Lecture 20
Thur. 11.03.2016

Introduction to classical oscillation and resonance

(Ch. 1 of Unit 4)

1D forced-damped-harmonic oscillator equations and Green's function solutions

Linear harmonic oscillator equation of motion.

Linear damped-harmonic oscillator equation of motion.

Frequency retardation and amplitude damping

Figure of oscillator merit (the 5% solution $3/\Gamma$ and other numbers)

Linear forced-damped-harmonic oscillator equation of motion.

Phase lag and amplitude resonance amplification

Figure of resonance merit: Quality factor $q=\omega_0/2\Gamma$

Properties of Green's function solutions and their mathematical/physical behavior

Transient solutions vs. Steady State solutions

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

Quality factors: Beat, lifetimes, and uncertainty

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

Common Lorentzian (a.k.a. Witch of Agnesi)

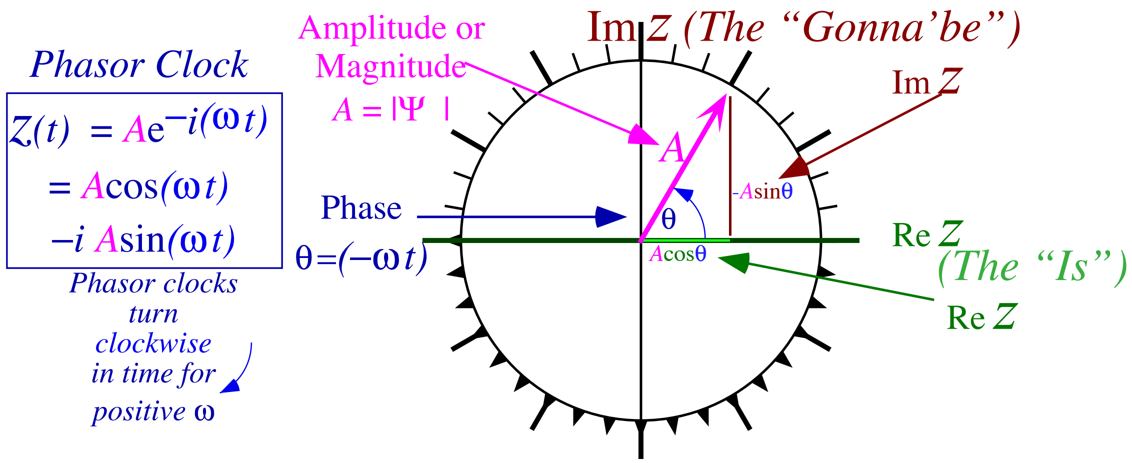
Smith Charts

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

driven by external stimulating force → $F_{stimulus}(t) = eE(t)$

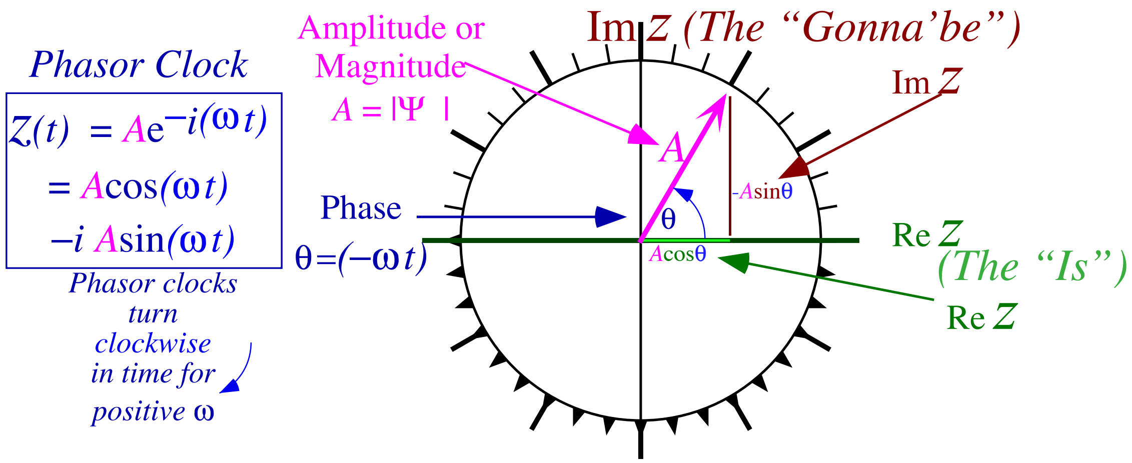
held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad (k = \omega_0^2 m),$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad (b = 2\Gamma m)$

Linear *harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{restore}}{m}$$



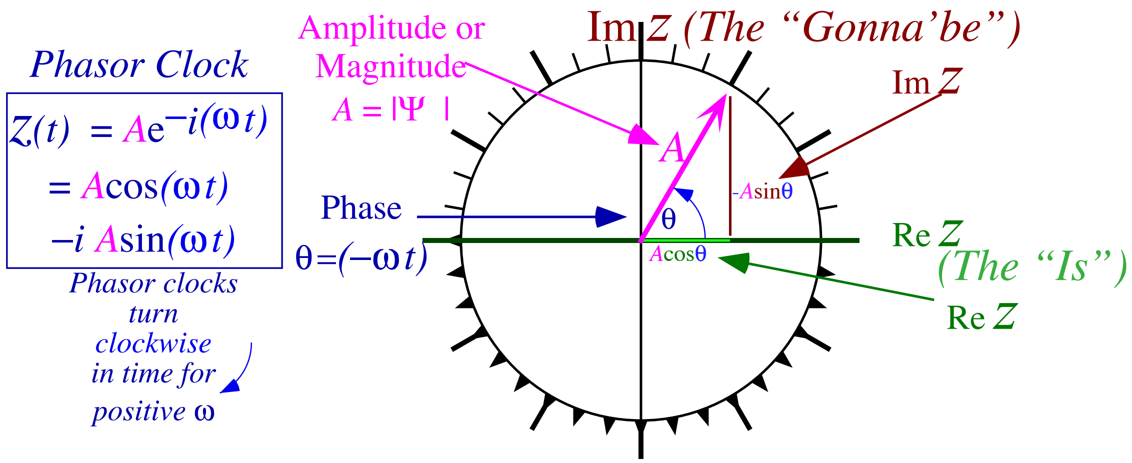
$$\frac{d^2z}{dt^2} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

Linear *harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = \qquad F_{restore}$$



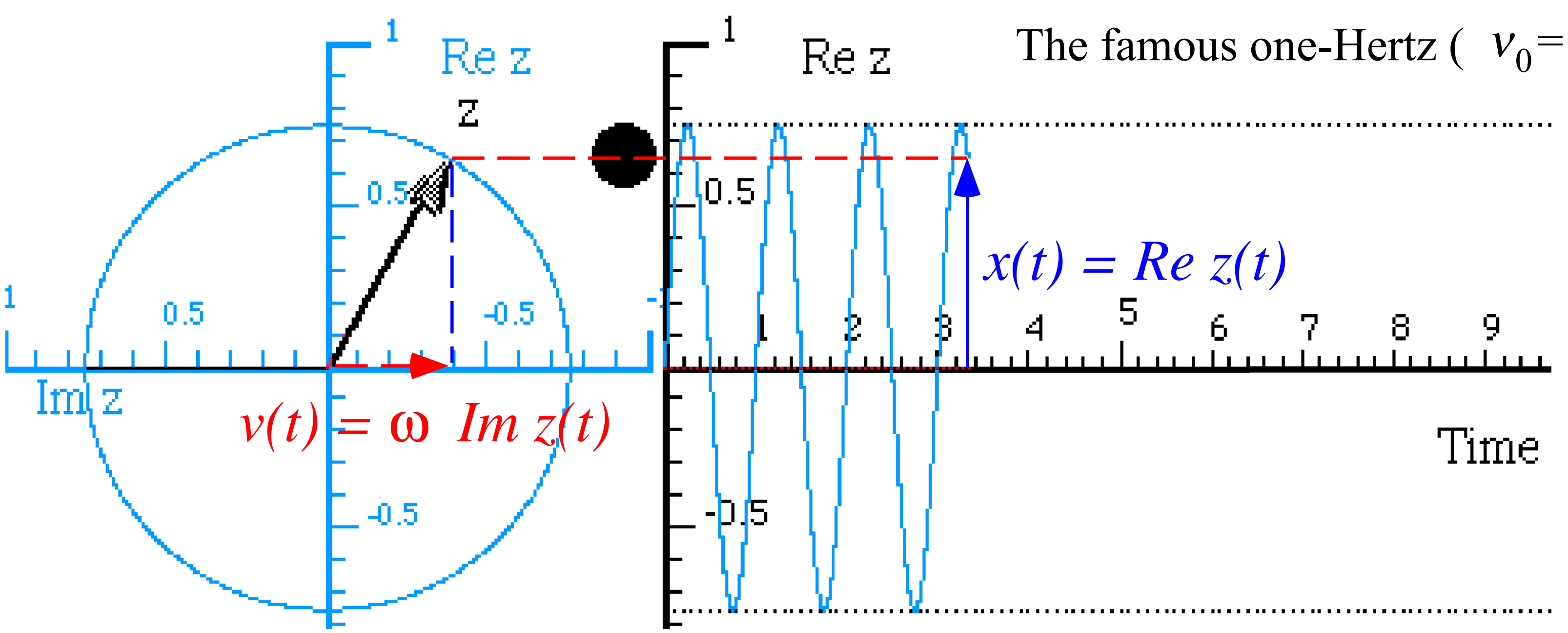
$$\frac{d^2z}{dt^2} = \frac{F_{restore}}{m}$$

$$\frac{d^2z}{dt^2} \qquad + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

The famous one-Hertz (ν_0=1/s. or: $\omega_0 = 2\pi = 6.2832$rad/s.) oscillator.



OscillIt Web Simulation

http://www.uark.edu/ua/modphys/markup/OscillItWeb.html

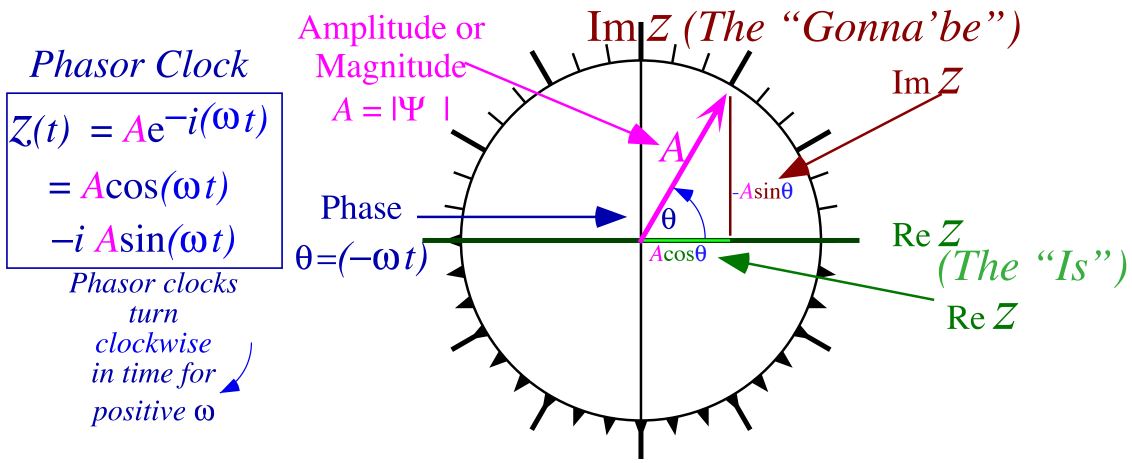
Fig. 3.2.2 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0$

Linear -*damped*-*harmonic* *oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$



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held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad (k = \omega_0^2 m),$

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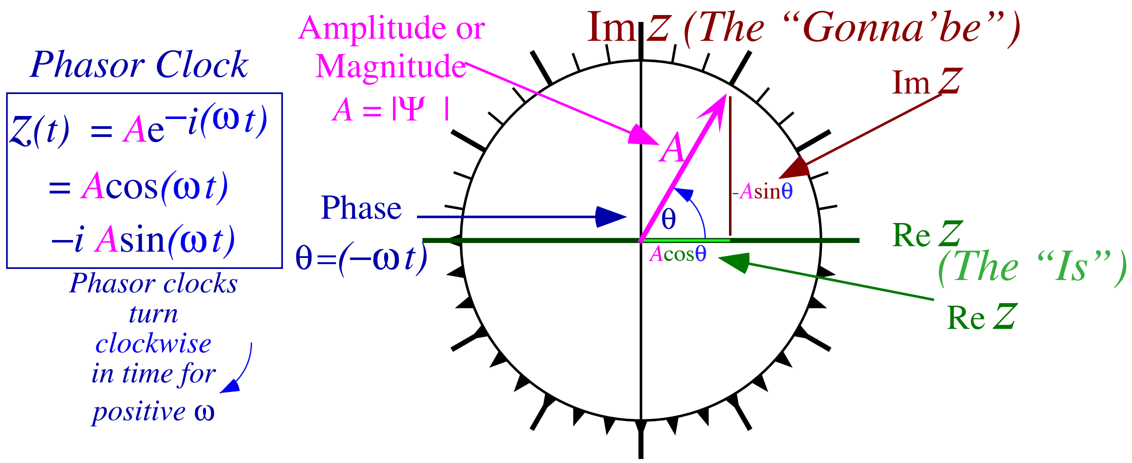

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force $\rightarrow$ $F_{restore} = -kz$

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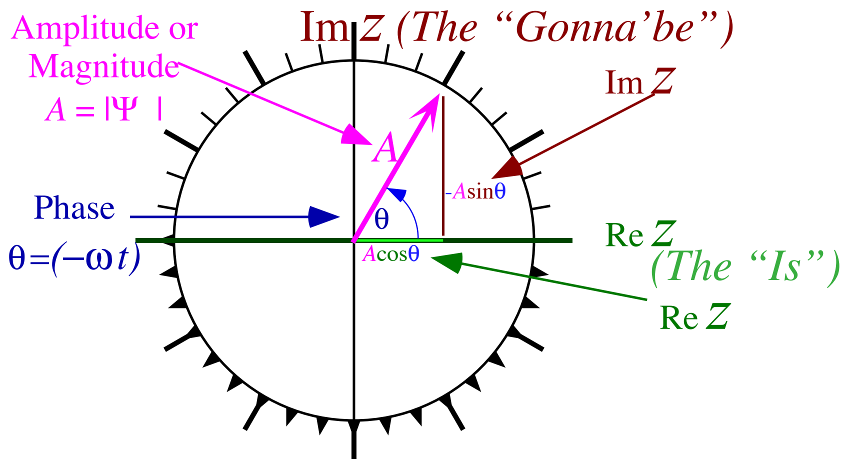
Phasor Clock

$Z(t) = Ae^{-i(\omega t)}$

$= A\cos(\omega t)$

$-i\, A\sin(\omega t)$

Phasor clocks turn clockwise in time for positive ω



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

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$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

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held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

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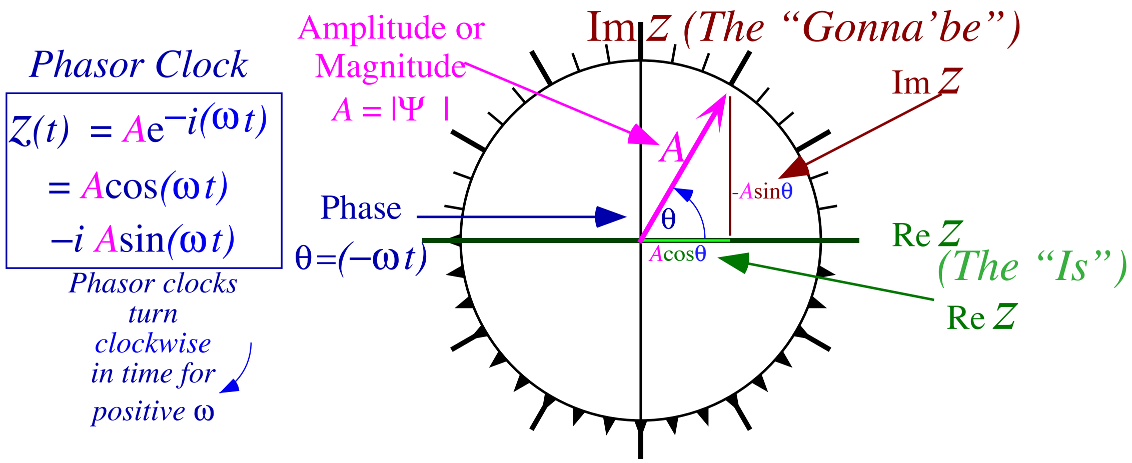

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$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

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Linear -damped-harmonic oscillator equation of motion.

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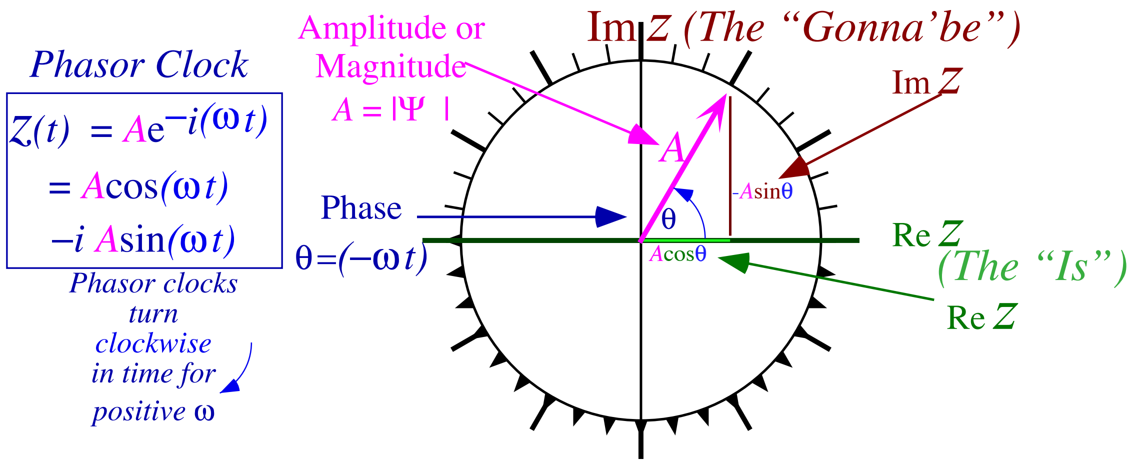

Trick:

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$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

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held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2-\Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2-\Gamma^2}\right)t}$$

$$= e^{-\Gamma t}e^{\pm i\sqrt{\omega_0^2-\Gamma^2}\,t}$$

$$= e^{-\Gamma t}e^{\pm i\omega_\Gamma t}$$

Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

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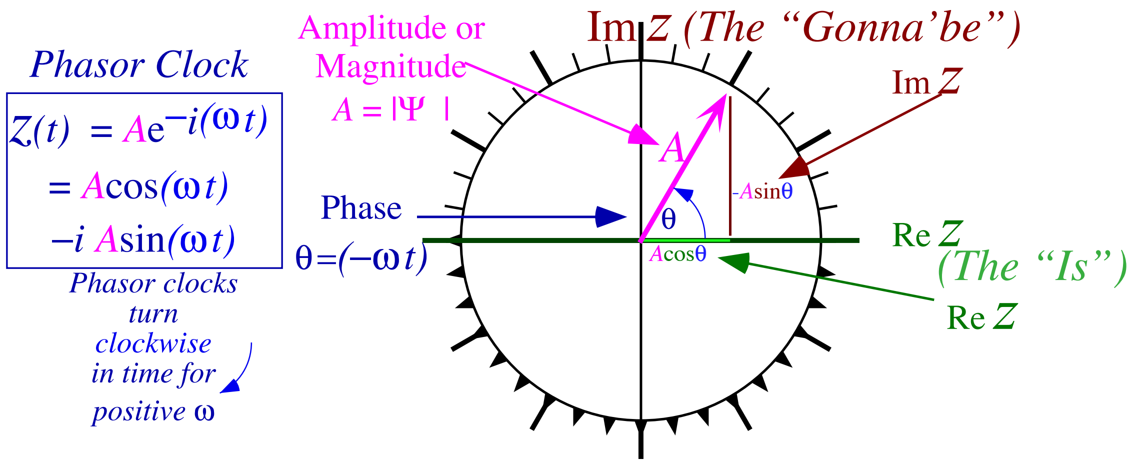

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Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm\sqrt{\omega_0^2-\Gamma^2}\right)t}$$

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$$= e^{-\Gamma t}e^{\pm i\sqrt{\omega_0^2-\Gamma^2}\,t}$$

$$= e^{-\Gamma t}e^{\pm i\omega_\Gamma t}$$

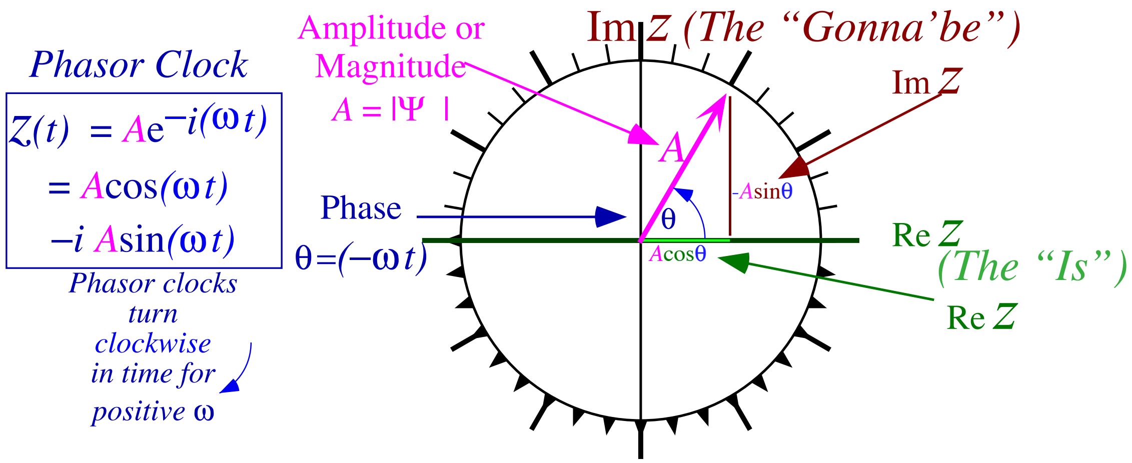
It oscillates at an angular frequency ω_Γ reduced slightly by .05% from ω_0 due to damping Γ =0.2.

$$\omega_\Gamma = \sqrt{\omega_0^2 - \Gamma^2} = \omega_0 - \tfrac{1}{2}(\Gamma^2/\omega_0) + ... = 6.2831853 - 0.003183 + .. = 6.280002 + ... = 6.280001$$

Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



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$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

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Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

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held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

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Solution:

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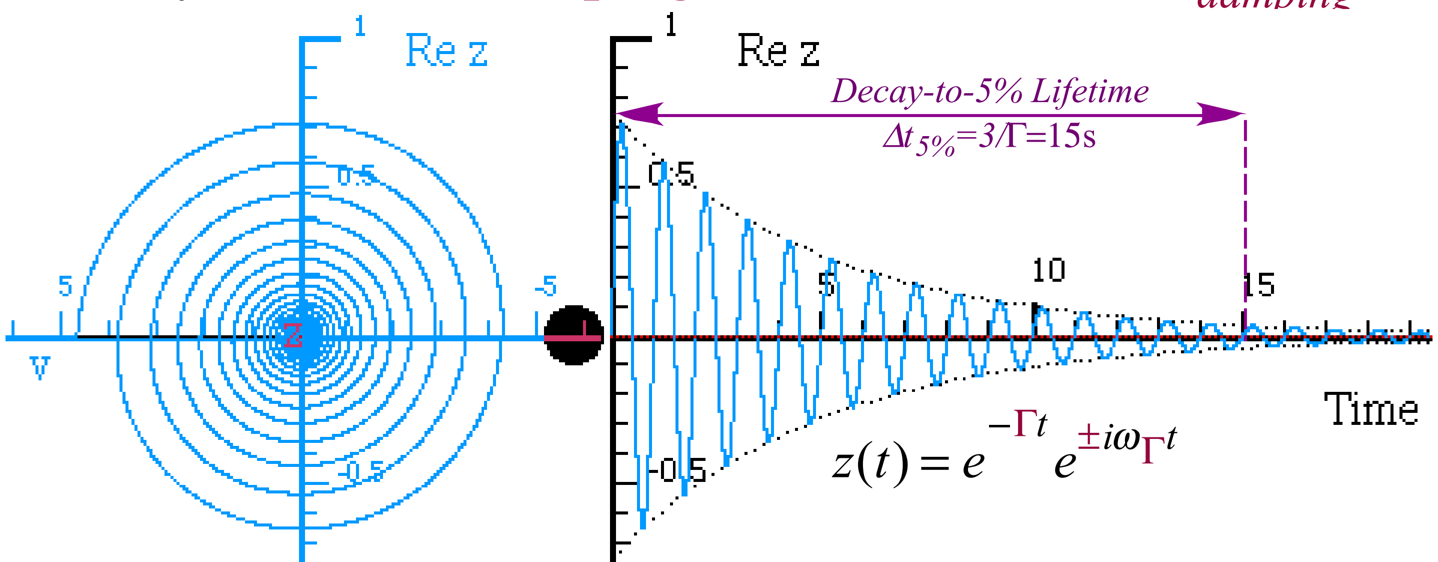


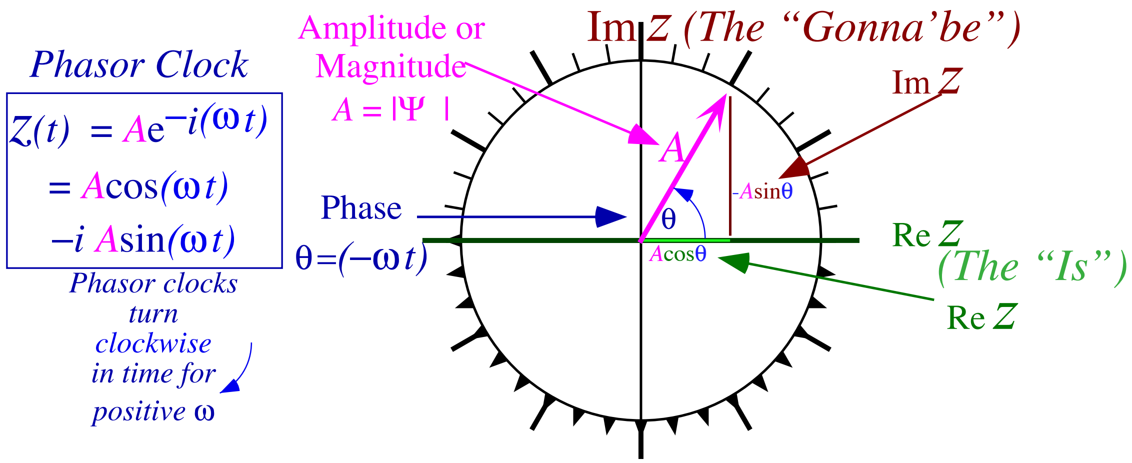
OscillIt Web Simulation

Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2 z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2 z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Oscillator Figures of Merit:

Time required to to reduce amplitude to 5%

Easy-to-recall *5%* approximation:

$$e^{-3} \cong 0.05$$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

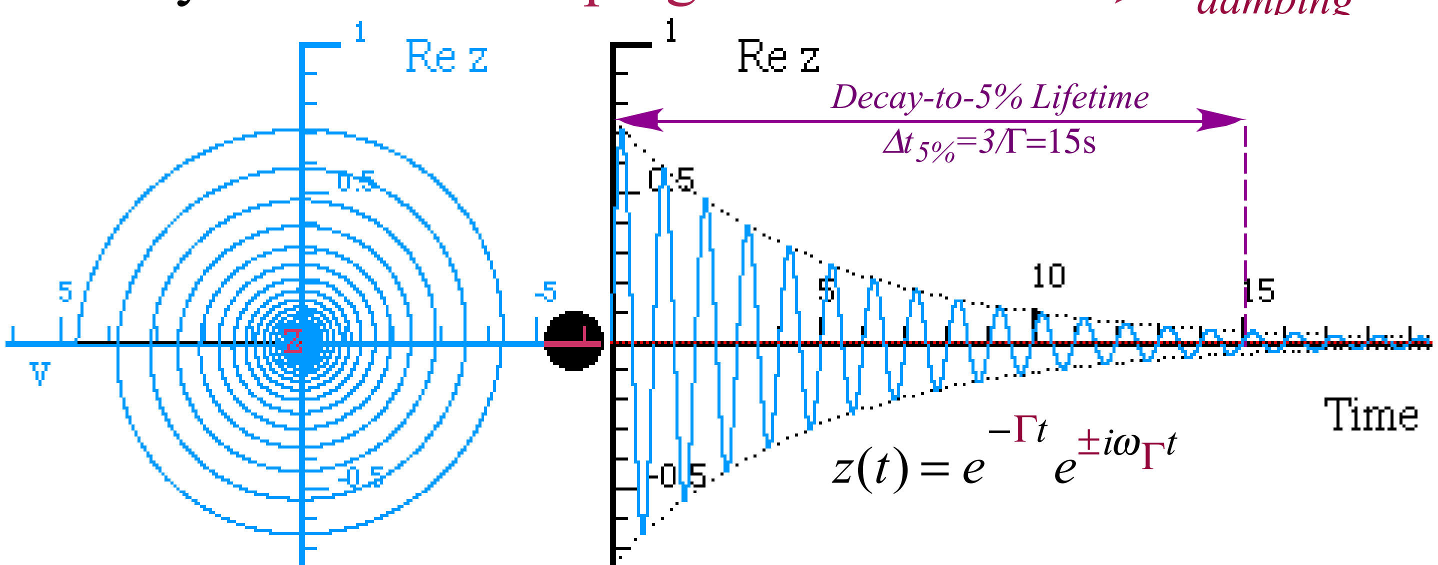


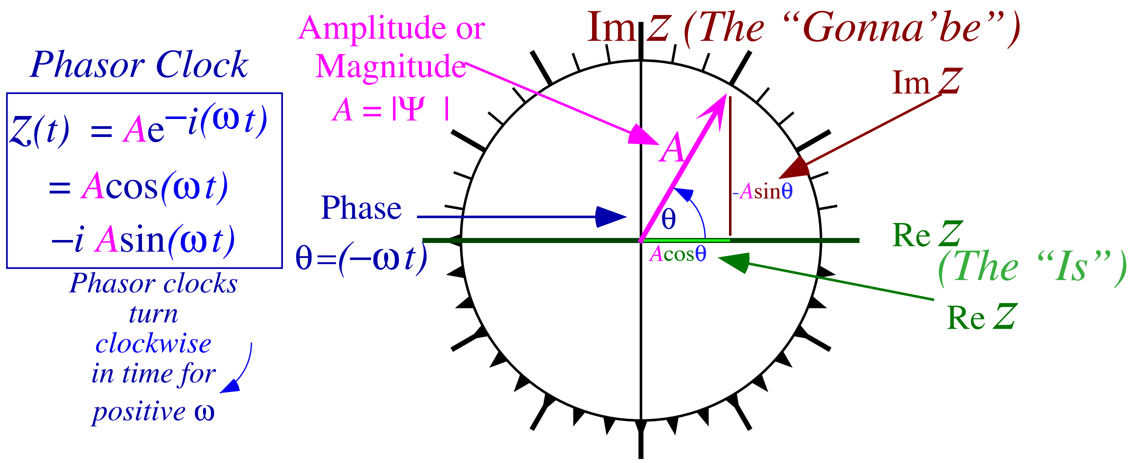
Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

OscillIt Web Simulation

Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

<u>Oscillator Figures of Merit:</u>

Time required to to reduce amplitude to 5% (or 4.321%)

Easy-to-recall *5%* approximation: $e^{-3} \cong 0.05$

More precise one: $e^{-\pi} \cong 0.04321$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$

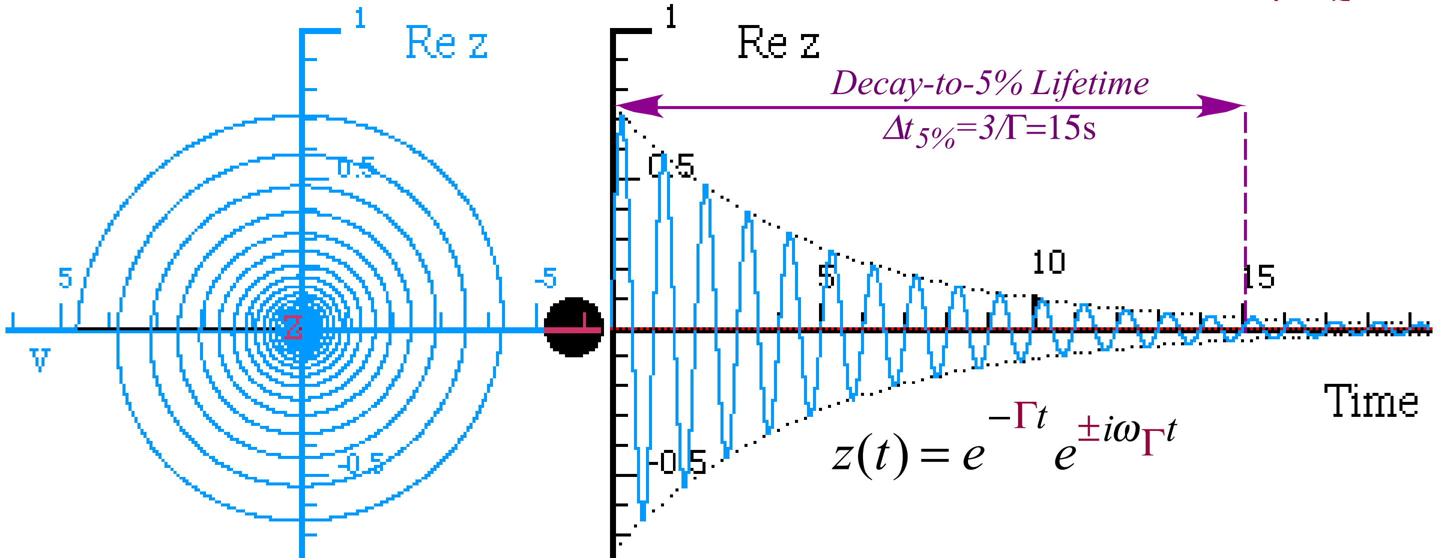


Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

[OscillIt Web Simulation]

Linear -damped-harmonic oscillator equation of motion.

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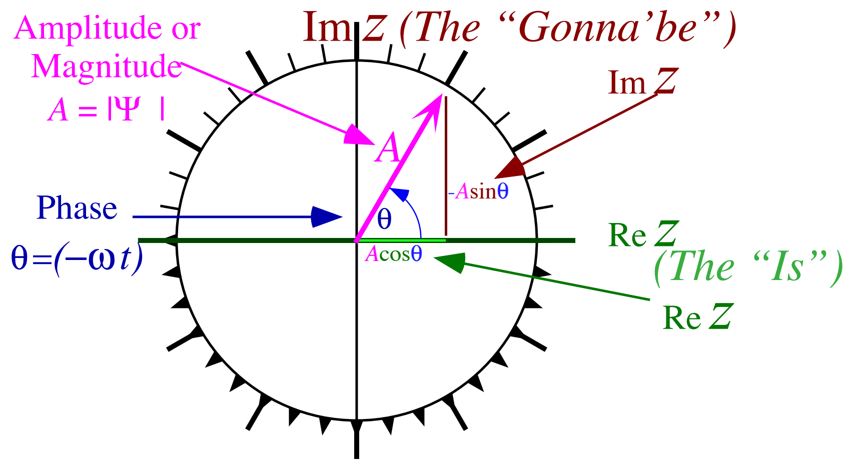
$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

Phasor Clock

$Z(t) = Ae^{-i(\omega t)}$
$= A\cos(\omega t)$
$-i\, A\sin(\omega t)$

Phasor clocks turn clockwise in time for positive ω



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

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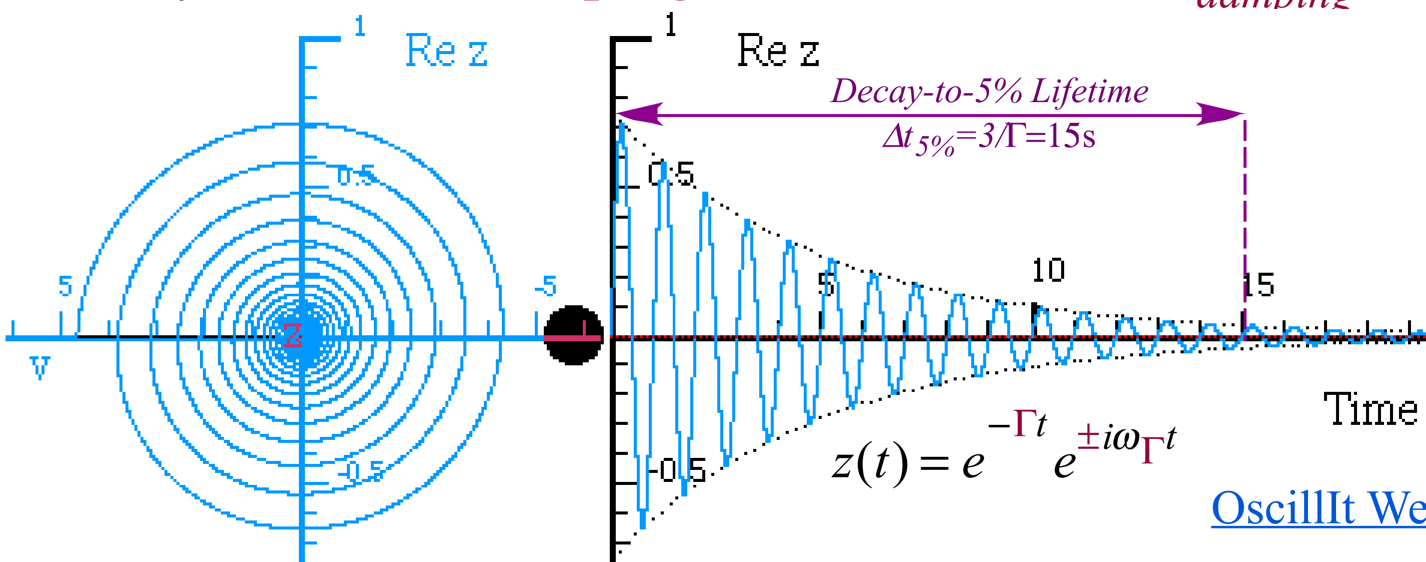


OscillIt Web Simulation

Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$

Oscillator Figures of Merit:

Number *N* of oscillations to reduce amplitude to 5% (or 4.321%)

Easy-to-recall *5%* approximation: More precise one:

$$e^{-3} \cong 0.05 \qquad e^{-\pi} \cong 0.04321$$

$$N_{5\%} = \frac{\omega_\Gamma \cdot t_{5\%}}{2\pi} = \frac{3\omega_\Gamma}{2\pi\Gamma} \sim \frac{\omega_\Gamma}{2\Gamma}$$

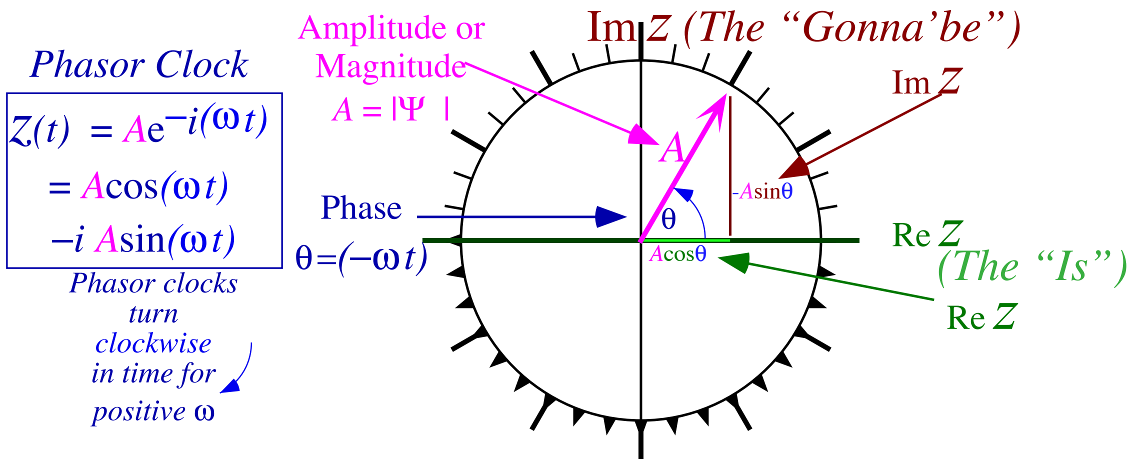
$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass m and charge e

driven by external stimulating force → $F_{stimulus}(t) = eE(t)$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

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Linear forced-damped-harmonic oscillator equation of motion.

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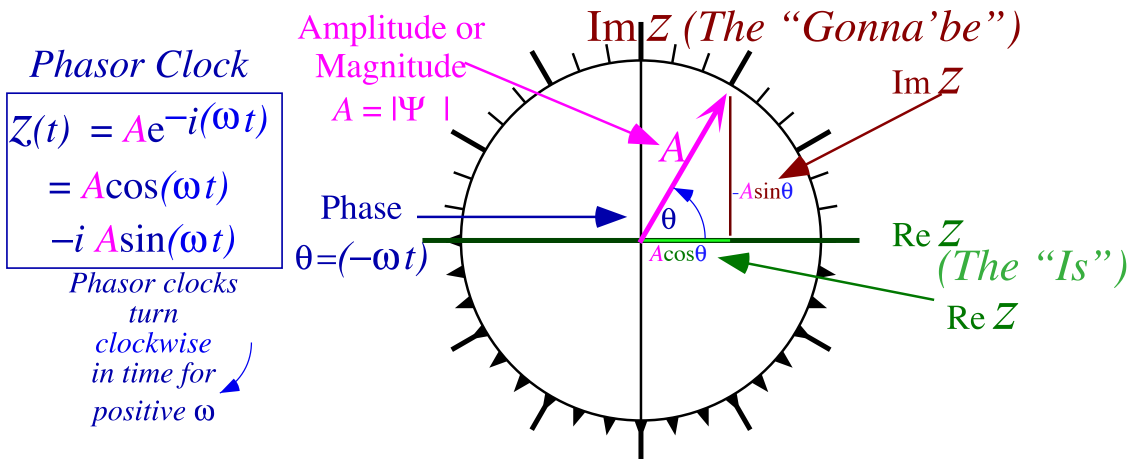

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

Linear forced-damped-harmonic oscillator equation of motion.

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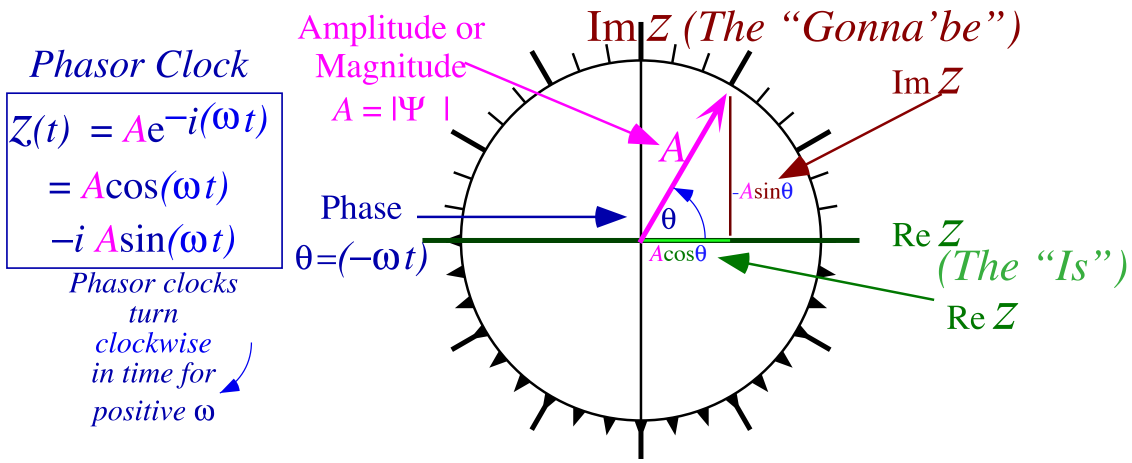

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$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

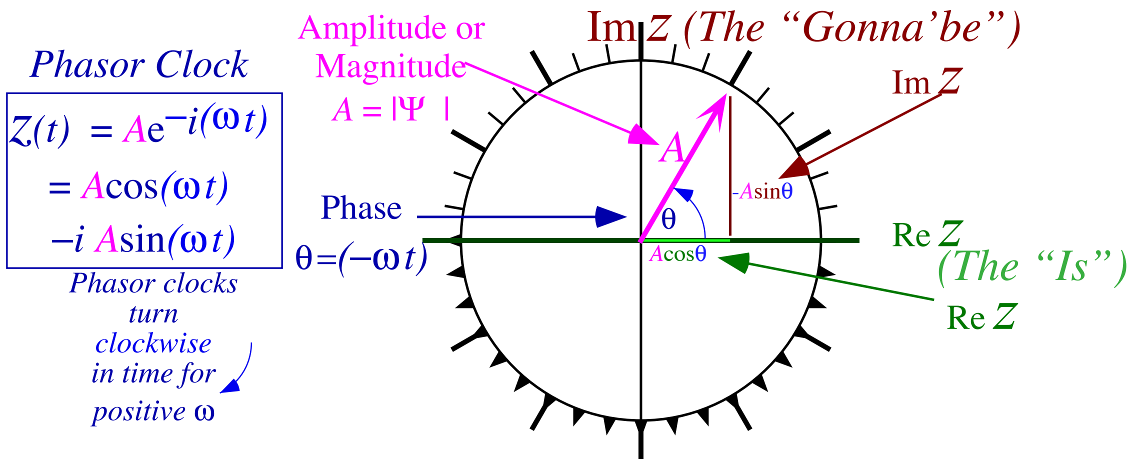
Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

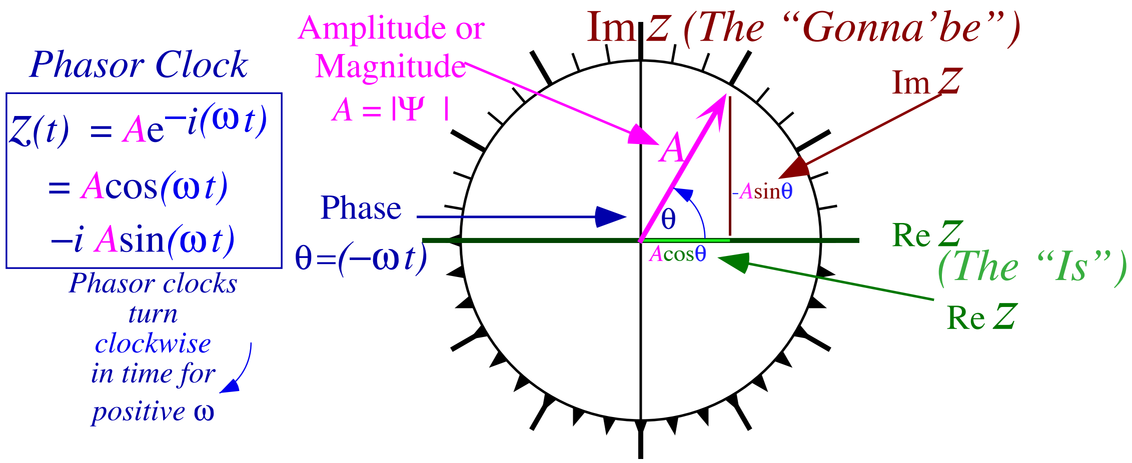
Pretty crazy?

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$



Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

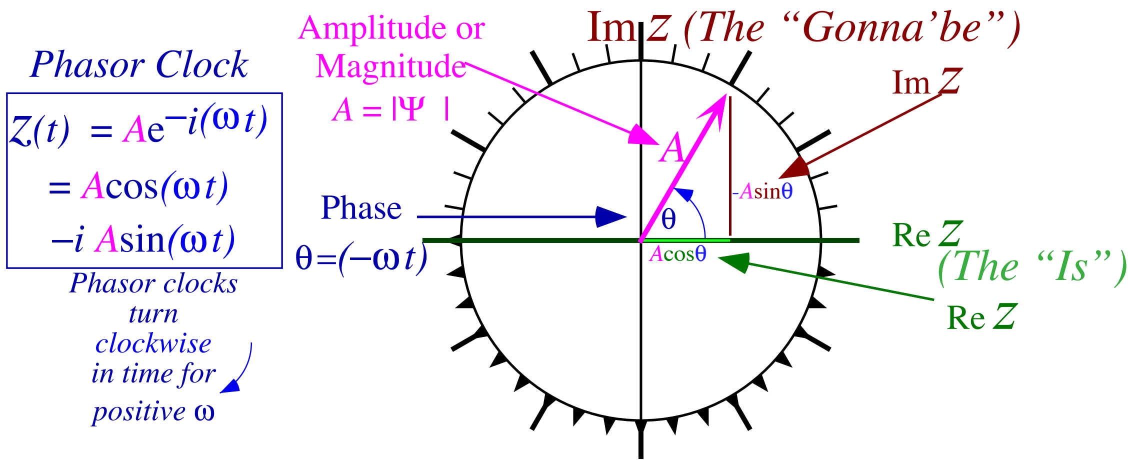
$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if
$a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}a_{stimulus}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s{}^2 - i2\Gamma\omega_s + \omega_0^2}a_s e^{-i\omega_s t}$$

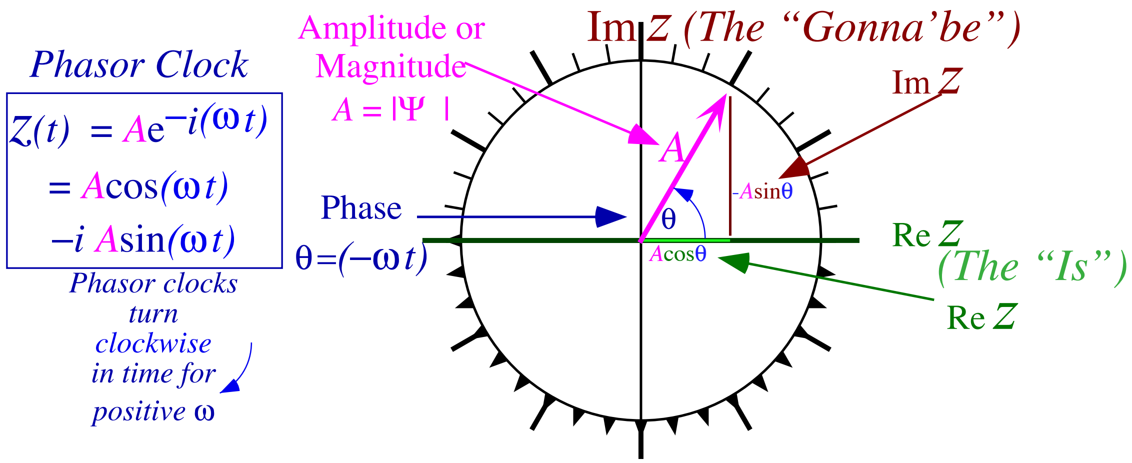
$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s{}^2 - i2\Gamma\omega_s}a_s e^{-i\omega_s t}$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

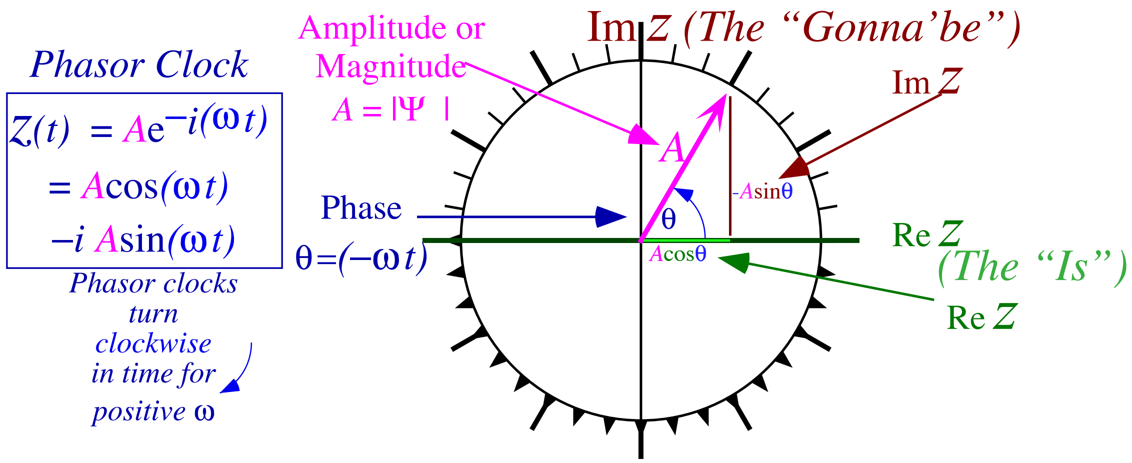
$$z_{stimulus} = \frac{1}{-\omega_s^{\ 2} - i2\Gamma\omega_s + \omega_0^2}a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^{\ 2} - i2\Gamma\omega_s}a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s) \cdot a_s$$

Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field)

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$:

$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

Pretty crazy? But not so crazy if $a_{stimulus}(t) = |a_{stimulus}|e^{-i\omega_{stimulus}t} = |a_s|e^{-i\omega_s t}$

$$z_{stimulus} = \frac{1}{-\omega_s^2 - i2\Gamma\omega_s + \omega_0^2}a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s}a_s e^{-i\omega_s t}$$

$$z_s = G_{\omega_0}(\omega_s)\cdot a_s$$

Green's Function for the F-D-H Oscillator (FDHO)

George Green (14 July 1793 – 31 May 1841)

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

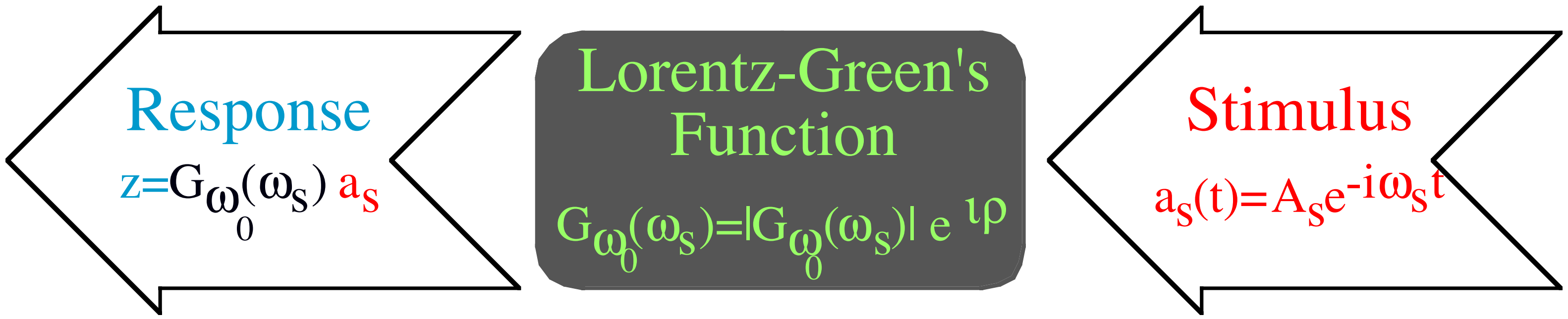


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\, G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of G:

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

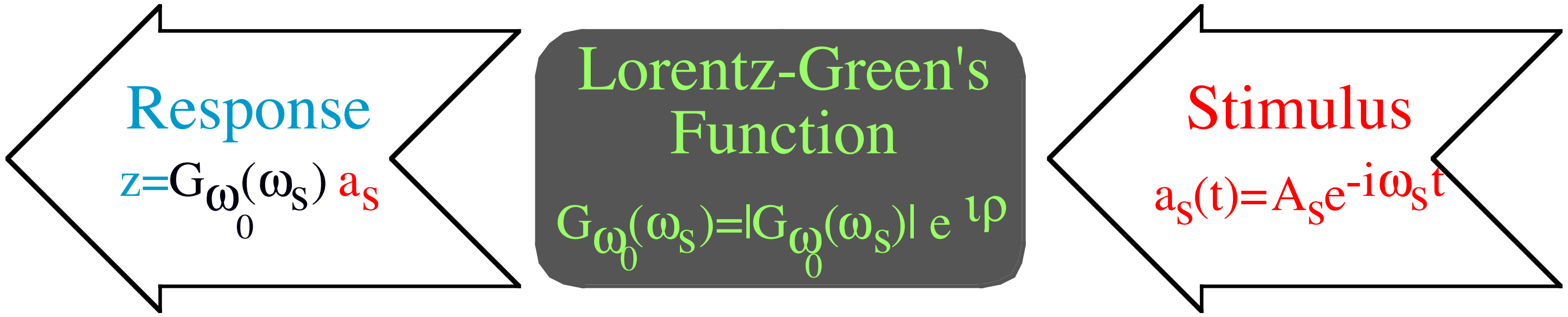


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \text{Re}\,G_{\omega_0}(\omega_s) + i\,\text{Im}\,G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of G: $\frac{1}{x-iy} = \frac{1}{x-iy}\frac{x+iy}{x+iy} = \frac{x+iy}{x^2+y^2}$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

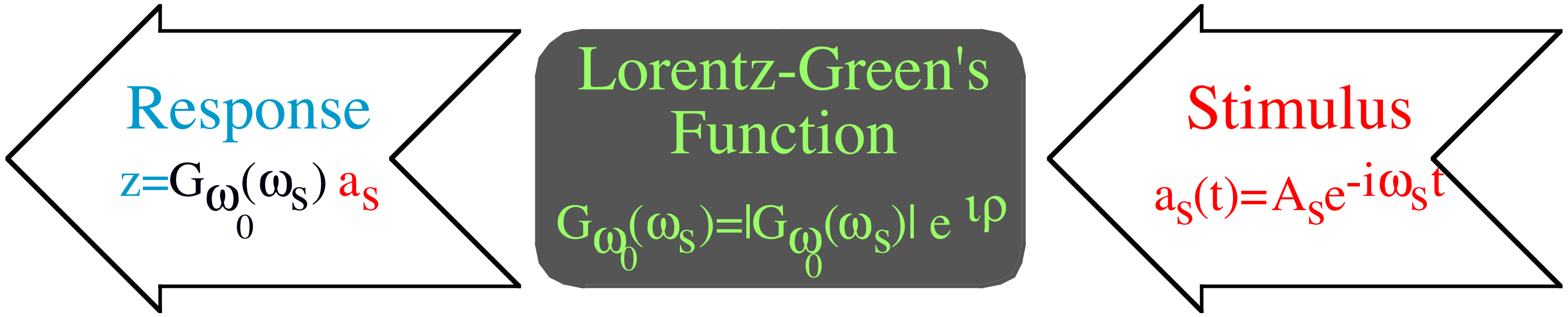


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}=\mathrm{Re}\,G_{\omega_0}(\omega_s)+i\,\mathrm{Im}\,G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of G:

$$\frac{1}{x-iy}=\frac{1}{x-iy}\frac{x+iy}{x+iy}=\frac{x+iy}{x^2+y^2}=\frac{x}{x^2+y^2}+i\frac{y}{x^2+y^2}$$

$$\mathrm{Re}\,G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

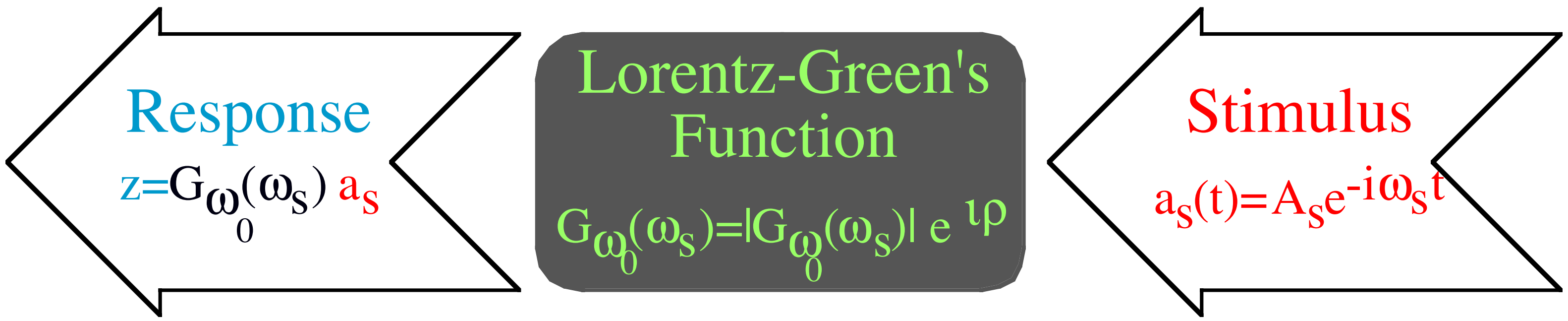


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}=\mathrm{Re}\,G_{\omega_0}(\omega_s)+i\,\mathrm{Im}\,G_{\omega_0}(\omega_s)=\left|G_{\omega_0}(\omega_s)\right|e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G:

$$\mathrm{Re}\,G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and polar angle ρ of the *polar form* of G:

$$\left|G_{\omega_0}(\omega_s)\right|=\frac{1}{\sqrt{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}}$$

$$\rho=\tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2-\omega_s^2}\right)$$

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

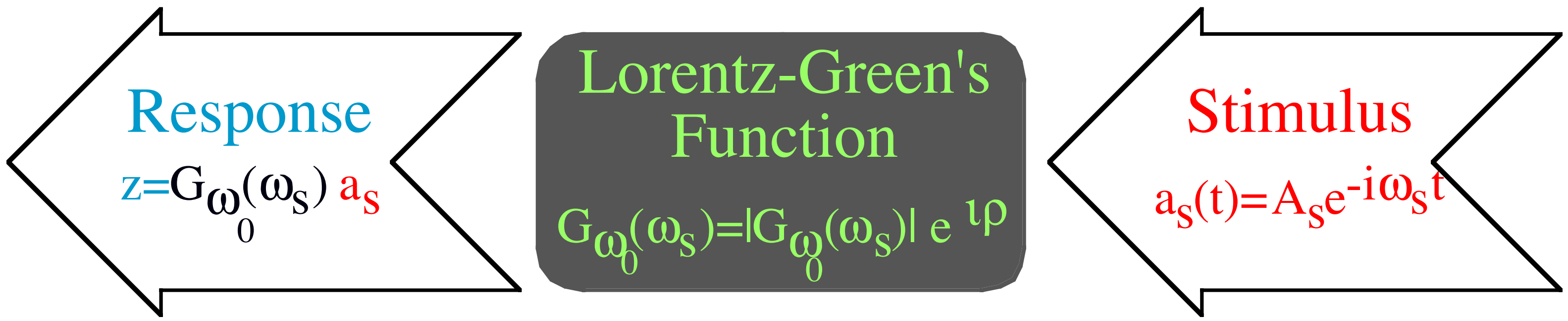


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\,G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\,G_{\omega_0}(\omega_s) = \left|G_{\omega_0}(\omega_s)\right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G:

$$\mathrm{Re}\,G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

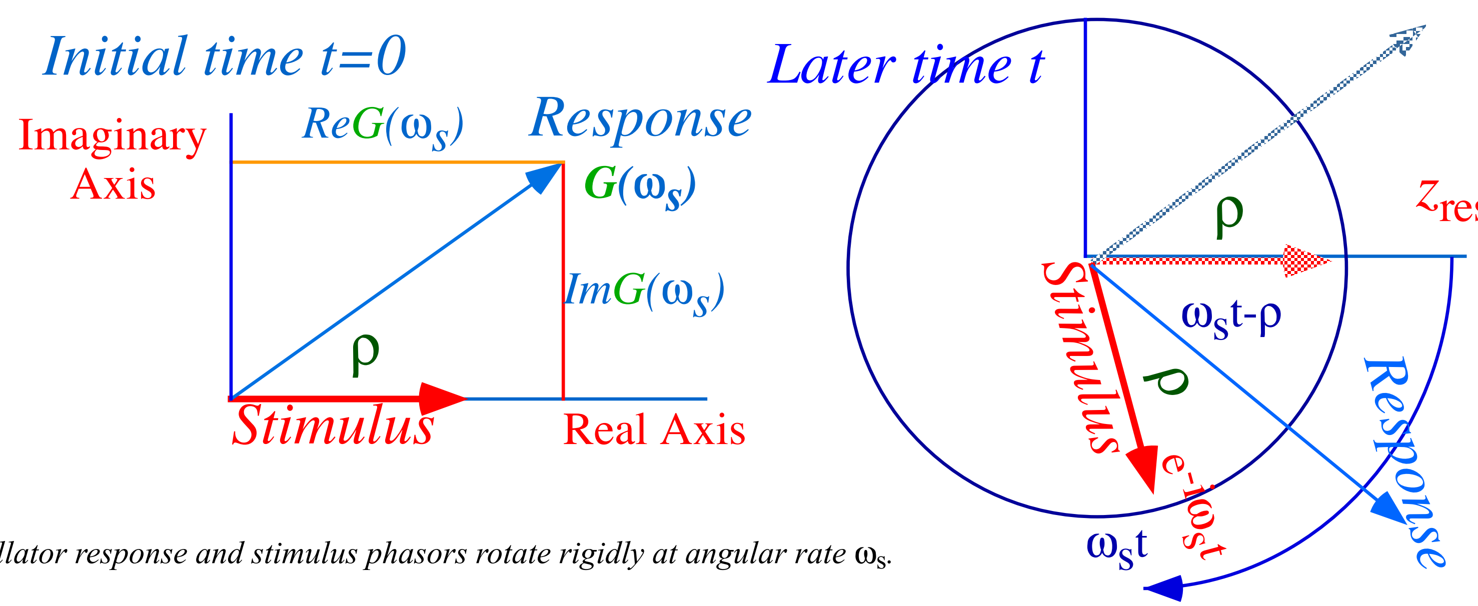
$$\mathrm{Im}\,G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of G:

$$\left|G_{\omega_0}(\omega_s)\right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

polar angle ρ is the *phase lag angle* ρ



$$z_{\text{response}}(t) = \left|G_{\omega_0}(\omega_s)\right| a(0)\, e^{-i(\omega_s t - \rho)}$$

Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s.

Green's Function for the FDHO (Forced-Damped-Harmonic Oscillator)

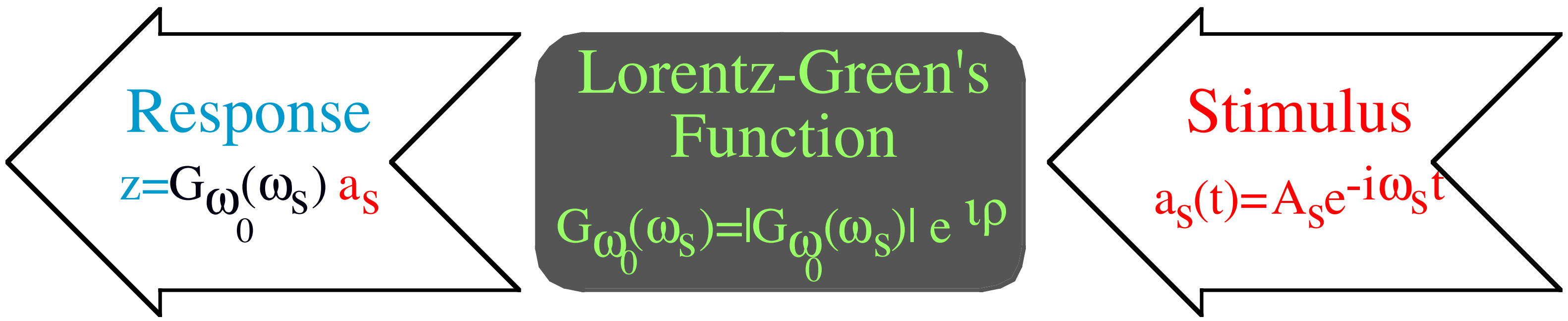


Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} = \mathrm{Re}\, G_{\omega_0}(\omega_s) + i\,\mathrm{Im}\, G_{\omega_0}(\omega_s) = \left|G_{\omega_0}(\omega_s)\right| e^{i\rho}$$

Real and imaginary parts of the *rectangular form* of G:

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of G:

$$\left|G_{\omega_0}(\omega_s)\right| = \frac{1}{\sqrt{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}}$$

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

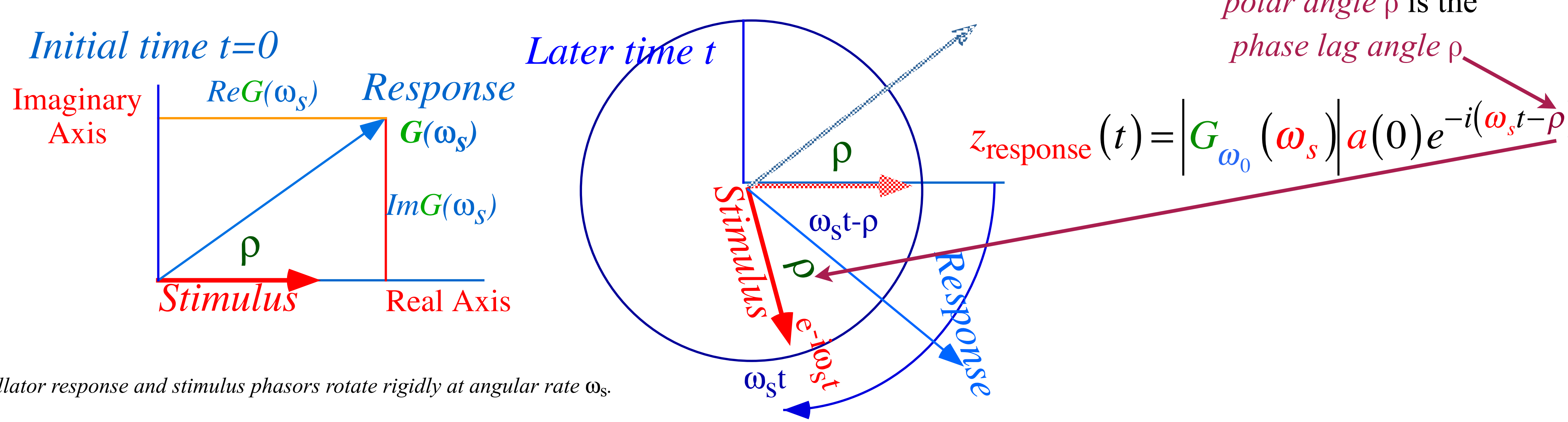
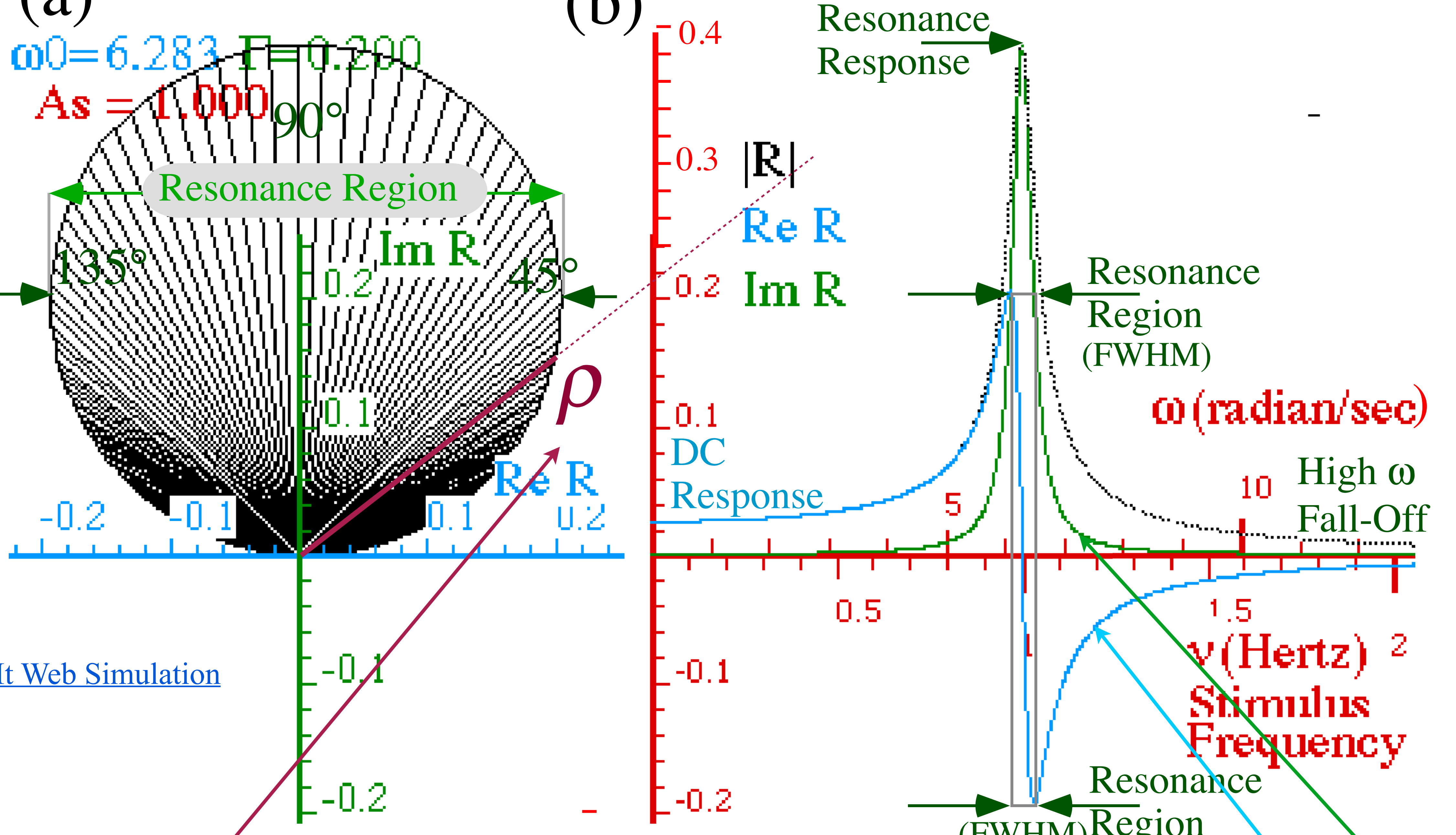


Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω_s.



OscillIt Web Simulation

Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

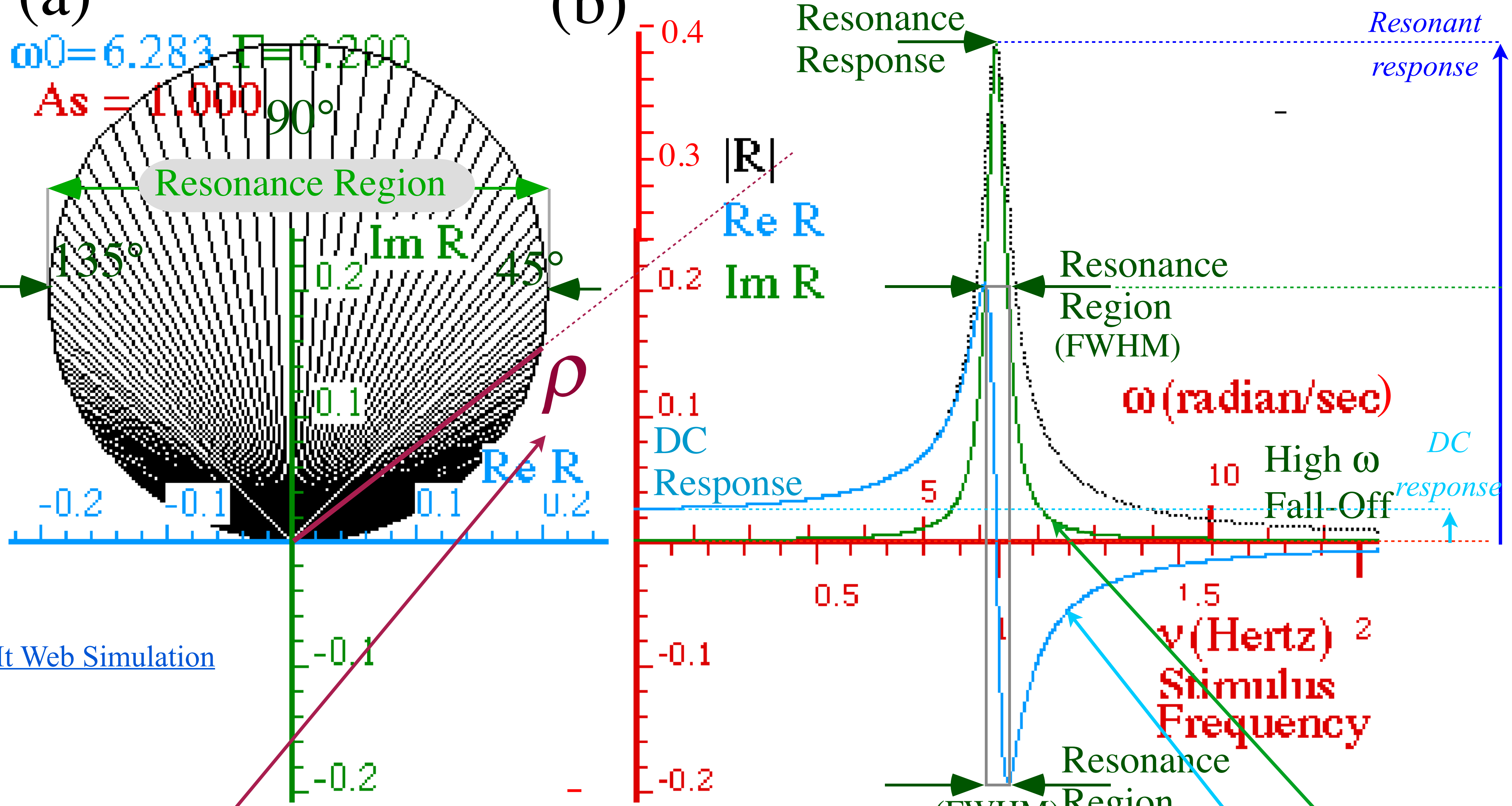
Real part

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Imaginary part

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

http://www.uark.edu/ua/modphys/markup/OscillItWeb.html



[OscillIt Web Simulation](http://www.uark.edu/ua/modphys/markup/OscillItWeb.html)

Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots

Phase lag angle

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

Real part

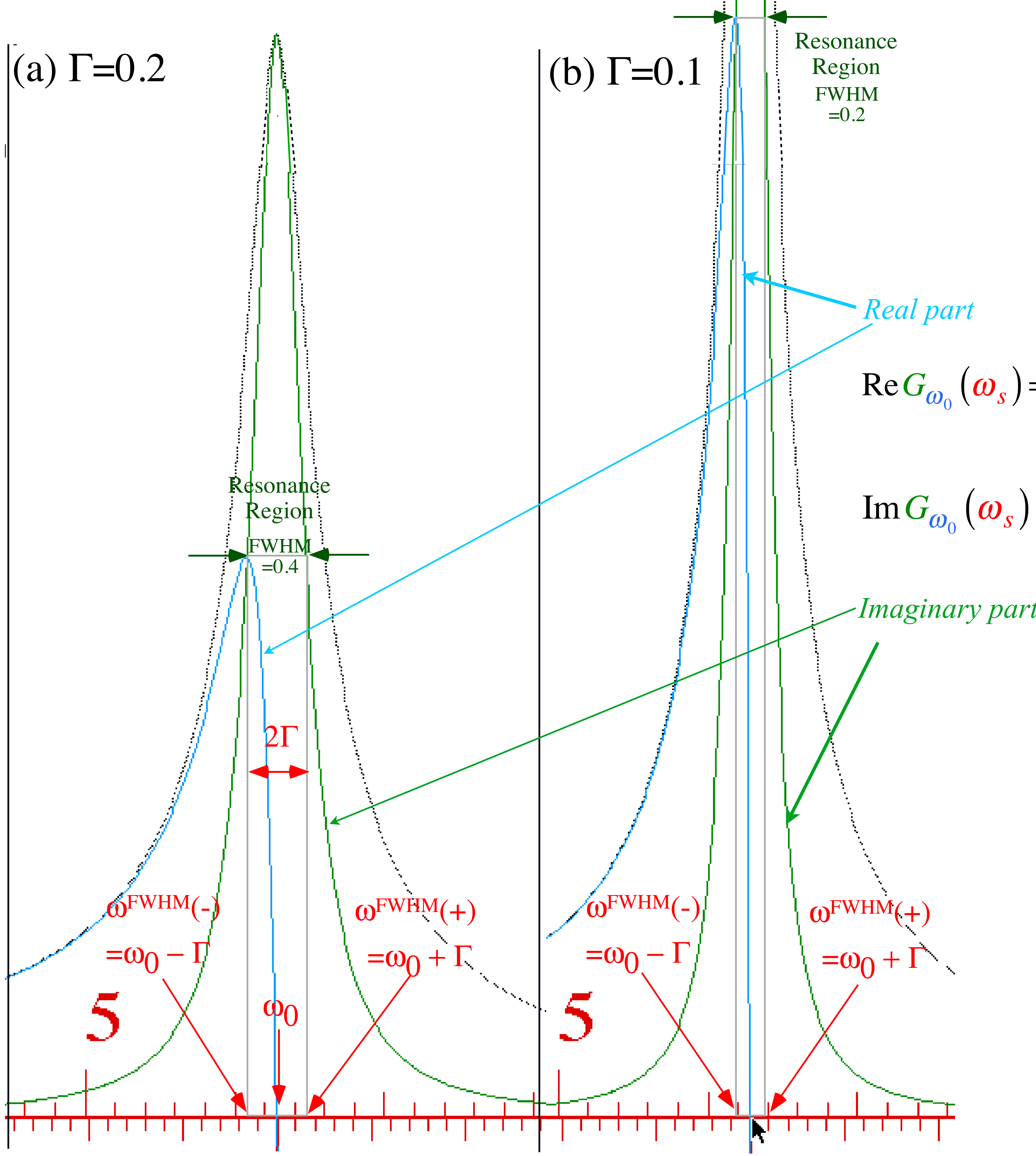
$$\mathrm{Re}\, G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

Imaginary part

$$\mathrm{Im}\, G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}(\omega_s = \omega_0)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

http://www.uark.edu/ua/modphys/markup/OscillItWeb.html



$$\mathrm{Re}\,G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

OscillIt Web Simulation

Fig. 4.2.7 Comparing Lorentz-Green resonance region for (a) Γ=0.2 and (b) Γ=0.1.
Maximum and minimum points of Re$G(\omega)$ *and inflection points of* Im$G(\omega)$ *are near region boundaries* $\omega^{FWHM}(\pm)=\omega_0\pm\Gamma$.

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s)a(0)e^{-i\omega_s t}$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s)a(0)e^{-i\omega_s t}$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s)a(0)e^{-i\omega_s t}$$

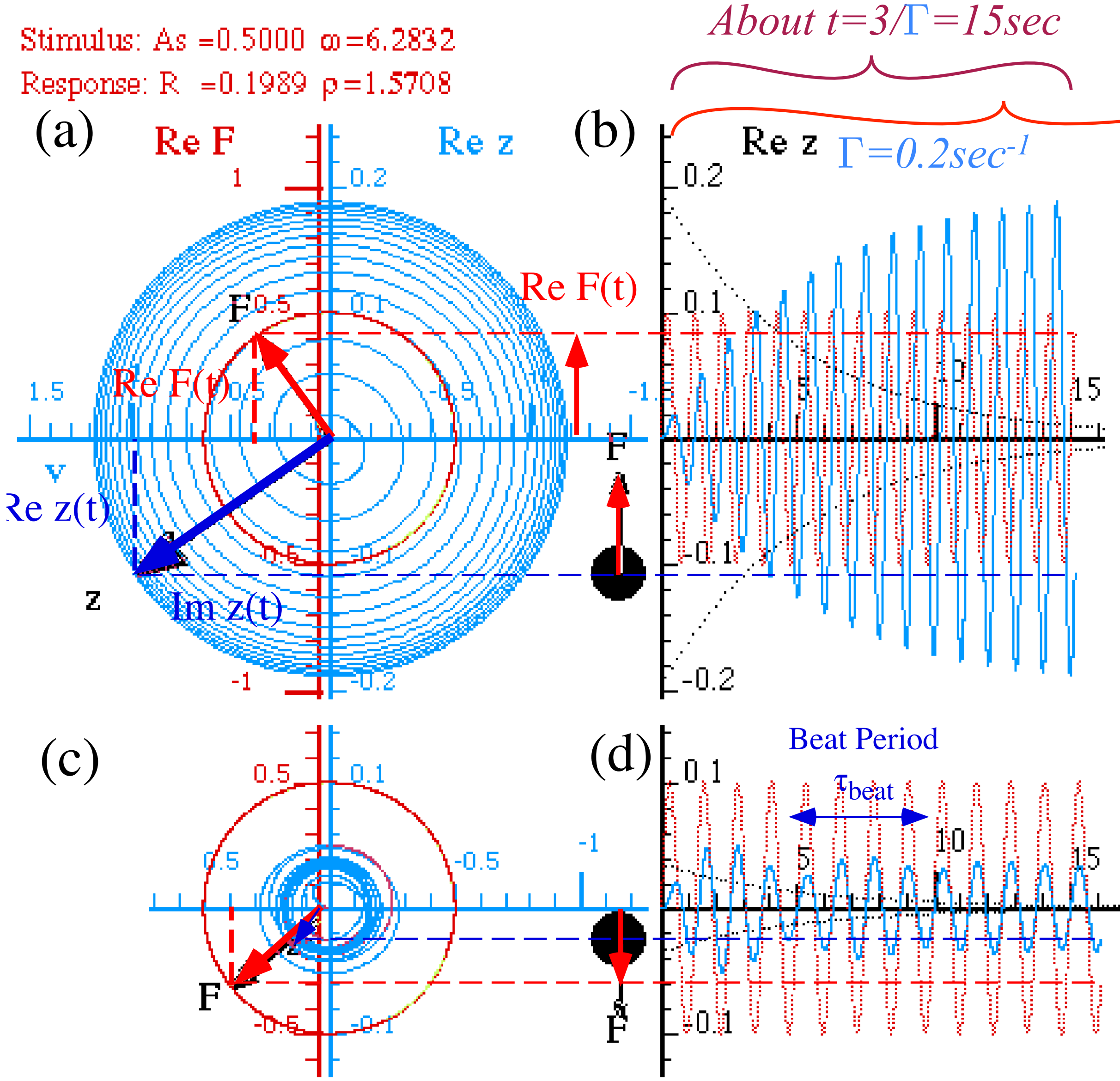
$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

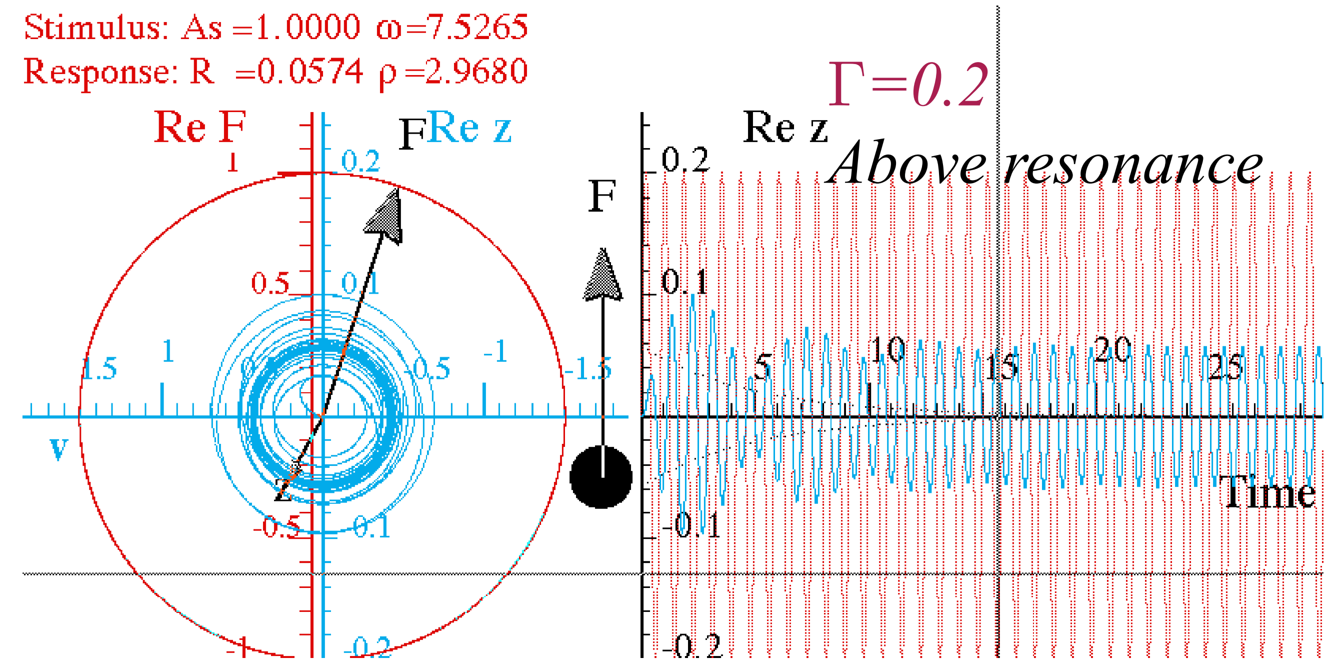


[OscillIt (On Resonance) Simulation]

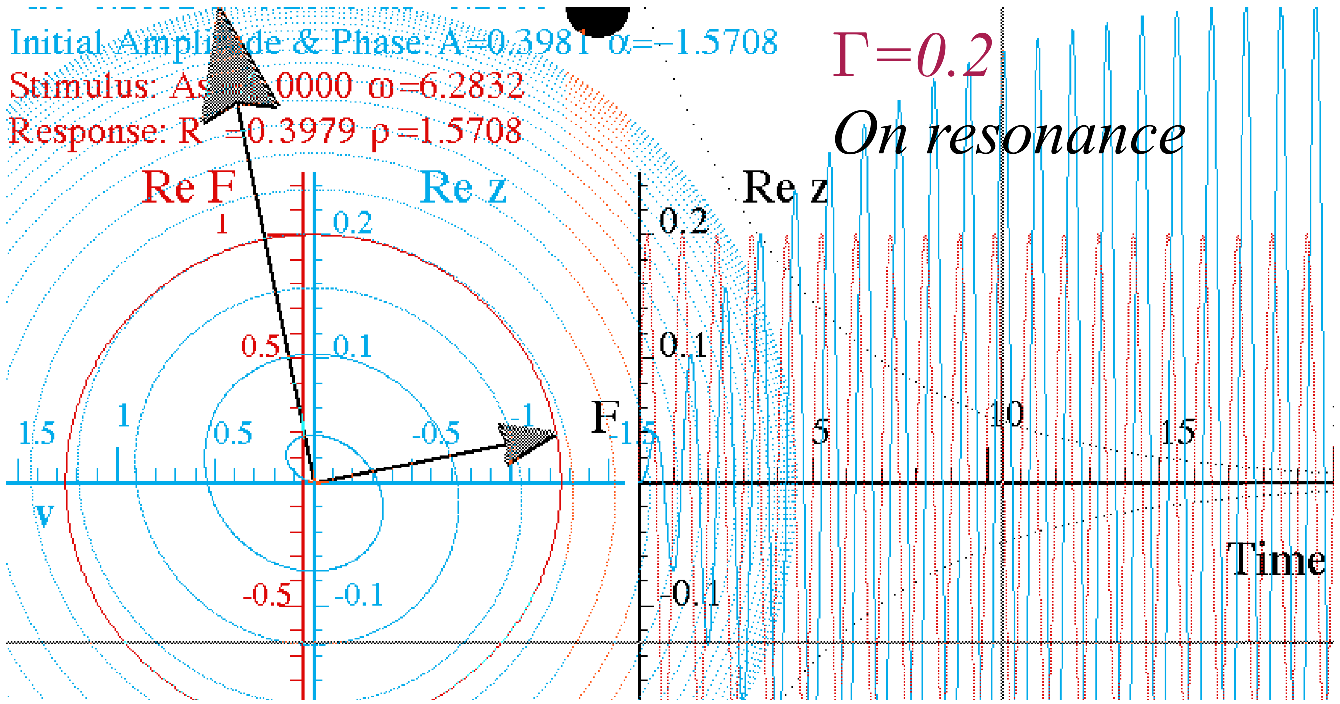
Fig. 4.2.8 On Resonance *(a)Response z-phasor lags ρ=90° behind stimulus F-phasor.*
($\omega_s=\omega_0=2\pi$, $\omega_0=2\pi$, and $\Gamma=0.2$). *(b) Time plots of* Re *z(t) and* Re *F(t)*

Fig. 4.2.8 Below Resonance *(c)Response z-phasor lags ρ=8.05° behind stimulus F-phasor.*
(ω_s=5.03,$\omega_0=2\pi$, and $\Gamma=0.2$). *(d) Time plots of* Re *z(t) and* Re *F(t). Beats are barely visible.*

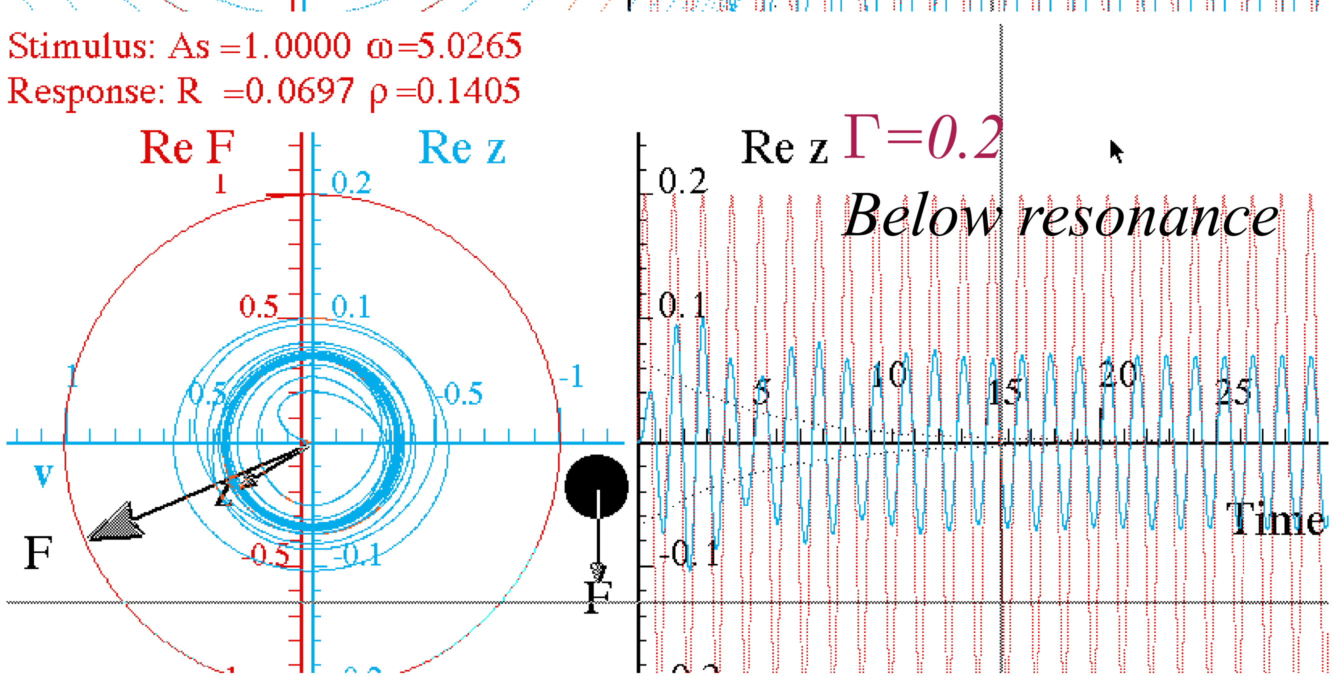
[OscillIt (Way Below Resonance) Simulation]



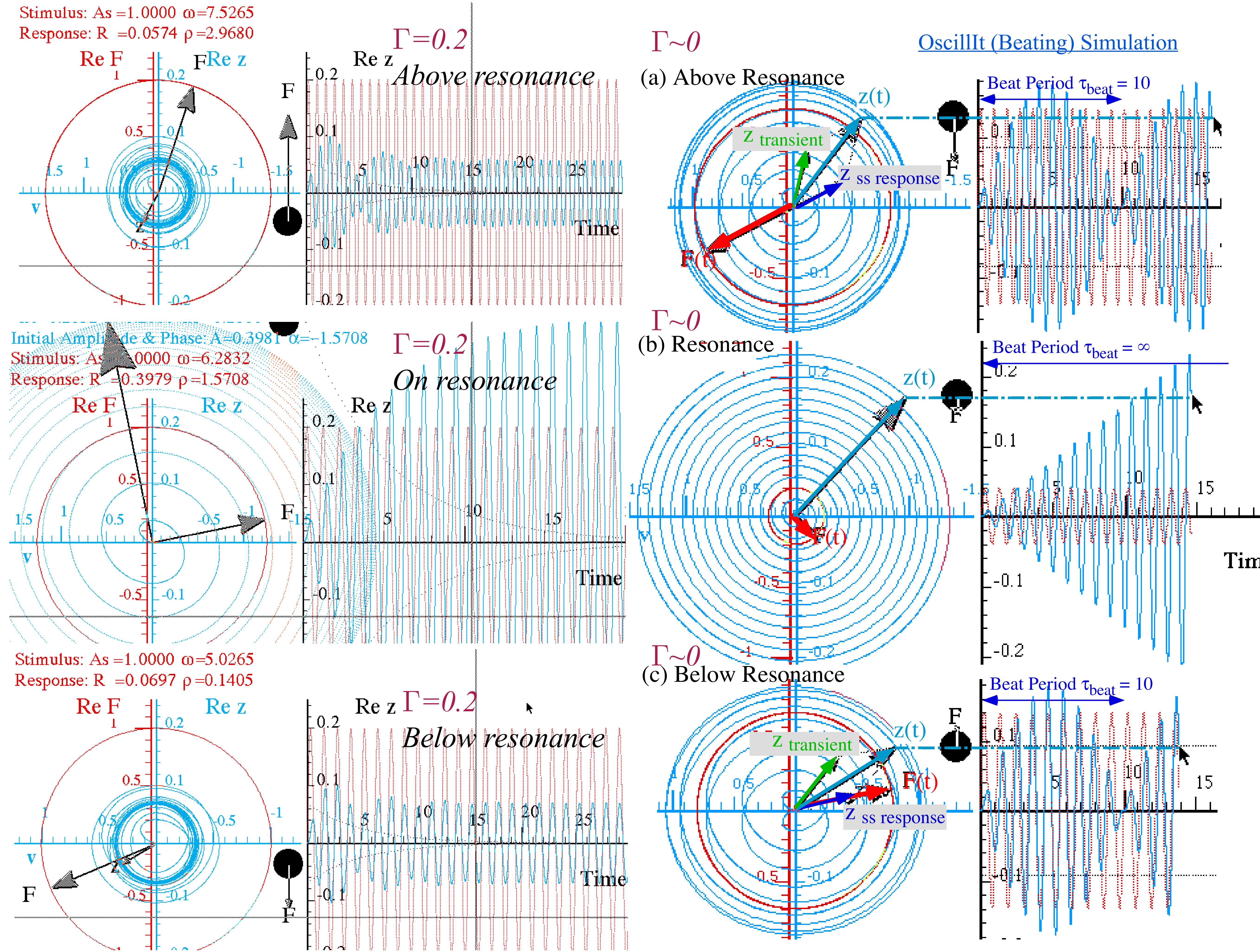
[OscillIt (Way Above Resonance) Simulation]



[OscillIt (On Resonance) Simulation]



[OscillIt (Way Below Resonance) Simulation]


Stimulus: As =1.0000 ω=7.5265
Response: R =0.0574 ρ=2.9680
Γ=0.2
Above resonance
Re F
Re z
Time
Initial Amplitude & Phase: A=0.3981 α=-1.5708
Stimulus: As =1.0000 ω=6.2832
Response: R =0.3979 ρ=1.5708
Γ=0.2
On resonance
Stimulus: As =1.0000 ω=5.0265
Response: R =0.0697 ρ=0.1405
Γ=0.2
Below resonance
Γ~0
OscillIt (Beating) Simulation
(a) Above Resonance
Beat Period τ_{beat} = 10
z(t)
Z transient
Z ss response
F(t)
Γ~0
(b) Resonance
Beat Period τ_{beat} = ∞
Γ~0
(c) Below Resonance
Beat Period τ_{beat} = 10

Lorentz-Green's Function for high quality FDHO

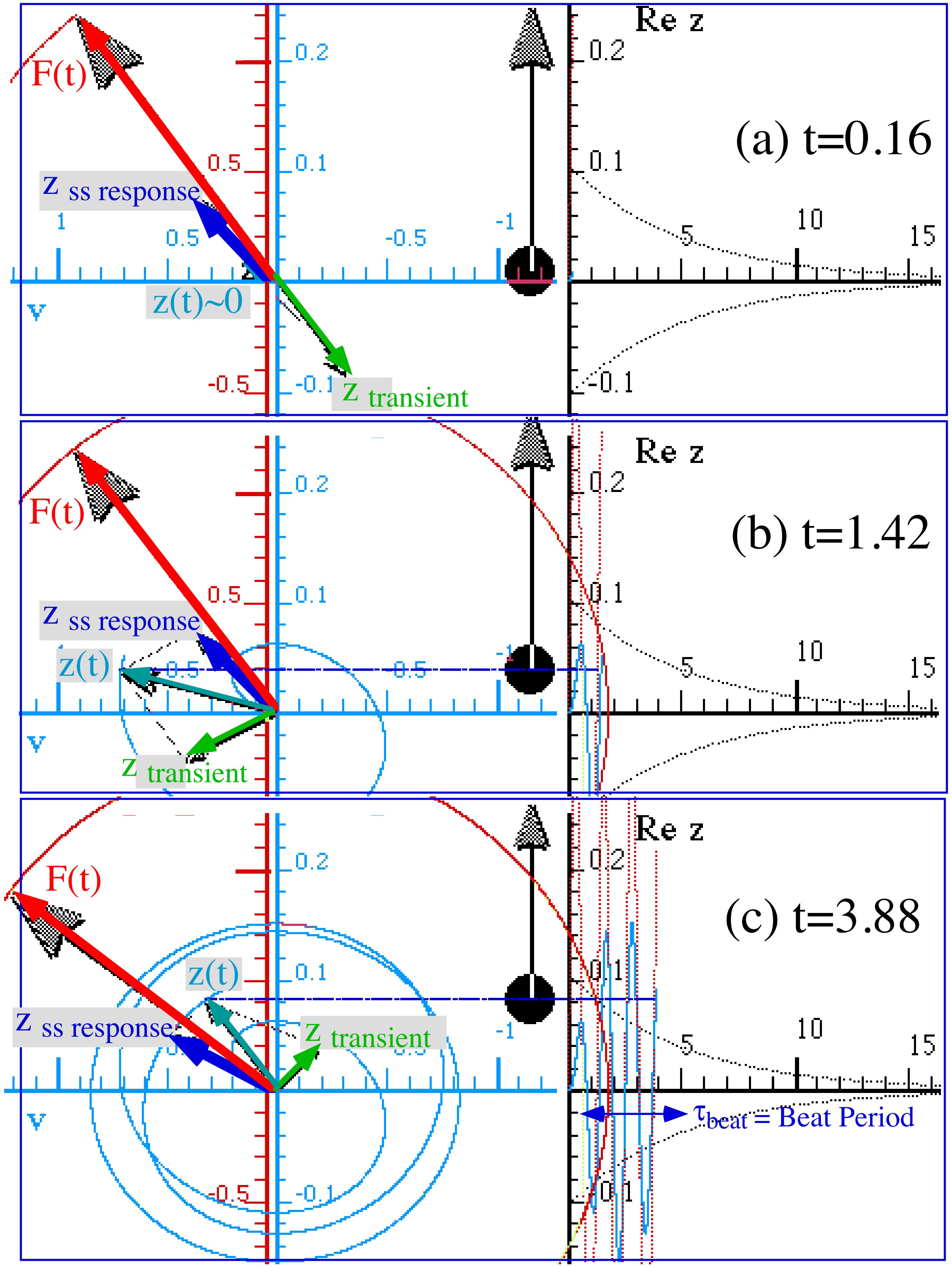


Fig. 4.2.9 Beat formation.

Transient phasor $z_{transient}$ catches up with F-phasor and passes it.

[OscillIt (Beating) Simulation]

Oscillator figures of merit: quality factors Q and q=2πQ

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s = \omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad (\textit{angular quality factor})$$

Amplification factor $q = \omega_0/2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0/2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

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$$\left(\begin{array}{c} t_{5\%} = 3/\Gamma = \textit{Lifetime} \\ \text{for decaying oscillator} \\ \text{to lose 95\% of} \\ \text{amplitude} \end{array}\right) \textit{times} \left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{array}{c} \text{number } n_{5\%} \\ \text{of oscillations} \\ \text{in a } t_{5\%} \textit{ Lifetime} \end{array}$$

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The "Heartbeat Count" measure of lifetime

Oscillator figures of merit: quality factors Q and q=2πQ

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The "Heartbeat Count" measure of lifetime

Energy decay
(proportional to the square of oscillator amplitude): $\left(e^{\Gamma t}\right)^2 = e^{-2\Gamma t}$

$$dE = -2\Gamma E$$

Oscillator figures of merit: quality factors Q and $q=2\pi Q$

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The "Heartbeat Count" measure of lifetime

Energy decay (proportional to the square of oscillator amplitude): $\left(e^{\Gamma t}\right)^2 = e^{-2\Gamma t}$ $\quad dE = -2\Gamma E$

$$\text{Relative amount of energy lost each cycle period} = \tau_0\left(\frac{-dE}{E}\right) = \frac{2\Gamma}{\upsilon_0} \equiv \frac{1}{Q} = \frac{2\pi}{q}$$

$$\left(\tau_0 = \frac{1}{\upsilon_0}\right)$$

$$Q = (\textit{Standard angular quality factor}) = \frac{q}{2\pi}$$

Oscillator figures of merit: Uncertainty 1/q

To see a beat we need $\tau_{\text{half-beat}}$ to be less than $\tau_{5\%}$ or 3/Γ. (Here we approximate π~3.0, again.)

$$\pi / |\omega_s - \omega_0| < 3 / \Gamma \qquad\qquad |\omega_s - \omega_0| > \Gamma$$

This means ω-detuning error is greater than or equal to the decay rate Γ.

Any detuning less than Γ is virtually undetectable.

Total ω uncertainty is ±Γ or twice Γ (that is: FWHM $\Delta\omega = 2\Gamma$).

The *relative frequency uncertainty* $\frac{2\Gamma}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \frac{1}{q} = \frac{\Delta\upsilon}{\upsilon_0}$

is the *inverse* of the *angular quality factor q*.

Linear frequency uncertainty is:

$$\Delta\upsilon = \Delta\omega / 2\pi = \Gamma / \pi$$

If we think of the 5% or 4.321% lifetime of a musical note as its time uncertainty Δt, then:

$$\Delta t \Delta\upsilon = 3 / \pi \approx 1$$

$$\Delta t = t_{5\%} = 3 / \Gamma \qquad\qquad \Delta t = t_{4.321\%} = \pi / \Gamma$$

Very precise measures of imprecision

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

Complex detuning-decay $\delta = \Delta - i\Gamma$ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

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$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \text{Re}\,L \quad + i\,\text{Im}\,L \quad = \frac{\Delta}{\Delta^2 + \Gamma^2} \quad + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = \quad |L|^2\,\Delta \quad + i\,|L|^2\,\Gamma$$

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$$= |L|\,e^{i\rho} = |L|\cos\rho + i\,|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where: } |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

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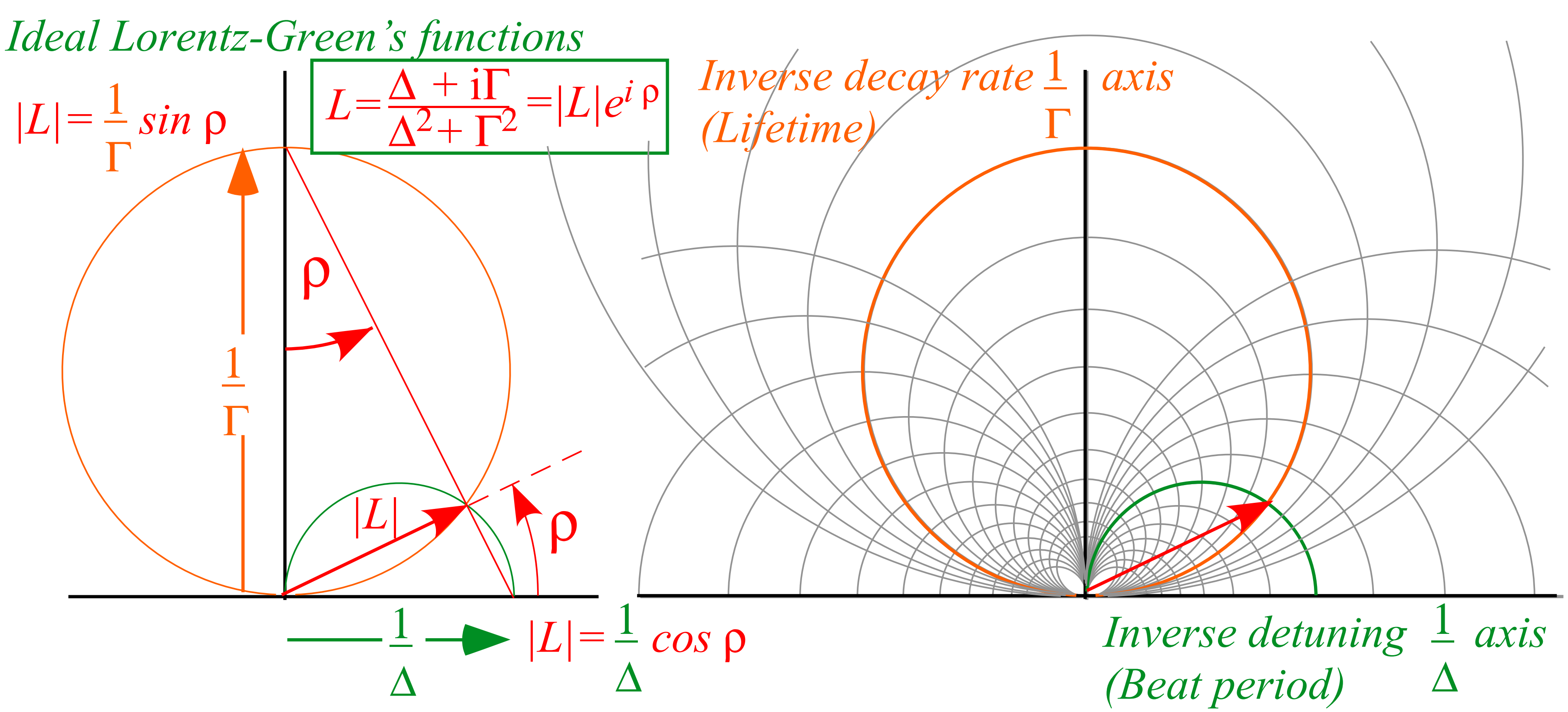
$$|L| = \frac{1}{\Gamma}\sin\rho$$

$$|L| = \frac{1}{\Delta}\cos\rho$$



Smith plots

Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0}L(\Delta - i\Gamma)$$

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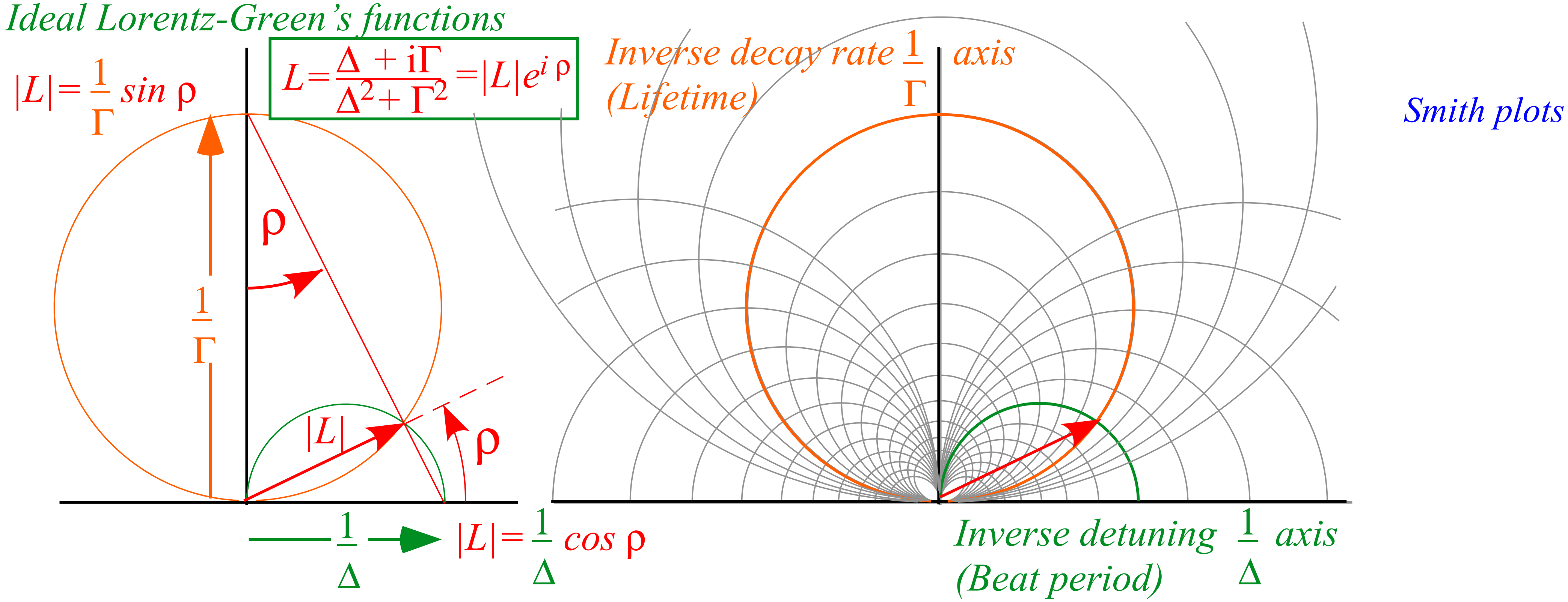


Fig. 4.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time $1/\Gamma$ *vs. beat-period* $1/\Delta$ *coordinates)*

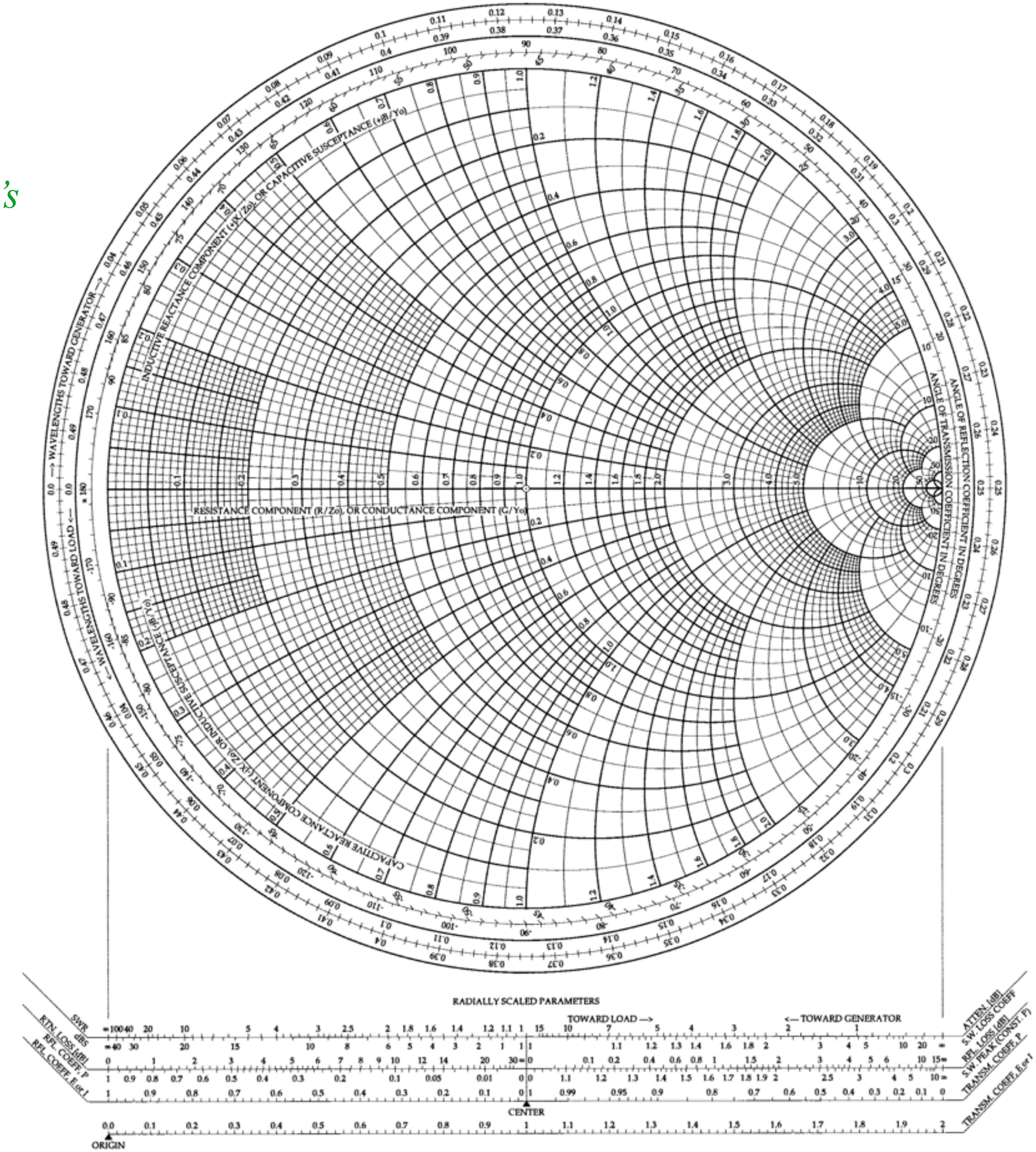
Constant Δ and Γ curves in Fig. 4.2.13 are orthogonal circles of *1/z- dipolar coordinates*. Recall Fig. 1.10.11.

SMITH CHART (Invented by Phillip H. Smith 1905-1987)

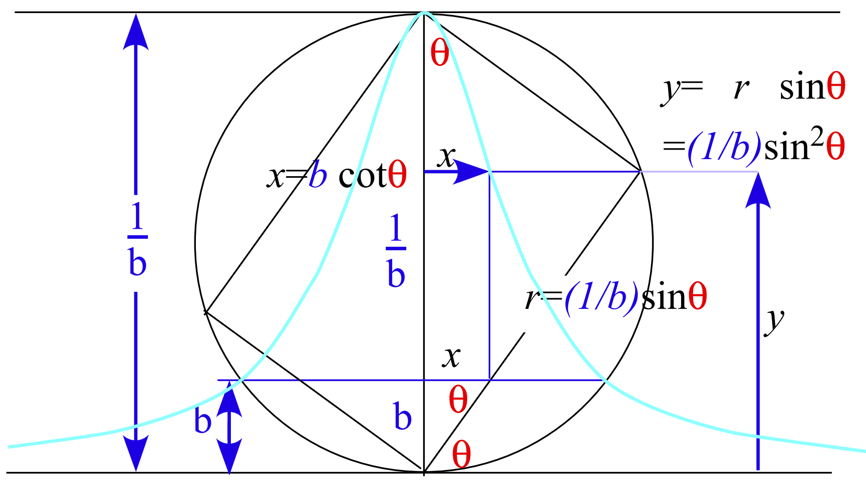
An FDHO Green's Function Slide rule

A plot of $f(z) = 1/z$

For wavy "Ohm's Laws"
$V = I \cdot Z$
$I = V/Z$



The Common Lorentzian (a.k.a. *The Witch of Agnesi*)



$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

$$y=\frac{b}{x^2+b^2}$$

Common Lorentzian function I.
(imaginary "absorbtive" part)

Maria Gaetana Agnesi

Born	May 16, 1718
Died	January 9, 1799 (aged 80)
Residence	Italy
Nationality	Italy
Fields	Mathematics

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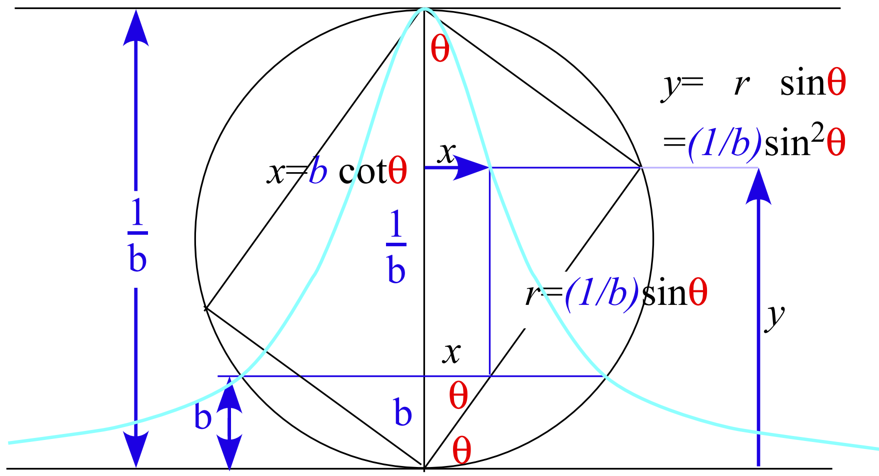

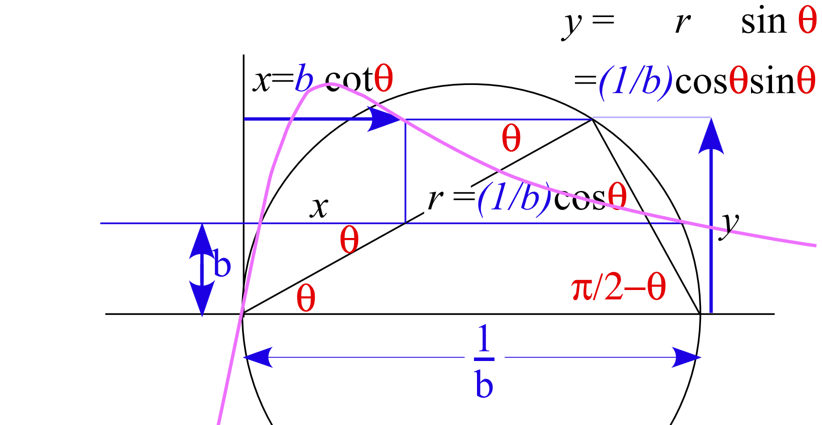

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(real "refractory" part)

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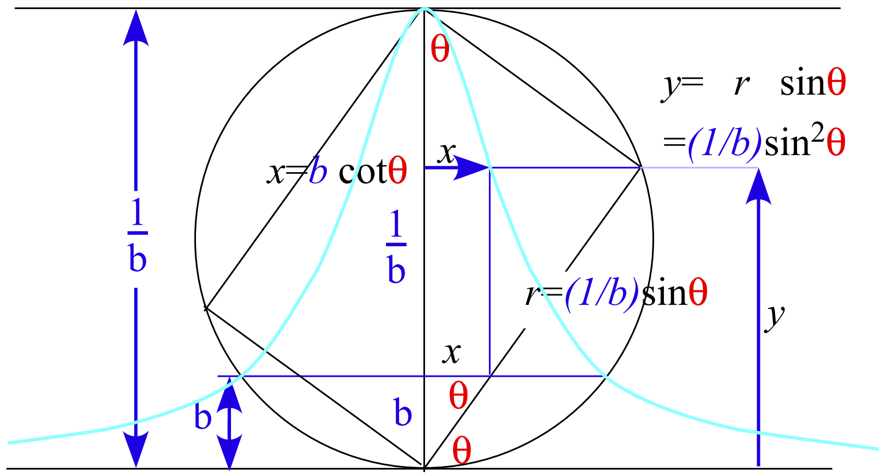

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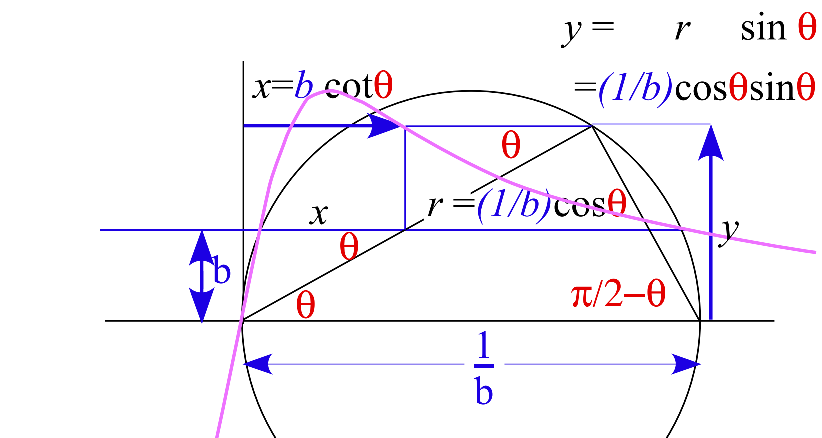

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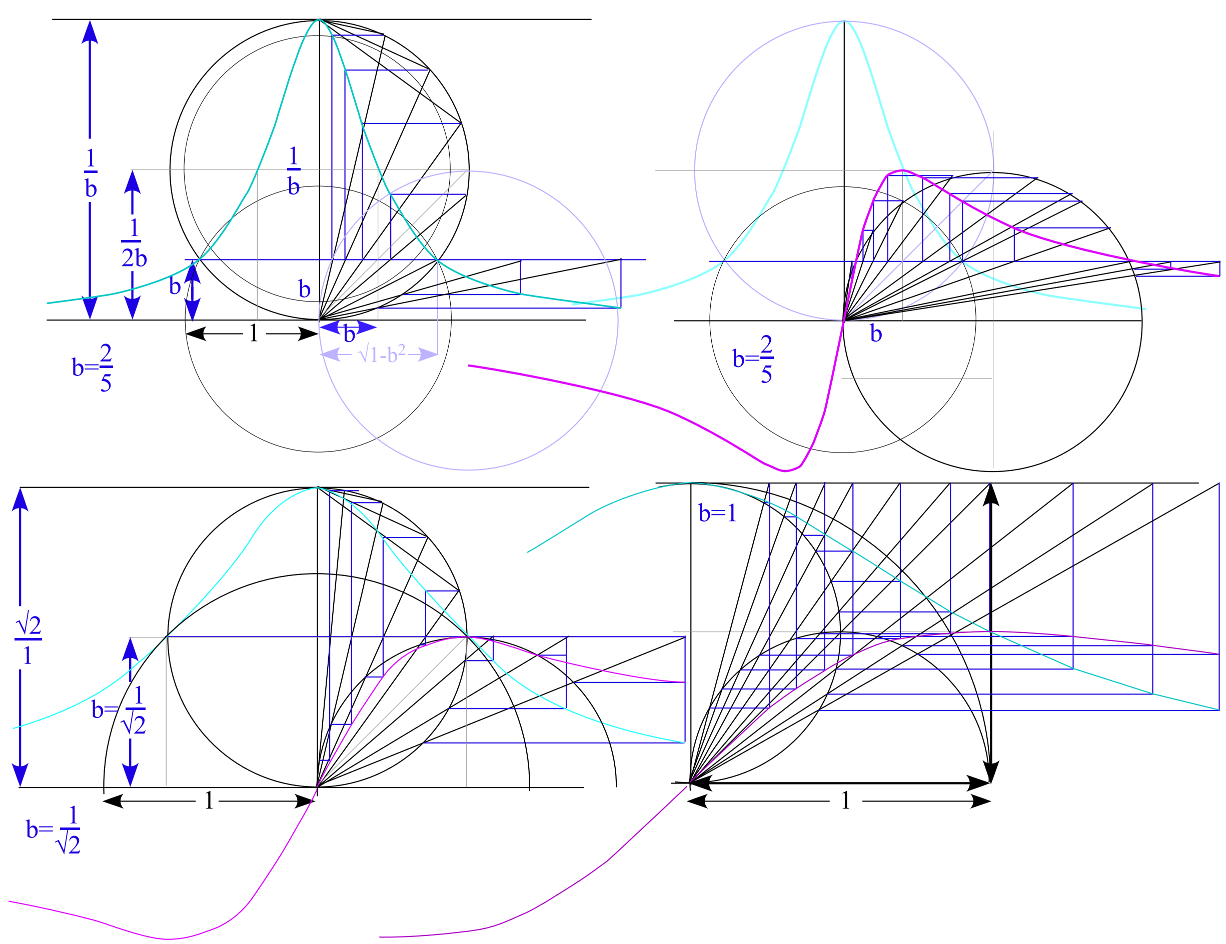

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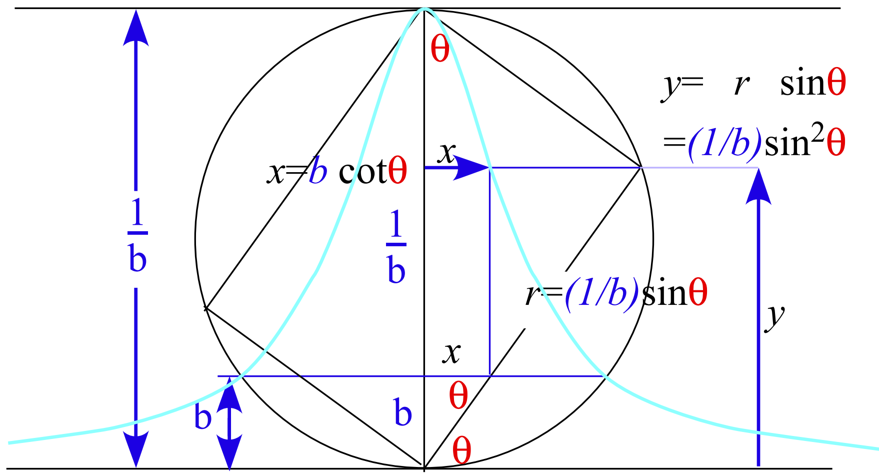

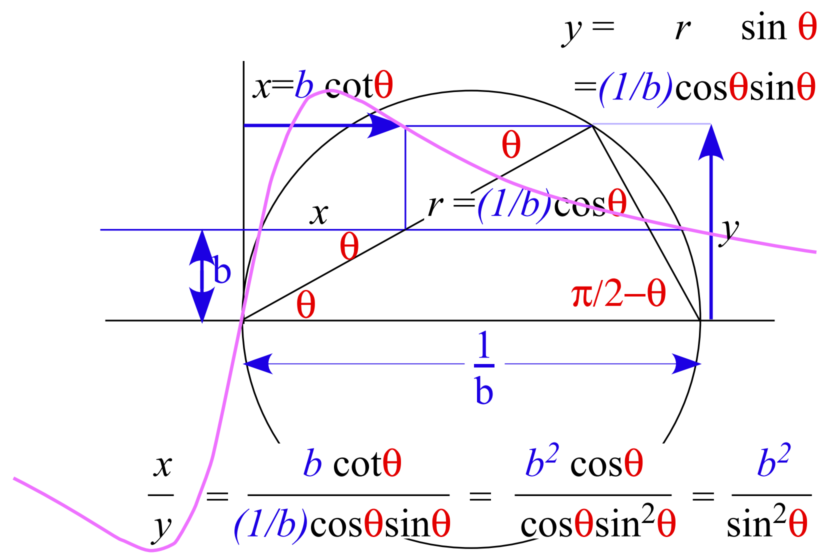

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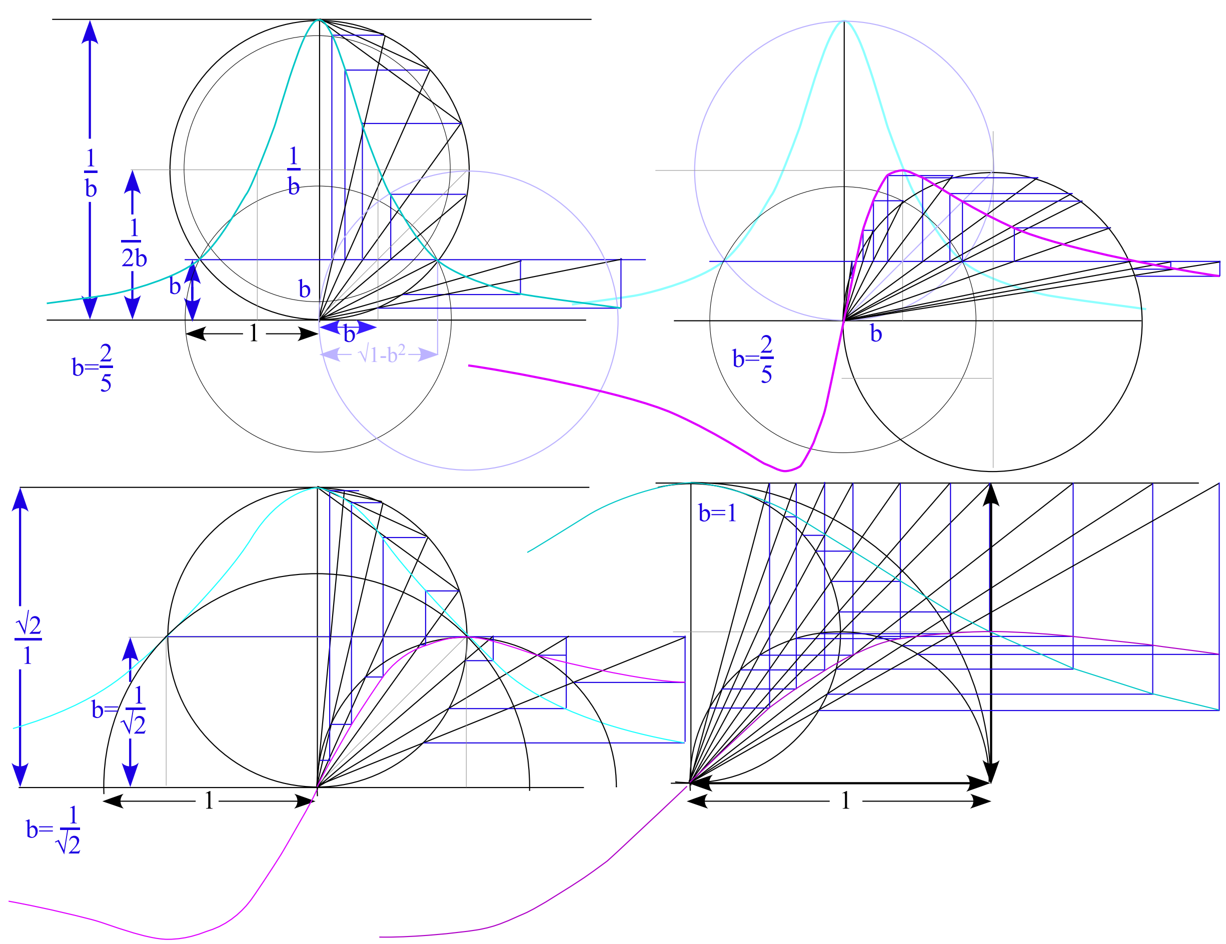
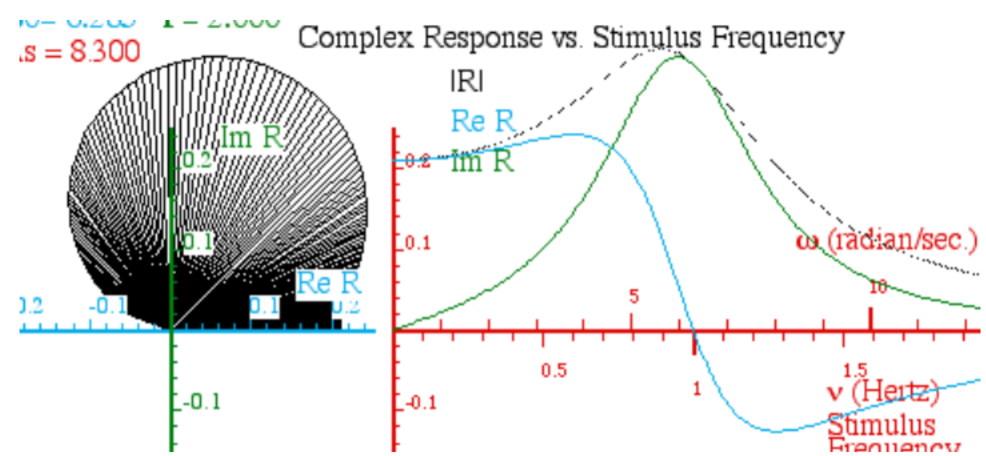
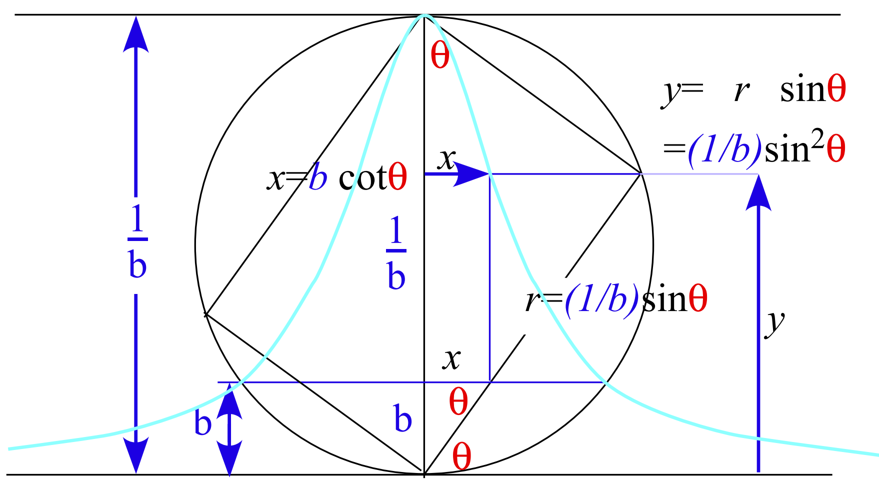


Compare ideal Lorentzians (Γ=0.2)
with a very non-ideal one (Γ=2)



[OscillIt Web Simulation - Lorentz Response Function (Γ = 0.2)]

The Common Lorentzian (a.k.a. *The Witch of Agnesi*)



$x^2=b^2\cot^2\theta=b^2\dfrac{\cos^2\theta}{\sin^2\theta}=b^2\dfrac{1-\sin^2\theta}{\sin^2\theta}=\dfrac{b^2}{\sin^2\theta}-b^2$

$x^2+b^2=\dfrac{b^2}{\sin^2\theta}=\dfrac{b}{y}$

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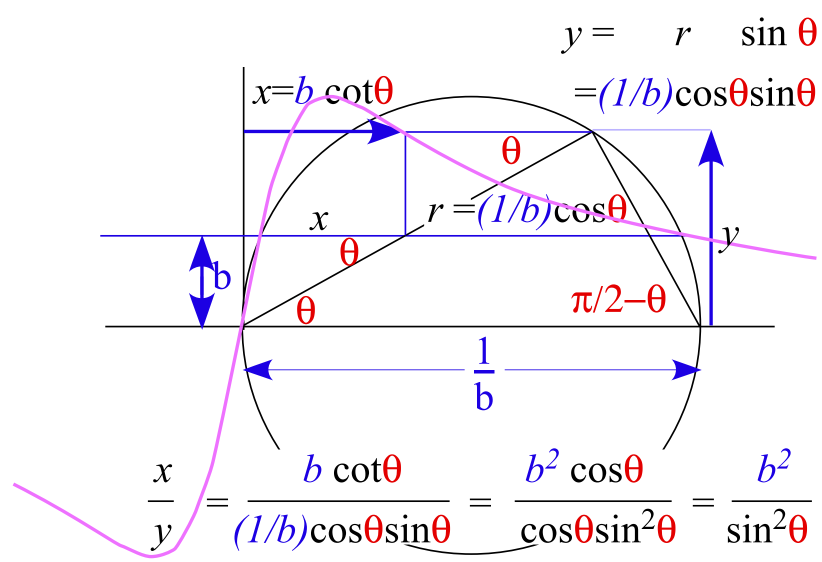
Common Lorentzian function I.
(imaginary "absorbtive" part)



$\dfrac{x}{y}=\dfrac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\dfrac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\dfrac{b^2}{\sin^2\theta}$

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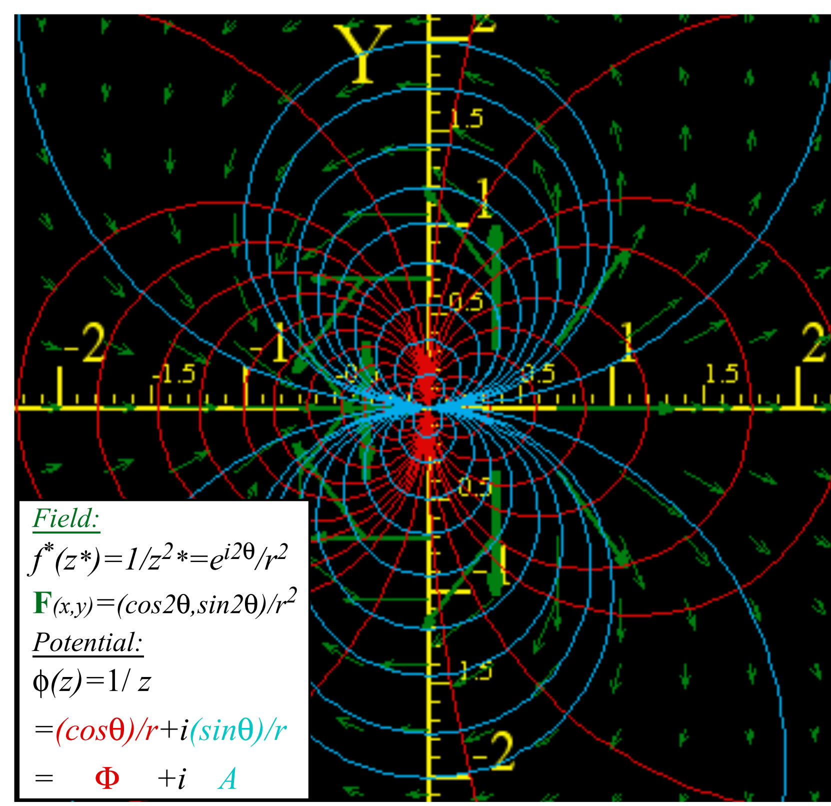


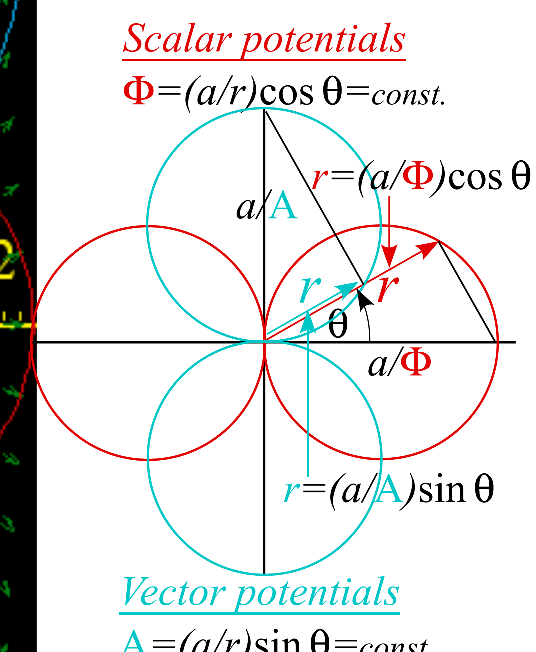
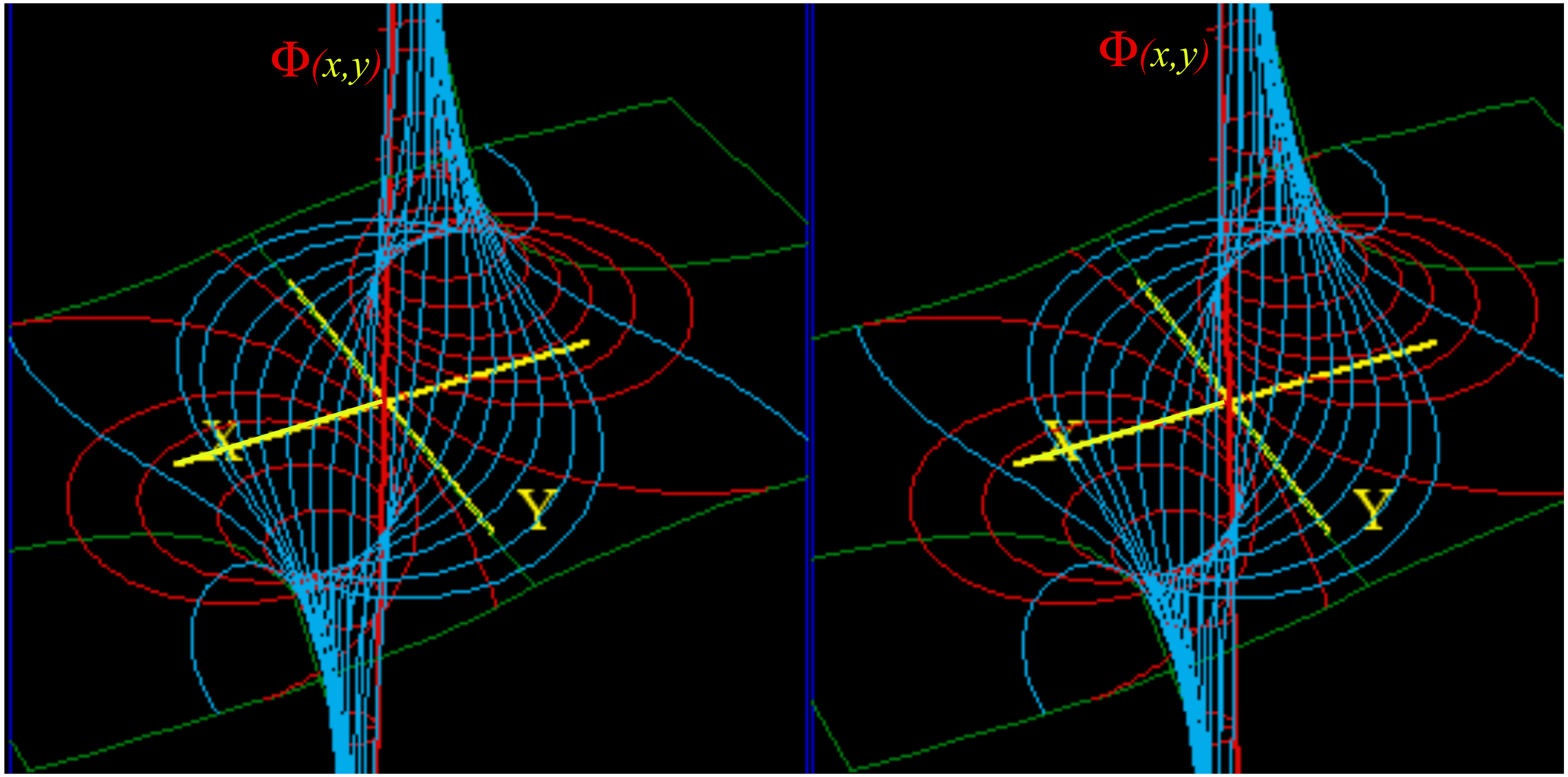
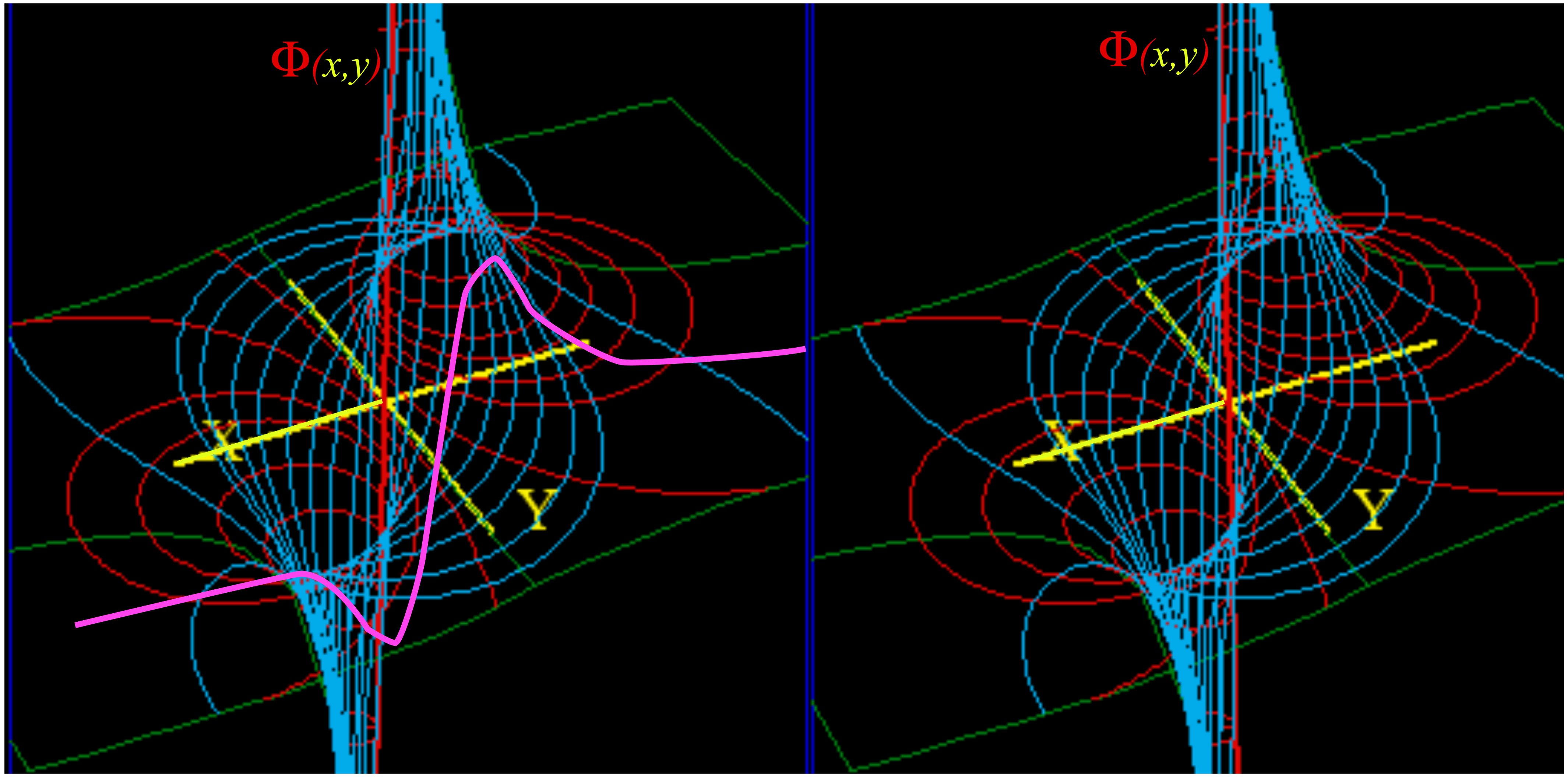


Fig. 10.11 Dipole **F***-field $f(z)=1/z^2$ and scalar potential (Φ=const.)-circles orthogonal to (A=const.)-circles.*



From: Fig. 1.10.12



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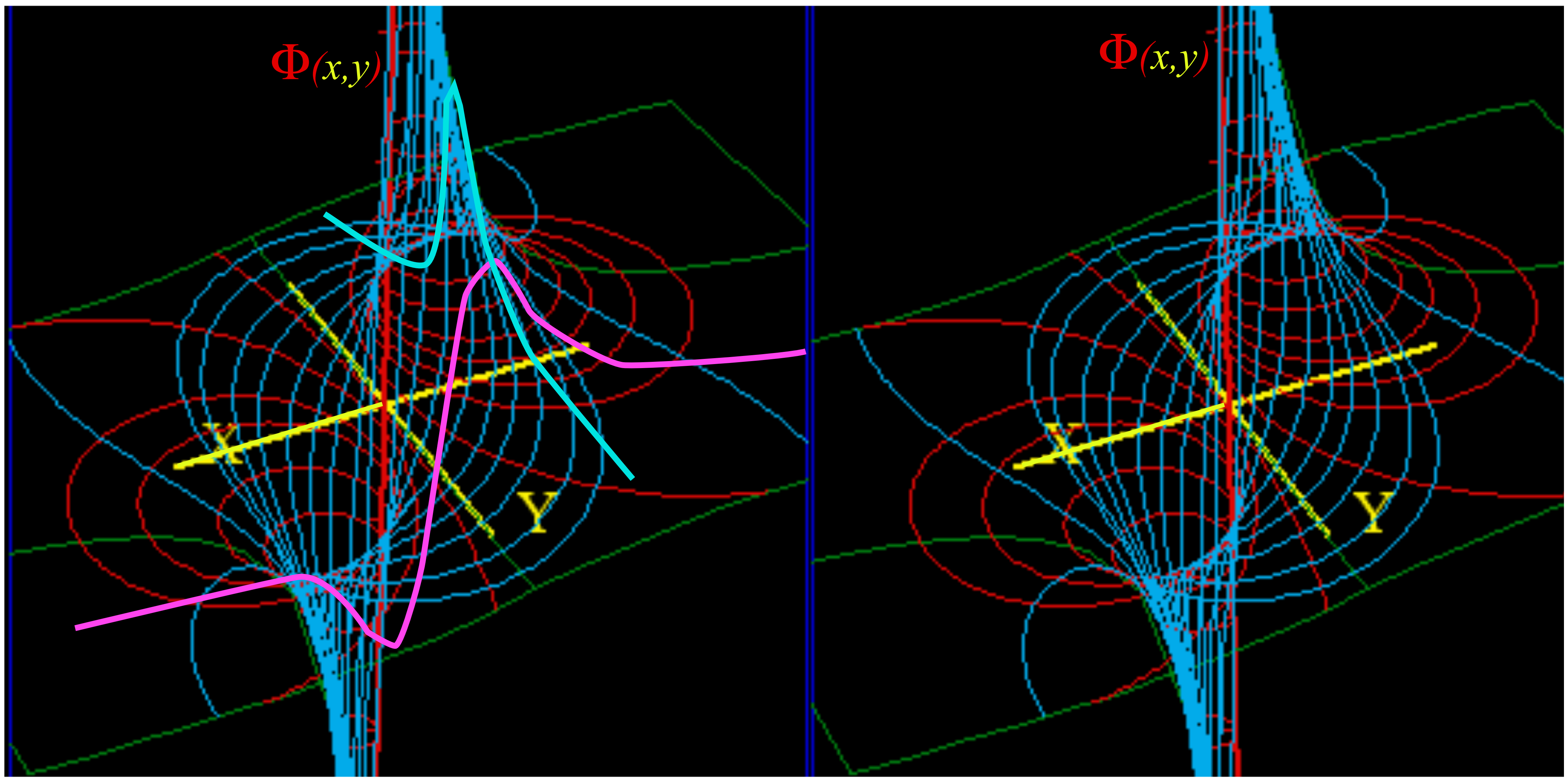
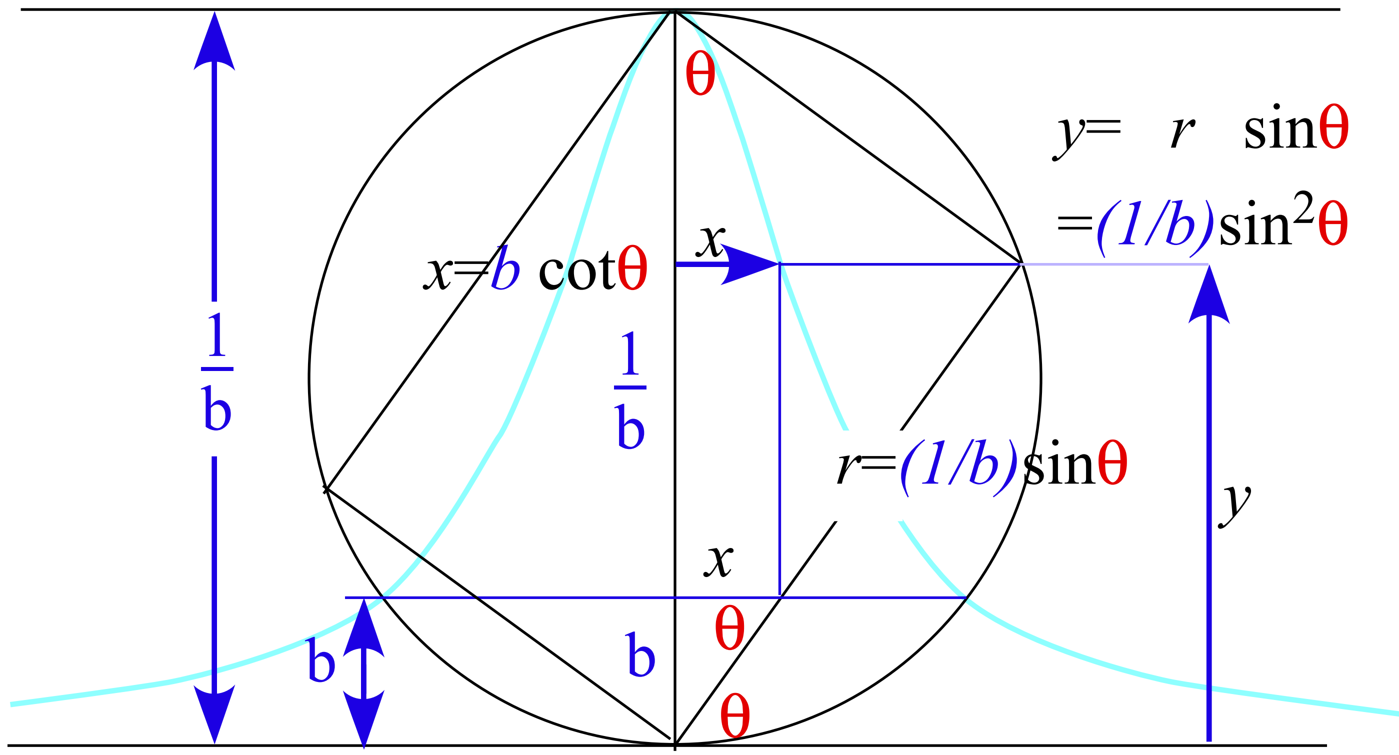


From: Fig. 1.10.12

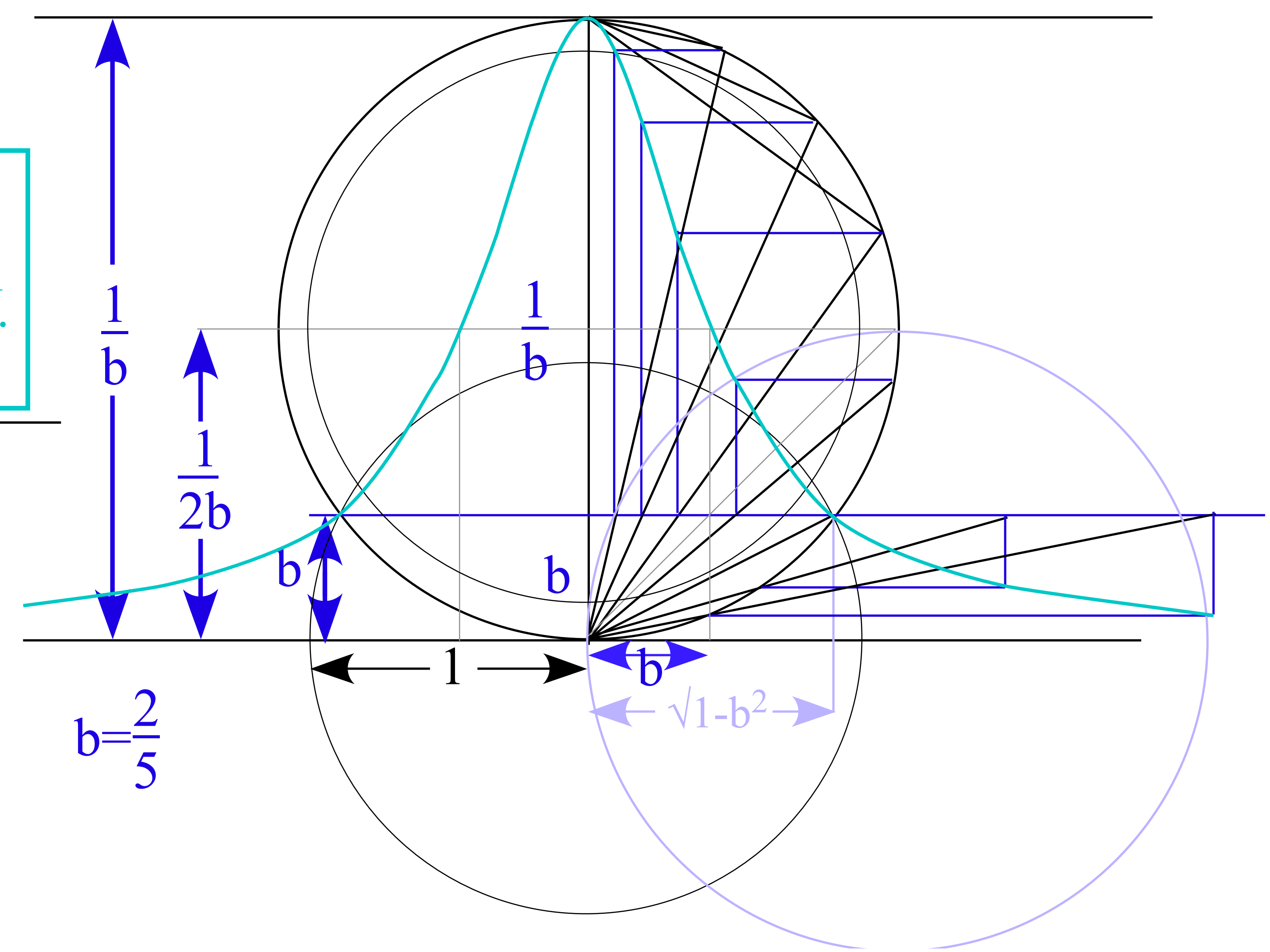


$$x^2 = b^2 \cot^2\theta = b^2 \frac{\cos^2\theta}{\sin^2\theta} = b^2 \frac{1-\sin^2\theta}{\sin^2\theta} = \frac{b^2}{\sin^2\theta} b^2$$

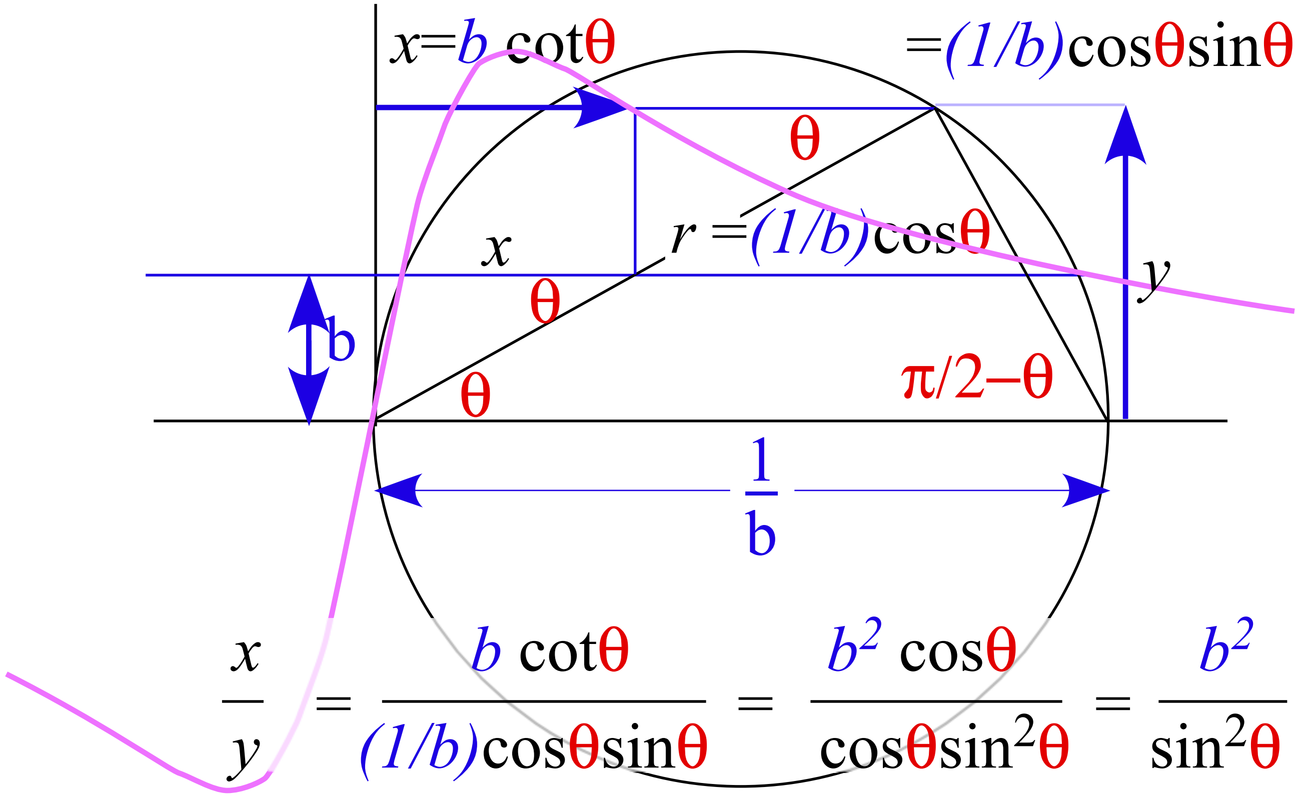
$$x^2 + b^2 = \frac{b^2}{\sin^2\theta} = \frac{b}{y}$$

$$y = \frac{b}{x^2+b^2}$$

Common Lorentzian function I.
(imaginary "absorbtive" part)



$$y = r \sin\theta = (1/b)\cos\theta\sin\theta$$

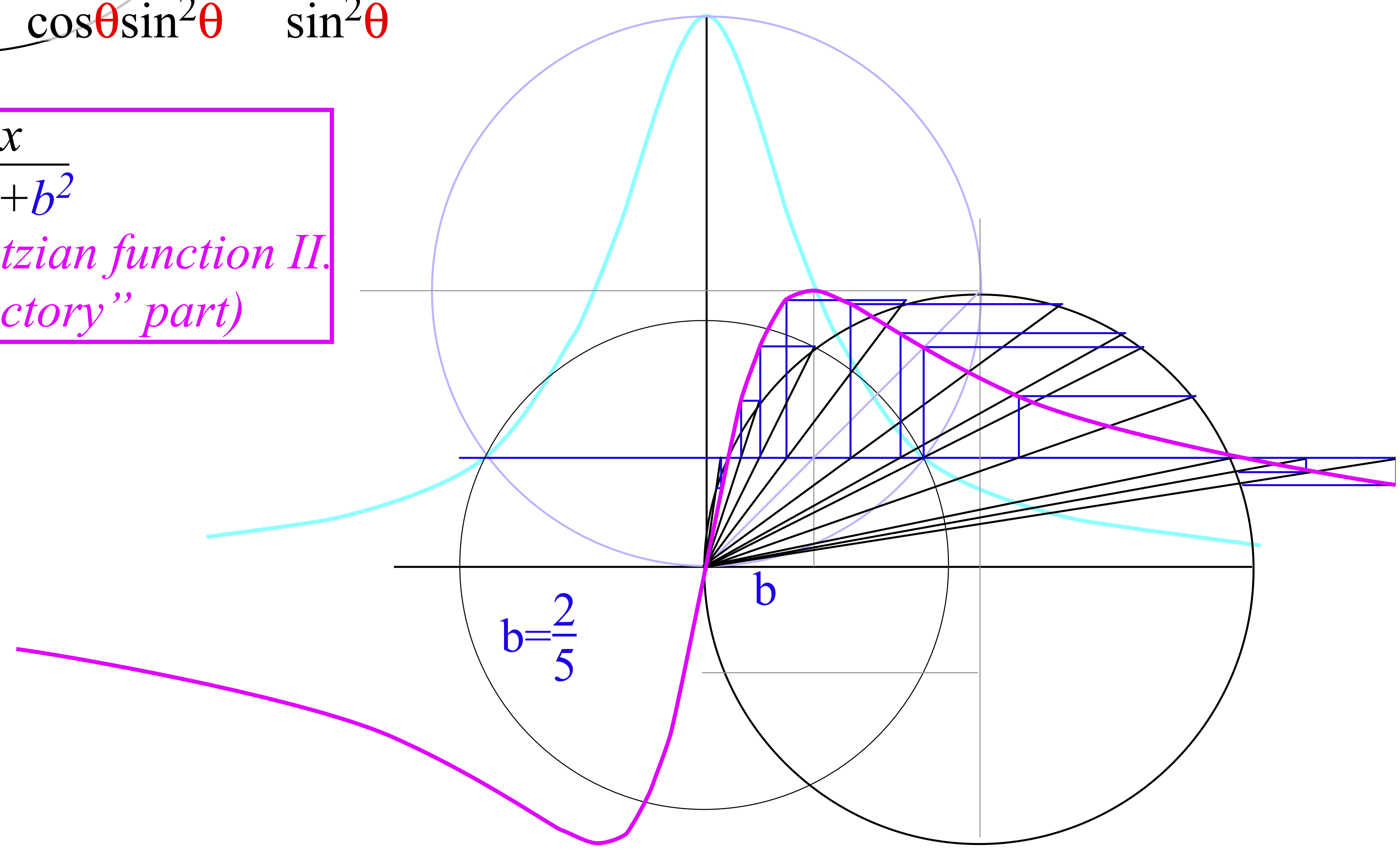


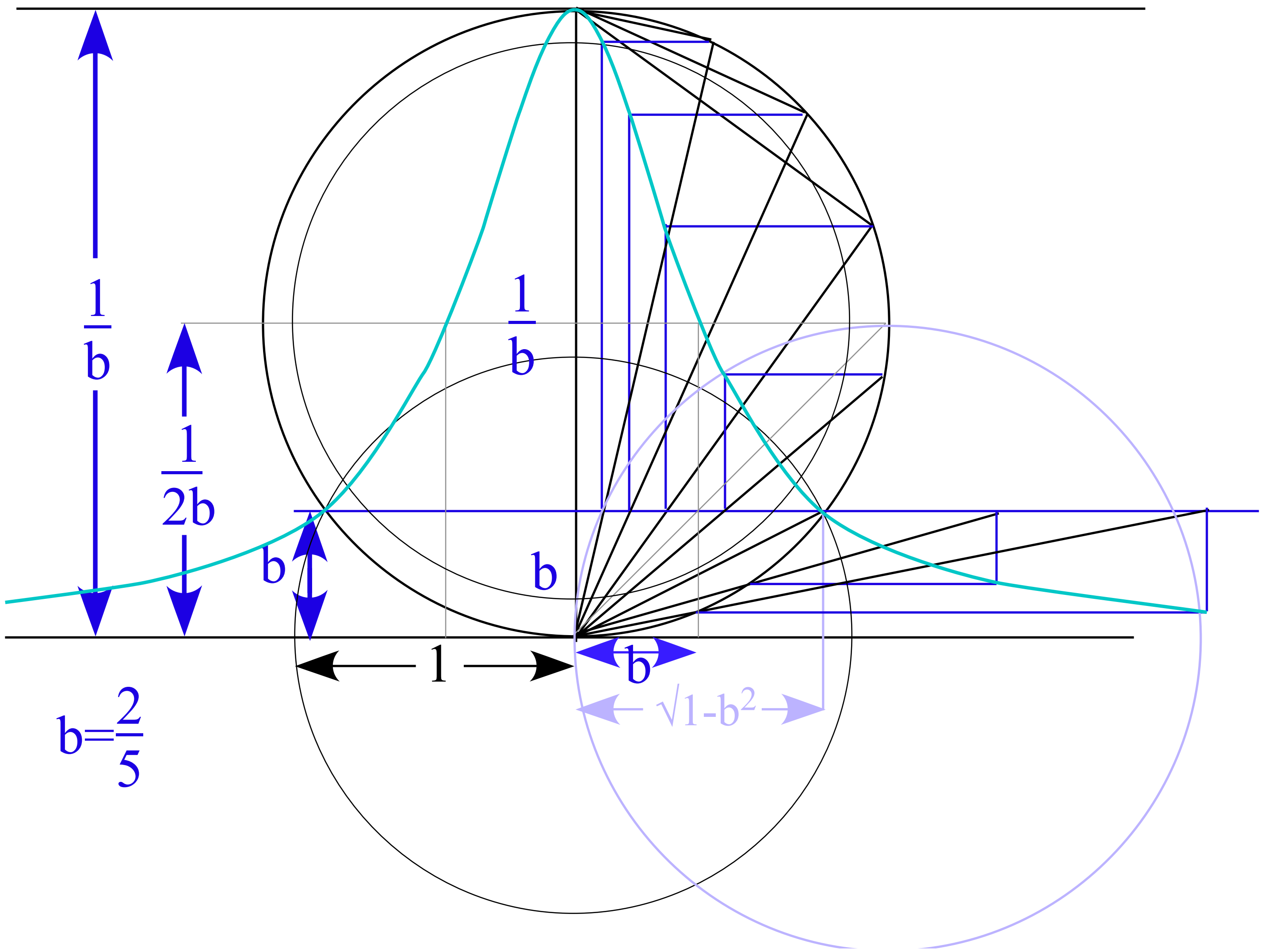
$$\frac{x}{y} = \frac{b\cot\theta}{(1/b)\cos\theta\sin\theta} = \frac{b^2\cos\theta}{\cos\theta\sin^2\theta} = \frac{b^2}{\sin^2\theta}$$

$$x^2+b^2 = \frac{b^2}{\sin^2\theta} = \frac{x}{y}$$

$$y = \frac{x}{x^2+b^2}$$

Common Lorentzian function II.
(real "refractory" part)





$\frac{1}{b}$
$\frac{1}{b}$
$\frac{1}{2b}$
b
b
1
b
$\sqrt{1-b^2}$
$b=\frac{2}{5}$