Lecture 18
Wed. 10.30.2019

Electromagnetic Lagrangian and charge-field mechanics (Ch. 2.8 of Unit 2)

Cycloidal geometry of flying levers

Practical poolhall application

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

Lagrangian for particle-in-($\mathbf{A}$,Φ)-potential

Hamiltonian for particle-in-($\mathbf{A}$,Φ)-potential

Canonical momentum in ($\mathbf{A}$,Φ) potential

Hamiltonian formulation

Hamilton's equations

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

*This mechanical analog of (E_x,B_z) field mimics **A**-field with tabletop **v**-field and the **E**-field with table slope*

YouTube Video of Analog to Syncrotron Motion

This Lecture's Reference Link Listing

Web Resources - front page
UAF Physics UTube channel

Quantum Theory for the Computer Age
Principles of Symmetry, Dynamics, and Spectroscopy
Classical Mechanics with a Bang!
Modern Physics and its Classical Foundations

2017 Group Theory for QM
2018 Adv CM
2018 AMOP
2019 Advanced Mechanics

Lectures #12 through #18

In reverse order

CoulIt Web Simulations:

Synchrotron Motion, Synchrotron Motion #2
Mechanical Analog to EM Motion (YouTube video)
iBall demo - Quasi-periodicity (YouTube video)

Trebuchet Web Simulations:

Default/Generic URL, Montezuma's Revenge, Seige of Kenilworth,
"Flinger",
Position Space (Course), Position Space (Fine)

Wacky Waving Solid Metal Arm Flailing Chaos Pendulum - Scooba_Steeve-yt-2015
Triple Double-Pendulum - Cohen-yt-2008
Punkin Chunkin - TheArmchairCritic-2011
Jersey Team Claims Title in Punkin Chunkin - sussexcountyonline-1999
Shooting range for medieval siege weapons. Anybody knows? - twcenter.net/forums
The Trebuchet - Chevedden-SciAm-1995
NOVA Builds a Trebuchet

Recent Articles of Interest:

Springer handbook on Molecular Symmetry and Dynamics - Ch_32 - Molecular Symmetry
Synthetic_Chiral_Light_for_Efficient_Control_of_Chiral_Light–Matter_Interaction_-_Ayuso-np-2019
A_Semi-Classical_Approach_to_the_Calculation_of_Highly_Excited_Rotational_Energies for …
Asymmetric-Top_Molecules_-_Schmiedt-pccp-2017
Quantum_Chaos_-_An_Introduction_-_Stockmann-cup-2006, Review by E. Heller
Tunable and broadband coherent perfect absorption by ultrathin blk phos metasurfaces - Guo-josab-2019
Quantum_Supremacy_Using_a_Programmable_Superconducting_Processor_-_Arute-n-2019
Vortex Detection in Vector Fields Using Geometric Algebra - Pollock-aaca-2013.pdf

An assist from *Physics Girl* (YouTube Channel):

Posted this year:
How to Make VORTEX RINGS in a Pool

Crazy pool vortex (new inclusion with more background)
Crazy pool vortex - pg-yt-2014

Posting with the best visuals:
Fun with Vortex Rings in the Pool - pg-yt-2014

Pirelli Relativity Challenge (Introduction level) - *Visualizing Waves:*

Using Earth as a clock,
Tesla's AC Phasors ,
Phasors using complex numbers.
CM wBang Unit 1 - Chapter 10, pdf_page=135
Calculus of exponentials, logarithms, and complex fields,

RelaWavity Web Simulation - Unit Circle and Hyperbola (Mixed labeling)
Smith Chart, Invented by Phillip H. Smith (1905-1987)

Select, exciting, and related Research

Clifford_Algebra_And_The_Projective_Model_Of_Homogeneous_Metric_Spaces -
Foundations - Sokolov-x-2013
Geometric Algebra 3 - Complex Numbers - MacDonald-yt-2015
Biquaternion -Complexified Quaternion- Roots of -1 - Sangwine-x-2015
An_Introduction_to_Clifford_Algebras_and_Spinors_-_Vaz-Rocha-op-2016
Unified View on Complex Numbers and Quaternions- Bongardt-wcmms-2015
Complex Functions and the Cauchy-Riemann Equations - complex2 - Friedman-columbia-2019
An_sp-hybridized_Molecular_Carbon_Allotrope-_cyclo-18-carbon_-_Kaiser-s-2019
An_Atomic-Scale_View_of_Cyclocarbon_Synthesis_-_Maier-s-2019
Discovery_Of_Topological_Weyl_Fermion_Lines_And_Drumhead_Surface_States_in_a_
Room_Temperature_Magnet_-_Belopolski-s-2019
"Weyl"ing_away_Time-reversal_Symmetry_-_Neto-s-2019
Non-Abelian_Band_Topology_in_Noninteracting_Metals_-_Wu-s-2019
What_Industry_Can_Teach_Academia_-_Mao-s-2019

RoVib-_quantum_state_resolution_of_the_C60_fullerene_-_Changala-Ye-s-2019 (Alt)
A_Degenerate_Fermi_Gas_of_Polar_molecules_-_DeMarco-s-2019

Excerpts (Page 44-47 in *Preliminary Draft*) from the
Geometric Algebra- A Guided Tour through Space and Time - Reimer-www-2019

Running Reference Link Listing

Lectures #11 through #7

In reverse order

Eric J Heller Gallery:

Main portal, Consonance and Dissonance II, Bessel 21, Chladni

The Semiclassical Way to Molecular Spectroscopy - Heller-acs-1981
Quantum_dynamical_tunneling_in_bound_states_-_Davis-Heller-jcp-1981

Pendulum Web Simulation
Cycloidulum Web Simulation

Links to previous lecture: *Page=74*, *Page=75*, *Page=79*

Pendulum Web Sim

Cycloidulum Web Sim

***JerkIt* Web Simulations:** *Basic/Generic: Inverted*, *FVPlot*

CMwithBang Lecture 8, page=20
WWW.sciencenewsforstudents.org: Cassini - Saturnian polar vortex

“RelaWavity” Web Simulations:
2-CW laser wave, Lagrangian vs Hamiltonian,
Physical Terms Lagrangian L(u) vs Hamiltonian H(p)
CoulIt Web Simulation of the Volcanoes of Io
BohrIt Multi-Panel Plot:
Relativistically shifted Time-Space plots of 2 CW light waves

BoxIt Web Simulations:

Generic/Default
Most Basic A-Type
Basic A-Type w/reference lines
Basic A-Type A-Type with Potential energy
A-Type with Potential energy and Stokes Plot
A-Type w/3 time rates of change
A-Type w/3 time rates of change with Stokes Plot
B-Type (A=1.0, B=-0.05, C=0.0, D=1.0)

RelaWavity Web Elliptical Motion Simulations:

Orbits with b/a=0.125
Orbits with b/a=0.5
Orbits with b/a=0.7
Exegesis with b/a=0.125
Exegesis with b/a=0.5
Exegesis with b/a=0.7
Contact Ellipsometry

***CoulIt* Web Simulations:**

Basic/Generic
Exploding Starlet
Volcanoes of Io (Color Quantized)

***JerkIt* Web Simulations:**

Basic/Generic
Catcher in the Eye - IHO with Linear Hooke perturbation - Force-potential-Velocity Plot

***OscillatorPE* Web Simulation:**

Coulomb-Newton-Inverse_Square,
Hooke-Isotropic Harmonic,
Pendulum-Circular_Constraint

AMOP Ch 0 Space-Time Symmetry - 2019
Seminar at Rochester Institute of Optics, Aux. slides-2018

NASA Astronomy Picture of the Day -
Io: The Prometheus Plume *(Just Image)*
NASA Galileo - *Io's Alien Volcanoes*
New Horizons - *Volcanic Eruption Plume on Jupiter's moon IO*
NASA Galileo - A Hawaiian-Style Volcano on Io

Pirelli Site: Phasors animimation
CMwithBang Lecture #6, page=70 (9.10.18)

Select, exciting, and related Research & Articles of Interest:

Burning a hole in reality—design for a new laser may be powerful enough to pierce space-time - Sumner-KOS-2019
Trampoline mirror may push laser pulse through fabric of the Universe - Lee-ArsTechnica-2019
Achieving_Extreme_Light_Intensities_using_Optically_Curved_Relativistic_Plasma_Mirrors_-_Vincenti-prl-2019

A_Soft_Matter_Computer_for_Soft_Robots_-_Garrad-sr-2019
Correlated_Insulator_Behaviour_at_Half-Filling_in_Magic-Angle_Graphene_Superlattices_-_cao-n-2018

*Sorting_ultracold_atoms_in_a_three-dimensional_optical_lattice_in_a_*
realization_of_Maxwell's_Demon - Kumar-n-2018
Synthetic_three-dimensional_atomic_structures_assembled_atom_by_atom - Barredo-n-2018

Older ones:
Wave-particle_duality_of_C60_molecules - Arndt-ltn-1999
Optical_Vortex_Knots_-_One_Photon__At_A_Time - Tempone-Wiltshire-Sr-2018
*Baryon_Deceleration_by_Strong_Chromofields_in_Ultrarelativistic_*,
Nuclear_Collisions - Mishustin-PhysRevC-2007, *APS Link & Abstract*
Hadronic Molecules - Guo-x-2017
Hidden-charm_pentaquark_and_tetraquark_states - Chen-pr-2016

Running Reference Link Listing

Lectures #6 through #1

In reverse order

RelaWavity Web Simulation: Contact Ellipsometry
BoxIt Web Simulation: Elliptical Motion (A-Type)
CMwBang Course: Site Title Page
Pirelli Relativity Challenge: Describing Wave Motion With Complex Phasors
UAF Physics UTube channel

Velocity Amplification in Collision Experiments Involving Superballs - Harter, 1971
MIT OpenCourseWare: High School/Physics/Impulse and Momentum
Hubble Site: Supernova - SN 1987A

BounceIt Web Animation - Scenarios:
Generic Scenario: 2-Balls dropped no Gravity (7:1) - V vs V Plot (Power=4)
1-Ball dropped w/Gravity=0.5 w/Potential Plot: Power=1, Power=4
7:1 - V vs V Plot: Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=4
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1
3-Ball Stack (10:3:1) w/Newton plot (y vs t) - Power=1 w/Gaps
4-Ball Stack (27:9:3:1) w/Newton plot (y vs t) - Power=4
4-Newton's Balls (1:1:1:1) w/Newtonian plot (y vs t) - Power=4 w/Gaps
6-Ball Totally Inelastic (1:1:1:1:1:1) w/Gaps: Newtonian plot (t vs x), V6 vs V5 plot
5-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Minkowski plot (t vs x1) w/Gaps
1-Ball Totally Inelastic Pile-up w/ 5-Stationary-Balls - Vx2 vs Vx1 plot w/Gaps

BounceItIt Web Animation - Scenarios:
49:1 y vs t, 49:1 V2 vs V1, 1:500:1 - 1D Gas Model w/ faux restorative force (Cool),
1:500:1 - 1D Gas (Warm), 1:500:1 - 1D Gas Model (Cool, Zoomed in),
Farey Sequence - Wolfram
Fractions - Ford-AMM-1938
Monstermash BounceItIt Animations:
1000:1 - V2 vs V1, 1000:1 with t vs x - Minkowski Plot
Quantum Revivals of Morse Oscillators and Farey-Ford Geometry - Li-Harter-2013
Quantum_Revivals_of_Morse_Oscillators_and_Farey-Ford_Geometry - Li-Harter-cpl-2015
Quant. Revivals of Morse Oscillators and Farey-Ford Geom. - Harter-Li-CPL-2015 (Publ.)
Velocity_Amplification_in_Collision_Experiments_Involving_Superballs-Harter-1971
WaveIt Web Animation - Scenarios:
Quantum_Carpet, Quantum_Carpet_wMBars,
Quantum_Carpet_BCar, Quantum_Carpet_BCar_wMBars
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-JMS-2001
Wave Node Dynamics and Revival Symmetry in Quantum Rotors - Harter-jms-2001 (Publ.)

BounceIt Dual plots
$m_1:m_2 = 3:1$
v_2 vs v_1 and V_2 vs V_1, $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$
y_2 vs y_1 plots: $(v_1, v_2)=(1, 0.1)$, $(v_1, v_2)=(1, 0)$, $(v_1, v_2)=(1, -1)$
Estrangian plot V_2 vs V_1: $(v_1, v_2)=(0, 1)$, $(v_1, v_2)=(1, -1)$

$m_1:m_2 = 4:1$
v2 vs v1, y2 vs y1
$m_1:m_2 = 100:1$, $(v_1, v_2)=(1, 0)$: *V2 vs V1 Estrangian plot, y2 vs y1 plot*

With g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
M_1=49, M_2=1 with Newtonian time plot
M_1=49, M_2=1 with V_2 vs V_1 plot
Example with friction
Low force constant with drag displaying a Pass-thru, Fall-Thru, Bounce-Off
m1:m2= 3:1 and (v1, v2) = (1, 0) Comparison with Estrangian

AJP article on superball dynamics
AAPT Summer Reading List
Scitation.org - AIP publications
HarterSoft Youtube Channel

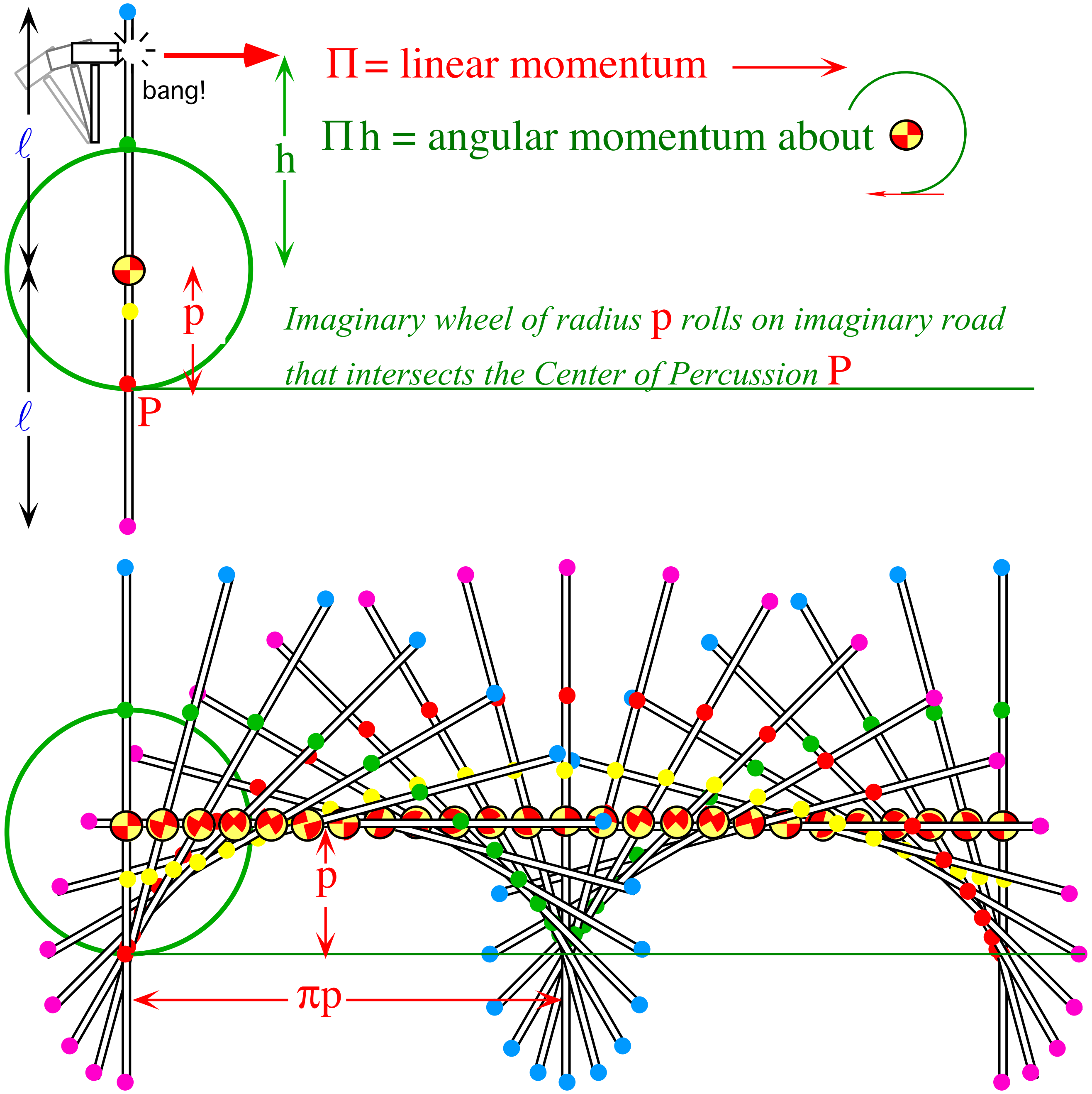
X2 paper: *Velocity Amplification in Collision Experiments Involving Superballs - Harter, et. al. 1971 (pdf)*
Car Collision Web Simulator: https://modphys.hosted.uark.edu/markup/CMMotionWeb.html
Superball Collision Web Simulator: https://modphys.hosted.uark.edu/markup/BounceItWeb.html; with Scenarios: 1007
BounceIt web simulation with g=0 and 70:10 mass ratio
With non zero g, velocity dependent damping and mass ratio of 70:35
Elastic Collision Dual Panel Space vs Space: Space vs Time (Newton) , Time vs. Space(Minkowski)
Inelastic Collision Dual Panel Space vs Space: Space vs Time (Newton), Time vs. Space(Minkowski)
Matrix Collision Simulator: M_1=49, M_2=1 V_2 vs V_1 plot <<Under Construction>>

More Advanced QM and classical references will *soon* be available through our: ***Mechanics References Page***
(Now in Development)

Cycloidal geometry of flying levers

Practical poolhall application

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$



Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

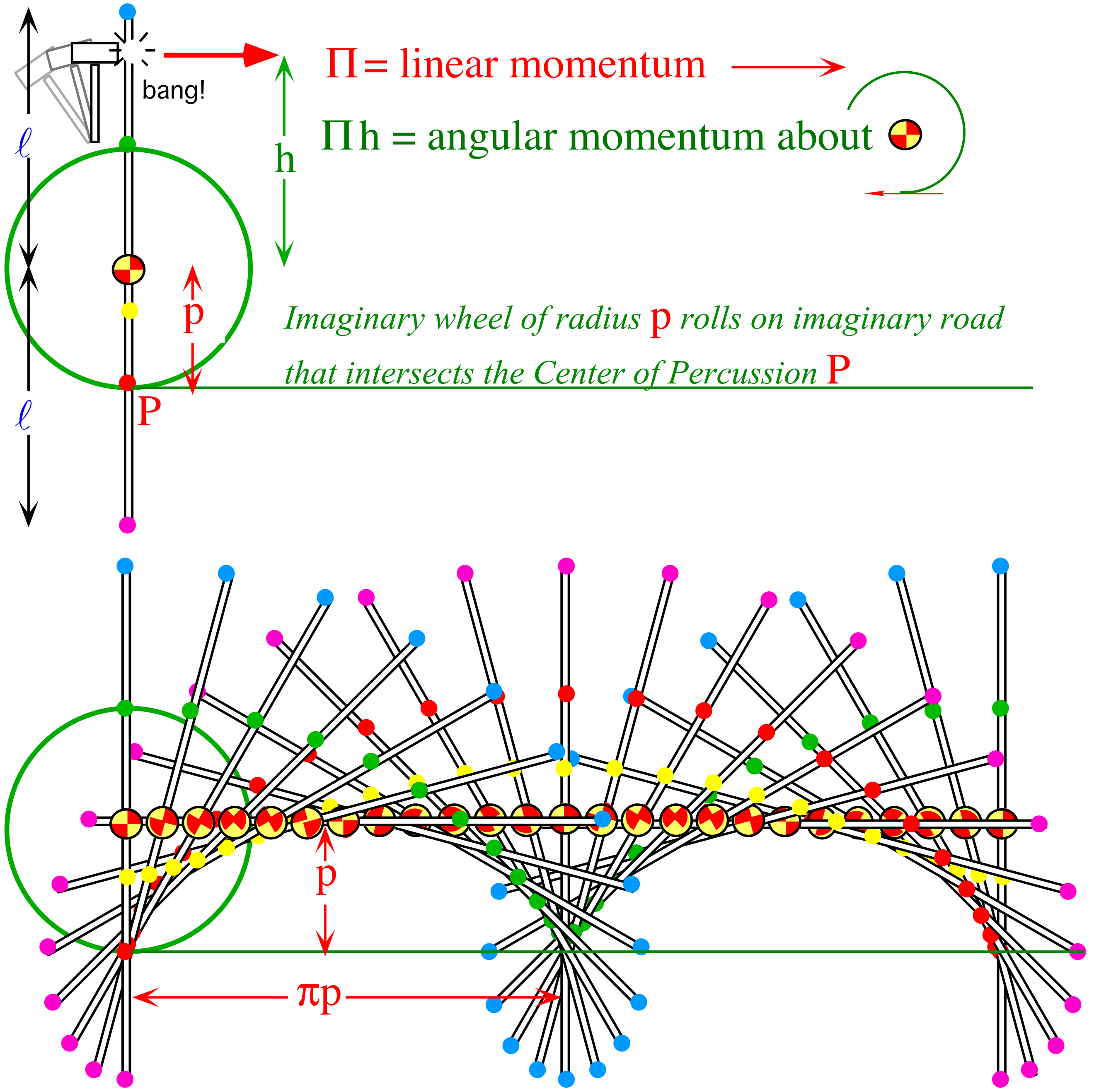
Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$I = \int_0^\ell \rho r^2\,dr = \frac{\rho r^3}{3}\Big|_0^\ell = \frac{\rho \ell^3}{3} = M\frac{\ell^2}{3}$$

$M = \rho\ell$ (Mass is linear density times length)



Imaginary wheel of radius p rolls on imaginary road that intersects the Center of Percussion P

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

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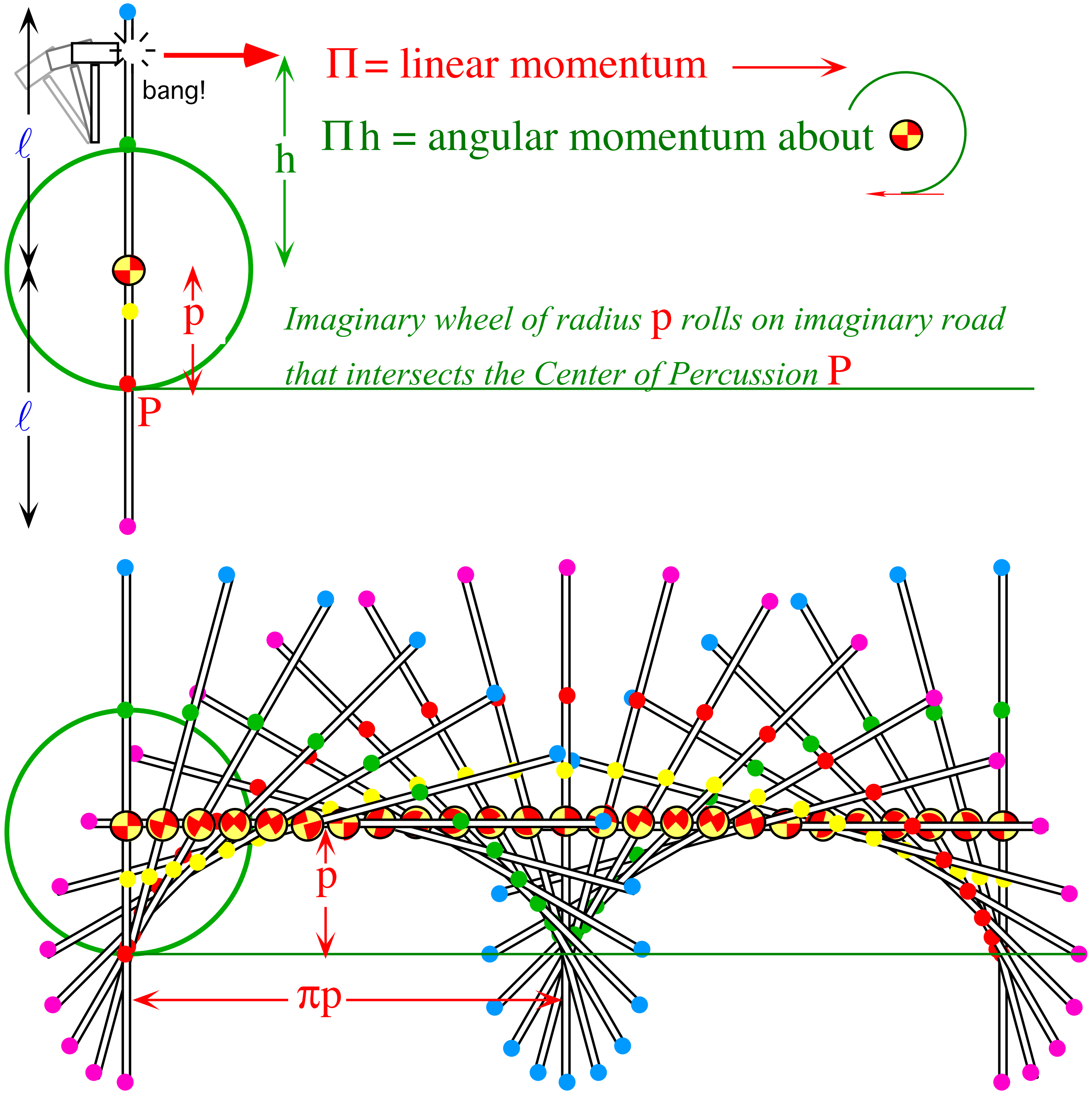
$$\omega = \Lambda / I \quad (=3\Lambda /(M \ell^2) \text{ for stick})$$
$$= h\Pi / I \quad (=3h\Pi/(M \ell^2) \text{for stick})$$



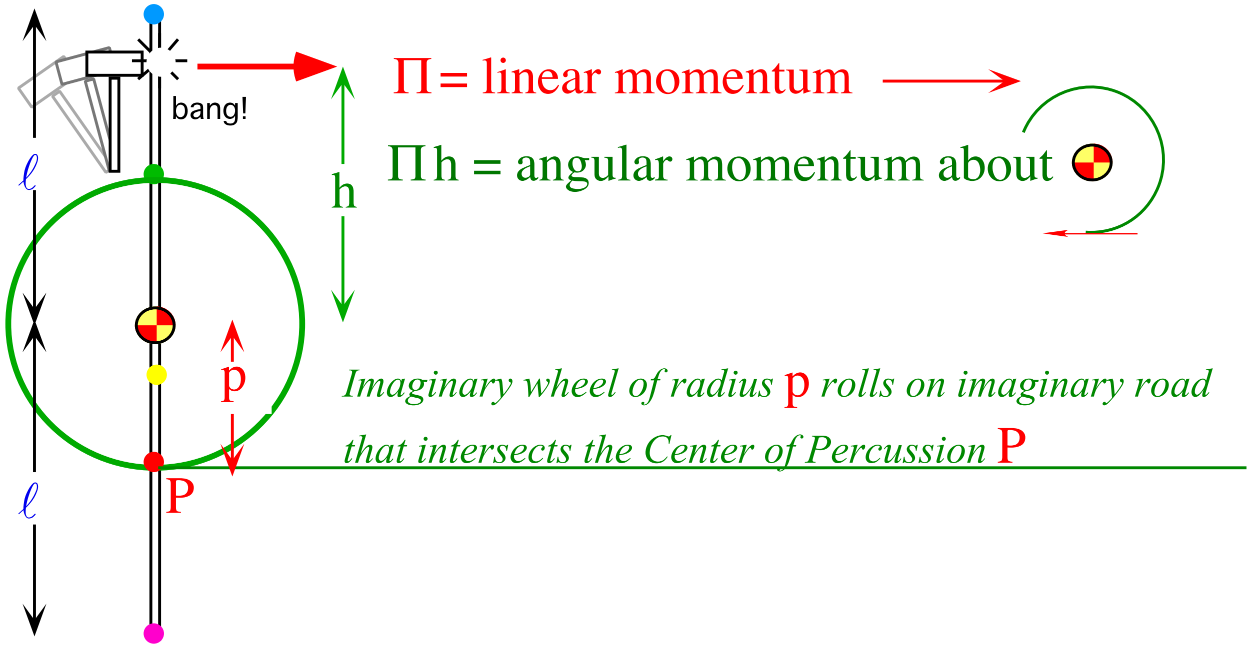
Π = linear momentum

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center you give it some linear momentum Π and some angular momentum $\Lambda = h\cdot\Pi$



Π = linear momentum

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Resulting angular velocity ω about the center is angular momentum Λ divided by moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda /(M\,\ell^2)\ \text{for stick})$$

$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

One point P, or *center of percussion* (CoP), is on the wheel where speed $p\omega$ due to rotation just cancels translational speed V_{Center} of stick.

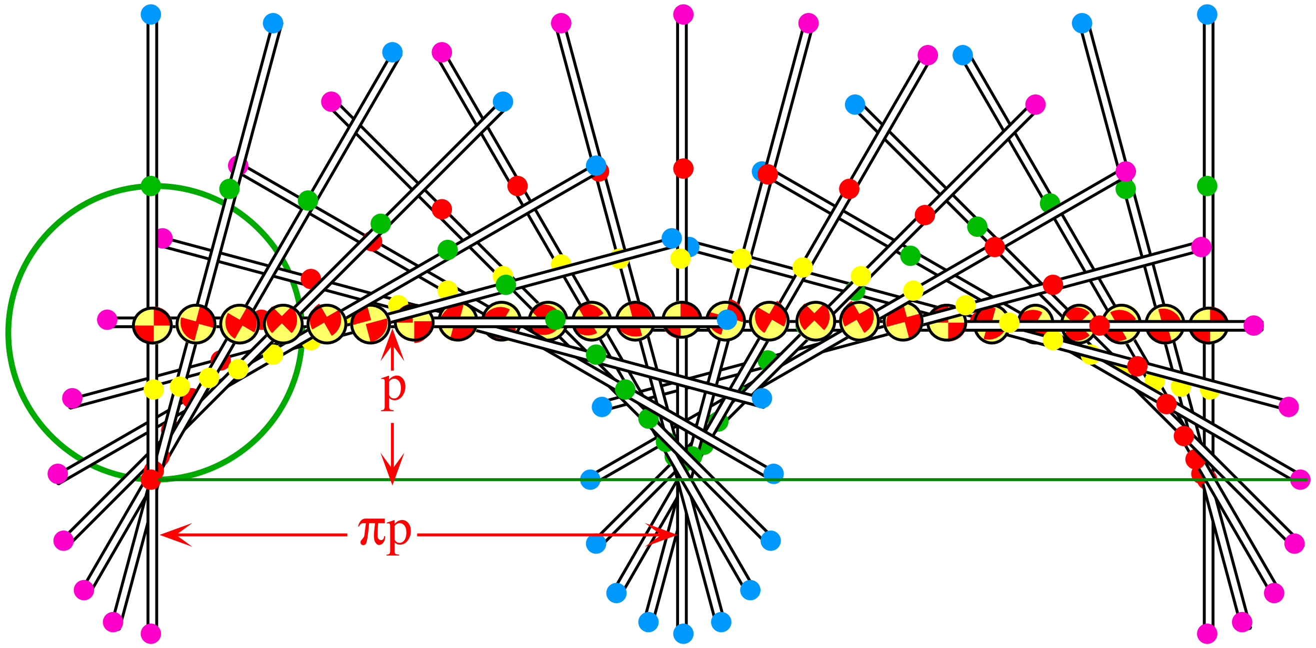


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$

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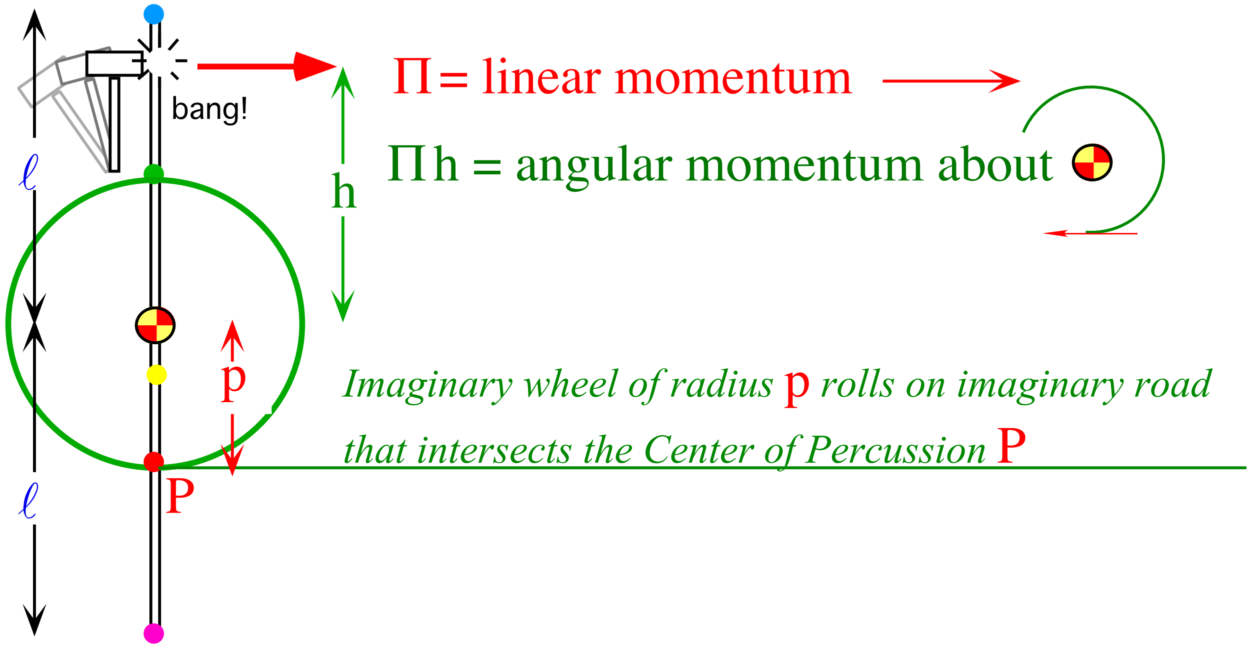

Π = linear momentum

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

One point P, or *center of percussion* (CoP), is
on the wheel where speed $p\omega$ due to rotation
just cancels translational speed V_{Center} of stick.

$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

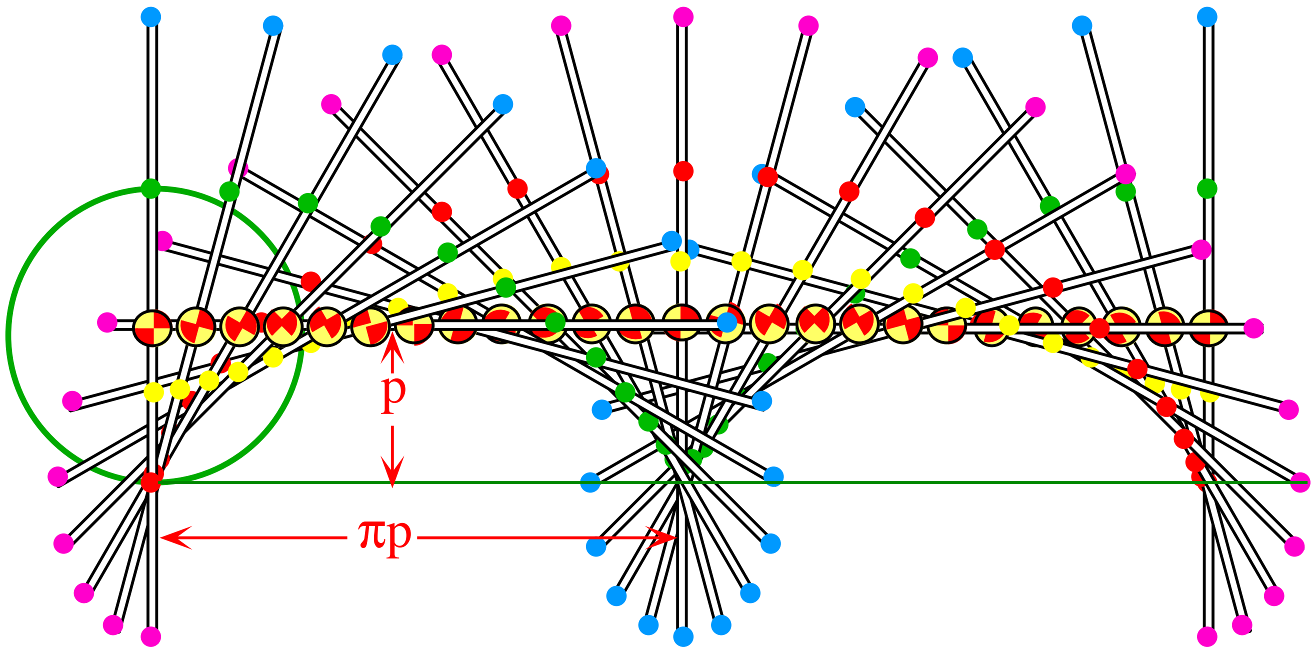

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
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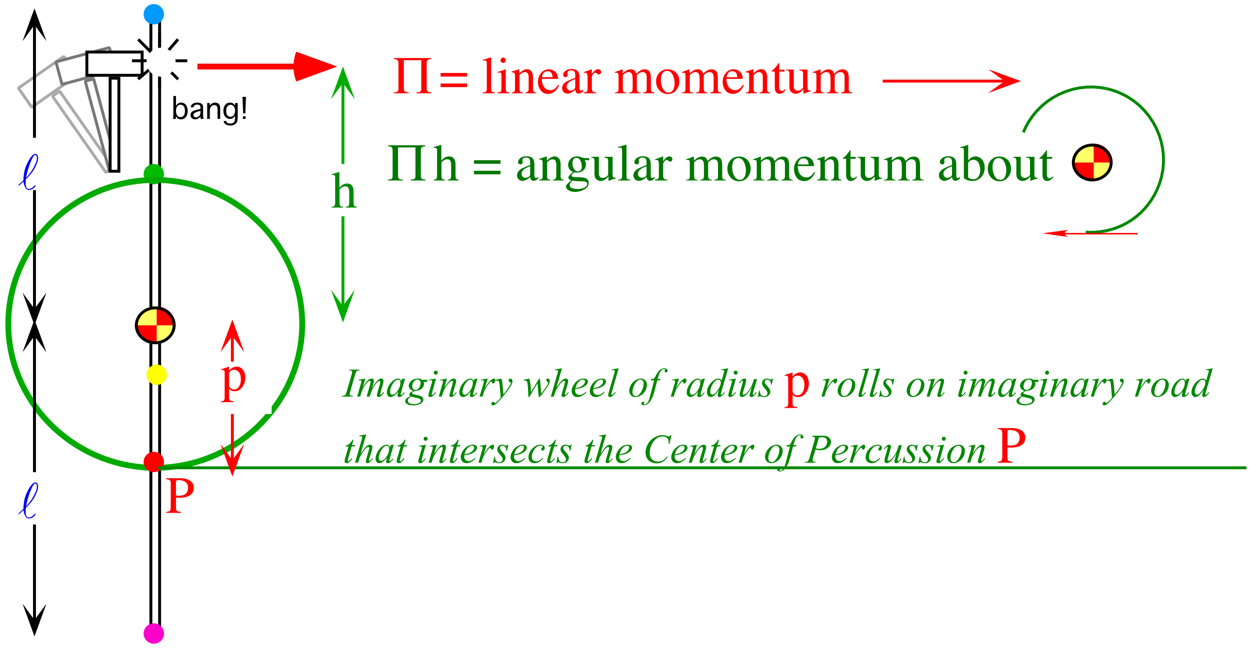
bang!

Π = linear momentum

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda /(M\,\ell^2)\ \text{for stick})$$

$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

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$$\Pi /M = V_{Center} = |p\omega| = p \cdot h\Pi/I$$

$$I /M = \qquad = \qquad = p \cdot h$$

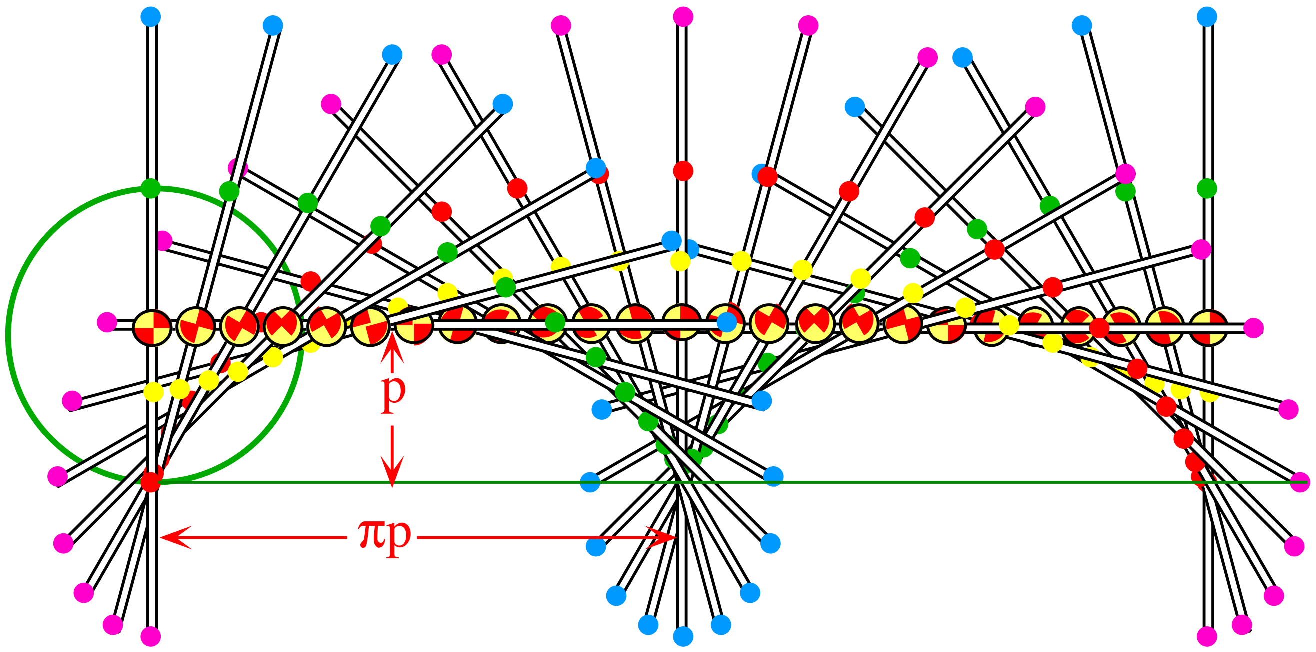
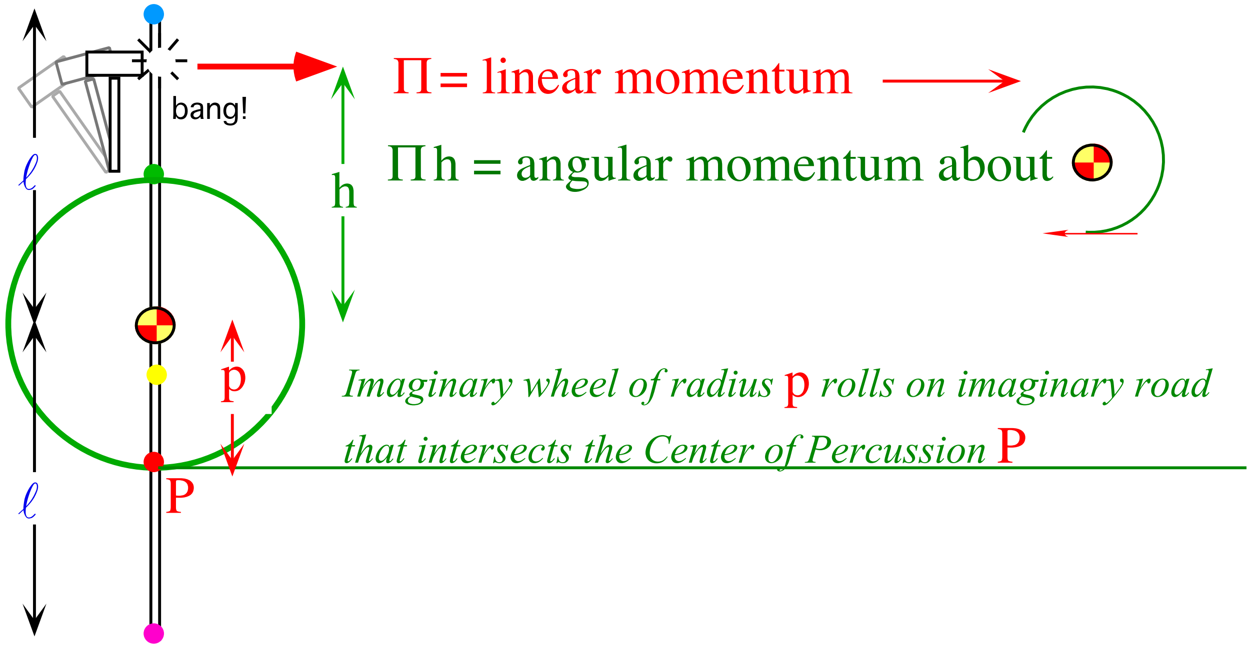


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h\cdot\Pi$



Π = linear momentum →

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Resulting angular velocity ω about the center is angular momentum Λ divided by moment of inertia $I = M\,\ell^2/3$ of the stick.

$$\omega = \Lambda / I \qquad (=3\Lambda/(M\,\ell^2)\ \text{ for stick})$$

$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

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$$\Pi/M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

$$I/M = \qquad = \qquad = p\cdot h \qquad or:\ p=I/(Mh)$$

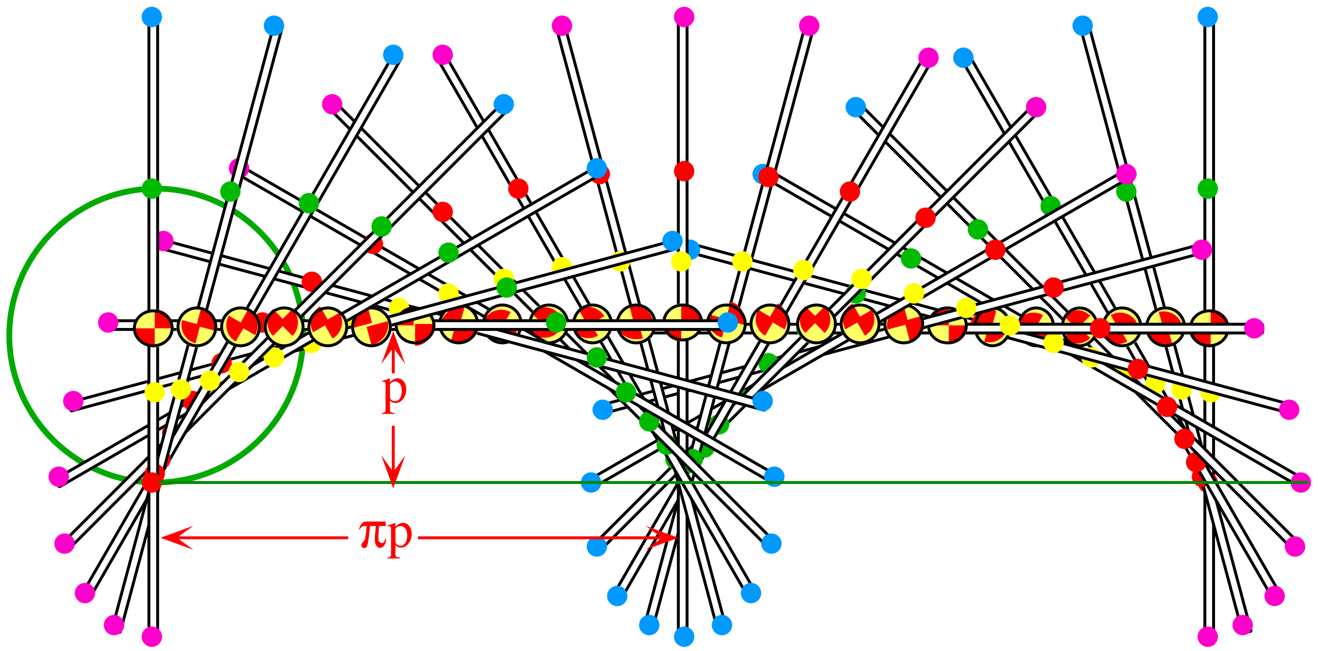

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
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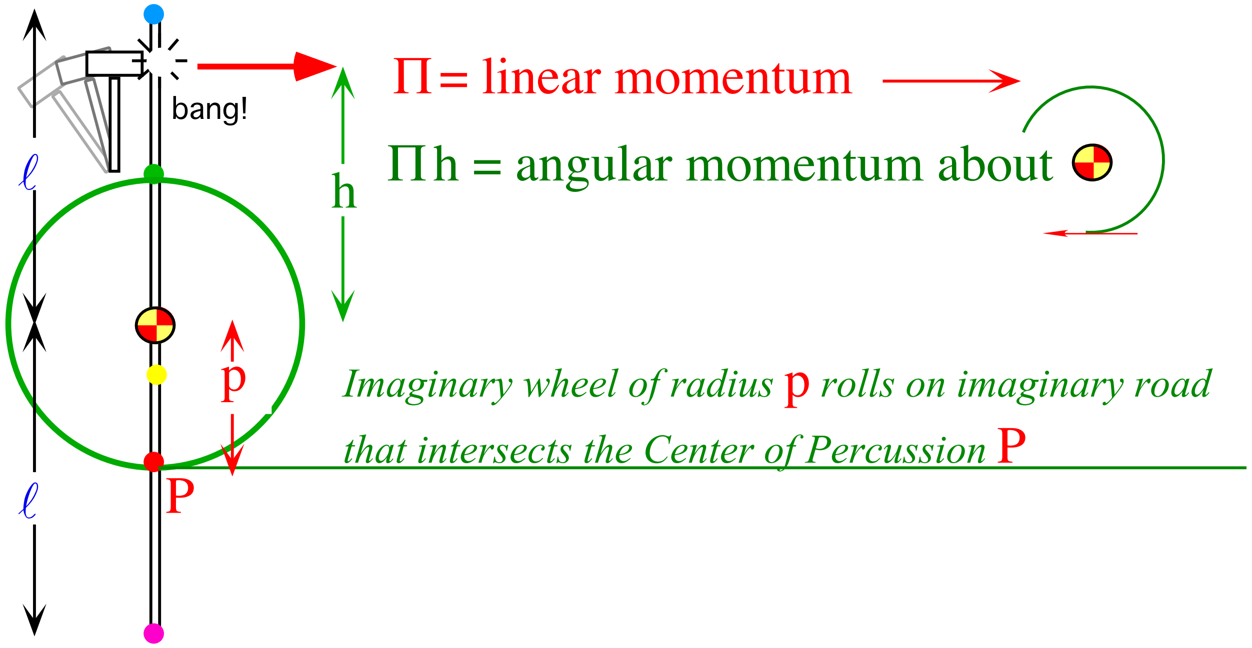

Π = linear momentum

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Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

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$$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)\text{for stick})$$

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$$\Pi / M = V_{Center} = |p\omega| = p\cdot h\Pi/I$$

$$I/M = \qquad = \qquad = p\cdot h \qquad \text{or: } p=I/(Mh)$$

P follows a normal cycloid made by a circle
of radius $p=I/(Mh)$ rolling on an imaginary road
thru point P in direction of Π.

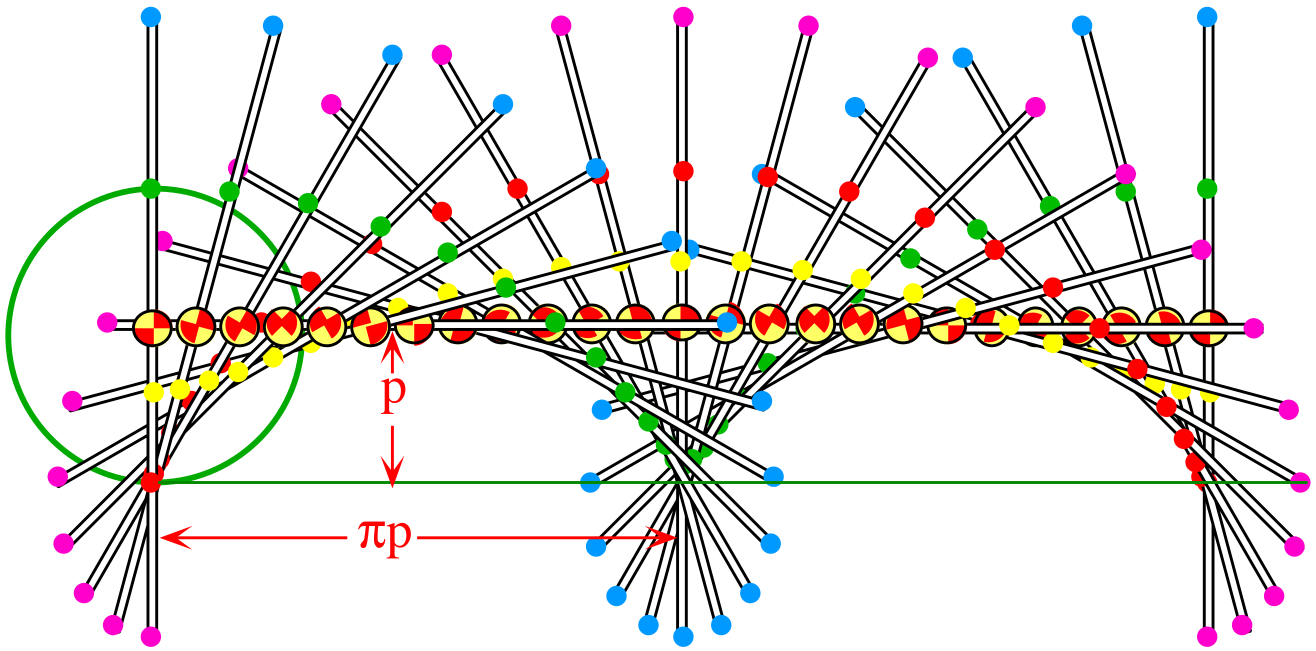
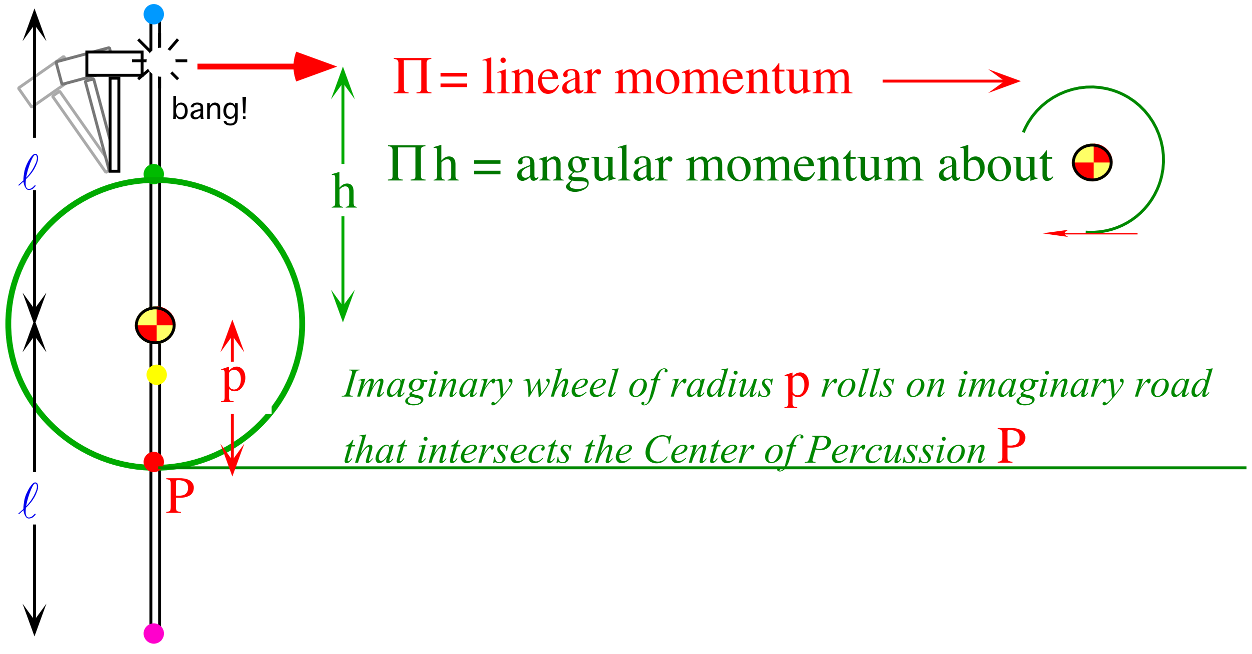


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

If you hammer a stick at a point h meters from its center
you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$



Π = linear momentum →

Πh = angular momentum about

Imaginary wheel of radius p *rolls on imaginary road that intersects the Center of Percussion* P

Resulting angular velocity ω about the center
is angular momentum Λ divided by
moment of inertia $I = M\,\ell^2/3$ of the stick.

$\omega = \Lambda / I \quad (=3\Lambda /(M\,\ell^2)$ for stick)

$= h\Pi / I \quad (=3h\Pi/(M\,\ell^2)$for stick)

One point P, or *center of percussion* (CoP), is
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$\Pi /M = V_{Center} = |p\omega| = p \cdot h\Pi/I$

$I /M = \quad = \quad = p \cdot h \qquad$ *or:* $p=I/(Mh)$

P follows a normal cycloid made by a circle
of radius $p=I/(Mh)$ rolling on an imaginary road
thru point P in direction of Π.

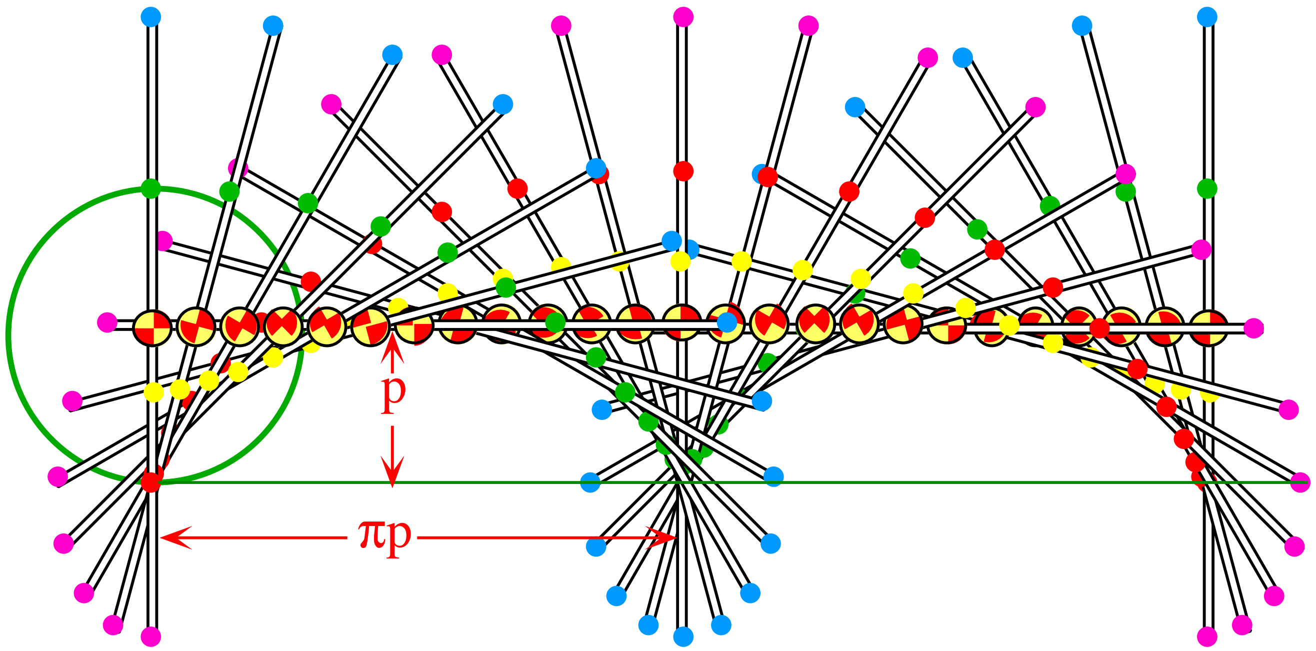


Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

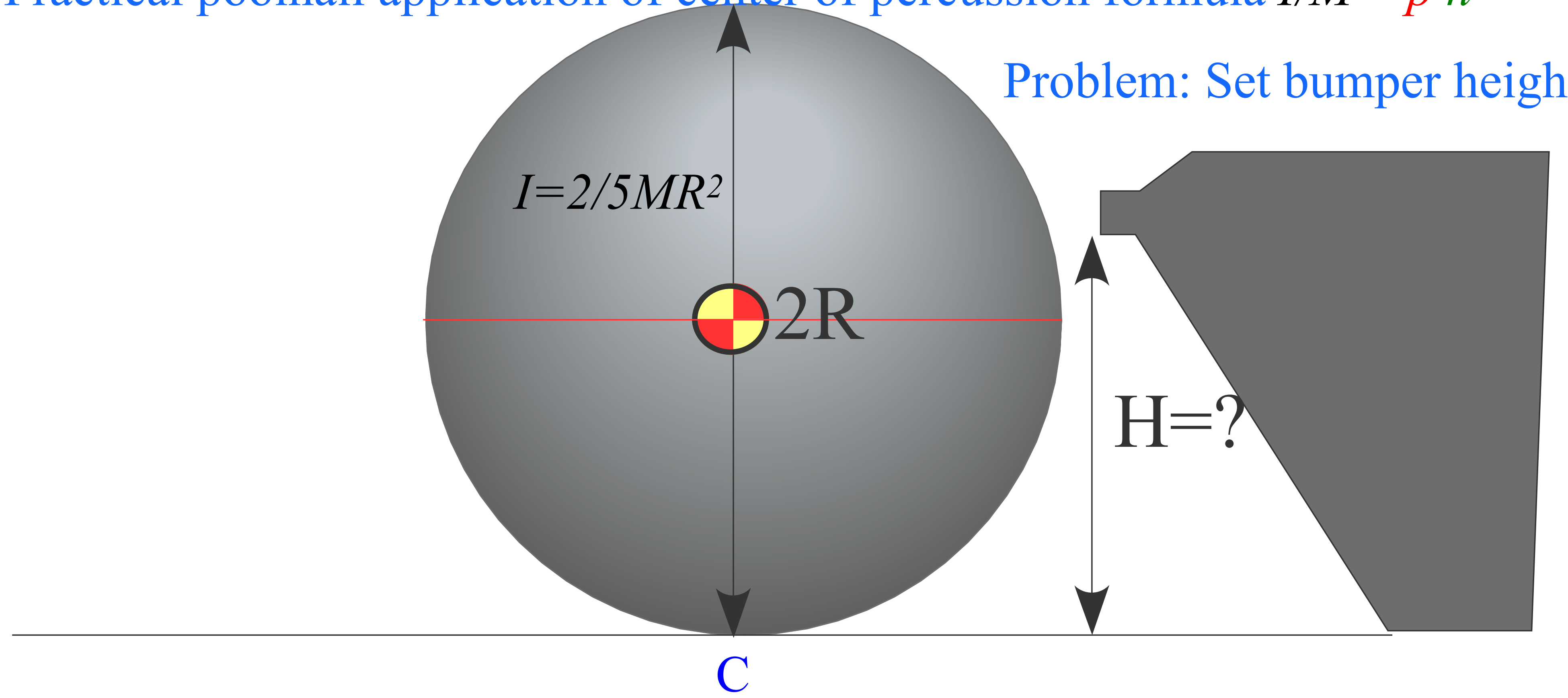
The *percussion radius* $p = \ell^2/3h$ is of the CoP point
that has no velocity just after hammer hits at h.

Cycloidal geometry of flying levers

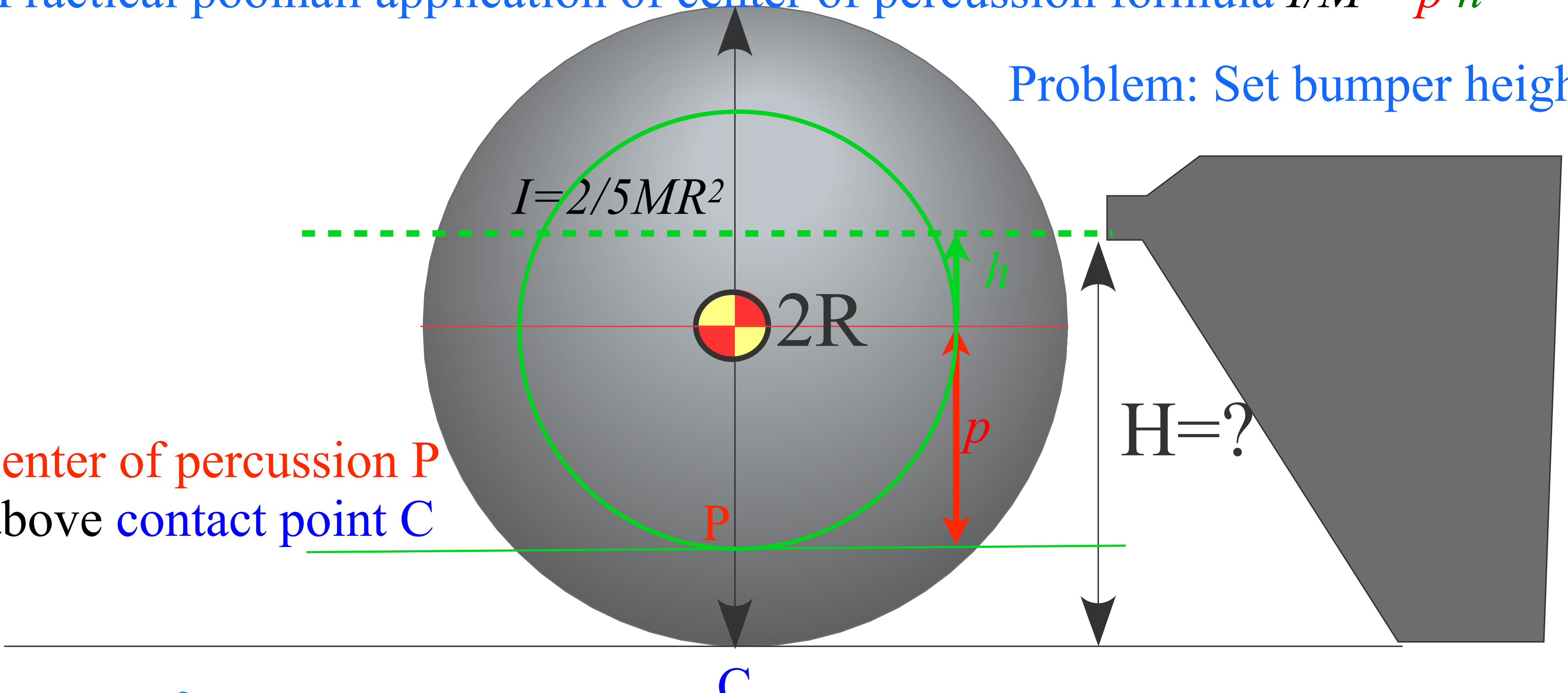
Practical poolhall application

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



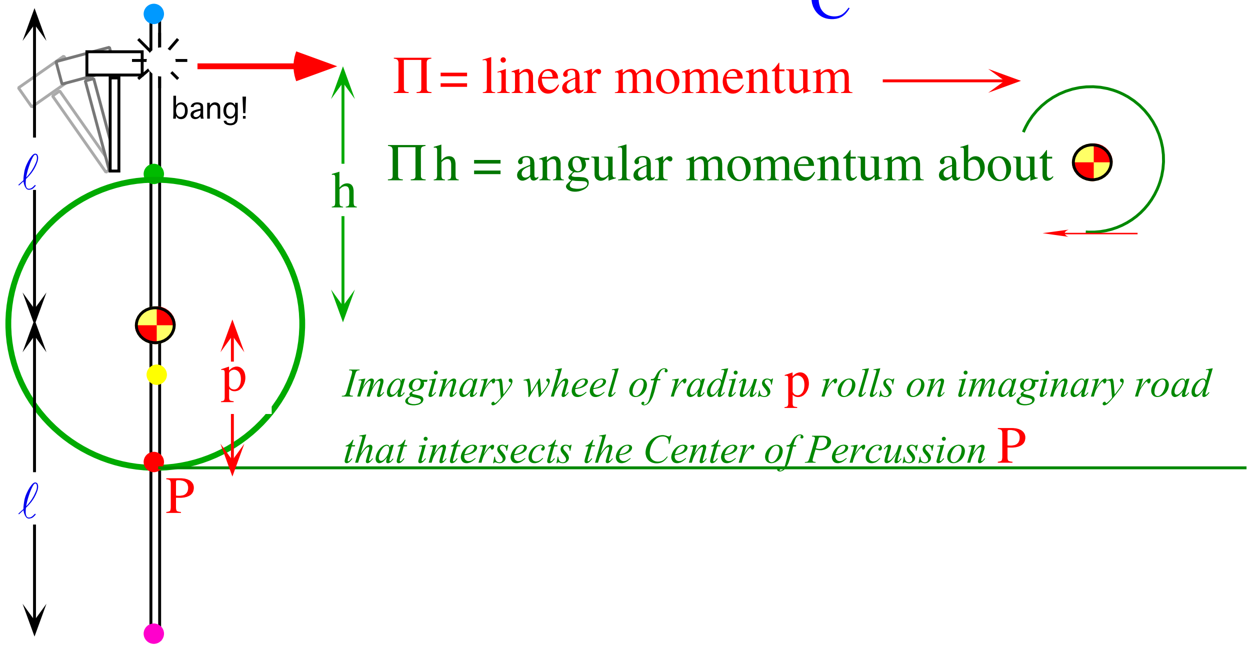
Practical poolhall application of center of percussion formula $I/M = p\cdot h$



Problem: Set bumper height H so ball does not skid.

Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
above contact point C



Π = linear momentum

Πh = angular momentum about ⊕

Imaginary wheel of radius p *rolls on imaginary road*
that intersects the Center of Percussion P

$I/M = p\cdot h$

Practical poolhall application of center of percussion formula $I/M = p\cdot h$

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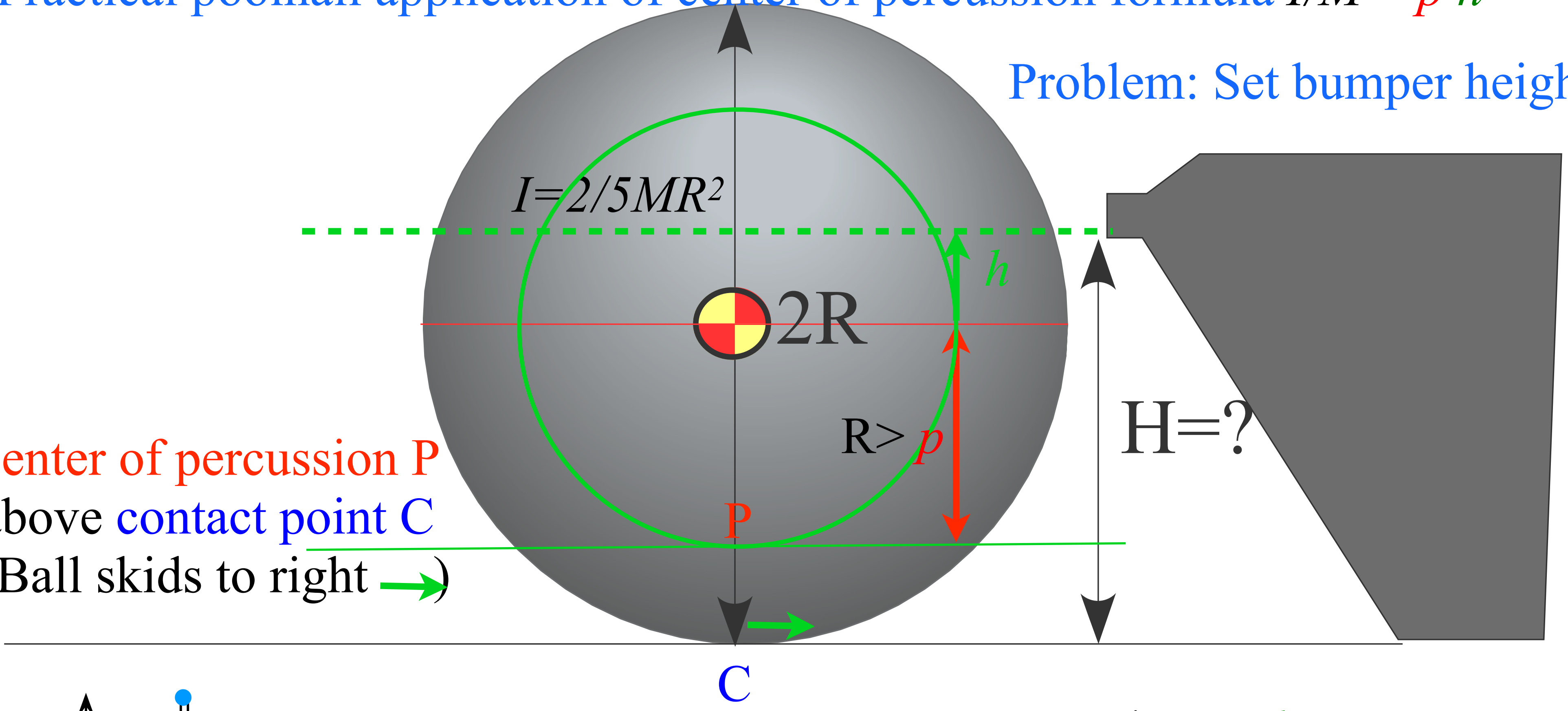
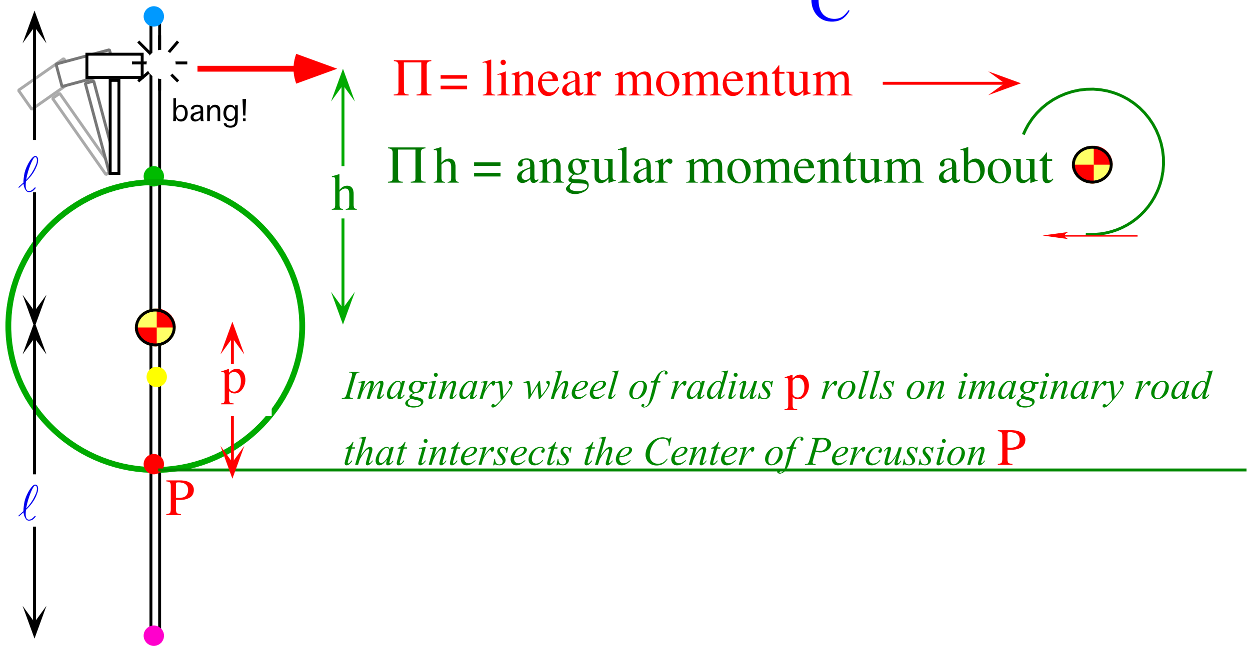


center of percussion P above contact point C (Ball skids to right →)

$I/M = p\cdot h$



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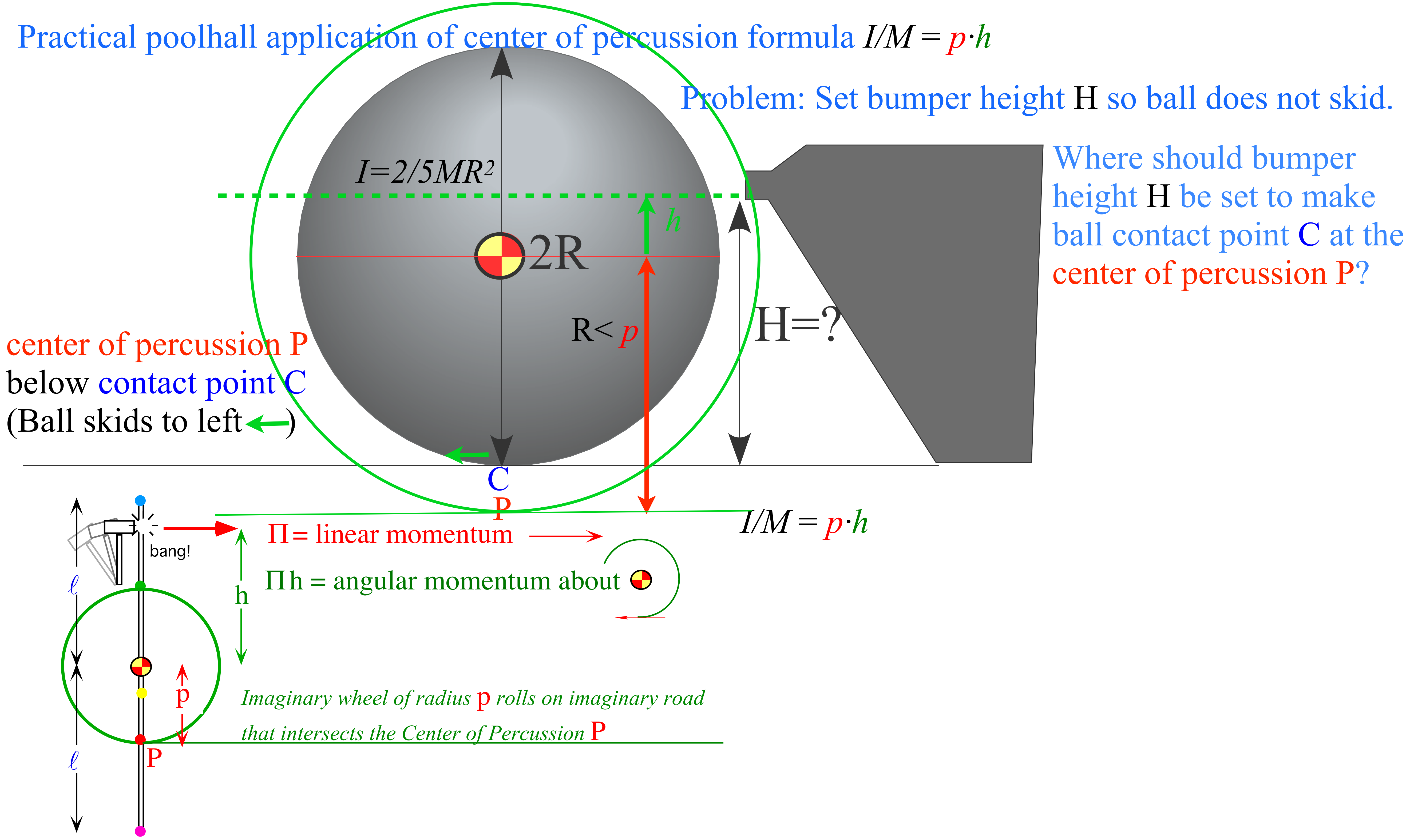

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Where should bumper height H be set to make ball contact point C at the center of percussion P?

center of percussion P
below contact point C
(Ball skids to left ←)

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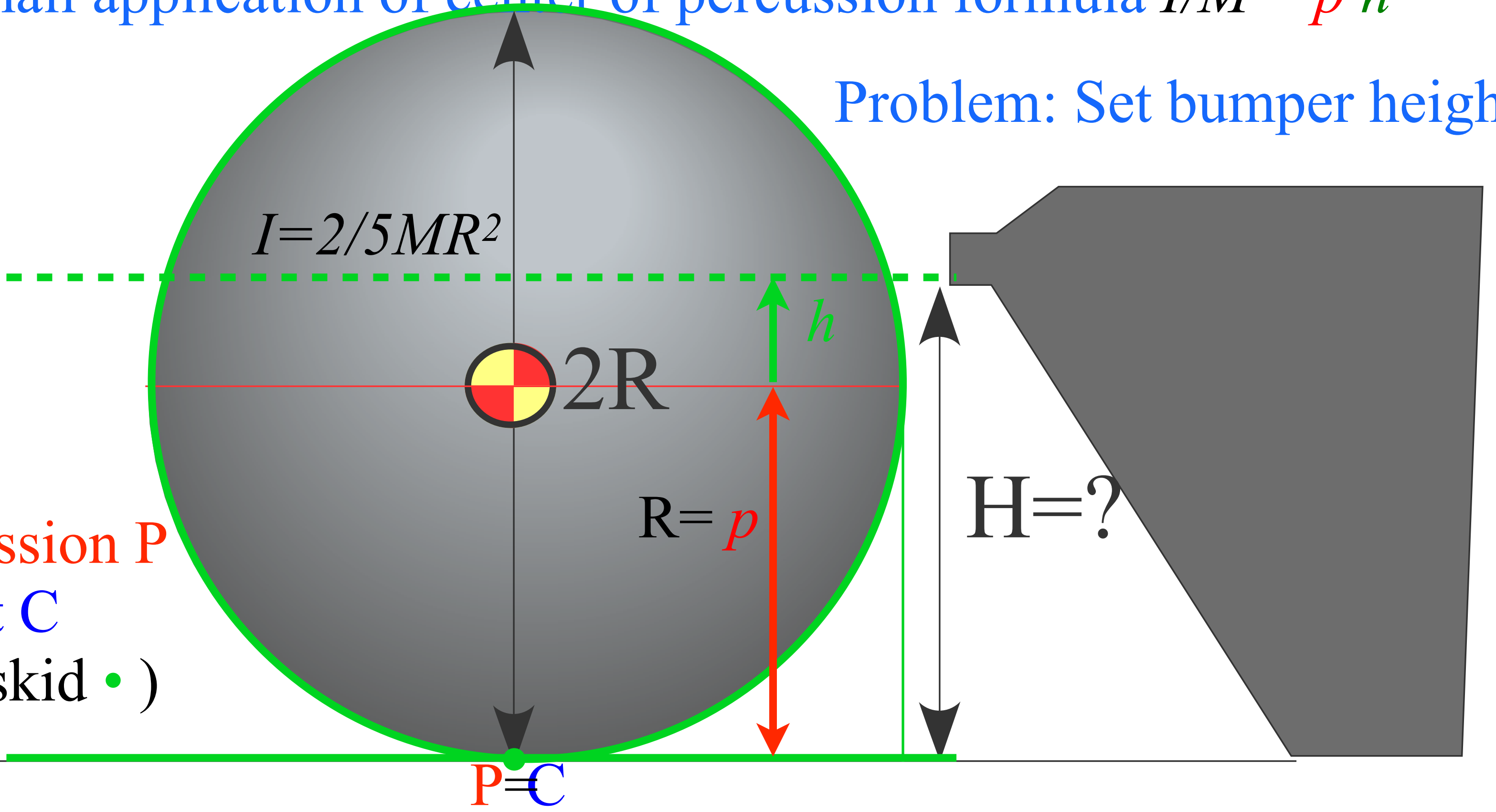

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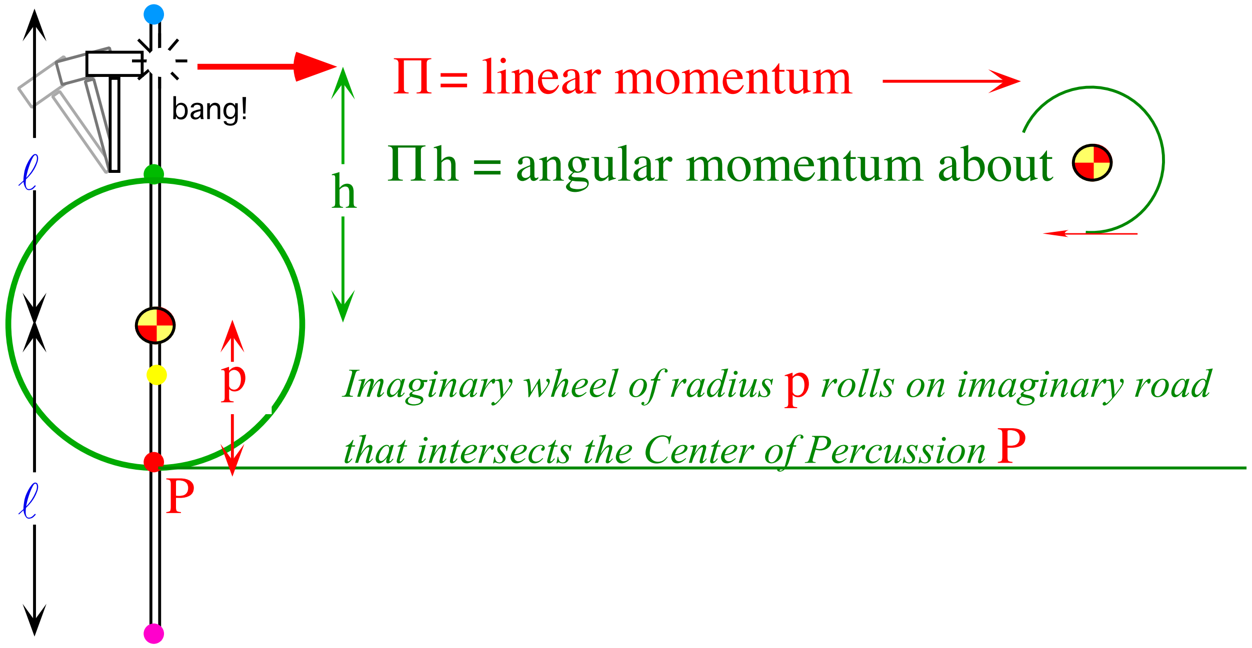
center of percussion P
at contact point C
(Ball does not skid •)



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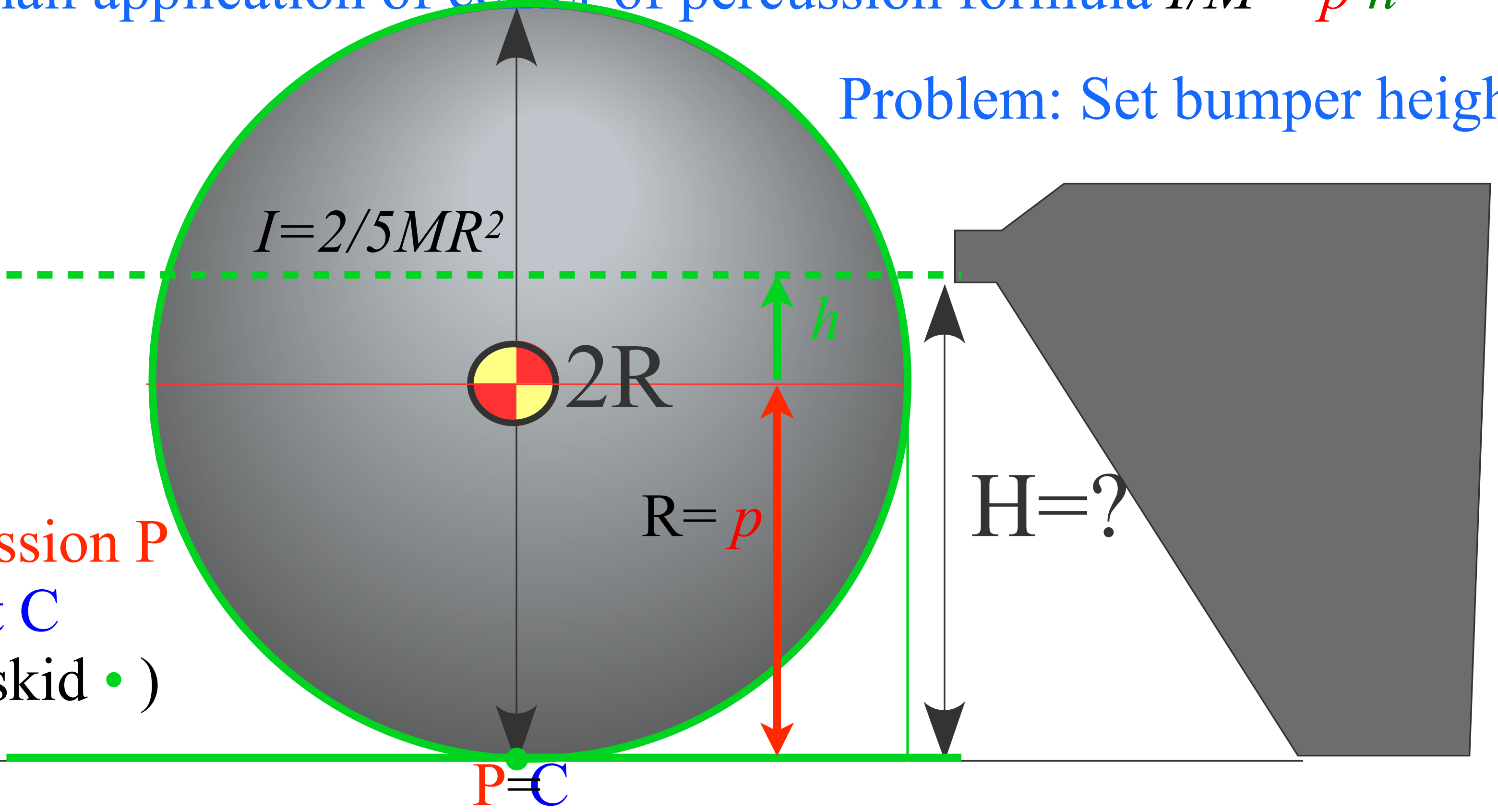

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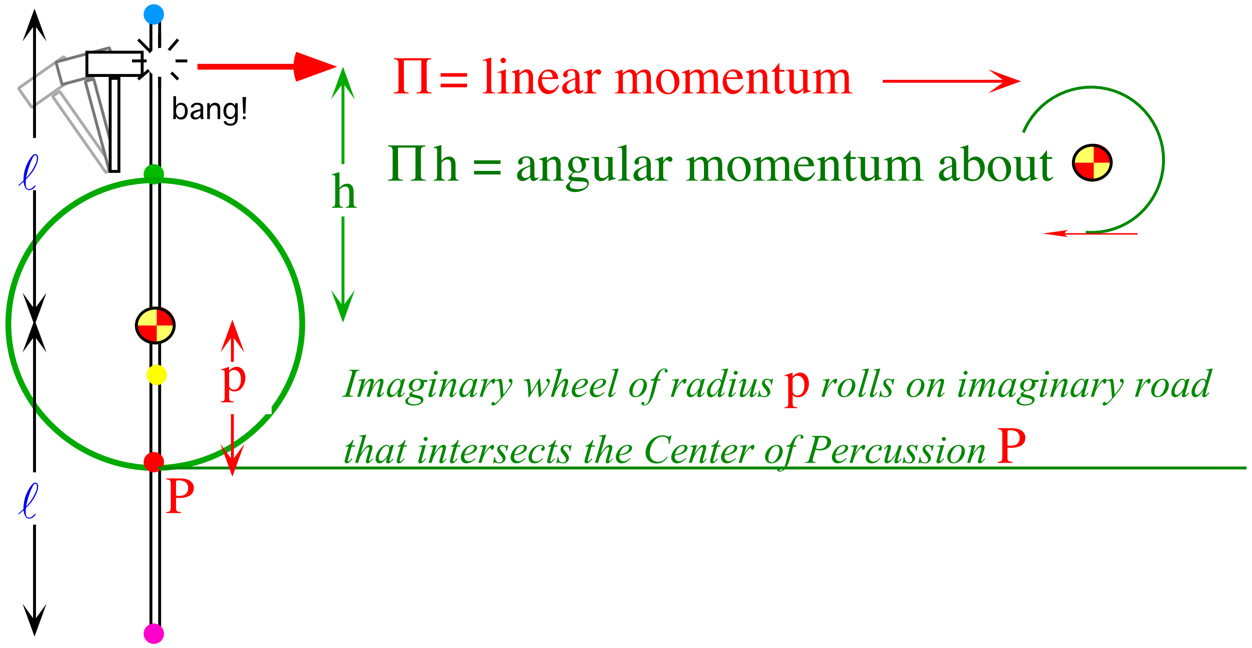

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$$= 2/5MR^2/MR$$

$$= 2/5R$$

For: H= $R+h =7/10(2R)$ ball does not skid.

(70% of ball diameter)

Charge mechanics in electromagnetic fields

- → *Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential*
- *Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential*
- *Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential*
    - *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
    - *Hamiltonian formulation*
    - *Hamilton's equations*

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Electric field **E** and magnetic field **B**

scalar potential field $\Phi=\Phi(\mathbf{r},t)$

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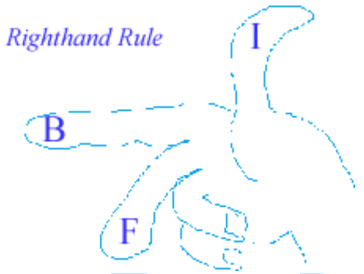

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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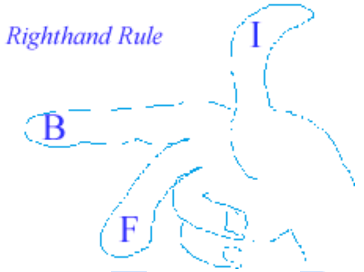

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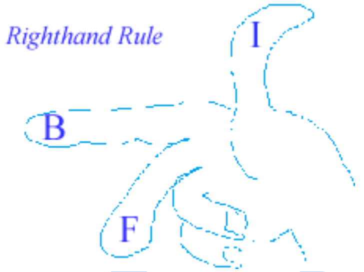

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Doing a double-cross

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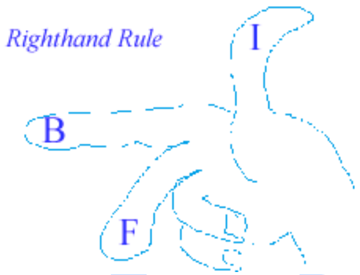

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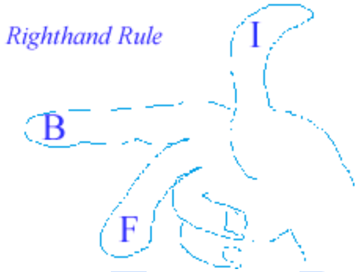

F=*q*v×B=I×B

Doing a double-cross

ε_{ijk}-Tensor analysis of **v**×(**∇**×**A**)

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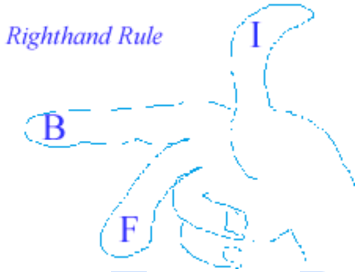

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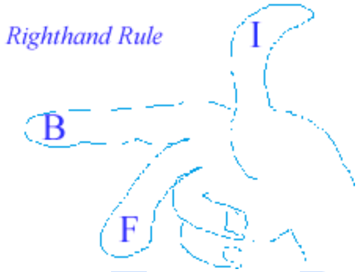

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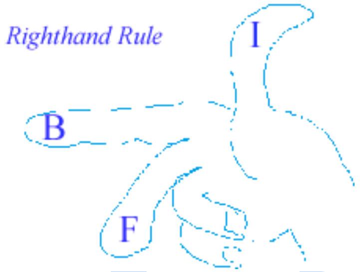

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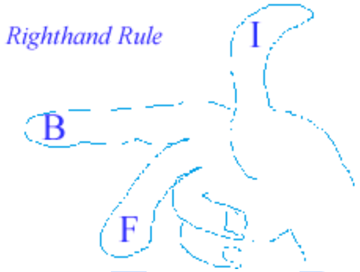

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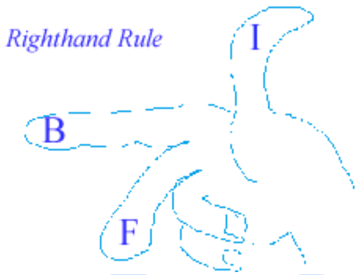

F=*q***v**×**B**=**I**×**B**

Doing a double-cross

ε_{ijk}-Tensor analysis of **v**×(**∇**×**A**)

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ for } \textit{even} \text{ permutaion of } i<j<k \\ 0 \quad \text{if any of } i,j,k \text{ are equal} \\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k \end{cases}$$

$$\varepsilon_{ijk} = \varepsilon_{ikj} = \varepsilon_{kij} = 1$$
$$= -\varepsilon_{jik} = -\varepsilon_{jki} = -\varepsilon_{kji}$$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k = \varepsilon_{kij} v_i \left(\nabla\times\mathbf{A}\right)_j$$
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$$= \varepsilon_{kij}\varepsilon_{abj} \quad v_i\left(\partial_a A_b\right)$$
$$= (\delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}) v_i\left(\partial_a A_b\right)$$
$$= \delta_{ka}\delta_{ib} v_i\left(\partial_a A_b\right) - \delta_{kb}\delta_{ia} v_i\left(\partial_a A_b\right)$$
$$= \quad v_b\left(\partial_k A_b\right) \quad - v_a\left(\partial_a A_k\right)$$
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Converting back to Gibbs’s **bold** notation involves *tensors* like ∇**A** and ∇**v**.

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
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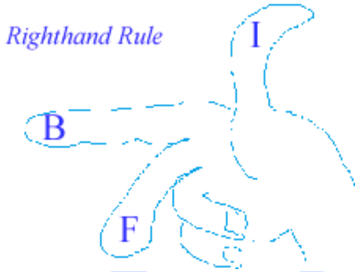

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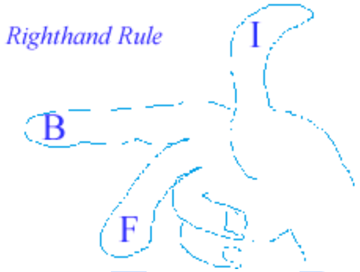

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Summary of Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

Tensor index notation helps to distinguish $(\nabla\mathbf{A})\bullet\mathbf{v}$, $\mathbf{v}\bullet(\nabla\mathbf{A})$, and $\nabla(\mathbf{A}\bullet\mathbf{v}) = (\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}$.

$$\left[(\nabla\mathbf{A})\bullet\mathbf{v}\right]_k = \frac{\partial A_j}{\partial x_k} v_j$$

$$= (\partial_k A_j) v_j$$

$$\left[\mathbf{v}\bullet(\nabla\mathbf{A})\right]_k = v_j \frac{\partial A_k}{\partial x_j}$$

$$= (v_j \partial_j A_k)$$

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Charge mechanics in electromagnetic fields

- *Vector analysis for particle-in-*$(\mathbf{A},\Phi)$*-potential*
- → *Lagrangian for particle-in-*$(\mathbf{A},\Phi)$*-potential*
- *Hamiltonian for particle-in-*$(\mathbf{A},\Phi)$*-potential*
    - *Canonical momentum in* $(\mathbf{A},\Phi)$ *potential*
    - *Hamiltonian formulation*
    - *Hamilton's equations*

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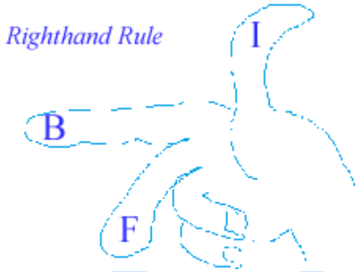

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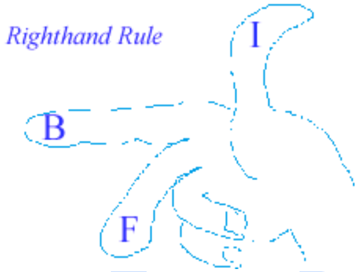

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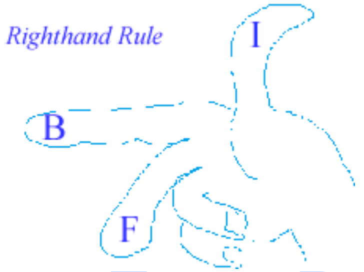

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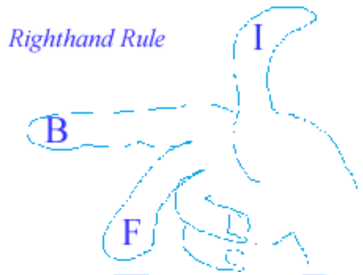

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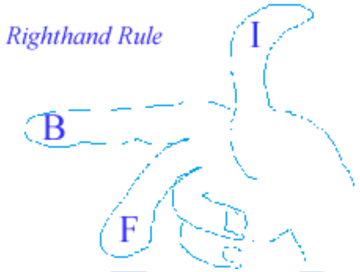

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Begin
Lagrange
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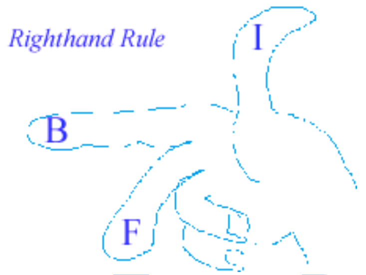

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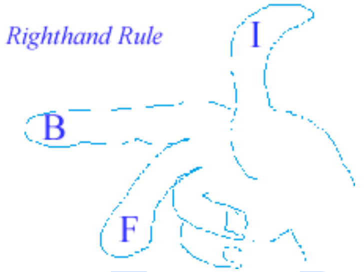

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$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \boxed{\frac{\partial \mathbf{A}}{\partial t}} + \nabla(\mathbf{v}\bullet\mathbf{A}) - \boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}\right]$$

Chain rule expansion of vector potential total *t*-derivative:

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial x}\dot{x} + \frac{\partial\mathbf{A}}{\partial y}\dot{y} + \frac{\partial\mathbf{A}}{\partial z}\dot{z} + \frac{\partial\mathbf{A}}{\partial t} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\bullet\nabla)\mathbf{A}$$

so: $$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\bullet\nabla)\mathbf{A}$$

"streaming" derivative

Begin
Lagrange
trickery:

$$m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$m\frac{d\mathbf{v}}{dt} = e\left[-\nabla\Phi + \nabla(\mathbf{v}\bullet\mathbf{A}) - \frac{\partial \mathbf{A}}{\partial t} - (\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

$$= e\left[-\nabla(\Phi - \mathbf{v}\bullet\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right]$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} = \frac{d}{dt}\left(-e\mathbf{A}\right) \qquad -\nabla(e\Phi - \mathbf{v}\bullet e\mathbf{A})$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} = \frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) - \nabla(e\Phi - \mathbf{v}\bullet e\mathbf{A})$$

$$\left(\textit{This step requires that}: \frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right) = 0\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) = -e\frac{d\mathbf{A}}{dt}$$

Lagrangian for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10^{-19}Coulombs

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

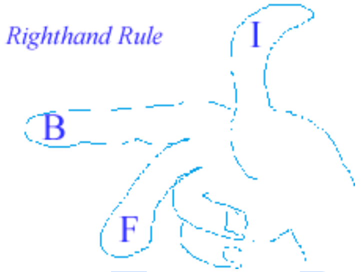
Electric field **E** and magnetic field **B**

scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

vector potential field $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$

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Chain rule expansion of vector potential total *t*-derivative: $\frac{d\mathbf{A}}{dt}=\frac{\partial\mathbf{A}}{\partial x}\dot{x}+\frac{\partial\mathbf{A}}{\partial y}\dot{y}+\frac{\partial\mathbf{A}}{\partial z}\dot{z}+\frac{\partial\mathbf{A}}{\partial t}=\boxed{\frac{\partial\mathbf{A}}{\partial t}+(\mathbf{v}\bullet\nabla)\mathbf{A}}$

Begin Lagrange trickery:

$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

$$=e\left[-\nabla(\Phi-\mathbf{v}\bullet\mathbf{A})\ -\ \frac{d\mathbf{A}}{dt}\right]$$

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"streaming" derivative

$$m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\left(-e\mathbf{A}\right)\qquad-\nabla(e\Phi-\mathbf{v}\bullet e\mathbf{A})$$

This step requires that : $\frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right)=0$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla(e\Phi-\mathbf{v}\bullet e\mathbf{A})$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

This step requires that :

$$\nabla\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)\equiv\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)=\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)$$

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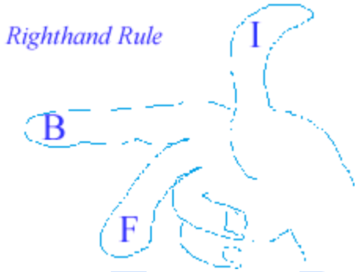

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This step requires that: $\frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right)=0$

$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla(e\Phi-\mathbf{v}\bullet e\mathbf{A})$$

This step requires that:

$\nabla\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)\equiv\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}=\frac{\partial}{\partial\mathbf{r}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}\qquad=\qquad\frac{\partial L}{\partial\mathbf{r}}$$

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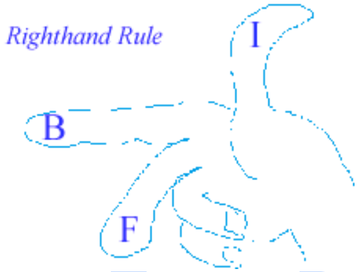

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This step requires that: $\frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right)=0$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla(e\Phi-\mathbf{v}\bullet e\mathbf{A})$$

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This step requires that:

$$\nabla\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)\equiv\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

$$\underbrace{\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}_{\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}}=\underbrace{\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}_{\frac{\partial L}{\partial\mathbf{r}}}$$

Lagrangian has a *linear* velocity term *e***v•A** in addition to the usual quadratic $KE=mv^2/2$ and $PE=e\Phi$.

$$L=L(\mathbf{r},\mathbf{v},t)=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

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Electric field **E** and magnetic field **B**

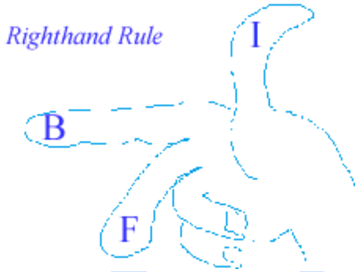
scalar potential field **Φ=Φ(r**,*t)*

vector potential field **A=A(r**,*t)*

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



F=*q*v×B=I×B

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}=e\left[-\nabla\Phi-\boxed{\frac{\partial\mathbf{A}}{\partial t}}+\nabla(\mathbf{v}\bullet\mathbf{A})-\boxed{(\mathbf{v}\bullet\nabla)\mathbf{A}}\right]$$

Chain rule expansion of vector potential total *t*-derivative:

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$$m\frac{d\mathbf{v}}{dt}=e\left[-\nabla\Phi+\nabla(\mathbf{v}\bullet\mathbf{A})-\frac{\partial\mathbf{A}}{\partial t}-(\mathbf{v}\bullet\nabla)\mathbf{A}\right]=e\left[-\nabla(\Phi-\mathbf{v}\bullet\mathbf{A})-\frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}=\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)-\nabla\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)$$

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$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}\qquad=\qquad\frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian has a *linear* velocity term *e***v•A** in addition to the usual quadratic *KE=mv^2/2* and *PE=e*Φ.

$$L=L(\mathbf{r},\mathbf{v},t)=\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

Charge mechanics in electromagnetic fields

- *Vector analysis for particle-in-*$(\boldsymbol{A},\Phi)$*-potential*
- *Lagrangian for particle-in-*$(\boldsymbol{A},\Phi)$*-potential*
- *Hamiltonian for particle-in-*$(\boldsymbol{A},\Phi)$*-potential*
  - → *Canonical momentum in* $(\boldsymbol{A},\Phi)$ *potential*
  - *Hamiltonian formulation*
  - *Hamilton's equations*

*Hamiltonian for particle-in-(**A**,Φ)-potential*

Lagrangian has a *linear* velocity term $e\mathbf{v}\bullet\mathbf{A}$ in addition to the usual quadratic $KE=mv^2/2$ and $PE=e\Phi$.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

*Canonical momentum in (**A**,Φ) potential*

Canonical momentum is defined by L's **v**-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t)$$

Hamiltonian for particle-in-(A,Φ)-potential

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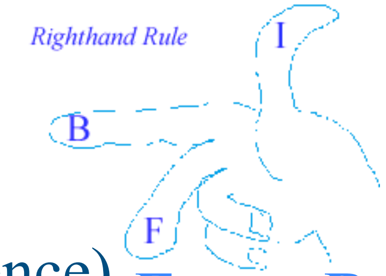

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Canonical momentum in (A,Φ) potential

Canonical momentum is defined by L's $\mathbf{v}$-derivative (…scalar Φ has no $\mathbf{v}$-dependence)

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right) = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - e\Phi(\mathbf{r},t)\right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \qquad = \quad m\mathbf{v} \qquad For\ \mathbf{A}(\mathbf{r},t)=0$$

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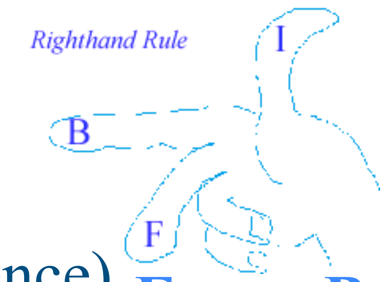

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Lagrangian is usual form $L= T - V$ with electric (scalar) potential $V=\mathbf{\Phi}(\mathbf{r},t)$

if magnetic (vector) potential $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$ is zero everywhere.

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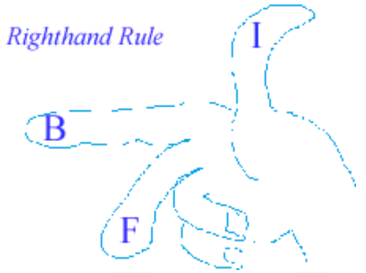

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Then canonical momentum is usual form: $\mathbf{p} = m\mathbf{v}$ $\quad (For\ \mathbf{A}(\mathbf{r},t)=0)$

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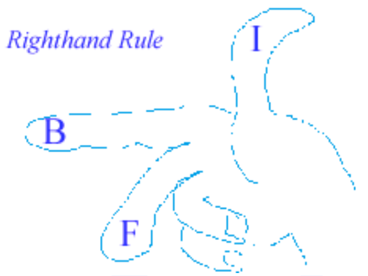

F=*q*v×B=I×B

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Canonical momentum is defined by *L*'s **v**-derivative

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<u>if</u> magnetic (vector) potential $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$ is zero everywhere.

Then canonical momentum is usual form: $\mathbf{p} = m\mathbf{v}$ $\quad (For\ \mathbf{A}(\mathbf{r},t)=0)$

Otherwise vector potential term **-v**•*e***A** leads to an extraordinary *canonical momentum:* $\mathbf{p} = m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)$.

Particle momentum $m\mathbf{v}$ is not canonical, but related to *canonical* **p** as follows: $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

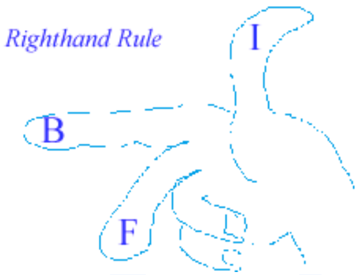
Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- → *Hamiltonian formulation*
- *Hamilton's equations*

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.

$$H = \sum_{\mu} \dot{q}^{\mu} p_{\mu} - L = \mathbf{v} \bullet \mathbf{p} - L = \mathbf{v} \bullet \left(m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \right) - \left(\frac{1}{2} m\mathbf{v} \bullet \mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v} \bullet e\mathbf{A}(\mathbf{r},t) \right) \right)$$



$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = \mathbf{I} \times \mathbf{B}$

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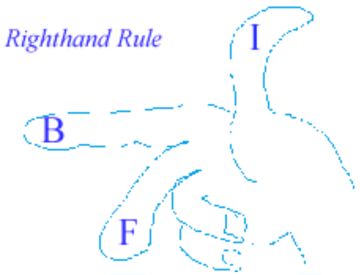

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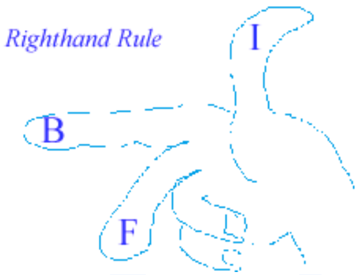

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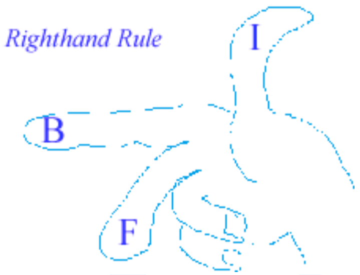

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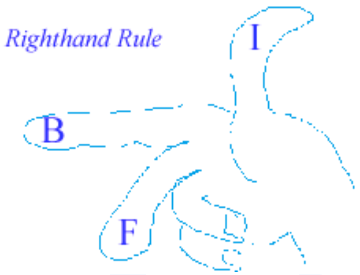

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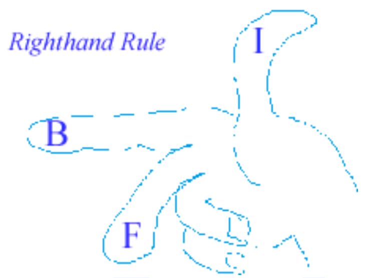

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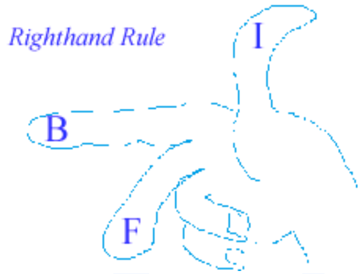

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Charge mechanics in electromagnetic fields

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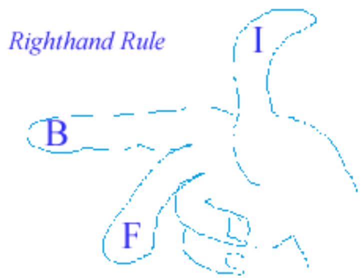

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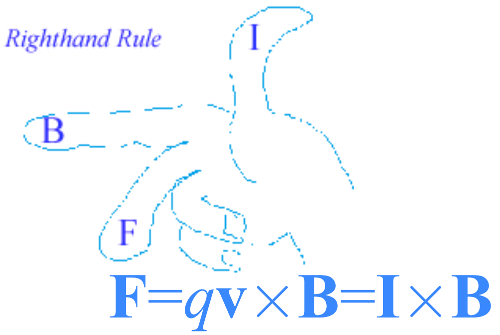

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$$\frac{\partial \mathbf{A}}{\partial t} = \frac{d\mathbf{A}}{dt} - (\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$\dot{p}_a = m\dot{v}_a + e\dot{A}_a = e\left(v_\mu \frac{\partial A_\mu}{\partial x_a} + \overbrace{\boxed{\dot{A}_a - v_\mu \frac{\partial A_a}{\partial x_\mu}}} + E_a \right)$$

$$\frac{\partial A_a}{\partial t} = \boxed{\dot{A}_a - \sum_\mu v_\mu \frac{\partial A_a}{\partial x_\mu}}$$

$$m\dot{v}_a = e\left(\underbrace{v_\mu \frac{\partial A_\mu}{\partial x_a} - v_\mu \frac{\partial A_a}{\partial x_\mu}} + E_a \right)$$

$$m\dot{\mathbf{v}} = e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right) + \mathbf{E} \right) = e(\mathbf{v}\times\mathbf{B}+\mathbf{E})$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

Hamilton's equations for charged particle in fields

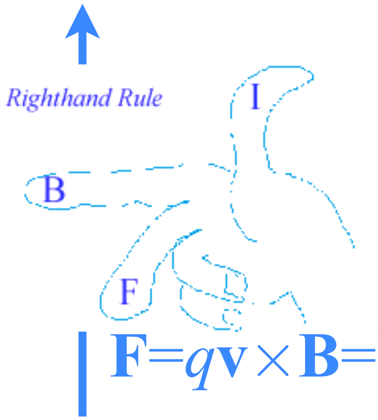
Hamiltonian is explicit function of ***momentum*** **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p}-e\mathbf{A}(\mathbf{r},t)$.

$$H=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p}-e\mathbf{A}(\mathbf{r},t)\right)+e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$

(Expanded)



F=$q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Hamilton's **v** equation:

$$\mathbf{v}=\dot{\mathbf{r}}=\frac{\partial H}{\partial\mathbf{p}}=\frac{\mathbf{p}-e\mathbf{A}(\mathbf{r},t)}{m}$$

(Just copies particle velocity relation.)

Hamilton's d**p**$/dt$ equation:
(In index notation.)

$$\dot{p}_a=-\frac{\partial H}{\partial x_a}=-\frac{\partial}{\partial x_a}\frac{\left(p_\mu-eA_\mu\right)\left(p_\mu-eA_\mu\right)}{2m}-e\frac{\partial\Phi}{\partial x_a}$$

$$\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$$

$$E_a=-\frac{\partial\Phi}{\partial x^a}-\frac{\partial A_a}{\partial t}$$

$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=+\underbrace{\frac{\left(p_\mu-eA_\mu\right)}{m}}_{}e\frac{\partial A_\mu}{\partial x_a}\quad-e\frac{\partial\Phi}{\partial x_a}$$

$$-\frac{\partial\Phi}{\partial x^a}=\frac{\partial A_a}{\partial t}+E_a$$

$$\dot{p}_a=m\dot{v}_a+e\dot{A}_a=e\left(v_\mu\frac{\partial A_\mu}{\partial x_a}+\overbrace{\frac{\partial A_a}{\partial t}+E_a}^{}\right)$$

$$\frac{\partial\mathbf{A}}{\partial t}=\frac{d\mathbf{A}}{dt}-(\mathbf{v}\bullet\nabla)\mathbf{A}$$

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$$\frac{\partial A_a}{\partial t}=\dot{A}_a-\sum_\mu v_\mu\frac{\partial A_a}{\partial x_\mu}$$

...and now

we come back

full circle...

$$m\dot{v}_a=e\left(\underbrace{v_\mu\frac{\partial A_\mu}{\partial x_a}-v_\mu\frac{\partial A_a}{\partial x_\mu}}_{}+E_a\right)$$

$$m\dot{\mathbf{v}}=e\left(\mathbf{v}\times\left(\nabla\times\mathbf{A}\right)+\mathbf{E}\right)=e(\mathbf{v}\times\mathbf{B}+\mathbf{E})$$

$$\mathbf{B}=\nabla\times\mathbf{A}$$

$\mathbf{v}\times(\nabla\times\mathbf{A})=\mathbf{v}\bullet(\nabla\mathbf{A})-(\mathbf{v}\bullet\nabla)\mathbf{A}$ *for particle mechanics*

Crossed E and B field mechanics

→ *Classical Hall-effect and cyclotron orbit orbit equations*

Vector theory vs. complex variable theory

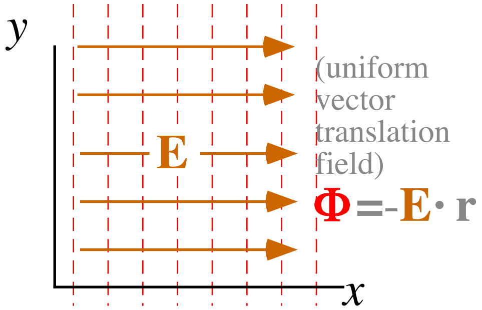
Mechanical analog of cyclotron and FBI rule

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\mathbf{\Phi}(\mathbf{r}) = -\mathbf{E} \bullet \mathbf{r}, \qquad -\nabla\mathbf{\Phi}(\mathbf{r}) = -\nabla\left(-\mathbf{E} \bullet \mathbf{r}\right) = \mathbf{E} = const.$$

Fig. 2.4.1.



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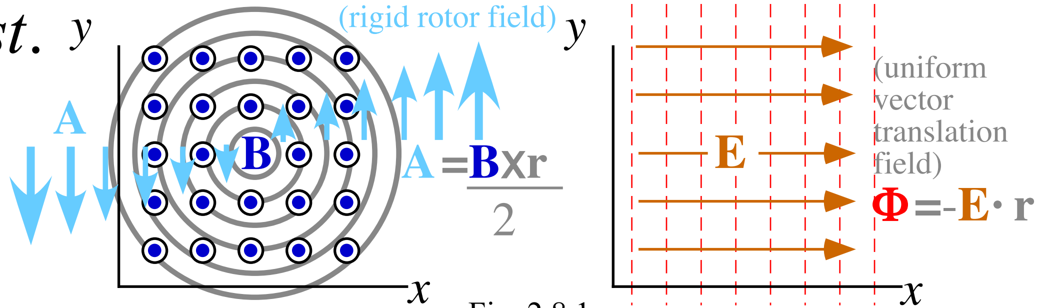

Fig. 2.8.1.

*This mechanical analog of (E_x,B_z) field mimics **A**-field with tabletop **v**-field*

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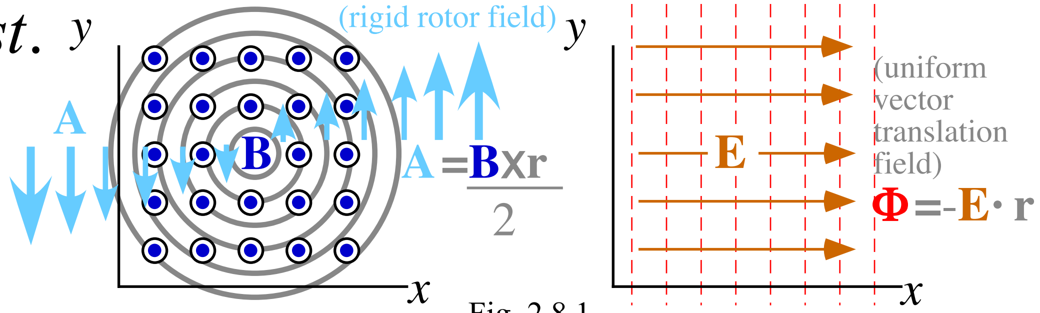
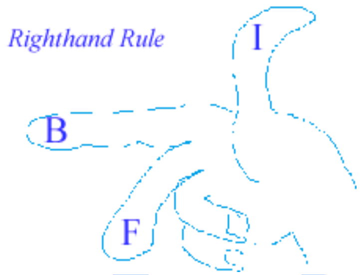


Fig. 2.8.1.

Newtonian electromagnetic equations of motion:

$$m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B}.$$



$$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$$

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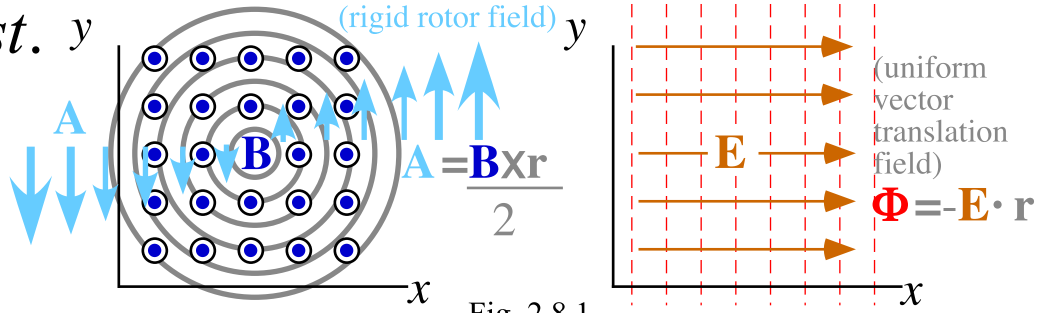

Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

Crossed E and B field mechanics

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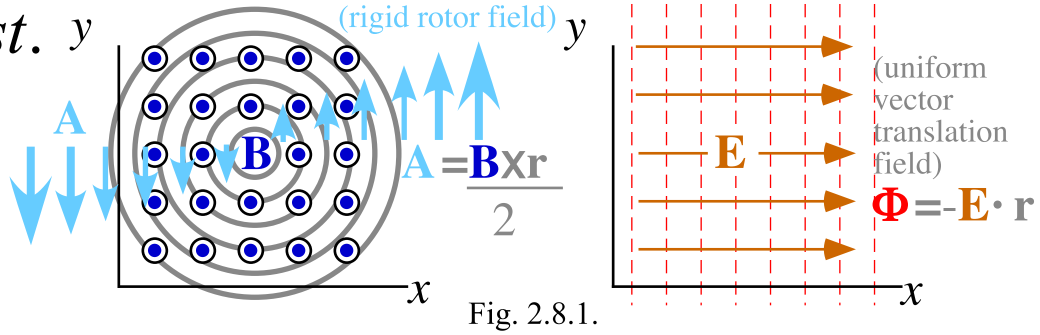

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where: $\hat{\mathbf{e}}_\mathbf{x}\times\hat{\mathbf{e}}_\mathbf{z} = -\hat{\mathbf{e}}_\mathbf{x}$ and: $\hat{\mathbf{e}}_\mathbf{y}\times\hat{\mathbf{e}}_\mathbf{z} = \hat{\mathbf{e}}_\mathbf{x}$

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→ *Vector theory vs. complex variable theory*

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Cycloid geometry and flying sticks

Practical poolhall application

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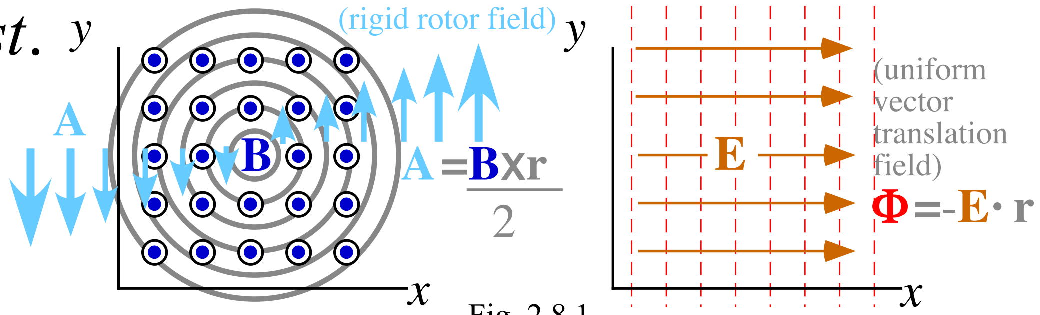

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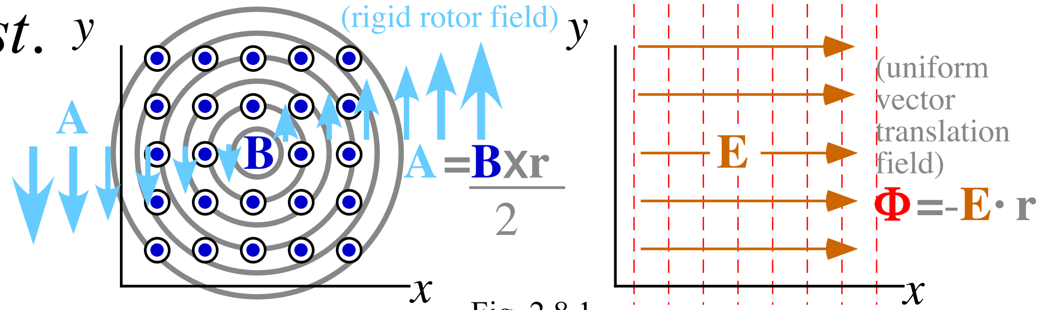

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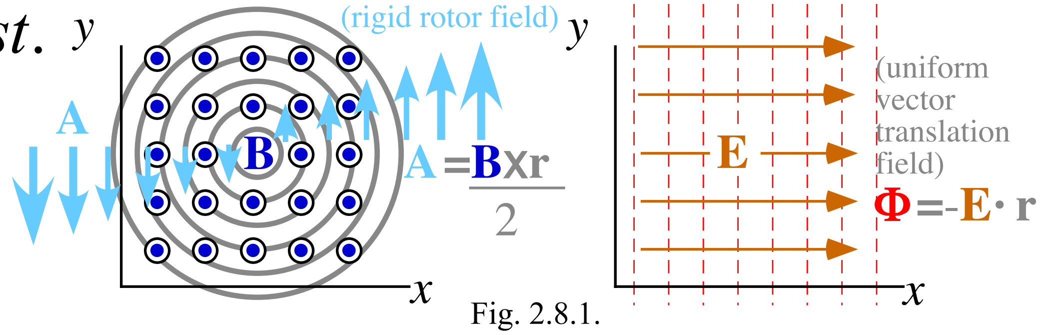

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Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

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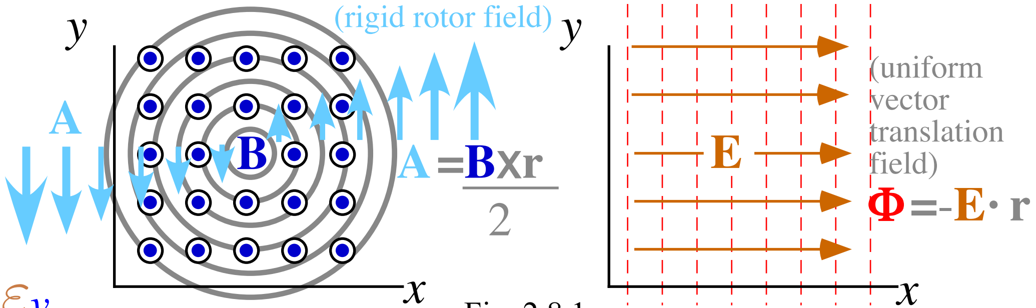

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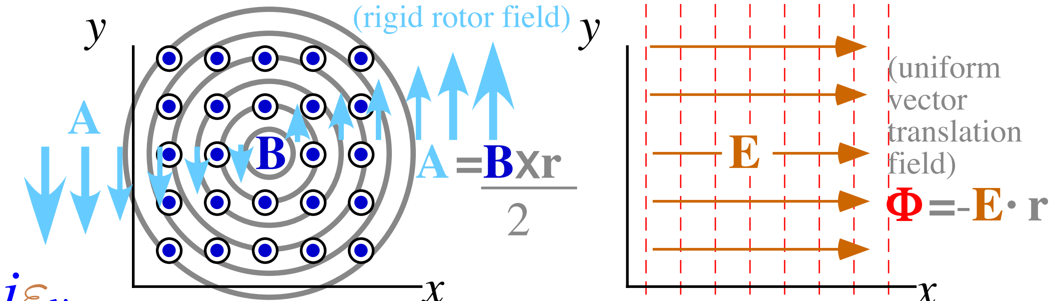

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$$\boxed{\dot V(t) = \dot v(t) + \dot\beta = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t)} \qquad \text{where : } \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\begin{array}{l}\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling}\end{array}}$$

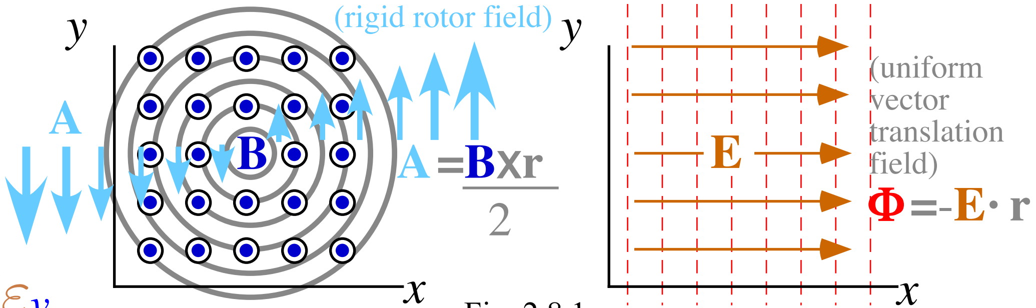

Fig. 2.8.1.

Complex variable velocity: $v = v_x + iv_y$ and electric field: $\varepsilon = \varepsilon_x + i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} \quad = \quad \varepsilon \quad - \quad iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_\mathbf{x} \rightarrow 1 \qquad \text{and : } \hat{\mathbf{e}}_\mathbf{y} \rightarrow i = \sqrt{-1}$$

A velocity transformation $V(t) = v(t) + \beta$ cancels constant ε-field to give an equation: $\boxed{\dot{V} = (const.)V}$

$$\boxed{\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t)} \qquad \text{where : } \boxed{\beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}}$$

An exponential $V(t) = e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

$$v(t) + \beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0) + \beta)$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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Shorthand Labeling

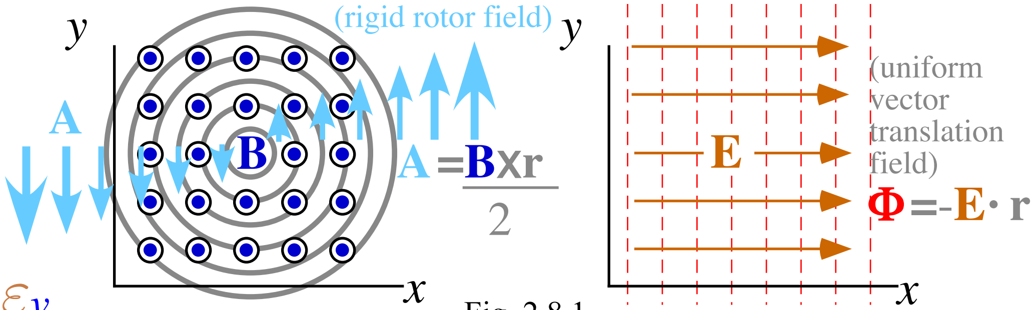

Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ and electric field: $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot v_x + i\dot v_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

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complex form

$$v(t) + \beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0) + \beta) \qquad \text{or:} \qquad v(t) = e^{-iB\cdot t}(v(0) + \beta) - \beta = e^{-iB\cdot t}(v(0) + i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

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$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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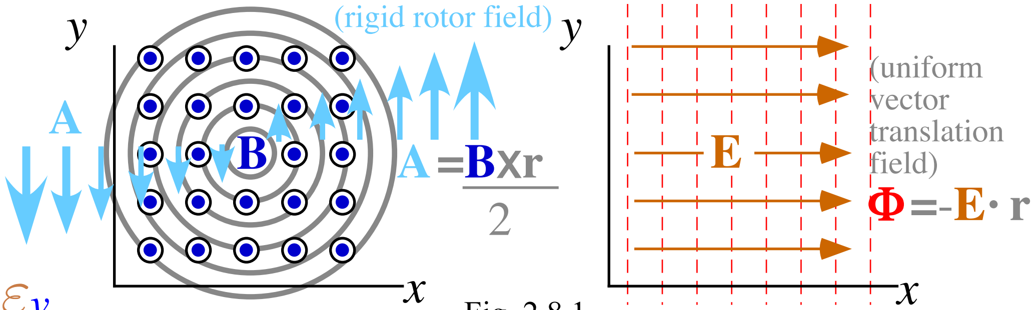

Fig. 2.8.1.

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$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Crossed E and B field mechanics (Solution by complex variables)

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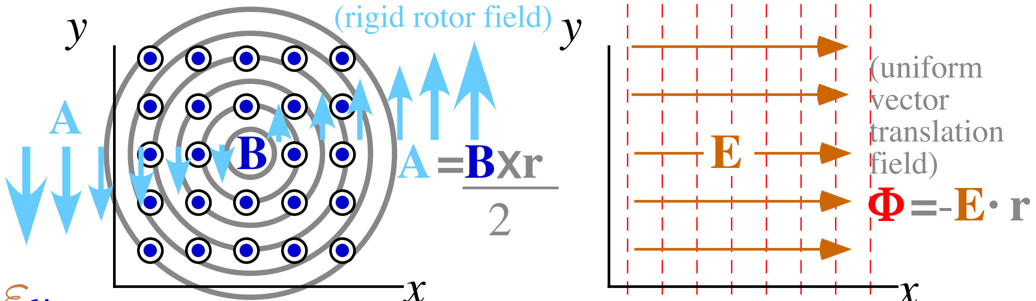

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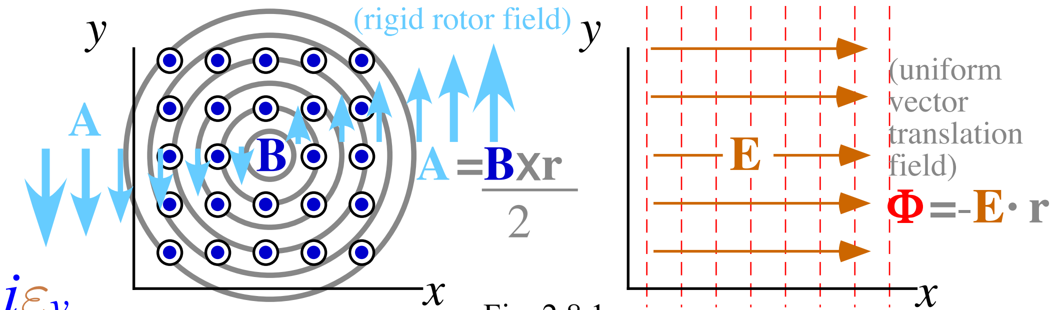

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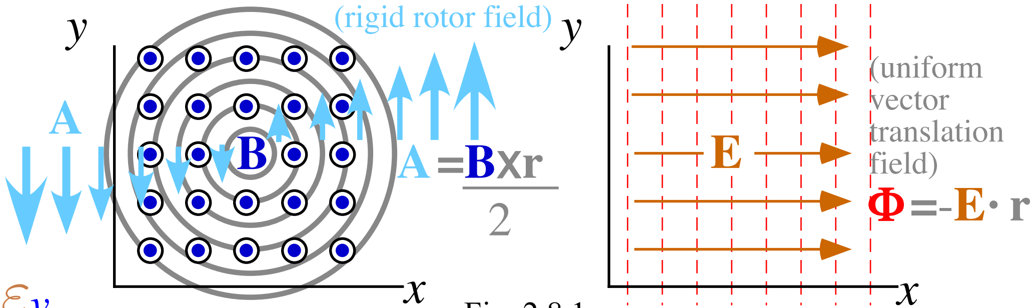
complex form

Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\begin{array}{l}\dot{\mathbf{v}}=\frac{e}{m}\mathbf{E}+\mathbf{v}\times\frac{e}{m}\mathbf{B}=\boldsymbol{\varepsilon}+\mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z\\ \varepsilon_x=\frac{e}{m}E_x \qquad \varepsilon_y=\frac{e}{m}E_y \qquad B=\frac{e}{m}B_z\\ \textit{Shorthand Labeling}\end{array}}$$



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$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

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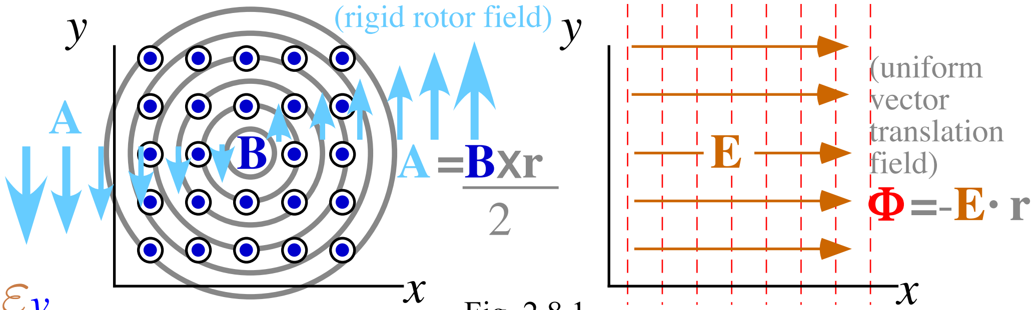
Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t)=\int v(t)dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2}) \quad \textit{complex form}$$

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\boxed{\begin{array}{l}\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z \\ \varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z \\ \textit{Shorthand Labeling}\end{array}}$$



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$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_\mathbf{x} \to 1 \qquad \text{and : } \hat{\mathbf{e}}_\mathbf{y} \to i = \sqrt{-1}$$

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An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

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complex form

$$x(t)+iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

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complex form

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

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vector form

Integrating *v(t)* yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

complex form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

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vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

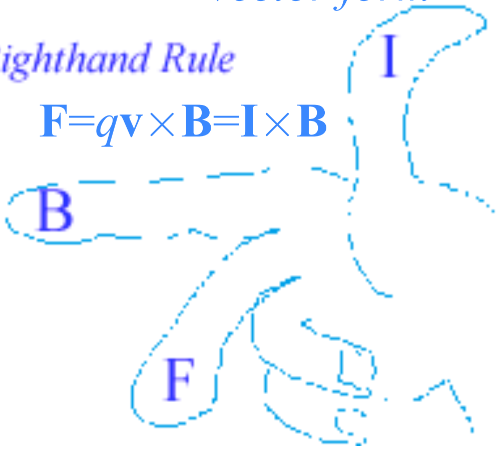
$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}\left(v(0)+i\frac{\varepsilon}{B}\right)-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-\left(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2}\right)$$

complex form

$$x(t)+iy(t)=e^{-iB\cdot t}\left(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2}\right)-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

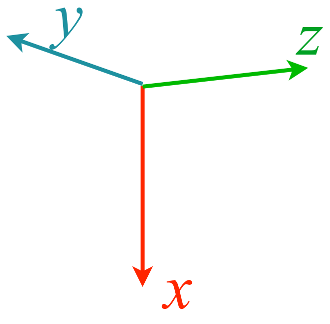
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form



Righthand Rule

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$



Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta}=\varepsilon-iBv=\varepsilon-iB(V(t)-\beta)=-iBV(t)$$

where : $\boxed{\beta=-\frac{\varepsilon}{iB}=i\frac{\varepsilon}{B}}$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

$$v(t)+\beta=V(t)=e^{-iB\cdot t}V(0)=e^{-iB\cdot t}(v(0)+\beta)$$

or: $$v(t)=e^{-iB\cdot t}(v(0)+\beta)-\beta=e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

complex form

Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

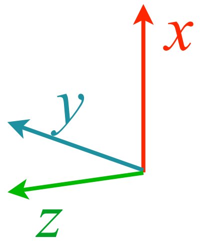
vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$

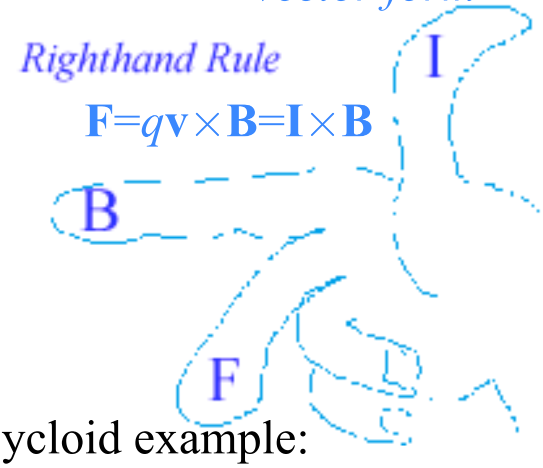
complex form

$$x(t)+iy(t)=\quad e^{-iB\cdot t}\quad(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})\quad-i\frac{\varepsilon}{B}\cdot t\quad+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$



$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form



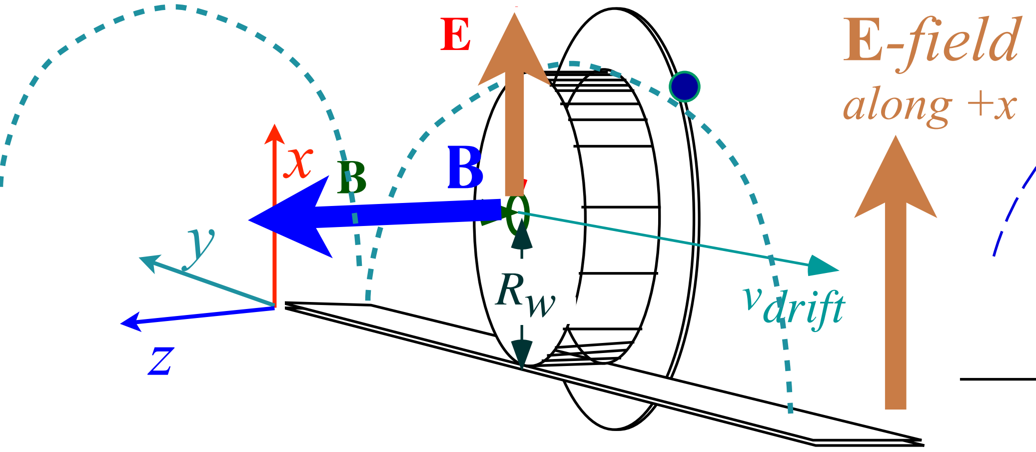
Righthand Rule

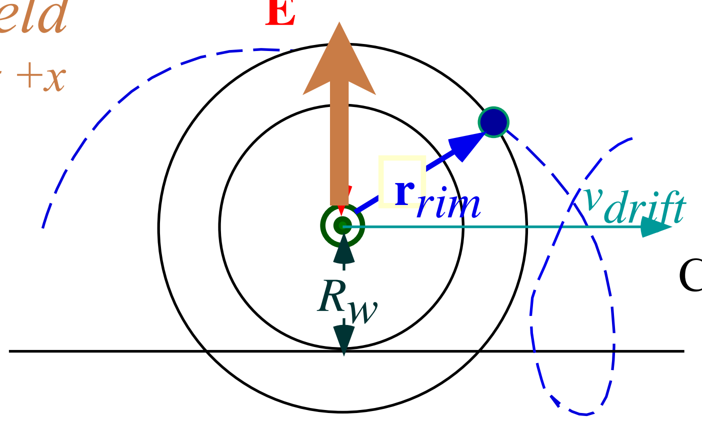
F=q**v**×**B**=**I**×**B**

Cycloid example:
initial $(x(0)\,,\,y(0))=(0,0)$
and $(v_x(0),v_y(0))=(0,0)$

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=$ is on rim of a wheel of radius $R_W=E/B^2$

$$\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{E}{B^2} \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ -\frac{E}{B}t \end{pmatrix}+\begin{pmatrix} \frac{E}{B^2} \\ 0 \end{pmatrix}$$





Constant v_{drift} in -y-direction
$v_{drift}=-E/B$

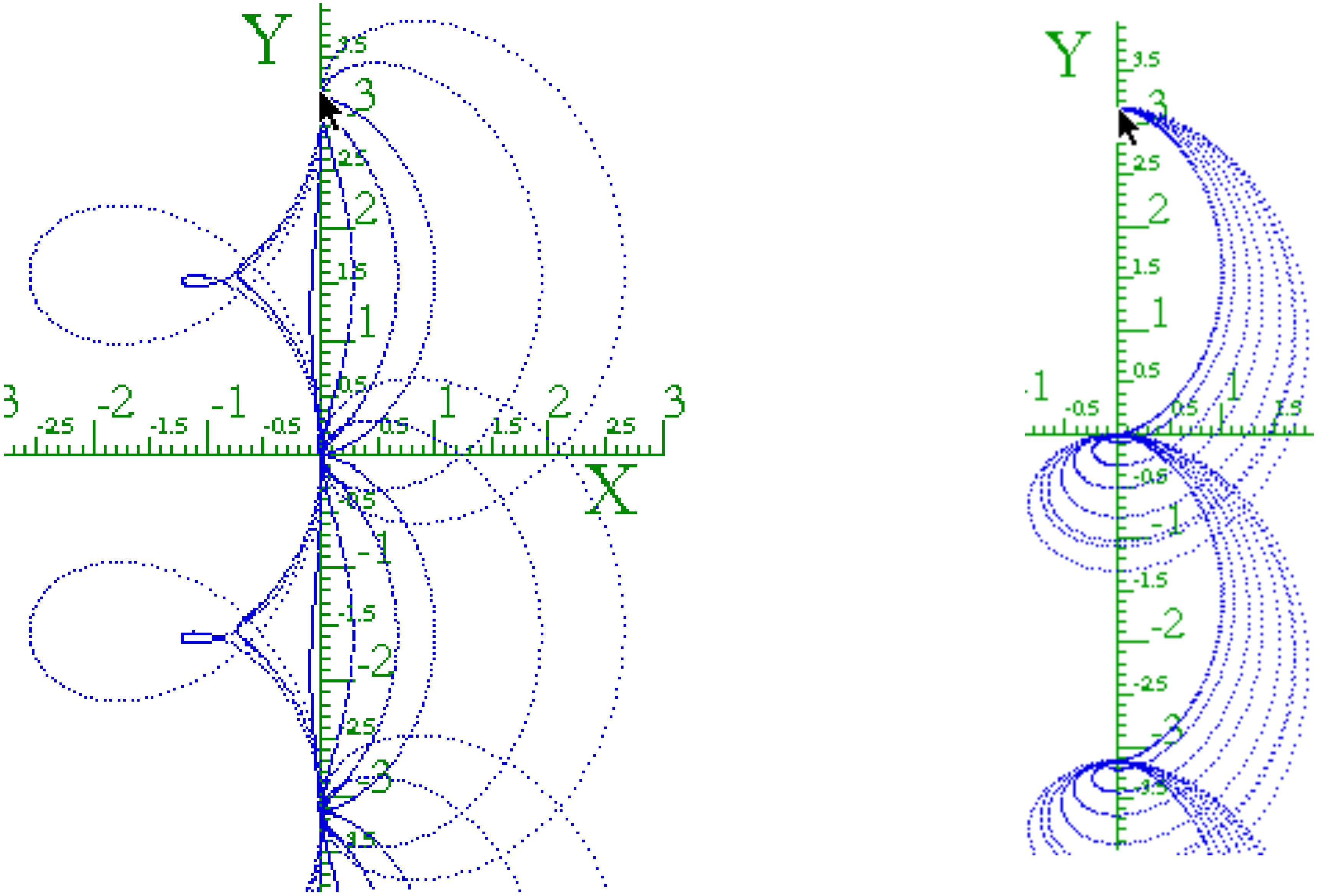


Fig. 2.8.2 Trajectories of unit charge and mass in magnetic and electric fields (E=1/2, B=1)

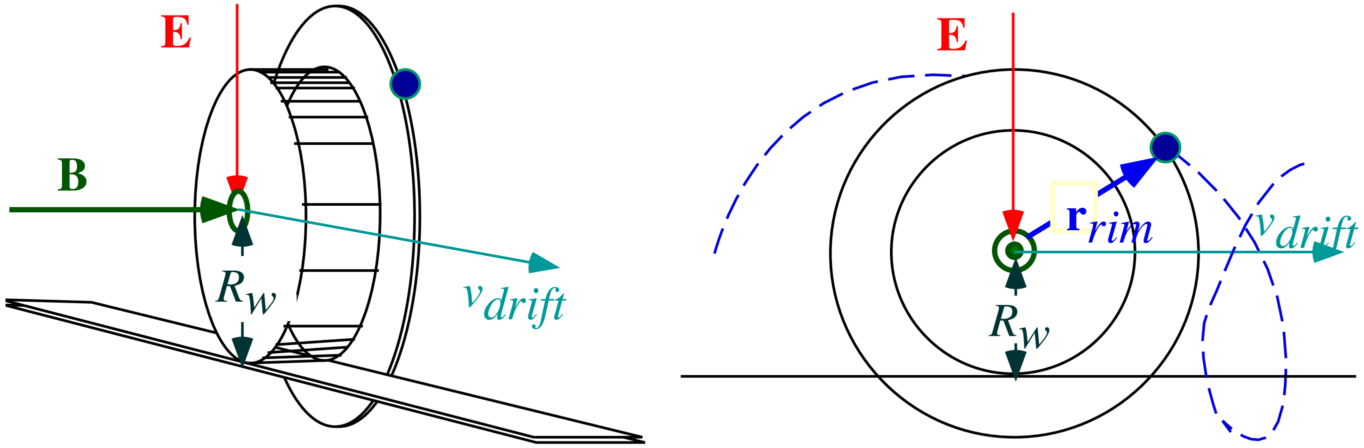
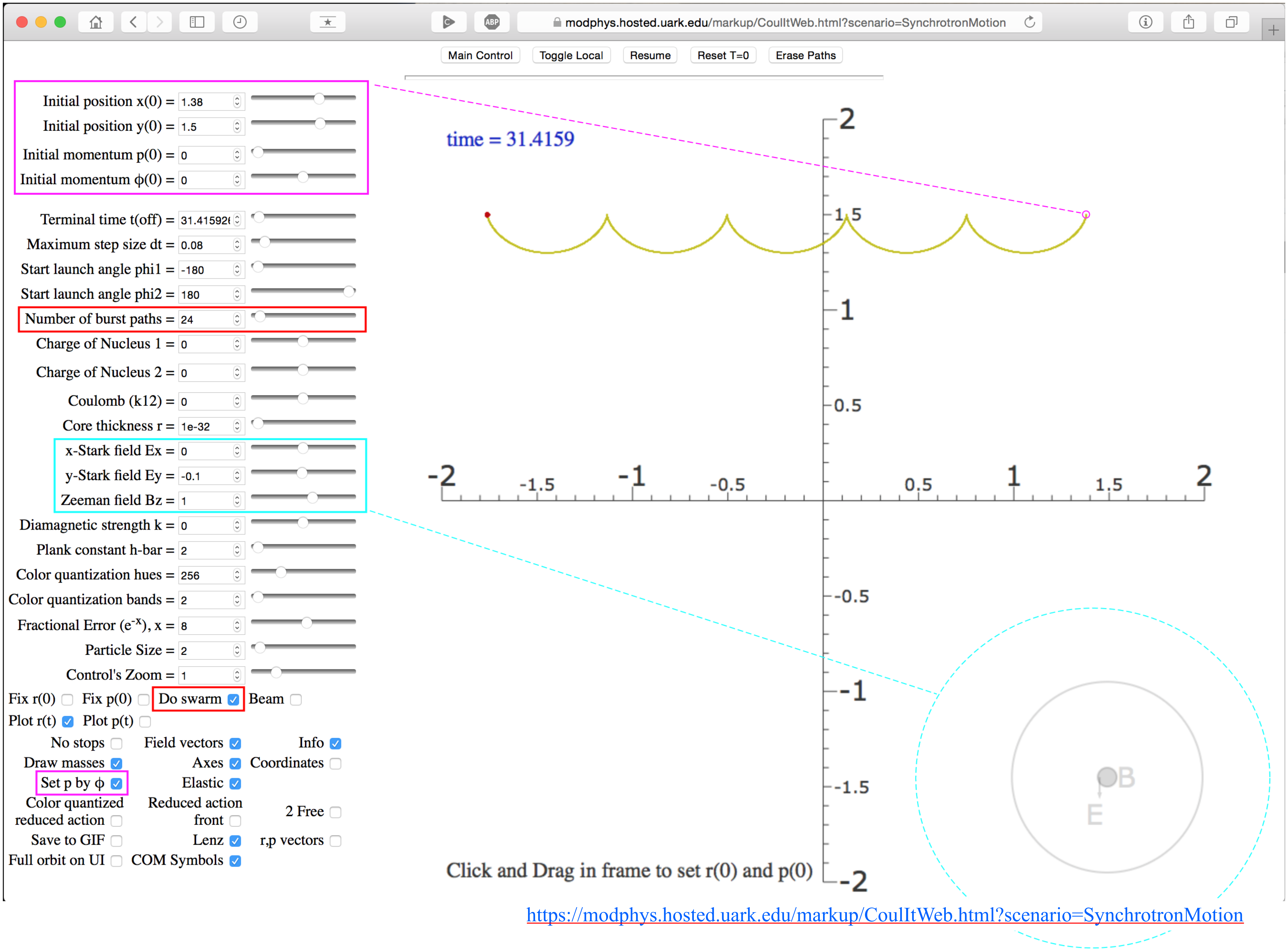


Fig. 2.8.3 Rolling railroad wheel and rim analogy for cyclotron orbits

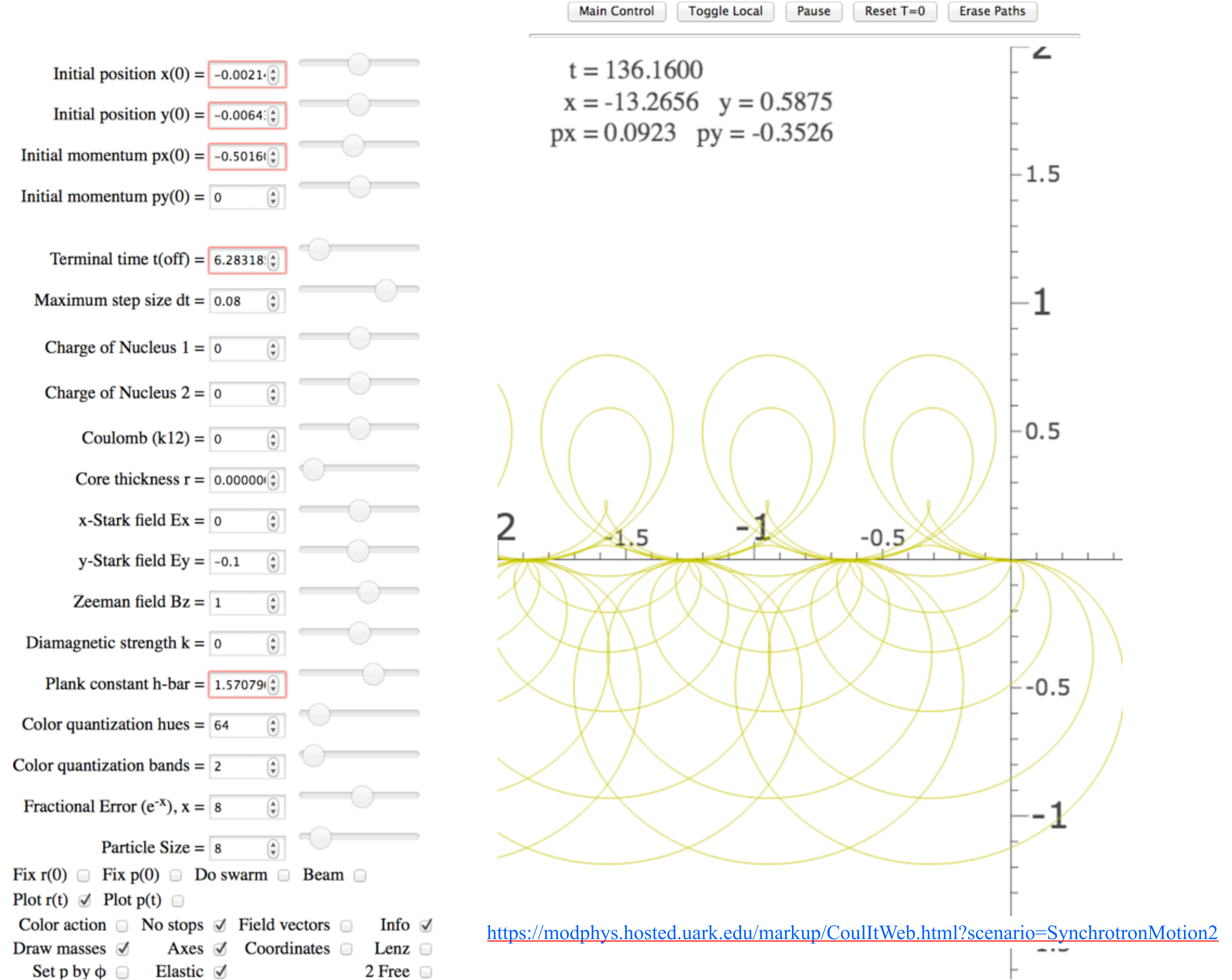


Initial position x(0) = 1.38
Initial position y(0) = 1.5
Initial momentum p(0) = 0
Initial momentum φ(0) = 0

Terminal time t(off) = 31.415926
Maximum step size dt = 0.08
Start launch angle phi1 = -180
Start launch angle phi2 = 180
Number of burst paths = 24
Charge of Nucleus 1 = 0
Charge of Nucleus 2 = 0
Coulomb (k12) = 0
Core thickness r = 1e-32
x-Stark field Ex = 0
y-Stark field Ey = -0.1
Zeeman field Bz = 1
Diamagnetic strength k = 0
Plank constant h-bar = 2
Color quantization hues = 256
Color quantization bands = 2
Fractional Error (e^{-x}), x = 8
Particle Size = 2
Control's Zoom = 1

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☑ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
No stops ☐ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐
Set p by φ ☑ Elastic ☑
Color quantized reduced action ☐ Reduced action front ☐ 2 Free ☐
Save to GIF ☐ Lenz ☑ r,p vectors ☐
Full orbit on UI ☐ COM Symbols ☑

https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion



Initial position x(0) = -0.0021

Initial position y(0) = -0.0064

Initial momentum px(0) = -0.5016

Initial momentum py(0) = 0

Terminal time t(off) = 6.28318

Maximum step size dt = 0.08

Charge of Nucleus 1 = 0

Charge of Nucleus 2 = 0

Coulomb (k12) = 0

Core thickness r = 0.00000

x-Stark field Ex = 0

y-Stark field Ey = -0.1

Zeeman field Bz = 1

Diamagnetic strength k = 0

Plank constant h-bar = 1.57079

Color quantization hues = 64

Color quantization bands = 2

Fractional Error (e^{-x}), x = 8

Particle Size = 8

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐

Plot r(t) ☑ Plot p(t) ☐

Color action ☐ No stops ☑ Field vectors ☐ Info ☑

Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☐

Set p by ϕ ☐ Elastic ☑ 2 Free ☐

t = 136.1600

x = -13.2656 y = 0.5875

px = 0.0923 py = -0.3526

https://modphys.hosted.uark.edu/markup/CoulItWeb.html?scenario=SynchrotronMotion2

Crossed E and B field mechanics

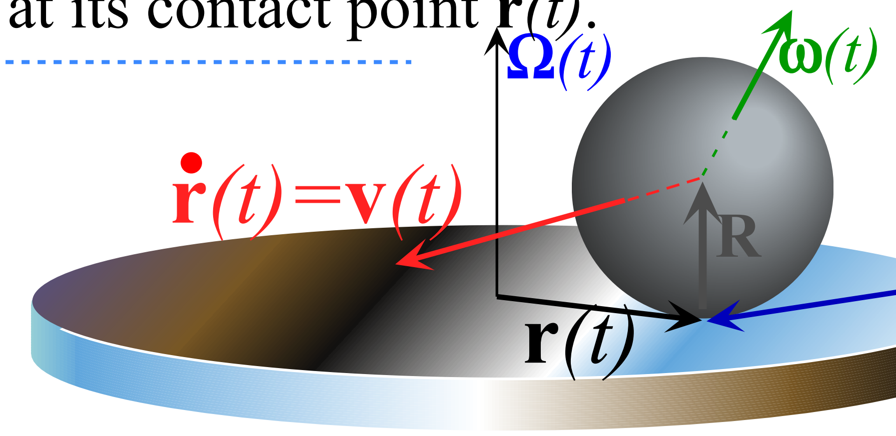
Classical Hall-effect and cyclotron orbit equations

Vector theory vs. complex variable theory

→ *Mechanical analog of cyclotron and FBI rule*

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\omega(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



$$\mathbf{F}\mathbf{x}\mathrm{R}=I\,\dot{\omega}(t)$$

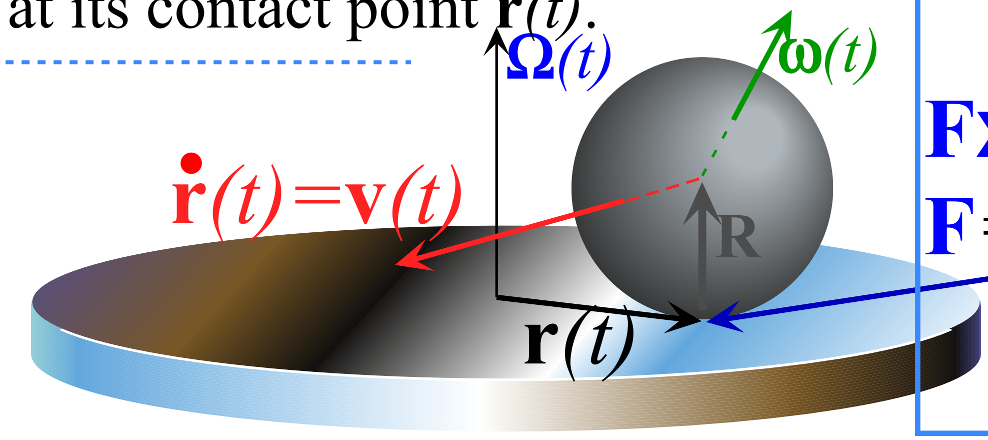
$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

YouTube Video of Analog to Syncrotron Motion

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\omega(t)\times\mathbf{R}$) equals table surface velocity $\mathbf{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\omega}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\omega}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Turntable turning at constant angular velocity $\mathbf{\Omega}=\Omega\hat{\mathbf{z}}$.

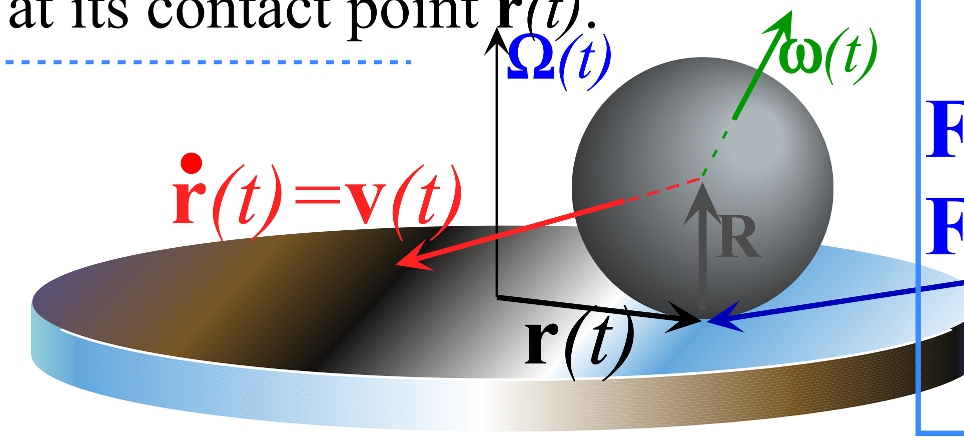
Torque-and-F=ma
equations of motion:

$$
\begin{aligned}
I\dot{\omega}(t) &= \mathbf{F}(t)\times\mathbf{R}\\
&= m\dot{\mathbf{v}}(t)\times\mathbf{R}\\
&= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R
\end{aligned}
$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

Equations of Motion:

rotation $Torque=\mathbf{F}\times\mathbf{R}=I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

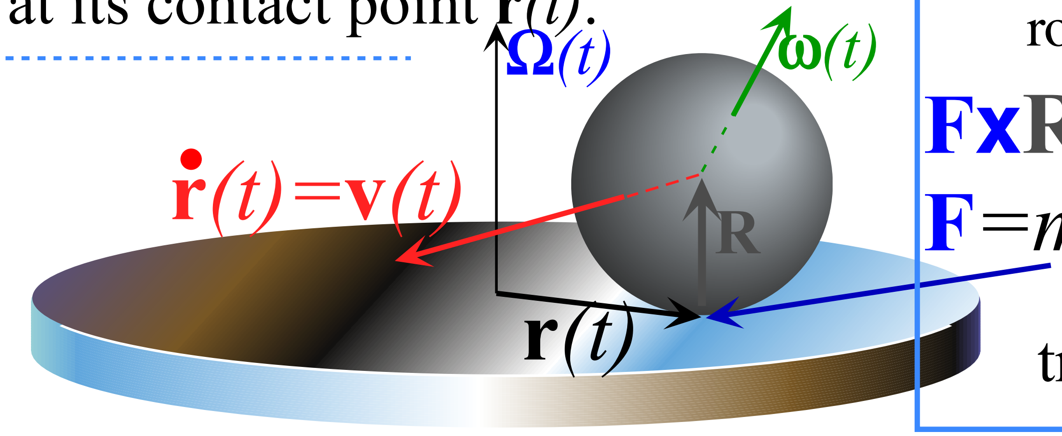
translation $Force=\mathbf{F}=m\dot{\mathbf{v}}$

*Torque-and-*F=ma
equations of motion:

$$\begin{aligned} I\dot{\boldsymbol{\omega}}(t) &= \mathbf{F}(t)\times\mathbf{R} \\ &= m\dot{\mathbf{v}}(t)\times\mathbf{R} \\ &= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R \end{aligned}$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque=\mathbf{F}\times\mathbf{R}=I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation $Force=\mathbf{F}=m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

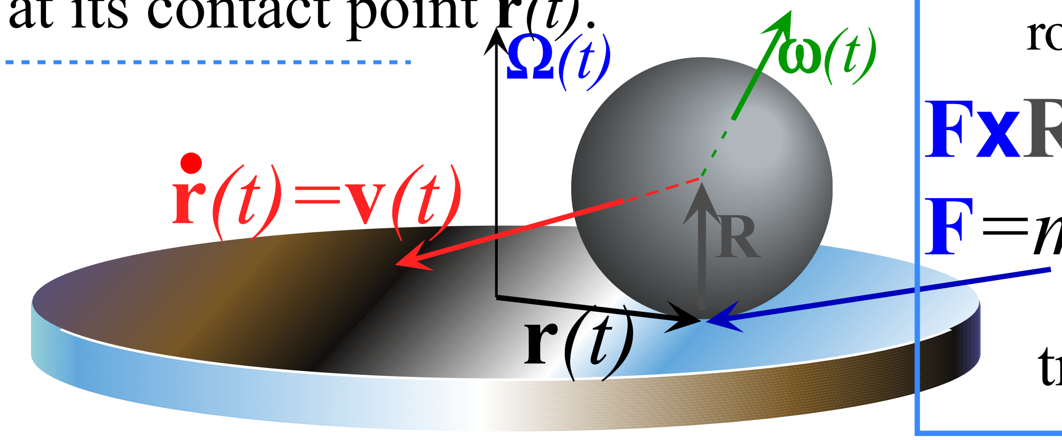
$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R}\quad=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma
equations of motion:

$$\begin{aligned}I\dot{\boldsymbol{\omega}}(t)&=\mathbf{F}(t)\times\mathbf{R}\\&=m\dot{\mathbf{v}}(t)\times\mathbf{R}\\&=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R\end{aligned}$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R} = \boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t) = \boldsymbol{\Omega}\times\mathbf{r}(t) + \boldsymbol{\omega}(t)\times\mathbf{R} \quad = \boldsymbol{\Omega}\times\mathbf{r}(t) + \boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\dot{\mathbf{r}}(t) + \dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R = \boldsymbol{\Omega}\times\mathbf{v}(t) + \dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

*Torque-and-*F=ma
equations of motion:

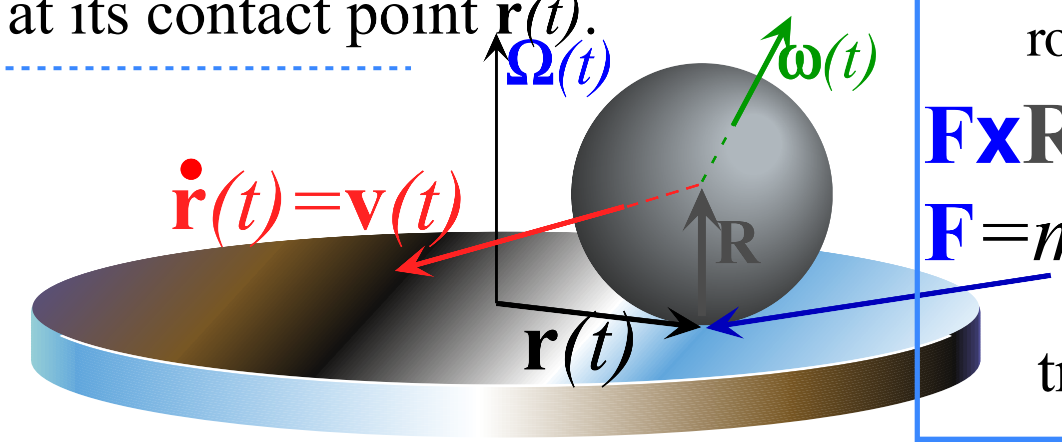
$$I\dot{\boldsymbol{\omega}}(t) = \mathbf{F}(t)\times\mathbf{R}$$

$$= m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

$$= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation $Force = \mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R}\quad=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

Torque-and-F=ma equations of motion:

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

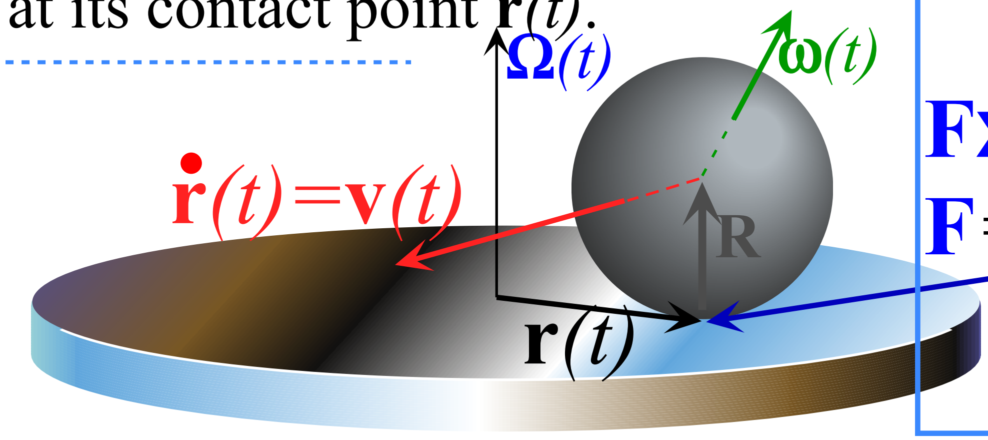
$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\;+\quad\dot{\boldsymbol{\omega}}(t)\quad\times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

Equations of Motion:

rotation *Torque* $=\mathbf{F}\times\mathbf{R}=I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation *Force*$=\mathbf{F}=m\dot{\mathbf{v}}$

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R}\quad=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

*Torque-and-*F=ma *equations of motion:*

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\ +\quad\dot{\boldsymbol{\omega}}(t)\quad\times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

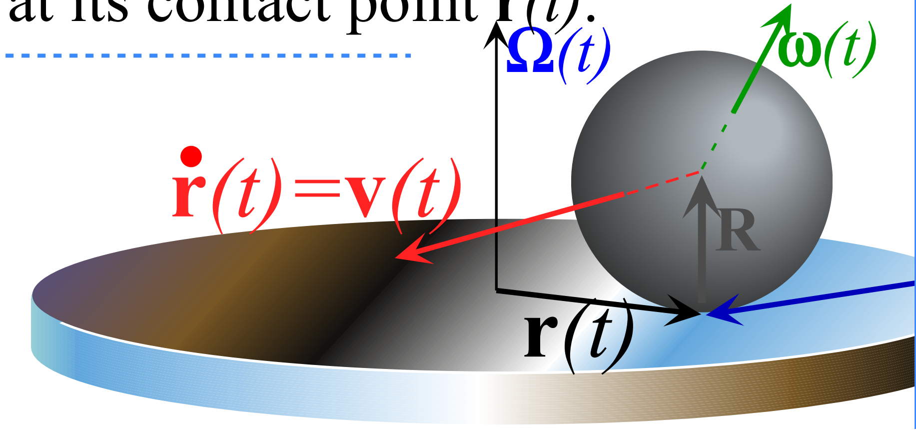
$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

with: $\mathbf{B}=\dfrac{m\dot{\mathbf{v}}(t)}{I}$ and: $\mathbf{A}=\hat{\mathbf{z}}R=\mathbf{C}$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Equations of Motion:

rotation *Torque* $= \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation *Force*$=\mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint

Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

*Torque-and-*F=ma *equations of motion:*

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

$$=m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R$$

$$\mathbf{v}(t)=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\mathbf{R}\quad=\boldsymbol{\Omega}\times\mathbf{r}(t)+\boldsymbol{\omega}(t)\times\hat{\mathbf{z}}R$$

Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)\;+\;\dot{\boldsymbol{\omega}}(t)\;\times\hat{\mathbf{z}}R$$

use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

with: $\mathbf{B}=\dfrac{m\dot{\mathbf{v}}(t)}{I}$ and: $\mathbf{A}=\hat{\mathbf{z}}R=\mathbf{C}$

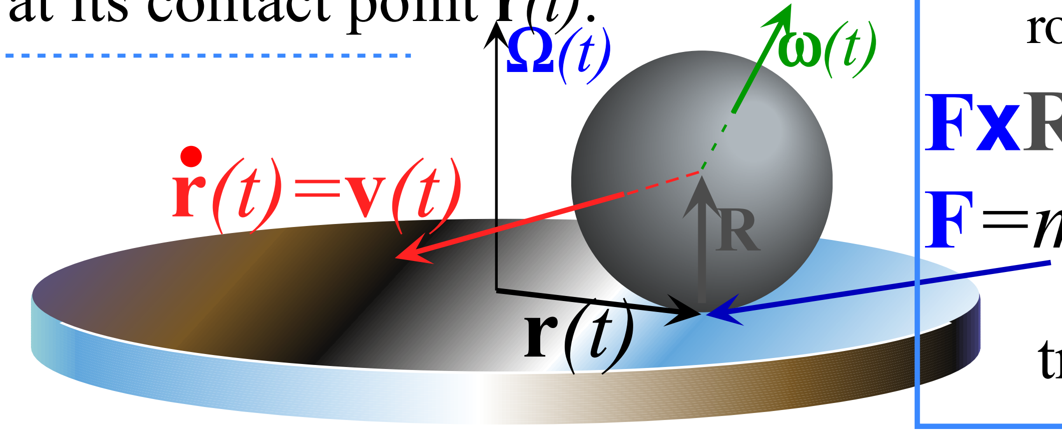
$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\bullet\hat{\mathbf{z}}R}{I}\hat{\mathbf{z}}R-\frac{mR^2}{I}\dot{\mathbf{v}}(t)$$

($\mathbf{v}(t)$ *always normal to* $\hat{\mathbf{z}}$)

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)$-$\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



Equations of Motion:

rotation *Torque* $= \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

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Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

Torque-and-F=ma equations of motion:

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use: $\dot{\boldsymbol{\omega}}(t)=\dfrac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

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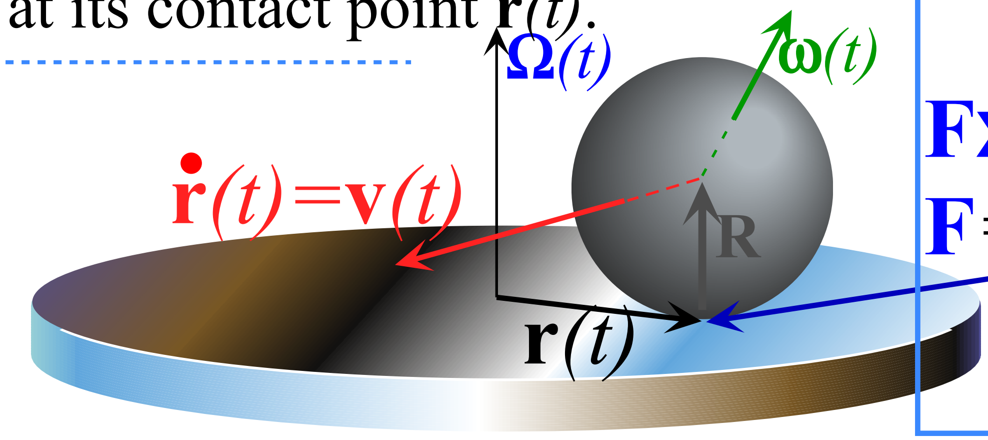

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$$\left(1+\frac{mR^2}{I}\right)\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)$$

or :

$\mathbf{F}=\mathbf{B}\times\mathbf{v}$ *mechanical analog:*

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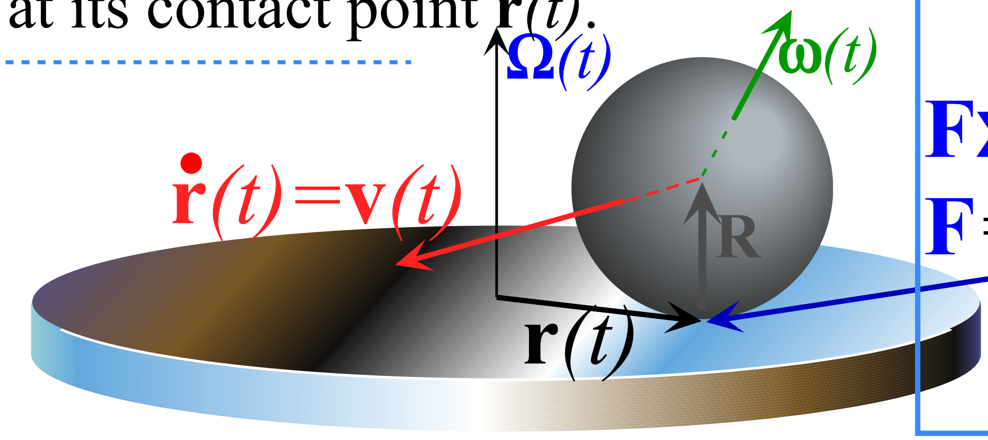

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$m\mathbf{a}=e\mathbf{B}\times\mathbf{v}$ mechanical analog:

$$\dot{\mathbf{v}}(t)=\frac{\boldsymbol{\Omega}}{1+\frac{mR^2}{I}}\times\mathbf{v}(t)$$

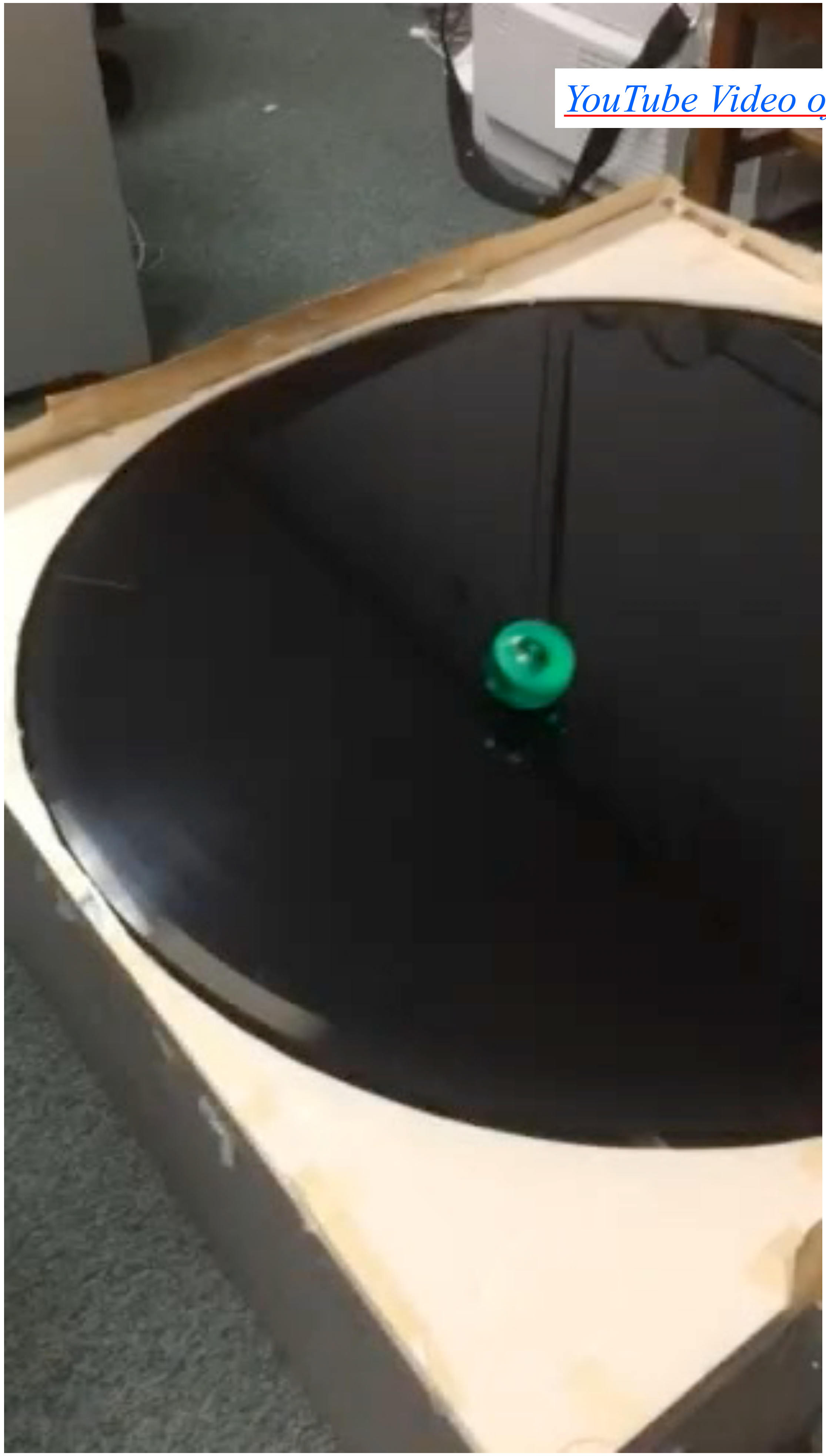
Mechanical analog cyclotron frequency

$$\omega=\frac{e}{m}B=\frac{\boldsymbol{\Omega}}{1+\frac{mR^2}{I}}$$

$\omega=\frac{2}{7}\boldsymbol{\Omega}$ for:$\frac{I}{mR^2}=\frac{2}{5}$ ●

$=\frac{2}{5}\boldsymbol{\Omega}$ for:$\frac{I}{mR^2}=\frac{2}{3}$ ○

[YouTube Video of Analog to Syncrotron Motion]



YouTube Video of Analog to Syncrotron Motion

Solid ball has 2 orbits
as table turns 7 rotations

Mechanical analog
cyclotron frequency

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Thats all folks!