Lecture 18
Thur. 10.26.2017

Electromagnetic Lagrangian and charge-field mechanics (Ch. 2.8 of Unit 2)

Charge mechanics in electromagnetic fields

*Vector analysis for particle-in-(**A**,Φ)-potential*
*Lagrangian for particle-in-(**A**,Φ)-potential*
*Hamiltonian for particle-in-(**A**,Φ)-potential*
*Canonical momentum in (**A**,Φ) potential*
Hamiltonian formulation
Hamilton's equations

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit orbit equations
Vector theory vs. complex variable theory
Mechanical analog of cyclotron and FBI rule
Cycloidal ruler&compass geometry
Cycloidal geometry of flying levers
Practical poolhall application

*This mechanical analog of (E_x,B_z) field mimics **A**-field with tabletop **v**-field*

Charge mechanics in electromagnetic fields

→ *Vector analysis for particle-in-(**A**,Φ)-potential*

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- *Canonical momentum in (**A**,Φ) potential*
- *Hamiltonian formulation*
- *Hamilton's equations*

Vector analysis for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
$e = -1.602176 \cdot 10^{-19}$ *Coulombs*

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Electric field **E** and magnetic field **B**

scalar potential field $\mathbf{\Phi}=\mathbf{\Phi}(\mathbf{r},t)$

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$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}$$

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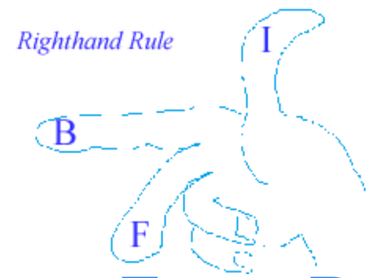

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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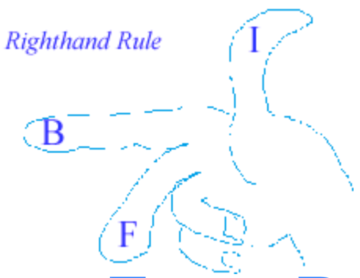

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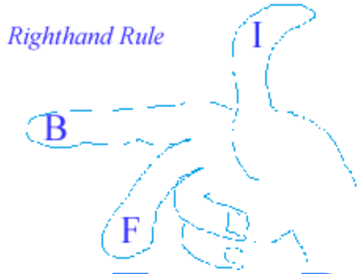

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Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k = \varepsilon_{kij} v_i (\nabla\times\mathbf{A})_j$$

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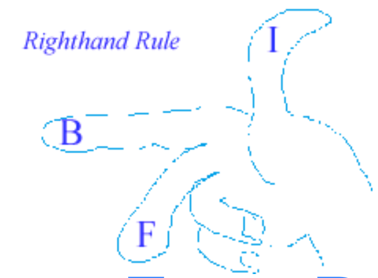

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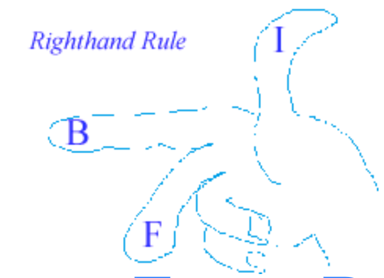

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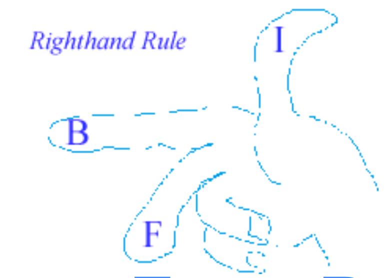

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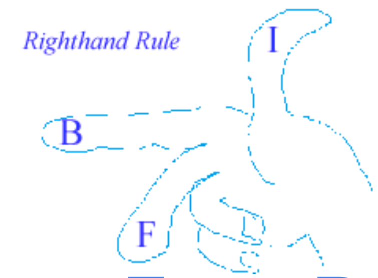

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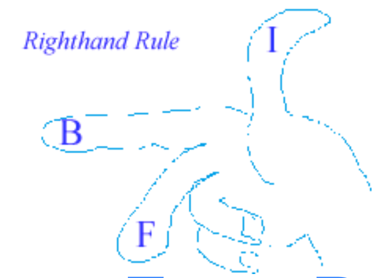

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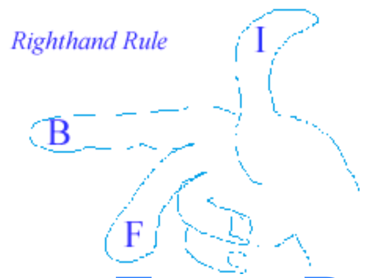

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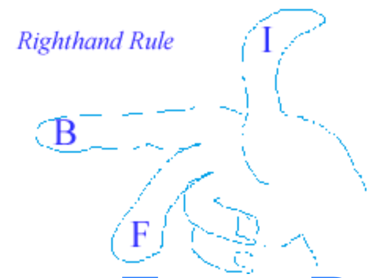

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Converting back to Gibbs's **bold** notation involves *tensors* like $\nabla\mathbf{A}$ and $\nabla\mathbf{v}$.

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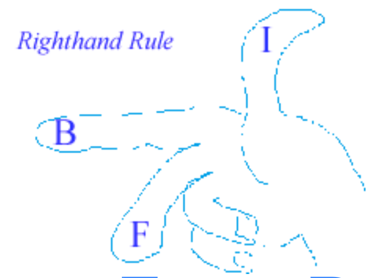

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&= v_b\left(\partial_k A_b\right) - v_a\left(\partial_a A_k\right) \\
&= \left(\partial_k A_b\right) v_b - v_a\left(\partial_a A_k\right) = (\nabla\mathbf{A})\bullet\mathbf{v} - \mathbf{v}\bullet\nabla\mathbf{A} \\
&= \partial_k\left(A_b v_b\right) - \left(\partial_k v_b\right) A_b - v_a\left(\partial_a A_k\right) = \nabla(\mathbf{A}\bullet\mathbf{v}) - (\nabla\mathbf{v})\bullet\mathbf{A} - \mathbf{v}\bullet\nabla\mathbf{A}
\end{aligned}$$

Applying Levi-Civita ε-identity:

$$\varepsilon_{kij}\varepsilon_{abj} = \delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}$$

Converting back to Gibbs's **bold** notation involves *tensors* like ∇**A** and ∇**v**.

Newtonian mechanics has *no explicit dependence* of position **r** and velocity **v**.

r-partial derivative of **v** (or vice-versa) is identically zero.

$$\partial_k v^j \equiv 0 \quad \textit{iff}: \quad \nabla\mathbf{v} = \frac{\partial\mathbf{v}}{\partial\mathbf{r}} = \mathbf{0}$$

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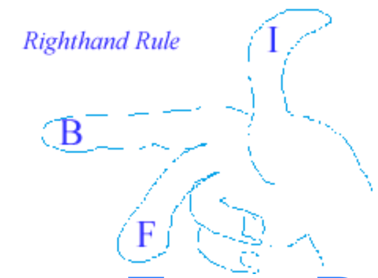

$$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$$

Doing a double-cross

ε_{ijk}-Tensor analysis of $\mathbf{v}\times(\nabla\times\mathbf{A})$

$$\varepsilon_{ijk}=\begin{cases}+1 \text{ for } \textit{even} \text{ permutaion of } i<j<k\\ 0 \quad \text{if any of } i,j,k \text{ are equal}\\ -1 \text{ for } \textit{odd} \text{ permutaion of } i<j<k\end{cases}$$

$$\varepsilon_{ijk}=\varepsilon_{ikj}=\varepsilon_{kij}=1$$

$$=-\varepsilon_{jik}=-\varepsilon_{jki}=-\varepsilon_{kji}$$

$$\left[\mathbf{v}\times(\nabla\times\mathbf{A})\right]_k=\varepsilon_{kij}v_i\left(\nabla\times\mathbf{A}\right)_j$$

$$=\varepsilon_{kij}v_i\left(\varepsilon_{abj}\left(\partial_a A_b\right)\right)$$

$$=\varepsilon_{kij}\varepsilon_{abj}\quad v_i\left(\partial_a A_b\right)$$

$$=(\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia})v_i\left(\partial_a A_b\right)$$

$$=\delta_{ka}\delta_{ib}v_i\left(\partial_a A_b\right)-\delta_{kb}\delta_{ia}v_i\left(\partial_a A_b\right)$$

$$=\quad v_b\left(\partial_k A_b\right)\quad -v_a\left(\partial_a A_k\right)$$

$$=\quad\left(\partial_k A_b\right)v_b\quad -v_a\left(\partial_a A_k\right)=(\nabla\mathbf{A})\bullet\mathbf{v}-\mathbf{v}\bullet\nabla\mathbf{A}$$

$$=\partial_k\left(A_b v_b\right)-\left(\partial_k v_b\right)A_b-v_a\left(\partial_a A_k\right)=\nabla(\mathbf{A}\bullet\mathbf{v})-(\nabla\mathbf{v})\bullet\mathbf{A}-\mathbf{v}\bullet\nabla\mathbf{A}$$

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Summary of Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

Tensor index notation helps to distinguish $(\nabla\mathbf{A})\bullet\mathbf{v}$, $\mathbf{v}\bullet(\nabla\mathbf{A})$, and $\nabla(\mathbf{A}\bullet\mathbf{v}) = (\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}$.

$$\left[(\nabla\mathbf{A})\bullet\mathbf{v}\right]_k = \frac{\partial A_j}{\partial x_k} v_j = (\partial_k A_j) v_j$$

$$\left[\mathbf{v}\bullet(\nabla\mathbf{A})\right]_k = v_j \frac{\partial A_k}{\partial x_j} = (v_j \partial_j A_k)$$

$$\left[\nabla(\mathbf{A}\bullet\mathbf{v})\right]_k = \left[(\nabla\mathbf{A})\bullet\mathbf{v} + (\nabla\mathbf{v})\bullet\mathbf{A}\right]_k$$

$$\partial_k\left(A_b v_b\right) = \left(\partial_k v_b\right) A_b - \left(\partial_k v_a\right) A_a$$

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Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

→ *Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential*

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
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- *Hamilton's equations*

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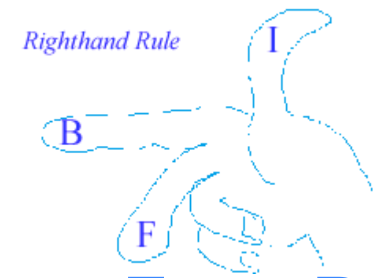

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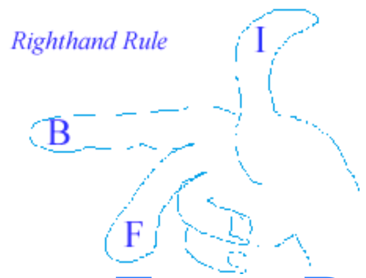

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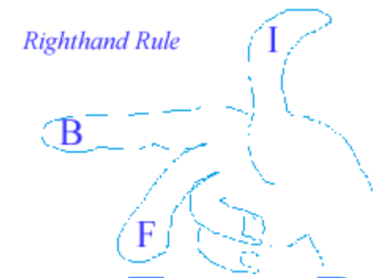

F=*q***v**×**B**=**I**×**B**

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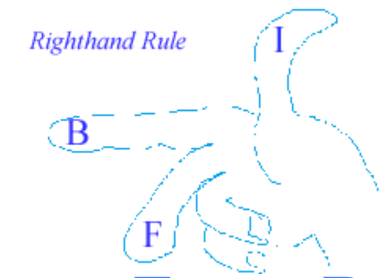

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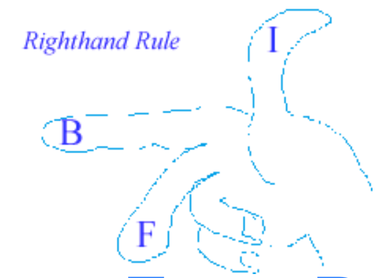

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$$m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\frac{\partial}{\partial \mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} \qquad \frac{d}{dt}\frac{\partial}{\partial \mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} = \frac{d}{dt}\left(-e\mathbf{A}\right) \qquad -\nabla(e\Phi - \mathbf{v}\bullet e\mathbf{A}) \qquad -e\frac{d\mathbf{A}}{dt}$$

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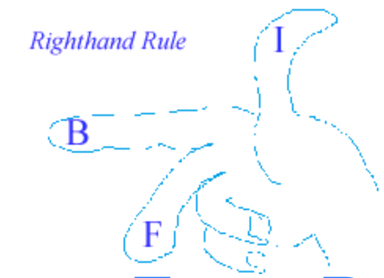

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Inserting Φ-term that $\partial_{\mathbf{v}}$ zeros :

$\left(\textit{This step requires that}: \frac{\partial}{\partial\mathbf{v}}\left(e\Phi\right) = 0\right)$ $\left(\textit{and}: \frac{\partial}{\partial\mathbf{v}}\left(\mathbf{v}\bullet e\mathbf{A}\right) = e\mathbf{A}\right)$

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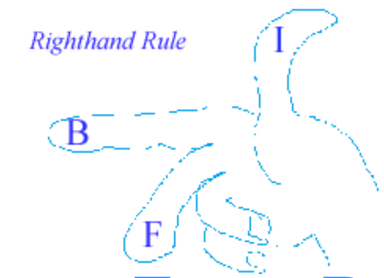

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$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})\right) = \frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})\right)$$

This step requires that :

$$\nabla\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right) \equiv \frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right) = 0$$

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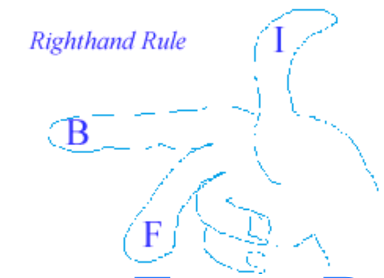

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$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi-\mathbf{v}\bullet e\mathbf{A}\right)=-e\frac{d\mathbf{A}}{dt}$$

$$\underbrace{\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}_{\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}}=\underbrace{\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-(e\Phi-\mathbf{v}\bullet e\mathbf{A})\right)}_{\frac{\partial L}{\partial\mathbf{r}}}$$

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right)=0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}}=\frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian for particle-in-(A,Φ)-potential

So-called *pondermotive* form for Newton's *F=ma* equation for a mass *m* of charge *e*.

electronic charge:
e =-1.602176·10-19Coulombs

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Electric field **E** and magnetic field **B**

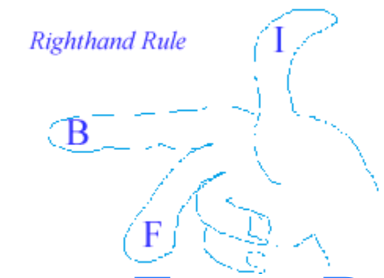
scalar potential field **Φ=Φ(r,*t*)**

vector potential field **A=A(r,*t*)**

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times\left(\nabla\times\mathbf{A}\right)\right]$$



$$\mathbf{F} = q\mathbf{v}\times\mathbf{B} = \mathbf{I}\times\mathbf{B}$$

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = e\left[-\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t} + \nabla(\mathbf{v}\bullet\mathbf{A}) - (\mathbf{v}\bullet\nabla)\mathbf{A}\right]$$

Chain rule expansion of vector potential total *t*-derivative:

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial x}\dot{x} + \frac{\partial\mathbf{A}}{\partial y}\dot{y} + \frac{\partial\mathbf{A}}{\partial z}\dot{z} + \frac{\partial\mathbf{A}}{\partial t} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\bullet\nabla)\mathbf{A}$$

$$m\frac{d\mathbf{v}}{dt} = e\left[-\nabla\Phi + \nabla(\mathbf{v}\bullet\mathbf{A}) - \frac{\partial\mathbf{A}}{\partial t} - (\mathbf{v}\bullet\nabla)\mathbf{A}\right] = e\left[-\nabla(\Phi - \mathbf{v}\bullet\mathbf{A}) - \frac{d\mathbf{A}}{dt}\right]$$

$$m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} = \frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) - \nabla\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right)$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\left(e\Phi - \mathbf{v}\bullet e\mathbf{A}\right) = -e\frac{d\mathbf{A}}{dt}$$

$$\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})\right)} = \frac{\partial}{\partial\mathbf{r}}\underbrace{\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - (e\Phi - \mathbf{v}\bullet e\mathbf{A})\right)}$$

$$\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\mathbf{v}} = \frac{\partial L}{\partial\mathbf{r}}$$

Lagrangian has a *linear* velocity term *e***v•A** in addition to the usual quadratic *KE=mv²/2* and *PE=e*Φ.

$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\mathbf{A}$,Φ)-potential

Lagrangian for particle-in-($\mathbf{A}$,Φ)-potential

Hamiltonian for particle-in-($\mathbf{A}$,Φ)-potential

→ *Canonical momentum in ($\mathbf{A}$,Φ) potential*

Hamiltonian formulation

Hamilton's equations

*Hamiltonian for particle-in-(**A**,Φ)-potential*

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*Canonical momentum in (**A**,Φ) potential*

Canonical momentum is defined by L's **v**-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\boldsymbol{\Phi}(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t)$$

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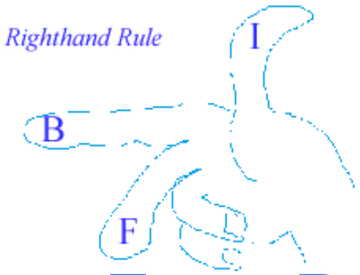

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

*Canonical momentum in (**A**,Φ) potential*

Canonical momentum is defined by L's $\mathbf{v}$-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right) = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - e\Phi(\mathbf{r},t)\right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

$$\mathbf{p} = m\mathbf{v} + e\mathbf{A}(\mathbf{r},t) \qquad = \quad m\mathbf{v} \qquad For\ \mathbf{A}(\mathbf{r},t)=0$$

Hamiltonian for particle-in-(A,Φ)-potential

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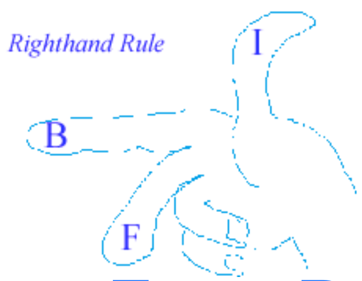

$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Canonical momentum in (A,Φ) potential

Canonical momentum is defined by L's $\mathbf{v}$-derivative

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right) = \frac{\partial}{\partial \mathbf{v}}\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - e\Phi(\mathbf{r},t)\right)_{For\ \mathbf{A}(\mathbf{r},t)=0}$$

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Lagrangian is usual form $L= T - V$ with electric (scalar) potential $V=\Phi(\mathbf{r},t)$
if magnetic (vector) potential $\mathbf{A}=\mathbf{A}(\mathbf{r},t)$ is zero everywhere.

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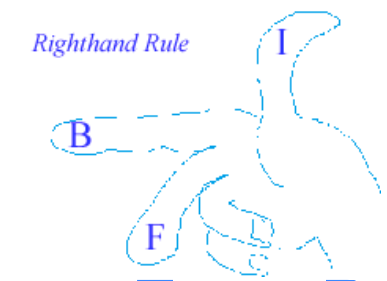

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Canonical momentum in ($\mathbf{A}$,Φ) potential

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Then canonical momentum is usual form: $\mathbf{p} = m\mathbf{v}$ $\qquad (For\ \mathbf{A}(\mathbf{r},t)=0)$

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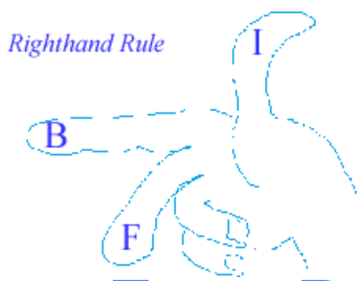
$$L = L(\mathbf{r},\mathbf{v},t) = \frac{1}{2}m\mathbf{v}\bullet\mathbf{v} - \left(e\Phi(\mathbf{r},t) - \mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)$$



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Then canonical momentum is usual form: $\mathbf{p} = m\mathbf{v}$ $\qquad (For\ \mathbf{A}(\mathbf{r},t)=0)$

Otherwise vector potential term $-\mathbf{v}\bullet e\mathbf{A}$ leads to an extraordinary *canonical momentum:* $\mathbf{p} = m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)$.
Particle momentum $m\mathbf{v}$ is not canonical, but related to *canonical* $\mathbf{p}$ as follows: $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

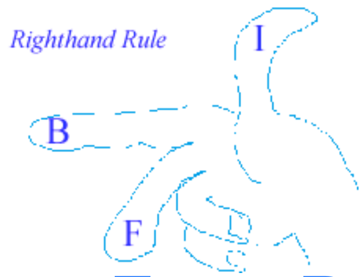
Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

Hamiltonian for particle-in-($\boldsymbol{A}$,Φ)-potential

- *Canonical momentum in ($\boldsymbol{A}$,Φ) potential*
- → *Hamiltonian formulation*
- *Hamilton's equations*

Hamiltonian for charged particle in fields

The Hamiltonian function of the Legendre-Poincare form is the following.



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

$$H=\sum_{\mu}\dot{q}^{\mu}p_{\mu}-L=\mathbf{v}\bullet\mathbf{p}-L=\mathbf{v}\bullet\left(m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)\right)-\left(\frac{1}{2}m\mathbf{v}\bullet\mathbf{v}-\left(e\Phi(\mathbf{r},t)-\mathbf{v}\bullet e\mathbf{A}(\mathbf{r},t)\right)\right)$$

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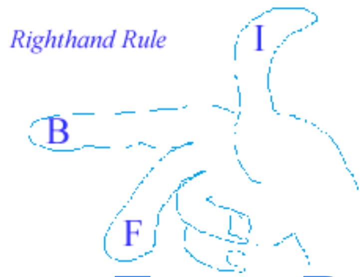

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(Only correct numerically!)

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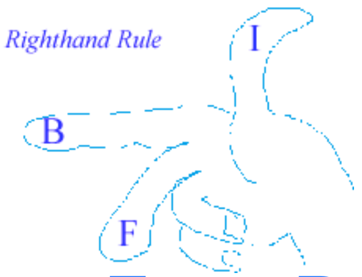

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Hamiltonian for charged particle in fields

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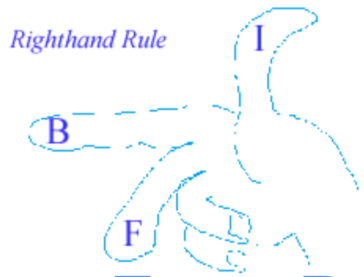

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Hamiltonian for charged particle in fields

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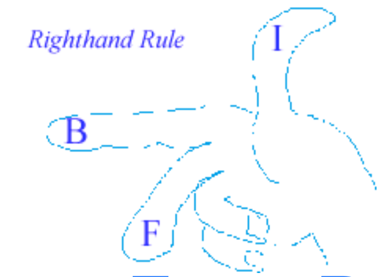

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(Correct formally and numerically)

Hamiltonian for charged particle in fields

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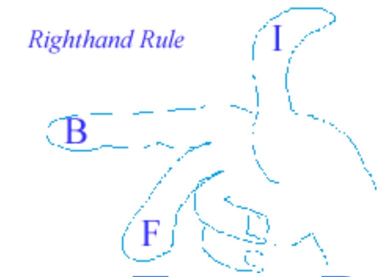

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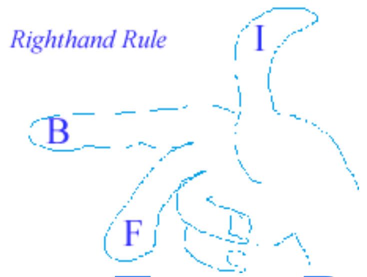
(Correct formally and numerically)

$$H=\frac{\mathbf{p}\bullet\mathbf{p}}{2m}-\frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A}+\mathbf{A}\bullet\mathbf{p}\right)+\frac{e^{2}}{2m}\mathbf{A}\bullet\mathbf{A}+e\Phi(\mathbf{r},t)$$



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Charge mechanics in electromagnetic fields

Vector analysis for particle-in-($\boldsymbol{A}$,Φ)-potential

Lagrangian for particle-in-($\boldsymbol{A}$,Φ)-potential

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- *Hamiltonian formulation*
- → *Hamilton's equations*

Hamilton's equations for charged particle in fields

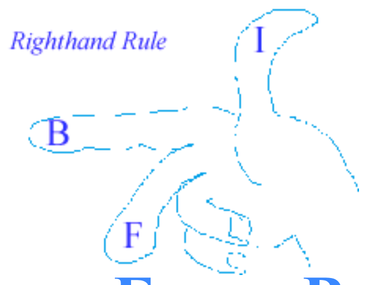
Hamiltonian is explicit function of ***momentum*** **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) \bullet \left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) + e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

$$H = \frac{\mathbf{p} \bullet \mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p} \bullet \mathbf{A} + \mathbf{A} \bullet \mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A} \bullet \mathbf{A} + e\Phi(\mathbf{r},t)$$

(Expanded)



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Hamiltonian for charged particle in fields

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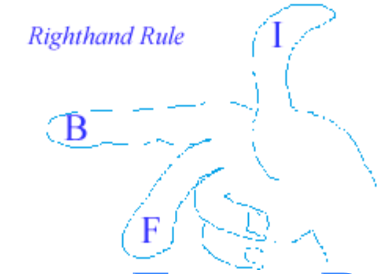
(Correct formally and numerically)

$$H = \frac{\mathbf{p}\bullet\mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A} + \mathbf{A}\bullet\mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A} + e\Phi(\mathbf{r},t)$$

(Expanded)



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

Hamilton's **v** equation: $$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$$

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) + e\Phi(\mathbf{r},t)$$ (Correct formally and numerically)

$$H = \frac{\mathbf{p}\bullet\mathbf{p}}{2m} - \frac{e}{2m}\left(\mathbf{p}\bullet\mathbf{A} + \mathbf{A}\bullet\mathbf{p}\right) + \frac{e^2}{2m}\mathbf{A}\bullet\mathbf{A} + e\Phi(\mathbf{r},t)$$ *(Expanded)*

Hamilton's **v** equation: $$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$$ *(Just copies particle velocity relation.)*

Hamilton's equations for charged particle in fields

Hamiltonian is explicit function of *momentum* **p**. Must replace velocity **v** using $m\mathbf{v}=\mathbf{p} - e\mathbf{A}(\mathbf{r},t)$.

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right)\bullet\left(\mathbf{p} - e\mathbf{A}(\mathbf{r},t)\right) + e\Phi(\mathbf{r},t)$$

(Correct formally and numerically)

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Hamilton's **v** equation: $\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p} - e\mathbf{A}(\mathbf{r},t)}{m}$ *(Just copies particle velocity relation.)*

Hamilton's $d\mathbf{p}/dt$ equation:
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$m\mathbf{v}+e\mathbf{A}(\mathbf{r},t)=\mathbf{p}$ ------------------

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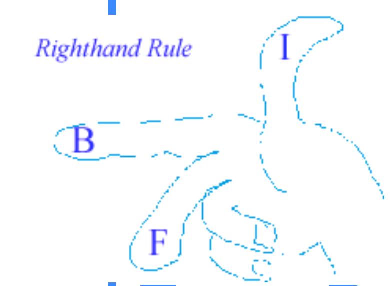
(Expanded)



$\mathbf{F}=q\mathbf{v}\times\mathbf{B}=\mathbf{I}\times\mathbf{B}$

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$$\mathbf{B} = \nabla\times\mathbf{A}$$

...and now

we come back

full circle...

$\mathbf{v}\times(\nabla\times\mathbf{A}) = \mathbf{v}\bullet(\nabla\mathbf{A}) - (\mathbf{v}\bullet\nabla)\mathbf{A}$ *for particle mechanics*

Crossed E and B field mechanics

→ *Classical Hall-effect and cyclotron orbit orbit equations*

Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

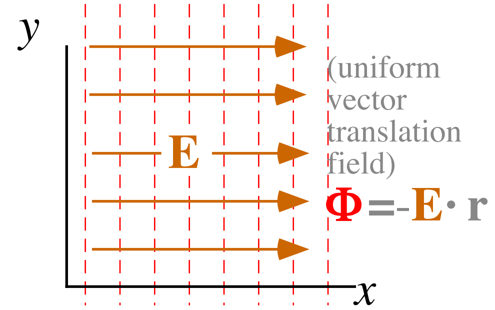
Practical poolhall application

Crossed E and B field mechanics

A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\Phi(\mathbf{r}) = -\mathbf{E} \bullet \mathbf{r}, \qquad -\nabla\Phi(\mathbf{r}) = -\nabla\left(-\mathbf{E} \bullet \mathbf{r}\right) = \mathbf{E} = const.$$

Fig. 2.4.1.



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A constant **B** field has a vector potential field **A** that resembles a disc spinning counter-clockwise around the **B** axis.

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{B}\times\mathbf{r}, \qquad \nabla\times\mathbf{A}(\mathbf{r}) = \nabla\times\left(\frac{1}{2}\mathbf{B}\times\mathbf{r}\right) = \mathbf{B} = const.$$

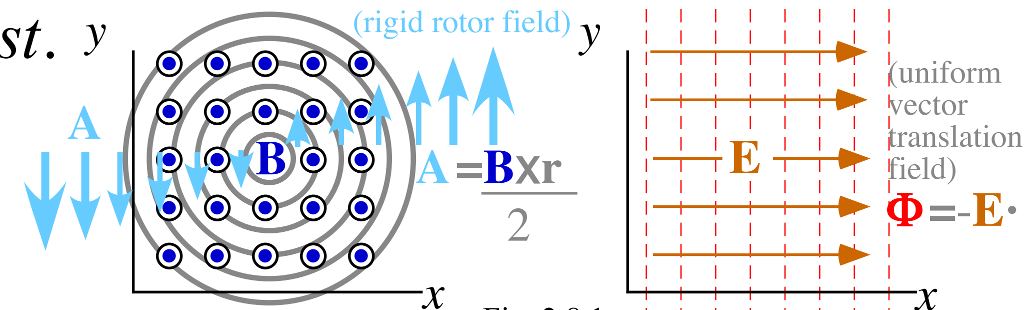

Fig. 2.8.1.

*This mechanical analog of (E_x,B_z) field mimics **A**-field with tabletop **v**-field*

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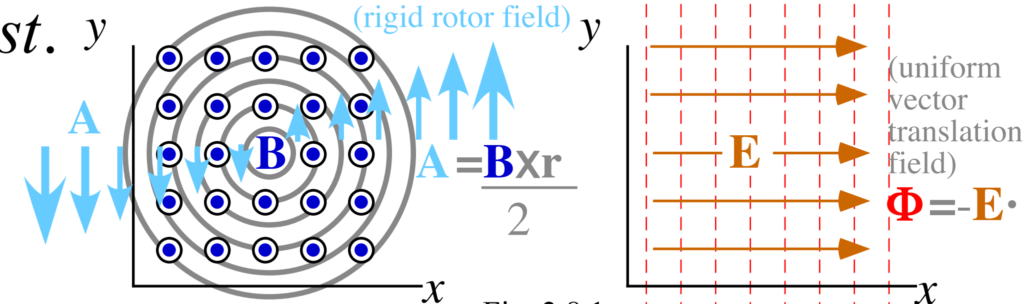
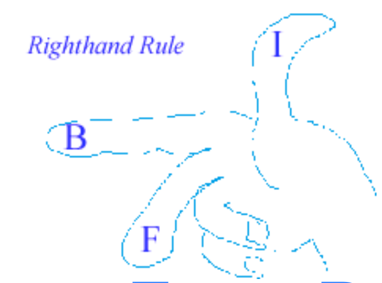


Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B}.$$



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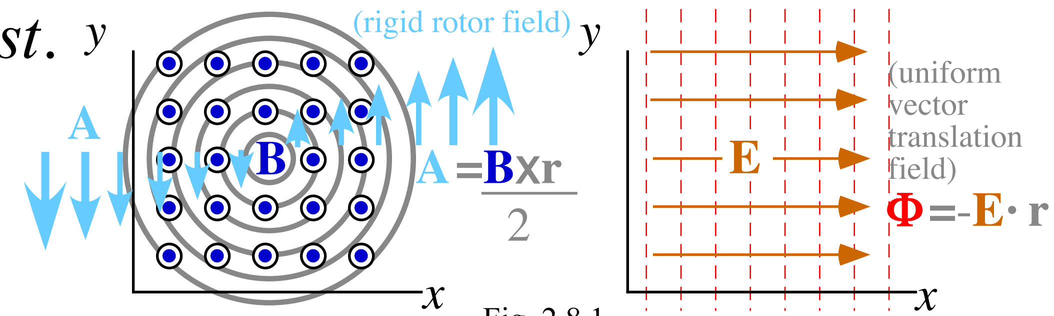

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$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

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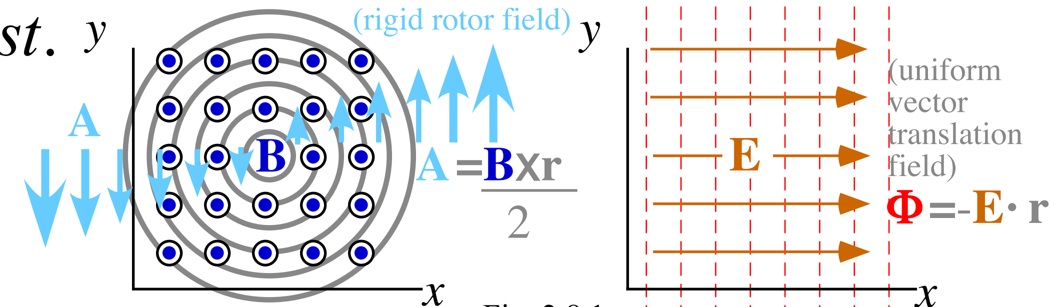

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Shorthand Labeling

Gibb's notation:

$$\dot{\mathbf{v}} \quad = \quad \boldsymbol{\varepsilon} \quad + \quad \mathbf{v} \quad \times B\hat{\mathbf{e}}_\mathbf{z}$$

$$\dot{v}_x\hat{\mathbf{e}}_\mathbf{x} + \dot{v}_y\hat{\mathbf{e}}_\mathbf{y} = \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} + (v_x\hat{\mathbf{e}}_\mathbf{x} + v_y\hat{\mathbf{e}}_\mathbf{y})\times B\hat{\mathbf{e}}_\mathbf{z}$$

$$= \varepsilon_x\hat{\mathbf{e}}_\mathbf{x} + \varepsilon_y\hat{\mathbf{e}}_\mathbf{y} - Bv_x\hat{\mathbf{e}}_\mathbf{y} + Bv_y\hat{\mathbf{e}}_\mathbf{x}$$

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Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit equations

→ *Vector theory vs. complex variable theory*

Mechanical analog of cyclotron and FBI rule

Cycloid geometry and flying sticks

Practical poolhall application

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A constant **E** field has a scalar potential field **Φ** with constant gradient.

$$\boldsymbol{\Phi}(\mathbf{r}) = -\mathbf{E} \bullet \mathbf{r}, \qquad -\nabla\boldsymbol{\Phi}(\mathbf{r}) = -\nabla\left(-\mathbf{E}\bullet\mathbf{r}\right) = \mathbf{E} = const.$$

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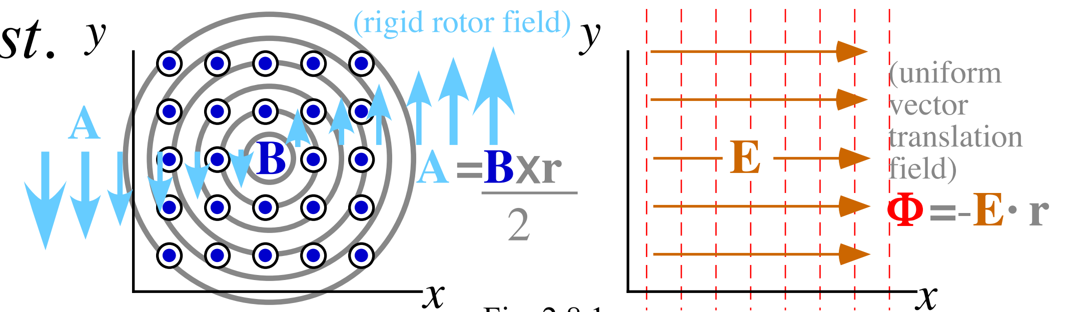

Fig. 2.8.1.

Newtonian electromagnetic equations of motion: $m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

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Shorthand Labeling

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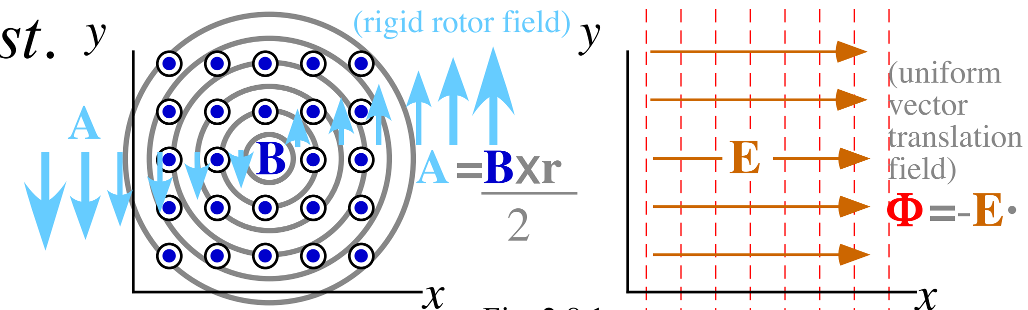

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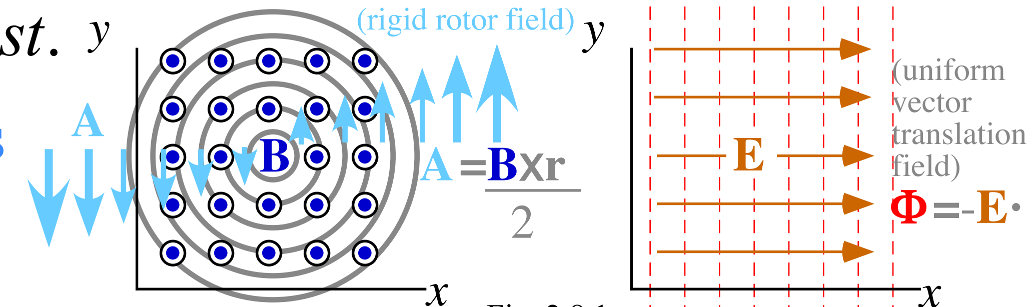

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Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

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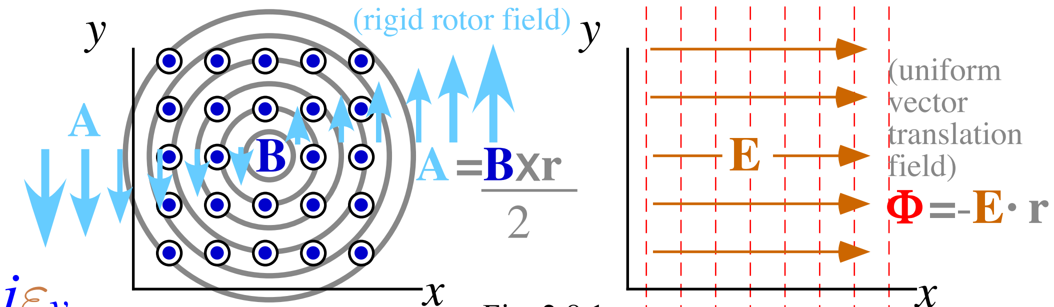

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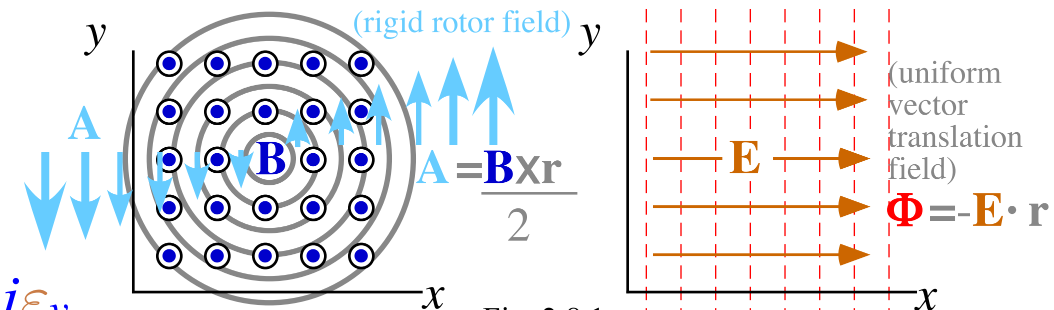

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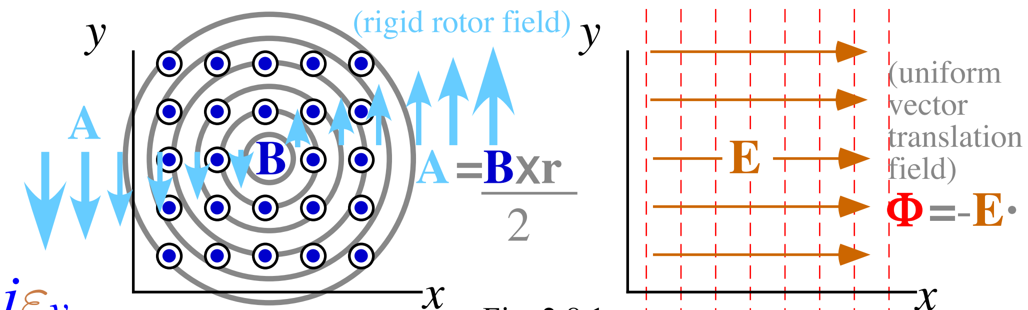

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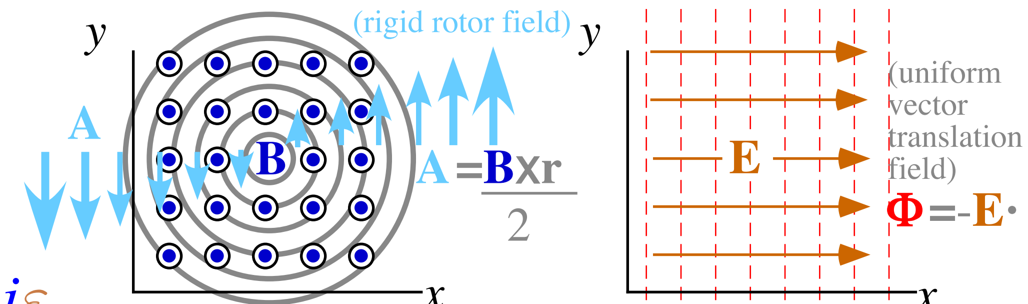

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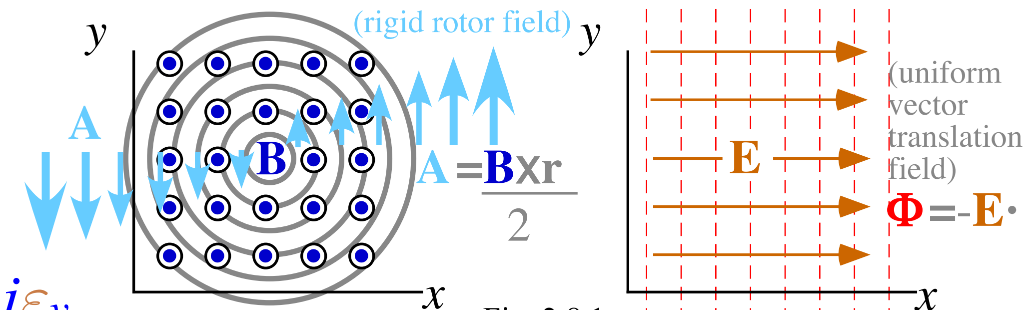

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Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} v_x(0) - \frac{\varepsilon_y}{B} \\ v_y(0) + \frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

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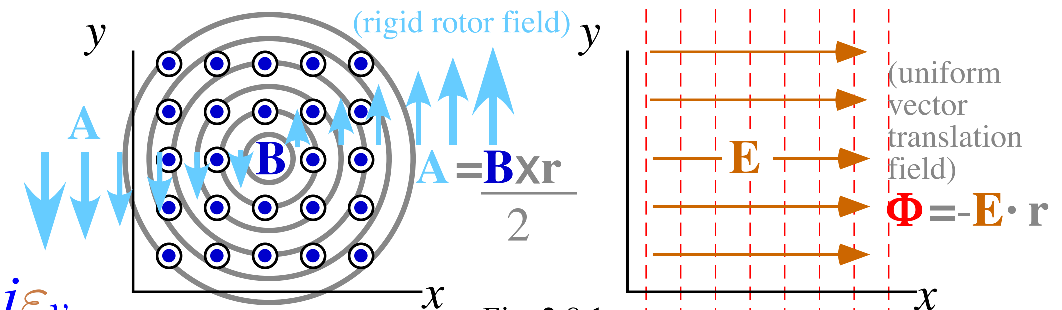

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Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

Crossed E and B field mechanics (Solution by complex variables)

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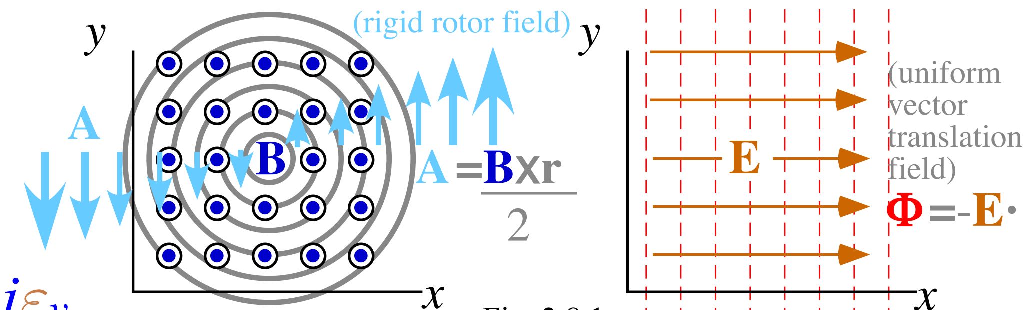

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Expanding e^{-iBt}, $v=v_x+iv_y$, and $\varepsilon=\varepsilon_x+i\varepsilon_y$ reveals x (Real) and y (Imaginary) components

$$\begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} v_x(0)-\frac{\varepsilon_y}{B} \\ v_y(0)+\frac{\varepsilon_x}{B} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B} \\ -\frac{\varepsilon_x}{B} \end{pmatrix}$$

vector form

Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.

$$q(t) = \int v(t)\,dt = \frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}\cdot t + Const. \quad \text{where: } Const. = q(0) - (\frac{v(0)}{-iB} - \frac{\varepsilon}{B^2})$$

complex form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \boldsymbol{\varepsilon} + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

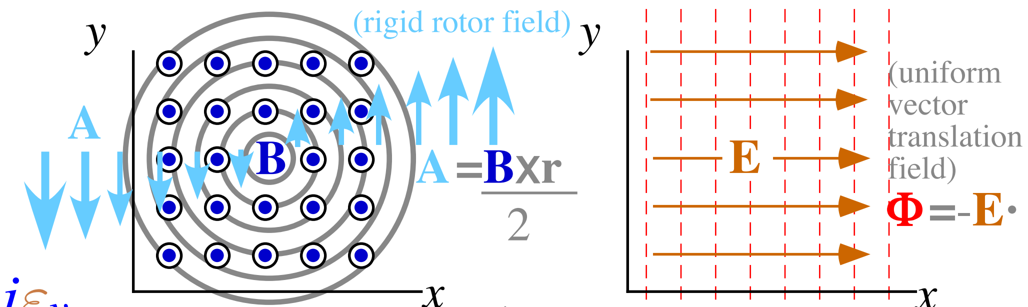

Fig. 2.8.1.

Complex variable velocity: $v=v_x+iv_y$ *and electric field:* $\varepsilon=\varepsilon_x+i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements : } \hat{\mathbf{e}}_\mathbf{x} \to 1 \quad \text{and : } \hat{\mathbf{e}}_\mathbf{y} \to i = \sqrt{-1}$$

A velocity transformation $V(t)=v(t)+\beta$ cancels constant ε-field to give an equation: $\dot{V}=(const.)V$

$$\dot{V}(t)=\dot{v}(t)+\dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t)-\beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t)+\beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0)+\beta) \quad \text{or:} \quad v(t)= e^{-iB\cdot t}(v(0)+\beta)-\beta = e^{-iB\cdot t}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}$$

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$$x(t)+iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{\mathbf{v}} = \frac{e}{m}\mathbf{E} + \mathbf{v}\times\frac{e}{m}\mathbf{B} = \varepsilon + \mathbf{v}\times\frac{e}{m}B\hat{\mathbf{e}}_Z$$

$$\varepsilon_x = \frac{e}{m}E_x \qquad \varepsilon_y = \frac{e}{m}E_y \qquad B = \frac{e}{m}B_z$$

Shorthand Labeling

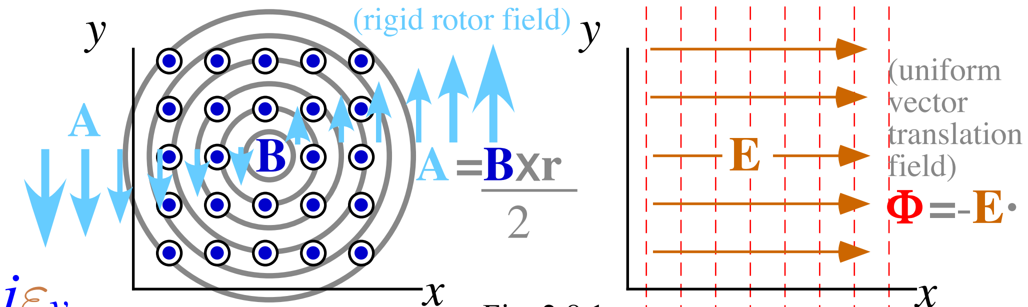

Fig. 2.8.1.

Complex variable velocity: $v = v_x + iv_y$ *and electric field:* $\varepsilon = \varepsilon_x + i\varepsilon_y$

$$\dot{v}_x + i\dot{v}_y = \varepsilon_x + i\varepsilon_y - iBv_x + Bv_y = \varepsilon_x + i\varepsilon_y - iB(v_x + iv_y)$$

$$\dot{v} = \varepsilon - iBv \qquad \text{with replacements} : \hat{\mathbf{e}}_{\mathbf{x}} \rightarrow 1 \quad \text{and} : \hat{\mathbf{e}}_{\mathbf{y}} \rightarrow i = \sqrt{-1}$$

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$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where} : \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

An exponential $V(t)=e^{-iBt}V(0)$ solution results: e^{-iBt} is a clockwise 2D rotation.

complex form

$$v(t) + \beta = V(t) = e^{-iB\cdot t}V(0) = e^{-iB\cdot t}(v(0) + \beta) \quad \text{or:} \quad v(t) = e^{-iB\cdot t}(v(0) + \beta) - \beta = e^{-iB\cdot t}(v(0) + i\frac{\varepsilon}{B}) - i\frac{\varepsilon}{B}$$

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complex form

$$x(t) + iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

Move last part of this calculation UP↑

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where : } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

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complex form

$$x(t) + iy(t) = e^{-iB\cdot t}(i\frac{v(0)}{B} - \frac{\varepsilon}{B^2}) - i\frac{\varepsilon}{B}\cdot t + x(0) + iy(0) - i\frac{v(0)}{B} + \frac{\varepsilon}{B^2}$$

complex form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix} \begin{pmatrix} -\frac{v_y(0)}{B} - \frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B} - \frac{\varepsilon_y}{B^2} \end{pmatrix} + \begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix} + \begin{pmatrix} x(0) + \frac{v_y(0)}{B} + \frac{\varepsilon_x}{B^2} \\ y(0) - \frac{v_x(0)}{B} + \frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form

Crossed E and B field mechanics (Solution by complex variables)

$$\dot{V}(t) = \dot{v}(t) + \dot{\beta} = \varepsilon - iBv = \varepsilon - iB(V(t) - \beta) = -iBV(t) \qquad \text{where: } \beta = -\frac{\varepsilon}{iB} = i\frac{\varepsilon}{B}$$

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complex form

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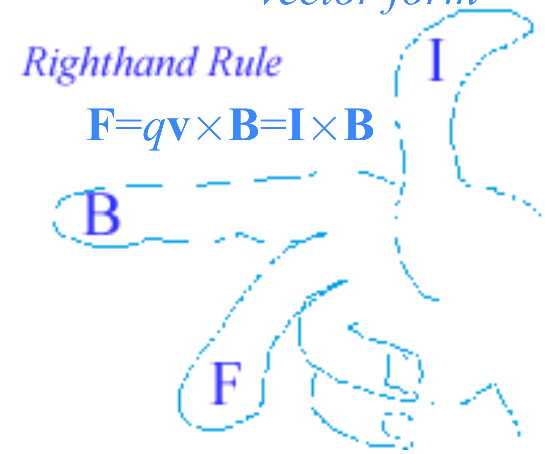
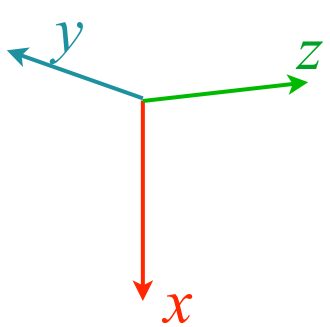
vector form





Crossed E and B field mechanics (Solution by complex variables)

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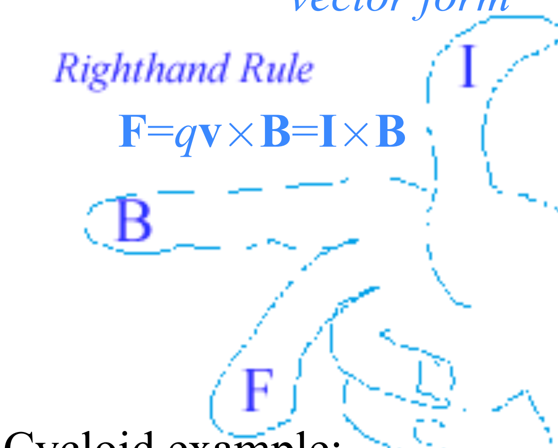
Integrating $v(t)$ yields complex coordinate $q=x+iy$ affected by both ε_x and ε_y.



$$q(t)=\int v(t)\,dt=\frac{e^{-iB\cdot t}}{-iB}(v(0)+i\frac{\varepsilon}{B})-i\frac{\varepsilon}{B}\cdot t+Const. \quad \text{where: } Const.=q(0)-(\frac{v(0)}{-iB}-\frac{\varepsilon}{B^2})$$

complex form

$$x(t)+iy(t)=e^{-iB\cdot t}(i\frac{v(0)}{B}-\frac{\varepsilon}{B^2})-i\frac{\varepsilon}{B}\cdot t+x(0)+iy(0)-i\frac{v(0)}{B}+\frac{\varepsilon}{B^2}$$

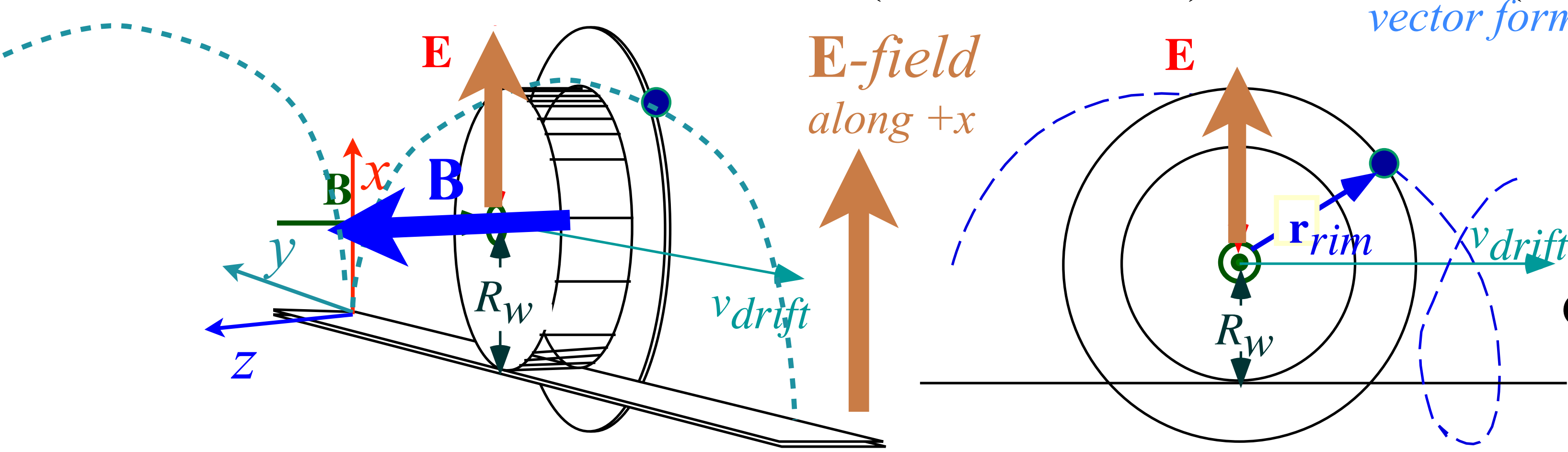
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{v_y(0)}{B}-\frac{\varepsilon_x}{B^2} \\ \frac{v_x(0)}{B}-\frac{\varepsilon_y}{B^2} \end{pmatrix}+\begin{pmatrix} \frac{\varepsilon_y}{B}t \\ -\frac{\varepsilon_x}{B}t \end{pmatrix}+\begin{pmatrix} x(0)+\frac{v_y(0)}{B}+\frac{\varepsilon_x}{B^2} \\ y(0)-\frac{v_x(0)}{B}+\frac{\varepsilon_y}{B^2} \end{pmatrix}$$

vector form

Cycloid example:
initial $(x(0), y(0))=(0,0)$
and $(v_x(0), v_y(0))=(0,0)$

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}=$ is on rim of a wheel of radius $R_W=E/B^2$

$$\begin{pmatrix} \cos B\cdot t & \sin B\cdot t \\ -\sin B\cdot t & \cos B\cdot t \end{pmatrix}\begin{pmatrix} -\frac{E}{B^2} \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ -\frac{E}{B}t \end{pmatrix}+\begin{pmatrix} \frac{E}{B^2} \\ 0 \end{pmatrix}$$



Constant v_{drift} in -y-direction

$v_{drift}=-E/B$

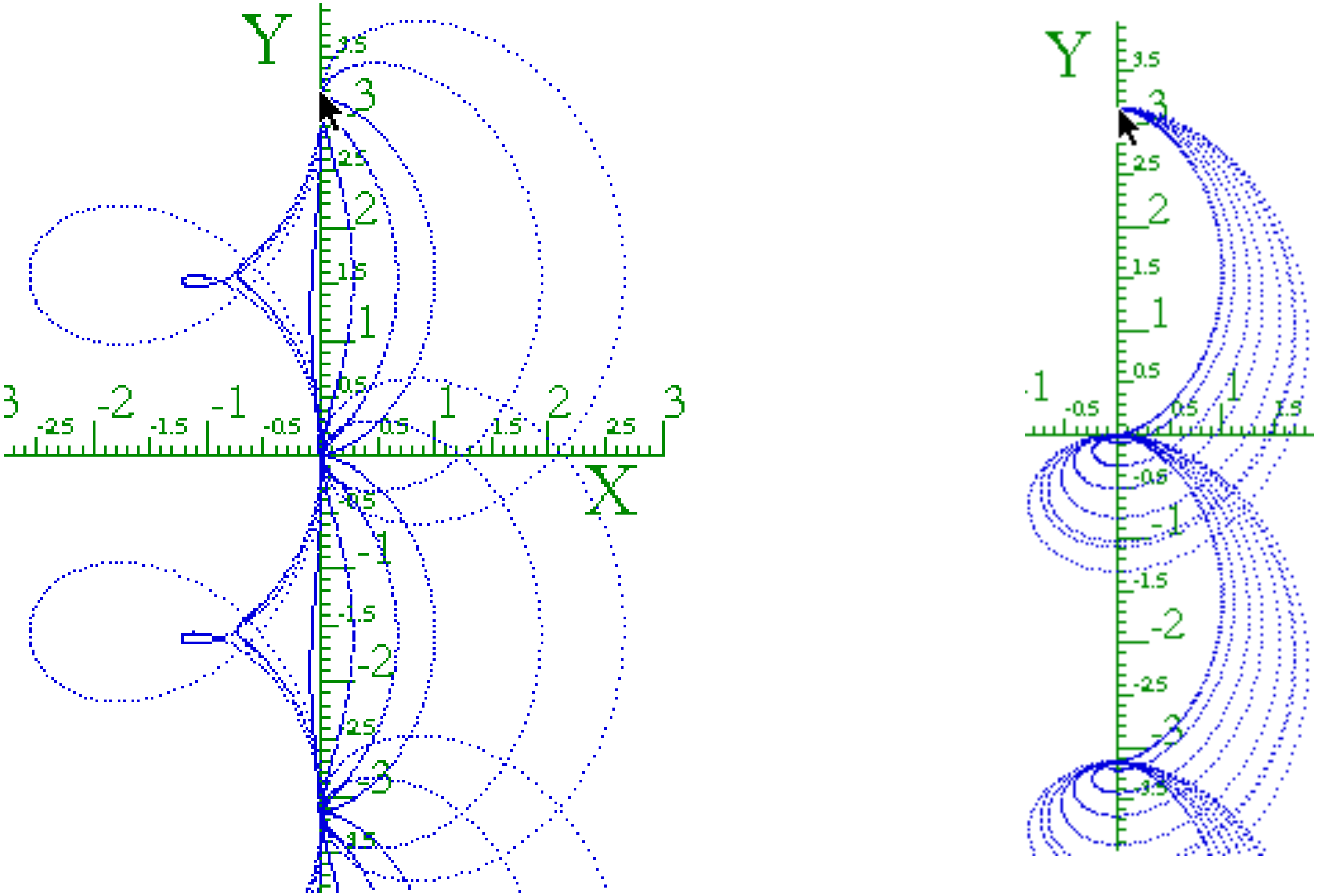


Fig. 2.8.2 Trajectories of unit charge and mass in magnetic and electric fields (E=1/2, B=1)

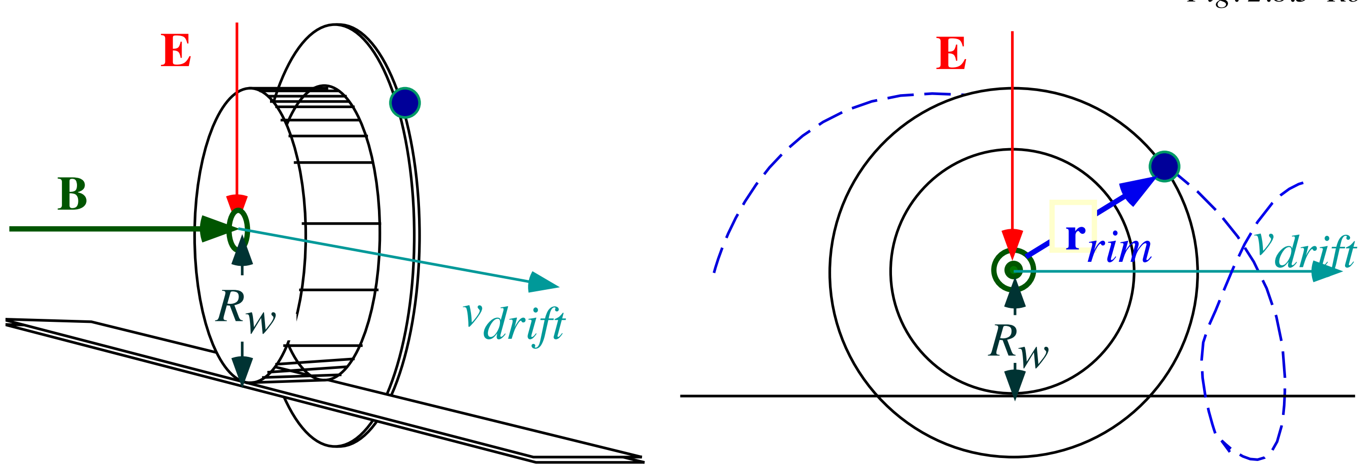
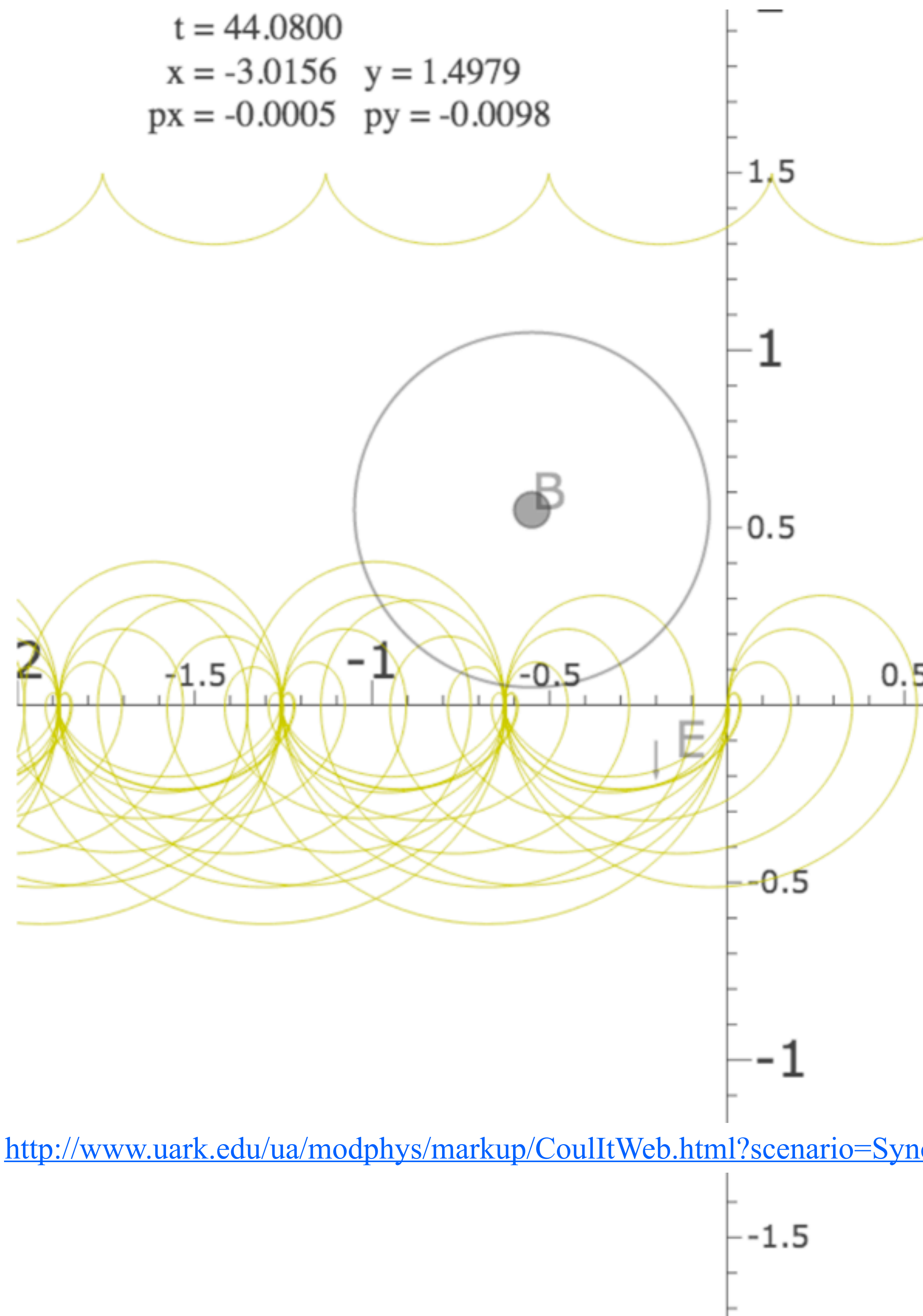


Fig. 2.8.3 Rolling railroad wheel and rim analogy for cyclotron orbits

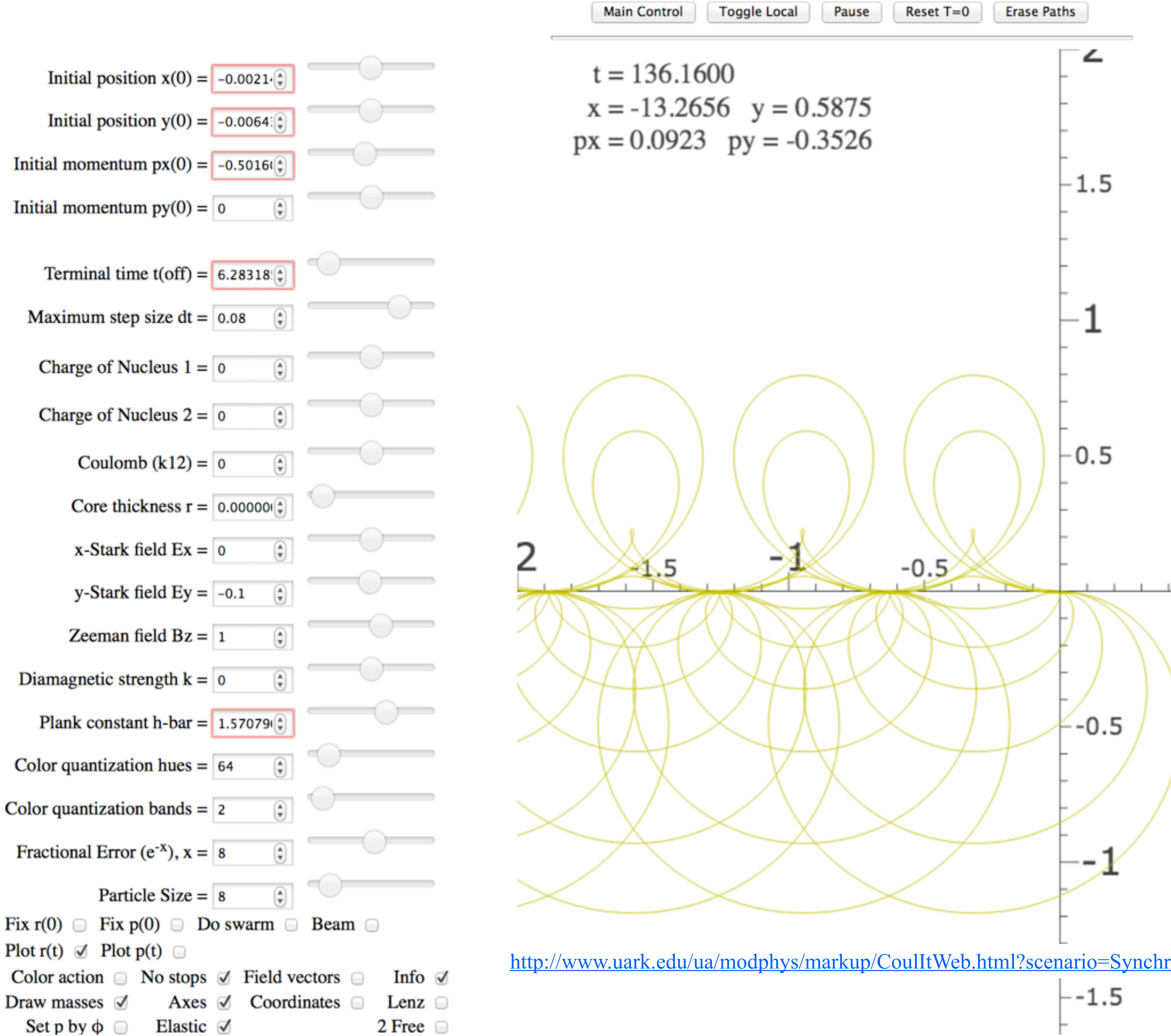
Initial position x(0) = 1.382631
Initial position y(0) = 1.49839
Initial momentum px(0) = 0
Initial momentum py(0) = 0

Terminal time t(off) = 6.28318
Maximum step size dt = 0.08
Charge of Nucleus 1 = 0
Charge of Nucleus 2 = 0
Coulomb (k12) = 0
Core thickness r = 0.000001
x-Stark field Ex = 0
y-Stark field Ey = -0.1
Zeeman field Bz = 1
Diamagnetic strength k = 0
Plank constant h-bar = 1.57079
Color quantization hues = 64
Color quantization bands = 2
Fractional Error (e^{-x}), x = 8
Particle Size = 8

Fix r(0) ☐ Fix p(0) ☐ Do swarm ☐ Beam ☐
Plot r(t) ☑ Plot p(t) ☐
Color action ☐ No stops ☑ Field vectors ☑ Info ☑
Draw masses ☑ Axes ☑ Coordinates ☐ Lenz ☐
Set p by ϕ ☐ Elastic ☑ 2 Free ☐
Save to GIF ☐



http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=SynchrotronMotion



http://www.uark.edu/ua/modphys/markup/CoulItWeb.html?scenario=SynchrotronMotion2

Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbit equations

Vector theory vs. complex variable theory

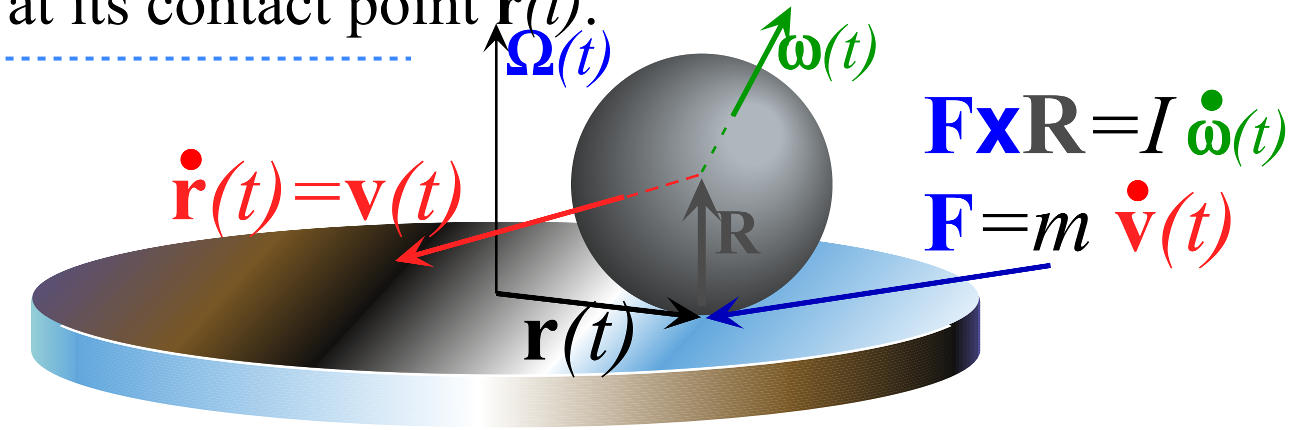
→ *Mechanical analog of cyclotron and FBI rule*

Cycloid geometry and flying sticks

Practical poolhall application

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

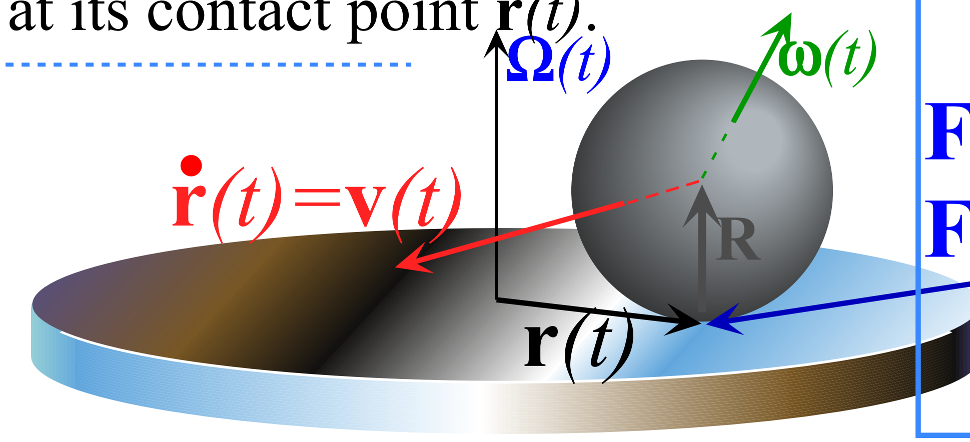


Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

YouTube Video of Analog to Syncrotron Motion

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.



Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

Equations of Motion:

rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$\mathbf{F}\mathbf{x}\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$

$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

translation $Force=\mathbf{F} = m\dot{\mathbf{v}}$

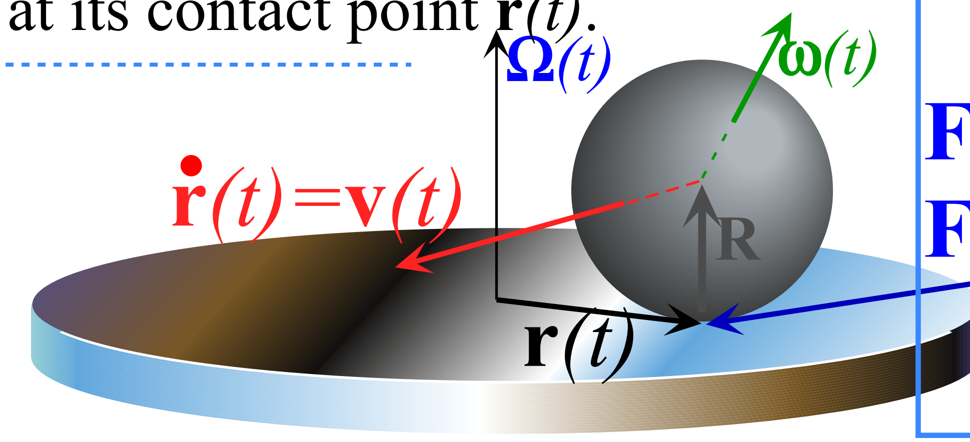
*Torque-and-*F=ma
equations of motion:

$$
\begin{aligned}
I\dot{\boldsymbol{\omega}}(t) &= \mathbf{F}(t)\times\mathbf{R} \\
&= m\dot{\mathbf{v}}(t)\times\mathbf{R} \\
&= m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R
\end{aligned}
$$

Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Rolling Constraint



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$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$

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Turntable turning at constant angular velocity $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$.

No-slipping: $\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}=\boldsymbol{\Omega}\times\mathbf{r}(t)$ *(where:* $\mathbf{R}=R\hat{\mathbf{z}}$ and $\boldsymbol{\Omega}=\Omega\hat{\mathbf{z}}$ *are constant.)*

*Torque-and-*F=ma
equations of motion:

$$I\dot{\boldsymbol{\omega}}(t)=\mathbf{F}(t)\times\mathbf{R}$$
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Mechanical analog of cyclotron and FBI rule

Velocity vector of the ball contact point ($\mathbf{v}(t)-\boldsymbol{\omega}(t)\times\mathbf{R}$) equals table surface velocity $\boldsymbol{\Omega}\times\mathbf{r}(t)$ at its contact point $\mathbf{r}(t)$.

Equations of Motion:

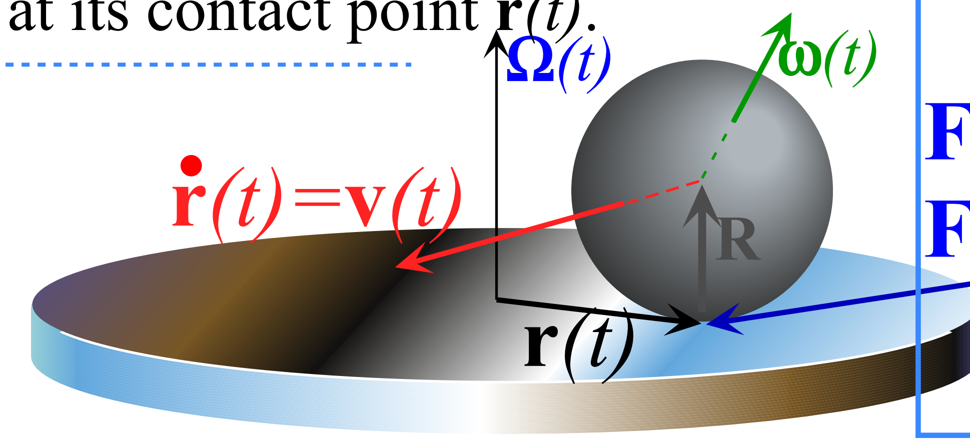
rotation $Torque = \mathbf{F}\times\mathbf{R} = I\dot{\boldsymbol{\omega}}$

$$\mathbf{F}\times\mathbf{R}=I\,\dot{\boldsymbol{\omega}}(t)$$

$$\mathbf{F}=m\,\dot{\mathbf{v}}(t)$$

translation $Force = \mathbf{F} = m\dot{\mathbf{v}}$

Rolling Constraint



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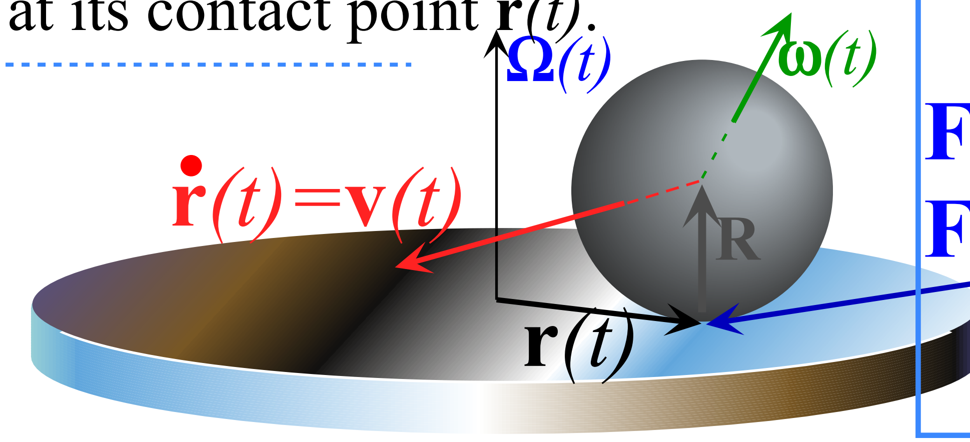

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Do time-derivative

$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\dot{\mathbf{r}}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R=\boldsymbol{\Omega}\times\mathbf{v}(t)+\dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

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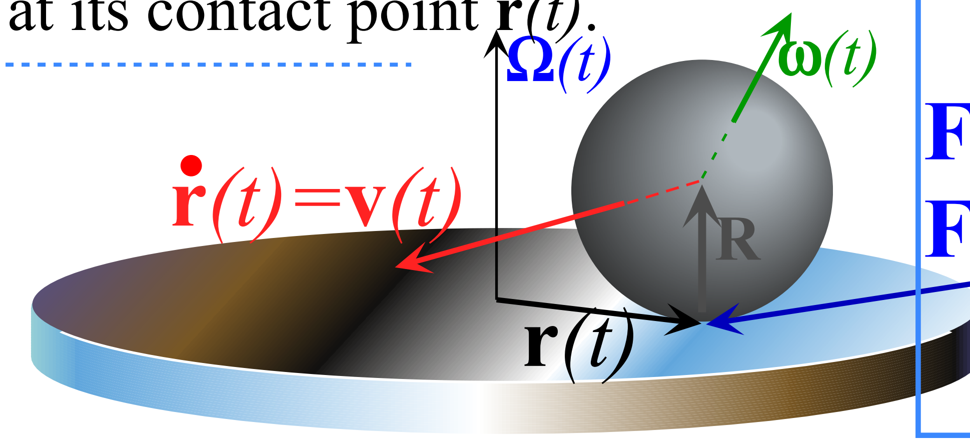

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$$\dot{\mathbf{v}}(t) = \boldsymbol{\Omega}\times\dot{\mathbf{r}}(t) + \dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R = \boldsymbol{\Omega}\times\mathbf{v}(t) + \dot{\boldsymbol{\omega}}(t)\times\hat{\mathbf{z}}R$$

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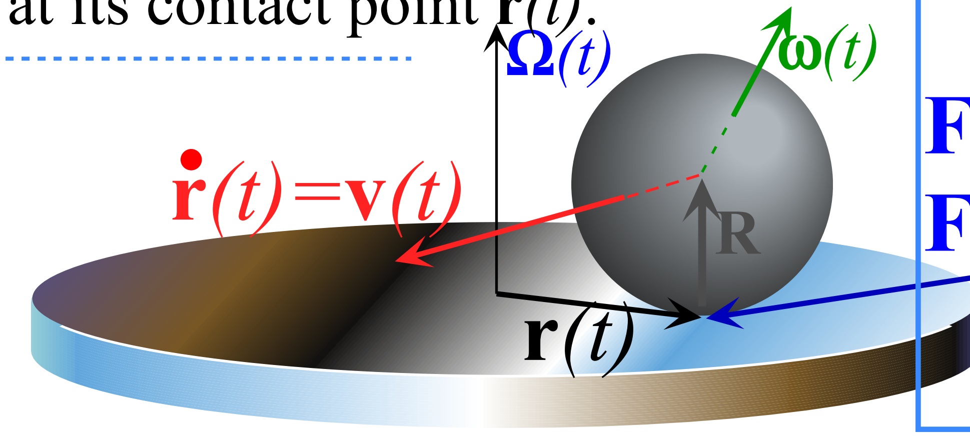

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Torque-and-F=ma equations of motion:

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use: $$\dot{\boldsymbol{\omega}}(t)=\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}$$

$$=m\dot{\mathbf{v}}(t)\times\mathbf{R}$$

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$$\dot{\mathbf{v}}(t)=\boldsymbol{\Omega}\times\mathbf{v}(t)+\frac{m\dot{\mathbf{v}}(t)\times\hat{\mathbf{z}}R}{I}\times\hat{\mathbf{z}}R$$

use: $(\mathbf{B}\times\mathbf{C})\times\mathbf{A}=(\mathbf{A}\bullet\mathbf{B})\mathbf{C}-(\mathbf{A}\bullet\mathbf{C})\mathbf{B}$

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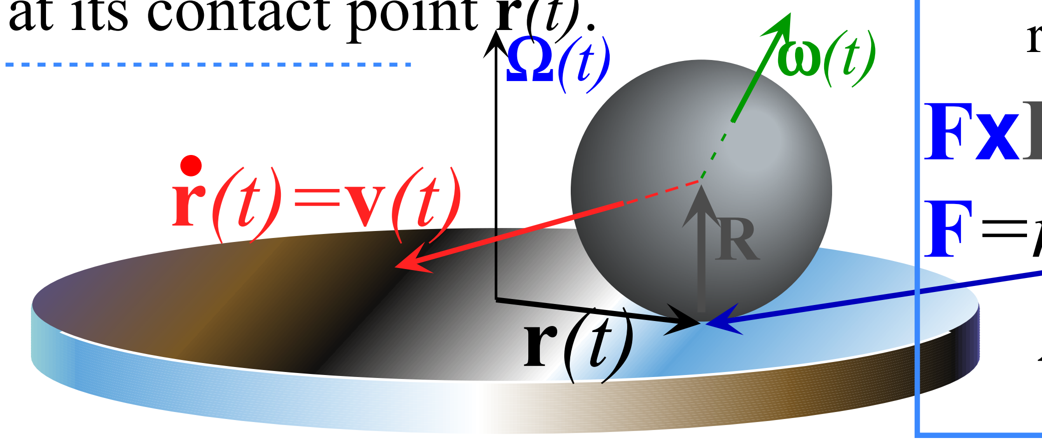

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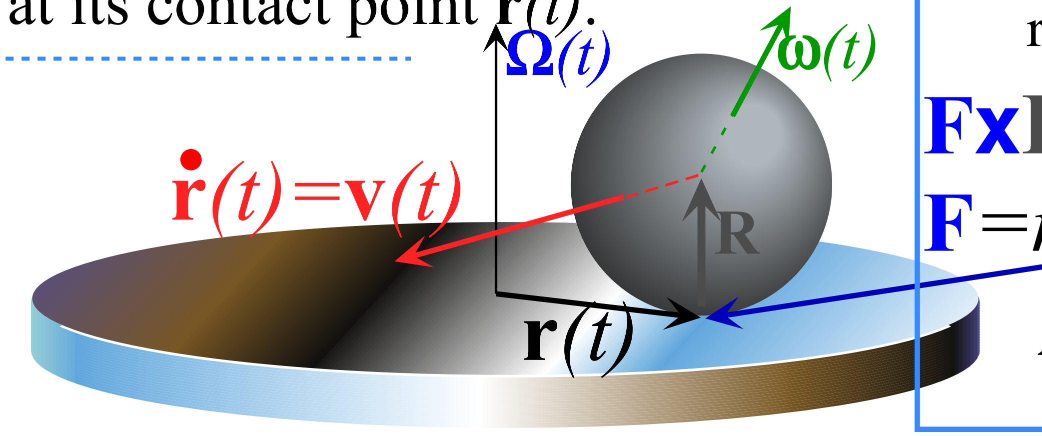

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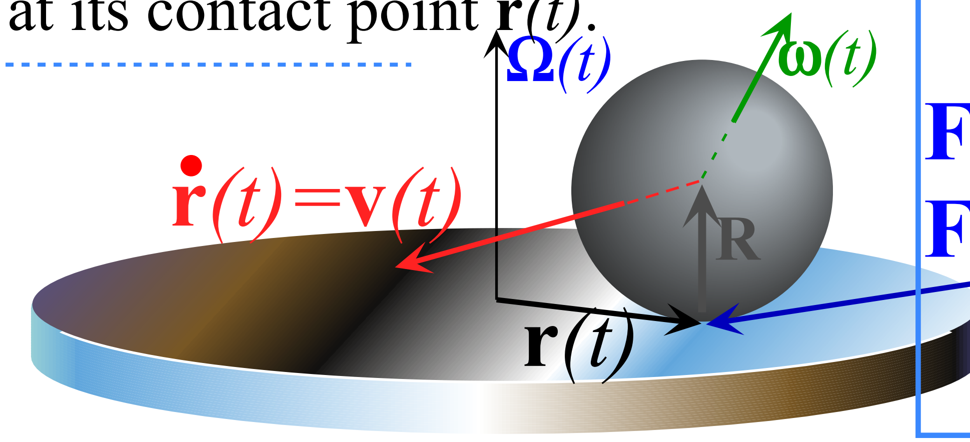

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or :

F=B×v *mechanical analog:*

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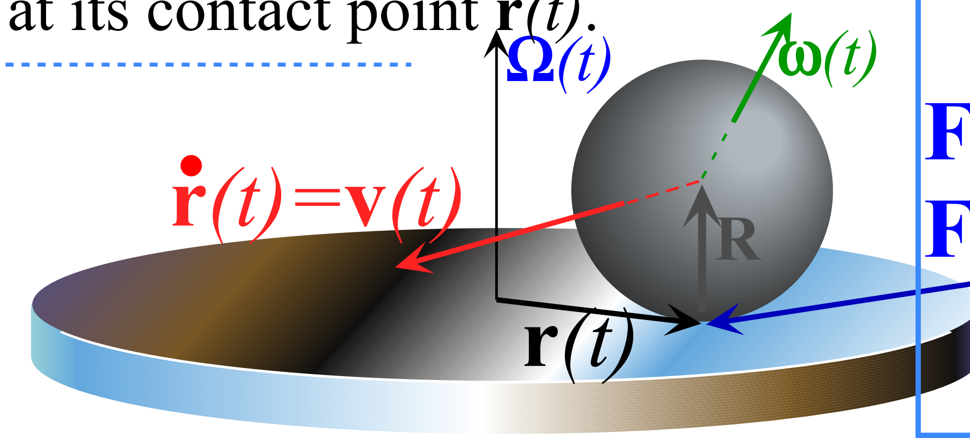

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or :

$m\mathbf{a}=e\mathbf{B}\times\mathbf{v}$ mechanical analog:

$$\dot{\mathbf{v}}(t)=\frac{\boldsymbol{\Omega}}{1+\frac{mR^2}{I}}\times\mathbf{v}(t)$$

Mechanical analog cyclotron frequency

$$\omega=\frac{e}{m}B=\frac{\boldsymbol{\Omega}}{1+\frac{mR^2}{I}}$$

$\omega=\frac{2}{7}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2}=\frac{2}{5}$

$=\frac{2}{5}\boldsymbol{\Omega}$ for: $\frac{I}{mR^2}=\frac{2}{3}$

[YouTube Video of Analog to Syncrotron Motion]

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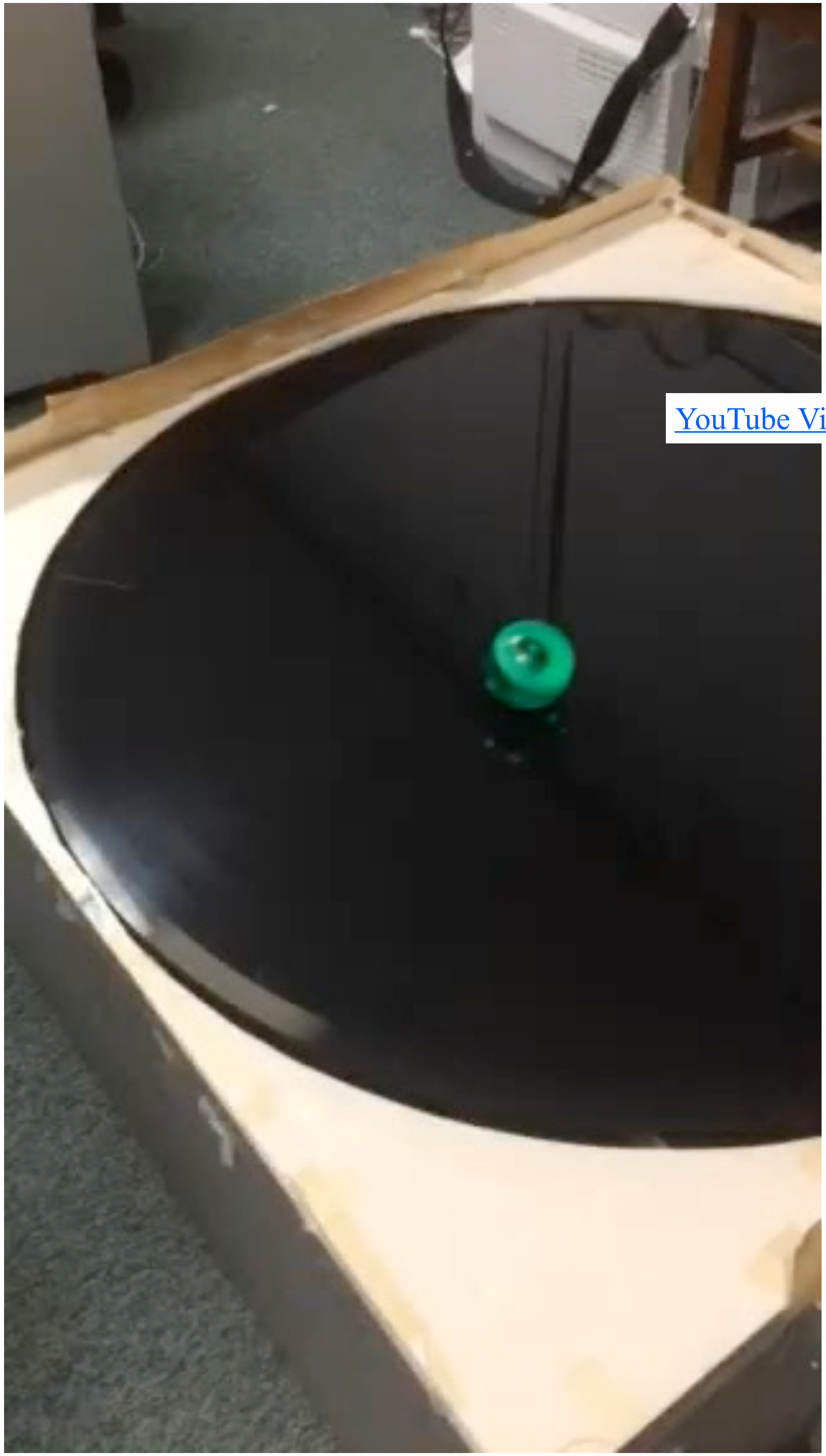

Solid ball has 2 orbits as table turns 7 rotations

Mechanical analog cyclotron frequency

$$\omega = \frac{e}{m} B = \frac{\boldsymbol{\Omega}}{1 + \frac{mR^2}{I}}$$

$$\omega = \frac{2}{7}\boldsymbol{\Omega} \text{ for: } \frac{I}{mR^2} = \frac{2}{5}$$

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Crossed E and B field mechanics

Classical Hall-effect and cyclotron orbits

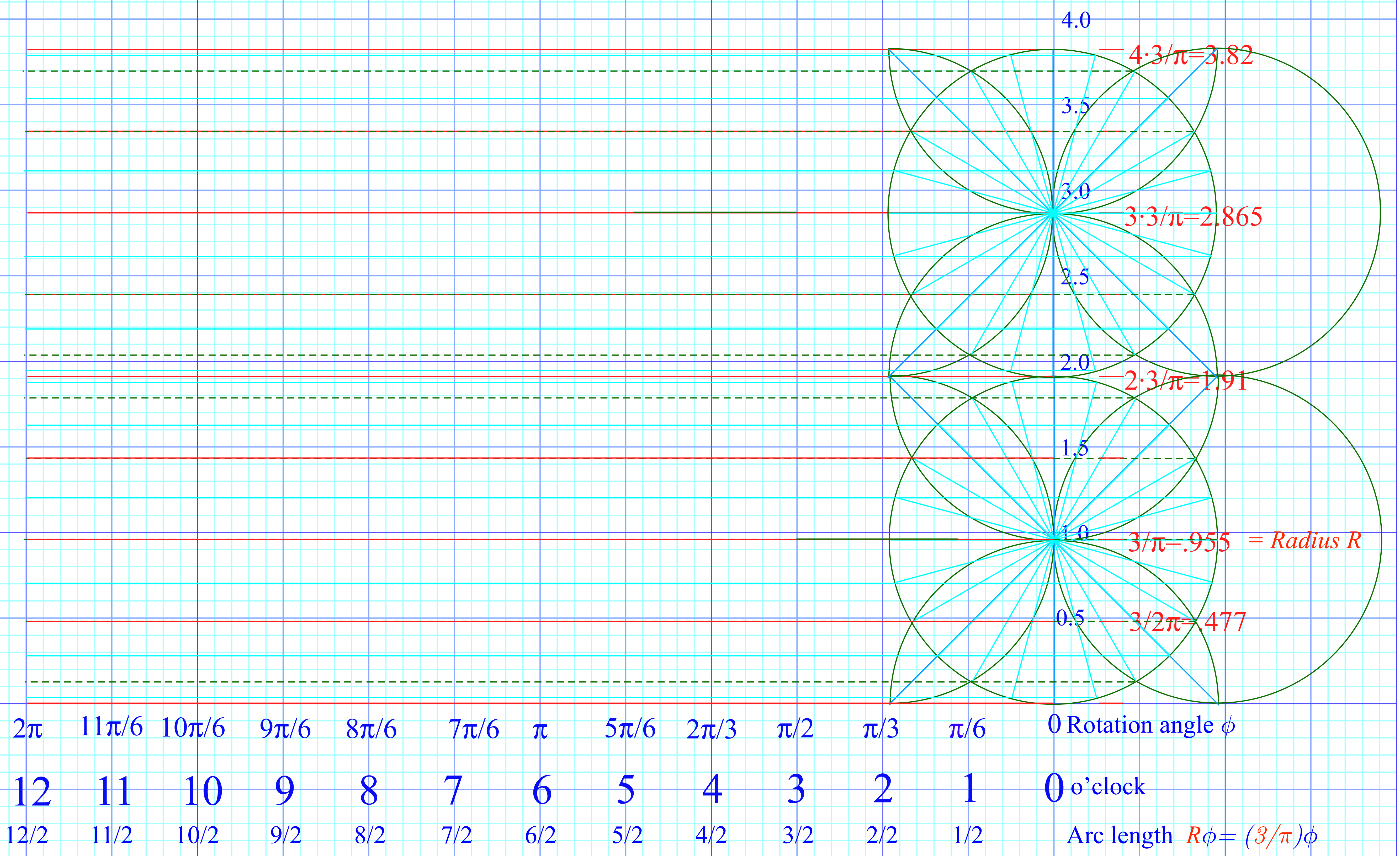
Vector theory vs. complex variable theory

Mechanical analog of cyclotron and FBI rule

→ *Cycloid geometry and flying sticks*

Practical poolhall application

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi= m\pi/n$ is a rational length of rolled -out circumference $R\phi= (3/\pi)m\pi/n=3m/n$.



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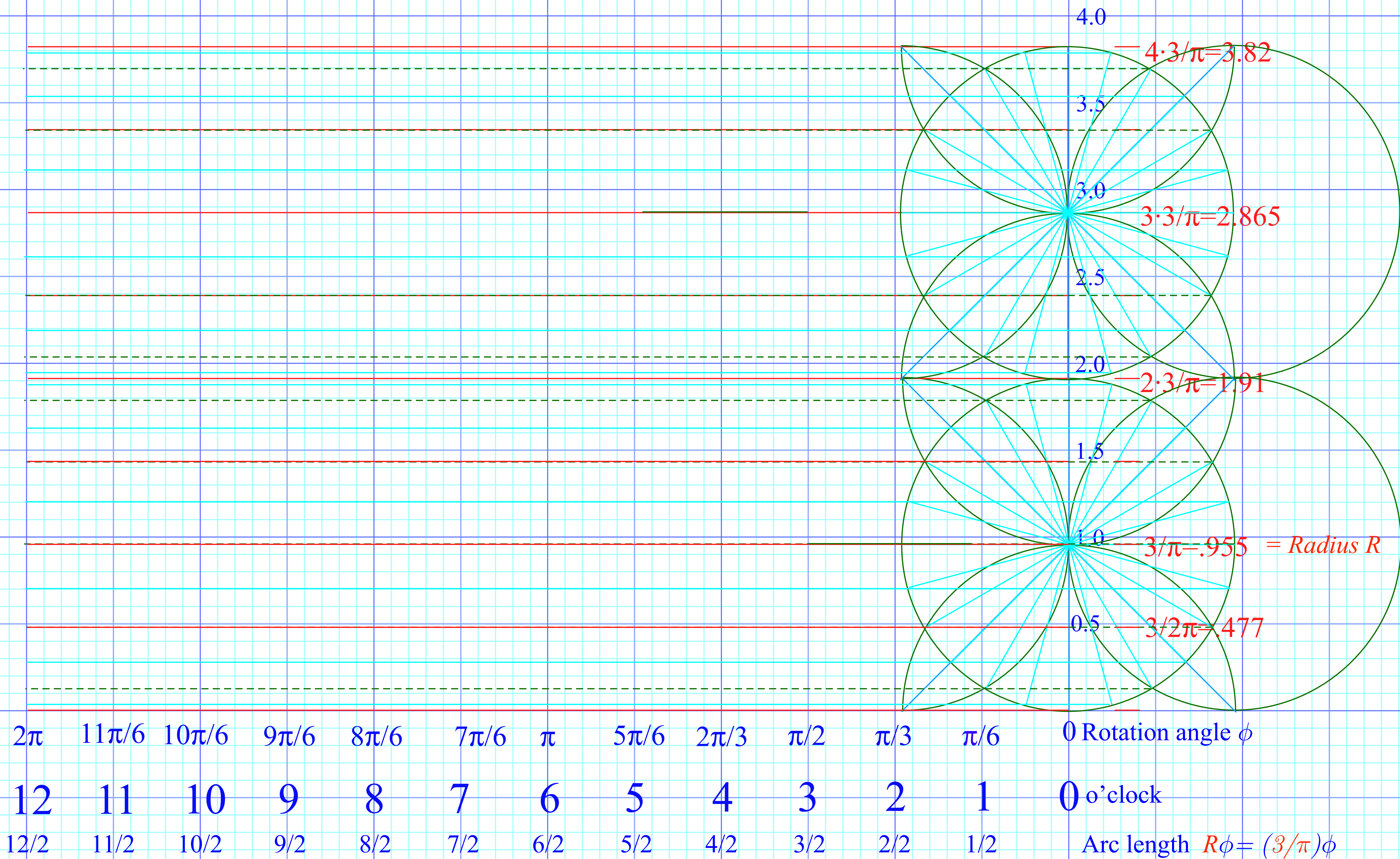

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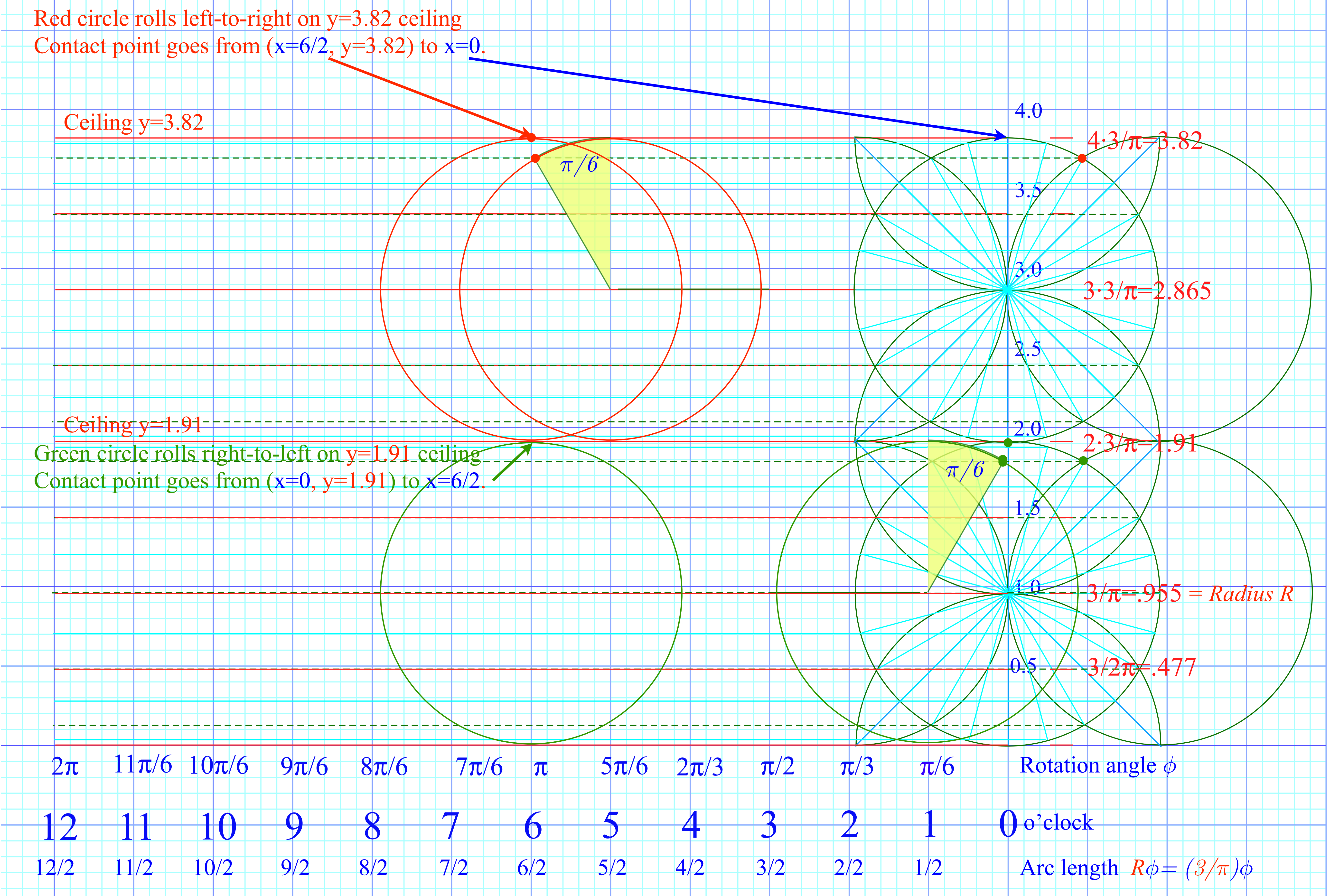

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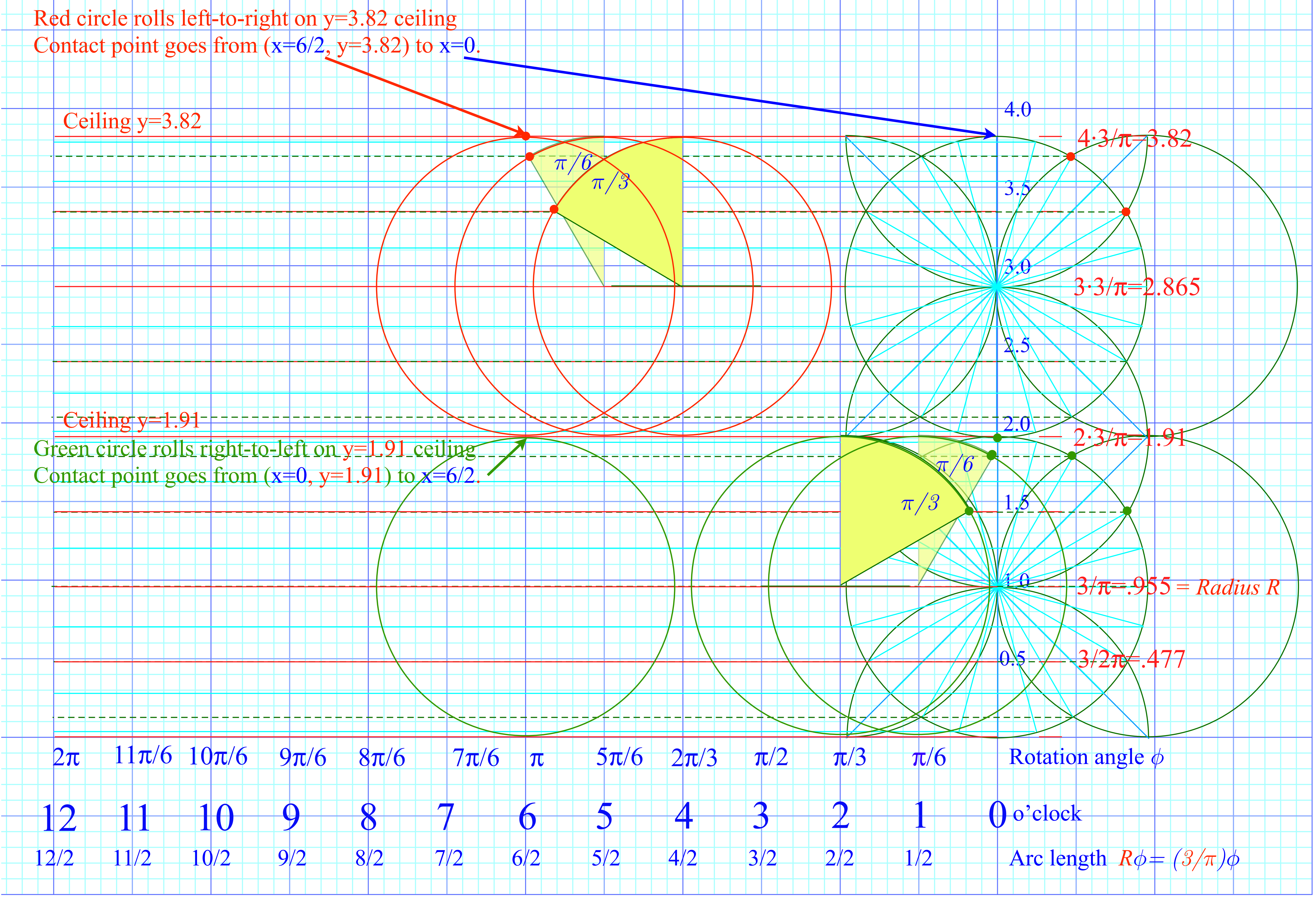

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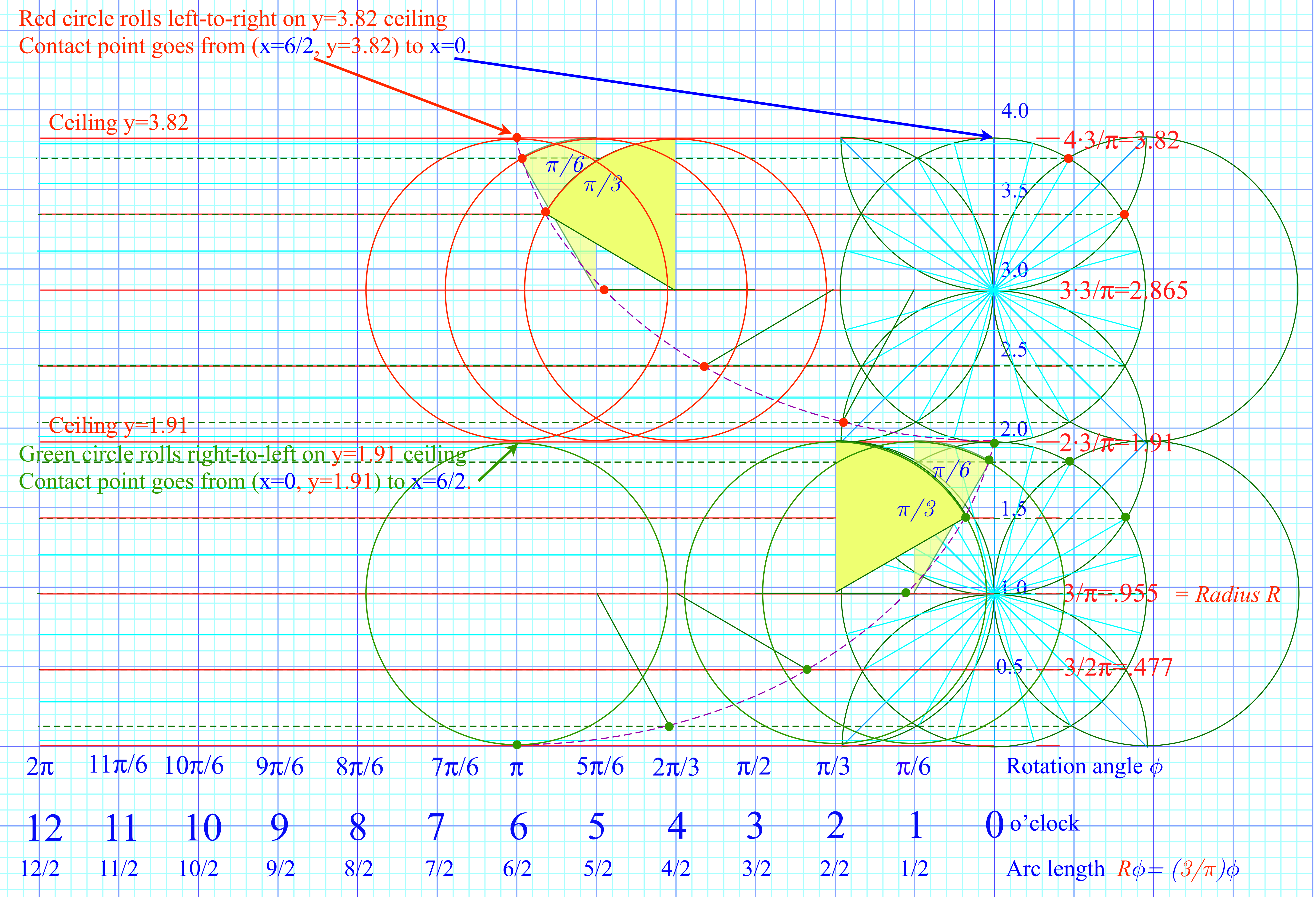

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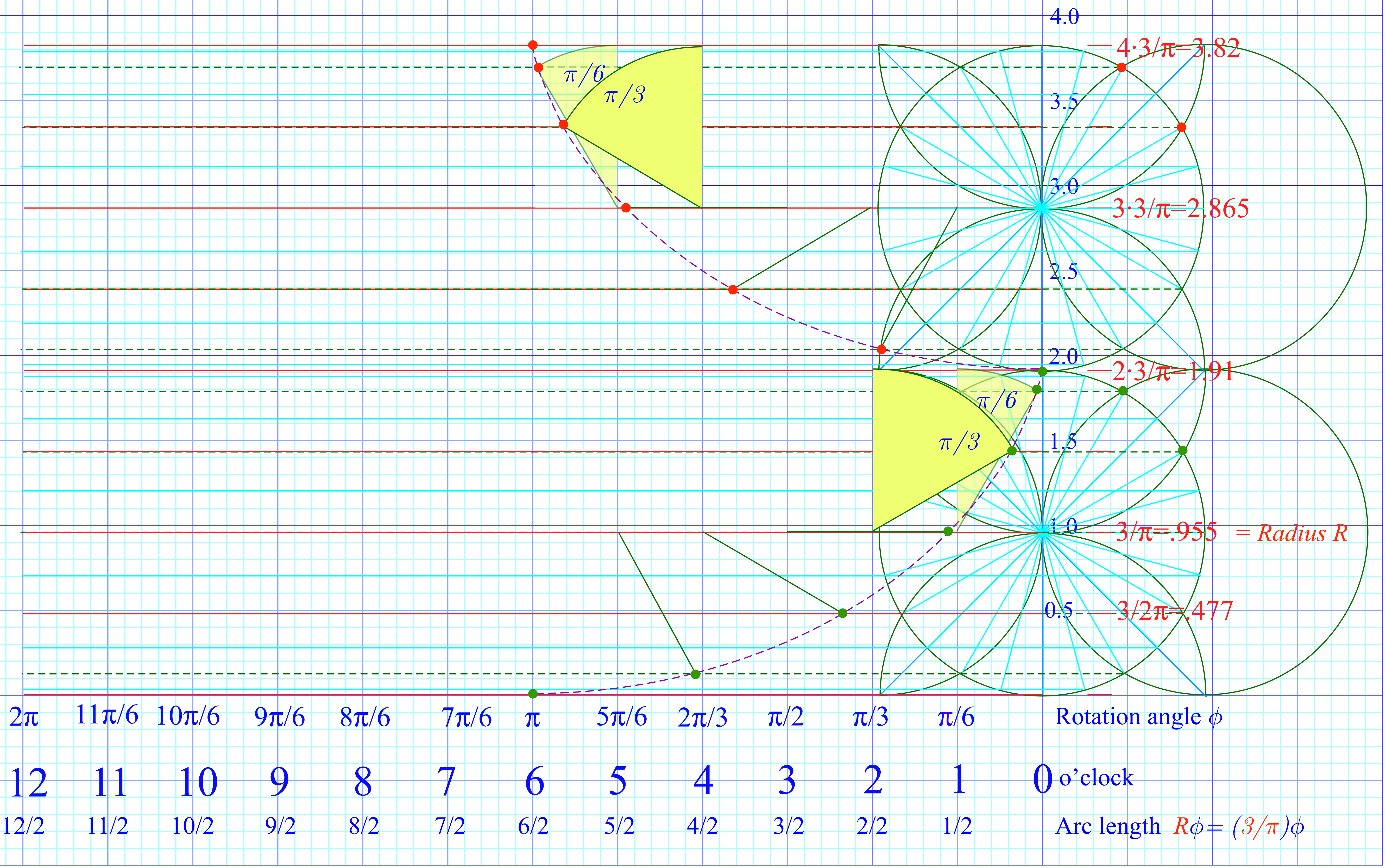

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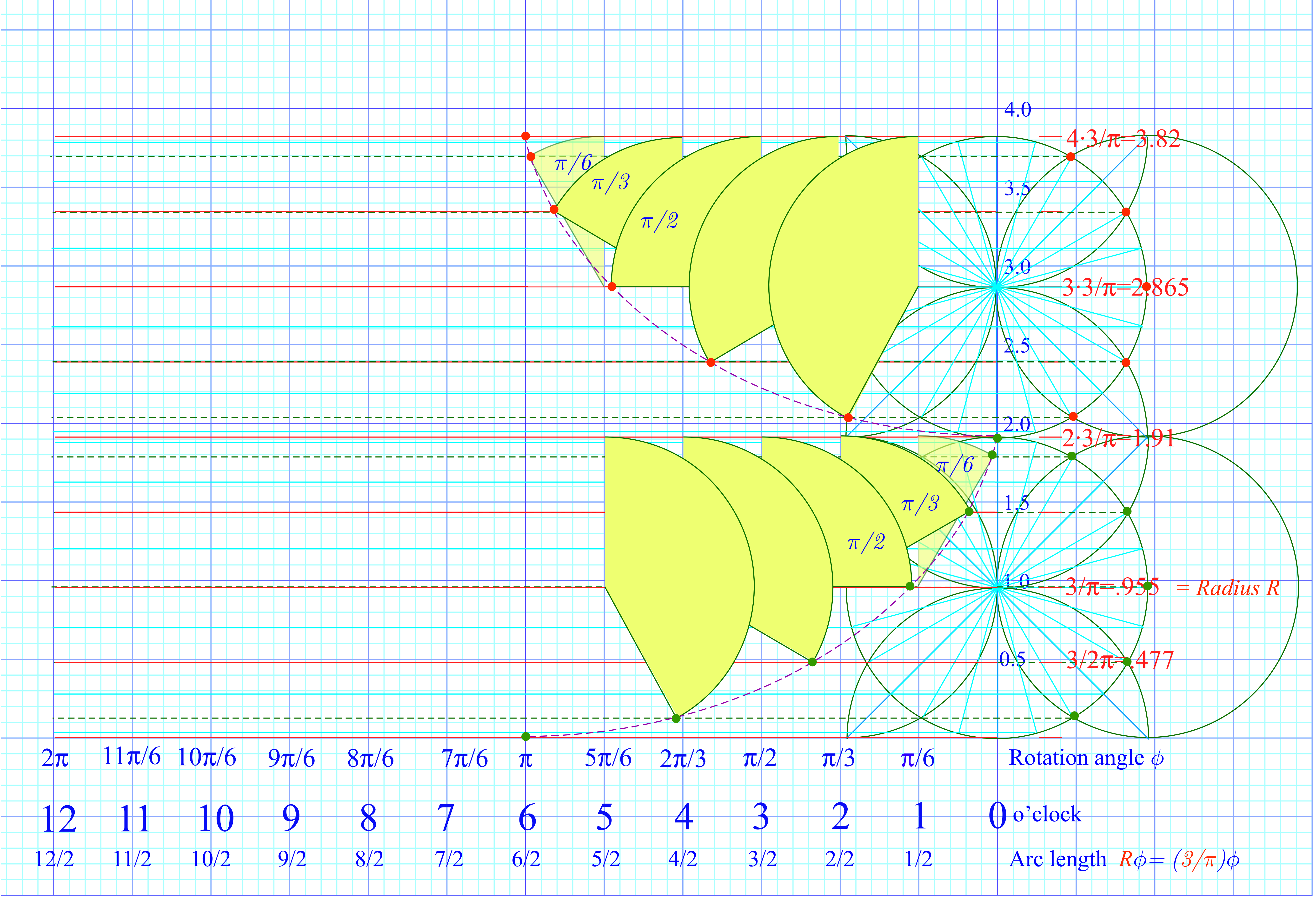

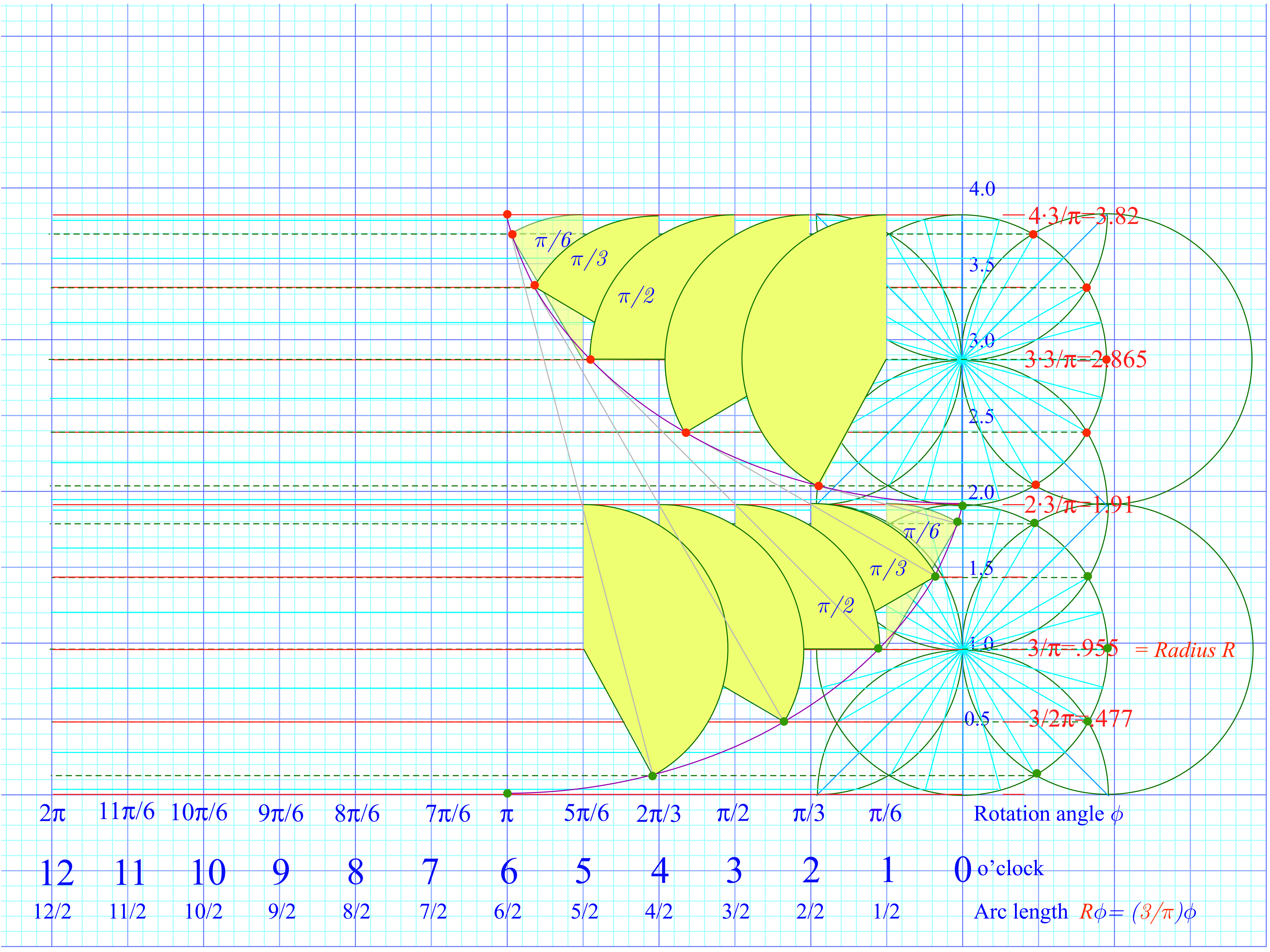
4.0
4·3/π=3.82
π/6
π/3
π/2
3.5
3.0
3·3/π=2.865
2.5
2.0
2·3/π=1.91
π/6
π/3
1.5
π/2
1.0
3/π=.955 = Radius R
0.5
3/2π=.477
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6
Rotation angle φ
12 11 10 9 8 7 6 5 4 3 2 1 0
o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2
Arc length Rφ= (3/π)φ

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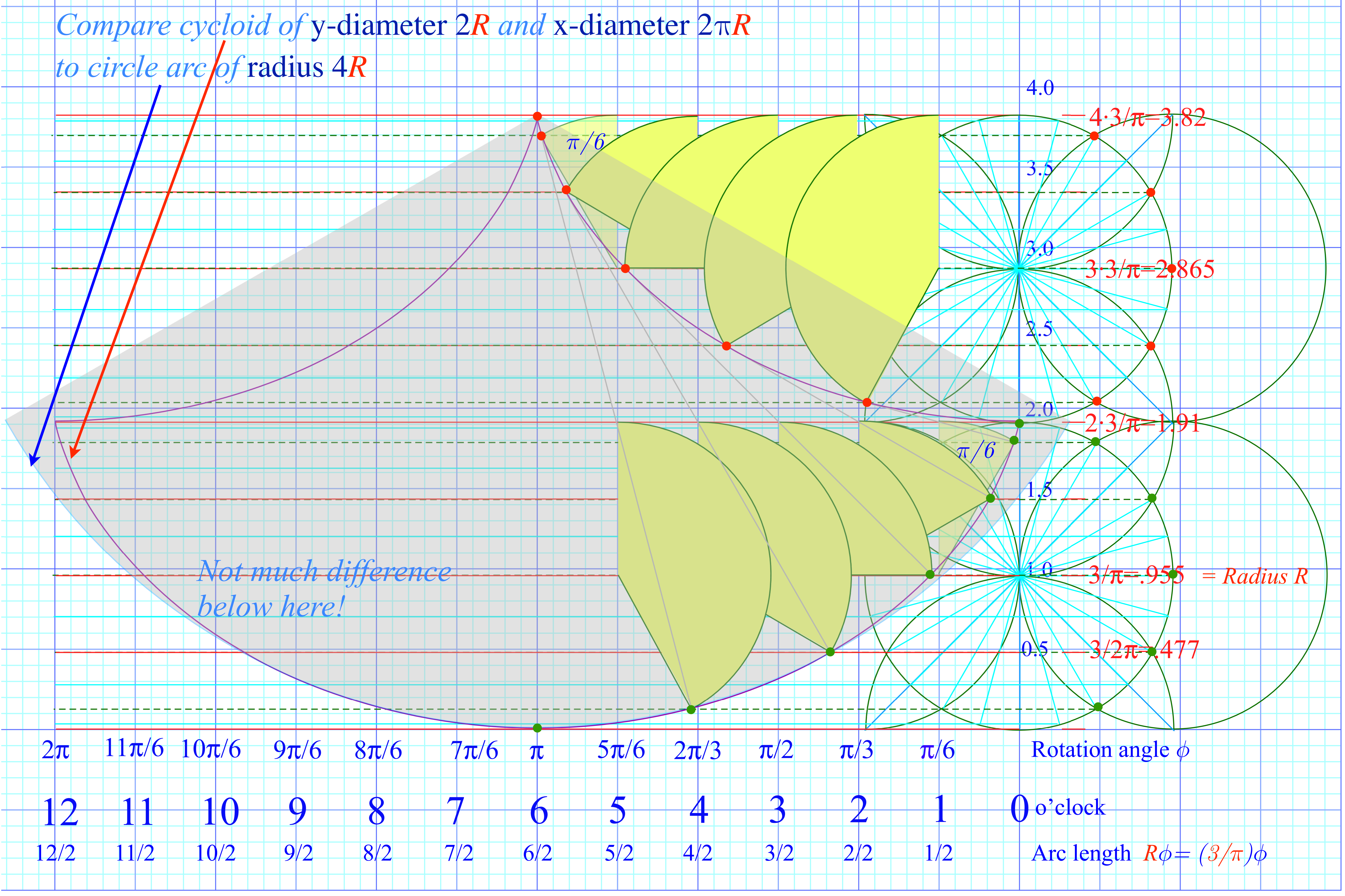

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you give it some linear momentum Π
and some angular momentum $\Lambda = h \cdot \Pi$

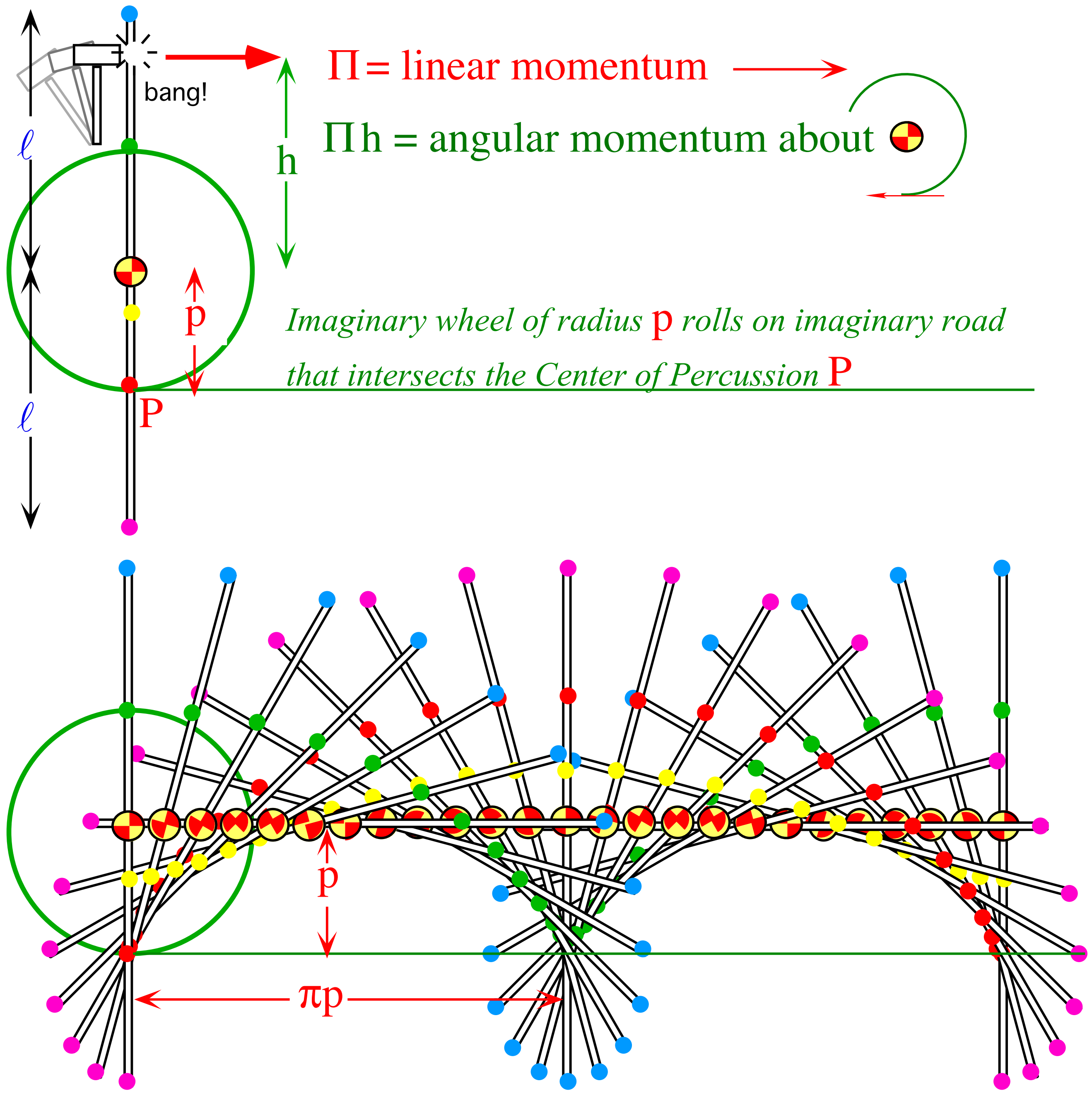

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

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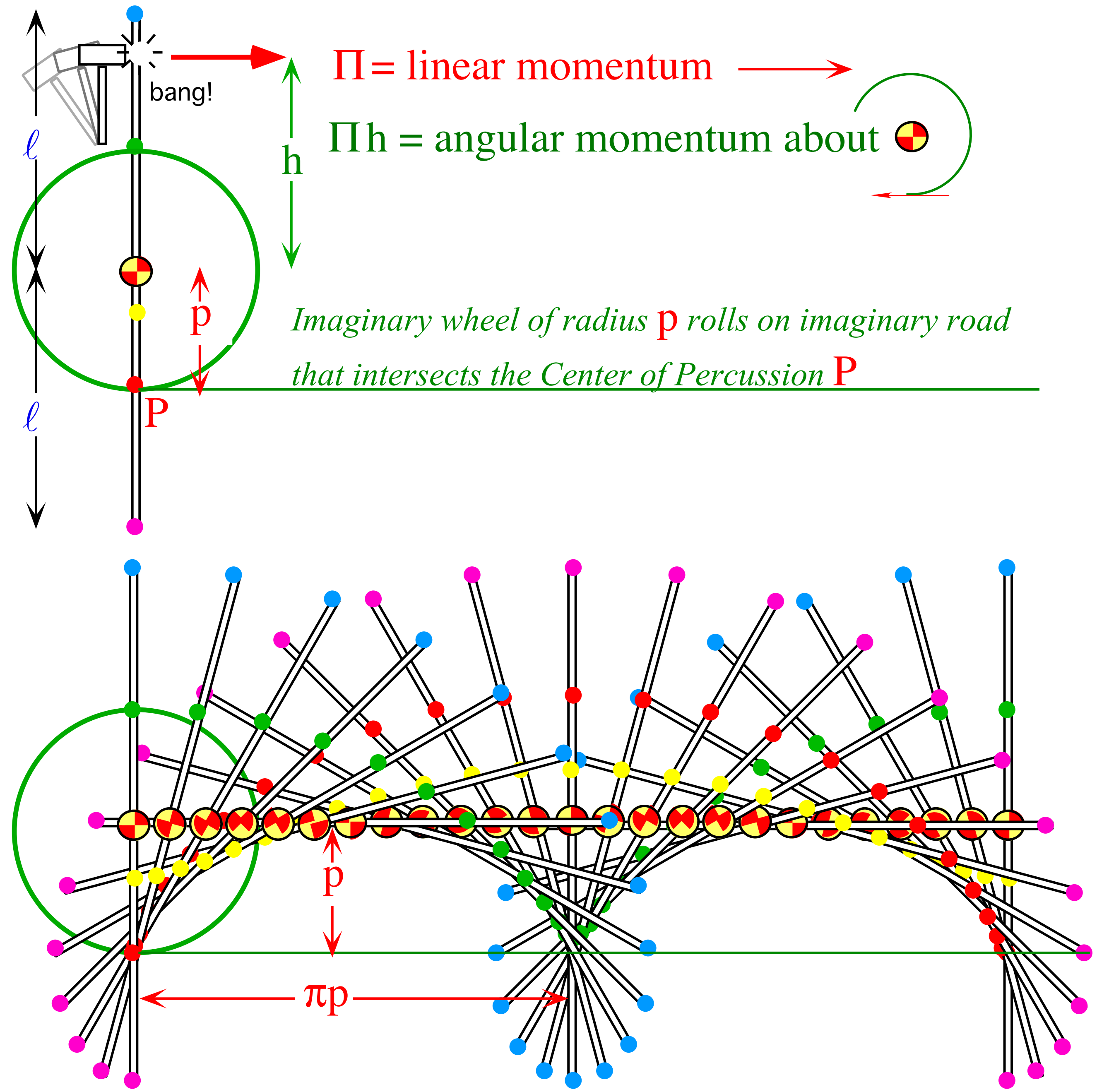

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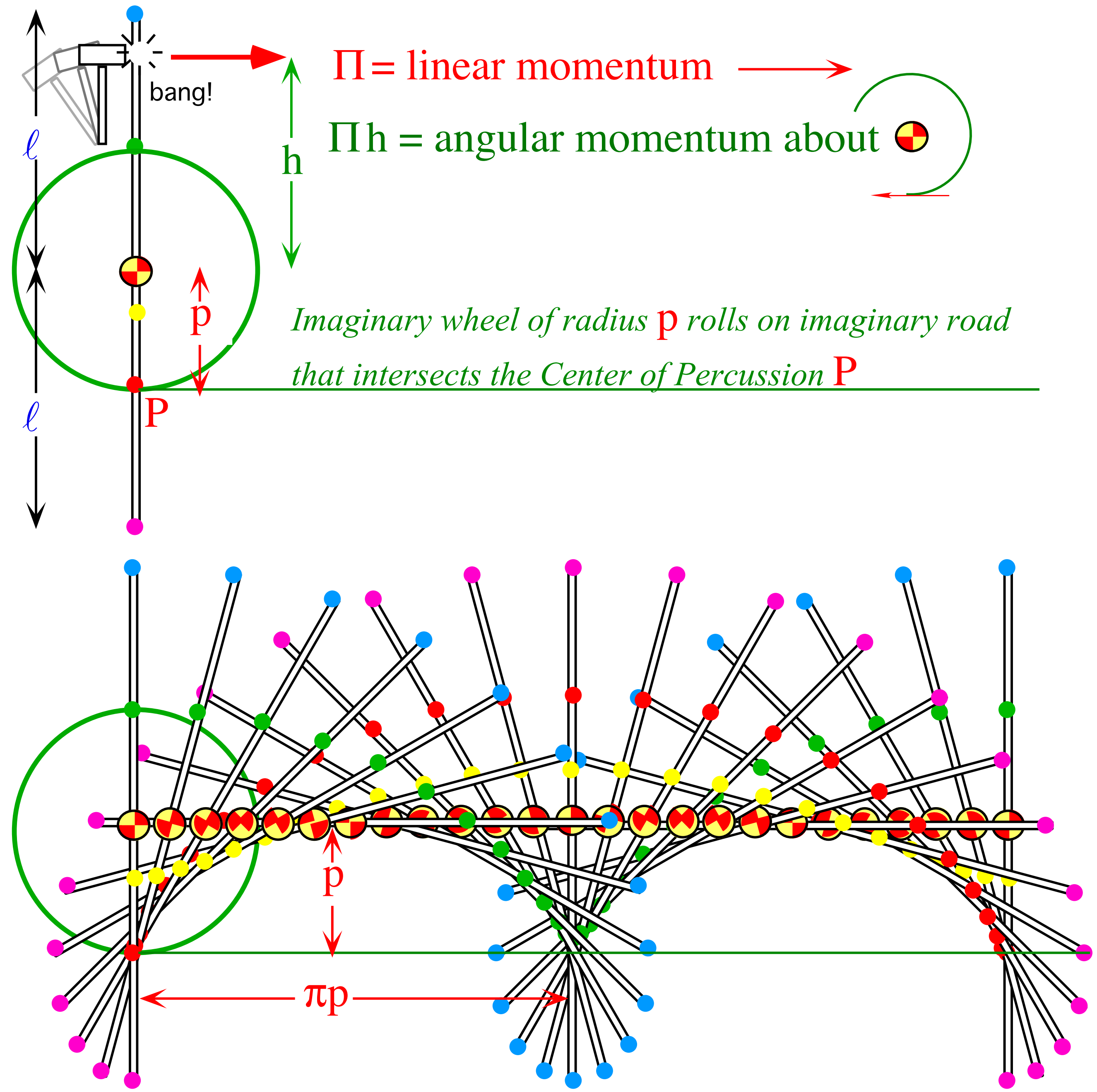

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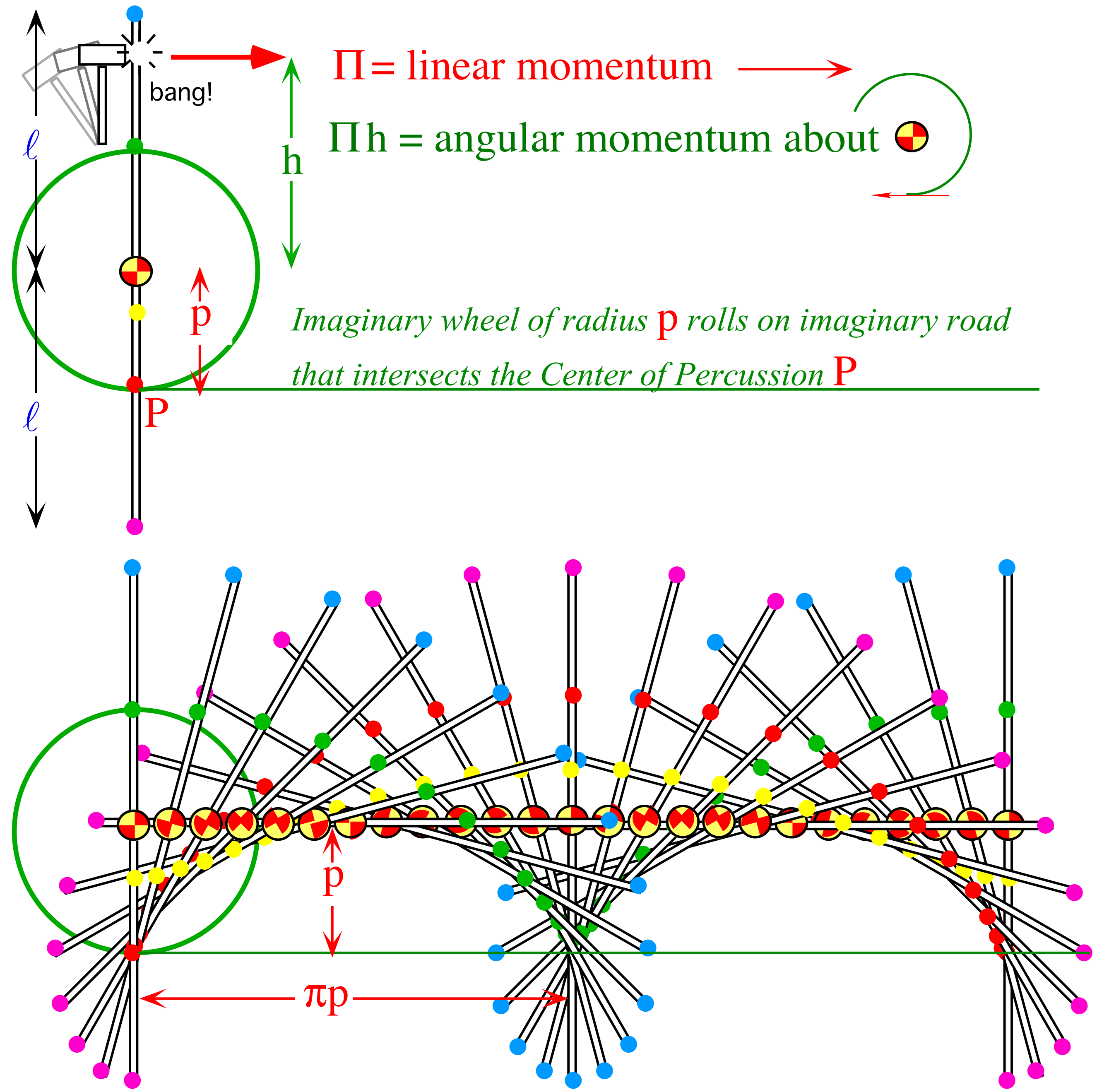

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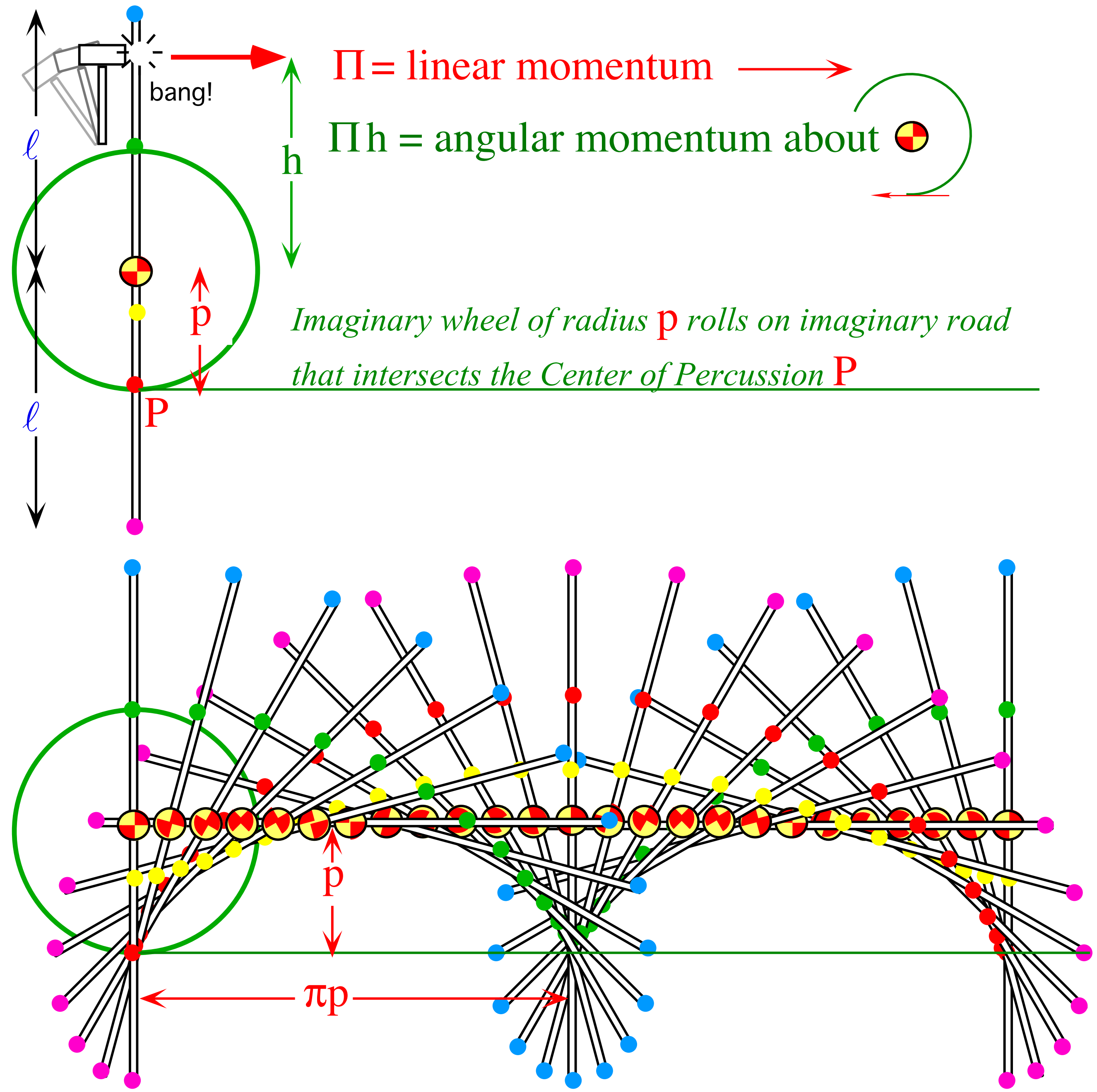

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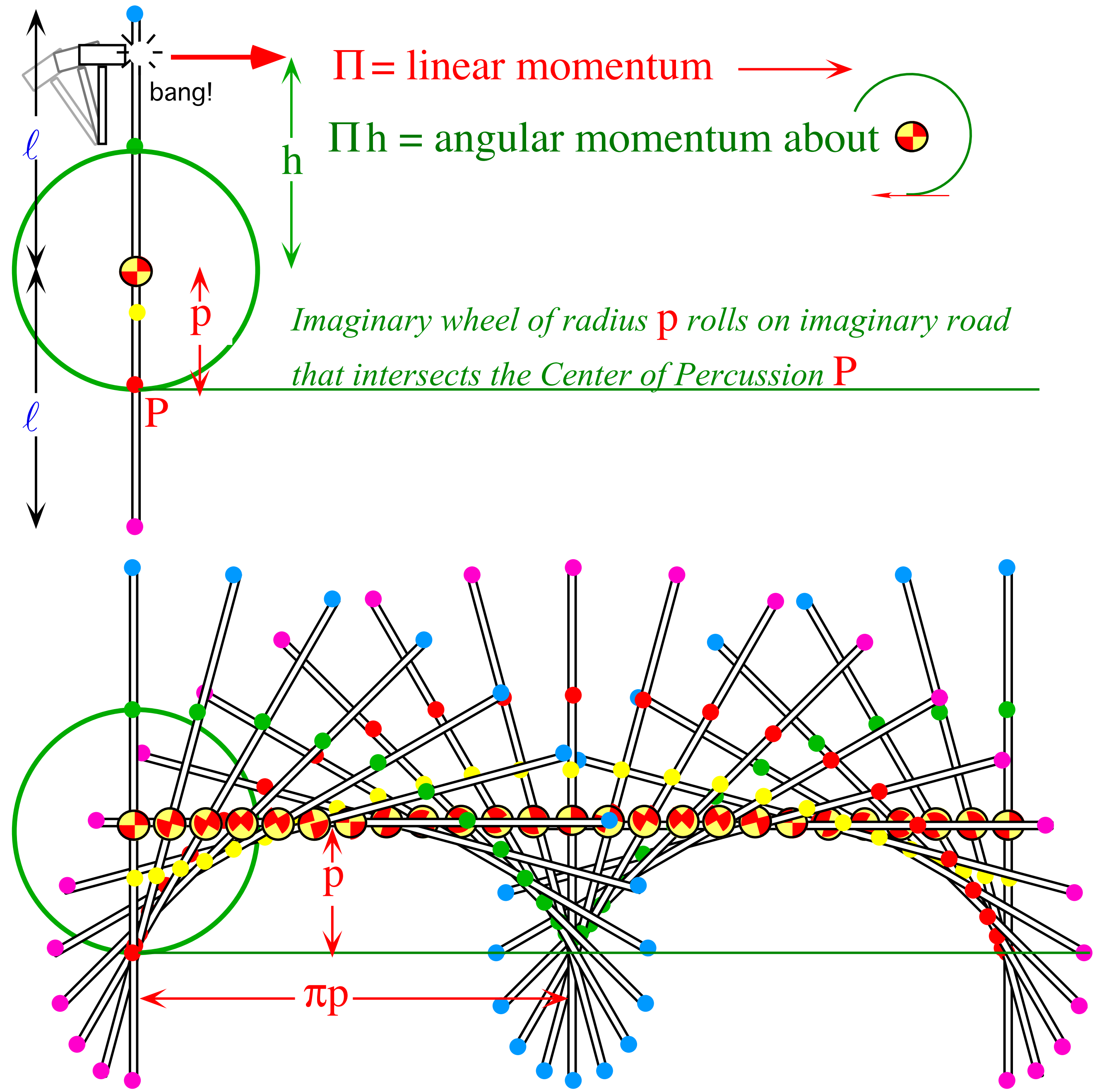

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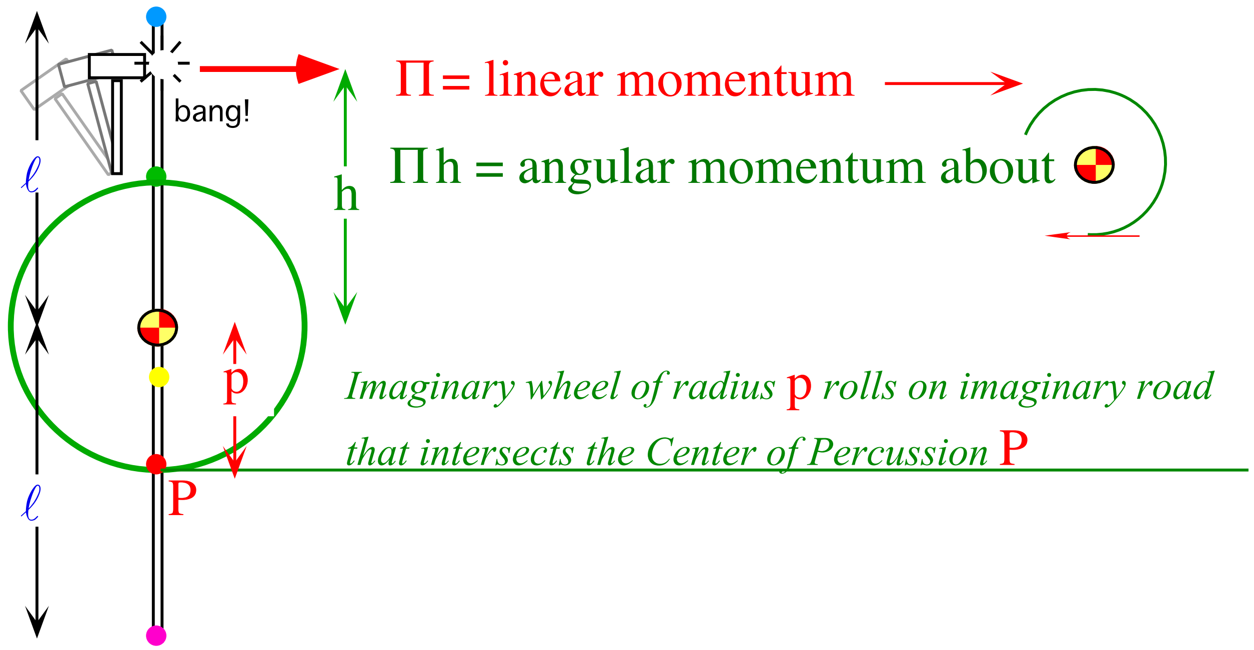

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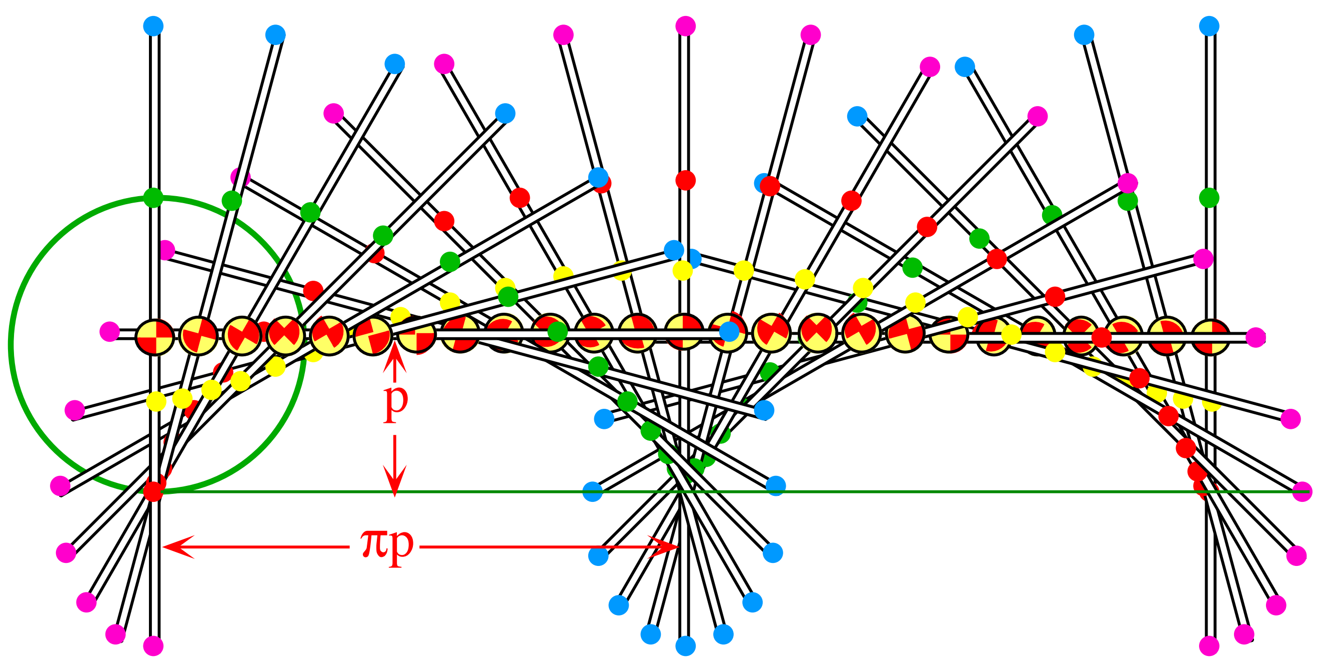
$$\Pi/M = V_{Center} = |p\omega| = p \cdot h\Pi/I$$
$$I/M = \quad = \quad = p \cdot h \qquad \text{or: } p = I/(Mh)$$



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$\omega = \Lambda / I$ $\quad (=3\Lambda/(M\ell^2))$ for stick)

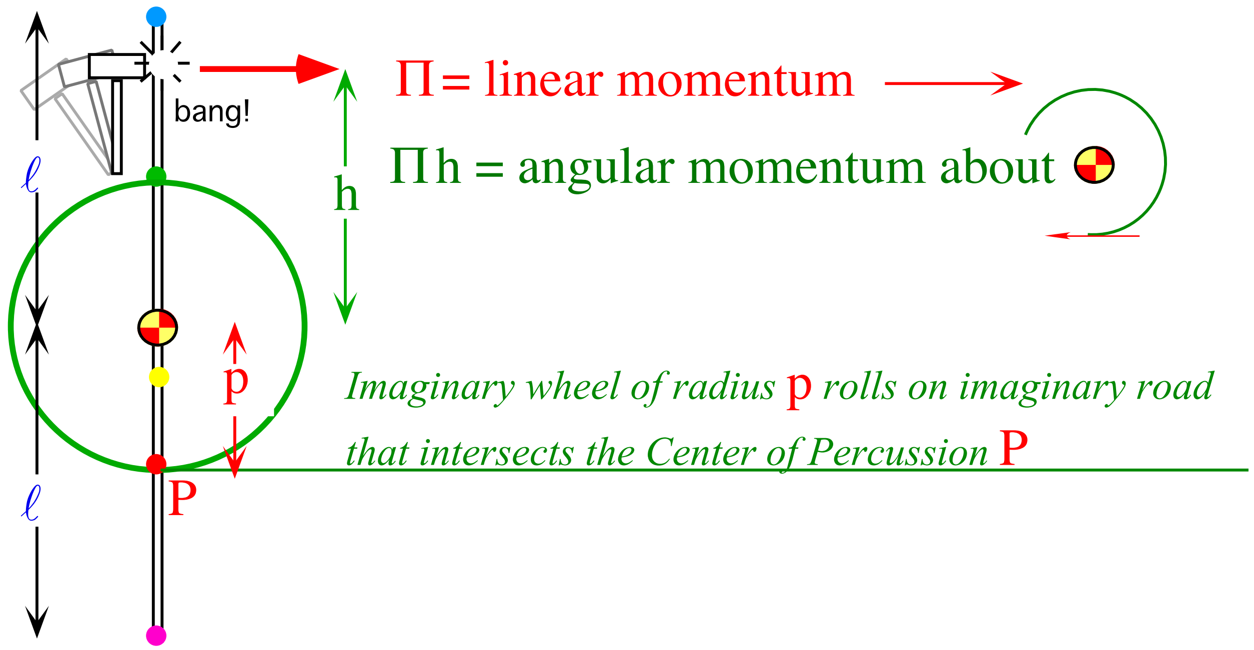
$= h\Pi / I$ $\quad (=3h\Pi/(M\ell^2))$for stick)

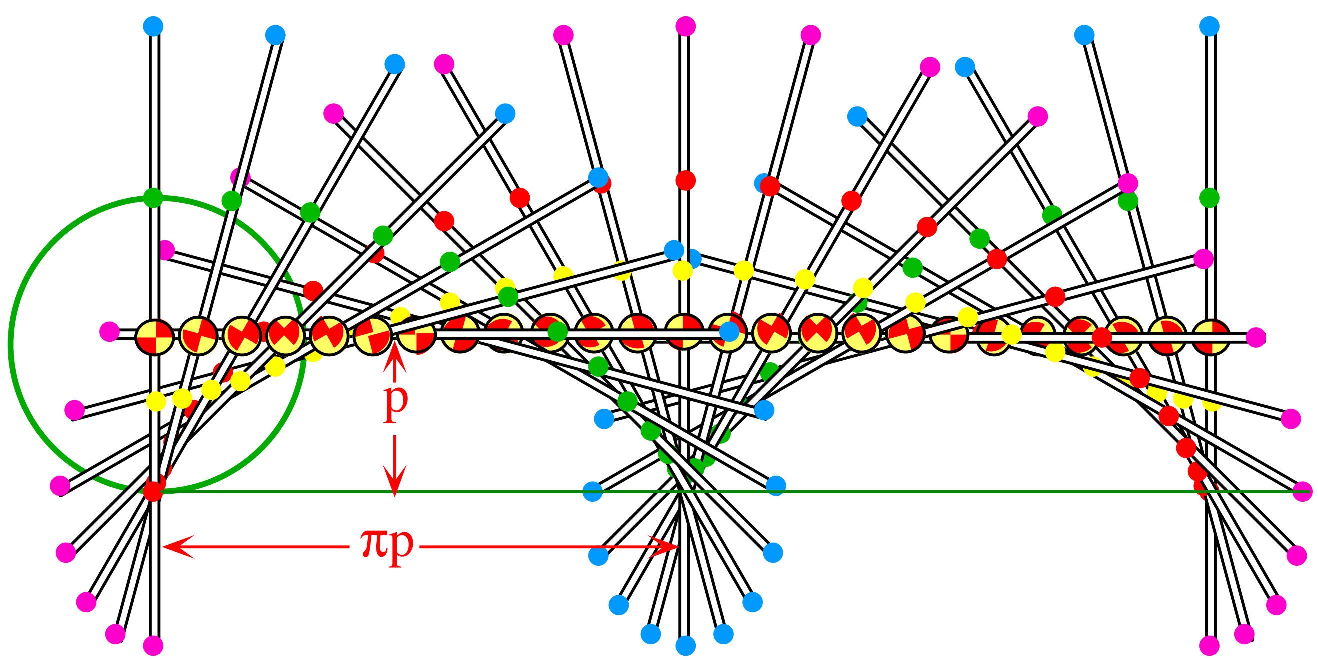
One point P, or *center of percussion* (CoP), is on the wheel where speed $p\omega$ due to rotation just cancels translational speed V_{Center} of stick.

$\Pi / M = V_{Center} = |p\omega| = p \cdot h\Pi / I$

$I / M =$ $\quad =$ $\quad = p \cdot h$ $\qquad$ *or:* $p = I/(Mh)$

P follows a normal cycloid made by a circle of radius $p=I/(Mh)$ rolling on an imaginary road thru point P in direction of Π.





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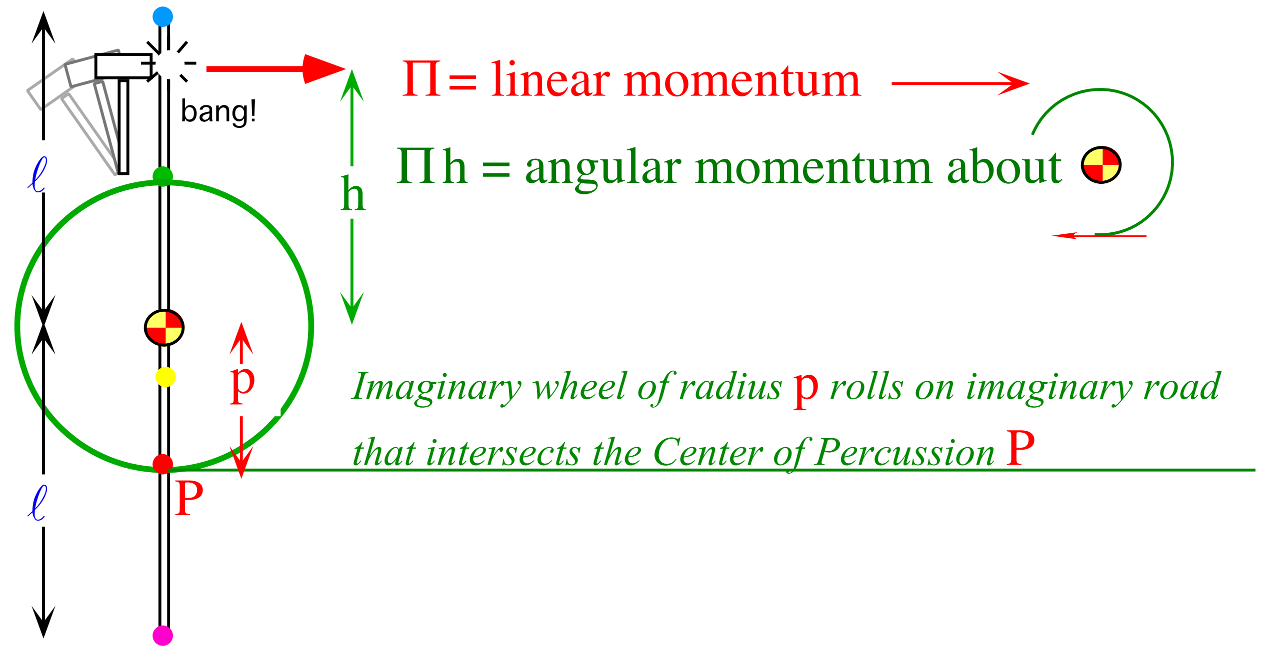

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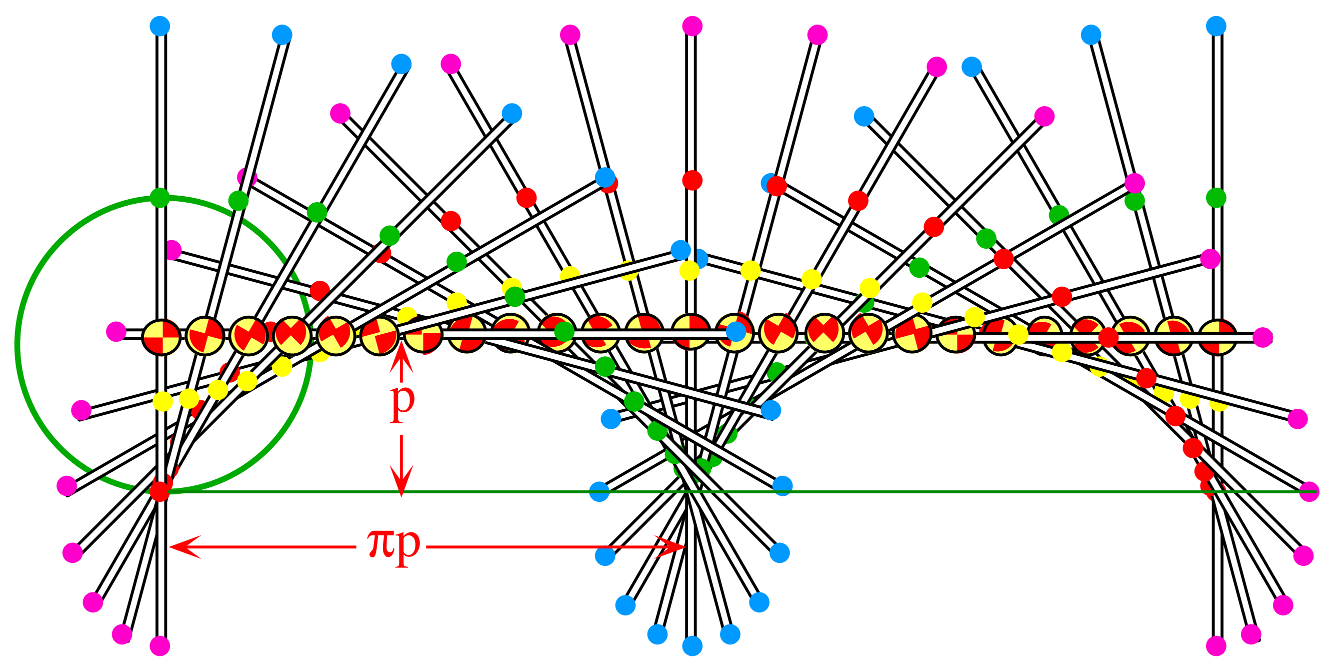

Fig. 2.A.1 Cycloidic paths due to hitting a stationary stick.

The *percussion radius* $p = \ell^2/3h$ is of the CoP point
that has no velocity just after hammer hits at h.

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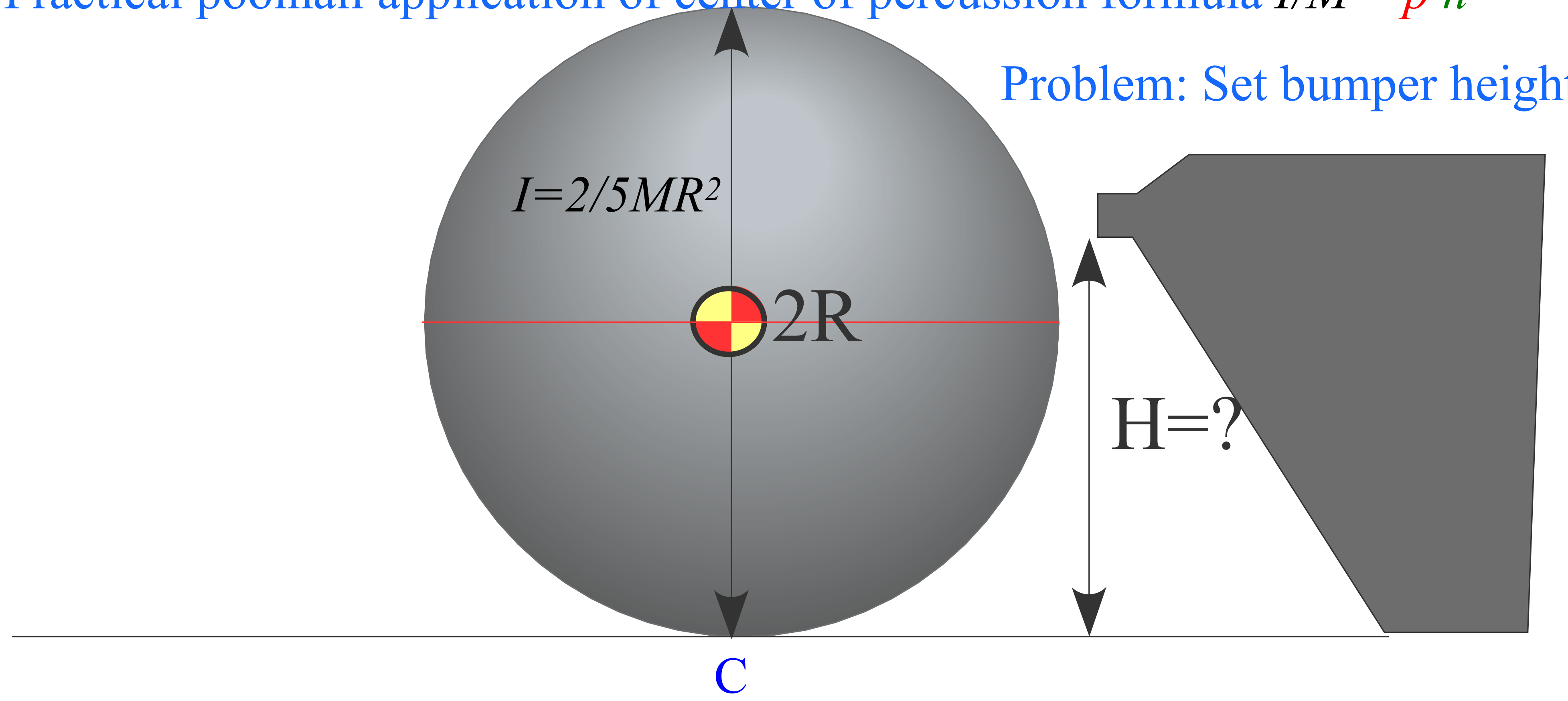
→ *Practical poolhall application*

Practical poolhall application of center of percussion formula $I/M = p \cdot h$

Problem: Set bumper height H so ball does not skid.



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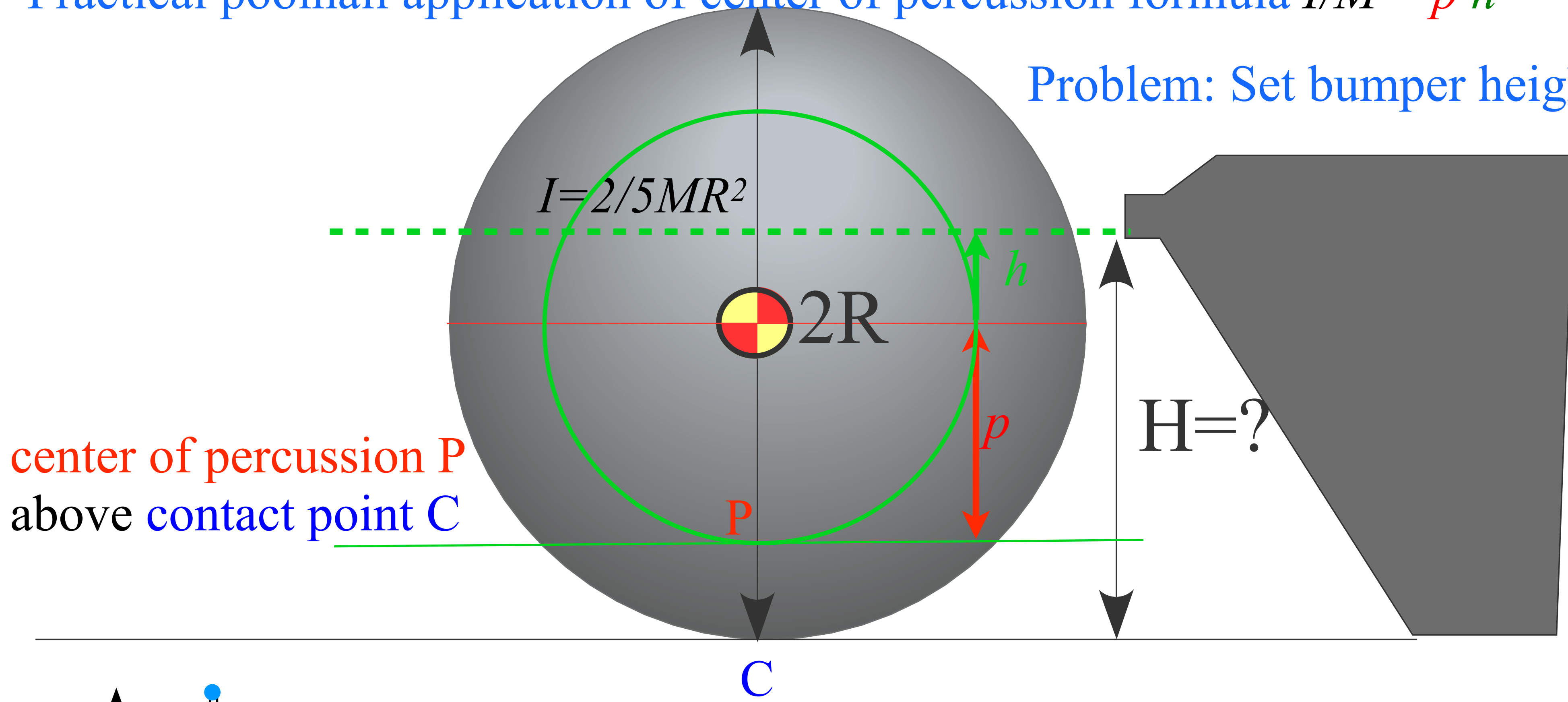

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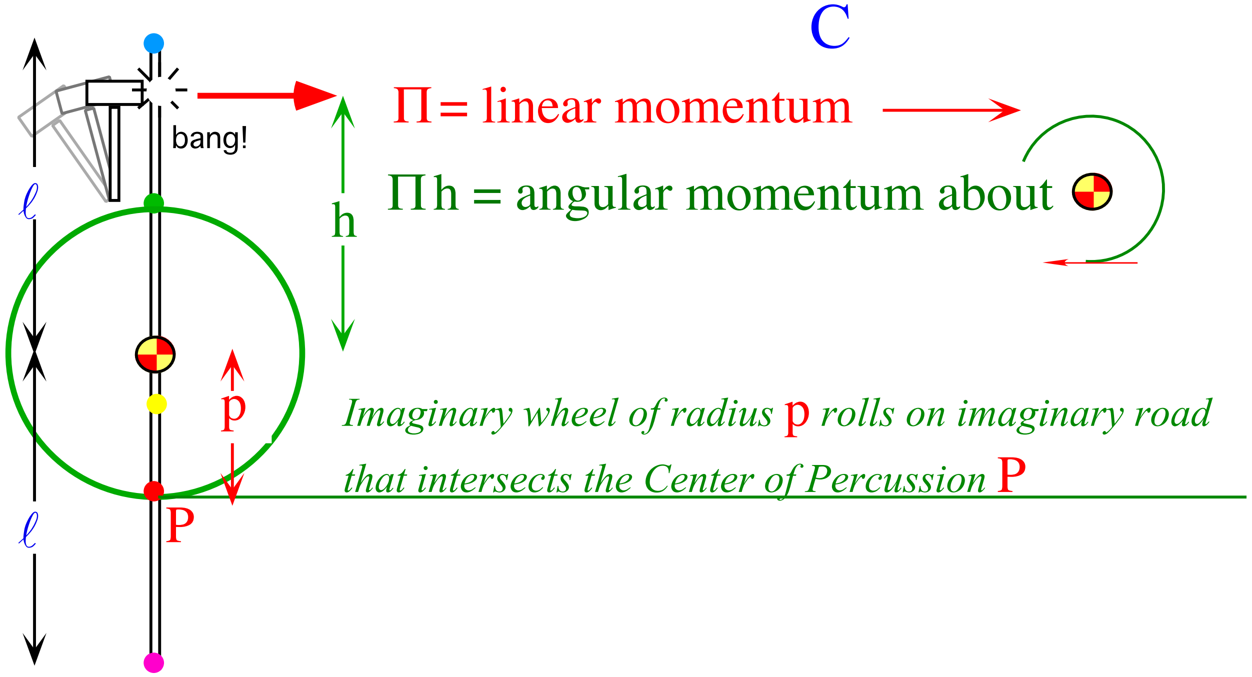

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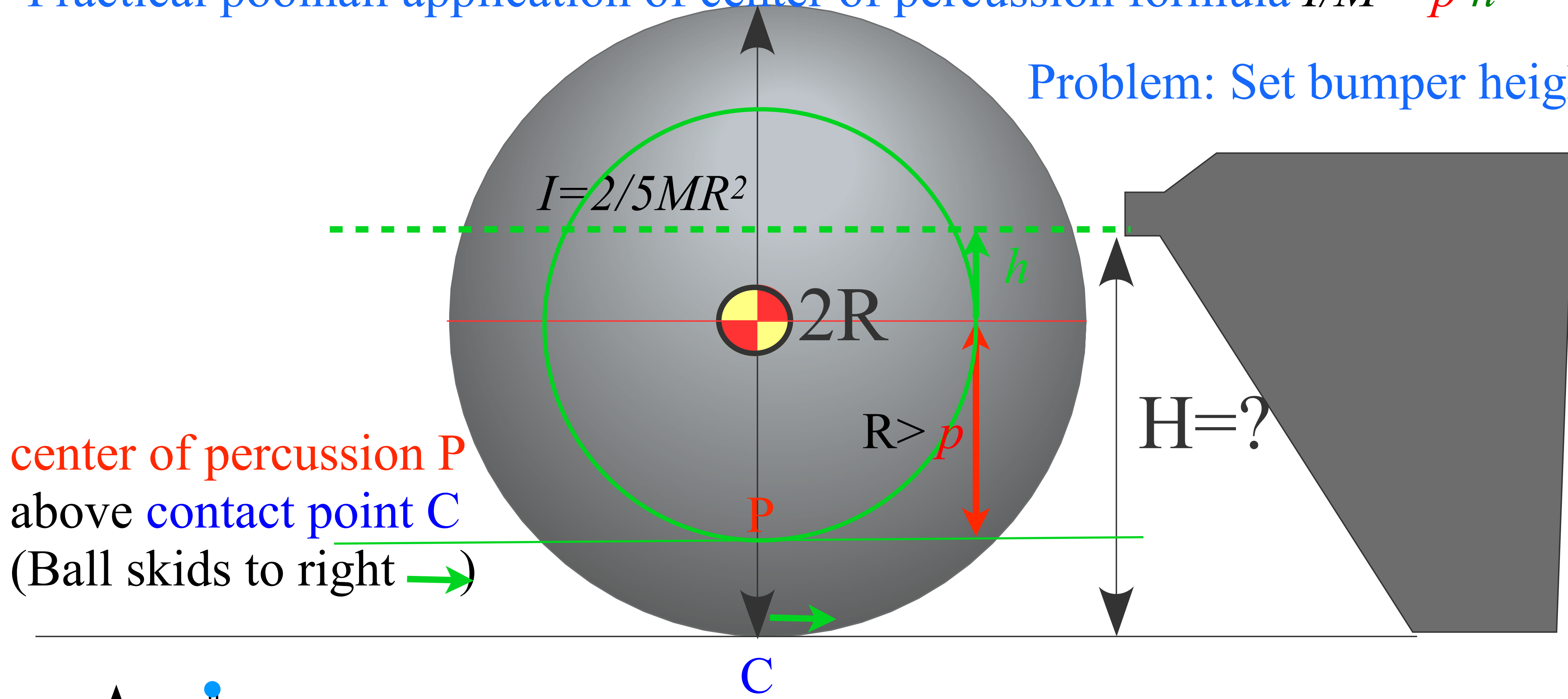

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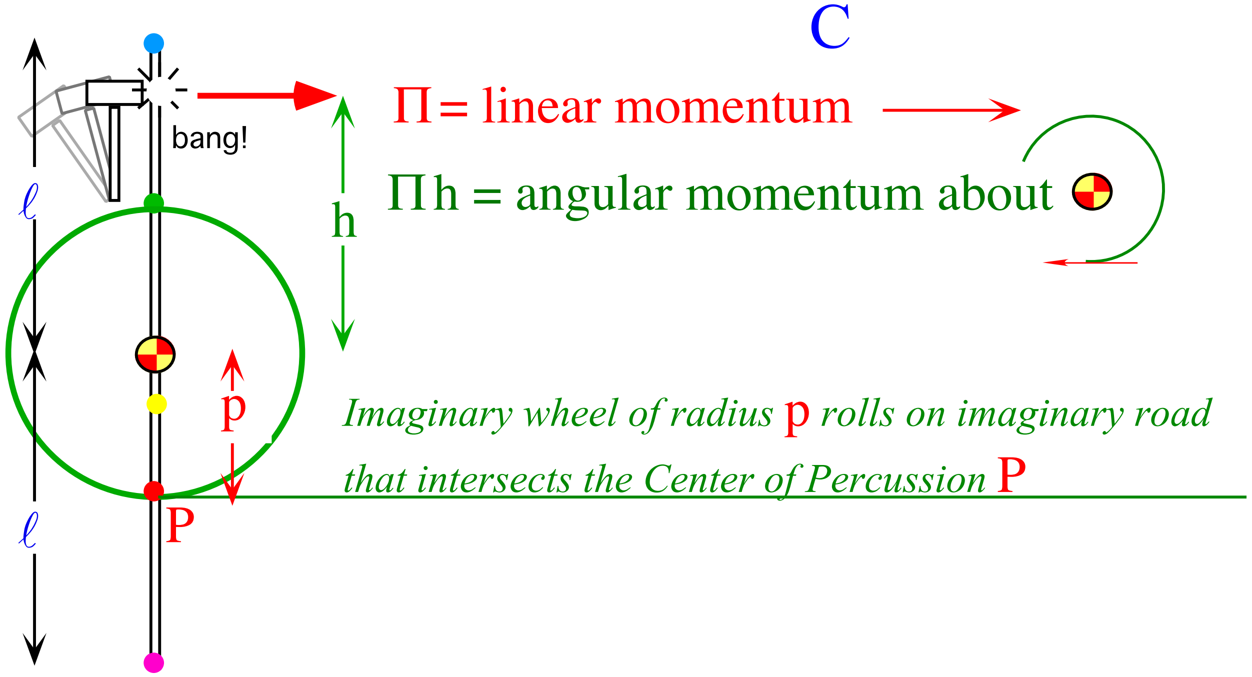

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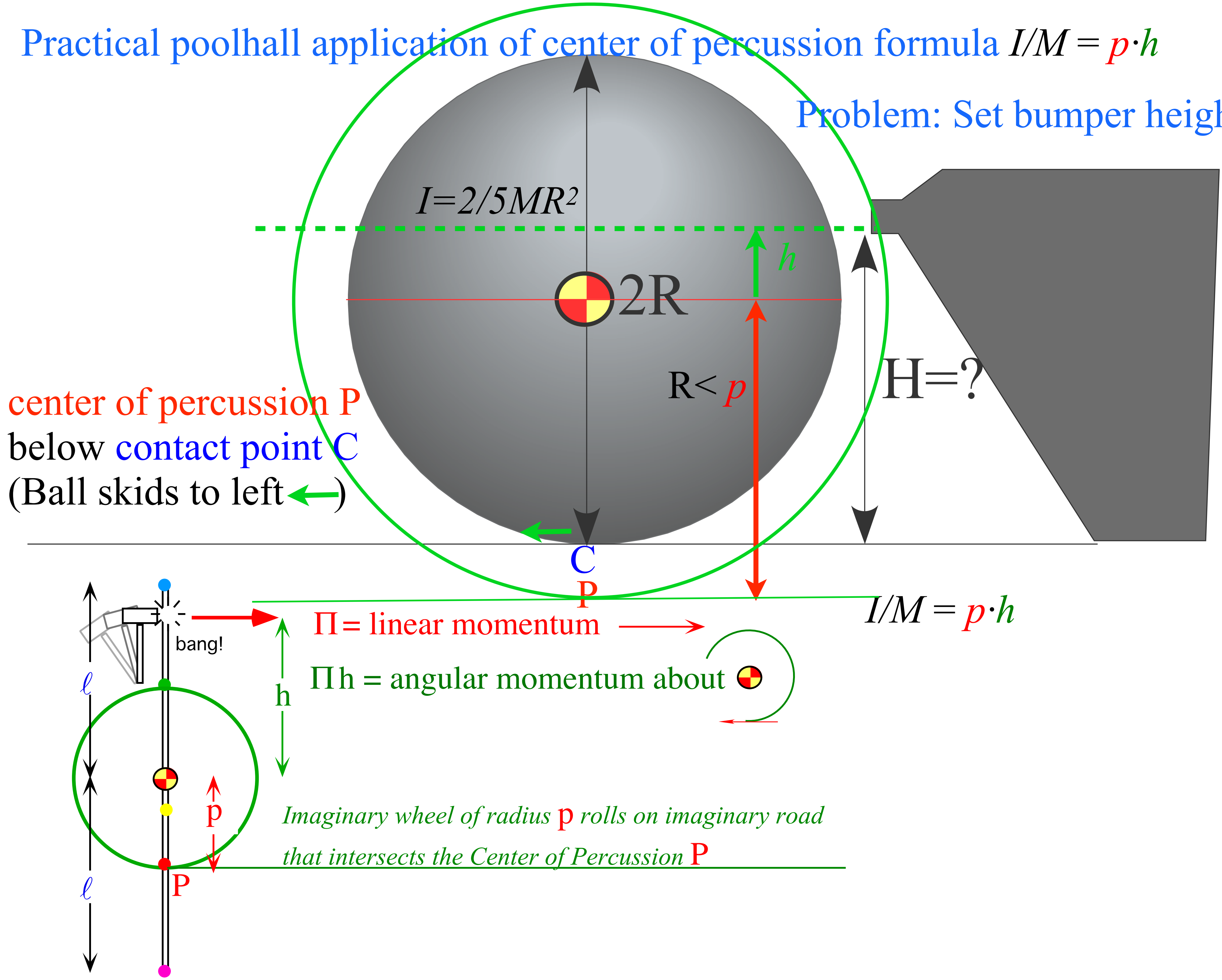

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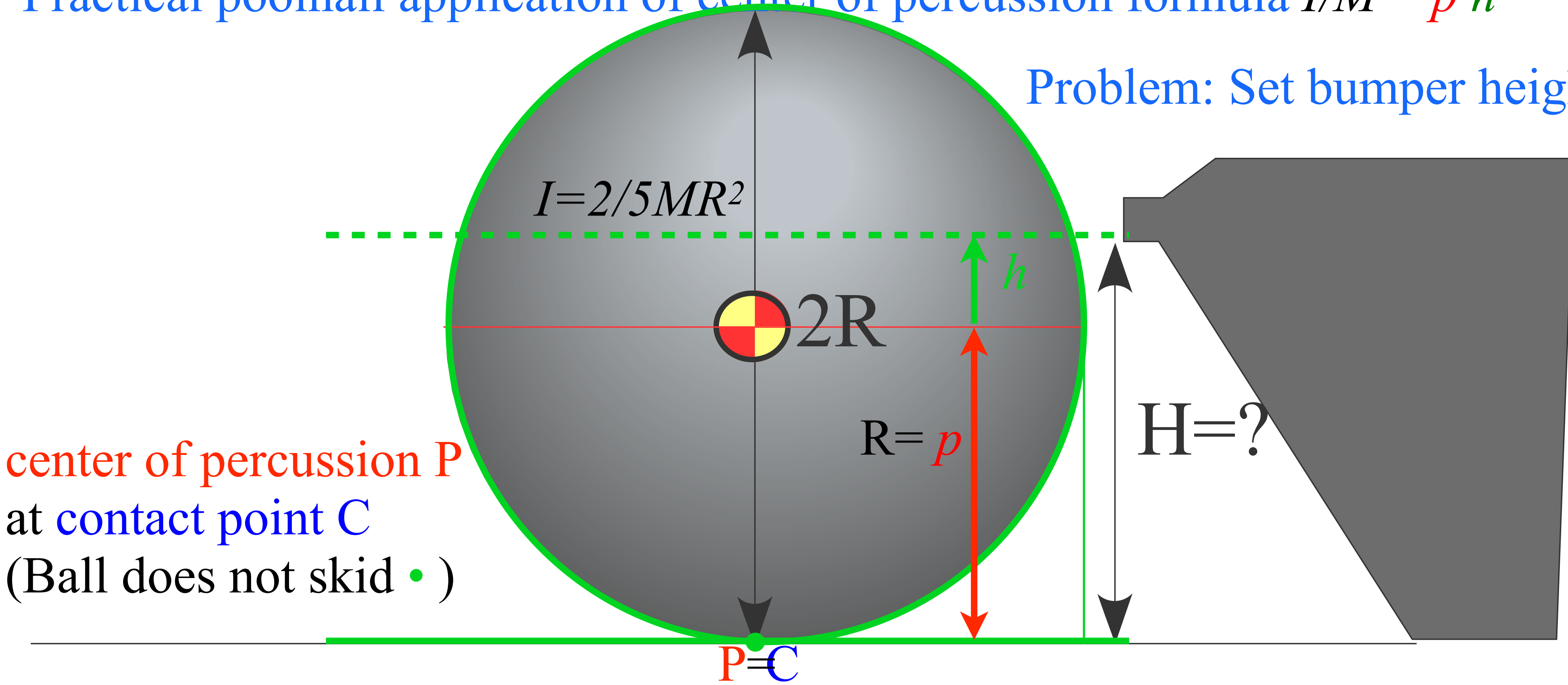

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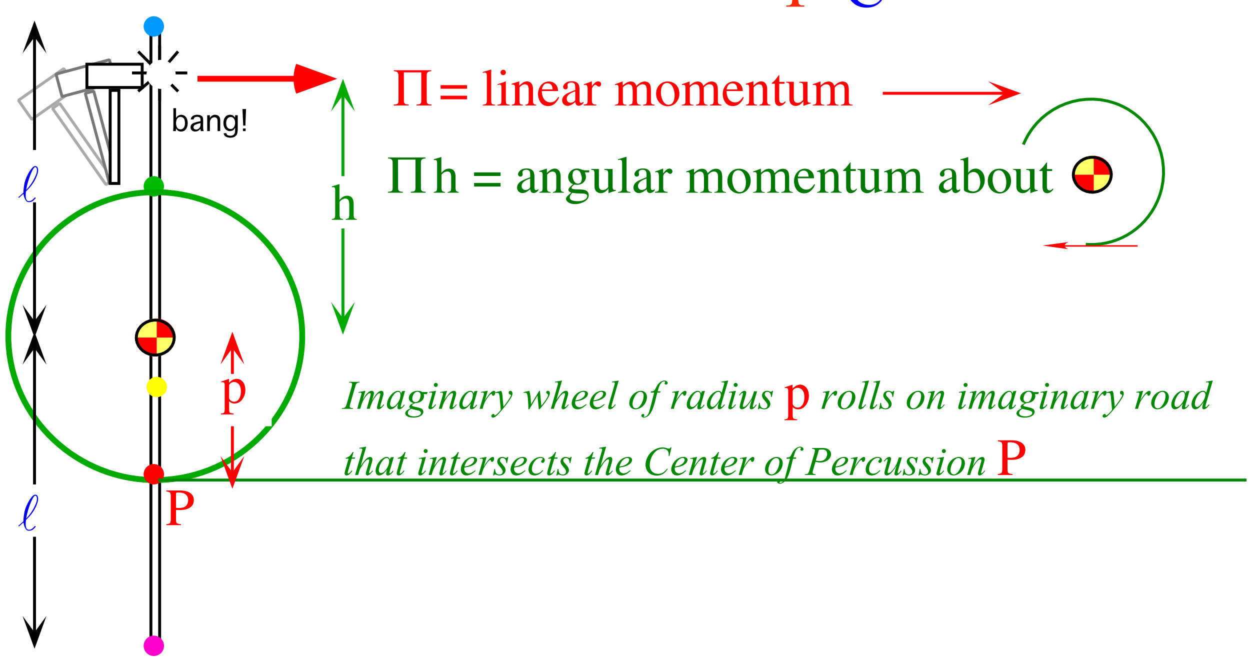

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$h = I/Mp = I/MR$ (For $R = p$)

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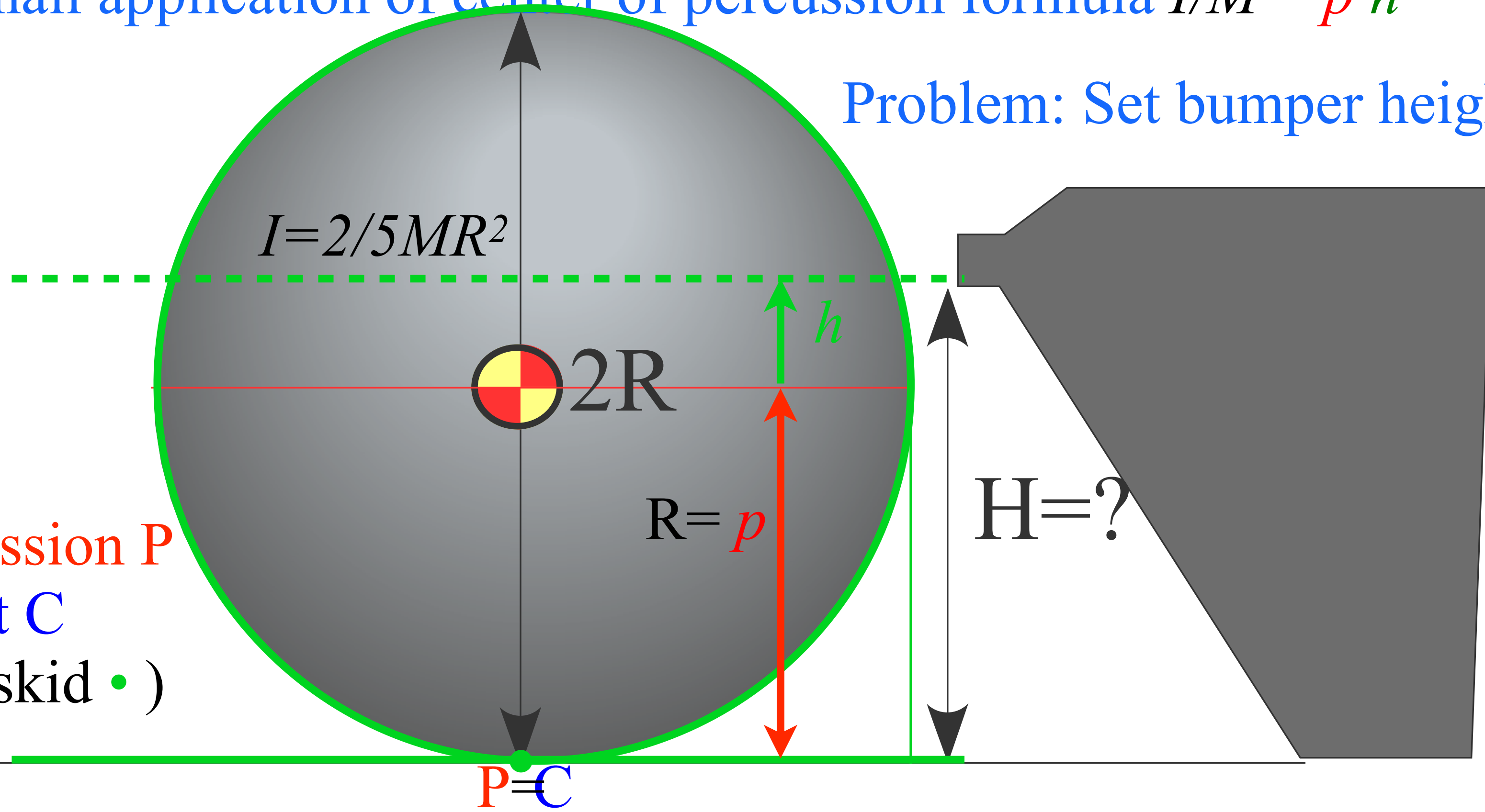

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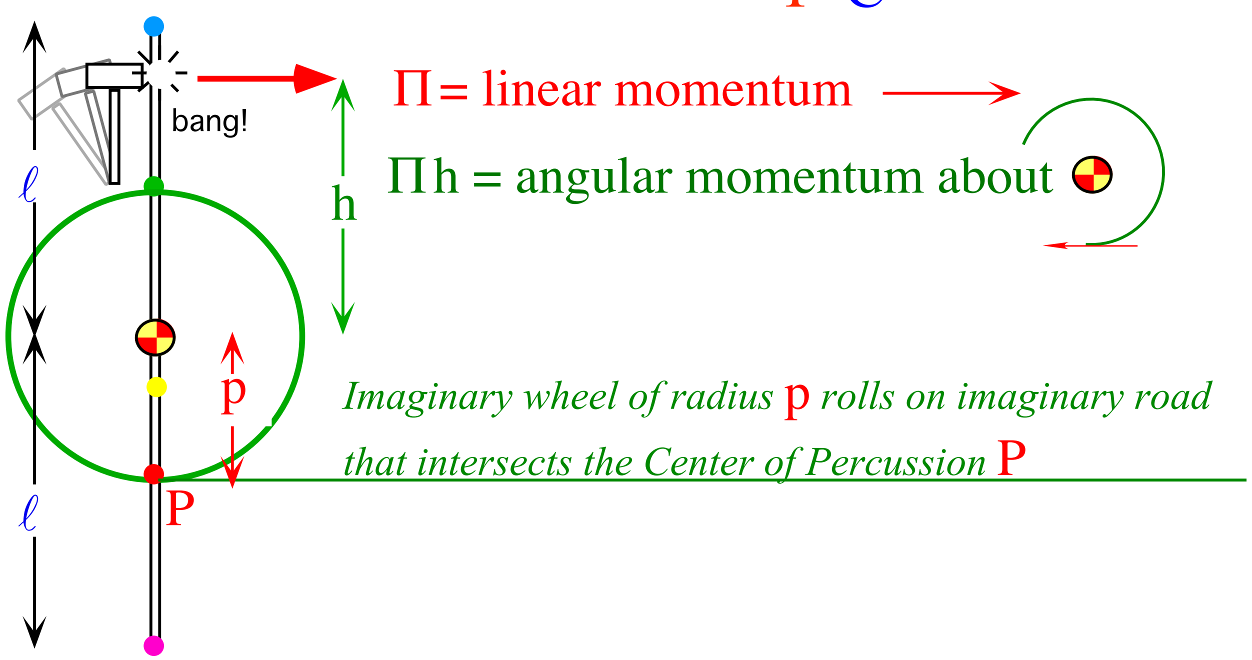
center of percussion P
at contact point C
(Ball does not skid •)



$I/M = p \cdot h$

$h = I/Mp = I/MR$ (For $R = p$)

$= 2/5MR^2/MR$

$= 2/5R$

For: $H = R + h = 7/10(2R)$ ball does not skid.

Thats all folks!