Reimann-Christoffel equations and covariant derivative (Ch. 4-7 of Unit 3)

Separation of GCC Equations: Effective Potentials

Small radial oscillations
2D Spherical pendulum or “Bowl-Bowling”
Cycloidal ruler & compass geometry
Cycloid as brachistichrone with various geometries
Cycloid as tautochrone
Cycloidulum vs Pendulum
Cycloidal geometry of flying levers
Practical poolhall application
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Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \phi^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{Numerically correct ONLY!}) \]

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_r^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{Formally and Numerically correct}) \]
Separation of GCC Equations: Effective Potentials (For isotropic $H(r,p_r,\phi,p_\phi)$)

$$
H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad \text{(Numerically correct ONLY!)}
$$

$$
= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad \text{(Formally and Numerically correct)}
$$

Potential $V$ is \textit{isotropic} (cylindrical) function of radius $\rho$. ($V = V(\rho)$)

$H$ has no explicit $\phi$–dependence and the $\phi$–momenta is constant.

$$
m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu
$$
Separation of GCC Equations: Effective Potentials (For isotropic $H(r,p_r,\phi,p_\phi)$)

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \phi^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{Numerically correct ONLY!})$$

$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{Formally and Numerically correct})$$

Potential $V$ is \textit{isotropic} (cylindrical) function of radius $\rho$. ($V = V(\rho)$)

$H$ has no explicit $\phi$–dependence and the $\phi$–momenta is constant.

$$m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

If $H$ has no explicit $z$–dependence then the $z$–momenta is constant, too.

$$m \dot{z} = p_z = \text{const.} = k$$
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} \dot{\rho}^2 + V \quad \text{(Numerically correct ONLY!)} \]

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m \rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad \text{(Formally and Numerically correct)} \]

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( V = V(\rho) \)

If \( H \) has no explicit \( z \)–dependence then the \( z \)–momenta is constant, too.

\[ m \dot{\rho}^2 = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

\[ m \dot{z} = p_z = \text{const.} = k \]
Separation of GCC Equations: Effective Potentials

\[
H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 - \frac{1}{2} m \rho \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V 
\]

\[
= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_{\rho}^2 + \frac{1}{2m} p_{\phi}^2 + \frac{1}{2m} p_{z}^2 + V 
\]

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \((V = V(\rho))\)

\( H \) has no explicit \( \phi \)-dependence and the \( \phi \)-momenta is constant.

\[
m \rho^2 \dot{\phi} = p_{\phi} = \text{const.} = \mu
\]

\[
H = \frac{1}{2m} p_{\rho}^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}
\]

If \( H \) has no explicit \( z \)-dependence then the \( z \)-momenta is constant, too.

\[
m \dot{z} = p_z = \text{const.} = k
\]

\((\text{Let } k = 0)\)
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \]  
\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m} p_\phi^2 + \frac{1}{2m} p_z^2 + V \]  
\[ (\text{Numerically correct ONLY!}) \]

\[ (\text{Formally and Numerically correct}) \]

Potential \( V \) is *isotropic* (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)

\( H \) has no explicit \( \phi \)--dependence and the \( \phi \)--momenta is constant.

\[ m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.

\[ H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.} \]

If \( H \) has no explicit \( z \)--dependence

then the \( z \)--momenta is constant, too.

\[ m \dot{z} = p_z = \text{const.} = k \]

\( (\text{Let } k = 0) \)
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \phi^2 + \frac{1}{2} m \dot{z}^2 + V \]  

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p^2 + \frac{1}{2m} p^2 + \frac{1}{2m} p^2 + V \]

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)

\( H \) has no explicit \( \phi \)--dependence and the \( \phi \)--momenta is constant.

\[ m \rho^2 \phi = p_\phi = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.

\[ H = \frac{1}{2m} p^2 + V^{\text{eff}}(\rho) = E = \text{const.} \]

An effective potential \( V^{\text{eff}}(\rho) \) has a centrifugal barrier.

\[ V^{\text{eff}}(\rho) = \frac{\mu^2}{2m \rho^2} + V(\rho) \]

If \( H \) has no explicit \( z \)--dependence then the \( z \)--momenta is constant, too.

\[ m \dot{z} = p_z = \text{const.} = k \]

(\( \text{Let } k = 0 \))
Separation of GCC Equations: Effective Potentials

\[ H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \phi^2 + \frac{1}{2} m \dot{z}^2 + V \quad \text{(Numerically correct ONLY!)} \]

\[ = \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_{\rho}^2 + \frac{1}{2m} \rho \phi^2 + \frac{1}{2m} p_{z}^2 + V \quad \text{(Formally and Numerically correct)} \]

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)

\( H \) has no explicit \( \phi \)-dependence and the \( \phi \)-momenta is constant.

\[ m \rho^2 \dot{\phi} = p_{\phi} = \text{const.} = \mu \]

\[ H = \frac{1}{2m} p_{\rho}^2 + \frac{\mu^2}{2m \rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.} \]

Symmetry reduces problem to a one-dimensional form.

\[ H = \frac{1}{2m} p_{\rho}^2 + V_{\text{eff}}(\rho) = E = \text{const.} \]

An effective potential \( V_{\text{eff}}(\rho) \) has a centrifugal barrier.

\[ V_{\text{eff}}(\rho) = \frac{\mu^2}{2m \rho^2} + V(\rho) \]

Velocity relations:

\[ \dot{\phi} = \mu / \left( m \rho^2 \right) \quad \dot{\rho} = \frac{d \rho}{dt} = \frac{\partial H}{\partial p_{\rho}} = \frac{p_{\rho}}{m} = \pm \sqrt{\frac{2}{m} \left( E - V_{\text{eff}}(\rho) \right)} \]

If \( H \) has no explicit \( z \)-dependence then the \( z \)-momenta is constant, too.

\[ m \dot{z} = p_{z} = \text{const.} = k \]

(Let \( k = 0 \))
Separation of GCC Equations: Effective Potentials

\[
H = \frac{1}{2} \sum_{m} \dot{q}_m \ddot{q}_m + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V
\]

(Numerically correct ONLY!)

\[
= \frac{1}{2} \sum_{m} \sum_{n} \gamma_{m n} p_m p_n + V = \frac{1}{2} m p_\rho^2 + \frac{1}{2} m p_\phi^2 + \frac{1}{2} m p_z^2 + V
\]

(Formally and Numerically correct)

Potential \( V \) is isotropic (cylindrical) function of radius \( \rho \). \( (V = V(\rho)) \)
\( H \) has no explicit \( \phi \)–dependence and the \( \phi \)–momenta is constant.

\[
m\dot{\rho}^2 \phi = p_\phi = \text{const.} = \mu
\]

\[
H = \frac{1}{2} m p_\rho^2 + \frac{\mu^2}{2 m \rho^2} + \frac{k^2}{2 m} + V(\rho) = E = \text{const.}
\]

Symmetry reduces problem to a one-dimensional form.

\[
H = \frac{1}{2} m p_\rho^2 + V_{\text{eff}}(\rho) = E = \text{const.}
\]

An effective potential \( V_{\text{eff}}(\rho) \) has a centrifugal barrier.

\[
V_{\text{eff}}(\rho) = \frac{\mu^2}{2 m \rho^2} + V(\rho)
\]

Velocity relations:
\[
\dot{\phi} = \frac{\mu}{m \rho^2} \quad \dot{\rho} = \frac{d \rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm \sqrt{\frac{2}{m} \left( E - V_{\text{eff}}(\rho) \right)}
\]

Equations solved by a quadrature integral for time versus radius.

\[
\int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d \rho}{\sqrt{2 m \left( E - V_{\text{eff}}(\rho) \right)}} = \text{Travel time } \rho_0 \text{ to } \rho_1 = t_1 - t_0
\]

If \( H \) has no explicit \( z \)–dependence then the \( z \)–momenta is constant, too.

\[
m \dot{z} = p_z = \text{const.} = k
\]

(Let \( k = 0 \))

Thursday, November 6, 2014
Separation of GCC Equations: Effective Potentials

Small radial oscillations

2D Spherical pendulum or “Bowl-Bowling”

Cycloidal ruler&compass geometry

Cycloid as brachistichrone

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Cycloidulum vs Pendulum

Cycloidal geometry of flying levers

Practical poolhall application
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[ \frac{dV_{eff}(\rho)}{d\rho} \bigg|_{\rho_0} = 0 , \quad \text{with:} \quad \frac{d^2V_{eff}}{d\rho^2} \bigg|_{\rho_0} > 0 . \]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[ V_{eff}(\rho) = V_{eff}(\rho_0) + 0 + \frac{1}{2} (\rho - \rho_0)^2 \frac{d^2V_{eff}}{d\rho^2} \bigg|_{\rho_0} \]

Fig. 2.7.4  Phase paths around fixed points (a) Stable point (b) Unstable saddle point
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[
\frac{dV_{\text{eff}}(\rho)}{d\rho}_{\rho_{\text{stable}}} = 0, \quad \text{with:} \quad \frac{d^2V_{\text{eff}}}{d\rho^2}_{\rho_{\text{stable}}} > 0.
\]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[
V_{\text{eff}}(\rho) = V_{\text{eff}}(\rho_{\text{stable}}) + 0 + \frac{1}{2} (\rho - \rho_{\text{stable}})^2 \frac{d^2V_{\text{eff}}}{d\rho^2}_{\rho_{\text{stable}}}
\]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[
k_{\text{eff}} = \frac{d^2V_{\text{eff}}}{d\rho^2}_{\rho_{\text{stable}}}
\]

\[
\omega_{\rho_{\text{stable}}} = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \frac{d^2V_{\text{eff}}}{d\rho^2}_{\rho_{\text{stable}}}}
\]
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[ \left. \frac{dV^{\text{eff}}(\rho)}{d\rho} \right|_{\rho_{\text{stable}}} = 0 , \quad \text{with:} \quad \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}} > 0 . \]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[ V^{\text{eff}}(\rho) = V^{\text{eff}}(\rho_{\text{stable}}) + 0 + \frac{1}{2} (\rho - \rho_{\text{stable}})^2 \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}} \]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[ k^{\text{eff}} = \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}} \]

\[ \omega_{\rho_{\text{stable}}} = \frac{\sqrt{k^{\text{eff}}}}{m} = \sqrt{\frac{1}{m} \left. \frac{d^2V^{\text{eff}}}{d\rho^2} \right|_{\rho_{\text{stable}}}} \]

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

\[ \frac{\omega_{\rho_{\text{stable}}}}{\omega_{\phi}} = \frac{\omega_{\rho_{\text{stable}}}}{\phi(\rho_{\text{stable}})} = \frac{n_\rho}{n_\phi} \quad \iff \text{Orbit is closed-periodic} \]
Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

\[ \frac{dV_{\text{eff}}(\rho)}{d\rho} \bigg|_{\rho_{\text{stable}}} = 0, \quad \text{with:} \quad \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} > 0. \]

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

\[ V_{\text{eff}}(\rho) = V_{\text{eff}}(\rho_{\text{stable}}) + 0 + \frac{1}{2} (\rho - \rho_{\text{stable}})^2 \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} \]

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

\[ k_{\text{eff}} = \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}} \quad \omega_{\rho_{\text{stable}}} = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{1}{m} \frac{d^2V_{\text{eff}}}{d\rho^2} \bigg|_{\rho_{\text{stable}}}} \]

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

\[ \frac{\omega_{\rho_{\text{stable}}}}{\omega_\phi} = \frac{\omega_{\rho_{\text{stable}}}}{\phi(\rho_{\text{stable}})} = \frac{n_\rho}{n_\phi} \Leftrightarrow \text{Orbit is closed-periodic} \]

Some generic shapes resulting from various ratios \( n_\rho : n_\phi \)
Separation of GCC Equations: Effective Potentials

Small radial oscillations

2D Spherical pendulum or “Bowl-Bowling”

Cycloidal ruler&compass geometry

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2D Spherical pendulum or “Bowl-Bowling”

Spherical coordinates: \( \{ q^1 = r, q^2 = \theta, q^3 = \phi \} \) obvious choice:

\[
x = x^1 = rsin\theta \cos\phi, \quad y = x^2 = rsin\theta \sin\phi, \quad z = x^3 = r\cos\theta,
\]
2D Spherical pendulum or "Bowl-Bowling"

Spherical coordinates: \( \{q^1=r, q^2=\theta, q^3=\phi\} \) obvious choice:

\[
x = x^1 = r \sin \theta \cos \phi, \quad y = x^2 = r \sin \theta \sin \phi, \quad z = x^3 = r \cos \theta,
\]

Jacobian matrices and determinants:

\[
J = \begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{pmatrix} = \\
\begin{pmatrix}
\sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
\sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
\cos \theta & -r \sin \theta & 0
\end{pmatrix}
\]

Reduced to cylindrical coordinates:

\[
\det J = \det J^T = \frac{\partial \{xyz\}}{\partial \{r\theta\phi\}} = r^2 \sin \theta
\]

\[
\theta = \pi/2, \quad r = \rho \Rightarrow r^2 \sin \theta \rightarrow \rho^2
\]
2D Spherical pendulum or “Bowl-Bowling”

Spherical coordinates: \( \{q^1=r, q^2=\theta, q^3=\phi \} \) obvious choice:

\[
x = x^1 = r \sin \theta \cos \phi, \quad y = x^2 = r \sin \theta \sin \phi, \quad z = x^3 = r \cos \theta,
\]

Jacobian matrices and determinants:

\[
J = \begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{pmatrix}
\]

Reduced to cylindrical coordinates:

\[
\det J = \det J^T = \frac{\partial \{xyz\}}{\partial \{r\theta\phi\}} = r^2 \sin \theta \quad \text{to} \quad \rho^2
\]

Covariant metric \( g_{\mu\nu} \) is matrix product \( g = J^T \cdot J \) of Jacobian and its transpose. OCC g’s are diagonal.

Covariant: \( g_{rr} = E_r \cdot E_r = 1, \quad g_{\theta\theta} = E_\theta \cdot E_\theta = r^2, \quad g_{\phi\phi} = E_\phi \cdot E_\phi = r^2 \sin^2 \theta, \)

Contravariant: \( g^{rr} = 1, \quad g^{\theta\theta} = 1/r^2, \quad g^{\phi\phi} = 1/r^2 \sin^2 \theta. \)
2D Spherical pendulum or “Bowl-Bowling”

Spherical coordinates: \( \{q^1=r, q^2=\theta, q^3=\phi \} \) obvious choice:
\[
\begin{align*}
  x &= x^1=r\sin\theta \cos\phi, \\
  y &= x^2=r\sin\theta \sin\phi, \\
  z &= x^3=r\cos\theta,
\end{align*}
\]

Jacobian matrices and determinants:
\[
J = \begin{bmatrix}
  \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
  \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
  \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{bmatrix} = \begin{bmatrix}
  \sin\theta \cos\phi & r \cos\theta \cos\phi & -r \sin\theta \sin\phi \\
  \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\
  \cos\theta & -r \sin\theta & 0
\end{bmatrix}
\]

Reduced to cylindrical coordinates:
\[
\det J = \det J^T = \frac{\partial \{xyz\}}{\partial \{r\theta\phi\}} = r^2 \sin\theta \rightarrow \rho^2
\]

Covariant metric \( g_{\mu\nu} \) is matrix product \( g = J^T \cdot J \) of Jacobian and its transpose. OCC g’s are diagonal.

Covariant:
\[
g_{rr} = E_r \cdot E_r = 1, \quad g_{\theta\theta} = E_\theta \cdot E_\theta = r^2, \quad g_{\phi\phi} = E_\phi \cdot E_\phi = r^2 \sin^2\theta,
\]

Contravariant:
\[
g^{rr} = 1, \quad g^{\theta\theta} = 1/r^2, \quad g^{\phi\phi} = 1/r^2 \sin^2\theta.
\]

\( \text{Lagrangian form} \)
\( \text{Hamiltonian form} \)
\[
T = \frac{m}{2} (g_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 + g_{\phi\phi} \dot{\phi}^2) = \frac{1}{2m} (g^{rr} p_r^2 + g^{\theta\theta} p_\theta^2 + g^{\phi\phi} p_\phi^2)
\]
\[
= \frac{1}{2} (\gamma_{rr} \dot{r}^2 + \gamma_{\theta\theta} \dot{\theta}^2 + \gamma_{\phi\phi} \dot{\phi}^2) = \frac{1}{2} \left( \gamma^{rr} p_r^2 + \gamma^{\theta\theta} p_\theta^2 + \gamma^{\phi\phi} p_\phi^2 \right)
\]
\[
= \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2) = \frac{1}{2m} (p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta})
\]
2D Spherical pendulum or “Bowl-Bowling”

Spherical coordinates: \( \{ q^1=r, q^2=\theta, q^3=\phi \} \) obvious choice:
\[ x=x^1=r \sin \theta \cos \phi, \ y=x^2=r \sin \theta \sin \phi, \ z=x^3=r \cos \theta, \]

Jacobian matrices and determinants:
\[
\begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{bmatrix}
\]

Reduced to cylindrical coordinates:
\[
J = \begin{pmatrix}
\sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
\sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
\cos \theta & -r \sin \theta & 0
\end{pmatrix}
\]
\[
\begin{pmatrix}
\cos \phi & 0 & -\rho \sin \phi \\
\sin \phi & 0 & \rho \cos \phi \\
0 & -\rho & 0
\end{pmatrix}
\]
\[
\det J = \det J^T = \frac{\partial \{x,y,z\}}{\partial \{r,\theta,\phi\}} = r^2 \sin \theta \frac{\rho}{r=\pi/2} \rightarrow \rho^2
\]

Covariant metric \( g_{\mu \nu} \) is matrix product \( g=J^T \cdot J \) of Jacobian and its transpose. OCC \( g \)'s are diagonal.

Covariant: \( g_{rr}=E_r \cdot E_r=1, \ g_{\theta \theta}=E_{\theta} \cdot E_{\theta}=r^2, \ g_{\phi \phi}=E_{\phi} \cdot E_{\phi}=r^2 \sin^2 \theta, \)

Contravariant: \( g^{rr}=1, \ g^{\theta \theta}=1/r^2, \ g^{\phi \phi}=1/r^2 \sin^2 \theta. \)

Lagrangian form
\[
T = \frac{1}{2} (g_{rr} \dot{r}^2 + g_{\theta \theta} \dot{\theta}^2 + g_{\phi \phi} \dot{\phi}^2)
\]
\[
= \frac{1}{2m} (g^{rr} p_r^2 + g^{\theta \theta} p_{\theta}^2 + g^{\phi \phi} p_{\phi}^2)
\]
\[
= \frac{1}{2} \left( \gamma_{rr} \dot{r}^2 + \gamma_{\theta \theta} \dot{\theta}^2 + \gamma_{\phi \phi} \dot{\phi}^2 \right)
\]
\[
= \frac{1}{2} \left( \gamma^{rr} p_r^2 + \gamma^{\theta \theta} p_{\theta}^2 + \gamma^{\phi \phi} p_{\phi}^2 \right)
\]
\[
= \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right)
\]
\[
= \frac{1}{2} \left( p_r^2 + \frac{p_{\theta}^2}{r^2} + \frac{p_{\phi}^2}{r^2 \sin^2 \theta} \right)
\]

Hamiltonian form
\[
Spherical coordinates with constant radius \( r \)
implies conserved azimuthal momentum:
\[
p_{\phi} \equiv \frac{\partial T}{\partial \dot{\phi}} = m (R^2 \sin^2 \theta) \dot{\phi} = \text{const.}
\]
2D Spherical pendulum or “Bowl-Bowling”

Spherical coordinates: \{ q^1=r, q^2=\theta, q^3=\phi \} obvious choice:
\[ x=x^1=r\sin\theta \cos\phi, \quad y=x^2=r\sin\theta \sin\phi, \quad z=x^3=r\cos\theta, \]

Jacobi matrices and determinants:

\[
J = \begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\
\frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}
\end{bmatrix}
\]

Reduced to cylindrical coordinates:

\[
\det J = \det J^T = \frac{\partial \{xyz\}}{\partial \{r\theta\phi\}} = r^2 \sin \theta \frac{\partial \{xyz\}}{\partial \{r\theta\phi\}} = \rho^2
\]

Covariant metric \( g_{\mu\nu} \) is matrix product \( g=J^T \cdot J \) of Jacobian and its transpose. OCC \( g\)'s are diagonal.

Covariant: \( g_{rr}=E_r \cdot E_r = 1, \quad g_{\theta\theta}=E_\theta \cdot E_\theta = r^2, \quad g_{\phi\phi}=E_\phi \cdot E_\phi = r^2 \sin^2 \theta, \)

Contravariant: \( g^{rr}=1, \quad g^{\theta\theta}=1/r^2, \quad g^{\phi\phi}=1/r^2 \sin^2 \theta. \)

(Lagrangian form) \quad (Hamiltonian form)

\[
T = \frac{m}{2} (g_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 + g_{\phi\phi} \dot{\phi}^2) = \frac{1}{2m} (g^{rr} p_r^2 + g^{\theta\theta} p_{\theta}^2 + g^{\phi\phi} p_{\phi}^2)
\]

\[
= \frac{1}{2} (\gamma_{rr} \dot{r}^2 + \gamma_{\theta\theta} \dot{\theta}^2 + \gamma_{\phi\phi} \dot{\phi}^2) = \frac{1}{2} (\gamma^{rr} p_r^2 + \gamma^{\theta\theta} p_{\theta}^2 + \gamma^{\phi\phi} p_{\phi}^2)
\]

\[
= \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) = \frac{1}{2m} (p_r^2 + p_{\theta}^2 + p_{\phi}^2)
\]

Spherical coordinates with constant radius \( r \) implies conserved azimuthal momentum:

\[
p_{\phi} \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}} = m(R^2 \sin^2 \theta) \dot{\phi} = \text{const}.
\]

Total Energy from Hamiltonian \( E=T+V(\text{gravity})=\text{const}. \):

\[
E = \frac{mR^2}{2} \dot{\theta}^2 + V_{\text{effective}}(\theta) = \frac{mR^2}{2} \dot{\theta}^2 + \frac{p_{\phi}^2}{2mR^2 \sin^2 \theta} + mgR \cos \theta = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + g \cos \theta
\]

Let: \( \alpha = \frac{mR^2}{2}, \quad \delta = \frac{p_{\phi}^2}{2mR^2}, \quad g = mgR \) where: \( p_{\phi} = mR^2 \sin^2 \theta \dot{\phi} \)
2D Spherical pendulum or “Bowl-Bowling”

Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$ : 

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V_{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_{\phi}^2}{2mR^2}$, $\gamma = mgR$ where: $p_{\phi} = mR^2 \sin^2 \theta (\dot{\phi})$
2D Spherical pendulum or “Bowl-Bowling”

Total Energy from Hamiltonian $E=T+V(gravity)=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let:

$$\alpha = \frac{mR^2}{2}, \quad \delta = \frac{p^2_\phi}{2mR^2}, \quad \gamma = mgR$$

where:

$$p^2_\phi = mR^2 \sin^2 \theta (\dot{\phi})$$

Equilibrium point of stable orbit

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2 \delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2 p^2_\phi \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$
Total Energy from Hamiltonian $E=T+V(gravity)=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let:

$$\alpha = \frac{mR^2}{2}, \quad \delta = \frac{p_\phi^2}{2mR^2}, \quad \gamma = mgR \quad \text{where:} \quad p_\phi = mR^2 \sin^2 \theta(\dot{\phi})$$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = -\frac{2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$\left(\omega_\theta^{\text{equil}}\right)^2 = \frac{1}{mR^2} \left. \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$
2D Spherical pendulum or “Bowl-Bowling”

Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V_{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = m g R$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV_{\text{effective}}(\theta)}{d\theta} = \frac{-2 \delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2 p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or} \quad \dot{\phi}^2_{\text{equil}} = -\frac{g}{R \cos \theta_{\text{equil}}}$$

(Polar angle librational frequency $\omega^2_{\theta \text{equil}}$ is related to azimuthal frequency $\dot{\phi}^2_{\text{equil}}$.)
Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{m R^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let:

$$\alpha = \frac{m R^2}{2}, \quad \delta = \frac{p_\phi}{2 m R^2}, \quad \gamma = m g R$$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2 \delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2 p_\phi^2}{2 m R^2 \sin^3 \theta} - m g R \sin \theta$$

$$0 = (m R^2 \sin \theta) \dot{\phi}^2 \cos \theta - m g R \sin \theta \quad \text{or} \quad \dot{\phi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$$

V-Derivative for small oscillation frequency:

$$\frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} = -\gamma \cos \theta + \frac{2 \delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2 \delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2 \delta \frac{\sin^2 \theta + 3 \cos^2 \theta}{\sin^4 \theta}$$

$$= -m g R \cos \theta + \frac{2 (m R^2 \sin^2 \theta \phi)^2}{2 m R^2 \sin^4 \theta} \left( 1 + 2 \cos^2 \theta \right)$$

$$= -m g R \cos \theta + m R^2 \dot{\phi}^2 \left( 1 + 2 \cos^2 \theta \right)$$

(Polar angle librational frequency $\omega_{\theta_{\text{equil}}}$ is related to azimuthal frequency $\dot{\phi}_{\text{equil}}^2$.)
2D Spherical pendulum or “Bowl-Bowling”

Total Energy from Hamiltonian $E=T+V(gravity)=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p^{2}_\phi}{2mR^2}$, $\gamma = mgR$ where: $p^{2}_\phi = mR^2 \sin^2 \theta(\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = -2 \frac{\delta \cos \theta}{\sin^3 \theta} - \frac{\gamma \sin \theta}{\sin^3 \theta} = 0 = \frac{-2 p^{2}_\phi \cos \theta}{2 m R^2 \sin^3 \theta} - mgR \sin \theta$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta$$

Or: $\dot{\varphi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$ (Polar angle librational frequency $\omega^{\text{equil}}_\theta$ is related to azimuthal frequency $\dot{\varphi}_{\text{equil}}^2$)

**V-Derivative for small oscillation frequency:**

$$\frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} = -\gamma \cos \theta + \frac{2 \delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2 \delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2 \delta \frac{\sin^2 \theta + 3 \cos^2 \theta}{\sin^4 \theta}$$

$$= -mgR \cos \theta + \frac{2 \left( mR^2 \sin^2 \theta \dot{\varphi} \right)^2}{2mR^2} \frac{1 + 2 \cos^2 \theta}{\sin^4 \theta}$$

$$= -mgR \cos \theta + mR^2 \dot{\varphi}^2 \left( 1 + 2 \cos^2 \theta \right)$$

At equilibrium:

$$\frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} \bigg|_{\text{equil}} = -mgR \cos \theta_{\text{equil}} + mR^2 \left( -\frac{g}{R \cos \theta_{\text{equil}}} \right) \left( 1 + 3 \cos^2 \theta_{\text{equil}} \right)$$

$$= -\frac{mgR}{\cos \theta_{\text{equil}}} \left( 1 + 3 \cos^2 \theta_{\text{equil}} \right)$$
2D Spherical pendulum or “Bowl-Bowling”

Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V_{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV_{\text{effective}}(\theta)}{d\theta} = -\frac{2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or:} \quad \dot{\phi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$$

(Polar angle librational frequency $\omega_{\theta_{\text{equil}}}$ is related to azimuthal frequency $\dot{\phi}_{\text{equil}}^2$.)

V-Derivative for small oscillation frequency:

$$\frac{d^2V_{\text{effective}}(\theta)}{d\theta^2} = -\gamma \cos \theta + \frac{2\delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2\delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2\delta \frac{\sin^2 \theta + 3 \cos^2 \theta}{\sin^4 \theta}$$

$$= -mgR \cos \theta + \frac{2(mR^2 \sin^2 \theta \phi)^2}{2mR^2 \sin^4 \theta} \frac{1+2\cos^2 \theta}{\sin^4 \theta}$$

$$= -mgR \cos \theta + mR^2 \phi^2 \left(1+2\cos^2 \theta\right)$$

At equilibrium:

$$\frac{d^2V_{\text{effective}}(\theta)}{d\theta^2}_{\text{equil}} = -mgR \cos \theta_{\text{equil}} + mR^2 \left(\frac{g}{R \cos \theta_{\text{equil}}}\right) \left(1+2\cos^2 \theta_{\text{equil}}\right)$$

$$\left(\omega_{\theta_{\text{equil}}}^2 / (\dot{\phi}_{\text{equil}}^2)\right) = \left(1+3\cos^2 \theta_{\text{equil}}\right)$$
2D Spherical pendulum or “Bowl-Bowling”

Total Energy from Hamiltonian $E = T + V(\text{gravity}) = \text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let:

$$\alpha = \frac{mR^2}{2}, \quad \delta = \frac{p^2}{2mR^2}, \quad \gamma = mgR$$

where:

$$p = mR^2 \sin^2 \theta (\dot{\phi})$$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = -\frac{2 \delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = -\frac{2p^2}{2mR^2} \cos \theta - mgR \sin \theta$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or} \quad \dot{\phi}^2_{\text{equil}} = -\frac{g}{R \cos \theta_{\text{equil}}}$$

(Polar angle librational frequency $\omega^2_{\text{equil}}$ is related to azimuthal frequency $\dot{\phi}^2_{\text{equil}}$.)

At equilibrium:

$$\left(\omega^2_{\text{equil}}\right)^2 = \left. \frac{1}{mR^2} \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$

$$\frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \bigg|_{\text{equil}} = -mgR \cos \theta_{\text{equil}} + mR^2 \left(-\frac{g}{R \cos \theta_{\text{equil}}} \right) \left(1 + 3 \cos^2 \theta_{\text{equil}} \right)$$

$$\left(\omega^2_{\text{equil}}\right)^2 / (\dot{\phi}^2_{\text{equil}}) = (1 + 3 \cos^2 \theta_{\text{equil}})$$

V-Derivative for small oscillation frequency:

$$\frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} = -\gamma \cos \theta + \frac{2 \delta \sin \theta}{\sin^3 \theta} + 3 \cdot \frac{2 \delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2 \delta \frac{\sin^2 \theta + 3 \cos^2 \theta}{\sin^4 \theta}$$

$$= -mgR \cos \theta + \frac{2 \left(mR^2 \sin^2 \theta \right) \dot{\phi}^2}{2mR^2} \frac{1 + 2 \cos^2 \theta}{\sin^4 \theta}$$

$$= -mgR \cos \theta + mR^2 \dot{\phi}^2 \frac{1 + 2 \cos^2 \theta}{\sin^4 \theta}$$

At bottom $\theta \to \pi$ the ratio of in-out $\omega_{\theta}$ to circle $\omega_{\phi}$ approaches $2:1$

At equator $\theta \to \pi/2$ the ratio approaches $1:1$. 

$$\omega_{\theta} : \omega_{\phi} \sim 2 \quad 2 > \omega_{\theta} : \omega_{\phi} > 1 \quad \omega_{\theta} : \omega_{\phi} \sim 1$$

$$\text{prograde precession of nodes} \quad \text{retrograde precession of nodes}$$
Total Energy from Hamiltonian \( E=T+V(gravity)=\text{const.} \) :

\[
E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta
\]

Let: \( \alpha = \frac{mR^2}{2} \), \( \delta = \frac{p_\phi^2}{2mR^2} \), \( \gamma = mgR \) where: \( p_\phi = mR^2 \sin^2 \theta (\dot{\phi}) \)

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

\[
\frac{dV^{\text{effective}}(\theta)}{d\theta} = -2\delta \cos \theta \sin^3 \theta - \gamma \sin \theta = 0 = -2p_\phi^2 \cos \theta \frac{1}{2mR^2 \sin^3 \theta} - mgR \sin \theta
\]

\[
0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or} \quad \dot{\phi}^2_{\text{equil}} = -\frac{g}{R \cos \theta_{\text{equil}}}
\]

(Polar angle librational frequency \( \omega^\text{equil}_\theta \) is related to azimuthal frequency \( \dot{\phi}^2_{\text{equil}} \).

At equilibrium:

\[
\left(\omega^\text{equil}_\theta\right)^2 = \frac{1}{mR^2} \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2}_{\text{equil}}
\]

V-Derivative for small oscillation frequency:

\[
\frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} = -\gamma \cos \theta + \frac{2\delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2 \delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2\delta \sin^2 \theta + 3\cos^2 \theta
\]

\[
= -mgR \cos \theta + \frac{2 (mR^2 \sin^2 \theta \dot{\phi})^2}{2mR^2} \frac{1+2\cos^2 \theta}{\sin^4 \theta}
\]

\[
= -mgR \cos \theta + mR^2 \dot{\phi}^2 \frac{1+2\cos^2 \theta}{\sin^4 \theta}
\]

At bottom \( \theta \rightarrow \pi \) the ratio of in-out \( \omega_\theta \) to circle \( \omega_\phi \) approaches 2:1
At equator \( \theta \rightarrow \pi/2 \) the ratio approaches 1:1.

Ratio is between 2 and 1
(Usually irrational non-closed orbit).
(2:1 is like 2D IHO, but 1:1 is like coulomb orbit.)
Separation of GCC Equations: Effective Potentials

Small radial oscillations
2D Spherical pendulum or “Bowl-Bowling”
Cycloidal ruler & compass geometry
Cycloid as brachistichrone
Cycloid as tautochrone
Cycloidulum vs Pendulum
Cycloidal geometry of flying levers
Practical poolhall application
Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)\frac{m\pi}{n} = \frac{3m}{n}$. 

Arc length $R\phi = \left(\frac{3}{\pi}\right)\phi$
Here the radius is plotted as an irrational $R = \frac{3}{\pi} \approx 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left( \frac{3}{\pi} \right) m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} \approx 1.91$.
Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} = 1.91$.

Red circle rolls left-to-right on $y = 3.82$ ceiling.
Contact point goes from $(x = 6/2, y = 3.82)$ to $x = 0$.

Green circle rolls right-to-left on $y = 1.91$ ceiling.
Contact point goes from $(x = 0, y = 1.91)$ to $x = 6/2$.
Here the radius is plotted as an irrational \( R = \frac{3}{\pi} = 0.955 \) length so rolling by rational angle \( \phi = \frac{m\pi}{n} \) is a rational length of rolled-out circumference \( R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n} \). Diameter is \( 2R = \frac{6}{\pi} = 1.91 \)

Red circle rolls left-to-right on \( y = 3.82 \) ceiling
Contact point goes from \( (x = 6/2, y = 3.82) \) to \( x = 0 \).

Green circle rolls right-to-left on \( y = 1.91 \) ceiling
Contact point goes from \( (x = 0, y = 1.91) \) to \( x = 6/2 \).

Rotation angle \( \phi \)

Arc length \( R\phi = \left(\frac{3}{\pi}\right)\phi \)
Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} = 1.91$

Red circle rolls left-to-right on $y=3.82$ ceiling
Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.

Ceiling $y=3.82$

Green circle rolls right-to-left on $y=1.91$ ceiling
Contact point goes from $(x=0, y=1.91)$ to $x=6/2$.

Ceiling $y=1.91$

---

Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} = 1.91$

Red circle rolls left-to-right on $y=3.82$ ceiling
Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.

Ceiling $y=3.82$

Green circle rolls right-to-left on $y=1.91$ ceiling
Contact point goes from $(x=0, y=1.91)$ to $x=6/2$.

Ceiling $y=1.91$

---

Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} = 1.91$

Red circle rolls left-to-right on $y=3.82$ ceiling
Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.

Ceiling $y=3.82$

Green circle rolls right-to-left on $y=1.91$ ceiling
Contact point goes from $(x=0, y=1.91)$ to $x=6/2$.

Ceiling $y=1.91$

---

Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} = 1.91$

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Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.

Ceiling $y=3.82$

Green circle rolls right-to-left on $y=1.91$ ceiling
Contact point goes from $(x=0, y=1.91)$ to $x=6/2$.

Ceiling $y=1.91$

---

Here the radius is plotted as an irrational $R = \frac{3}{\pi} = 0.955$ length so rolling by rational angle $\phi = \frac{m\pi}{n}$ is a rational length of rolled-out circumference $R\phi = \left(\frac{3}{\pi}\right)m\pi/n = \frac{3m}{n}$. Diameter is $2R = \frac{6}{\pi} = 1.91$

Red circle rolls left-to-right on $y=3.82$ ceiling
Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.

Ceiling $y=3.82$

Green circle rolls right-to-left on $y=1.91$ ceiling
Contact point goes from $(x=0, y=1.91)$ to $x=6/2$.

Ceiling $y=1.91$
Rotation angle $\phi$

Arc length $R\phi = \left(\frac{3}{\pi}\right)\phi$

$\frac{\pi}{6}$
$\frac{\pi}{3}$

$\frac{2\pi}{6}$
$\frac{4\pi}{6}$
$\frac{5\pi}{6}$

$\frac{2\pi}{3}$

$\frac{\pi}{2}$

$\frac{\pi}{3}$

$\frac{2\pi}{3}$

$\frac{\pi}{6}$

$12/2$
$11/2$
$10/2$
$9/2$
$8/2$
$7/2$
$6/2$
$5/2$
$4/2$
$3/2$
$2/2$
$1/2$

12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock

Radius $R = \frac{3\pi}{955}$

4:00 - $3\frac{\pi}{191} = 1.91$

3:00 - $3\frac{\pi}{477} = 2.865$

2:00 - $2\frac{\pi}{3} = 3.82$

1:00 - $\frac{\pi}{2} = 4.0$

0:00 - $\frac{\pi}{6} = 0.5$

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Rotation angle $\phi$

Arc length $R\phi = (3/\pi)\phi$
Separation of GCC Equations: Effective Potentials

Small radial oscillations
2D Spherical pendulum or “Bowl-Bowling”
Cycloidal ruler&compass geometry
   Cycloid as \textit{brachistichrone}
   Cycloid as \textit{tautochrone}
   Cycloidulum vs Pendulum
Cycloidal geometry of flying levers
   Practical poolhall application
The brachistichrone or minimum-time curve for a particle falling in a uniform gravitational potential. Its solution gives that of another problem, the tautochrone or equal-time period curve of Huygens. Energy conservation gives velocity $v$ from gravitational $g$. Elapsed travel time $t$ is to be minimized.

$$\frac{ds}{dt} = v = \sqrt{2gy}$$

$$t = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int dy \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \int L \, dy$$
The \textit{brachistichrone} or minimum-time curve for a particle falling in a uniform gravitational potential. Its solution gives that of another problem, the \textit{tautochrone} or equal-time period curve of Huygens. Energy conservation gives velocity $v$ from gravitational $g$. Elapsed travel time $t$ is to be minimized.

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\frac{ds}{dt} = v = \sqrt{2gy}
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t = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int dy \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}} = \int L \, dy
\]

A “pseudo-momentum” $p_x$ for “pseudo-Lagrange” $L$ in $y$-integral is constant if $L$ is $x$-independent.

\[
p_x = \text{const.} = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'} \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}} = \frac{x'}{\sqrt{2gy\sqrt{1 + x'^2}}} = \frac{1}{y' \sqrt{2gy\sqrt{1 + 1/y'^2}}} \quad \text{where: } x' = \frac{dx}{dy} = \frac{1}{y'}
\]
The *brachistichrone* or minimum-time curve for a particle falling in a uniform gravitational potential. Its solution gives that of another problem, the *tautochrone* or equal-time period curve of Huygens. Energy conservation gives velocity $v$ from gravitational $g$. Elapsed travel time $t$ is to be minimized.

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$$p_x = \text{const.} = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'} \sqrt{1+x'^2} = \frac{x'}{\sqrt{2gy\sqrt{1+x'^2}}} = \frac{1}{y'\sqrt{2gy\sqrt{1+1/y'^2}}}$$

where: $x' = \frac{dx}{dy} = \frac{1}{y'}$

Change variables from $y$ to velocity $v$ to simplify using:

$$v^2 = 2gy, \quad dy = \frac{vdv}{g}, \quad y' = \frac{v}{g} \frac{dv}{dx}$$
The *brachistichrone* or minimum-time curve for a particle falling in a uniform gravitational potential. Its solution gives that of another problem, the *tautochrone* or equal-time period curve of Huygens. Energy conservation gives velocity $v$ from gravitational $g$. Elapsed travel time $t$ is to be minimized.

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\frac{ds}{dt} = v = \sqrt{2gy}
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t = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int dy \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \int L dy
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A “pseudo-momentum” $p_x$ for “pseudo-Lagrange” $L$ in $y$-integral is constant if $L$ is $x$-independent.

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\]

where: $x' = \frac{dx}{dy} = \frac{1}{y'}$

Change variables from $y$ to velocity $v$ to simplify using: $v^2 = 2gy$, $dy = \frac{v dv}{g}$, $y' = \frac{v}{g} \frac{dv}{dx}$

\[
p_x = \frac{1}{\sqrt{2gy\sqrt{y'^2+1}}} = \frac{1}{\sqrt{v^2 \left( \frac{dv}{dx} \right)^2 + 1}}
\]

is: $p_x^2 v^2 = \frac{1}{\sqrt{v^2 \left( \frac{dv}{dx} \right)^2 + 1}}$

is: $\frac{v^2}{g^2} \left( \frac{dv}{dx} \right)^2 = \frac{1}{p_x^2 v^2} - 1 = \frac{1 - p_x^2 v^2}{p_x^2 v^2}$
The **brachistichrone** or **minimum-time curve** for a particle falling in a uniform gravitational potential. Its solution gives that of another problem, the **tautochrone** or **equal-time period curve** of Huygens. Energy conservation gives velocity $v$ from gravitational $g$. Elapsed travel time $t$ is to be minimized.

$$\frac{ds}{dt} = v = \sqrt{2gy}$$

$$t = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int dy \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \int y'^2 \frac{dy}{2gy}$$

A “pseudo-momentum” $p_x$ for “pseudo-Lagrange” $L$ in $y$-integral is constant if $L$ is $x$-independent.

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Change variables from $y$ to velocity $v$ to simplify using: $v^2 = 2gy$, $dy = \frac{vdv}{g}$, $y' = \frac{v}{dx}$

$$p_x = \frac{1}{\sqrt{2gy\sqrt{y'^2+1}}} = \frac{1}{v \sqrt{\frac{g^2}{v^2} \left( \frac{dv}{dx} \right)^2 + 1}}$$

is: $p_x^2 v^2 = \frac{1}{v^2 \left( \frac{dv}{dx} \right)^2 + 1}$

$$v^2 \left( \frac{dv}{dx} \right)^2 = \frac{1}{p_x^2 v^2} - 1 = \frac{1 - p_x^2 v^2}{p_x^2 v^2}$$

$$(\frac{dv}{dx})^2 = \frac{g^2}{v^2} \frac{1 - p_x^2 v^2}{p_x^2 v^2} = \frac{g^2}{v^2} \frac{p_x^{-2} - v^2}{v^2}$$

becomes: $\frac{dv}{dx} = \frac{g}{v^2} \sqrt{p_x^{-2} - v^2}$ and integral: $\int \frac{v^2 dv}{g \sqrt{a^2 - v^2}} = \int \frac{dv}{dx}$ where: $a^2 = p_x^{-2}$

An elementary integral results and suggests an elementary substitution $v = a \cos \theta$. 

*Thursday, November 6, 2014*
The brachistichrone or minimum-time curve for a particle falling in a uniform gravitational potential. Its solution gives that of another problem, the tautochrone or equal-time period curve of Huygens. Energy conservation gives velocity $v$ from gravitational $g$. Elapsed travel time $t$ is to be minimized.

$$\frac{ds}{dt} = v = \sqrt{2gy}$$

$$t = \int dt = \int \frac{ds}{\sqrt{2gy}} = \int dy \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \int L
dy$$

A “pseudo-momentum” $p_x$ for “pseudo-Lagrange” $L$ in $y$-integral is constant if $L$ is $x$-independent.

$$p_x = \text{const} = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'} \sqrt{1+x'^2} = \frac{x'}{\sqrt{2gy\sqrt{1+x'^2}}} = \frac{1}{y'}\sqrt{2gy\sqrt{1+1/y'^2}}$$

where: $x' = \frac{dx}{dy} = \frac{1}{y'}$

Change variables from $y$ to velocity $v$ to simplify using: $v^2 = 2gy$, $dy = \frac{v dv}{g}$, $y' = \frac{v}{g} \frac{dv}{dx}$

$$p_x = \frac{1}{\sqrt{2gy\sqrt{y'^2+1}}} = \frac{1}{\sqrt{v^2 \left( \frac{dv}{dx} \right)^2 +1}}$$

$$p_x^2 v^2 = \frac{1}{\frac{v^2 \left( \frac{dv}{dx} \right)^2 +1}{\left( \frac{dv}{dx} \right)^2}} = \frac{1}{\frac{g^2}{v^2} - \frac{p_x^2 v^2}{v^2}}$$

$$(\frac{dv}{dx})^2 = \frac{g^2}{v^2} \frac{1-p_x^2 v^2}{p_x^2 v^2} = \frac{g^2}{v^2} \frac{p_x^{-2} - v^2}{v^2}$$

becomes: $\frac{dv}{dx} = \frac{g}{v^2} \sqrt{p_x^{-2} - v^2}$ and integral: $\int \frac{v^2 dv}{g\sqrt{a^2 - v^2}} = \int dx$ where: $a^2 = p_x^{-2}$

An elementary integral results and suggests an elementary substitution $v = a \cos \theta$.

$$\int \frac{a^2 \cos^2 \theta \, \sin \theta \, d\theta}{g \sin \theta} = \int \frac{a^2}{g} \cos^2 \theta \, d\theta = \int dx = \frac{a^2}{2g} \left(1+\cos 2\theta \right) d\theta = -R \left(2\theta + \sin 2\theta \right)$$

where: $R = \frac{a^2}{4g}$

$$v^2 = 2gy = a^2 \cos^2 \theta$$

gives: $y = \frac{a^2}{2g} \cos^2 \theta = R \left(1+\cos 2\theta \right)$
Separation of GCC Equations: Effective Potentials

Small radial oscillations
2D Spherical pendulum or “Bowl-Bowling”
Cycloidal ruler&compass geometry
Cycloid as brachistichrone (With interesting linear dynamics)
Cycloid as tautochrone
Cycloidulum vs Pendulum
Cycloidal geometry of flying levers
Practical poolhall application
Some extraordinary properties of the cycloid are related to the constant $p_x$ (pseudo-momentum)

$$
p_x = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'} \frac{\sqrt{1+x'^2}}{2gy} = \frac{x'}{\sqrt{2gy \sqrt{1+x'^2}}} = \frac{1}{\sqrt{2gy \sqrt{y'^2} + 1}}
$$

where: $x' = \frac{dx}{dy} = \frac{1}{y'}$ and: $p_x^2 = \frac{1}{4Rg}$

$$\frac{1}{p_x^2} = const. = 2gy\left(y'^2 + 1\right) = v^2 \sec^2 \theta = a^2$$
Some extraordinary properties of the cycloid are related to the constant $p_x$ (pseudo-momentum)

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p_x = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'} \frac{\sqrt{1 + x'^2}}{2gy} = \frac{x'}{\sqrt{2gy} \sqrt{1 + x'^2}} = \frac{1}{\sqrt{2gy} \sqrt{y'^2 + 1}}
\]

where: \( x' = \frac{dx}{dy} = \frac{1}{y'} \) and: \( p_x^2 = \frac{1}{4Rg} \)

\[
\frac{1}{p_x^2} = const. = 2gy\left(y'^2 + 1\right) = v^2 \sec^2 \theta = a^2
\]

**t-derivatives of \((x,y)\) give \(v \text{ vs } \phi = 2\theta\)**: \( v^2 = x^2 + y^2 = \phi^2 \left[ (R + R\cos \phi)^2 + (-R\sin \phi)^2 \right] = 2R\phi^2(1 + \cos \phi) = 4R^2\phi^2 \cos^2 \theta \)
Some extraordinary properties of the cycloid are related to the constant $p_x$ (pseudo-momentum)

$$p_x = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'} \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \frac{x'}{\sqrt{2gy \sqrt{1+x'^2}}} = \frac{1}{\sqrt{2gy \sqrt{y'^2 + 1}}}$$

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$$\frac{1}{p_x^2} = \text{const.} = 2gy(y'^2 + 1) = v^2 \sec^2 \theta = a^2$$

$t$-derivatives of $(x,y)$ give $v$ vs $\phi = 2\theta$:

$$v^2 = x^2 + y^2 = \dot{\phi}^2 \left[ (R + R \cos \phi)^2 + (-R \sin \phi)^2 \right] = 2R\dot{\phi}^2 (1 + \cos \phi) = 4R^2 \dot{\phi}^2 \cos^2 \theta$$

The circle starting at $\phi = \pi = 2\theta$ turns at a constant rate $\dot{\phi} = \omega$ and moves at a constant velocity $v = \omega R$.

$$\frac{1}{p_x} = a = \sqrt{4gR} = 4R\dot{\phi} = 8R\theta \quad \text{or} \quad \omega = \dot{\phi} = \sqrt{\frac{g}{4R}}$$
Some extraordinary properties of the cycloid are related to the constant $p_x$ (pseudo-momentum)

$$p_x = \frac{\partial L}{\partial x'} = \frac{\partial}{\partial x'} \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} = \frac{x'}{\sqrt{2gy} \left( 1 + x'^2 \right)} = \frac{1}{\sqrt{2gy} \left( y'^2 + 1 \right)}$$

where: $x' = \frac{dx}{dy} = \frac{1}{y'}$ and: $p_x^2 = \frac{1}{4Rg}$

\[
\frac{1}{p_x^2} = \text{const.} = 2gy(y'^2 + 1) = v^2 \sec^2 \theta = a^2
\]

t-derivatives of $(x,y)$ give $v = 2\theta : v^2 = x'^2 + y'^2 = \dot{\phi}^2 \left[ (R + R\cos \phi)^2 + (-R\sin \phi)^2 \right] = 2R\dot{\phi}^2(1 + \cos \phi) = 4R^2\dot{\phi}^2 \cos^2 \theta$

The circle starting at $\phi = \pi = 2\theta$ turns at a constant rate $\dot{\phi} = \omega$ and moves at a constant velocity $v = \omega R$.

\[
\frac{1}{p_x} = a = \sqrt{4gR} = 4R\dot{\phi} = 8R\dot{\theta} \quad \text{or:} \quad \omega = \dot{\phi} = \sqrt{\frac{g}{4R}}
\]

This relates to the arc length of the cycloid from bottom $(\theta = 0)$ to a point at angle $\theta < \pi/2$ or $\phi < \pi$.

\[
s = \int_0^\theta v \, dt = \int_0^\theta 2R\omega \cos \theta \, dt = \int_0^\theta 2R(\omega/\dot{\theta}) \cos \theta \, d\theta = 4R \sin \theta
\]
Separation of GCC Equations: Effective Potentials

Small radial oscillations
2D Spherical pendulum or “Bowl-Bowling”
Cycloidal ruler & compass geometry

- Cycloid as *brachistichrone*  (With interesting curvature geometry)
- Cycloid as *tautochrone*

Cycloidulum vs Pendulum
Cycloidal geometry of flying levers
Practical poolhall application
Arc length \( s \) is indicated by a segment \( hh \) of length \( 2h = 4R \sin \theta \) left hand Fig. 7.3.4 below. That is precisely the length of unwound string between points \( m' \) and \( m'' \), and between points \( m' \) and \( m \), is a segment \( h'h' \) of length \( 2h' = 4R \cos \theta \) unwound from middle cycloid.
Arc length $s$ is indicated by a segment $hh$ of length $2h = 4R \sin \theta$ left hand Fig. 7.3.4 below. That is precisely the length of unwound string between points $m'$ and $m''$, and between points $m'$ and $m$, is a segment $h'h'$ of length $2h' = 4R \cos \theta$ unwound from middle cycloid.

Segment $hh$ is the radius of curvature $r_c(m') = 2h = 4R \sin \theta$ of the $m'$ cycloid and the points $m'$ or $m''$ are centers of curvature for circular arcs around unwinding points $m''$ or $m'$, respectively.
Arc length $s$ is indicated by a segment $hh$ of length $2h=4R\sin\theta$ left hand Fig. 7.3.4 below. That is precisely the length of unwound string between points $m'$ and $m''$, and between points $m'$ and $m$, is a segment $h'h'$ of length $2h'=4R\cos\theta$ unwound from middle cycloid.

Segment $hh$ is the radius of curvature $r_c(m') = 2h = 4R\sin\theta$ of the $m'$ cycloid and the points $m'$ or $m''$ are centers of curvature for circular arcs around unwinding points $m''$ or $m'$, respectively.

Three wheels roll synchronically on their respective ceilings. As point $m$ approaches the top of its cycloid, point $m'$ approaches $m$ so that curvature becomes infinite. ($k=1/r_c \rightarrow \infty$ as $\theta \rightarrow \pi/2$.)
Arc length \( s \) is indicated by a segment \( hh \) of length \( 2h=4R\sin\theta \) left hand Fig. 7.3.4 below. That is precisely the length of unwound string between points \( m' \) and \( m'' \), and between points \( m' \) and \( m \), is a segment \( h'h' \) of length \( 2h'=4R\cos\theta \) unwound from middle cycloid.

Segment \( hh \) is the *radius of curvature* \( r_c(m') = 2h=4R\sin\theta \) of the \( m' \) cycloid and the points \( m' \) or \( m'' \) are *centers of curvature* for circular arcs around unwinding points \( m'' \) or \( m' \), respectively.

Three wheels roll synchronically on their respective ceilings. As point \( m \) approaches the top of its cycloid, point \( m' \) approaches \( m \) so that curvature becomes infinite. ( \( k=1/r_c \to \infty \) as \( \theta \to \pi/2 \).)

Figure 7.3.5 shows circular arcs fitting a cycloid. The largest arc and one with the least curvature \( k_c = 1/(4R) \) is a circle of radius \( r_c = 4R \) that surrounds the entire cycloid. This is the path of a simple circular pendulum. The figure shows that the circle deviates only slightly from the cycloid with the greatest deviation near the tips of the cycloid where curvature blows up.
Separation of GCC Equations: Effective Potentials

Small radial oscillations
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The cycloid path has the unique ability to guarantee the same frequency \( \omega = \sqrt{g/4R} \) for any amplitude \( \theta_0 \) of oscillation within the range \( \{-\pi/2 < \theta_0 < \pi/2\} \) between cycloid tips.
The cycloid path has the unique ability to guarantee the same frequency $\omega = \sqrt{(g/4R)}$ for any amplitude $\theta_0$ of oscillation within the range $\{-\pi/2 < \theta_0 < \pi/2\}$ between cycloid tips.

The circular pendulum frequency $\omega = \sqrt{(g/\ell)}$ holds only for small amplitudes $\theta << 1$.

The time integral below varies with $\theta_0$ in the range $\{-\pi/2 < \theta_0 < \pi/2\}$.

$$t_{1/4} = \int_{s_0}^{0} \frac{ds}{\sqrt{2g(y - y_0)}} = \int_{\theta_0}^{0} \frac{4R \cos \theta \ d\theta}{\sqrt{2gR(\cos 2\theta - \cos 2\theta_0)}} = \sqrt{\frac{4R}{g}} \int_{\theta_0}^{0} \frac{\cos \theta \ d\theta}{\sqrt{\sin^2 \theta_0 - \sin^2 \theta}}$$
The cycloid path has the unique ability to guarantee the same frequency \( \omega = \sqrt{(g/4R)} \) for any amplitude \( \theta_0 \) of oscillation within the range \( \{-\pi/2 < \theta_0 < \pi/2\} \) between cycloid tips.

The circular pendulum frequency \( \omega = \sqrt{(g/\ell)} \) holds only for small amplitudes \( \theta << 1 \).

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\[
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\]

Arc length \( s = 4R \sin \theta \) and cycloid height \( y = R(1 + \cos 2\theta) \) are used above.

To finish integral for a 1/4-period we set: \( \sin \theta = \sin \theta_0 \sin \alpha \) below.

\[
t_{1/4} = \sqrt{\frac{4R}{g}} \int_0^{\alpha = \pi/2} \frac{\sin \theta_0 \cos \alpha \ d\alpha}{\sin \theta_0 \sqrt{1 - \sin^2 \alpha}} = \frac{\pi}{2} \sqrt{\frac{4R}{g}}
\]
The cycloid path has the unique ability to guarantee the same frequency \( \omega = \sqrt{\frac{g}{4R}} \) for any amplitude \( \theta_0 \) of oscillation within the range \(-\pi/2 < \theta_0 < \pi/2\) between cycloid tips.

The circular pendulum frequency \( \omega = \sqrt{\frac{g}{\ell}} \) holds only for small amplitudes \( \theta \ll 1 \).

The time integral below varies with \( \theta_0 \) in the range \(-\pi/2 < \theta_0 < \pi/2\).

\[
t_{1/4} = \int_{s_0}^{0} \frac{ds}{\sqrt{2g(y - y_0)}} = \int_{0}^{\theta_0} \frac{4R \cos \theta \, d\theta}{\sqrt{2gR(\cos 2\theta - \cos 2\theta_0)}} = \sqrt{\frac{4R}{g}} \int_{0}^{\theta_0} \frac{\cos \theta \, d\theta}{\sqrt{\sin^2 \theta_0 - \sin^2 \theta}}
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To finish integral for a 1/4-period we set: \( \sin \theta = \sin \theta_0 \sin \alpha \) below.

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t_{1/4} = \sqrt{\frac{4R}{g}} \int_{0}^{\pi/2} \frac{\sin \theta_0 \cos \alpha \, d\alpha}{\sin \theta_0 \sqrt{1 - \sin^2 \alpha}} = \frac{\pi}{2} \sqrt{\frac{4R}{g}}
\]

A cycloid has a full period of \( t_1 = 2\pi \sqrt{\ell/g} \) for all \( \theta_0 \). Even for large \( \theta_0 \) the “cycloidulum” matches the period of a simple circular (\( \ell = 4R \))-pendulum at small \( \theta_0 \).
Separation of GCC Equations: Effective Potentials

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  - Cycloid as tautochrone
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  - Practical poolhall application
Simple circular ($\ell=4R$)-pendulum is harmonic only at small $\theta_0$. 

http://www.uark.edu/ua/modphys/markup/PendulumWeb.html
Huygen’s *Cycloidulum* ($\ell=4R$) is harmonic at all $\theta_0$ in range $-\pi$ to $+\pi$. Angular frequency is exactly:

$$\omega = \sqrt{\frac{g}{\ell}} = \sqrt{\frac{g}{4R}}$$
Separation of GCC Equations: Effective Potentials

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If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$.

![Diagram of cycloidal paths](image)

Fig. 2.A.1 Cycloidal paths due to hitting a stationary stick.
If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity $\omega$ about the center is angular momentum $\Lambda$ divided by moment of inertia $I = M \ell^2/3$ of the stick.

![Diagram of cycloidal paths due to hitting a stationary stick.](image)

**Fig. 2.A.1 Cycloidal paths due to hitting a stationary stick.**
If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$.

Resulting angular velocity $\omega$ about the center is angular momentum $\Lambda$ divided by moment of inertia $I = M \ell^2 / 3$ of the stick.

$$\omega = \frac{\Lambda}{I} \quad (= \frac{3\Lambda}{M \ell^2} \text{ for stick})$$

$$= \frac{h\Pi}{I} \quad (= \frac{3h\Pi}{M \ell^2} \text{ for stick})$$

Fig. 2.A.1 Cycloidal paths due to hitting a stationary stick.
If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$

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$$\omega = \frac{\Lambda}{I} \quad (= \frac{3\Lambda}{M \ell^2} \text{ for stick})$$
$$= \frac{h \Pi}{I} \quad (= \frac{3h \Pi}{M \ell^2} \text{ for stick})$$

One point $P$, or *center of percussion (CoP)*, is on the wheel where speed $p \omega$ due to rotation just cancels translational speed $V_{Center}$ of stick.

*Fig. 2.A.1 Cycloidal paths due to hitting a stationary stick.*
If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$.

Resulting angular velocity $\omega$ about the center is angular momentum $\Lambda$ divided by moment of inertia $I = M \ell^2 / 3$ of the stick.

$$\omega = \frac{\Lambda}{I} \quad (= 3\Lambda / (M \ell^2) \text{ for stick})$$

$$= \frac{h \Pi}{I} \quad (= 3h \Pi / (M \ell^2) \text{ for stick})$$

One point $P$, or center of percussion (CoP), is on the wheel where speed $p \omega$ due to rotation just cancels translational speed $V_{Center}$ of stick.

$$\frac{\Pi}{M} = V_{Center} = |p \omega| = p \cdot h \Pi / I$$

Fig. 2.A.1 Cycloidal paths due to hitting a stationary stick.
If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$.

Resulting angular velocity $\omega$ about the center is angular momentum $\Lambda$ divided by moment of inertia $I = M \ell^2/3$ of the stick.

$$\omega = \frac{\Lambda}{I} \quad (= \frac{3\Lambda}{M \ell^2} \text{ for stick})$$
$$= \frac{h \Pi}{I} \quad (= \frac{3h \Pi}{M \ell^2} \text{ for stick})$$

One point $P$, or center of percussion (CoP), is on the wheel where speed $p \omega$ due to rotation just cancels translational speed $V_{\text{Center}}$ of stick.

$$\frac{\Pi}{M} = V_{\text{Center}} = |p \omega| = p \cdot h \frac{\Pi}{I}$$
$$\frac{I}{M} = = p \cdot h$$
If you hammer a stick at a point \( h \) meters from its center you give it some linear momentum \( \Pi \) and some angular momentum \( \Lambda = h \cdot \Pi \)

Resulting angular velocity \( \omega \) about the center is angular momentum \( \Lambda \) divided by moment of inertia \( I = M \ell^2/3 \) of the stick.

\[
\omega = \frac{\Lambda}{I} \quad (= \frac{3\Lambda}{(M \ell^2)} \text{ for stick})
\]
\[
= \frac{h\Pi}{I} \quad (= \frac{3h\Pi}{(M \ell^2)} \text{ for stick})
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One point \( P \), or *center of percussion* (CoP), is on the wheel where speed \( p\omega \) due to rotation just cancels translational speed \( V_{\text{Center}} \) of stick.

\[
\frac{\Pi}{M} = V_{\text{Center}} = |p\omega| = p \cdot h\frac{\Pi}{I}
\]
\[
\frac{I}{M} = \pi p = p \cdot h \quad \text{or: } p = \frac{I}{(Mh)}
\]
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One point \( P \), or center of percussion (CoP), is on the wheel where speed \( p\omega \) due to rotation just cancels translational speed \( V_{Center} \) of stick.

\[
\frac{\Pi}{M} = V_{Center} = |p\omega| = p\cdot h\Pi/I
\]

\[
\frac{I}{M} = p\cdot h \quad \text{or: } p = I/(Mh)
\]

\( P \) follows a normal cycloid made by a circle of radius \( p = I/(Mh) \) rolling on an imaginary road thru point \( P \) in direction of \( \Pi \).

Fig. 2.A.1 Cycloidal paths due to hitting a stationary stick.
If you hammer a stick at a point $h$ meters from its center you give it some linear momentum $\Pi$ and some angular momentum $\Lambda = h \cdot \Pi$

Resulting angular velocity $\omega$ about the center is angular momentum $\Lambda$ divided by moment of inertia $I = M \ell^2/3$ of the stick.

$$\omega = \frac{\Lambda}{I} \quad (=3\frac{\Lambda}{M \ell^2} \text{ for stick})$$

$$= \frac{h \Pi}{I} \quad (=3\frac{h \Pi}{M \ell^2} \text{ for stick})$$

One point $P$, or center of percussion (CoP), is on the wheel where speed $p \omega$ due to rotation just cancels translational speed $V_{Center}$ of stick.

$$\frac{\Pi}{M} = V_{Center} = |p \omega| = p \cdot h \frac{\Pi}{I}$$

$$I/M = \frac{\Pi}{M} = p \cdot h \quad \text{or: } p = I/(Mh)$$

$P$ follows a normal cycloid made by a circle of radius $p = I/(Mh)$ rolling on an imaginary road thru point $P$ in direction of $\Pi$.

The percussion radius $p = \ell^2/3h$ is of the CoP point that has no velocity just after hammer hits at $h$. Fig. 2.A.1 Cycloidal paths due to hitting a stationary stick.
Separation of GCC Equations: Effective Potentials

Small radial oscillations
2D Spherical pendulum or “Bowl-Bowling”
Cycloidal ruler & compass geometry
Cycloid as brachistichrone
Cycloid as tautochrone
Cycloidulum vs Pendulum
Cycloidal geometry of flying levers

Practical poolhall application
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height $H$ so ball does not skid.
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height $H$ so ball does not skid.

Where should bumper height $H$ be set to make ball contact point $C$ at the center of percussion $P$?

Center of percussion $P$ above contact point $C$
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

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Center of percussion $P$ above contact point $C$
(Ball skids to right)
Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height $H$ so ball does not skid.

Where should bumper height $H$ be set to make ball contact point $C$ at the center of percussion $P$?

center of percussion $P$ below contact point $C$
(Ball skids to left)

$I = \frac{2}{5}MR^2$

$H = ?$

$R < p$

$I/M = p \cdot h$

$\Pi = \text{linear momentum}$

$\Pi h = \text{angular momentum about}$

$\text{Imaginary wheel of radius } p \text{ rolls on imaginary road that intersects the Center of Percussion } P$
Problem: Set bumper height \( H \) so ball does not skid.

Where should bumper height \( H \) be set to make ball contact point \( C \) at the center of percussion \( P \)?

Practical poolhall application of center of percussion formula \( I/M = p \cdot h \)

Center of percussion \( P \) at contact point \( C \)
(Ball does not skid •)

\[ I = 2/5MR^2 \]

\[ R = p \]

\[ H =? \]

\[ h = I/Mp = I/MR \]  
(For \( R = p \) )
 Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height $H$ so ball does not skid.

Where should bumper height $H$ be set to make ball contact point $C$ at the center of percussion $P$?

center of percussion $P$ at contact point $C$
(Ball does not skid • )

$I = \frac{2}{5}MR^2$

$R = p$

$I/M = p \cdot h$

$h = I/Mp = I/MR$

(For $R = p$)

$= \frac{2}{5}MR^2/MR$

$= \frac{2}{5}R$

For: $H = R + h = 7/10(2R)$ ball does not skid.

$\Pi = \text{linear momentum}$

$\Pi h = \text{angular momentum about}$

Imaginary wheel of radius $p$ rolls on imaginary road that intersects the Center of Percussion $P$
Thats all folks!