# Lecture 20
## Thur. 11.02 thru Tue. 11.07 2017

# *Introduction to classical oscillation and resonance*

*(Ch. 1 of Unit 4 )*

*1D forced-damped-harmonic oscillator equations and Green's function solutions*

- *Linear harmonic oscillator equation of motion.*
- *Linear damped-harmonic oscillator equation of motion.*
  - *Frequency retardation and amplitude damping*
  - *Figure of oscillator merit (the 5% solution $3/\Gamma$ and other numbers)*
- *Linear forced-damped-harmonic oscillator equation of motion.*
  - *Phase lag and amplitude resonance amplification*
  - *Figure of resonance merit: (angular) Quality factor $q=\omega_0/2\Gamma$*

*Properties of Green's function solutions and their mathematical/physical behavior*

- *Transient solutions vs. Steady State solutions*

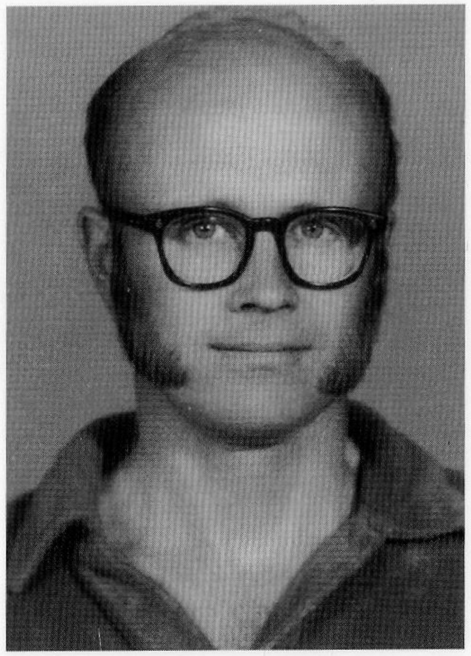
*Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)*

- *Quality factors: Beat, lifetimes, and uncertainty*

*Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

- *Common Lorentzian (a.k.a. Witch of Agnesi)*
- *Smith Charts*

> *Without resonance…*
> *…we are all blind, deaf, and dumb.*
>
> *Anonymous*



PAULINIA, BRASIL 1976

THE SPEED OF LIGHT IS
299,792,458 METERS PER SECOND!

# -- The Purest Light and a Resonance Hero – Ken Evenson (1932-2002) --

Ken Evenson

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When travelers punch up their GPS coordinates they owe a debt of gratitude to an under sung hero who, alongside his colleagues and students, often toiled 18 hour days deep inside a laser laboratory lit only by the purest light in the universe.

Ken was an "Indiana Jones" of modern physics. While he may never have been called "Montana Ken," such a name would describe a real life hero from Bozeman, Montana, whose extraordinary accomplishments in many ways surpass the fictional characters in cinematic thrillers like *Raiders of the Lost Arc*.

Indeed, there were some exciting real life moments shared by his wife Vera, one together with Ken in a canoe literally inches from the hundred-foot drop-off of Brazil's largest waterfall. But, such outdoor exploits, of which Ken had many, pale in the light of an in-the-lab brilliance and courage that profoundly enriched the world.

Ken is one of few researchers and perhaps the only physicist to be twice listed in the *Guinness Book of Records*. The listings are not for jungle exploits but for his lab's highest frequency measurement and for a speed of light determination that made $c$ many times more precise due to his lab's pioneering work with John Hall in laser resonance and metrology[†].

The meter-kilogram-second (mks) system of units underwent a redefinition largely because of these efforts. Thereafter, the speed of light $c$ was set to $299{,}792{,}458 ms^{-1}$. The meter was defined in terms of $c$, instead of the other way around since his time precision had so far trumped that for distance. Without such resonance precision, the Global Positioning System (GPS), the first large-scale wave space-time coordinate system, would not be possible.

Ken's courage and persistence at the Time and Frequency Division of the Boulder Laboratories in the National Bureau of Standards (now the National Institute of Standards and Technology or NIST) are legendary as are his railings against boneheaded administrators who seemed bent on thwarting his best efforts. Undaunted, Ken's lab painstakingly exploited the resonance properties of metal-insulator diodes, and succeeded in literally counting the waves of near-infrared radiation and eventually visible light itself.

Those who knew Ken miss him terribly. But, his indelible legacy resonates today as ultra-precise atomic and molecular wave and pulse quantum optics continue to advance and provide heretofore unimaginable capability. Our quality of life depends on their metrology through the Quality and Finesse of the resonant oscillators that are the heartbeats of our technology.

Before being taken by Lou Gehrig's disease, Ken began ultra-precise laser spectroscopy of unusual molecules such as $HO_2$, the radical cousin of the more common $H_2O$. Like Ken, such radical molecules affect us as much or more than better known ones. But also like Ken, they toil in obscurity, illuminated only by the purest light in the universe.

In 2005 the Nobel Prize in physics was awarded to Glauber, Hall, and Hensch[††] for laser optics and metrology.

[†] K. M. Evenson, J.S. Wells, F.R. Peterson, B.L. Danielson, G.W. Day, R.L. Barger and J.L. Hall,
Phys. Rev. Letters 29, 1346(1972).

[††] *The Nobel Prize in Physics, 2005*. http://nobelprize.org/

# *Linear forced-damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$

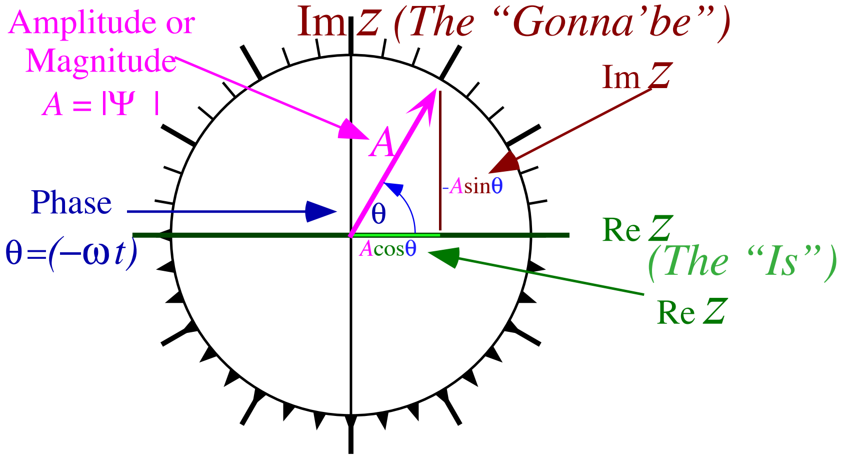
*Phasor Clock*

$Z(t) = Ae^{-i(\omega t)}$

$= A\cos(\omega t)$

$-i\,A\sin(\omega t)$

*Phasor clocks turn clockwise in time for positive ω*



$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

driven by external stimulating force → $F_{stimulus}(t) = eE(t)$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

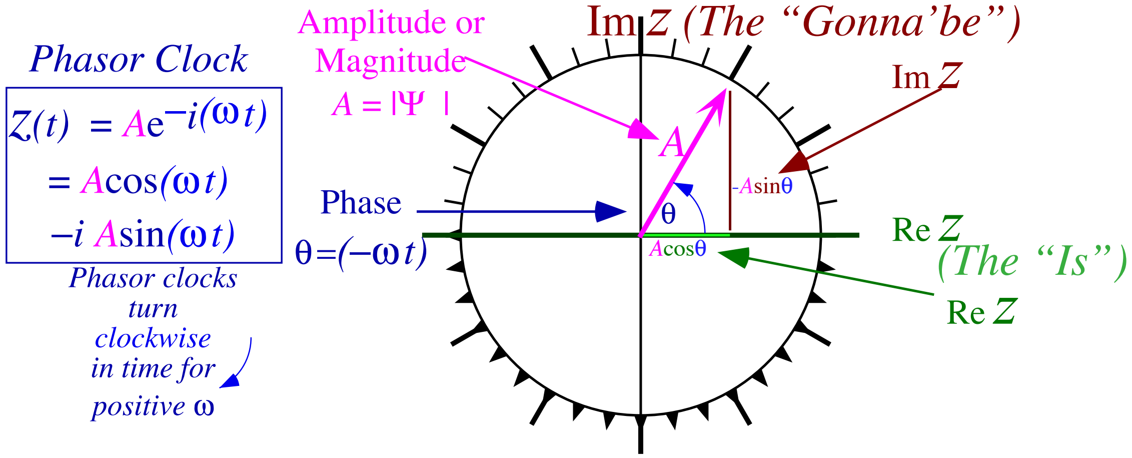
retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad \left(b = 2\Gamma m\right)$

# Linear harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = \quad F_{restore}$$

$$\frac{d^2z}{dt^2} = \quad \frac{F_{restore}}{m}$$

$$\frac{d^2z}{dt^2} \quad + \omega_0^2 z = \quad 0$$



Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force ⟶ $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

# Linear harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{restore}$$

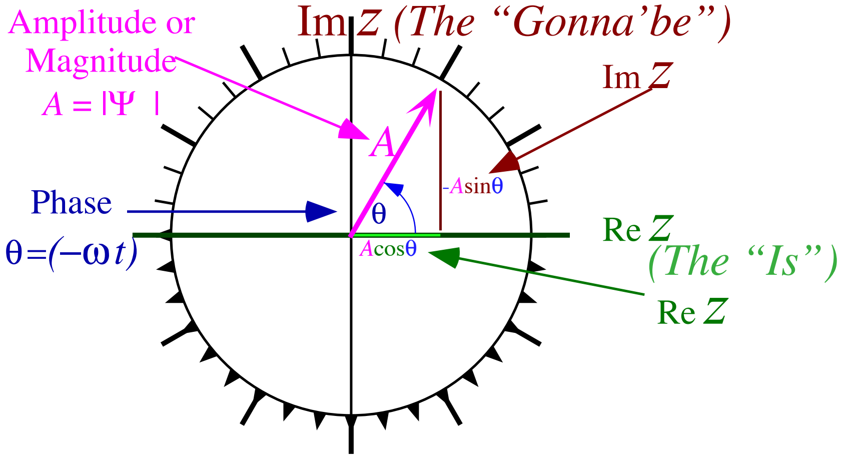
$$\frac{d^2z}{dt^2} = \frac{F_{restore}}{m}$$

$$\frac{d^2z}{dt^2} + \omega_0^2 z = 0$$

**Phasor Clock**

$Z(t) = Ae^{-i(\omega t)}$
$= A\cos(\omega t)$
$-i\, A\sin(\omega t)$

*Phasor clocks turn clockwise in time for positive ω*



Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

The famous one-Hertz ( $\nu_0$=1/s. or: $\omega_0 = 2\pi = 6.2832$rad/s. ) oscillator.

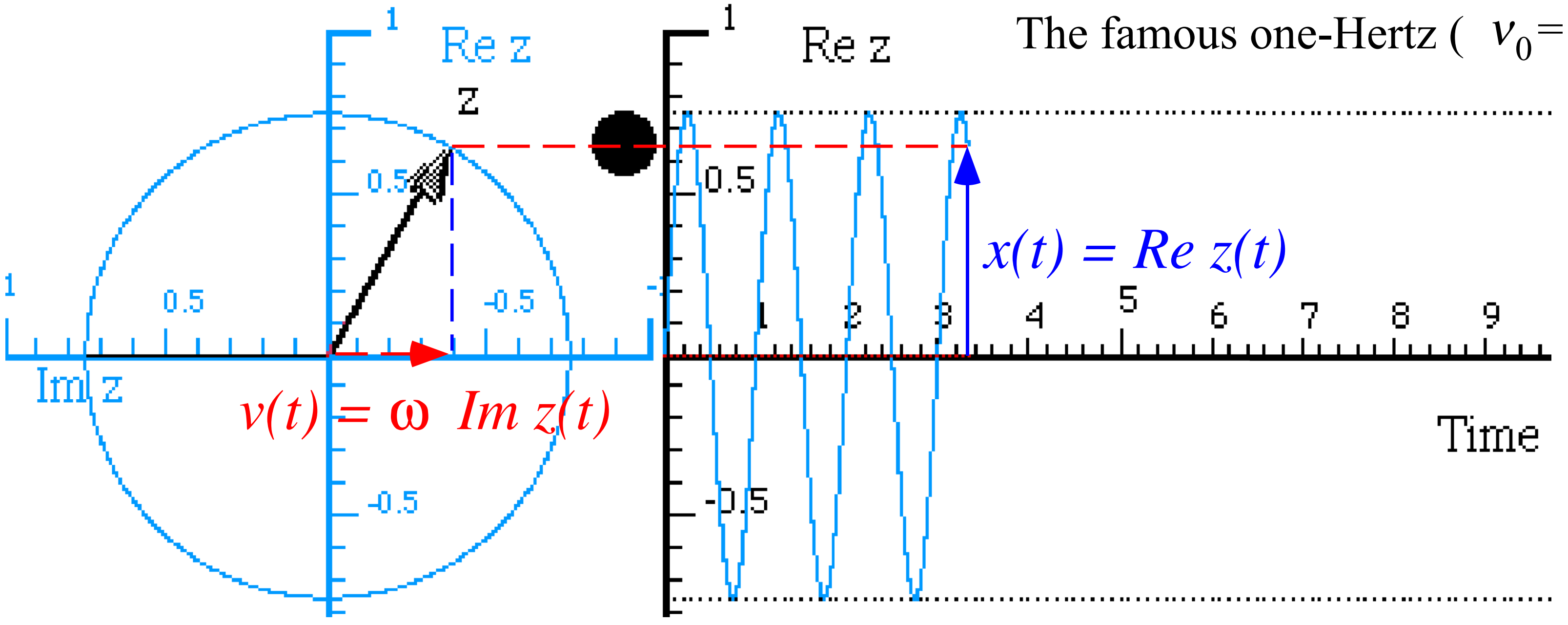


*Fig. 3.2.2 Phasor z and corresponding coordinate versus time plot for $\omega_0$=2π and Γ=0*

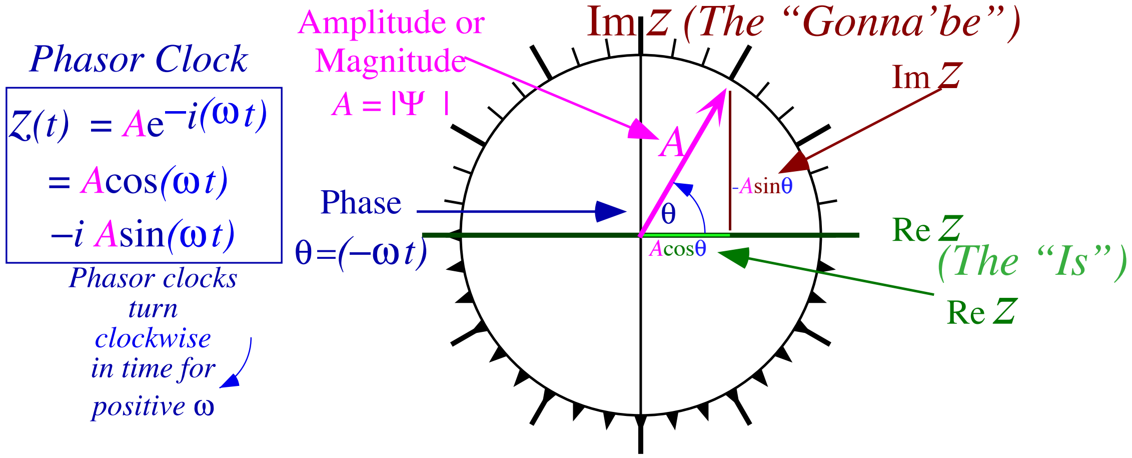
OscillIt Web Simulation

OscillIt Web Simulation (Generic):
http://www.uark.edu/ua/modphys/markup/OscillItWeb.html

# Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

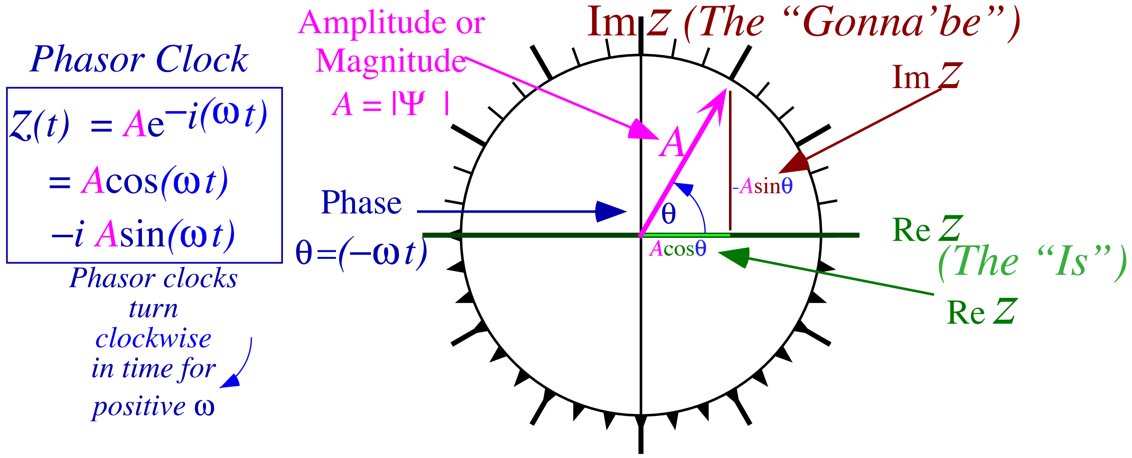
held back by a harmonic (linear) restoring force → $F_{restore} = -kz, \quad \left(k = \omega_0^2 m\right),$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}, \quad \left(b = 2\Gamma m\right)$

# *Linear -damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

# *Linear -damped-harmonic oscillator equation of motion.*

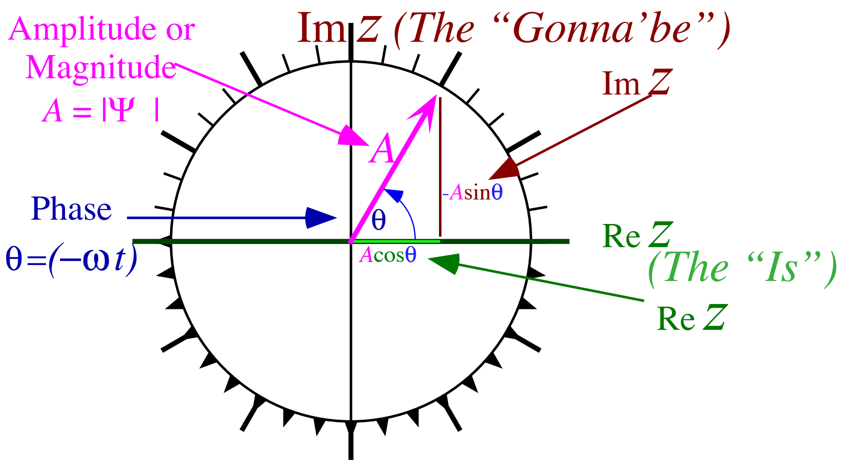
$$F_{total}(t)=m\frac{d^2z}{dt^2}=F_{damping}+F_{restore}$$

$$\frac{d^2z}{dt^2}=\frac{F_{damping}}{m}+\frac{F_{restore}}{m}$$

*Phasor Clock*

$$Z(t)=Ae^{-i(\omega t)}$$
$$=A\cos(\omega t)$$
$$-i\,A\sin(\omega t)$$

*Phasor clocks turn clockwise in time for positive ω*



$$\frac{d^2z}{dt^2}+2\Gamma\frac{dz}{dt}+\omega_0^2z=0$$

Trick:
Set: $z=z(t)=Ae^{-i\omega t}$

$$\left[(-i\omega)^2+2\Gamma(-i\omega)+\omega_0^2\right]e^{-i\omega t}=0$$

$$\omega^2+2i\Gamma\omega-\omega_0^2=0$$

Solve for: $\omega=\omega_\pm$

$$\omega_\pm=\frac{-2i\Gamma\pm\sqrt{-4\Gamma^2+4\omega_0^2}}{2}$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

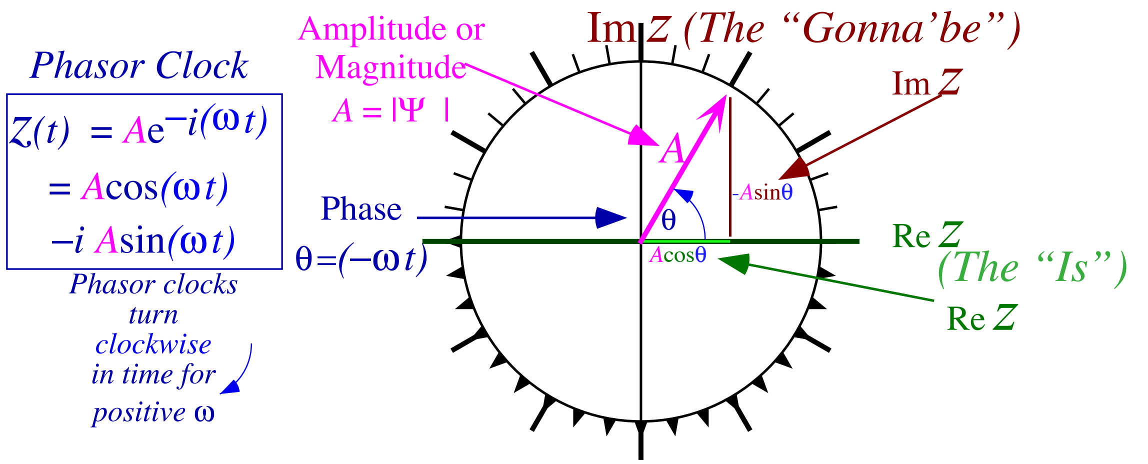
held back by a harmonic (linear) restoring force → $F_{restore}=-kz$

retarded by frictional damping force → $F_{damping}=-b\frac{dz}{dt}$

# *Linear damped-harmonic oscillator equation of motion.*

$$F_{total}(t)=m\frac{d^2z}{dt^2}=F_{damping}+F_{restore}$$

$$\frac{d^2z}{dt^2}=\frac{F_{damping}}{m}+\frac{F_{restore}}{m}$$



*Phasor Clock*

$Z(t)=Ae^{-i(\omega t)}$

$=A\cos(\omega t)$

$-i\,A\sin(\omega t)$

*Phasor clocks turn clockwise in time for positive ω*

Trick:
Set: $z=z(t)=Ae^{-i\omega t}$

$$\frac{d^2z}{dt^2}+2\Gamma\frac{dz}{dt}+\omega_0^2z=0$$

$$\left[(-i\omega)^2+2\Gamma(-i\omega)+\omega_0^2\right]e^{-i\omega t}=0$$

$$\omega^2+2i\Gamma\omega-\omega_0^2=0$$

Solve for: $\omega=\omega_\pm$

$$\omega_\pm=\frac{-2i\Gamma\pm\sqrt{-4\Gamma^2+4\omega_0^2}}{2}$$

$$=-i\Gamma\pm\sqrt{\omega_0^2-\Gamma^2}$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore}=-kz$

retarded by frictional damping force → $F_{damping}=-b\frac{dz}{dt}$

# Linear -damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

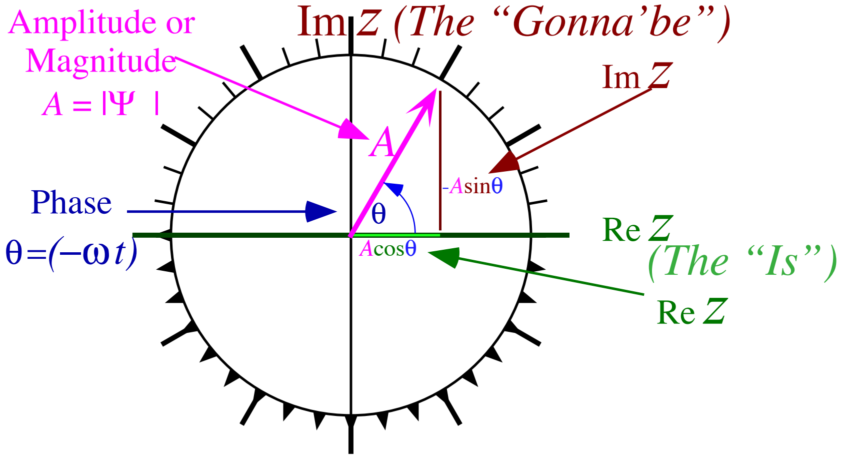
*Phasor Clock*

$Z(t) = Ae^{-i(\omega t)}$

$= A\cos(\omega t)$

$-i\,A\sin(\omega t)$

*Phasor clocks turn clockwise in time for positive ω*



Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$

$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$

$$= e^{-\Gamma t}e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}\,t}$$

$$= e^{-\Gamma t}e^{\pm i\omega_\Gamma t}$$

# *Linear damped-harmonic oscillator equation of motion.*

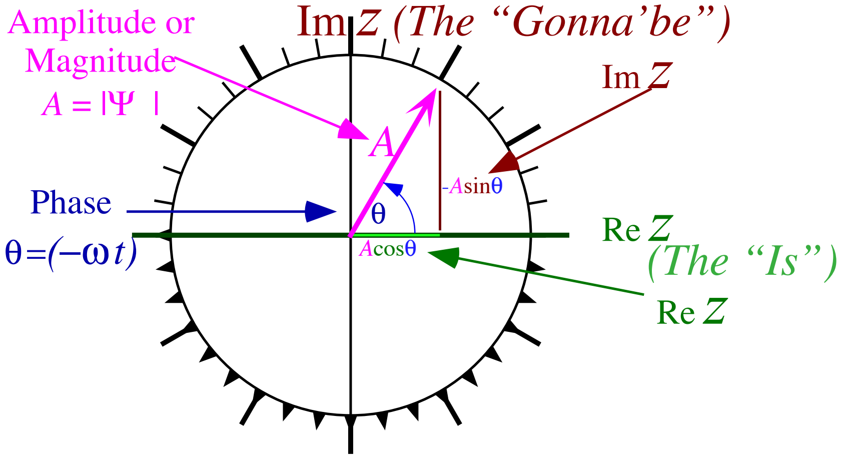
$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$

*Phasor Clock*

$$Z(t) = Ae^{-i(\omega t)}$$
$$= A\cos(\omega t)$$
$$-i\,A\sin(\omega t)$$

*Phasor clocks turn clockwise in time for positive ω*



Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Trick:
Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2}$$
$$= -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}\right)t}$$
$$= e^{\left(-\Gamma \pm i\sqrt{\omega_0^2 - \Gamma^2}\right)t}$$
$$= e^{-\Gamma t}\,e^{\pm i\sqrt{\omega_0^2 - \Gamma^2}\,t}$$
$$= e^{-\Gamma t}\,e^{\pm i\omega_\Gamma t}$$

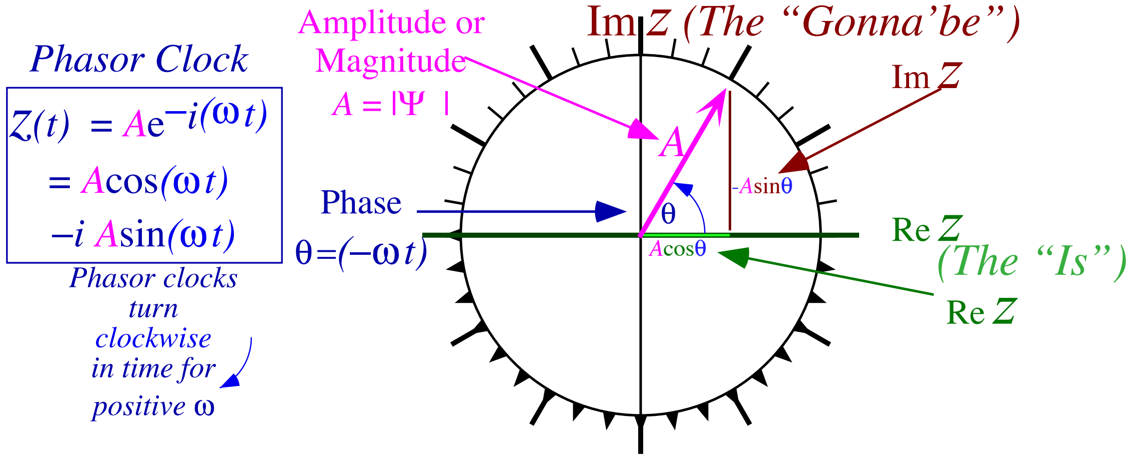
It oscillates at an angular frequency $\omega_\Gamma$ reduced slightly by .05% from $\omega_0$ due to damping $\Gamma$ =0.2.

$$\omega_\Gamma = \sqrt{\omega_0^2 - \Gamma^2} = \omega_0 - \tfrac{1}{2}(\Gamma^2/\omega_0) + \ldots = 6.2831853 - 0.003183 + .. = 6.280002 + \ldots = 6.280001$$

# *Linear -damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$Z(t) = Ae^{-i(\omega t)} = A\cos(\omega t) - i\,A\sin(\omega t)$$

*Phasor clocks turn clockwise in time for positive ω*

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force

retarded by frictional damping force

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

$$F_{restore} = -kz$$

$$F_{damping} = -b\frac{dz}{dt}$$

Trick:

Set: $z=z(t)= Ae^{-i\omega t}$

$$\left[(-i\omega)^2 + 2\Gamma(-i\omega) + \omega_0^2\right]e^{-i\omega t} = 0$$

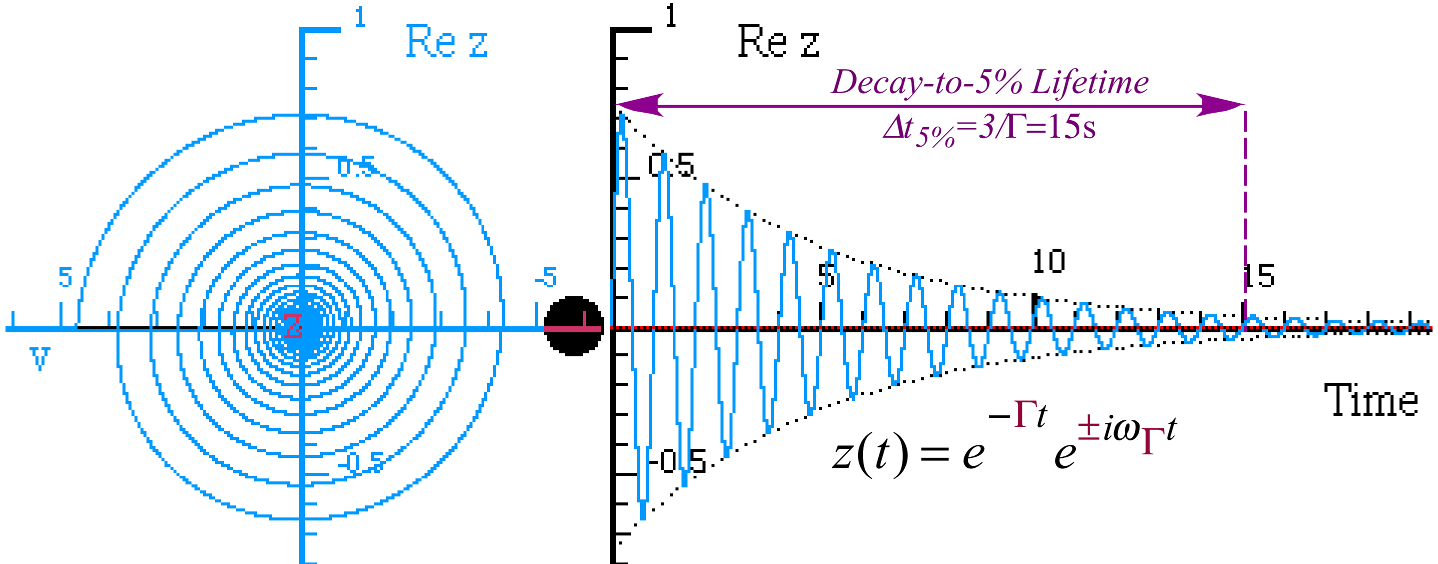
$$\omega^2 + 2i\Gamma\omega - \omega_0^2 = 0$$

Solve for: $\omega = \omega_\pm$

$$\omega_\pm = \frac{-2i\Gamma \pm \sqrt{-4\Gamma^2 + 4\omega_0^2}}{2} = -i\Gamma \pm \sqrt{\omega_0^2 - \Gamma^2}$$

Solution:

$$z(t) = e^{-i\left(-i\Gamma \pm \sqrt{\omega_0^2-\Gamma^2}\right)t} = e^{\left(-\Gamma \pm i\sqrt{\omega_0^2-\Gamma^2}\right)t} = e^{-\Gamma t}e^{\pm i\sqrt{\omega_0^2-\Gamma^2}\,t} = e^{-\Gamma t}e^{\pm i\omega_\Gamma t}$$



$\Delta t_{5\%} = 3/\Gamma = 15\text{s}$

$$z(t) = e^{-\Gamma t}e^{\pm i\omega_\Gamma t}$$

*Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$*

OscillIt Web Simulation

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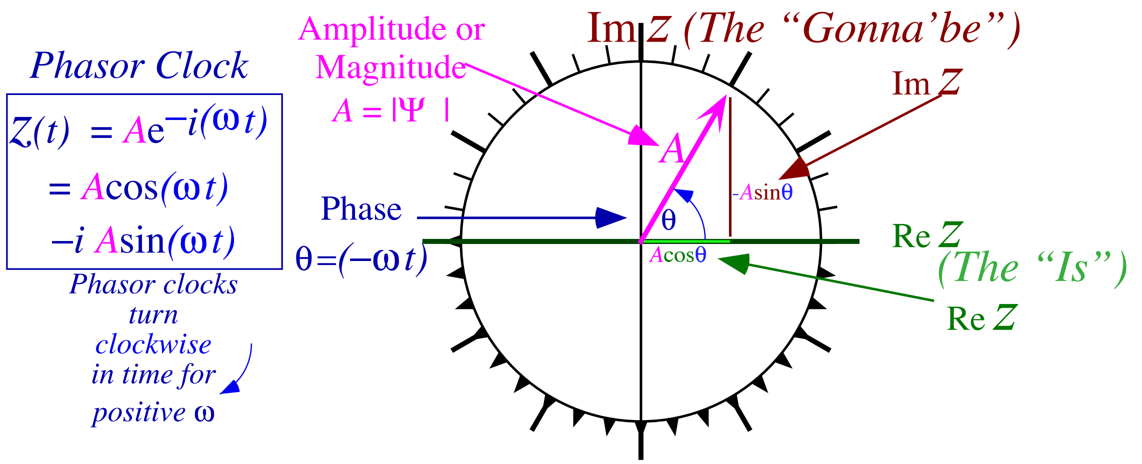
# Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$


Phasor Clock

$$\begin{aligned} Z(t) &= Ae^{-i(\omega t)} \\ &= A\cos(\omega t) \\ &\quad -i\,A\sin(\omega t) \end{aligned}$$

Phasor clocks turn clockwise in time for positive ω

$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

## Oscillator Figures of Merit:

Time required to to reduce amplitude to 5%

Easy-to-recall 5% approximation:

$$e^{-3} \cong 0.05$$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

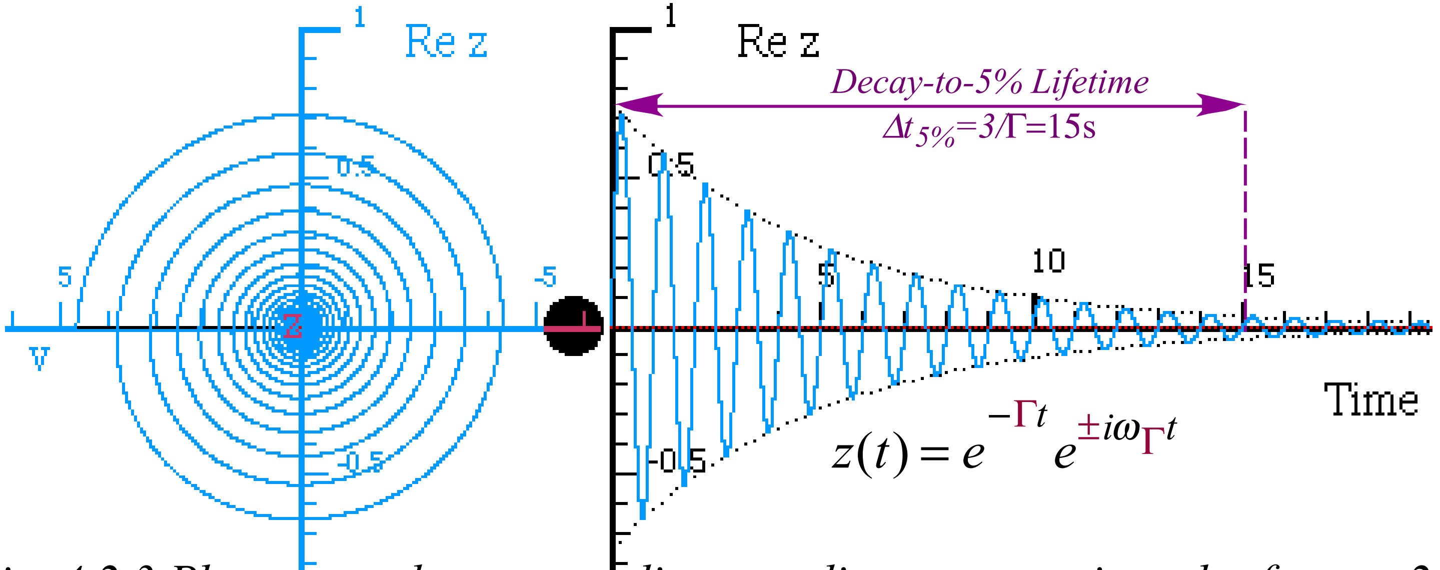


*Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$*

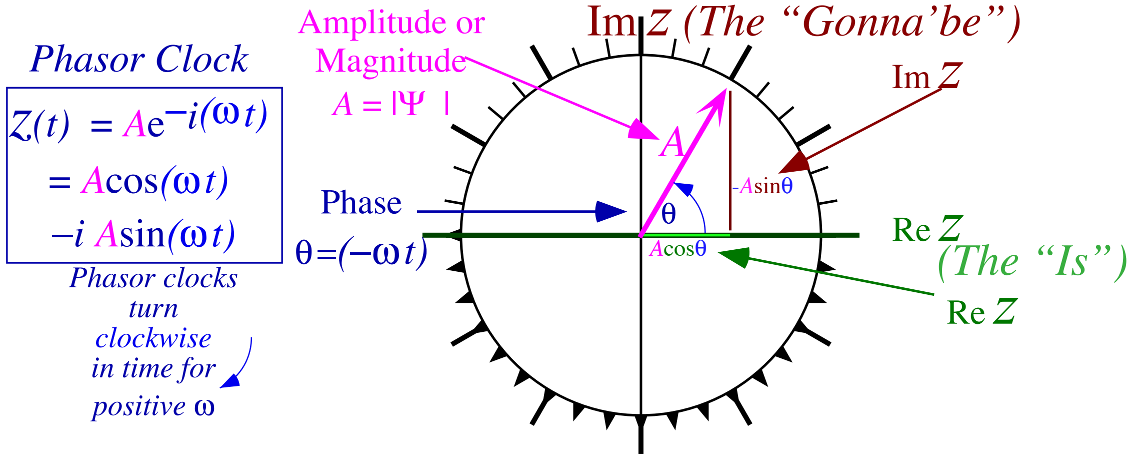
http://www.uark.edu/ua/modphys/markup/OscillItWeb.html

OscillIt Web Simulation

# *Linear damped-harmonic oscillator equation of motion.*

$$F_{total}(t) = m\frac{d^2z}{dt^2} = F_{damping} + F_{restore}$$

$$\frac{d^2z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m}$$



$$\frac{d^2z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = 0$$

Coordinate *z=z(t)* is the response coordinate for a particle of mass *m* and charge *e*

held back by a harmonic (linear) restoring force → $F_{restore} = -kz$

retarded by frictional damping force → $F_{damping} = -b\frac{dz}{dt}$

## Oscillator Figures of Merit:

Time required to
to reduce amplitude
to 5%      (or 4.321%)

Easy-to-recall *5%* approximation:

$$e^{-3} \cong 0.05$$

$$t_{5\%} = \frac{3}{\Gamma} = \frac{3}{0.2} = 15$$

More precise one:

$$e^{-\pi} \cong 0.04321$$

$$t_{4.321\%} = \frac{\pi}{\Gamma} = \frac{\pi}{0.2} = 15.708$$

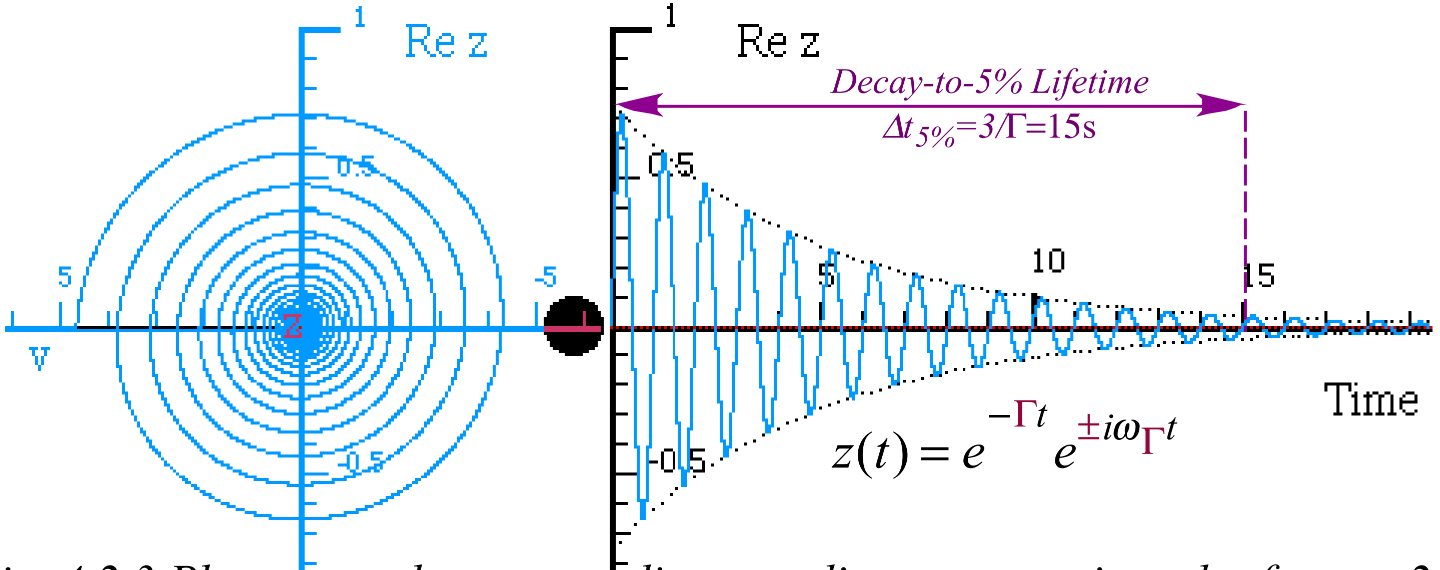


*Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0$=2π and Γ=0.2*

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OscillIt Web Simulation

# Linear damped-harmonic oscillator equation of motion.

$$F_{total}(t)=m\frac{d^2z}{dt^2}=F_{damping}+F_{restore}$$

$$\frac{d^2z}{dt^2}=\frac{F_{damping}}{m}+\frac{F_{restore}}{m}$$

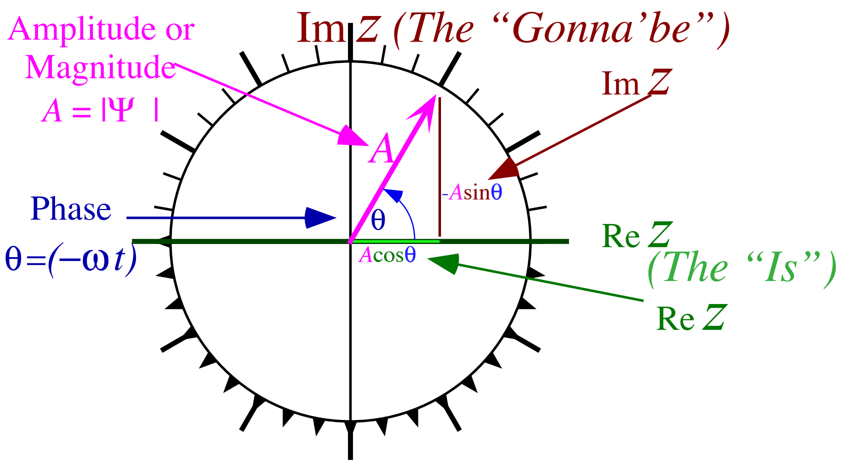
*Phasor Clock*

$Z(t) = Ae^{-i(\omega t)}$

$= A\cos(\omega t)$

$-i\,A\sin(\omega t)$

*Phasor clocks turn clockwise in time for positive ω*



$$\frac{d^2z}{dt^2}+2\Gamma\frac{dz}{dt}+\omega_0^2z=0$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

held back by a harmonic (linear) restoring force → $F_{restore}=-kz$

retarded by frictional damping force → $F_{damping}=-b\frac{dz}{dt}$

## Oscillator Figures of Merit:

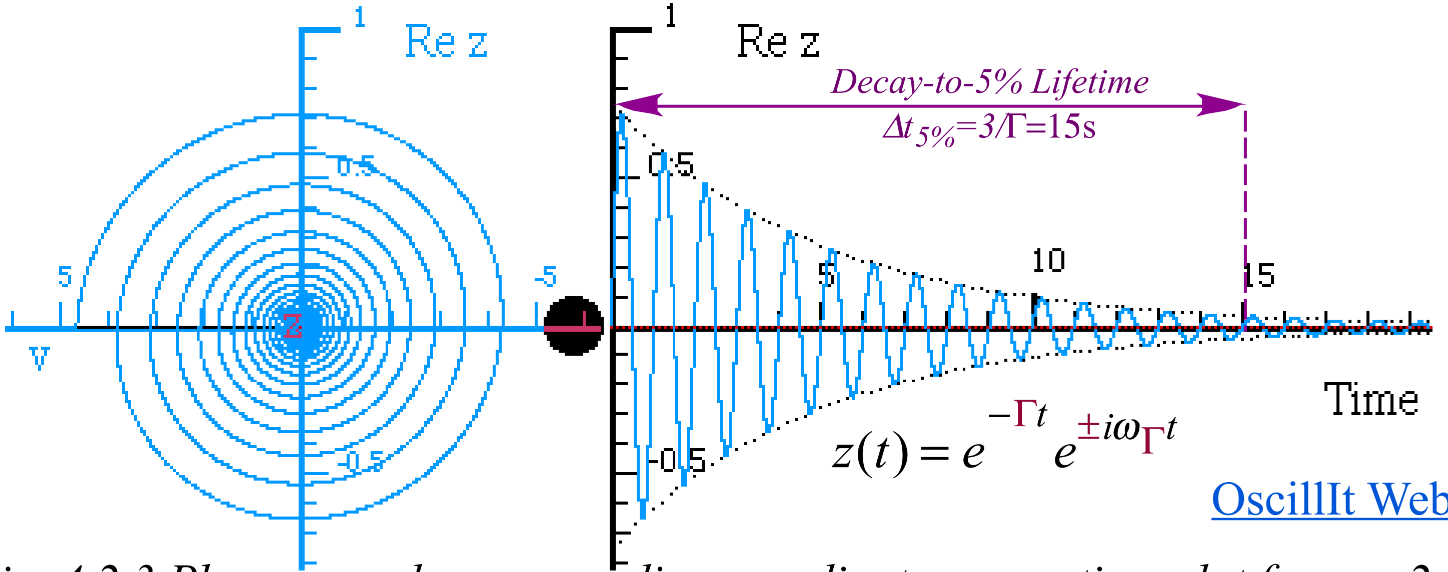
Number $N$ of oscillations to reduce amplitude to 5% (or 4.321%)

Easy-to-recall *5%* approximation: $e^{-3}\cong 0.05$

More precise one: $e^{-\pi}\cong 0.04321$

$$N_{5\%}=\frac{\omega_\Gamma\cdot t_{5\%}}{2\pi}=\frac{3\omega_\Gamma}{2\pi\Gamma}\sim\frac{\omega_\Gamma}{2\Gamma}$$

$$t_{4.321\%}=\frac{\pi}{\Gamma}=\frac{\pi}{0.2}=15.708$$



$z(t)=e^{-\Gamma t}e^{\pm i\omega_\Gamma t}$

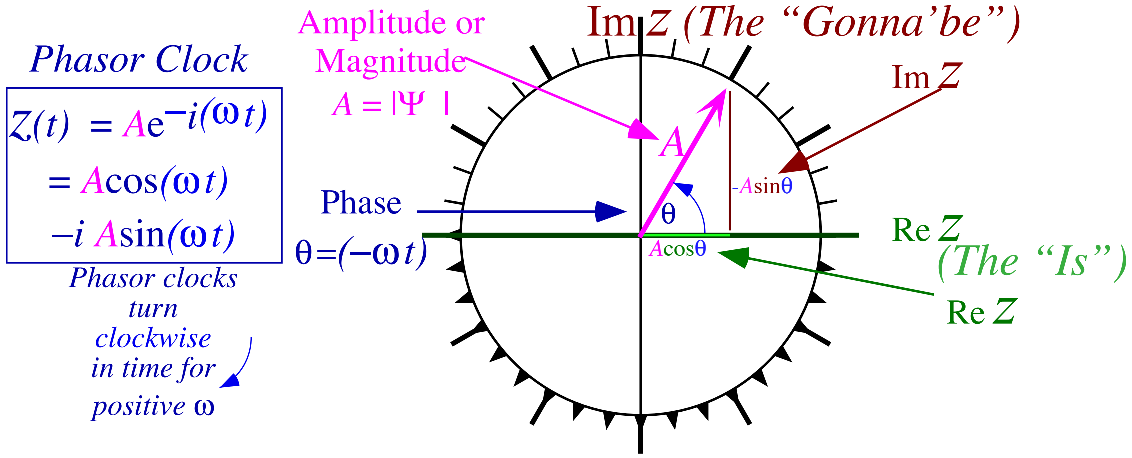
OscillIt Web Simulation

*Fig. 4.2.3 Phasor z and corresponding coordinate versus time plot for $\omega_0=2\pi$ and $\Gamma=0.2$*

http://www.uark.edu/ua/modphys/markup/OscillItWeb.html

# *Linear forced-damped-harmonic oscillator equation of motion.*

$$F_{total}(t)=m\frac{d^2z}{dt^2}=F_{damping}+F_{restore}+F_{stimulus}$$



$$\frac{d^2z}{dt^2}=\frac{F_{damping}}{m}+\frac{F_{restore}}{m}+\frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus}=a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2z}{dt^2}+2\Gamma\frac{dz}{dt}+\omega_0^2z=a_{stimulus}=\frac{e}{m}E(t)$$

Coordinate $z=z(t)$ is the response coordinate for a particle of mass $m$ and charge $e$

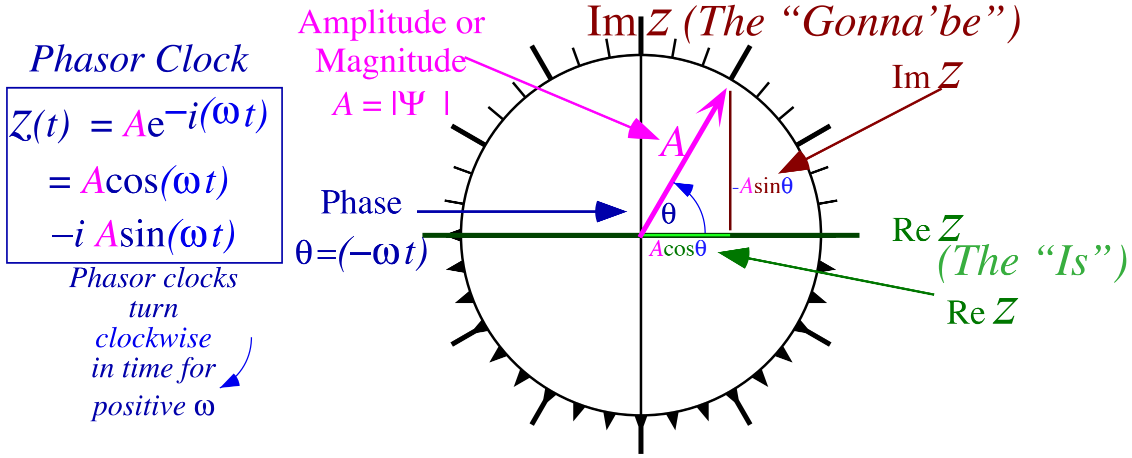
driven by external stimulating force $\longrightarrow$ $F_{stimulus}(t)=eE(t)$

held back by a harmonic (linear) restoring force $\longrightarrow$ $F_{restore}=-kz,\quad \left(k=\omega_0^2m\right),$

retarded by frictional damping force $\longrightarrow$ $F_{damping}=-b\dfrac{dz}{dt},\quad (b=2\Gamma m)$

# Linear forced-damped-harmonic oscillator equation of motion.

$$F_{total}(t) = m\frac{d^2 z}{dt^2} = F_{damping} + F_{restore} + F_{stimulus}$$



*Phasor Clock*

$$\boxed{\begin{aligned} Z(t) &= Ae^{-i(\omega t)} \\ &= A\cos(\omega t) \\ &\quad -i\, A\sin(\omega t) \end{aligned}}$$

*Phasor clocks turn clockwise in time for positive ω*

$$\frac{d^2 z}{dt^2} = \frac{F_{damping}}{m} + \frac{F_{restore}}{m} + \frac{F_{stimulus}}{m}$$

Stimulating acceleration $a_{stimulus} = a(t)$ due to stimulating force $F_{stimulus}(t)$ (Typically **E**-field )

$$\frac{d^2 z}{dt^2} + 2\Gamma\frac{dz}{dt} + \omega_0^2 z = a_{stimulus} = \frac{e}{m}E(t)$$

Solving for $z_{stimulus}(t)$ given $a_{stimulus}$ :

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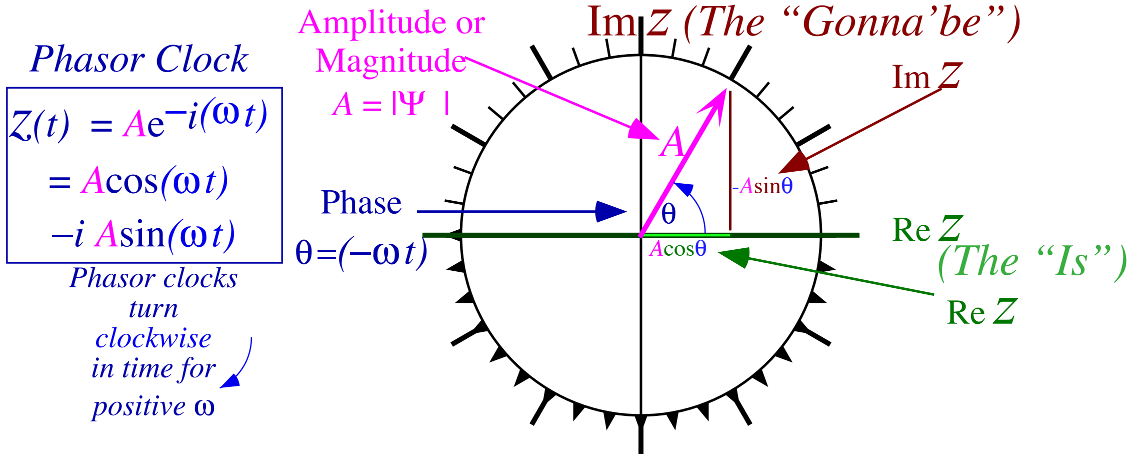

$$Z(t) = Ae^{-i(\omega t)} = A\cos(\omega t) - i\,A\sin(\omega t)$$

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$$\left(\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2\right)z = a_{stimulus}$$

$$z = \frac{1}{\frac{d^2}{dt^2} + 2\Gamma\frac{d}{dt} + \omega_0^2}a_{stimulus}$$

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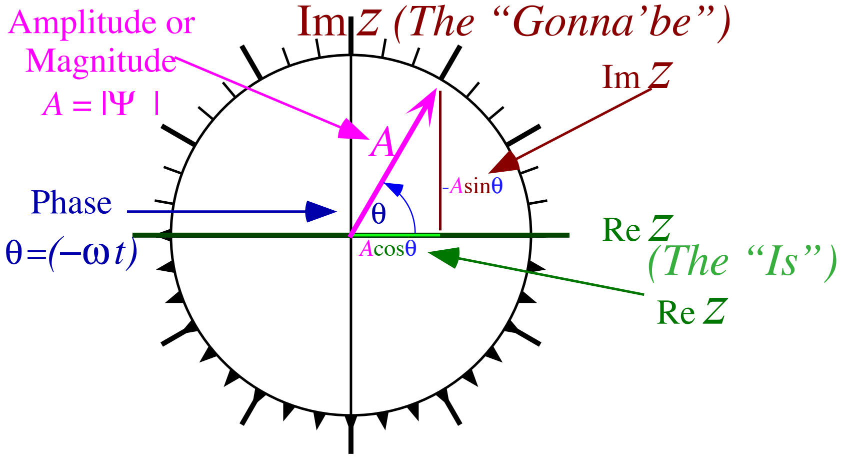

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Pretty crazy?

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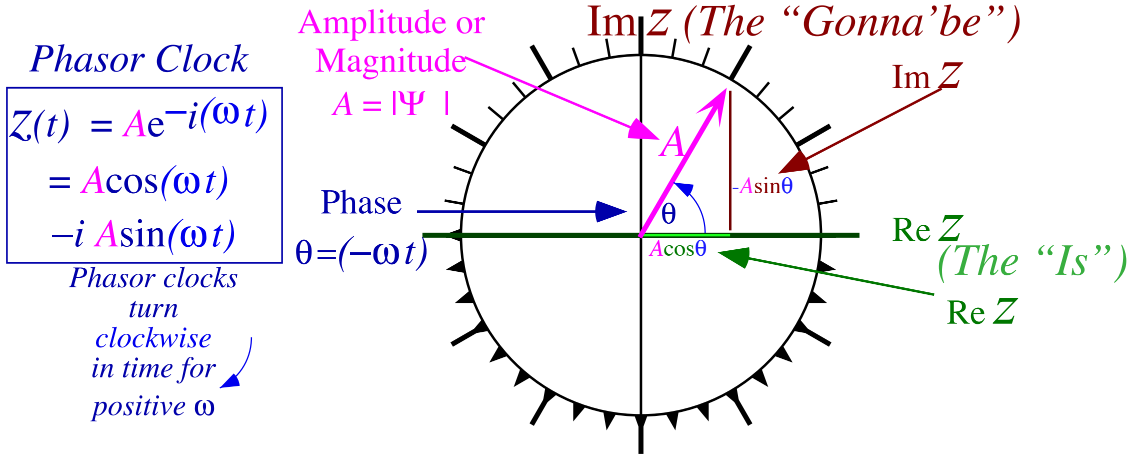

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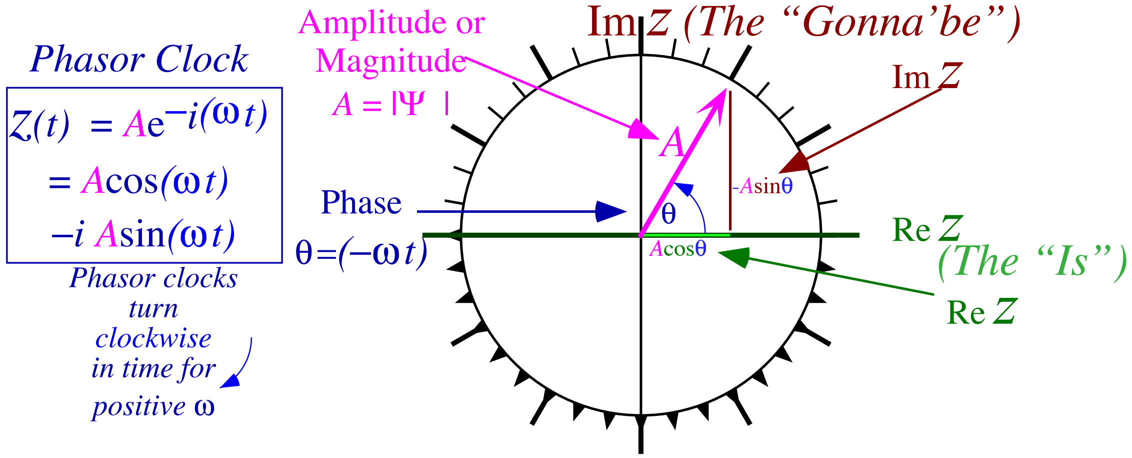

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$$z_{stimulus} = \frac{1}{-\omega_s^{\ 2} - i2\Gamma\omega_s + \omega_0^2}\,a_s e^{-i\omega_s t}$$

$$z_s e^{-i\omega_s t} = \frac{1}{\omega_0^2 - \omega_s^{\ 2} - i2\Gamma\omega_s}\,a_s e^{-i\omega_s t}$$

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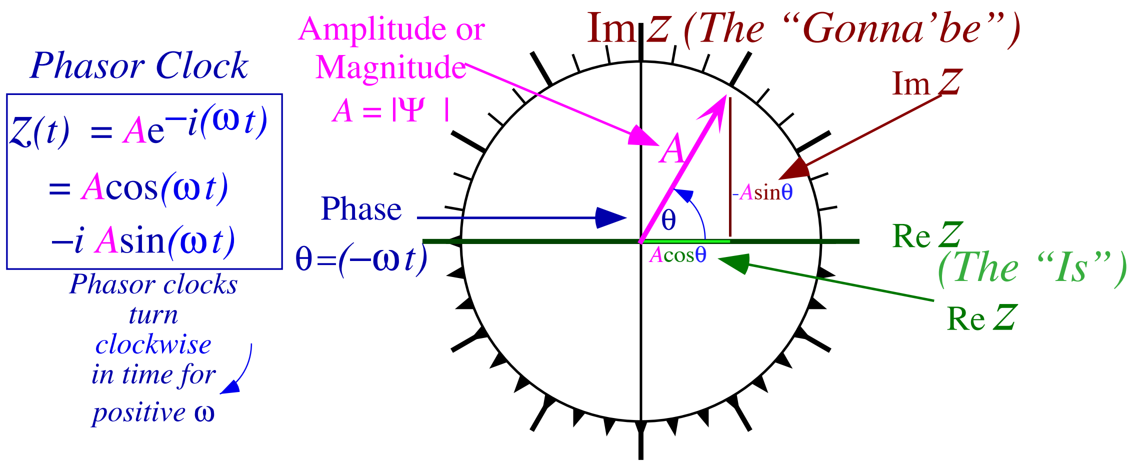

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$$z_s = G_{\omega_0}(\omega_s) \cdot a_s$$

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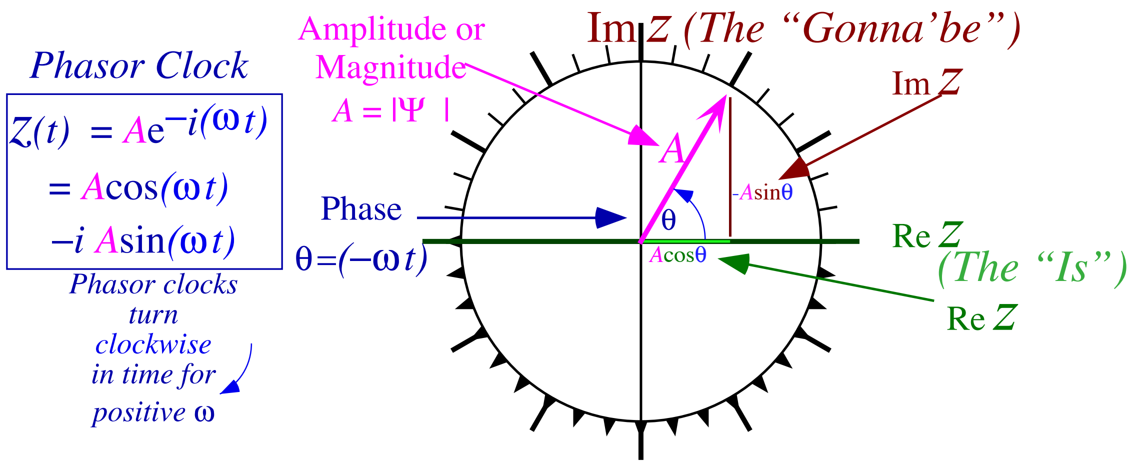

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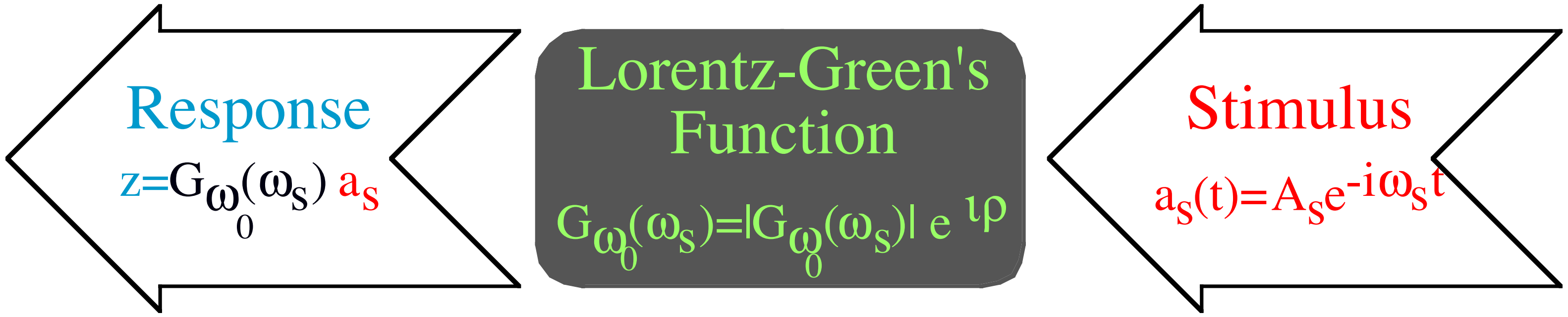
$$z_s = \boxed{G_{\omega_0}(\omega_s)}\cdot a_s$$

*Green's Function* for the F-D-H Oscillator (FDHO)

**George Green** (14 July 1793 – 31 May 1841)

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)



Response

$z=G_{\omega_0}(\omega_s)\,a_s$

Lorentz-Green's Function

$G_{\omega_0}(\omega_s)=|G_{\omega_0}(\omega_s)|\,e^{i\rho}$

Stimulus

$a_s(t)=A_s e^{-i\omega_s t}$

*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}=\mathrm{Re}\,G_{\omega_0}(\omega_s)+i\,\mathrm{Im}\,G_{\omega_0}(\omega_s)$$

Real and imaginary parts of the *rectangular form* of $G$:

Hendrik A. Lorentz

July 18, 1853. - February 4, 1928

# *Green’s Function* for the FDHO (Forced-Damped-Harmonic Oscillator)

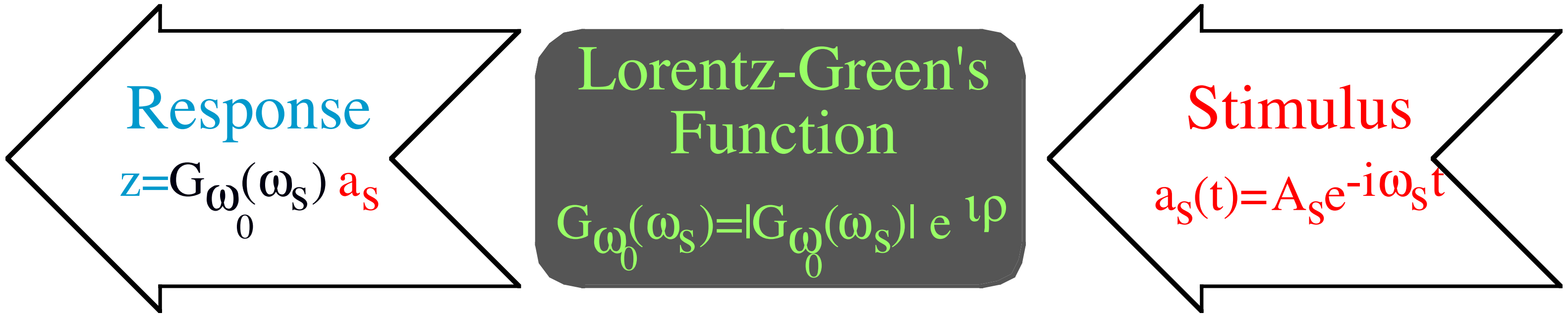


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*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

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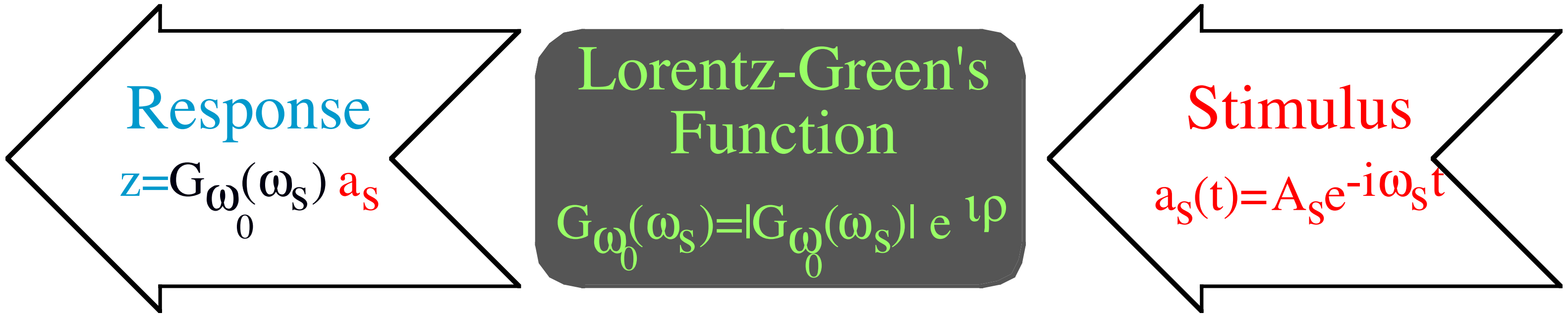
Real and imaginary parts of the *rectangular form* of *G*: $\frac{1}{x-iy}=\frac{1}{x-iy}\frac{x+iy}{x+iy}=\frac{x+iy}{x^2+y^2}$

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)



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Real and imaginary parts of the *rectangular form* of $G$:

$$\frac{1}{x-iy}=\frac{1}{x-iy}\frac{x+iy}{x+iy}=\frac{x+iy}{x^2+y^2}=\frac{x}{x^2+y^2}+i\frac{y}{x^2+y^2}$$

$$\mathrm{Re}\,G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)

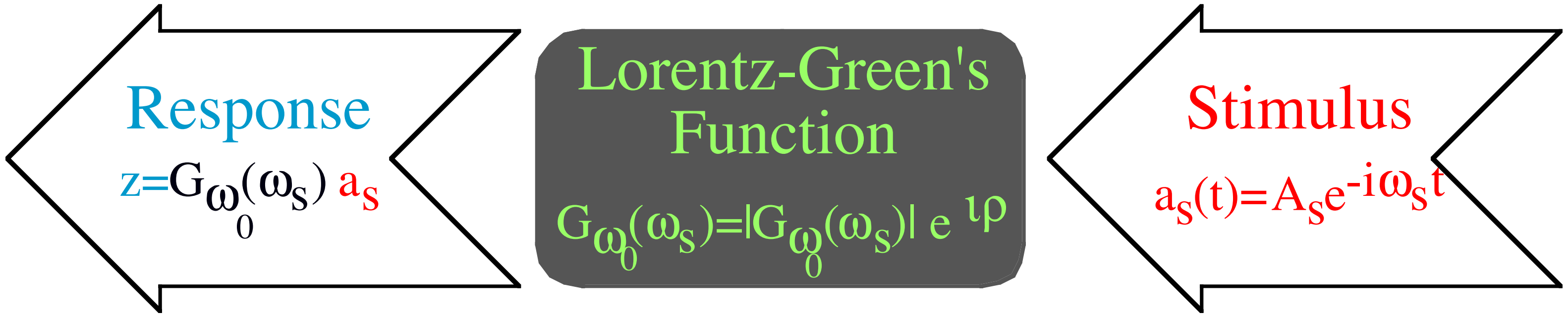

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*Fig. 4.2.4 Black-box diagram of oscillator response to monochromatic stimulus*

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Real and imaginary parts of the *rectangular form* of *G*:

$$\operatorname{Re}G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

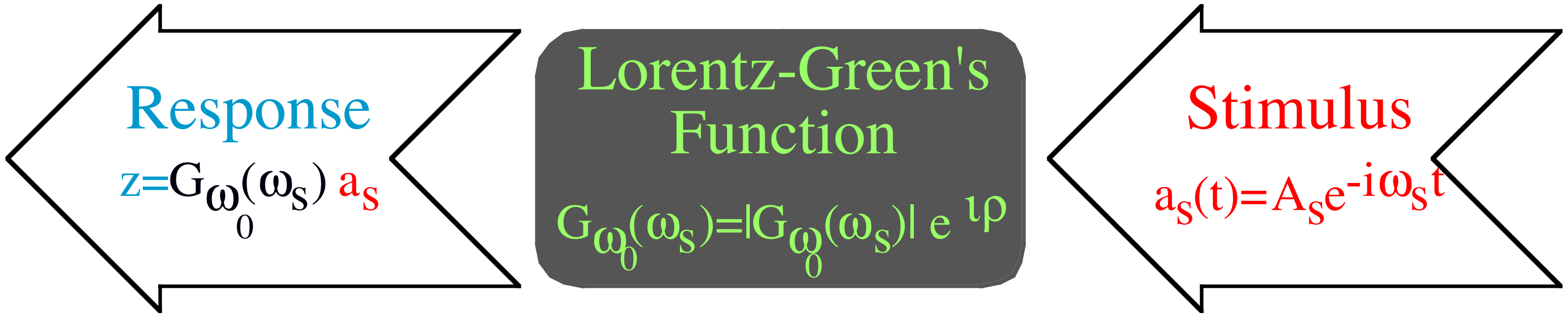
$$\operatorname{Im}G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and polar angle $\rho$ of the *polar form* of *G*:

$$\left|G_{\omega_0}(\omega_s)\right|=\frac{1}{\sqrt{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}}$$

$$\rho=\tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2-\omega_s^2}\right)$$

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)



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Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of $G$:

$$\left|G_{\omega_0}(\omega_s)\right|=\frac{1}{\sqrt{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}}$$

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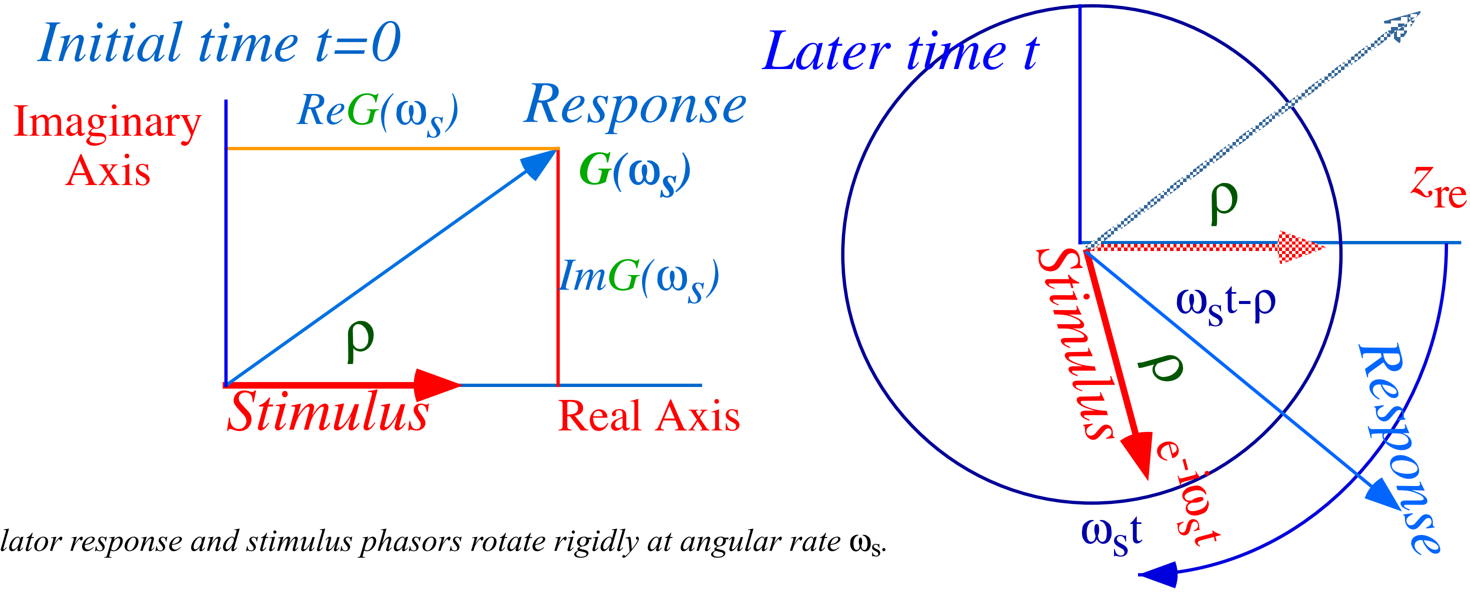
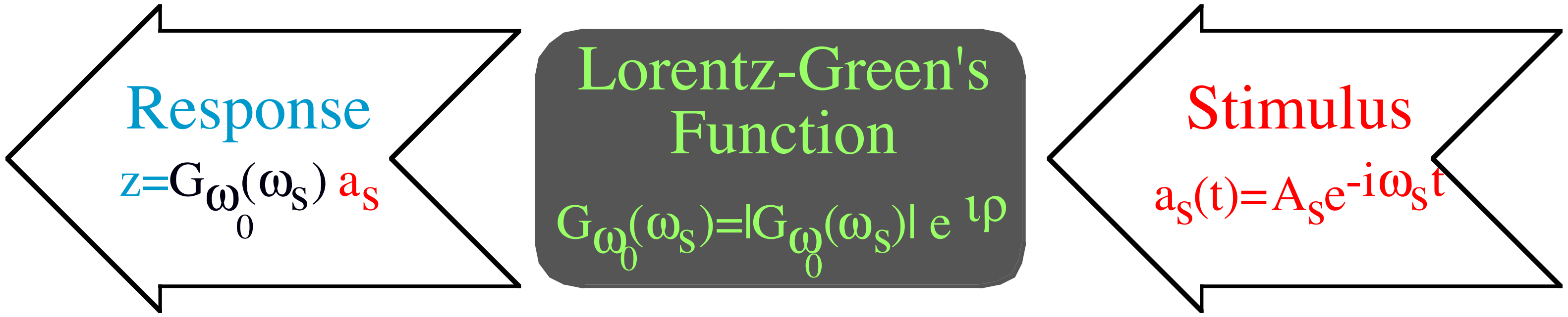


*polar angle* ρ is the *phase lag angle* ρ

$$z_{\text{response}}(t)=\left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

*Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate* $\omega_s$.

# *Green's Function* for the FDHO (Forced-Damped-Harmonic Oscillator)



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Magnitude $\left|G_{\omega_0}(\omega_s)\right|$ and *polar angle* ρ of the *polar form* of $G$:

$$\left|G_{\omega_0}(\omega_s)\right|=\frac{1}{\sqrt{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}}$$

$$\rho=\tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2-\omega_s^2}\right)$$

*polar angle* ρ is the

*phase lag angle* ρ

$$z_{\text{response}}(t)=\left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

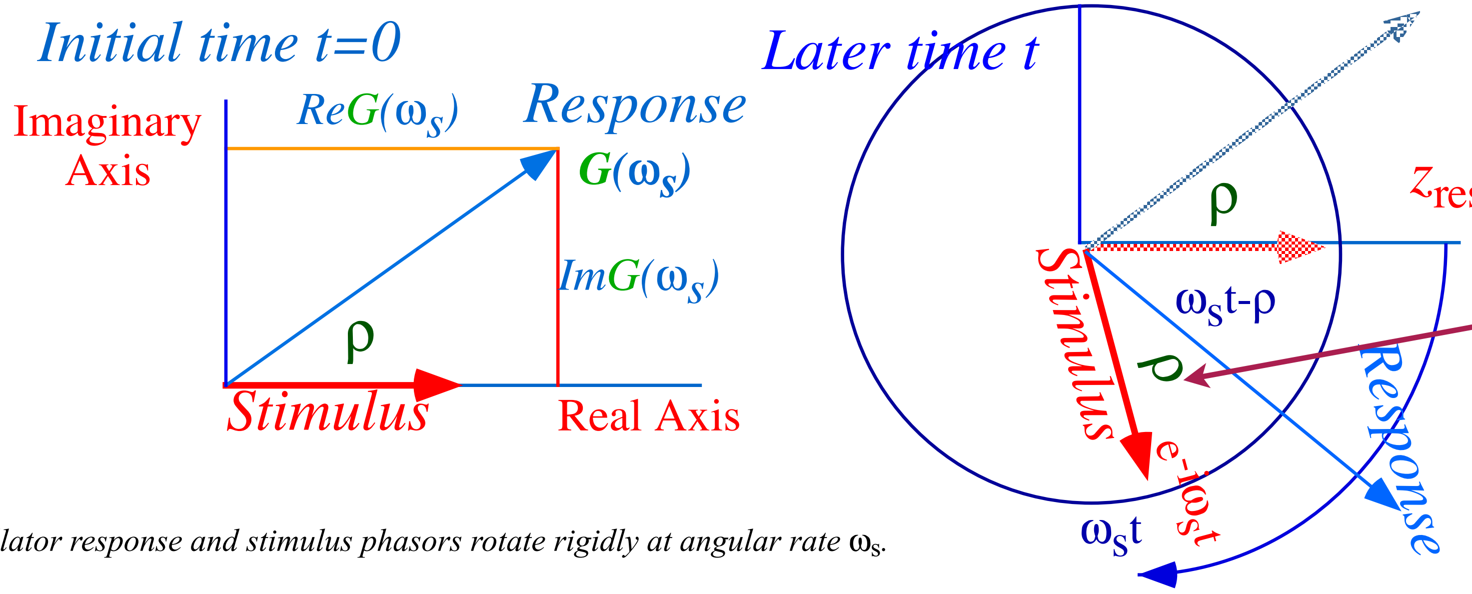
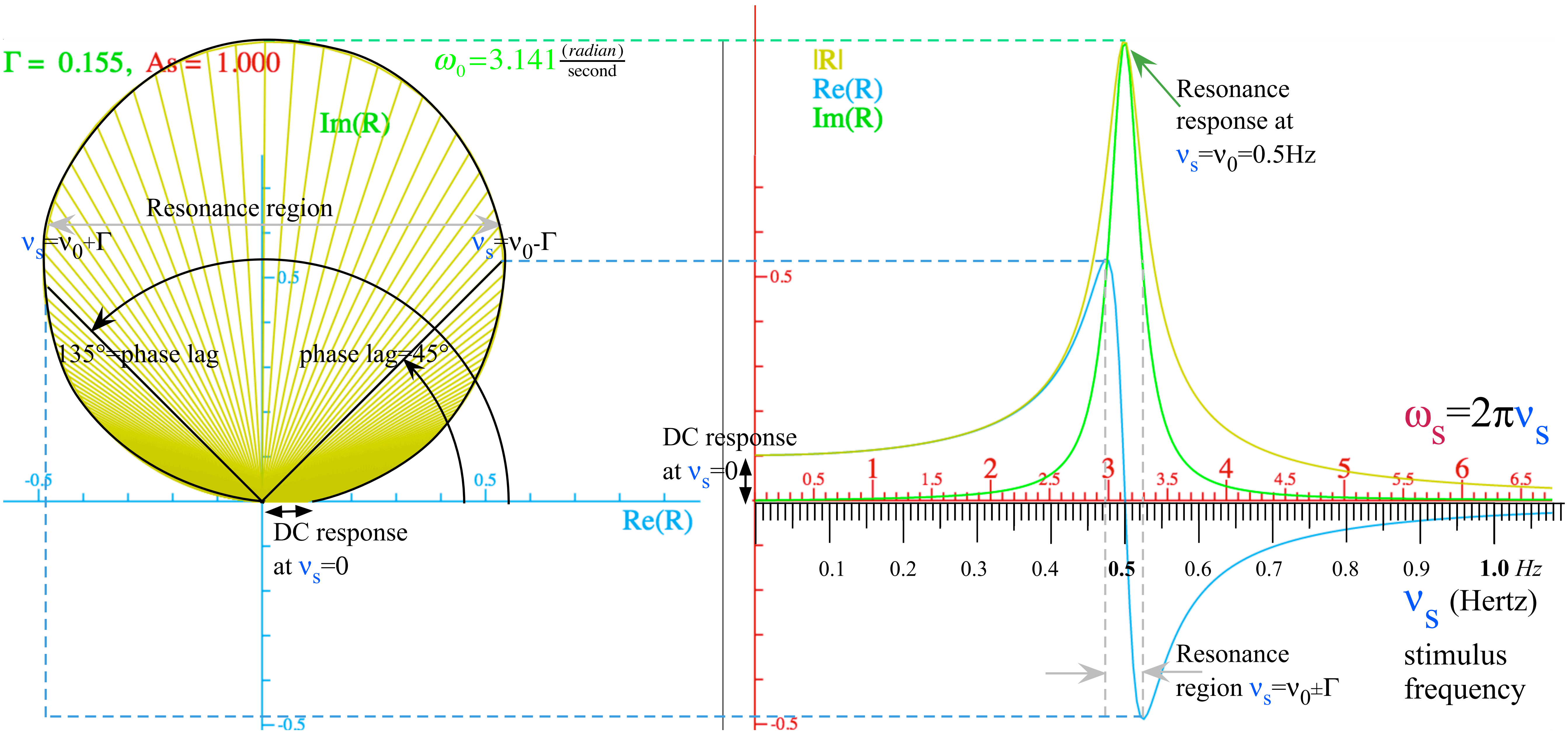


*Fig. 4.2.5 Oscillator response and stimulus phasors rotate rigidly at angular rate ω*$_s$.

# *Lorentz-Green's function for* $\nu_0 = 0.5\,Hz$ *or* $: \omega_0 = \pi \frac{(radian)}{\text{second}}$



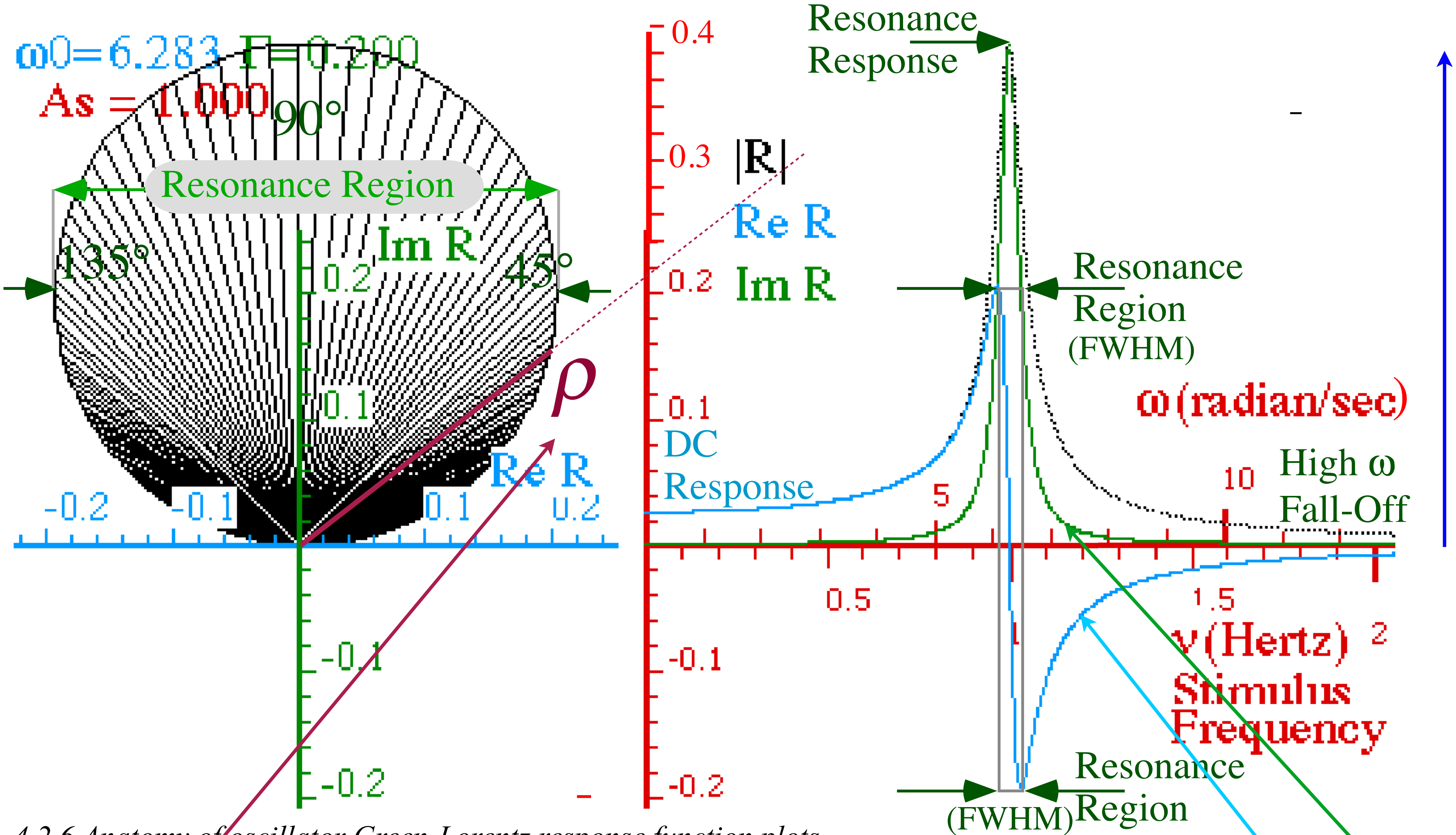


*Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots*

*Phase lag angle*

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2-\omega_s^2}\right)$$

$$\operatorname{Re} G_{\omega_0}(\omega_s) = \frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

*Real part*

$$\operatorname{Im} G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

*Imaginary part*

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s=\omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \qquad \textit{(angular quality factor)}$$

http://www.uark.edu/ua/modphys/markup/OscillItWeb.html

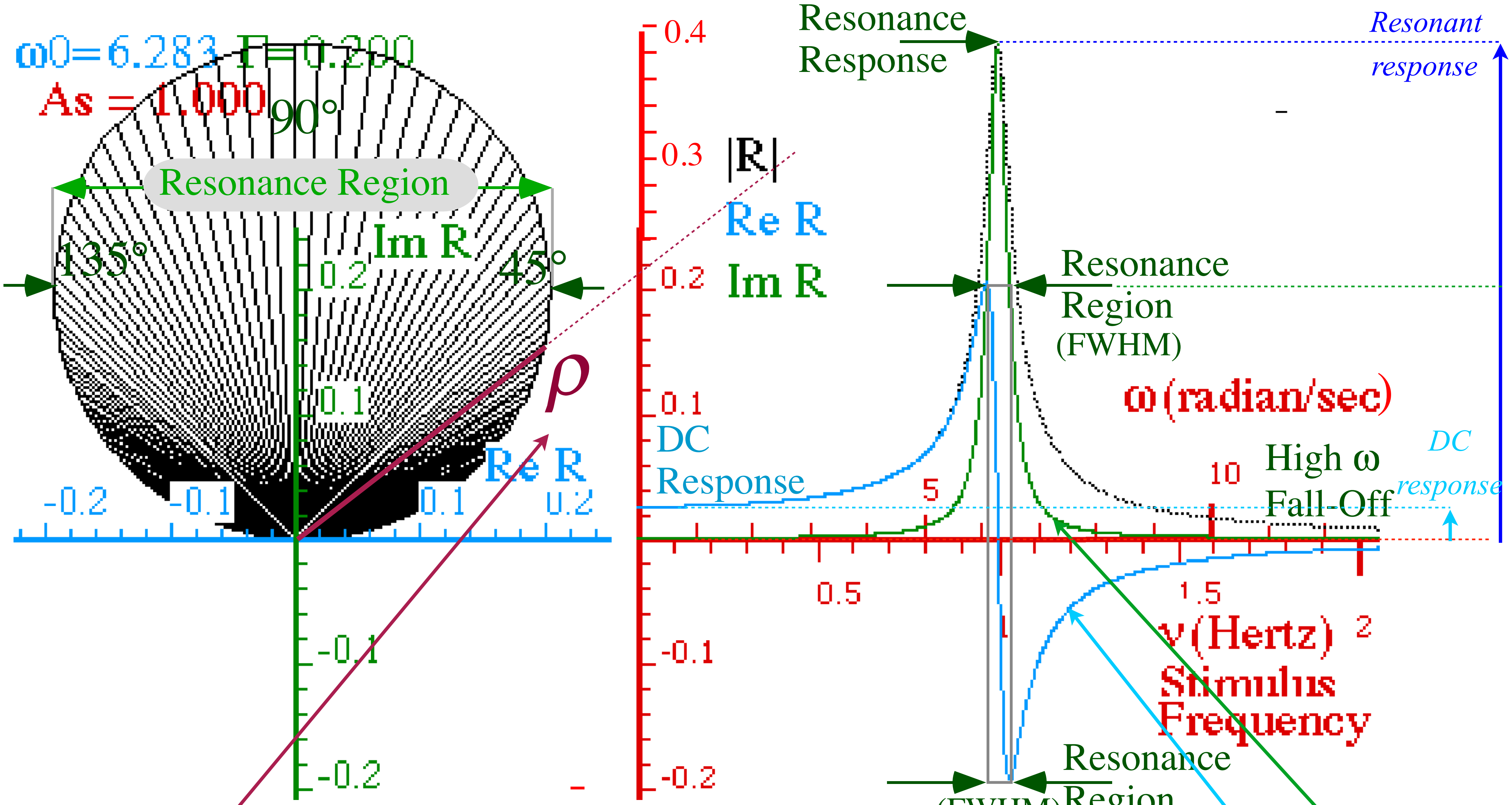


*Fig. 4.2.6 Anatomy of oscillator Green-Lorentz response function plots*

*Phase lag angle*

$$\rho = \tan^{-1}\left(\frac{2\Gamma\omega_s}{\omega_0^2 - \omega_s^2}\right)$$

*Real part*

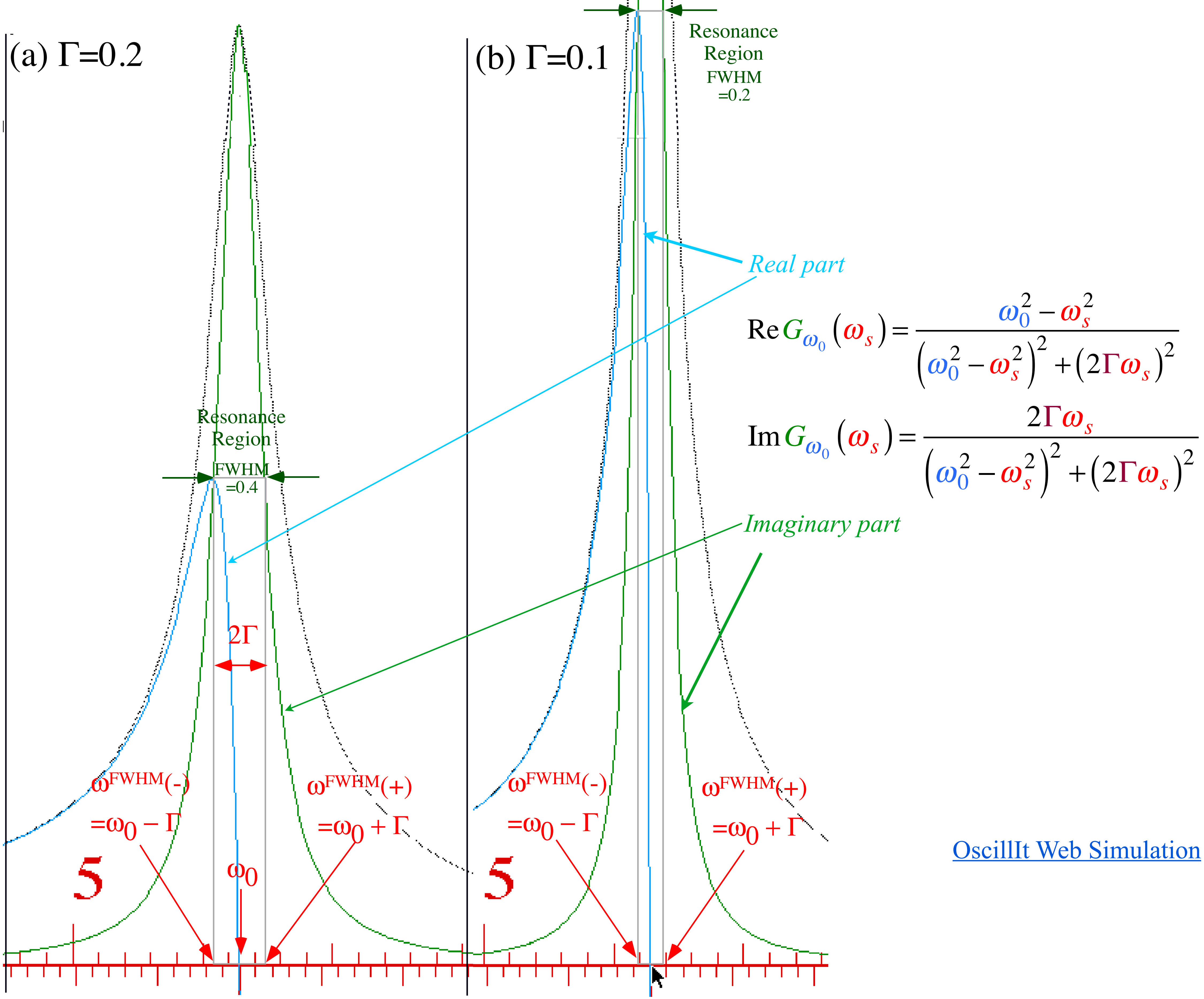
$$\operatorname{Re} G_{\omega_0}(\omega_s) = \frac{\omega_0^2 - \omega_s^2}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

*Imaginary part*

$$\operatorname{Im} G_{\omega_0}(\omega_s) = \frac{2\Gamma\omega_s}{\left(\omega_0^2 - \omega_s^2\right)^2 + \left(2\Gamma\omega_s\right)^2}$$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s = \omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \quad \textit{(angular quality factor)}$$

http://www.uark.edu/ua/modphys/markup/OscillItWeb.html



$$\mathrm{Re}\,G_{\omega_0}(\omega_s)=\frac{\omega_0^2-\omega_s^2}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

$$\mathrm{Im}\,G_{\omega_0}(\omega_s)=\frac{2\Gamma\omega_s}{\left(\omega_0^2-\omega_s^2\right)^2+\left(2\Gamma\omega_s\right)^2}$$

OscillIt Web Simulation

*Fig. 4.2.7 Comparing Lorentz-Green resonance region for (a) Γ=0.2 and (b) Γ=0.1.*
*Maximum and minimum points of* ReG(ω) *and inflection points of* ImG(ω) *are near region boundaries* $\omega^{FWHM}(\pm)=\omega_0\pm\Gamma$.

# *Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)*

$$z(t) = z_{\text{transient}}(t) + z_{\text{response}}(t) \equiv z_{\text{decaying}}(t) + z_{\text{steady state}}(t)$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + G_{\omega_0}(\omega_s)a(0)e^{-i\omega_s t}$$

$$= Ae^{-\Gamma t}e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right|a(0)e^{-i(\omega_s t-\rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

# *Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)*

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$$= Ae^{-\Gamma t} e^{-i\omega_\Gamma t} + \left|G_{\omega_0}(\omega_s)\right| a(0)\, e^{-i(\omega_s t - \rho)}$$

Known as "homogeneous" solution (no force)
Let's you set initial values or boundary conditions

Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

# *Complete Green's Solution for the FDHO (Forced-Damped-Harmonic Oscillator)*

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Let's you set initial values or boundary conditions

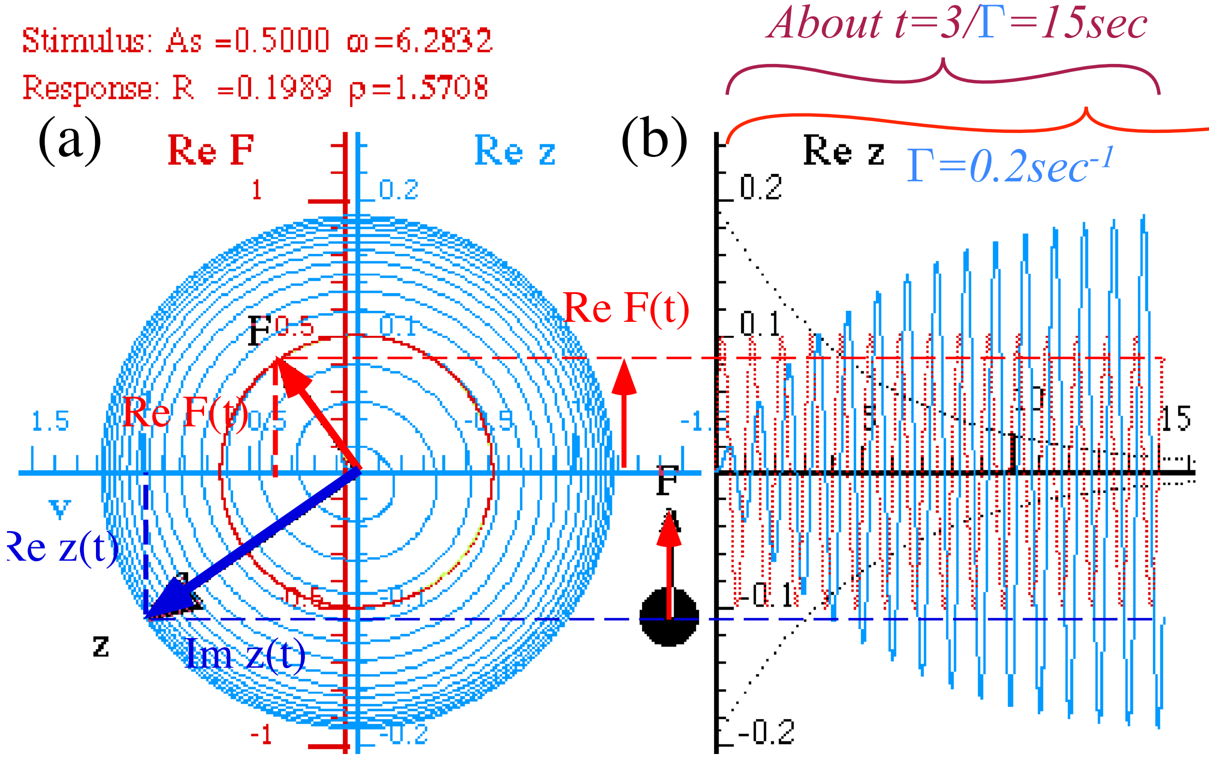
Known as *Transient* solution since it dies-off as time advances past initial conditions

Known as "*in*homogeneous" solution
Not function of initial values. Marches to stimulus only.

Known as *Steady State* solution since it is present as long as stimulus is.

*About t=3/Γ=15sec*

*About t = forever*



[OscillIt (On Resonance) Simulation]

*Fig. 4.2.8 On Resonance (a)Response z-phasor lags ρ=90° behind stimulus F-phasor.*

*($\omega_s=\omega_0=2\pi$, $\omega_0=2\pi$ , and $\Gamma=0.2$). (b) Time plots of* Re *z(t) and* Re *F(t)*

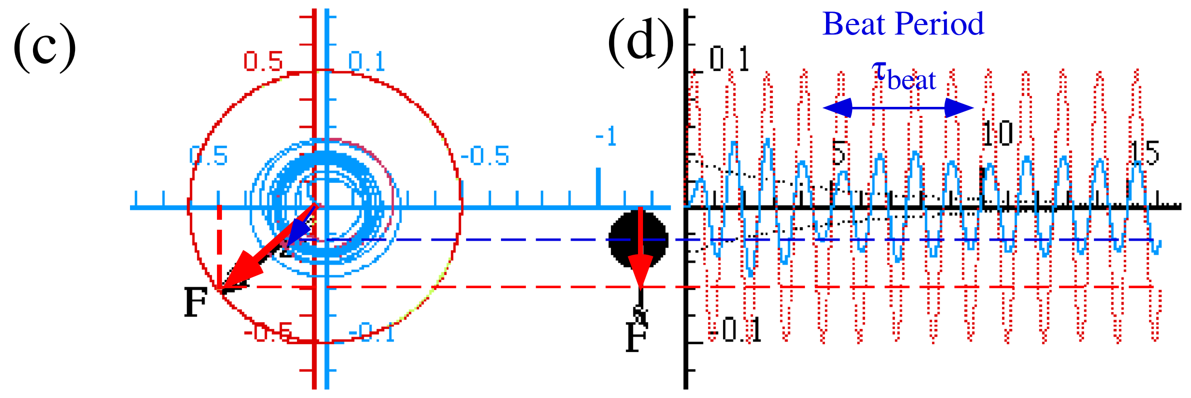
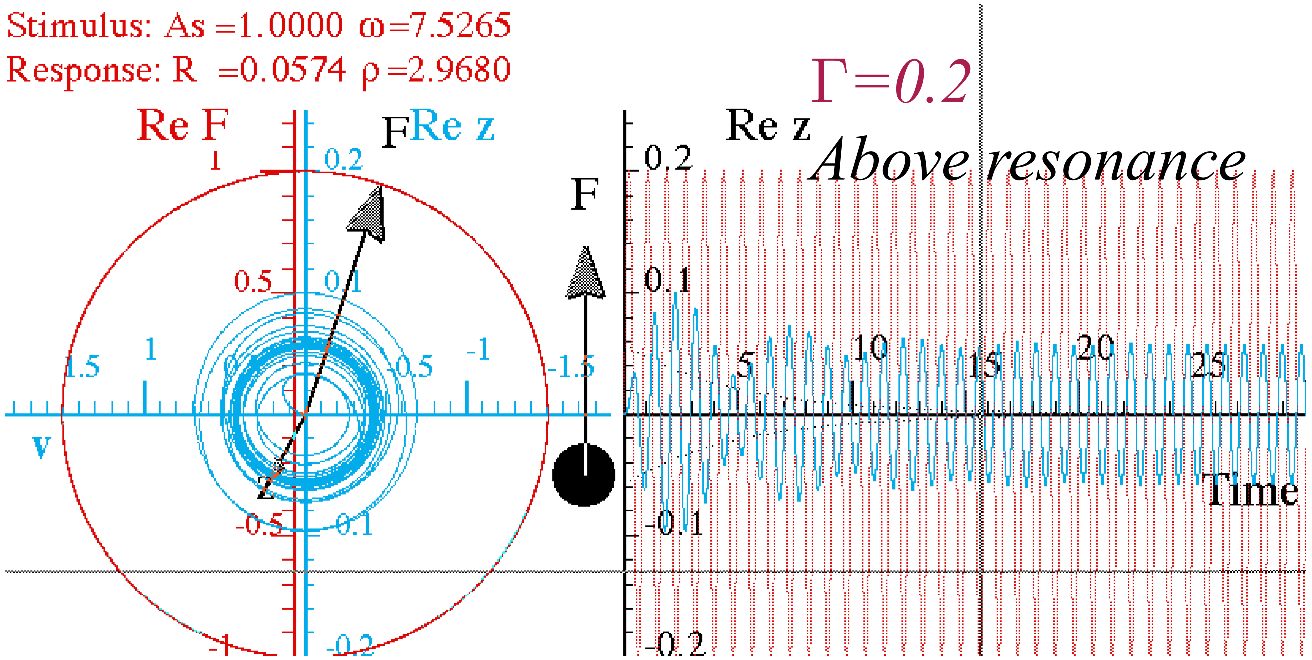


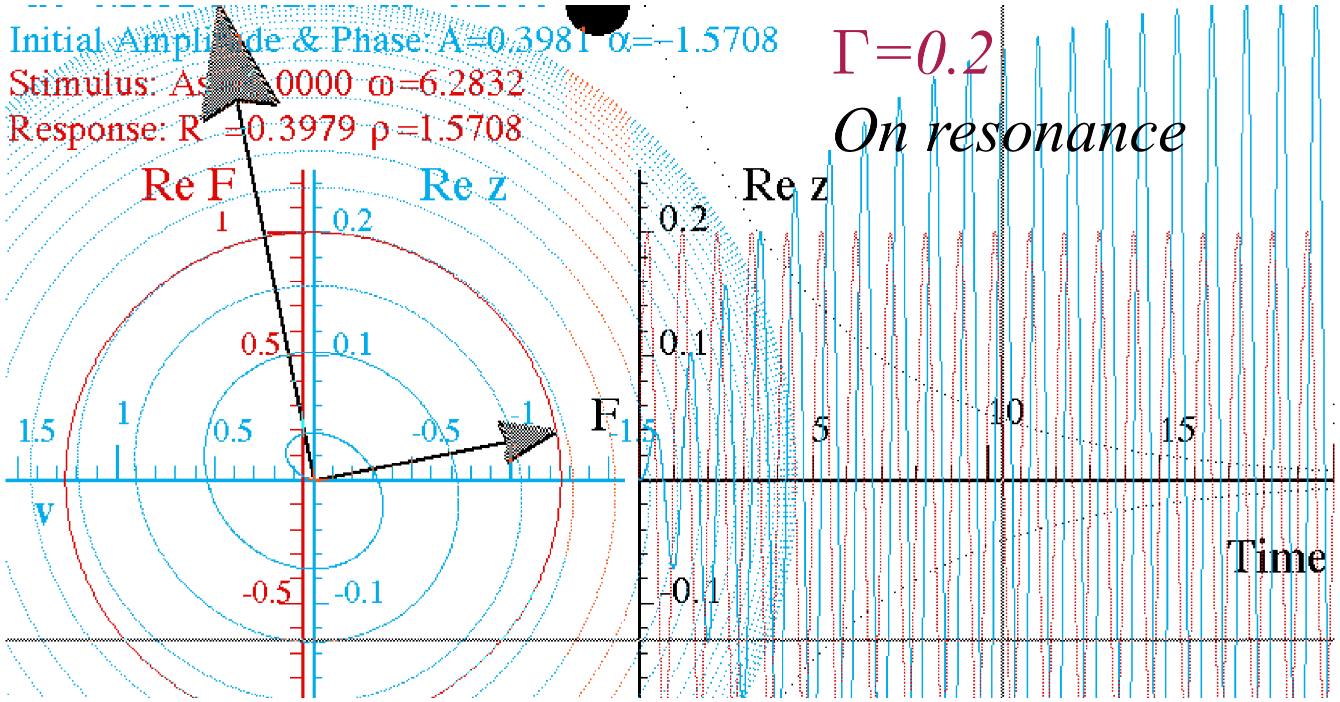
*Fig. 4.2.8 Below Resonance (c)Response z-phasor lags ρ=8.05° behind stimulus F-phasor.*

*($\omega_s=5.03$,$\omega_0=2\pi$ , and $\Gamma=0.2$). (d) Time plots of* Re *z(t) and* Re *F(t). Beats are barely visible.*

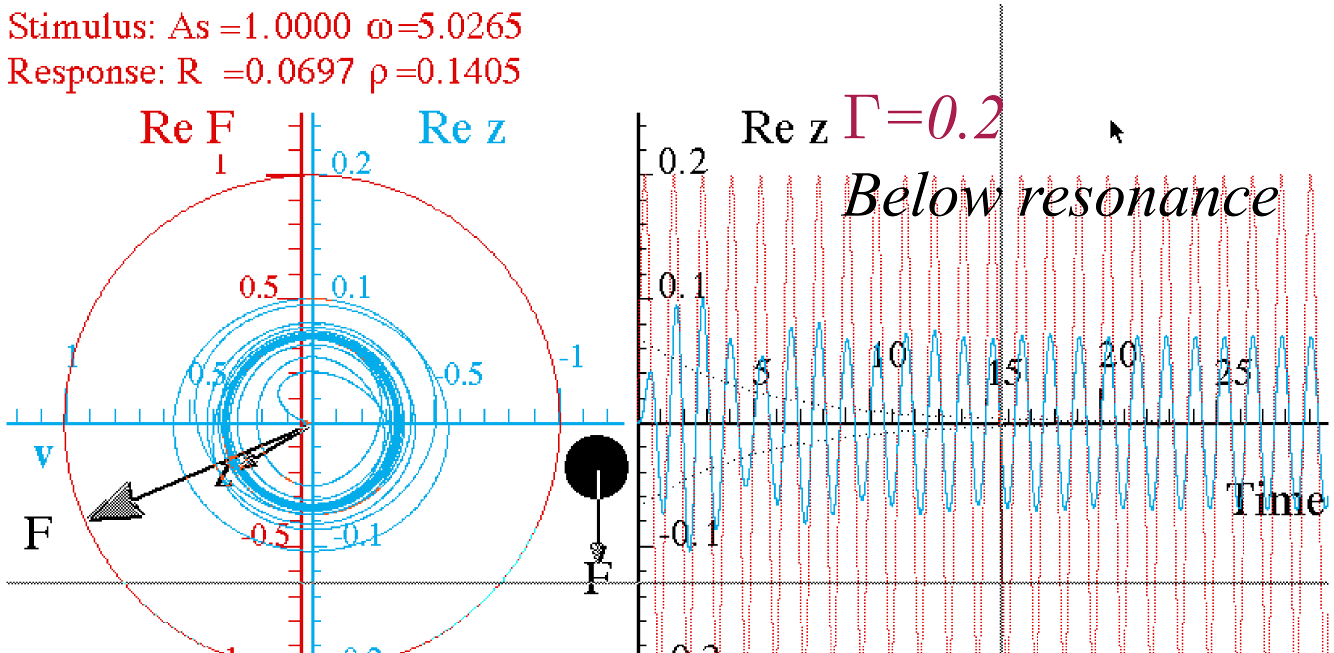
[OscillIt (Way Below Resonance) Simulation]



[OscillIt (Way Above Resonance) Simulation]

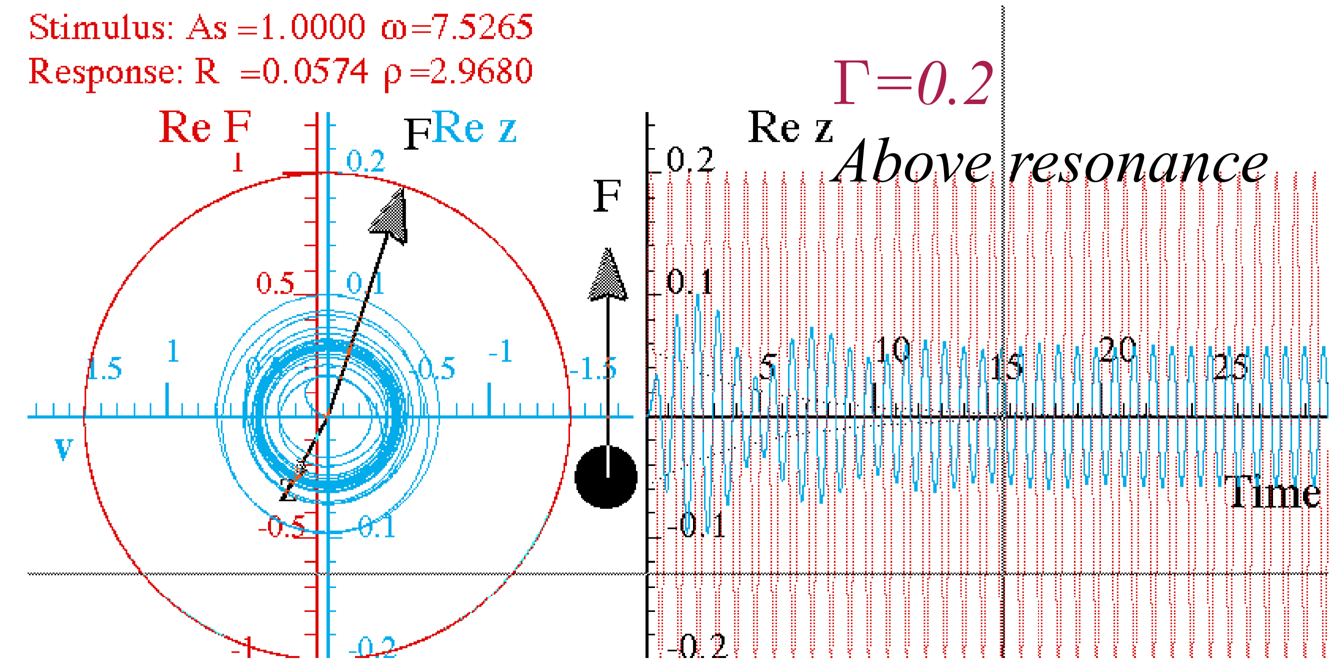


[OscillIt (On Resonance) Simulation]



[OscillIt (Way Below Resonance) Simulation]

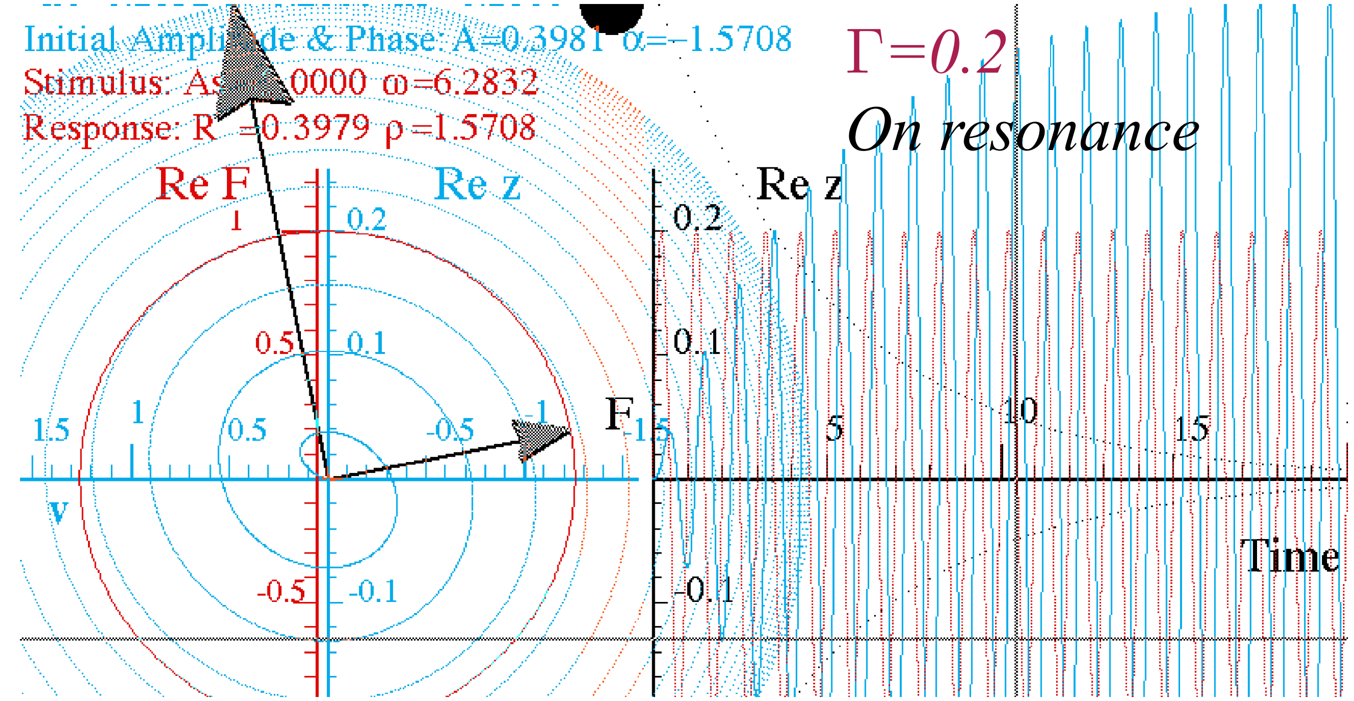
## OscillIt (Beating) Simulation



Stimulus: As =1.0000 ω=7.5265
Response: R =0.0574 ρ=2.9680

Γ=0.2

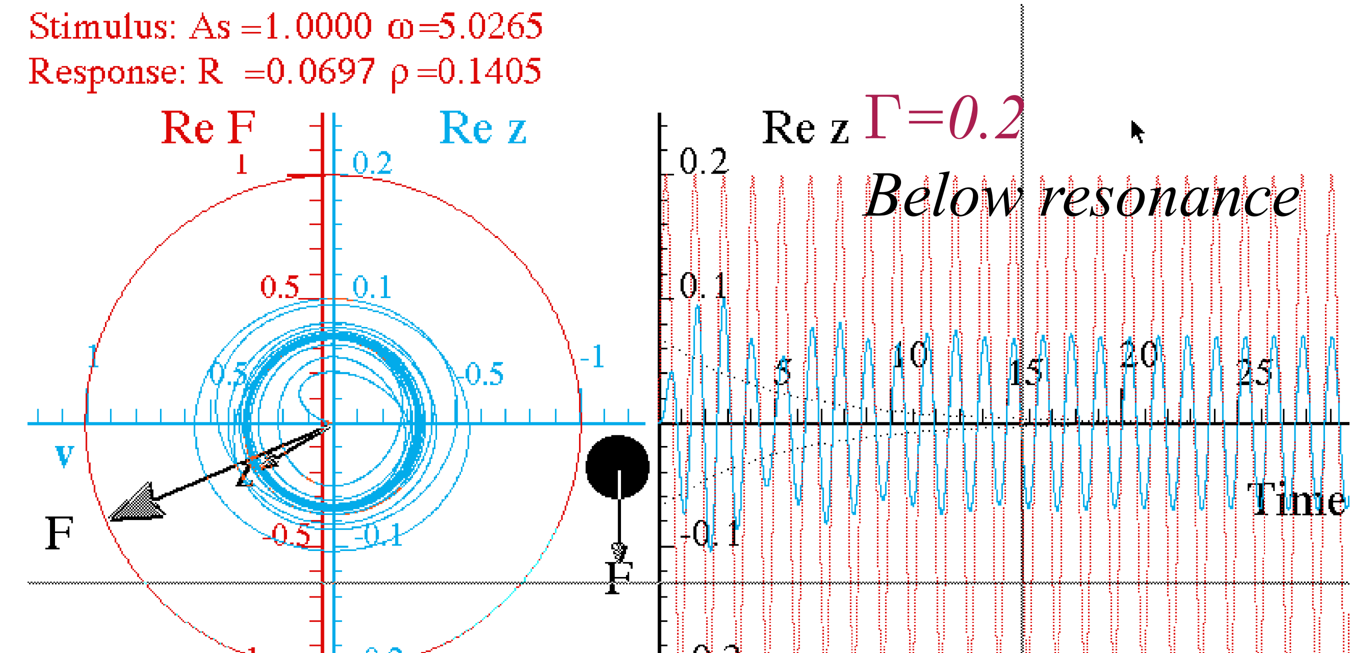
*Above resonance*



Initial Amplitude & Phase: A=0.3981 α=-1.5708
Stimulus: As =1.0000 ω=6.2832
Response: R =0.3979 ρ=1.5708

Γ=0.2

*On resonance*



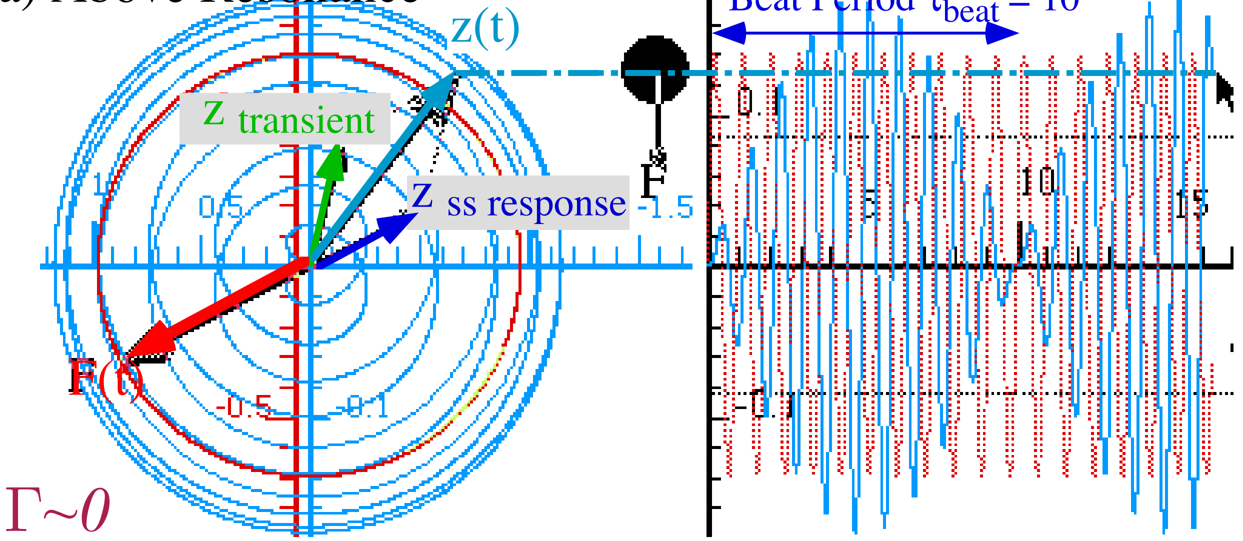
Stimulus: As =1.0000 ω=5.0265
Response: R =0.0697 ρ=0.1405

Γ=0.2

*Below resonance*

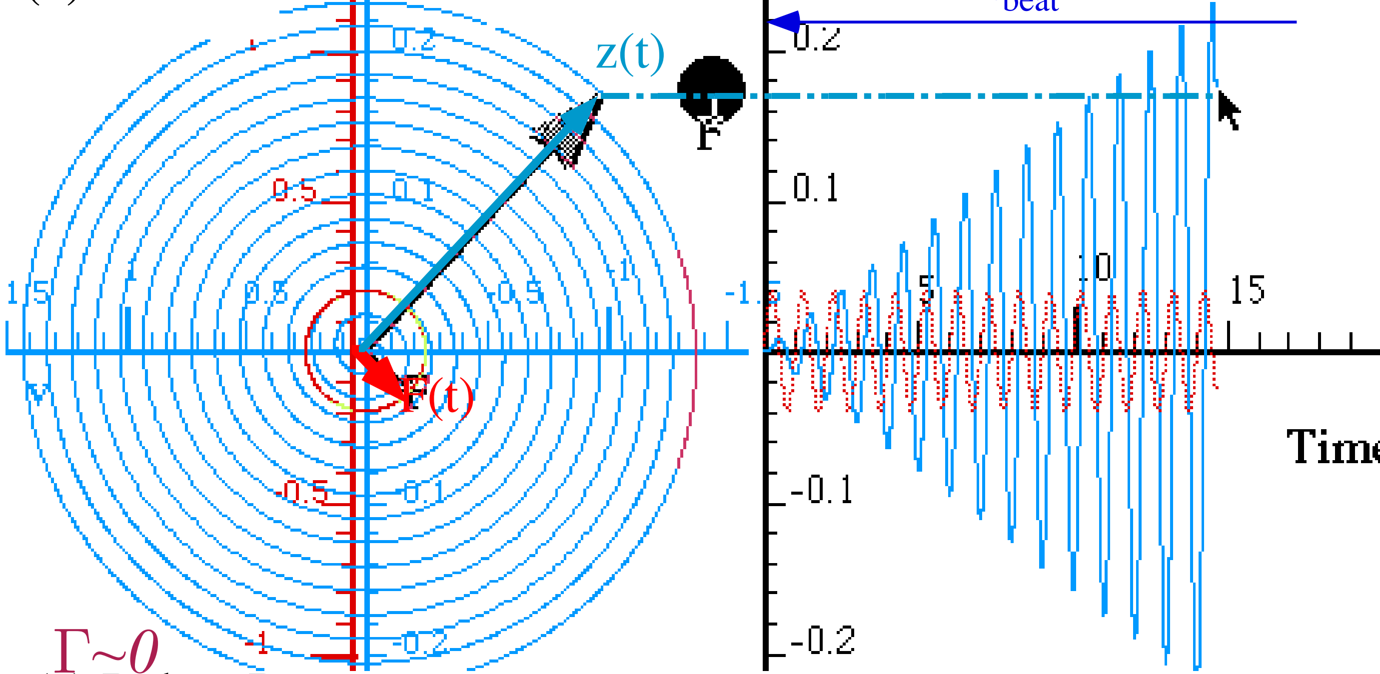
Γ~0

(a) Above Resonance



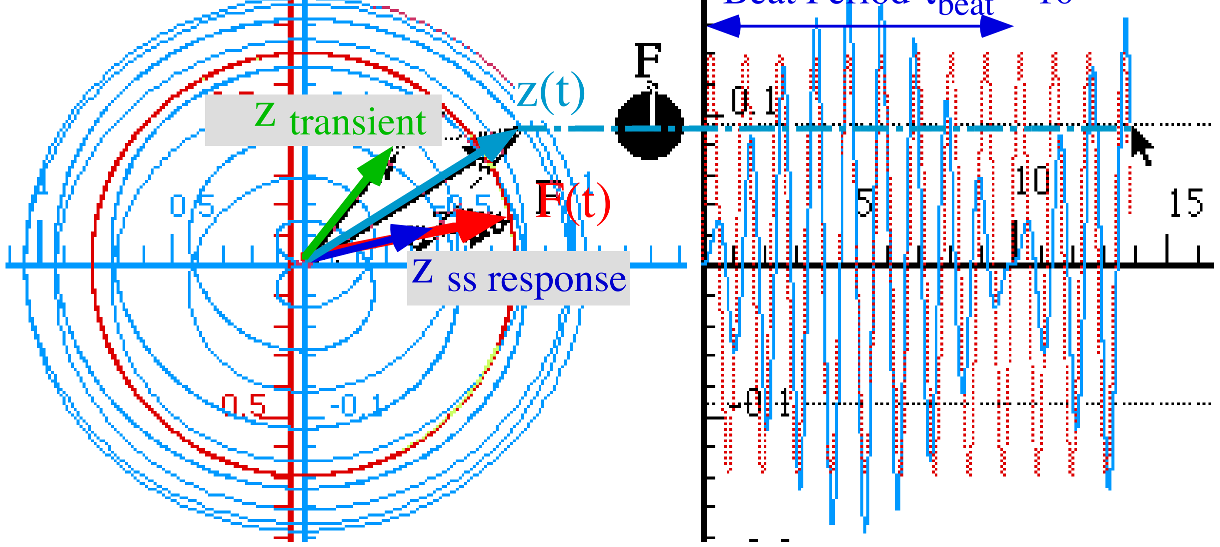
Γ~0

(b) Resonance



Γ~0

(c) Below Resonance



# *Lorentz-Green's Function for high quality FDHO*

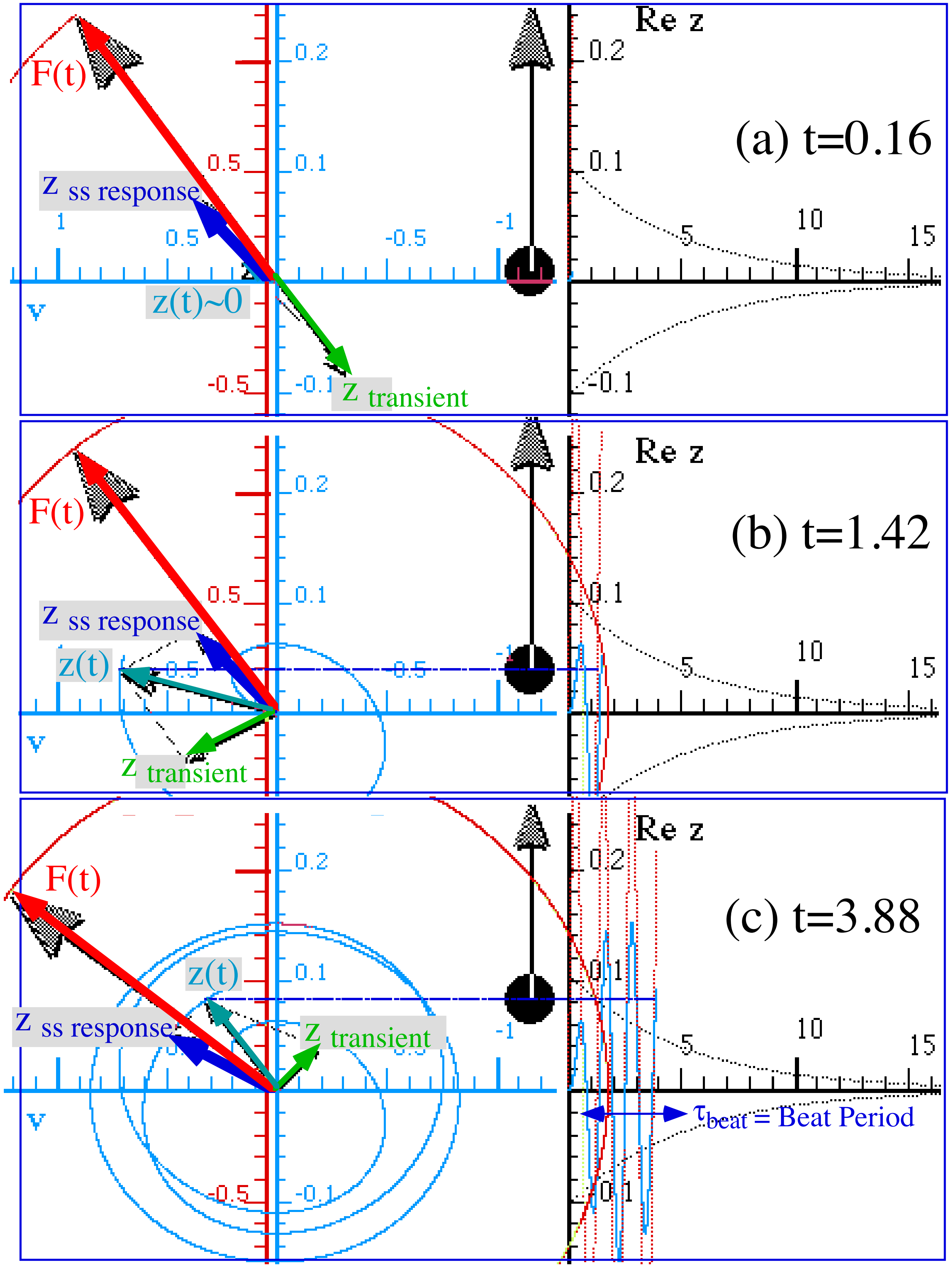


*Fig. 4.2.9 Beat formation.*

*Transient phasor $z_{transient}$ catches up with F-phasor and passes it.*

OscillIt (Beating) Simulation

# Oscillator figures of merit: quality factors Q and $q=2\pi Q$

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s = \omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \qquad (\textit{angular quality factor})$$

Amplification factor $q = \omega_0/2\Gamma$

Natural oscillation frequency is approximately $\upsilon_0 = \omega_0/2\pi$ (for $\omega_0 >> \Gamma$ we have $\omega_0 \sim \omega_\Gamma$).

# Oscillator figures of merit: quality factors Q and q=2πQ

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$$\overset{t_{5\%} = 3/\Gamma = \textit{Lifetime}}{\left(\begin{matrix}\text{for decaying oscillator}\\ \text{to lose 95\% of}\\ \text{amplitude}\end{matrix}\right)} \textit{times} \left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{matrix}\text{number } n_{5\%}\\ \text{of oscillations}\\ \text{in a } t_{5\%} \textit{ Lifetime}\end{matrix}$$

# Oscillator figures of merit: quality factors Q and q=2πQ

$$AAF = \frac{\text{Resonant response}}{\text{DC response}} = \frac{\left|G_{\omega_0}\left(\omega_s = \omega_0\right)\right|}{\left|G_{\omega_0}(0)\right|} = \frac{1/\left(2\Gamma\omega_0\right)}{1/\omega_0^2} = \frac{\omega_0}{2\Gamma} \equiv q \qquad (\textit{angular quality factor})$$

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$$n_{5\%} = t_{5\%}\upsilon_0 = \frac{3}{\Gamma}\cdot\frac{\omega_0}{2\pi} \cong \frac{\omega_0}{2\Gamma} = q$$

The “Heartbeat Count”
measure of lifetime

# Oscillator figures of merit: quality factors Q and q=2πQ

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The “Heartbeat Count”
measure of lifetime

Energy decay
(proportional to the square of oscillator amplitude): $\left(e^{\Gamma t}\right)^2 = e^{-2\Gamma t}$

$$dE = -2\Gamma E$$

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Amplification factor $q = \omega_0/2\Gamma$

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$$\left(\begin{array}{c}\text{for decaying oscillator}\\ \text{to lose 95\% of}\\ \text{amplitude}\end{array}\right)\textit{times}\left(\upsilon_0 = \frac{\omega_0}{2\pi}\right) = \begin{array}{c}\text{number } n_{5\%}\\ \text{of oscillations}\\ \text{in a } t_{5\%}\ \textit{Lifetime}\end{array}$$

$$n_{5\%} = t_{5\%}\upsilon_0 = \frac{3}{\Gamma}\cdot\frac{\omega_0}{2\pi} \cong \frac{\omega_0}{2\Gamma} = q$$

The “Heartbeat Count”
measure of lifetime

Energy decay
(proportional to the square of oscillator amplitude): $\left(e^{\Gamma t}\right)^2 = e^{-2\Gamma t}$

$$dE = -2\Gamma E$$

$$\begin{array}{r}\text{Relative amount}\\ \text{of energy lost}\\ \text{each cycle period}\end{array} = \tau_0\left(\frac{-dE}{E}\right) = \frac{2\Gamma}{\upsilon_0} \equiv \frac{1}{Q} = \frac{2\pi}{q}$$

$$\left(\tau_0 = \frac{1}{\upsilon_0}\right)$$

$$Q = (\textit{Standard angular quality factor}) = \frac{q}{2\pi}$$

# Oscillator figures of merit: Uncertainty 1/q

To see a beat we need $\tau_{\text{half-beat}}$ to be less than $\tau_{5\%}$ or 3/Γ. (Here we approximate π~3.0, again.)

$$\pi / \left|\omega_s - \omega_0\right| < 3/\Gamma \qquad\qquad \left|\omega_s - \omega_0\right| > \Gamma$$

This means ω-detuning error is greater than or equal to the decay rate Γ.

Any detuning less than Γ is virtually undetectable.
Total ω uncertainty is ±Γ or twice Γ (that is: FWHM $\Delta\omega = 2\Gamma$ ).

Linear frequency uncertainty is:

$$\Delta\upsilon = \Delta\omega / 2\pi = \Gamma / \pi$$

The *relative frequency uncertainty*

$$\frac{2\Gamma}{\omega_0} = \frac{\Delta\omega}{\omega_0} = \frac{1}{q} = \frac{\Delta\upsilon}{\upsilon_0}$$

is the *inverse* of the *angular quality factor q*.

If we think of the 5% or 4.321% lifetime of a musical note
as its time uncertainty Δt, then:

$$\Delta t \Delta\upsilon = 3/\pi \approx 1$$

$$\Delta t = t_{5\%} = 3/\Gamma \qquad\qquad \Delta t = t_{4.321\%} = \pi / \Gamma$$

*Very precise measures of imprecision*

# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow{\omega_s \to \omega_0} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

*Complex detuning-decay* δ=Δ-*i*Γ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

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*Complex detuning-decay* δ=Δ-*i*Γ variable δ is defined with the *real detuning* $\Delta = \omega_0 - \omega_s$

$$L(\Delta - i\Gamma) = \frac{1}{\Delta - i\Gamma} = \mathrm{Re}\,L \quad + i\,\mathrm{Im}\,L \quad = \quad \frac{\Delta}{\Delta^2 + \Gamma^2} \quad + i\frac{\Gamma}{\Delta^2 + \Gamma^2} = \quad |L|^2\,\Delta \quad + i\,|L|^2\,\Gamma$$

# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

$$G_{\omega_0}(\omega_s) = \frac{1}{\omega_0^2 - \omega_s^2 - i2\Gamma\omega_s} \xrightarrow[\omega_s \to \omega_0]{} \frac{1}{2\omega_s}\frac{1}{\omega_0 - \omega_s - i\Gamma} \approx \frac{1}{2\omega_0}\frac{1}{\Delta - i\Gamma} = \frac{1}{2\omega_0} L(\Delta - i\Gamma)$$

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$$= |L|e^{i\rho} = |L|\cos\rho + i\,|L|\sin\rho = \frac{\cos\rho}{\sqrt{\Delta^2 + \Gamma^2}} + i\frac{\sin\rho}{\sqrt{\Delta^2 + \Gamma^2}} \quad \text{where:} |L| = \frac{1}{\sqrt{\Delta^2 + \Gamma^2}}$$

# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

$$G_{\omega_0}(\omega_s)=\frac{1}{\omega_0^2-\omega_s^2-i2\Gamma\omega_s}\xrightarrow[\omega_s\to\omega_0]{}\frac{1}{2\omega_s}\frac{1}{\omega_0-\omega_s-i\Gamma}\approx\frac{1}{2\omega_0}\frac{1}{\Delta-i\Gamma}=\frac{1}{2\omega_0}L(\Delta-i\Gamma)$$

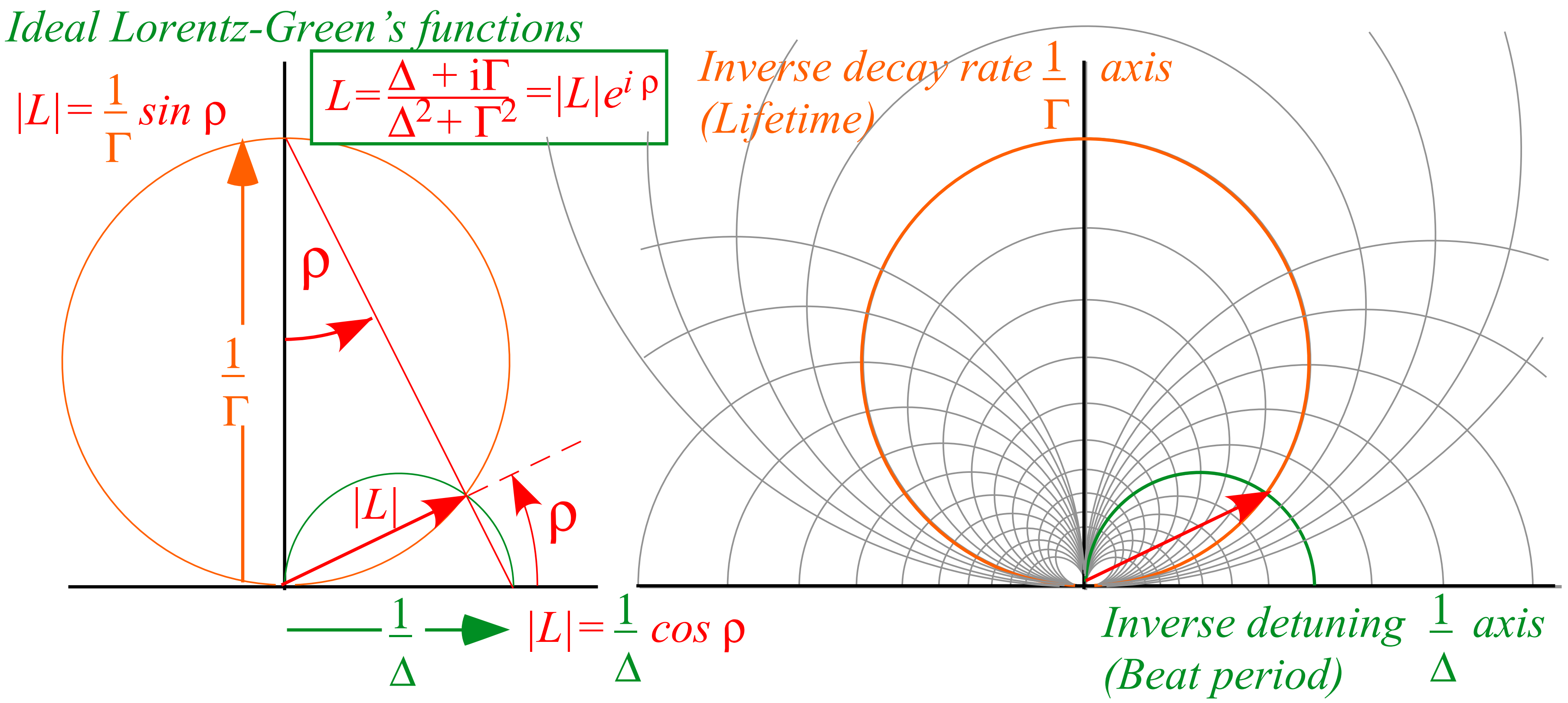
*Complex detuning-decay* $\delta=\Delta-i\Gamma$ variable $\delta$ is defined with the *real detuning* $\Delta=\omega_0-\omega_s$

$$L(\Delta-i\Gamma)=\frac{1}{\Delta-i\Gamma}=\mathrm{Re}\,L\quad+i\,\mathrm{Im}\,L\quad=\frac{\Delta}{\Delta^2+\Gamma^2}\quad+i\frac{\Gamma}{\Delta^2+\Gamma^2}=\quad|L|^2\,\Delta\quad+i\,|L|^2\,\Gamma$$

$$=|L|e^{i\rho}=|L|\cos\rho+i\,|L|\sin\rho=\frac{\cos\rho}{\sqrt{\Delta^2+\Gamma^2}}+i\frac{\sin\rho}{\sqrt{\Delta^2+\Gamma^2}}\quad\text{where:}\,|L|=\frac{1}{\sqrt{\Delta^2+\Gamma^2}}$$

$$|L|=\frac{1}{\Gamma}\sin\rho$$

$$|L|=\frac{1}{\Delta}\cos\rho$$



*Smith plots*

# *Approximate Lorentz-Green's Function for high quality FDHO (Quantum propagator)*

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*Complex detuning-decay* δ=Δ-*i*Γ variable δ is defined with the *real detuning* $\Delta=\omega_0-\omega_s$

$$L(\Delta-i\Gamma)=\frac{1}{\Delta-i\Gamma}=\mathrm{Re}\,L+i\,\mathrm{Im}\,L=\frac{\Delta}{\Delta^2+\Gamma^2}+i\frac{\Gamma}{\Delta^2+\Gamma^2}=|L|^2\,\Delta+i|L|^2\,\Gamma$$

$$=|L|e^{i\rho}=|L|\cos\rho+i|L|\sin\rho=\frac{\cos\rho}{\sqrt{\Delta^2+\Gamma^2}}+i\frac{\sin\rho}{\sqrt{\Delta^2+\Gamma^2}}\quad\text{where: }|L|=\frac{1}{\sqrt{\Delta^2+\Gamma^2}}$$

$$|L|=\frac{1}{\Gamma}\sin\rho$$

$$|L|=\frac{1}{\Delta}\cos\rho$$

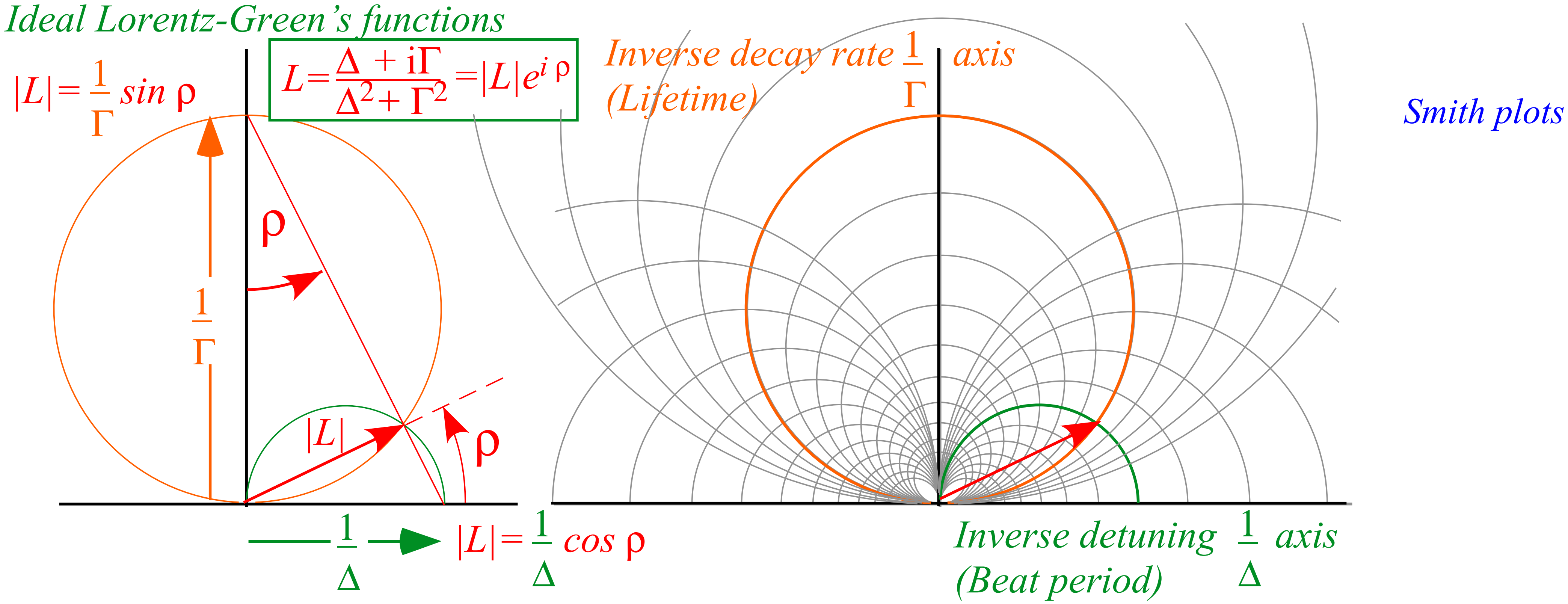


*Fig. 4.2.13 Ideal Lorentzian in inverse rate space. (Smith life-time* 1/Γ *vs. beat-period* 1/Δ *coordinates)*

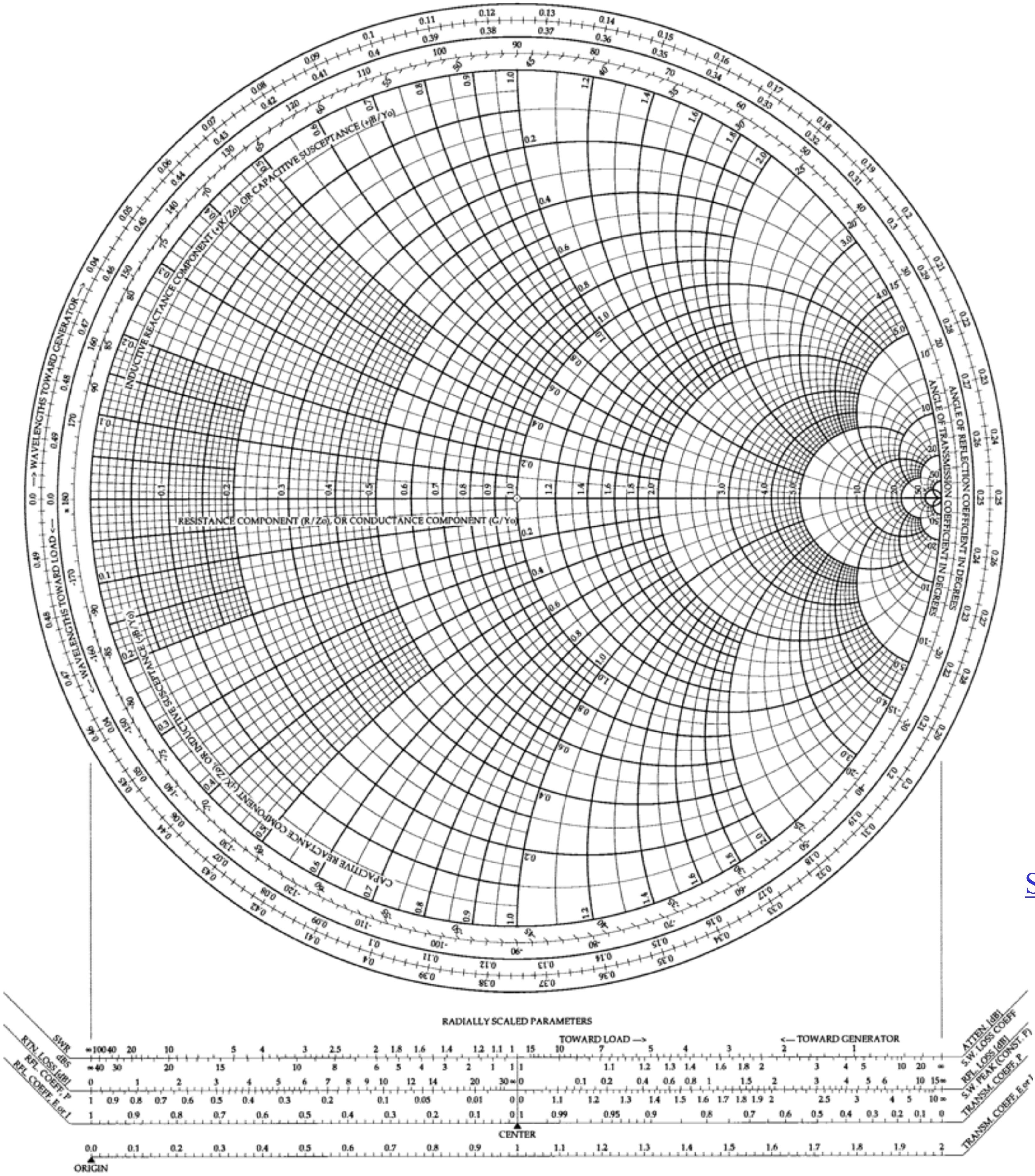
Constant Δ and Γ curves in Fig. 4.2.13 are orthogonal circles of *1/z- dipolar coordinates*. Recall Fig. 1.10.11.

SMITH CHART (Invented by Phillip H. Smith 1905-1987)

*An FDHO Green's Function Slide rule*

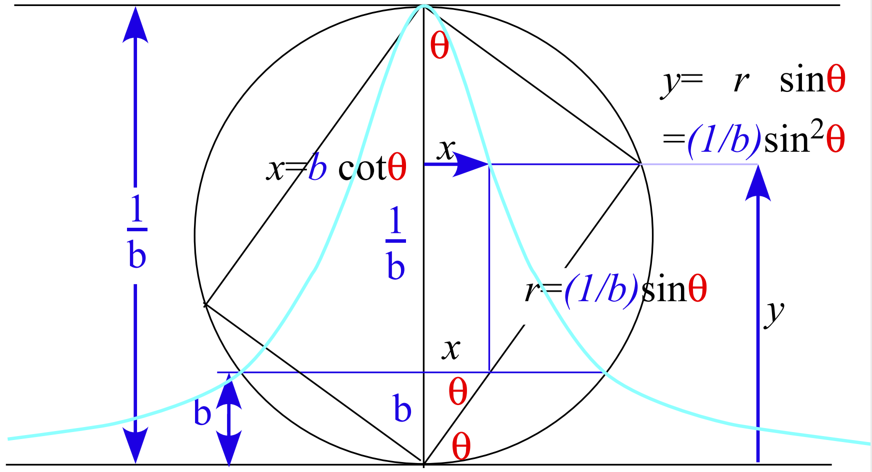
*A plot of f(z) =1/z*

*For wavy "Ohm's Laws"*
*V =I·Z*
*I=V/Z*



Smith plot: Graph paper

# The Common Lorentzian (a.k.a. *The Witch of Agnesi*)



$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

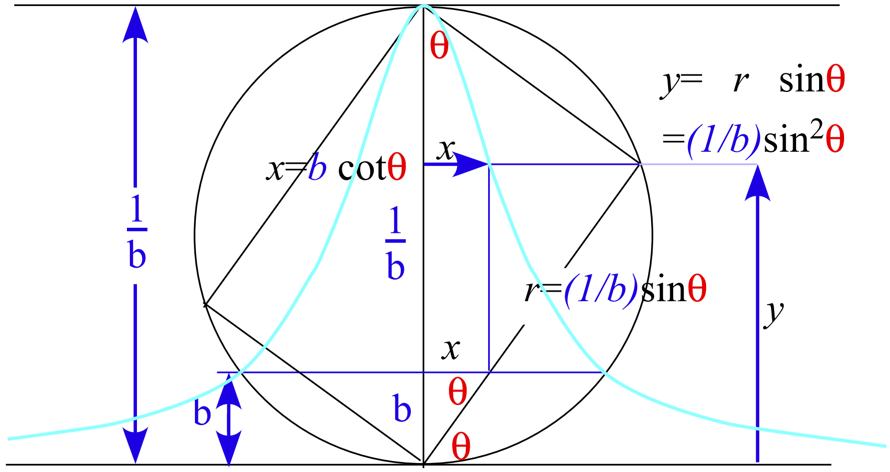
$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*

**Maria Gaetana Agnesi**

| | |
|---|---|
| **Born** | May 16, 1718 |
| **Died** | January 9, 1799 (aged 80) |
| **Residence** | Italy |
| **Nationality** | Italy |
| **Fields** | Mathematics |

# The Common Lorentzian (a.k.a. The Witch of Agnesi)

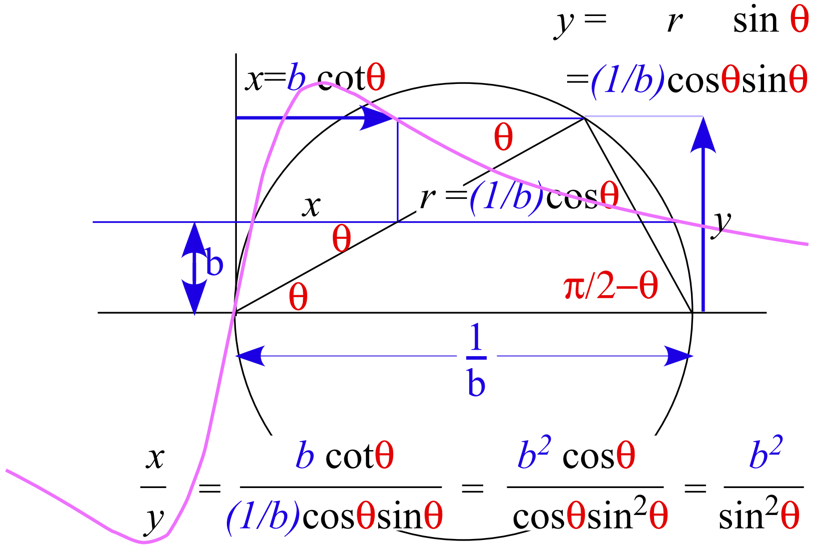


$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

$$y=\frac{b}{x^2+b^2}$$

Common Lorentzian function I.
(imaginary “absorbtive” part)



$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

$$y=\frac{x}{x^2+b^2}$$

Common Lorentzian function II.
(real “refractory” part)

**Maria Gaetana Agnesi**

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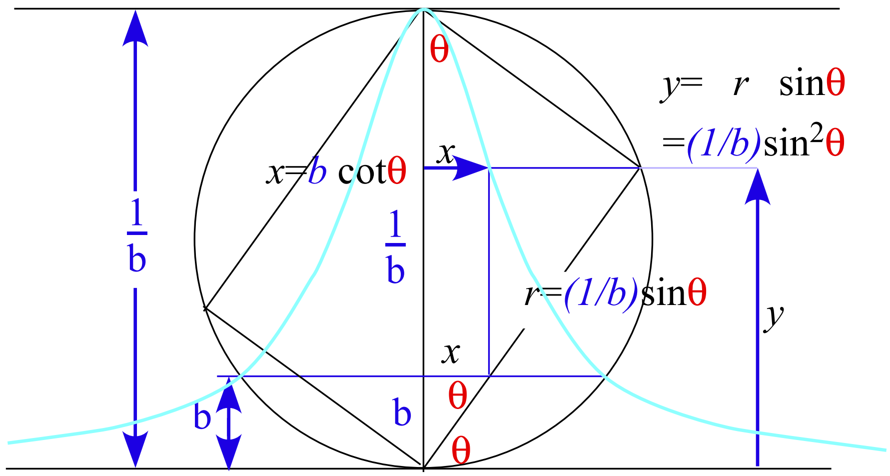
# The Common Lorentzian (a.k.a. *The Witch of Agnesi*)

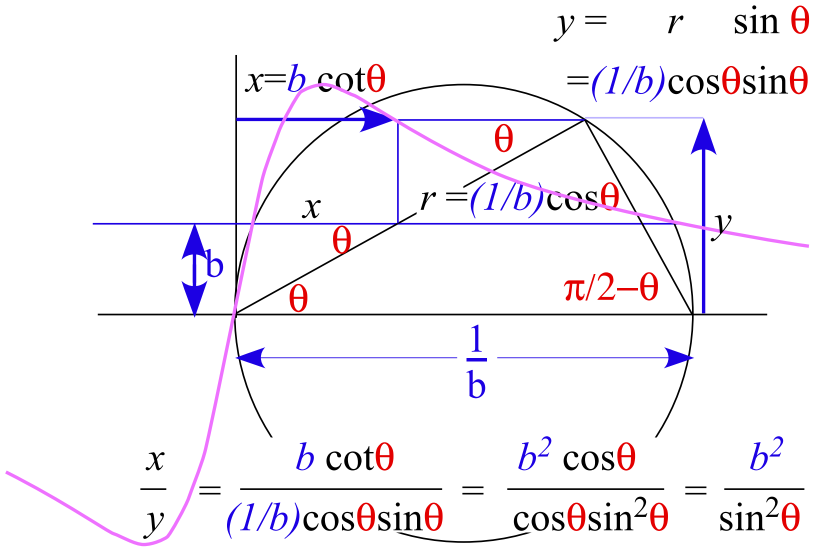


$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*

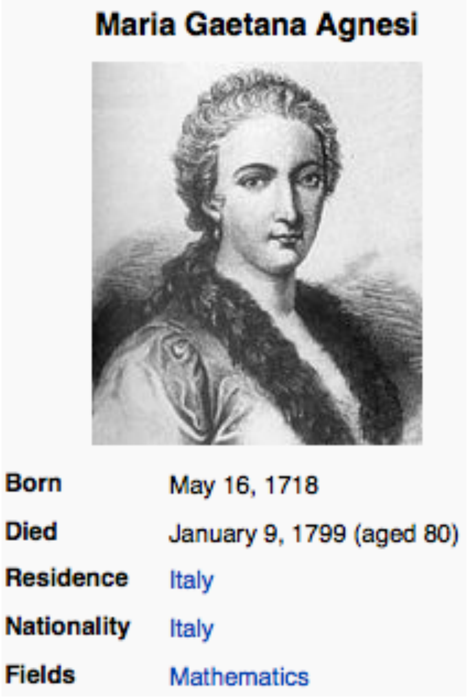


$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

$$y=\frac{x}{x^2+b^2}$$

*Common Lorentzian function II.*
*(real "refractory" part)*



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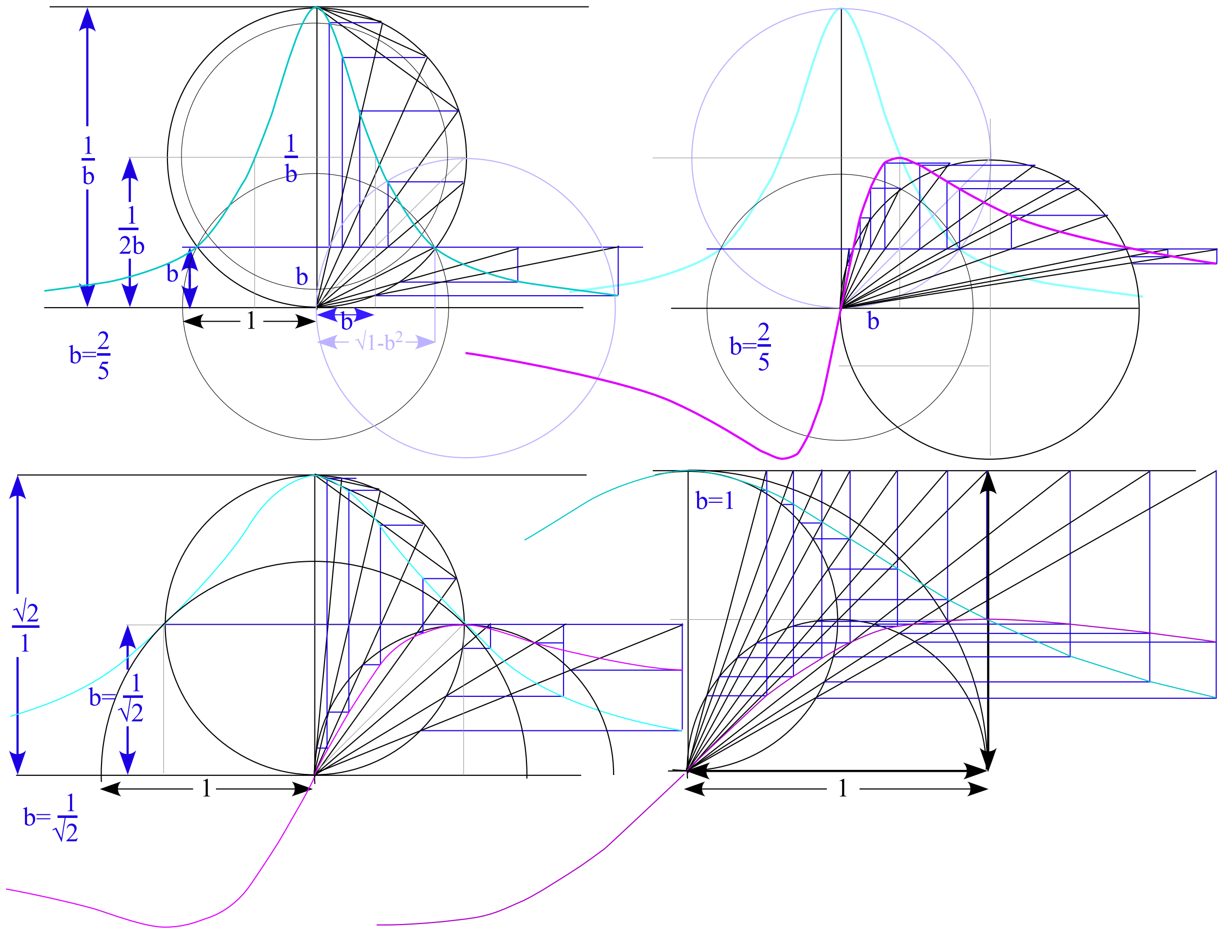
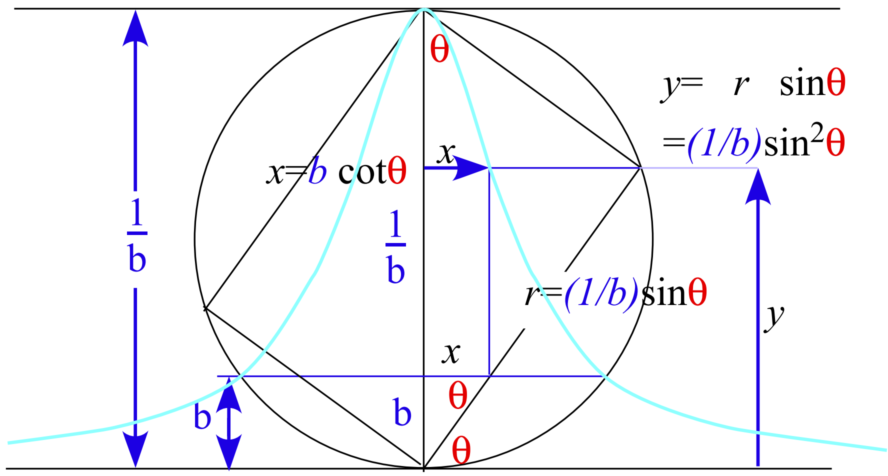
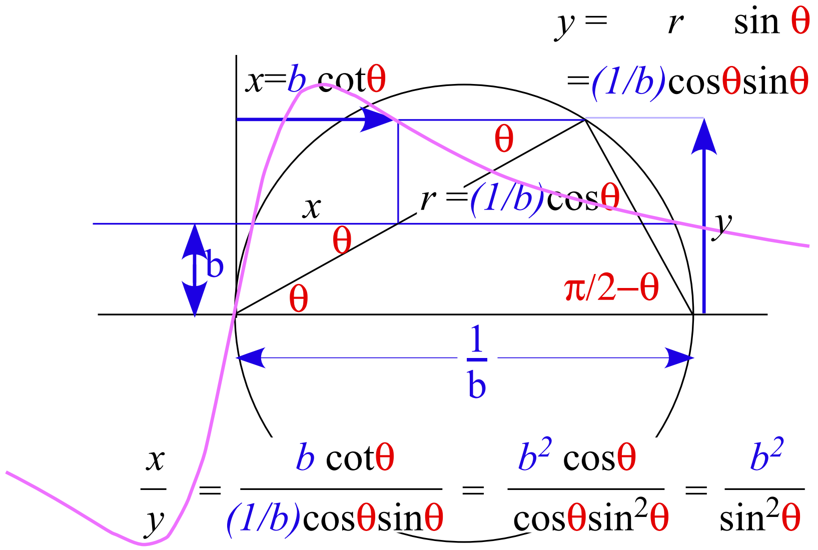


# The Common Lorentzian (a.k.a. *The Witch of Agnesi*)





$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}-b^2$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{b}{y}$$

$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

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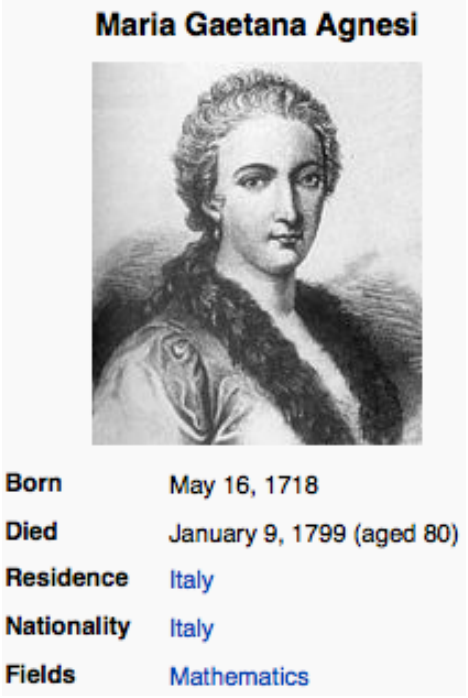
*Common Lorentzian function II.*
*(real "refractory" part)*



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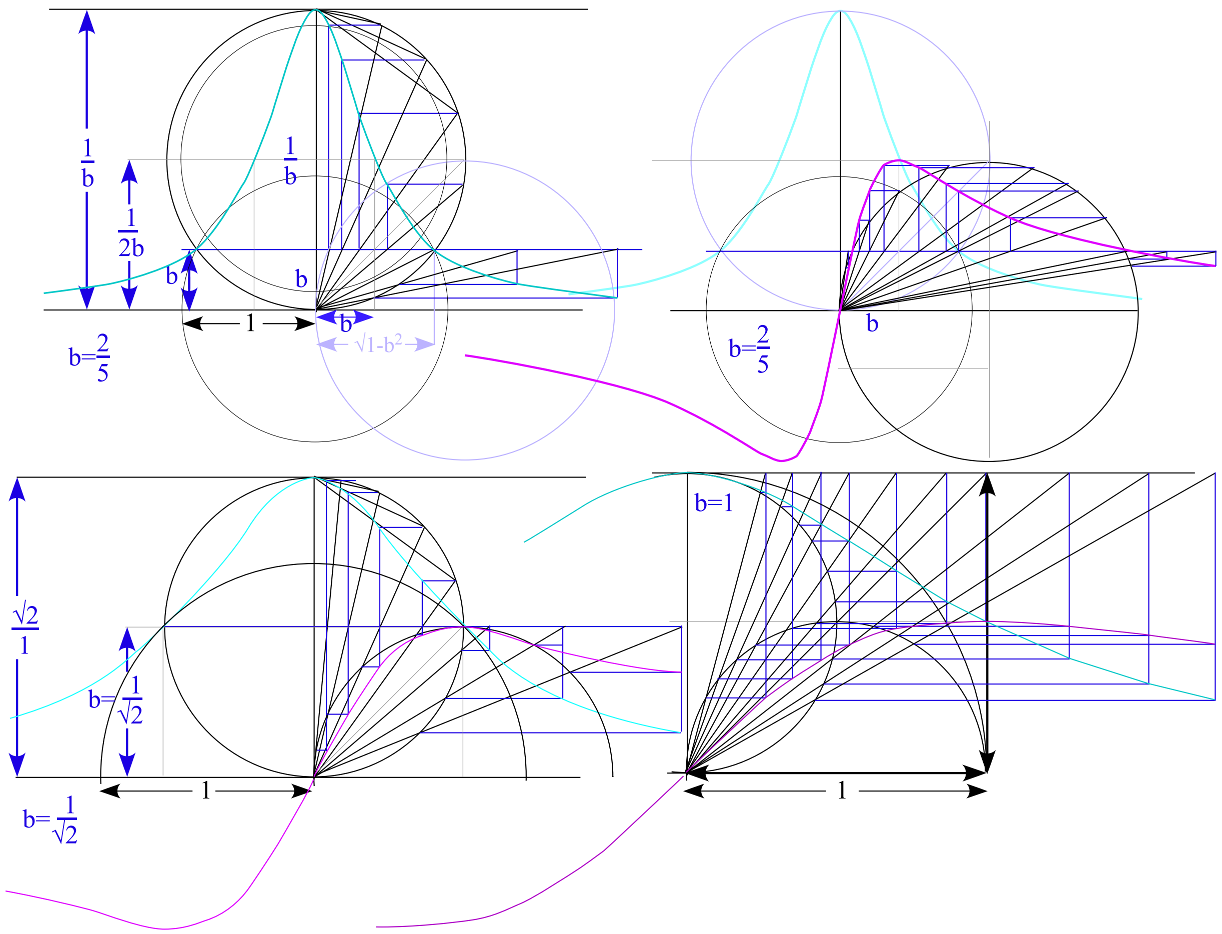
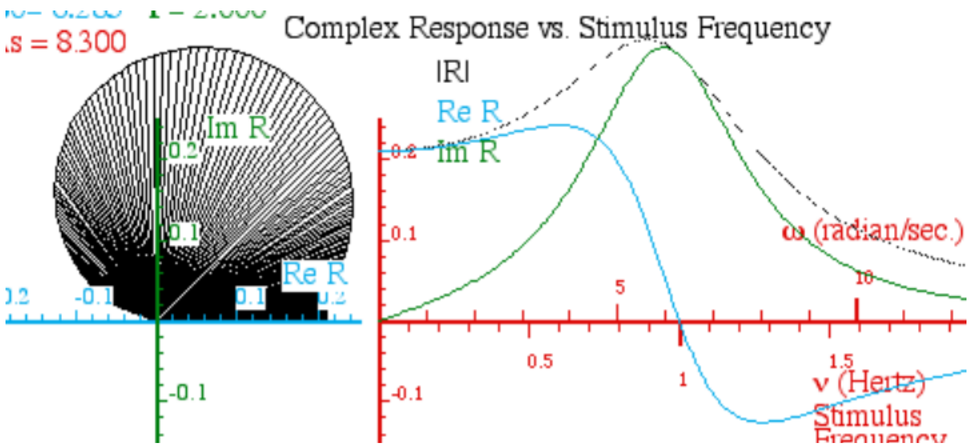
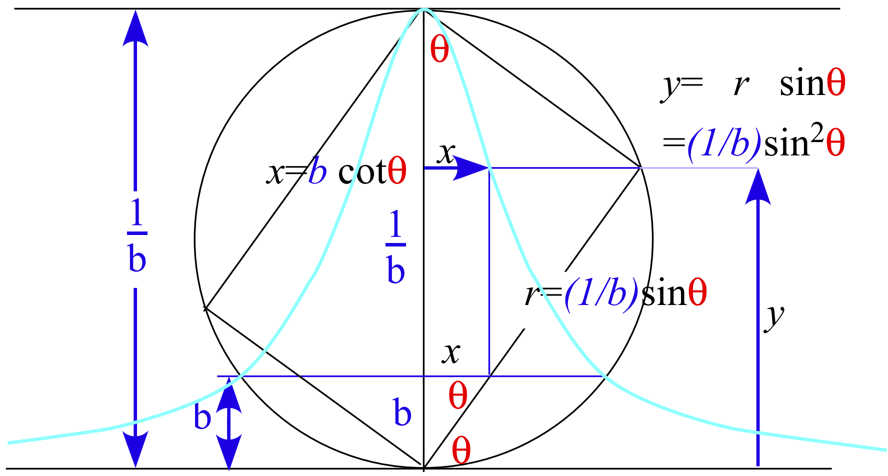


*Compare ideal Lorentzians (Γ=0.2)*
*with a very non-ideal one (Γ=2)*



OscillIt Web Simulation - Lorentz Response Function (Γ = 0.2)

# The Common Lorentzian (a.k.a. *The Witch of Agnesi*)



$$x^2=b^2\cot^2\theta=b^2\frac{\cos^2\theta}{\sin^2\theta}=b^2\frac{1-\sin^2\theta}{\sin^2\theta}=\frac{b^2}{\sin^2\theta}b^2$$

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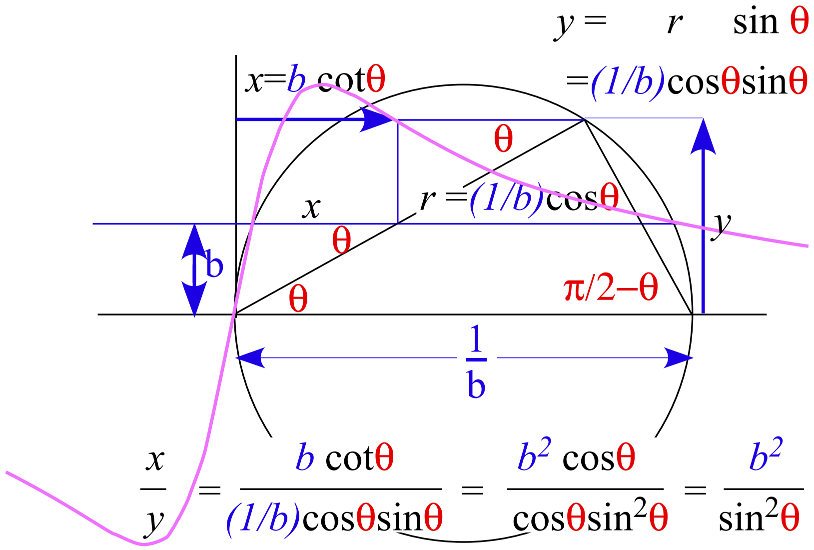
$$y=\frac{b}{x^2+b^2}$$

*Common Lorentzian function I.*
*(imaginary "absorbtive" part)*

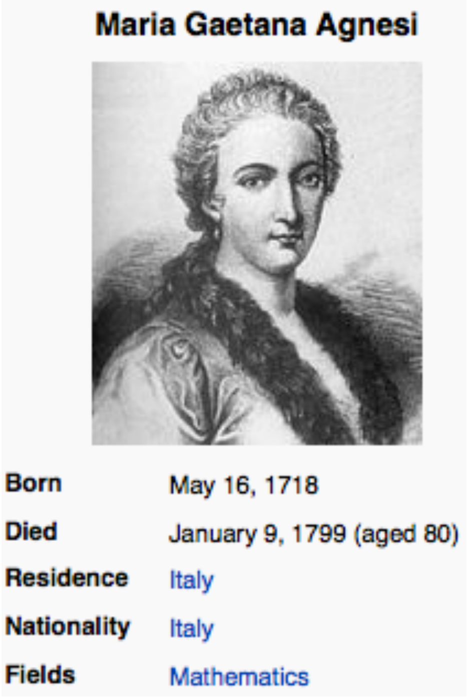


$$\frac{x}{y}=\frac{b\cot\theta}{(1/b)\cos\theta\sin\theta}=\frac{b^2\cos\theta}{\cos\theta\sin^2\theta}=\frac{b^2}{\sin^2\theta}$$

$$x^2+b^2=\frac{b^2}{\sin^2\theta}=\frac{x}{y}$$

$$y=\frac{x}{x^2+b^2}$$

*Common Lorentzian function II.*
*(real "refractory" part)*



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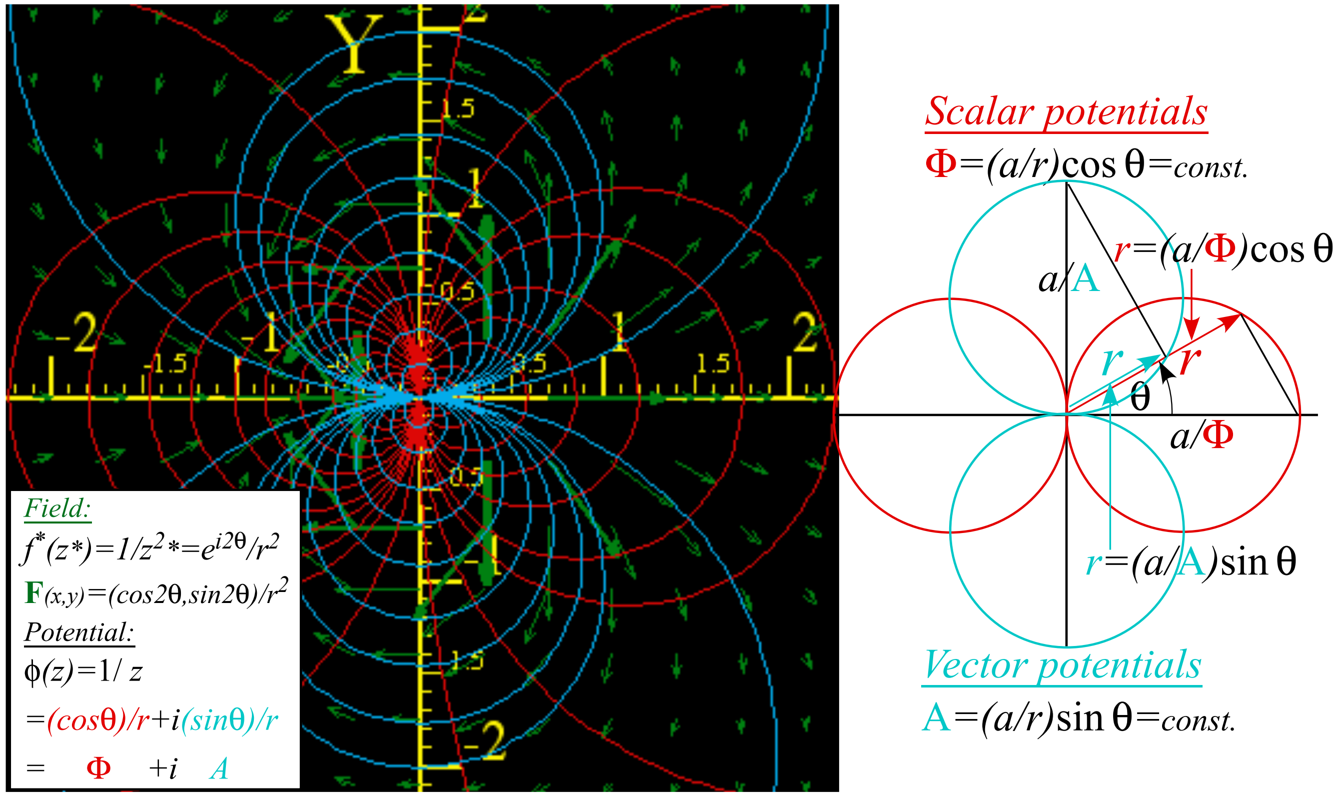
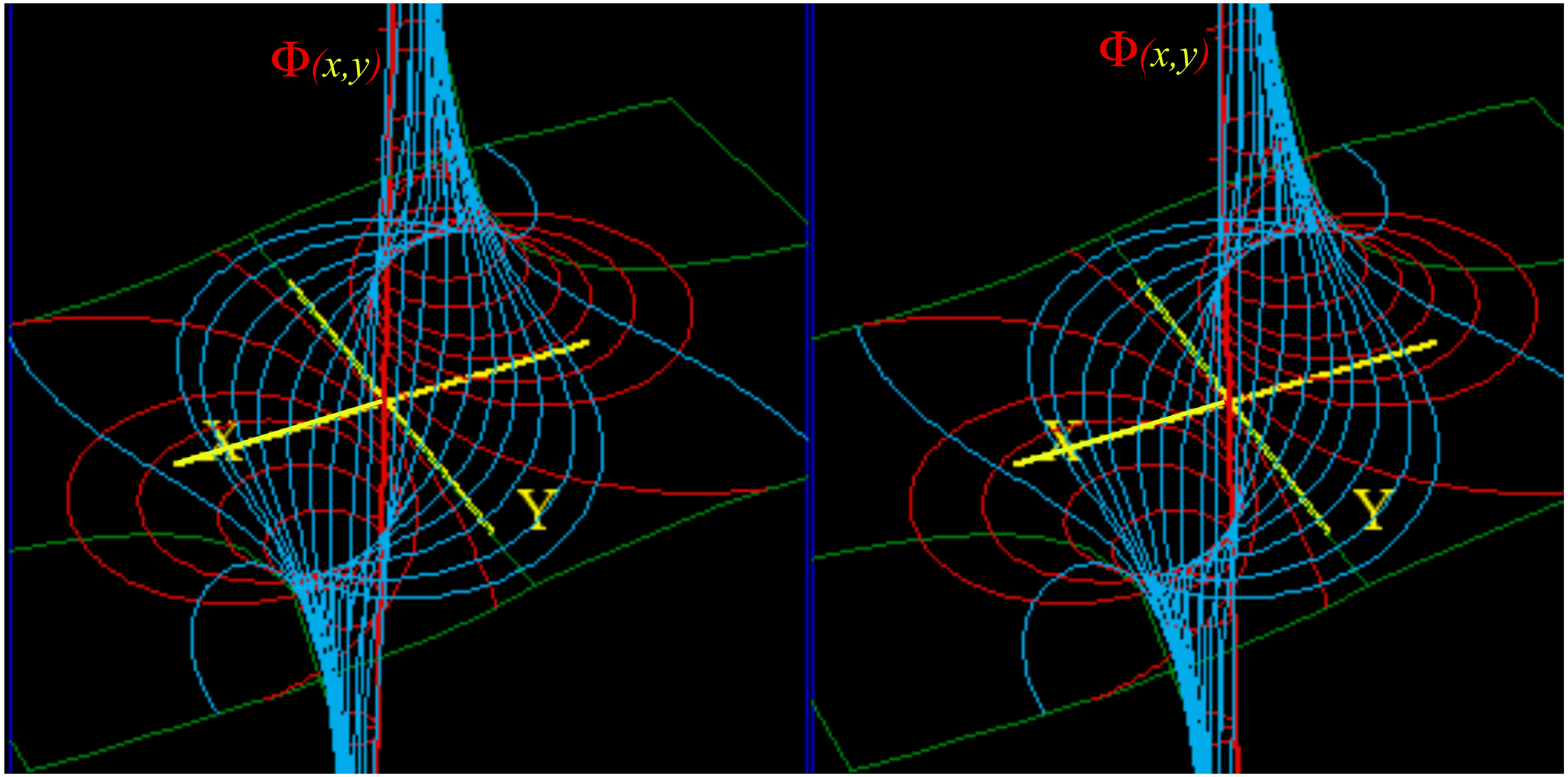


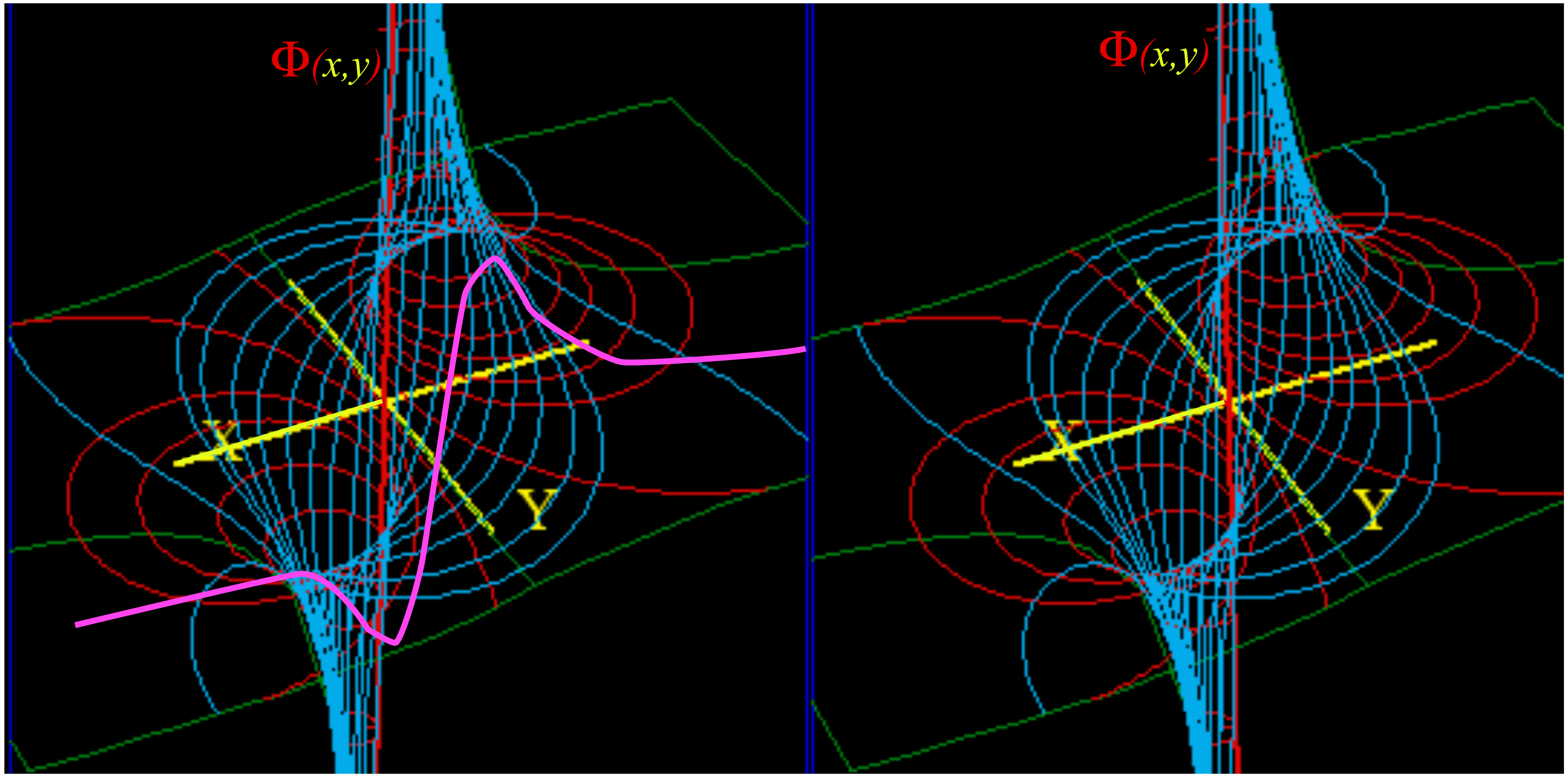
*Field:*
$f^*(z^*)=1/z^{2*}=e^{i2\theta}/r^2$
$\mathbf{F}(x,y)=(\cos 2\theta,\sin 2\theta)/r^2$
*Potential:*
$\phi(z)=1/z$
$=(\cos\theta)/r+i(\sin\theta)/r$
$=\Phi+i\,A$

*Scalar potentials*
$\Phi=(a/r)\cos\theta=const.$
$r=(a/\Phi)\cos\theta$
$r=(a/A)\sin\theta$
*Vector potentials*
$A=(a/r)\sin\theta=const.$

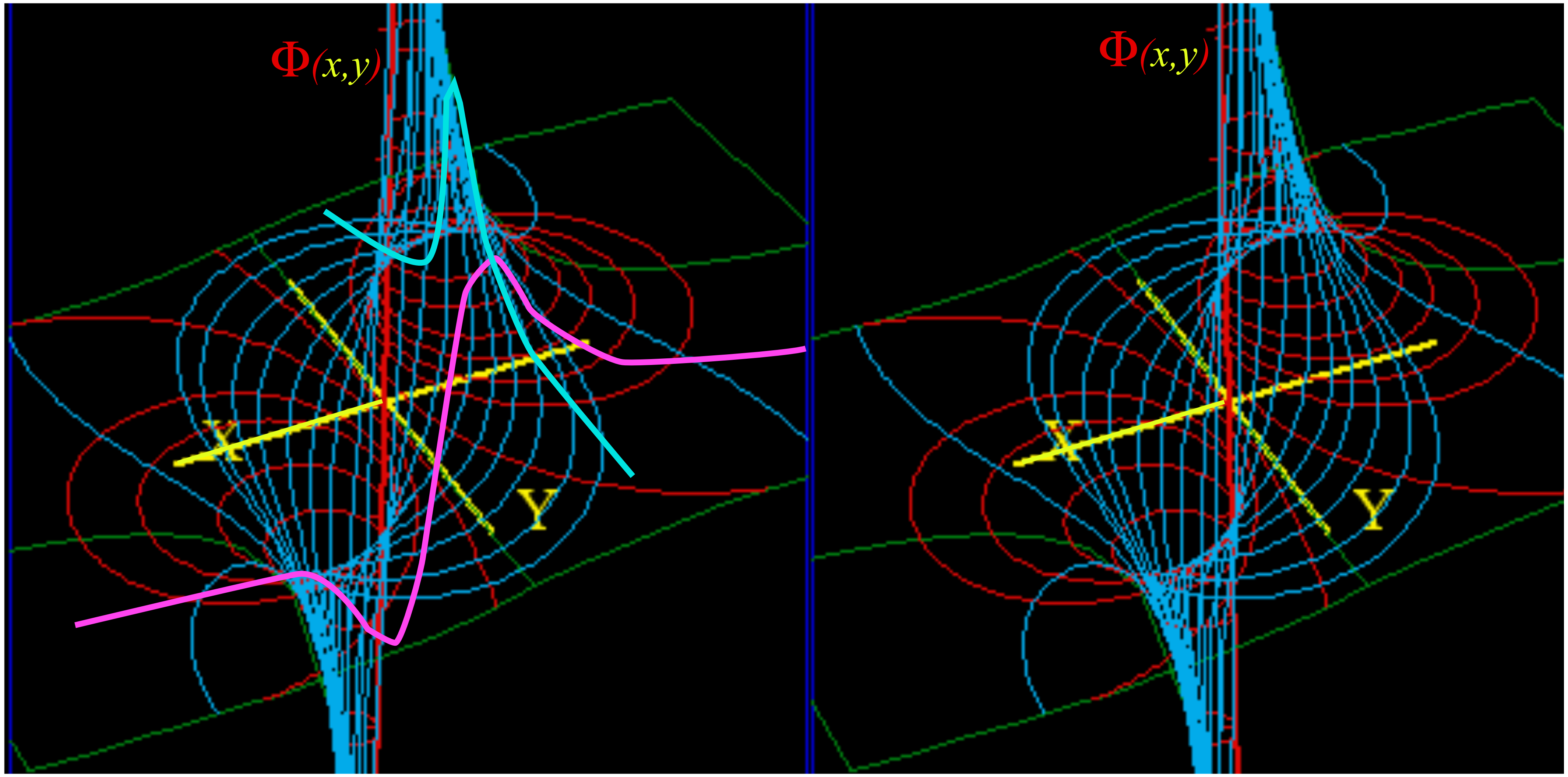
*Fig. 10.11 Dipole* **F***-field* $f(z)=1/z^2$ *and scalar potential (*$\Phi$*=const.)-circles orthogonal to (*$A$*=const.)-circles.*



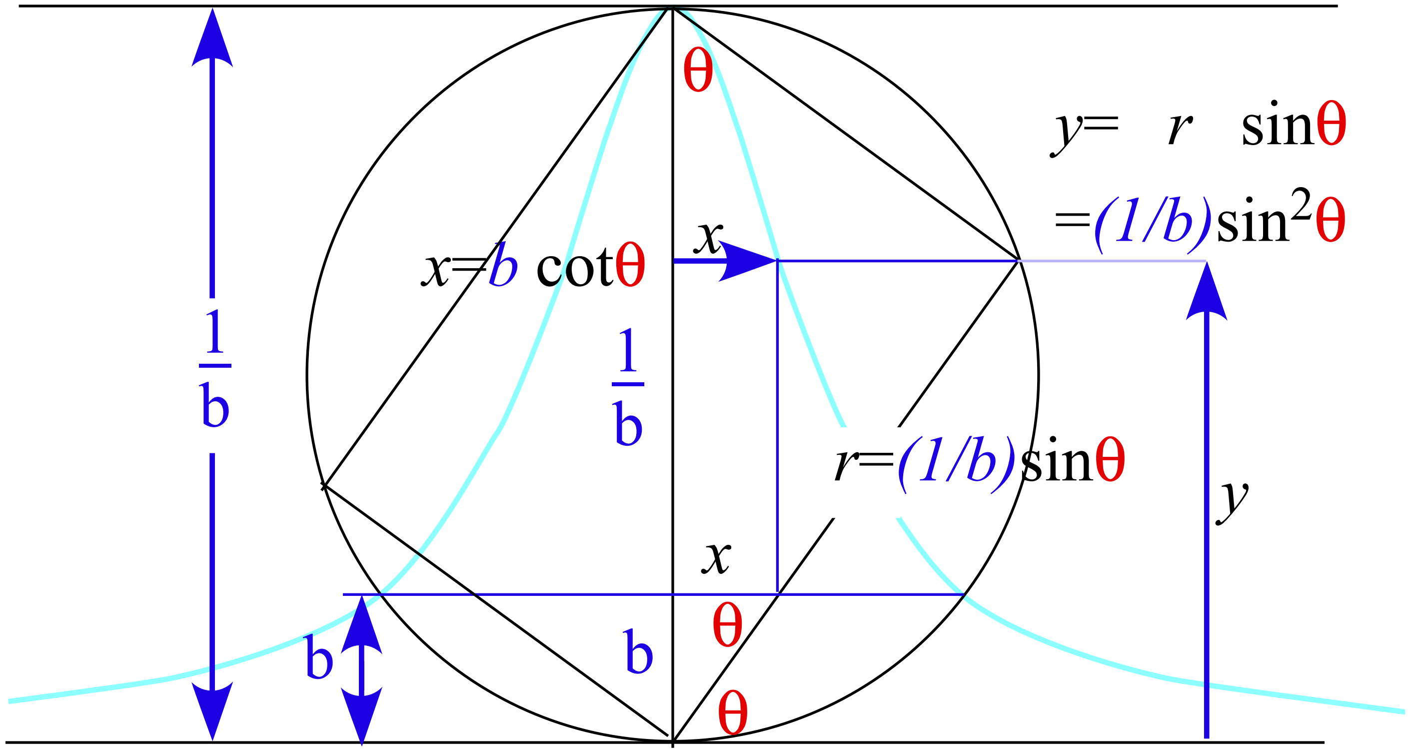
From: Fig. 1.10.12



From: Fig. 1.10.12



From: Fig. 1.10.12



$$x^2 = b^2 \cot^2\theta = b^2 \frac{\cos^2\theta}{\sin^2\theta} = b^2 \frac{1-\sin^2\theta}{\sin^2\theta} = \frac{b^2}{\sin^2\theta} b^2$$

$$x^2 + b^2 = \frac{b^2}{\sin^2\theta} = \frac{b}{y}$$

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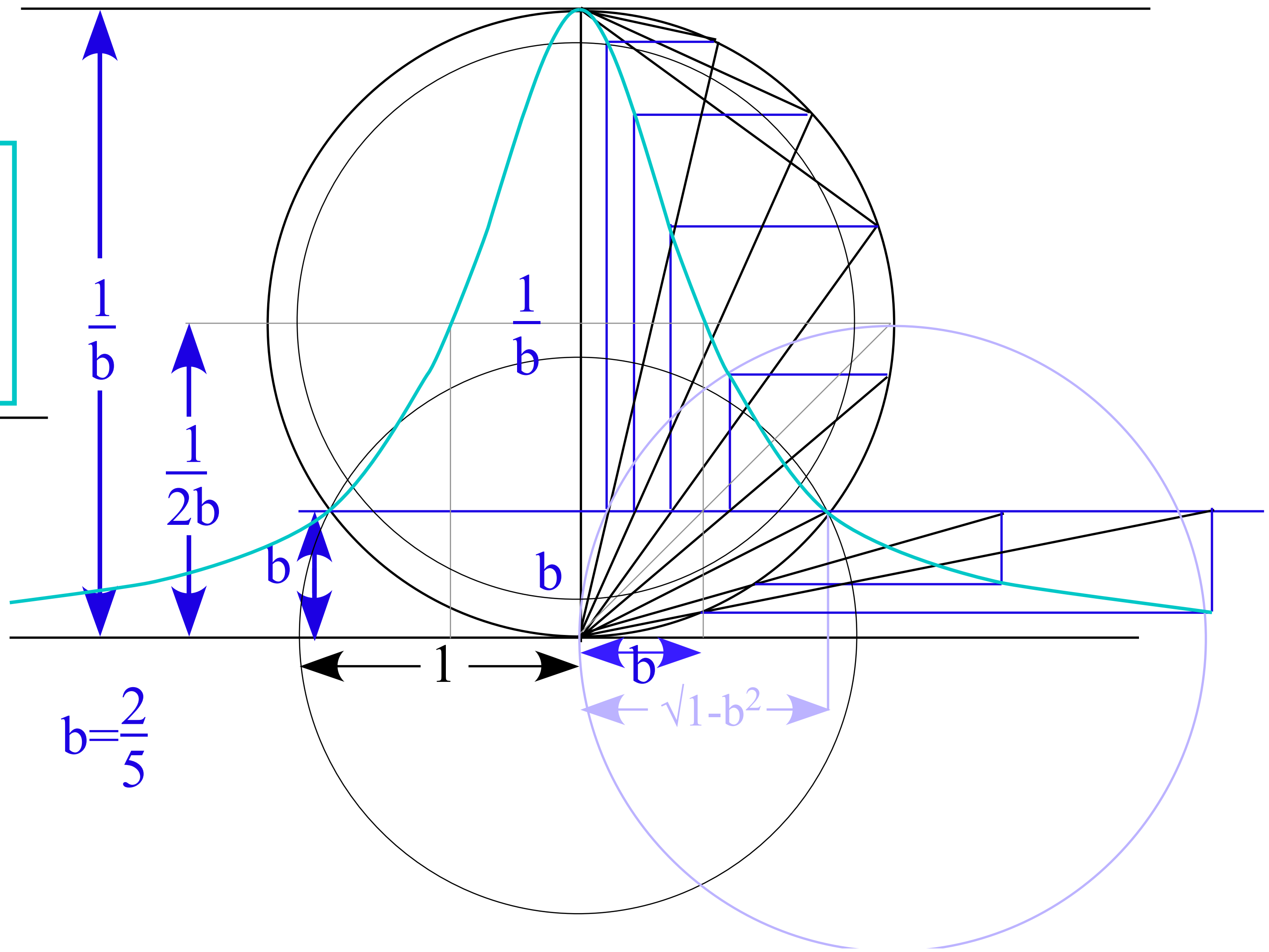
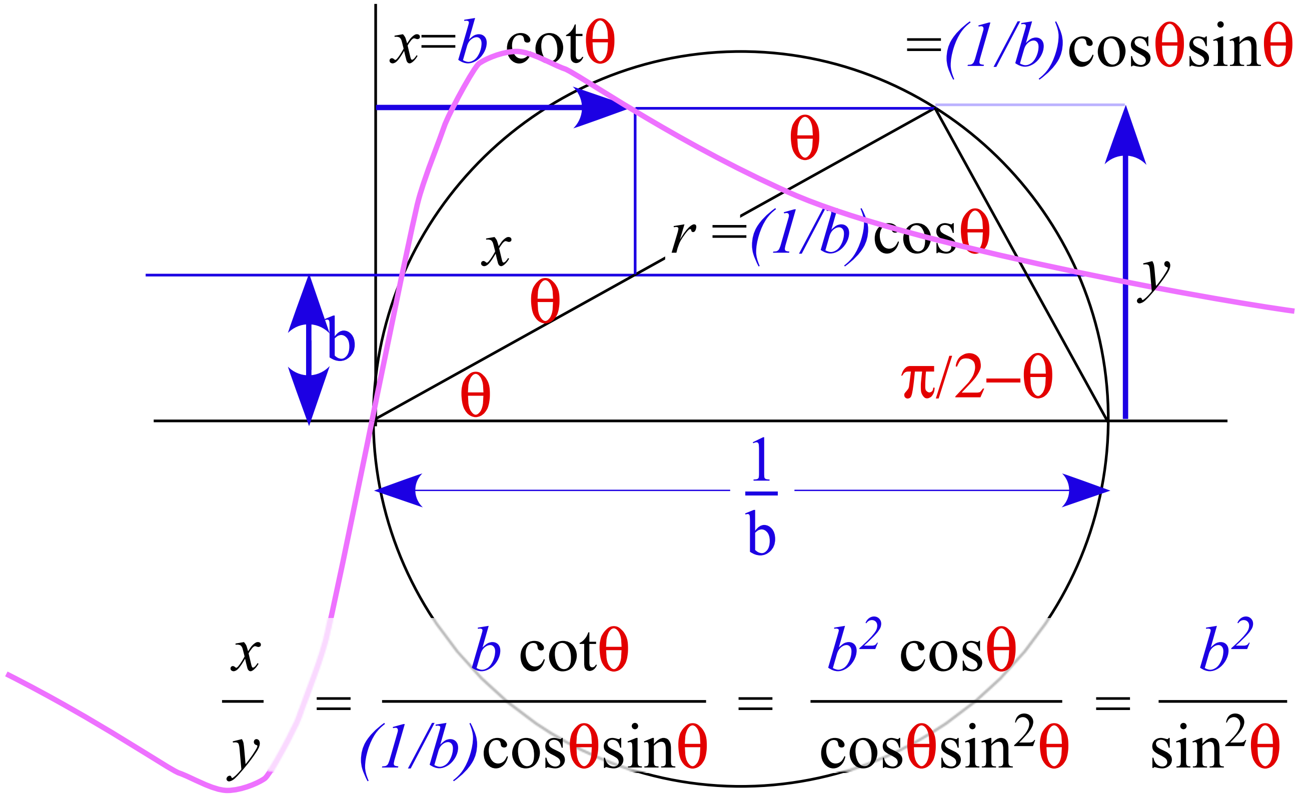
*Common Lorentzian function I.*
*(imaginary “absorbtive” part)*



$$y = r\sin\theta$$

$$= (1/b)\cos\theta\sin\theta$$



$$\frac{x}{y} = \frac{b\cot\theta}{(1/b)\cos\theta\sin\theta} = \frac{b^2\cos\theta}{\cos\theta\sin^2\theta} = \frac{b^2}{\sin^2\theta}$$

$$x^2+b^2 = \frac{b^2}{\sin^2\theta} = \frac{x}{y}$$

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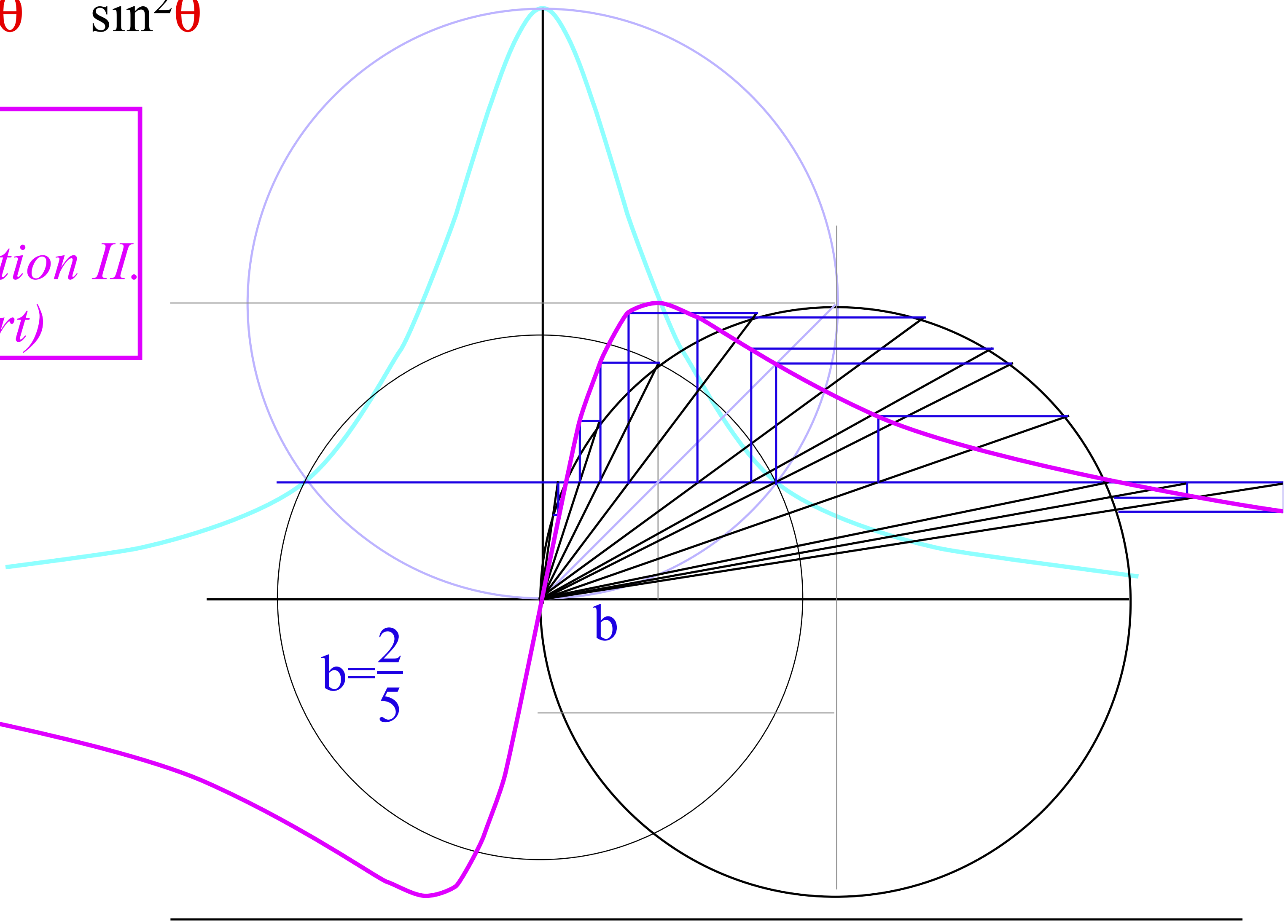
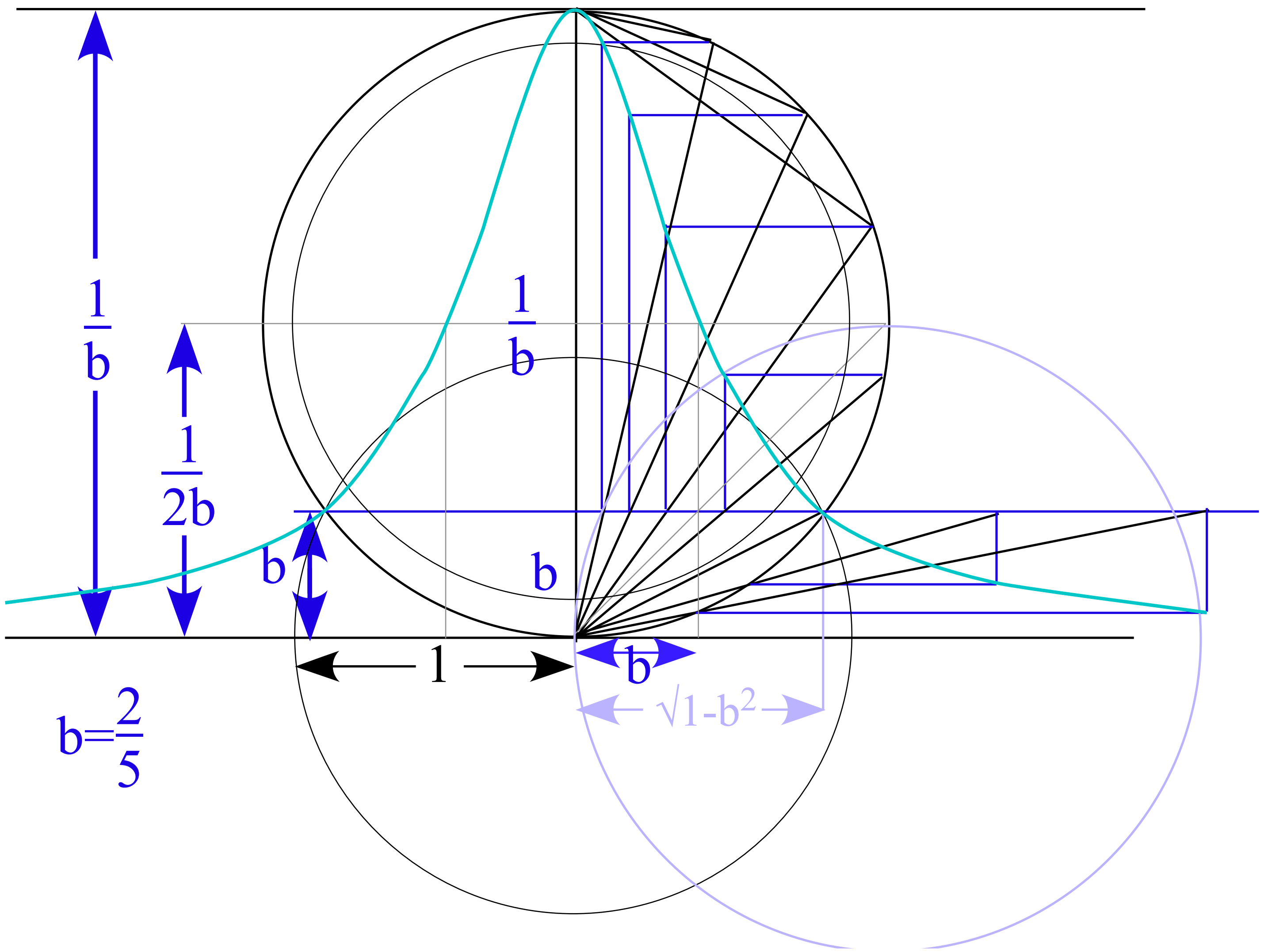
*Common Lorentzian function II.*

*(real "refractory" part)*




1/b
1/b
1/(2b)
b
b
1
b
√1-b²
b=2/5