# Lecture 20
## Mon. 11.04 thru Wed. 11.06 2019

# *Introduction to classical oscillation and resonance*

*(Ch. 1 of Unit 4 )*

*1D forced-damped-harmonic oscillator equations and Green's function solutions*

*Linear harmonic oscillator equation of motion.*

*Linear damped-harmonic oscillator equation of motion.*

*Frequency retardation and amplitude damping*

*Figure of oscillator merit (the 5% solution $3/\Gamma$ and other numbers)*

*Linear forced-damped-harmonic oscillator equation of motion.*

*Phase lag and amplitude resonance amplification*

*Figure of resonance merit: (angular) Quality factor $q=\omega_0/2\Gamma$*

*Properties of Green's function solutions and their mathematical/physical behavior*

*Transient solutions vs. Steady State solutions*

*Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)*

*Quality factors: Beat, lifetimes, and uncertainty*

*Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

*Common Lorentzian (a.k.a. Witch of Agnesi)*

*Smith Charts (Graph paper)*

# *This Lecture's Reference Link Listing*

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
**2019 Advanced Mechanics**

***Lectures #12 through #20***

*In reverse order*

Smith Chart, Invented by Phillip H. Smith (1905-1987)
Nobelprize.org
2005 Physics Award

***Intra-lecture links***: *Page=18*, *Page=35*, *Page=47*

**OscillIt Web Simulations:**
Default/Generic, Weakly Damped #18,
Forced : Way below resonance,On resonance
Way above resonance,Underdamped
Complex Response Plot

**CoulIt Web Simulations:**
Stark-Coulomb : Bound-state motion in parabolic coordinates
Molecular Ion : Bound-state motion in hyperbolic coordinates
Synchrotron Motion, Synchrotron Motion #2
Mechanical Analog to EM Motion (YouTube video)
iBall demo - Quasi-periodicity (YouTube video)

**Trebuchet Web Simulations:**
Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

**Recent Articles of Interest:**
Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
Synthetic_Chiral_Light_for_Efficient_Control_of_Chiral_Light–Matter_Interaction_-_Ayuso-np-2019
A_Semi-Classical_Approach_to_the_Calculation_of_Highly_Excited_Rotational_Energies for …
Asymmetric-Top_Molecules_-_Schmiedt-pccp-2017
Quantum_Chaos_-_An_Introduction_-_Stockmann-cup-2006, Review by E. Heller
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

**Pirelli Relativity Challenge (Introduction level)** - *Visualizing Waves:*
Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10, pdf_page=135
Calculus of exponentials, logarithms, and complex fields,
RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)

**Select, exciting, and related Research**
Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces -
Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_
Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019
RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

**An assist from *Physics Girl* (YouTube Channel):**
*How to Make VORTEX RINGS in a Pool*
*Crazy pool vortex - pg-yt-2014*
*Fun with Vortex Rings in the Pool - pg-yt-2014*

**Excerpts** (Page 44-47 in *Preliminary Draft*) from the
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019

# *Running Reference Link Listing*

## *Lectures #11 through #7*

*In reverse order*

**Eric J Heller Gallery:**
Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

**Links to previous lecture:** *Page=74*, *Page=75*, *Page=79*
*Pendulum Web Sim*
*Cycloidulum Web Sim*

***JerkIt* Web Simulations:** *Basic/Generic: Inverted*, *FVPlot*

CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

"RelaWavity" Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

***BoxIt Web Simulations:***
*Generic/Default*
*Most Basic A-Type*
*Basic A-Type w/reference lines*
*Basic A-Type A-Type with Potential energy*
*A-Type with Potential energy and Stokes Plot*
*A-Type w/3 time rates of change*
*A-Type w/3 time rates of change with Stokes Plot*
*B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)*

***RelaWavity Web Elliptical Motion Simulations:***
*Orbits with b/a=0.125*
*Orbits with b/a=0.5*
*Orbits with b/a=0.7*
*Exegesis with b/a=0.125*
*Exegesis with b/a=0.5*
*Exegesis with b/a=0.7*
*Contact Ellipsometry*

***CoulIt* Web Simulations:**
*Basic/Generic*
*Exploding Starlet*
*Volcanoes of Io (Color Quantized)*

***JerkIt* Web Simulations:**
*Basic/Generic*
*Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot*

***OscillatorPE* Web Simulation:**
*Coulomb-Newton-Inverse_Square*,
*Hooke-Isotropic Harmonic*,
*Pendulum-Circular_Constraint*

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io
*Pirelli Site: Phasors animimation*
*CMwithBang Lecture #6, page=70 (9.10.18)*

**Select, exciting, and related Research & Articles of Interest:**
*Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019*
*Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019*
*Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019*
*A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019*
*Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018*
*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
*Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_*,
*Nuclear_Collisions - Mishustin-PhysRevC-2007*, *APS Link & Abstract*
*Hadronic Molecules - Guo-x-2017*
*Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016*

# Running Reference Link Listing

## Lectures #6 through #1

*In reverse order*

*RelaWavity Web Simulation: Contact Ellipsometry*
*BoxIt Web Simulation: Elliptical Motion (A-Type)*
*CMwBang Course: Site Title Page*
*Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors*
UAF Physics UTube channel

*Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971*
*MIT OpenCourseWare: High School/Physics/Impulse and Momentum*
*Hubble Site: Supernova - SN 1987A*

***BounceItIt Web Animation - Scenarios:***
*49:1 y vs t*, *49:1 V2 vs V1*, *1:500:1 - 1D Gas Model w/ faux restorative force (Cool)*,
*1:500:1 - 1D Gas (Warm)*, *1:500:1 - 1D Gas Model (Cool, Zoomed in)*,
*Farey Sequence - Wolfram*
*Fractions - Ford-AMM-1938*
***Monstermash BounceItIt Animations:***
*1000:1 - V2 vs V1*, *1000:1 with t vs x - Minkowski Plot*
*Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013*
*Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015*
*Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)*
*Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971*
***WaveIt Web Animation - Scenarios:***
*Quantum_Carpet*, *Quantum_Carpet_wMBars*,
*Quantum_Carpet_BCar*, *Quantum_Carpet_BCar_wMBars*
*Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001*
*Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)*

***BounceIt Web Animation - Scenarios:***
*Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)*
*1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4*
*7:1 - V vs V Plot: Power=1*
*3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4*
*3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1*
*3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps*
*4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4*
*4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps*
*6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot*
*5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps*
*1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps*

***BounceIt Dual plots***
**$m_1:m_2 = 3:1$**
*$v_2$ vs $v_1$ and $V_2$ vs $V_1$*, *$(v_1, v_2)=(1, 0.1)$*, *$(v_1, v_2)=(1, 0)$*
*$y_2$ vs $y_1$ plots:* *$(v_1, v_2)=(1, 0.1)$*, *$(v_1, v_2)=(1, 0)$*, *$(v_1, v_2)=(1, -1)$*
Estrangian plot *$V_2$ vs $V_1$:* *$(v_1, v_2)=(0, 1)$*, *$(v_1, v_2)=(1, -1)$*
**$m_1:m_2 = 4:1$**
*v2 vs v1*, *y2 vs y1*
**$m_1:m_2 = 100:1$,** *$(v_1, v_2)=(1, 0)$:* *V2 vs V1 Estrangian plot*, *y2 vs y1 plot*

*With g=0 and 70:10 mass ratio*
*With non zero g, velocity dependent damping and mass ratio of 70:35*
*$M_1$=49, $M_2$=1 with Newtonian time plot*
*$M_1$=49, $M_2$=1 with $V_2$ vs $V_1$ plot*
*Example with friction*
*Low force constant with drag displaying a Pass-thru*, *Fall-Thru*, *Bounce-Off*
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

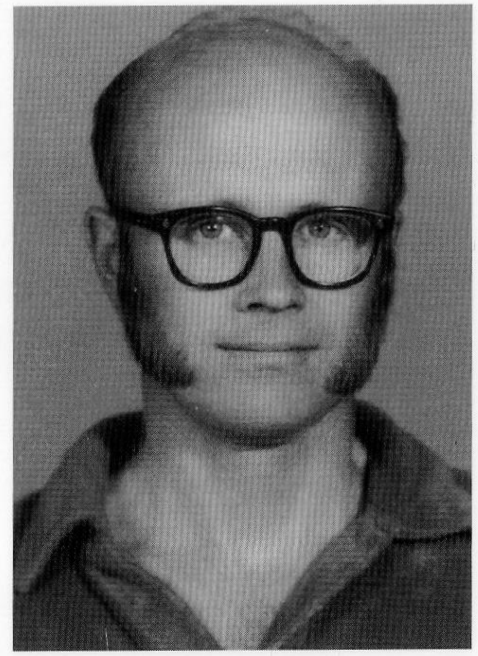
*AJP article on superball dynamics*
*AAPT Summer Reading List*
*Scitation.org - AIP publications*
*HarterSoft Youtube Channel*

X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
*Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html*
*Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007*
*BounceIt web simulation with g=0 and 70:10 mass ratio*
*With non zero g, velocity dependent damping and mass ratio of 70:35*
*Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)*
*Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)*
*Matrix Collision Simulator: $M_1$=49, $M_2$=1 $V_2$ vs $V_1$ plot* <<*Under Construction*>>

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
*(Now in Development)*

*Without resonance…*
*…we are all blind, deaf, and dumb.*

*Anonymous*



PAULINIA, BRASIL 1976

THE SPEED OF LIGHT IS
299,792,458 METERS PER SECOND!

# -- The Purest Light and a Resonance Hero – Ken Evenson (1932-2002) --

Ken Evenson

When travelers punch up their GPS coordinates they owe a debt of gratitude to an under sung hero who, alongside his colleagues and students, often toiled 18 hour days deep inside a laser laboratory lit only by the purest light in the universe.

Ken was an "Indiana Jones" of modern physics. While he may never have been called "Montana Ken," such a name would describe a real life hero from Bozeman, Montana, whose extraordinary accomplishments in many ways surpass the fictional characters in cinematic thrillers like *Raiders of the Lost Arc*.

Indeed, there were some exciting real life moments shared by his wife Vera, one together with Ken in a canoe literally inches from the hundred-foot drop-off of Brazil's largest waterfall. But, such outdoor exploits, of which Ken had many, pale in the light of an in-the-lab brilliance and courage that profoundly enriched the world.

Ken is one of few researchers and perhaps the only physicist to be twice listed in the *Guinness Book of Records*. The listings are not for jungle exploits but for his lab's highest frequency measurement and for a speed of light determination that made $c$ many times more precise due to his lab's pioneering work with John Hall in laser resonance and metrology[†].

The meter-kilogram-second (mks) system of units underwent a redefinition largely because of these efforts. Thereafter, the speed of light $c$ was set to $299{,}792{,}458 ms^{-1}$. The meter was defined in terms of $c$, instead of the other way around since his time precision had so far trumped that for distance. Without such resonance precision, the Global Positioning System (GPS), the first large-scale wave space-time coordinate system, would not be possible.

Ken's courage and persistence at the Time and Frequency Division of the Boulder Laboratories in the National Bureau of Standards (now the National Institute of Standards and Technology or NIST) are legendary as are his railings against boneheaded administrators who seemed bent on thwarting his best efforts. Undaunted, Ken's lab painstakingly exploited the resonance properties of metal-insulator diodes, and succeeded in literally counting the waves of near-infrared radiation and eventually visible light itself.

Those who knew Ken miss him terribly. But, his indelible legacy resonates today as ultra-precise atomic and molecular wave and pulse quantum optics continue to advance and provide heretofore unimaginable capability. Our quality of life depends on their metrology through the Quality and Finesse of the resonant oscillators that are the heartbeats of our technology.

Before being taken by Lou Gehrig's disease, Ken began ultra-precise laser spectroscopy of unusual molecules such as $HO_2$, the radical cousin of the more common $H_2O$. Like Ken, such radical molecules affect us as much or more than better known ones. But also like Ken, they toil in obscurity, illuminated only by the purest light in the universe.

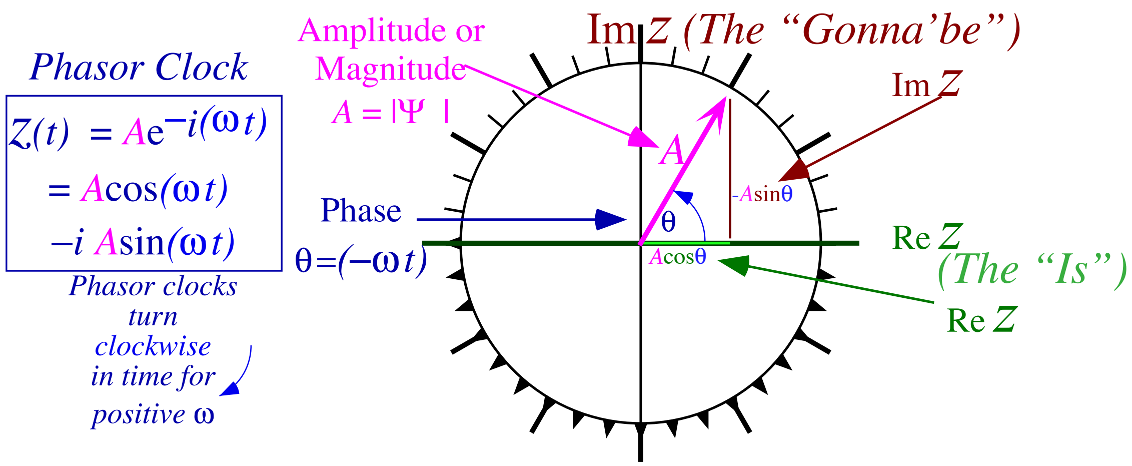
In 2005 the Nobel Prize in physics was awarded to Glauber, Hall, and Hensch[††] for laser optics and metrology.

† K. M. Evenson, J.S. Wells, F.R. Peterson, B.L. Danielson, G.W. Day, R.L. Barger and J.L. Hall,
Phys. Rev. Letters 29, 1346(1972).

†† *The Nobel Prize in Physics, 2005*. http://nobelprize.org/

# *Linear forced-damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$




$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}$=*a(t)* due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

driven by external stimulating force → $F_{stimulus}(t) = eE(t)$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

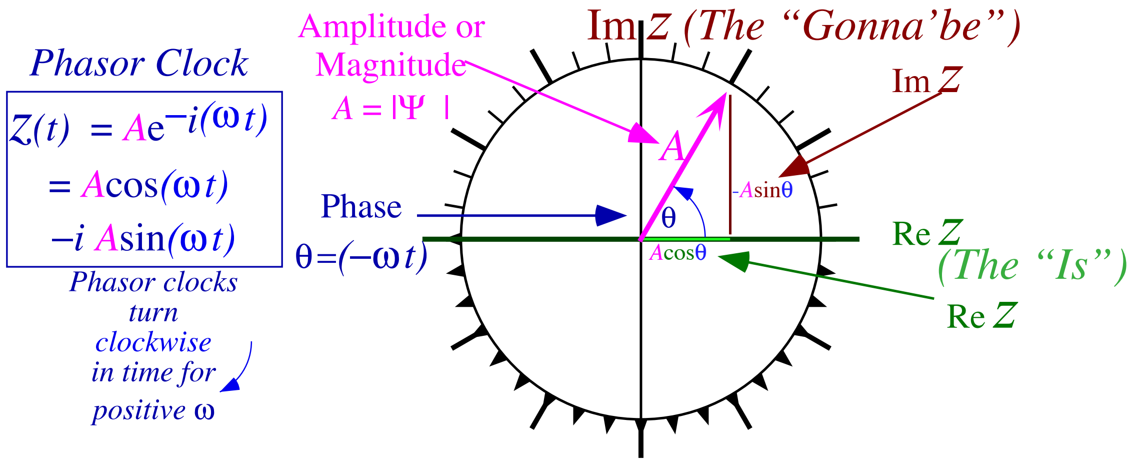
retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad \left(b = 2\Gamma m\right)$

# Linear harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{restore}}{m}$$

$$\frac{d^2z}{dt^2} + \omega_0^2 z = 0$$



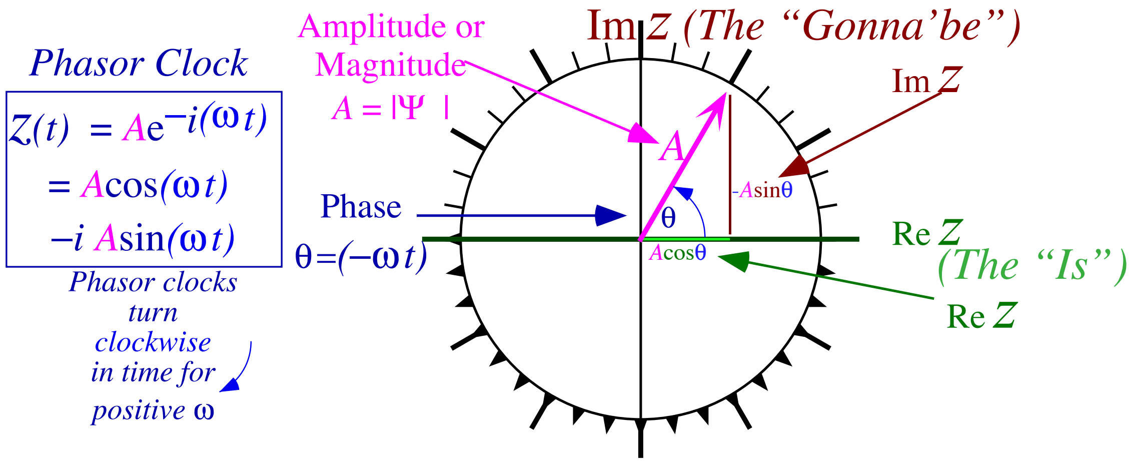

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

# Linear harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = \qquad F_{restore}$$

$$\frac{d^2z}{dt^2} = \qquad \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} \qquad + \omega_0^2 z = \quad 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force $\longrightarrow$ $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

The famous one-Hertz ( $\nu_0$=1/s. or: $\omega_0 = 2\pi = 6.2832$rad/s. ) oscillator.

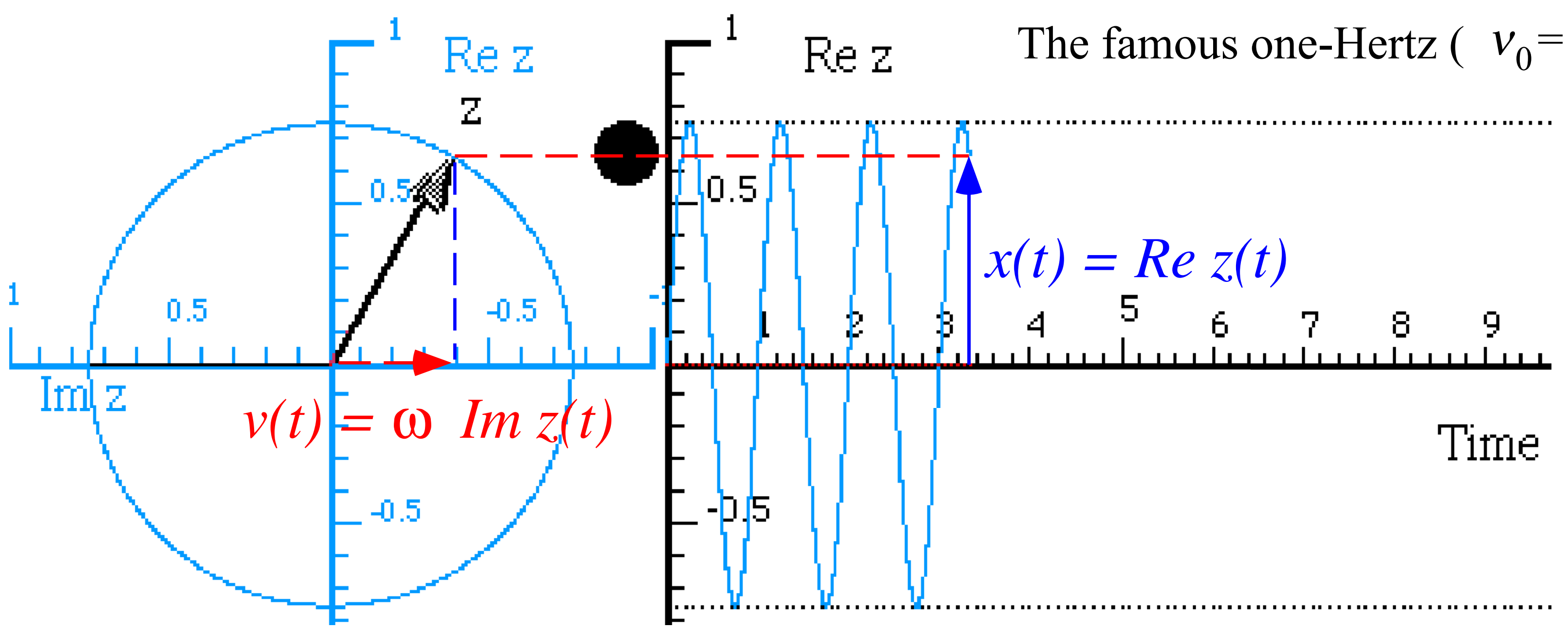



*Fig. 3.2.2 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0$*

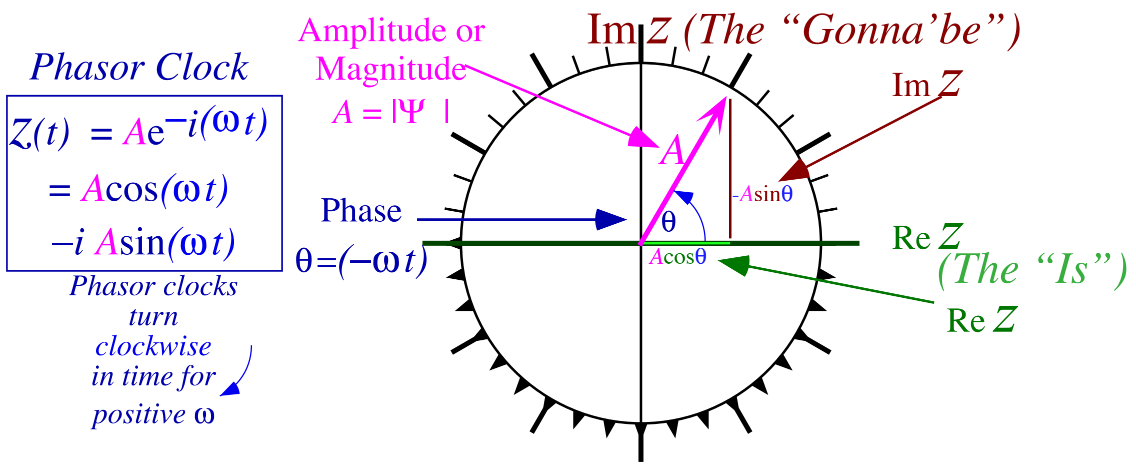
OscillIt Web Simulation

OscillIt Web Simulation (Generic):
https://modphys.hosted.uark.edu/markup/OscillItWeb.html

# *Linear -damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

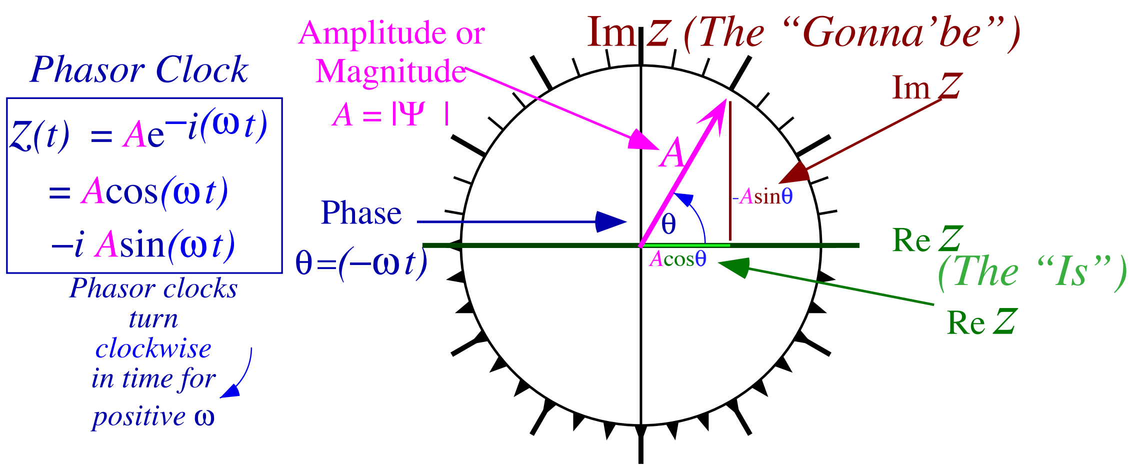
held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad \left(b = 2\Gamma m\right)$

# *Linear damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:
Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

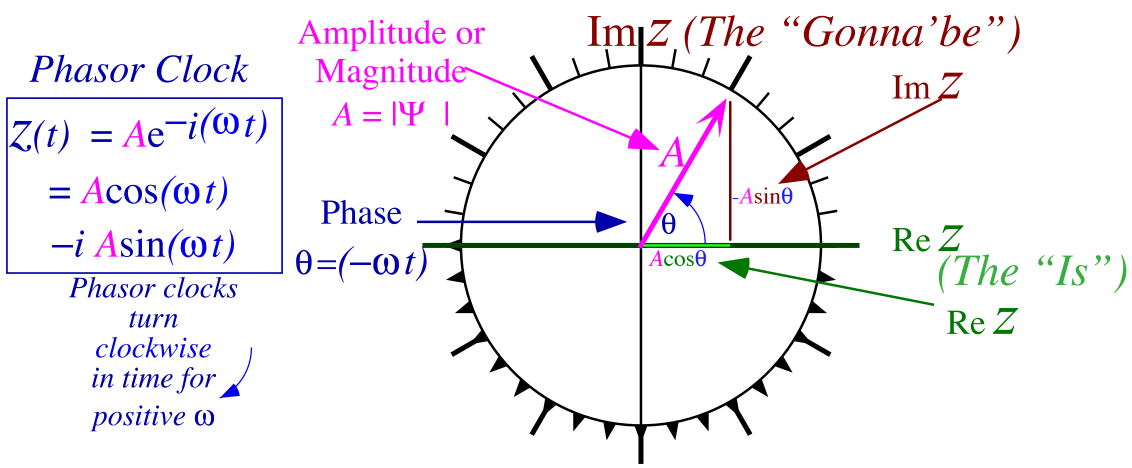
held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

# *Linear -damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

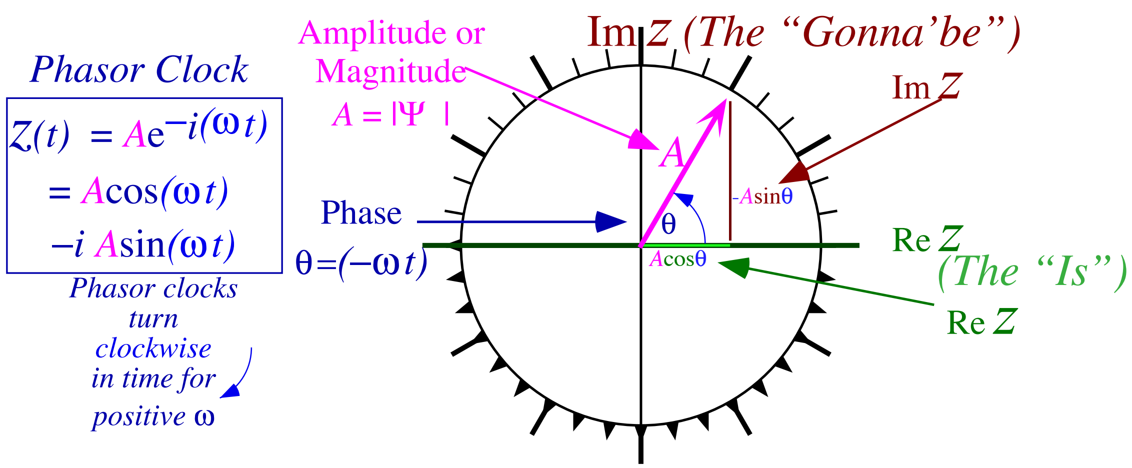
held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

# *Linear* -*damped*-*harmonic* *oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

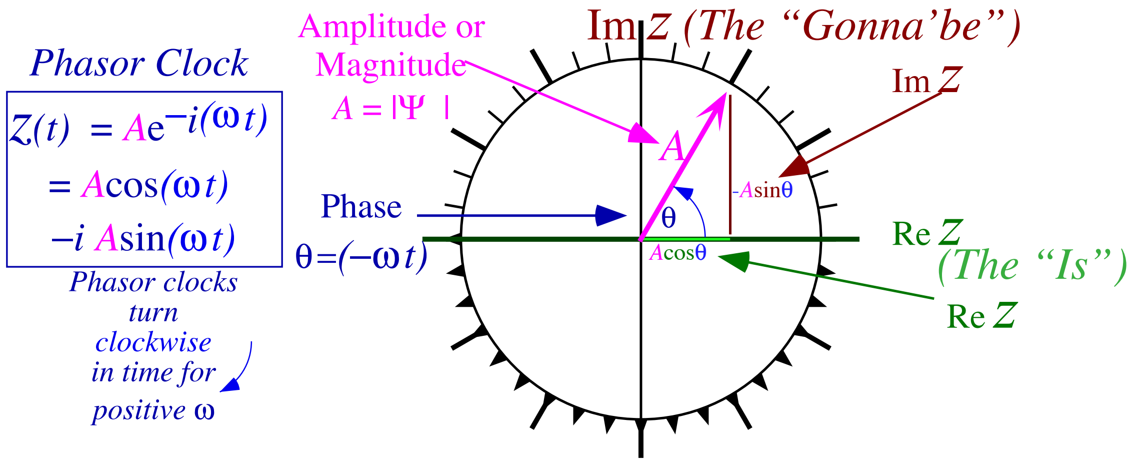
held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

# Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_{\pm}$

$$\omega_{\pm} = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2} = -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

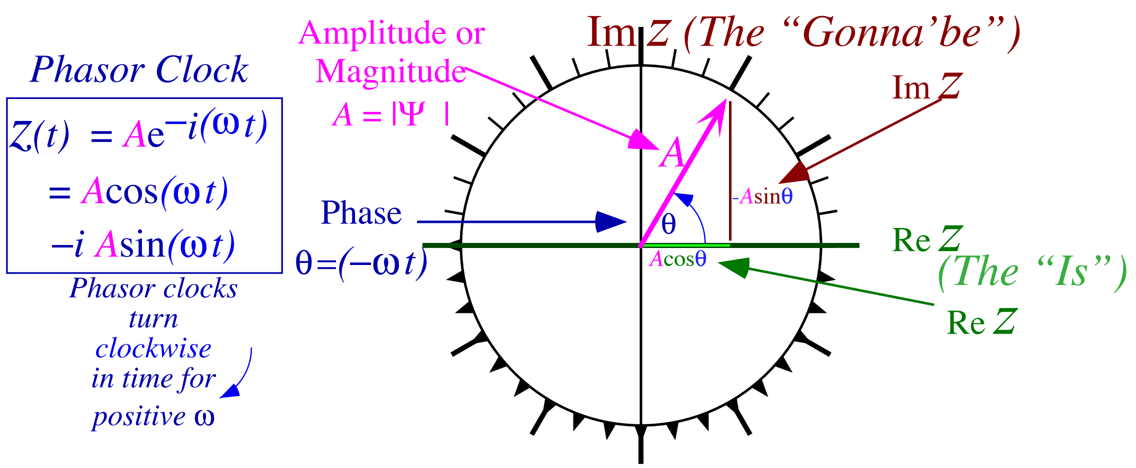
$$= e^{-\Gamma t}e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}t}$$

$$= e^{-\Gamma t}e^{\pm i\omega_\Gamma t}$$

# *Linear damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:
Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{-\Gamma t} e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}\, t}$$

$$= e^{-\Gamma t} e^{\pm i\omega_\Gamma t}$$

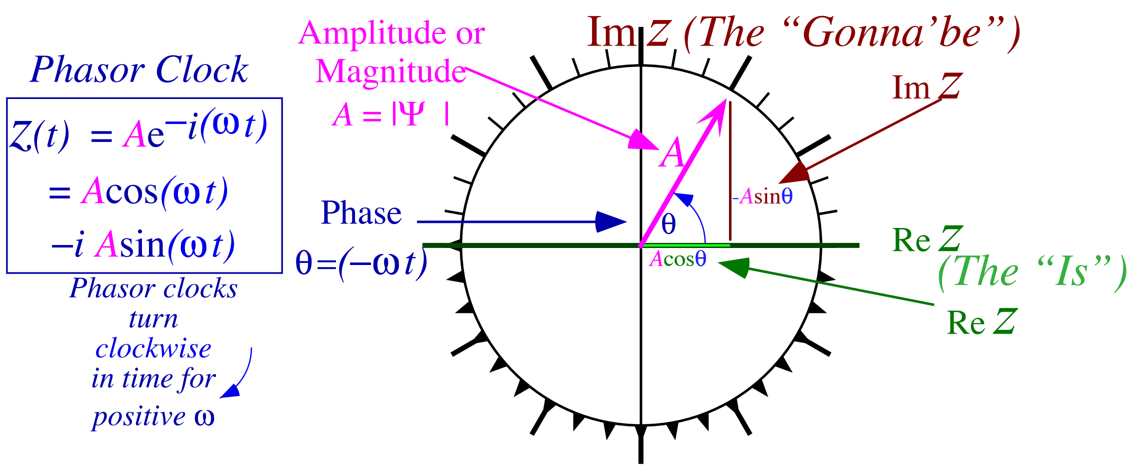
It oscillates at an angular frequency $\omega_\Gamma$ reduced slightly by .05% from $\omega_0$ due to damping $\Gamma$ =0.2.

$$\omega_\Gamma = \sqrt{\omega_0^2 - \Gamma^2} = \omega_0 - \tfrac{1}{2}(\Gamma^2/\omega_0) + \ldots = 6.2831853 - 0.003183 + .. = 6.280002 + \ldots = 6.280001$$

# Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2} = -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

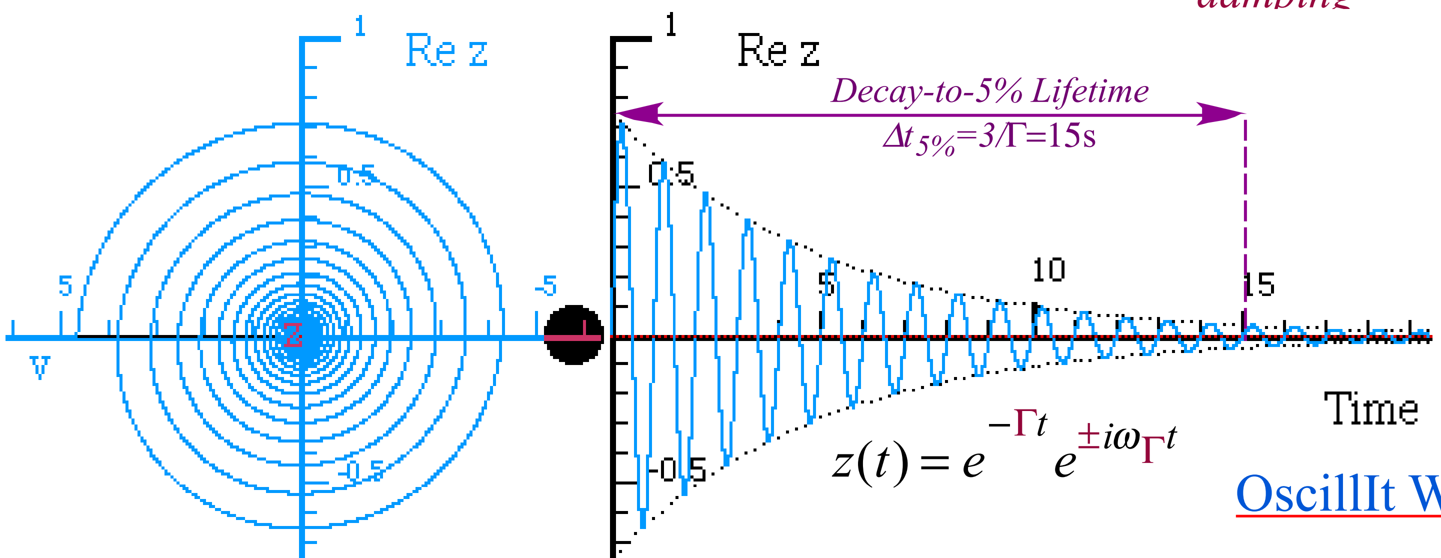
Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm\sqrt{\omega_0^2-\Gamma^2}\right)t} = e^{\left(-\Gamma \pm i\sqrt{\omega_0^2-\Gamma^2}\right)t} = e^{-\Gamma t}e^{\pm i\sqrt{\omega_0^2-\Gamma^2}\,t} = e^{-\Gamma t}e^{\pm i\omega_\Gamma t}$$



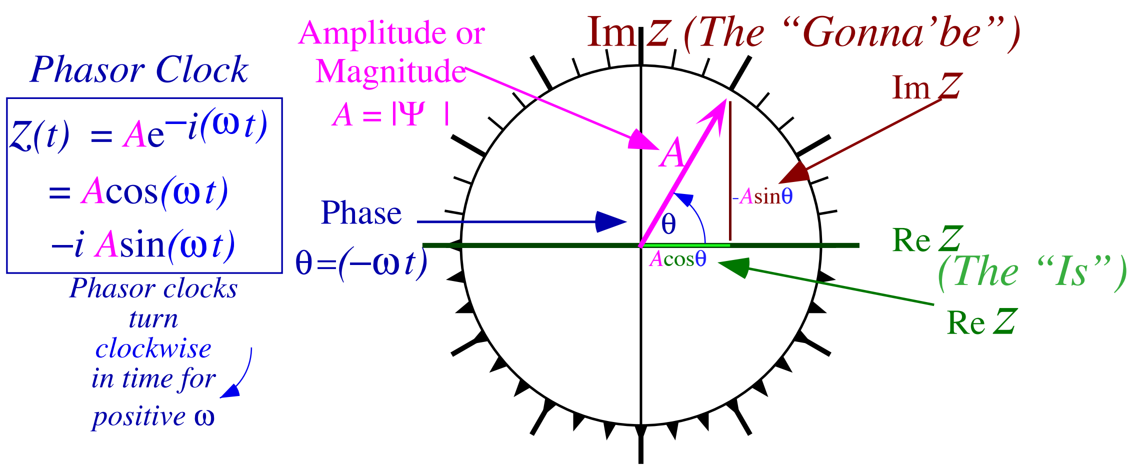

OscillIt Web Simulation

*Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$*

# *Linear damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

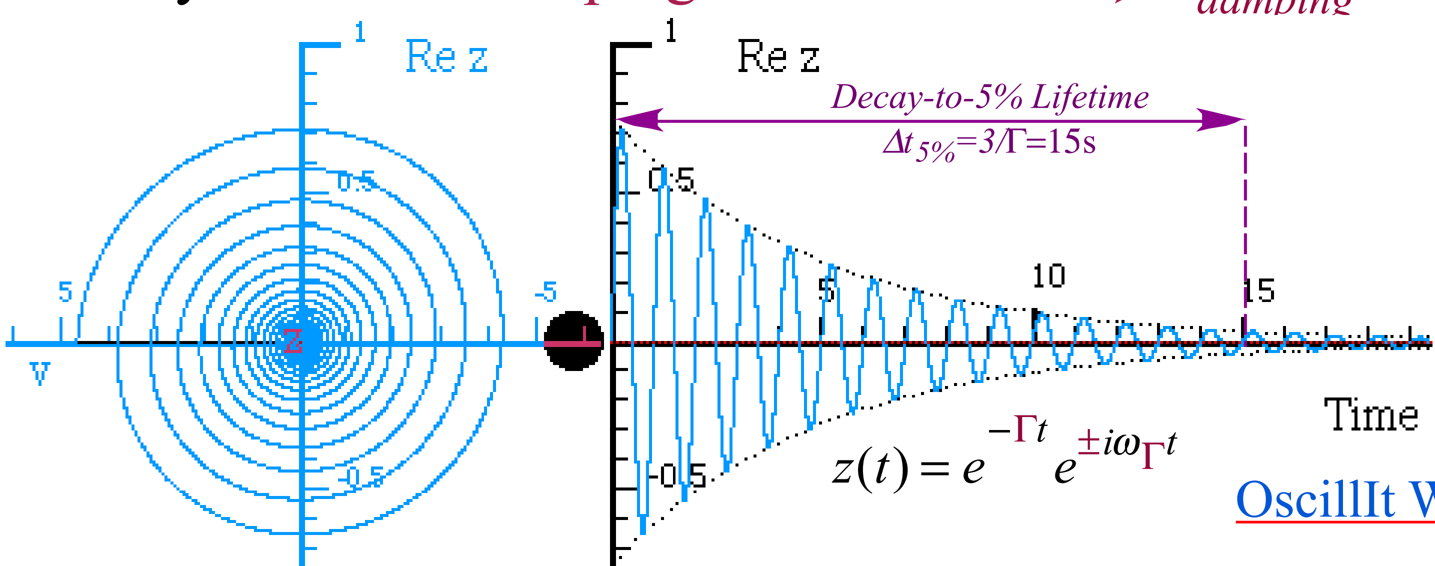
## Oscillator Figures of Merit:

Time required to to reduce amplitude to 5%

Easy-to-recall *5%* approximation:

$$e^{-3} \cong 0.05$$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$



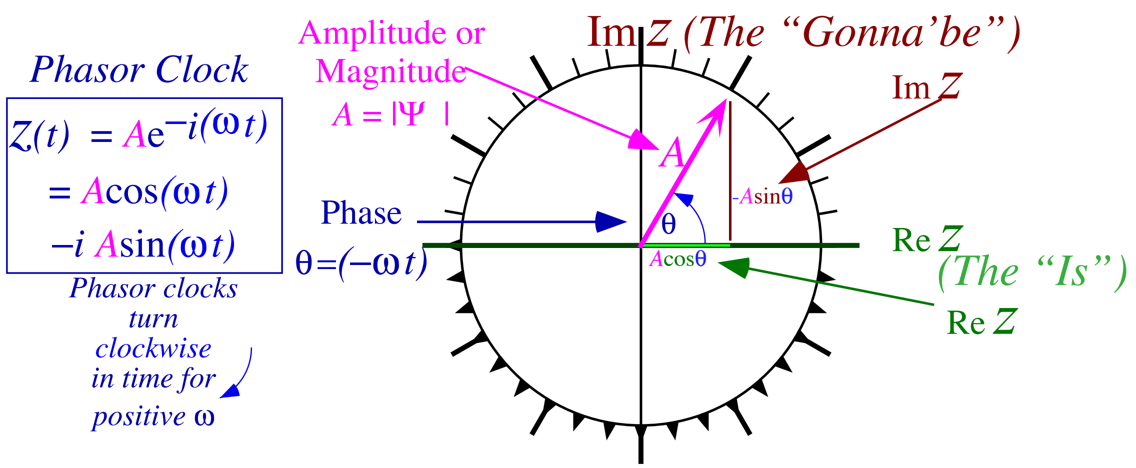

OscillIt Web Simulation

*Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$*

# *Linear damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

## Oscillator Figures of Merit:

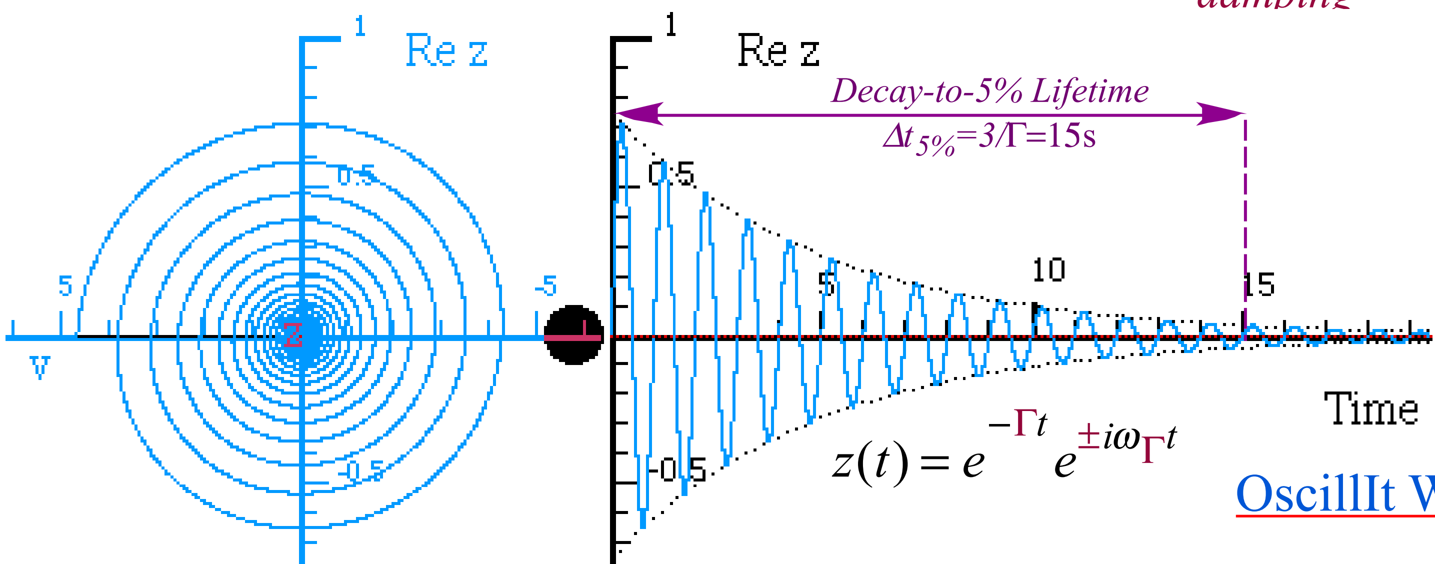
Time required to to reduce amplitude to 5% (or 4.321%)

Easy-to-recall *5%* approximation: $e^{-3} \cong 0.05$

More precise one: $e^{-\pi} \cong 0.04321$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$



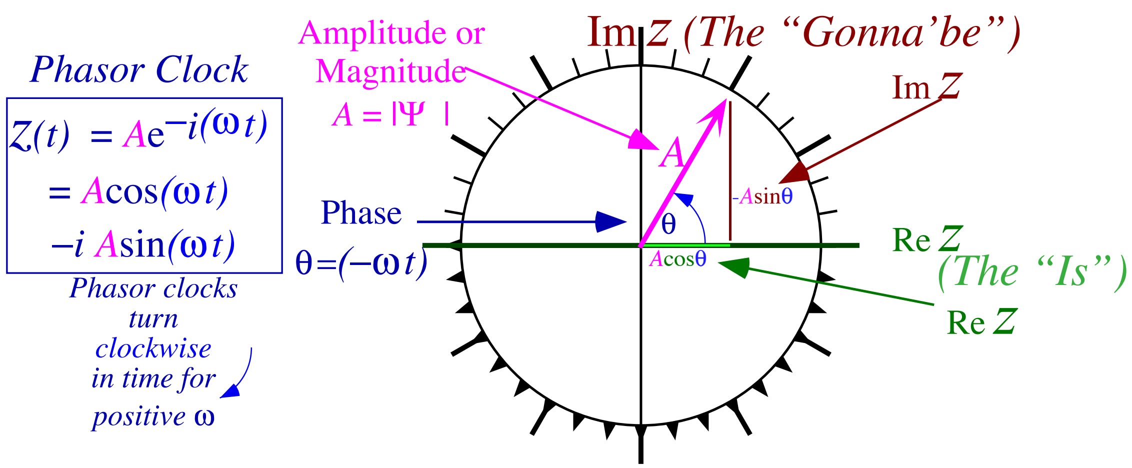

OscillIt Web Simulation

*Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$*

# *Linear* *-damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$




$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Click to see p.35.

Click to see p.47.

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

## Oscillator Figures of Merit:

Number $N$ of oscillations to reduce amplitude to 5% (or 4.321%)

Easy-to-recall 5% approximation:

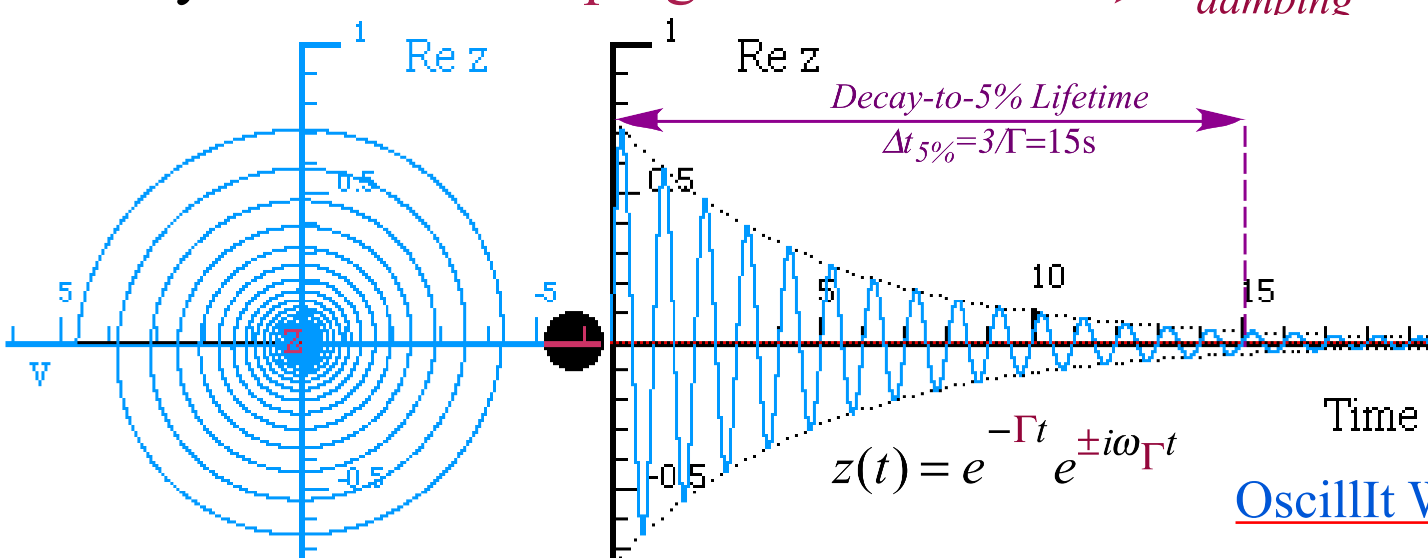
$$e^{-3} \cong 0.05$$

More precise one:

$$e^{-\pi} \cong 0.04321$$

$$N_{5\%} = \frac{\omega_\Gamma \cdot t_{5\%}}{2\pi} = \frac{3\omega_\Gamma}{2\pi\Gamma} \sim \frac{\omega_\Gamma}{2\Gamma}$$

$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$



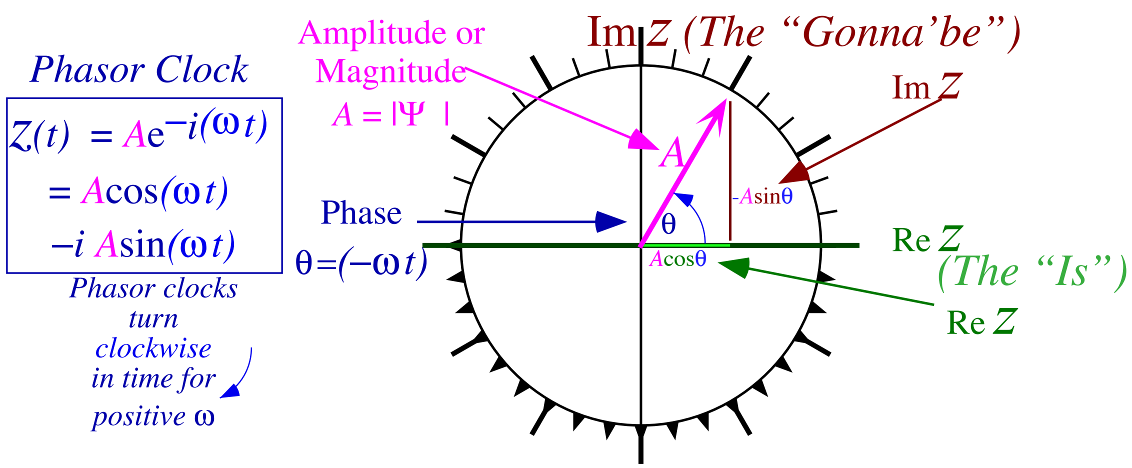

OscillIt Web Simulation

*Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$*

# *Linear forced-damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$




Stimulating acceleration $a_{stimulus}$=*a(t)* due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

driven by external stimulating force → $F_{stimulus}(t) = eE(t)$

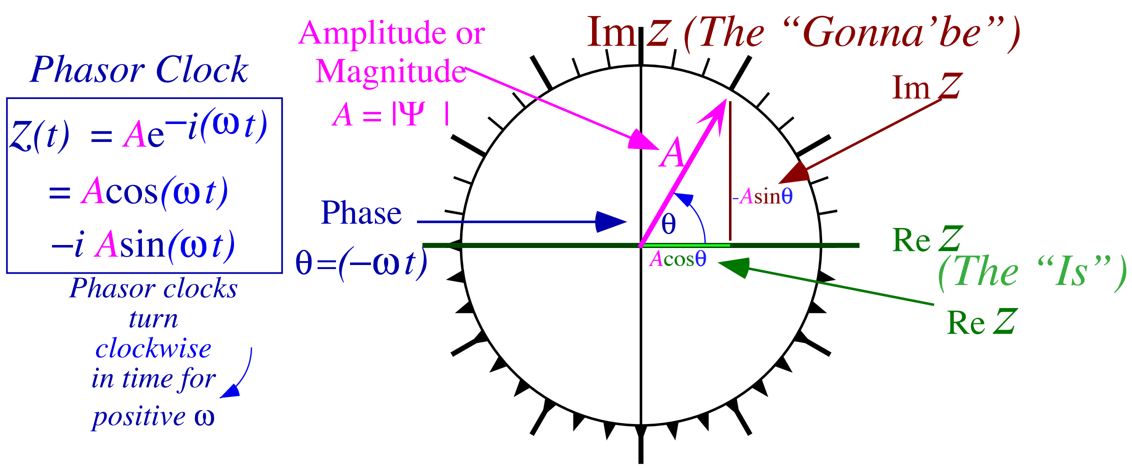
held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad \left(b = 2\Gamma m\right)$

# Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2 z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$



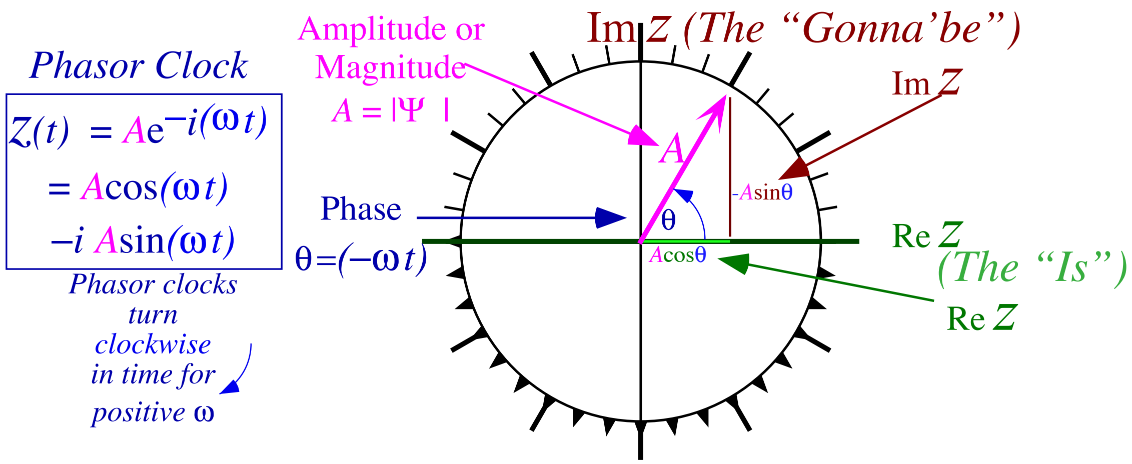

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2 z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$ :

# Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$




$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

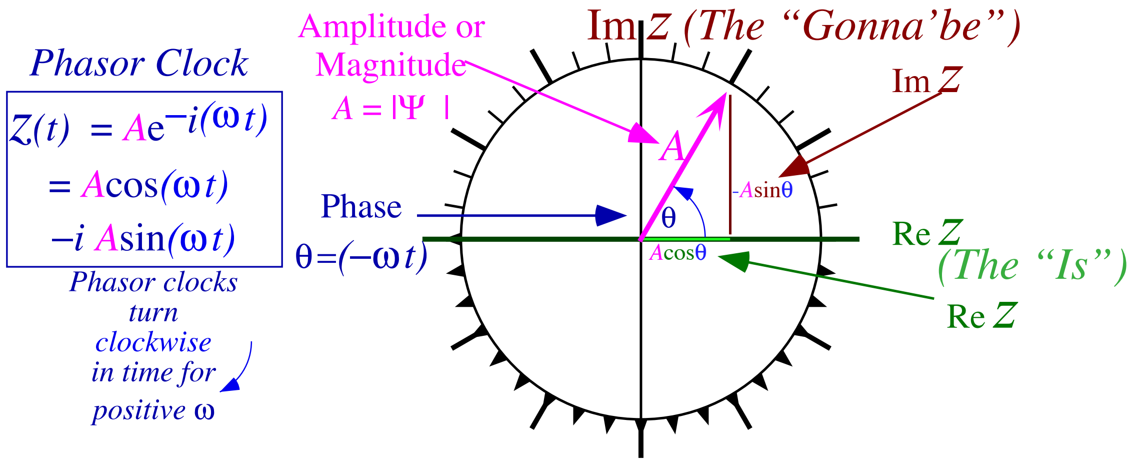
Solving for $z_{stimulus}(t)$ given $a_{stimulus}$ :

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}a_{stimulus}$$

# Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$




$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$ :

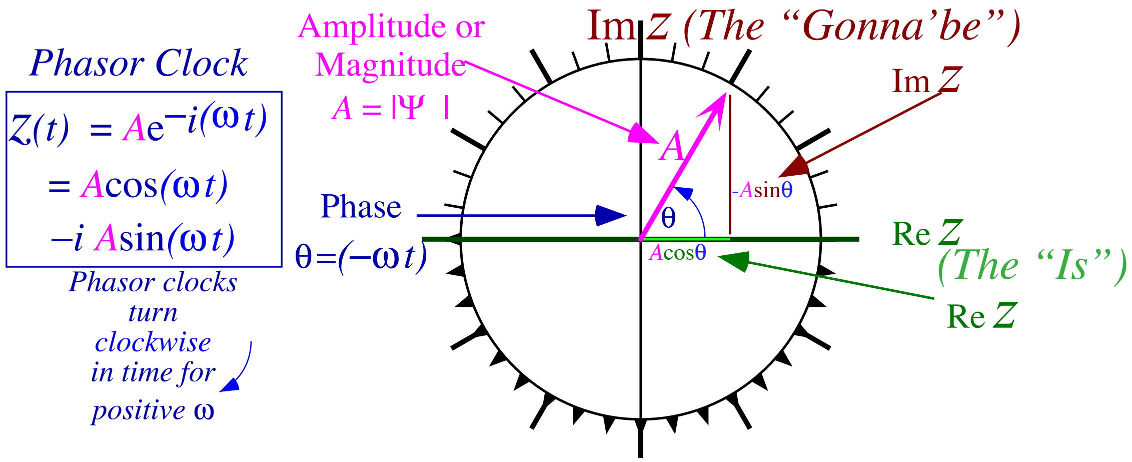
$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right) z = a_{stimulus}$$

Pretty crazy?

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2} a_{stimulus}$$

# Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$




$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$ :

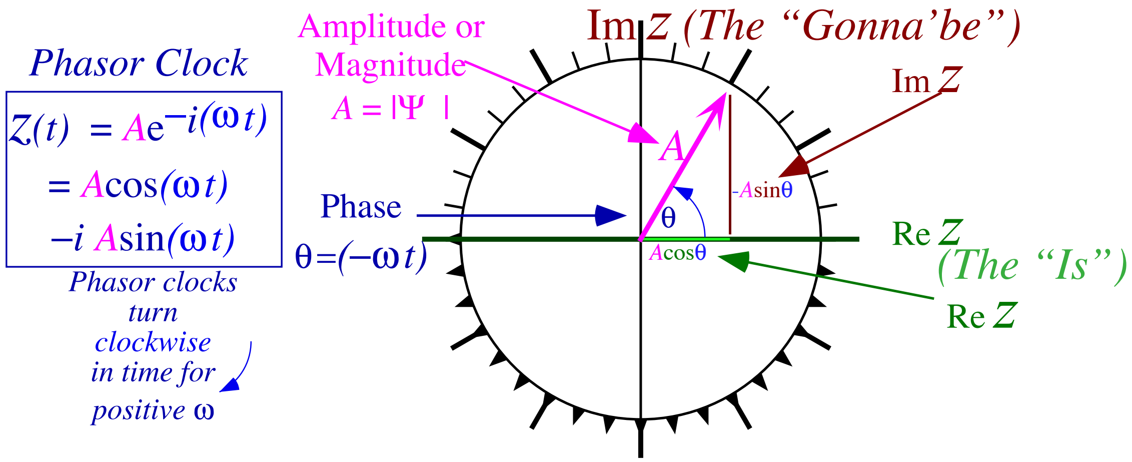
$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if
$a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}a_{stimulus}$$

# Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$




$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$ :

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

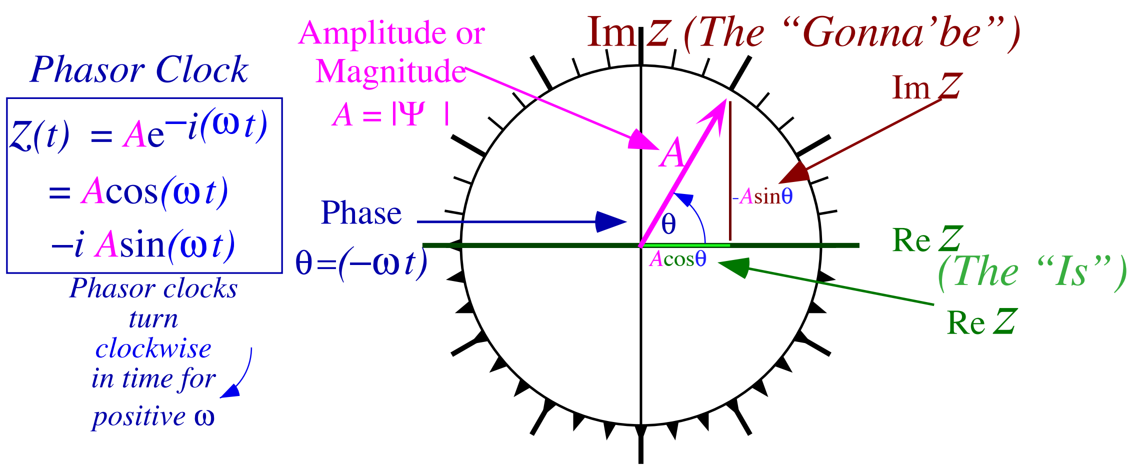
Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s{}^2 - i2\Gamma\omega_s + \omega_0^2}a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s{}^2 - i2\Gamma\omega_s}a_s e^{-i\omega_s t}$$

# Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$




$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$ :

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if
$a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

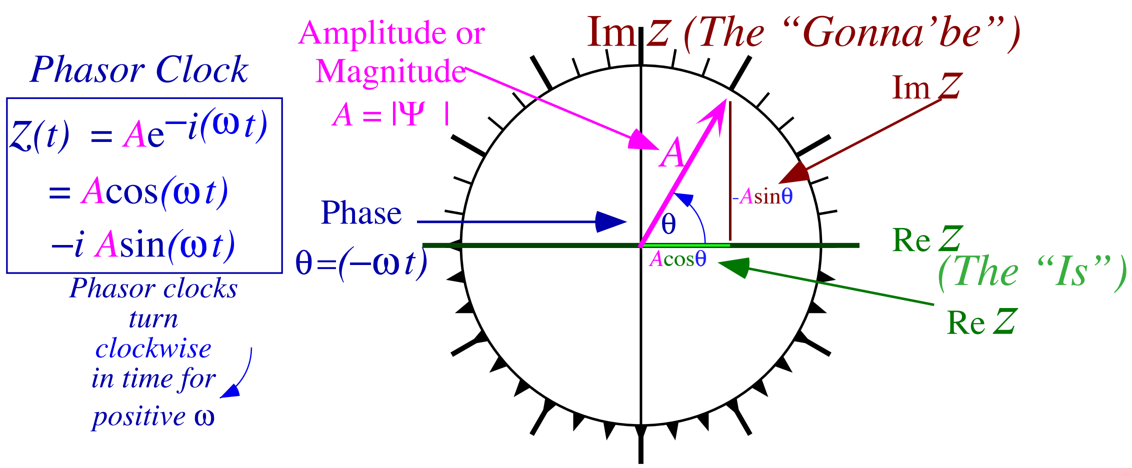
$$z_{stimulus} = \frac{1}{-\omega_s{}^2 - i2\Gamma\omega_s + \omega_0^2} a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s{}^2 - i2\Gamma\omega_s} a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s) \cdot a_s$$

# Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2 z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$




$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2 z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$ :

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right) z = a_{stimulus}$$

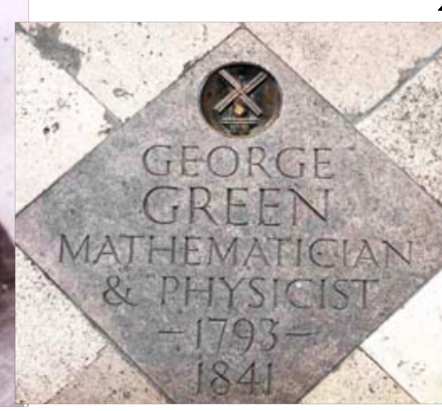
Pretty crazy? But not so crazy if
$a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s^{\,2} - i2\Gamma\omega_s + \omega_0^2} a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^{\,2} - i2\Gamma\omega_s} a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s) \cdot a_s$$

*Green's Function* for the F-D-H Oscillator (FDHO)




**George Green** (14 July 1793 – 31 May 1841)

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)

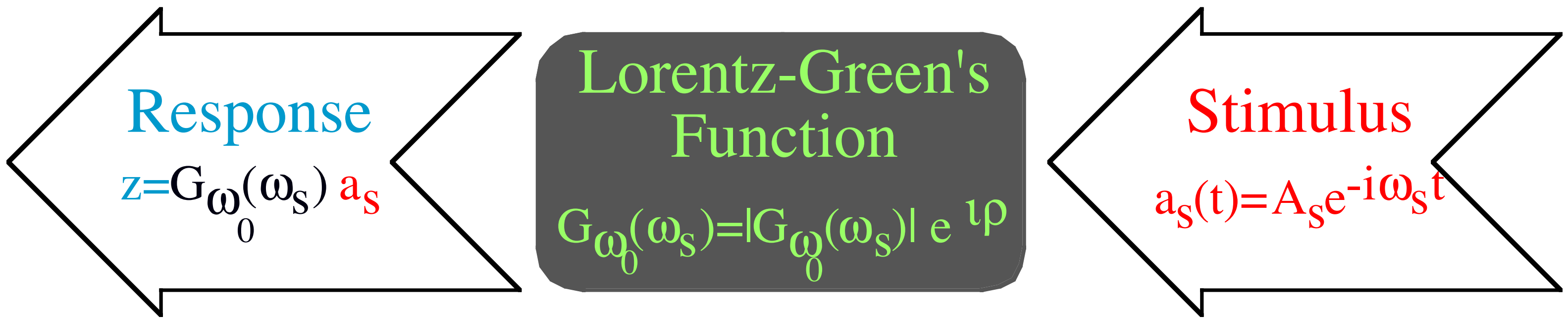



*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\, G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of $G$:

Hendrik A. Lorentz

July 18, 1853. - February 4, 1928

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)

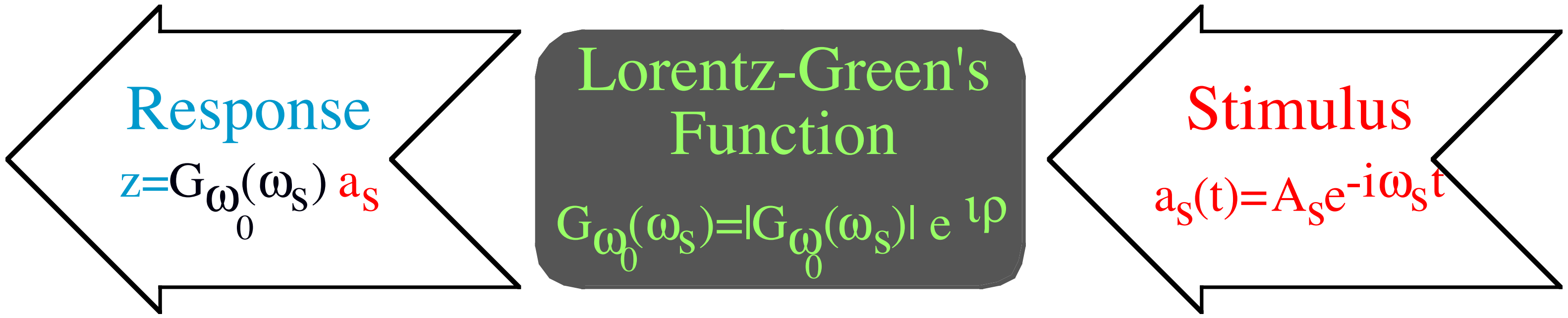



*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\, \mathrm{Im}\, G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of $G$: $\frac{1}{x-iy} = \frac{1}{x-iy}\frac{x+iy}{x+iy} = \frac{x+iy}{x^2+y^2}$

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)

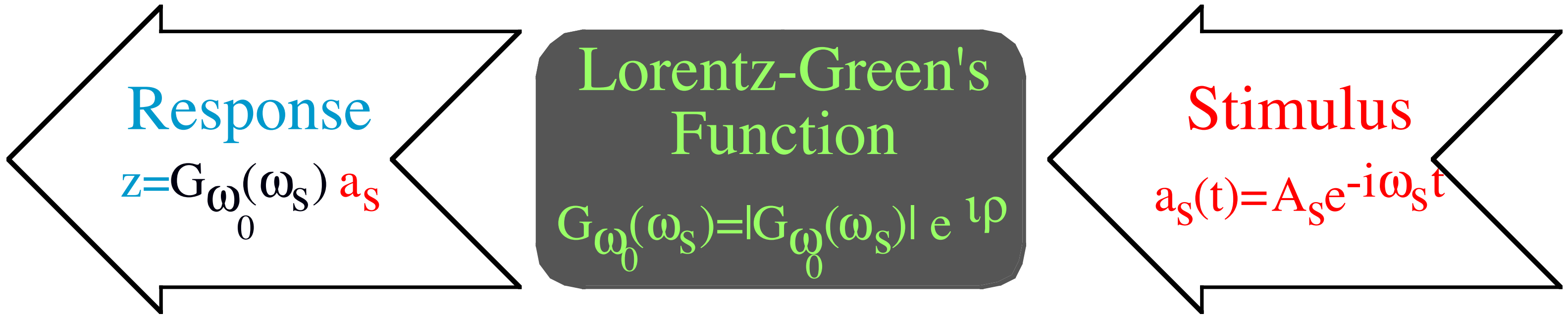



*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \text{Re}\, G_{\omega_0}(\omega_s) + i\, \text{Im}\, G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of $G$:

$$\frac{1}{x-iy} = \frac{1}{x-iy}\frac{x+iy}{x+iy} = \frac{x+iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

$$\text{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\text{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)

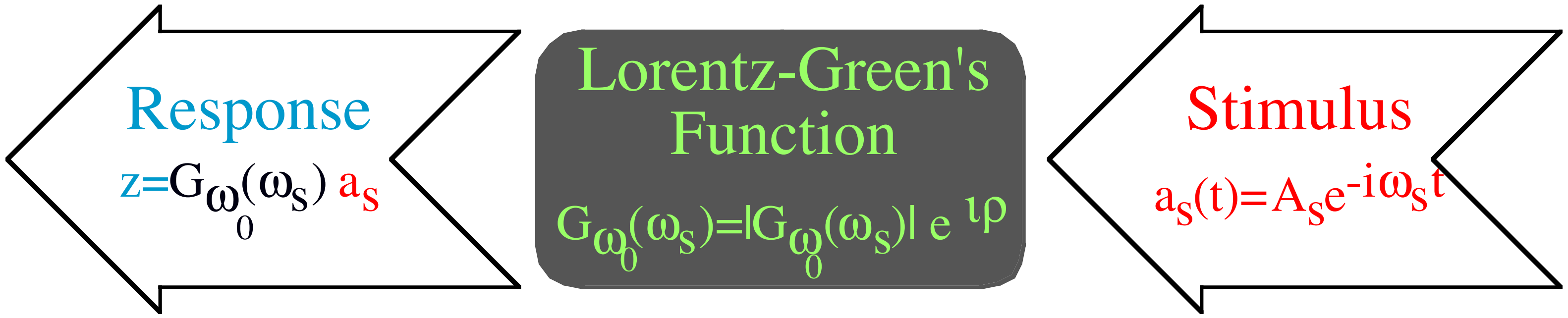



*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}=\mathrm{Re}\,G_{\omega_0}(\omega_s)+i\,\mathrm{Im}\,G_{\omega_0}(\omega_s)=\left|G_{\omega_0}(\omega_s)\right|e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of $G$:

$$\mathrm{Re}\,G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{(\omega_0^2-\omega_s^2)^2+(2\Gamma\omega_s)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{(\omega_0^2-\omega_s^2)^2+(2\Gamma\omega_s)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and polar angle ρ of the *polar form* of $G$:

$$\left|G_{\omega_0}(\omega_s)\right|=\frac{1}{\sqrt{(\omega_0^2-\omega_s^2)^2+(2\Gamma\omega_s)^2}}$$

$$\rho=\tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2-\omega_s^2}\right)$$

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)

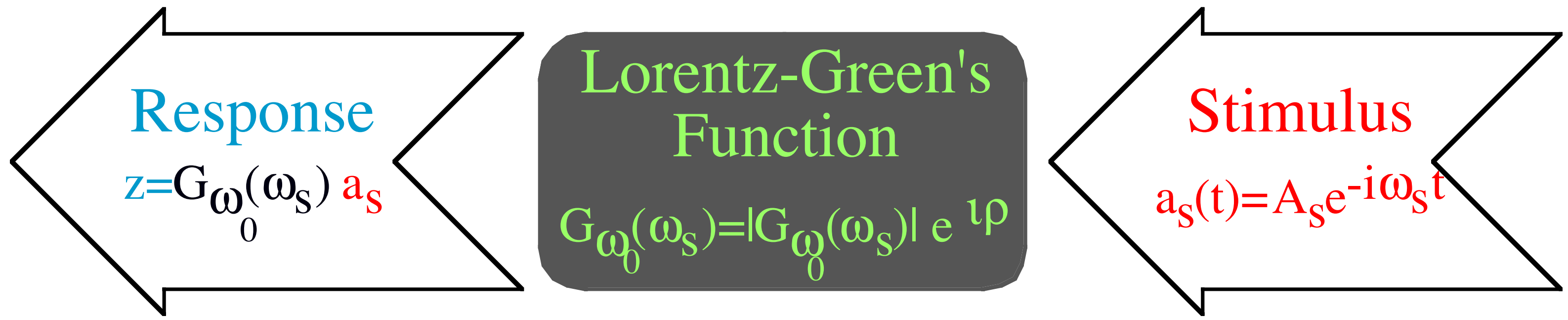



*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\, G_{\omega_0}(\omega_s) = \left|G_{\omega_0}(\omega_s)\right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of $G$:

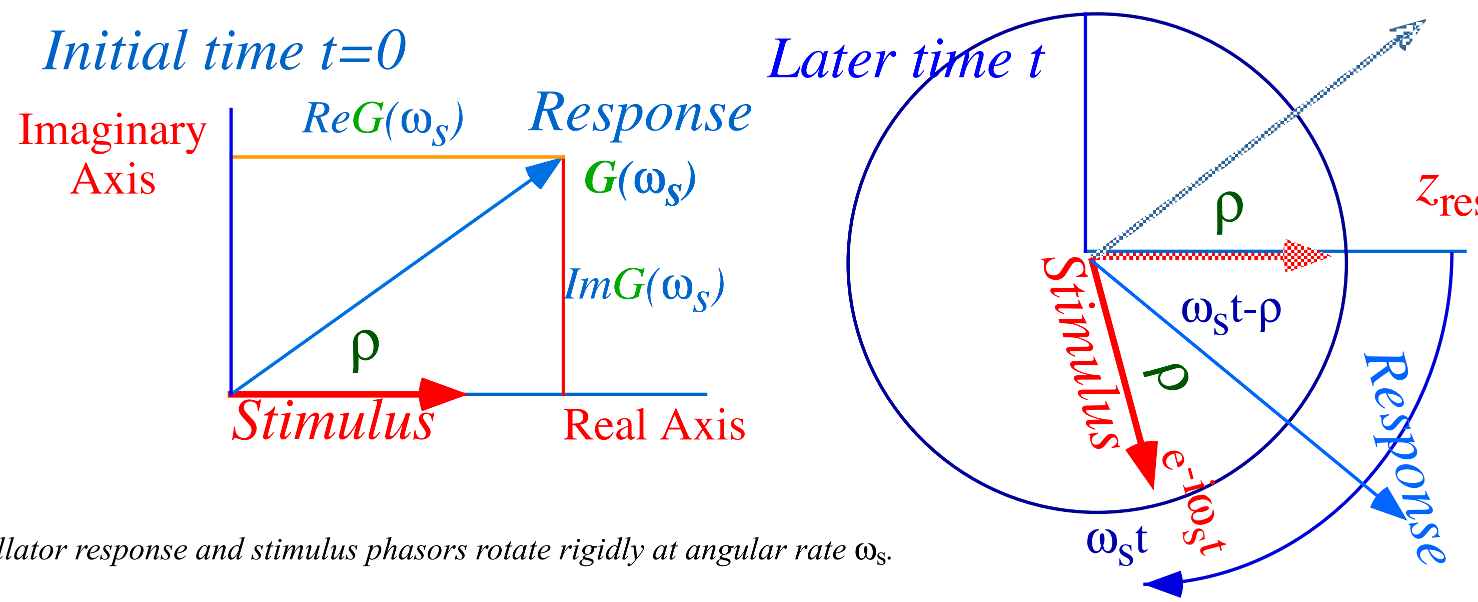
$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of $G$:

$$\left|G_{\omega_0}(\omega_s)\right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$




*polar angle* ρ is the *phase lag angle* ρ

$$z_{\text{response}}(t) = \left|G_{\omega_0}(\omega_s)\right| a(0)\, e^{-i(\omega_s t - \rho)}$$

*Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate* $\omega_s$.

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)

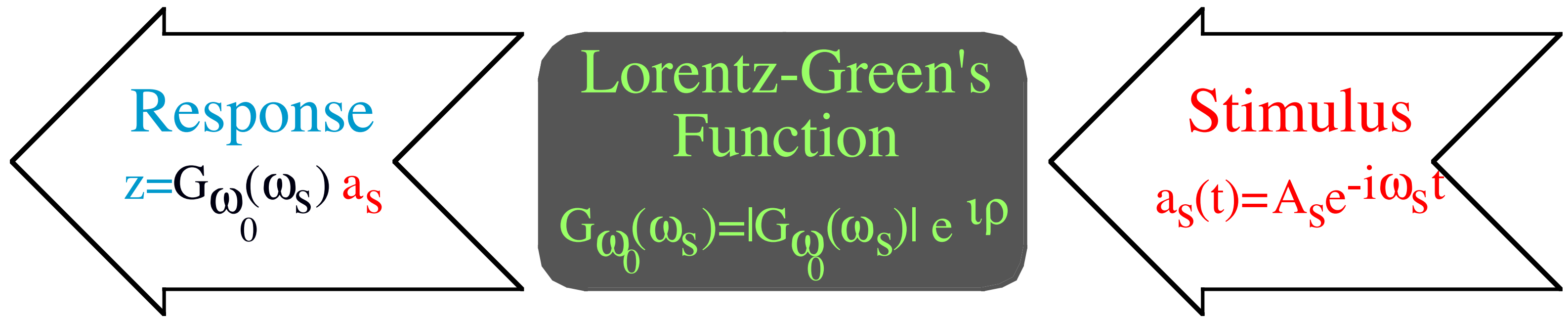



*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\, G_{\omega_0}(\omega_s) = \left|G_{\omega_0}(\omega_s)\right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of $G$:

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of $G$:

$$\left|G_{\omega_0}(\omega_s)\right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

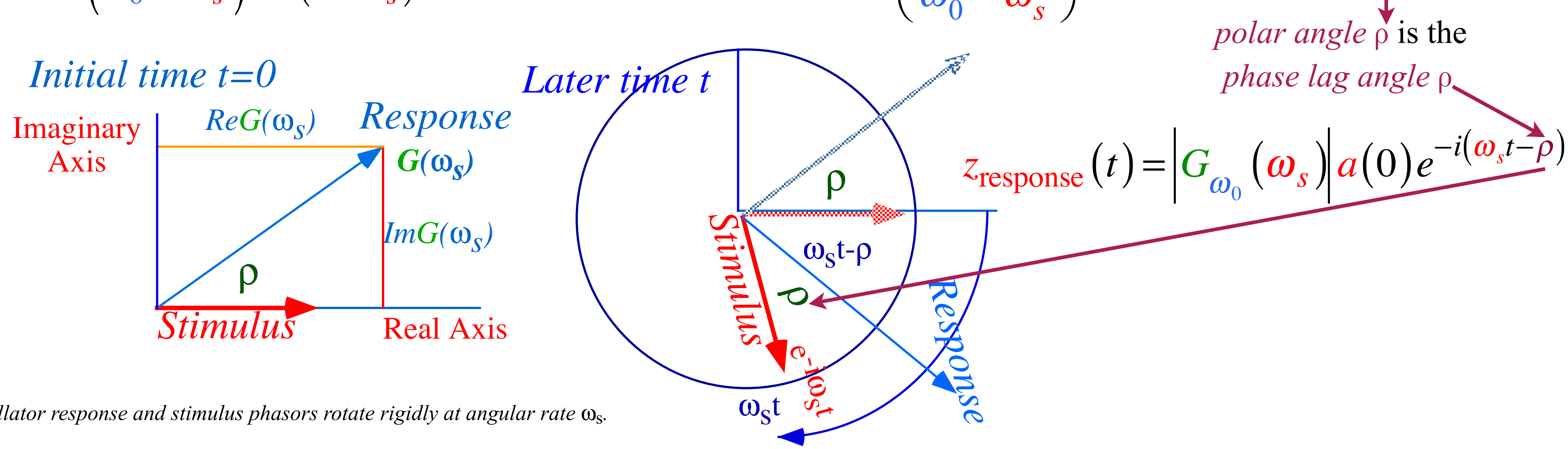
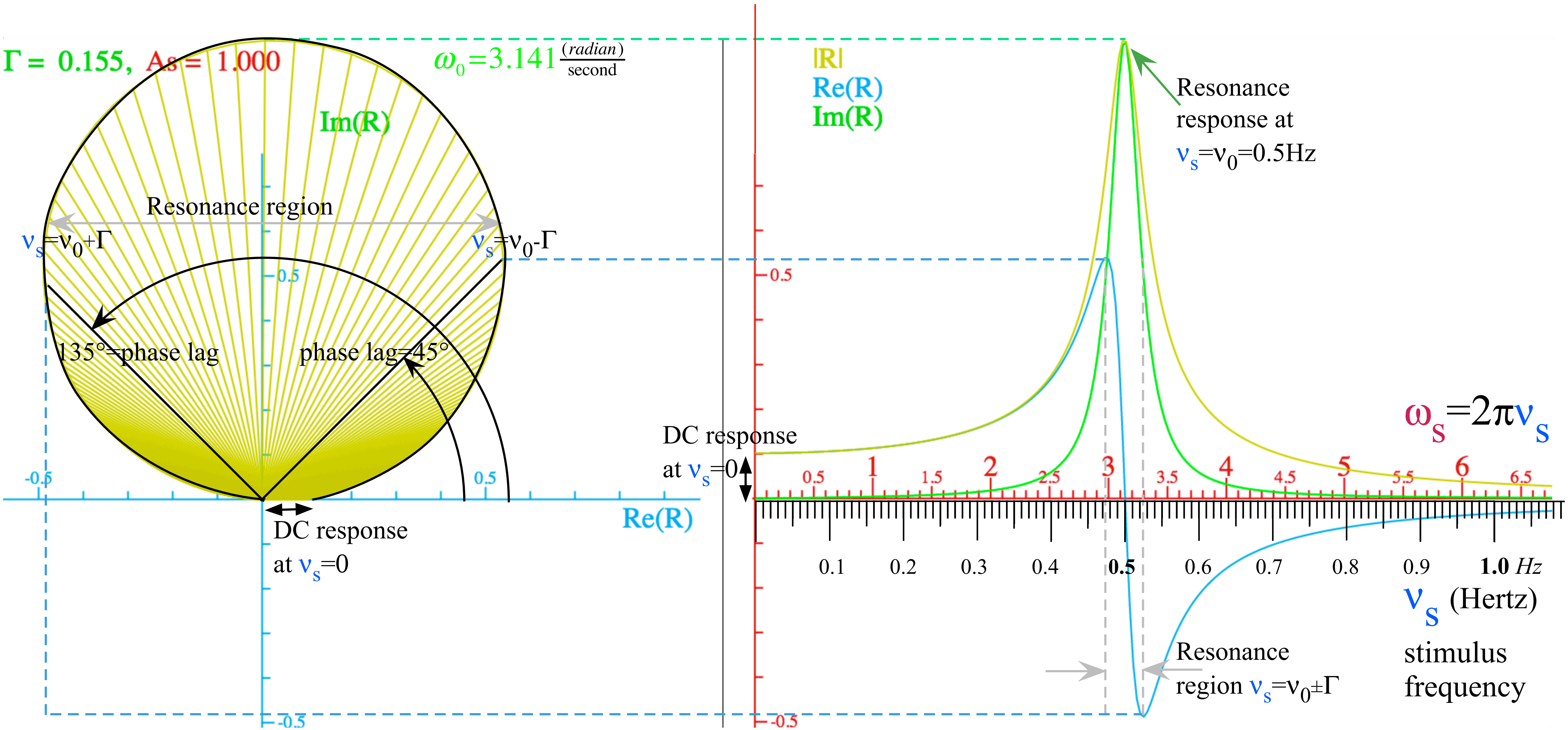



*Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate* $\omega_s$.

$$Lorentz\text{-}Green\text{'s function for } \nu_0 = 0.5\,Hz \text{ or}: \omega_0 = \pi \frac{(radian)}{\text{second}}$$




OscillIt Web Simulation: Lorentz Response Function

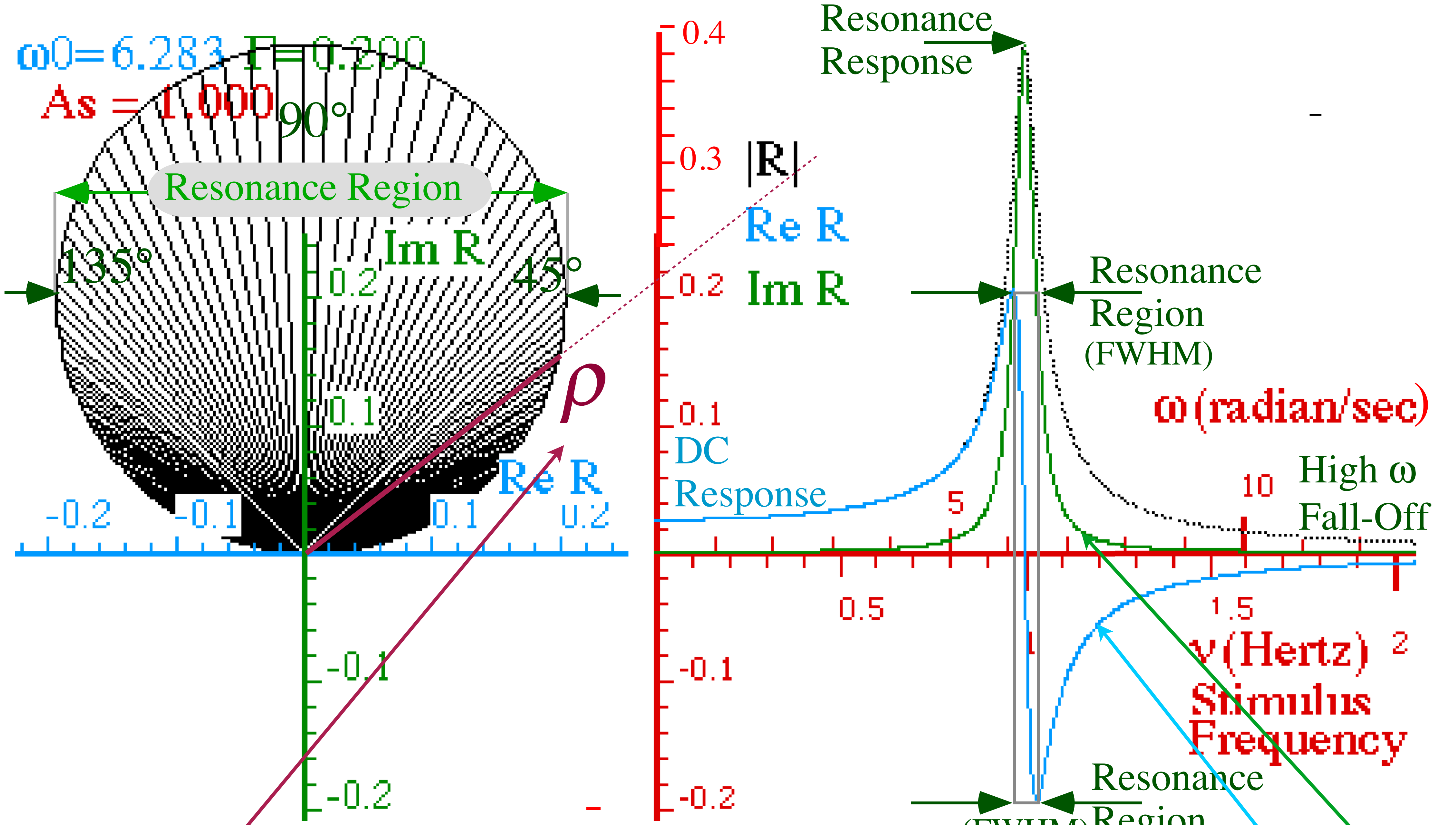



*Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots*

*Phase lag angle*

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

*Real part*

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

*Imaginary part*

$$Q\ factor = \frac{\nu_0}{2\Gamma} = \frac{q}{2\pi}$$

*Amplitude Amplification Factor*

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \textit{(angular quality factor)}$$

OscillIt Web Simulation: Lorentz Response Function

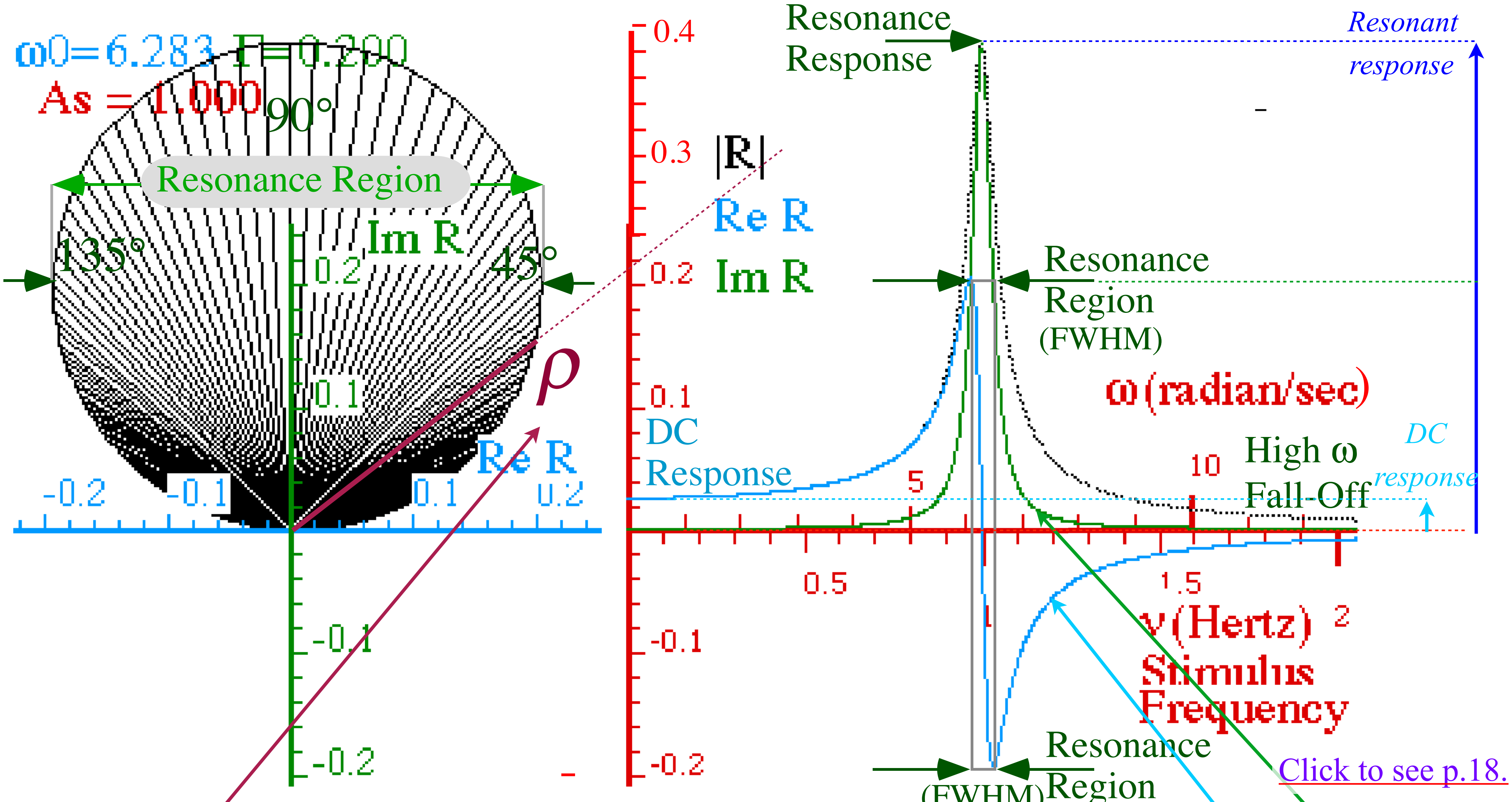



*Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots*

[Click to see p.18.](#)

[Click to see p.47.](#)

*Real part*

*Imaginary part*

*Phase lag angle*

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

$$\operatorname{Re} G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\operatorname{Im} G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$Q\ factor = \frac{\nu_0}{2\Gamma} = \frac{q}{2\pi}$$

*Amplitude Amplification Factor*

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \textit{(angular quality factor)}$$

[OscillIt Web Simulation: Lorentz Response Function](#)

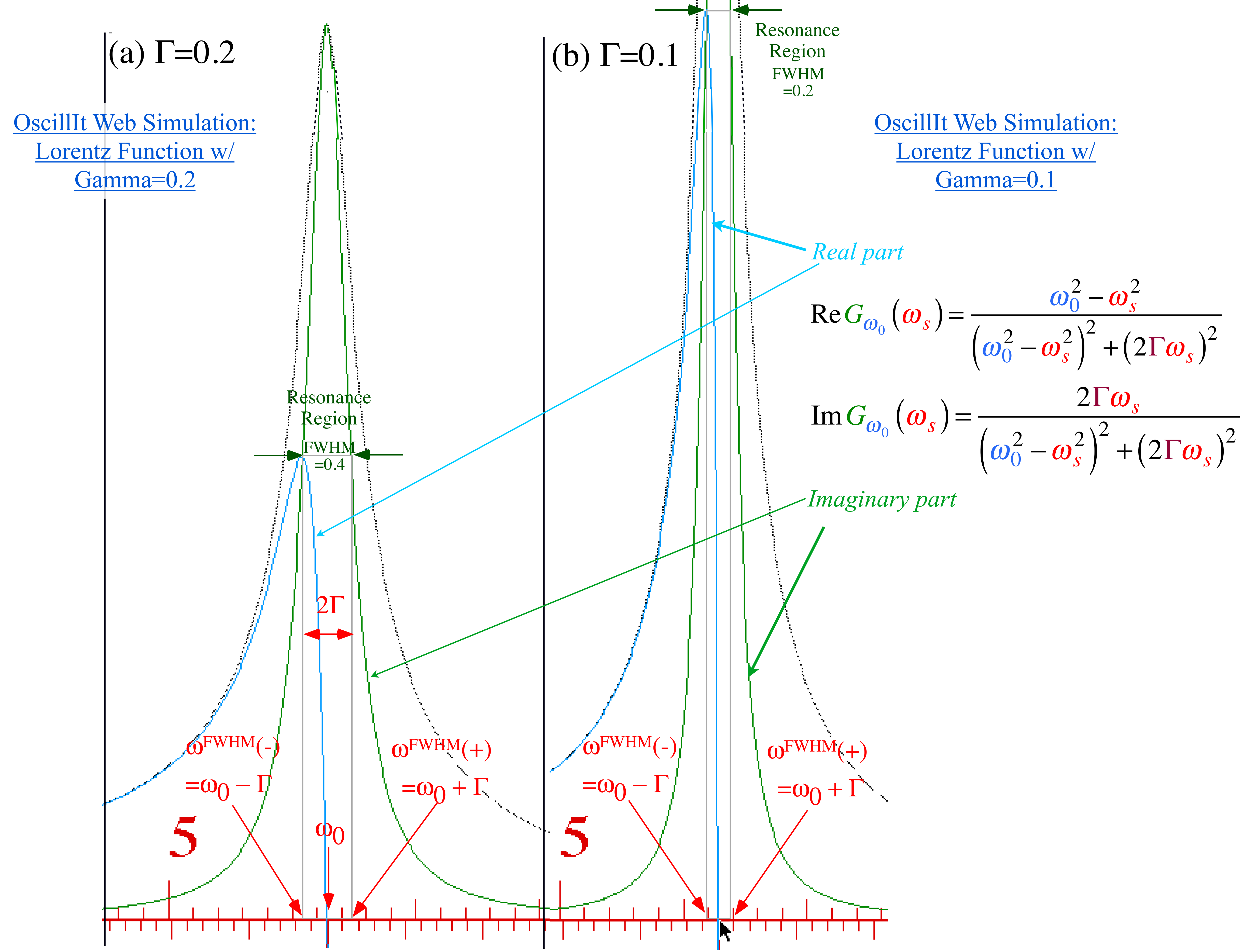



*Fig. 4.2.7 Comparing Lorentz-Green resonance region for (a) Γ=0.2 and (b) Γ=0.1.*
*Maximum and minimum points of* ReG(ω) *and inflection points of* ImG(ω) *are near region boundaries* $\omega^{FWHM}(\pm)=\omega_0\pm\Gamma$.

*Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)*

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s)a(0)e^{-i\omega_s t}$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

*Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)*

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s)a(0)e^{-i\omega_s t}$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

# *Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)*

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t} e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s) a(0) e^{-i\omega_s t}$$

$$= Ae^{-\Gamma t} e^{-i\omega_\Gamma t} + \left| G_{\omega_0}(\omega_s) \right| a(0) e^{-i(\omega_s t - \rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

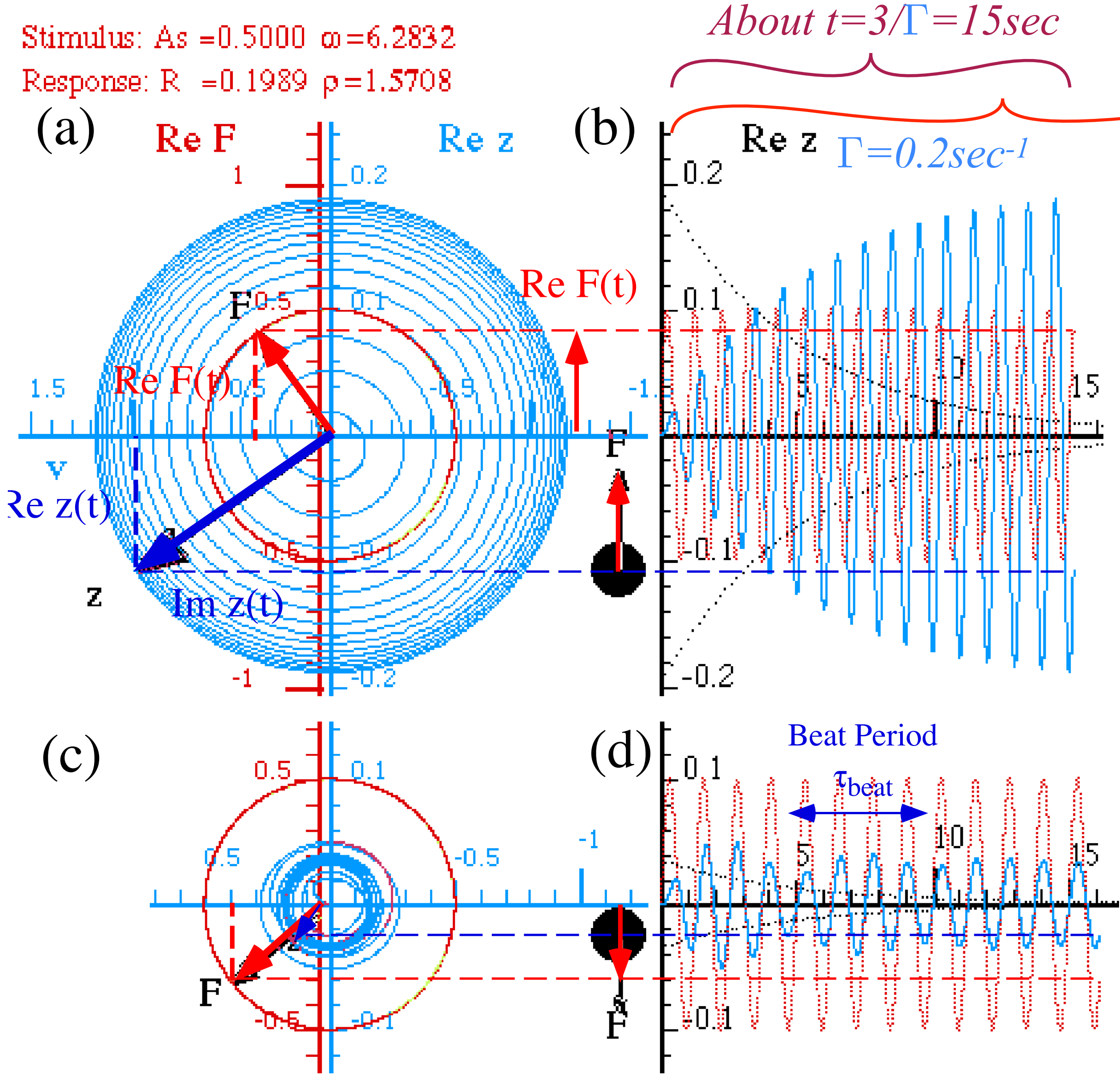
Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

*About t=3/Γ=15sec*

*About t = forever*

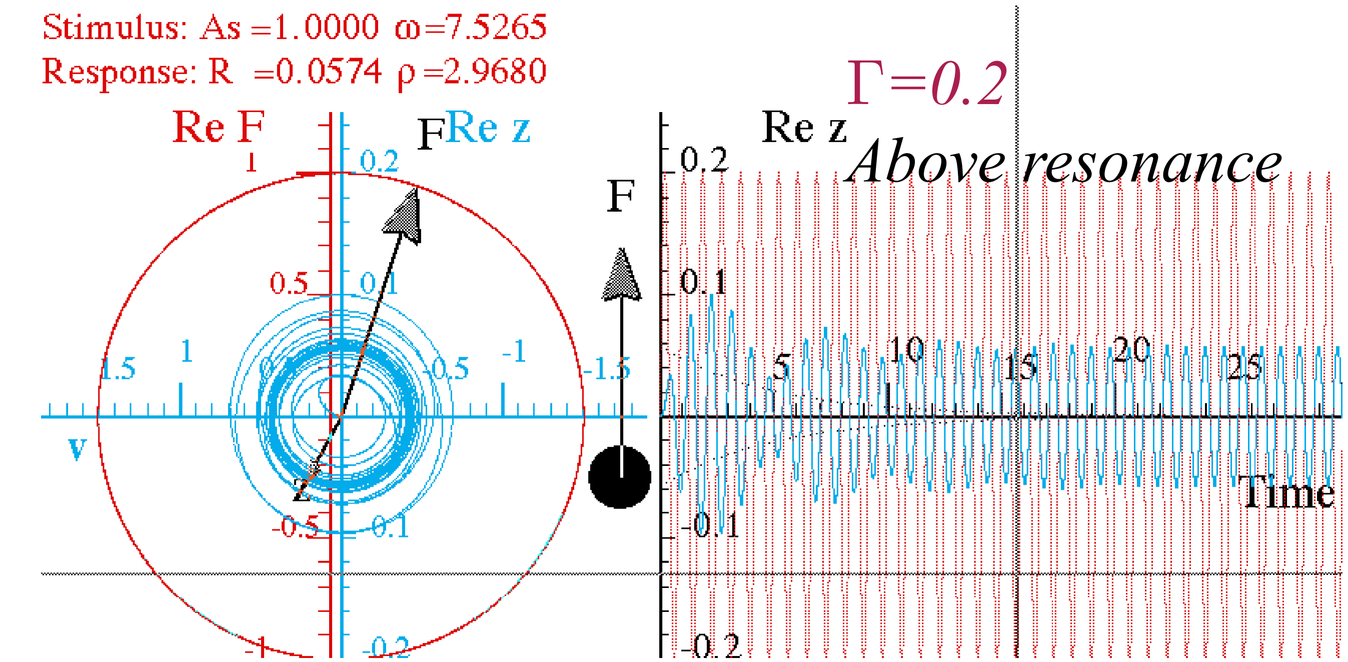



[OscillIt (On Resonance) Simulation]

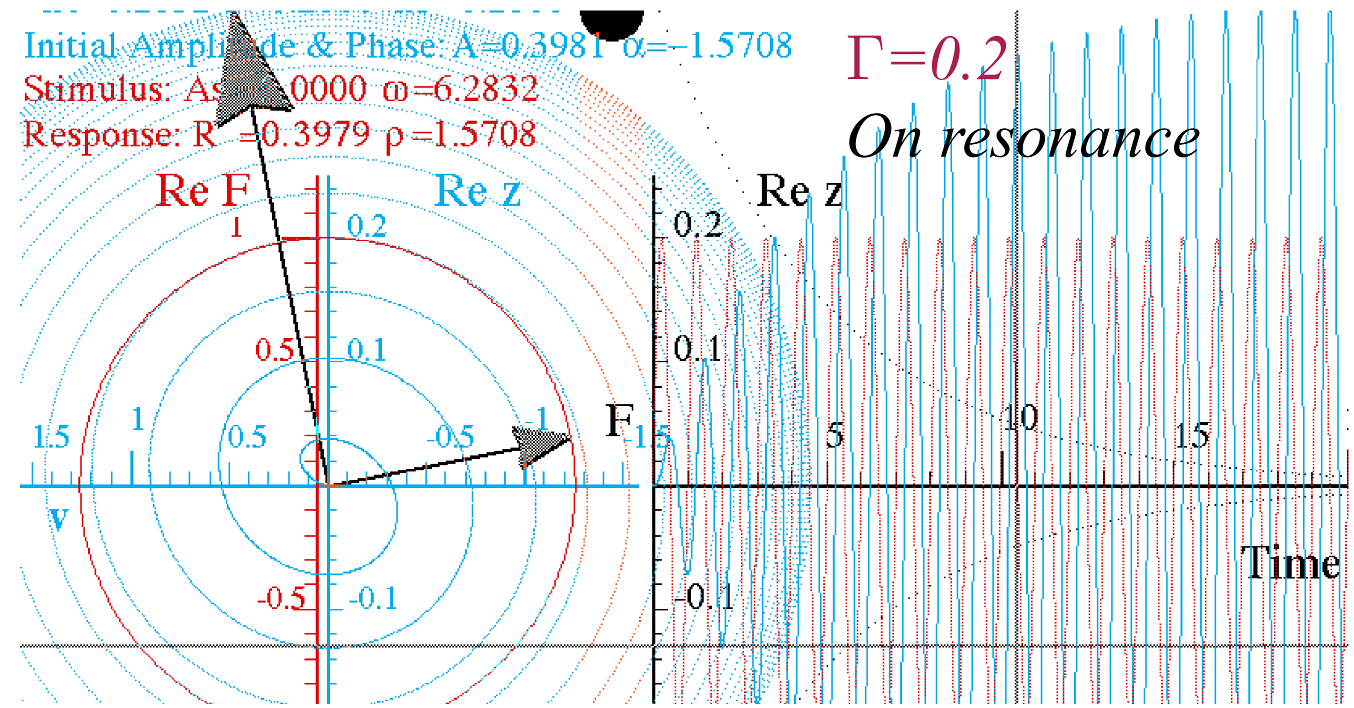
*Fig. 4.2.8 On Resonance (a)Response z-phasor lags ρ=90° behind stimulus F-phasor. ($\omega_s=\omega_0=2\pi$, $\omega_0=2\pi$, and $\Gamma=0.2$). (b) Time plots of* Re *z(t) and* Re *F(t)*

*Fig. 4.2.8 Below Resonance (c)Response z-phasor lags ρ=8.05° behind stimulus F-phasor. ($\omega_s=5.03$, $\omega_0=2\pi$, and $\Gamma=0.2$). (d) Time plots of* Re *z(t) and* Re *F(t). Beats are barely visible.*

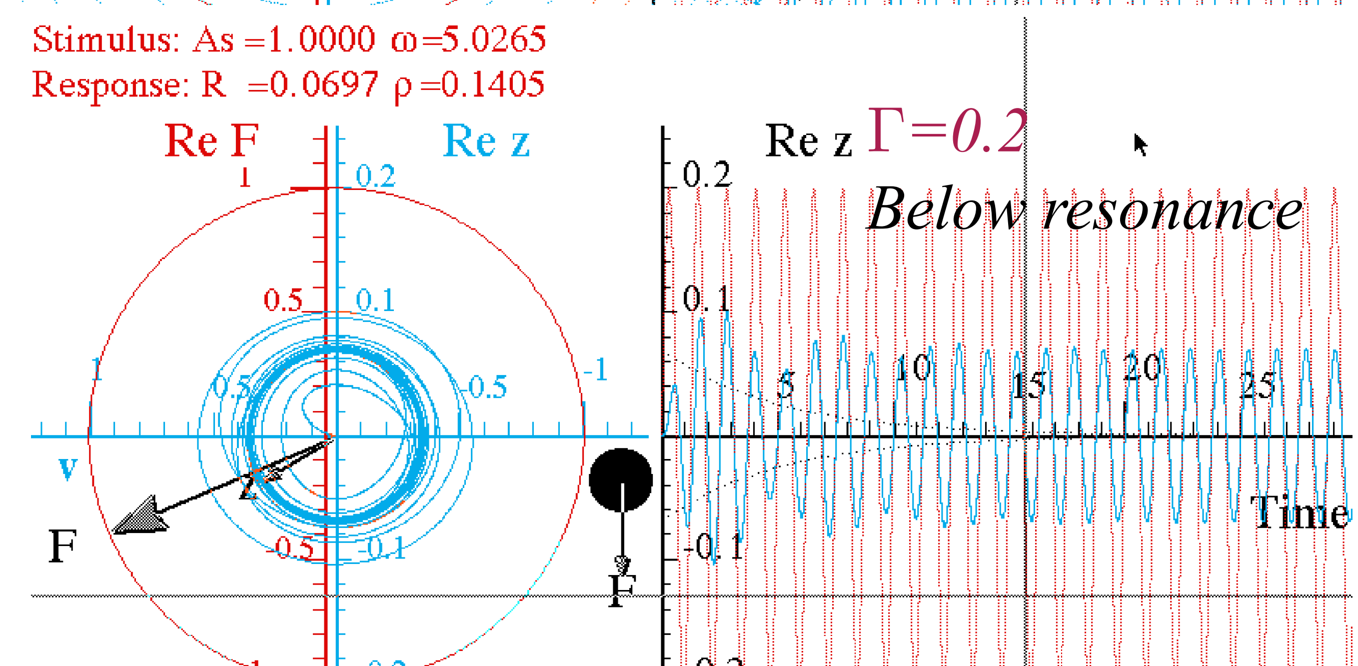
[OscillIt (Way Below Resonance) Simulation]



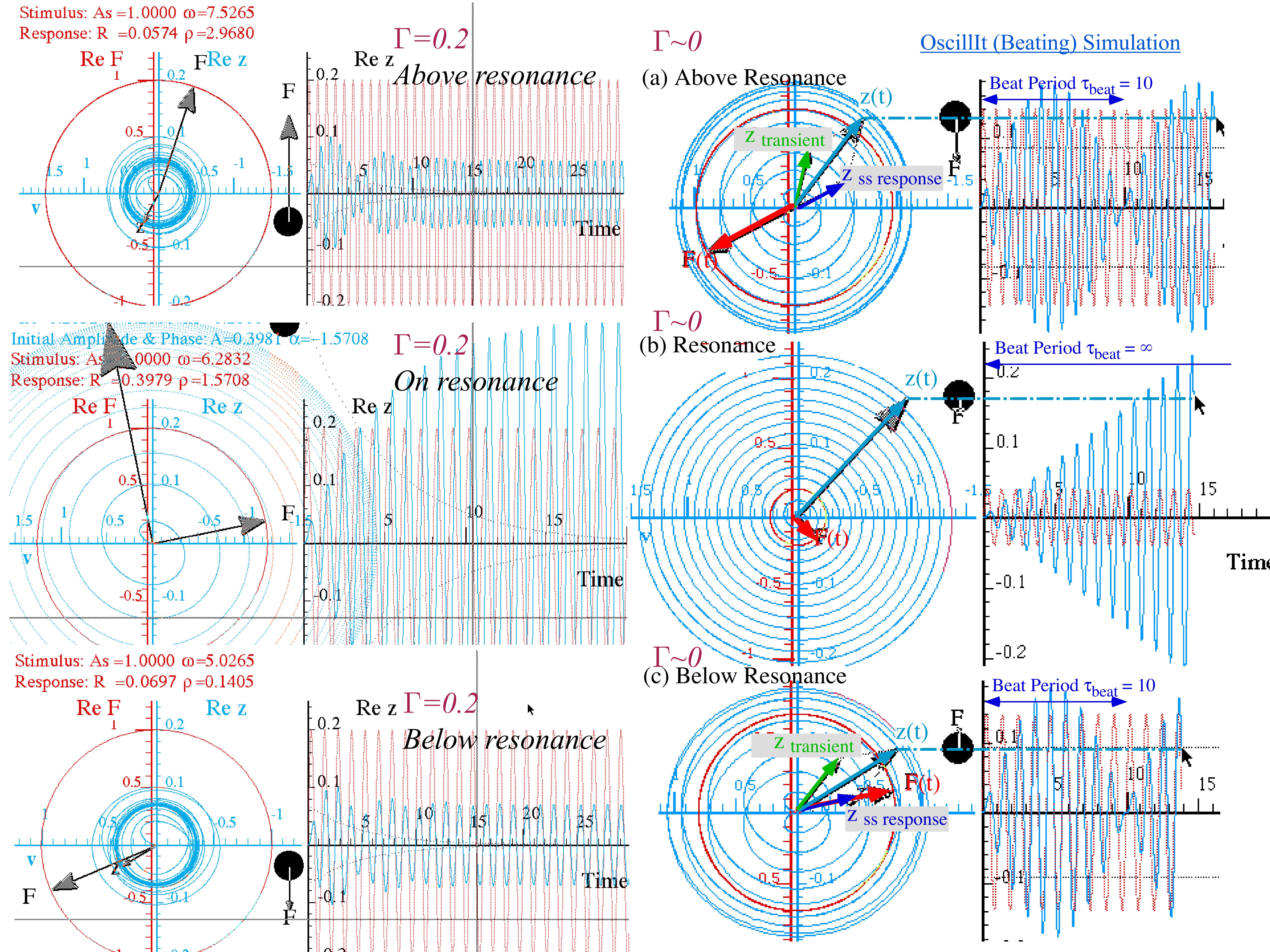

[OscillIt (Way Above Resonance) Simulation]




[OscillIt (On Resonance) Simulation]




[OscillIt (Way Below Resonance) Simulation]



OscillIt (Beating) Simulation
Stimulus: As =1.0000 ω=7.5265
Response: R =0.0574 ρ=2.9680
Γ=0.2
Above resonance
Γ~0
(a) Above Resonance
z(t)
z transient
z ss response
F(t)
Beat Period τbeat = 10
Initial Amplitude & Phase: A=0.3981 α=-1.5708
Stimulus: As =1.0000 ω=6.2832
Response: R =0.3979 ρ=1.5708
Γ=0.2
On resonance
(b) Resonance
Beat Period τbeat = ∞
Stimulus: As =1.0000 ω=5.0265
Response: R =0.0697 ρ=0.1405
Γ=0.2
Below resonance
(c) Below Resonance
Beat Period τbeat = 10
Re F
Re z
Time

# *Lorentz-Green's Function for high quality FDHO*

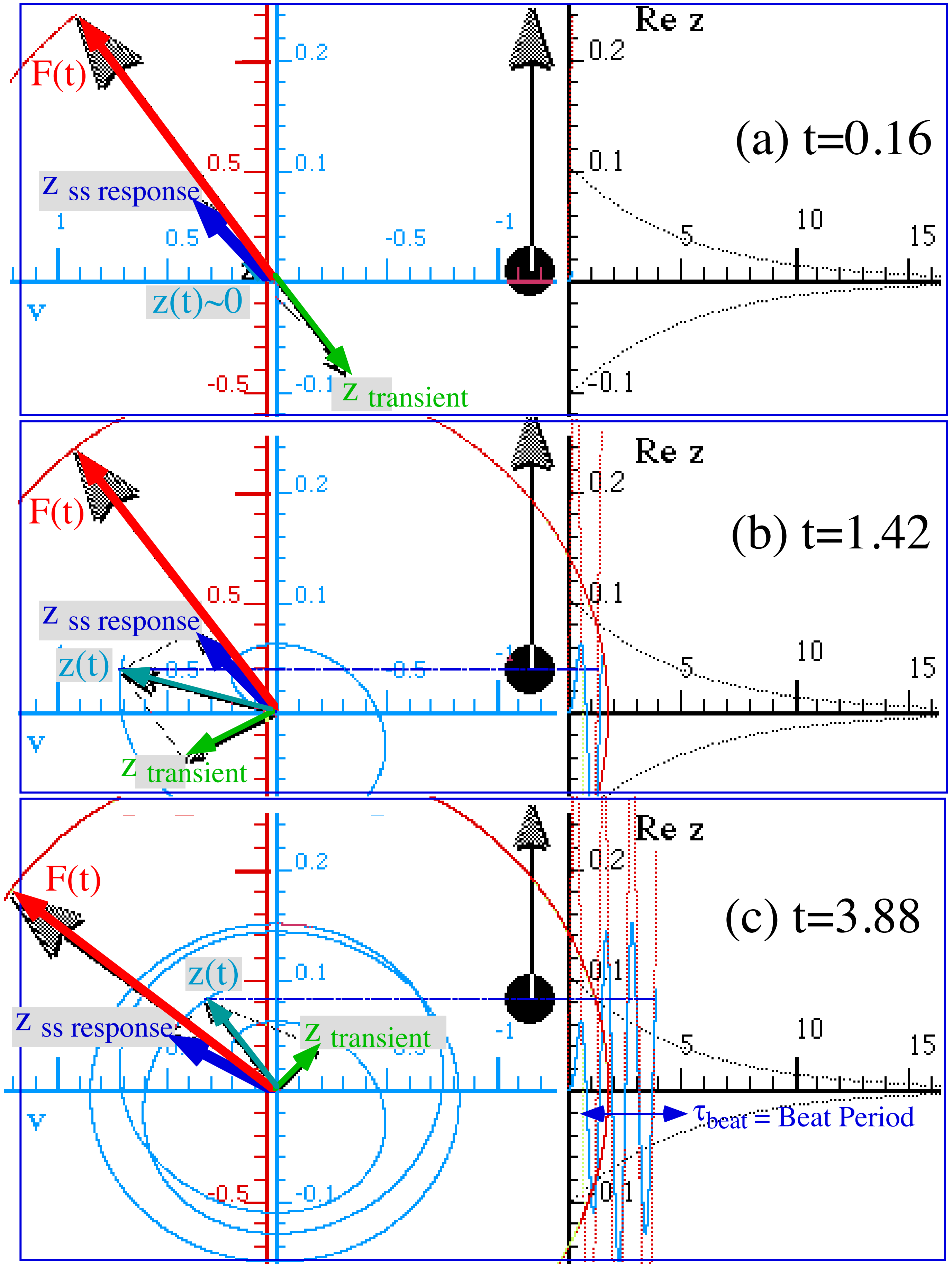



*Fig. 4.2.9 Beat formation.*

*Transient phasor $z_{transient}$ catches up with F-phasor and passes it.*

OscillIt (Beating) Simulation

# Oscillator figures of merit: quality factors Q and q=2πQ

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s = \omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

Amplification factor $q = \omega_0/2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0/2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

# Oscillator figures of merit: quality factors Q and q=2πQ

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \qquad (\textit{angular quality factor})$$

Amplification factor $q = \omega_0/2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0/2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$ ).

$$\begin{pmatrix} t_{5\%} = 3/\Gamma = \textit{Lifetime} \\ \text{for decaying oscillator} \\ \text{to lose 95\% of} \\ \text{amplitude} \end{pmatrix}_{times} \left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{matrix} \text{number } n_{5\%} \\ \text{of oscillations} \\ \text{in a } t_{5\%} \textit{ Lifetime} \end{matrix}$$

# Oscillator figures of merit: quality factors Q and q=2πQ

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

Amplification factor $q = \omega_0 / 2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0 / 2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

$$\underbrace{\begin{pmatrix}\text{for decaying oscillator}\\ \text{to lose 95\% of}\\ \text{amplitude}\end{pmatrix}}_{t_{5\%} = 3/\Gamma = \textit{Lifetime}} \textit{times} \left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{matrix}\text{number } n_{5\%}\\ \text{of oscillations}\\ \text{in a } t_{5\%} \textit{ Lifetime}\end{matrix}$$

$$n_{5\%} = t_{5\%}\upsilon_0 = \frac{3}{\Gamma} \cdot \frac{\omega_0}{2\pi} \cong \frac{\omega_0}{2\Gamma} = q$$

The "Heartbeat Count" measure of lifetime

# Oscillator figures of merit: quality factors Q and $q=2\pi Q$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \qquad (\textit{angular quality factor})$$

Amplification factor $q = \omega_0/2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0/2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

$$\begin{pmatrix} t_{5\%} = 3/\Gamma = \textit{Lifetime} \\ \text{for decaying oscillator} \\ \text{to lose 95\% of} \\ \text{amplitude} \end{pmatrix} \textit{times} \left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{matrix} \text{number } n_{5\%} \\ \text{of oscillations} \\ \text{in a } t_{5\%} \textit{ Lifetime} \end{matrix}$$

$$n_{5\%} = t_{5\%}\upsilon_0 = \frac{3}{\Gamma} \cdot \frac{\omega_0}{2\pi} \cong \frac{\omega_0}{2\Gamma} = q$$

The "Heartbeat Count" measure of lifetime

Energy decay (proportional to the square of oscillator amplitude): $\left(e^{\Gamma t}\right)^2 = e^{-2\Gamma t}$

$$dE = -2\Gamma E$$

# Oscillator figures of merit: quality factors Q and q=2πQ

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/(2\Gamma\omega_0)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

Amplification factor $q = \omega_0/2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0/2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

$t_{5\%} = 3/\Gamma = Lifetime$ (for decaying oscillator to lose 95% of amplitude) *times* $\left(\upsilon_0 = \frac{\omega_0}{2\pi}\right)$ = number $n_{5\%}$ of oscillations in a $t_{5\%}$ *Lifetime*

$$n_{5\%} = t_{5\%}\upsilon_0 = \frac{3}{\Gamma}\cdot\frac{\omega_0}{2\pi} \cong \frac{\omega_0}{2\Gamma} = q$$

The "Heartbeat Count" measure of lifetime

Energy decay (proportional to the square of oscillator amplitude): $\left(e^{\Gamma t}\right)^2 = e^{-2\Gamma t}$ $\qquad dE = -2\Gamma E$

Relative amount of energy lost each cycle period $= \tau_0\left(\frac{-dE}{E}\right) = \frac{2\Gamma}{\upsilon_0} \equiv \frac{1}{Q} = \frac{2\pi}{q}$ $\quad \left(\tau_0 = \frac{1}{\upsilon_0}\right)$

$$Q = (\textit{Standard amplitude quality factor}) = \frac{q}{2\pi}$$

Click to see p.18.

Click to see p.35.

# Oscillator figures of merit: Uncertainty 1/q

To see a beat we need $\tau_{\text{half-beat}}$ to be less than $\tau_{5\%}$ or 3/Γ. (Here we approximate π~3.0, again.)

$$\pi / |\omega_s - \omega_0| < 3/\Gamma \qquad\qquad |\omega_s - \omega_0| > \Gamma$$

This means ω-detuning error is greater than or equal to the decay rate Γ.

Any detuning less than Γ is virtually undetectable.

Total ω uncertainty is ±Γ or twice Γ (that is: FWHM $\Delta\omega = 2\Gamma$ ).

Linear frequency uncertainty is:

$$\Delta\upsilon = \Delta\omega / 2\pi = \Gamma / \pi$$

The *relative frequency uncertainty* $\frac{2\Gamma}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \frac{1}{q} = \frac{\Delta\upsilon}{\upsilon_0}$

is the *inverse* of the *angular quality factor q*.

$$Q = (\textit{Standard amplitude quality factor}) = \frac{q}{2\pi}$$

If we think of the 5% or 4.321% lifetime of a musical note as its time uncertainty Δt, then:

$$\Delta t \Delta\upsilon = 3/\pi \approx 1$$

$$\Delta t = t_{5\%} = 3/\Gamma \qquad\qquad \Delta t = t_{4.321\%} = \pi / \Gamma$$

*Very precise measures of imprecision*

# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow{\omega_s \to \omega_0} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

*Complex detuning-decay* $\delta = \Delta - i\Gamma$ variable $\delta$ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

*Complex detuning-decay* $\delta = \Delta - i\Gamma$ variable $\delta$ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\,L + i\,\mathrm{Im}\,L = \frac{\Delta}{\Delta^2 + \Gamma^2} + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = |L|^2\,\Delta + i\,|L|^2\,\Gamma$$

# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

*Complex detuning-decay* $\delta = \Delta - i\Gamma$ variable $\delta$ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\, L \quad + i\,\mathrm{Im}\, L \quad = \quad \frac{\Delta}{\Delta^2 + \Gamma^2} \quad + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = \quad |L|^2 \Delta \quad + i|L|^2 \Gamma$$

$$= |L| e^{i\rho} = |L|\cos\rho + i|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where:} |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

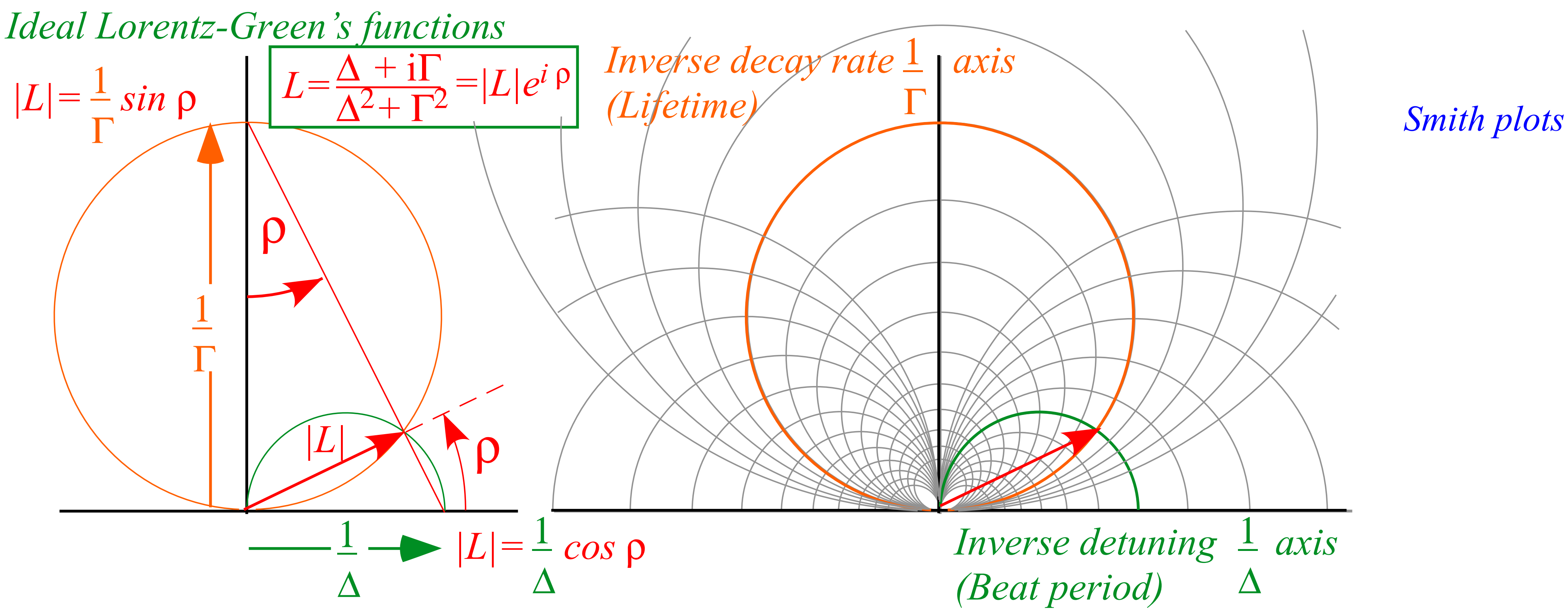
*Complex detuning-decay* $\delta = \Delta - i\Gamma$ variable $\delta$ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\,L + i\,\mathrm{Im}\,L = \frac{\Delta}{\Delta^2 + \Gamma^2} + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = |L|^2\,\Delta + i\,|L|^2\,\Gamma$$

$$= |L|\,e^{i\rho} = |L|\cos\rho + i\,|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

$$|L| = \frac{1}{\Gamma}\sin\rho$$

$$|L| = \frac{1}{\Delta}\cos\rho$$



# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

*Complex detuning-decay* $\delta = \Delta - i\Gamma$ variable $\delta$ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\,L \quad + i\,\mathrm{Im}\,L \quad = \frac{\Delta}{\Delta^2 + \Gamma^2} \quad + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = \quad |L|^2\,\Delta \quad + i\,|L|^2\,\Gamma$$

$$= |L|\,e^{i\rho} = |L|\cos\rho + i\,|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

$$|L| = \frac{1}{\Gamma}\sin\rho$$

$$|L| = \frac{1}{\Delta}\cos\rho$$

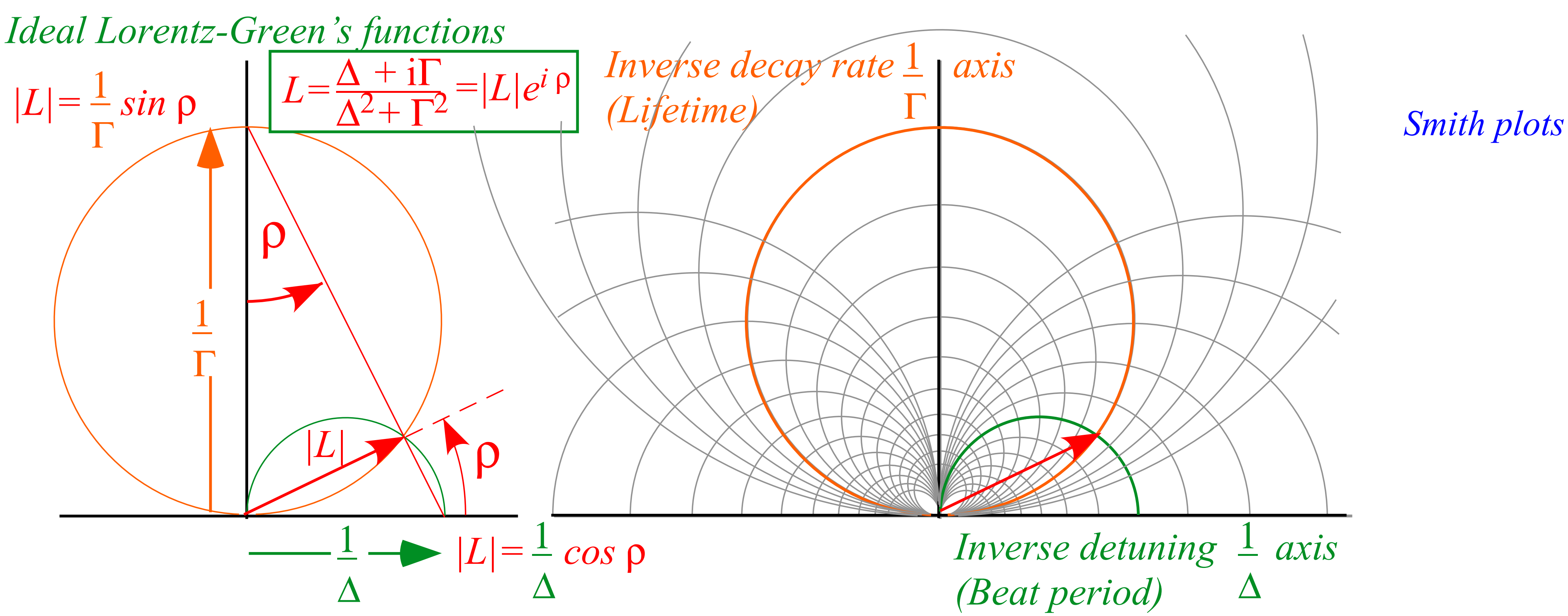



*Fig. 4.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time* $1/\Gamma$ *vs. beat-period* $1/\Delta$ *coordinates)*

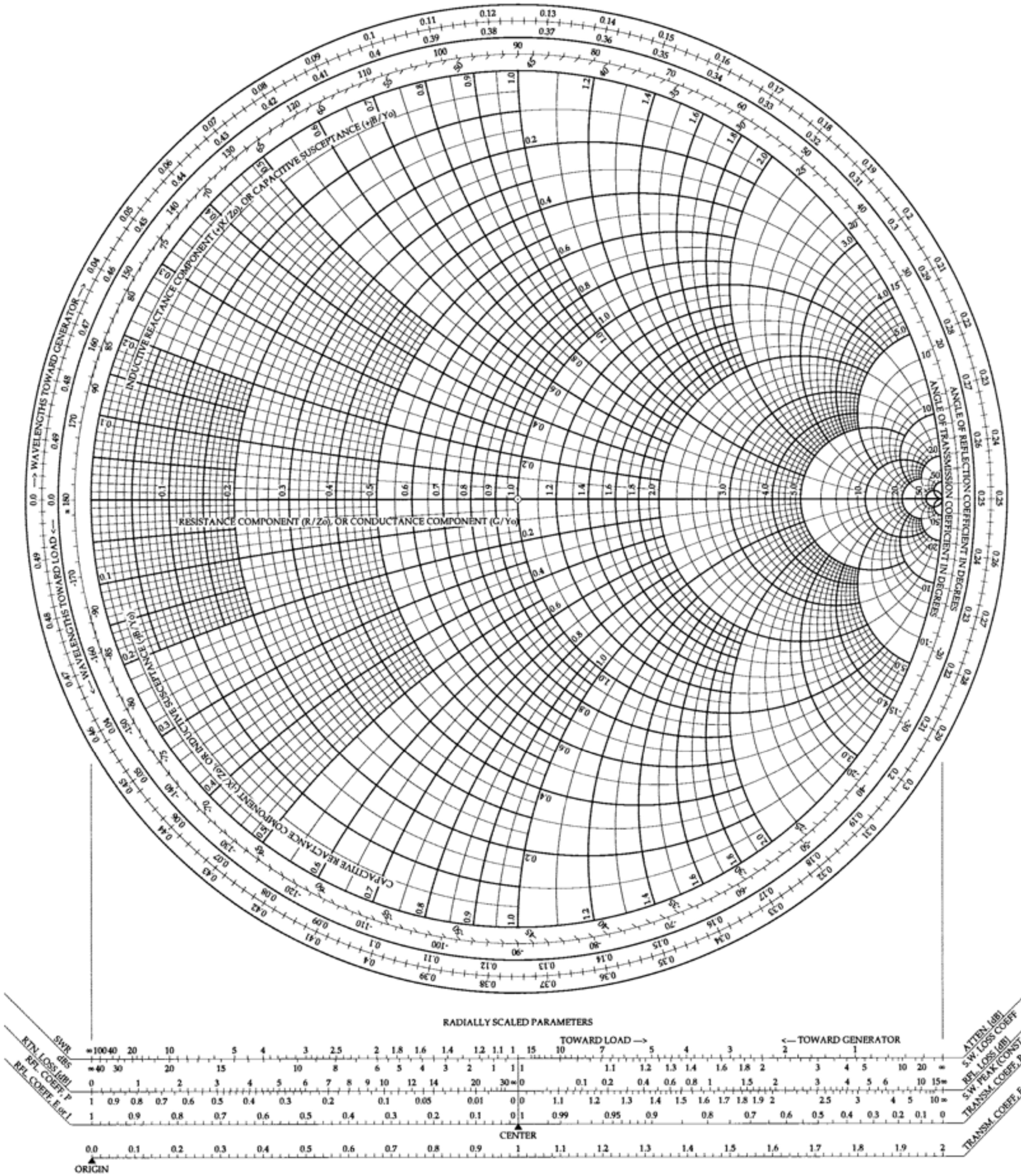
Constant $\Delta$ and $\Gamma$ curves in Fig. 4.2.13 are orthogonal circles of *1/z- dipolar coordinates*. Recall Fig. 1.10.11.

SMITH CHART (Invented by Phillip H. Smith 1905-1987)

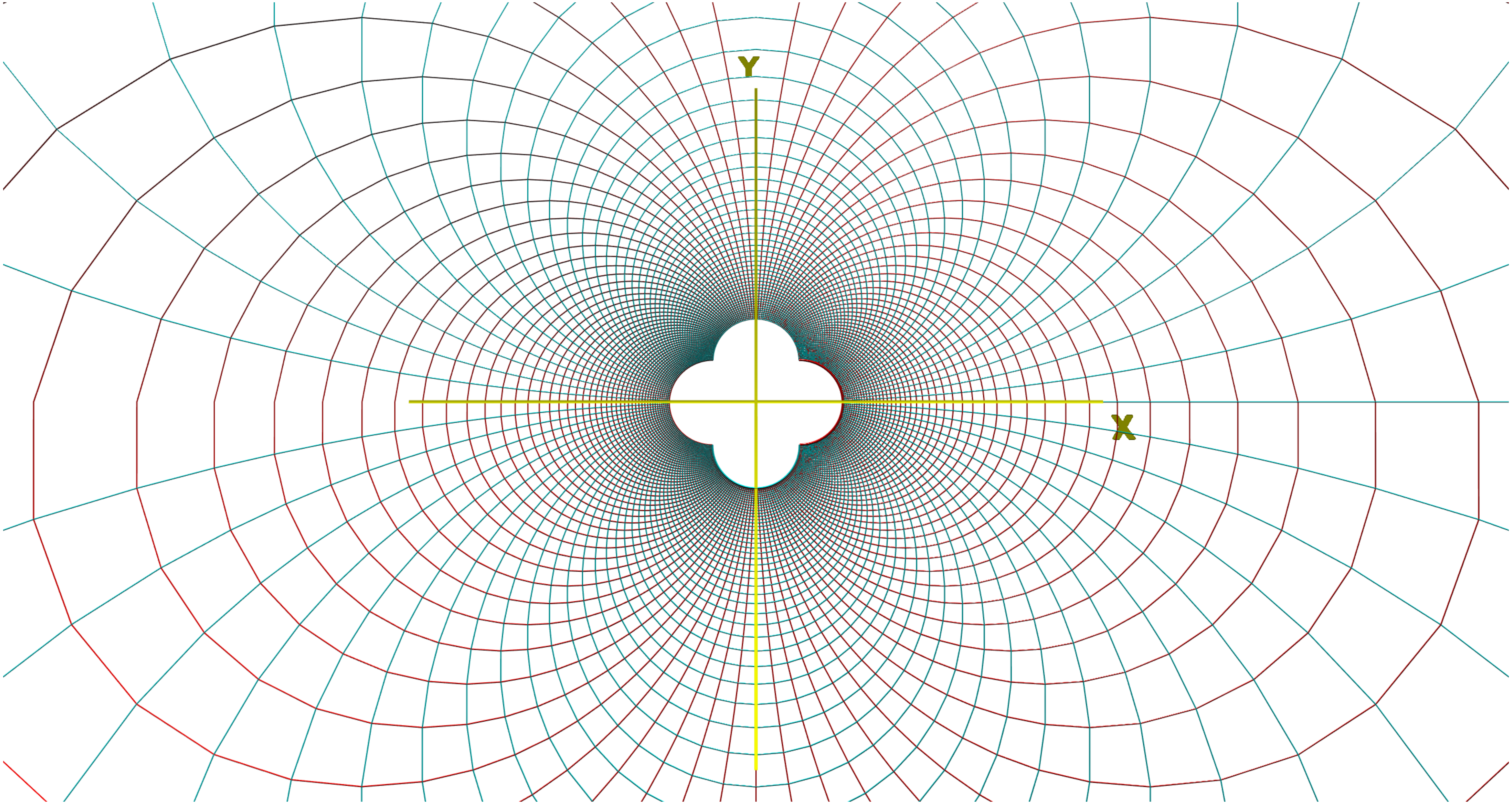
*An FDHO Green's Function Slide rule*

*A plot of f(z) =1/z*

*For wavy "Ohm's Laws" V =I·Z I=V/Z*




Smith plot: Graph paper



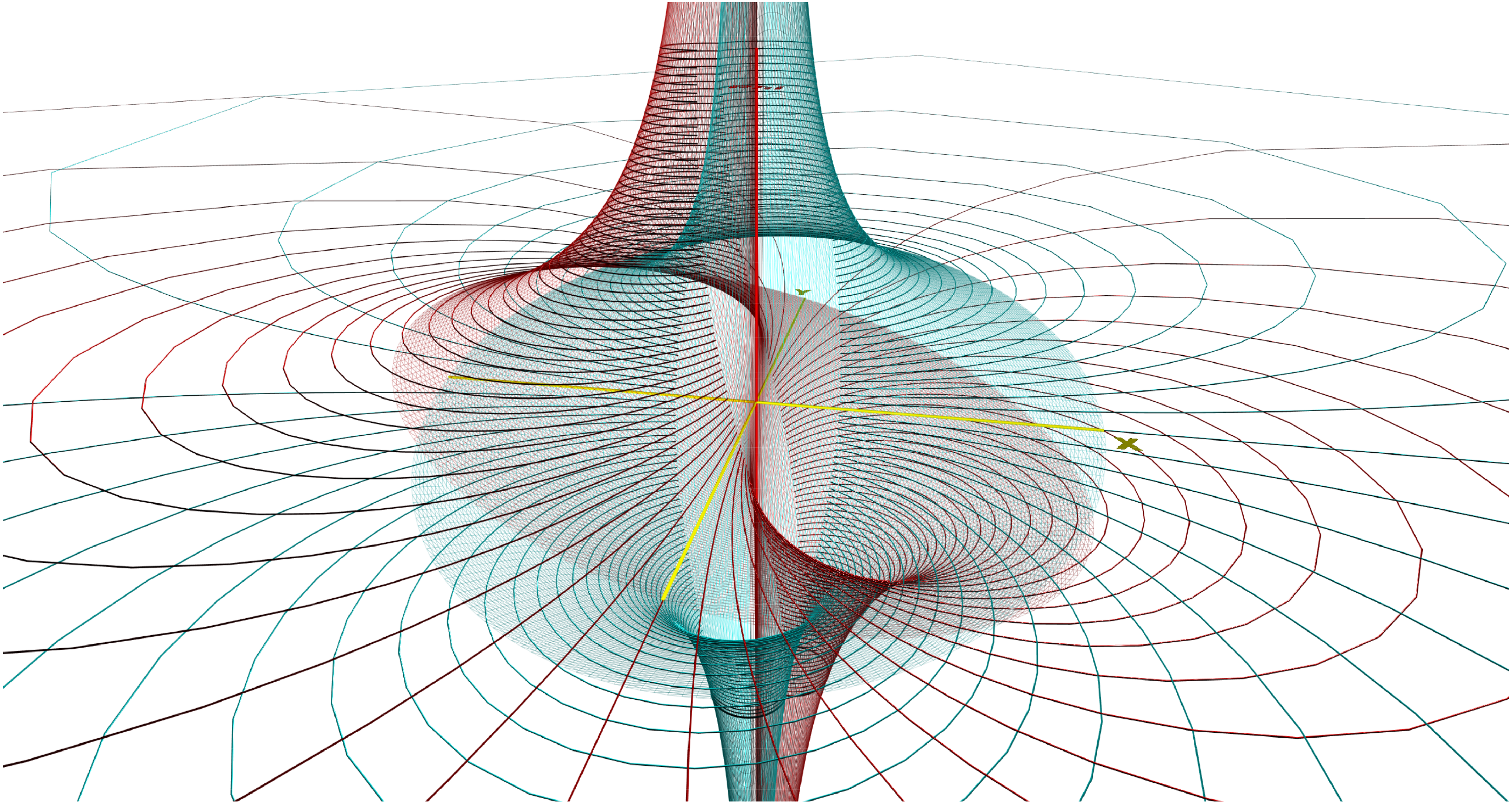

https://modphys.hosted.uark.edu/video/AnalyIt_0-3.webm

https://modphys.hosted.uark.edu/video/AnalyIt_4-1.webm

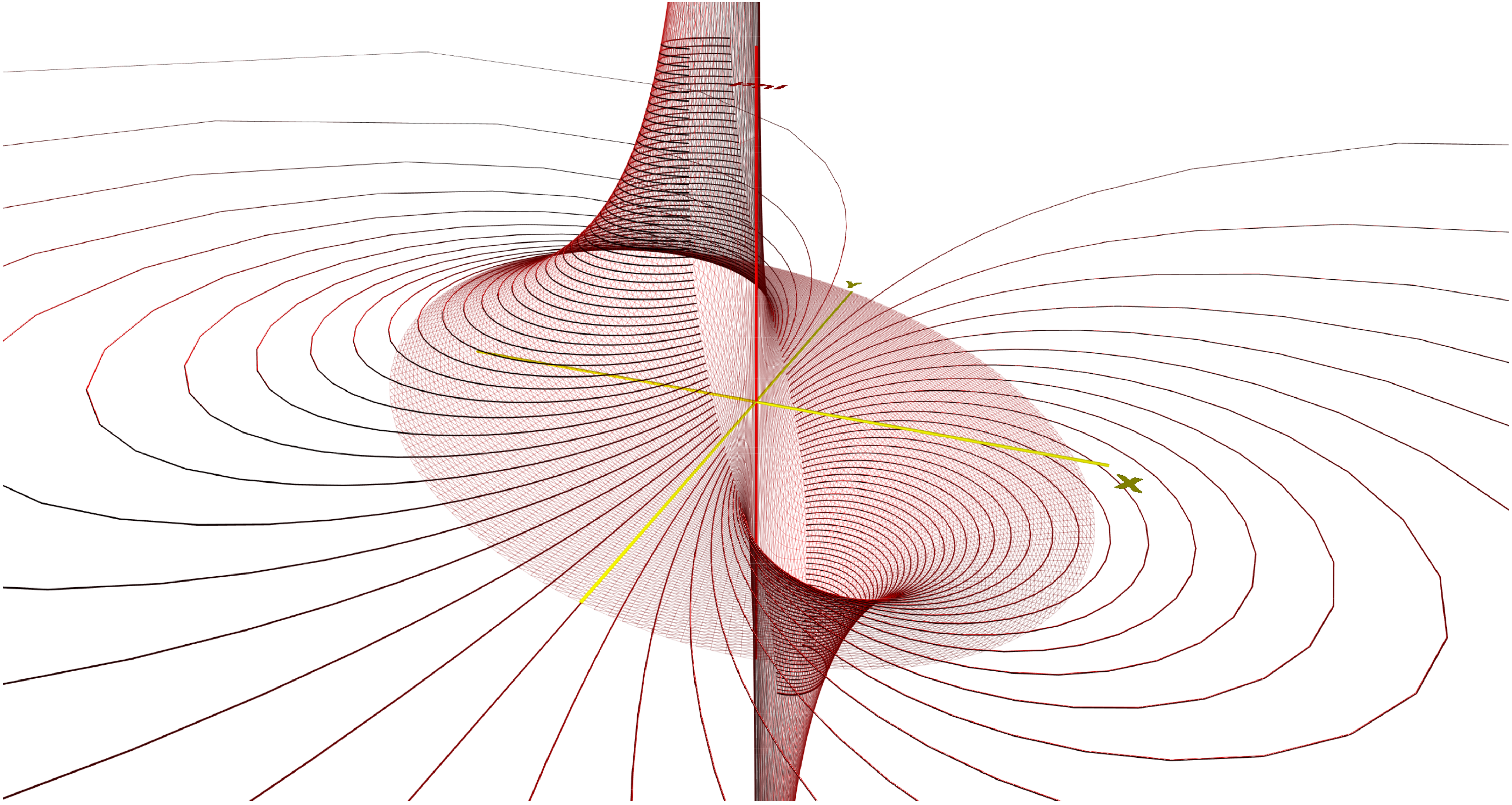
https://modphys.hosted.uark.edu/video/AnalyIt_0-2.webm

https://modphys.hosted.uark.edu/video/AnalyIt_3-1.webm

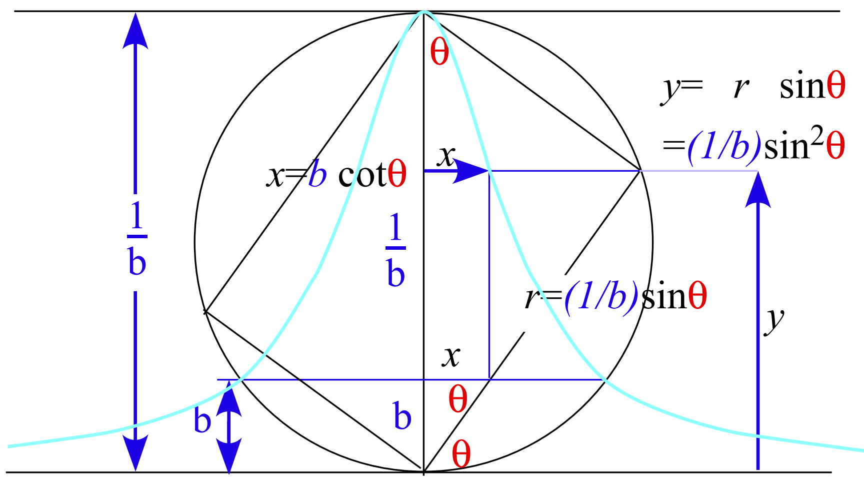
https://modphys.hosted.uark.edu/video/AnalyIt_1-1.webm



y
x


y
x

# *The Common Lorentzian (a.k.a.* The Witch of Agnesi*)*




$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

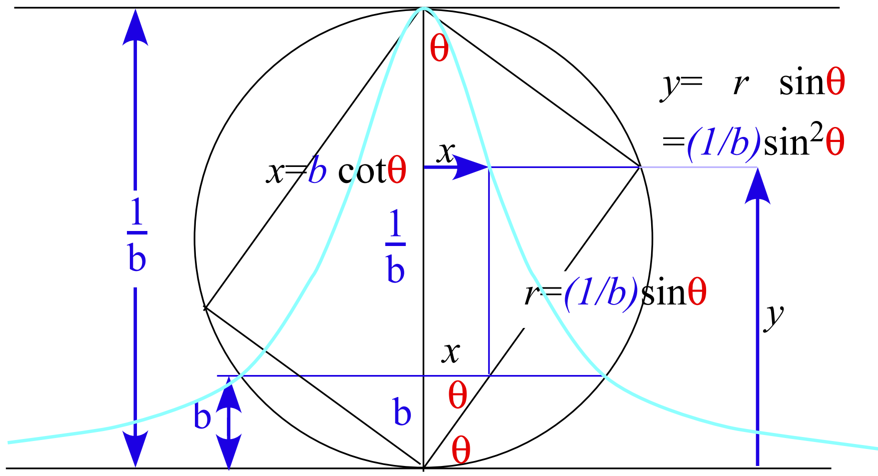
$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*

**Maria Gaetana Agnesi**

| | |
|---|---|
| **Born** | May 16, 1718 |
| **Died** | January 9, 1799 (aged 80) |
| **Residence** | Italy |
| **Nationality** | Italy |
| **Fields** | Mathematics |

# *The Common Lorentzian (a.k.a.* The Witch of Agnesi*)*

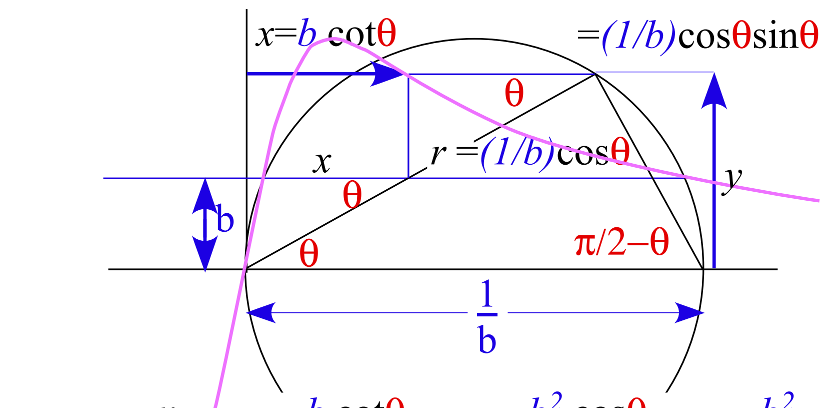



$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*




$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

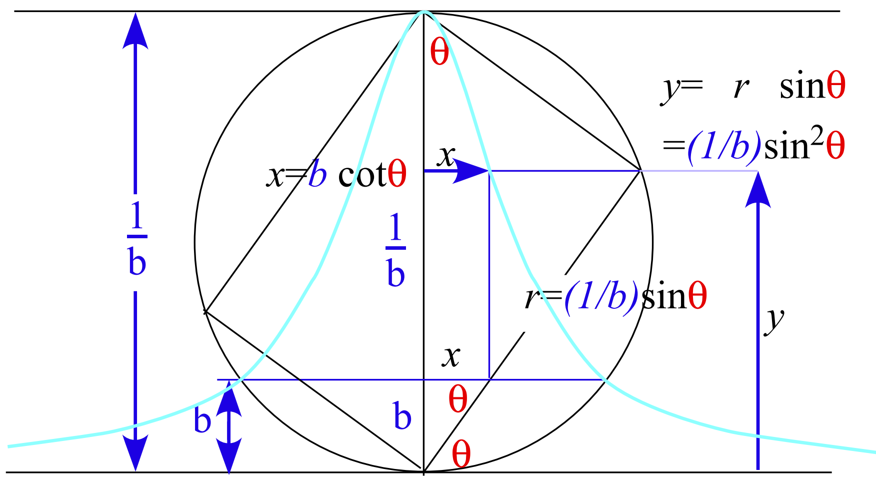
$$y=\frac{x}{x^2+b^2}$$

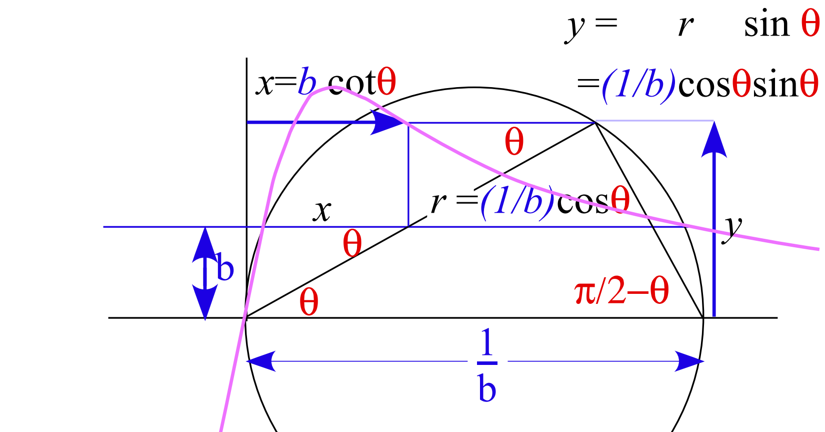
*Common Lorentzian function II.*
*(real "refractory" part)*

**Maria Gaetana Agnesi**

| | |
|---|---|
| **Born** | May 16, 1718 |
| **Died** | January 9, 1799 (aged 80) |
| **Residence** | Italy |
| **Nationality** | Italy |
| **Fields** | Mathematics |

# *The Common Lorentzian (a.k.a.* The Witch of Agnesi*)*




$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}\,b^2$$

$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*

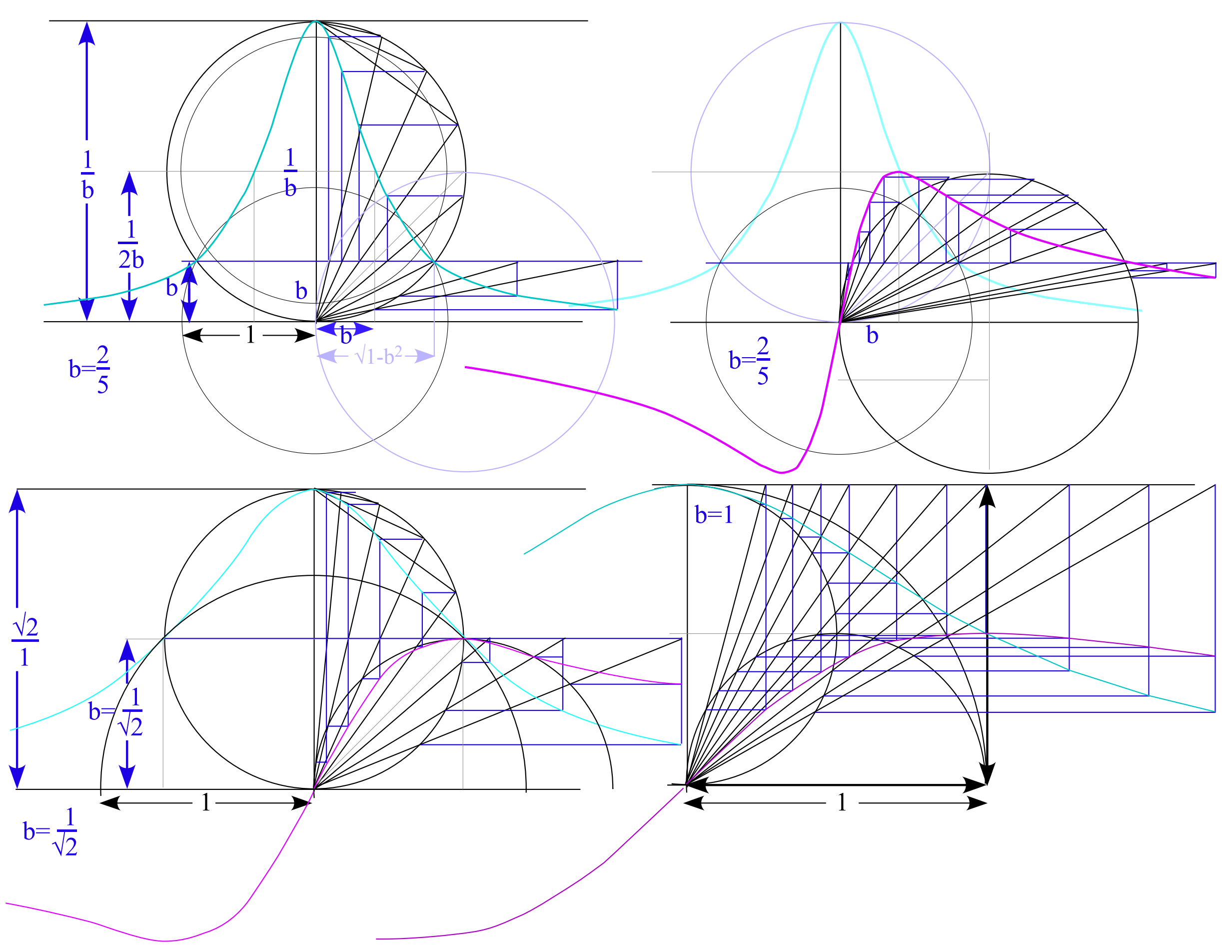
$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

$$y=\frac{x}{x^2+b^2}$$

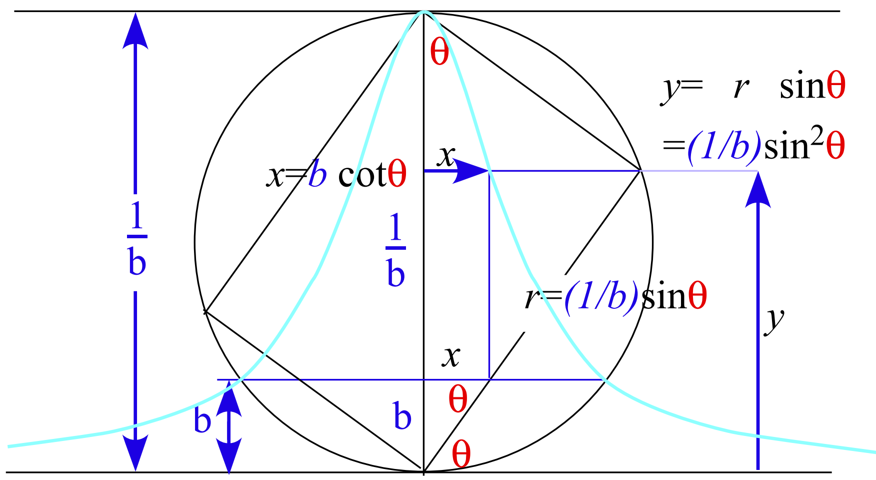
*Common Lorentzian function II.*
*(real "refractory" part)*

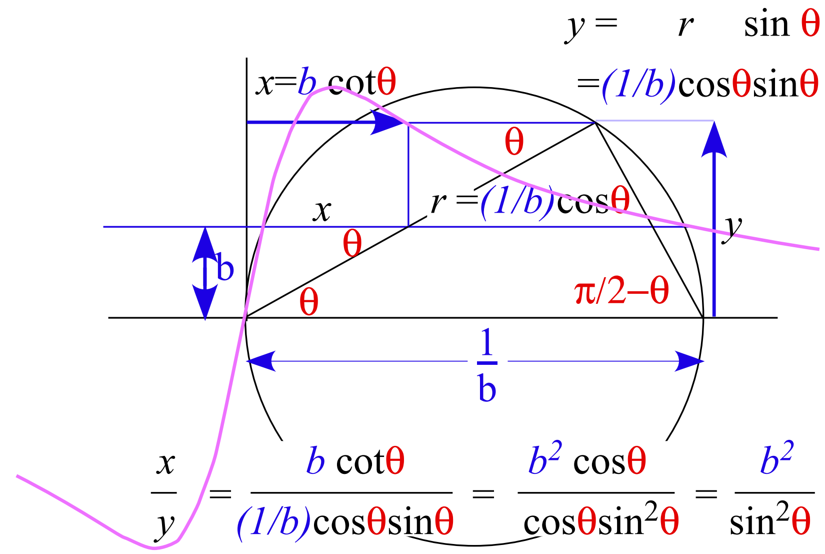
**Maria Gaetana Agnesi**

| | |
|---|---|
| **Born** | May 16, 1718 |
| **Died** | January 9, 1799 (aged 80) |
| **Residence** | Italy |
| **Nationality** | Italy |
| **Fields** | Mathematics |



# *The Common Lorentzian (a.k.a.* The Witch of Agnesi*)*




$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

**Maria Gaetana Agnesi**

| | |
|---|---|
| Born | May 16, 1718 |
| Died | January 9, 1799 (aged 80) |
| Residence | Italy |
| Nationality | Italy |
| Fields | Mathematics |

$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}-b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

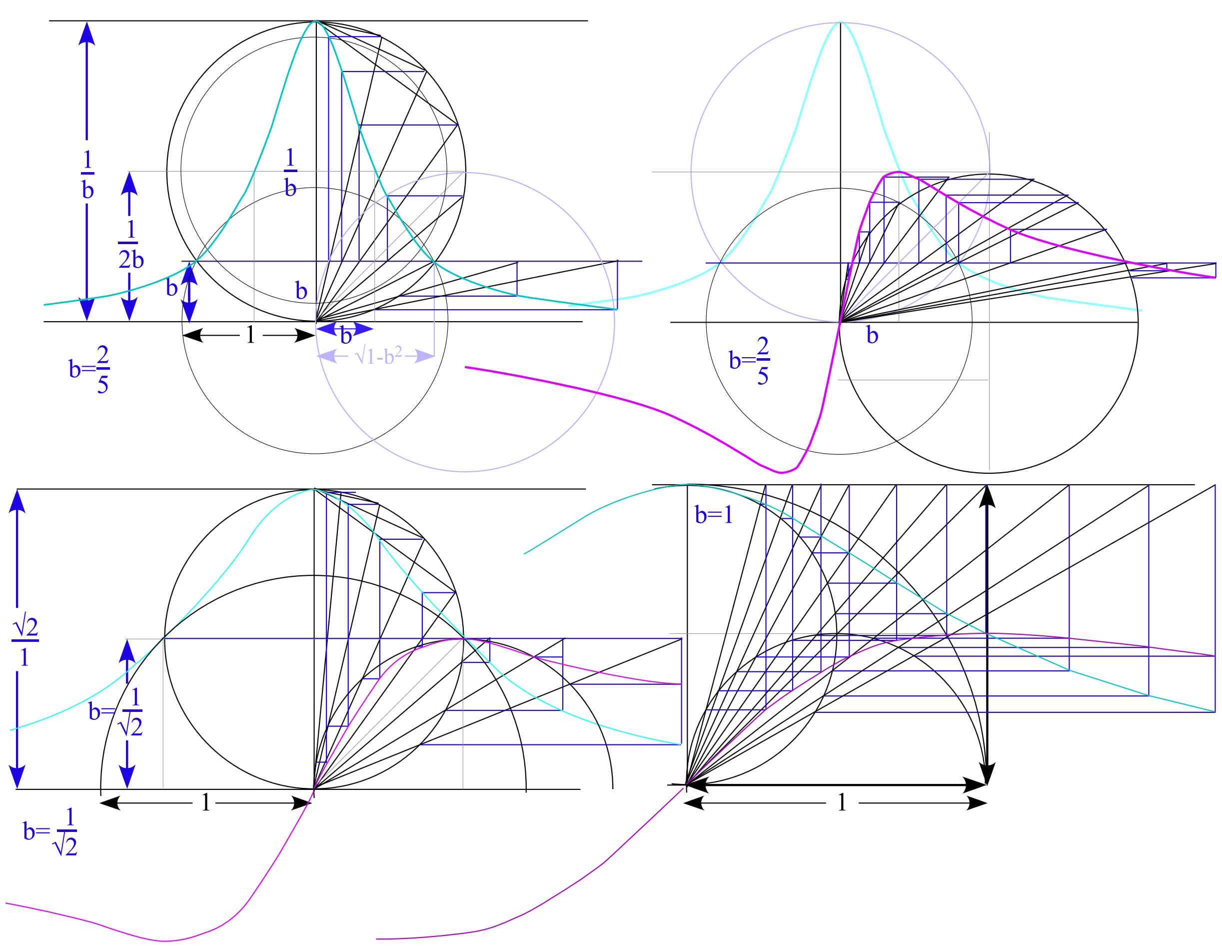
$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

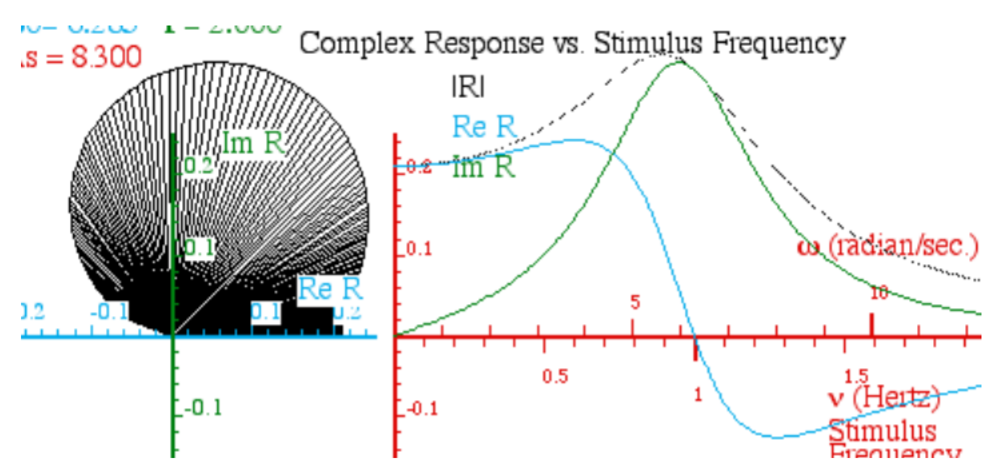
$$y=\frac{x}{x^2+b^2}$$

*Common Lorentzian function II.*
*(real "refractory" part)*

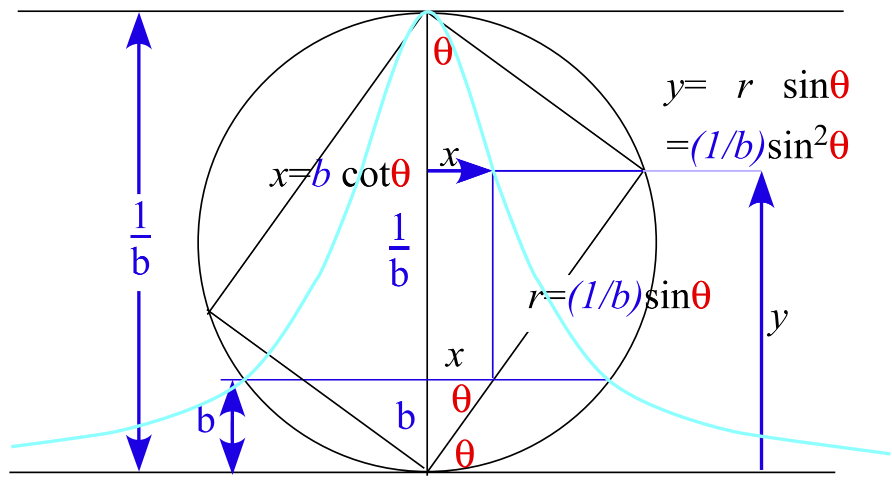


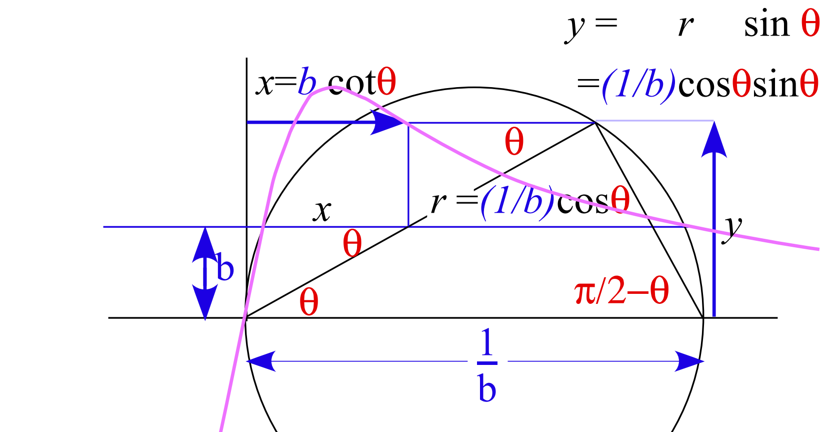

Underlined below are links to the OscillIt Web Simulations

*Compare ideal Lorentzians (Γ=0.2)*
*with a very non-ideal one (Γ=2)*



# *The Common Lorentzian (a.k.a.* The Witch of Agnesi*)*




$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

$$y=\frac{x}{x^2+b^2}$$

*Common Lorentzian function II.*
*(real "refractory" part)*

**Maria Gaetana Agnesi**

| | |
|---|---|
| **Born** | May 16, 1718 |
| **Died** | January 9, 1799 (aged 80) |
| **Residence** | Italy |
| **Nationality** | Italy |
| **Fields** | Mathematics |

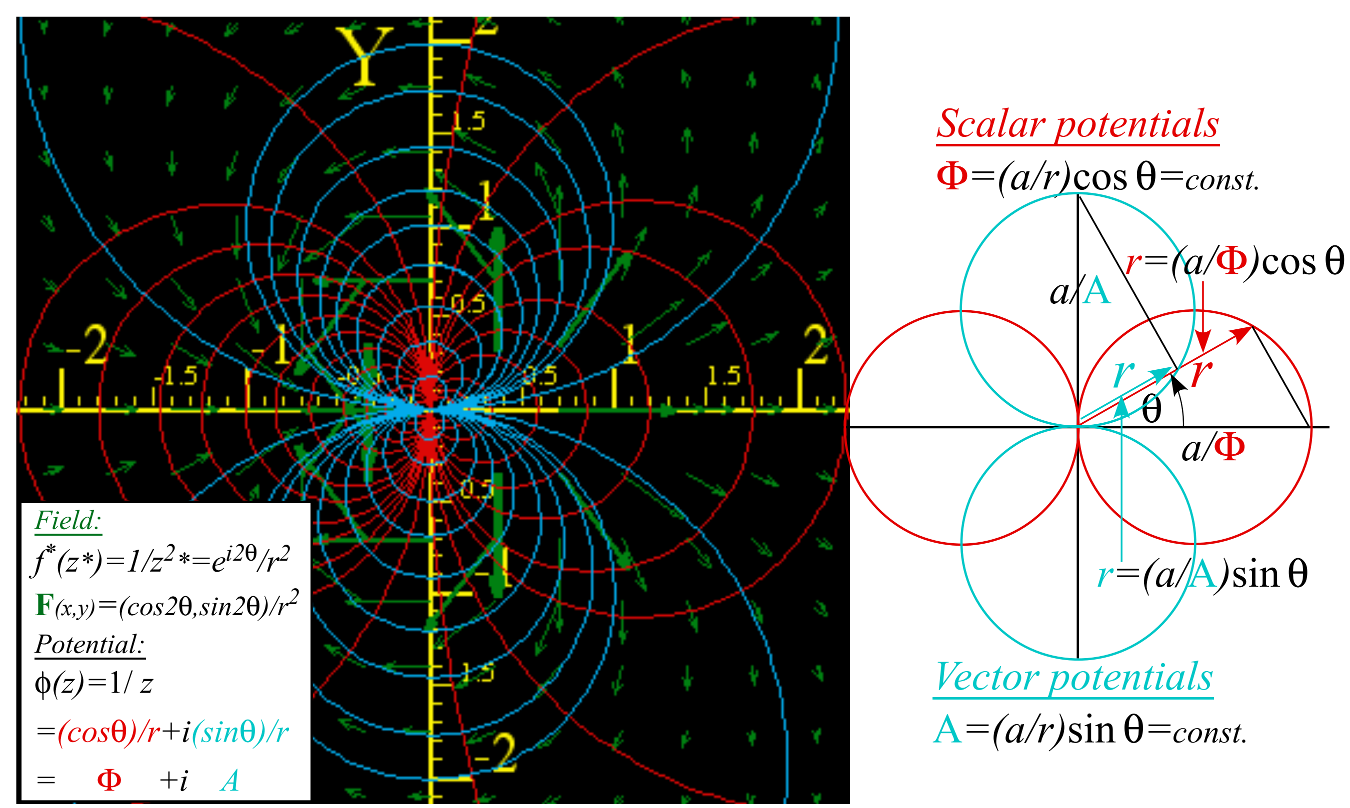
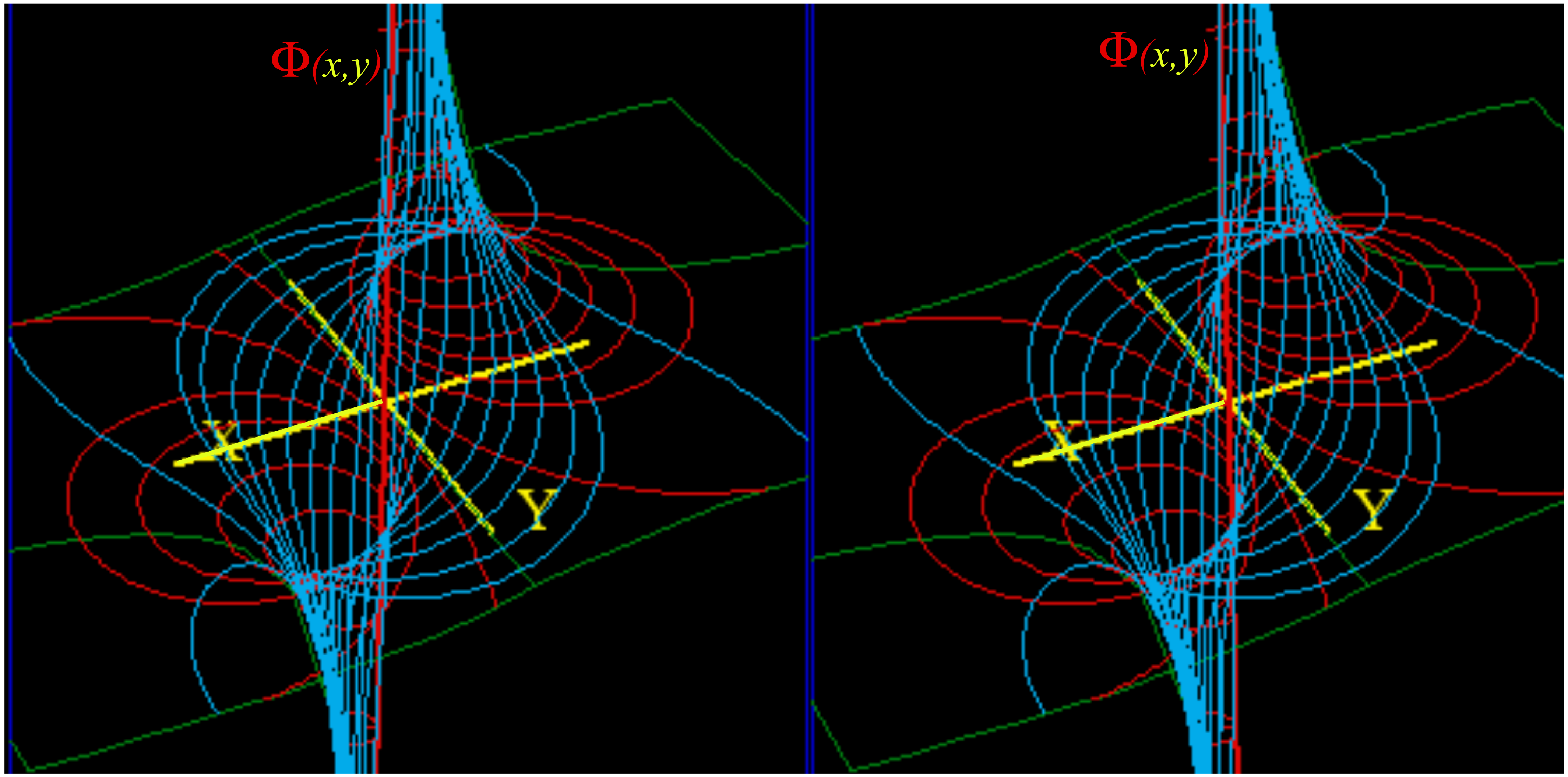
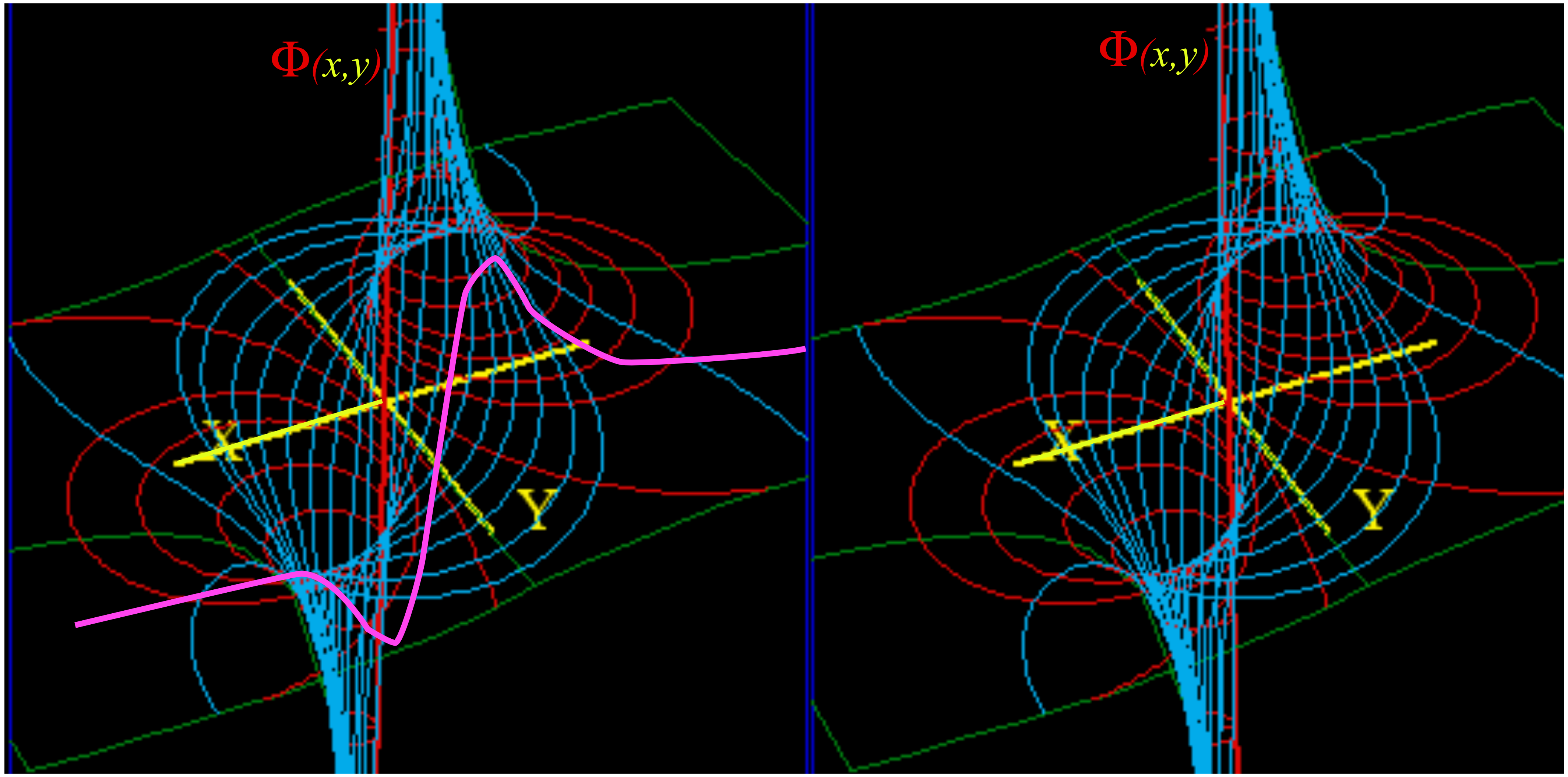


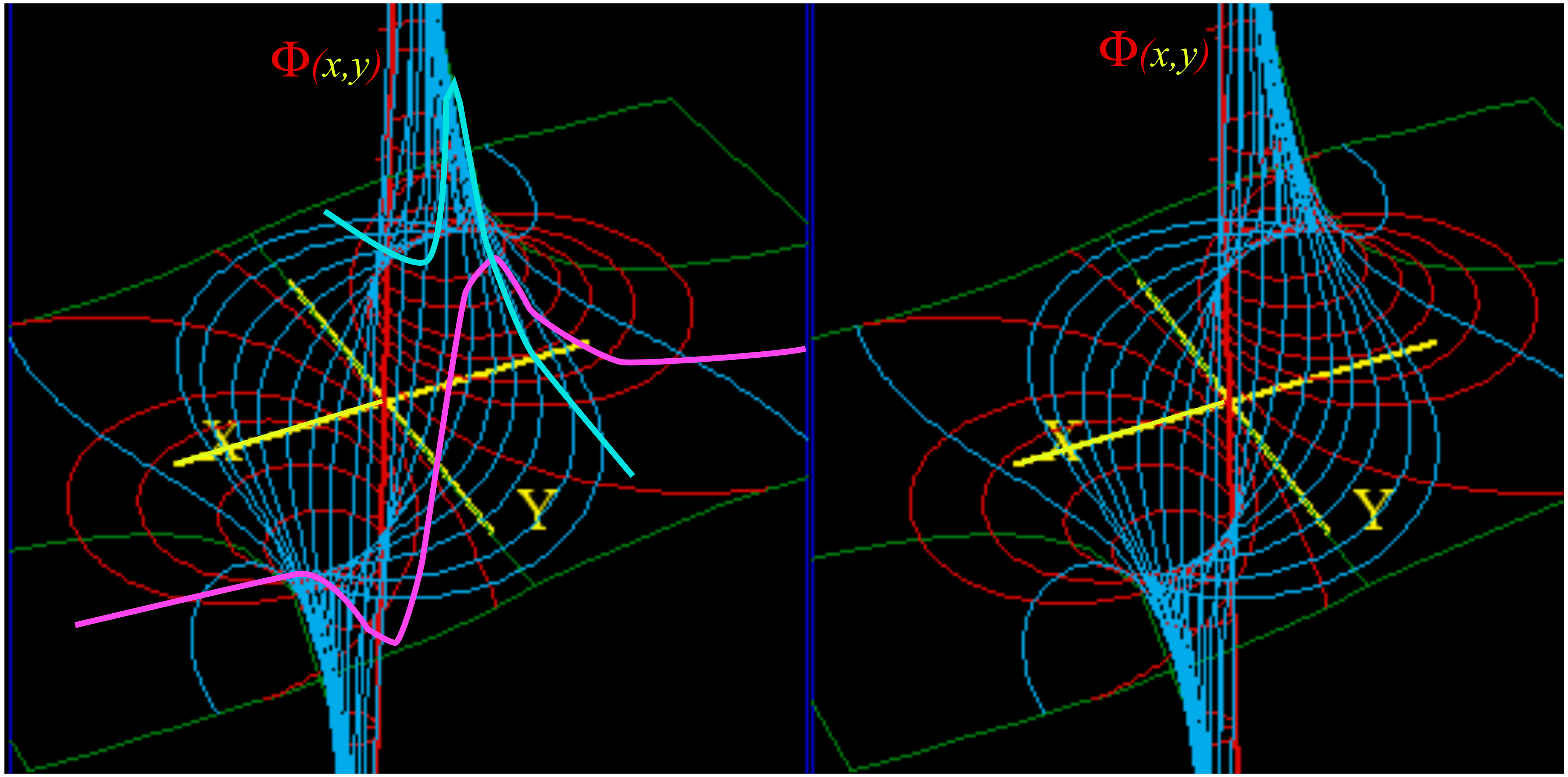

*Fig. 10.11 Dipole* **F***-field* $f(z)=1/z^2$ *and scalar potential (*$\Phi$*=const.)-circles orthogonal to (*$A$*=const.)-circles.*



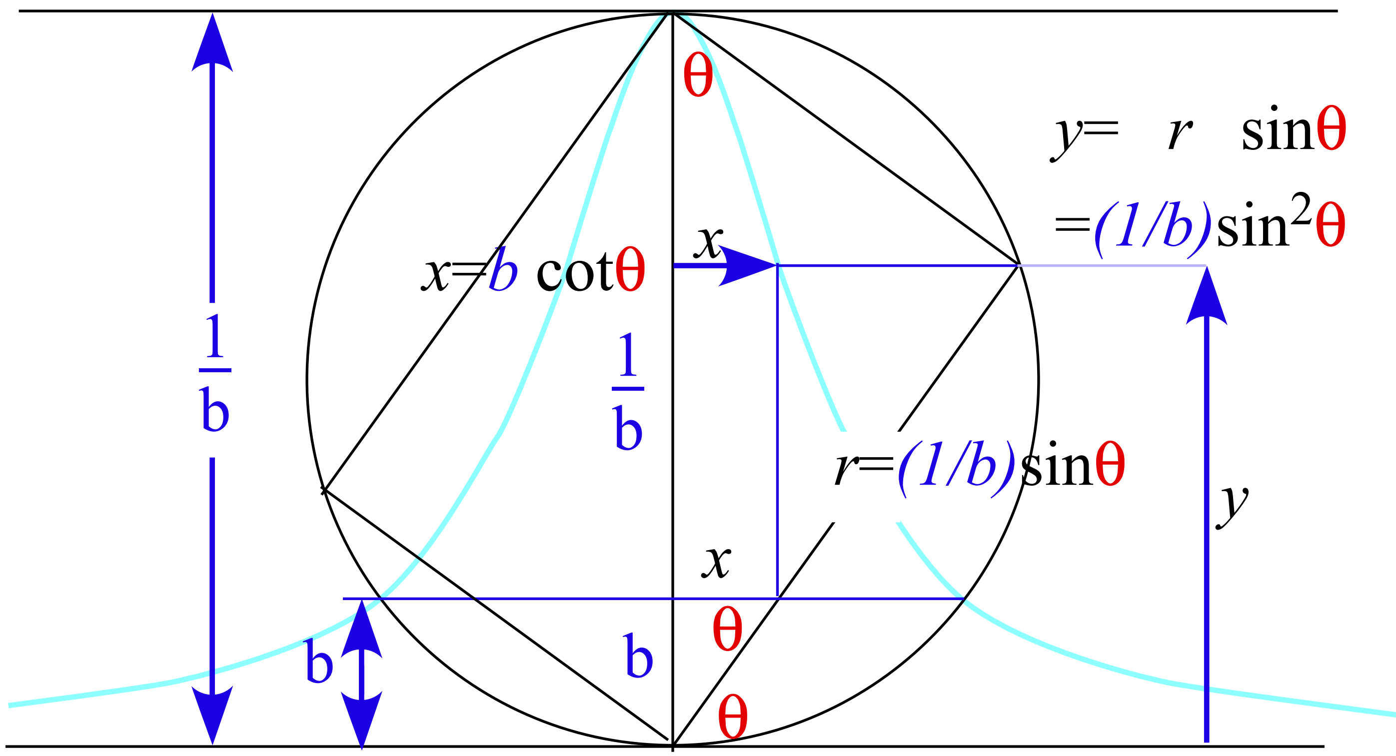

From: Fig. 1.10.12




From: Fig. 1.10.12




From: Fig. 1.10.12

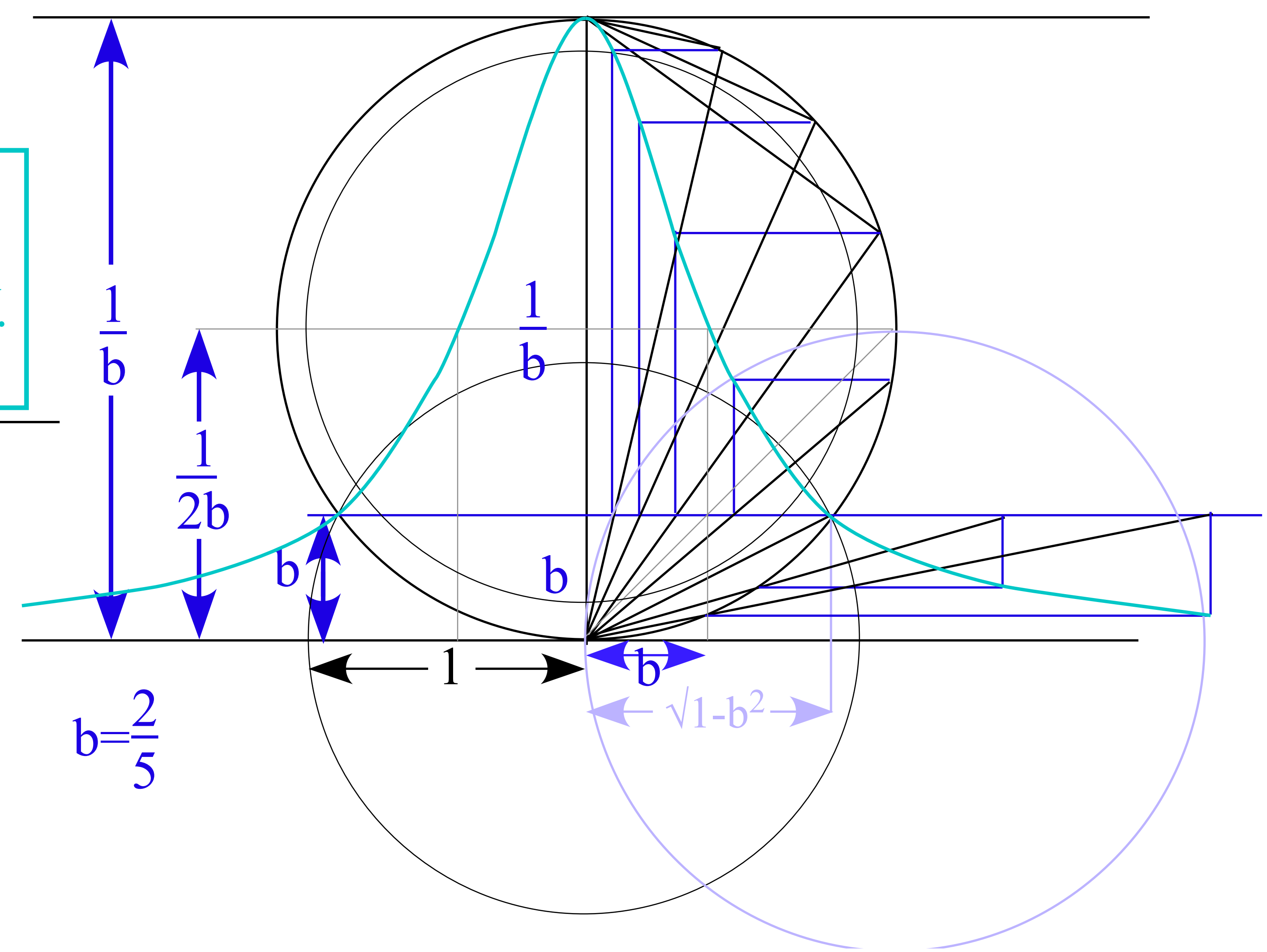



$$x^2 = b^2 \cot^2\theta = b^2 \frac{\cos^2\theta}{\sin^2\theta} = b^2 \frac{1-\sin^2\theta}{\sin^2\theta} = \frac{b^2}{\sin^2\theta} b^2$$

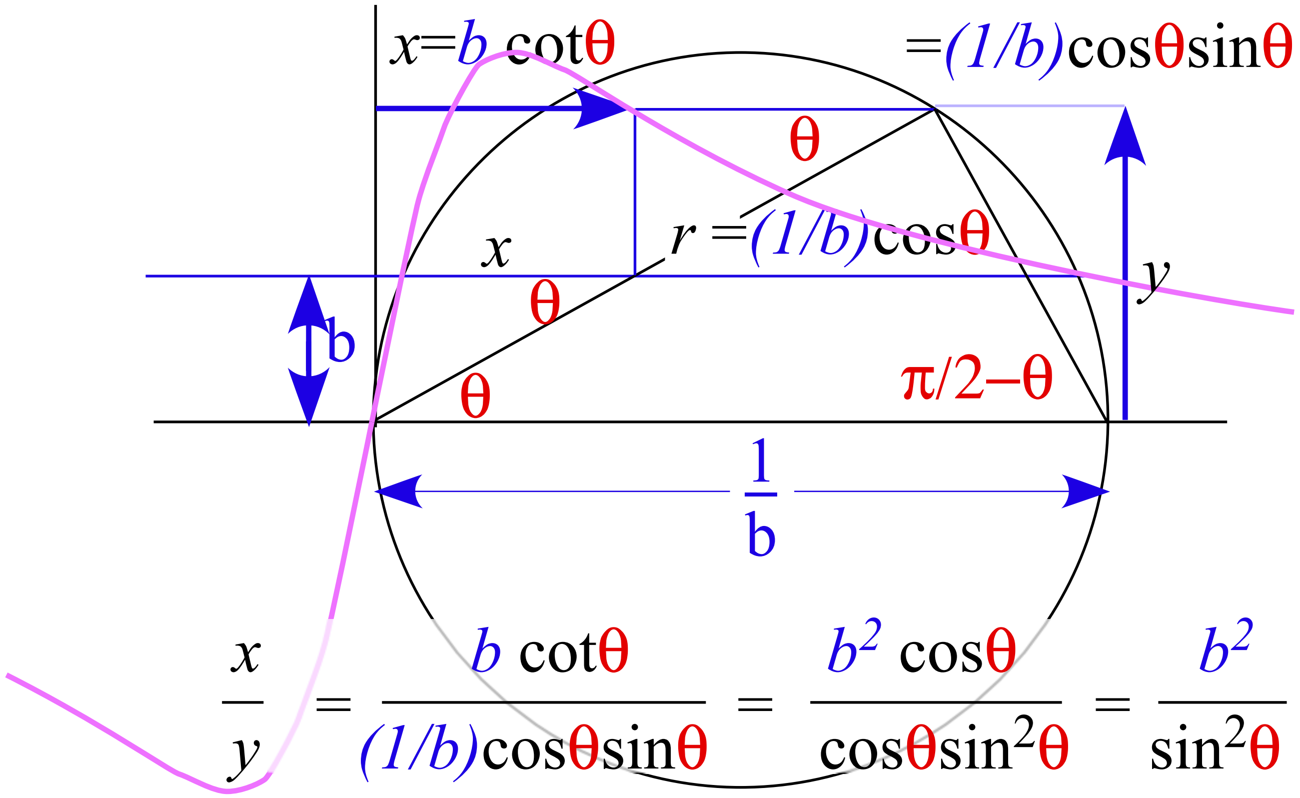
$$x^2 + b^2 = \frac{b^2}{\sin^2\theta} = \frac{b}{y}$$

$$y = \frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary “absorbtive” part)*



$$y = r\sin\theta = (1/b)\cos\theta\sin\theta$$

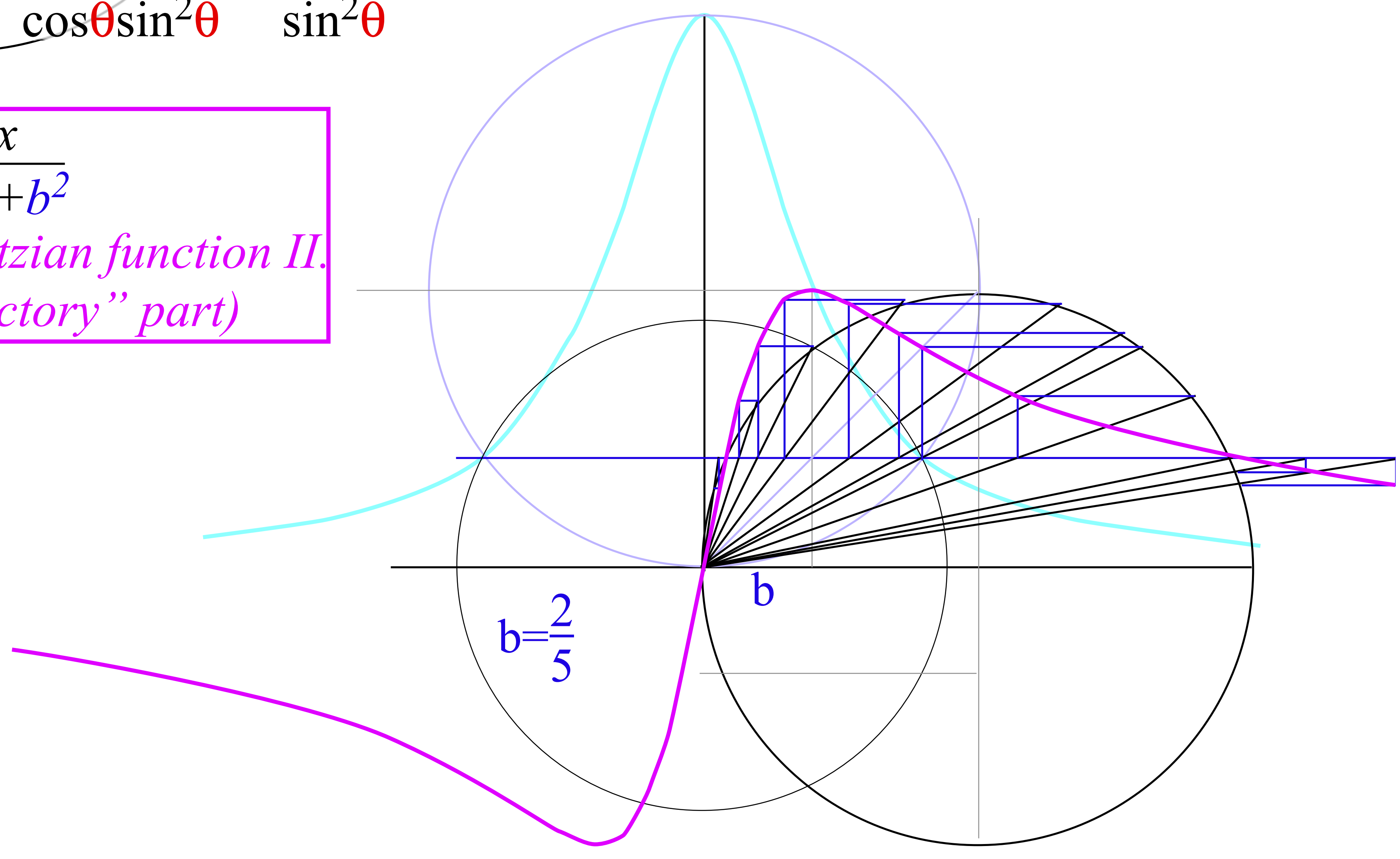


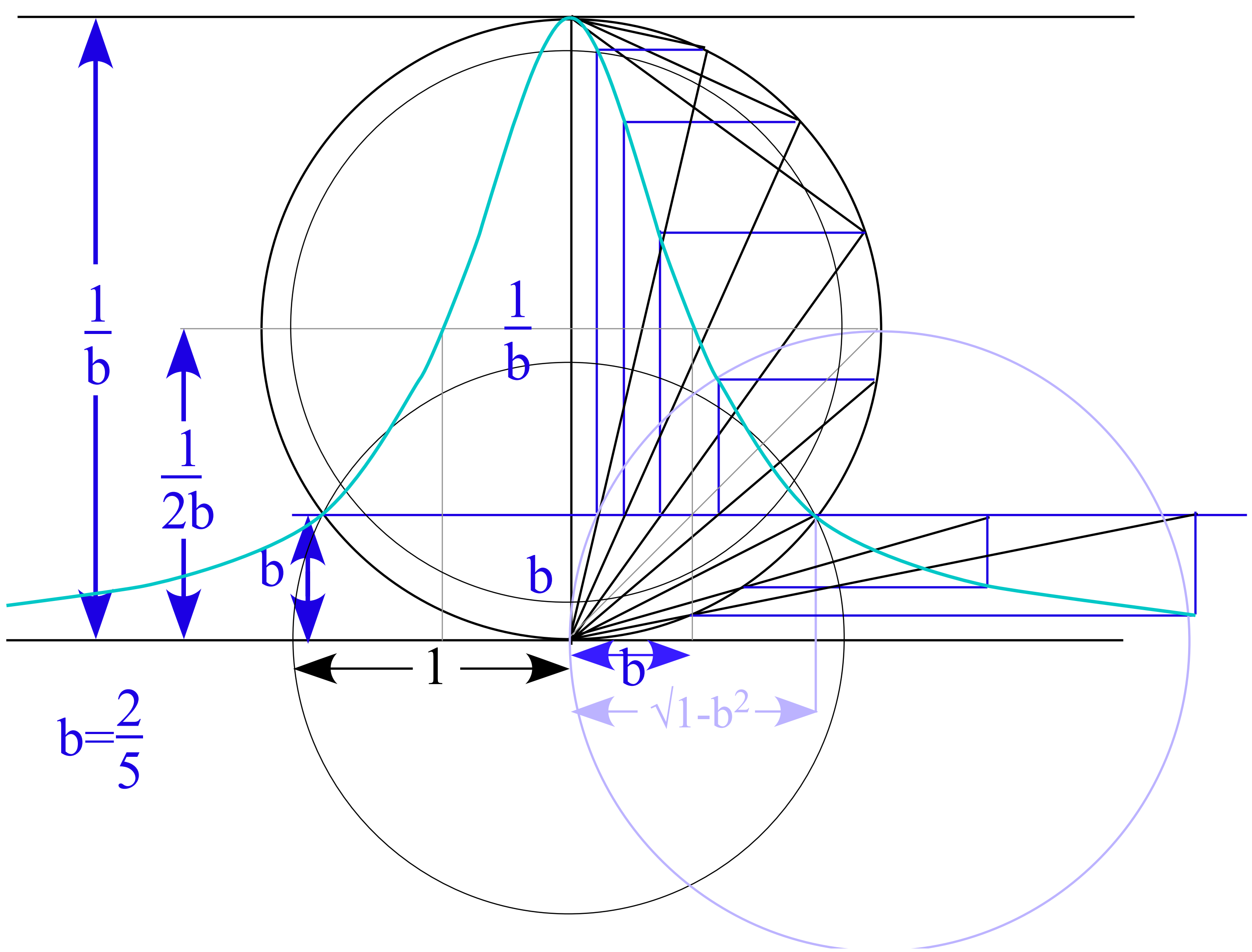

$$\frac{x}{y} = \frac{b\cot\theta}{(1/b)\cos\theta\sin\theta} = \frac{b^2\cos\theta}{\cos\theta\sin^2\theta} = \frac{b^2}{\sin^2\theta}$$

$$x^2+b^2 = \frac{b^2}{\sin^2\theta} = \frac{x}{y}$$

$$y = \frac{x}{x^2+b^2}$$

*Common Lorentzian function II.*
*(real "refractory" part)*





1/b
1/b
1/2b
b
b
1
b
√1-b²
b=2/5