Lecture 22
Tue. 11.17.2015

Introduction to Spinor-Vector resonance dynamics

(Ch. 2-4 of Unit 4 Ch. 6-7 of Unit 6)

Review: *2D harmonic oscillator equations with Lagrangian and matrix forms*

ANALOGY: *2-State Schrodinger:* $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials (“Crazy-Thing”-Theorem)

Geometry of evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The “mysterious” factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

Euler’s state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$*, and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Spin-1/2 (2D-complex spinor) case

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A=S_Z$*, Balance* $S_B=S_X$*, and Chirality* $S_C=S_Y$

Polarization ellipse and spinor state dynamics

The “Great Spectral Avoided-Crossing” and A-to-B-to-A symmetry breaking

See also: QTCA Lect. 9(2.12) p.61-103 for polarization ellipsometry

2D harmonic oscillator equations

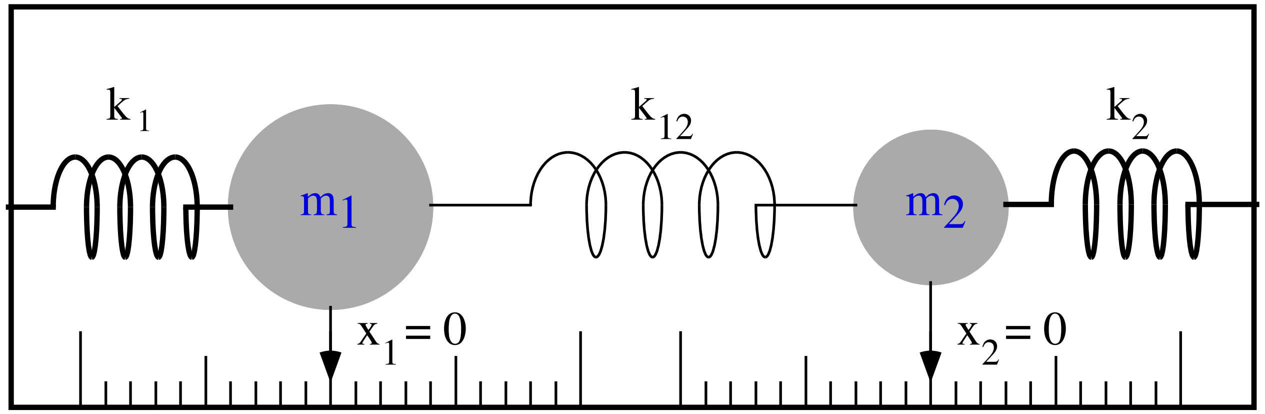
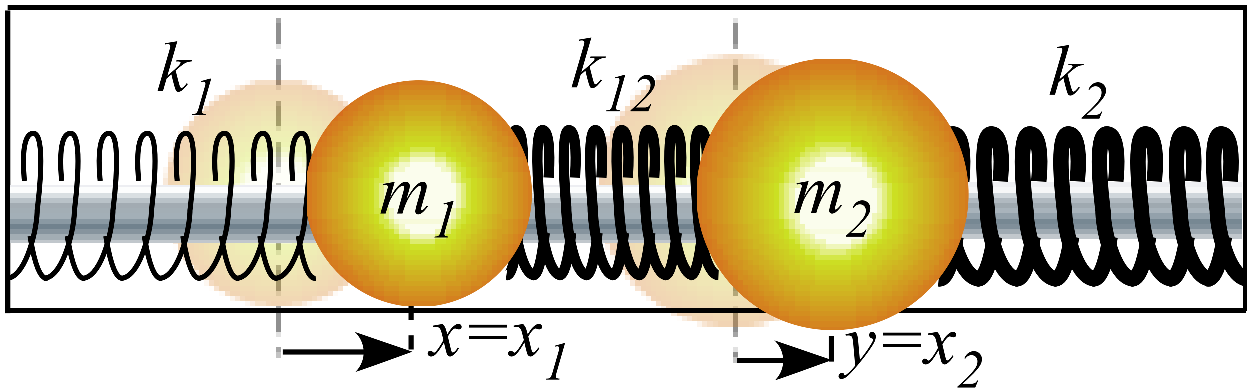


Fig. 3.3.1 Two 1-dimensional coupled oscillators



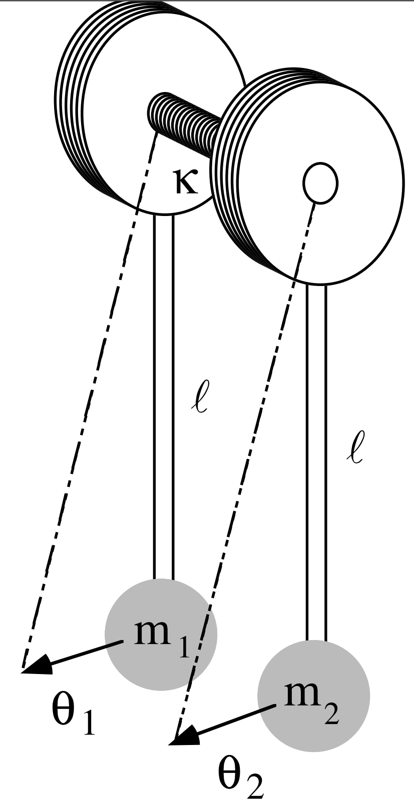


Fig. 3.3.2 Coupled pendulums

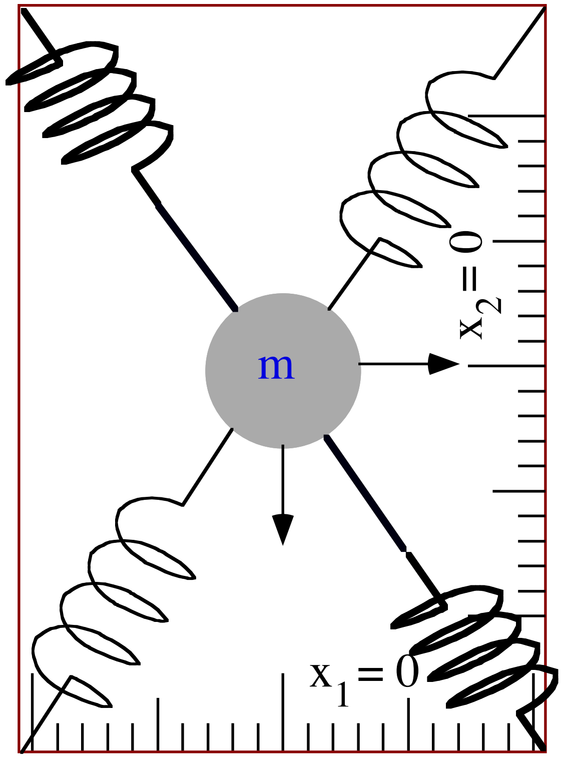


Fig. 3.3.3 One 2-dimensional coupled oscillator

(Review of Lect. 23)

2D HO kinetic energy $T(v_1, v_2)$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$= \frac{1}{2}\langle \dot{\mathbf{x}}|\mathbf{M}|\dot{\mathbf{x}}\rangle$$

2D HO potential energy $V(x_1, x_2)$

$$V = \frac{1}{2}(k_1 + k_{12})x_1^2 - k_{12}x_1x_2 + \frac{1}{2}(k_2 + k_{12})x_2^2$$

$$= \frac{1}{2}\langle \mathbf{x}|\mathbf{K}|\mathbf{x}\rangle \quad \text{where: } \mathbf{K} = \begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}$$

Lagrangian $L=T-V$

2D HO Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1 = F_1 = -\frac{\partial V}{\partial x_1} = -(k_1 + k_{12})x_1 + k_{12}x_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2 = F_2 = -\frac{\partial V}{\partial x_2} = k_{12}x_1 - (k_2 + k_{12})x_2$$

2D HO Matrix operator equations

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Matrix operator notation:

$$\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$$

2D harmonic oscillator equation solutions *(Review of Lect. 23)*

1. May rewrite equation $\mathbf{M}\bullet|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$ in *acceleration* matrix form: $|\ddot{\mathbf{x}}\rangle = -\mathbf{A}|\mathbf{x}\rangle$ where: $\mathbf{A} = \mathbf{M}^{-1}\bullet\mathbf{K}$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}^{-1} \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \frac{k_1+k_{12}}{m_1} & \frac{-k_{12}}{m_1} \\ \frac{-k_{12}}{m_2} & \frac{k_2+k_{12}}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2. Need to find *eigenvectors* $|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, \ldots$ of acceleration matrix such that: $\mathbf{A}|\mathbf{e}_n\rangle = \varepsilon_n|\mathbf{e}_n\rangle = \omega_n^2|\mathbf{e}_n\rangle$

Then equations decouple to: $|\ddot{\mathbf{e}}_n\rangle = -\mathbf{A}|\mathbf{e}_n\rangle = -\varepsilon_n|\mathbf{e}_n\rangle = -\omega_n^2|\mathbf{e}_n\rangle$ where ε_n is an *eigenvalue*

and ω_n is an *eigenfrequency*

Note eigenvalue is square of eigenfrequency

To introduce eigensolutions we take a simple case of unit masses *(m_1=1=m_2)*

So equation of motion is simply: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$

Eigenvectors $|\mathbf{x}\rangle = |\mathbf{e}_n\rangle$ are in special directions where $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}|\mathbf{x}\rangle$ is in same direction as $|\mathbf{x}\rangle$

(Review of Lect. 23)

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H_{jk} matrix must obey: $(H_{jk})^* = H_{kj}$

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$$\begin{pmatrix} i\dot{x}_1 - \dot{p}_1 \\ i\dot{x}_2 - \dot{p}_2 \end{pmatrix} = \begin{pmatrix} Ax_1 + Bx_2 + Cp_2 + iAp_1 + iBp_2 - iCx_2 \\ Bx_1 + Dx_2 - Cp_1 + iBp_1 + iDp_2 + iCx_1 \end{pmatrix}$$

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Finally a 2^{nd} time derivative (Assume *constant* $A, B, D,$ and *let* $C=0$) gives 2^{nd}-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle=-\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1&=A\dot{p}_1+B\dot{p}_2-C\dot{x}_2\\&=-A\left(Ax_1+Bx_2+Cp_2\right)-B\left(Bx_1+Dx_2-Cp_1\right)-C\left(Bp_1+Dp_2+Cx_1\right)\\&=-\left(A^2+B^2+C^2\right)x_1-\left(AB+BD\right)x_2-C\left(A+D\right)p_2\end{aligned}$$

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For constant $A,B,C,$ and D

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$$\dot{x}_2 = \frac{\partial H_c}{\partial p_2} = Bp_1 + Dp_2 + Cx_1 \qquad \dot{p}_2 = -\frac{\partial H_c}{\partial x_2} = -\left(Bx_1 + Dx_2 - Cp_1\right)$$

Finally a 2nd time derivative (Assume *constant* *A, B, D,* and *let C=0*) gives 2nd-order classical Newton-Hooke-like equation: $|\ddot{\mathbf{x}}\rangle = -\mathbf{K}\bullet|\mathbf{x}\rangle$

$$\begin{aligned}\ddot{x}_1 &= A\dot{p}_1 + B\dot{p}_2 - C\dot{x}_2 \\ &= -A\left(Ax_1 + Bx_2 + Cp_2\right) - B\left(Bx_1 + Dx_2 - Cp_1\right) - C\left(Bp_1 + Dp_2 + Cx_1\right) \\ &= -\left(A^2 + B^2 + C^2\right)x_1 - \left(AB + BD\right)x_2 - C\left(A+D\right)p_2\end{aligned}$$

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For constant A,B,C, and D

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} A^2 + B^2 & AB + BD \\ AB + BD & B^2 + D^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For C=0 Is form of 2D Hooke harmonic oscillator

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus Classical 2D-HO: $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

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➡ *Hamilton-Pauli spinor symmetry (* $\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic
Spinor vector algebra like complex vector algebra
Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

Geometry of evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space
2D Spinor vs 3D vector rotation
NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

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...current-carrier...

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*

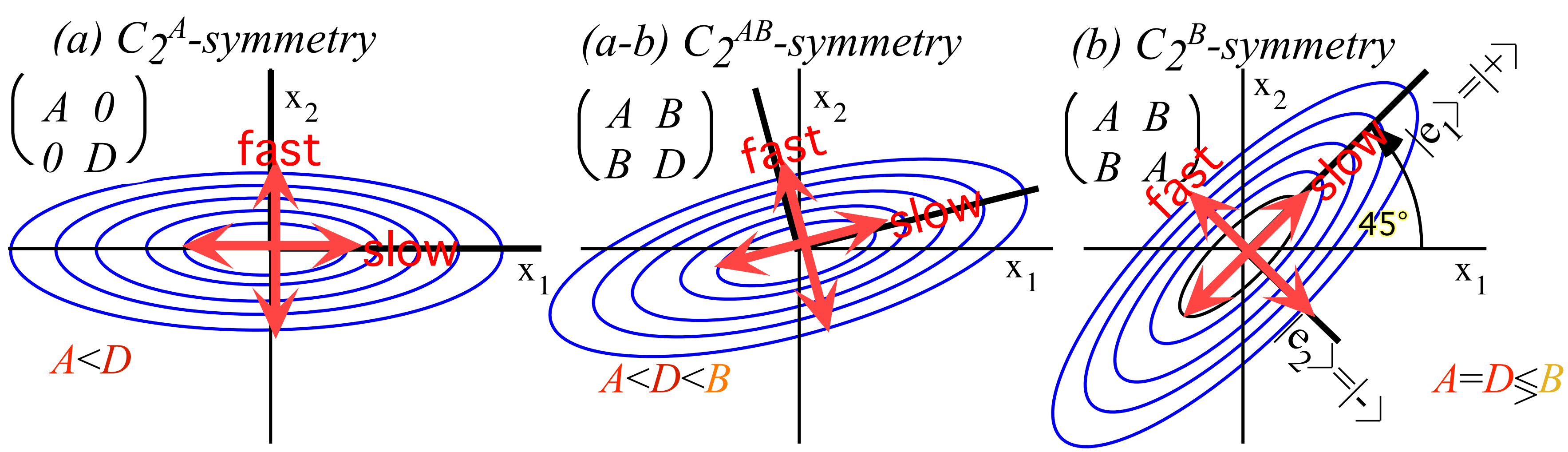

Fig. 3.4.1 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral U(2)system.

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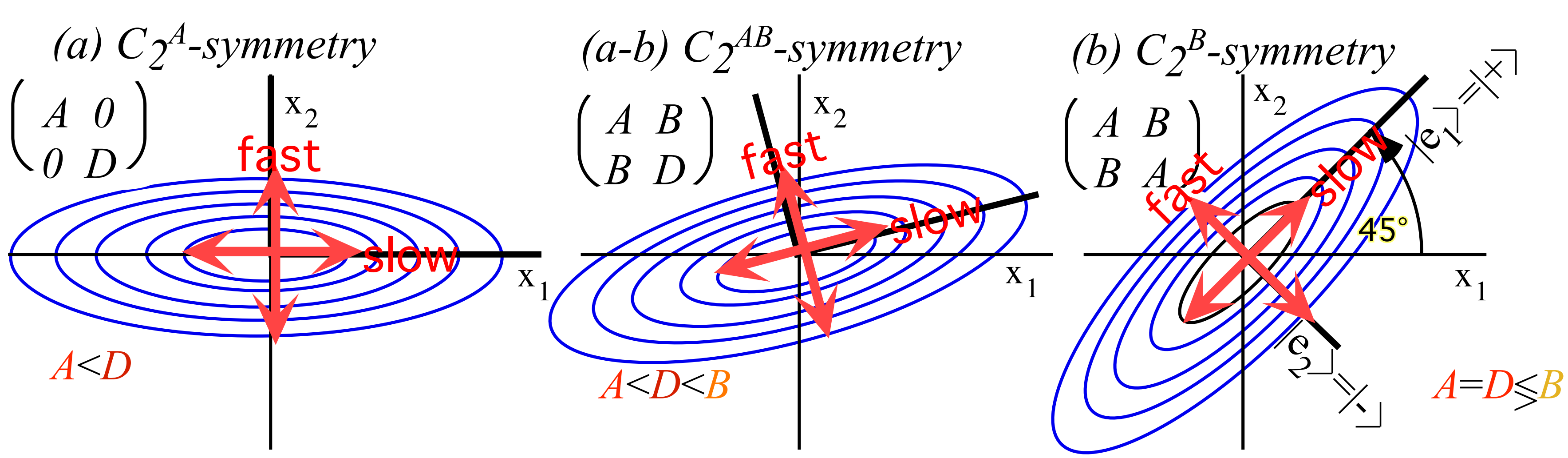

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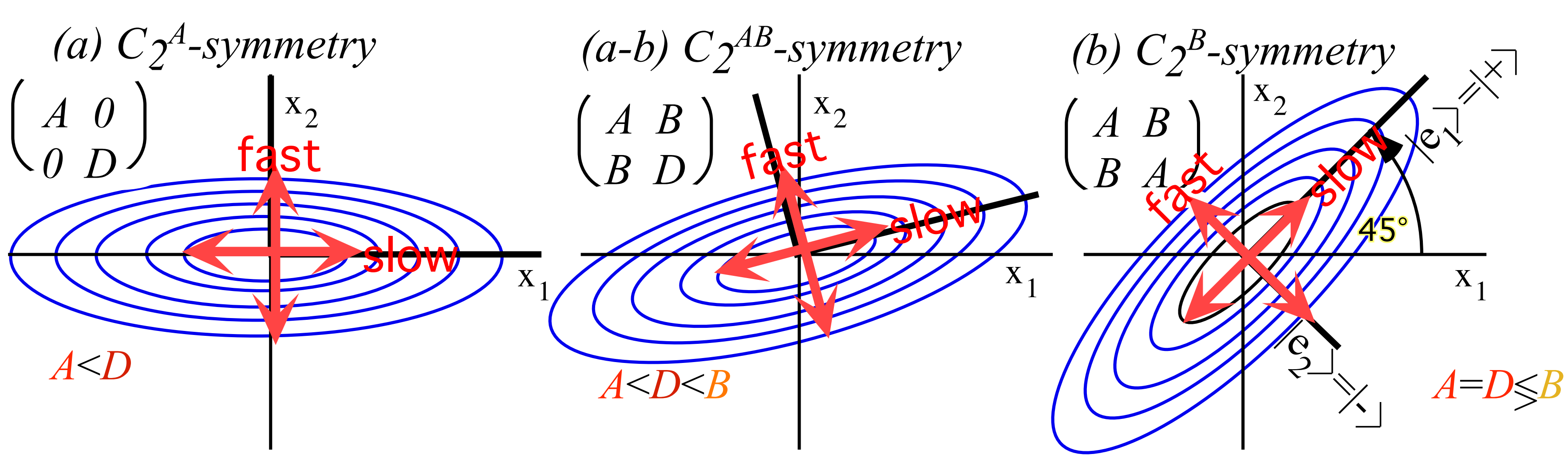

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Each Hamilton quaternion squares to *negative*-**1** ($\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-\mathbf{1}$) like imaginary number $i^2=-1$. (They make up the Quaternion group.)

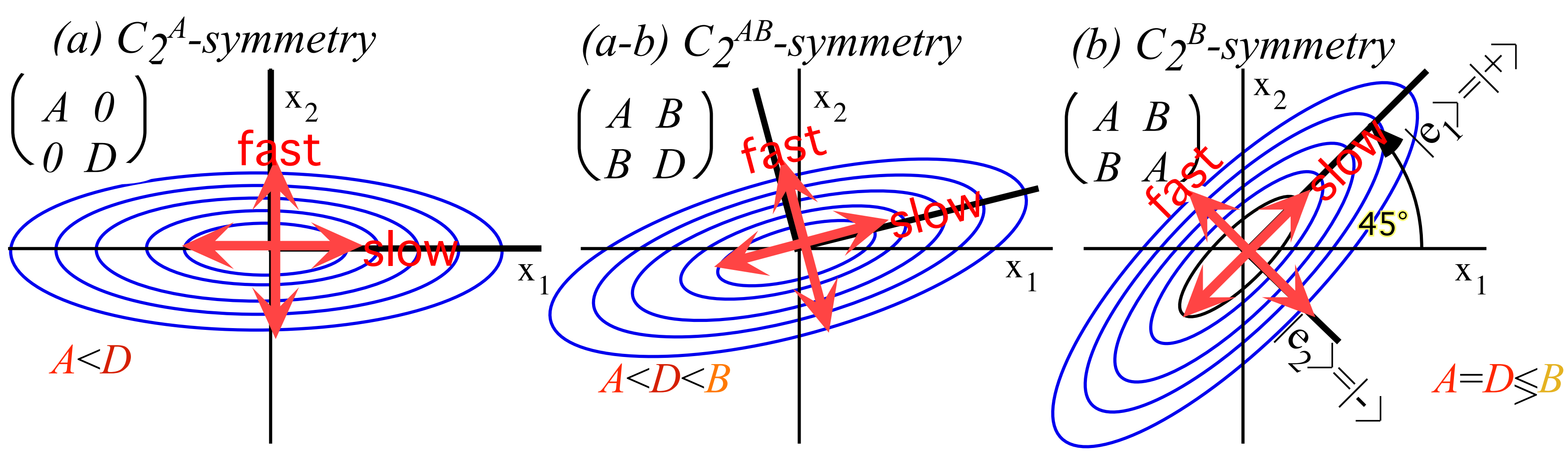


Fig. 3.4.1 Potentials for (a) C_2^A-asymmetric-diagonal, (ab) C_2^{AB}-mixed , (b) C_2^B-bilateral U(2)system.

ABCD Symmetry operator analysis and U(2) spinors

Decompose the Hamiltonian operator **H** into four *ABCD symmetry operators*
(Labeled to provide dynamic mnemonics as well as colorful analogies)

$$\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + D\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A\mathbf{e}_{11} + B\boldsymbol{\sigma}_B + C\boldsymbol{\sigma}_C + D\mathbf{e}_{22}$$

$$= \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{H} = \frac{A-D}{2}\boldsymbol{\sigma}_A + B\,\boldsymbol{\sigma}_B + C\,\boldsymbol{\sigma}_C + \frac{A+D}{2}\boldsymbol{\sigma}_0$$

...current-carrier...

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex-Coriolis-cyclotron-curly...)*

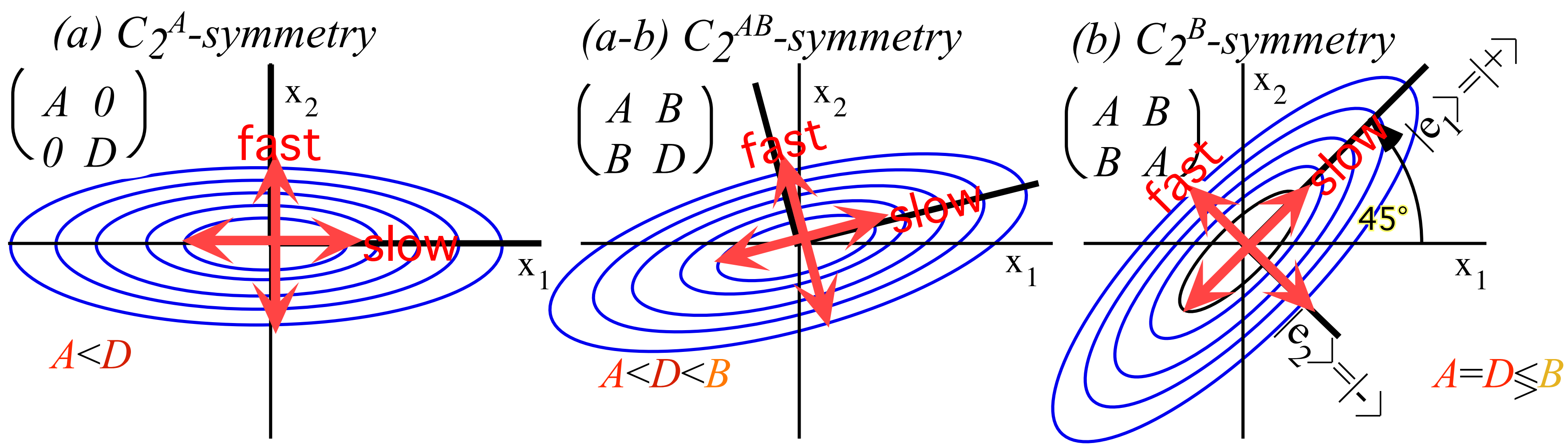
The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are best known as *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$ developed in 1927.

Wolfgang Pauli (1927) Zur Quantenmechanik des magnetischen Elektrons *Zeitschrift für Physik* (43) 601-623

In 1843 Hamilton invents *quaternions* $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. σ_μ related by *i*-factor: $\{\sigma_1=\mathbf{1}=\sigma_0, i\sigma_B=\mathbf{i}=i\sigma_X, i\sigma_C=\mathbf{j}=i\sigma_Y, i\sigma_A=\mathbf{k}=i\sigma_Z\}$.

Each Hamilton quaternion squares to *negative*-**1** ($\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=-\mathbf{1}$) like imaginary number $i^2=-1$. (They make up the Quaternion group.)

Each Pauli σ_μ squares to *positive*-**1** ($\sigma_X^2=\sigma_Y^2=\sigma_Z^2=+\mathbf{1}$) (Each makes a cyclic C_2 group $C_2^A=\{\mathbf{1},\sigma_A\}$, $C_2^B=\{\mathbf{1},\sigma_B\}$, or $C_2^C=\{\mathbf{1},\sigma_C\}$.)



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ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry (**σ***-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

➜ *Derive* **σ***-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

➜ *Spinor arithmetic* *like* *complex arithmetic*

Spinor vector algebra *like* *complex vector algebra*

Spinor exponentials *like* *complex exponentials* *("Crazy-Thing"-Theorem)*

Geometry of evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$|\Psi(t)\rangle = e^{-i\mathbf{H}t}|\Psi(0)\rangle$$

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}t}$$

$\sigma_A = \sigma_Z$ $\sigma_B = \sigma_X$ $\sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0 t} = e^{-i\vec{\sigma}\bullet\vec{\omega}t} e^{-i\omega_0 t} \text{ where: } \vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t \text{ and: } \omega_0 = \frac{A+D}{2}$$

ABCD Time evolution operator

For constant A,B,C, and D

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions **i**$=-i\sigma_X$, **j**$=-i\sigma_Y$, and **k**$=-i\sigma_Z$ powerful.

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$$\begin{matrix} & a_X\sigma_X a_X\sigma_X & +a_X\sigma_X a_Y\sigma_Y & +a_X\sigma_X a_Z\sigma_Z & & a_X a_X\sigma_X\sigma_X & +a_X a_Y\sigma_X\sigma_Y & +a_X a_Z\sigma_X\sigma_Z \\ = & +a_Y\sigma_Y a_X\sigma_X & +a_Y\sigma_Y a_Y\sigma_Y & +a_Y\sigma_Y a_Z\sigma_Z & = & +a_Y a_X\sigma_Y\sigma_X & +a_Y a_Y\sigma_Y\sigma_Y & +a_Y a_Z\sigma_Y\sigma_Z \\ & +a_Z\sigma_Z a_X\sigma_X & +a_Z\sigma_Z a_Y\sigma_Y & +a_Z\sigma_Z a_Z\sigma_Z & & +a_Z a_X\sigma_Z\sigma_X & +a_Z a_Y\sigma_Z\sigma_Y & +a_Z a_Z\sigma_Z\sigma_Z \end{matrix}$$

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$$\sigma_Z \;\square\; \sigma_X$$
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_Y$$

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So: $\sigma_a^2 = \mathbf{1}$

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(the off-diagonal terms are struck through in cancelling pairs)

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$$= \begin{matrix} a_Xb_X\mathbf{1} & +a_Xb_Y\sigma_X\sigma_Y & -a_Xb_Z\sigma_Z\sigma_X \\ -a_Yb_X\sigma_X\sigma_Y & +a_Yb_Y\mathbf{1} & +a_Yb_Z\sigma_Y\sigma_Z \\ +a_Zb_X\sigma_Z\sigma_X & -a_Zb_X\sigma_Y\sigma_Z & +a_Zb_Z\mathbf{1} \end{matrix} = (a_Xb_X + a_Yb_Y + a_Zb_Z)\mathbf{1} \begin{matrix} +i(a_Yb_Z - a_Zb_Y)\sigma_X \\ +i(a_Zb_X - a_Xb_Z)\sigma_Y \\ +i(a_Xb_Y - a_Yb_X)\sigma_Z \end{matrix}$$

$$\sigma_Z \cdot \sigma_X: \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} = i\sigma_Y$$

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$$\begin{array}{llll} & a_X b_X \mathbf{1} & +a_X b_Y \sigma_X\sigma_Y & -a_X b_Z \sigma_Z\sigma_X \\ = & -a_Y b_X \sigma_X\sigma_Y & +a_Y b_Y \mathbf{1} & +a_Y b_Z \sigma_Y\sigma_Z \\ & +a_Z b_X \sigma_Z\sigma_X & -a_Z b_X \sigma_Y\sigma_Z & +a_Z b_Z \mathbf{1}\end{array} = (a_X b_X + a_Y b_Y + a_Z b_Z)\mathbf{1} \begin{array}{l} +i(a_Y b_Z - a_Z b_Y)\sigma_X \\ +i(a_Z b_X - a_X b_Z)\sigma_Y \\ +i(a_X b_Y - a_Y b_X)\sigma_Z\end{array}$$

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(Recall (1.10.29). in complex variable unit.)

$$A^*B = (A_X + iA_Y)^*(B_X + iB_Y) = (A_X - iA_Y)(B_X + iB_Y)$$
$$= (A_XB_X + A_YB_Y) + i(A_XB_Y - A_YB_X) = (\mathbf{A}\bullet\mathbf{B}) + i(\mathbf{A}\times\mathbf{B})_Z$$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle=\mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x}=-\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H}=\omega_\mu\boldsymbol{\sigma}_\mu$

➔ *Derive* $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

➔ *Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)*

Geometry of evolution (or revolution) operator $\mathbf{U}=e^{-i\mathbf{H}t}=e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

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This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_Z\sigma_Z \doteq (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$$e^{-i\varphi} = 1\ \cos\varphi - i\ \sin\varphi$$

generalizes to:

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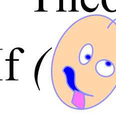
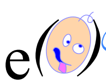
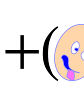
The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{(\ )\varphi}$  = **1** cos φ + () sin φ

This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi \varphi}$ for any $\sigma_\varphi \varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_Z\sigma_Z \doteqdot (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$$e^{-i\varphi} = 1\ \cos\varphi - i\ \sin\varphi$$

generalizes to:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi - i\ \sigma_\varphi \sin\varphi$$

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ so matrix exponential becomes powerful.

ABCD Time evolution operator

$$e^{-i\mathbf{H}t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}t}$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

$$= e^{-i\sigma_\varphi \varphi} e^{-i\omega_0 t} = e^{-i\vec{\sigma}\bullet\vec{\omega}t} e^{-i\omega_0 t} \text{ where: } \vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t \text{ and: } \omega_0 = \frac{A+D}{2}$$

$$= (\mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi)e^{-i\omega_0 t}$$

For constant A,B,C, and D

Symmetry relations make spinors $\sigma_X = \sigma_B$, $\sigma_Y = \sigma_C$, and $\sigma_Z = \sigma_A$ or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

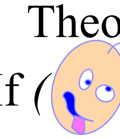
Hamilton is able to generalize Euler's complex rotation operators $e^{+i\varphi}$ and $e^{-i\varphi}$. (Recall (1.10.17).)

$$e^{-i\varphi} = 1 + (-i\varphi) + \frac{1}{2!}(-i\varphi)^2 + \frac{1}{3!}(-i\varphi)^3 + \frac{1}{4!}(-i\varphi)^4 \cdots = [1 - \frac{1}{2!}\varphi^2 + \frac{1}{4!}\varphi^4 \cdots] = [\cos\varphi]$$

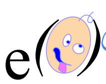
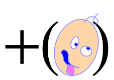
$$- i(\varphi + \frac{1}{3!}\varphi^3 \cdots) \quad - i(\sin\varphi)$$

Note even powers of *(-i)* are ±*1* and odd powers of *(-i)* are ±*i*.: $(-i)^0 = +1,\ (-i)^1 = -i,\ (-i)^2 = -1,\ (-i)^3 = +i,\ (-i)^4 = +1,\ (-i)^5 = -i,\ etc.$

The Crazy Thing Theorem:

If ()^2 = **-1**

Then:

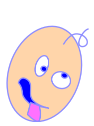
e$^{(}$  $^{)\varphi}$ = $\mathbf{1}\cos\varphi$ + ()$\sin\varphi$

Hamilton replaces *(-i)* with $-i\sigma_\varphi$ in the $e^{-i\varphi}$ power series above to get a sequence of terms just like it.

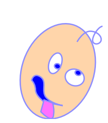
$$(-i\sigma_\varphi)^0 = +\mathbf{1},\ (-i\sigma_\varphi)^1 = -i\sigma_\varphi,\ (-i\sigma_\varphi)^2 = -\mathbf{1},\ (-i\sigma_\varphi)^3 = +i\sigma_\varphi,\ (-i\sigma_\varphi)^4 = +\mathbf{1},\ (-i\sigma_\varphi)^5 = -i\sigma_\varphi,\ etc.$$

This allows Hamilton to generalize Euler's rotation $e^{-i\varphi}$ to $e^{-i\sigma_\varphi\varphi}$ for any $\sigma_\varphi\varphi = (\boldsymbol{\sigma}\bullet\vec{\varphi}) = \varphi_A\sigma_A + \varphi_B\sigma_B + \varphi_Z\sigma_Z \doteq (\boldsymbol{\sigma}\bullet\hat{\varphi})\varphi$

$e^{-i\varphi} = 1\ \cos\varphi\ -\ i\ \sin\varphi$ *generalizes to:* $e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi\ -\ i\ \sigma_\varphi \sin\varphi$

Here:  = *-i*

Crazy thing is just -√-1

Here:  $= -i\sigma_\varphi = -i(\boldsymbol{\sigma}\bullet\hat{\varphi}) = -i\frac{(\boldsymbol{\sigma}\bullet\vec{\varphi})}{\varphi}$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

Spinor arithmetic like complex arithmetic

Spinor vector algebra like complex vector algebra

Spinor exponentials like complex exponentials ("Crazy-Thing"-Theorem)

➤ *Geometry of evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}t}\left|\Psi(0)\right\rangle$$

ABCD Time evolution operator

For constant A,B,C, and D

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$$e^{-i\mathbf{H}t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}t - iC\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})t} = e^{-i\omega_0 t}\left(\mathbf{1}\cos\omega t - i\sigma_\varphi \sin\omega t\right)$$

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$$e^{-i\varphi} = 1\cos\varphi - i\sin\varphi$$

generalizes to:

$$e^{-i\sigma_\varphi\varphi} = \mathbf{1}\cos\varphi - i\,\sigma_\varphi \sin\varphi$$

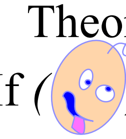
where: $\vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\sin\varphi_A$$

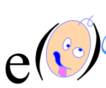
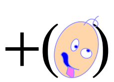
$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1:
A or Z rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{(}$  $^{)\varphi}$ = **1** $\cos\varphi$ + () $\sin\varphi$

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$$e^{-i\varphi} = 1\ \cos\varphi \ - \ i\ \sin\varphi$$

generalizes to:

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$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

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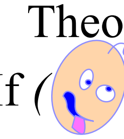
Example 1:
A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

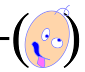
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2:
C or Y rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e$^{()\varphi}$ = **1** cos φ + () sin φ

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$$e^{-i\varphi} = 1\ \cos\varphi\ -\ i\ \sin\varphi$$

generalizes to:

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$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

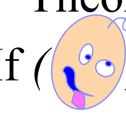
Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

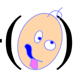
$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

The Crazy Thing Theorem:

If ()^2 = -**1**

Then:

e()$^{\varphi}$ = $\mathbf{1}\cos\varphi$ + ()$\sin\varphi$

Example 3: Any $\varphi = \omega t$-axial rotation

Let: $\vec{\varphi} = \vec{\omega}\cdot t$

$$e^{-i(\sigma\bullet\vec{\varphi})t} = \mathbf{1}\cos\varphi - i\sigma_\varphi \sin\varphi = \mathbf{1}\cos\varphi - i\ (\sigma\bullet\hat{\varphi})\sin\varphi$$

$$= \mathbf{1}\cos\varphi\ - i\ (\sigma_A\hat{\varphi}_A)\sin\varphi\ - i\ (\sigma_B\hat{\varphi}_B)\sin\varphi\ - i\ (\sigma_C\hat{\varphi}_C)\sin\varphi$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\hat{\varphi}_A \sin\varphi - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{\varphi}_B \sin\varphi - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\hat{\varphi}_C \sin\varphi$$

$$= \begin{pmatrix} \cos\varphi - i\hat{\varphi}_A \sin\varphi & (-i\hat{\varphi}_B - \hat{\varphi}_C)\sin\varphi \\ (-i\hat{\varphi}_B + \hat{\varphi}_C)\sin\varphi & \cos\varphi + i\hat{\varphi}_A \sin\varphi \end{pmatrix}$$

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$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}t}\left|\Psi(0)\right\rangle$$

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$$e^{-i\mathbf{H}t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}t - iC\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})t} = e^{-i\omega_0 t}\left(\mathbf{1}\cos\omega t - i\sigma_\varphi \sin\omega t\right)$$

$\sigma_A = \sigma_Z$ $\sigma_B = \sigma_X$ $\sigma_C = \sigma_Y$

$$e^{-i\varphi} = 1\cos\varphi - i\sin\varphi$$

generalizes to:

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where: $\vec{\varphi} = \begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t = \begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\varphi_A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_A - i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\varphi_A$$

$$= \begin{pmatrix} \cos\varphi_A - i\sin\varphi_A & 0 \\ 0 & \cos\varphi_A - i\sin\varphi_A \end{pmatrix} = \begin{pmatrix} e^{-i\varphi_A} & 0 \\ 0 & e^{i\varphi_A} \end{pmatrix}$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\varphi_C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\varphi_C - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\varphi_C$$

$$= \begin{pmatrix} \cos\varphi_C & -\sin\varphi_C \\ \sin\varphi_C & \cos\varphi_C \end{pmatrix}$$

Example 2: C or Y rotation

We test these operators by making them rotate each other....

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

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$$e^{-i\mathbf{H}t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}t} = e^{-i\frac{A-D}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}t - iB\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}t - iC\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})t} = e^{-i\omega_0 t}\left(\mathbf{1}\cos\omega t - i\sigma_{\varphi}\sin\omega t\right)$$

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$$e^{-i\varphi} = 1\ \cos\varphi \ -\ i\ \sin\varphi$$

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$$e^{-i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\varphi_A} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\sin\varphi_A$$
$$= \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0\\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix}e^{-i\varphi_A} & 0\\ 0 & e^{i\varphi_A}\end{pmatrix}$$

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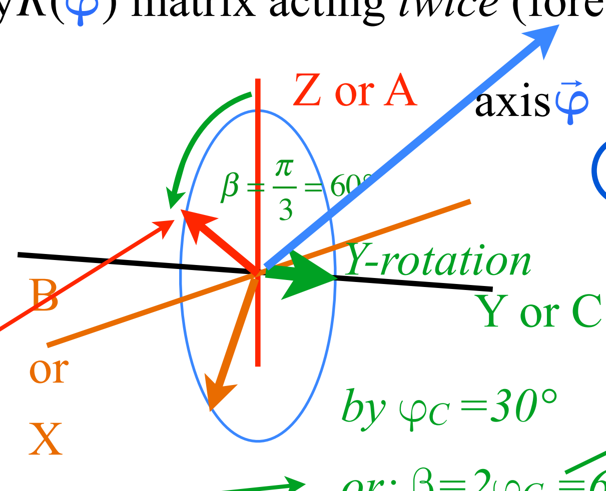
$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

Example 2: C or Y rotation

3D axis vector $\vec{\varphi} = \vec{\omega}\cdot t$ corresponds to generator $\sigma_{\varphi} = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_{\varphi}\varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_{\varphi}\sin\varphi$ about axis $\vec{\varphi}$.

Any 2-by-2 σ_{μ}-matrix may be rotated by any $R(\vec{\varphi})$ matrix acting *twice* (fore-and-aft^{-1}) to give: $\sigma_{\mu}^{(\vec{\varphi}\text{-rotated})} = R(\vec{\varphi})\sigma_{\mu}R^{-1}(\vec{\varphi}) = R(\vec{\varphi})\sigma_{\mu}R^{\dagger}(\vec{\varphi})$

$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$
$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$
$$= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sin 2\varphi_C$$
$$= \boldsymbol{\sigma}_A\cos 2\varphi_C + \boldsymbol{\sigma}_B\sin 2\varphi_C$$



The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

$$\hat{\varphi} = \begin{pmatrix}\hat{\varphi}_A \\ \hat{\varphi}_B \\ \hat{\varphi}_C\end{pmatrix} = \begin{pmatrix}\varphi_A \\ \varphi_B \\ \varphi_C\end{pmatrix}\frac{1}{\sqrt{\varphi_A^2 + \varphi_B^2 + \varphi_C^2}} = \begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix}\frac{1}{\sqrt{\omega_A^2 + \omega_B^2 + \omega_C^2}}$$

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}t}\left|\Psi(0)\right\rangle$$

ABCD Time evolution operator

For constant A,B,C, and D

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}t} = e^{-i\frac{A-D}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}t - iB\begin{pmatrix}0&1\\1&0\end{pmatrix}t - iC\begin{pmatrix}0&-i\\i&0\end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})t} = e^{-i\omega_0 t}\left(\mathbf{1}\cos\omega t - i\sigma_\varphi \sin\omega t\right)$$

$\boldsymbol{\sigma}_A = \boldsymbol{\sigma}_Z$ $\boldsymbol{\sigma}_B = \boldsymbol{\sigma}_X$ $\boldsymbol{\sigma}_C = \boldsymbol{\sigma}_Y$

$$e^{-i\varphi} = 1\cos\varphi - i\sin\varphi$$

generalizes to:

$$e^{-i\sigma_\varphi \varphi} = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$$

where: $\vec{\varphi} = \begin{pmatrix}\varphi_A\\ \varphi_B\\ \varphi_C\end{pmatrix} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A\\ \omega_B\\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2}\\ B\\ C\end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$

$$e^{-i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\varphi_A} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_A - i\begin{pmatrix}1&0\\0&-1\end{pmatrix}\sin\varphi_A = \begin{pmatrix}\cos\varphi_A - i\sin\varphi_A & 0\\ 0 & \cos\varphi_A - i\sin\varphi_A\end{pmatrix} = \begin{pmatrix}e^{-i\varphi_A} & 0\\ 0 & e^{i\varphi_A}\end{pmatrix}$$

Example 1: A or Z rotation

$$e^{-i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\varphi_C} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\cos\varphi_C - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\varphi_C = \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

Example 2: C or Y rotation

3D axis vector $\vec{\varphi} = \vec{\omega}\cdot t$ corresponds to generator $\sigma_\varphi = \sigma_A\hat{\varphi}_A + \sigma_B\hat{\varphi}_B + \sigma_C\hat{\varphi}_C$ of *rotation* $e^{-i\sigma_\varphi\varphi} = R(\vec{\varphi}) = \mathbf{1}\cos\varphi - i\sigma_\varphi\sin\varphi$ about axis $\vec{\varphi}$.

Any 2-by-2 σ_μ-matrix may be rotated by any $R(\vec{\varphi})$ matrix acting *twice* (fore-and-aft^{-1}) to give: $\sigma_\mu^{(\vec{\varphi}\text{-rotated})} = R(\vec{\varphi})\sigma_\mu R^{-1}(\vec{\varphi}) = R(\vec{\varphi})\sigma_\mu R^\dagger(\vec{\varphi})$

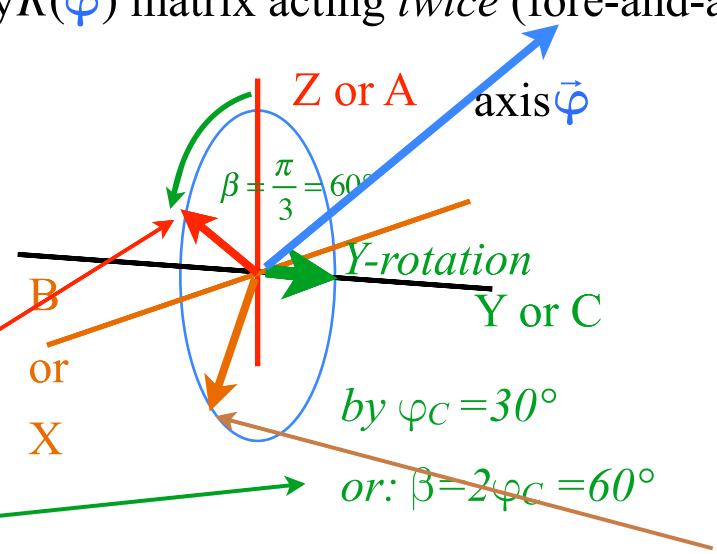
$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_A\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}\cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\\ 2\sin\varphi_C\cos\varphi_C & \sin^2\varphi_C - \cos^2\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos 2\varphi_C + \begin{pmatrix}0&1\\1&0\end{pmatrix}\sin 2\varphi_C$$

$$= \boldsymbol{\sigma}_A\cos 2\varphi_C + \boldsymbol{\sigma}_B\sin 2\varphi_C$$



$$\mathbf{R}(\varphi_C)\cdot\boldsymbol{\sigma}_B\cdot\mathbf{R}^{-1}(\varphi_C)$$

$$= \begin{pmatrix}\cos\varphi_C & -\sin\varphi_C\\ \sin\varphi_C & \cos\varphi_C\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\cos\varphi_C & \sin\varphi_C\\ -\sin\varphi_C & \cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}-2\sin\varphi_C\cos\varphi_C & \cos^2\varphi_C - \sin^2\varphi_C\\ \cos^2\varphi_C - \sin^2\varphi_C & 2\sin\varphi_C\cos\varphi_C\end{pmatrix}$$

$$= \begin{pmatrix}-1&0\\0&1\end{pmatrix}\sin 2\varphi_C + \begin{pmatrix}0&1\\1&0\end{pmatrix}\cos 2\varphi_C$$

$$= -\boldsymbol{\sigma}_A\sin 2\varphi_C + \boldsymbol{\sigma}_B\cos 2\varphi_C$$

The *3D*-rotation is by 2φ, *twice* the *2D* angle φ.

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

Spinor arithmetic	*like*	*complex arithmetic*
Spinor vector algebra	*like*	*complex vector algebra*
Spinor exponentials	*like*	*complex exponentials*

Geometry of evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

➔ *The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space*

2D Spinor vs 3D vector rotation

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \omega_0 \sigma_0 + \omega_A \sigma_A + \omega_B \sigma_B + \omega_C \sigma_C = \omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma} = \omega_0 \mathbf{1} + \omega \boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1 = \boldsymbol{\sigma}_0, \sigma_B = \sigma_X, \sigma_C = \sigma_Y, \sigma_A = \sigma_Z\}$

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Notation for 2D Spinor space

$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\sigma_\omega$$

$$=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0^{th} component unchanged

components A, B, C switch 1/2-factor from ω-velocity to S-momentum

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

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$$=\omega_0\sigma_0+\omega_A\sigma_A+\omega_B\sigma_B+\omega_C\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$=\Omega_0\mathbf{1}+\Omega_A\mathbf{S}_A+\Omega_B\mathbf{S}_B+\Omega_C\mathbf{S}_C=\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

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The $\{\mathbf{1}, \mathsf{S}_A, \mathsf{S}_B, \mathsf{S}_C\}$ are the *Jordan-Angular-Momentum operators* $\{\mathbf{1}=\sigma_0, \mathsf{S}_B=\mathsf{S}_X, \mathsf{S}_C=\mathsf{S}_Y, \mathsf{S}_A=\mathsf{S}_Z\}$
(Often labeled $\{\mathsf{J}_X, \mathsf{J}_Y, \mathsf{J}_Z\}$)

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

$$\mathbf{H}=\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$=\omega_0\ \sigma_0 + \omega_A\ \sigma_A + \omega_B\ \sigma_B + \omega_C\ \sigma_C = \omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

Notation for 2D Spinor space

$$=\Omega_0\ \mathbf{1} + \Omega_A\ \mathbf{S}_A + \Omega_B\ \mathbf{S}_B + \Omega_C\ \mathbf{S}_C = \Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}}$$

$$=\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}+(A-D)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}+2B\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}+2C\begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

Notation for 3D Vector space

0th component unchanged *components A, B, C switch 1/2-factor from ω-velocity to S-momentum*

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

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Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

$$e^{-i\mathbf{H}t}=e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix}t}=e^{-i(\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma})t}=e^{-i\omega_0 t}e^{-i\vec{\omega}\bullet\vec{\sigma}t}=e^{-i\omega_0 t}e^{-i\boldsymbol{\sigma}_\omega\omega t}=e^{-i\omega_0 t}\left(\mathbf{1}\cos\omega t-i\boldsymbol{\sigma}_\omega\sin\omega t\right)$$

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Notation for 2D Spinor space

$$=\omega_0\,\sigma_0+\omega_A\,\sigma_A+\omega_B\,\sigma_B+\omega_C\,\sigma_C=\omega_0\sigma_0+\vec{\omega}\bullet\vec{\sigma}=\omega_0\mathbf{1}+\omega\boldsymbol{\sigma}_\omega$$

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Notation for 2D Spinor space

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$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})t}=e^{-i\Omega_0 t}e^{-i\,\vec{\Omega}t\bullet\mathbf{S}}=e^{-i\Omega_0 t}\left(\mathbf{1}\cos\frac{\Omega t}{2}-i\boldsymbol{\sigma}_\omega\sin\frac{\Omega t}{2}\right)$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

Notation for 2D Spinor space

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Notation for 3D Vector space

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"Crank" vector

$$\vec{\varphi}=\begin{pmatrix} \varphi_A \\ \varphi_B \\ \varphi_C \end{pmatrix}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$$

Notation for 2D Spinor space

where: $\vec{\varphi}=\vec{\omega}\cdot t=\begin{pmatrix} \omega_A \\ \omega_B \\ \omega_C \end{pmatrix}\cdot t=\begin{pmatrix} \frac{A-D}{2} \\ B \\ C \end{pmatrix}\cdot t$ and: $\omega_0=\frac{A+D}{2}$

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$$=e^{-i(\Omega_0\mathbf{1}+\vec{\Omega}\bullet\vec{\mathbf{S}})t}=e^{-i\Omega_0 t}e^{-i\vec{\Omega}t\bullet\mathbf{S}}=e^{-i\Omega_0 t}\left(\mathbf{1}\cos\frac{\Omega t}{2}-i\sigma_\omega\sin\frac{\Omega t}{2}\right)$$

"Crank" vector

$$\vec{\Theta}=\begin{pmatrix} \Theta_A \\ \Theta_B \\ \Theta_C \end{pmatrix}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$$

Notation for 3D Vector space

where: $\vec{\Theta}=\vec{\Omega}\cdot t=\begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix}\cdot t=\begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}\cdot t$ and: $\Omega_0=\frac{A+D}{2}$

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2{}_t\mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

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Derive **σ**-*exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

Spinor arithmetic	*like*	*complex arithmetic*
Spinor vector algebra	*like*	*complex vector algebra*
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Geometry of evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu\omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

➔ *2D Spinor vs 3D vector rotation*

NMR Hamiltonian: 3D Spin Moment **m** *in* **B** *field*

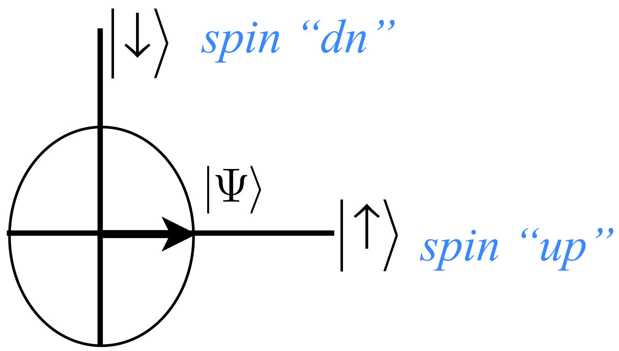
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor $\{|\uparrow\rangle,|\downarrow\rangle\}$*-space (complex)*

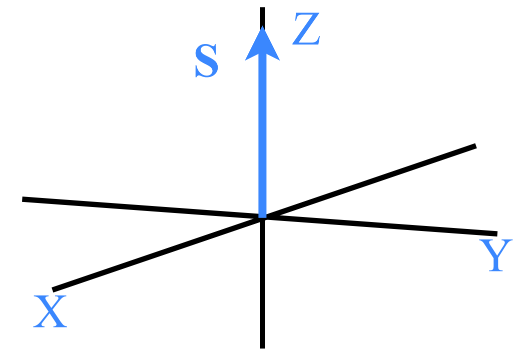
R(3):3D Spin Vector $\{S_X, S_Y, S_Z\}$*-space (real)*

State vector $|\Psi\rangle = |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$

Spin vector $\mathbf{S} = |X\rangle\langle X|\mathbf{S}\rangle + |Y\rangle\langle Y|\mathbf{S}\rangle + |Z\rangle\langle Z|\mathbf{S}\rangle$



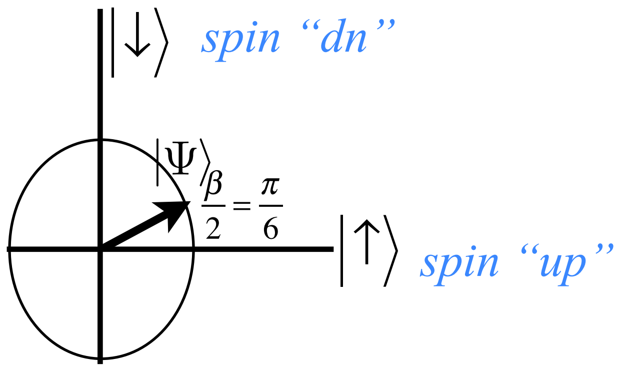
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$



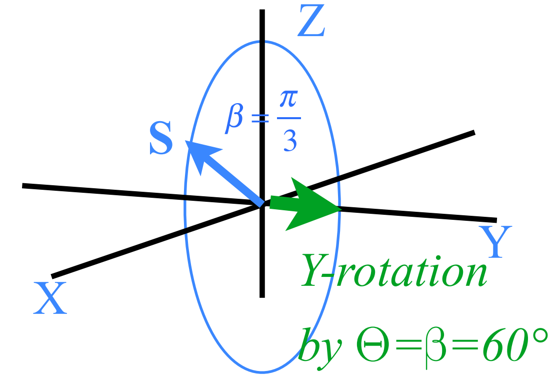
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2} & -\sin\frac{\beta}{2}\\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

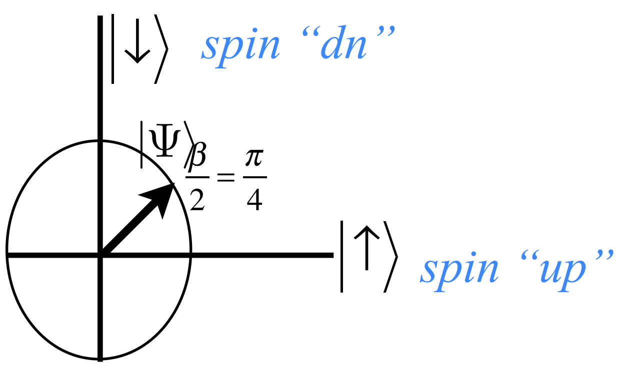
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$



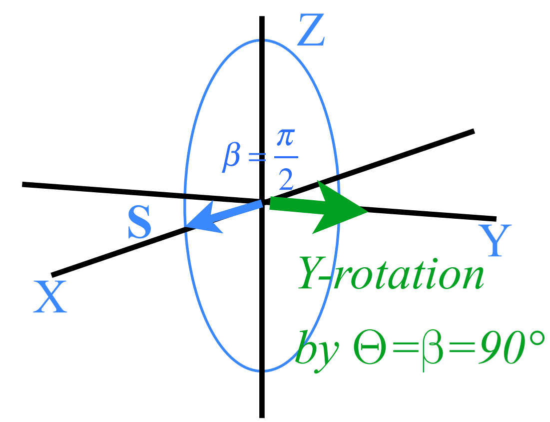
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$



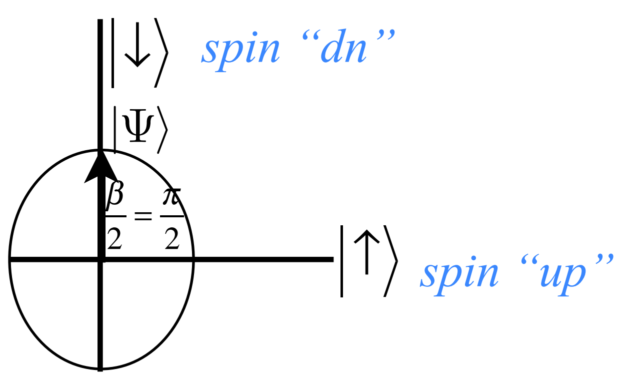
$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2\\ 0\\ 1/2\end{pmatrix}$$



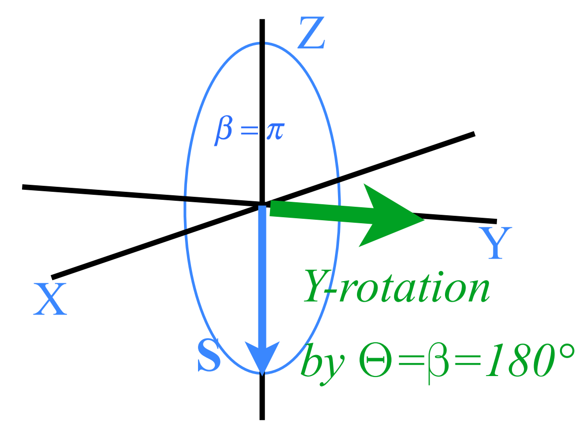
$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$



$$\begin{pmatrix}\Psi_\uparrow\\ \Psi_\downarrow\end{pmatrix} = \begin{pmatrix}\cos\frac{\beta}{2}\\ \sin\frac{\beta}{2}\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$



$$\begin{pmatrix}S_X\\ S_Y\\ S_Z\end{pmatrix} = \begin{pmatrix}\sin\beta\\ 0\\ \cos\beta\end{pmatrix} = \begin{pmatrix}0\\ 0\\ -1\end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast"

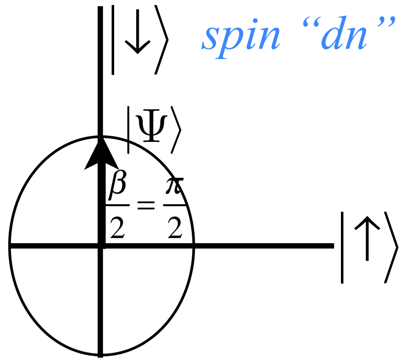
The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

U(2):2D Spinor $\{|\uparrow\rangle, |\downarrow\rangle\}$*-space (complex)*

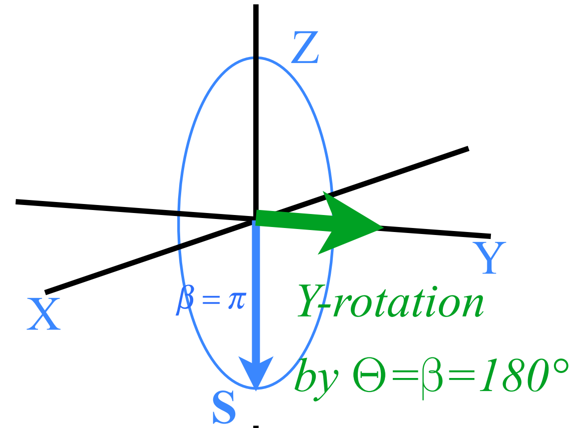
R(3):3D Spin Vector $\{S_X, S_Y, S_Z\}$*-space (real)*

State vector $|\Psi\rangle = |\uparrow\rangle\langle\uparrow|\Psi\rangle + |\downarrow\rangle\langle\downarrow|\Psi\rangle$

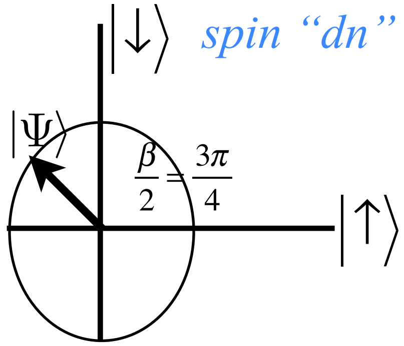
Spin vector $\mathbf{S} = |X\rangle\langle X|\mathbf{S}\rangle + |Y\rangle\langle Y|\mathbf{S}\rangle + |Z\rangle\langle Z|\mathbf{S}\rangle$



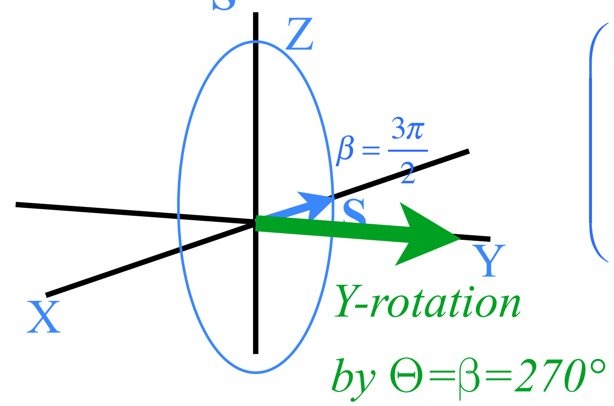
$$\begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



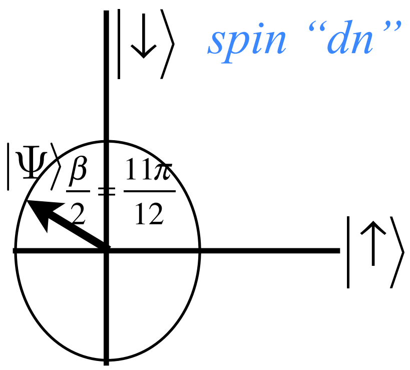
$$\begin{pmatrix} S_X \\ S_Y \\ S_Z \end{pmatrix} = \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$



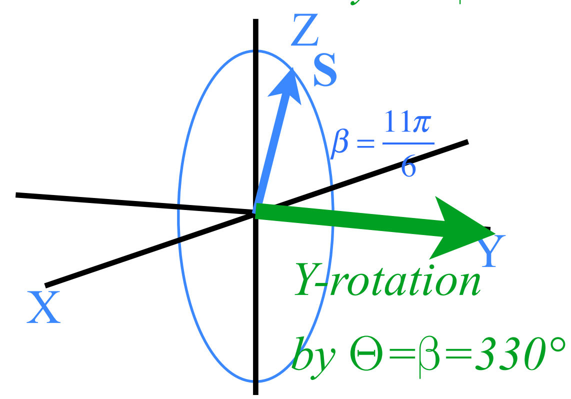
$$\begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$



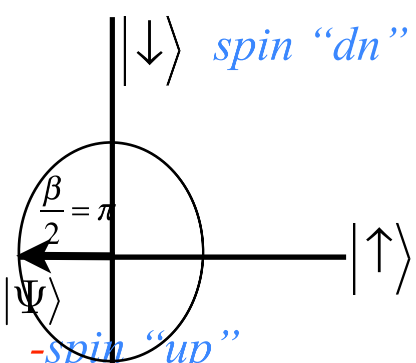
$$\begin{pmatrix} S_X \\ S_Y \\ S_Z \end{pmatrix} = \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$



$$\begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} \frac{-\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$



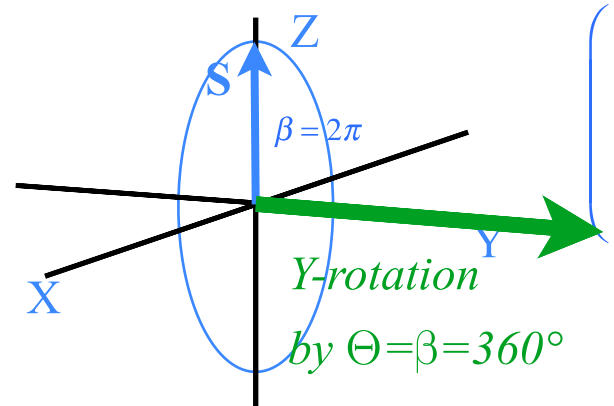
$$\begin{pmatrix} S_X \\ S_Y \\ S_Z \end{pmatrix} = \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix} =$$



with π-phase

$$\begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

(Only "half-way" home after 2π =360° rotation)



$$\begin{pmatrix} S_X \\ S_Y \\ S_Z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Life in 2D Spinor space is "Half-Fast" and needs Θ=4π=720° to return to original state

ANALOGY: 2-State Schrodinger: $i\hbar\partial_t|\Psi(t)\rangle = \mathbf{H}|\Psi(t)\rangle$ versus *Classical 2D-HO:* $\partial^2_t \mathbf{x} = -\mathbf{K}\bullet\mathbf{x}$

Hamilton-Pauli spinor symmetry ($\boldsymbol{\sigma}$*-expansion in ABCD-Types)* $\mathbf{H} = \omega_\mu \boldsymbol{\sigma}_\mu$

Derive $\boldsymbol{\sigma}$*-exponential time evolution (or revolution) operator* $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

Spinor arithmetic	*like*	*complex arithmetic*
Spinor vector algebra	*like*	*complex vector algebra*
Spinor exponentials	*like*	*complex exponentials*

Geometry of evolution (or revolution) operator $\mathbf{U} = e^{-i\mathbf{H}t} = e^{-i\boldsymbol{\sigma}_\mu \omega_\mu t}$

The "mysterious" factors of 2 (or 1/2): 2D Spinor vs 3D Spin Vector space

2D Spinor vs 3D vector rotation

➔ *NMR Hamiltonian: 3D Spin Moment* **m** *in* **B** *field*

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X - igB_Y \\ gB_X + igB_Y & -gB_Z \end{pmatrix} = gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= gB_Z \;\sigma_A \;+\; gB_X\;\sigma_X \;+ gB_Y\;\sigma_Y \;=\vec{\omega}\bullet\vec{\sigma}=\omega\sigma_\omega$$

Notation for
2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

Hamiltonian for NMR: 3D Spin Moment Vector $\mathbf{m}=(m_x, m_y, m_z)$ *in field* $\mathbf{B}=(B_x, B_y, B_z)$

$$\mathbf{H}=\mathbf{m}\bullet\mathbf{B}=g\,\sigma\bullet\mathbf{B}=\begin{pmatrix} gB_Z & gB_X-igB_Y \\ gB_X+igB_Y & -gB_Z \end{pmatrix}=gB_Z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+gB_X\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+gB_Y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

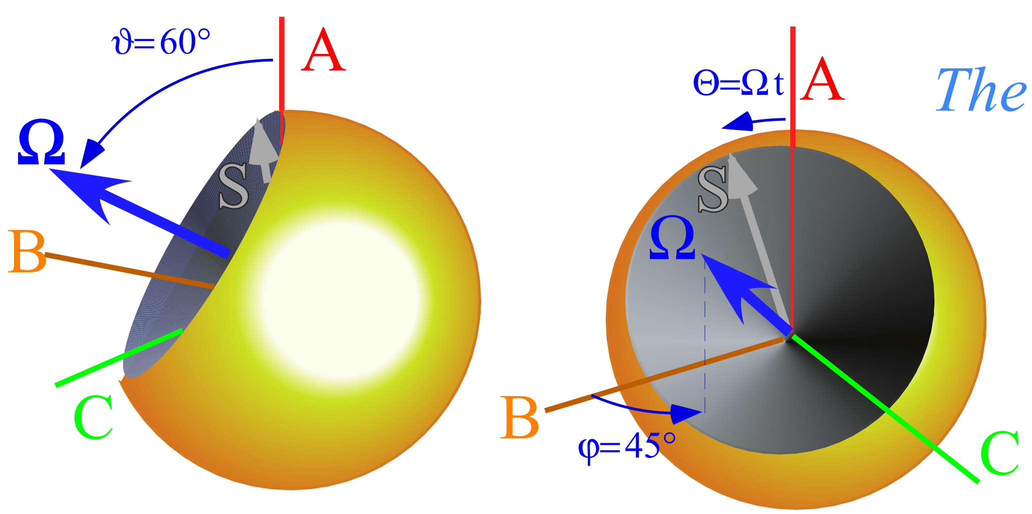
$$= gB_Z\ \sigma_A + gB_X\ \sigma_X + gB_Y\ \sigma_Y = \vec{\omega}\bullet\vec{\sigma}=\omega\sigma_\omega$$

Notation for 2D Spinor space

Symmetry archetypes: *A (Asymmetric-diagonal)*| *B (Bilateral-balanced)*| *C (Chiral-circular-complex...)*

The $\{\sigma_1, \sigma_A, \sigma_B, \sigma_C\}$ are the well known *Pauli-spin operators* $\{\sigma_1=\sigma_0, \sigma_B=\sigma_X, \sigma_C=\sigma_Y, \sigma_A=\sigma_Z\}$

Notation for 3D Vector space



The driving $\boldsymbol{\Theta}=\boldsymbol{\Omega}t$ *vector is defined by the ABCD of Hamiltonian* **H** *.*

The driven spin vector **S** *defines the state. But, how?*

Fig. 3.4.2 Two views of Hamilton crank vector $\Omega(\varphi,\vartheta)$ *whirling Stokes state vector* ***S*** *in ABC-space.*

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

➔ *Spin-1 (3D-real vector) case*

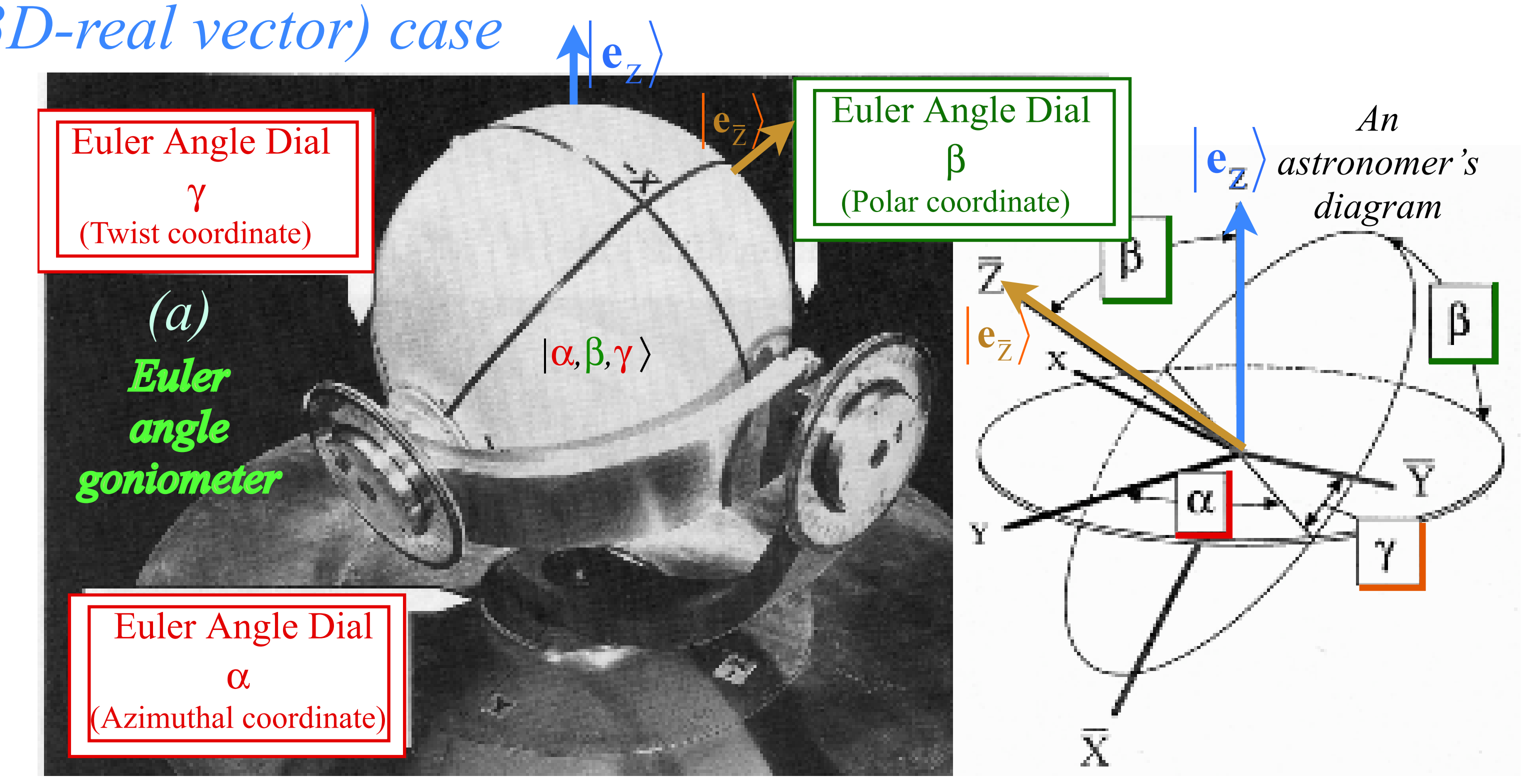
Spin-1/2 (2D-complex spinor) case

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6

See also Lects. 8-9
QTofCA Ch. 10A-B
Grp. Th. in QM 5093

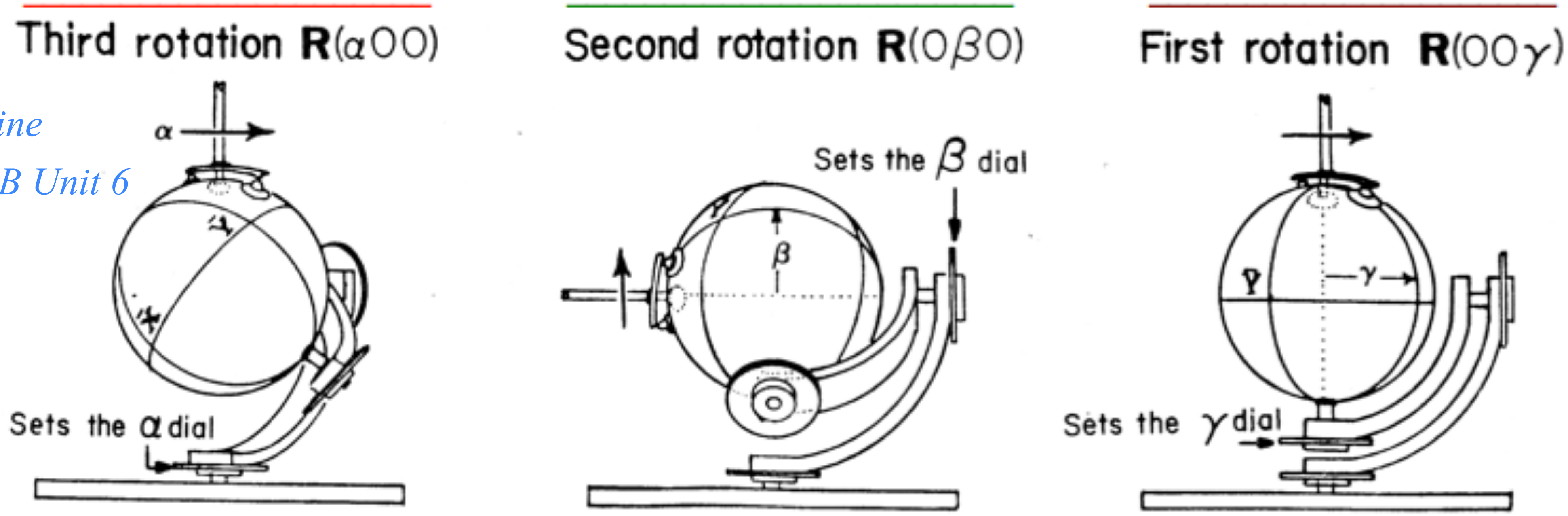


Under Construction!
Web based U(2) Calculator - Euler State

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6



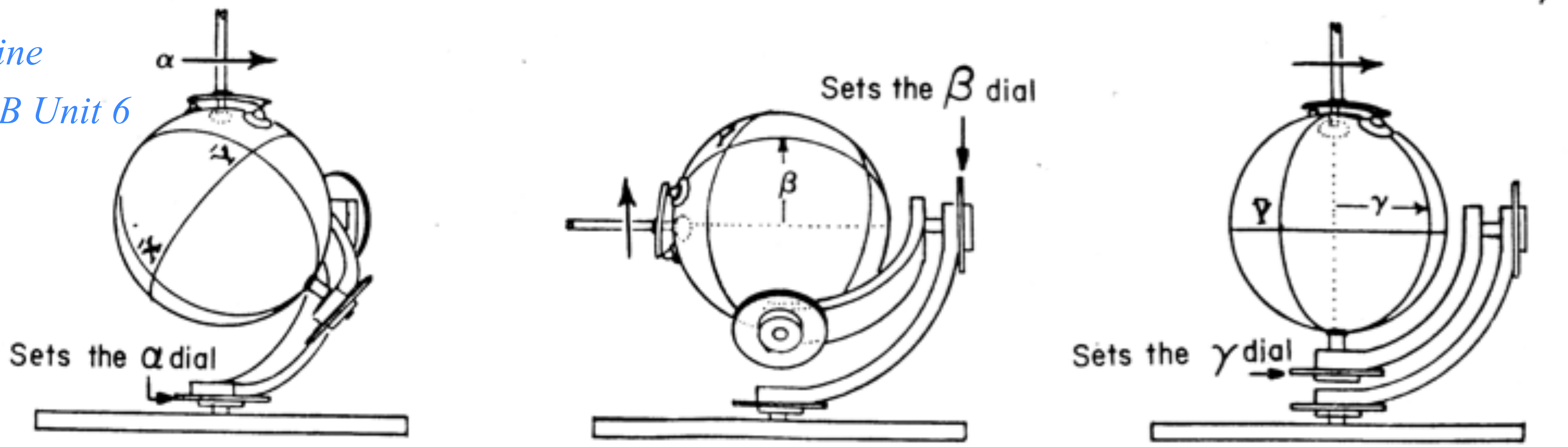
$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \qquad \langle R(0\beta 0)\rangle \qquad \langle R(00\gamma)\rangle$$

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, and $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6

Third rotation $\mathbf{R}(\alpha 00)$ | Second rotation $\mathbf{R}(0\beta 0)$ | First rotation $\mathbf{R}(00\gamma)$



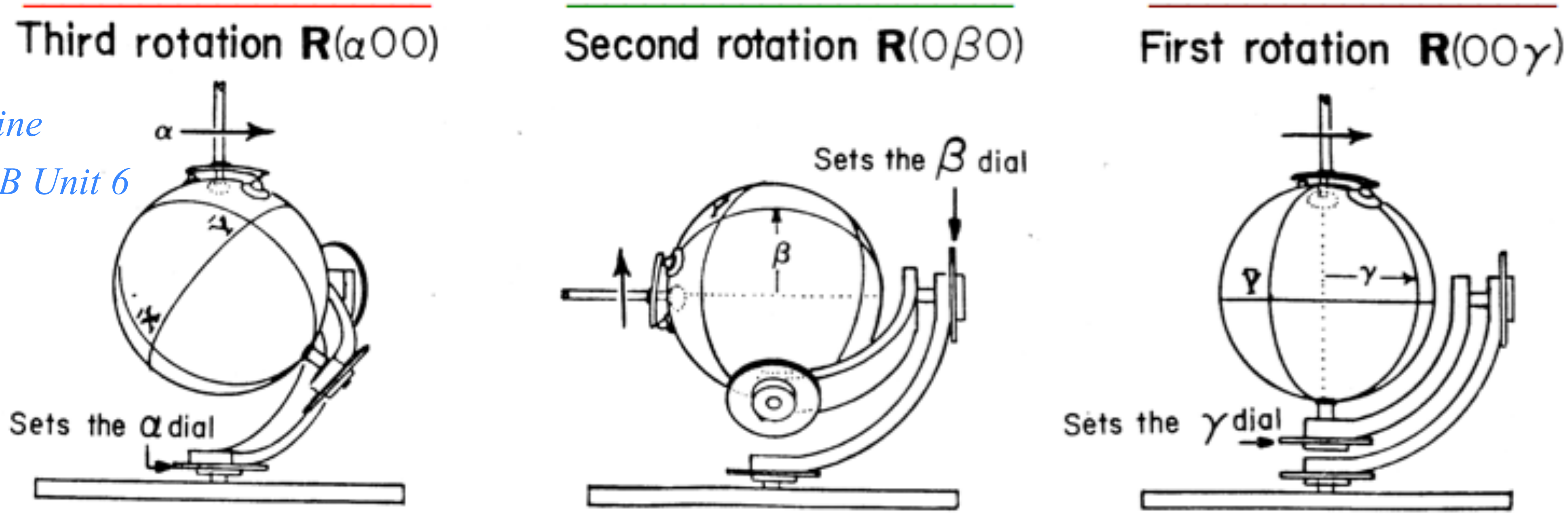
$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \quad \langle R(0\beta 0)\rangle \quad \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

$$\left(\langle \mathbf{e}_A | R(\alpha\beta\gamma) | \mathbf{e}_B\rangle\right) = \begin{matrix} \langle \mathbf{e}_X| \\ \langle \mathbf{e}_Y| \\ \langle \mathbf{e}_Z| \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \boxed{\cos\alpha\sin\beta} \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \boxed{\sin\alpha\sin\beta} \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \boxed{\cos\beta} \end{pmatrix}$$

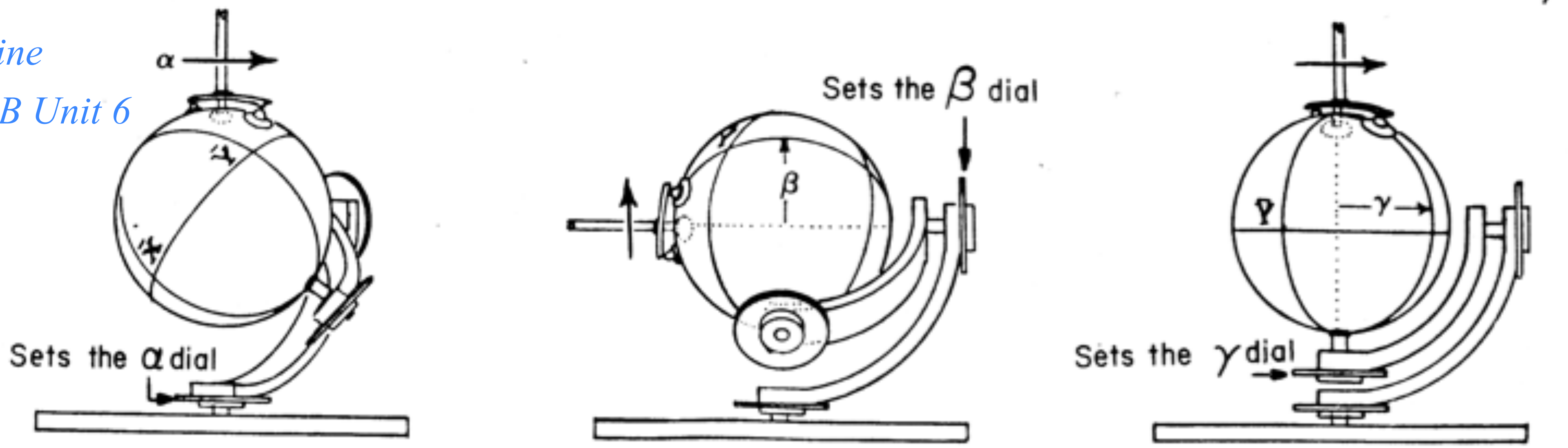
Note lab-frame polar coordinates of Z-body vector $|\mathbf{e}_{\bar{Z}}\rangle$

Euler's rotation state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

Euler Angle machine discussed in CMwB Unit 6

Third rotation $\mathbf{R}(\alpha 00)$ — Second rotation $\mathbf{R}(0\beta 0)$ — First rotation $\mathbf{R}(00\gamma)$



$$\langle R(\alpha\beta\gamma)\rangle = \langle R(\alpha 00)\rangle \langle R(0\beta 0)\rangle \langle R(00\gamma)\rangle$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|\mathbf{e}_{\bar{X}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_X\rangle \qquad |\mathbf{e}_{\bar{Y}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Y\rangle \qquad |\mathbf{e}_{\bar{Z}}\rangle = R(\alpha\beta\gamma)|\mathbf{e}_Z\rangle$$

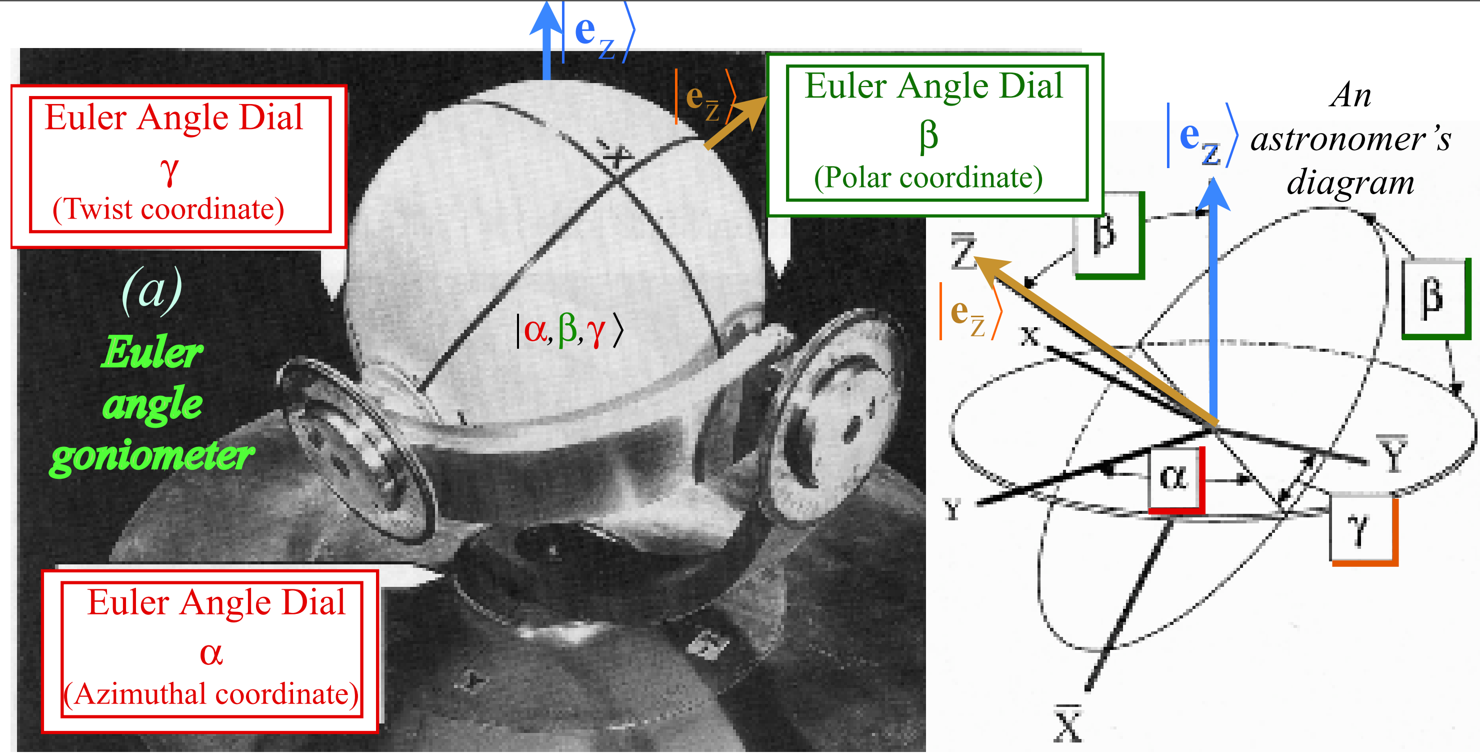
$$\left(\langle \mathbf{e}_A|R(\alpha\beta\gamma)|\mathbf{e}_B\rangle\right) = \begin{matrix} \langle\mathbf{e}_X| \\ \langle\mathbf{e}_Y| \\ \langle\mathbf{e}_Z| \end{matrix} \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta \end{pmatrix}$$

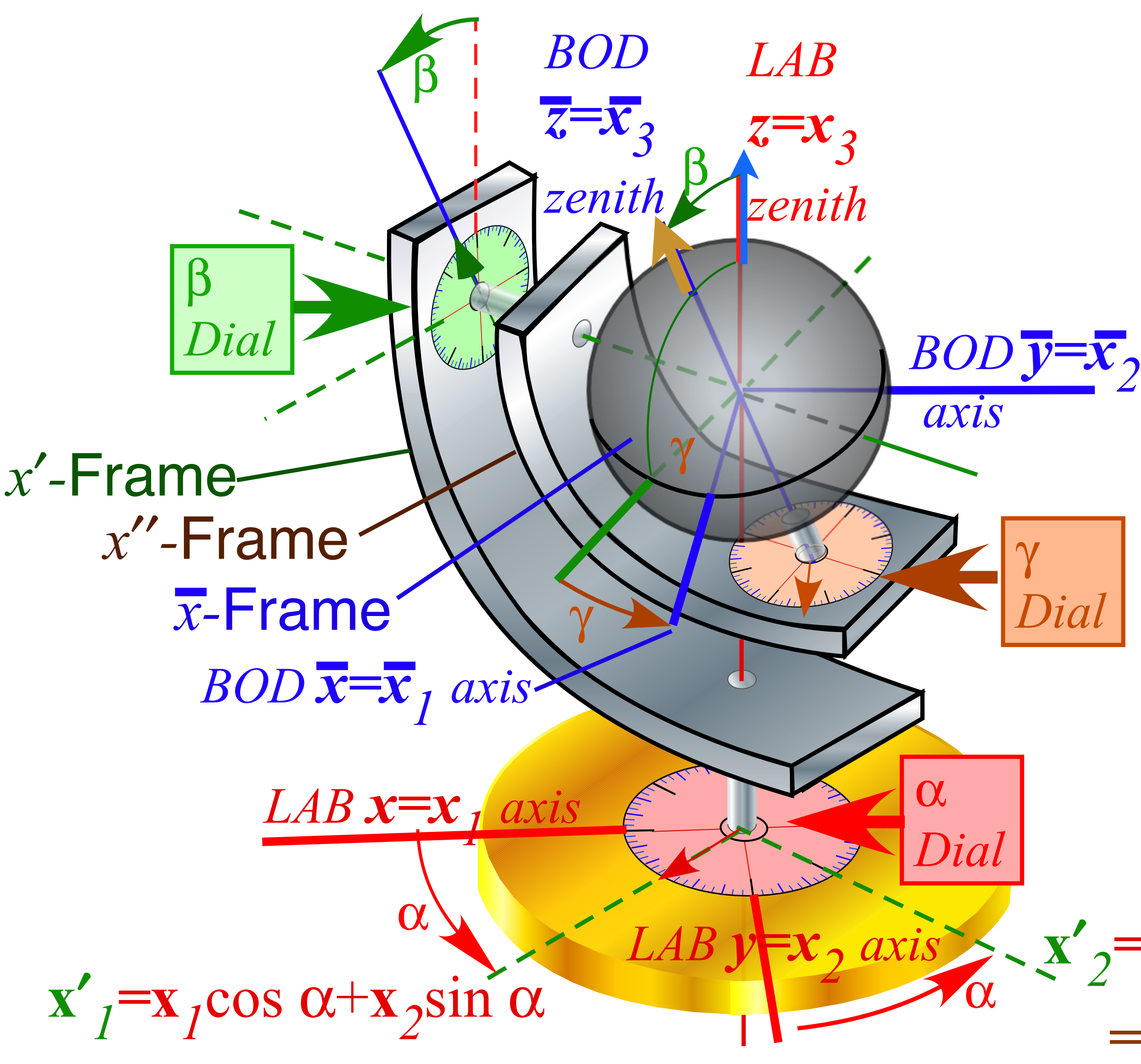
Note lab-frame polar coordinates of Z-body vector $|\mathbf{e}_{\bar{Z}}\rangle$

...and body-frame polar coordinates of Z-lab $|\mathbf{e}_Z\rangle$

Euler Angle machine discussed in CMwB Unit 6

See also Lects. 8-9
QTofCA Ch. 10A-B
Grp. Th. in QM 5093

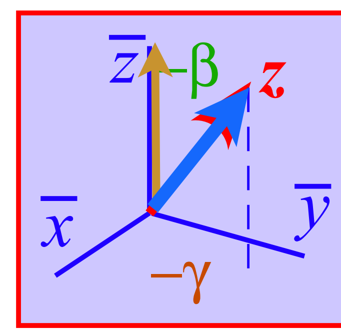




$\mathbf{x'}_1=\mathbf{x}_1\cos\alpha+\mathbf{x}_2\sin\alpha$

$\mathbf{x'}_2=-\mathbf{x}_1\sin\alpha+\mathbf{x}_2\cos\alpha$

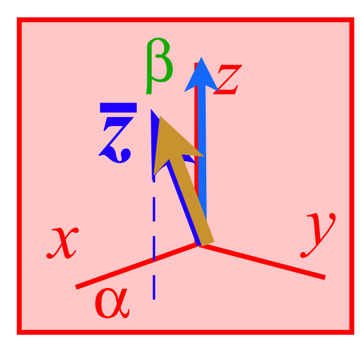
$=\bar{\mathbf{x}}_1\sin\gamma+\bar{\mathbf{x}}_2\cos\gamma$



BOD frame view

Polar angles of LAB zenith $z=x_3$ are

(azimuth angle $=-\gamma$, polar angle $=-\beta$)



LAB frame view

Polar angles of BOD zenith $\bar{z}=\bar{x}_3$ are

(azimuth angle $=\alpha$, polar angle $=\beta$)

Euler's state definition using rotations $\mathbf{R}(\alpha,0,0)$, $\mathbf{R}(0,\beta,0)$, *and* $\mathbf{R}(0,0,\gamma)$

Spin-1 (3D-real vector) case

➡ *Spin-1/2 (2D-complex spinor) case*

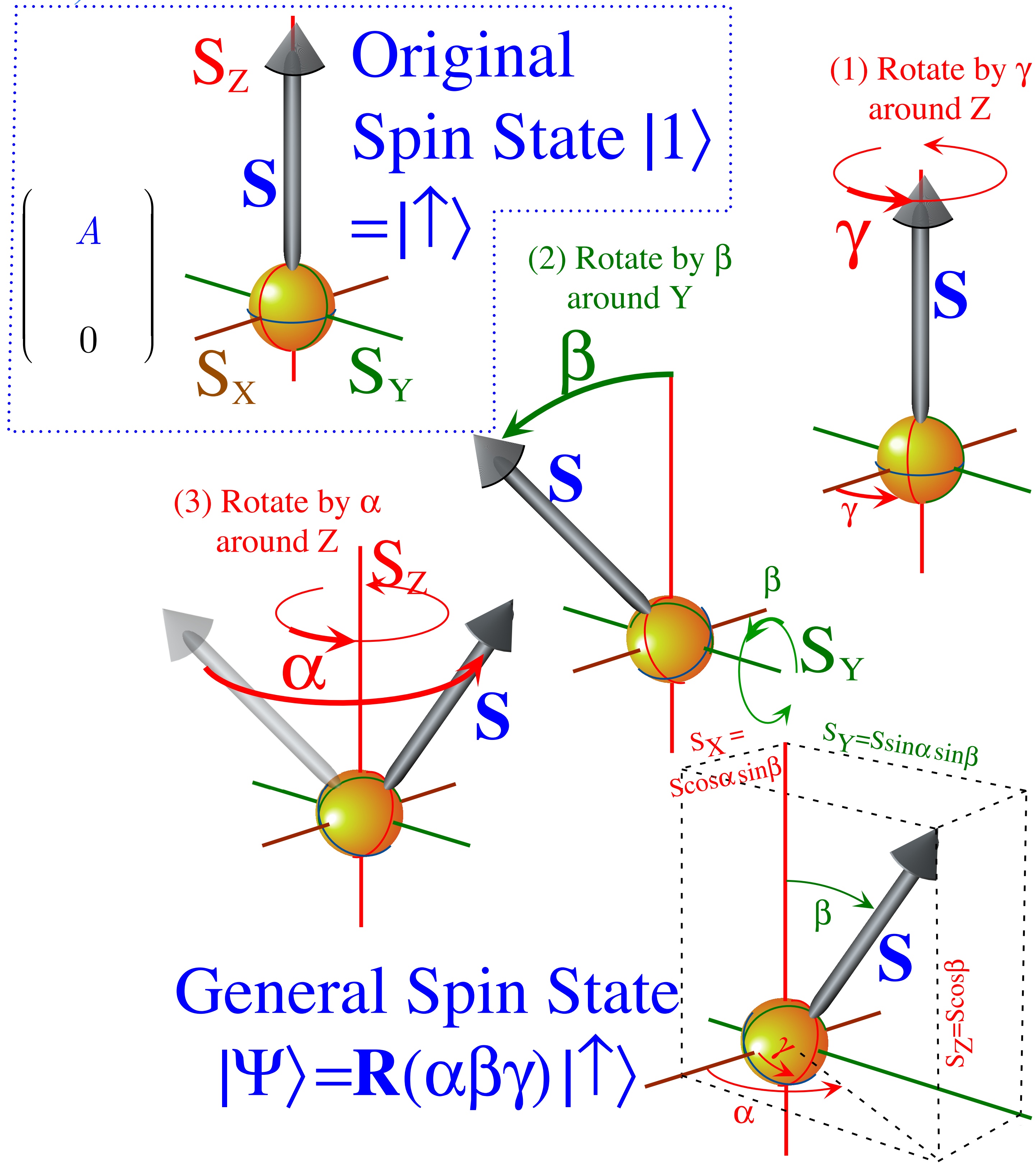
Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$\left|a\right\rangle = \mathbf{R}\left(\alpha\beta\gamma\right)\left|\uparrow\right\rangle$$

$$= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle$$

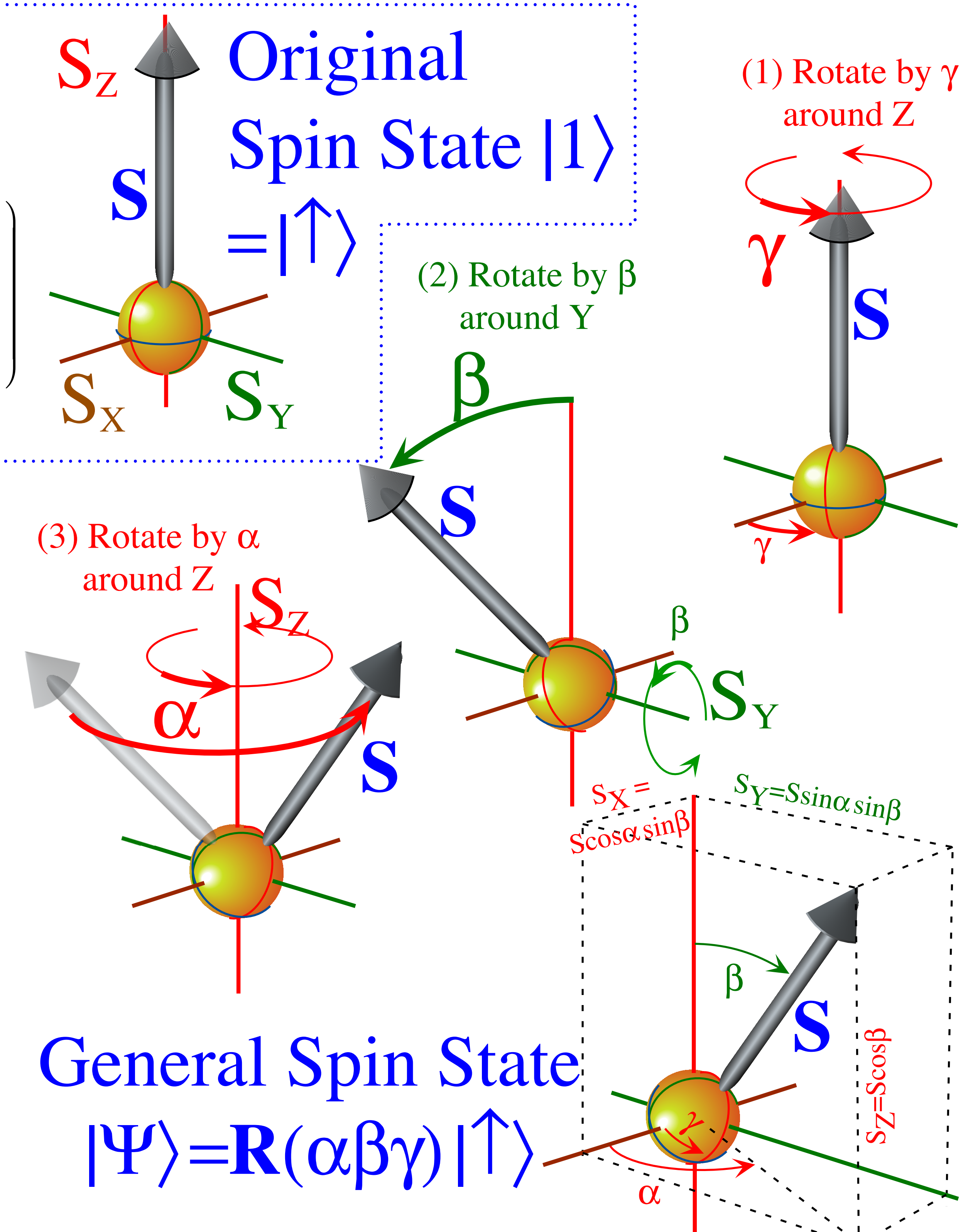
$$= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}\begin{pmatrix} A \\ 0 \end{pmatrix}$$



Euler's rotation state definition using rotations **R**$(\alpha,0,0)$, **R**$(0,\beta,0)$, *and* **R**$(0,0,\gamma)$

Spin-1/2 (2D-complex spinor) case

$$
\begin{aligned}
\left|a\right\rangle &= \mathbf{R}(\alpha\beta\gamma)\left|\uparrow\right\rangle \\
&= \mathbf{R}[\alpha \text{ about } Z]\cdot\mathbf{R}[\beta \text{ about } Y]\cdot\mathbf{R}[\gamma \text{ about } Z]\left|\uparrow\right\rangle \\
&= \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}
\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}
\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}
\begin{pmatrix} A \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}
\begin{pmatrix} A \\ 0 \end{pmatrix} \\
&= A\begin{pmatrix} e^{-i\frac{\alpha}{2}}\cos\frac{\beta}{2} \\ e^{i\frac{\alpha}{2}}\sin\frac{\beta}{2} \end{pmatrix} e^{-i\frac{\gamma}{2}} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}
\end{aligned}
$$



3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

➡ *Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$*

Polarization ellipse and spinor state dynamics

The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Each point $\{E_1, E_2\}$ defines *2D-HO phase space* or analogous Ψ-space given by 2D amplitude array:
This defines real *3D spin vector* (S_A, S_B, S_C) "pointing" to a polarization ellipse or state.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$\text{Asymmetry } S_A = \frac{1}{2}\left(a\middle|\sigma_A\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_1 - a_2^* a_2\right] = \frac{1}{2}\left[x_1^2 + p_1^2 - x_2^2 - p_2^2\right]$$

$$\text{Balance } S_B = \frac{1}{2}\left(a\middle|\sigma_B\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right]$$

$$\text{Chirality } S_C = \frac{1}{2}\left(a\middle|\sigma_C\middle|a\right) = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{-i}{2}\left[a_1^* a_2 - a_2^* a_1\right] = \left[x_1 p_2 - x_2 p_1\right]$$

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Balance $S_B = \frac{1}{2}\langle a|\sigma_B|a\rangle = \frac{1}{2}\begin{pmatrix} a_1^* & a_2^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{2}\left[a_1^* a_2 + a_2^* a_1\right] = \left[p_1 p_2 + x_1 x_2\right] = I\left[-\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha-\gamma}{2} + \cos\frac{\alpha+\gamma}{2}\cos\frac{\alpha-\gamma}{2}\right]\cos\frac{\beta}{2}\sin\frac{\beta}{2} = \frac{I}{2}\cos\alpha\sin\beta$

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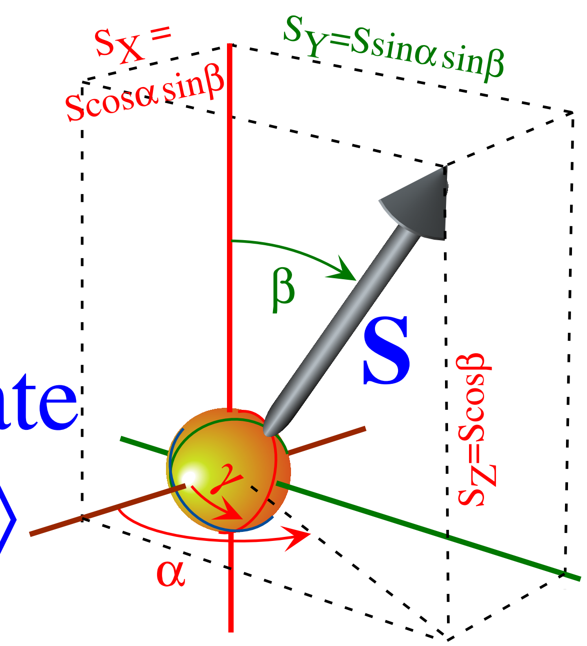
General Spin State

$|\Psi\rangle = \mathbf{R}(\alpha\beta\gamma)|\uparrow\rangle$



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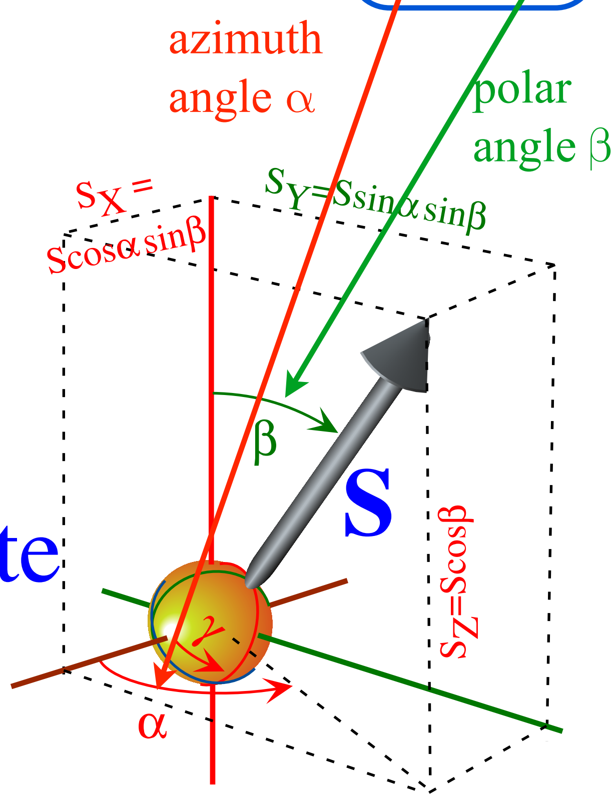

General Spin State
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Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).

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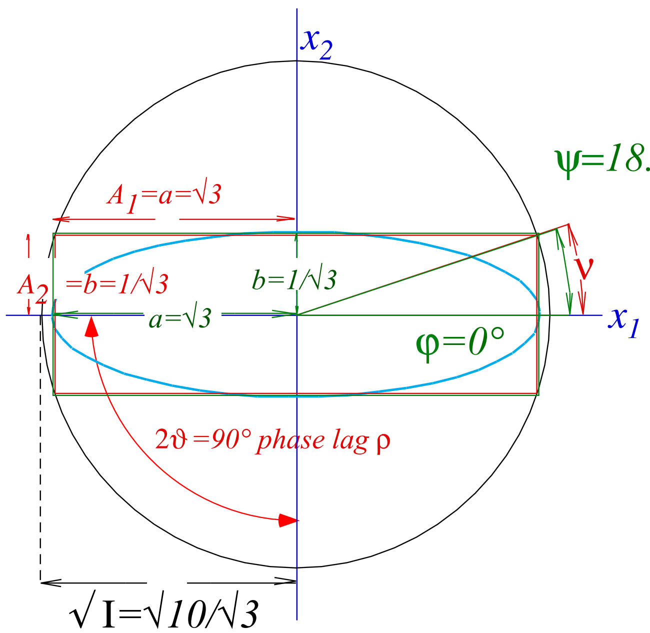

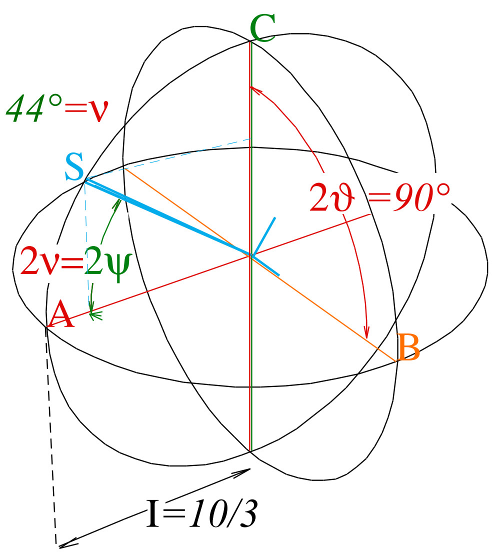

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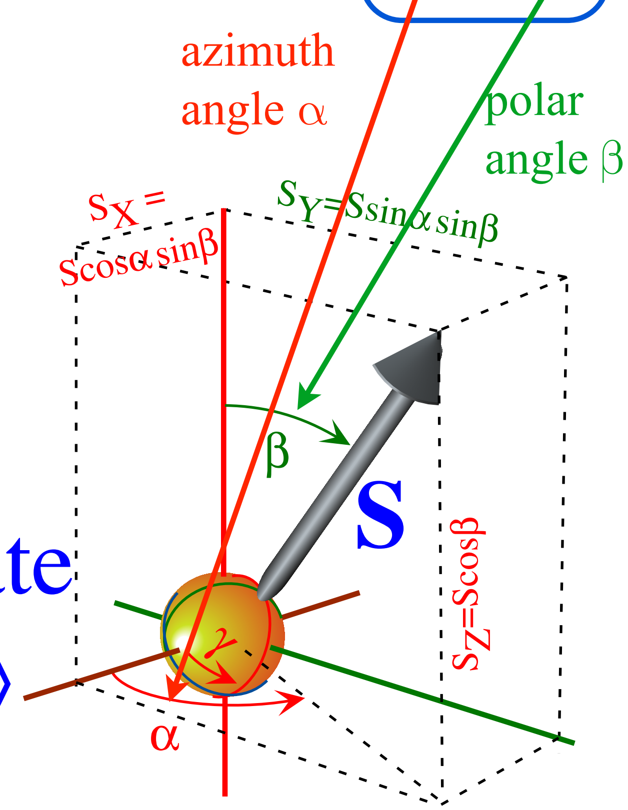

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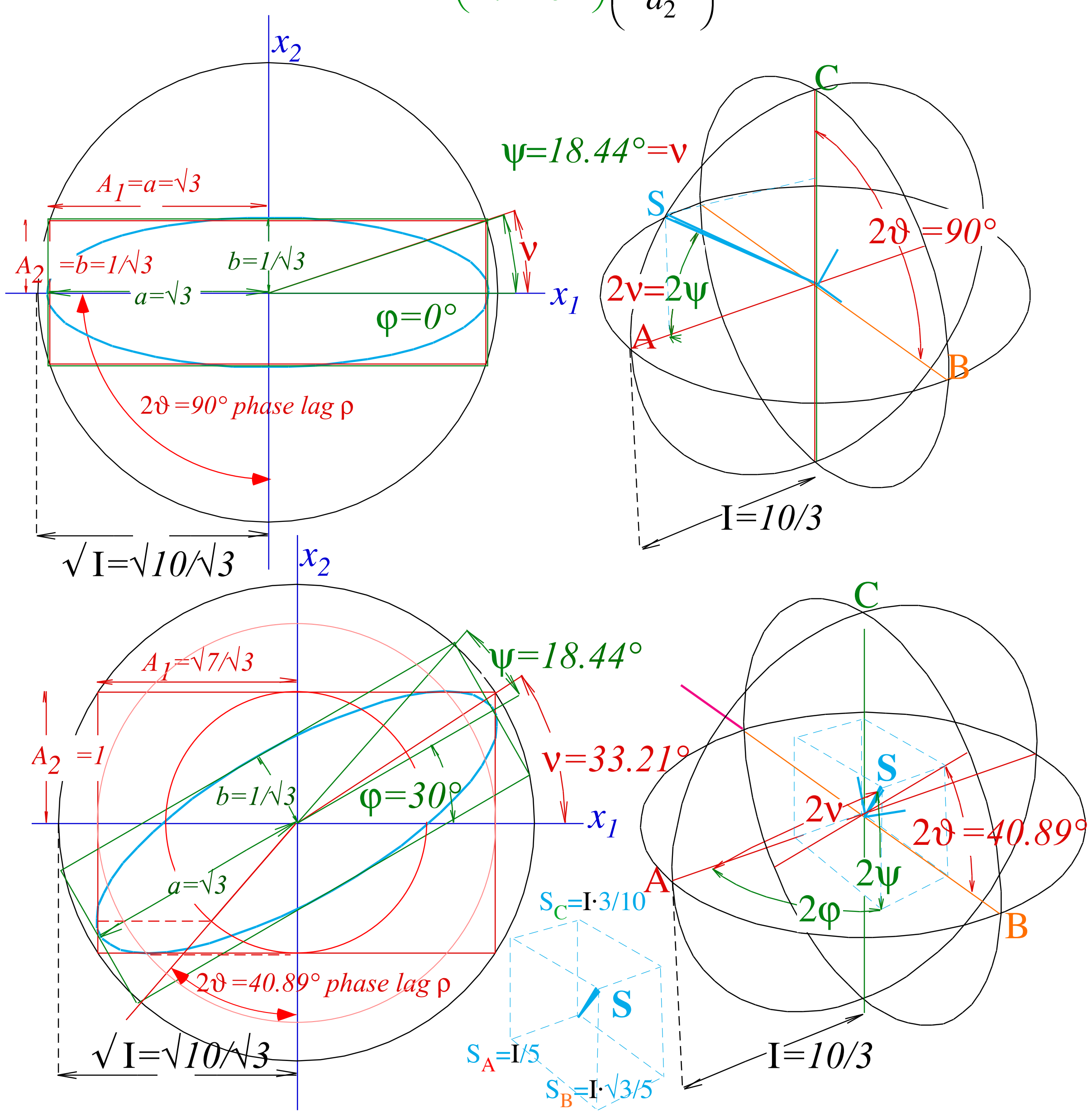

General Spin State

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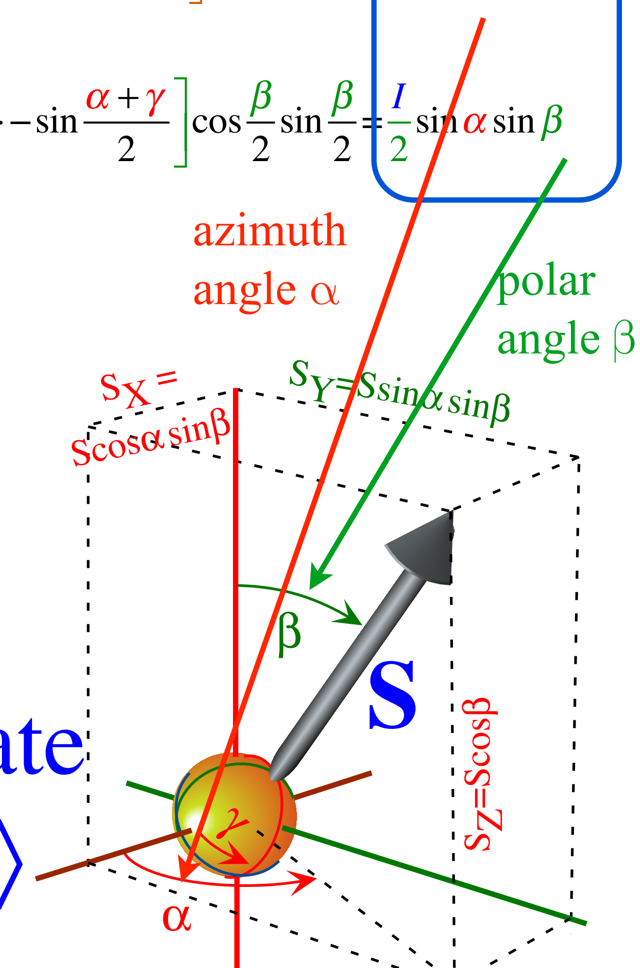

Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).

Polarization ellipse and spinor state dynamics

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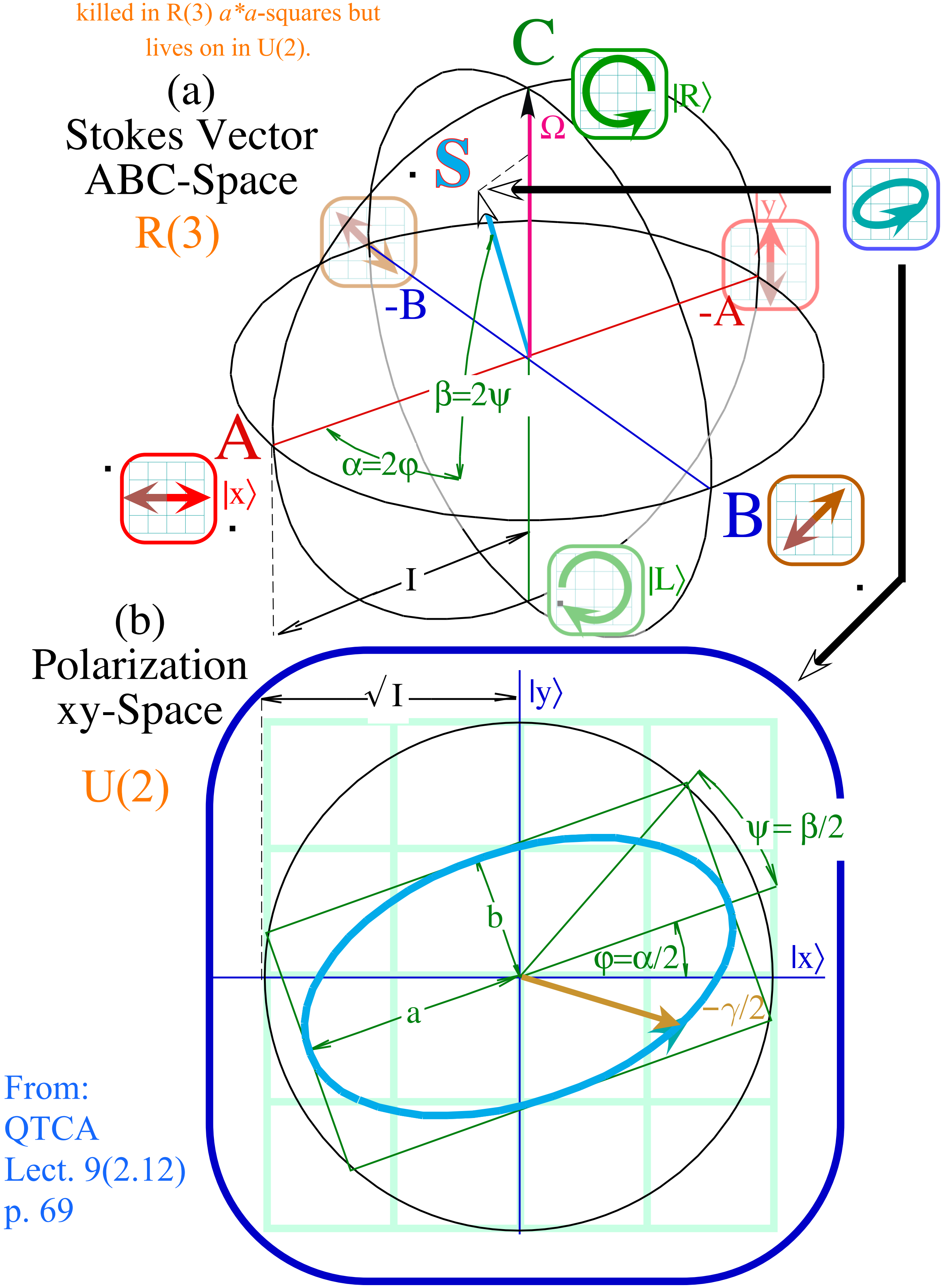

From:
QTCA
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p. 69

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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➤ *Polarization ellipse and spinor state dynamics*

The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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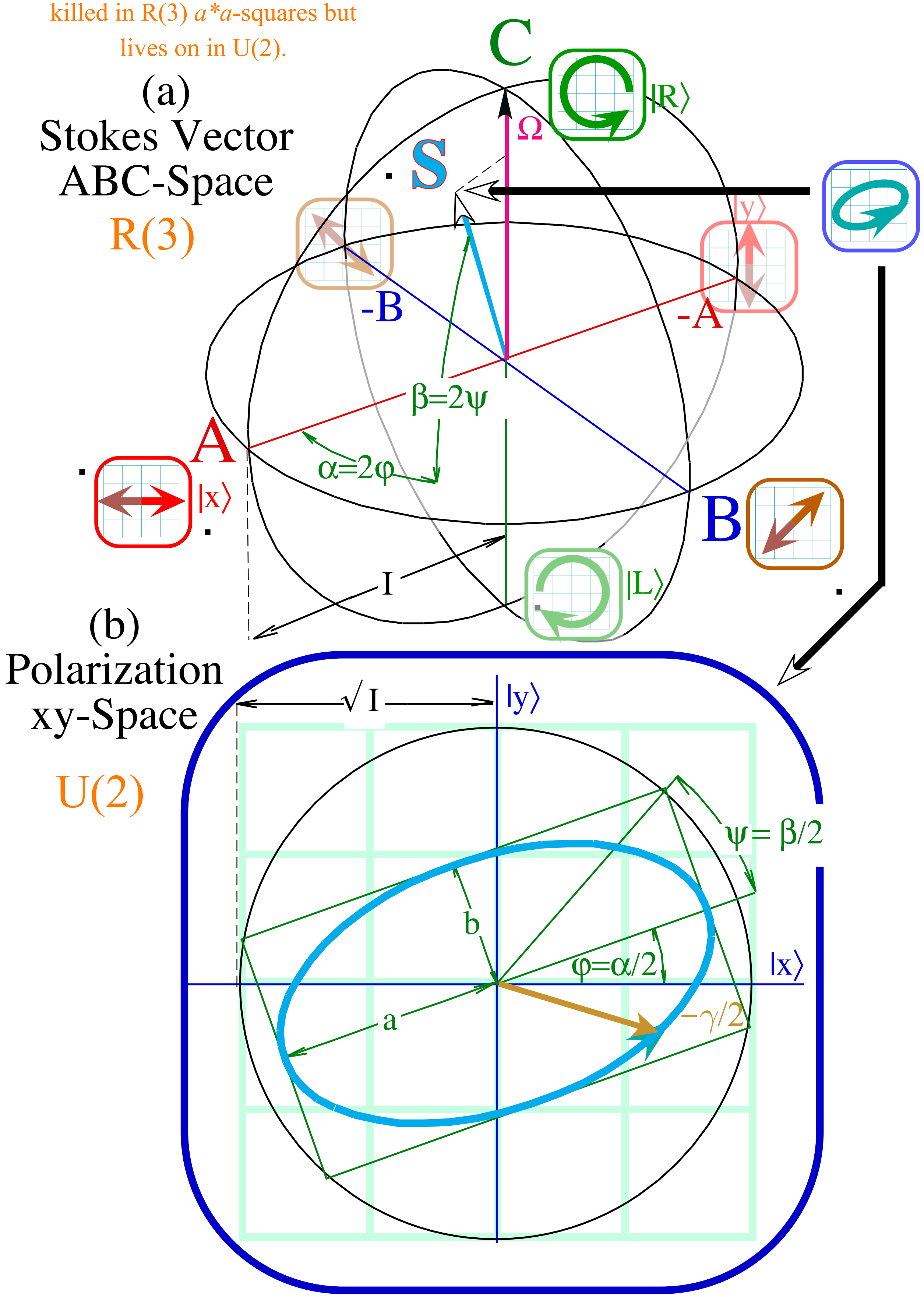
➤ *The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking*

Polarization ellipse and spinor state dynamics (A-Type motion)

Note phase or "gauge" angle γ is killed in R(3) a^*a-squares but lives on in U(2).



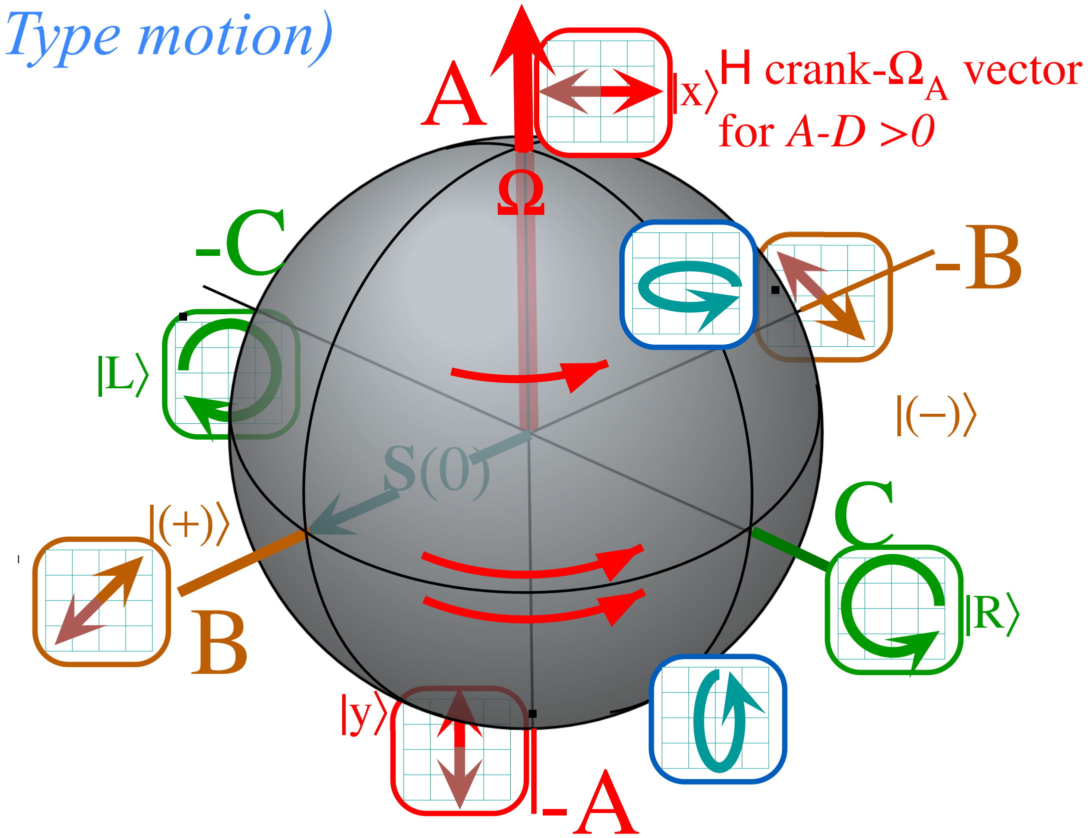


Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

The ABC's of U(2) dynamics (A-Type motion)

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\boldsymbol{\Omega}}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

From:
QTCA
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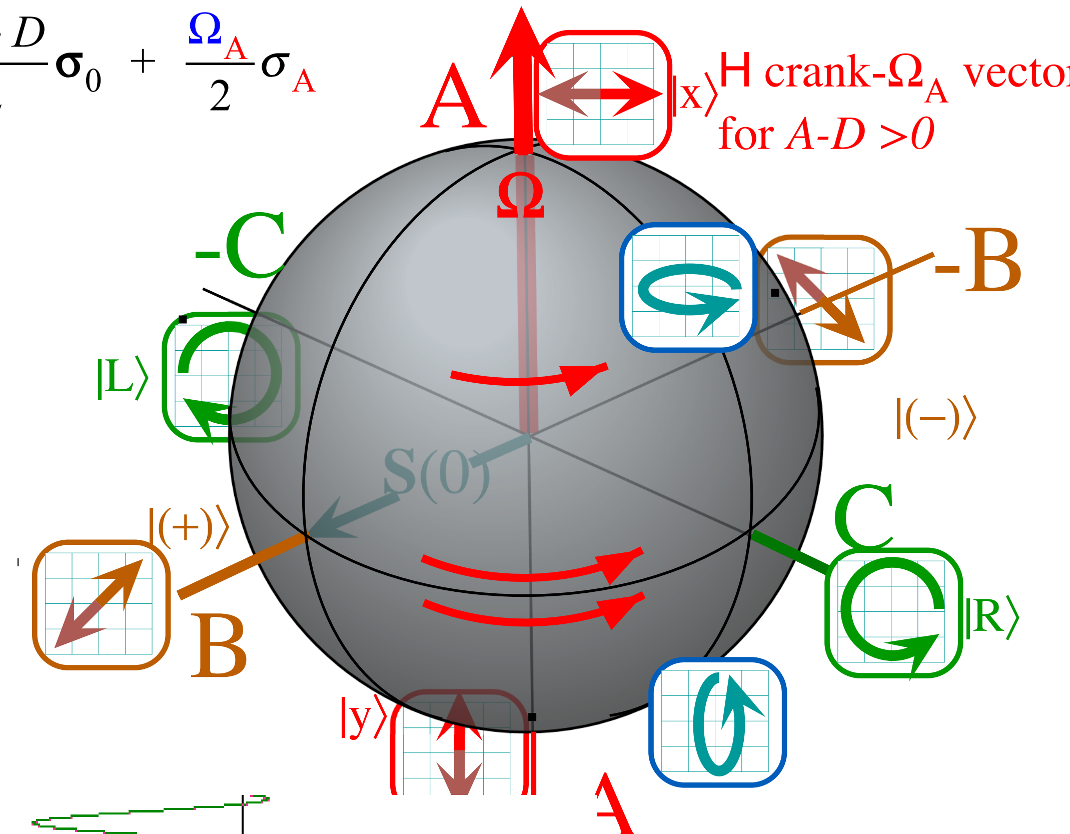
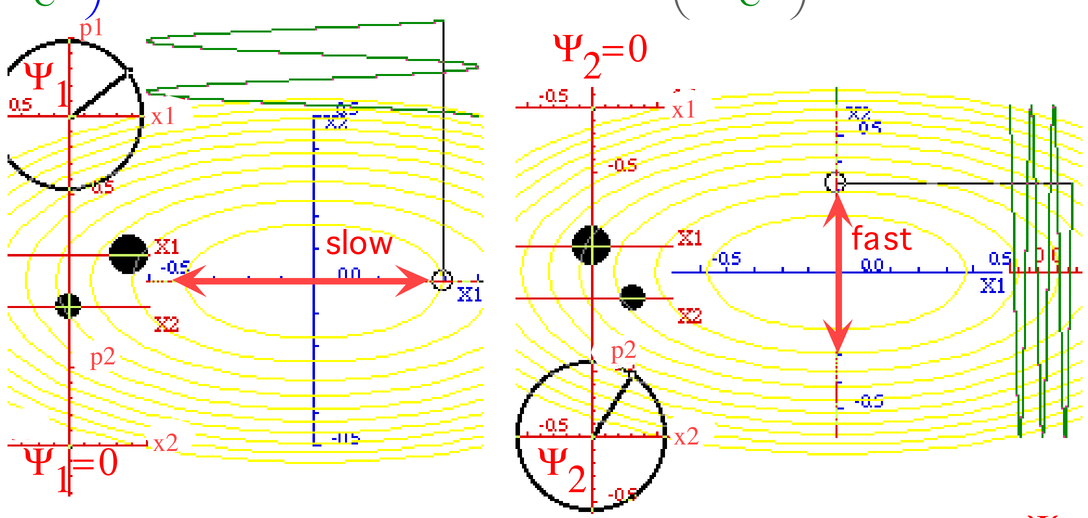
p.49

Asymmetric Diagonal A-Type motion

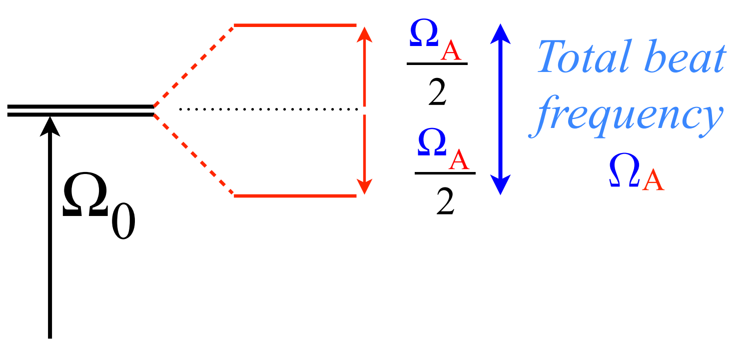
$$\begin{pmatrix} \langle 1|\mathbf{H}^A|1\rangle & \langle 1|\mathbf{H}^A|2\rangle \\ \langle 2|\mathbf{H}^A|1\rangle & \langle 2|\mathbf{H}^A|2\rangle \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A$$

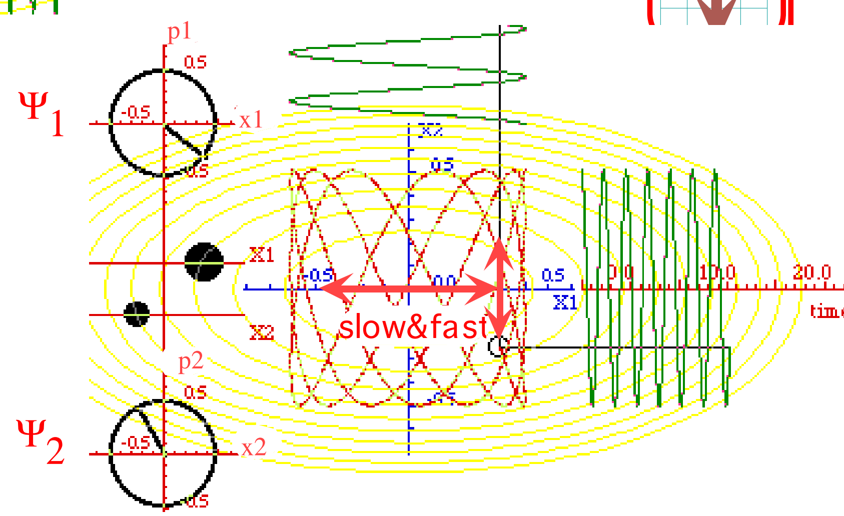
$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 0 \\ 0 \end{pmatrix} \qquad Eigen-Spin:\ \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} \pm S \\ 0 \\ 0 \end{pmatrix}$$





Beat dynamics:





BoxIt (A-Type)
Web Simulation

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

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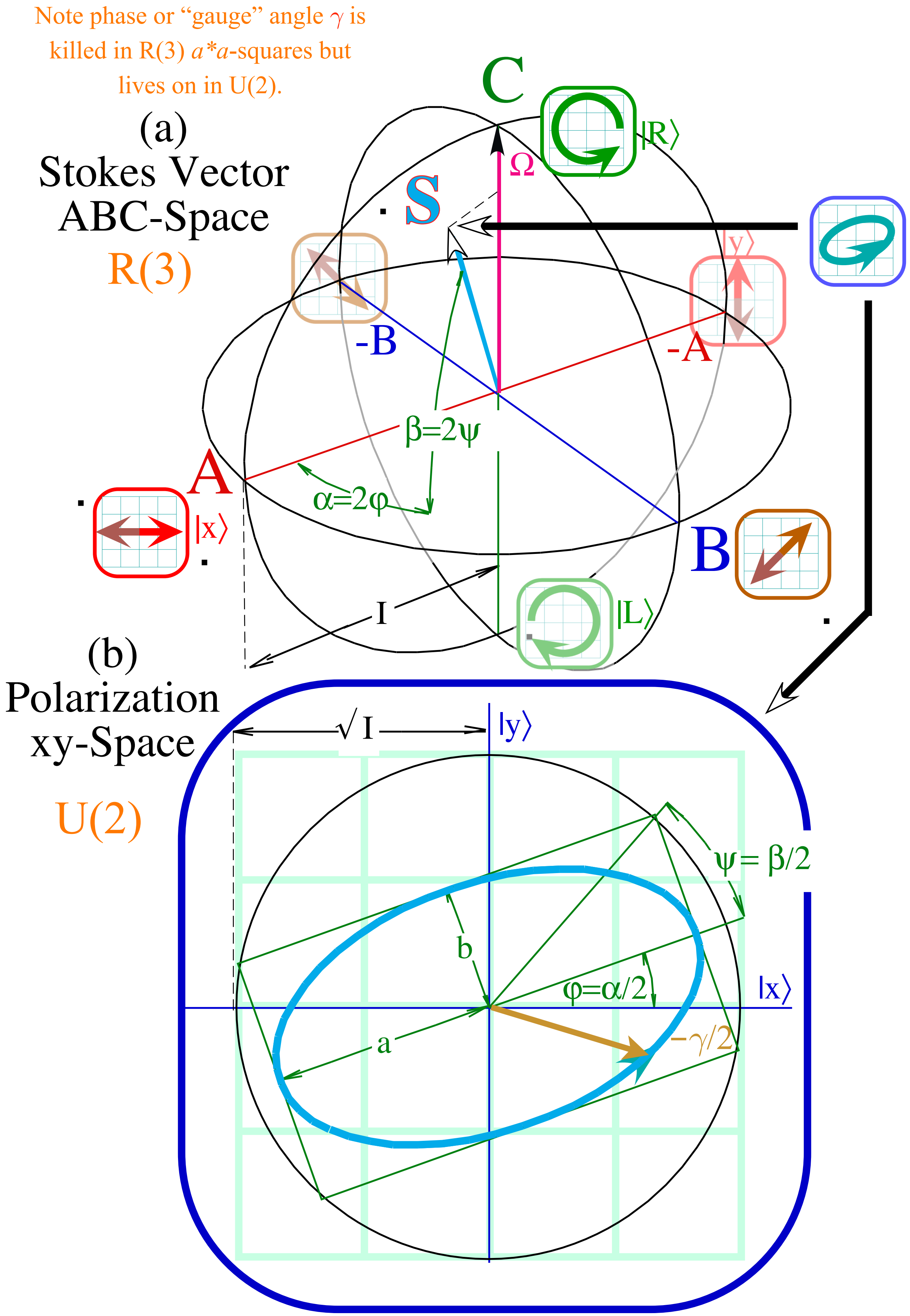
Polarization ellipse and spinor state dynamics

➤ *The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking*

Polarization ellipse and spinor state dynamics (B-Type motion)



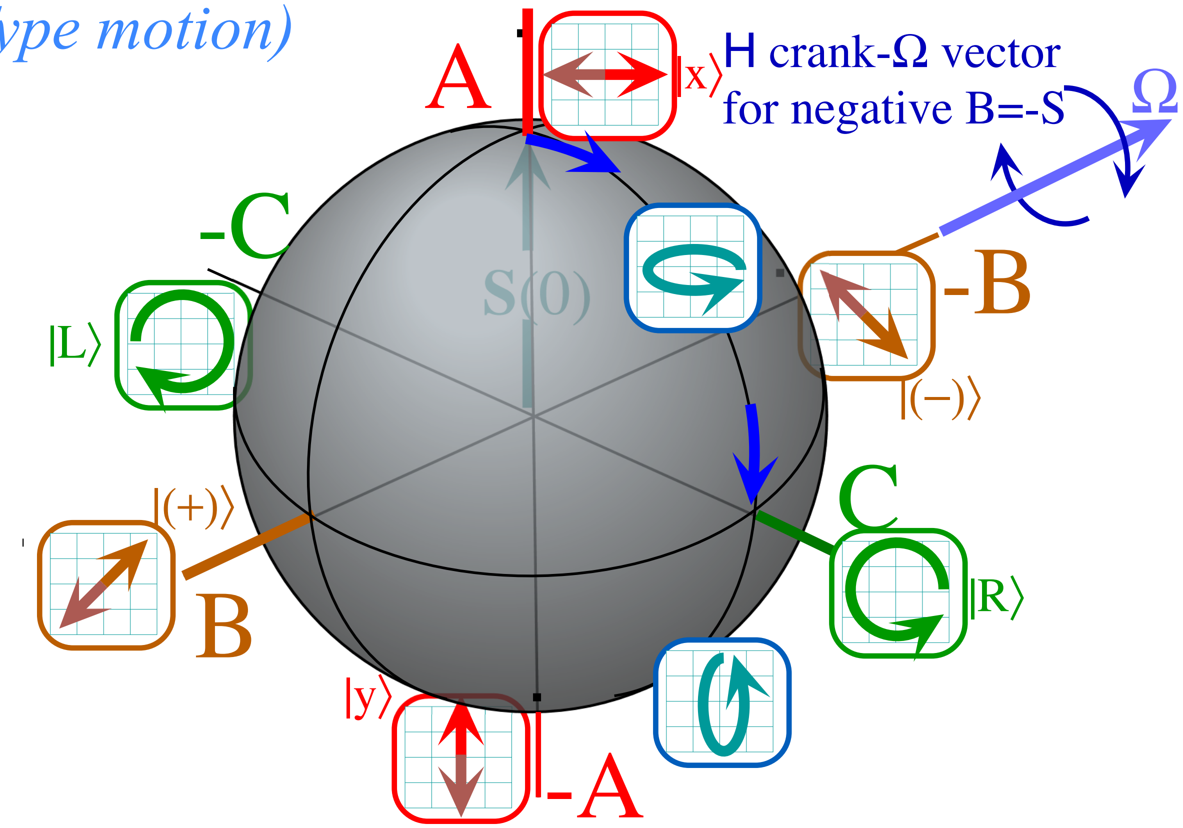


Fig. 3.4.6 Time evolution of a B-type beat. **S***-vector rotates from A to C to -A to -C and back to A.*

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

The ABC's of U(2) dynamics *(B-Type motion)*

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

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From:
QTCA
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p. 54

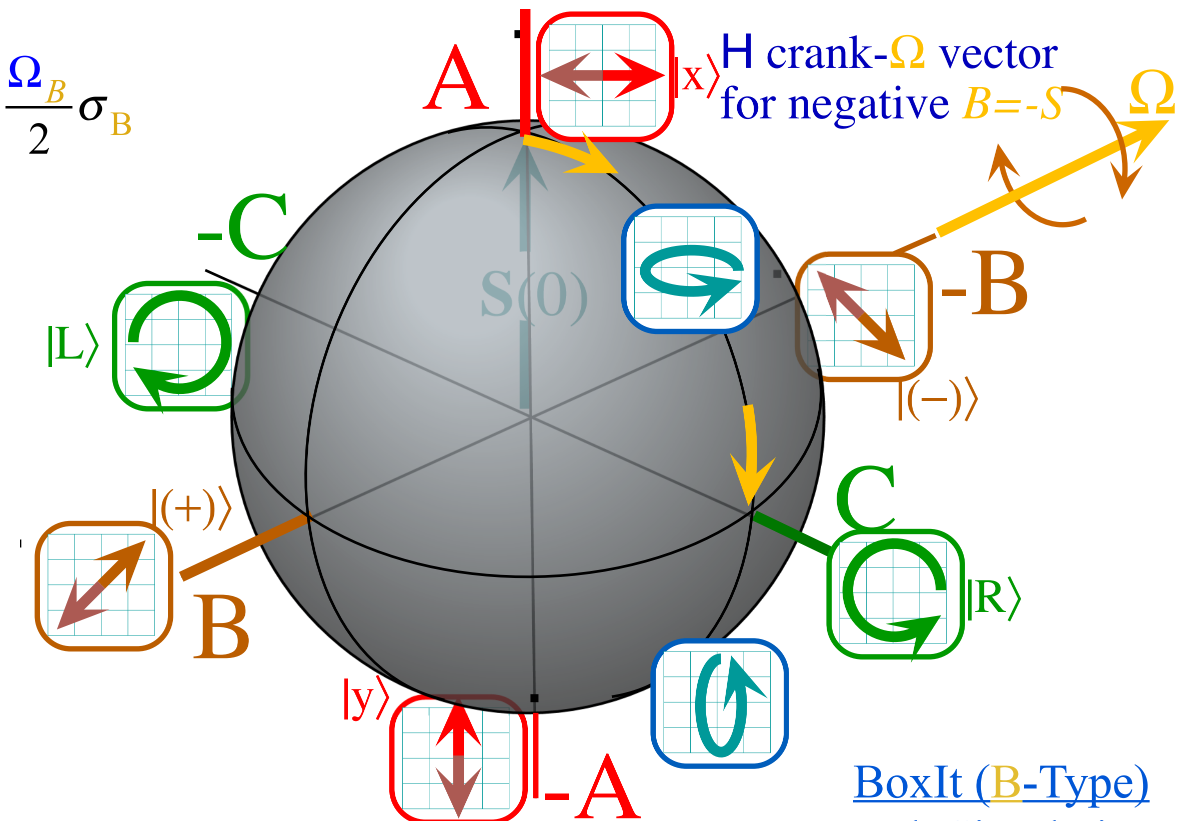
$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$
$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

Bilateral-Balanced B-Type motion

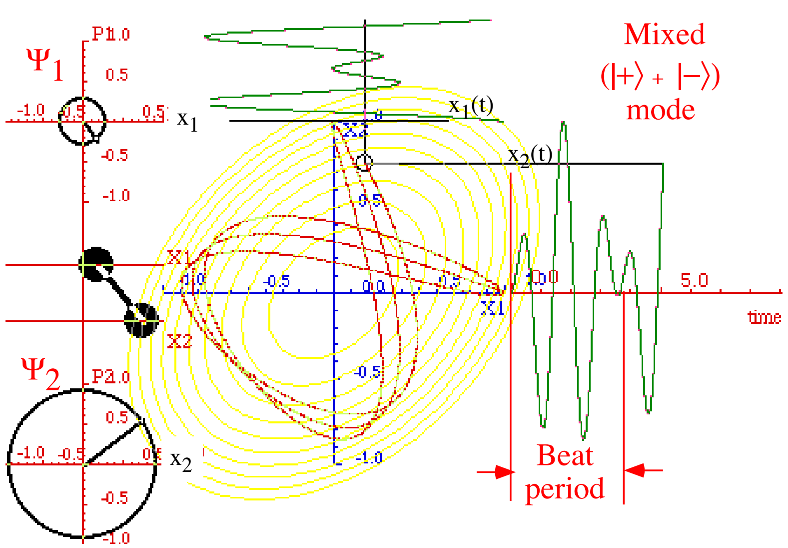
$$\begin{pmatrix} \langle 1|\mathbf{H}^B|1\rangle & \langle 1|\mathbf{H}^B|2\rangle \\ \langle 2|\mathbf{H}^B|1\rangle & \langle 2|\mathbf{H}^B|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & B \\ B & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B$$

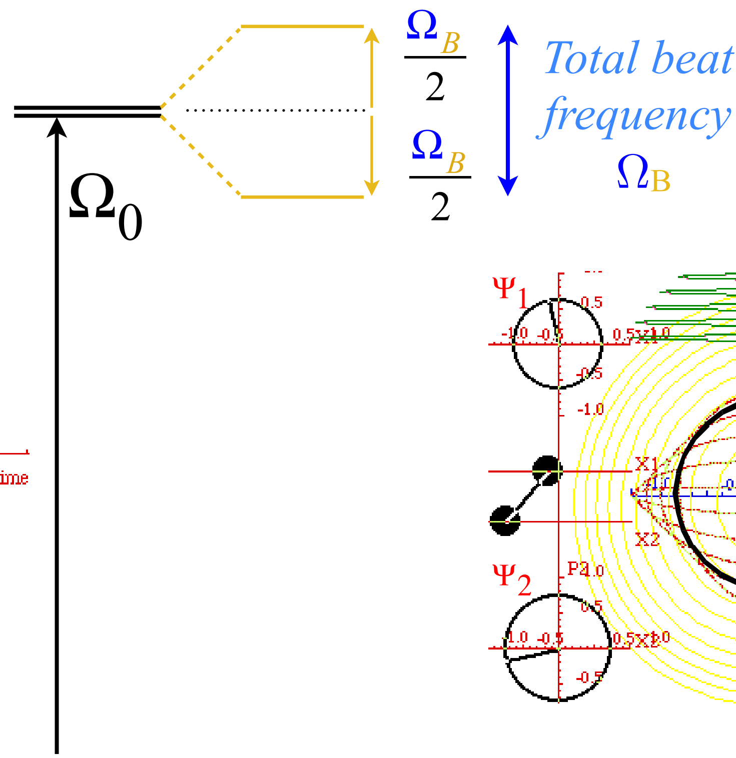
$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 2B \\ 0 \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ \pm S \\ 0 \end{pmatrix}$$

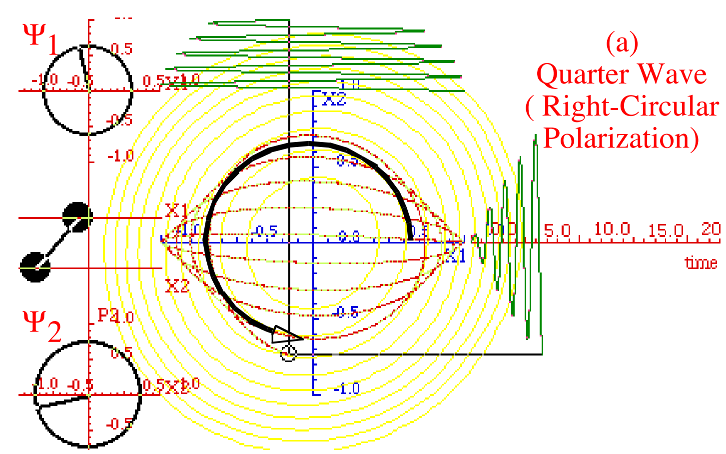


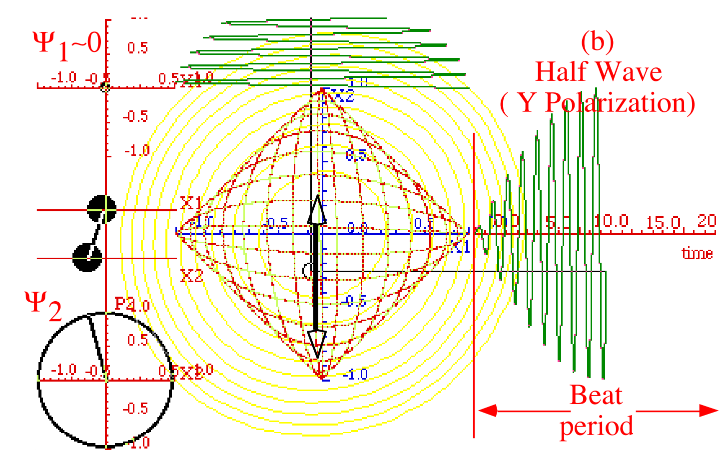
[BoxIt (B-Type) Web Simulation](#)

Beat dynamics:









3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

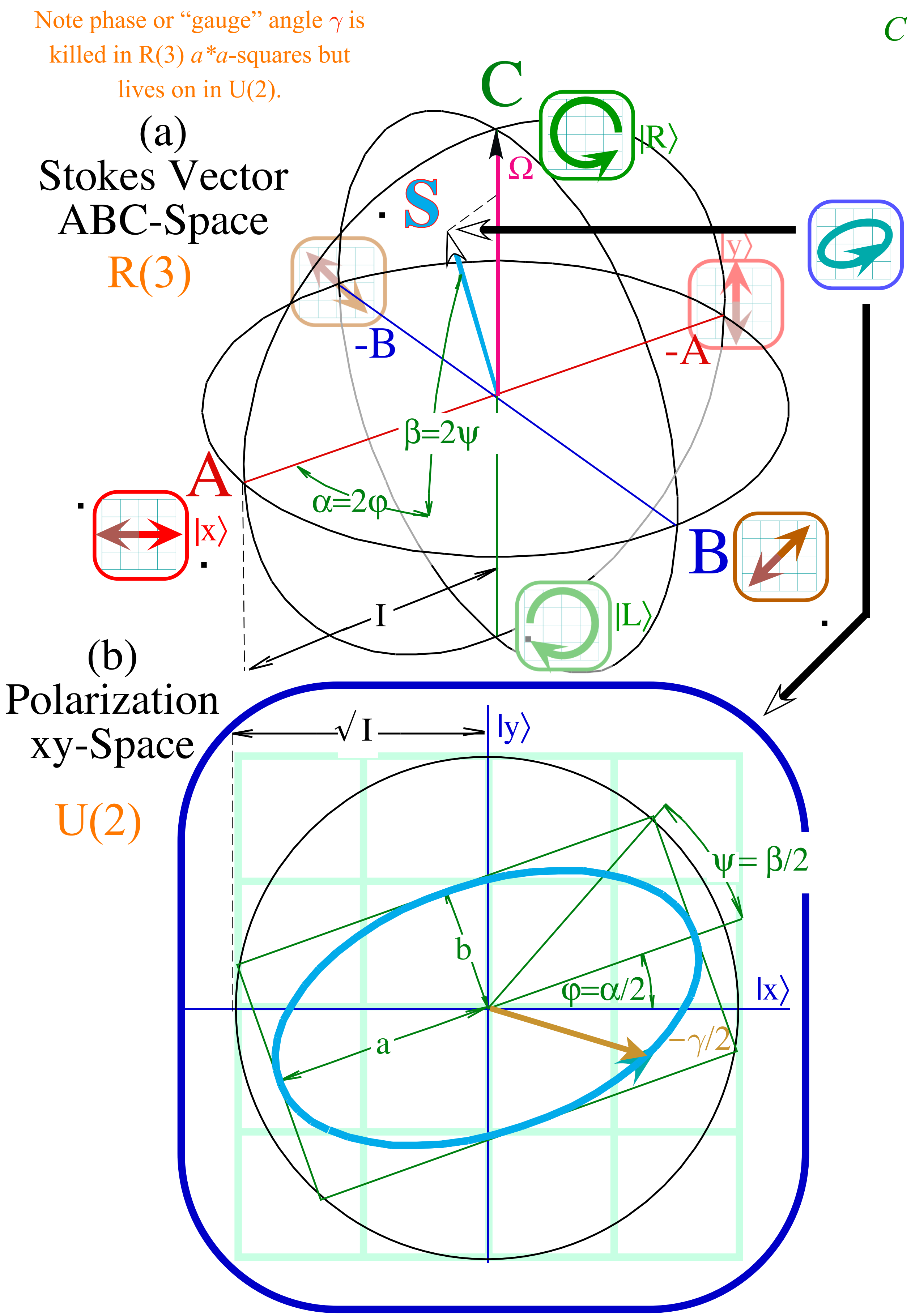
Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Polarization ellipse and spinor state dynamics

➤ *The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking*

Polarization ellipse and spinor state dynamics (C-Type motion)

C (Chiral-circular-complex-Coriolis-cyclotron-curly...current-carrier...)



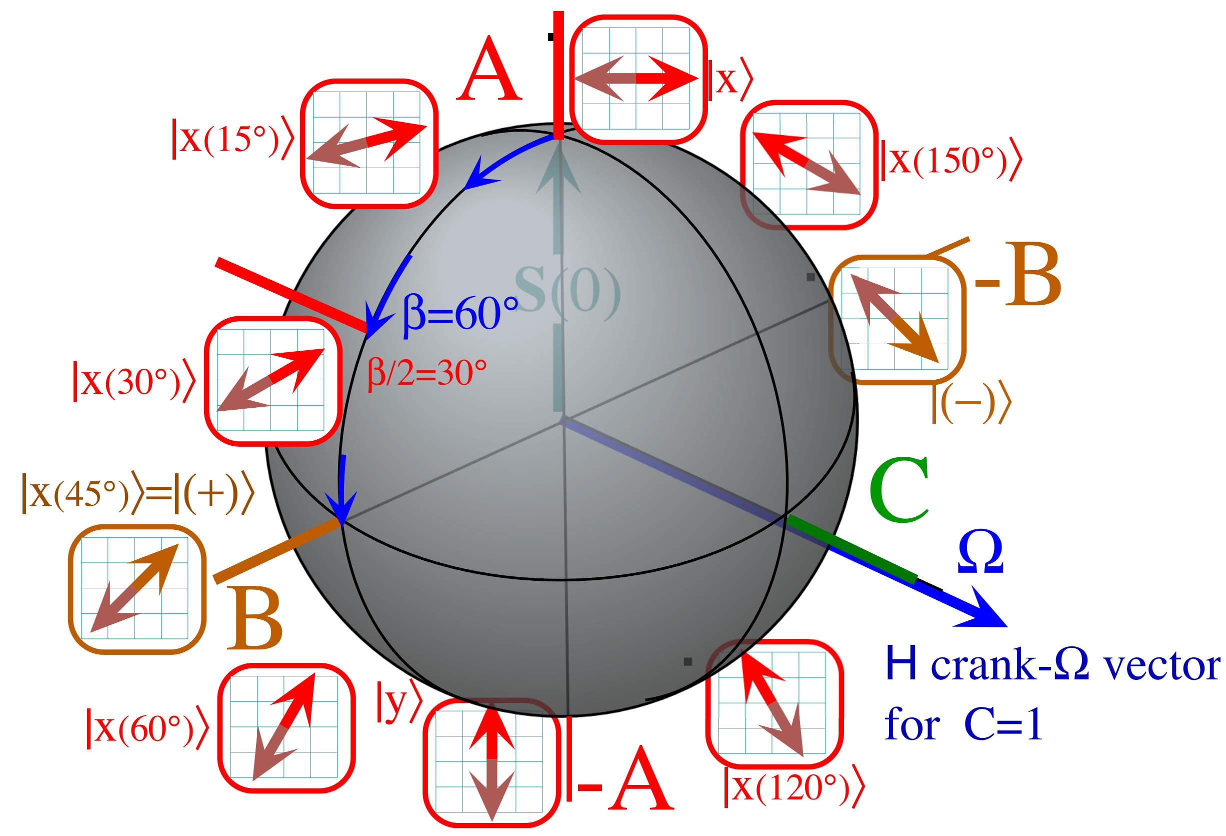


Fig. 3.4.7 Time evolution of a C-type beat. **S***-vector rotates from A to B to -A to -B and back to A.*

Fig. 3.4.5 Polarization variables (a) Stokes real-vector space (ABC) (b) Complex xy-spinor-space (x_1,x_2).

The ABC's of U(2) dynamics (C-Type motion)

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$

$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$

$$= \frac{A+D}{2}\boldsymbol{\sigma}_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

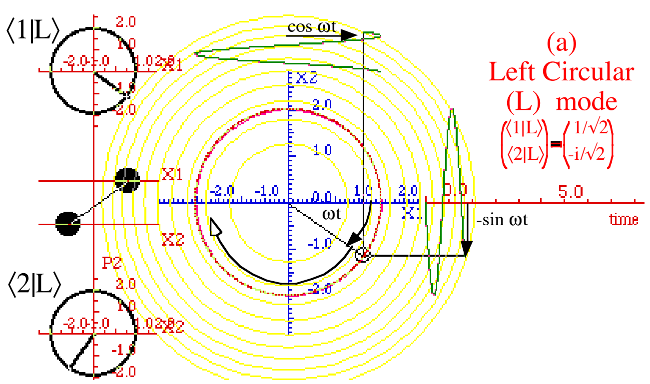
$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

From:
QTCA
Lect. 9(2.12)
p. 58

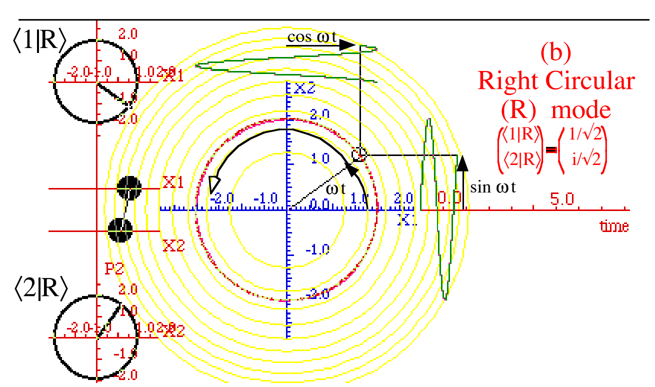
Circular-Coriolis... C-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^C|1\rangle & \langle 1|\mathbf{H}^C|2\rangle \\ \langle 2|\mathbf{H}^C|1\rangle & \langle 2|\mathbf{H}^C|2\rangle \end{pmatrix} = \begin{pmatrix} \Omega_0 & -iC \\ iC & \Omega_0 \end{pmatrix} = \Omega_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_C}{2}\sigma_C$$

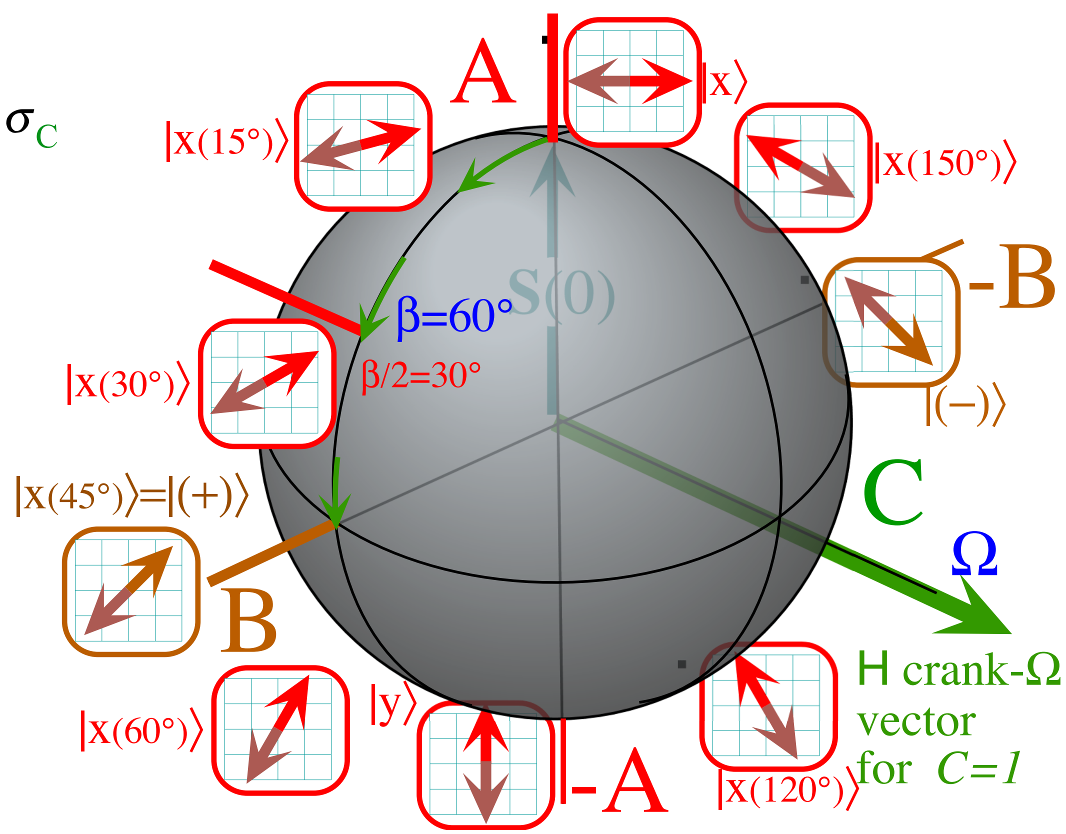
$$Crank: \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2C \end{pmatrix} \qquad Eigen-Spin: \vec{\mathbf{S}} = \begin{pmatrix} S_A \\ S_B \\ S_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm S \end{pmatrix}$$



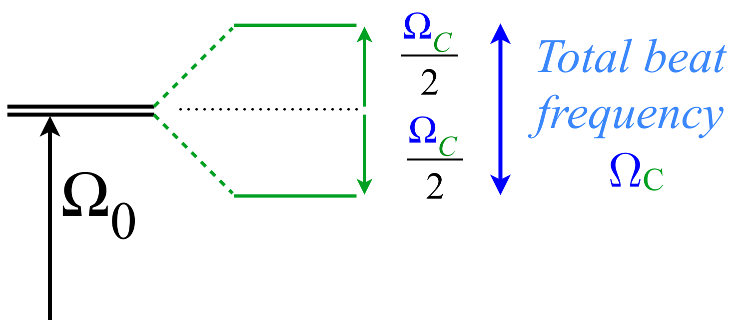
(a) Left Circular (L) mode $\begin{pmatrix}\langle 1|L\rangle \\ \langle 2|L\rangle\end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2}\end{pmatrix}$

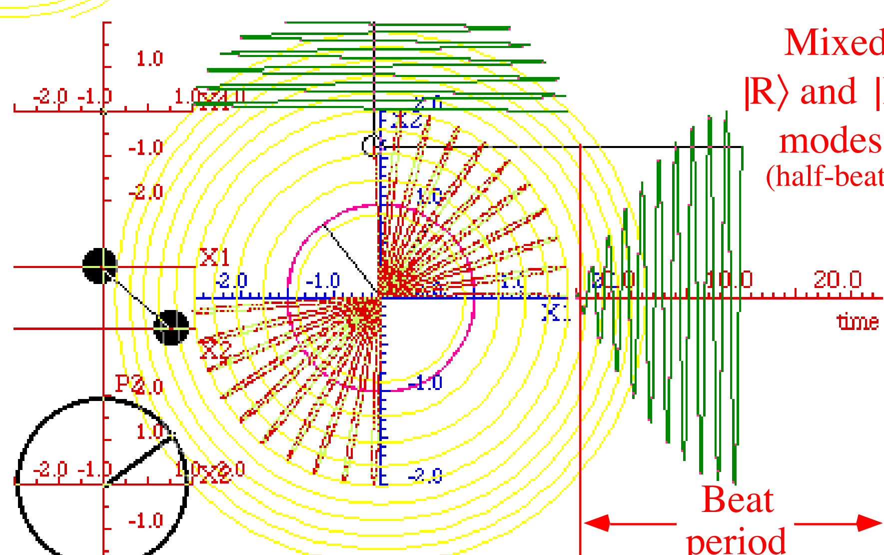


(b) Right Circular (R) mode $\begin{pmatrix}\langle 1|R\rangle \\ \langle 2|R\rangle\end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2}\end{pmatrix}$



Beat dynamics:





BoxIt (C-Type) Web Simulation

3D-real Stokes Vector defines 2D-HO polarization ellipses and spinor states

Asymmetry $S_A = S_Z$, Balance $S_B = S_X$, and Chirality $S_C = S_Y$

Polarization ellipse and spinor state dynamics

➤ *The "Great Spectral Avoided-Crossing" and A-to-B-to-A symmetry breaking*

The ABC's of U(2) dynamics-Mixed modes (AB-Type motion)

$$\rho = \frac{1}{2}N\mathbf{1} + \vec{\mathbf{S}}\bullet\boldsymbol{\sigma}$$
$$\mathbf{H} = \Omega_0\mathbf{1} + \frac{\vec{\Omega}}{2}\bullet\boldsymbol{\sigma}$$

$$\begin{pmatrix} \langle 1|\mathbf{H}|1\rangle & \langle 1|\mathbf{H}|2\rangle \\ \langle 2|\mathbf{H}|1\rangle & \langle 2|\mathbf{H}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B-iC \\ B+iC & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + C\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From:
QTCA
Lect. 9(2.12)
p.60

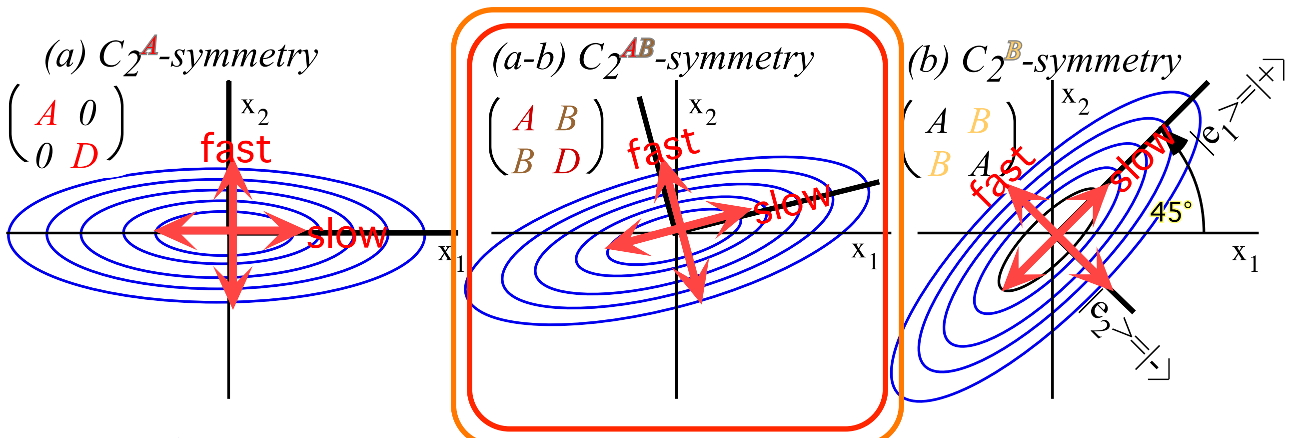
$$= \frac{A+D}{2}\mathbf{1} + B\,\sigma_B + C\,\sigma_C + \frac{A-D}{2}\sigma_A$$
$$= \frac{A+D}{2}\sigma_0 + \frac{\Omega_B}{2}\sigma_B + \frac{\Omega_C}{2}\sigma_C + \frac{\Omega_A}{2}\sigma_A$$

$$\vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 2C \end{pmatrix}$$

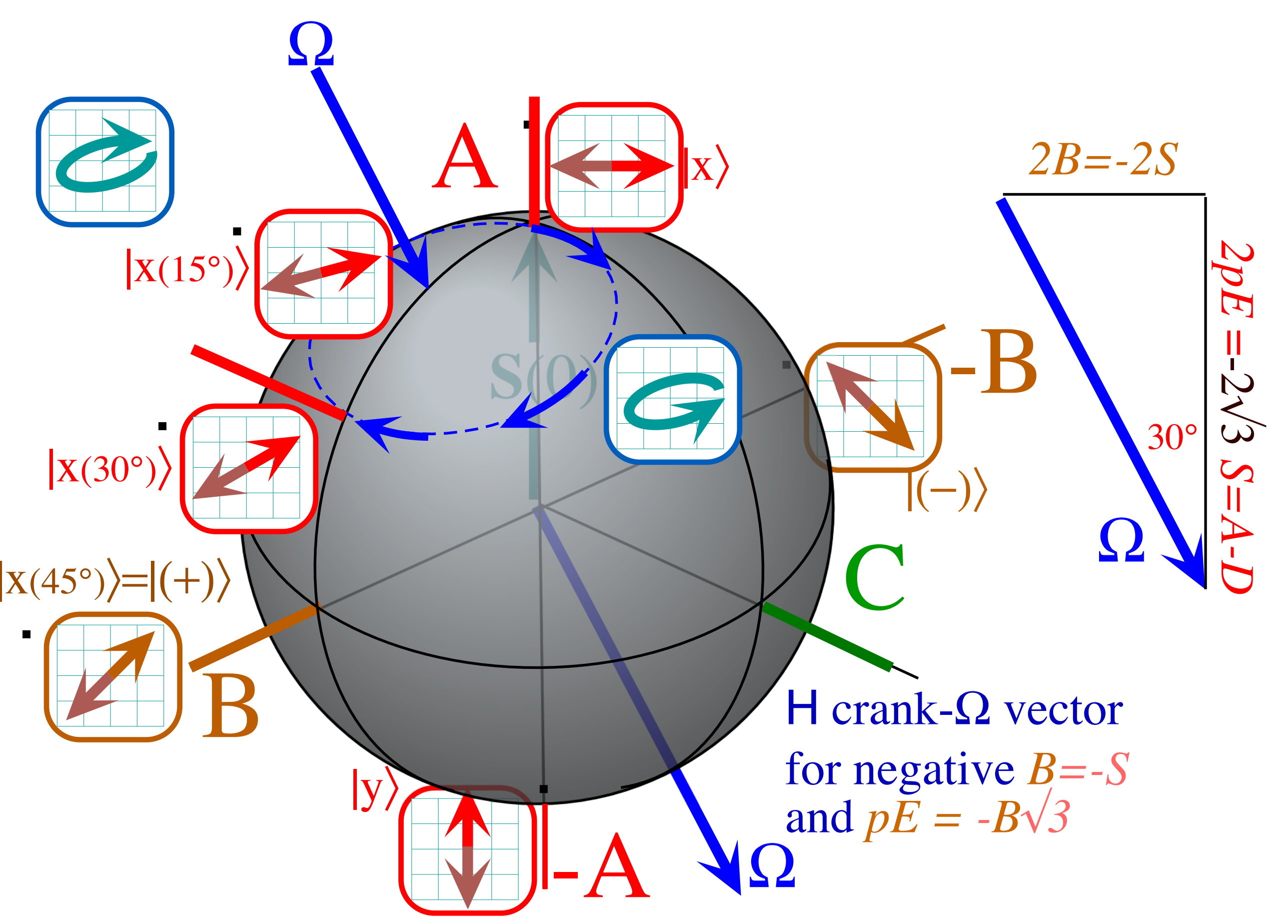
Tilted-plane polarization AB-Type motion

$$\begin{pmatrix} \langle 1|\mathbf{H}^{AB}|1\rangle & \langle 1|\mathbf{H}^{AB}|2\rangle \\ \langle 2|\mathbf{H}^{AB}|1\rangle & \langle 2|\mathbf{H}^{AB}|2\rangle \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Omega_0\boldsymbol{\sigma}_0 + \frac{\Omega_A}{2}\sigma_A + \frac{\Omega_B}{2}\sigma_B$$

$$Crank:\ \vec{\Omega} = \begin{pmatrix} \Omega_A \\ \Omega_B \\ \Omega_C \end{pmatrix} = \begin{pmatrix} A-D \\ 2B \\ 0 \end{pmatrix} \qquad Eigen-Spin:\vec{\mathbf{S}} = \pm S\,\hat{\Omega}$$



Beat dynamics:



BoxIt (AB-Type Motion)
Web Simulation

The Great Spectral "Avoided-Crossing"

A to B to A Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

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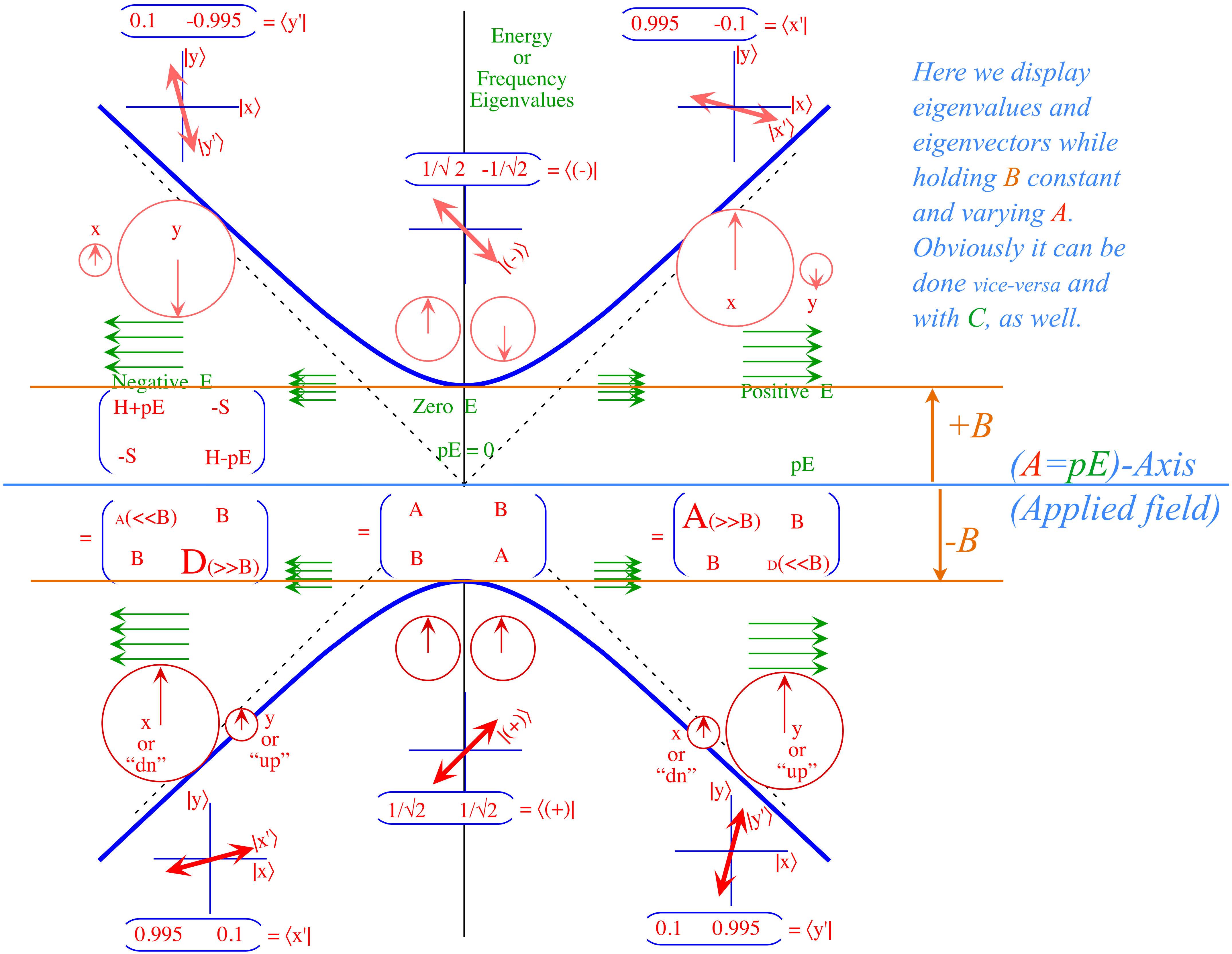


Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.

See also:
QTCA
Lect. 9(2.12)
p.61-66

The Great Spectral "Avoided-Crossing"

A to *B* to *A* Symmetry breaking described by hyperbolic eigenvalues of $A\boldsymbol{\sigma}_A + B\boldsymbol{\sigma}_B = \mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$

$\mathbf{H} = \begin{pmatrix} +A & B \\ B & -A \end{pmatrix}$ Secular equation: $\varepsilon^2 - 0 \cdot \varepsilon - (A^2 + B^2)$ gives *hyperbolic* energy levels: $\varepsilon = \pm\sqrt{A^2 + B^2}$

These on-and-off resonance effects are key to:

Laser QCD

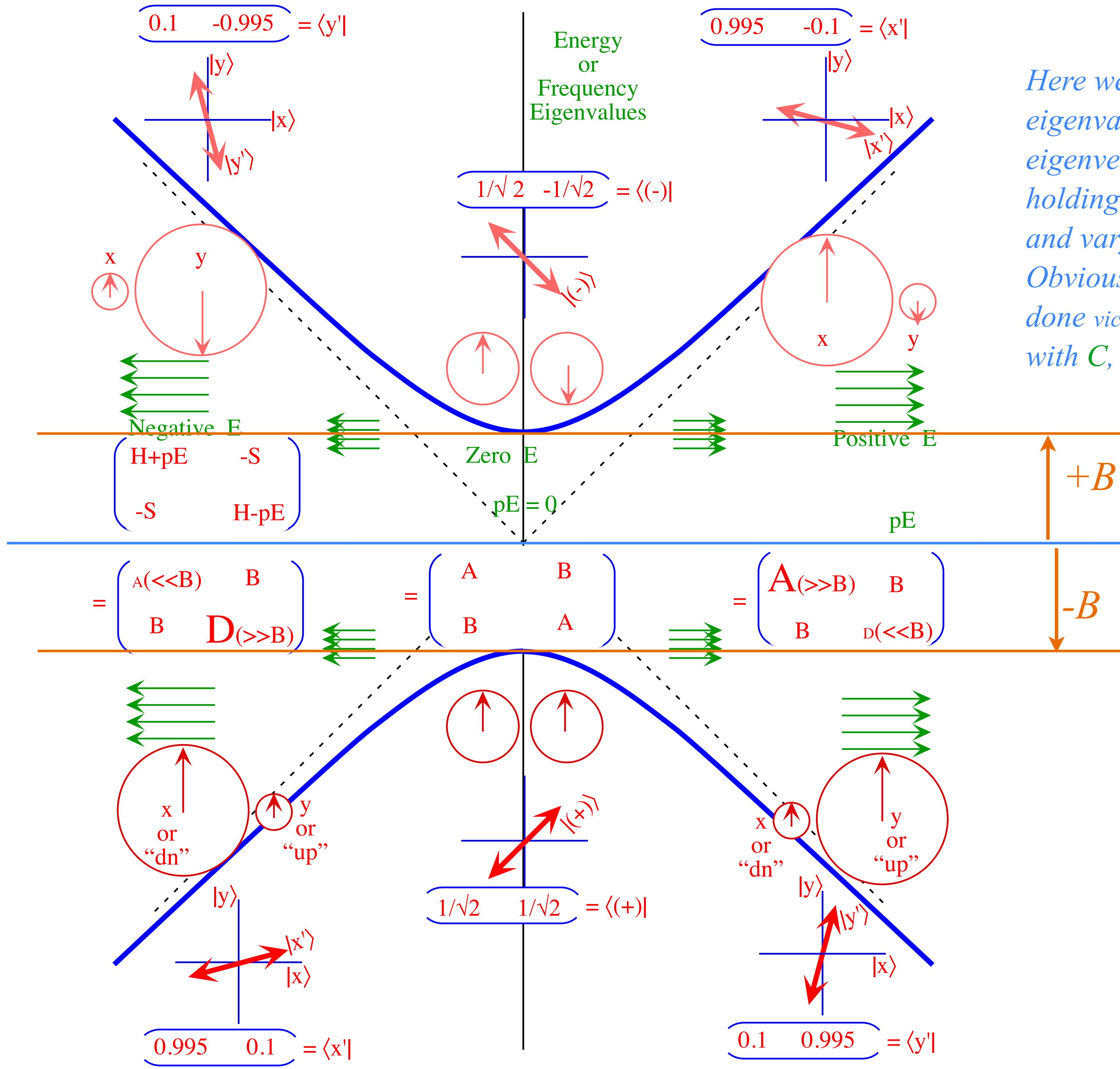
Relativistic QED

Quantum computing

Photosynthesis

and a whole lot of other things...

See also:
QTCA
Lect. 9(2.12)
p.61-66



(A=pE)-Axis
(Applied field)

Here we display eigenvalues and eigenvectors while holding B constant and varying A. Obviously it can be done vice-versa and with C, as well.

OBJECTIVE: Evaluate and (*most* important!) *visualize* matrix-exponent solutions.

ABCD Time evolution operator

$$\left|\Psi(t)\right\rangle = e^{-i\mathbf{H}t}\left|\Psi(0)\right\rangle$$

Hamilton generalized Euler's expansion $e^{-i\Omega t} = \cos\Omega t - i\sin\Omega t$ so matrix exponential becomes powerful.

$$e^{-i\mathbf{H}t} = e^{-i\begin{pmatrix} A & B-iC \\ B+iC & D\end{pmatrix}t} = e^{-i\frac{A-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}t - iB\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}t - iC\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}t - i\frac{A+D}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}t} = e^{-i(\omega_0\sigma_0 + \vec{\omega}\bullet\vec{\sigma})t} = e^{-i\omega_0 t}\left(\mathbf{1}\cos\omega t - i\sigma_\omega \sin\omega t\right)$$

$\sigma_A = \sigma_Z$ $\quad\sigma_B = \sigma_X$ $\quad\sigma_C = \sigma_Y$

where: $\vec{\varphi} = \vec{\omega}\cdot t = \begin{pmatrix}\omega_A \\ \omega_B \\ \omega_C\end{pmatrix}\cdot t = \begin{pmatrix}\frac{A-D}{2} \\ B \\ C\end{pmatrix}\cdot t$ and: $\omega_0 = \frac{A+D}{2}$ and: $\vec{\Theta} = \vec{\Omega}\cdot t = \begin{pmatrix}\Omega_A \\ \Omega_B \\ \Omega_C\end{pmatrix}\cdot t = \begin{pmatrix}A-D \\ 2B \\ 2C\end{pmatrix}\cdot t$ and: $\Omega_0 = \frac{A+D}{2}$

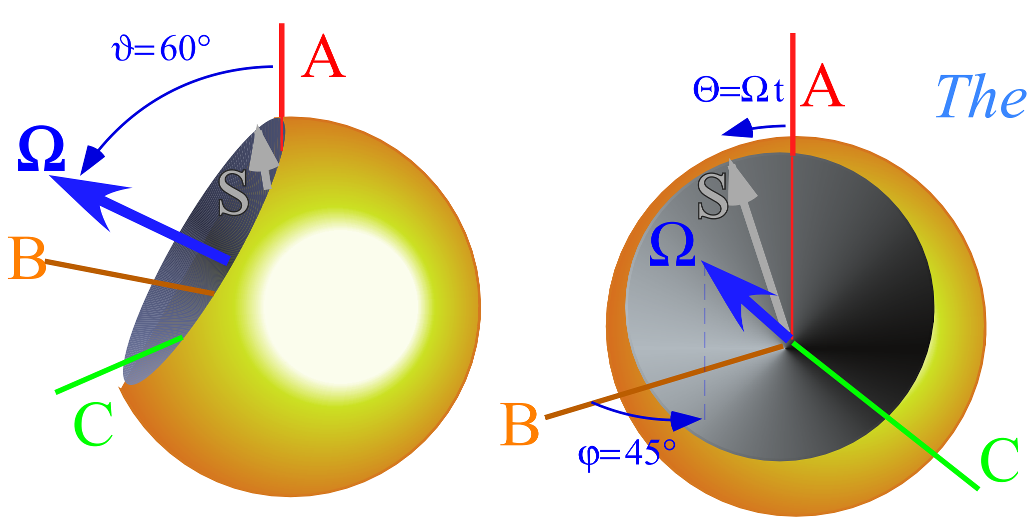
Symmetry relations make spinors σ_X, σ_Y, and σ_Z or quaternions $\mathbf{i} = -i\sigma_X$, $\mathbf{j} = -i\sigma_Y$, and $\mathbf{k} = -i\sigma_Z$ powerful.

3D crank vector $\vec{\Theta} = \vec{\Omega}\cdot t$ and *spin operator* $\mathbf{S}$ defines *3D ABC*-rotation with ratio $\frac{1}{2}$ or 2 between Θ_a and $\varphi_a = \frac{1}{2}\Theta_a$ or between $\mathbf{S}$ and $\sigma = 2\mathbf{S}$.

$$e^{-i\sigma\bullet\vec{\varphi}} = e^{-i\sigma\bullet\vec{\Theta}/2} = e^{-i\mathbf{S}\bullet\vec{\Theta}} = \mathbf{1}\cos\frac{\Theta}{2} - i(\sigma\bullet\hat{\Theta})\sin\frac{\Theta}{2} = \begin{pmatrix}\cos\frac{\Theta}{2} - i\hat{\Theta}_A\sin\frac{\Theta}{2} & (-i\hat{\Theta}_B - \hat{\Theta}_C)\sin\frac{\Theta}{2} \\ (-i\hat{\Theta}_B + \hat{\Theta}_C)\sin\frac{\Theta}{2} & \cos\frac{\Theta}{2} + i\hat{\Theta}_A\sin\frac{\Theta}{2}\end{pmatrix}$$

Example 3:
Any $\Theta = \Omega t$-axial rotation

2D angle : $\varphi = \frac{1}{2}\Theta$ *3D Crank vector* : $\vec{\Theta} = \Theta\hat{\Theta} = 2\varphi_a\hat{\mathbf{a}} = 2\vec{\varphi}$ *2D spin matrix* : $\mathbf{S} = \frac{1}{2}\sigma$

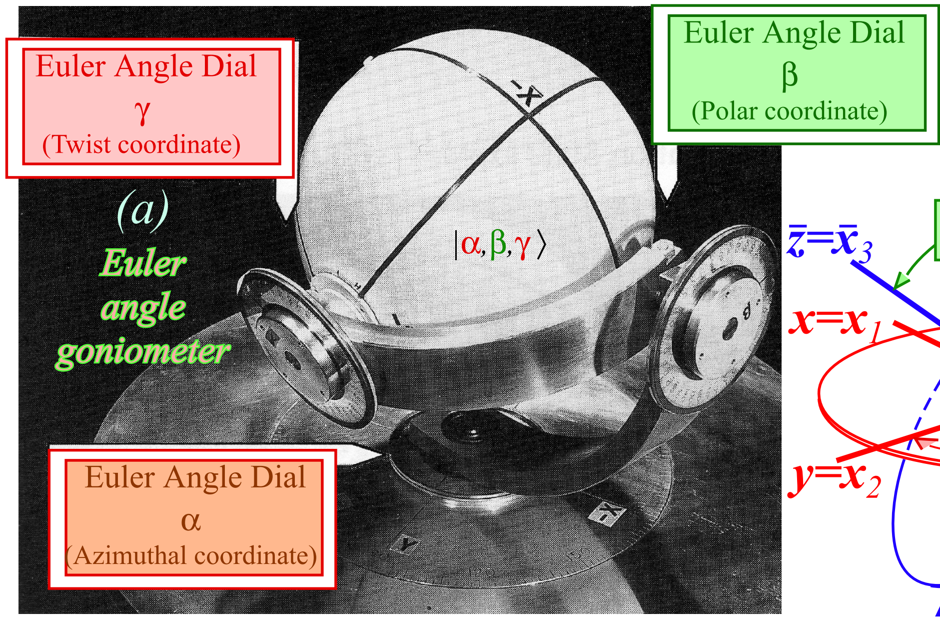


The driving $\Theta = \Omega t$ vector is defined by the ABCD of Hamiltonian **H** .

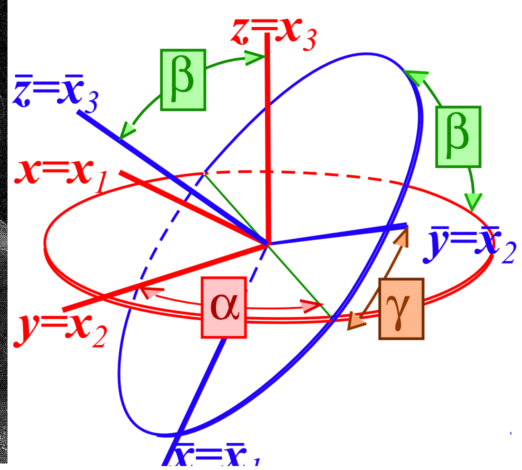
The driven spin vector **S** *defines the state. But, how?*

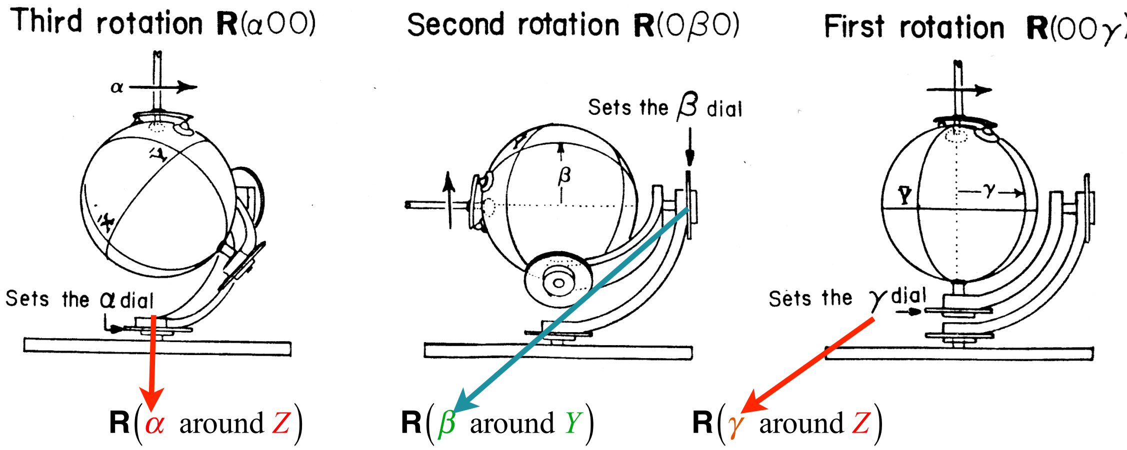
Fig. 3.4.2 Two views of Hamilton crank vector Ω(φ,ϑ) whirling Stokes state vector S in ABC-space.

Euler R(αβγ) versus *Darboux* R[φϑΘ]

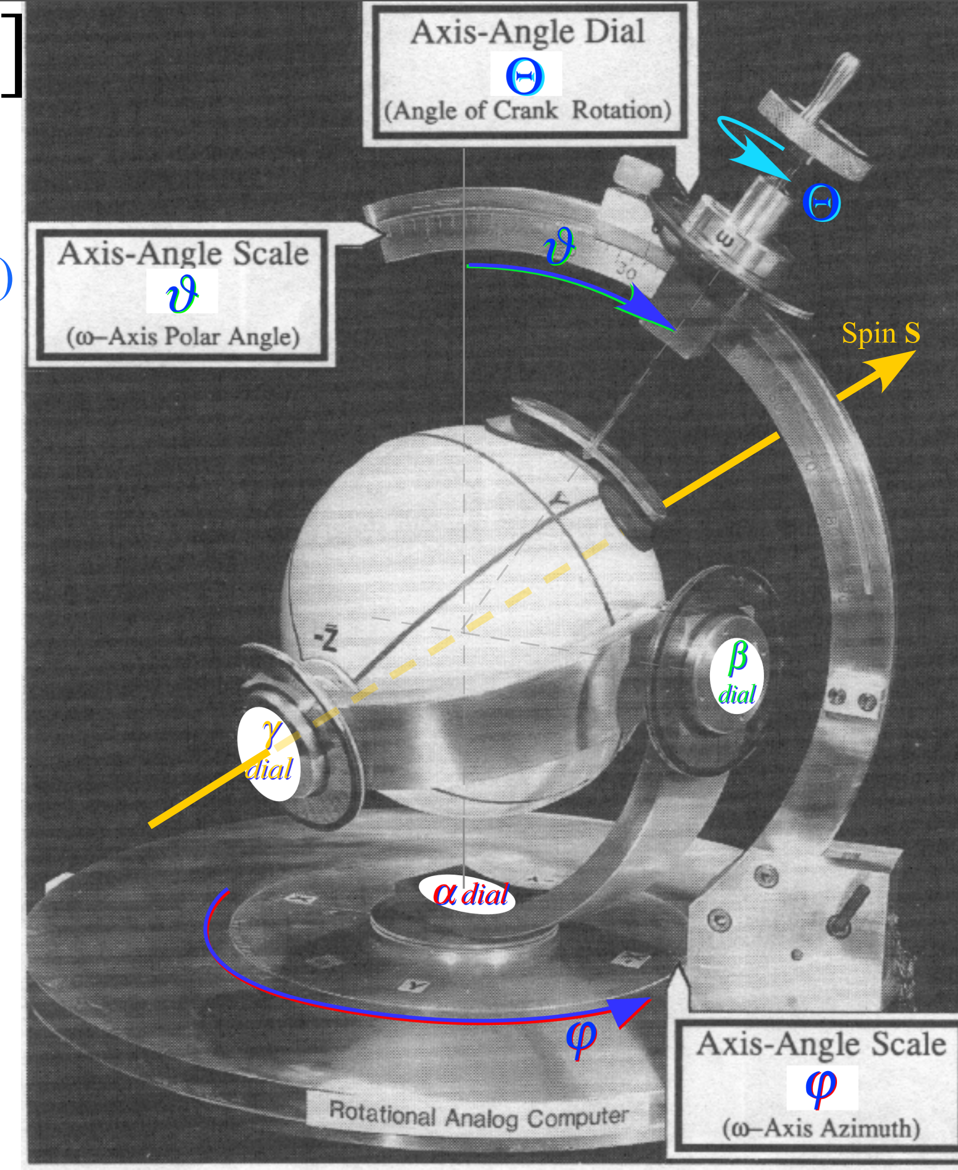


An astronomer's diagram





From:
QTCA
Lect. 9(2.12)
(See p.5-23 there)



$$\mathbf{R}(\alpha\beta\gamma)=\begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}=\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}=$$

$$=\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\cos\frac{\gamma-\alpha}{2}\sin\frac{\beta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\sin\frac{\alpha+\gamma}{2}\cos\frac{\beta}{2}$$

$$\mathbf{R}[\vec{\Theta}]=\begin{pmatrix} \cos\frac{\Theta}{2}-i\hat{\Theta}_Z\sin\frac{\Theta}{2} & -i\sin\frac{\Theta}{2}(\hat{\Theta}_X-i\hat{\Theta}_Y) \\ -i\sin\frac{\Theta}{2}(\hat{\Theta}_X+i\hat{\Theta}_Y) & \cos\frac{\Theta}{2}+i\hat{\Theta}_Z\sin\frac{\Theta}{2} \end{pmatrix}=\mathbf{R}[\varphi\vartheta\Theta]=e^{-i\mathbf{H}t}$$

$$=\cos\frac{\Theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}-i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\underbrace{\hat{\Theta}_X}_{\cos\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\underbrace{\hat{\Theta}_Y}_{\sin\varphi\ \sin\vartheta}\sin\frac{\Theta}{2}-i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\underbrace{\hat{\Theta}_Z}_{\cos\vartheta}\sin\frac{\Theta}{2}$$

$$=\begin{pmatrix} \cos\frac{\Theta}{2}-i\cos\vartheta\sin\frac{\Theta}{2} & -\sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta+i\cos\varphi\sin\vartheta) \\ \sin\frac{\Theta}{2}(\sin\varphi\sin\vartheta-i\cos\varphi\sin\vartheta) & \cos\frac{\Theta}{2}+i\cos\vartheta\sin\frac{\Theta}{2} \end{pmatrix}$$

Euler **R**(αβγ) is simpler to form than Θ-axis *Darboux* **R**[φϑΘ].

Euler *state definition* lets us relate **R**(αβγ) to **R**[φϑΘ] ...

$|\alpha\beta\gamma\rangle=\mathbf{R}(\alpha\beta\gamma)|000\rangle$ *(αβγ make better coordinates)*

$$\begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} x_1+ip_1 \\ x_2+ip_2 \end{pmatrix}$$

$x_1 = \cos[(\gamma+\alpha)/2]\cos\beta/2 = \cos\Theta/2$

$-p_2 = \sin[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_X\sin\Theta/2 = \cos\varphi\ \sin\vartheta\ \sin\Theta/2$

$x_2 = \cos[(\gamma-\alpha)/2]\sin\beta/2 = \hat{\Theta}_Y\sin\Theta/2 = \sin\varphi\ \sin\vartheta\ \sin\Theta/2$

$-p_1 = \sin[(\gamma+\alpha)/2]\cos\beta/2 = \hat{\Theta}_Z\sin\Theta/2 = \cos\vartheta\ \sin\Theta/2$

Ellipsometry using U(2) symmetry coordinates

Conventional amp-phase ellipse coordinates related to Euler Angles $(\alpha\beta\gamma)$

2D elliptic frequency ω orbit has amplitudes A_1 and A_2, and phase shifts ρ_1 and $\rho_2 = -\rho_1$.

$$\begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

$$x_1 = A_1 \cos(\omega t + \rho_1)$$
$$-p_1 = A_1 \sin(\omega t + \rho_1)$$
$$x_2 = A_2 \cos(\omega t - \rho_1)$$
$$-p_2 = A_2 \sin(\omega t - \rho_1)$$

Let: $A_1 = A\cos\beta/2$

$A_2 = A\sin\beta/2$

Real x_k and imaginary p_k parts of phasor amplitudes $a_k = x_k + ip_k$ depend on Euler angles $(\alpha\beta\gamma)$ and A.

$$x_1 = A\cos\beta/2\cos[(\gamma+\alpha)/2]$$
$$-p_1 = A\cos\beta/2\sin[(\gamma+\alpha)/2]$$
$$x_2 = A\sin\beta/2\cos[(\gamma-\alpha)/2]$$
$$-p_2 = A\sin\beta/2\sin[(\gamma-\alpha)/2]$$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

Let: $\omega t + \rho_1 = (\gamma+\alpha)/2$

$\omega t - \rho_1 = (\gamma-\alpha)/2$

$\tan\beta/2 = A_2/A_1 \qquad A^2 = A_1^2 + A_2^2$

$\alpha = 2\rho_1 \qquad \gamma = 2\omega \cdot t$

Euler parameters (α,β,γ, A) in terms of *amp-phase parameters* $(A_1, A_2, \omega t, \rho_1)$

$$\begin{pmatrix} Ae^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \\ Ae^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} A_1 e^{-i(\omega t+\rho_1)} \\ A_2 e^{-i(\omega t-\rho_1)} \end{pmatrix} = \begin{pmatrix} x_1 + ip_1 \\ x_2 + ip_2 \end{pmatrix}$$

See also:
QTCA
Lect. 9(2.12)
See pp. 96-104