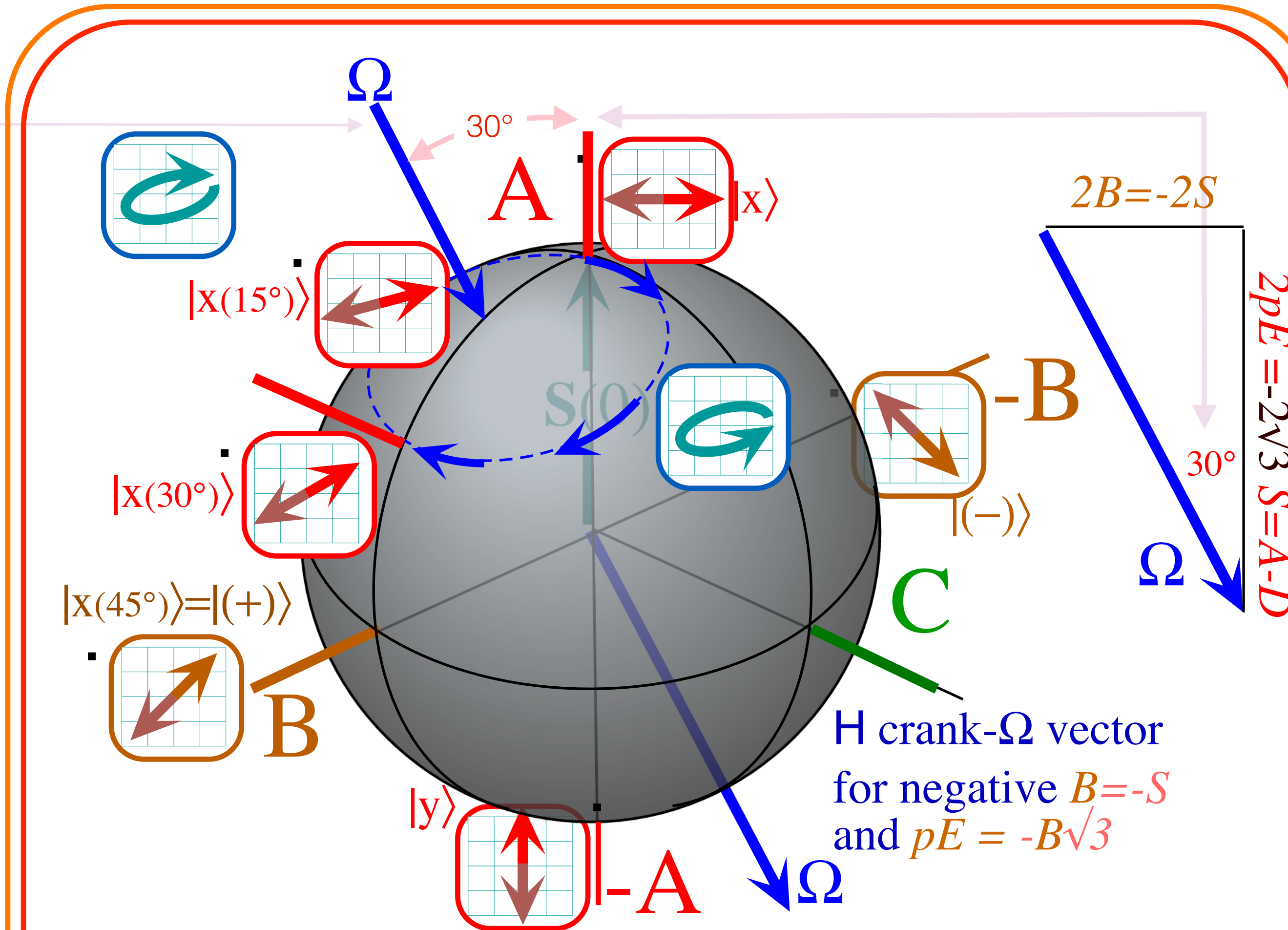
$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

Crank : $\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix}$ *Eigen−Spin* : $\vec{\mathbf{S}} = \pm S\cdot\hat{\Omega}$

Note the *relative* factor of 1/2



Beat dynamics:

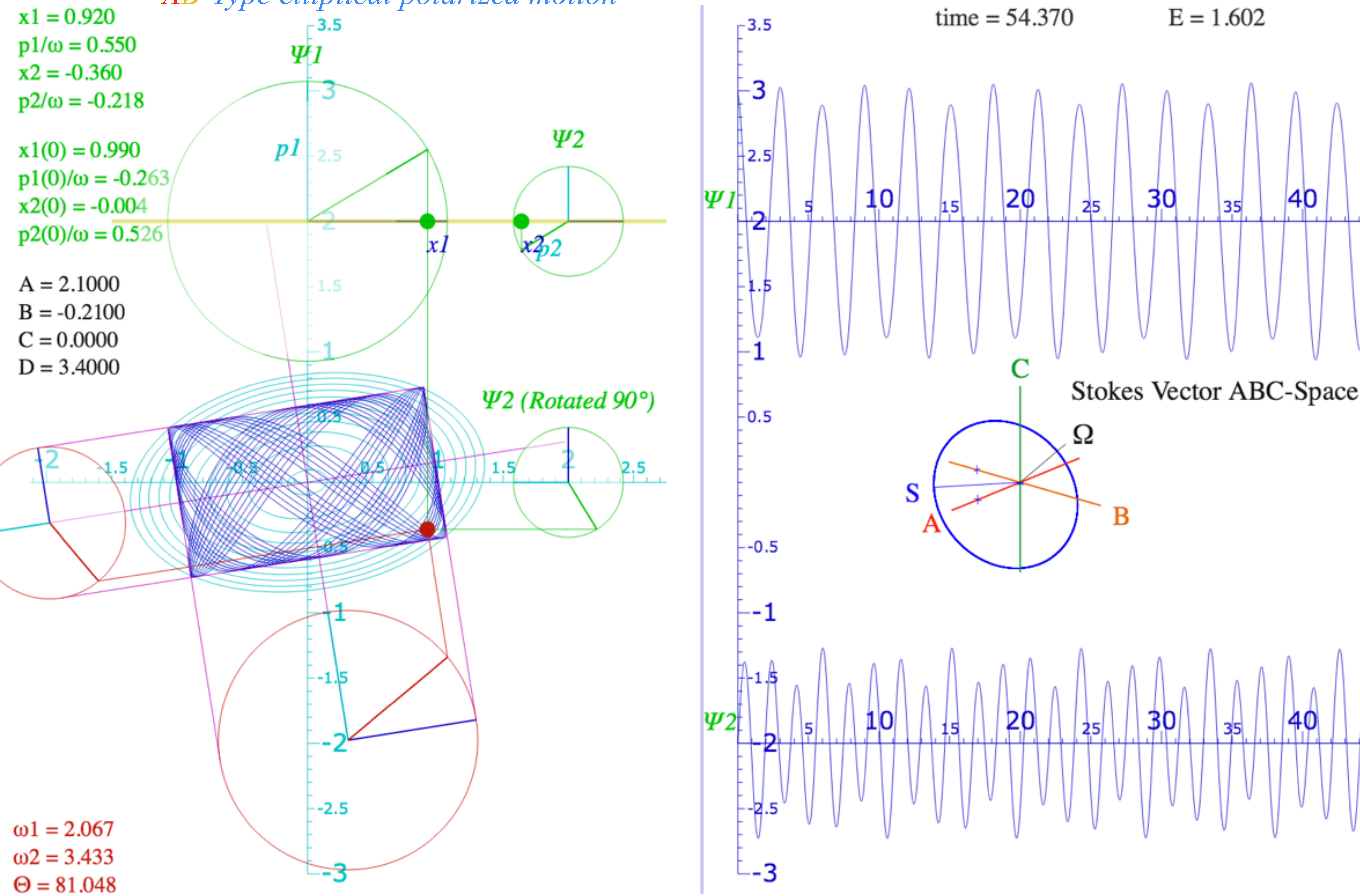


BoxIt Web Simulation: AB-Type Motion

Note: *In the Web App* the C axis is vertical

Older use: AB-Type mixed Motion: A=5.086, B=-0.27, C=0.0, D=2.024

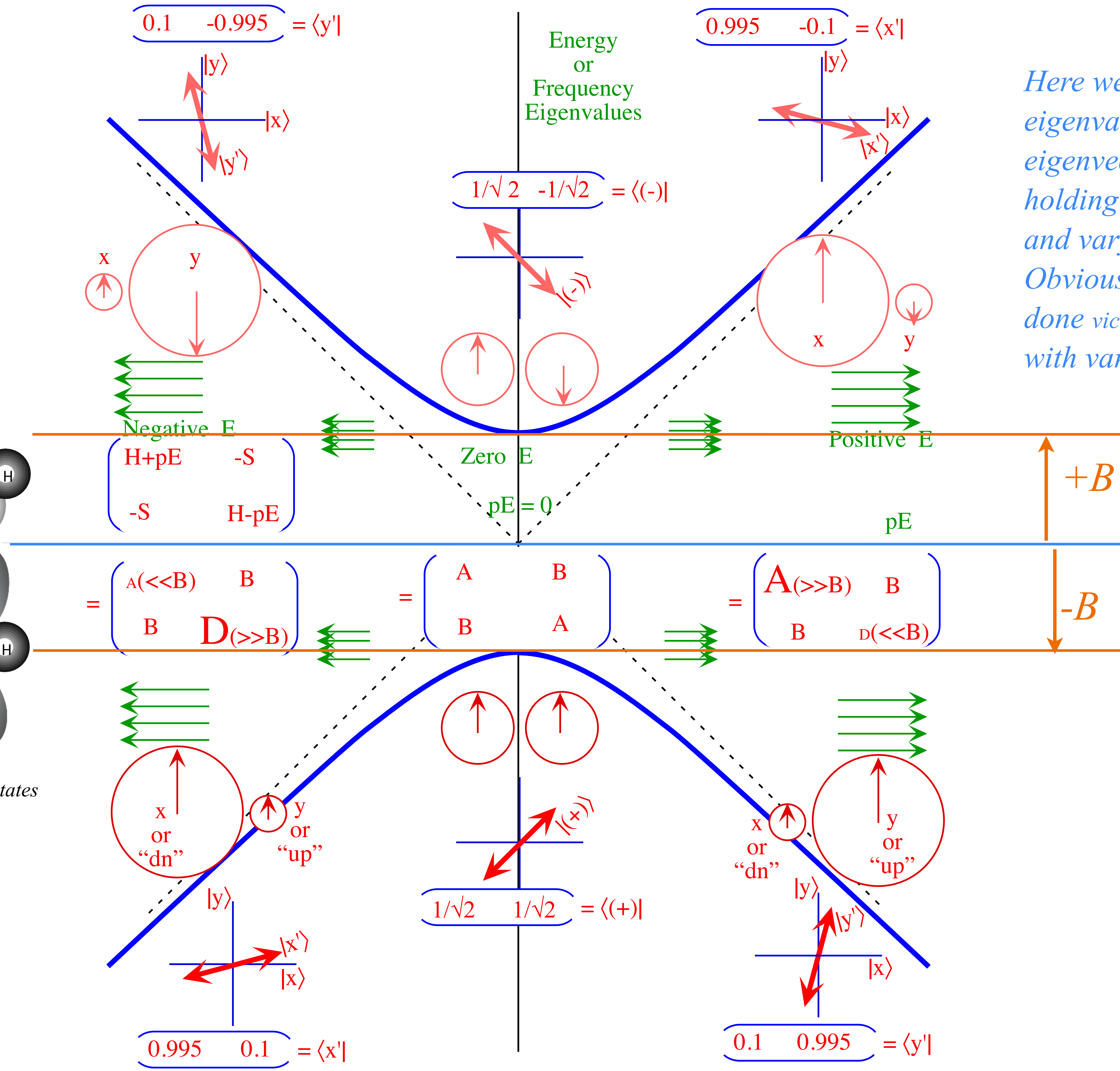
AB-Type elliptical polarized motion



[BoxIt Web Simulation: AB-Type with A=2.1; B=-0.21; D=3.4]

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$



Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

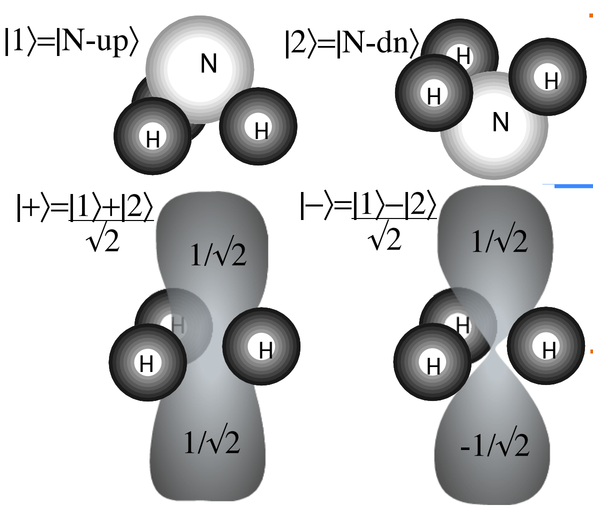


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

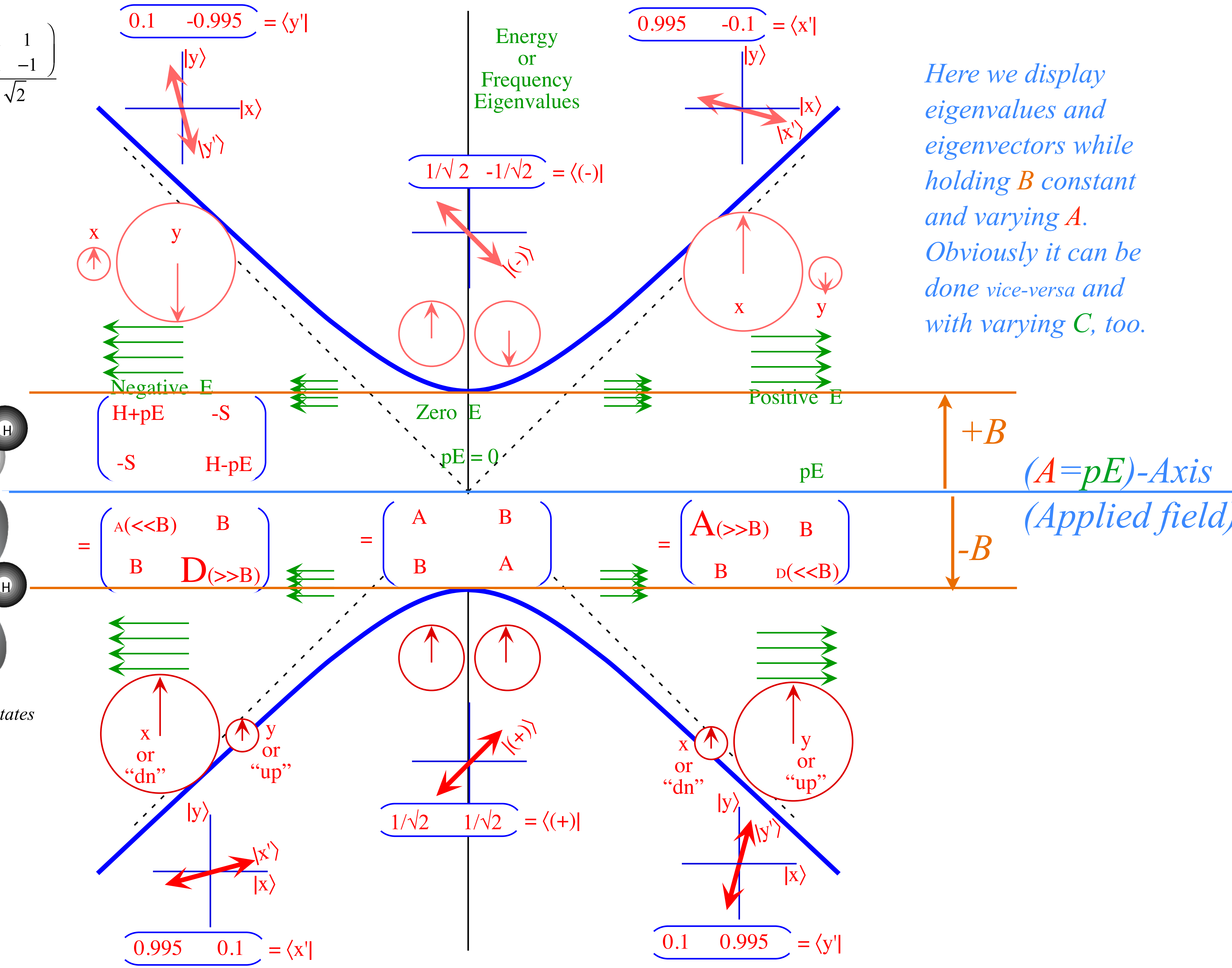
Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \quad \text{Secular equation: } \varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2) \quad \text{gives } \textit{hyperbolic} \text{ energy levels: } \varepsilon = \pm\sqrt{A^2 + B^2}$$

$$\mathbf{H}(B-basis) \qquad \mathbf{H}(A-basis)$$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$



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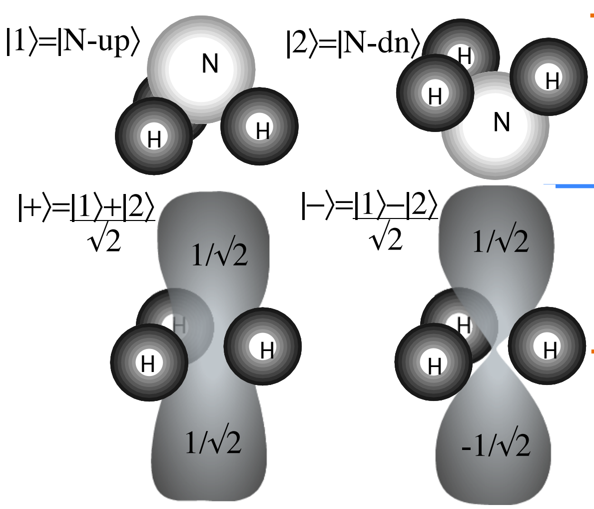

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$\mathbf{H}(B-basis)$ $\qquad$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

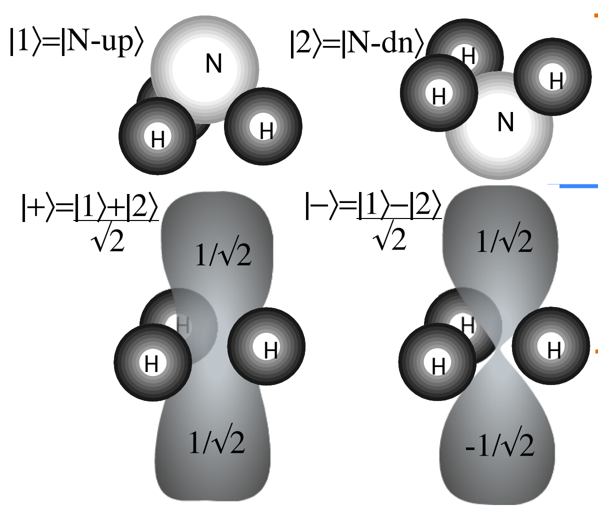


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

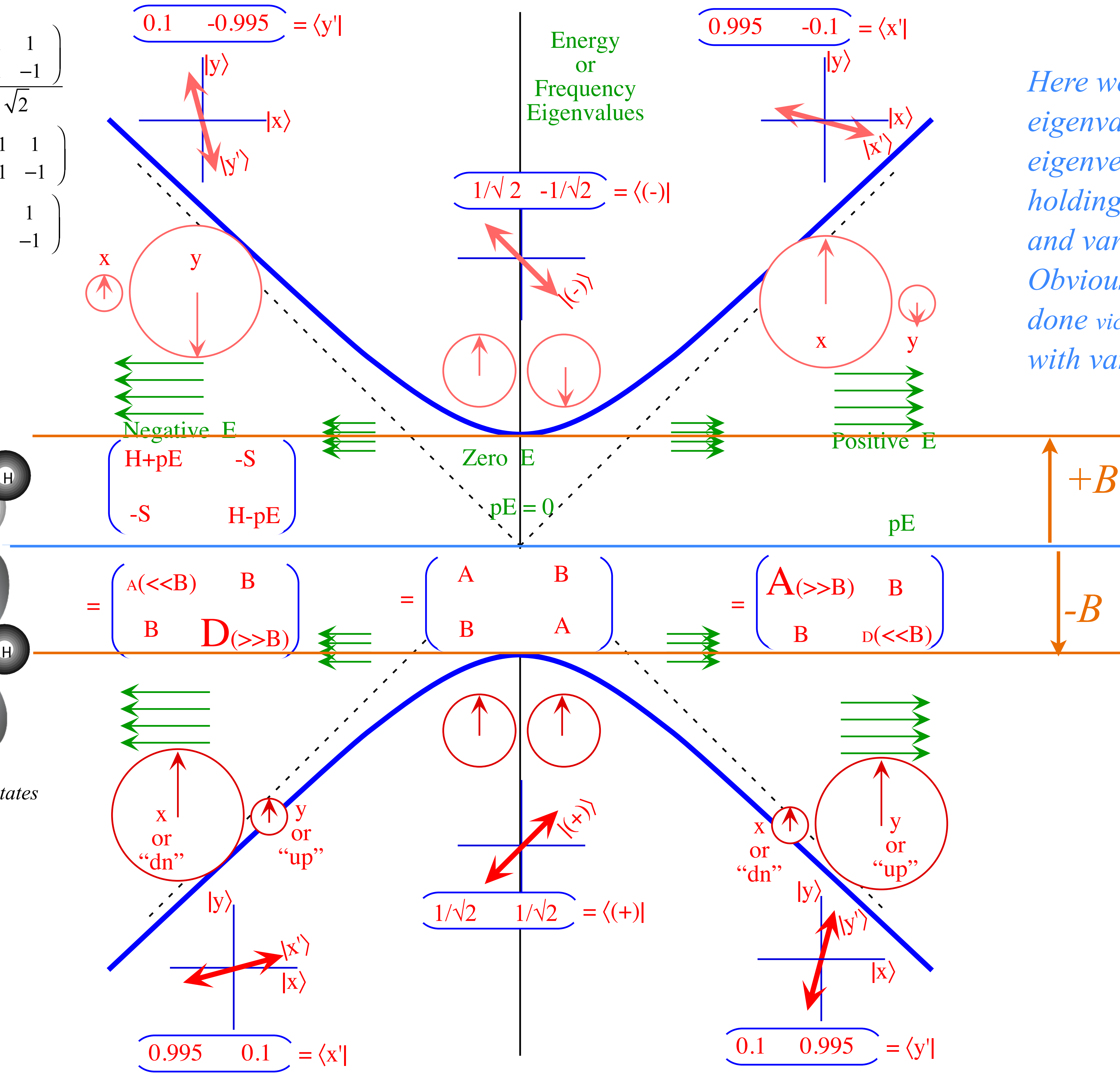


Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

(A=pE)-Axis
(Applied field)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \quad \text{Secular equation: } \varepsilon^2 - 0\cdot\varepsilon - (A^2 + B^2) \quad \text{gives } \textit{hyperbolic} \text{ energy levels: } \varepsilon = \pm\sqrt{A^2 + B^2}$$

$\mathbf{H}(B-basis)$ $\qquad$ $\mathbf{H}(A-basis)$

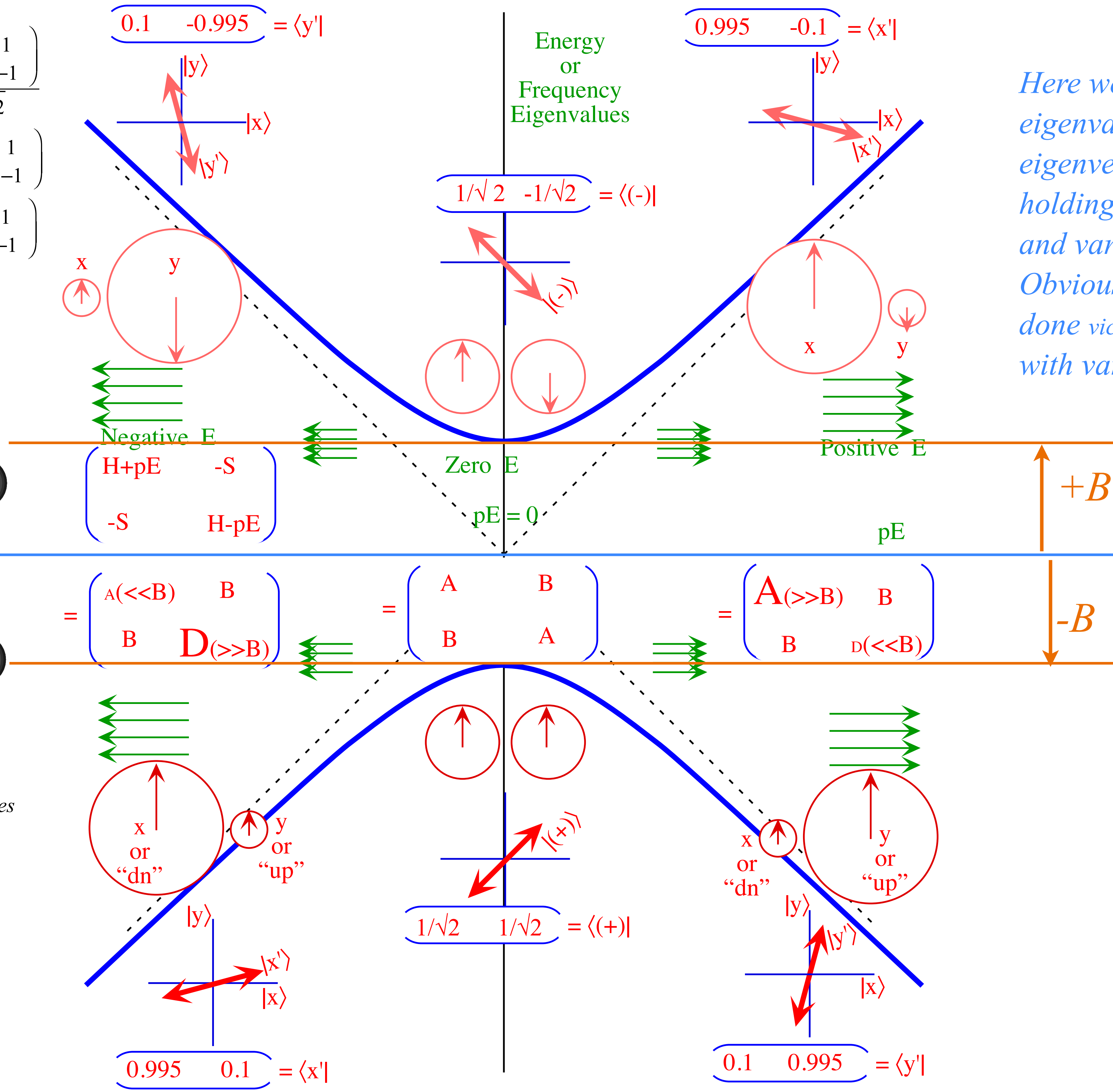
$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa *and with varying C, too.*



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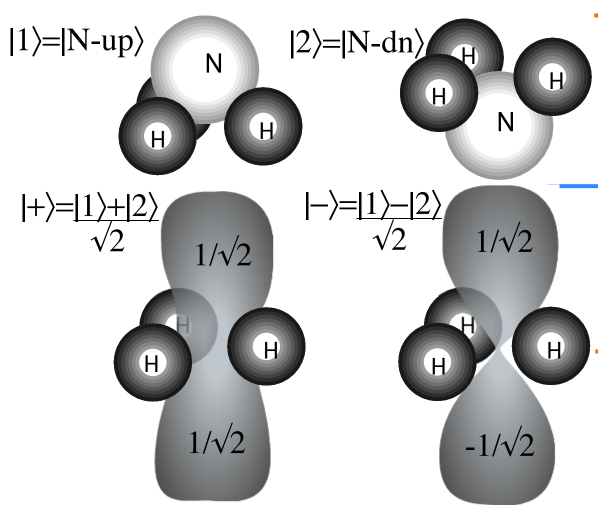

Fig. 10.3.2 Ammonia (NH_3) inversion states
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A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \quad \text{Secular equation: } \varepsilon^2 - 0\cdot\varepsilon - (A^2+B^2) \quad \text{gives } \textit{hyperbolic} \text{ energy levels: } \varepsilon = \pm\sqrt{A^2+B^2}$$

$\mathbf{H}(B-basis)$ $\qquad$ $\mathbf{H}(A-basis)$

$$\begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}} \begin{pmatrix} +A & B \\ B & -A \end{pmatrix} \frac{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} +A & B \\ B & -A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} +A+B & B-A \\ +A-B & B+A \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2B & 2A \\ 2A & -2B \end{pmatrix}$$

$$= \begin{pmatrix} +B & A \\ A & -B \end{pmatrix}$$

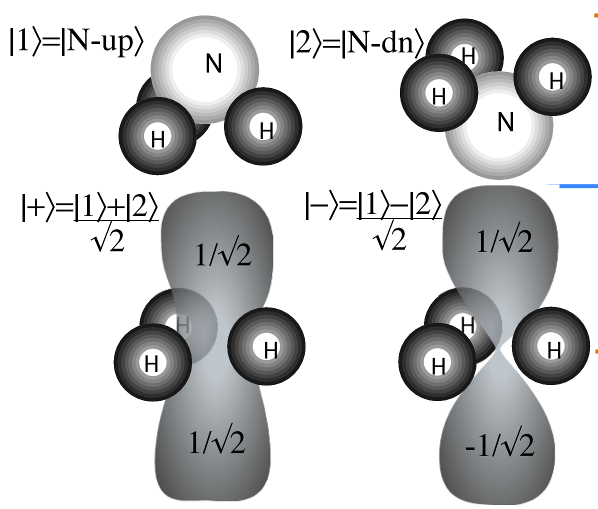
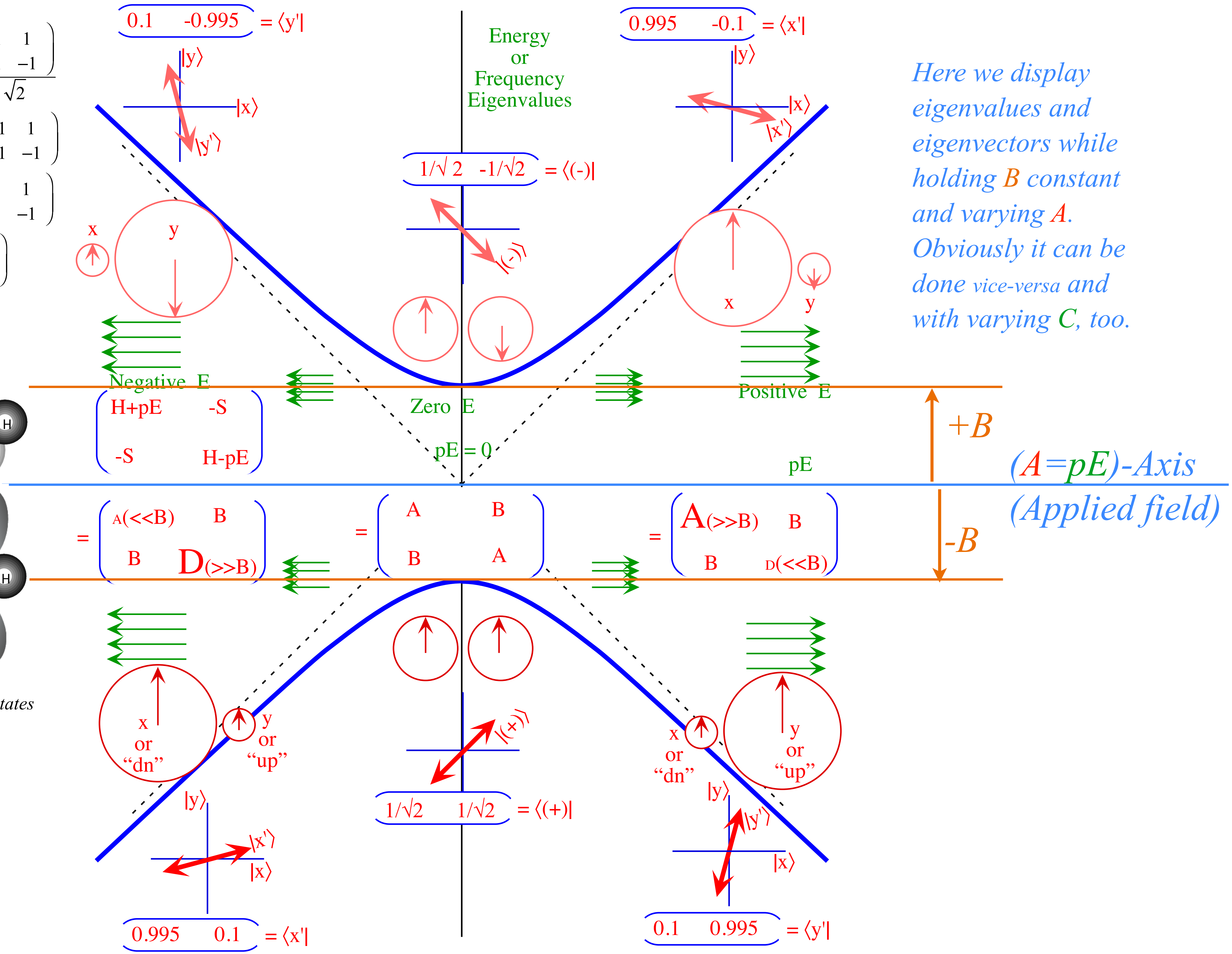


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with varying C, too.

Fig. 10.3.1 (b) Wigner avoided level crossing. (Fixed tunneling B=-S and variable A-D=pE field.)

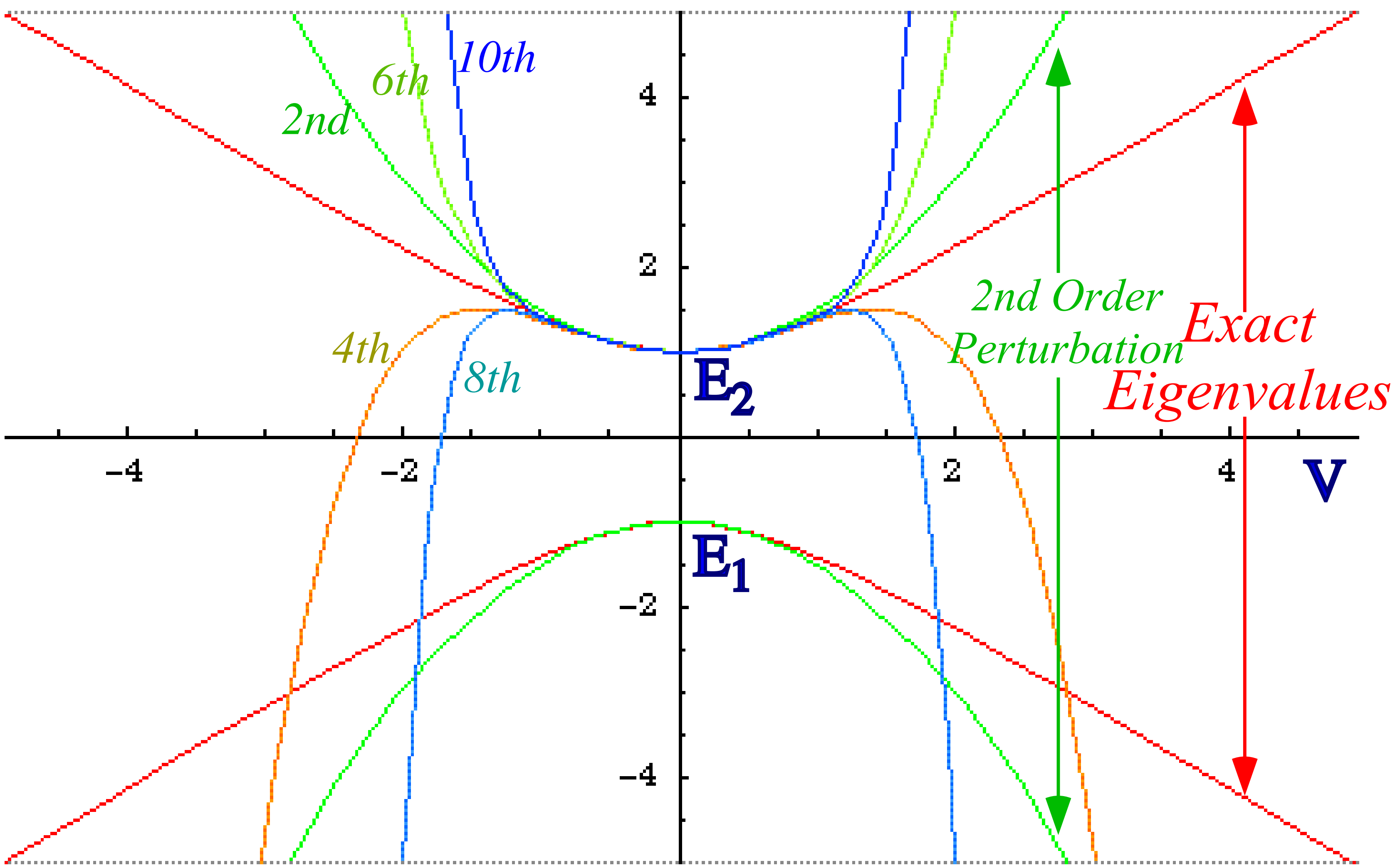
The failure of perturbation methods to get exact hyperbolic eigenvalues

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_1 & V \\ V & E_2 \end{pmatrix}$$

2nd order perturbation terms

$$\lambda_1 = E_1 + \frac{V^2}{E_1 - E_2} ,$$

$$\lambda_2 = E_2 + \frac{V^2}{E_2 - E_1} .$$



$$\lambda^2 - (Trace\mathbf{H})\lambda + \det|\mathbf{H}| = 0 = \lambda^2 - (E_1 + E_2)\lambda + (E_1 E_2 - V^2)$$

$$\lambda_{1,2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 + E_2)^2 - 4E_1E_2 + 4V^2}}{2} = \frac{E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4V^2}}{2} ,$$

Fig. 3.2.2 Comparison of exact vs. 2nd-order thru 10th-order perturbation approximations

$$E_2 = \frac{\Delta}{2} + \frac{V^2}{\Delta} - \frac{V^4}{\Delta^3} + \frac{V^6}{\Delta^5} - \frac{V^8}{\Delta^7} + \frac{V^{10}}{\Delta^9} \cdots \text{ , where: } \Delta = |E_1 - E_2|$$

A view of a conical intersection:

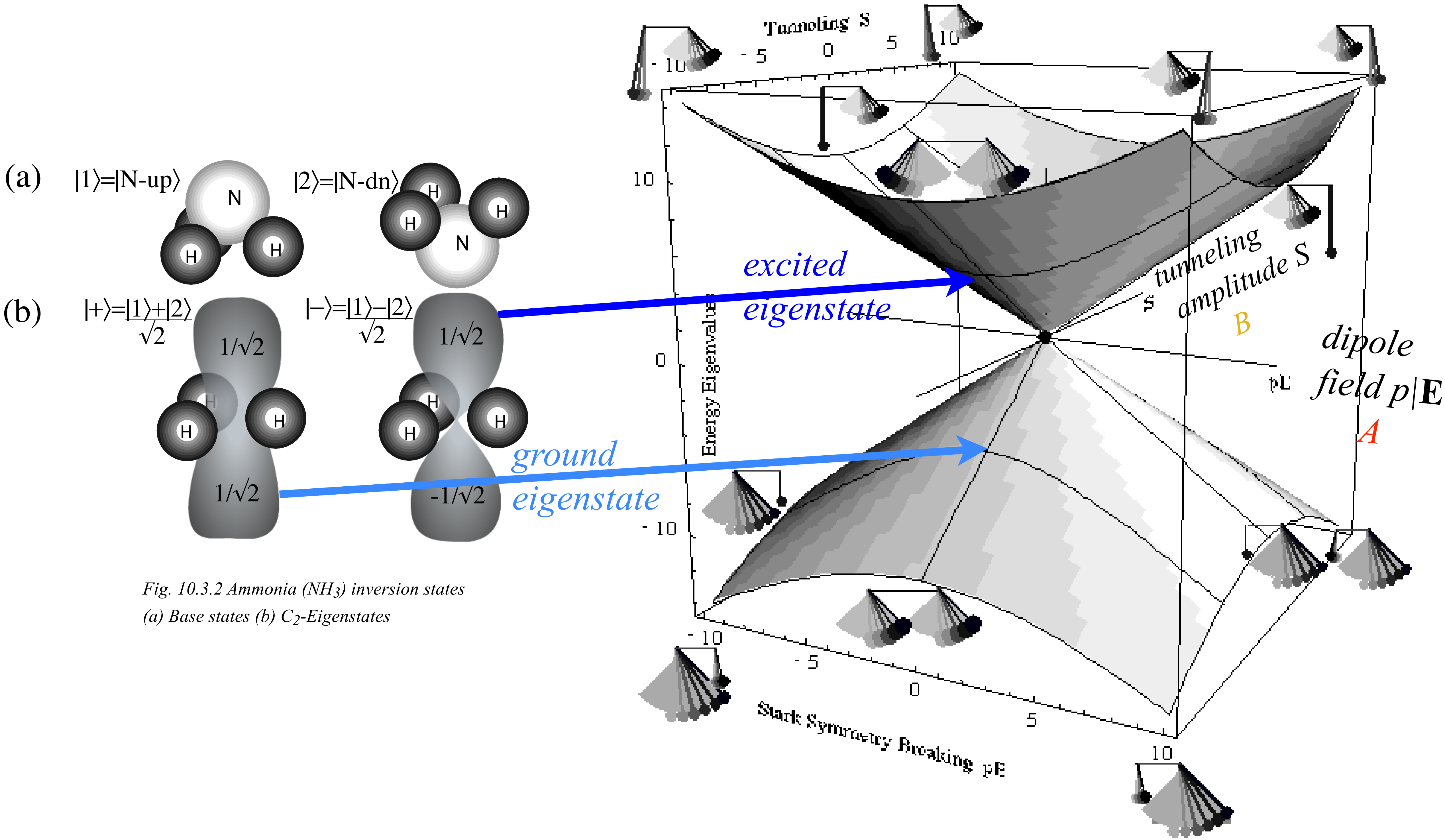


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates

10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

(Also known as a “Dirac-point”)

A view of a conical intersection: Any vertical cross-section is hyperbolic avoided-crossing

Once called a "Diablo"
Now called a "Dirac-Point"

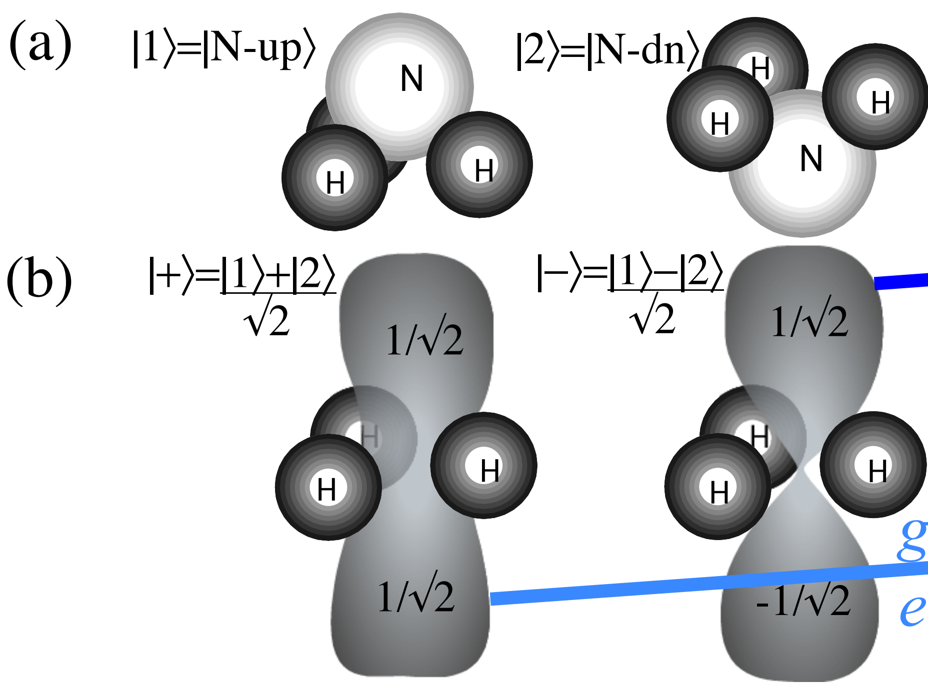
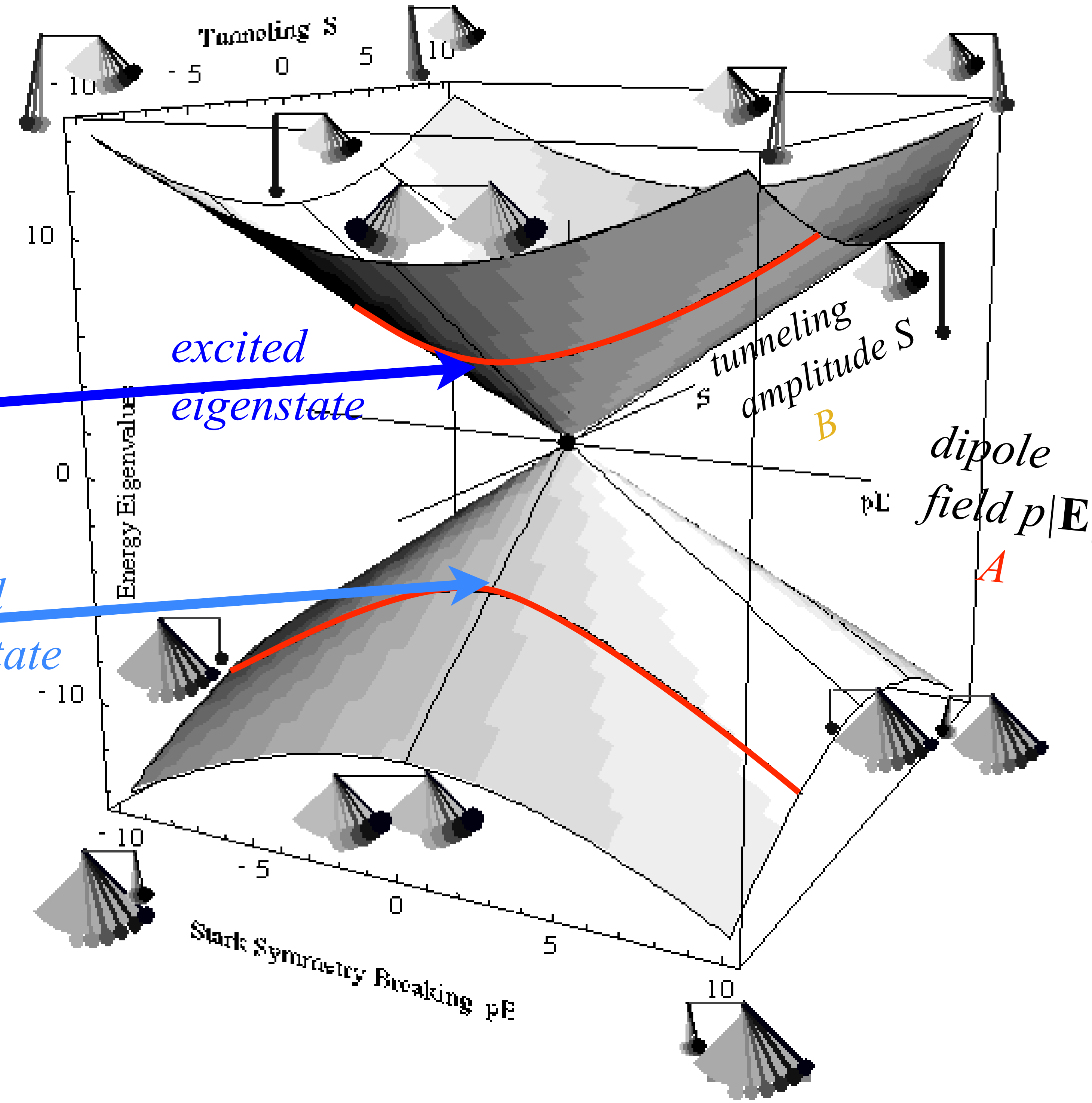


Fig. 10.3.2 Ammonia (NH_3) inversion states
(a) Base states (b) C_2-Eigenstates



10.3.1 (a) Two state eigenvalue "diablo" surfaces and conical intersection and pendulum eigenstates.

(Also known as a "Dirac-point")

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ and Darboux $\mathbf{R}[\varphi\vartheta\Theta]$ representations of U(2) and R(3)

Euler-defined state $|\alpha\beta\gamma\rangle$ described by Stoke's $\mathbf{S}$-vector, phasors, or ellipsometry
Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ and angular velocity $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$-vector
Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ derived from Darboux-defined $\mathbf{R}[\varphi\vartheta\Theta]$ and vice versa
Euler $\mathbf{R}(\alpha\beta\gamma)$ rotation $\Theta=0-4\pi$-sequence $[\varphi\vartheta]$ fixed (and "real-world" applications)

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes
Asymmetric-Diagonal A-Type motion
Bilateral-Balanced B-Type motion
Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes
AB-Type motion and Wigner's Avoided-Symmetry-Crossings
ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates
Conventional amp-phase ellipse coordinates
Euler Angle $(\alpha\beta\gamma)$ ellipse coordinates

ABC-Type elliptical polarized motion

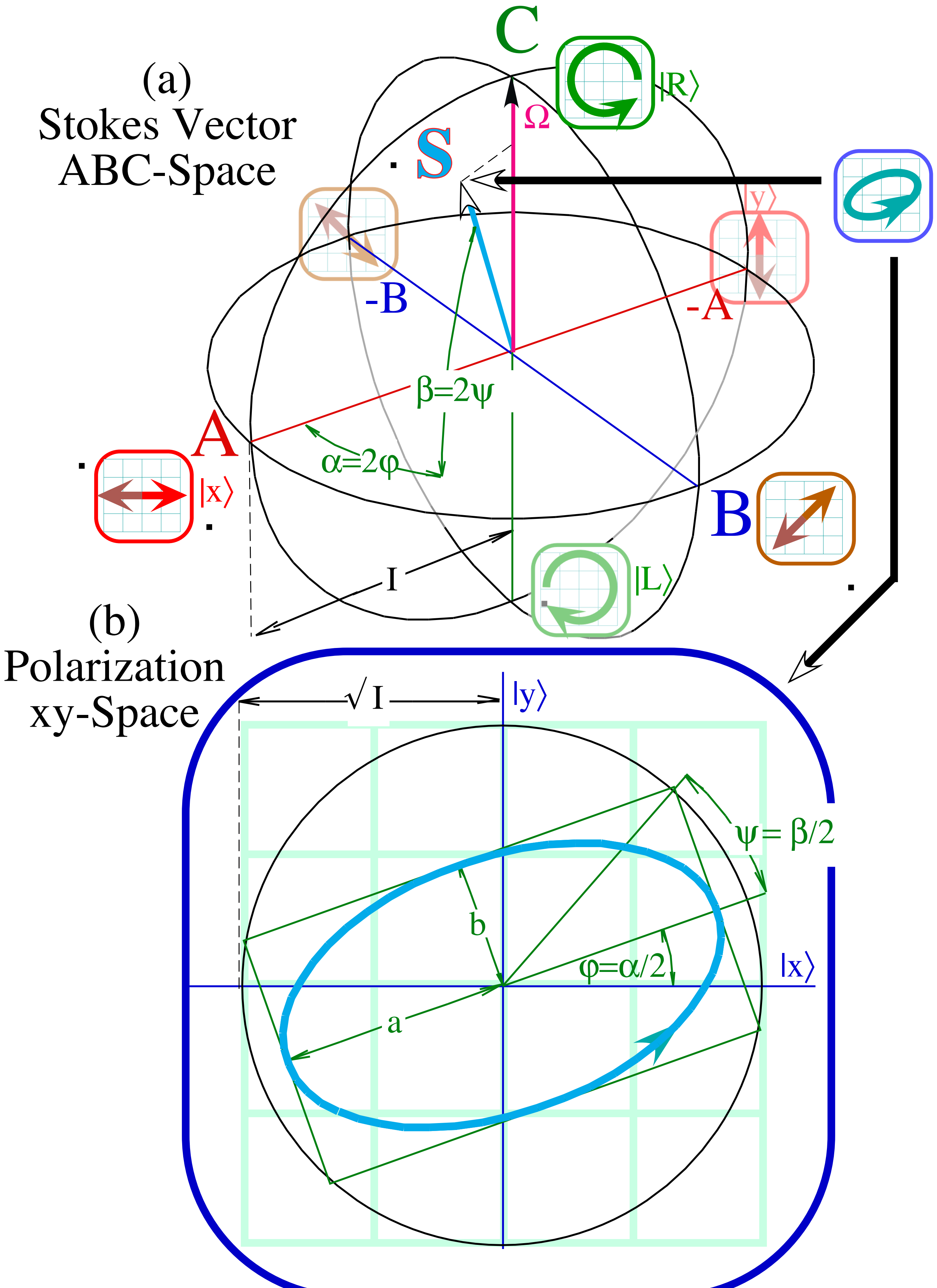
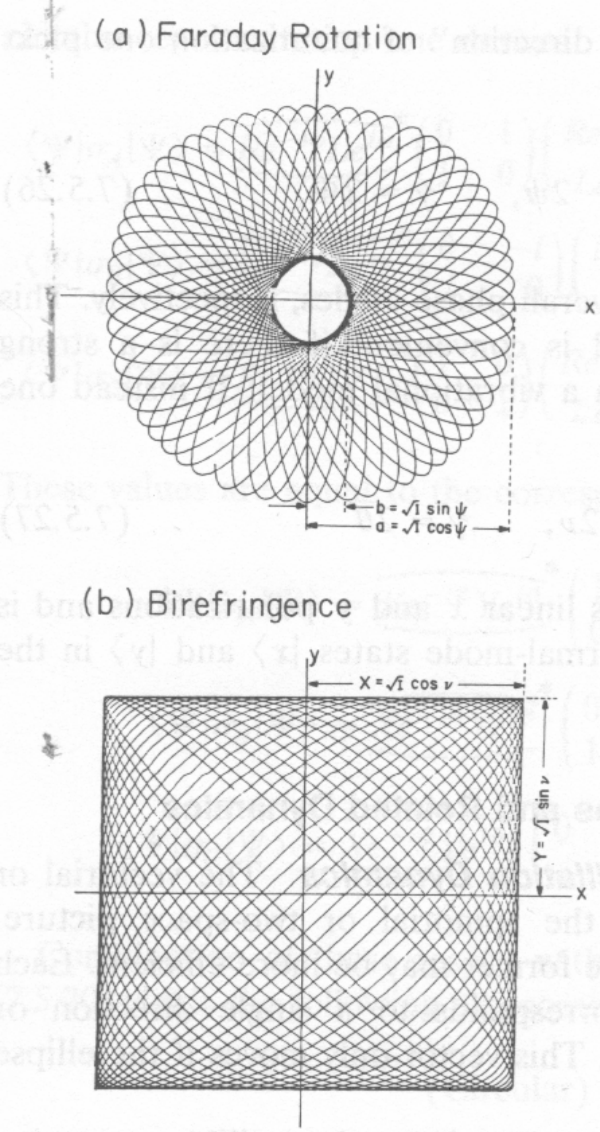


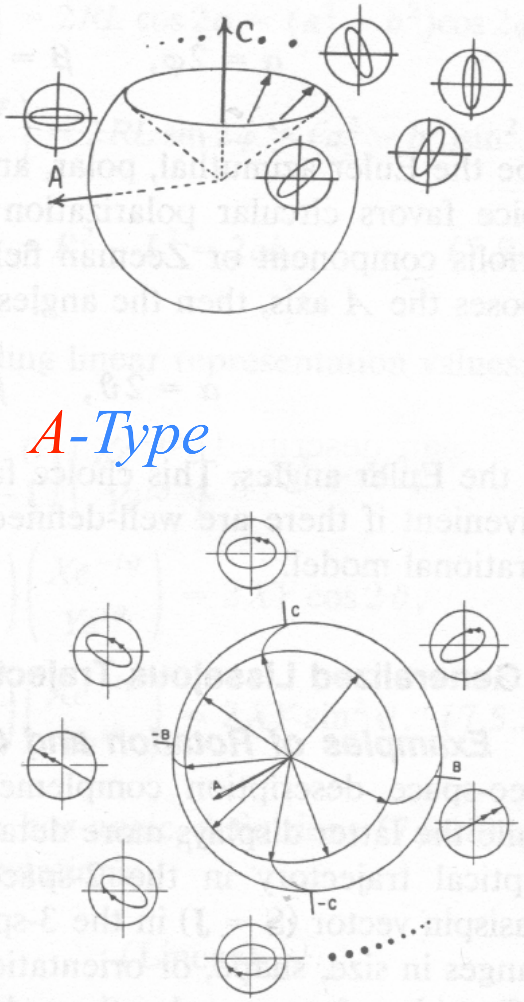
Fig. 10.B.3

Euler-like coordinates for
(a) R(3) spin vector
(b) U(2) polarization ellipse

ABC-Type elliptical polarized motion

(from Principles of Symmetry, Dynamics, and Spectroscopy)





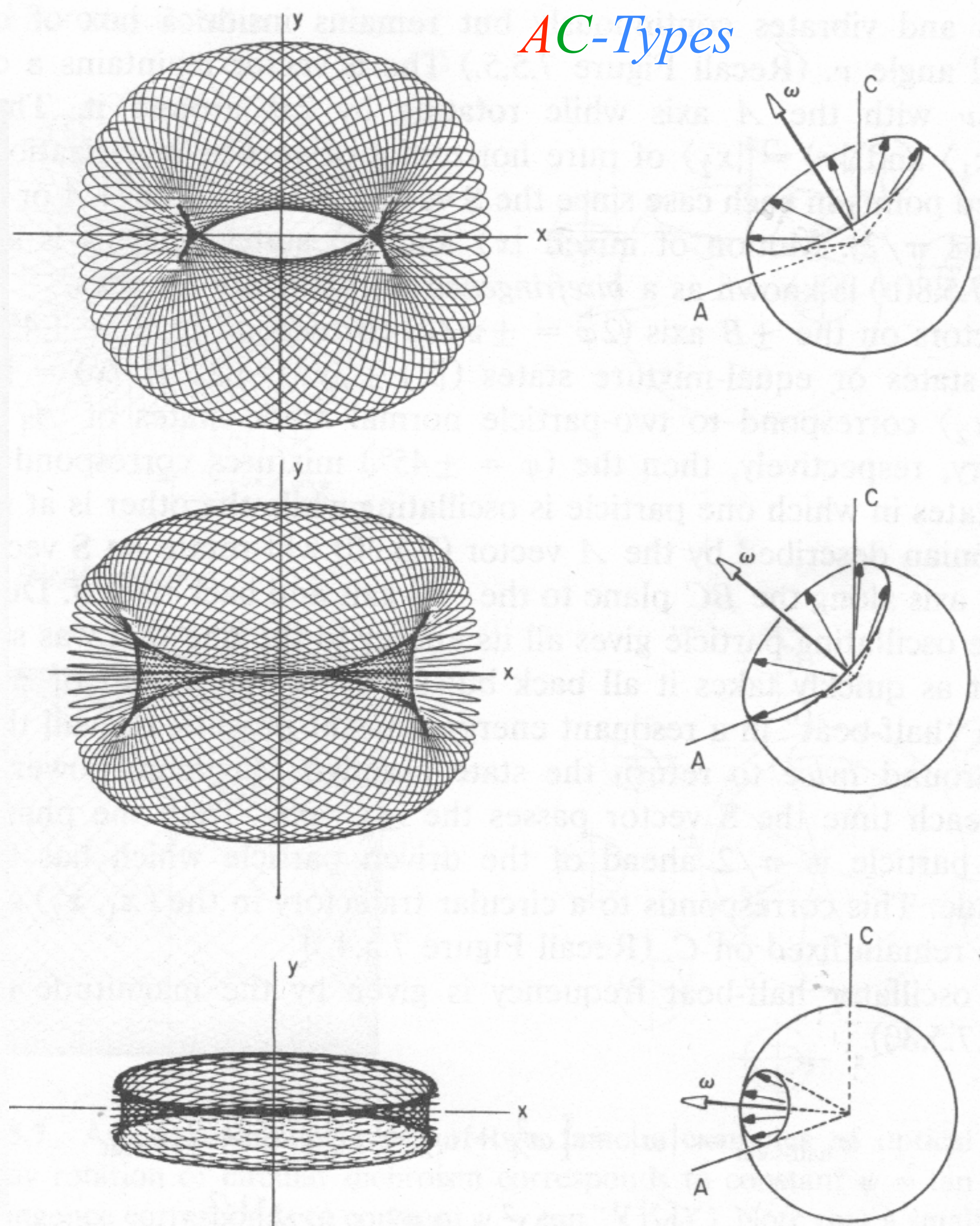
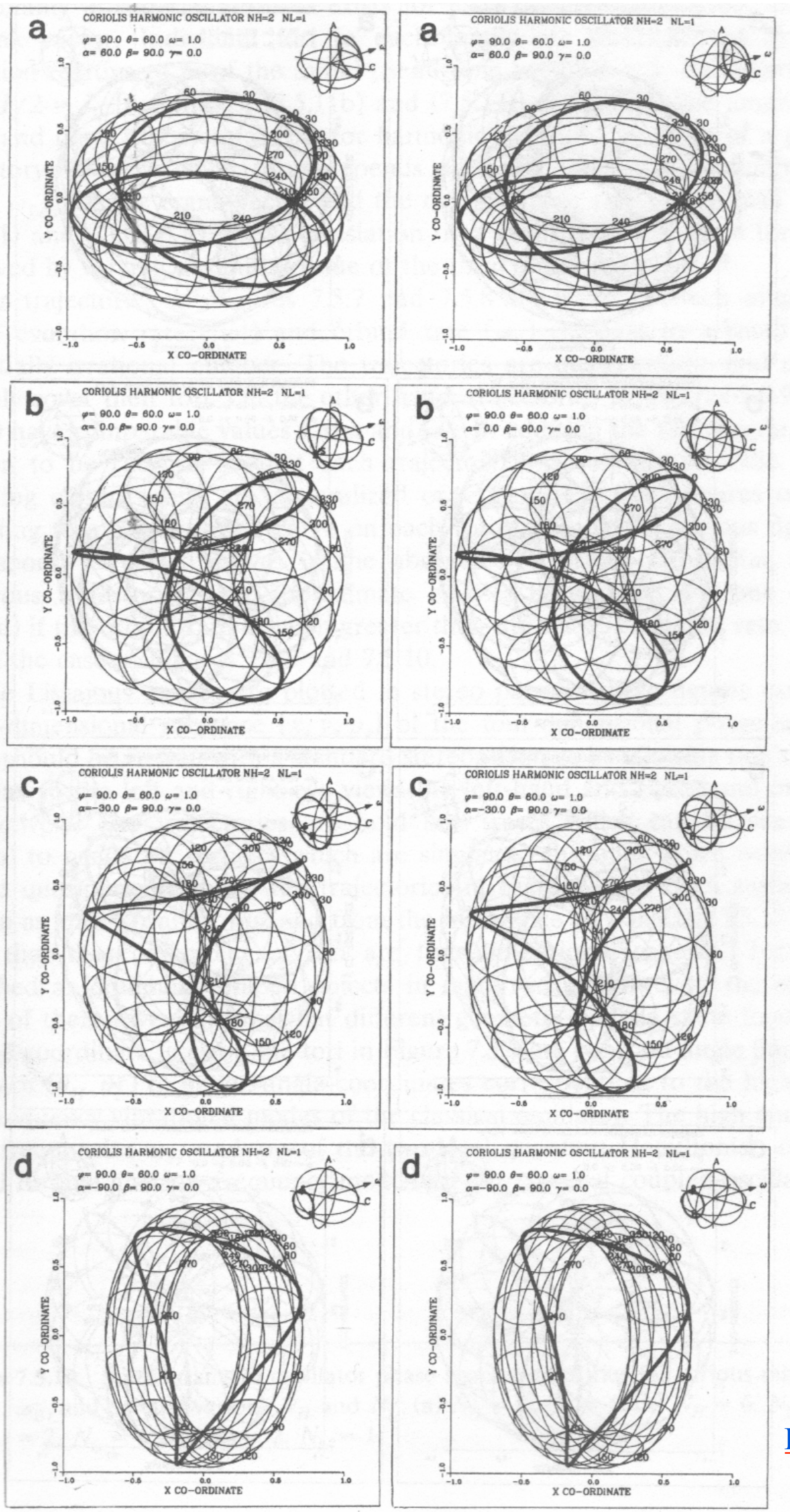
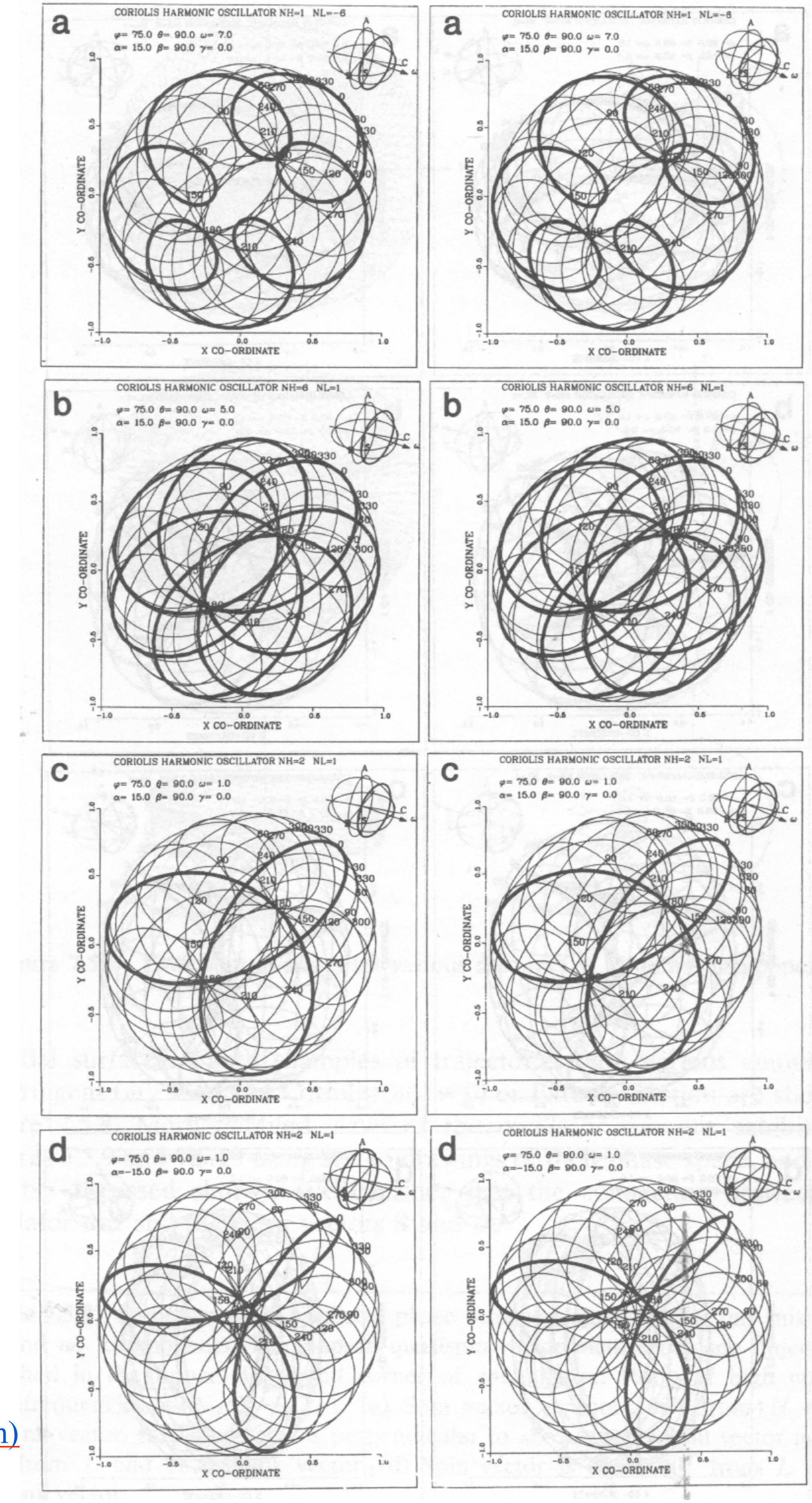


Figure 7.5.7 Analog computer plots of two famous examples of optical activity. (a) Faraday rotation or circular dichroism corresponds to constant $\psi = \tan^{-1}(b/a)$. (b) Birefringence corresponds to constant $\nu = \tan^{-1}(Y/X)$. Note that a small amount of birefringence is present in Figure 7.11(a); i.e., ψ oscillates slightly. Pure Faraday rotation is difficult to achieve on an analog computer.

7.5.8 Evolution of states for various mixtures of A and C components.

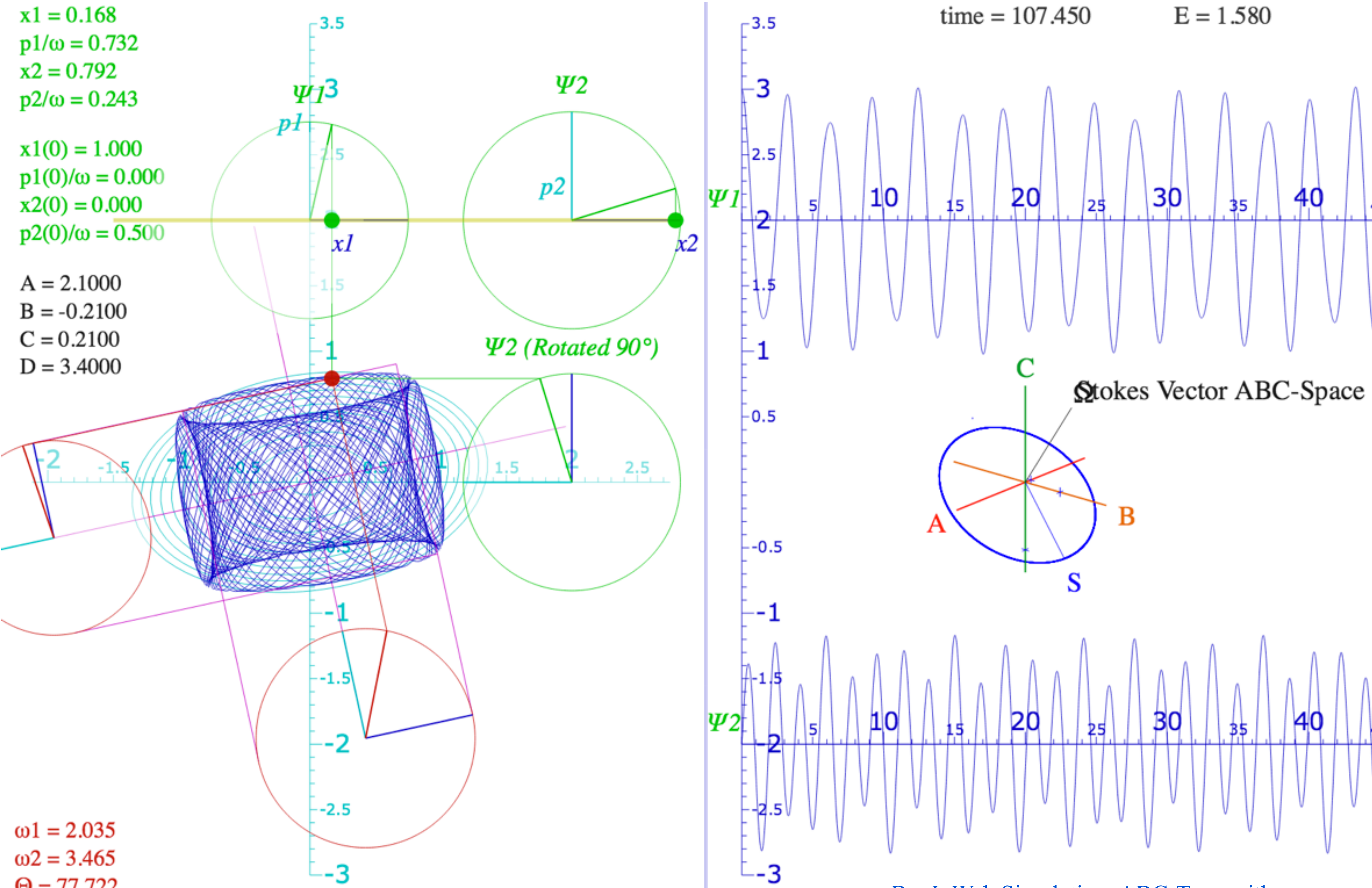


ABC-Type elliptical polarized dynamics



BoxIt (ABC-Motion)
Web Simulation

ABC-Type elliptical polarized motion



[BoxIt Web Simulation: ABC-Type with A=2.1; B=-0.21; C=0.21; D=3.4]

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\boldsymbol{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\boldsymbol{\Omega}(\varphi\vartheta)\cdot t=\boldsymbol{\Theta}$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\boldsymbol{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

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Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates and related to Euler Angles ($\alpha\beta\gamma$)

2D elliptic frequency ω orbit has amplitudes

A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

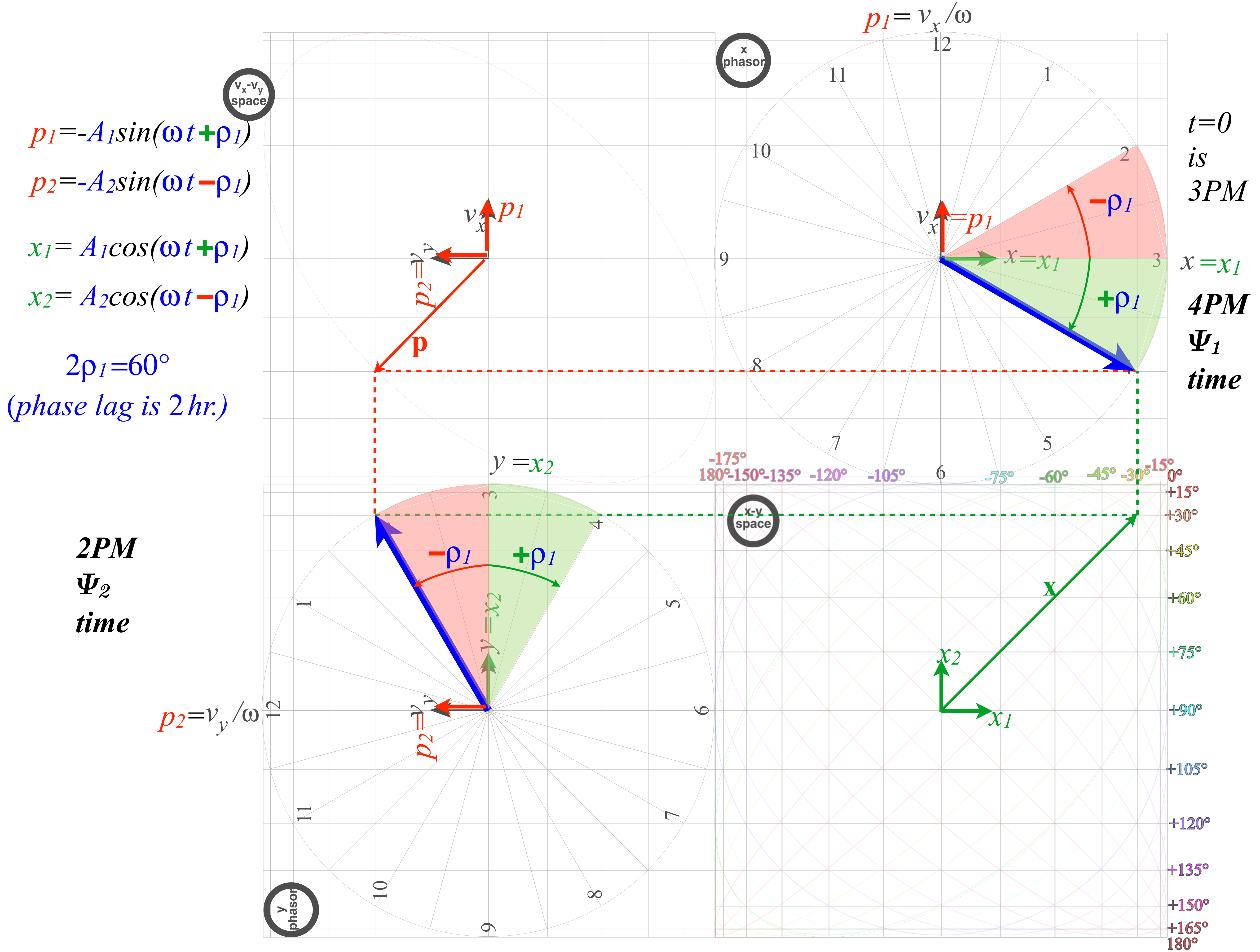
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$

$$x_2 = A_2 \cos(\omega t - \rho_1)$$

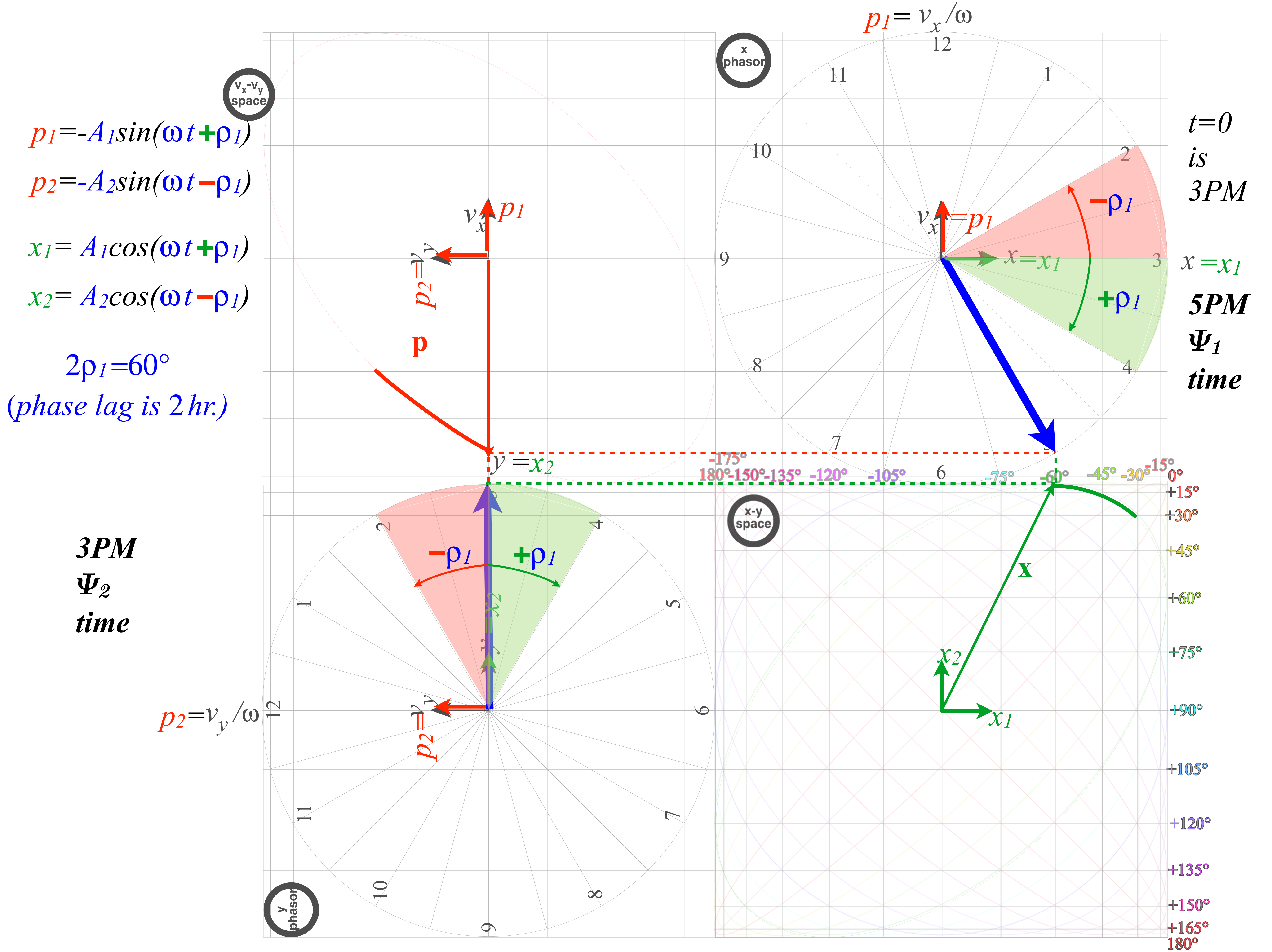
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Amp-phase parameters ($A_1, A_2, \omega t, \rho_1$)

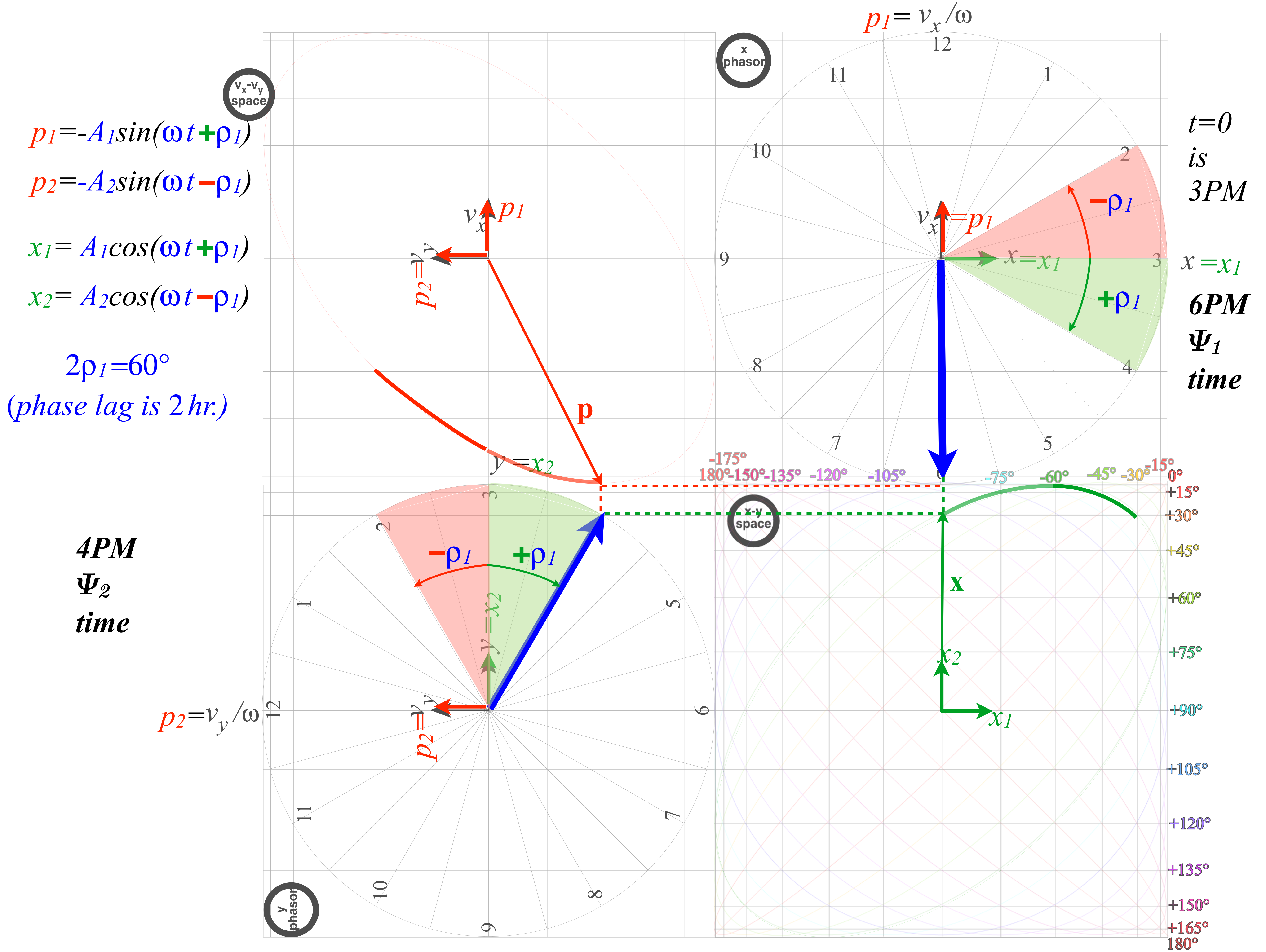
$$\begin{pmatrix} A_1 e^{-i(\omega t + \rho_1)} \\ A_2 e^{-i(\omega t - \rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + i p_1 \\ \\ x_2 + i p_2 \end{pmatrix}$$



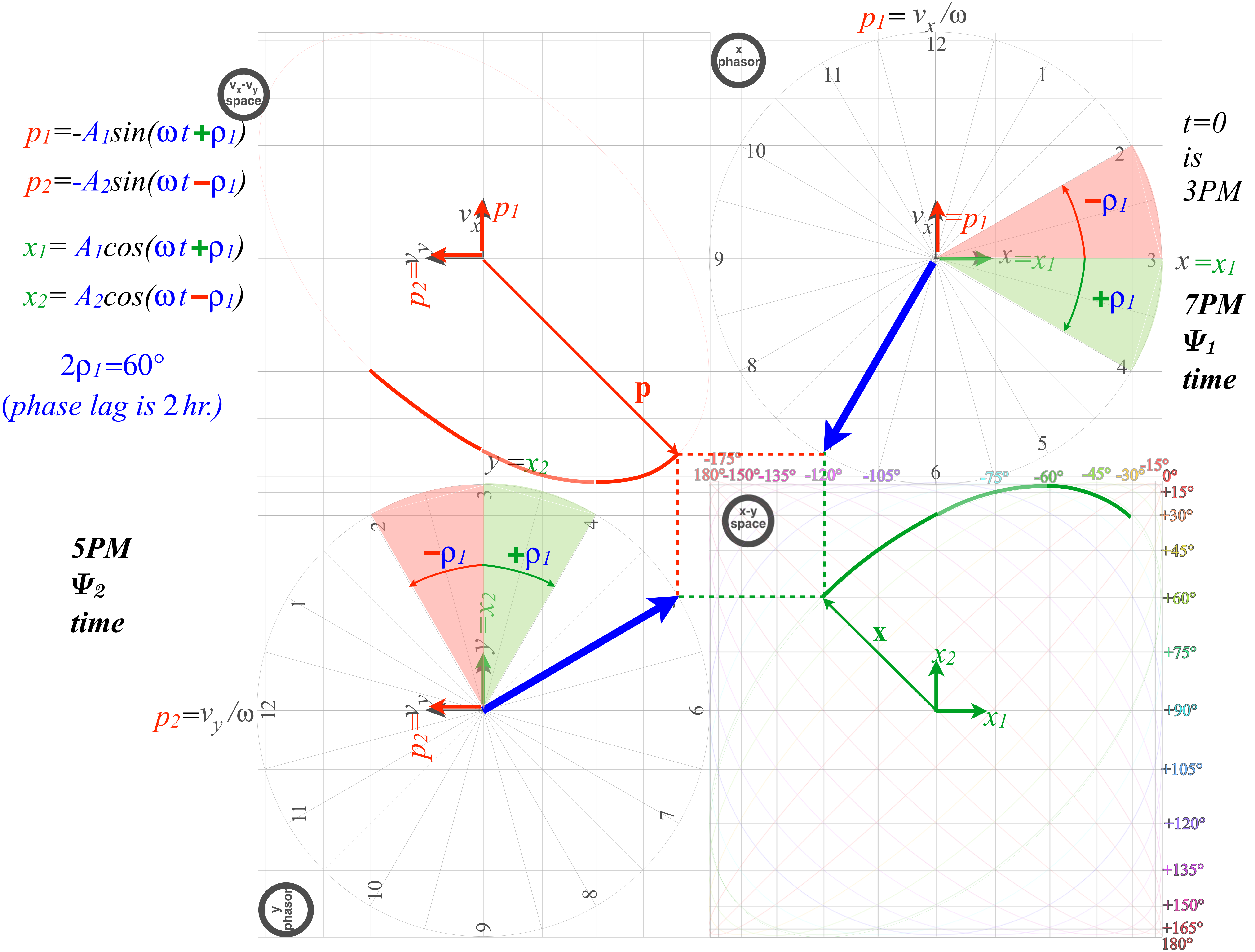
$p_1 = v_x/\omega$
x phasor
v_x-v_y space
p_1=-$A_1 sin(\omega t+\rho_1)$
p_2=-$A_2 sin(\omega t-\rho_1)$
$x_1 = A_1 cos(\omega t+\rho_1)$
$x_2 = A_2 cos(\omega t-\rho_1)$
$2\rho_1$=60°
(phase lag is 2 hr.)
v_x p_1
v_y
$p_2=$
p
$v_x=p_1$
$x=x_1$
$-\rho_1$
$+\rho_1$
t=0
is
3PM
4PM
Ψ_1
time
$y=x_2$
2PM
Ψ_2
time
$p_2=v_y/\omega$
y phasor
x-y space
x
x_2
x_1
-175° 180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°



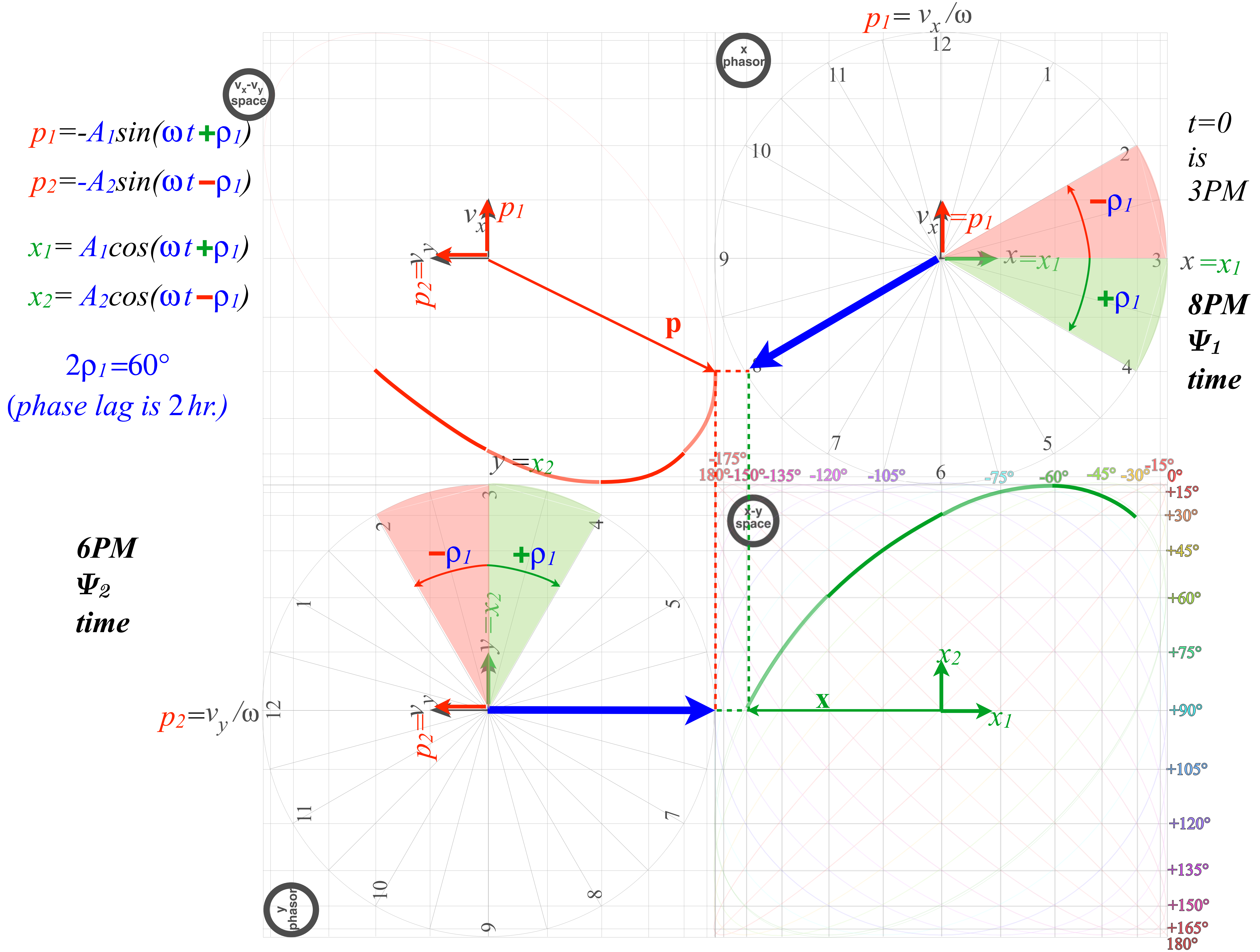
p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
x-y space
y phasor
p1= vx/ω
p2=vy/ω
t=0
is
3PM
5PM
Ψ1
time
3PM
Ψ2
time
−ρ1
+ρ1
p
x
x =x1
y =x2
vx=p1
p2=vy
x1
x2



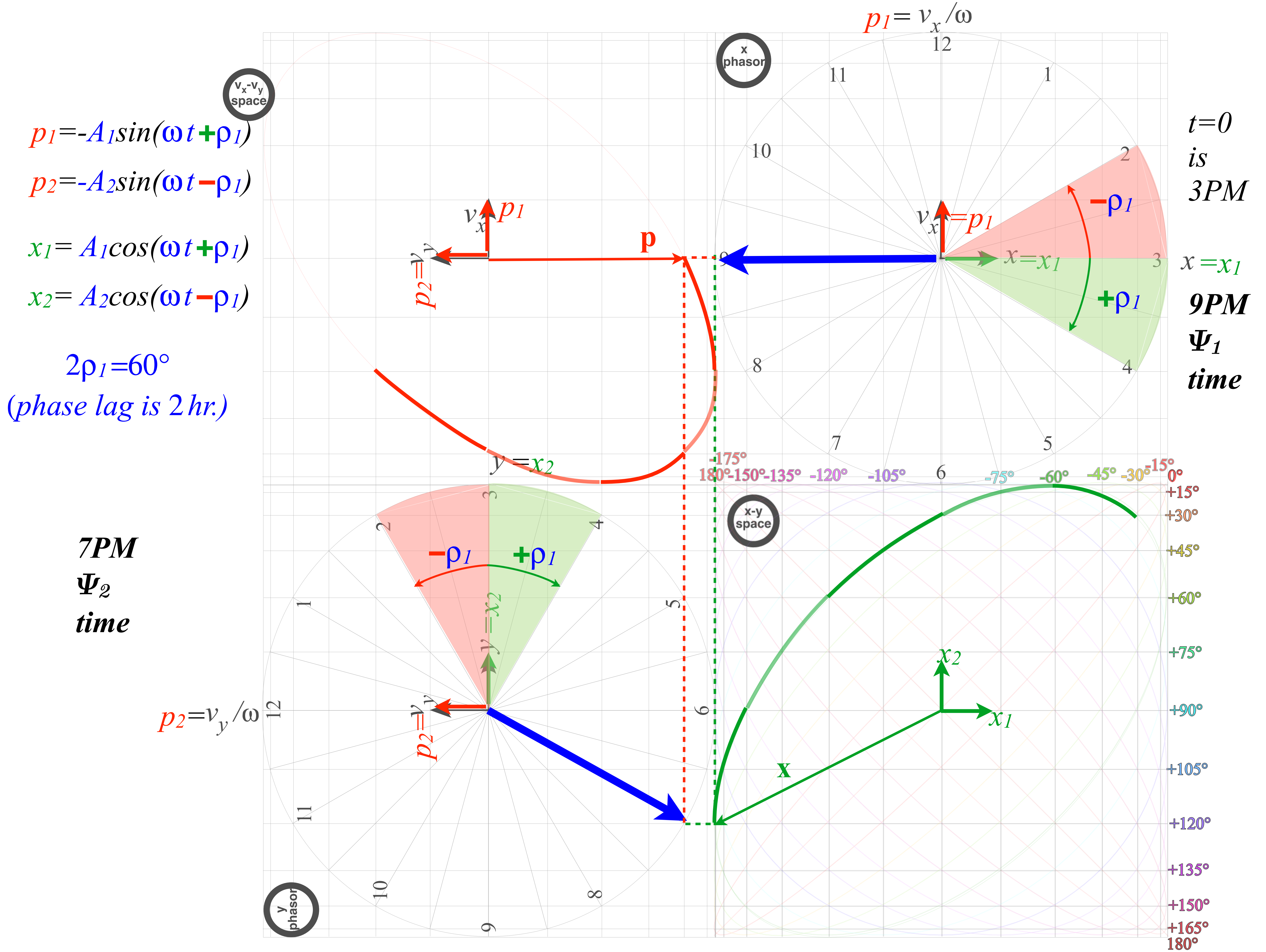
p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0
is
3PM
6PM
Ψ1
time
4PM
Ψ2
time
−ρ1
+ρ1
x=x1
y=x2
p
x
x1
x2
vx=p1
p2=vy
12
1
2
3
4
5
6
7
8
9
10
11
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°



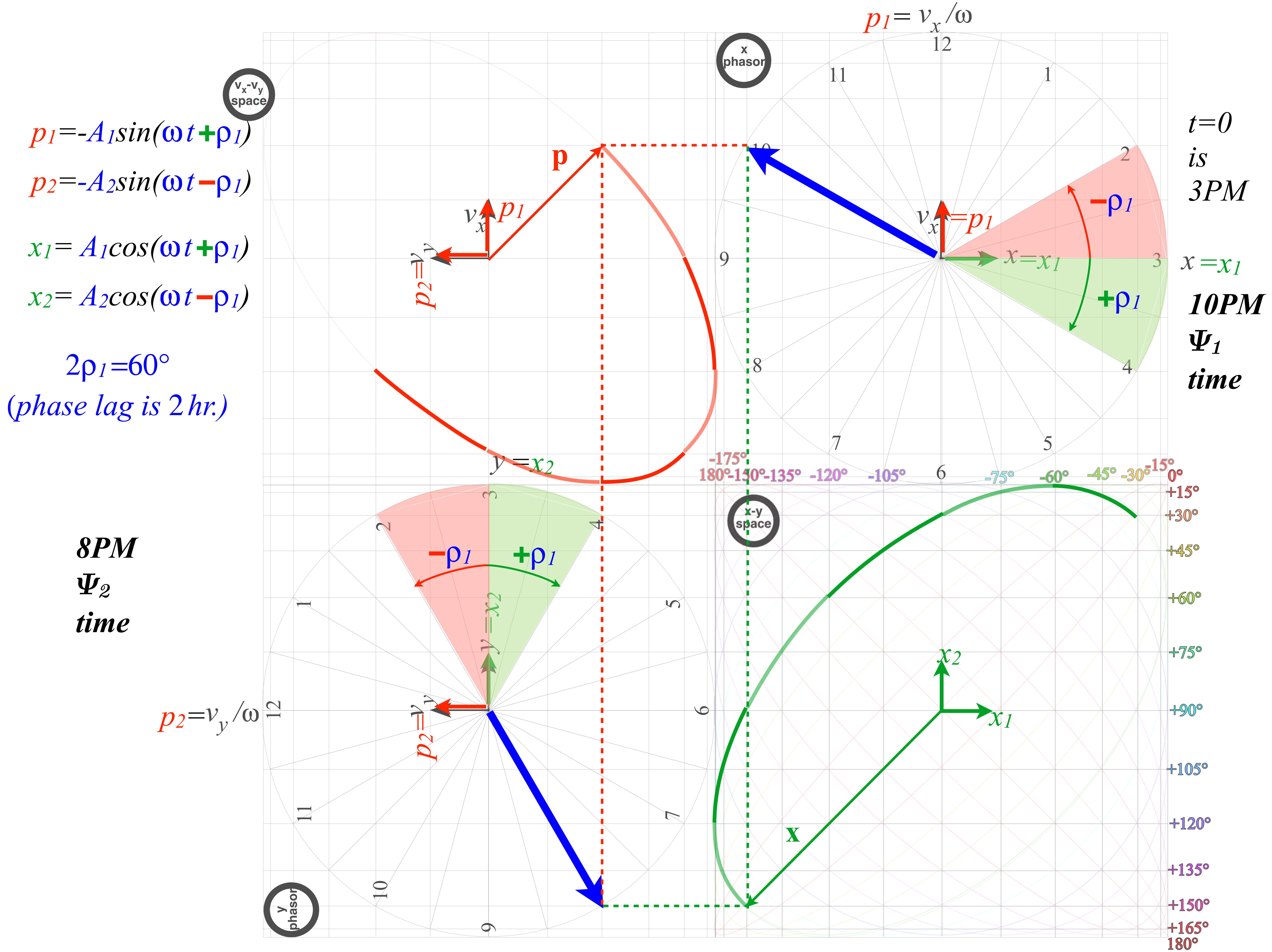
p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0
is
3PM
7PM
Ψ1
time
5PM
Ψ2
time
−ρ1
+ρ1
x =x1
y =x2
p
x
x1
x2
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°



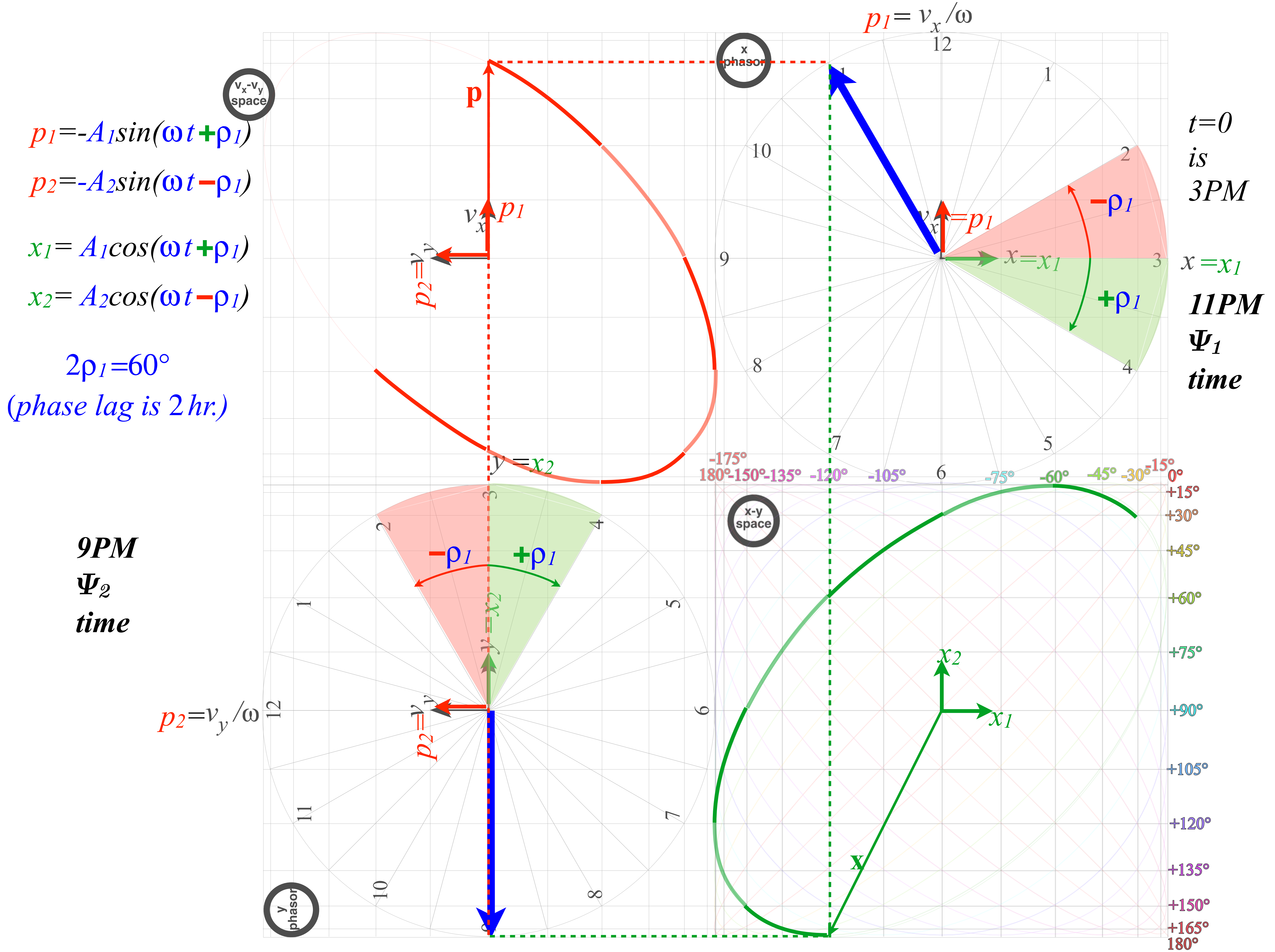
p1= vx /ω
x phasor
vx-vy space
p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
t=0
is
3PM
8PM
Ψ1
time
−ρ1
+ρ1
x =x1
vx=p1
p
y=x2
x-y space
6PM
Ψ2
time
p2=vy /ω
y phasor
x
x1
x2



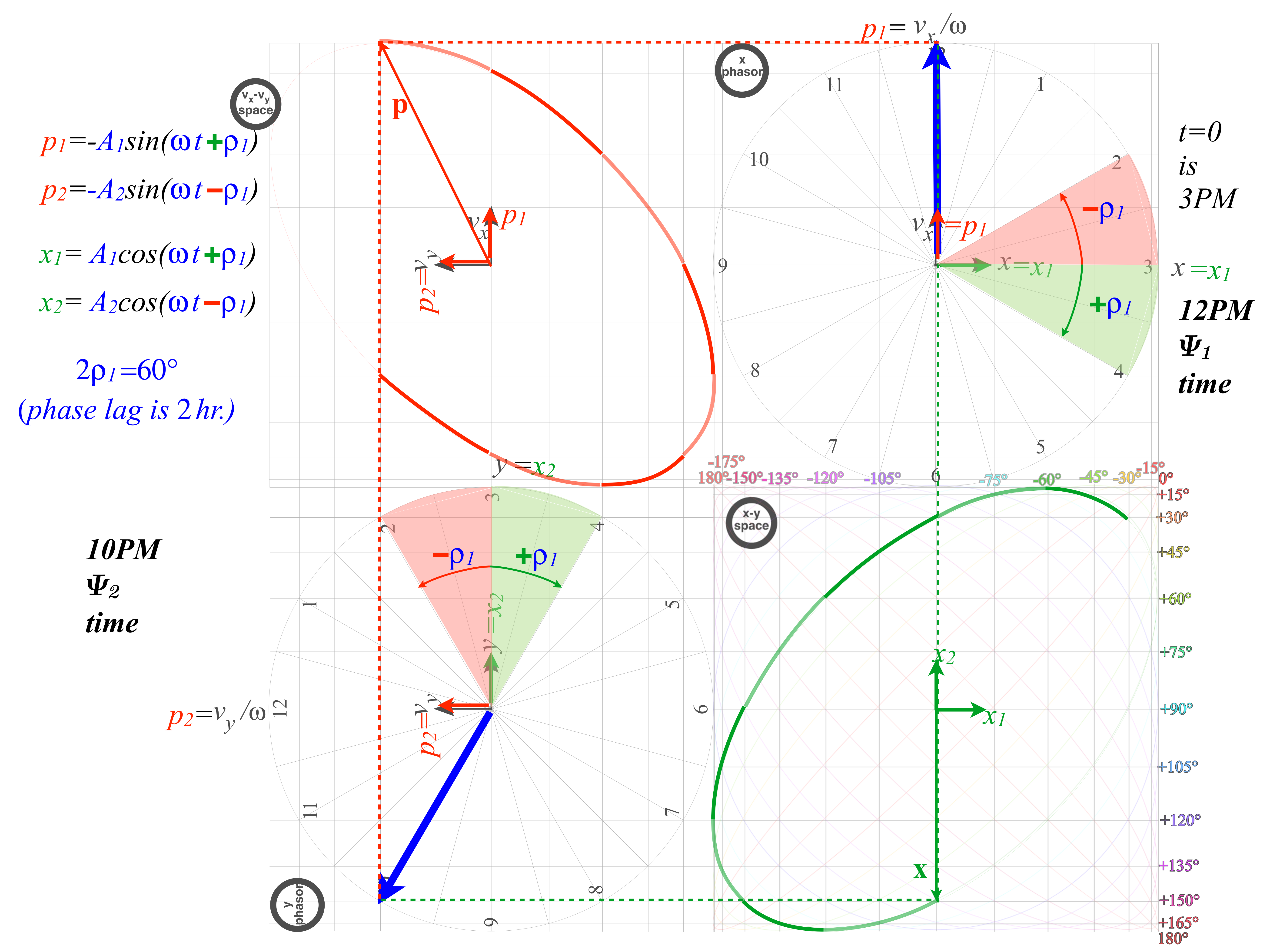
p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0
is
3PM
9PM
Ψ1
time
7PM
Ψ2
time
x =x1
y=x2
−ρ1
+ρ1
p
x
x1
x2



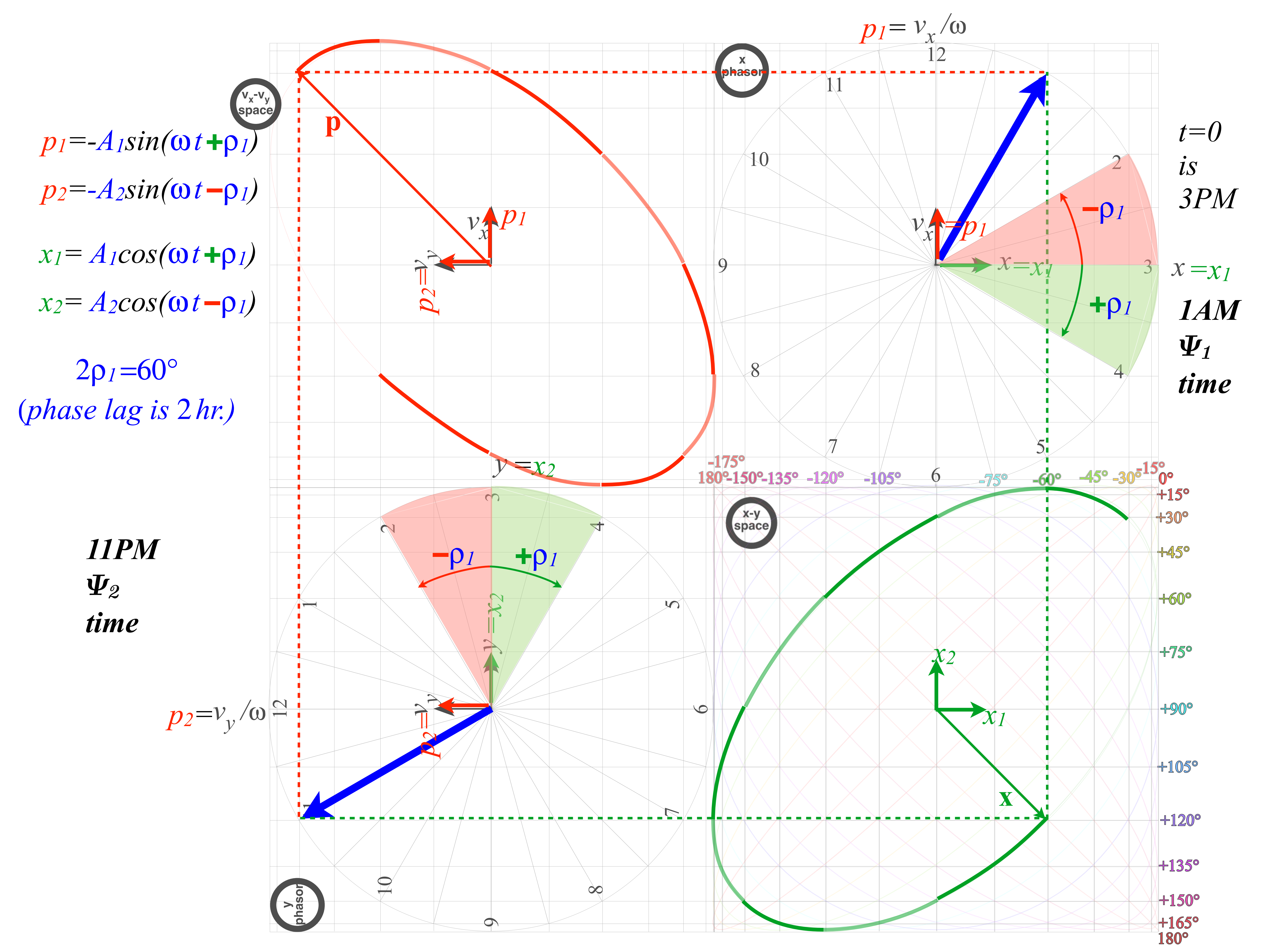
$p_1=-A_1\sin(\omega t+\rho_1)$
$p_2=-A_2\sin(\omega t-\rho_1)$
$x_1= A_1\cos(\omega t+\rho_1)$
$x_2= A_2\cos(\omega t-\rho_1)$
$2\rho_1=60°$
(phase lag is 2 hr.)
v_x-v_y space
x phasor
$p_1= v_x/\omega$
t=0 is 3PM
$-\rho_1$
$+\rho_1$
$x=x_1$
10PM
Ψ_1 time
p
v_x
p_1
$p_2=v_y$
$y=x_2$
8PM
Ψ_2 time
$p_2=v_y/\omega$
x-y space
x_2
x_1
x
y phasor
-175°
180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°



p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
p1= vx/ω
t=0
is
3PM
11PM
Ψ1
time
−ρ1
+ρ1
x=x1
y=x2
9PM
Ψ2
time
p2=vy/ω
y phasor
x-y space
x1
x2
X
p



$p_1=-A_1\sin(\omega t+\rho_1)$
$p_2=-A_2\sin(\omega t-\rho_1)$
$x_1= A_1\cos(\omega t+\rho_1)$
$x_2= A_2\cos(\omega t-\rho_1)$
$2\rho_1=60°$
(phase lag is 2 hr.)
v_x-v_y space
x phasor
y phasor
x-y space
$p_1= v_x/\omega$
$p_2=v_y/\omega$
p
$v_x=p_1$
$p_2=v_y$
$y=x_2$
$x=x_1$
$-\rho_1$
$+\rho_1$
t=0
is
3PM
12PM
Ψ_1
time
10PM
Ψ_2
time
x
x_1
x_2
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
+15°
+30°
+45°
+60°
+75°
+90°
+105°
+120°
+135°
+150°
+165°
180°



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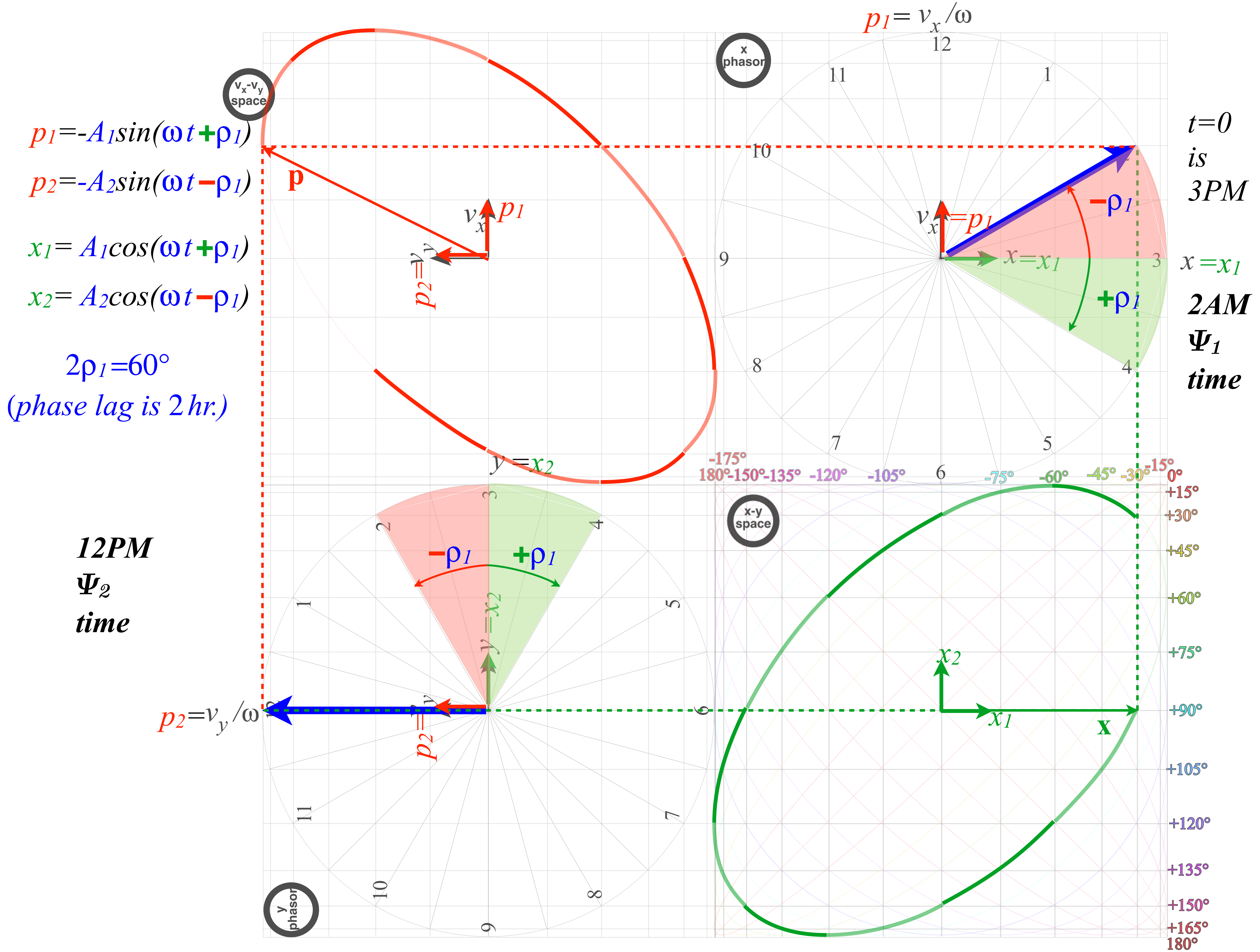
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0
is
3PM
1AM
Ψ1
time
11PM
Ψ2
time
−ρ1
+ρ1
x =x1
y =x2



vx-vy space
x phasor
y phasor
x-y space
p1=-A1sin(ωt+ρ1)
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x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
p1= vx/ω
t=0
is
3PM
x =x1
2AM
Ψ1
time
12PM
Ψ2
time
p
p2=vy/ω
y=x2
x1
x2
x
−ρ1
+ρ1
-175°
180°
-150°
-135°
-120°
-105°
-75°
-60°
-45°
-30°
-15°
0°
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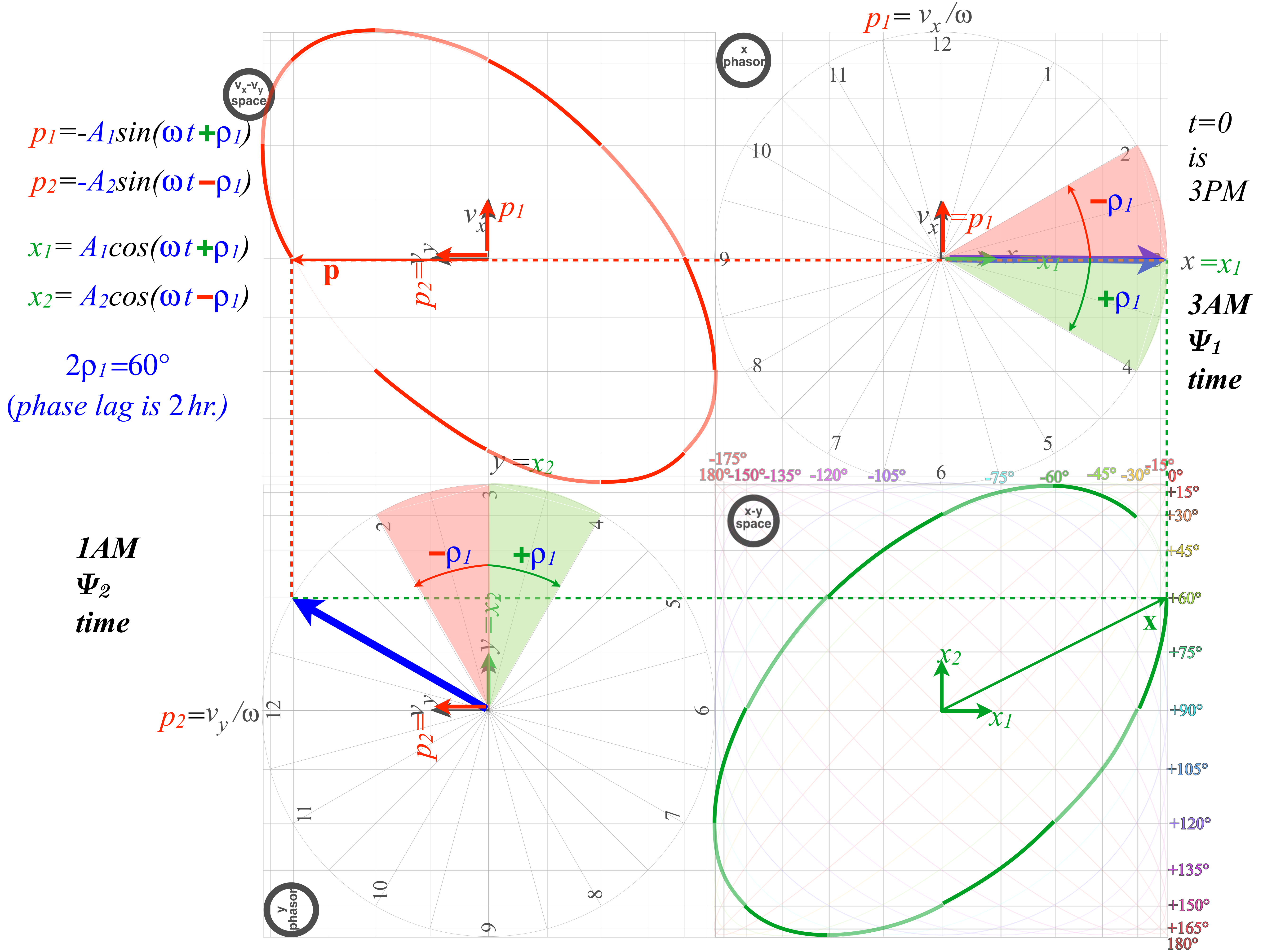
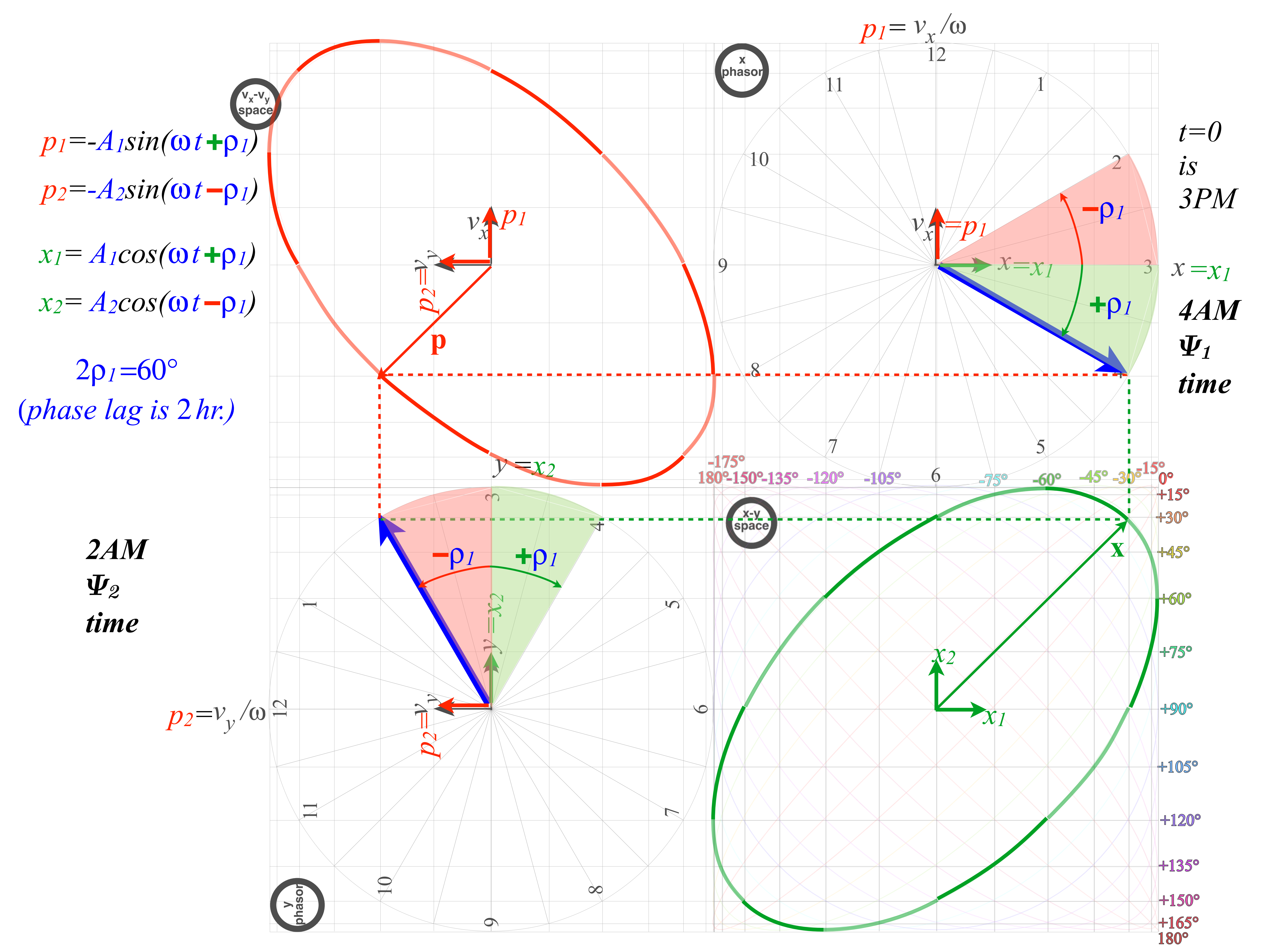
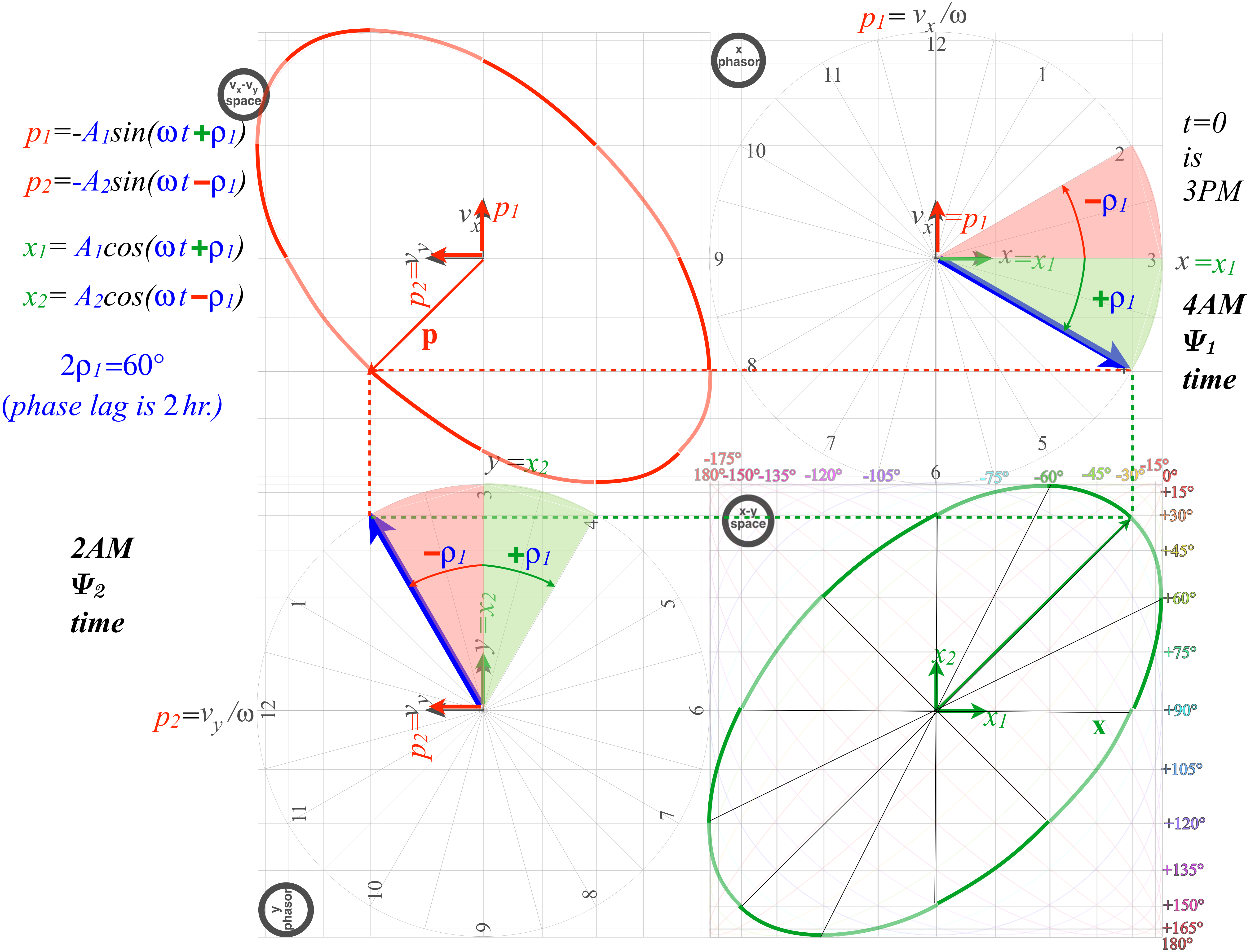
+150°
+165°
180°



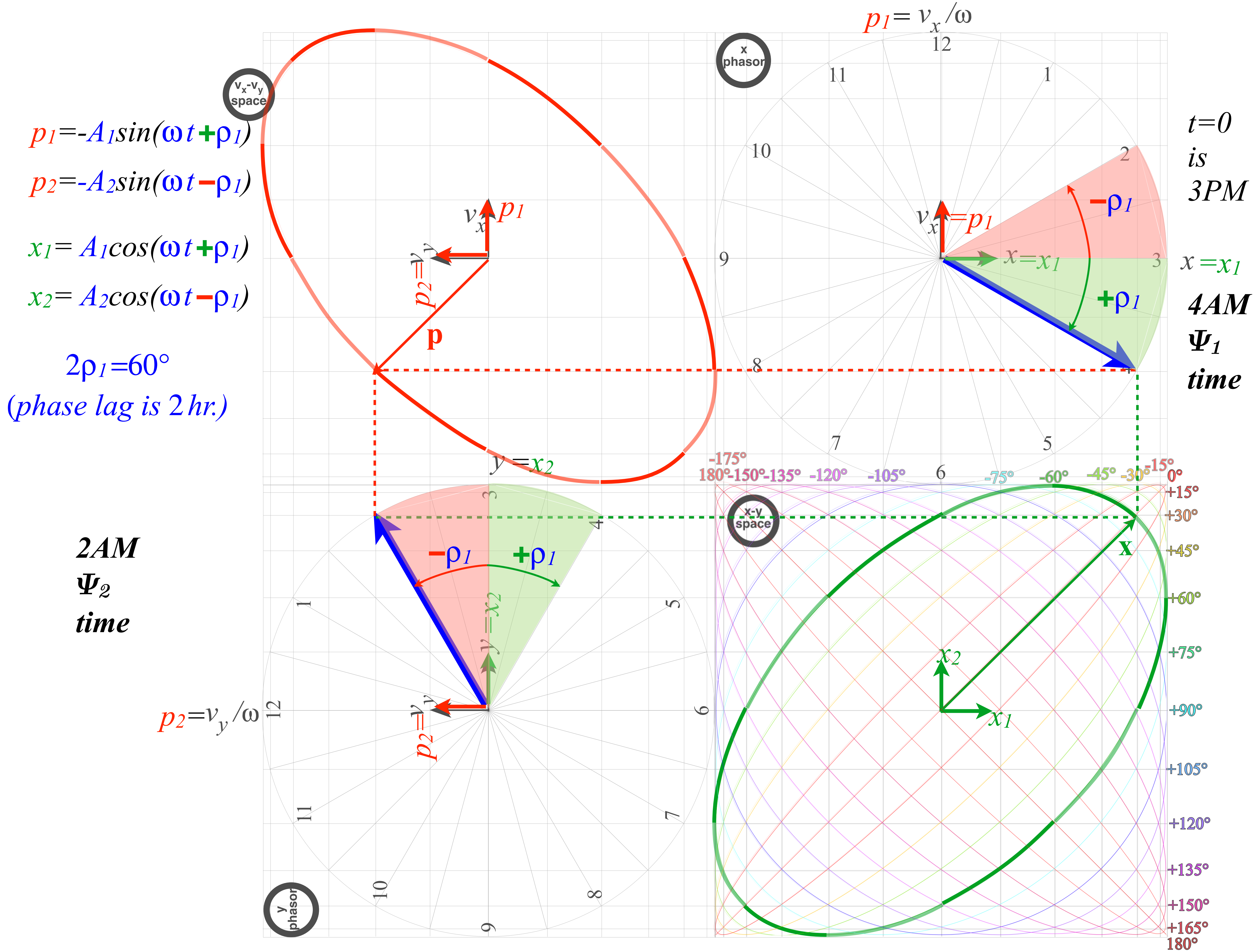
$p_1=-A_1\sin(\omega t+\rho_1)$
$p_2=-A_2\sin(\omega t-\rho_1)$
$x_1=A_1\cos(\omega t+\rho_1)$
$x_2=A_2\cos(\omega t-\rho_1)$
$2\rho_1=60°$
(phase lag is 2 hr.)
v_x-v_y space
x phasor
$p_1=v_x/\omega$
t=0 is 3PM
3AM Ψ_1 time
1AM Ψ_2 time
$x=x_1$
$y=x_2$
$p_2=v_y/\omega$
x-y space
y phasor



$p_1 = v_x/\omega$
v_x-v_y space
x phasor
$p_1 = -A_1 \sin(\omega t + \rho_1)$
$p_2 = -A_2 \sin(\omega t - \rho_1)$
$x_1 = A_1 \cos(\omega t + \rho_1)$
$x_2 = A_2 \cos(\omega t - \rho_1)$
$2\rho_1 = 60°$
(phase lag is 2 hr.)
v_x p_1
$p_2 = v_y$
p
$v_x = p_1$
$x = x_1$
$-\rho_1$
$+\rho_1$
$x = x_1$
t=0 is 3PM
4AM Ψ_1 time
$y = x_2$
2AM Ψ_2 time
x-y space
x
x_2
x_1
$y = x_2$
$p_2 = v_y/\omega$
y phasor
-175° 180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°



p1=-A1sin(ωt+ρ1)
p2=-A2sin(ωt−ρ1)
x1= A1cos(ωt+ρ1)
x2= A2cos(ωt−ρ1)
2ρ1=60°
(phase lag is 2 hr.)
vx-vy space
x phasor
y phasor
x-y space
p1= vx/ω
p2=vy/ω
t=0 is 3PM
4AM Ψ1 time
2AM Ψ2 time
x=x1
y=x2
−ρ1
+ρ1
p
x



$p_1 = v_x/\omega$
v_x-v_y space
x phasor
$p_1 = -A_1 \sin(\omega t + \rho_1)$
$p_2 = -A_2 \sin(\omega t - \rho_1)$
$x_1 = A_1 \cos(\omega t + \rho_1)$
$x_2 = A_2 \cos(\omega t - \rho_1)$
$2\rho_1 = 60°$
(phase lag is 2 hr.)
v_x p_1
$p_2 = v_y$
p
$-\rho_1$
$+\rho_1$
$x = x_1$
t=0
is
3PM
4AM
Ψ_1
time
$y = x_2$
2AM
Ψ_2
time
x-y space
y phasor
$p_2 = v_y/\omega$
x
x_2
x_1
-175° 180° -150° -135° -120° -105° -75° -60° -45° -30° -15° 0°
+15° +30° +45° +60° +75° +90° +105° +120° +135° +150° +165° 180°

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\mathbf{\Theta}$*-vector*

Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\mathbf{\Theta}=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2=-\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ \\ x_2+ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$

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Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

Ellipsometry using U(2) symmetry coordinates

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$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

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$$\begin{aligned} x_1 &= A_1 \cos(\omega t+\rho_1) \\ -p_1 &= A_1 \sin(\omega t+\rho_1) \\ x_2 &= A_2 \cos(\omega t-\rho_1) \\ -p_2 &= A_2 \sin(\omega t-\rho_1) \end{aligned}$$

Let: $A_1 = A\cos\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k=x_k+ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$\begin{aligned} x_1 &= A\cos\beta/2\cos[(\gamma+\alpha)/2] \\ -p_1 &= A\cos\beta/2\sin[(\gamma+\alpha)/2] \\ x_2 &= A\sin\beta/2\cos[(\gamma-\alpha)/2] \\ -p_2 &= A\sin\beta/2\sin[(\gamma-\alpha)/2] \end{aligned}$$

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$$x_1 = A \cos\beta/2 \cos[(\gamma+\alpha)/2]$$
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$$\tan\beta/2 = A_2/A_1 \qquad A^2 = A_1^2 + A_2^2$$

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Let: $\omega t + \rho_1 = (\gamma+\alpha)/2$

$\omega t - \rho_1 = (\gamma-\alpha)/2$

$$\alpha = 2\rho_1 \qquad \gamma = 2\omega\cdot t$$

Euler parameters (α,β,γ, A) in terms of *amp-phase parameters* ($A_1, A_2, \omega t, \rho_1$)

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

Reviewing fundamental Euler **R**$(\alpha\beta\gamma)$ *and Darboux* **R**$[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

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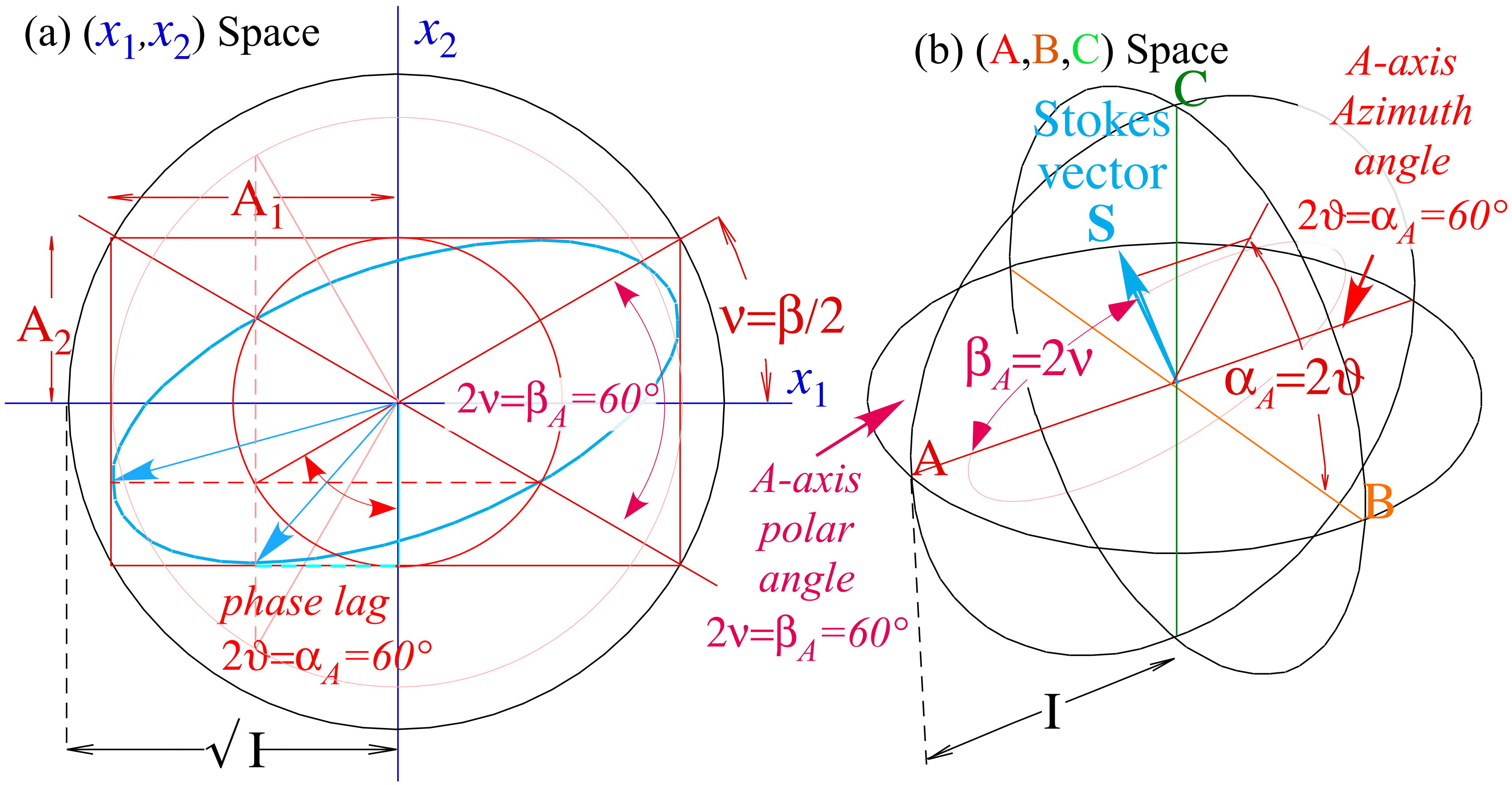
Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

The A-view in $\{x_1,x_2\}$-basis

Angles $\alpha_A=\rho_1-\rho_2=2\rho_1$, $\beta_A=2\tan^{-1}A_2/A_1$, $\gamma_A=2\omega\cdot t$
define ellipses with intensity $I=A^2=A_1^2+A_2^2$.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ e^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\omega t} = \begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$



A or *Z*-axis Euler angles

$\alpha=\alpha_A=\rho_1-\rho_2=2\rho_1=60°$

$\beta=\beta_A=2\tan^{-1}A_2/A_1=60°$

$\gamma_A=2\omega\cdot t$

The A-view in {x1,x2}-basis

Angles $\alpha_A = \rho_1 - \rho_2 = 2\rho_1$, $\beta_A = 2\tan^{-1} A_2/A_1$, $\gamma_A = 2\omega\cdot t$
define ellipses with intensity $I = A^2 = A_1^2 + A_2^2$.

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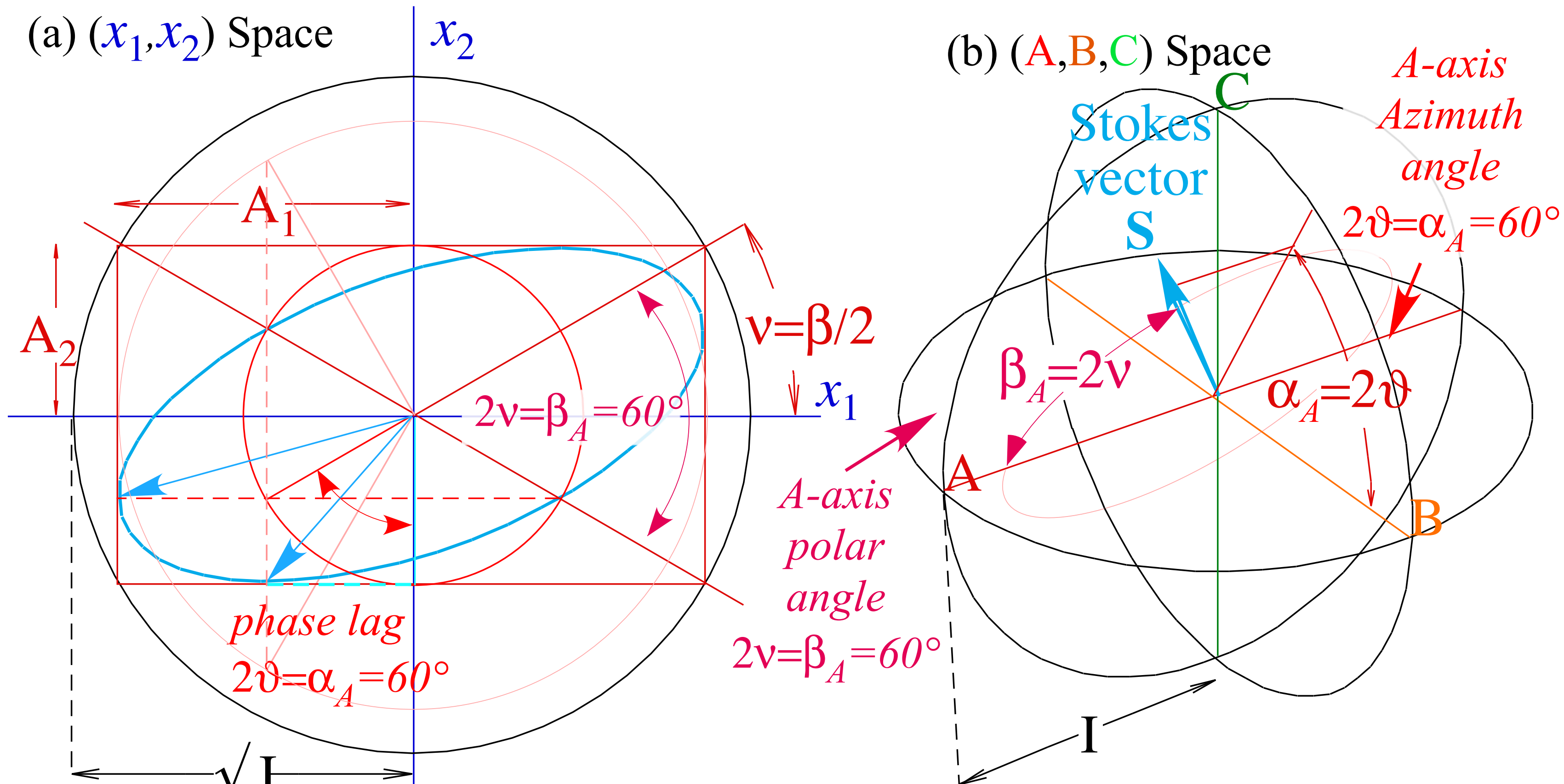

A or Z-axis Euler angles

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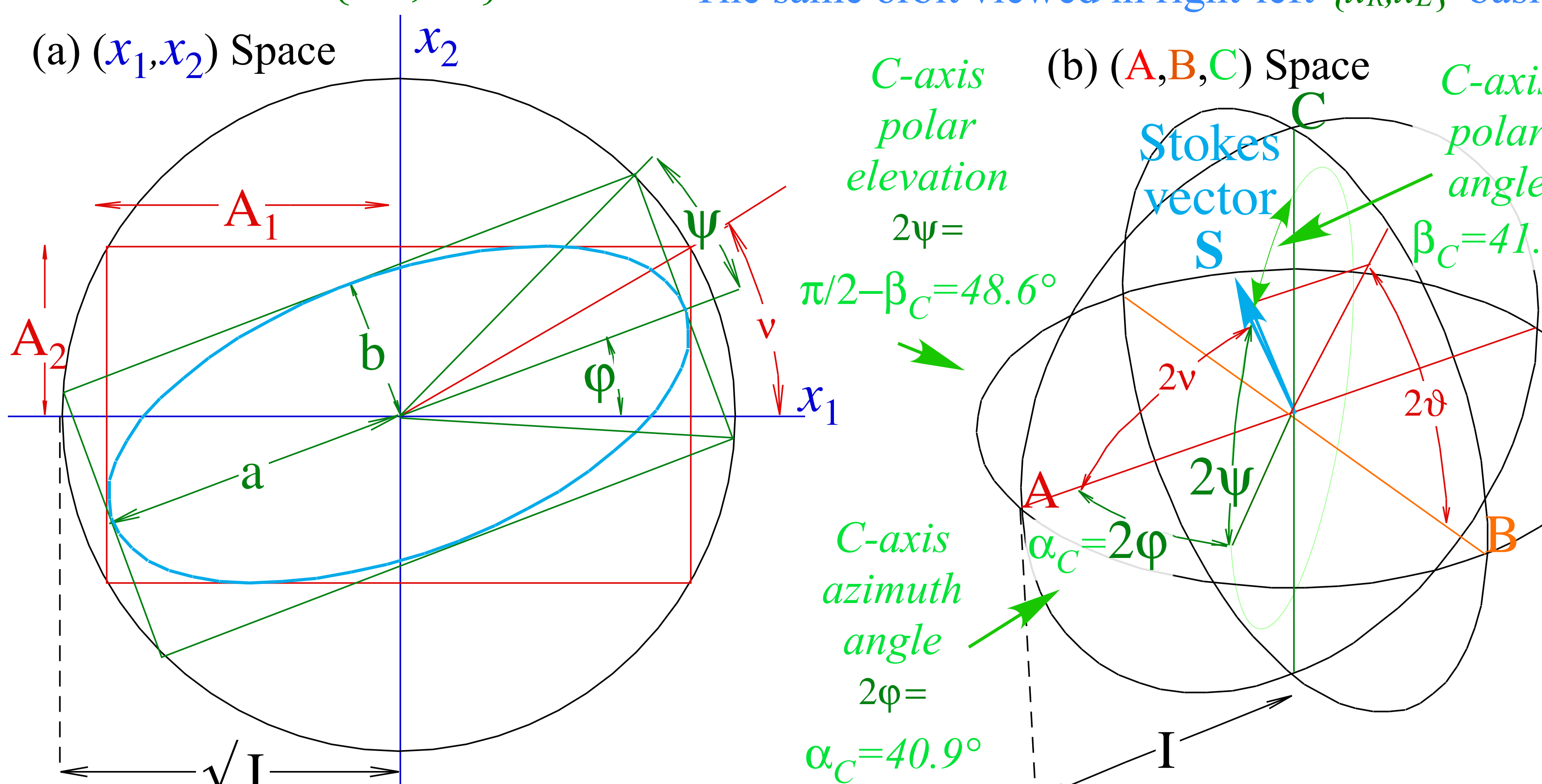
$\beta = \beta_A = 2\tan^{-1} A_2/A_1 = 60°$

$\gamma_A = 2\omega\cdot t$

The C-view in {xR,xL}-basis

The same orbit viewed in right-left $\{x_R, x_L\}$-basis of circular polarization with angles $(\alpha_C, \beta_C, \gamma_C)$.



$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$

Converting an A-based set of Stokes parameters into a C-based set or a B-based set involves cyclic permutation of A, B, and C polar formulas

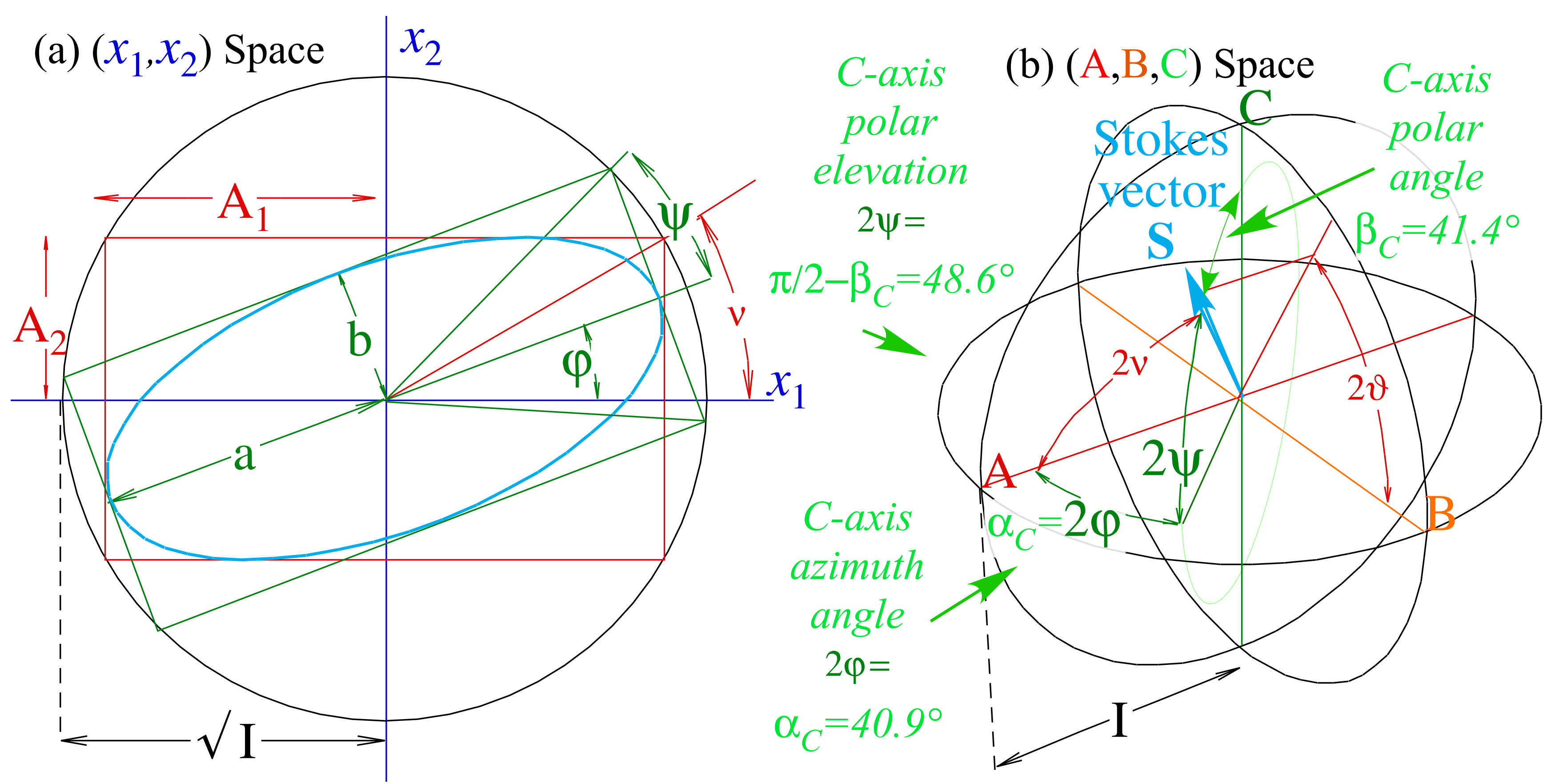
$$\textit{Asymmetry}\quad S_A=\frac{I}{2}\cos\beta_A \qquad\qquad =\frac{I}{2}\sin\alpha_B\sin\beta_B=\frac{I}{2}\cos\alpha_C\sin\beta_C$$

$$\textit{Balance}\quad S_B=\frac{I}{2}\cos\alpha_A\sin\beta_A=\frac{I}{2}\cos\beta_B \qquad\qquad =\frac{I}{2}\sin\alpha_C\sin\beta_C$$

$$\textit{Chirality}\quad S_C=\frac{I}{2}\sin\alpha_A\sin\beta_A=\frac{I}{2}\cos\alpha_B\sin\beta_B=\frac{I}{2}\cos\beta_C$$

The C-view in $\{x_R,x_L\}$-basis

The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C,\beta_C,\gamma_C)$.



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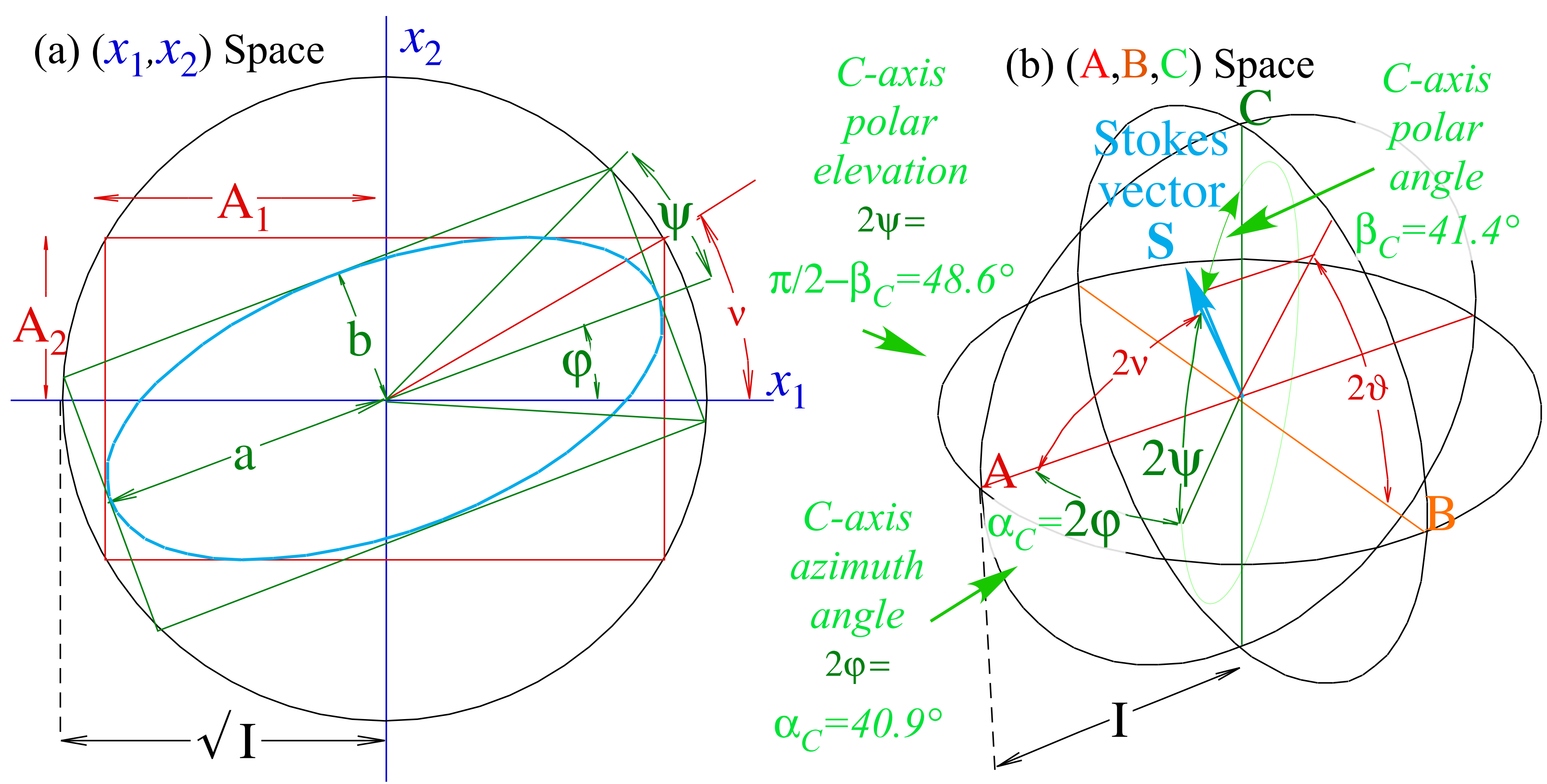
The same orbit viewed in right and left circular polarization $\{x_R,x_L\}$-bases using angles $(\alpha_C,\beta_C,\gamma_C)$.

Angles (α_C,β_C): C-axial polar angle β_C from above.

$$\sin\alpha_A \sin\beta_A = \cos\beta_C \qquad \text{or:} \quad \beta_C = \cos^{-1}(\sin\alpha_A \sin\beta_A) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\right) = 41.4^\circ$$



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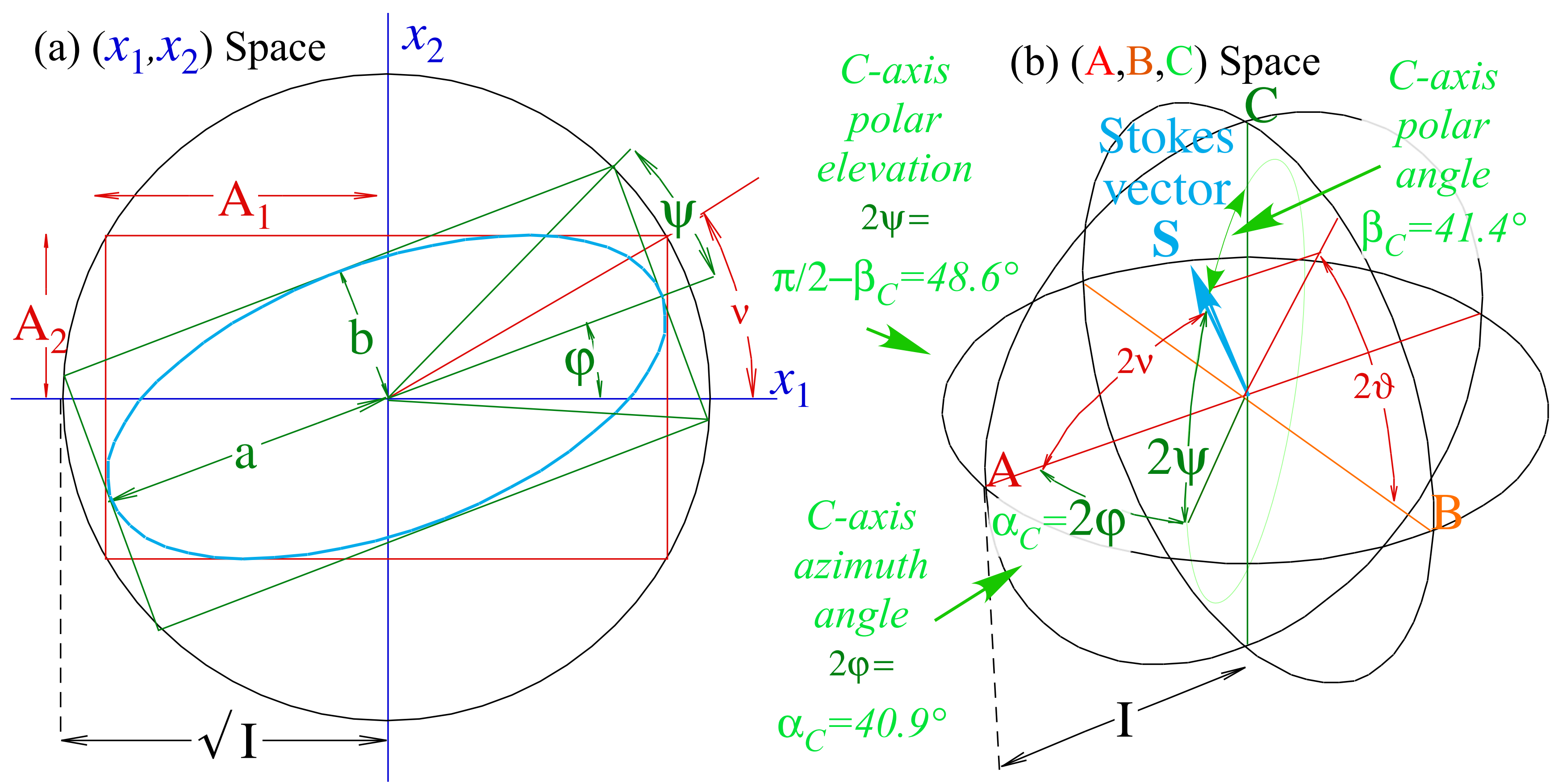
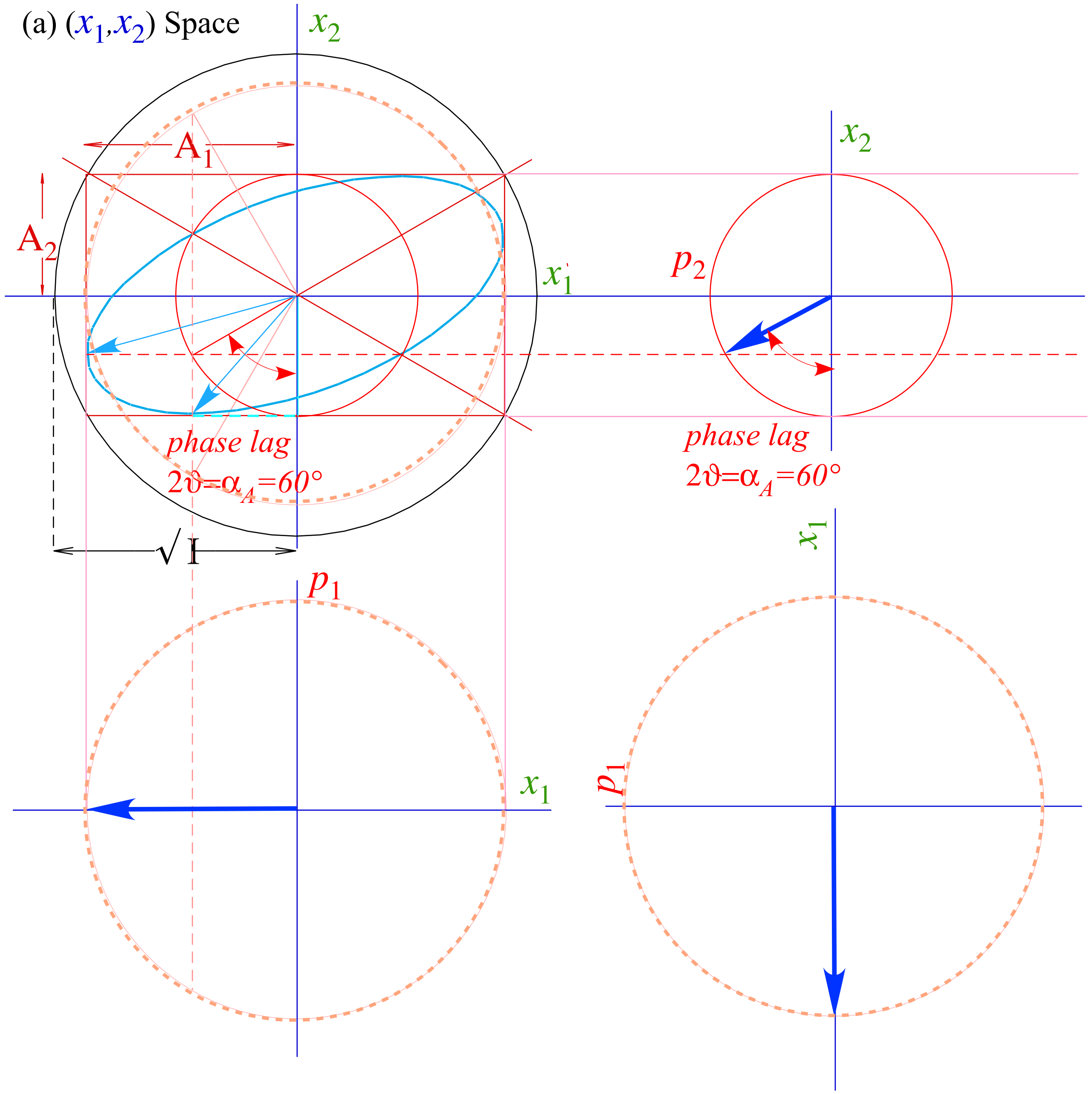
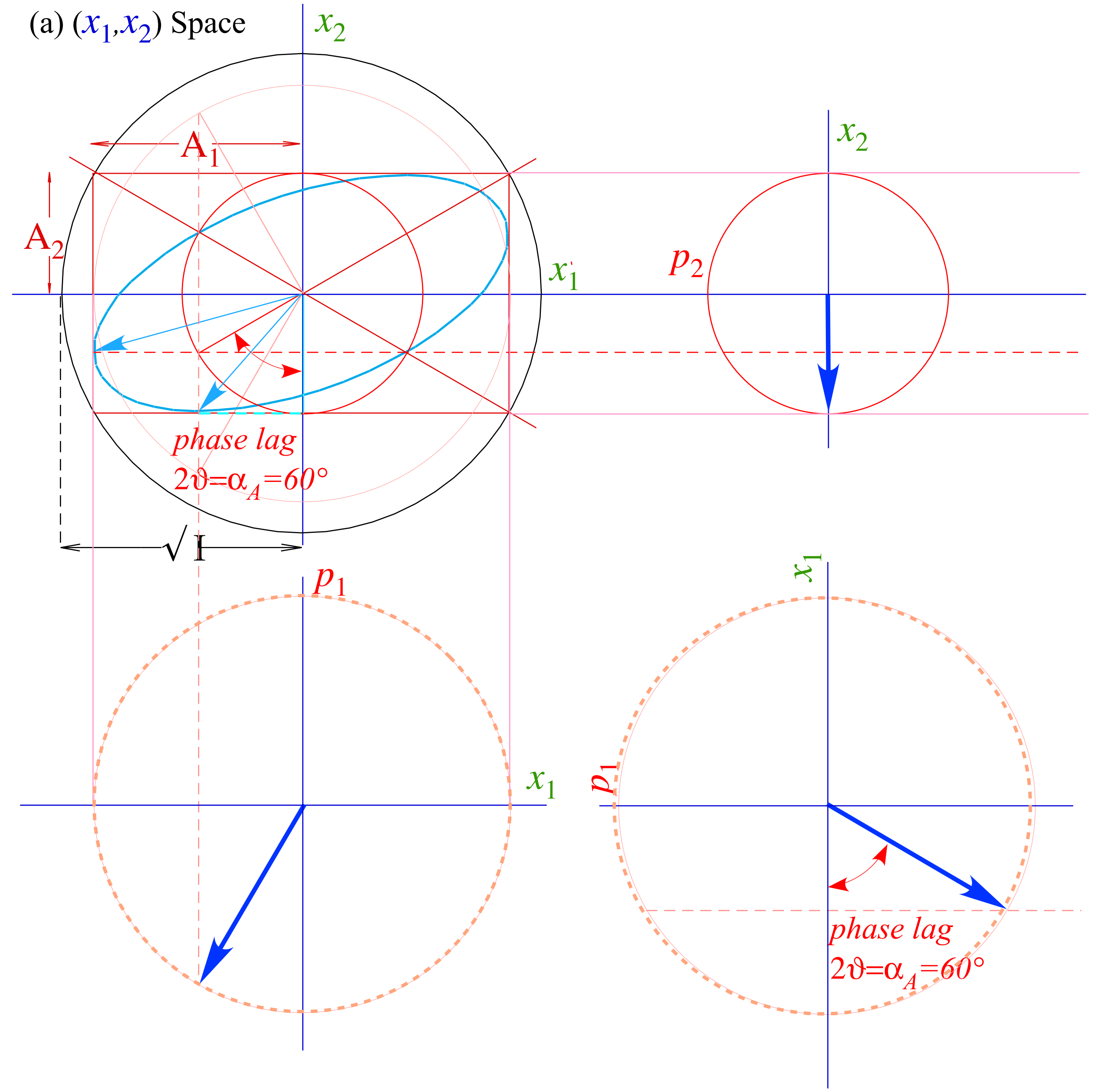
C-axis azimuth angle α_C relates to A-axis angles α_A and β_A . See $\alpha_C = 2\varphi$ below.

$$\frac{\cos\alpha_A \sin\beta_A}{\cos\beta_A} = \tan\alpha_C \qquad \text{or:}\quad \alpha_C = ATN2(\cos\alpha_A \sin\beta_A / \cos\beta_A) = ATN2\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{2} / \frac{1}{2}\right) = 40.9^\circ$$

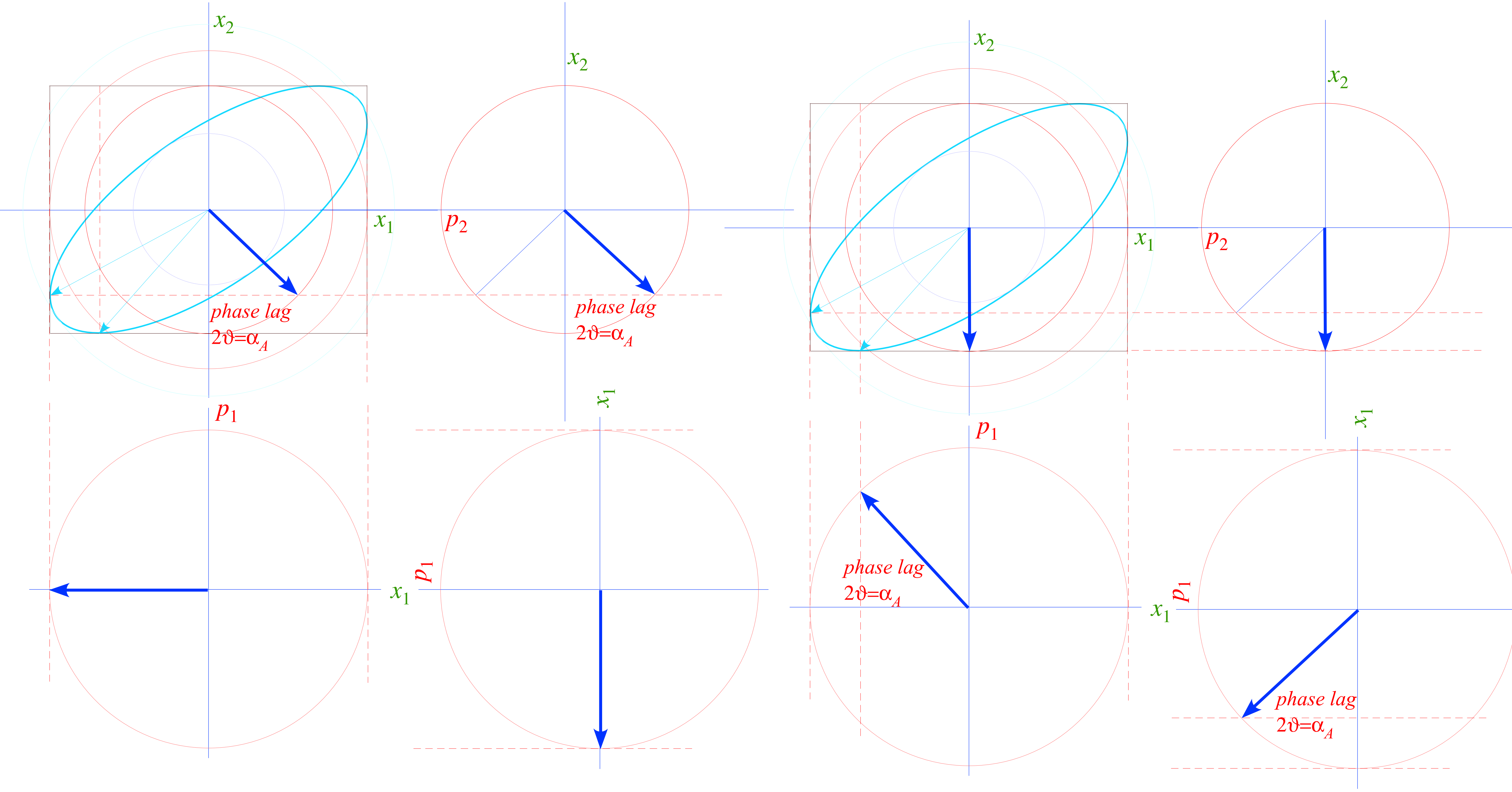




(a) (x_1,x_2) Space
x_2
x_1
A$_1$
A$_2$
p_2
p_1
phase lag
$2\vartheta=\alpha_A=60°$
$\sqrt{I}$



(a) (x_1,x_2) Space
x_2
x_1
A$_1$
A$_2$
p_2
p_1
phase lag
$2\vartheta=\alpha_A=60°$
$\sqrt{I}$

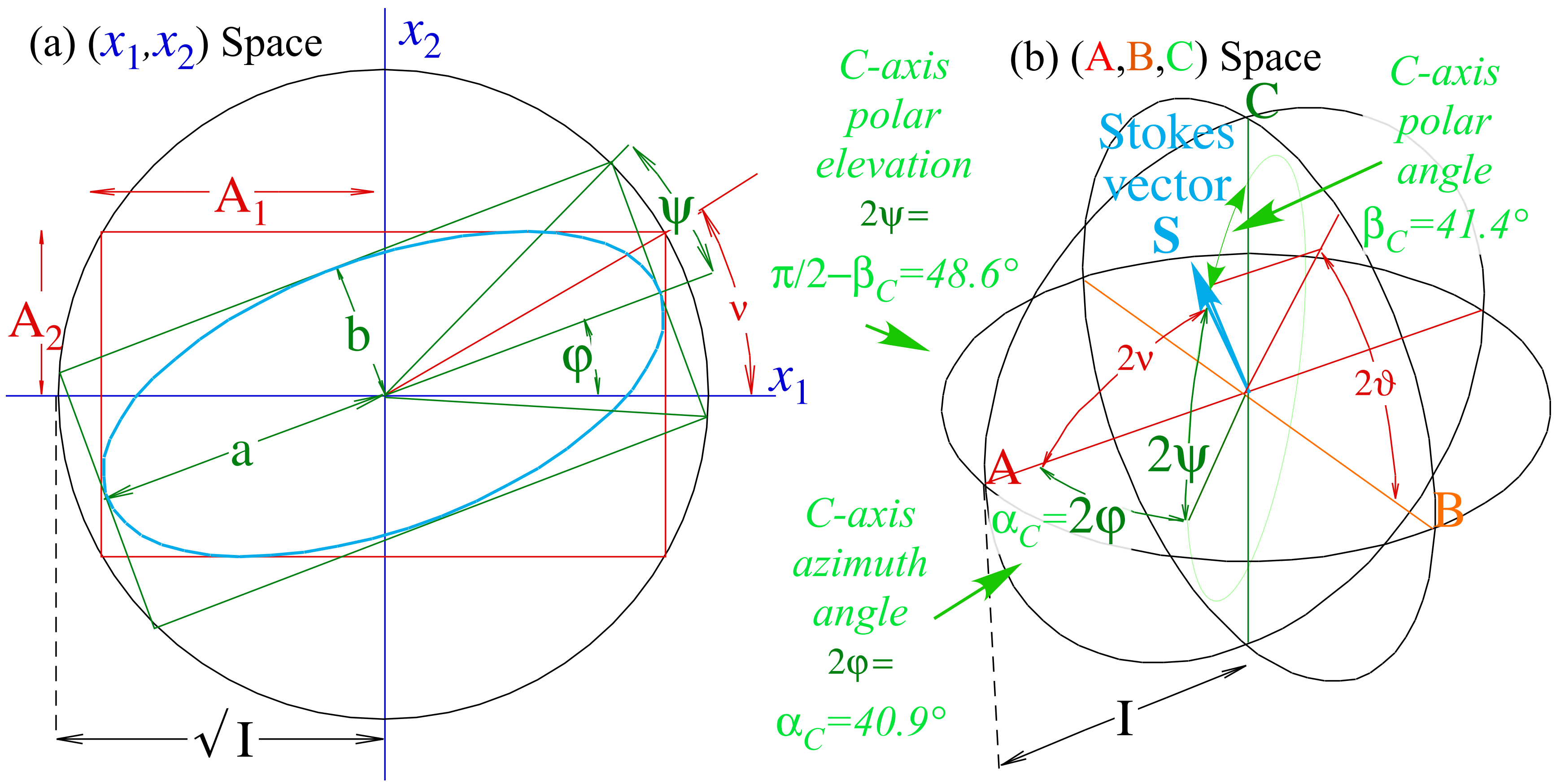


x2
x1
p2
phase lag
2ϑ=αA
x2
p2
phase lag
2ϑ=αA
p1
x1
x1
p1
x2
x1
phase lag
2ϑ=αA
x2
p2
p1
x1
phase lag
2ϑ=αA
x1
p1
phase lag
2ϑ=αA

The C-view in {xR,xL}-basis

The same orbit viewed in right and left circular polarization $\{x_R, x_L\}$-bases using angles $(\alpha_C, \beta_C, \gamma_C)$.

$$\begin{pmatrix} a_R \\ a_L \end{pmatrix} = A \begin{pmatrix} e^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ e^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_R + ip_R \end{pmatrix}$$



A 90° *B* –rotation $\mathbf{R}(\pi/4)\left|x_1\right\rangle = \left|x_R\right\rangle$ of axis *A* into *C* gets $(\alpha_C, \beta_C, \gamma_C)$ from $(\alpha_A, \beta_A, \gamma_A)$ all at once.

$$\begin{pmatrix} \cos\frac{\pi}{4} & i\sin\frac{\pi}{4} \\ i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} Ae^{-i\alpha_A/2}\cos\frac{\beta_A}{2} \\ Ae^{+i\alpha_A/2}\sin\frac{\beta_A}{2} \end{pmatrix} e^{-i\frac{\gamma_A}{2}} = \begin{pmatrix} Ae^{-i\alpha_C/2}\cos\frac{\beta_C}{2} \\ Ae^{+i\alpha_C/2}\sin\frac{\beta_C}{2} \end{pmatrix} e^{-i\frac{\gamma_C}{2}} = \begin{pmatrix} x_R + ip_R \\ x_L + ip_L \end{pmatrix}$$

Polarization ellipse and spinor state dynamics

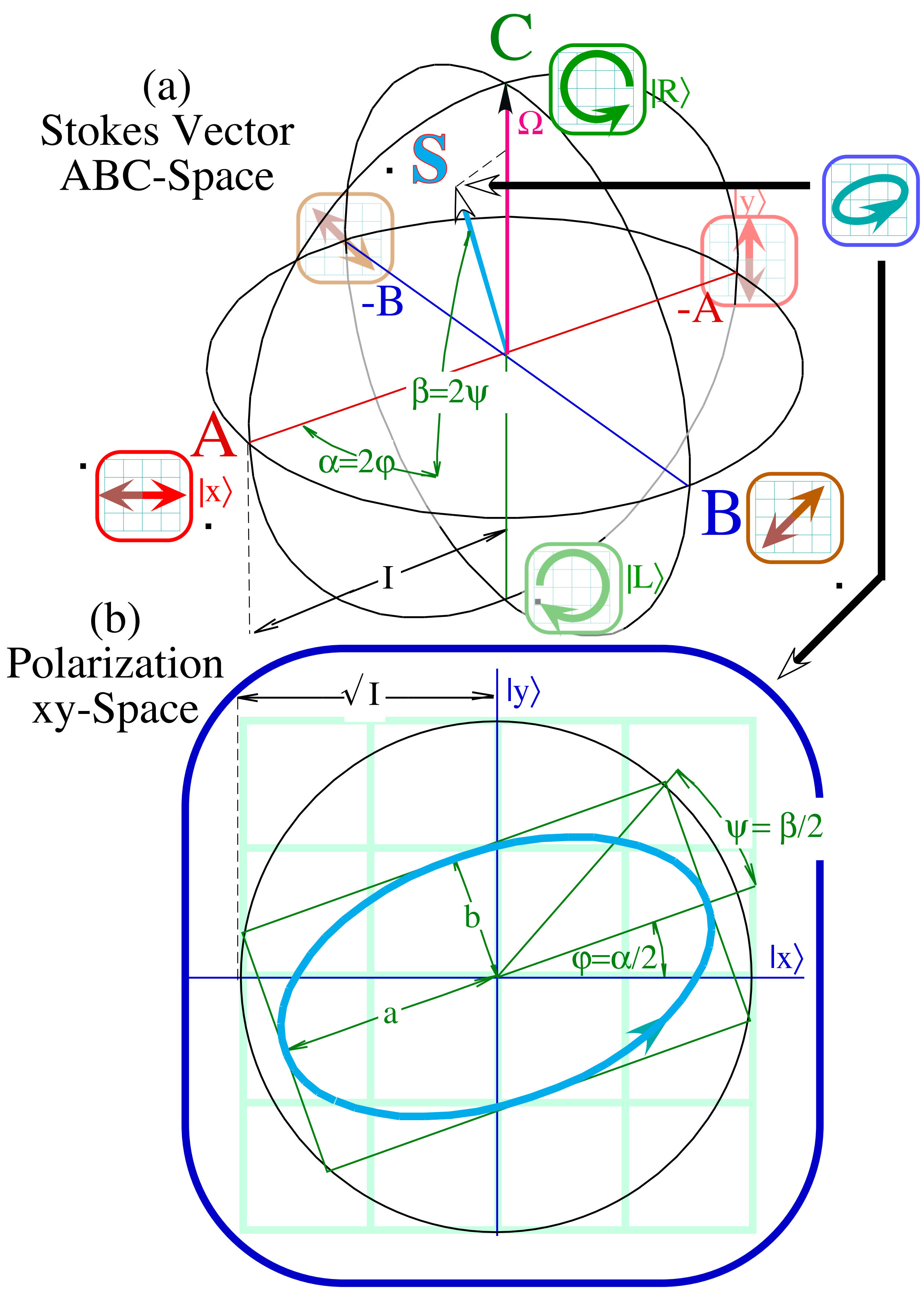


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

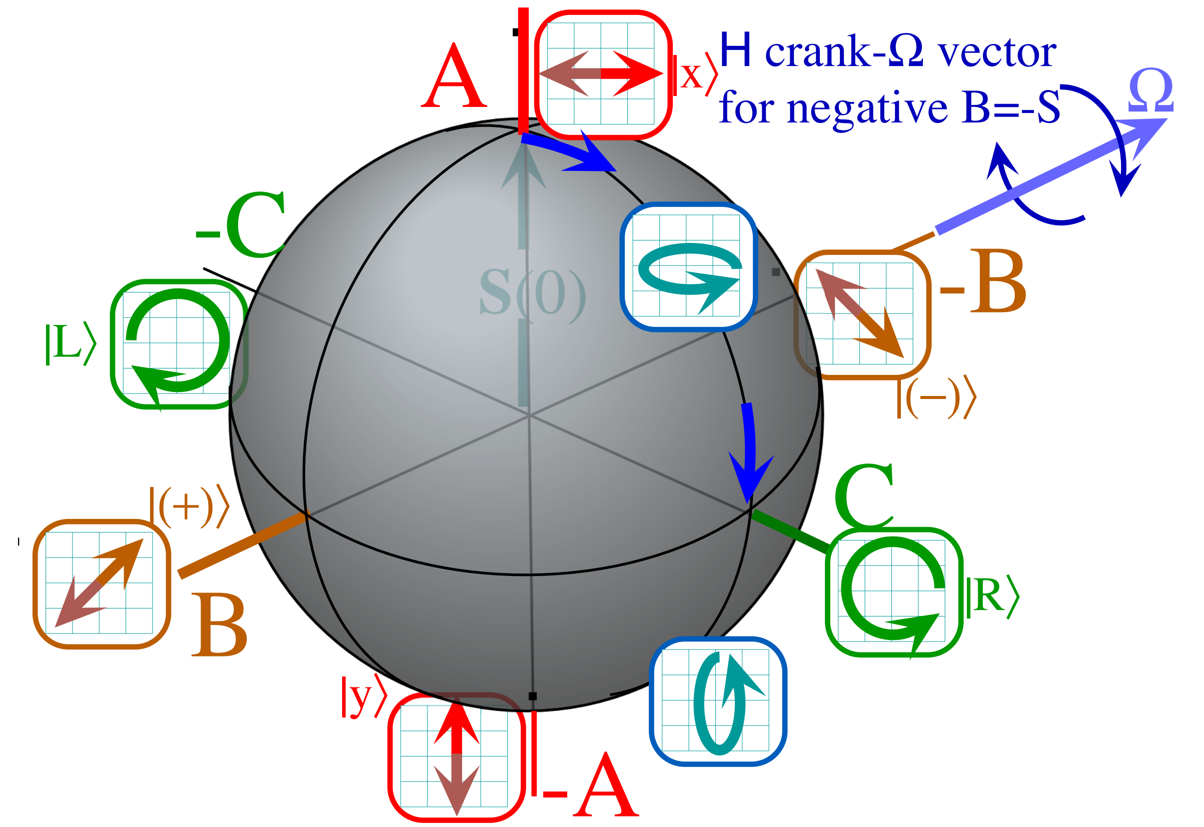
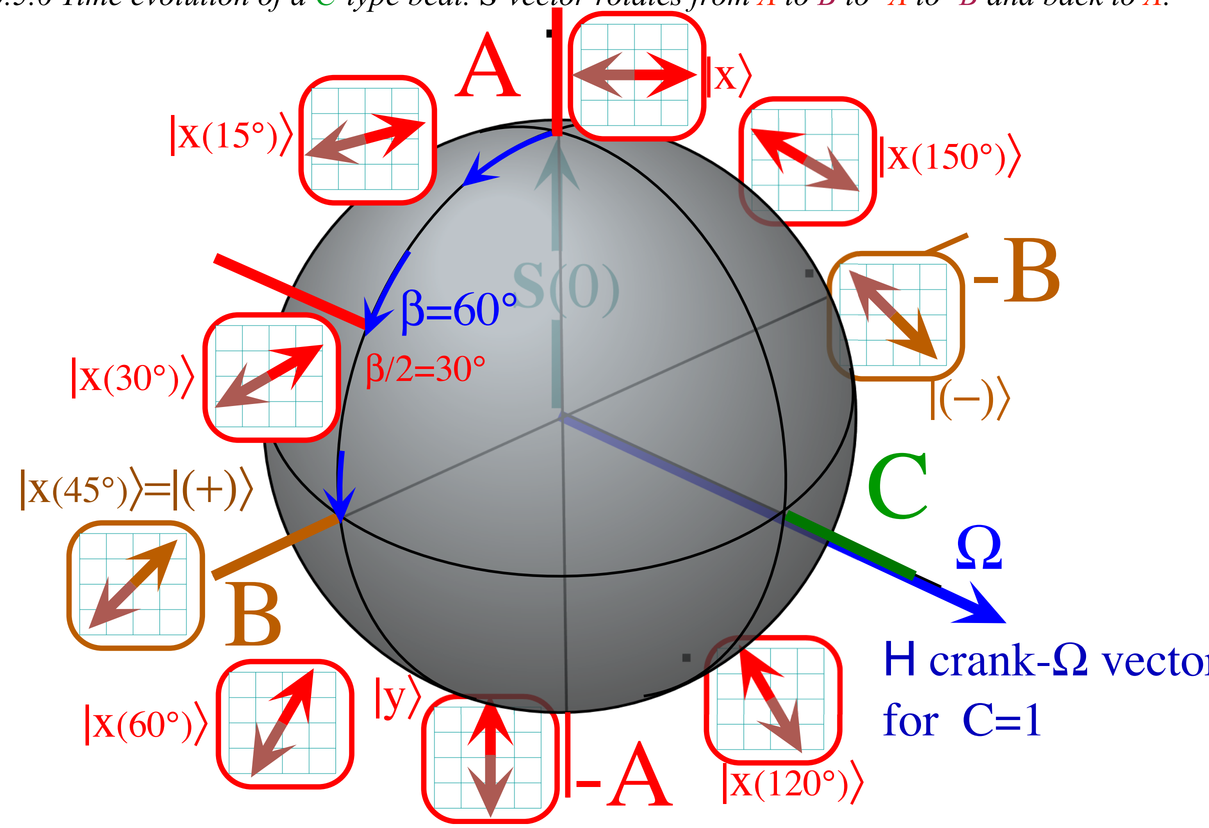


Fig. 10.5.5 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Fig. 10.5.6 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*



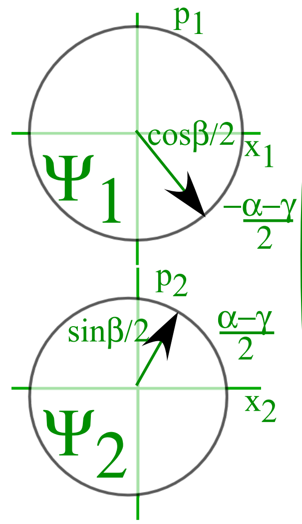
U(2) World : Complex 2D Spinors

Ellipsometry of U(2) states described by Two "Worlds" U(2) or R(3)

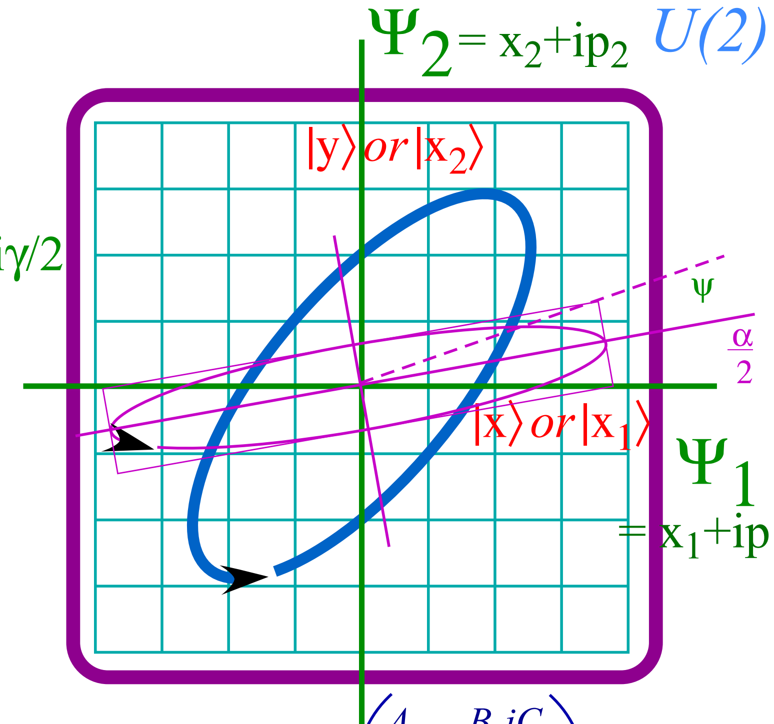
U(2) World labeled by two complex phasors and driven by complex operator

$$\mathbf{H} = \begin{pmatrix} A & B - iC \\ B + iC & D \end{pmatrix}$$

2-State ket $|\Psi\rangle =$



$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{N} e^{-i\alpha/2} \cos\beta/2 \\ \sqrt{N} e^{i\alpha/2} \sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$



R(3) World : Real 3D Vectors

$|\Psi\rangle$ State

Spin Vector S

$$\begin{pmatrix} S_B \\ S_C \\ S_A \end{pmatrix} = \begin{pmatrix} N\sin\beta\cos\alpha \\ N\sin\beta\sin\alpha \\ N\cos\beta \end{pmatrix} \frac{1}{2}$$

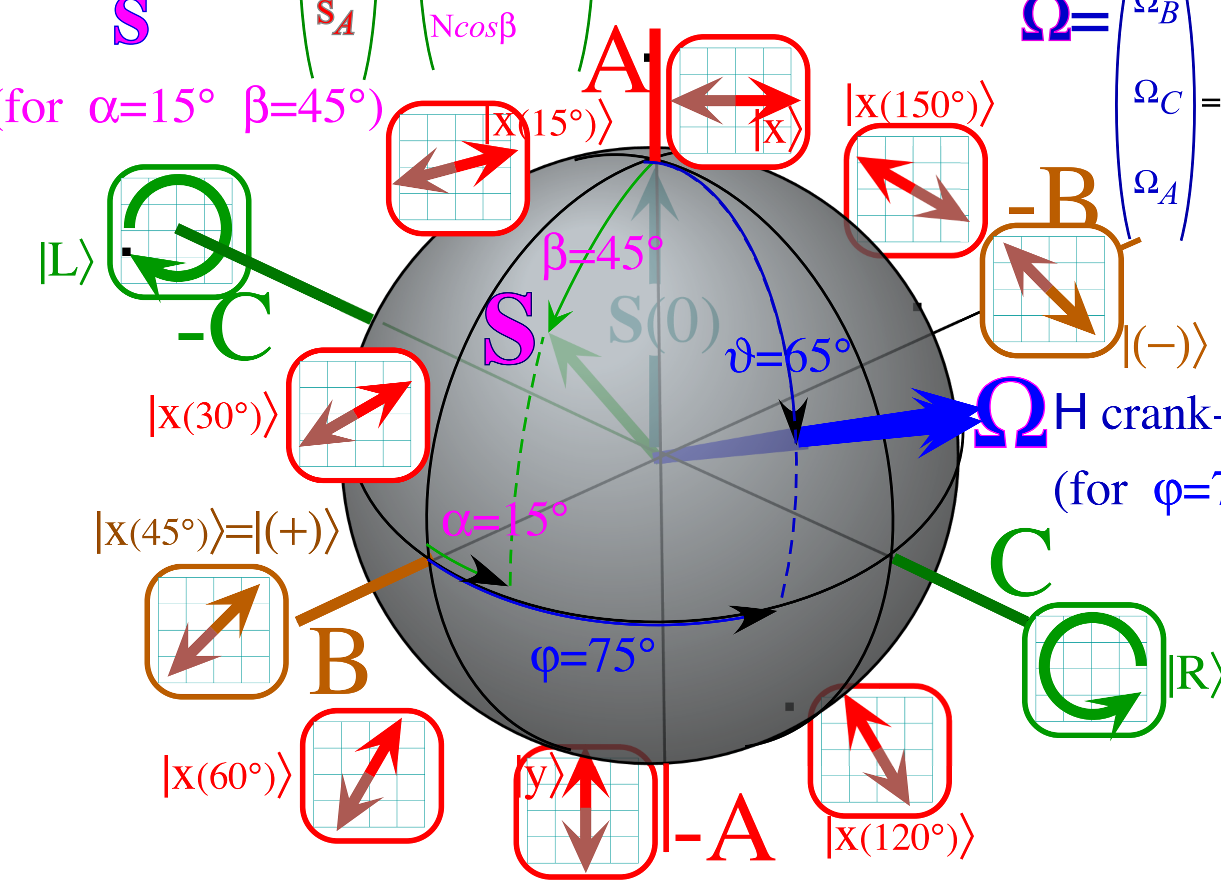
(for $\alpha=15°$ $\beta=45°$)

H-Operator $\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

Angular velocity

$$\boldsymbol{\Omega} = \begin{pmatrix} \Omega_B \\ \Omega_C \\ \Omega_A \end{pmatrix} = \begin{pmatrix} 2B \\ 2C \\ A-D \end{pmatrix} = \begin{pmatrix} \Omega\sin\vartheta\cos\varphi \\ \Omega\sin\vartheta\sin\varphi \\ \Omega\cos\vartheta \end{pmatrix}$$

R(3) World labeled by real 3-D "spin" vector **S** *of angular momentum and driven by real 3-D "spin" vector* **Ω** *of angular velocity*



H crank-Ω vector

(for $\varphi=75°$ $\vartheta=65°$)

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

- *Euler-defined state* $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*
- *Darboux defined Hamiltonian* $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\boldsymbol{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\boldsymbol{\Omega}(\varphi\vartheta)\cdot t=\boldsymbol{\Theta}$*-vector*
- *Euler-defined operator* $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*
- *Euler* $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

- *Asymmetric-Diagonal A-Type motion*
- *Bilateral-Balanced B-Type motion*
- *Circular-Coriolis… C-Type motion*

The ABC's of U(2) dynamics-Mixed modes

- *AB-Type motion and Wigner's Avoided-Symmetry-Crossings*
- *ABC-Type elliptical polarized motion*

Ellipsometry using U(2) symmetry and related coordinates

- *Conventional amp-phase ellipse coordinates*
- *Euler Angle* $(\alpha\beta\gamma)$ *ellipse coordinates*

Addenda: U(2) density matrix formalism

- *Bloch equation for density operator*

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1^2 + x_1^2 + p_2^2 + x_2^2\right)}_{4D\text{-}norm=1} \quad \textit{scaled by } \tfrac{1}{2}: \quad \frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^2 + x_1^2 - p_2^2 - x_2^2\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix} = \sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$

scaled by $\frac{1}{2}$: $\frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$

scaled by $\frac{1}{2}$: $S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = 2N\left(x_1x_2+p_1p_2\right)$$

scaled by $\frac{1}{2}$: $S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 \quad = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad = \frac{N}{2}\cos\alpha\,\sin\beta$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1{}^2 + x_1{}^2 + p_2{}^2 + x_2{}^2\right)}_{4D\text{-}norm=1} \quad \textit{scaled by } \tfrac{1}{2}: \quad \frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1{}^2 + x_1{}^2 - p_2{}^2 - x_2{}^2\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1 x_2 + p_1 p_2\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1 p_2 - x_2 p_1\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\sin\beta$$

$U(2)$ density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ)
and norm N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix} = \sqrt{N}\begin{pmatrix} e^{-i\alpha/2}\cos\beta/2 \\ e^{i\alpha/2}\sin\beta/2 \end{pmatrix} e^{-i\gamma/2}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1 = -\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2 = -\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle = N = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\underbrace{\left(p_1^{\,2} + x_1^{\,2} + p_2^{\,2} + x_2^{\,2}\right)}_{4D\text{-}norm=1} \quad \textit{scaled by } \tfrac{1}{2}: \quad \frac{1}{2}\left(|\Psi_1|^2 + |\Psi_2|^2\right) = \frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle = 2S_A = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = N\left(p_1^{\,2} + x_1^{\,2} - p_2^{\,2} - x_2^{\,2}\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_Z = S_A = \frac{1}{2}\left(|\Psi_1|^2 - |\Psi_2|^2\right) = \frac{N}{2}\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right) = \frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle = 2S_B = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1x_2 + p_1p_2\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_X = S_B = \mathrm{Re}\,\Psi_1^*\Psi_2 = N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle = 2S_C = \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 2N\left(x_1p_2 - x_2p_1\right) \quad \textit{scaled by } \tfrac{1}{2}: \quad S_Y = S_C = \mathrm{Im}\,\Psi_1^*\Psi_2 = N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \otimes \begin{pmatrix} \Psi_1^* & \Psi_2^* \end{pmatrix} = \begin{pmatrix} \Psi_1\Psi_1^* & \Psi_1\Psi_2^* \\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^* \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \Psi_1^*\Psi_1 & \Psi_2^*\Psi_1 \\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2 \end{pmatrix}$

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm* N *of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle=N=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}\quad \textit{scaled by } \tfrac{1}{2}:\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$$

$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

The *density operator* $\rho=|\Psi\rangle\langle\Psi|=$

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1 = \frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1 = S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2 = S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2 = \frac{1}{2}N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}$$

...2-by-2 *density operator* ρ

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α, β, γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\ x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\ e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
$p_2=-\sin[(\gamma-\alpha)/2]\sin\beta/2$

1/2 times **σ***-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

$$\langle\Psi|\mathbf{1}|\Psi\rangle= N =\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\underbrace{\left(p_1^2+x_1^2+p_2^2+x_2^2\right)}_{4D\text{-}norm=1}$$
scaled by $\frac{1}{2}$: $\quad \frac{1}{2}\left(|\Psi_1|^2+|\Psi_2|^2\right)=\frac{N}{2}$

$$\langle\Psi|\sigma_Z|\Psi\rangle=2S_A=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=N\left(p_1^2+x_1^2-p_2^2-x_2^2\right)$$
scaled by $\frac{1}{2}$: $\quad S_Z=S_A=\frac{1}{2}\left(|\Psi_1|^2-|\Psi_2|^2\right)=\frac{N}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)=\frac{N}{2}\cos\beta$

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scaled by $\frac{1}{2}$: $\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 \quad =N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\cos\alpha\sin\beta$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)$$
scaled by $\frac{1}{2}$: $\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2 \quad =N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2} \quad =\frac{N}{2}\sin\alpha\sin\beta$

The *density operator* ρ = |Ψ⟩⟨Ψ|=

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\otimes\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}=\begin{pmatrix}\Psi_1\Psi_1^* & \Psi_1\Psi_2^*\\ \Psi_2\Psi_1^* & \Psi_2\Psi_2^*\end{pmatrix}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix}=\begin{pmatrix}\Psi_1^*\Psi_1 & \Psi_2^*\Psi_1\\ \Psi_1^*\Psi_2 & \Psi_2^*\Psi_2\end{pmatrix}$$

$\rho_{11}=\Psi_1^*\Psi_1$ $=\frac{1}{2}N+S_Z$	$\rho_{12}=\Psi_2^*\Psi_1$ $=S_X-iS_Y,$
$\rho_{21}=\Psi_1^*\Psi_2$ $=S_X+iS_Y$	$\rho_{22}=\Psi_2^*\Psi_2$ $=\frac{1}{2}N-S_Z$

Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$=\underbrace{\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}}_{\rho}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\begin{pmatrix}0&1\\1&0\end{pmatrix}+S_Y\begin{pmatrix}0&-i\\i&0\end{pmatrix}+S_Z\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

...so state *density operator* ρ has **σ**-expansion

U(2) density operator approach to symmetry dynamics

Euler phase-angle coordinates (α,β,γ) *and norm N of quantum state* $|\Psi\rangle$

$$|\Psi\rangle=\begin{pmatrix}\Psi_1\\\Psi_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}x_1+ip_1\\x_2+ip_2\end{pmatrix}=\sqrt{N}\begin{pmatrix}e^{-i\alpha/2}\cos\beta/2\\e^{i\alpha/2}\sin\beta/2\end{pmatrix}e^{-i\gamma/2}$$

$x_1=\cos[(\gamma+\alpha)/2]\cos\beta/2$
$p_1=-\sin[(\gamma+\alpha)/2]\cos\beta/2$

$x_2=\cos[(\gamma-\alpha)/2]\sin\beta/2$
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1/2 times $\boldsymbol{\sigma}$*-operator expectation values* $\langle\Psi|\sigma_\mu|\Psi\rangle$ gives: Spin **S**-vector components:

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$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\\Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \text{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

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Norm: $N=\Psi1^*\Psi1+\Psi2^*\Psi2$

$$\rho=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\sigma_X}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{\sigma_Y}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\sigma_Z}$$

$$\rho=\frac{1}{2}N\,\mathbf{1}+S_X\,\boldsymbol{\sigma}_X+S_Y\,\boldsymbol{\sigma}_Y+S_Z\,\boldsymbol{\sigma}_Z=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has $\boldsymbol{\sigma}$-expansion

U(2) density operator approach to symmetry dynamics

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$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\\Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2=N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

$$\langle\Psi|\sigma_Y|\Psi\rangle=2S_C=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}\Psi_1\\\Psi_2\end{pmatrix}=2N\left(x_1p_2-x_2p_1\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_Y=S_C=\mathrm{Im}\,\Psi_1^*\Psi_2=N\sin\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\sin\alpha\sin\beta$$

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Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$\rho=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\sigma_X}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{\sigma_Y}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\sigma_Z}$$

$$\rho=\frac{1}{2}N\mathbf{1}+S_X\boldsymbol{\sigma}_X+S_Y\boldsymbol{\sigma}_Y+S_Z\boldsymbol{\sigma}_Z=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

...so state *density operator* ρ has **σ**-expansion like *Hamiltonian operator* **H**

$$\begin{pmatrix}A & B-iC\\B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\sigma_A}+B\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\sigma_B}+C\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}_{\sigma_C}$$

$$\mathbf{H}=\omega_0\sigma_0+\frac{\Omega_A}{2}\sigma_A+\frac{\Omega_B}{2}\sigma_B+\frac{\Omega_C}{2}\sigma_C=\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

U(2) density operator approach to symmetry dynamics

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$$\langle\Psi|\sigma_X|\Psi\rangle=2S_B=\begin{pmatrix}\Psi_1^* & \Psi_2^*\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=2N\left(x_1x_2+p_1p_2\right)\quad \textit{scaled by } \tfrac{1}{2}:\quad S_X=S_B=\mathrm{Re}\,\Psi_1^*\Psi_2 =N\cos\alpha\cos\frac{\beta}{2}\sin\frac{\beta}{2}=\frac{N}{2}\cos\alpha\sin\beta$$

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Norm: $N=\Psi_1^*\Psi_1+\Psi_2^*\Psi_2$

$$=\begin{pmatrix}\frac{1}{2}N+S_Z & S_X-iS_Y\\ S_X+iS_Y & \frac{1}{2}N-S_Z\end{pmatrix}=\frac{1}{2}N\begin{pmatrix}1&0\\0&1\end{pmatrix}+S_X\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+S_Y\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}+S_Z\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$\rho\quad=\frac{1}{2}N\ \mathbf{1}\ +S_X\ \boldsymbol{\sigma}_X\ +S_Y\ \boldsymbol{\sigma}_Y\ +S_Z\ \boldsymbol{\sigma}_Z\ =\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

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$$\begin{pmatrix}A & B-iC\\ B+iC & D\end{pmatrix}=\mathbf{H}=\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{A-D}{2}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}+B\underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}+C\underbrace{\begin{pmatrix}0&-i\\i&0\end{pmatrix}}$$

$$\mathbf{H}=\ \omega_0\ \sigma_0\ +\frac{\Omega_A}{2}\sigma_A\ +\frac{\Omega_B}{2}\sigma_B\ +\frac{\Omega_C}{2}\sigma_C\ =\omega_0\sigma_0+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H}=\ \Omega_0\ \mathbf{1}\ +\Omega_A\ \mathbf{S}_A\ +\Omega_B\ \mathbf{S}_B\ +\Omega_C\mathbf{S}_C\ =\Omega_0\mathbf{1}+\vec{\Omega}\bullet\mathbf{S}$$

$$\boxed{\begin{aligned}\rho&=\frac{1}{2}N\mathbf{1}+\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\\ \mathbf{H}&=\Omega_0\mathbf{1}+\frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}\end{aligned}}$$

Reviewing fundamental Euler $\mathbf{R}(\alpha\beta\gamma)$ *and Darboux* $\mathbf{R}[\varphi\vartheta\Theta]$ *representations of U(2) and R(3)*

Euler-defined state $|\alpha\beta\gamma\rangle$ *described by Stoke's* $\mathbf{S}$*-vector, phasors, or ellipsometry*

Darboux defined Hamiltonian $\mathbf{H}[\varphi\vartheta\Theta]=exp(-i\mathbf{\Omega}\bullet\mathbf{S})\cdot t$ *and angular velocity* $\mathbf{\Omega}(\varphi\vartheta)\cdot t=\Theta$*-vector*

Euler-defined operator $\mathbf{R}(\alpha\beta\gamma)$ *derived from Darboux-defined* $\mathbf{R}[\varphi\vartheta\Theta]$ *and vice versa*

Euler $\mathbf{R}(\alpha\beta\gamma)$ *rotation* $\Theta=0-4\pi$*-sequence* $[\varphi\vartheta]$ *fixed (and "real-world" applications)*

Quick U(2) way to find eigen-solutions for general 2-by-2 Hamiltonian $\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}$

The ABC's of U(2) dynamics-Archetypes

Asymmetric-Diagonal A-Type motion

Bilateral-Balanced B-Type motion

Circular-Coriolis… C-Type motion

The ABC's of U(2) dynamics-Mixed modes

AB-Type motion and Wigner's Avoided-Symmetry-Crossings

ABC-Type elliptical polarized motion

Ellipsometry using U(2) symmetry and related coordinates

Conventional amp-phase ellipse coordinates

Euler Angle $(\alpha\beta\gamma)$ *ellipse coordinates*

Addenda: U(2) density matrix formalism

Bloch equation for density operator

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2} N\mathbf{1} + \vec{\mathbf{S}} \bullet \boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0 \mathbf{1} + \frac{\vec{\Omega}}{2} \bullet \boldsymbol{\sigma}$$

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar\left|\dot{\Psi}\right\rangle = \mathbf{H}\left|\Psi\right\rangle, \quad \Leftarrow Daggar^{\dagger} \Rightarrow \quad -i\hbar\left\langle\dot{\Psi}\right| = \left\langle\Psi\right|\mathbf{H}$$

Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

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Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
$\rho^{\dagger} = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar\left|\dot{\Psi}\right\rangle\left\langle\Psi\right| + i\hbar\left|\Psi\right\rangle\left\langle\dot{\Psi}\right| = \mathbf{H}\left|\Psi\right\rangle\left\langle\Psi\right| - \left|\Psi\right\rangle\left\langle\Psi\right|\mathbf{H}$$

U(2) density operator approach to symmetry dynamics

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The result is called a *Bloch equation*.

$$\boxed{i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]}$$

U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

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Note: $\mathbf{H}^{\dagger} = \mathbf{H}$.
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The result is called a *Bloch equation*.

$$i\hbar\frac{\partial}{\partial t}\boldsymbol{\rho} = i\hbar\dot{\boldsymbol{\rho}} = \mathbf{H}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{H} = \left[\mathbf{H},\boldsymbol{\rho}\right]$$

Given $\boldsymbol{\rho}$ and $\mathbf{H}$ in terms *spin* $\mathbf{S}$-vector and *crank* $\boldsymbol{\Omega}$-vector:

$$\mathbf{H}\boldsymbol{\rho} = \left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)$$

$$-\boldsymbol{\rho}\mathbf{H} = \left(\frac{N}{2}\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\hbar\Omega_0\mathbf{1} + \frac{\hbar}{2}\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right) = \hbar\Omega_0\frac{N}{2}\mathbf{1} + \frac{N}{4}\hbar\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma} + \hbar\Omega_0\vec{\mathbf{S}}\bullet\boldsymbol{\sigma} + \frac{\hbar}{2}\left(\vec{\mathbf{S}}\bullet\boldsymbol{\sigma}\right)\left(\vec{\boldsymbol{\Omega}}\bullet\boldsymbol{\sigma}\right)$$

U(2) density operator approach to symmetry dynamics

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
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Bloch equation for density operator

Ket equation (time *forward*) and "daggered" bra-equation (time *reversed*).

$$i\hbar|\dot{\Psi}\rangle = \mathbf{H}|\Psi\rangle, \quad \Leftarrow Daggar^\dagger \Rightarrow \quad -i\hbar\langle\dot{\Psi}| = \langle\Psi|\mathbf{H}$$

Note: $\mathbf{H}^\dagger = \mathbf{H}$.
$\rho^\dagger = \rho$

Combining these gives a time derivative of the density operator $\rho = |\Psi\rangle\langle\Psi|$

$$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\dot{\rho} = i\hbar|\dot{\Psi}\rangle\langle\Psi| + i\hbar|\Psi\rangle\langle\dot{\Psi}| = \mathbf{H}|\Psi\rangle\langle\Psi| - |\Psi\rangle\langle\Psi|\mathbf{H}$$

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U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

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U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

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U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

$$\boldsymbol{\rho} = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
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U(2) density operator approach to symmetry dynamics

Bloch equation for density operator

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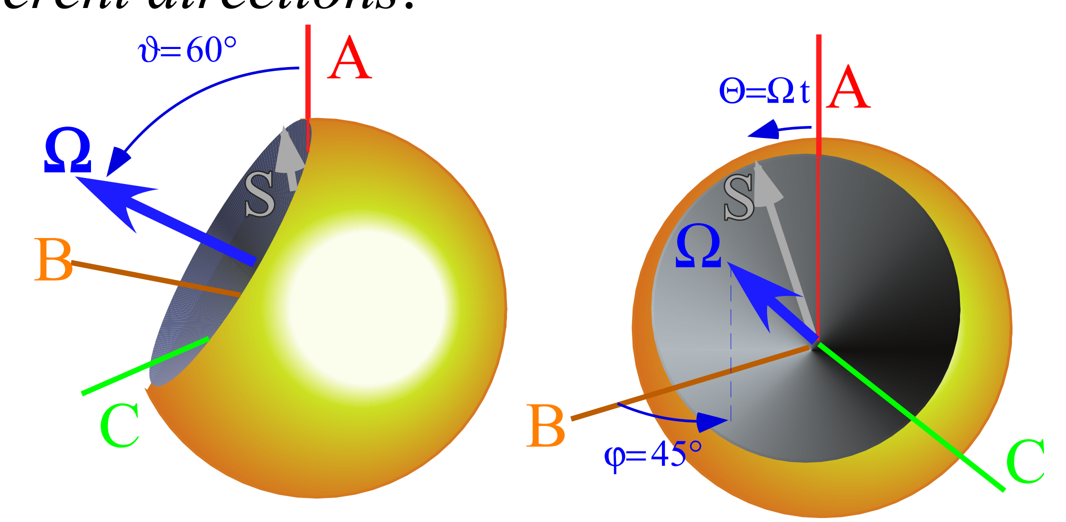

Factoring out $\bullet\boldsymbol{\sigma}$ gives a classical/quantum *gyro-precession equation*. $\dfrac{\partial\vec{\mathbf{S}}}{\partial t} = \dot{\vec{\mathbf{S}}} = \vec{\boldsymbol{\Omega}}\times\vec{\mathbf{S}}$