Lecture 17
Thur. 10.29.2015

Hamilton Equations for Trebuchet and Other Things
(Ch. 5-9 of Unit 2)

Review of Hamiltonian equation derivation (Elementary trebuchet)
- Hamiltonian definition from Lagrangian and γ_{mn} tensor
- Hamilton's equations and Poincare invariant relations
- Hamiltonian expression and contravariant γ^{mn} tensor

Hamiltonian energy and momentum conservation and symmetry coordinates
- Coordinate transformation helps reduce symmetric Hamiltonian
- Free-space trebuchet kinematics by symmetry
  - Algebraic approach
  - Direct approach and Superball analogy
- Trebuchet vs Flinger and sports kinematics
  - Many approaches to Mechanics

Chapter 1. The Trebuchet: A dream problem for Galileo?

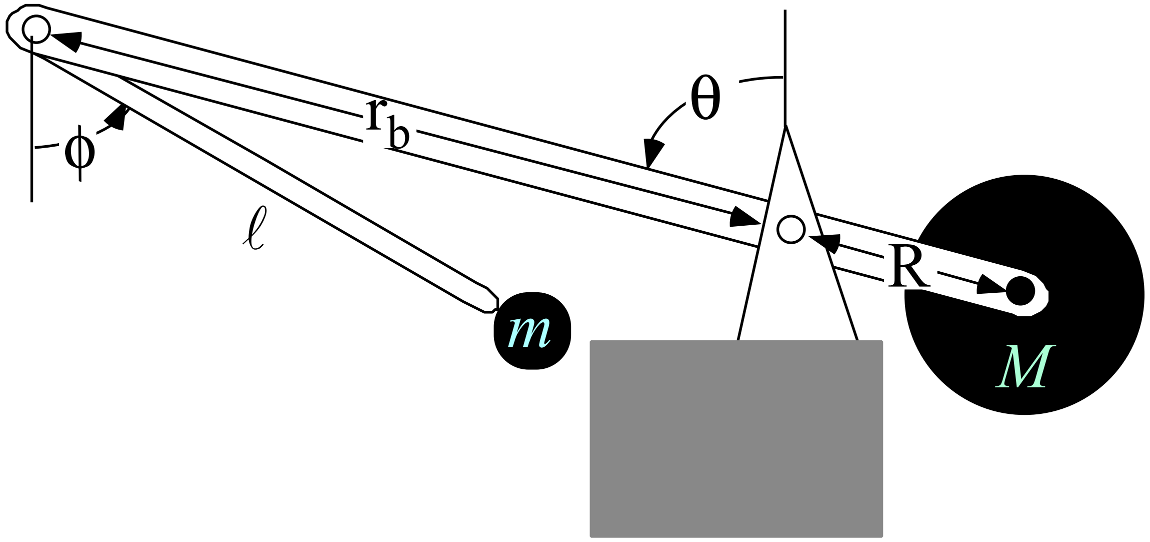
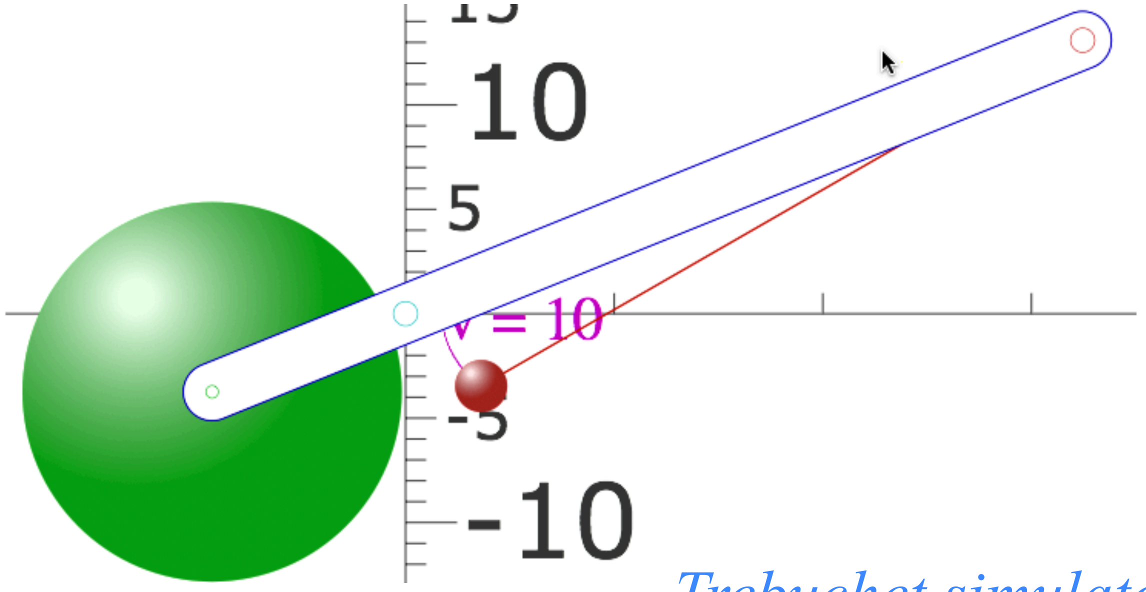
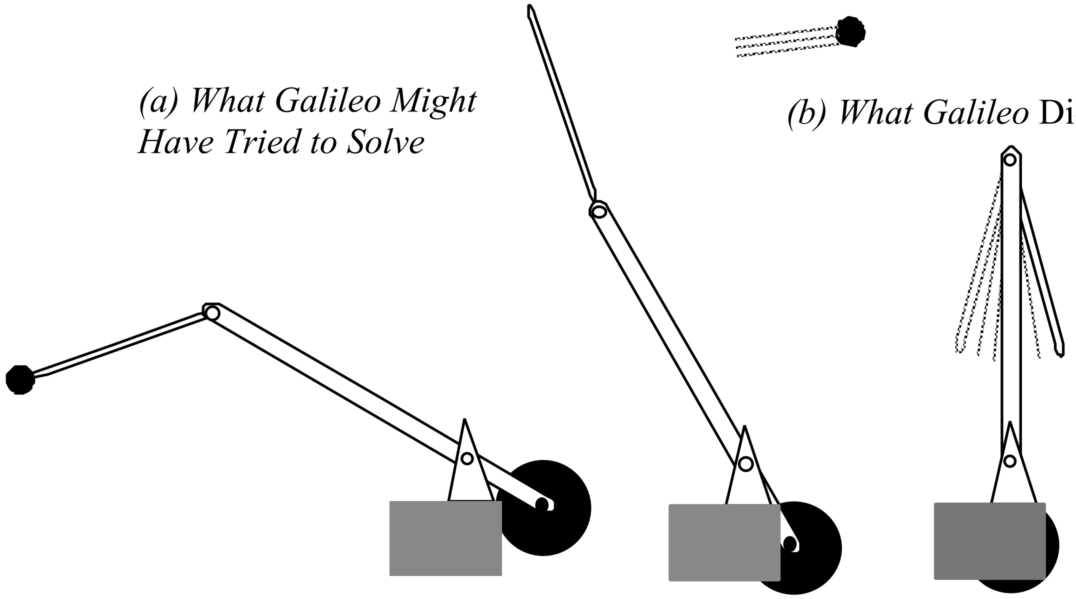


Fig. 2.1.1 An elementary ground-fixed trebuchet



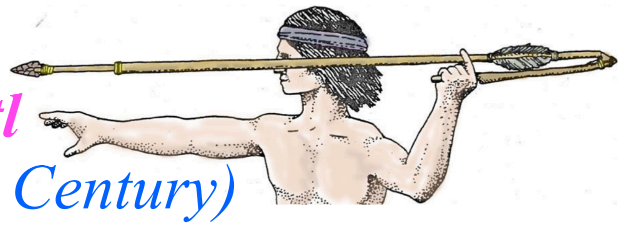
Trebuchet simulator

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html



(Simple pendulum dynamics)

Fig. 2.1.2 Galileo's (supposed) problem



The Atlatl

(Cahokia, IL 12th Century)

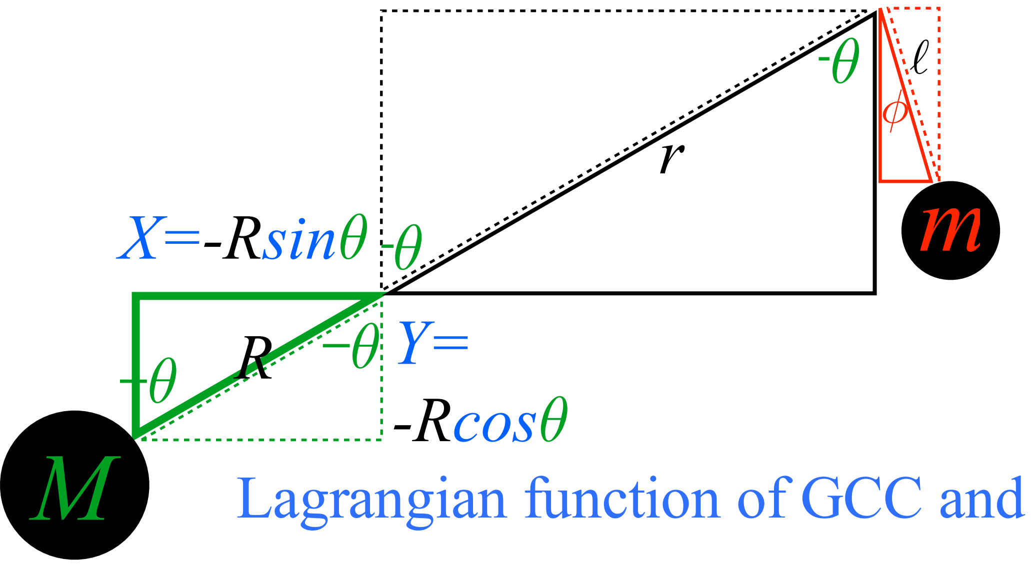
Review of Hamiltonian equation derivation (Elementary trebuchet)

→ *Hamiltonian definition from Lagrangian and* γ_{mn} *tensor*

Hamilton's equations and Poincare invariant relations

Hamiltonian expression and contravariant γ^{mn} *tensor*

$$\text{Total } KE = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

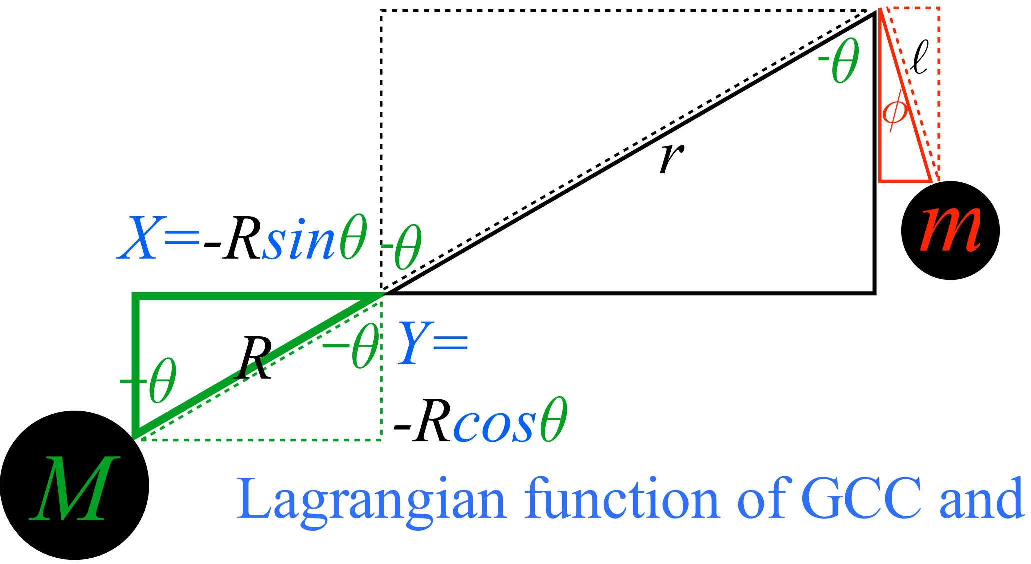
in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

$$dL\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = \frac{\partial L}{\partial\theta}d\theta + \frac{\partial L}{\partial\phi}d\phi + \frac{\partial L}{\partial\dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial\dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

1st differential chain

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



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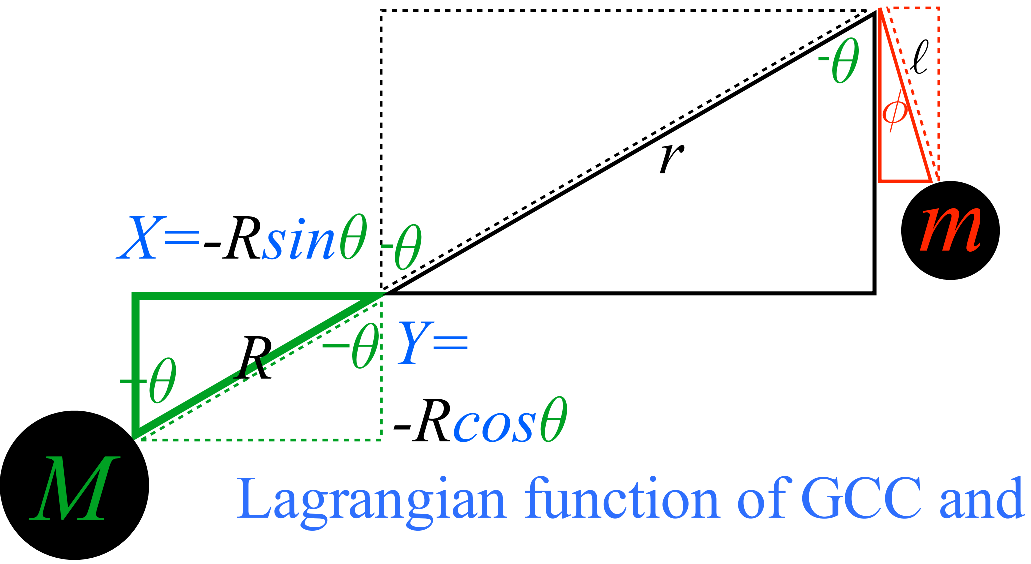
velocity chain

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



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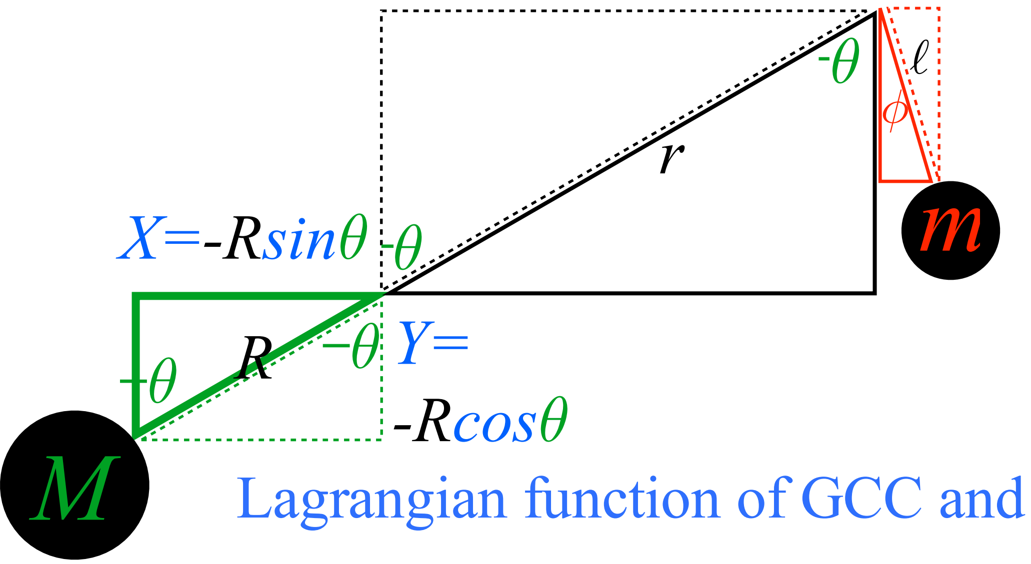

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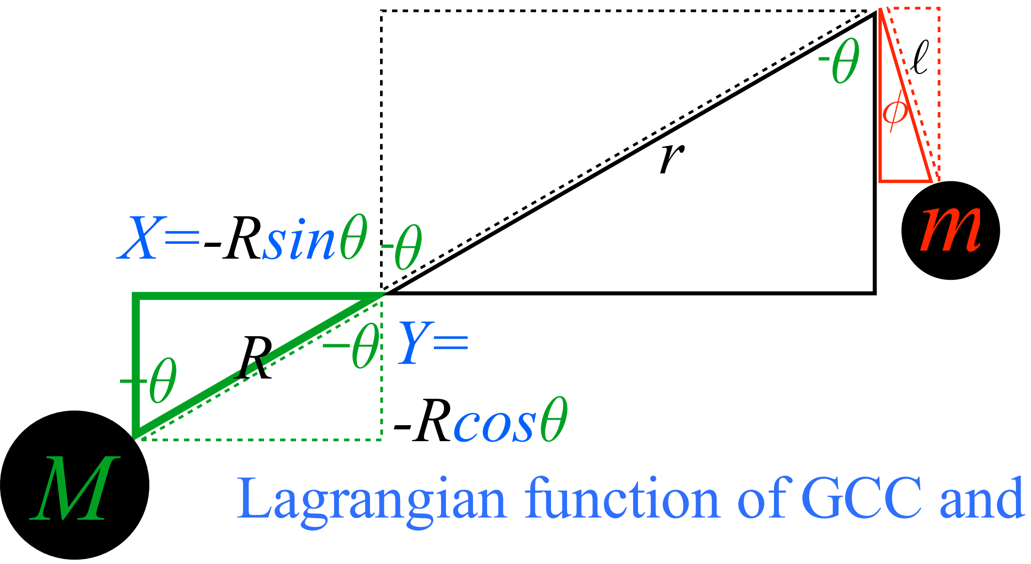

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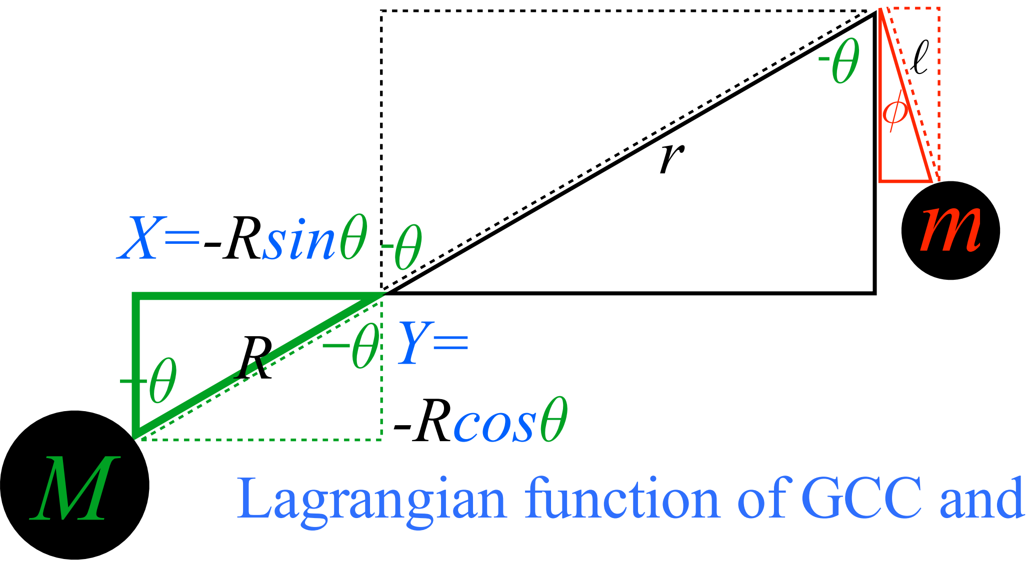

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Lagrange equations

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(Consolidating)

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(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

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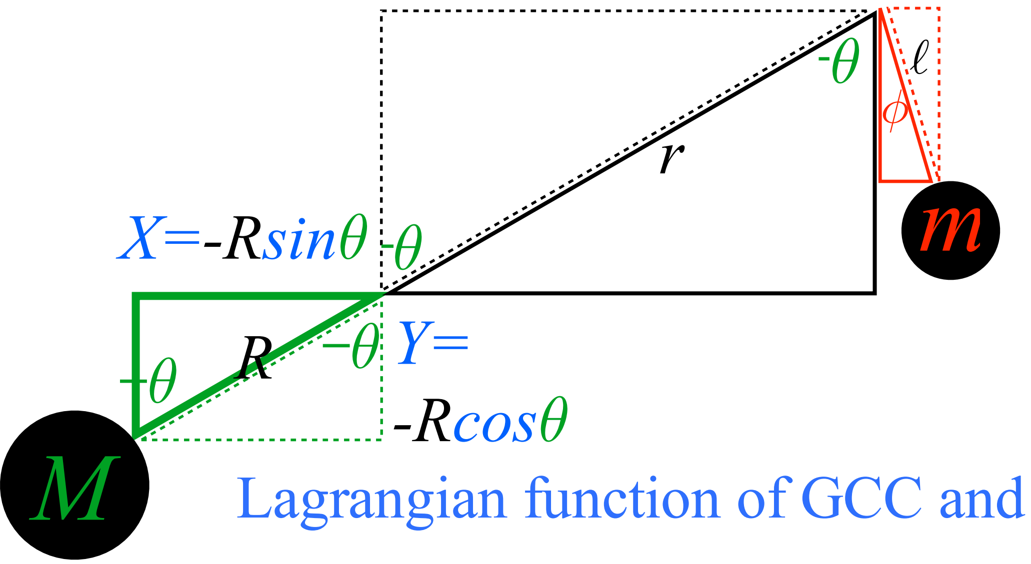

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velocity chain

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

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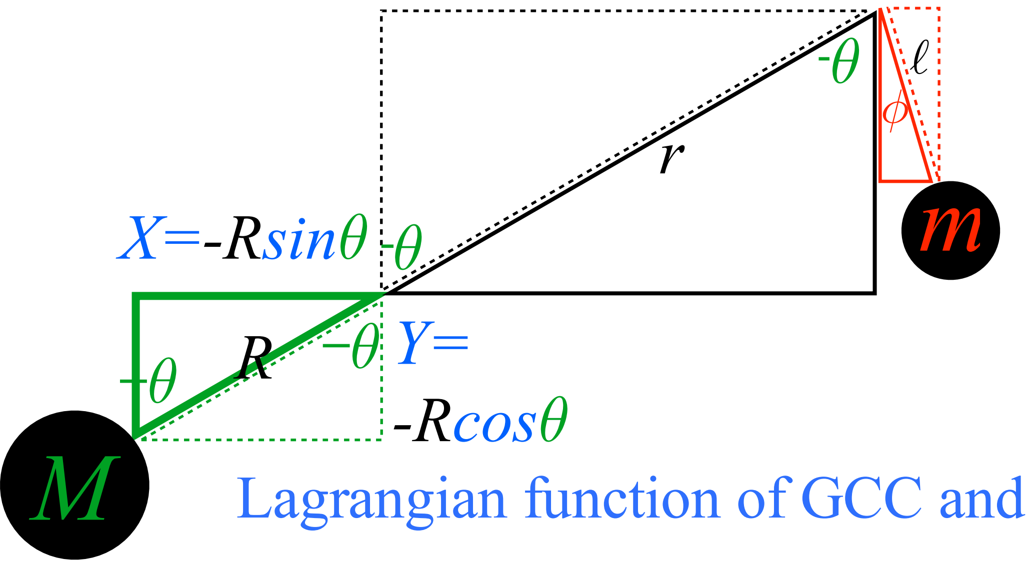

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$$\frac{\partial H}{\partial\theta} \overset{\downarrow}{=} -\frac{\partial L}{\partial\theta} \underset{\uparrow}{=} -\dot{p}_\theta$$

by assumed Lagrange functionality

by Lagrange equations

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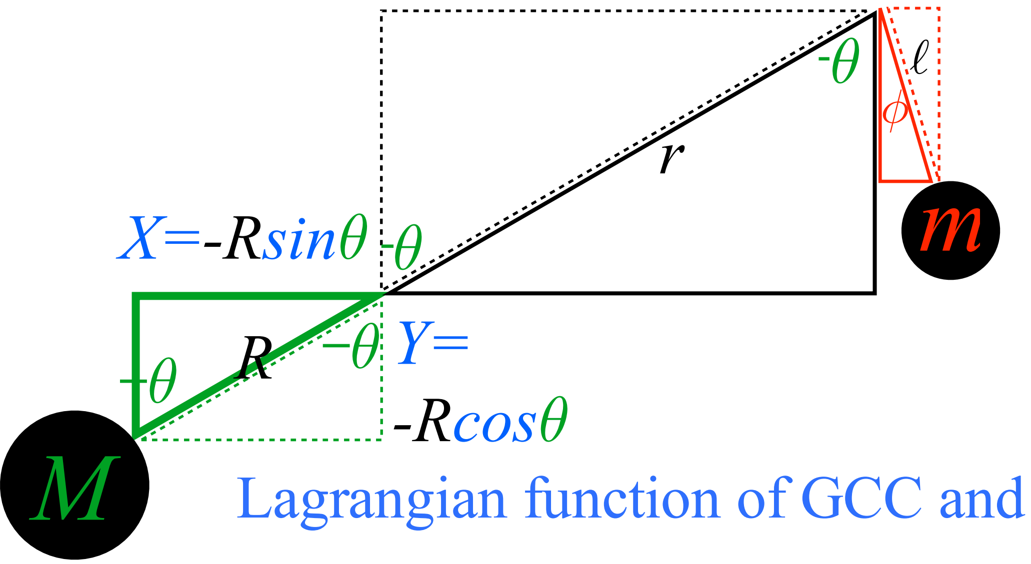


$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial\theta}d\theta + \frac{\partial L}{\partial\phi}d\phi + \frac{\partial L}{\partial\dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial\dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \frac{\partial L}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial\dot{\theta}}\frac{d\dot{\theta}}{dt} + \frac{\partial L}{\partial\dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

Defining the Hamiltonian function

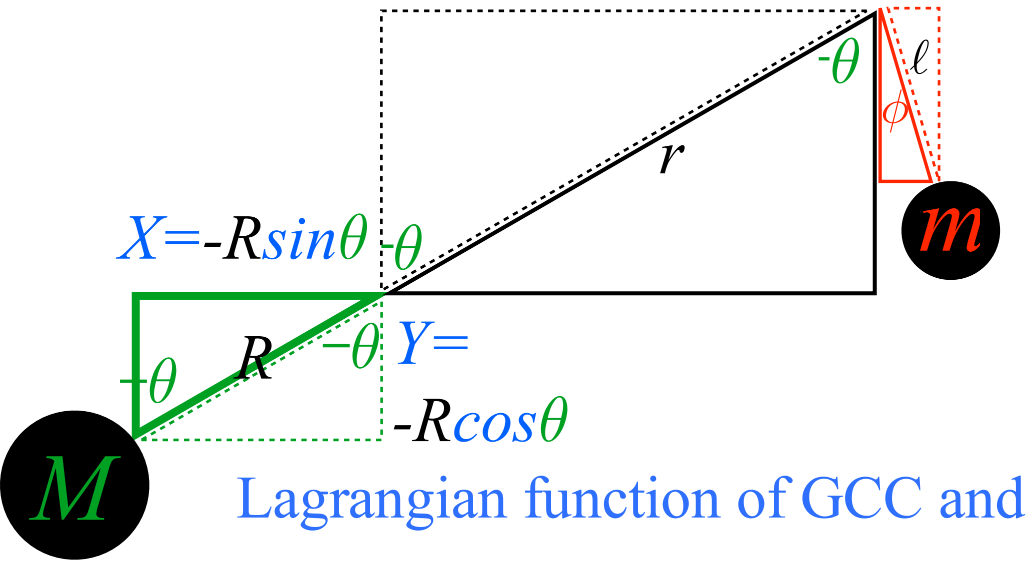
Hamiltonian function of GCC and **momenta**: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} = -\frac{\partial L}{\partial\theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} - \cancel{\frac{\partial L}{\partial p_\theta}} = \dot{\theta}$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



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$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Dynamic metric tensor

γ_{mn}

in GCC θ and ϕ

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

$$dL(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{\partial L}{\partial\theta}d\theta + \frac{\partial L}{\partial\phi}d\phi + \frac{\partial L}{\partial\dot{\theta}}d\dot{\theta} + \frac{\partial L}{\partial\dot{\phi}}d\dot{\phi} + \frac{\partial L}{\partial t}dt$$

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velocity chain

$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \overset{\downarrow}{\dot{p}_\theta}\frac{d\theta}{dt} + \overset{\downarrow}{\dot{p}_\phi}\frac{d\phi}{dt} + \overset{\downarrow}{p_\theta}\frac{d\dot{\theta}}{dt} + \overset{\downarrow}{p_\phi}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

Lagrange equations

$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

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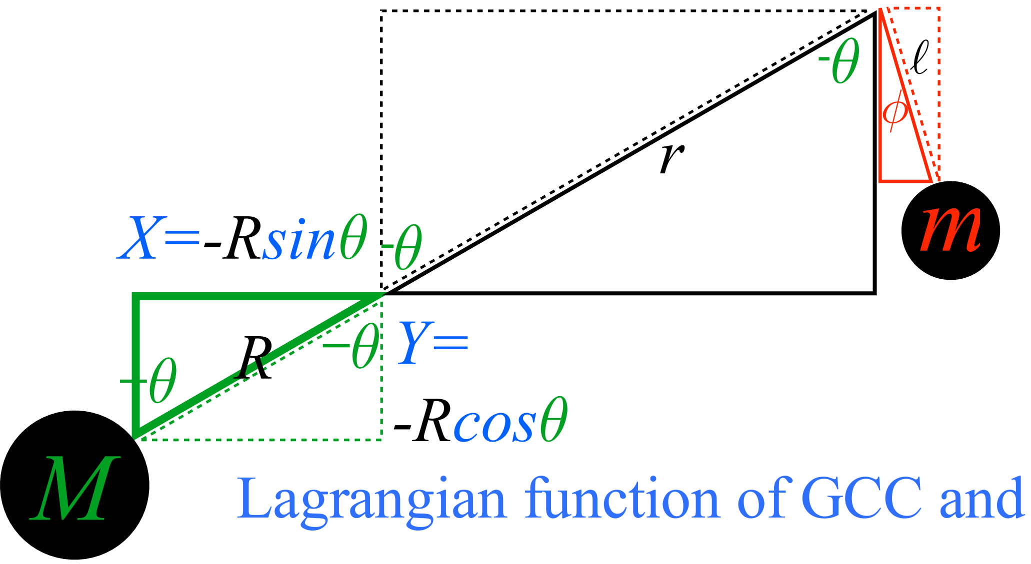
Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} \overset{\downarrow}{=} -\underset{\uparrow}{\frac{\partial L}{\partial\theta}} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} \overset{\downarrow}{=} \dot{\theta} - \underset{\uparrow}{\cancel{\frac{\partial L}{\partial p_\theta}}} = \dot{\theta} \qquad \frac{\partial H}{\partial\dot{\theta}} \overset{\downarrow}{=} p_\theta - \underset{\uparrow}{\frac{\partial L}{\partial\dot{\theta}}} = 0$$

by Lagrange equations

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



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Dynamic metric tensor

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$$\dot{L}(\theta,\phi,\dot{\theta},\dot{\phi},t) = \frac{dL}{dt} = \dot{p}_\theta\frac{d\theta}{dt} + \dot{p}_\phi\frac{d\phi}{dt} + p_\theta\frac{d\dot{\theta}}{dt} + p_\phi\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t}$$

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$$= \frac{dL}{dt} = \frac{d}{dt}\left(p_\theta\dot{\theta} + p_\phi\dot{\phi}\right) + \frac{\partial L}{\partial t}$$

(Consolidating)

$$\frac{d}{dt}\underbrace{\left(p_\theta\dot{\theta} + p_\phi\dot{\phi} - L\right)}_{} = -\frac{\partial L}{\partial t}$$

(Rearranging)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

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Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$

by assumed Lagrange functionality

$$\frac{\partial H}{\partial\theta} = -\dot{p}_\theta \qquad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \qquad \cancel{\frac{\partial H}{\partial\dot{\theta}} \equiv 0} \qquad \frac{\partial H}{\partial\phi} = -\dot{p}_\phi \qquad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \qquad \cancel{\frac{\partial H}{\partial\dot{\phi}} \equiv 0}$$

Hamilton's equations

by Lagrange equations

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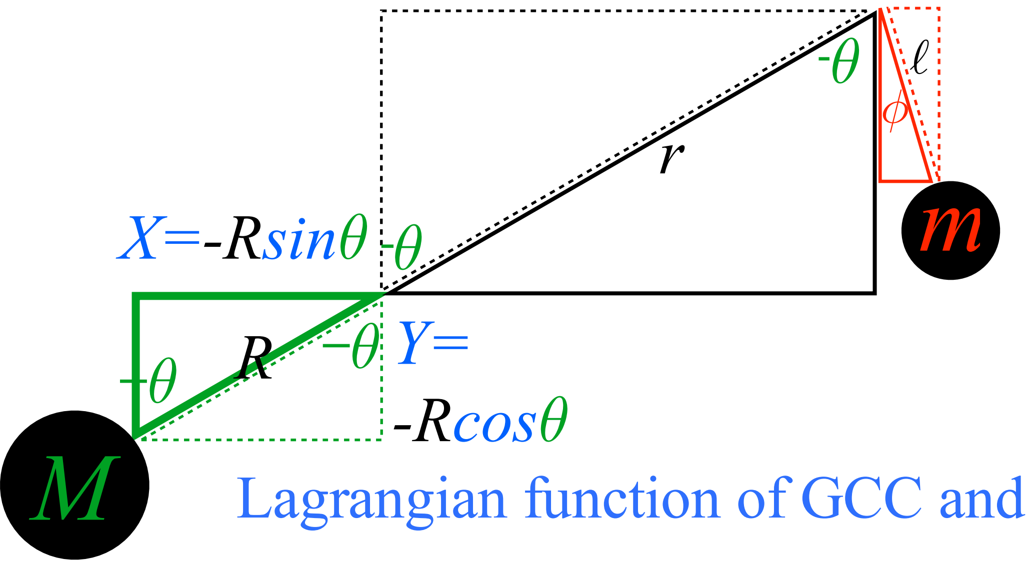

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(Rearranging)

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$

Defining the Hamiltonian function

Hamiltonian function of GCC and momenta: $\boxed{H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L}$ *Poincare-Legendre relation*

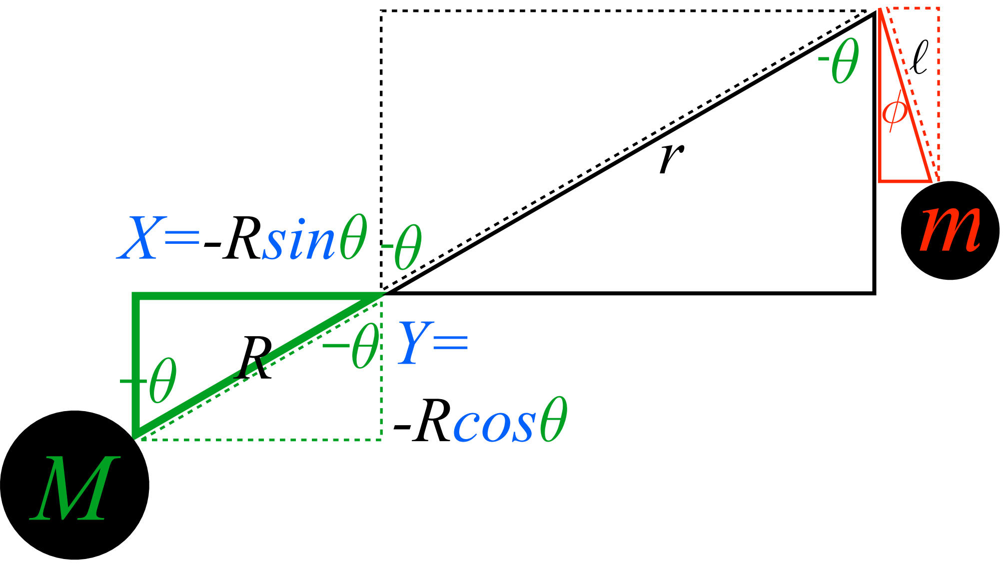
$$\boxed{\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \cancel{\frac{\partial H}{\partial \dot{\theta}}} = 0 \qquad \frac{\partial H}{\partial \phi} = -\dot{p}_\phi \quad \frac{\partial H}{\partial p_\phi} = \dot{\phi} \quad \cancel{\frac{\partial H}{\partial \dot{\phi}}} \equiv 0}$$

Hamilton's equations

Review of Hamiltonian equation derivation (Elementary trebuchet)

- *Hamiltonian definition from Lagrangian and γ_{mn} tensor*
- *Hamilton's equations and Poincare invariant relations*
- → *Hamiltonian expression and contravariant γ^{mn} tensor*

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



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$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

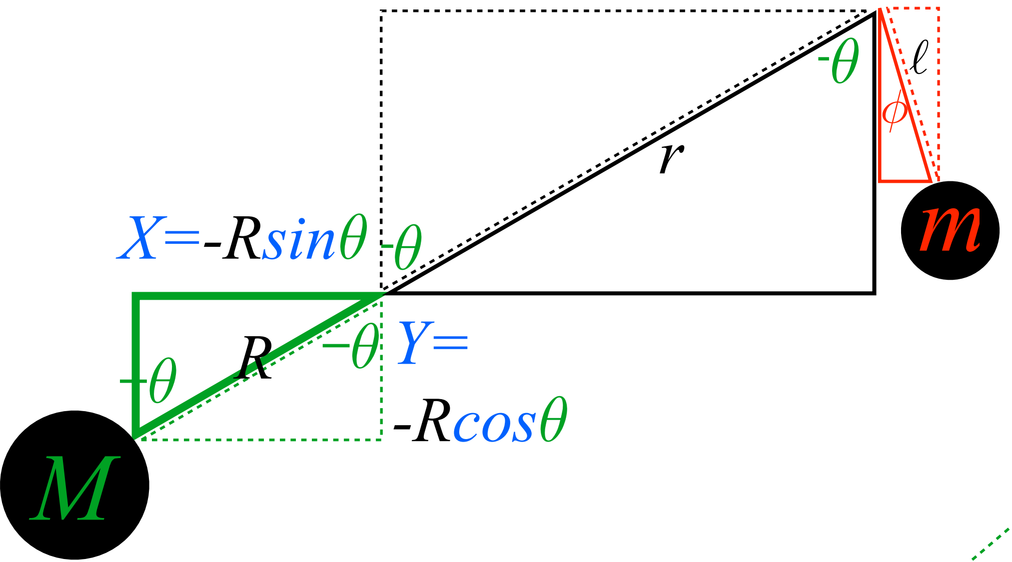
Lagrangian function of GCC and velocities: $L\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) = T\left(\theta,\phi,\dot{\theta},\dot{\phi},t\right) - V\left(\theta,\phi,t\right)$

Hamiltonian function of GCC and momenta: $H\left(\theta,\phi,p_\theta,p_\phi,t\right) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = \quad p_\theta\dot{\theta} \quad + \quad p_\phi\dot{\phi} \quad - T \quad + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

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Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

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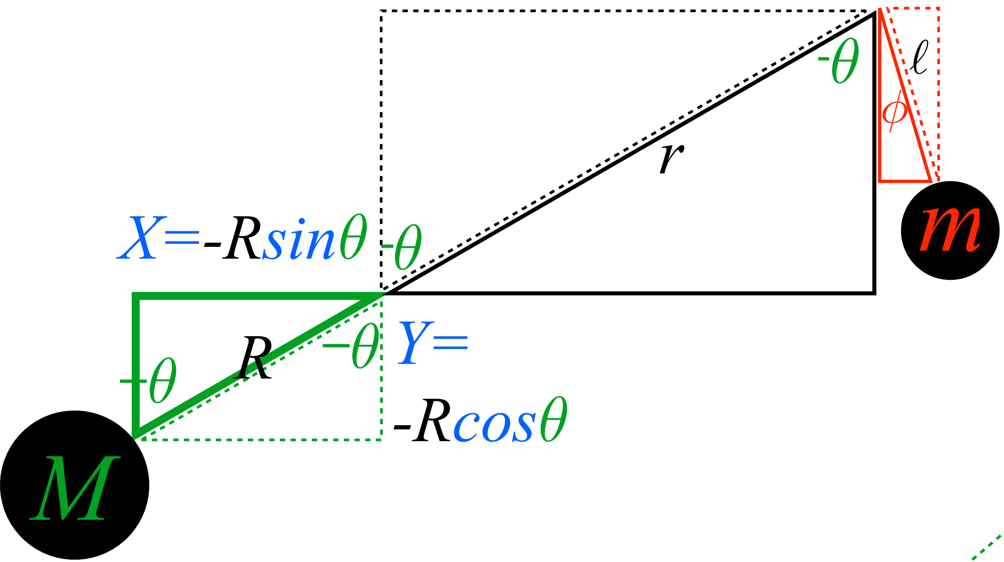
$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V$$

(Only correct numerically!)

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

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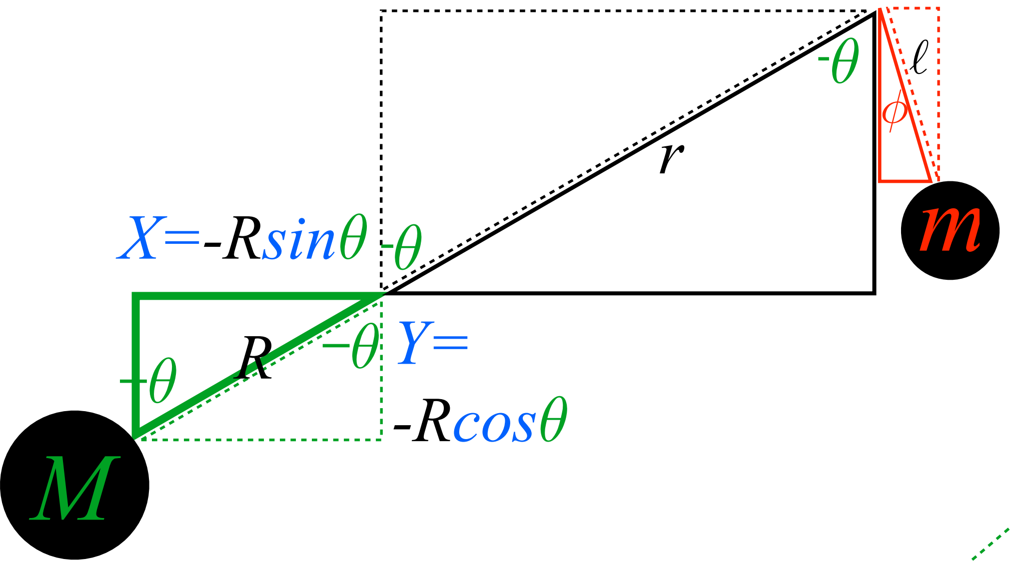
$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

(Only correct numerically!)

Hamiltonian must be explicit in momenta p_m

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$



$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix}p_\theta \\ p_\phi\end{pmatrix} = \begin{pmatrix}\gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix}\dot{\theta} \\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta \\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\,\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix}\begin{pmatrix}p_\theta \\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta,\phi,\dot{\theta},\dot{\phi},t) = T(\theta,\phi,\dot{\theta},\dot{\phi},t) - V(\theta,\phi,t)$

Hamiltonian function of GCC and momenta: $H(\theta,\phi,p_\theta,p_\phi,t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = (\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi})\dot{\theta} + (\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi})\dot{\phi} - \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V$$

$$H = \frac{1}{2}(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}) + V \qquad = T + V \equiv E$$

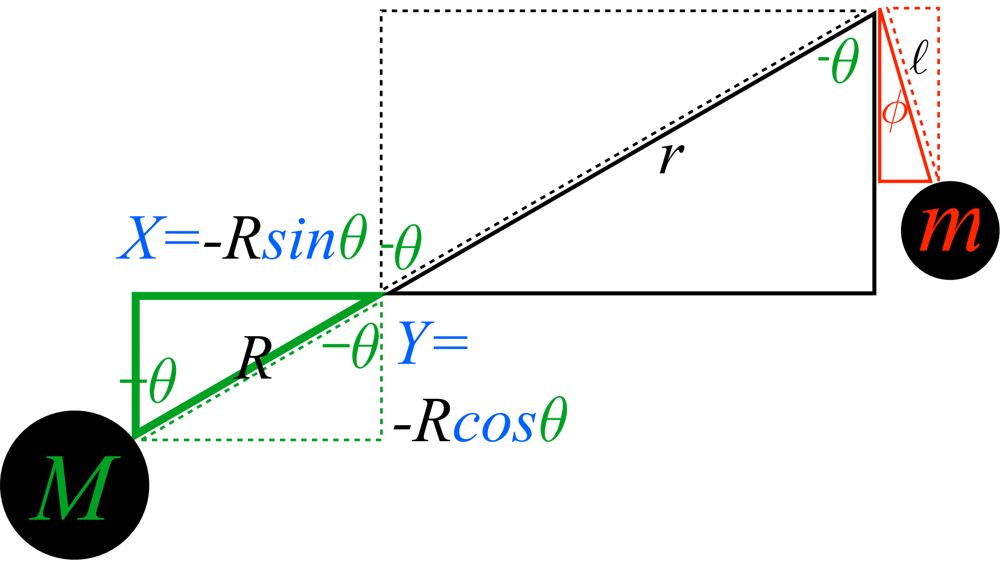
(Only correct numerically!)

Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi) + V = T + V$$

(Correct formally and numerically)

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2 \right] = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$



$$T = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \underbrace{\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta - \phi) \\ -mr\ell\cos(\theta - \phi) & m\ell^2 \end{pmatrix}} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \gamma_{\theta\theta} & \gamma_{\theta\phi} \\ \gamma_{\phi\theta} & \gamma_{\phi\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Covariant metric tensor

γ_{mn}

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2}\frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \overbrace{\begin{pmatrix} m\ell^2 & mr\ell\cos(\theta - \phi) \\ mr\ell\cos(\theta - \phi) & MR^2 + mr^2 \end{pmatrix}} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2\sin^2(\theta - \phi) \right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

Lagrangian function of GCC and velocities: $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t) = T(\theta, \phi, \dot{\theta}, \dot{\phi}, t) - V(\theta, \phi, t)$

Hamiltonian function of GCC and momenta: $H(\theta, \phi, p_\theta, p_\phi, t) = p_\theta\dot{\theta} + p_\phi\dot{\phi} - L$ *Poincare-Legendre relation*

$$H = p_\theta\dot{\theta} + p_\phi\dot{\phi} - T + V$$

$$H = \left(\gamma_{\theta\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\right)\dot{\theta} + \left(\gamma_{\phi\theta}\dot{\theta} + \gamma_{\phi\phi}\dot{\phi}\right)\dot{\phi} - \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V$$

$$H = \frac{1}{2}\left(\gamma_{\theta\theta}\dot{\theta}\dot{\theta} + \gamma_{\theta\phi}\dot{\phi}\dot{\theta} + \gamma_{\phi\theta}\dot{\theta}\dot{\phi} + \gamma_{\phi\phi}\dot{\phi}\dot{\phi}\right) + V \qquad = T + V \equiv E$$

(Only correct numerically!)

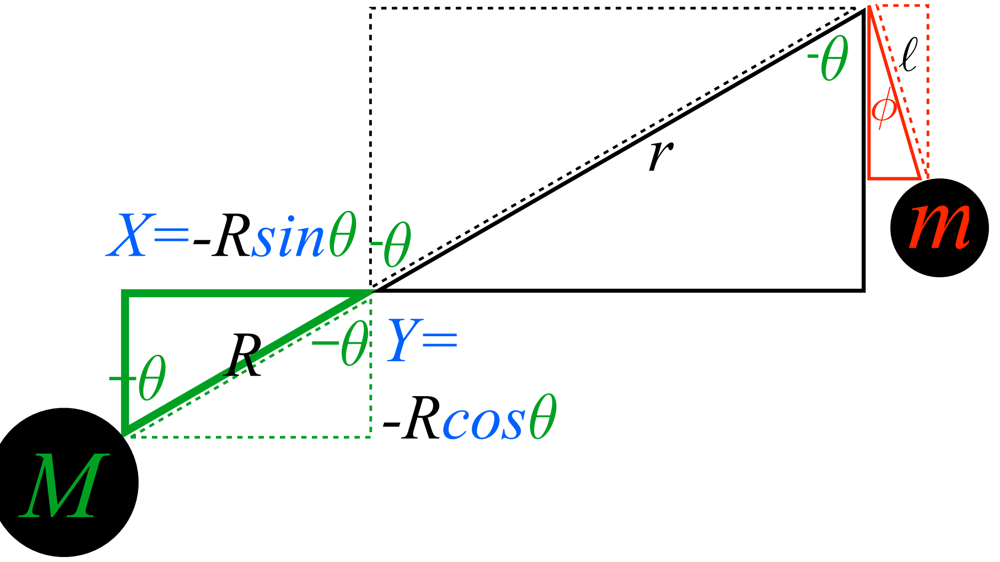
Hamiltonian must be explicit in momenta p_m

$$H = \frac{1}{2}\left(\gamma^{\theta\theta}p_\theta p_\theta + \gamma^{\theta\phi}p_\theta p_\phi + \gamma^{\phi\theta}p_\phi p_\theta + \gamma^{\phi\phi}p_\phi p_\phi\right) + V \quad = T + V$$

(Correct formally and numerically)

$$H = \frac{m\ell^2 p_\theta p_\theta + 2mr\ell\cos(\theta - \phi)p_\theta p_\phi + (MR^2 + mr^2)p_\phi p_\phi}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta - \phi) \right]} + V$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$\boxed{\begin{aligned}\dot\theta &= \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi\\ \dot\phi &= \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$T = \frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn}p_m p_n$$

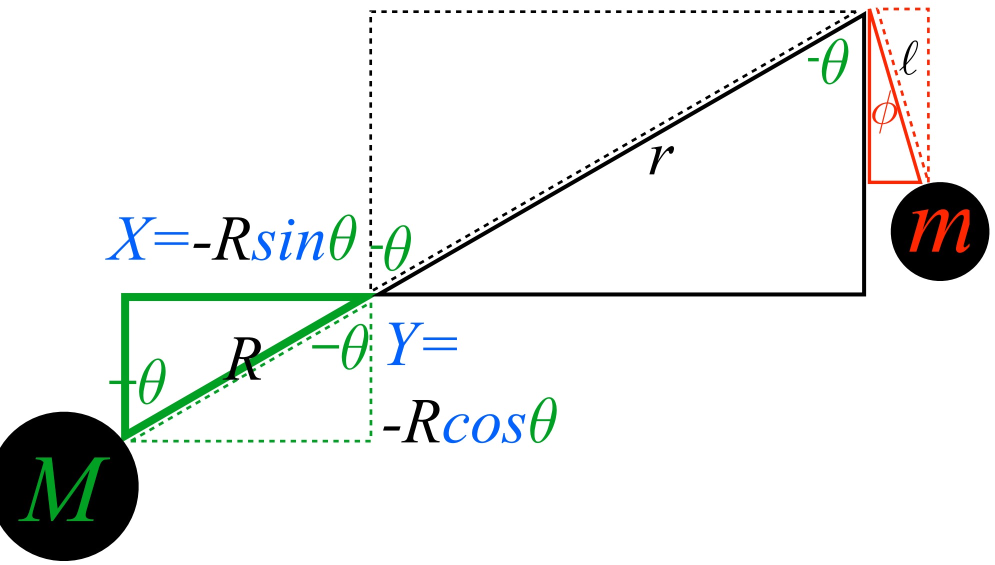
$$\boxed{\begin{aligned}p_\theta &= \gamma_{\theta\theta}\dot\theta + \gamma_{\theta\phi}\dot\phi\\ p_\phi &= \gamma_{\phi\theta}\dot\theta + \gamma_{\phi\phi}\dot\phi\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta}p_\theta + \gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta}p_\theta + \gamma^{\theta\phi}p_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot{\theta}&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot{\phi}&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot{\theta}+\gamma_{\theta\phi}\dot{\phi}\\ p_\phi&=\gamma_{\phi\theta}\dot{\theta}+\gamma_{\phi\phi}\dot{\phi}\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot{\theta}=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot{\phi}=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

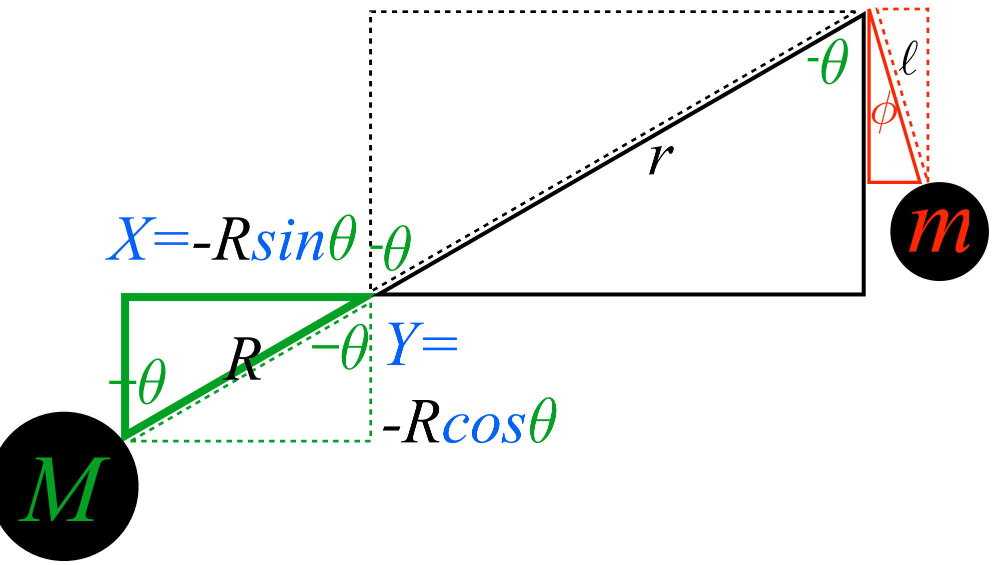
Momentum/force equations

$$\dot{p}_\theta=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}$$
$$=mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)+F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot{p}_\phi=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}$$
$$=\ \ \text{-}mr\ell\dot{\theta}\dot{\phi}\sin(\theta-\phi)+F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_m p_n$$

$$\boxed{\begin{aligned}\dot\theta&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot\phi&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot\theta+\gamma_{\theta\phi}\dot\phi\\ p_\phi&=\gamma_{\phi\theta}\dot\theta+\gamma_{\phi\phi}\dot\phi\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot\theta=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot\phi=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

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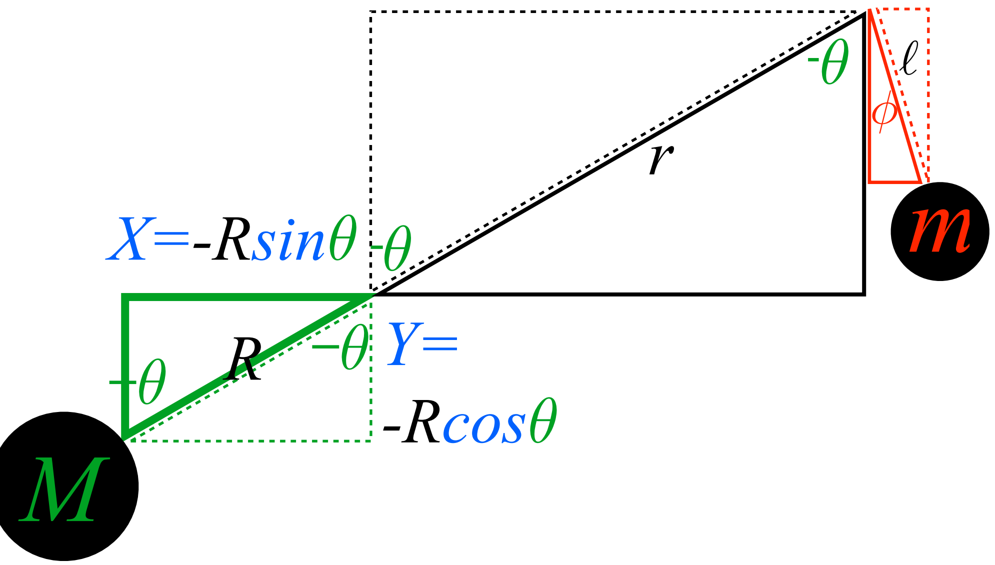
$$\dot p_\theta=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}$$
$$=mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\theta$$
$$=mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\theta$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\dot p_\phi=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}$$
$$=\ \text{-}mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\phi$$
$$=\text{-}mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\phi$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}=\begin{pmatrix}\gamma^{\theta\theta} & \gamma^{\theta\phi}\\ \gamma^{\phi\theta} & \gamma^{\phi\phi}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T=\frac{1}{2}\frac{\begin{pmatrix}p_\theta & p_\phi\end{pmatrix}\overbrace{\begin{pmatrix}m\ell^2 & mr\ell\cos(\theta-\phi)\\ mr\ell\cos(\theta-\phi) & MR^2+mr^2\end{pmatrix}}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}}{m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]}=\frac{1}{2}\gamma^{mn}p_mp_n$$

$$\boxed{\begin{aligned}\dot\theta&=\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi\\ \dot\phi&=\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi\end{aligned}}$$

$$\boxed{\begin{aligned}p_\theta&=\gamma_{\theta\theta}\dot\theta+\gamma_{\theta\phi}\dot\phi\\ p_\phi&=\gamma_{\phi\theta}\dot\theta+\gamma_{\phi\phi}\dot\phi\end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta}=\dot\theta=\ \gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi$$

$$\frac{\partial H}{\partial p_\phi}=\dot\phi=\ \gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi$$

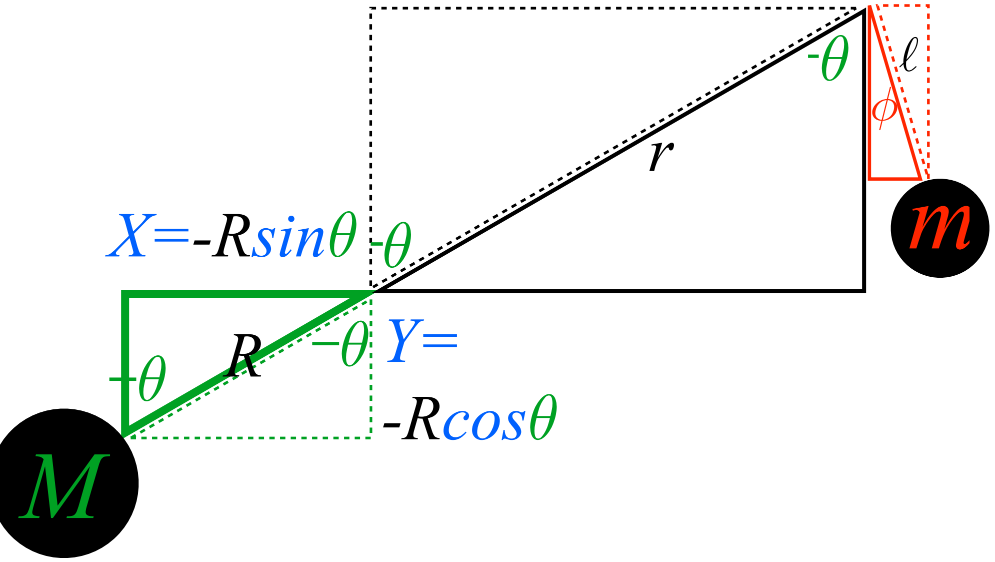
Momentum/force equations

$$\begin{aligned}\dot p_\theta&=-\frac{\partial H}{\partial\theta}=\frac{\partial L}{\partial\theta}=\frac{\partial T}{\partial\theta}-\frac{\partial V}{\partial\theta}\\ &=mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\theta\\ &=mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\theta\\ &=mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta}p_\theta^2+[\gamma^{\theta\theta}\gamma^{\phi\phi}+(\gamma^{\theta\phi})^2]p_\phi p_\theta+\gamma^{\theta\phi}\gamma^{\phi\phi}p_\phi^2)\sin(\theta-\phi)+F_\theta\end{aligned}$$

(May just use Lagrange results…
…but to be formally correct…
…must convert contra-velocities
to covariant momenta!)

$$\begin{aligned}\dot p_\phi&=-\frac{\partial H}{\partial\phi}=\frac{\partial L}{\partial\phi}=\frac{\partial T}{\partial\phi}-\frac{\partial V}{\partial\phi}\\ &=\ \text{-}mr\ell\dot\theta\dot\phi\sin(\theta-\phi)+F_\phi\\ &=\text{-}mr\ell(\gamma^{\theta\theta}p_\theta+\gamma^{\theta\phi}p_\phi)(\gamma^{\phi\theta}p_\theta+\gamma^{\phi\phi}p_\phi)\sin(\theta-\phi)+F_\phi\\ &=\text{-[messy factor]}\sin(\theta-\phi)+F_\phi\end{aligned}$$

Hamilton equations for elementary trebuchet



$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \gamma^{\theta\theta} & \gamma^{\theta\phi} \\ \gamma^{\phi\theta} & \gamma^{\phi\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

Contravariant metric tensor

γ^{mn}

$$T = \frac{1}{2} \frac{\begin{pmatrix} p_\theta & p_\phi \end{pmatrix} \begin{pmatrix} m\ell^2 & mr\ell\cos(\theta-\phi) \\ mr\ell\cos(\theta-\phi) & MR^2 + mr^2 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}}{m\ell^2\left[MR^2 + mr^2 \sin^2(\theta-\phi)\right]} = \frac{1}{2}\gamma^{mn} p_m p_n$$

$$\boxed{\begin{aligned} \dot\theta &= \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi \\ \dot\phi &= \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi \end{aligned}}$$

$$\boxed{\begin{aligned} p_\theta &= \gamma_{\theta\theta}\dot\theta + \gamma_{\theta\phi}\dot\phi \\ p_\phi &= \gamma_{\phi\theta}\dot\theta + \gamma_{\phi\phi}\dot\phi \end{aligned}}$$

Coordinate equations

$$\frac{\partial H}{\partial p_\theta} = \dot\theta = \gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi$$

$$\frac{\partial H}{\partial p_\phi} = \dot\phi = \gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi$$

Momentum/force equations

$$\begin{aligned} \dot p_\theta &= -\frac{\partial H}{\partial \theta} = \frac{\partial L}{\partial \theta} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} \\ &= mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\theta \\ &= mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi)\sin(\theta-\phi) + F_\theta \\ &= mr\ell(\gamma^{\theta\theta}\gamma^{\phi\theta} p_\theta^2 + [\gamma^{\theta\theta}\gamma^{\phi\phi} + (\gamma^{\theta\phi})^2] p_\phi p_\theta + \gamma^{\theta\phi}\gamma^{\phi\phi} p_\phi^2)\sin(\theta-\phi) + F_\theta \end{aligned}$$

$$\begin{aligned} \dot p_\phi &= -\frac{\partial H}{\partial \phi} = \frac{\partial L}{\partial \phi} = \frac{\partial T}{\partial \phi} - \frac{\partial V}{\partial \phi} \\ &= \text{-}mr\ell\dot\theta\dot\phi\sin(\theta-\phi) + F_\phi \\ &= \text{-}mr\ell(\gamma^{\theta\theta} p_\theta + \gamma^{\theta\phi} p_\phi)(\gamma^{\phi\theta} p_\theta + \gamma^{\phi\phi} p_\phi)\sin(\theta-\phi) + F_\phi \\ &= \text{-[messy factor]}\sin(\theta-\phi) + F_\phi \end{aligned}$$

A lesson on Hamiltonian “elegance”...
...may be very elegant formally...but may not be so elegant algebraically!

Hamiltonian energy and momentum conservation and symmetry coordinates

- ➡ *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \qquad + \pi/2$

Jacobian: $\qquad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot{\theta}_B\\ \dot{\phi}_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

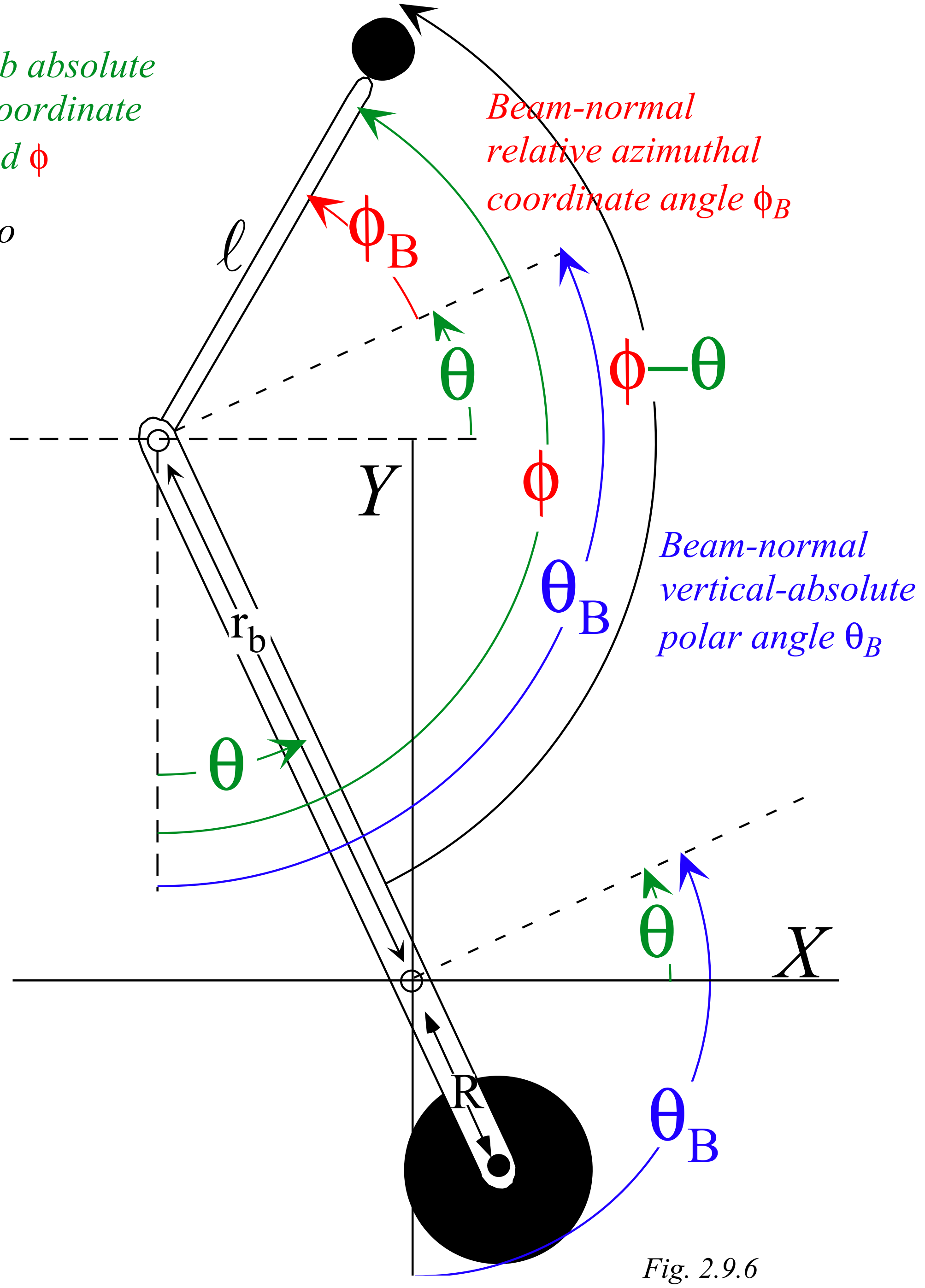


Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \qquad +\pi/2$

Jacobian: $\qquad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

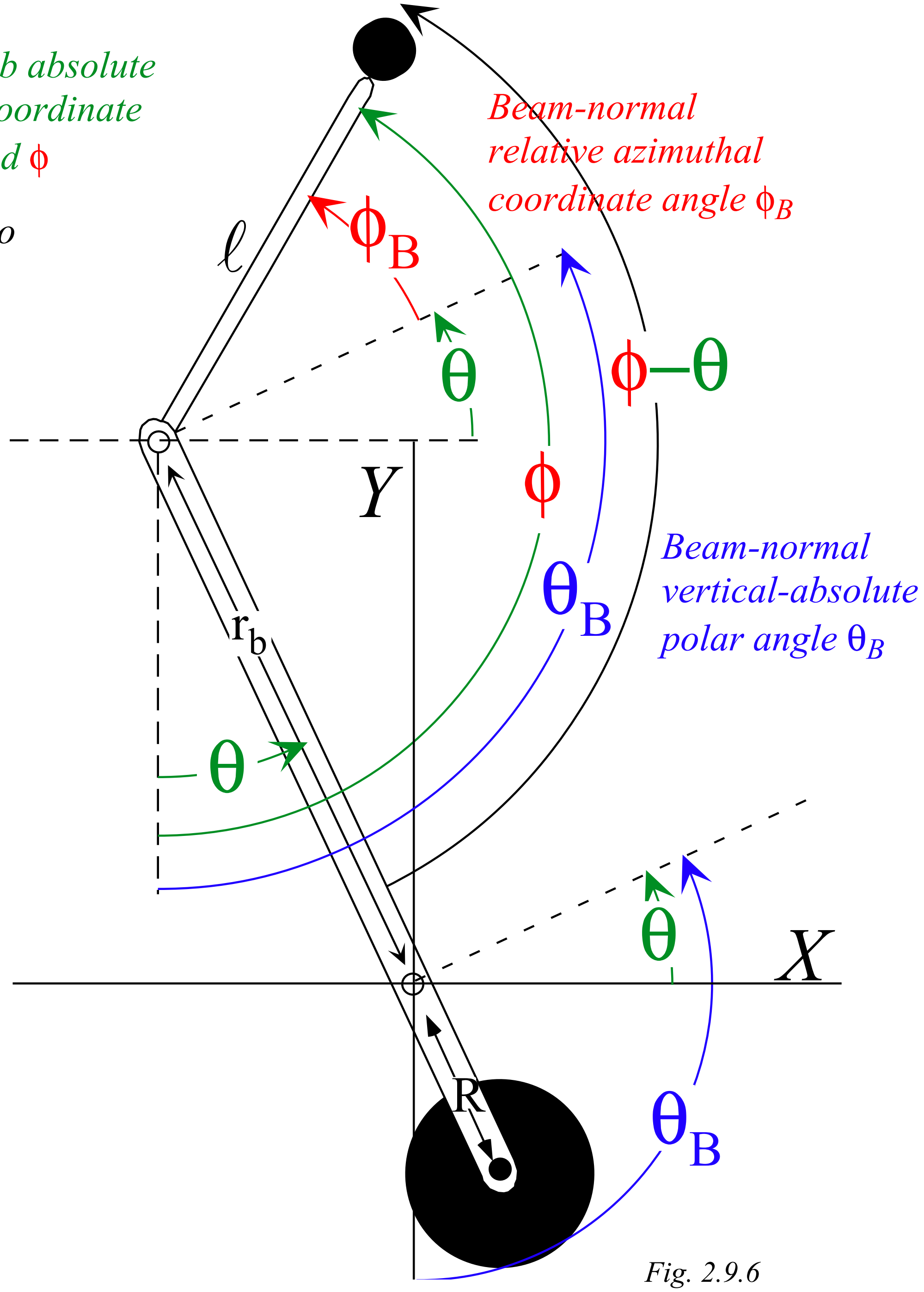


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\qquad \phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

$$\phi = \theta_B + \phi_B$$

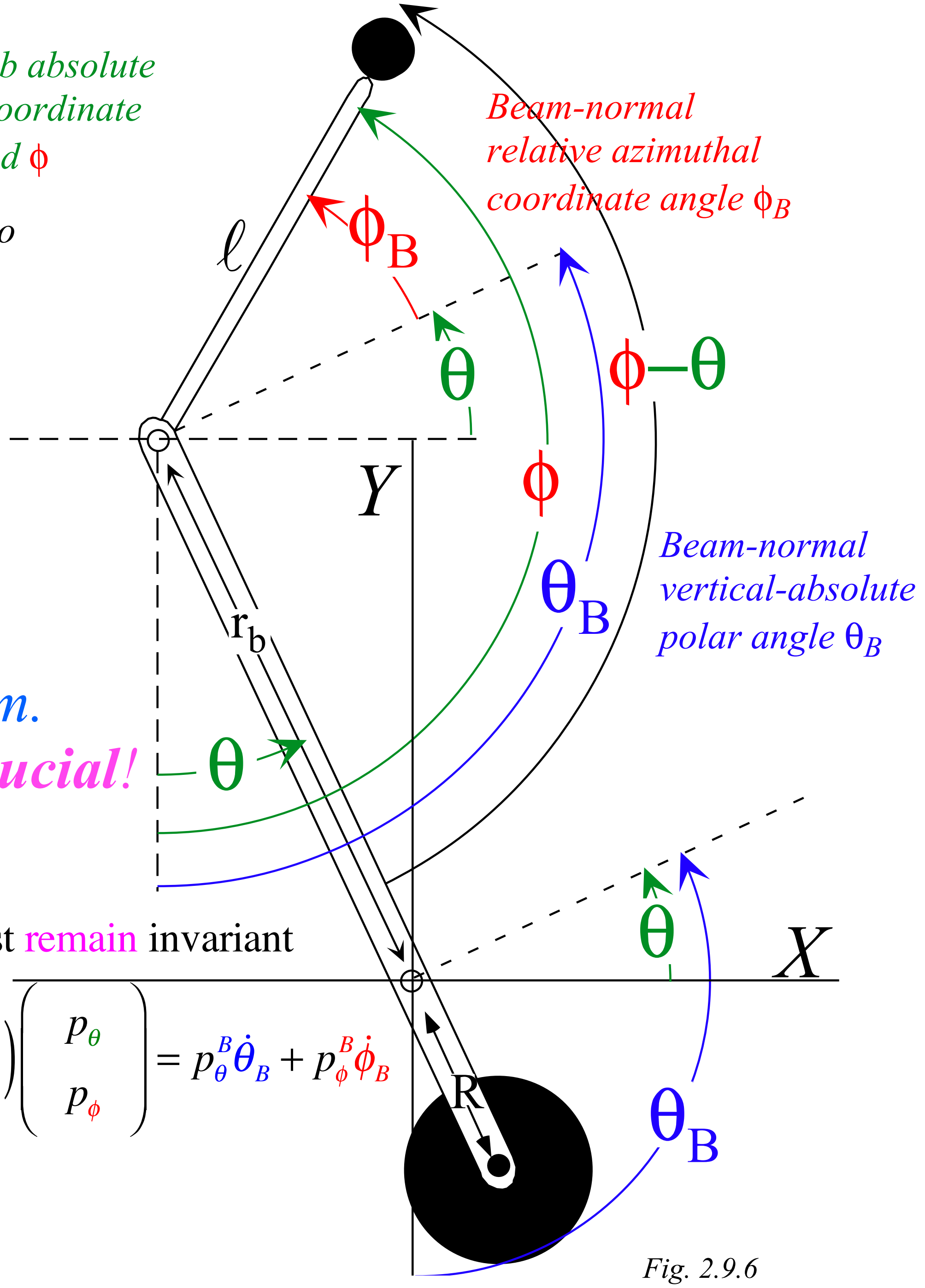
$$\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot\theta_B \\ \dot\phi_B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

Be careful with momentum.
*Poincare invariance is **crucial**!*



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B\dot\theta_B + p_\phi^B\dot\phi_B$$

Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B \ -\pi/2$

$\phi = \theta_B + \phi_B$

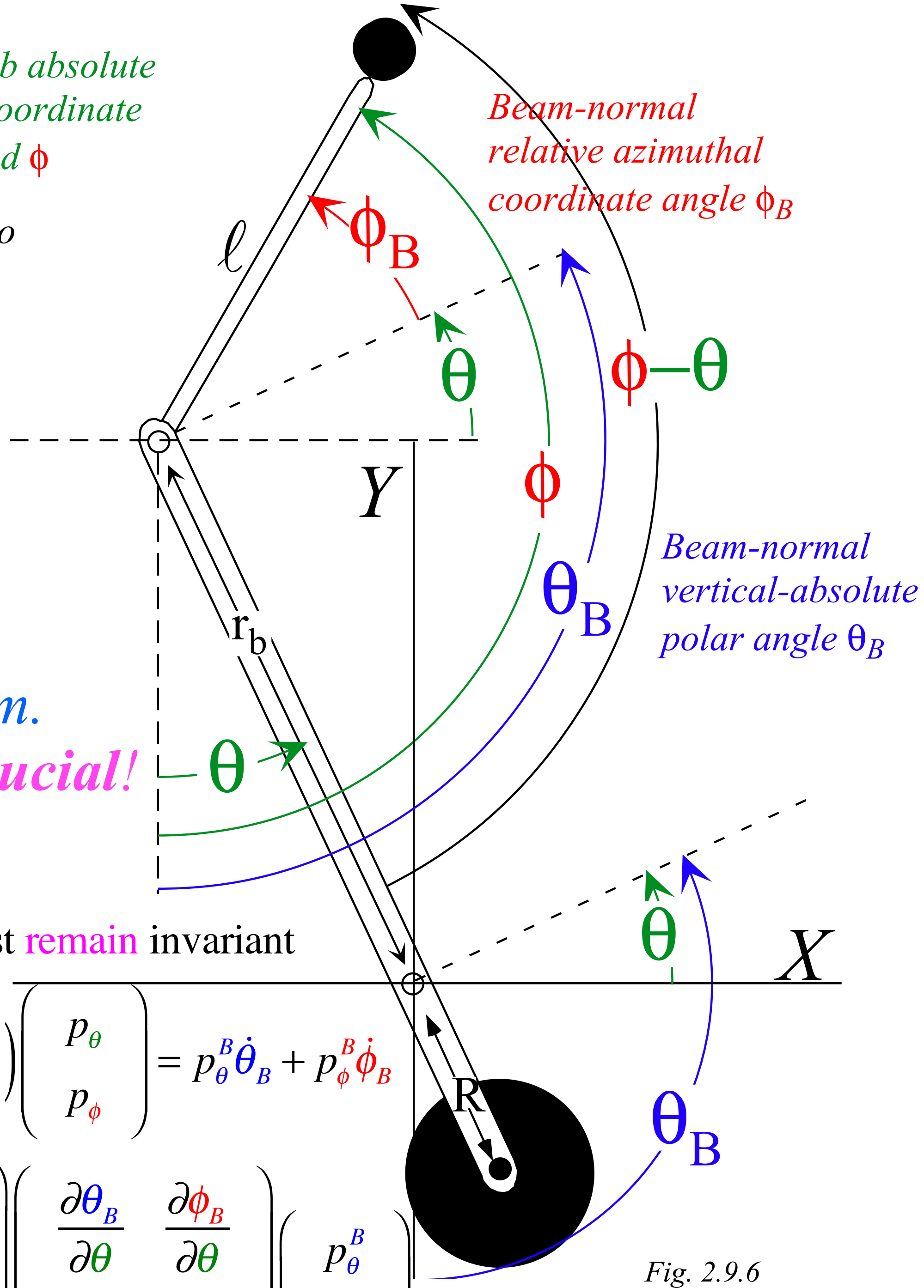
$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

$$\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$

Be careful with momentum.
*Poincare invariance is **crucial**!*



Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

$$\begin{pmatrix}\dot\theta_B & \dot\phi_B\end{pmatrix}\begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$

Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta \quad +\pi/2$

Jacobian: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

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p_m *transform is* TRANSPOSE INVERSE *to* q^m

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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$p_\phi = \quad p^B_\phi$

Be careful with momentum.
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$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

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Previous lab absolute trebuchet coordinate angles θ and φ

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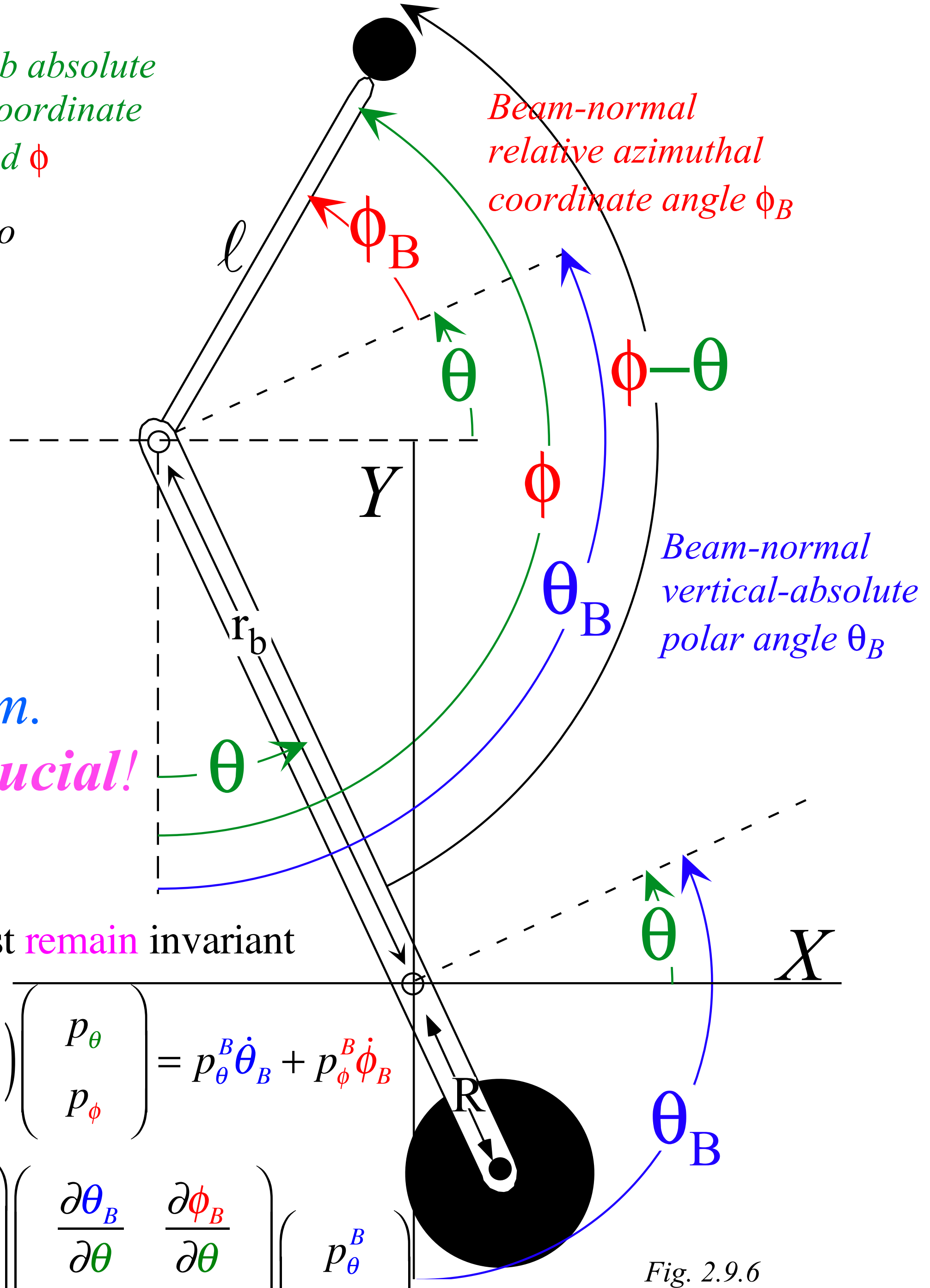

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p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}=\begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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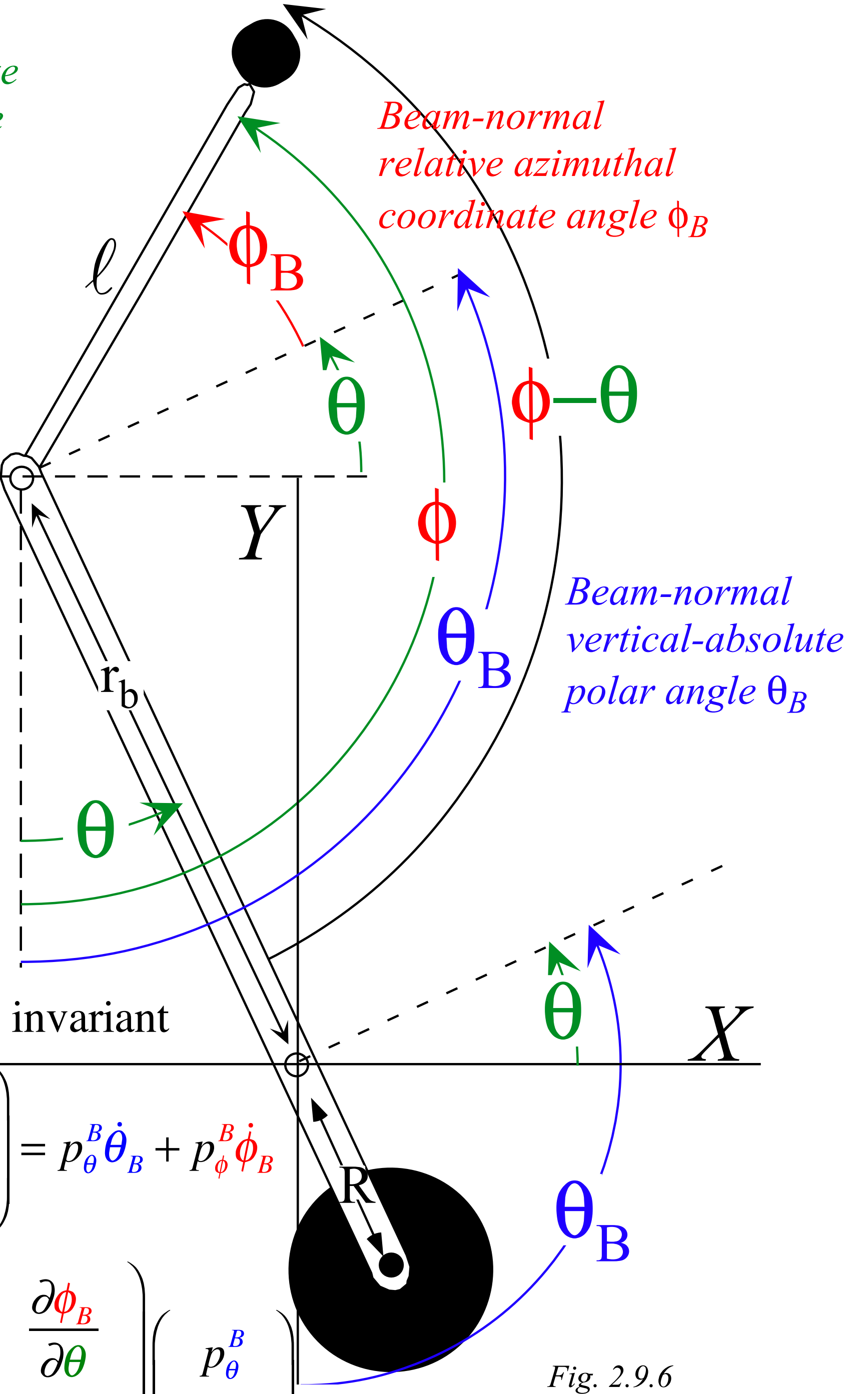
$$\begin{pmatrix}\dot\theta_B & \dot\phi_B\end{pmatrix}\begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$



Beam-normal relative azimuthal coordinate angle ϕ_B

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Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
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$$H=\frac{m\ell^2 p_\theta p_\theta + (MR^2+mr^2)p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta-\phi)}{2m\ell^2\left[MR^2+mr^2\sin^2(\theta-\phi)\right]} \qquad +V$$

Original (ϕ,θ) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi_B = \phi - \theta - \pi/2$ and $\theta_B = \theta$ $+\pi/2$

Jacobian: $\phi_B = -\theta + \phi - \pi/2$

$$\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi}\\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix}$$

Kajobian of inverse transform $\phi_B = \phi - \theta - \pi/2$ and $\theta = \theta_B$ $-\pi/2$

$\phi = \theta_B + \phi_B$

$$\begin{pmatrix}\dot\theta\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B}\\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\dot\theta_B\\ \dot\phi_B\end{pmatrix}$$

p_m transform is TRANSPOSE INVERSE *to q^m*

$$\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix} = \begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p^B_\theta - p^B_\phi$

$p_\phi = p^B_\phi$

Previous lab absolute trebuchet coordinate angles θ and φ

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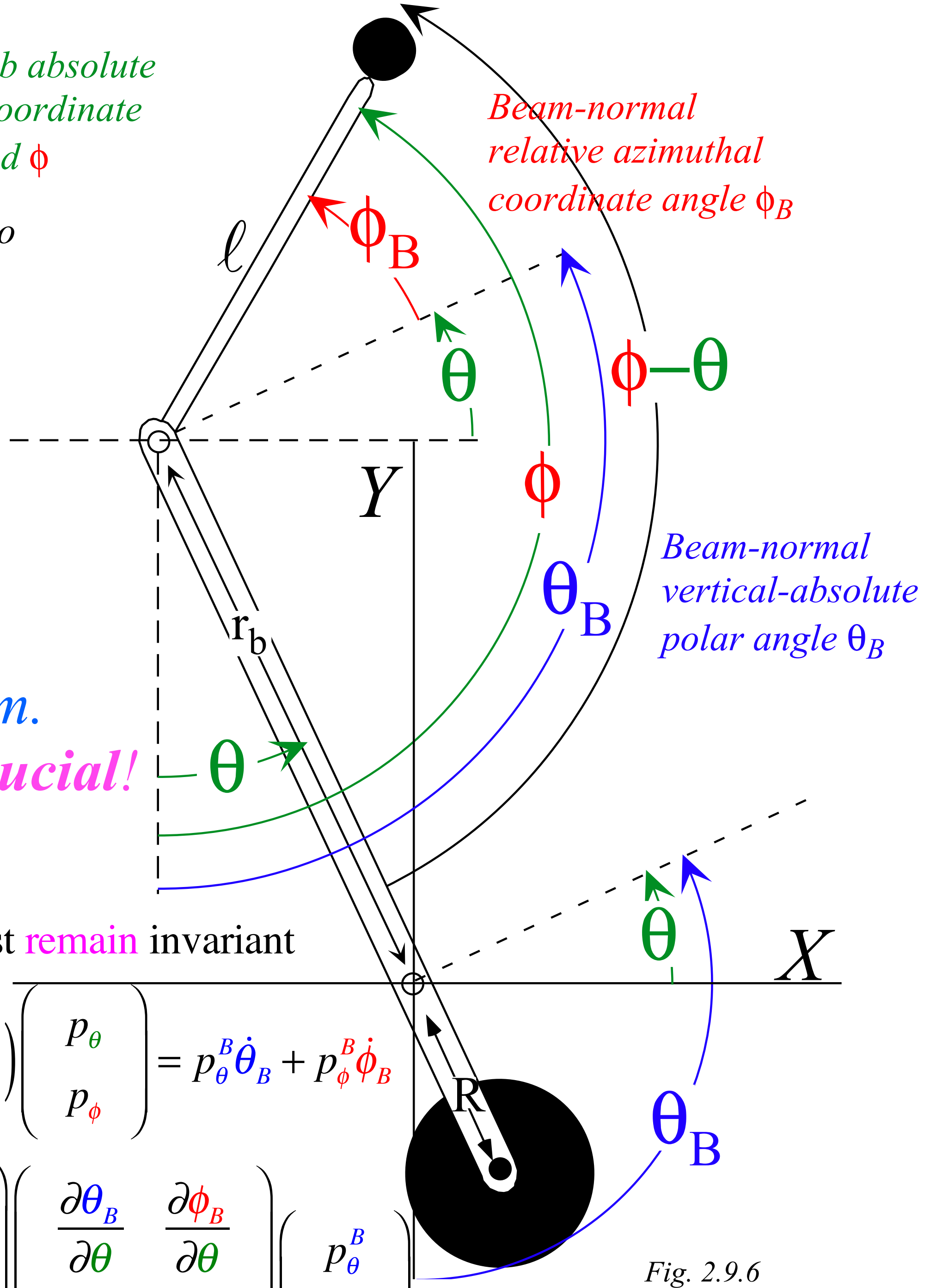

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Be careful with momentum.

Poincare invariance is ***crucial!***

Poincare invariant must remain invariant

$$p_\theta\dot\theta + p_\phi\dot\phi = \begin{pmatrix}\dot\theta & \dot\phi\end{pmatrix}\begin{pmatrix}p_\theta\\ p_\phi\end{pmatrix} = p^B_\theta\dot\theta_B + p^B_\phi\dot\phi_B$$

$$\begin{pmatrix}\dot\theta_B & \dot\phi_B\end{pmatrix}\begin{pmatrix}\frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B}\\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B}\end{pmatrix}\begin{pmatrix}\frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta}\\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi}\end{pmatrix}\begin{pmatrix}p^B_\theta\\ p^B_\phi\end{pmatrix}$$

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2)p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta - \phi)}{2m\ell^2\left[MR^2 + mr^2\sin^2(\theta - \phi)\right]} + V$$

Original (φ,θ) Hamiltonian

$$H = \frac{m\ell^2\left(p^B_\theta - p^B_\phi\right)^2 + \left(MR^2 + mr^2\right)\left(p^B_\phi\right)^2 - 2mr\ell p^B_\phi\left(p^B_\theta - p^B_\phi\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} + V$$

Transformed (ϕ_B,θ_B) Hamiltonian

Coordinate transformation helps reduce symmetric Hamiltonian

Define beam-relative angle $\phi B = \phi - \theta - \pi/2$ and $\theta B = \theta \quad +\pi/2$

Jacobian: $\phi B = -\theta + \phi - \pi/2$

$$\begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\theta_B}{\partial\phi} \\ \frac{\partial\phi_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

Kajobian of inverse transform $\phi B = \phi - \theta - \pi/2$ and $\theta = \theta B \; -\pi/2$

$\phi = \theta B + \phi B$

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\theta}{\partial\phi_B} \\ \frac{\partial\phi}{\partial\theta_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_B \\ \dot{\phi}_B \end{pmatrix}$$

p_m transform is **TRANSPOSE INVERSE** *to q^m*

$$\begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix} = \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix}$$

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Resulting momentum transform: $p_\theta = p_\theta^B \text{-} p_\phi^B$

$p_\phi = \quad p_\phi^B$

Be careful with momentum.

Poincare invariance is ***crucial!***

Poincare invariant must remain invariant

$$p_\theta\dot{\theta} + p_\phi\dot{\phi} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = p_\theta^B\dot{\theta}_B + p_\phi^B\dot{\phi}_B$$

$$\begin{pmatrix} \dot{\theta}_B & \dot{\phi}_B \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial\theta_B} & \frac{\partial\phi}{\partial\theta_B} \\ \frac{\partial\theta}{\partial\phi_B} & \frac{\partial\phi}{\partial\phi_B} \end{pmatrix} \begin{pmatrix} \frac{\partial\theta_B}{\partial\theta} & \frac{\partial\phi_B}{\partial\theta} \\ \frac{\partial\theta_B}{\partial\phi} & \frac{\partial\phi_B}{\partial\phi} \end{pmatrix} \begin{pmatrix} p_\theta^B \\ p_\phi^B \end{pmatrix}$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

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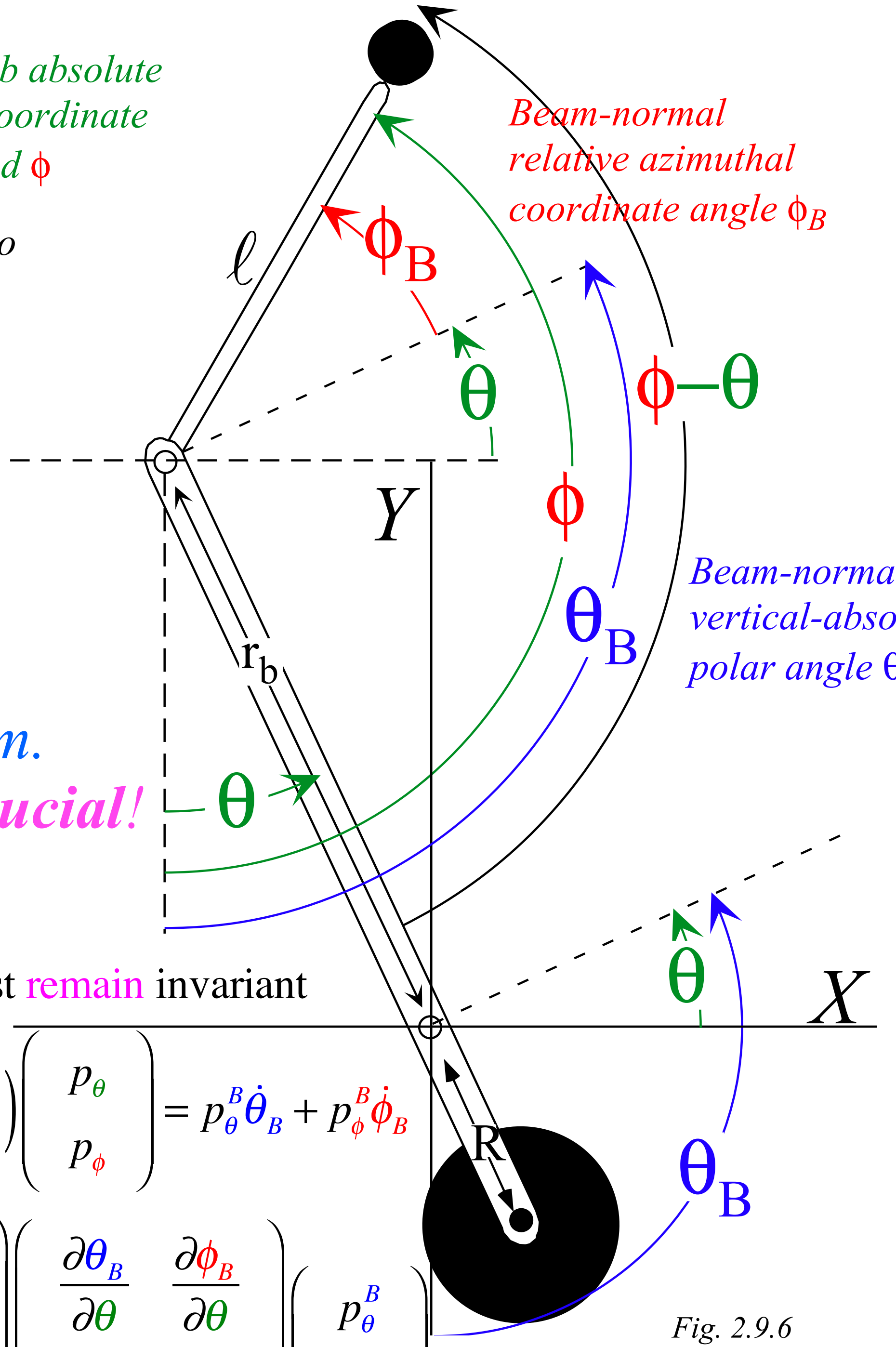


Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

$$H = \frac{m\ell^2 p_\theta p_\theta + (MR^2 + mr^2)p_\phi p_\phi + 2mr\ell\, p_\theta p_\phi \cos(\theta - \phi)}{2m\ell^2\left[MR^2 + mr^2 \sin^2(\theta - \phi)\right]} + (MR - mr)g\cos\theta + mg\ell\cos\phi$$

$$\boxed{\theta - \phi = -\pi/2 - \phi B}$$

$$H = \frac{m\ell^2\left(p_\theta^B - p_\phi^B\right)^2 + \left(MR^2 + mr^2\right)\left(p_\phi^B\right)^2 - 2mr\ell p_\phi^B\left(p_\theta^B - p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2 + mr^2\cos^2\phi_B\right]} - \left(MR - mr\right)g\sin\theta_B \; - mg\ell\cos\left(\phi_B + \theta_B\right)$$

$F_\theta = -MgR\sin\theta + mgr\sin\theta$

$F_\phi = -mg\ell\sin\phi$

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - → *Algebraic approach*
  - *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

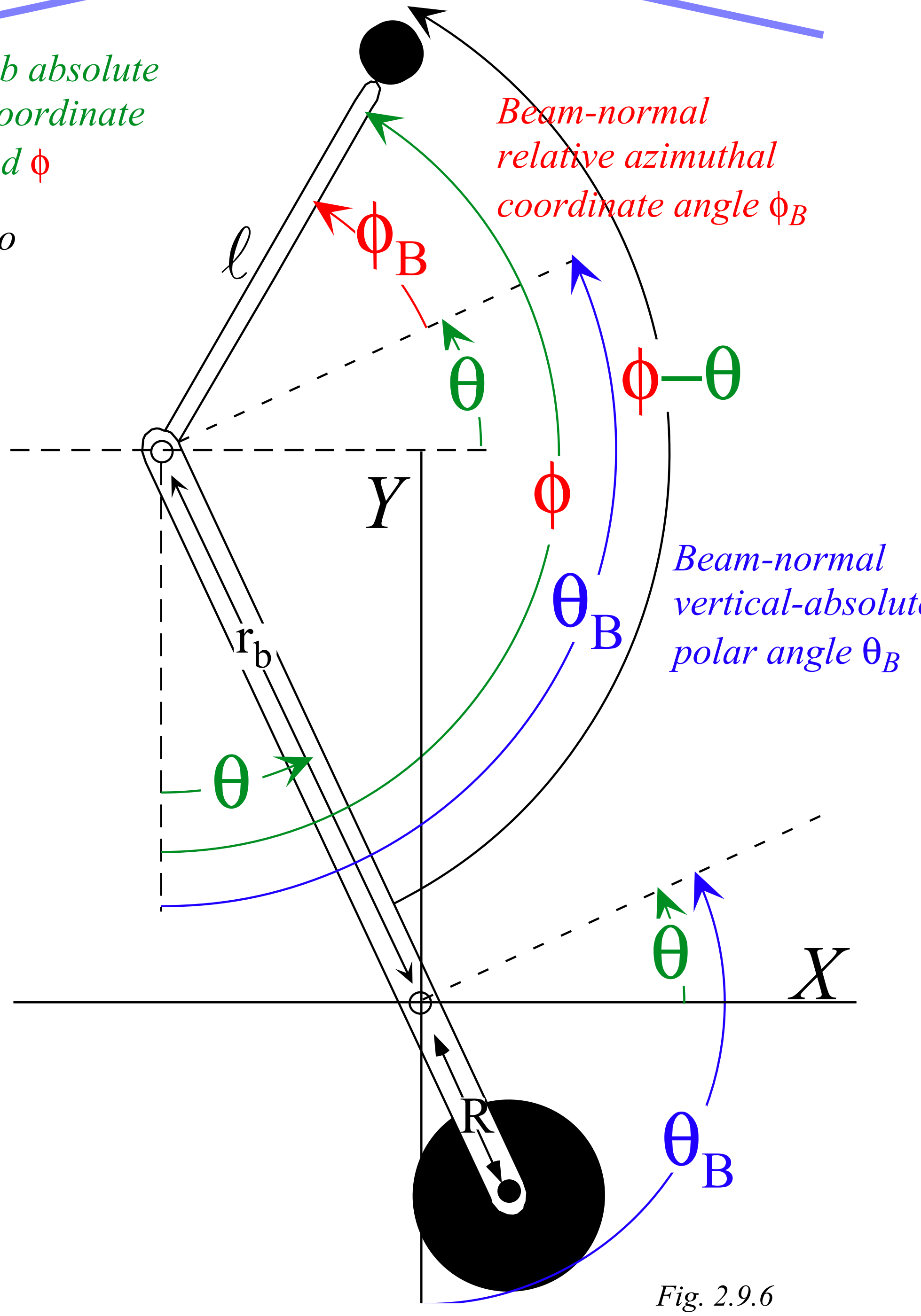
(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$



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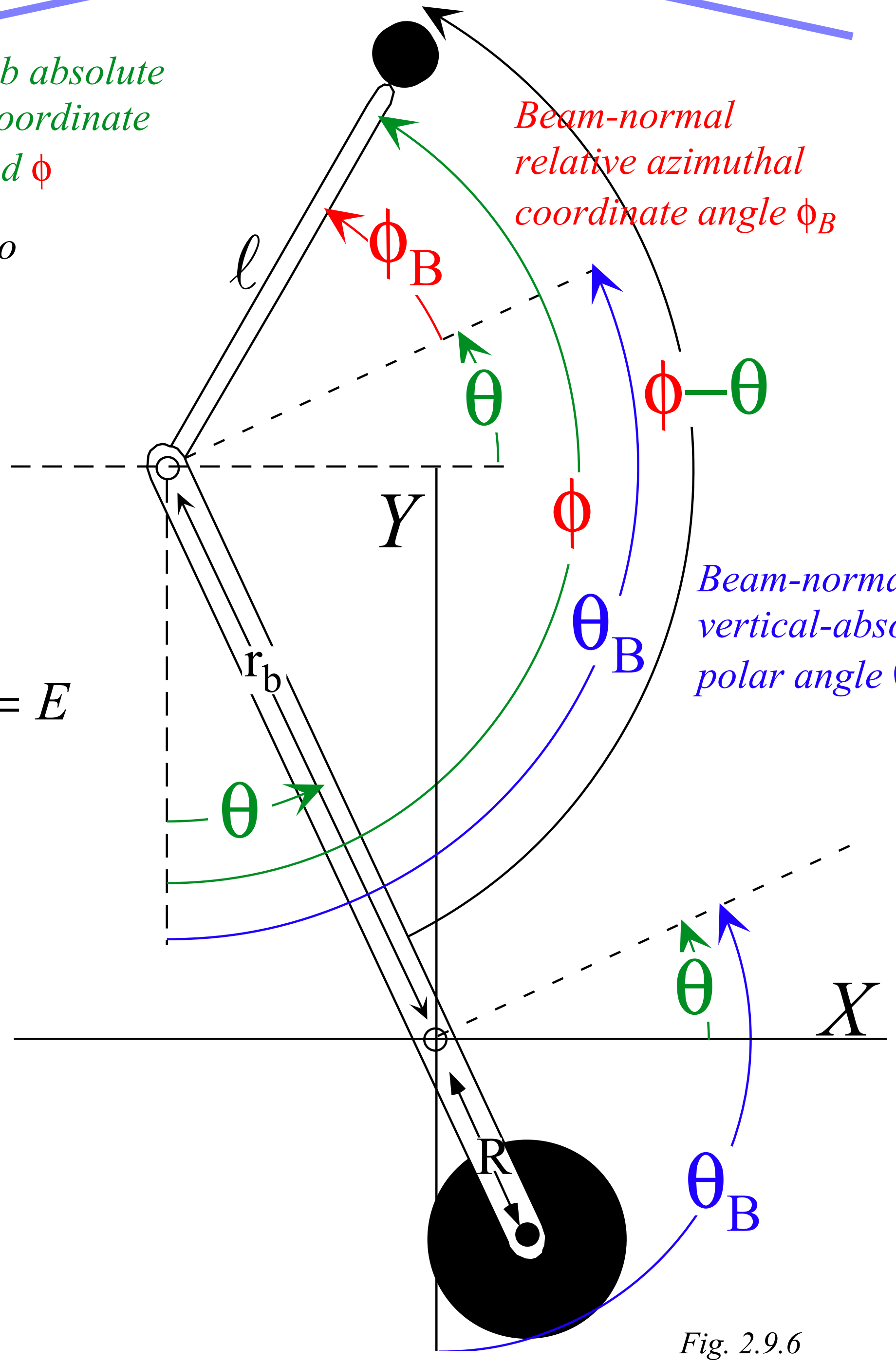

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

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(using last line of p. 36)

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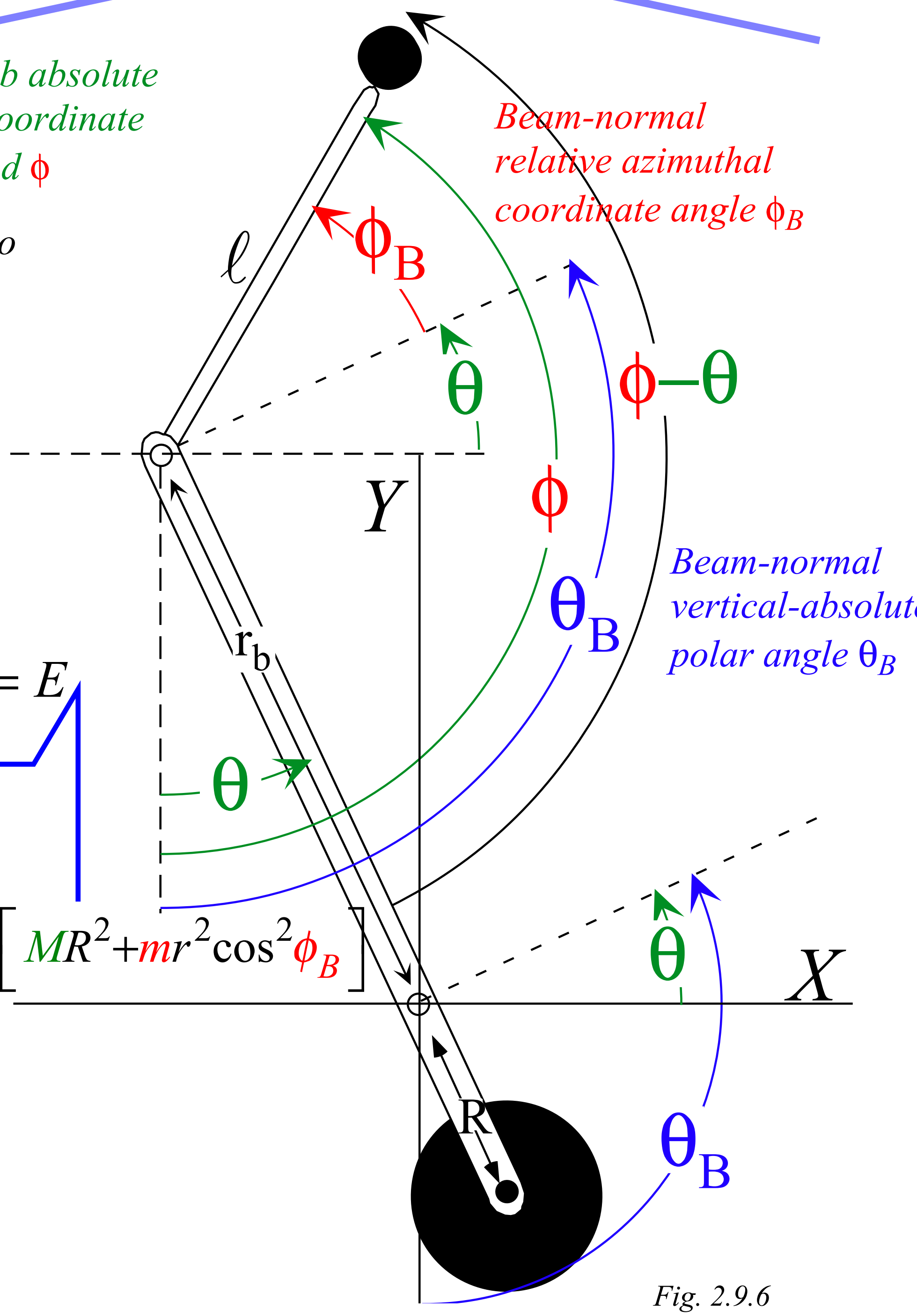
Rewrite H=E as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

Fig. 2.9.6

Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

Throwing-momentum p^B_ϕ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B_\theta$.

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

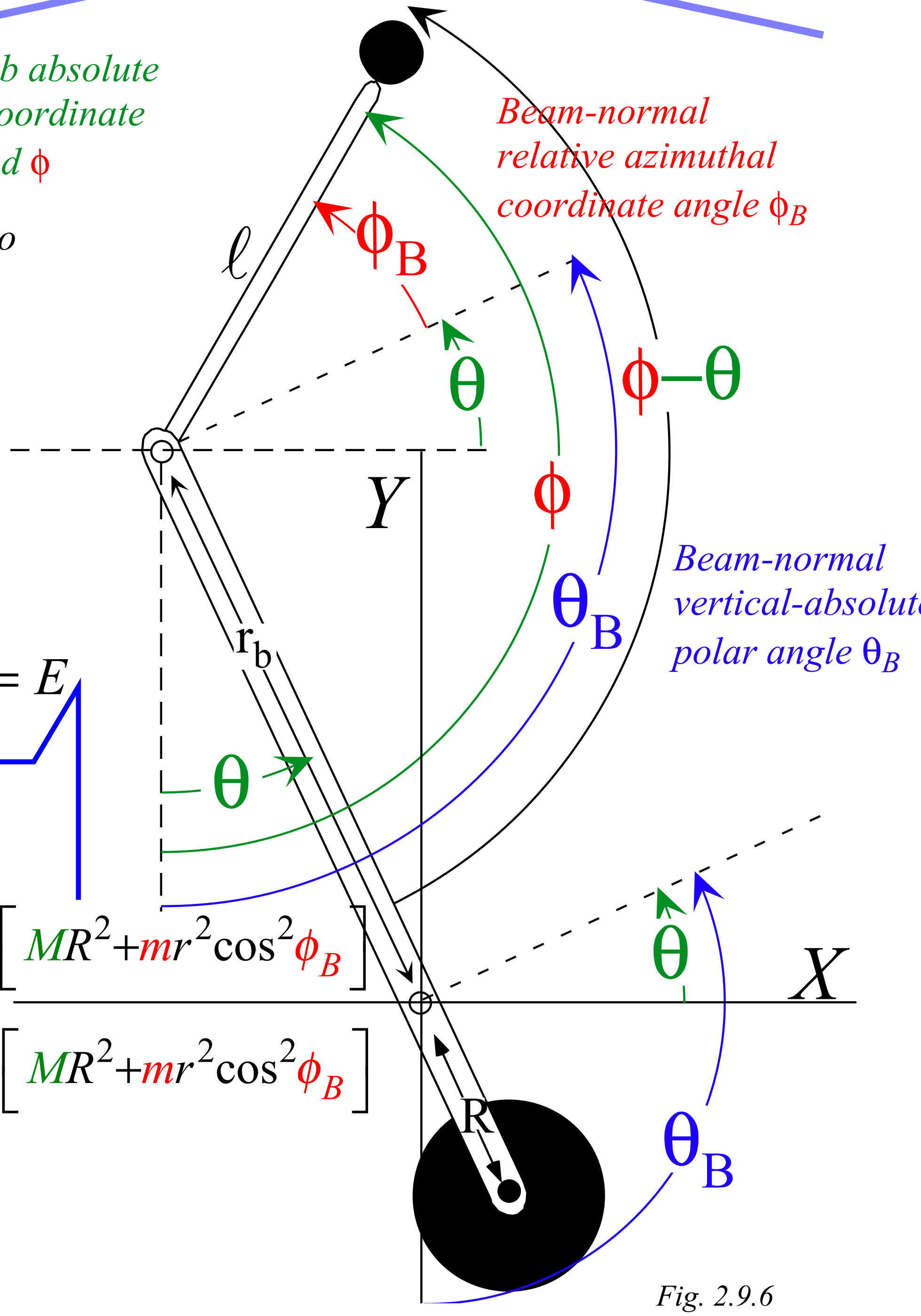
(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\xcancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ *and :* $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : H=const.=E



Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B *and* ϕ_B.

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolute polar angle θ_B

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite H=E as a quadratic equation in p_ϕ *:*

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) *relative coordinates for trebuchet.*

(Each value is positive.)

Throwing-momentum p^B_ϕ *is a function of beam-relative angle* ϕ_B*, total E, and* $\Lambda=p^B_\theta$ *.*

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\xcancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

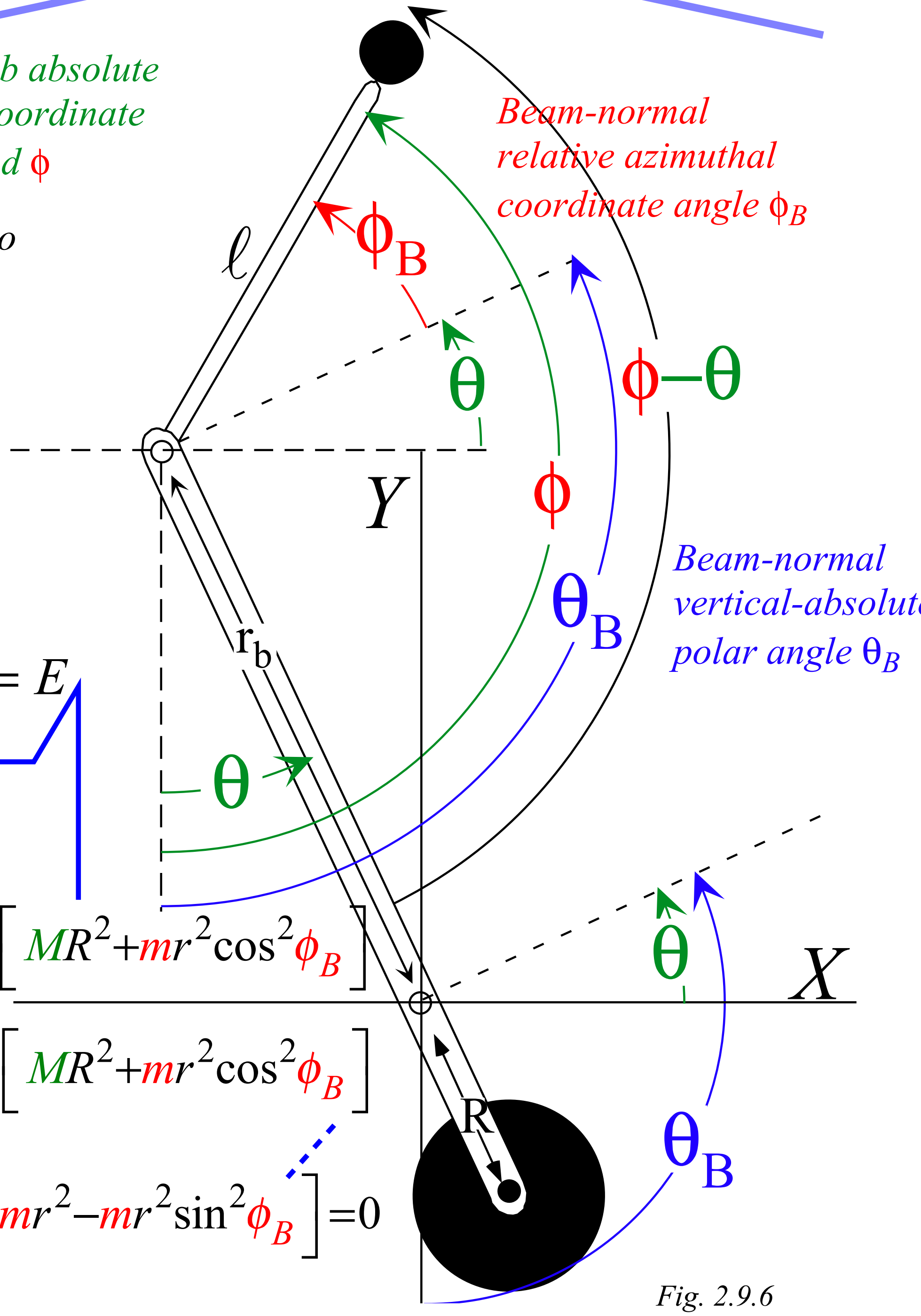


Fig. 2.9.6

Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.

(Each value is positive.)

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

Throwing-momentum p^B_ϕ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B_\theta$.

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}-\cancel{\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot p_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

$$\left(1+2(r/\ell)\sin\phi_B+J\right)(p_\phi^B)^2-2\Lambda\left((r/\ell)\sin\phi_B+1\right)(p_\phi^B)+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

Throwing-momentum $p^B{}_\phi$ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B{}_\theta$.

Previous lab absolute trebuchet coordinate angles θ and φ compared to new angles θ_B and ϕ_B.

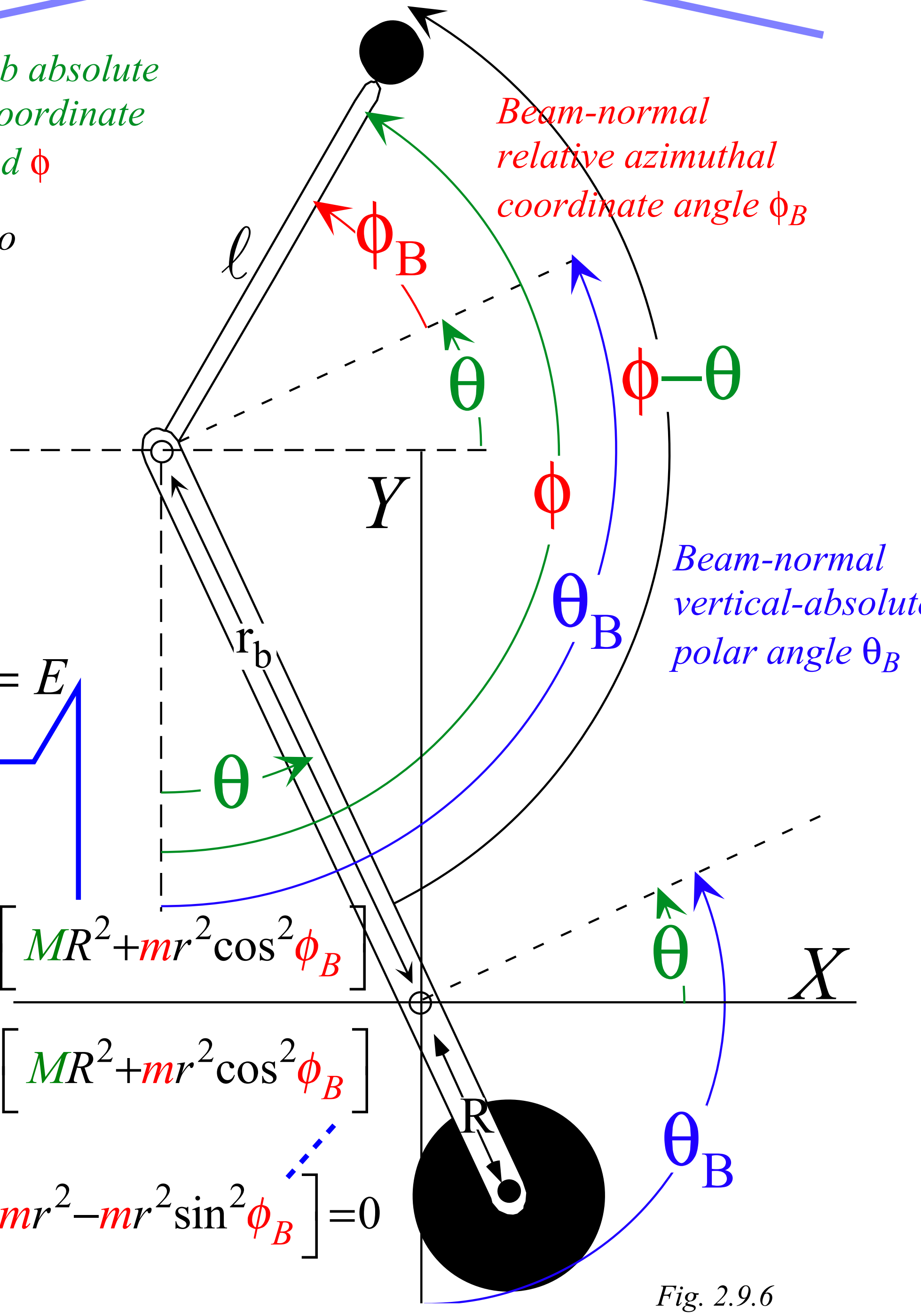


Fig. 2.9.6
Lab (θ,φ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet.
(Each value is positive.)

with: $J=\frac{MR^2+mr^2}{m\ell^2}$, $I=MR^2+mr^2$

Free-space trebuchet kinematics by symmetry: Algebraic approach

(using last line of p. 36)

(Assume zero-gravity)

$$H=\frac{m\ell^2\left(p_\theta^B-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(p_\theta^B-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}\cancel{-\left(MR-mr\right)g\sin\theta_B-mg\ell\cos\left(\phi_B+\theta_B\right)}$$

For zero-gravity H is not a function of θ_B

so : $\dot{p}_\theta^B=\frac{\partial H}{\partial\theta_B}=0$ and : $p_\theta^B=\Lambda=const.$

H is not an explicit function of t so : $H=const.=E$

$$H=\frac{m\ell^2\left(\Lambda-p_\phi^B\right)^2+\left(MR^2+mr^2\right)\left(p_\phi^B\right)^2-2mr\ell p_\phi^B\left(\Lambda-p_\phi^B\right)\sin\phi_B}{m\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]}=const.=E$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

new angles θ_B and ϕ_B.

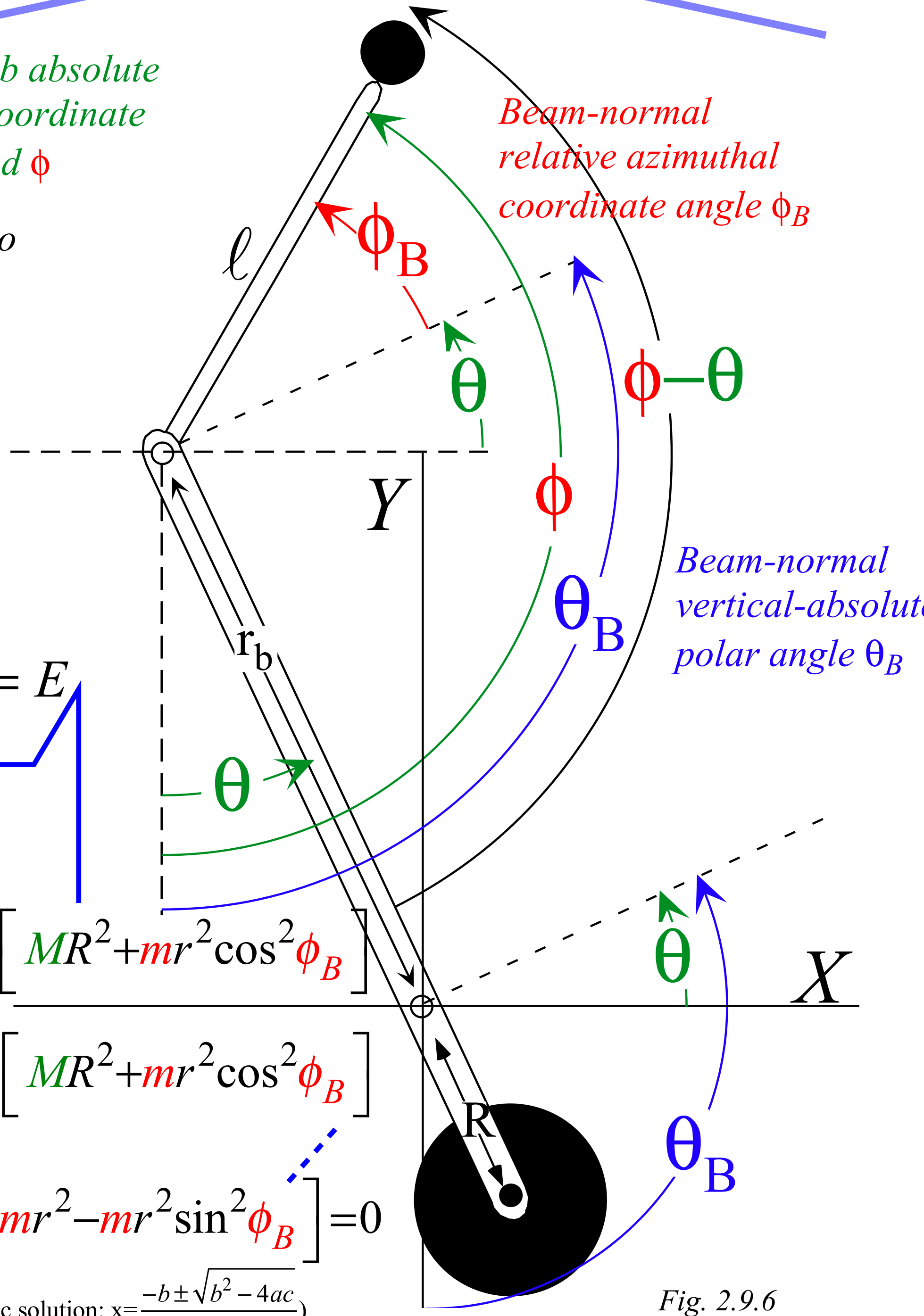


Fig. 2.9.6
Lab (θ,ϕ) and beam-normal (θ_B,ϕ_B) relative coordinates for trebuchet. (Each value is positive.)

Rewrite $H=E$ as a quadratic equation in p_ϕ :

$$m\ell^2\left(\Lambda^2-2\Lambda(p_\phi^B)+(p_\phi^B)^2\right)+\left(MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell(p_\phi^B)\left(\Lambda-p_\phi^B\right)\sin\phi_B=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$m\ell^2\Lambda^2-2m\ell^2\Lambda(p_\phi^B)+\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2mr\ell\Lambda\sin\phi_B(p_\phi^B)=Em\ell^2\left[MR^2+mr^2\cos^2\phi_B\right]$$

$$\left(m\ell^2+2mr\ell\sin\phi_B+MR^2+mr^2\right)(p_\phi^B)^2-2\Lambda\left(mr\ell\sin\phi_B+m\ell^2\right)(p_\phi^B)+m\ell^2\Lambda^2-Em\ell^2\left[MR^2+mr^2-mr^2\sin^2\phi_B\right]=0$$

$$\left(1+2(r/\ell)\sin\phi_B+J\right)(p_\phi^B)^2-2\Lambda\left((r/\ell)\sin\phi_B+1\right)(p_\phi^B)+\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]=0$$

(using quadratic solution: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$)

Throwing-momentum $p^B{}_\phi$ is a function of beam-relative angle ϕ_B, total E, and $\Lambda=p^B{}_\theta$.

$$p_\phi^B=\frac{2\Lambda\left((r/\ell)\sin\phi_B+1\right)\pm\sqrt{4\Lambda^2\left((r/\ell)\sin\phi_B+1\right)^2-4\left(1+2(r/\ell)\sin\phi_B+J\right)\left(\Lambda^2-E\left[I-mr^2\sin^2\phi_B\right]\right)}}{2\left(1+2(r/\ell)\sin\phi_B+J\right)}$$

with: $J=\frac{MR^2+mr^2}{m\ell^2}$, $I=MR^2+mr^2$

Hamiltonian energy and momentum conservation and symmetry coordinates

- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - ➡ *Direct approach and Superball analogy*
- *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

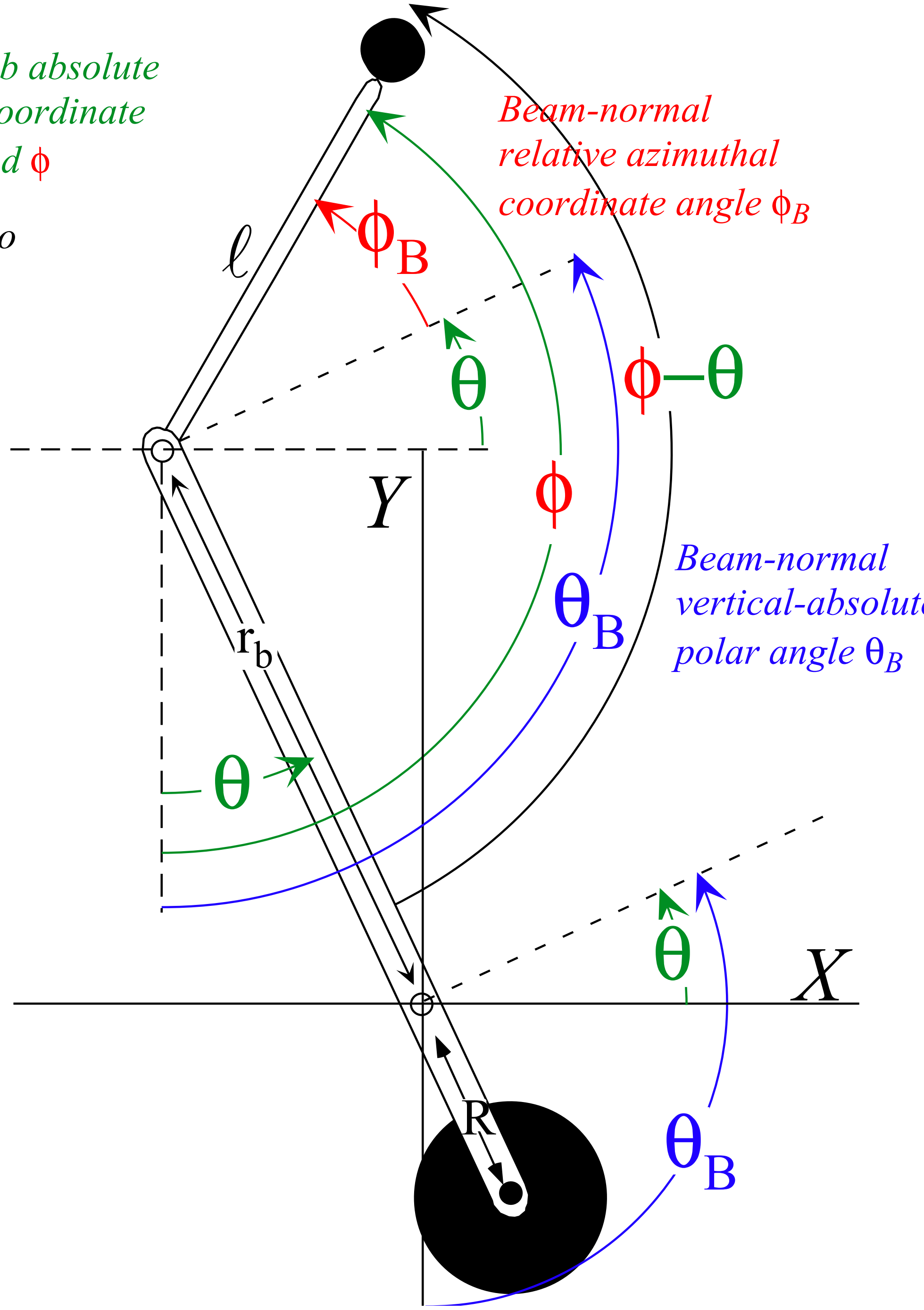
Energy for zero-gravity

(Assume zero-gravity)

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta - \phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta - \phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta - \phi)$$



Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

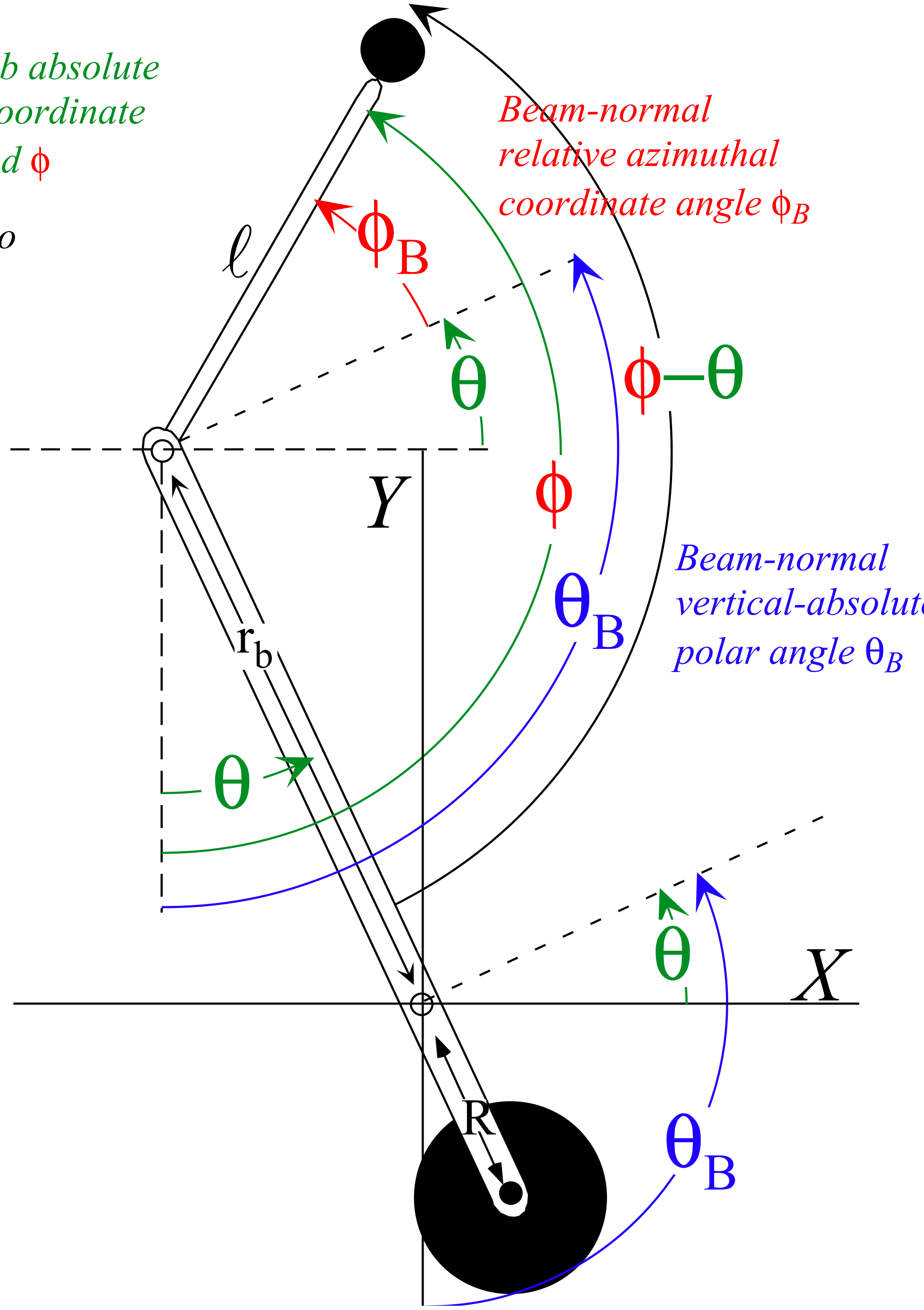
Transform to beam-relative coordinates and momenta

$$\theta = \theta_B \quad -\pi/2$$
$$\phi = \theta_B + \phi_B$$

$$\theta_B = \theta \quad +\pi/2$$
$$\phi_B = -\theta + \phi - \pi/2$$

$$p_\theta = p^B_\theta - p^B_\phi$$
$$p_\phi = \quad p^B_\phi$$

$$\theta - \phi = -\phi_B - \pi/2$$



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Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

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Transform to beam-relative coordinates and momenta

$$\boxed{\begin{aligned} \theta &= \theta_B - \pi/2 \\ \phi &= \theta_B + \phi_B \end{aligned}} \qquad \boxed{\begin{aligned} \theta_B &= \theta + \pi/2 \\ \phi_B &= -\theta + \phi - \pi/2 \end{aligned}} \qquad \boxed{\begin{aligned} p_\theta &= p_\theta^B - p_\phi^B \\ p_\phi &= p_\phi^B \end{aligned}}$$

$$\theta - \phi = -\phi_B - \pi/2$$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

Previous lab absolute trebuchet coordinate angles θ and φ

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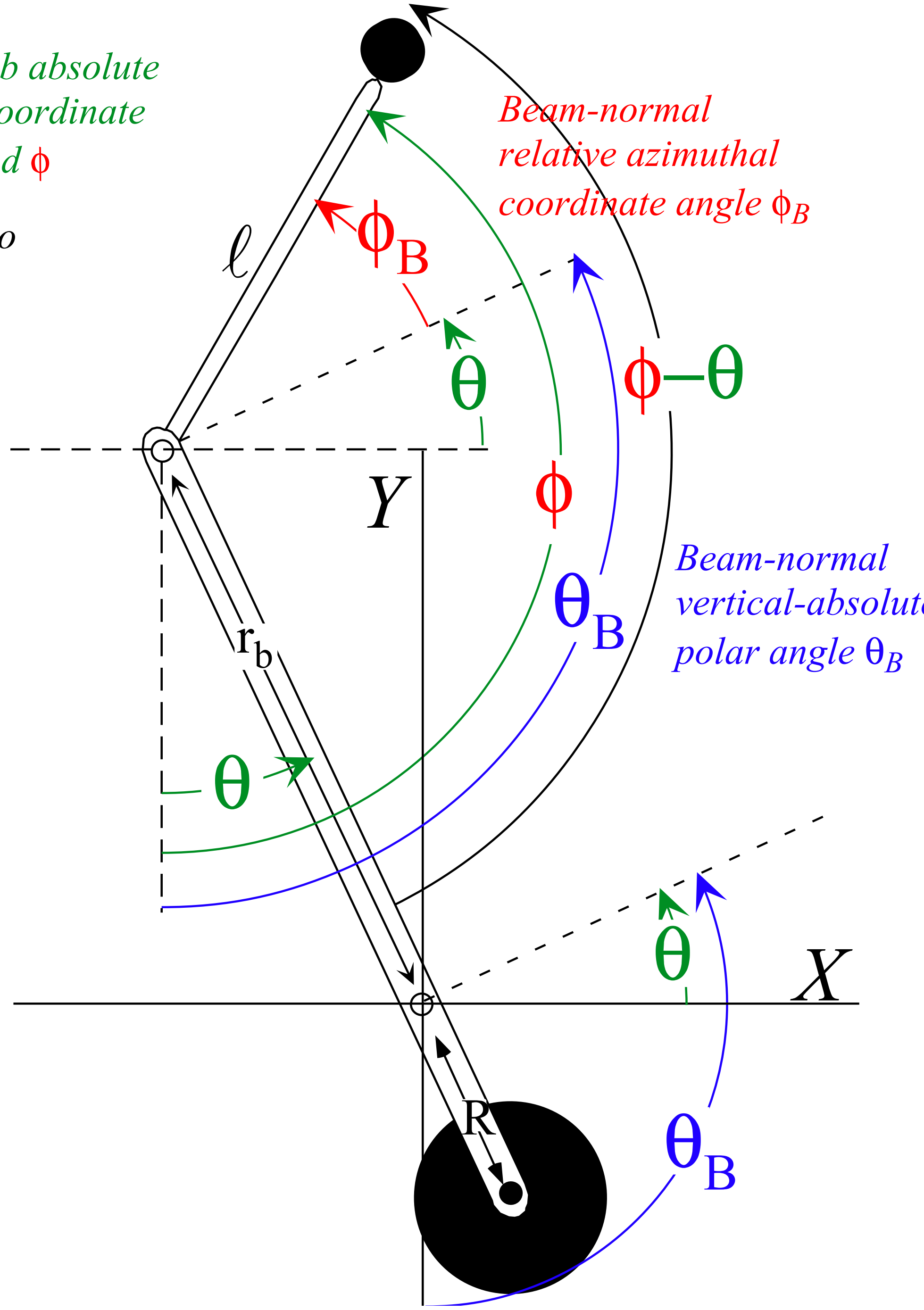

Beam-normal relative azimuthal coordinate angle ϕ_B

Beam-normal vertical-absolut polar angle θ_B

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

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Transform to beam-relative coordinates and momenta

$$\boxed{\begin{aligned}\theta &= \theta_B - \pi/2\\ \phi &= \theta_B + \phi_B\end{aligned}} \qquad \boxed{\begin{aligned}\theta_B &= \theta + \pi/2\\ \phi_B &= -\theta + \phi - \pi/2\end{aligned}} \qquad \boxed{\begin{aligned}p_\theta &= p_\theta^B - p_\phi^B\\ p_\phi &= p_\phi^B\end{aligned}}$$

$$\theta - \phi = -\phi_B - \pi/2$$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

$$p_\theta^B = \Lambda = const. = \quad p_\theta \quad + \quad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B\right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B\right)$$

Previous lab absolute trebuchet coordinate angles θ and ϕ

compared to

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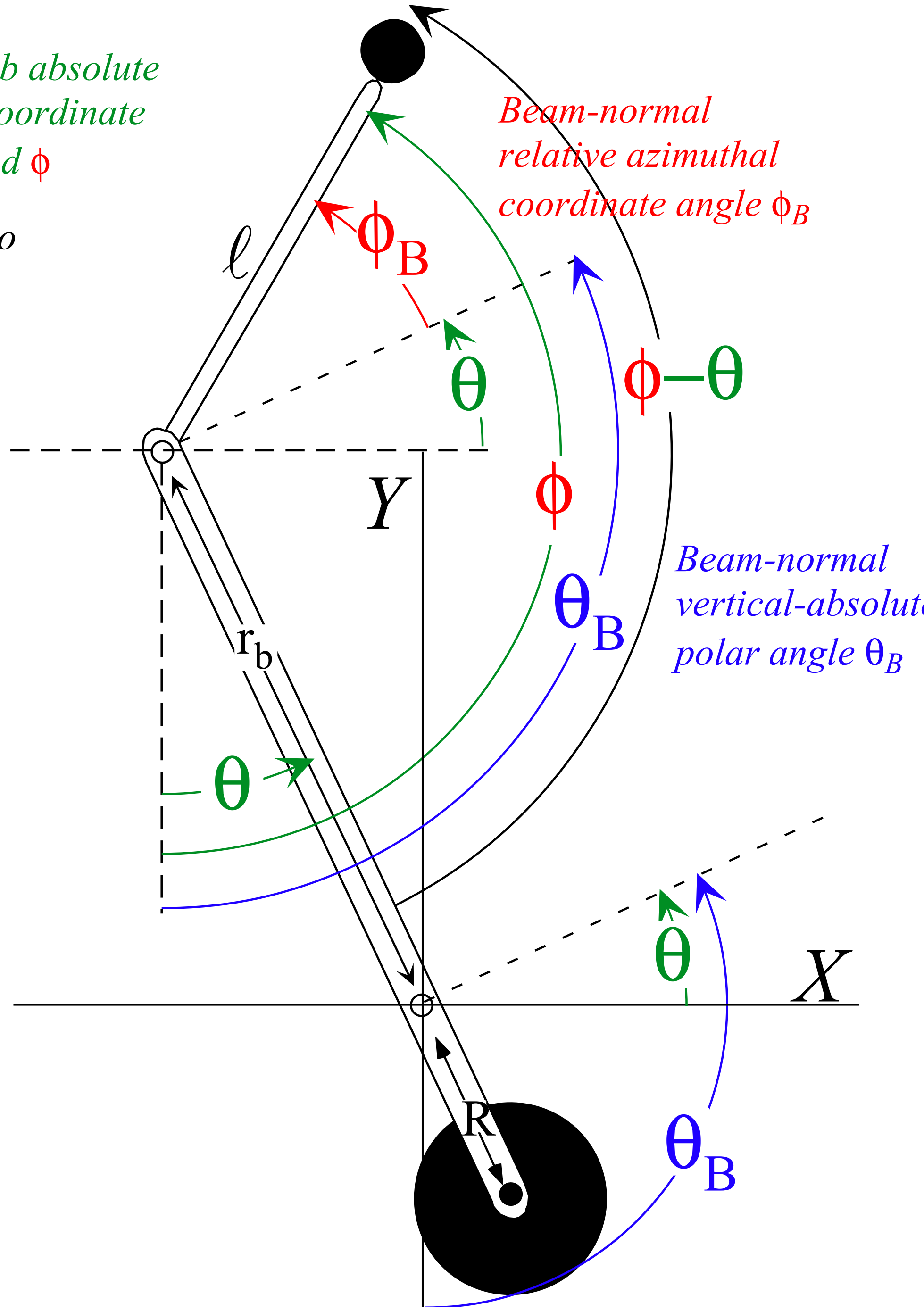

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

(Assume zero-gravity)

Energy for zero-gravity

$$\text{Total } KE = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

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$\theta = \theta_B \quad -\pi/2$

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$\theta_B = \theta \quad +\pi/2$

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$p_\theta = p_\theta^B - p_\phi^B$

$p_\phi = \quad p_\phi^B$

$\theta - \phi = -\phi_B - \pi/2$

$$2E = (MR^2 + mr^2)\dot{\theta}^2 + 2mr\ell\dot{\phi}\dot{\theta}\sin\phi_B + m\ell^2\dot{\phi}^2 = const.$$

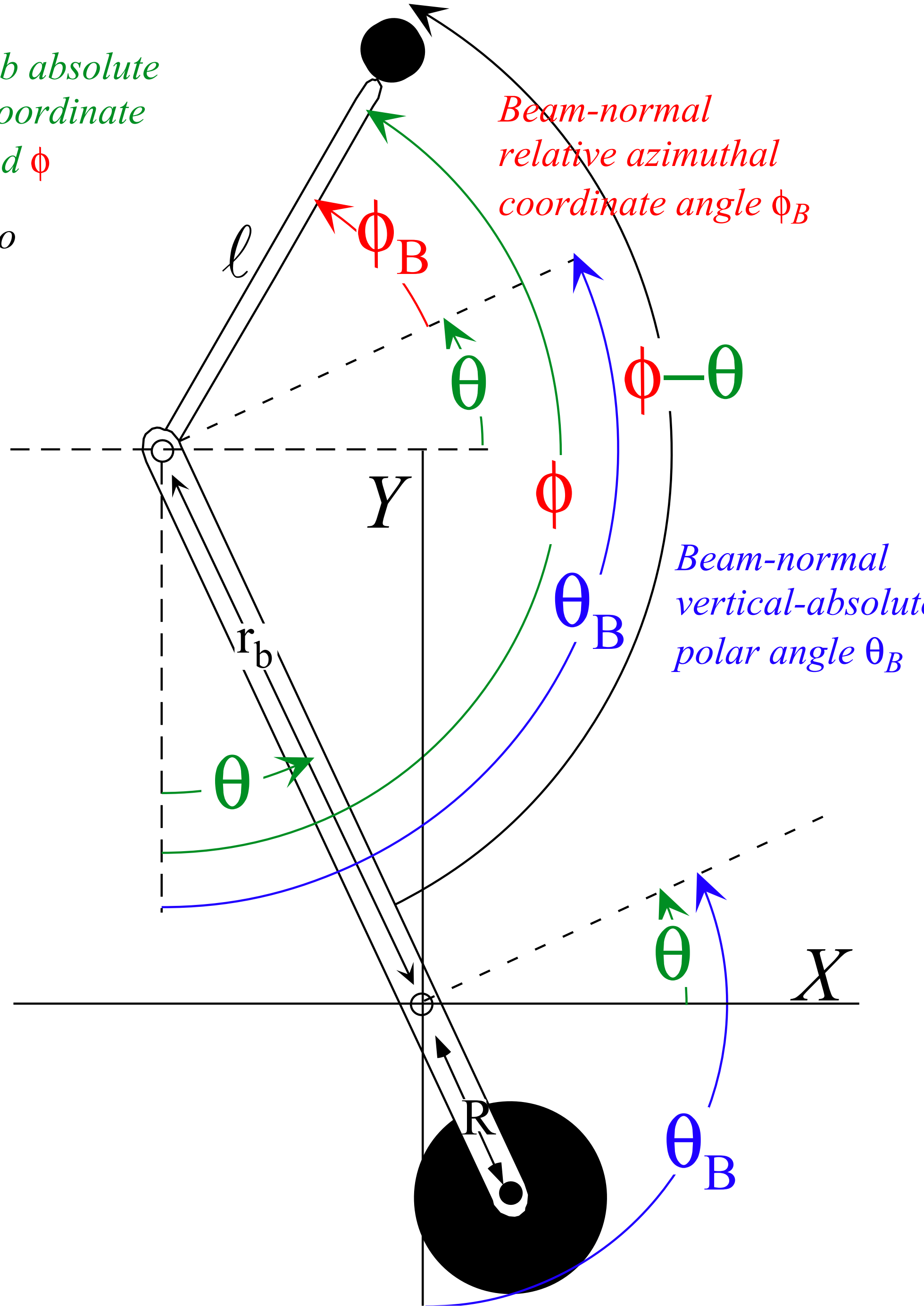
$$p_\theta^B = \Lambda = const. = \quad p_\theta \quad + \quad p_\phi$$

$$= \left((MR^2 + mr^2)\dot{\theta} + mr\ell\dot{\phi}\sin\phi_B \right) + \left(m\ell^2\dot{\phi} + mr\ell\dot{\theta}\sin\phi_B \right)$$

Previous lab absolute trebuchet coordinate angles θ and φ

compared to

new angles θ_B and ϕ_B.



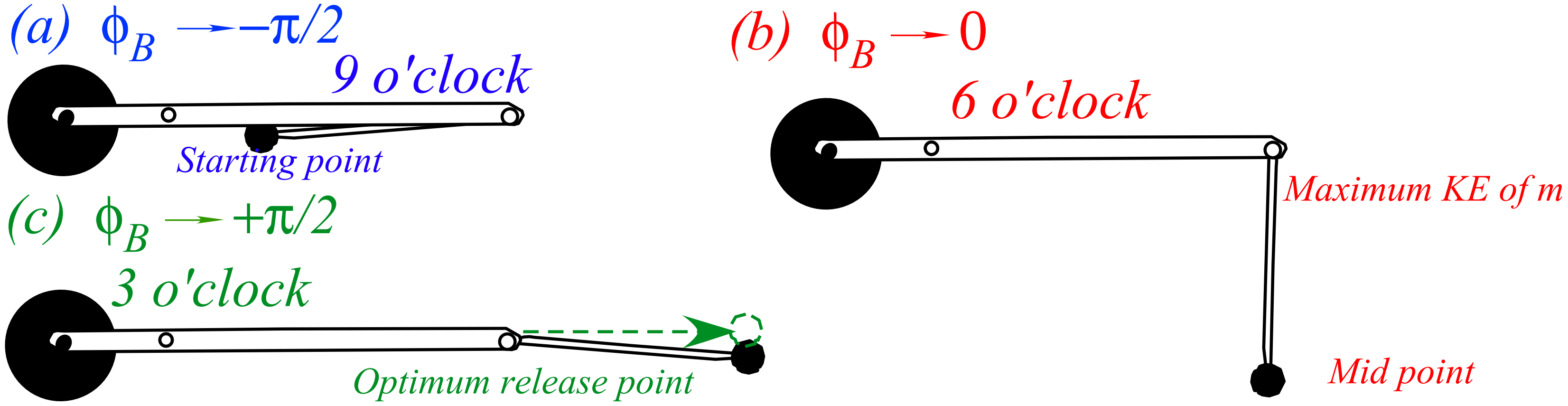
Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1 + \sin\phi_B\right)\left(\dot{\theta} + \dot{\phi}\right) \end{aligned}\right\} (\text{For: } r = \ell)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$

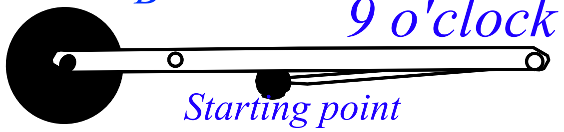


Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

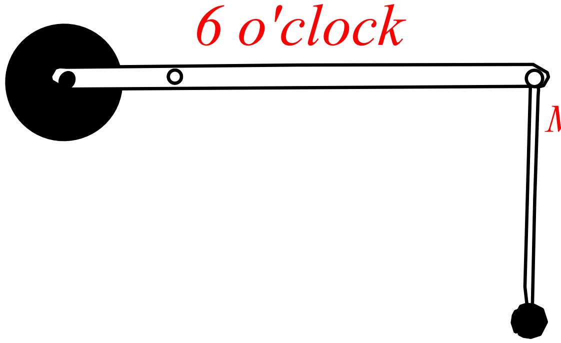
Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \quad \text{For: } r=\ell$$

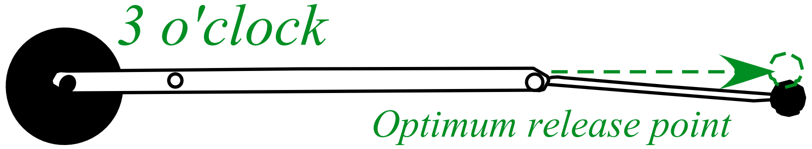
(a) $\phi_B \longrightarrow -\pi/2$



(b) $\phi_B \longrightarrow 0$



(c) $\phi_B \longrightarrow +\pi/2$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{aligned} 2E &= MR^2\dot{\theta}_{-\pi/2}^{\,2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \qquad or : \left\{ \begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right.$$

$\sin\phi_B$=-1

Conserved initial 2E

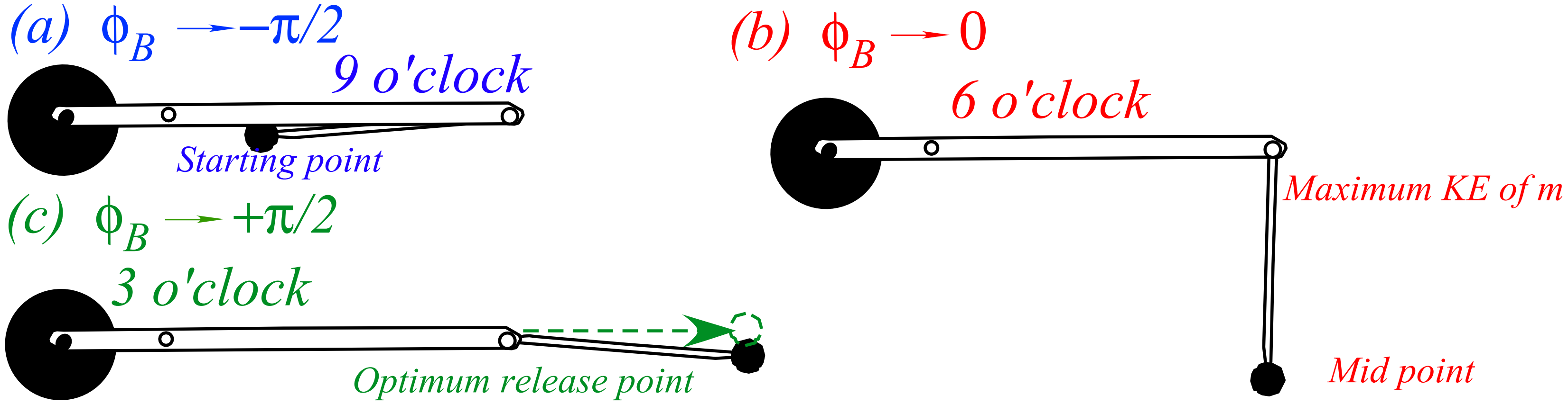
initial Λ

For: $\dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{array}{l} 2E=MR^2\dot{\theta}^2+mr^2\left(\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B+\dot{\phi}^2\right) \\ \Lambda=MR^2\dot{\theta}+mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{array}\right\} \begin{array}{l}\text{For:}\\ r=\ell\end{array}$$



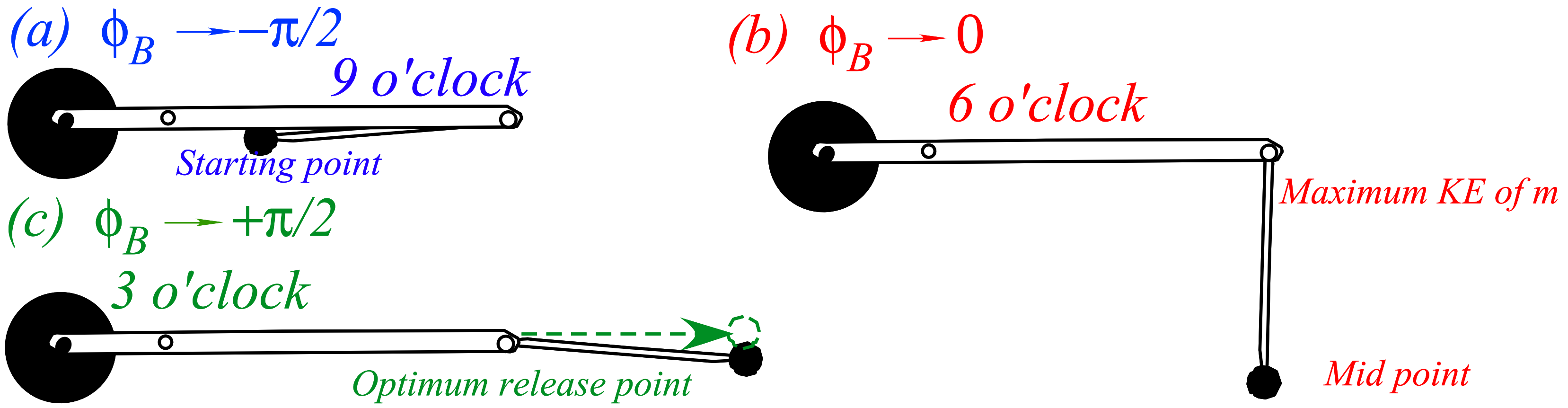
Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{array}{l}\phi_B=\dfrac{-\pi}{2}: \\ \sin\phi_B=\text{-1}\end{array}\left\{\begin{array}{c} 2E=MR^2\dot{\theta}_{-\pi/2}^2+mr^2\left(\dot{\phi}_{-\pi/2}-\dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda=MR^2\dot{\theta}_{-\pi/2} \end{array}\right. \quad or: \begin{array}{l}\text{Conserved}\\ \left\{\begin{array}{l}\text{initial } 2E \\ \quad 2E=MR^2\omega^2 \\ \quad \Lambda=MR^2\omega \\ \text{initial } \Lambda\end{array}\right.\end{array} \quad \text{For: } \dot{\theta}_{-\pi/2}=\omega=\dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{array}{l}\phi_B=0: \\ \sin\phi_B=0\end{array}\left\{\begin{array}{c} 2E=MR^2\dot{\theta}_0^2+mr^2\left(\dot{\phi}_0^2+\dot{\theta}_0^2\right) \\ \Lambda=MR^2\dot{\theta}_0+mr^2\left(\dot{\phi}_0+\dot{\theta}_0\right) \end{array}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{array}{l} 2E = MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda = MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{array}\right\} \begin{array}{l}\text{For:} \\ r=\ell\end{array}$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{array}{l}\phi_B = \dfrac{-\pi}{2}: \\ \sin\phi_B\text{=-1}\end{array}\left\{\begin{array}{l} 2E = MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot{\theta}_{-\pi/2} \end{array}\right. \quad or: \overset{\text{Conserved}}{\left\{\begin{array}{l} \text{initial 2E} \\ 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \\ \text{initial }\Lambda \end{array}\right.} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{array}{l}\phi_B = 0: \\ \sin\phi_B\text{=0}\end{array}\left\{\begin{array}{l} 2E = MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda = MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{array}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

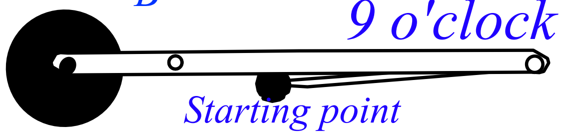
$$\begin{array}{l}\phi_B = \pi/2: \\ \sin\phi_B\text{=+1}\end{array}\left\{\begin{array}{l} 2E = MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 \overset{\text{Conserved}}{\overset{\text{initial 2E}}{=}} MR^2\omega^2 \\ \Lambda = MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) \underset{\text{initial }\Lambda}{=} MR^2\omega \end{array}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

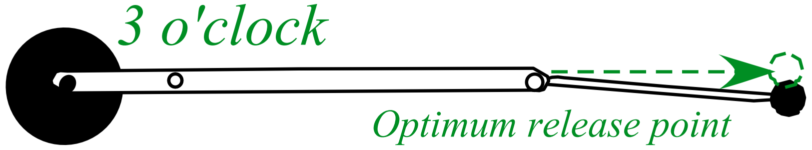
Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{array}{l} 2E=MR^2\dot{\theta}^2+mr^2\left(\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B+\dot{\phi}^2\right) \\ \Lambda = MR^2\dot{\theta}+mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{array}\right\} \begin{array}{l}\text{For:}\\ r=\ell\end{array}$$

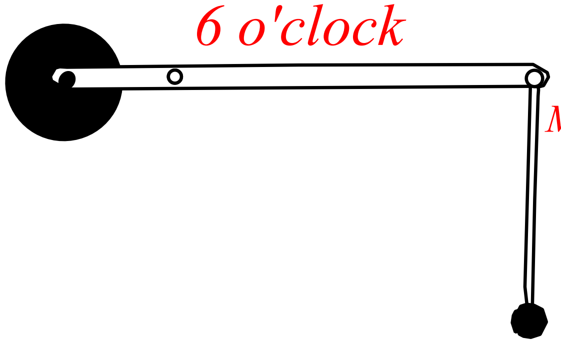
(a) $\phi_B \to -\pi/2$



(c) $\phi_B \to +\pi/2$



(b) $\phi_B \to 0$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{array}{l}\phi_B=\dfrac{-\pi}{2}: \\ \sin\phi_B\text{=-1}\end{array}\left\{\begin{array}{c} 2E=MR^2\dot{\theta}^2_{-\pi/2}+mr^2\left(\dot{\phi}_{-\pi/2}-\dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda=MR^2\dot{\theta}_{-\pi/2}\end{array}\right. \quad or: \overset{\text{Conserved}}{\left\{\begin{array}{l}\text{initial } 2E \\ 2E=MR^2\omega^2 \\ \Lambda=MR^2\omega \\ \text{initial } \Lambda\end{array}\right.} \quad \text{For: } \dot{\theta}_{-\pi/2}=\omega=\dot{\phi}_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{array}{l}\phi_B=0: \\ \sin\phi_B\text{=0}\end{array}\left\{\begin{array}{c} 2E=MR^2\dot{\theta}^2_0+mr^2\left(\dot{\phi}^2_0+\dot{\theta}^2_0\right) \\ \Lambda=MR^2\dot{\theta}_0+mr^2\left(\dot{\phi}_0+\dot{\theta}_0\right)\end{array}\right.$$

$$KE(m)=\frac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2+2\dot{\phi}\dot{\theta}\sin\phi_B\right)$$

$$=\left\{\begin{array}{ll} \dfrac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 & \left(\text{For: } \phi_B=-\dfrac{\pi}{2}\right) \\ \dfrac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) & \left(\text{For: } \phi_B=0\right) \\ \dfrac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 & \left(\text{For: } \phi_B=\dfrac{\pi}{2}\right)\end{array}\right.$$

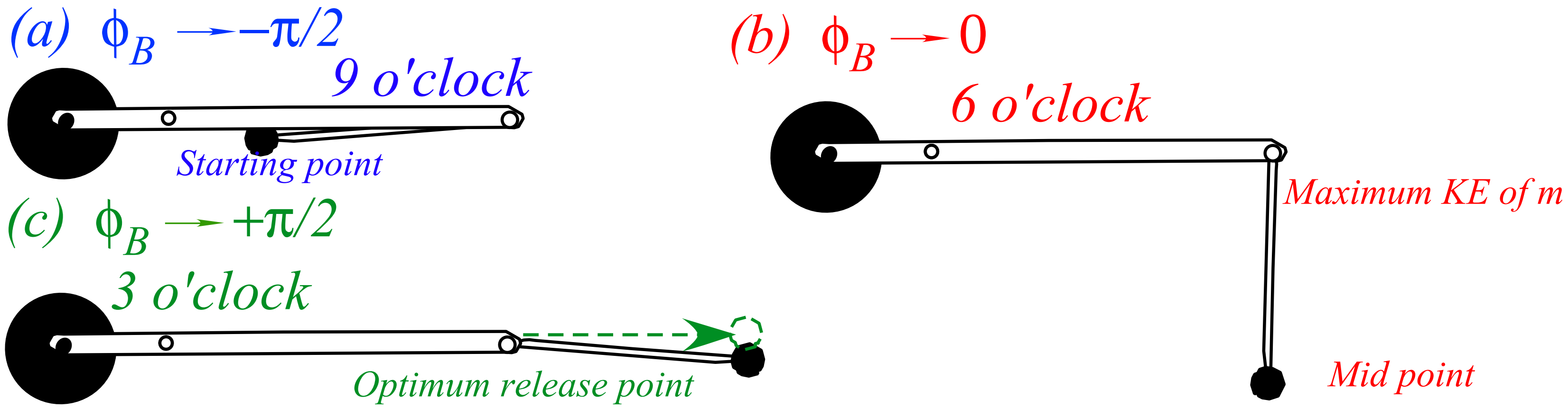
Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{array}{l}\phi_B=\pi/2: \\ \sin\phi_B\text{=+1}\end{array}\left\{\begin{array}{l} 2E=MR^2\dot{\theta}^2_{\pi/2}+mr^2\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right)^2 = \overset{\text{Conserved initial } 2E}{MR^2\omega^2} \\ \Lambda=MR^2\dot{\theta}_{\pi/2}+2mr^2\left(\dot{\phi}_{\pi/2}+\dot{\theta}_{\pi/2}\right) = \underset{\text{initial } \Lambda}{MR^2\omega}\end{array}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned}2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\text{ For: } r=\ell$$



Start at 9 o’clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B=\frac{-\pi}{2}:\ \sin\phi_B=-1\ \left\{\begin{aligned}2E&=MR^2\dot\theta_{-\pi/2}^2+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned}&\text{Conserved initial } 2E\\ &2E=MR^2\omega^2\\ &\Lambda=MR^2\omega\\ &\text{initial }\Lambda\end{aligned}\right.$$

For: $\dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$

Move to 6 o’clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0:\ \sin\phi_B=0\ \left\{\begin{aligned}2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

Move to 3 o’clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\phi_B=\pi/2:\ \sin\phi_B=+1\ \left\{\begin{aligned}2E&=MR^2\dot\theta_{\pi/2}^2+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2=MR^2\omega^2 \ \ (\text{Conserved initial } 2E)\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)=MR^2\omega\ \ (\text{initial }\Lambda)\end{aligned}\right.$$

$$\rightarrow\ \left(\omega^2-\dot\theta_{\pi/2}^2\right)=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

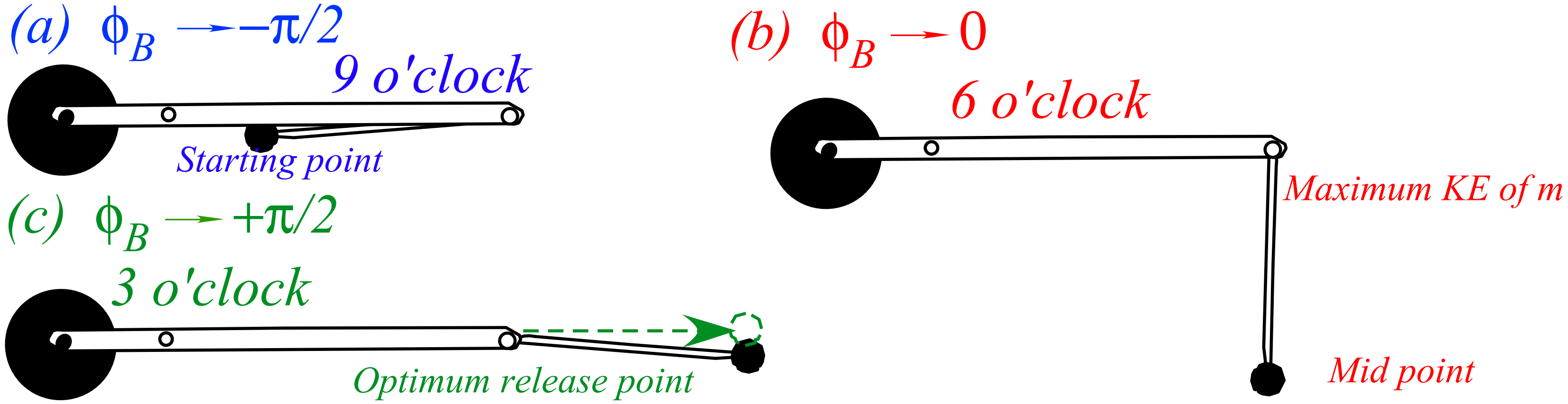
$$\rightarrow\ \left(\omega-\dot\theta_{\pi/2}\right)=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{aligned}&\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 &&\left(\text{For: }\phi_B=-\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) &&\left(\text{For: }\phi_B=0\right)\\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 &&\left(\text{For: }\phi_B=\frac{\pi}{2}\right)\end{aligned}\right.$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{array}{l}\phi_B=\dfrac{-\pi}{2}:\\ \sin\phi_B\text{=-1}\end{array}\left\{\begin{aligned}2E&=MR^2\dot\theta_{-\pi/2}^2+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned}&\text{Conserved initial }2E\\ &2E=MR^2\omega^2\\ &\Lambda=MR^2\omega\\ &\text{initial }\Lambda\end{aligned}\right.\quad\text{For: }\dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{array}{l}\phi_B=0:\\ \sin\phi_B\text{=0}\end{array}\left\{\begin{aligned}2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{aligned}&\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 &&\left(\text{For: }\phi_B=-\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) &&\left(\text{For: }\phi_B=0\right)\\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 &&\left(\text{For: }\phi_B=\frac{\pi}{2}\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{array}{l}\phi_B=\pi/2:\\ \sin\phi_B\text{=+1}\end{array}\left\{\begin{aligned}2E&=MR^2\dot\theta_{\pi/2}^2+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 &&\overset{\text{Conserved initial }2E}{=MR^2\omega^2}\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) &&\underset{\text{initial }\Lambda}{=MR^2\omega}\end{aligned}\right.$$

$$\rightarrow\left(\omega^2-\dot\theta_{\pi/2}^2\right)=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

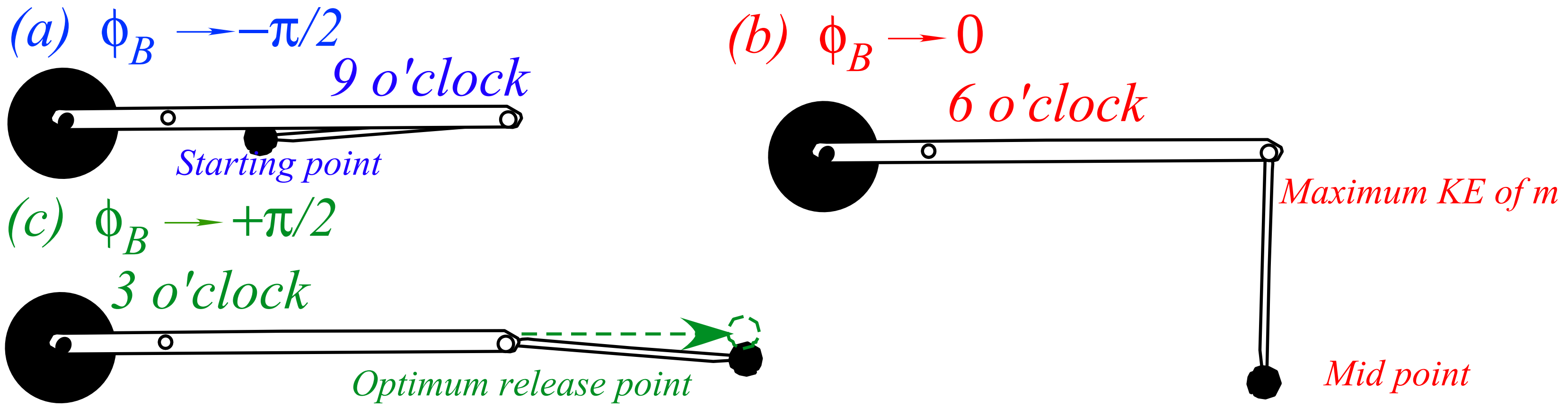
$$\rightarrow\left(\omega-\dot\theta_{\pi/2}\right)=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\longrightarrow\left(\omega+\dot\theta_{\pi/2}\right)=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B = \frac{-\pi}{2} : \left\{ \begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right.$$

$\sin\phi_B$=-1

Conserved initial 2E

$$or : \left\{ \begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right.$$

initial Λ

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B = 0 : \left\{ \begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$\sin\phi_B$=0

$KE(m) =$

$$\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right)$$

$$= \left\{ \begin{aligned} &\frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ &\frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) \quad \left(\text{For: } \phi_B = 0\right) \\ &\frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 \quad \left(\text{For: } \phi_B = \frac{\pi}{2}\right) \end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved initial 2E

$$\phi_B = \pi/2 : \left\{ \begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \end{aligned}\right.$$

initial Λ

$\sin\phi_B$=+1

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

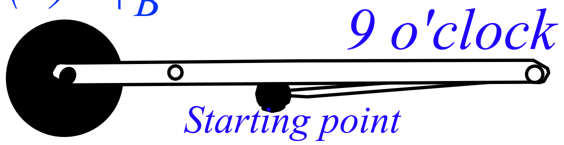
$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

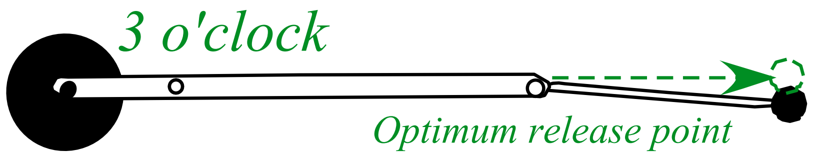
Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\} \begin{aligned}&\text{For:}\\ &r=\ell\end{aligned}$$

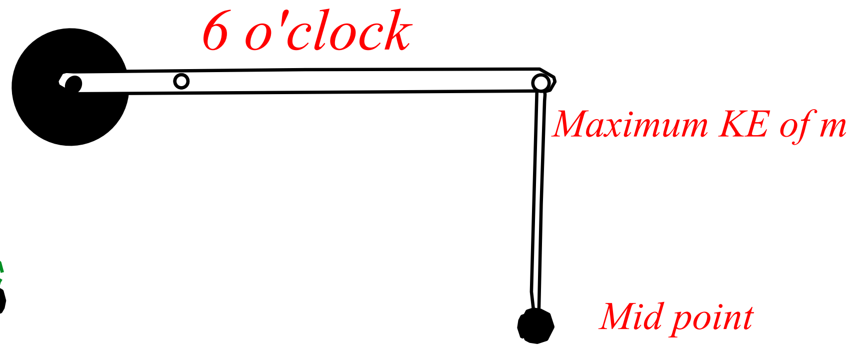
(a) $\phi_B \rightarrow -\pi/2$



(c) $\phi_B \rightarrow +\pi/2$



(b) $\phi_B \rightarrow 0$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{aligned}\phi_B = \frac{-\pi}{2}:\\ \sin\phi_B = -1\end{aligned}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2}\end{aligned}\right. \quad or: \left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega\end{aligned}\right.$$

Conserved initial 2E

initial Λ

For: $\dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{aligned}\phi_B = 0:\\ \sin\phi_B = 0\end{aligned}\left\{\begin{aligned} 2E &= MR^2\dot\theta_0^2 + mr^2\left(\dot\phi_0^2 + \dot\theta_0^2\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right)\end{aligned}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right) = \begin{cases} \frac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\frac{\pi}{2}\right) \\ \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \frac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \frac{\pi}{2}\right)\end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved initial 2E

$$\begin{aligned}\phi_B = \pi/2:\\ \sin\phi_B = +1\end{aligned}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega\end{aligned}\right.$$

initial Λ

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

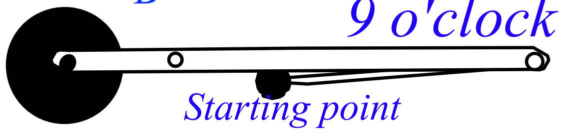
$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

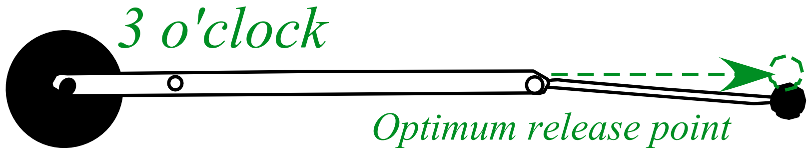
Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned}2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\quad \text{For: } r=\ell$$

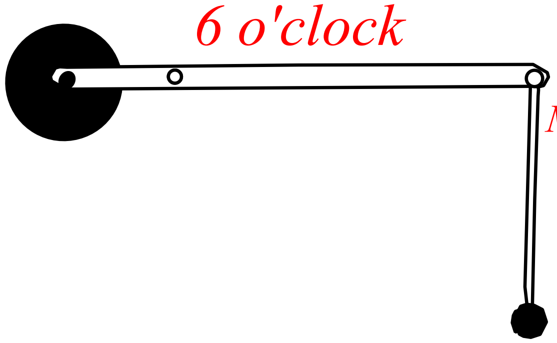
(a) $\phi_B \rightarrow -\pi/2$



(c) $\phi_B \rightarrow +\pi/2$



(b) $\phi_B \rightarrow 0$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B=\frac{-\pi}{2}:\left\{\begin{aligned}2E&=MR^2\dot\theta^2_{-\pi/2}+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned}&\text{initial } 2E\\ 2E&=MR^2\omega^2\\ \Lambda&=MR^2\omega\\ &\text{initial } \Lambda\end{aligned}\right.$$

$\sin\phi_B$=-1

Conserved

For: $\dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0:\left\{\begin{aligned}2E&=MR^2\dot\theta_0^2+mr^2\left(\dot\phi_0^2+\dot\theta_0^2\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

$\sin\phi_B$=0

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{aligned}&\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 &&\left(\text{For: }\phi_B=-\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) &&\left(\text{For: }\phi_B=0\right)\\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 &&\left(\text{For: }\phi_B=\frac{\pi}{2}\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

$$\phi_B=\pi/2:\left\{\begin{aligned}2E&=MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 &&\underset{\text{initial }2E}{=MR^2\omega^2}\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) &&\underset{\text{initial }\Lambda}{=MR^2\omega}\end{aligned}\right.$$

$\sin\phi_B$=+1

$$\left(\omega^2-\dot\theta^2_{\pi/2}\right)=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2$$

$$\left(\omega-\dot\theta_{\pi/2}\right)=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

divide 2E by Λ

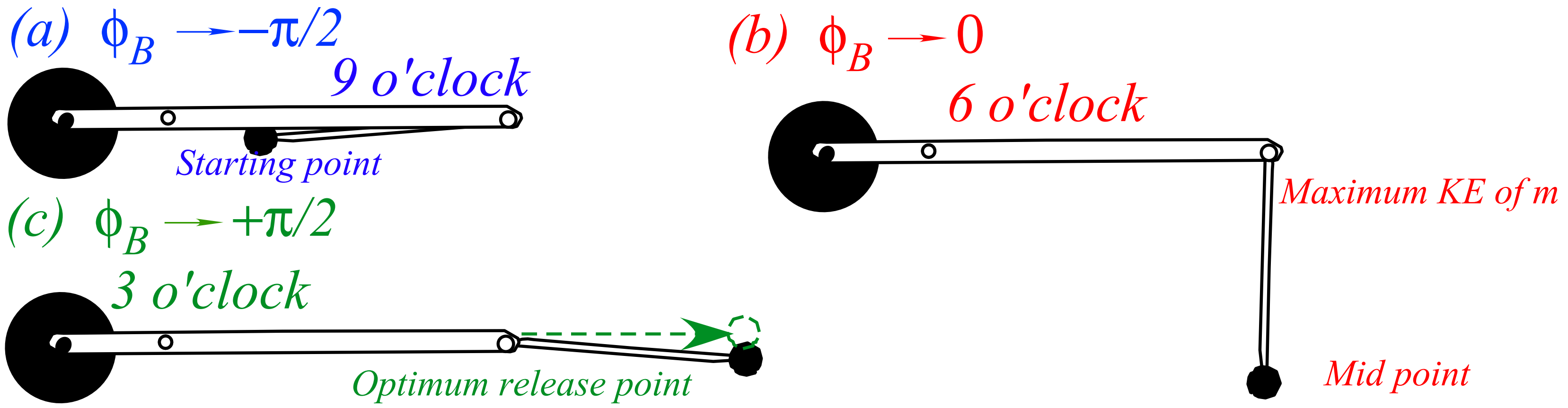
$$\left(\omega+\dot\theta_{\pi/2}\right)=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

substitute

$$\omega-\dot\theta_{\pi/2}=\frac{2mr^2}{MR^2}\left(2\omega+2\dot\theta_{\pi/2}\right)$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot{\theta}^2 + mr^2\left(\dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B + \dot{\phi}^2\right) \\ \Lambda &= MR^2\dot{\theta} + mr^2\left(1+\sin\phi_B\right)\left(\dot{\theta}+\dot{\phi}\right) \end{aligned}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with $\phi_B \sim -90°$ (beam r and throwing arm ℓ rotating together)

$$\begin{matrix}\phi_B = \dfrac{-\pi}{2}: \\ \sin\phi_B = -1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{-\pi/2} + mr^2\left(\dot{\phi}_{-\pi/2} - \dot{\phi}_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot{\theta}_{-\pi/2} \end{aligned}\right. \quad or: \begin{cases} \text{Conserved initial } 2E: & 2E = MR^2\omega^2 \\ & \Lambda = MR^2\omega \quad \text{initial } \Lambda \end{cases} \quad \text{For: } \dot{\theta}_{-\pi/2} = \omega = \dot{\phi}_{-\pi/2}$$

$$KE(m) = \frac{mr^2}{2}\left(\dot{\phi}^2 + \dot{\theta}^2 + 2\dot{\phi}\dot{\theta}\sin\phi_B\right) = \begin{cases} \dfrac{mr^2}{2}\left(\dot{\phi}-\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = -\dfrac{\pi}{2}\right) \\ \dfrac{mr^2}{2}\left(\dot{\phi}^2+\dot{\theta}^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \dfrac{mr^2}{2}\left(\dot{\phi}+\dot{\theta}\right)^2 & \left(\text{For: } \phi_B = \dfrac{\pi}{2}\right) \end{cases}$$

Move to 6 o'clock with $\phi_B \sim 0°$ (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0: \\ \sin\phi_B = 0\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}_0^2 + mr^2\left(\dot{\phi}_0^2 + \dot{\theta}_0^2\right) \\ \Lambda &= MR^2\dot{\theta}_0 + mr^2\left(\dot{\phi}_0 + \dot{\theta}_0\right) \end{aligned}\right.$$

Move to 3 o'clock with $\phi_B \sim +90°$ (beam r slowed, throwing arm ℓ releasing)

$$\begin{matrix}\phi_B = \pi/2: \\ \sin\phi_B = +1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot{\theta}^2_{\pi/2} + mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 &= MR^2\omega^2 \quad &\text{Conserved initial } 2E \\ \Lambda &= MR^2\dot{\theta}_{\pi/2} + 2mr^2\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) &= MR^2\omega \quad &\text{initial } \Lambda \end{aligned}\right.$$

$$\left(\omega^2 - \dot{\theta}^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)^2 \quad \xrightarrow{\text{divide } 2E \text{ by } \Lambda} \quad \left(\omega + \dot{\theta}_{\pi/2}\right) = \frac{1}{2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right)$$

$$\left(\omega - \dot{\theta}_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot{\phi}_{\pi/2} + \dot{\theta}_{\pi/2}\right) \quad \xleftarrow{\text{substitute}} \quad \boxed{\dot{\phi}_{\pi/2} = \dot{\theta}_{\pi/2} + 2\omega}$$

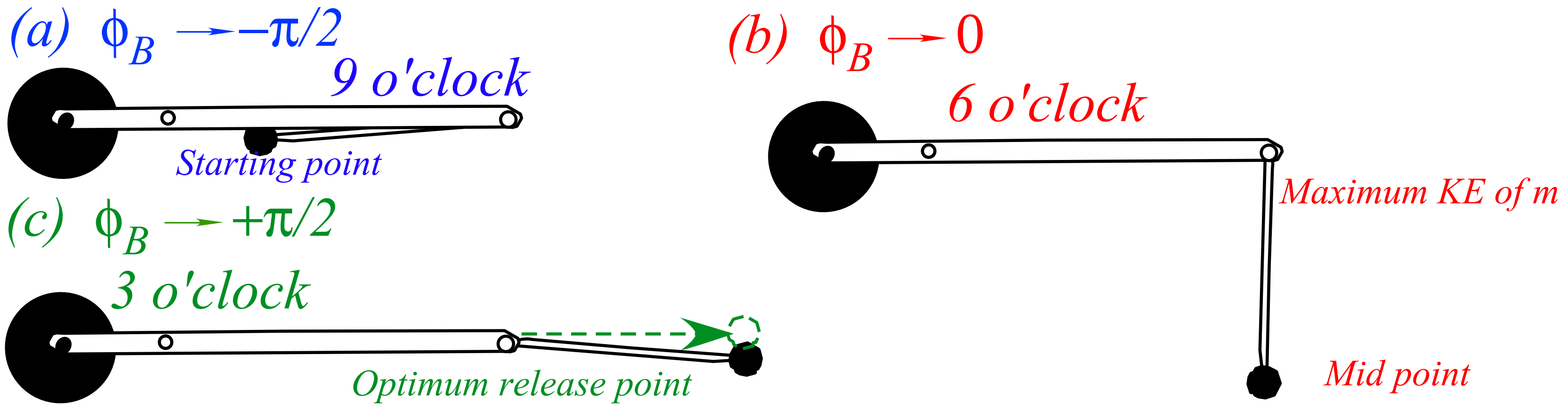
$$\omega - \dot{\theta}_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot{\theta}_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot{\theta}_{\pi/2} + \frac{4mr^2}{MR^2}\dot{\theta}_{\pi/2}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms r =ℓ (easier algebra)

$$\left.\begin{aligned}2E&=MR^2\dot\theta^2+mr^2\left(\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B+\dot\phi^2\right)\\ \Lambda&=MR^2\dot\theta+mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right)\end{aligned}\right\}\ \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\phi_B=\frac{-\pi}{2}:\ \sin\phi_B=-1\ \left\{\begin{aligned}2E&=MR^2\dot\theta^2_{-\pi/2}+mr^2\left(\dot\phi_{-\pi/2}-\dot\phi_{-\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{-\pi/2}\end{aligned}\right.\quad or:\left\{\begin{aligned}2E&=MR^2\omega^2\\ \Lambda&=MR^2\omega\end{aligned}\right.\quad \text{For: }\dot\theta_{-\pi/2}=\omega=\dot\phi_{-\pi/2}$$

Conserved initial 2E

initial Λ

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\phi_B=0:\ \sin\phi_B=0\ \left\{\begin{aligned}2E&=MR^2\dot\theta^2_0+mr^2\left(\dot\phi^2_0+\dot\theta^2_0\right)\\ \Lambda&=MR^2\dot\theta_0+mr^2\left(\dot\phi_0+\dot\theta_0\right)\end{aligned}\right.$$

$$KE(m)=\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2+2\dot\phi\dot\theta\sin\phi_B\right)=\left\{\begin{aligned}&\frac{mr^2}{2}\left(\dot\phi-\dot\theta\right)^2 &&\left(\text{For: }\phi_B=-\frac{\pi}{2}\right)\\ &\frac{mr^2}{2}\left(\dot\phi^2+\dot\theta^2\right) &&\left(\text{For: }\phi_B=0\right)\\ &\frac{mr^2}{2}\left(\dot\phi+\dot\theta\right)^2 &&\left(\text{For: }\phi_B=\frac{\pi}{2}\right)\end{aligned}\right.$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

Conserved

$$\phi_B=\pi/2:\ \sin\phi_B=+1\ \left\{\begin{aligned}2E&=MR^2\dot\theta^2_{\pi/2}+mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2 &&\underset{\text{initial 2E}}{=MR^2\omega^2}\ \longrightarrow\ \left(\omega^2-\dot\theta^2_{\pi/2}\right)=\frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)^2\\ \Lambda&=MR^2\dot\theta_{\pi/2}+2mr^2\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right) &&\underset{\text{initial }\Lambda}{=MR^2\omega}\ \longrightarrow\ \left(\omega-\dot\theta_{\pi/2}\right)=\frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)\end{aligned}\right.$$

$$\xrightarrow{\text{divide 2E by }\Lambda}\ \left(\omega+\dot\theta_{\pi/2}\right)=\frac{1}{2}\left(\dot\phi_{\pi/2}+\dot\theta_{\pi/2}\right)\ \downarrow\ \boxed{\dot\phi_{\pi/2}=\dot\theta_{\pi/2}+2\omega}$$

substitute

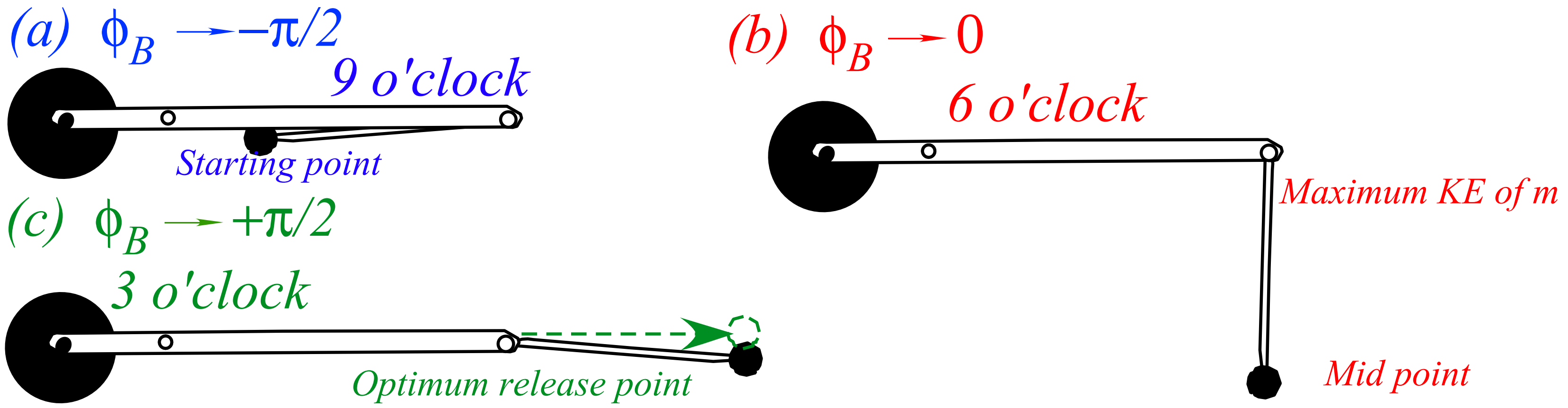
$$\omega-\dot\theta_{\pi/2}=\frac{2mr^2}{MR^2}\left(2\omega+2\dot\theta_{\pi/2}\right)$$

$$\omega-\frac{4mr^2}{MR^2}\omega=\dot\theta_{\pi/2}+\frac{4mr^2}{MR^2}\dot\theta_{\pi/2}\ \xrightarrow{\text{solve}}\ \boxed{\dot\theta_{\pi/2}=\frac{1-\dfrac{4mr^2}{MR^2}}{1+\dfrac{4mr^2}{MR^2}}\omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy

Case of equal arms $r=\ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$



Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{matrix}\phi_B = \dfrac{-\pi}{2} \\ \sin\phi_B = -1\end{matrix} : \begin{cases} 2E = MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda = MR^2\dot\theta_{-\pi/2} \end{cases} \quad or: \begin{cases} 2E = MR^2\omega^2 \\ \Lambda = MR^2\omega \end{cases} \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

Conserved

initial 2E

initial Λ

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right)$$

$$= \begin{cases} \dfrac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\dfrac{\pi}{2}\right) \\ \dfrac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \dfrac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \dfrac{\pi}{2}\right) \end{cases}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0 \\ \sin\phi_B = 0\end{matrix} : \begin{cases} 2E = MR^2\dot\theta^2_0 + mr^2\left(\dot\phi^2_0 + \dot\theta^2_0\right) \\ \Lambda = MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{matrix}\phi_B = \pi/2 \\ \sin\phi_B = +1\end{matrix} : \begin{cases} 2E = MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 = MR^2\omega^2 \\ \Lambda = MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) = MR^2\omega \end{cases}$$

Conserved

initial 2E

initial Λ

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

divide 2E by Λ

$$\left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega}$$

substitute

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2}$$

solve

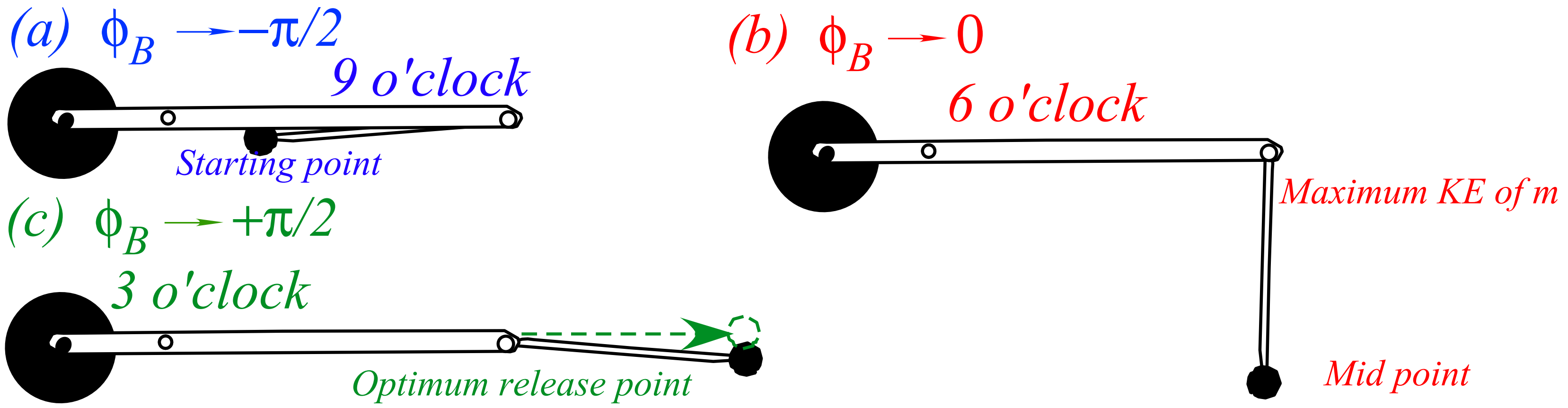
$$\boxed{\dot\theta_{\pi/2} = \frac{1 - \dfrac{4mr^2}{MR^2}}{1 + \dfrac{4mr^2}{MR^2}}\omega}$$

Large M≫m case

$$\boxed{\dot\phi_{\pi/2} = \omega + 2\omega = 3\omega}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Free-space trebuchet kinematics by symmetry: Direct approach and Superball analogy



Case of equal arms $r = \ell$ (easier algebra)

$$\left.\begin{aligned} 2E &= MR^2\dot\theta^2 + mr^2\left(\dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B + \dot\phi^2\right) \\ \Lambda &= MR^2\dot\theta + mr^2\left(1+\sin\phi_B\right)\left(\dot\theta+\dot\phi\right) \end{aligned}\right\} \text{For: } r=\ell$$

Start at 9 o'clock with ϕ_B~-90° (beam r and throwing arm ℓ rotating together)

$$\begin{matrix}\phi_B = \dfrac{-\pi}{2}: \\ \sin\phi_B = -1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{-\pi/2} + mr^2\left(\dot\phi_{-\pi/2} - \dot\phi_{-\pi/2}\right)^2 \\ \Lambda &= MR^2\dot\theta_{-\pi/2} \end{aligned}\right. \quad or: \overset{\text{Conserved initial } 2E}{\left\{\begin{aligned} 2E &= MR^2\omega^2 \\ \Lambda &= MR^2\omega \end{aligned}\right.}_{\text{initial } \Lambda} \quad \text{For: } \dot\theta_{-\pi/2} = \omega = \dot\phi_{-\pi/2}$$

Move to 6 o'clock with ϕ_B~ 0° (beam r slowing, throwing arm ℓ accelerating)

$$\begin{matrix}\phi_B = 0: \\ \sin\phi_B = 0\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_0 + mr^2\left(\dot\phi^2_0 + \dot\theta^2_0\right) \\ \Lambda &= MR^2\dot\theta_0 + mr^2\left(\dot\phi_0 + \dot\theta_0\right) \end{aligned}\right.$$

$$KE(m) = \frac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2 + 2\dot\phi\dot\theta\sin\phi_B\right)$$

$$= \begin{cases} \dfrac{mr^2}{2}\left(\dot\phi - \dot\theta\right)^2 & \left(\text{For: } \phi_B = -\dfrac{\pi}{2}\right) \\ \dfrac{mr^2}{2}\left(\dot\phi^2 + \dot\theta^2\right) & \left(\text{For: } \phi_B = 0\right) \\ \dfrac{mr^2}{2}\left(\dot\phi + \dot\theta\right)^2 & \left(\text{For: } \phi_B = \dfrac{\pi}{2}\right) \end{cases}$$

Move to 3 o'clock with ϕ_B~ +90° (beam r slowed, throwing arm ℓ releasing)

$$\begin{matrix}\phi_B = \pi/2: \\ \sin\phi_B = +1\end{matrix}\left\{\begin{aligned} 2E &= MR^2\dot\theta^2_{\pi/2} + mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2 \overset{\text{Conserved initial } 2E}{=} MR^2\omega^2 \\ \Lambda &= MR^2\dot\theta_{\pi/2} + 2mr^2\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right) \underset{\text{initial } \Lambda}{=} MR^2\omega \end{aligned}\right.$$

$$\left(\omega^2 - \dot\theta^2_{\pi/2}\right) = \frac{mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)^2$$

$$\left(\omega - \dot\theta_{\pi/2}\right) = \frac{2mr^2}{MR^2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\xrightarrow{\text{divide } 2E \text{ by } \Lambda} \left(\omega + \dot\theta_{\pi/2}\right) = \frac{1}{2}\left(\dot\phi_{\pi/2} + \dot\theta_{\pi/2}\right)$$

$$\boxed{\dot\phi_{\pi/2} = \dot\theta_{\pi/2} + 2\omega} \xrightarrow{\text{substitute}}$$

$$\omega - \dot\theta_{\pi/2} = \frac{2mr^2}{MR^2}\left(2\omega + 2\dot\theta_{\pi/2}\right)$$

$$\omega - \frac{4mr^2}{MR^2}\omega = \dot\theta_{\pi/2} + \frac{4mr^2}{MR^2}\dot\theta_{\pi/2} \xrightarrow{\text{solve}} \boxed{\dot\theta_{\pi/2} = \frac{1 - \dfrac{4mr^2}{MR^2}}{1 + \dfrac{4mr^2}{MR^2}}\omega}$$

Large $M \gg m$ case

$$\boxed{\dot\phi_{\pi/2} = \omega + 2\omega = 3\omega}$$

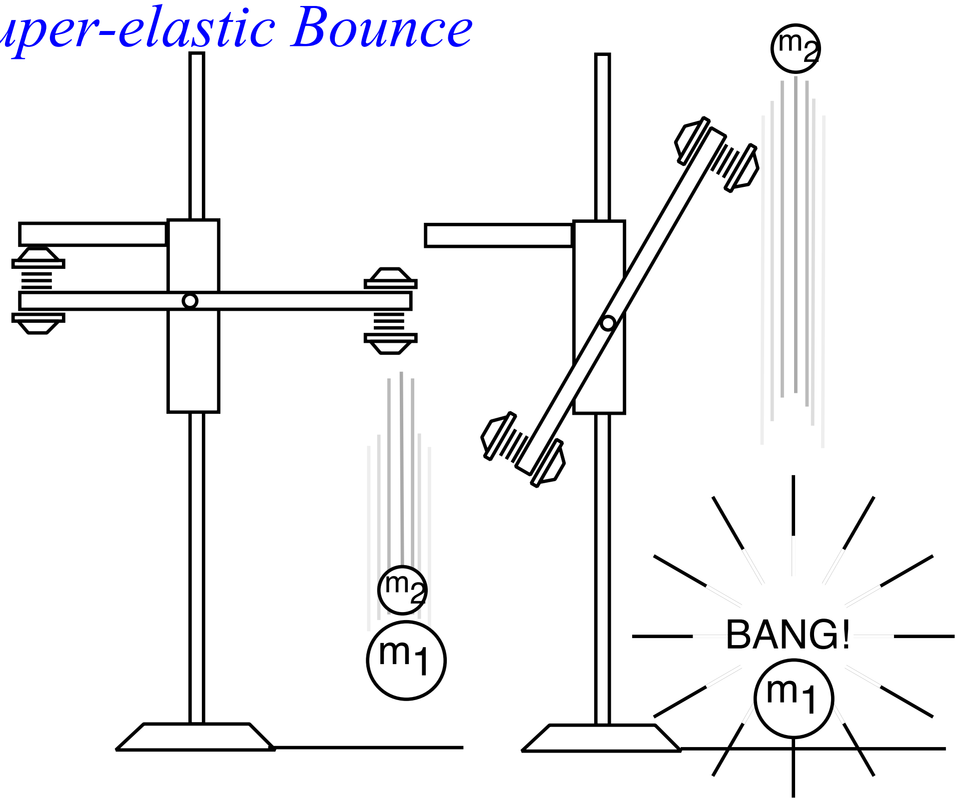
$$\boxed{\dot\theta_{\pi/2} = \frac{1-0}{1+0}\omega = \omega}$$

Optimum $MR^2 = 4mr^2$ case

$$\boxed{\dot\phi_{\pi/2} = 0 + 2\omega = 2\omega}$$

$$\boxed{\dot\theta_{\pi/2} = \frac{1-1}{1+1}\omega = 0}$$

Super-elastic Bounce

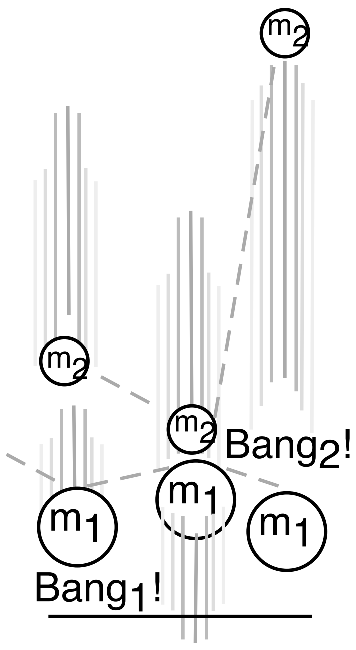


Analogous Superball Models

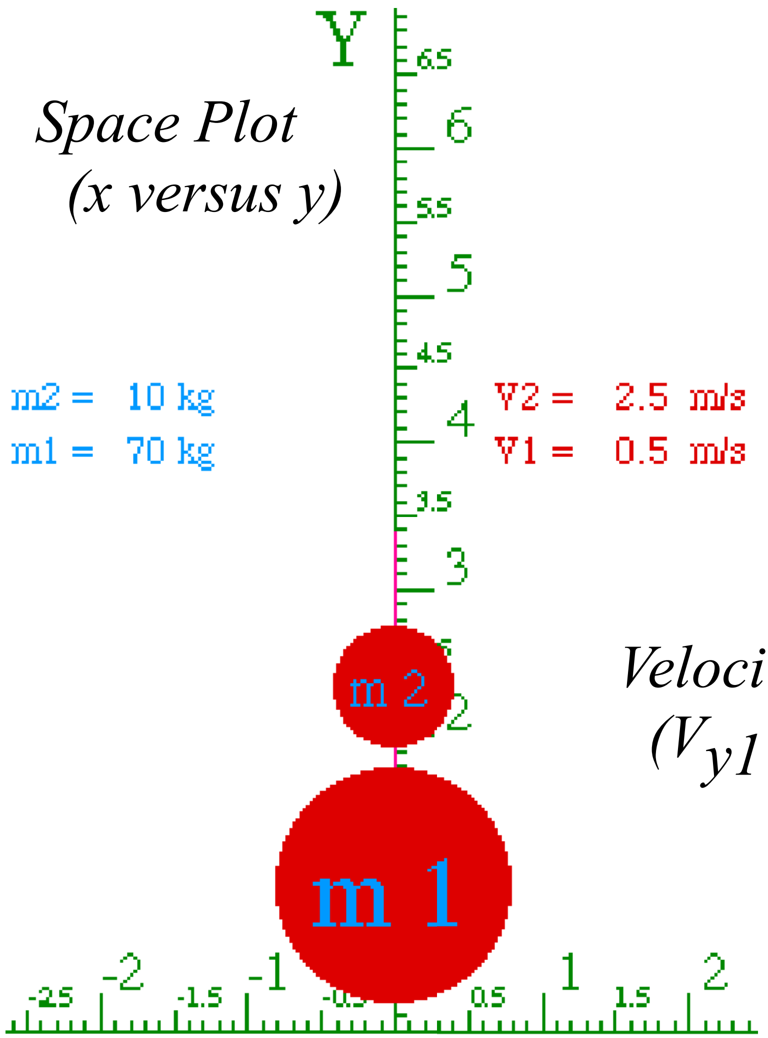
Similar in some ways to trebuchet models

Class of W. G. Harter, "Velocity Amplification in Collision Experiments Involving Superballs," Am. J. Phys. **39**, 656 (1971)
(A class project)

2-Bang Model



Space Plot
(x versus y)



Velocity Plot →
(V_{y1} versus V_{y2})

Optimal Throw

m_1:m_2 = 3:1

100% Energy Transfer

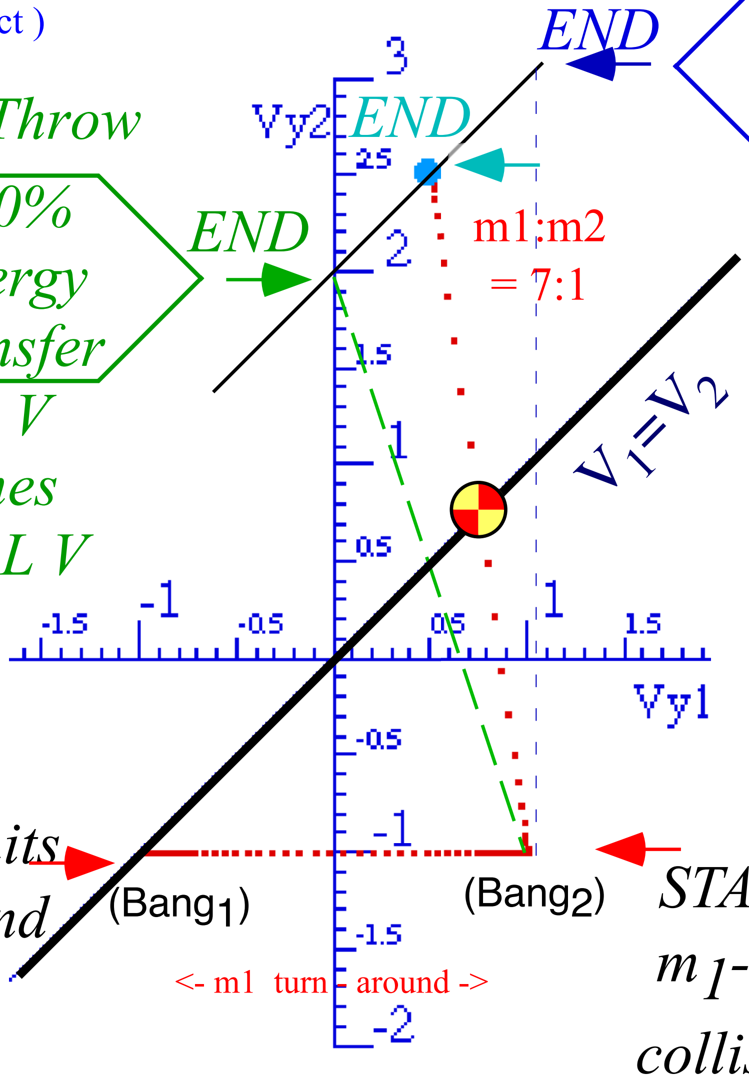
FINAL V is 2 times INITIAL V

Fastest Throw

0% Energy Transfer

m_1:m_2 = ∞:1

FINAL V is 3 times INITIAL V

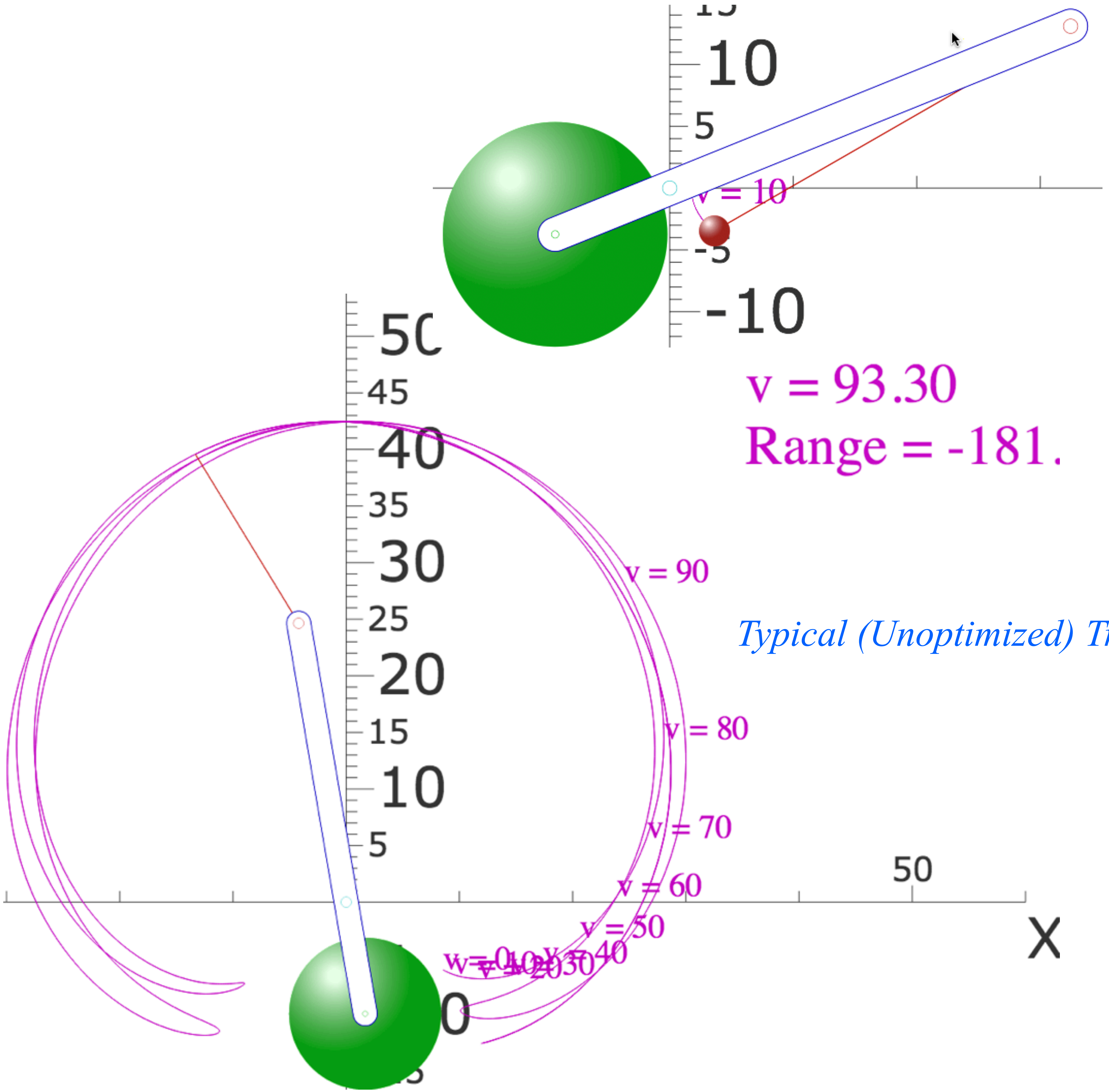


m_1 first hits ground

START m_1-m_2 collision

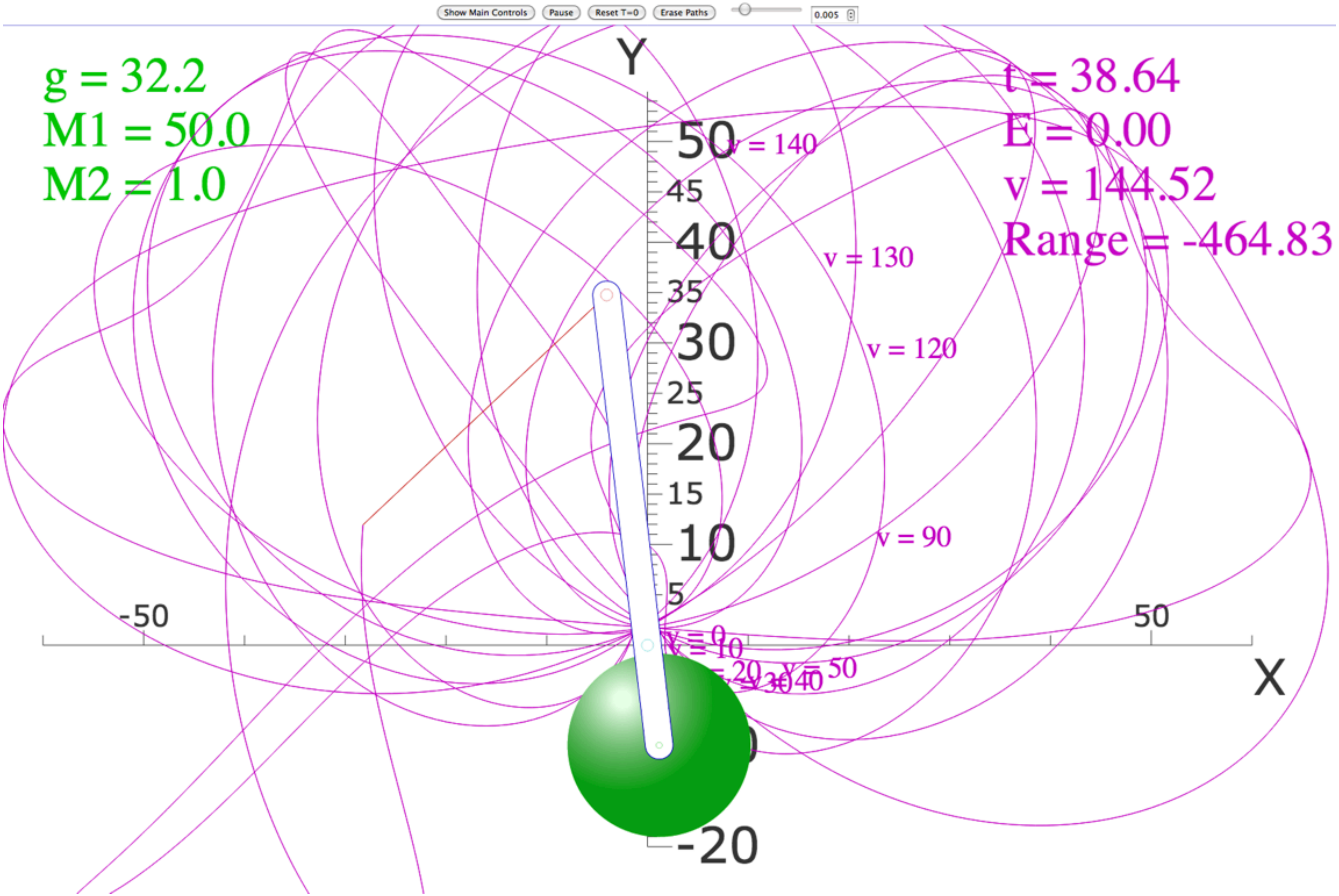
Graphic Solution

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html



Typical (Unoptimized) Trebuchet

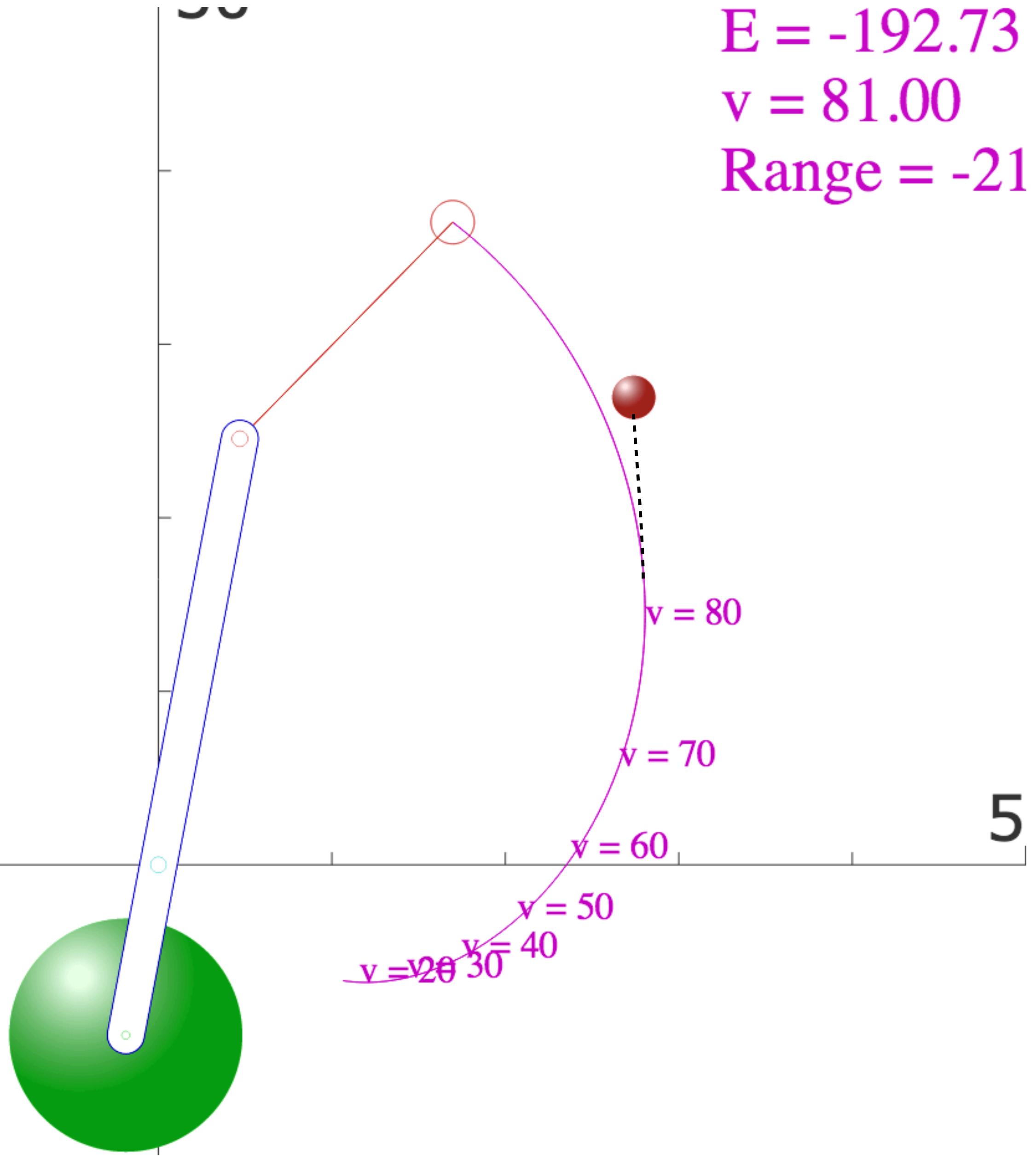
Trebuchet in Siege of Kenilworth 1215 ACE
(Re-enactment shown on NOVA-TV 2005)



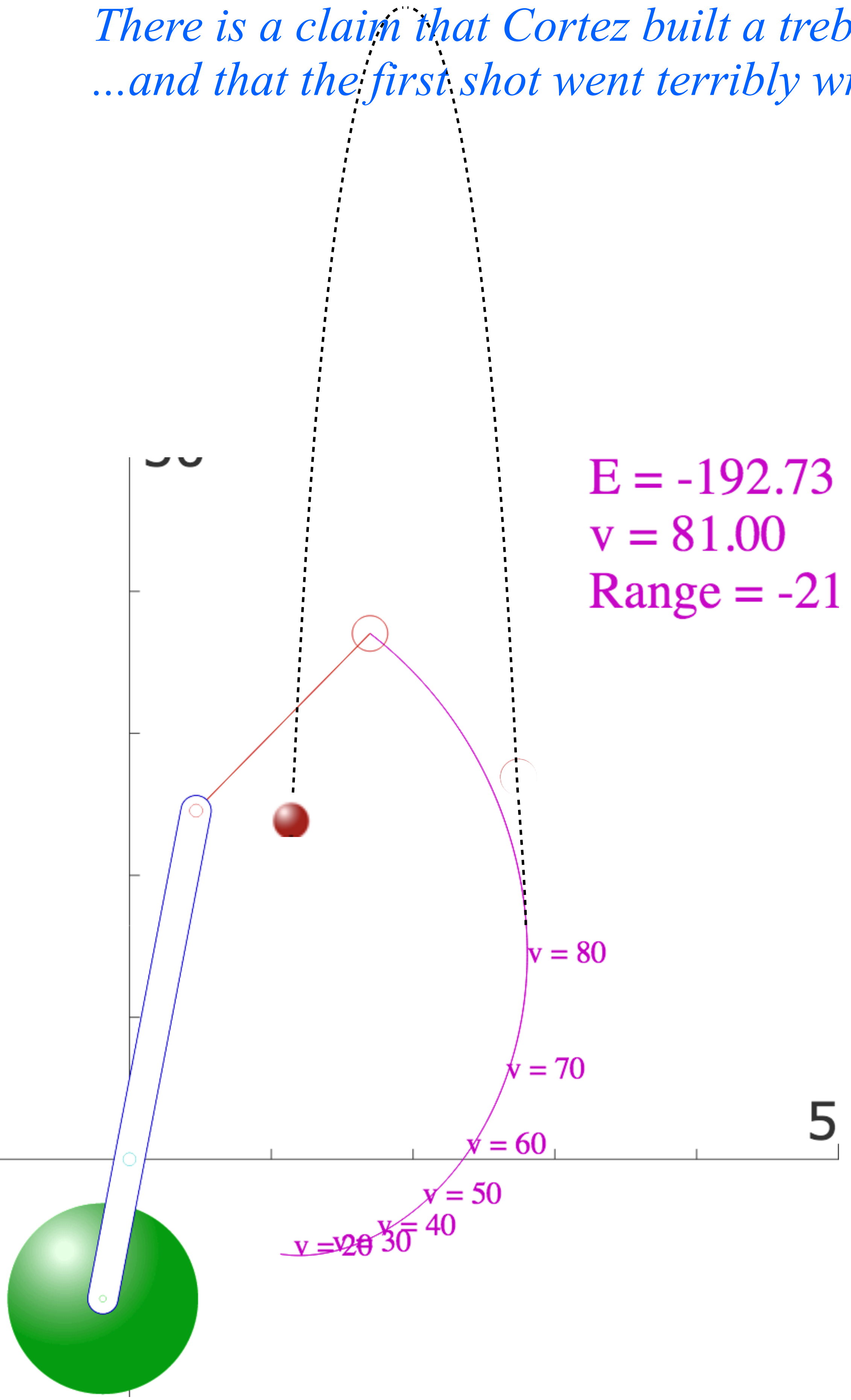
g = 32.2
M1 = 50.0
M2 = 1.0

t = 38.64
E = 0.00
v = 144.52
Range = -464.83

There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...

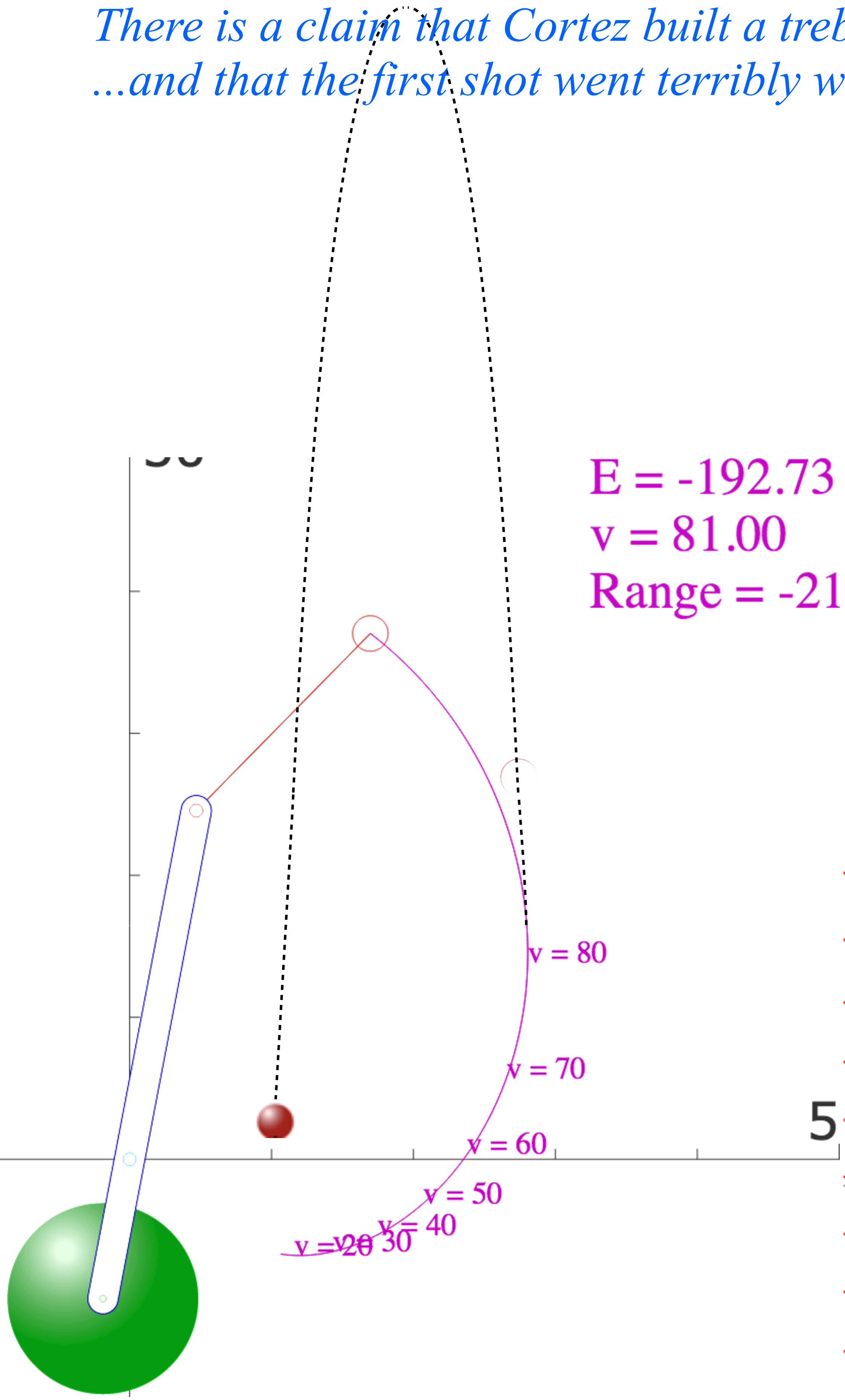


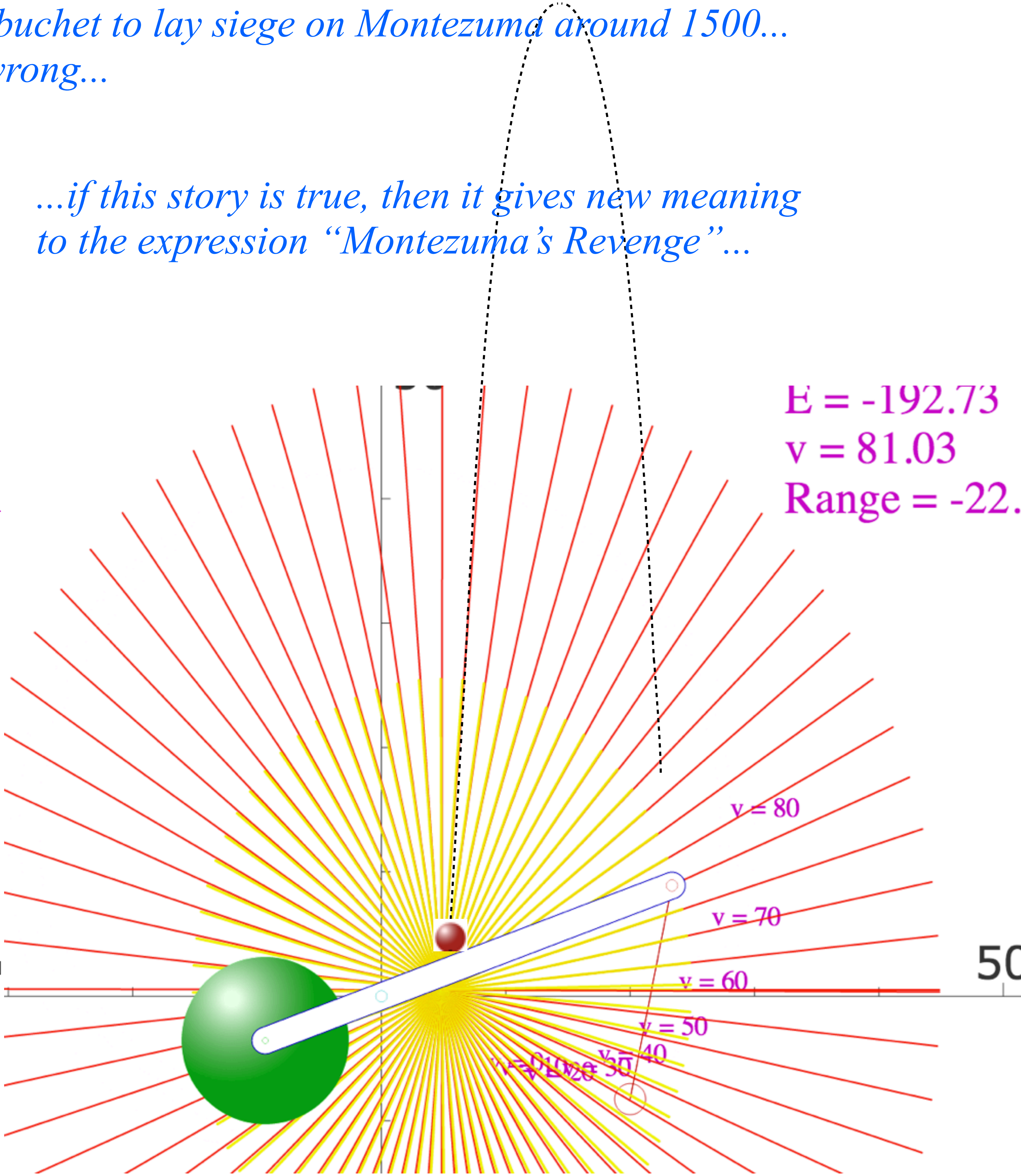
There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...



There is a claim that Cortez built a trebuchet to lay siege on Montezuma around 1500...
...and that the first shot went terribly wrong...

...if this story is true, then it gives new meaning to the expression "Montezuma's Revenge"...

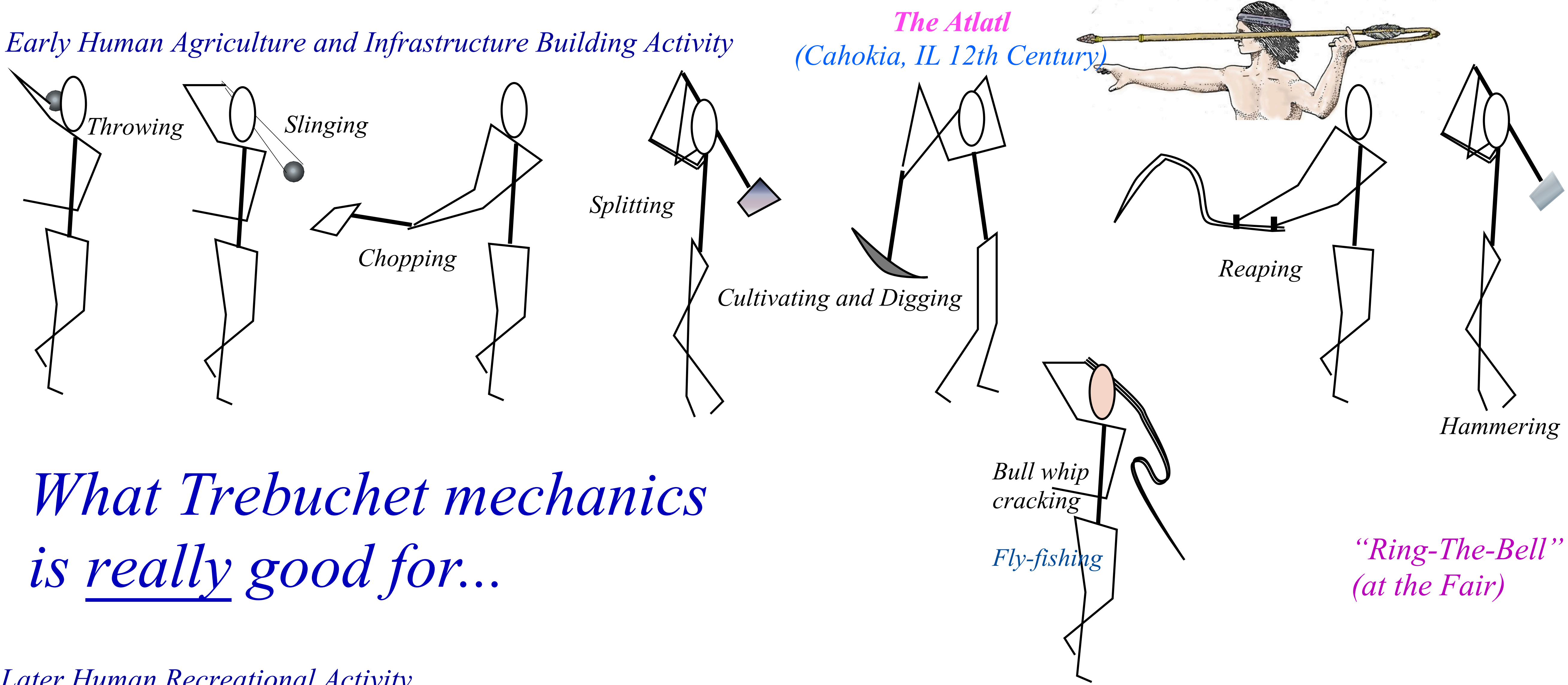




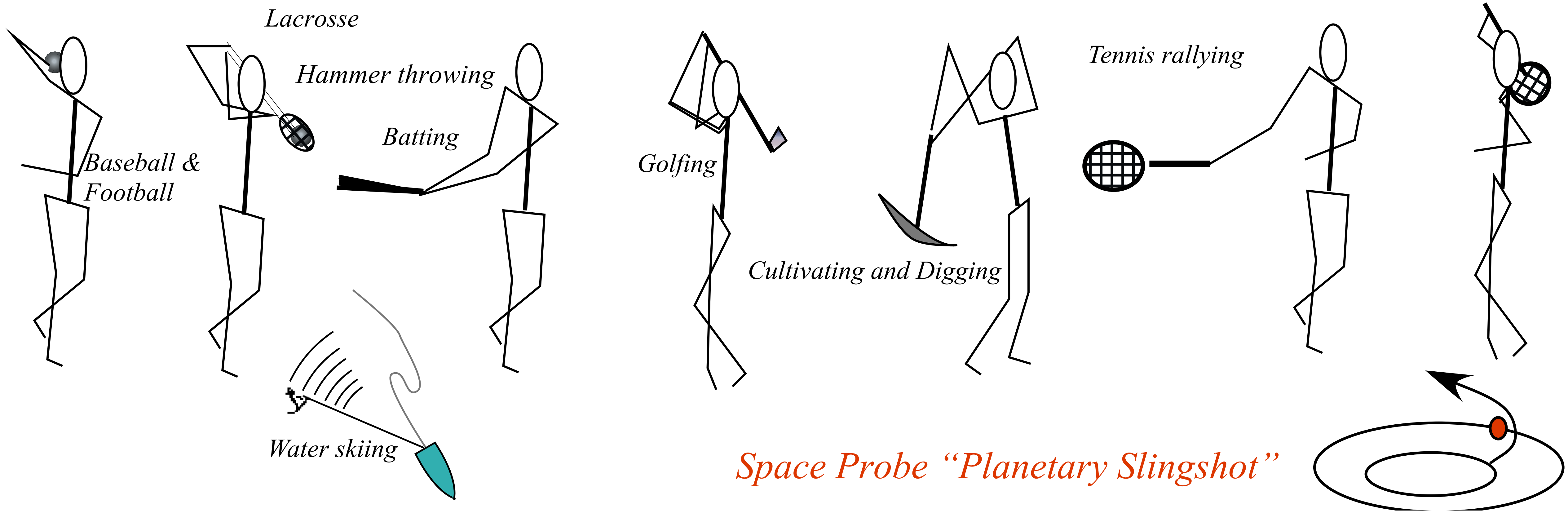
Hamiltonian energy and momentum conservation and symmetry coordinates
- *Coordinate transformation helps reduce symmetric Hamiltonian*
- *Free-space trebuchet kinematics by symmetry*
  - *Algebraic approach*
  - *Direct approach and Superball analogy*
- → *Trebuchet vs Flinger and sports kinematics*
  - *Many approaches to Mechanics*

What Trebuchet mechanics is *really* good for...

Early Human Agriculture and Infrastructure Building Activity

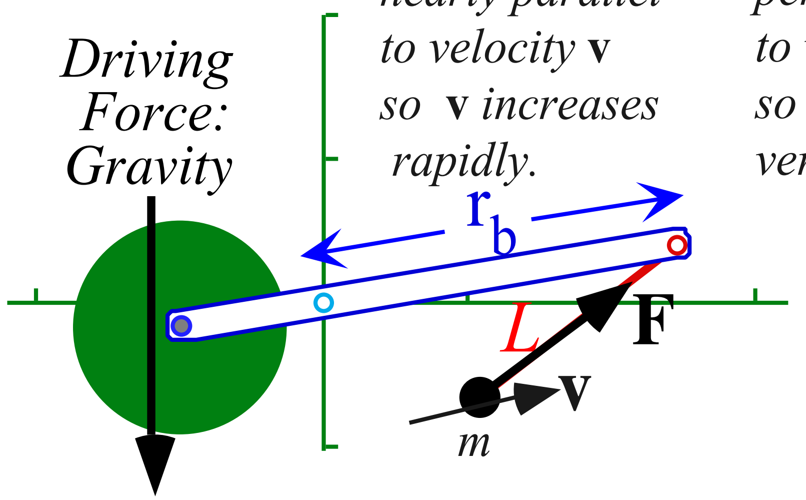


Later Human Recreational Activity



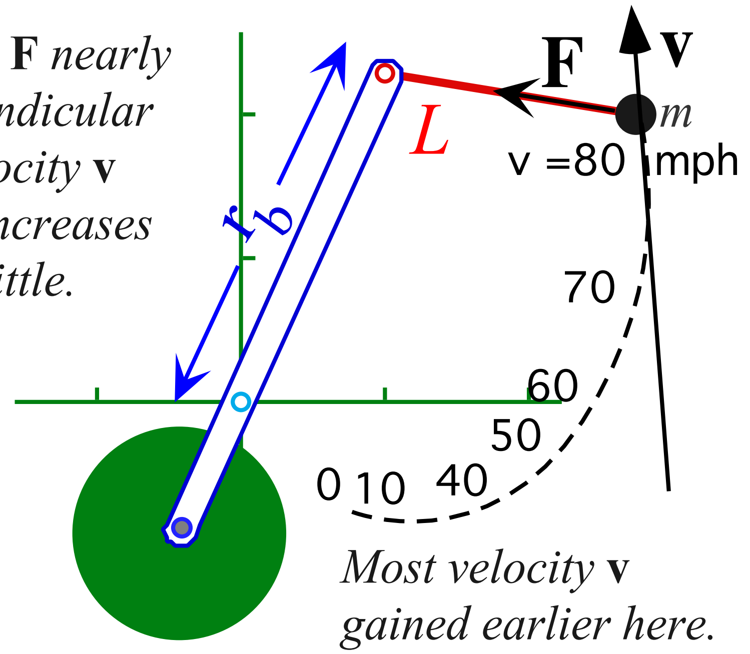
Trebuchet analogy with racquet swing - What we learn

Driving Force: Gravity



Large force **F** *nearly parallel to velocity* **v** *so* **v** *increases rapidly.*

Force **F** *nearly perpendicular to velocity* **v** *so* **v** *increases very little.*



F *mostly serves to steer m here.*

Most velocity **v** *gained earlier here.*

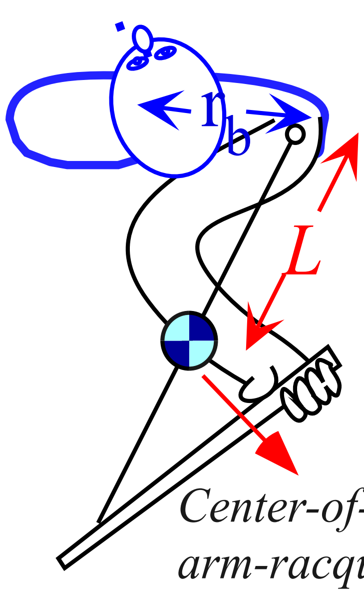
Early on

(Gain the energy/momentum)

Later on

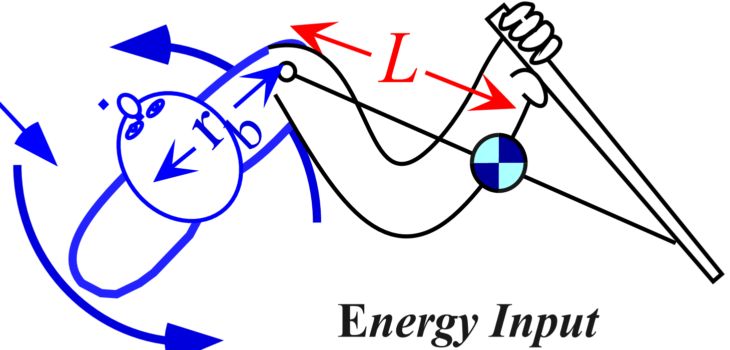
(Steer or guide)

Rotation of body r_b provides most of energy of arm-racquet lever L.



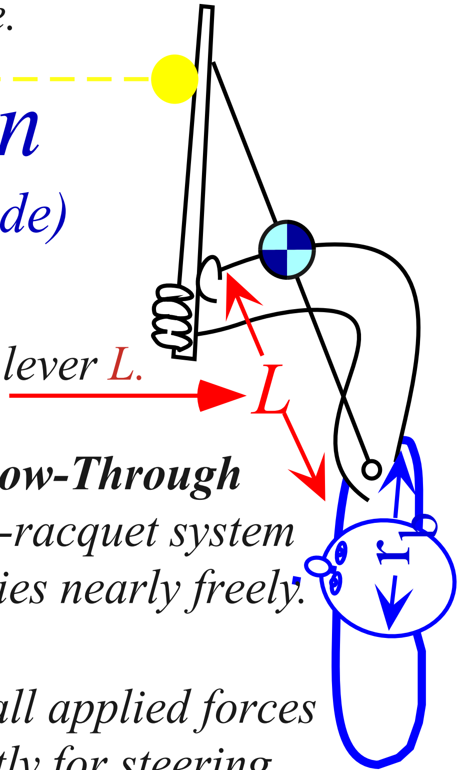
Preparation

Center-of-mass for semi-rigid arm-racquet system L *is "cocked."*



Energy Input

Most of speed gained early by arm-racquet system L.



Follow-Through

Arm-racquet system L *flies nearly freely.*

Small applied forces mostly for steering.

Ball hit occurs.

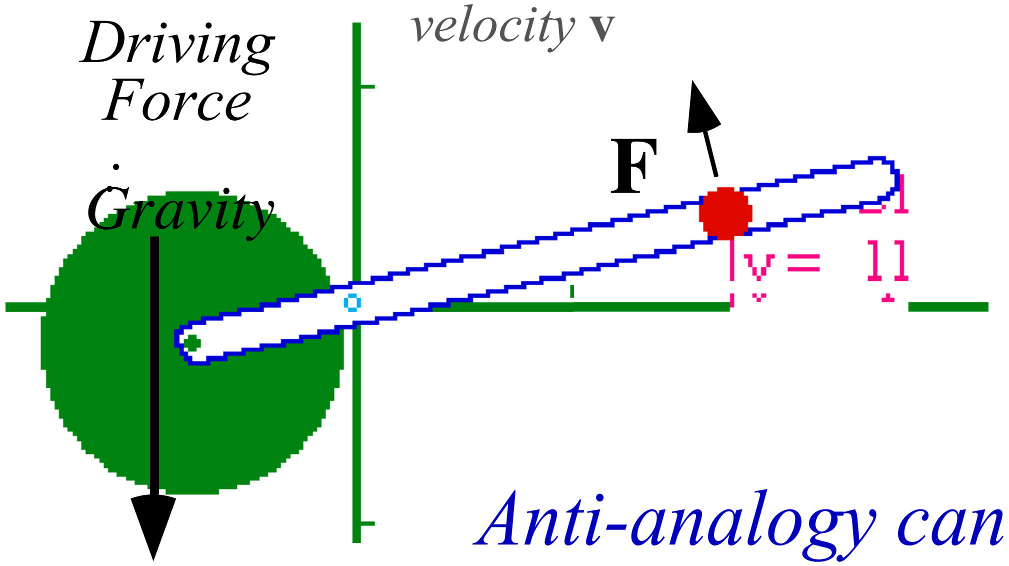
An Opposite to Trebuchet Mechanics- The "Flinger"

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

Early on

(Not much happening)

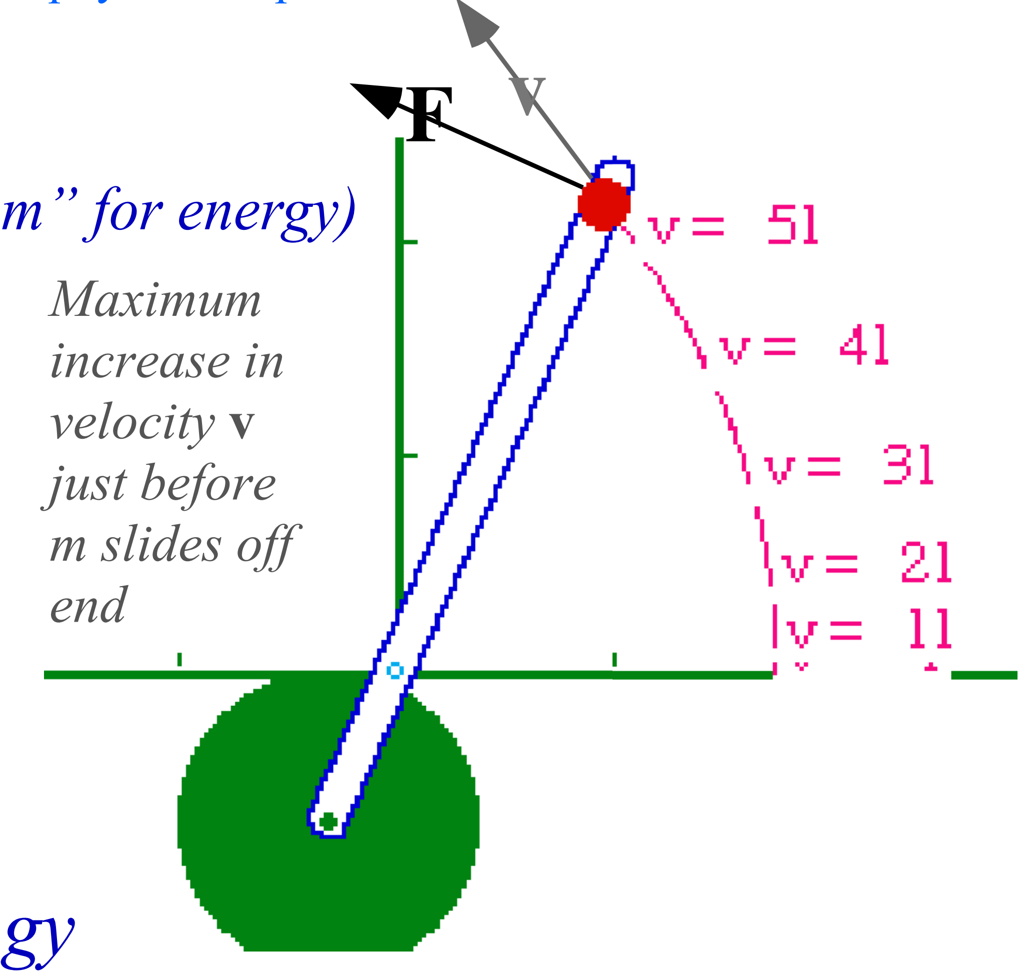
Not much increase in velocity **v**



Later on

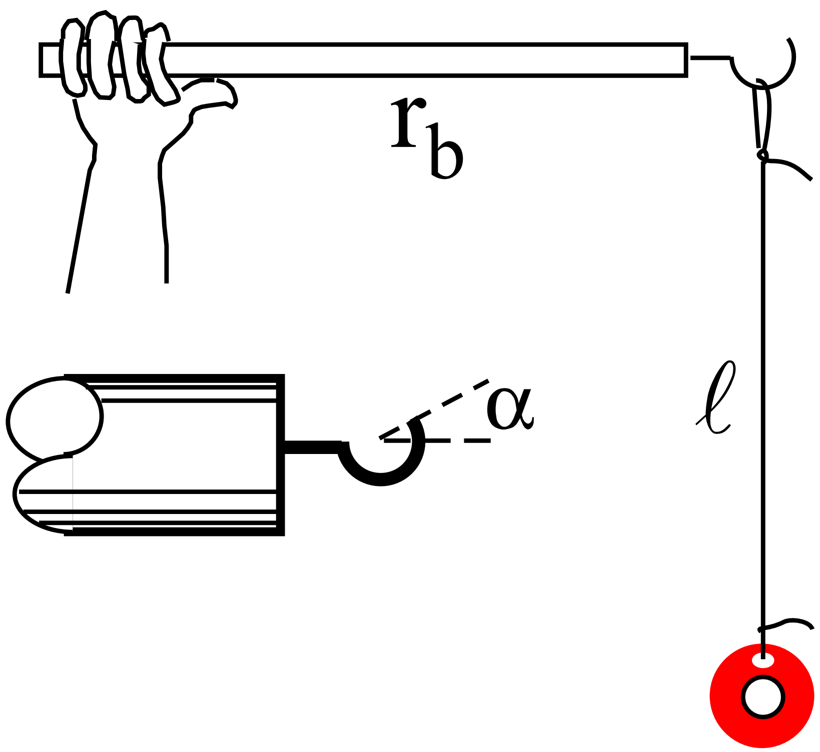
(Last-minute "cram" for energy)

Maximum increase in velocity **v** *just before m slides off end*



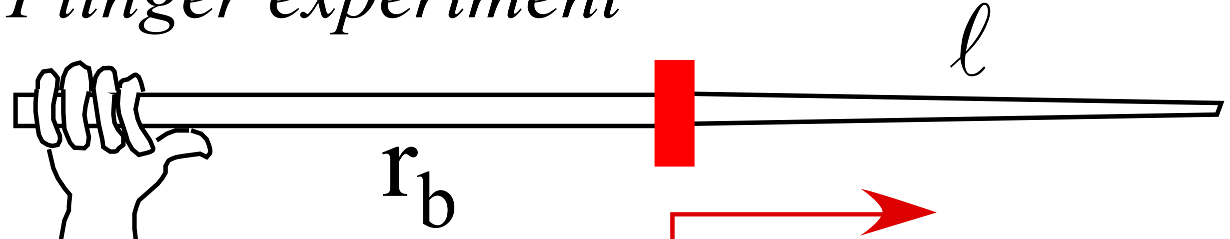
Anti-analogy can be useful pedagogy

Trebuchet-like experiment



skateboard wheel swings

Flinger experiment



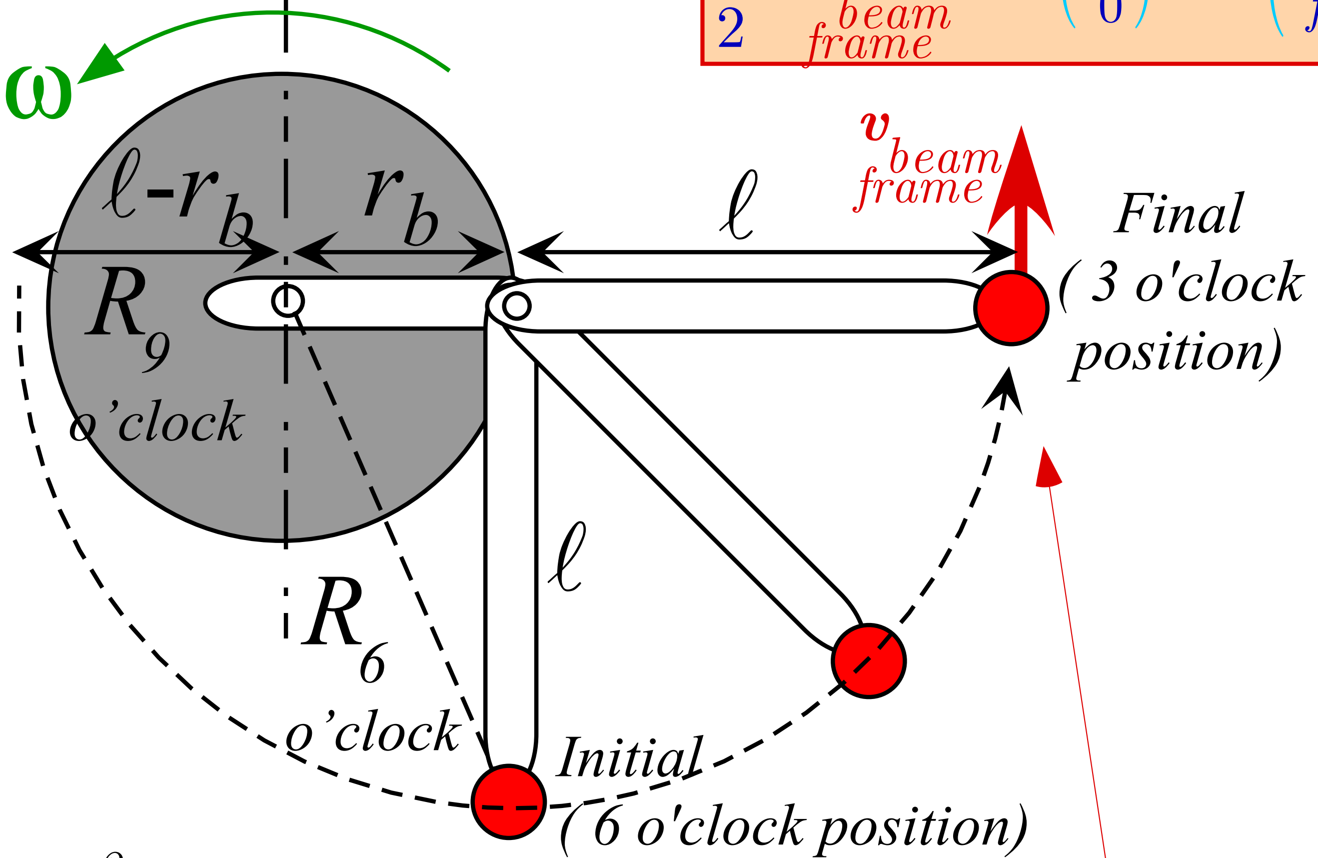
skateboard wheel slides on pool que-stick

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

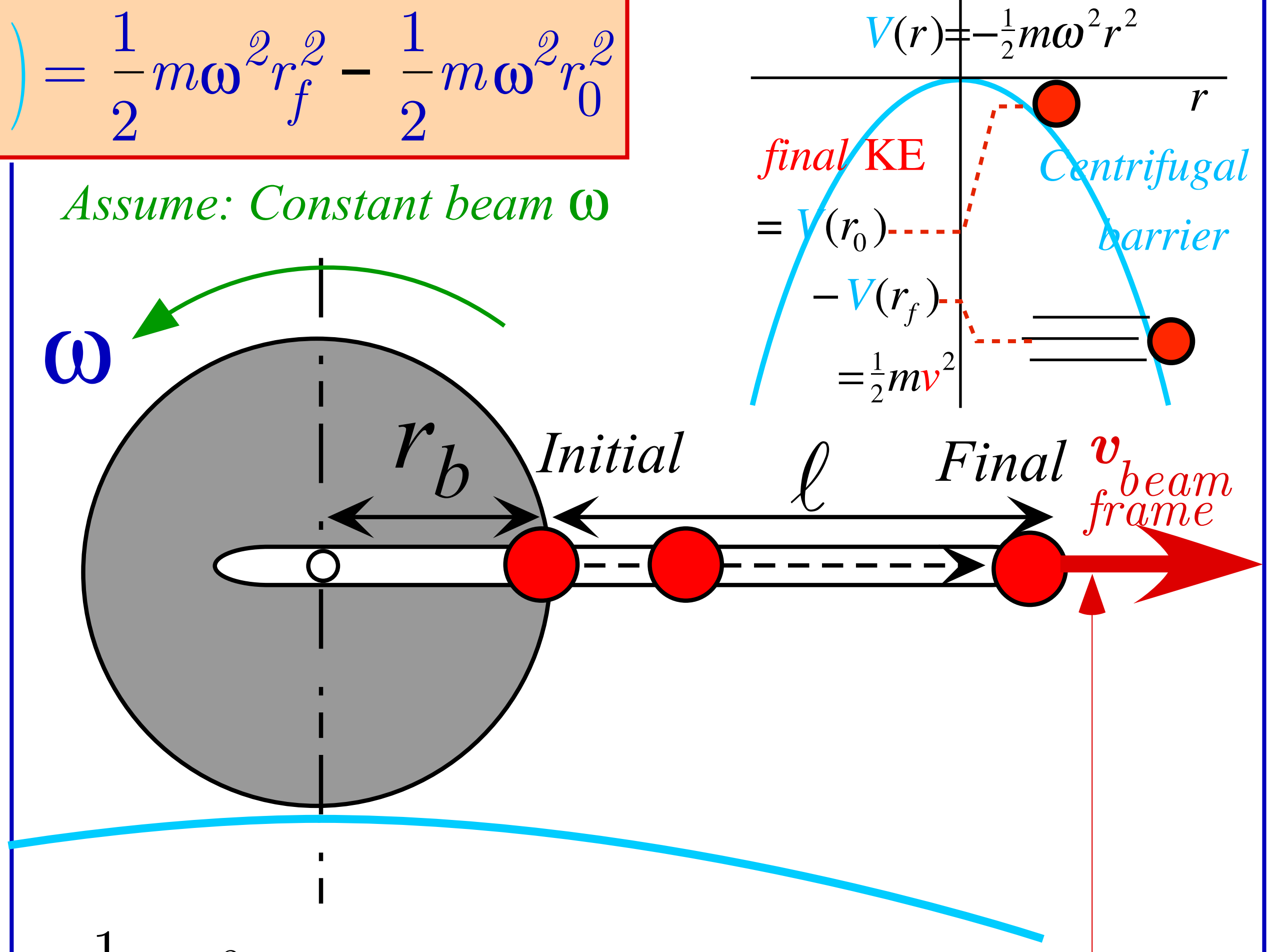
$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V\left(r_0\right) - V\left(r_f\right) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \textit{Final Trebuchet } KE_{\substack{beam\\frame}}$$

Assume: Constant beam ω

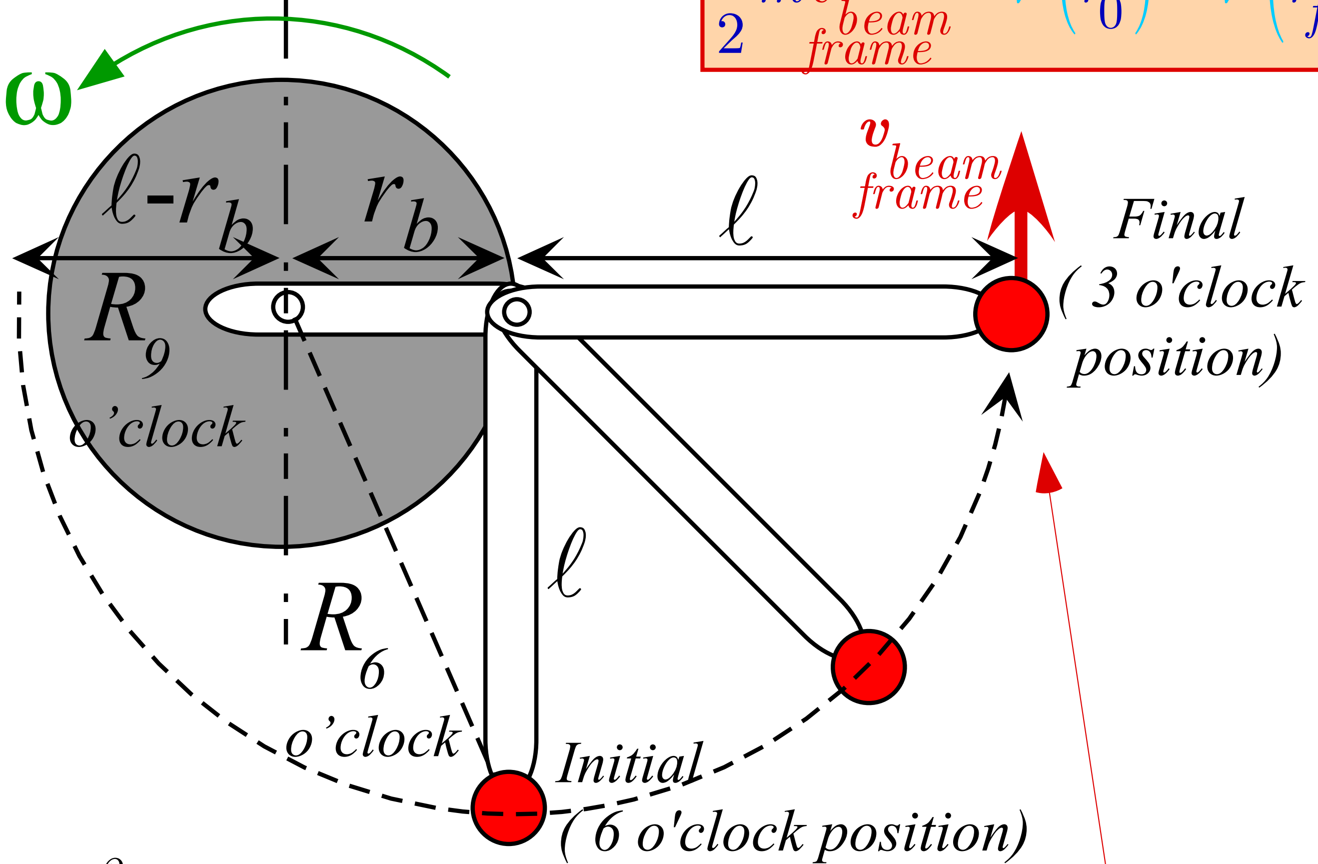


$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \textit{Final Flinger } KE_{\substack{beam\\frame}}$$

Trebuchet model in rotating beam frame

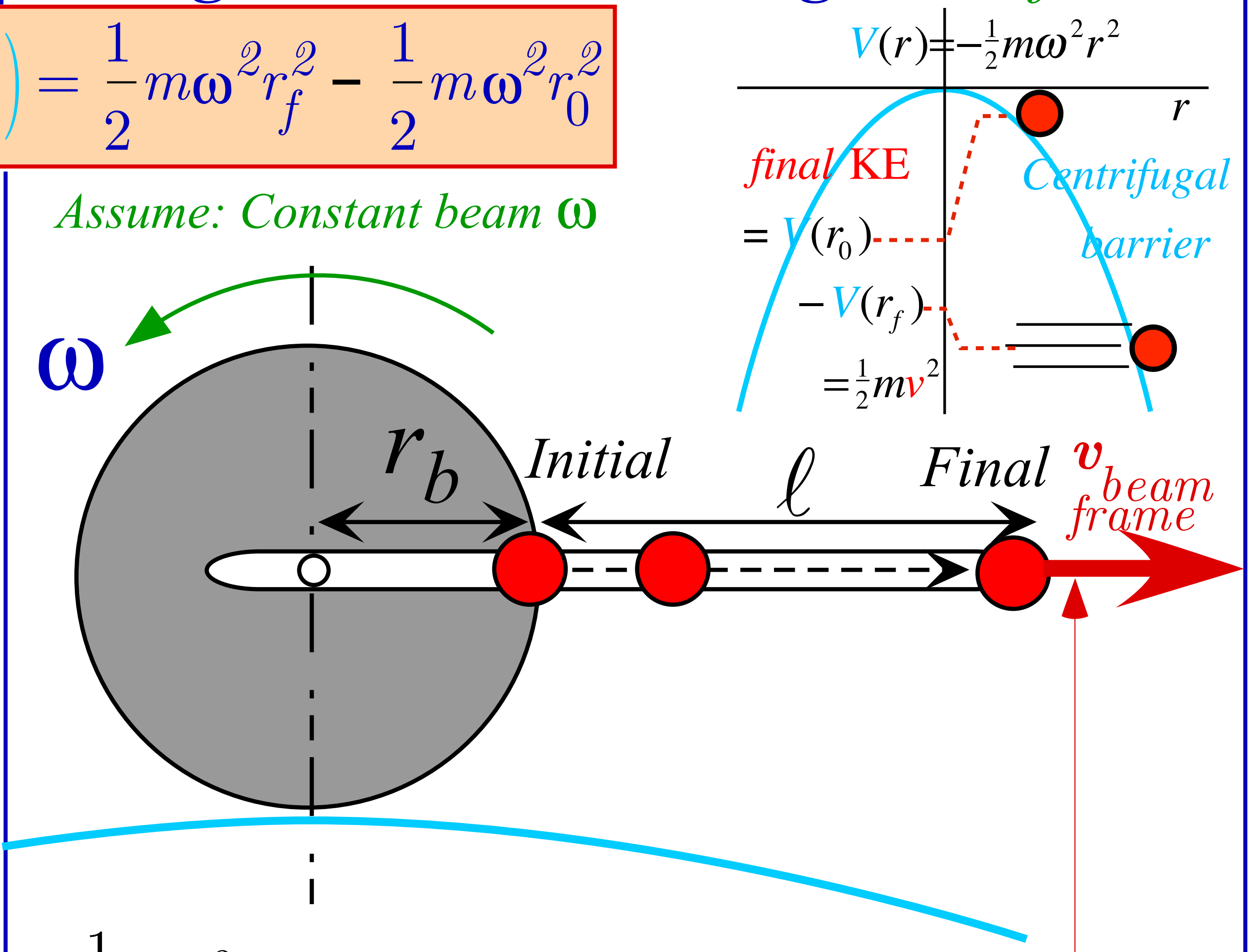
Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

$$\underset{Final}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{Initial}{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)} = \frac{1}{2}m\omega^2\left(2r_b\ \ell\right)$$



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

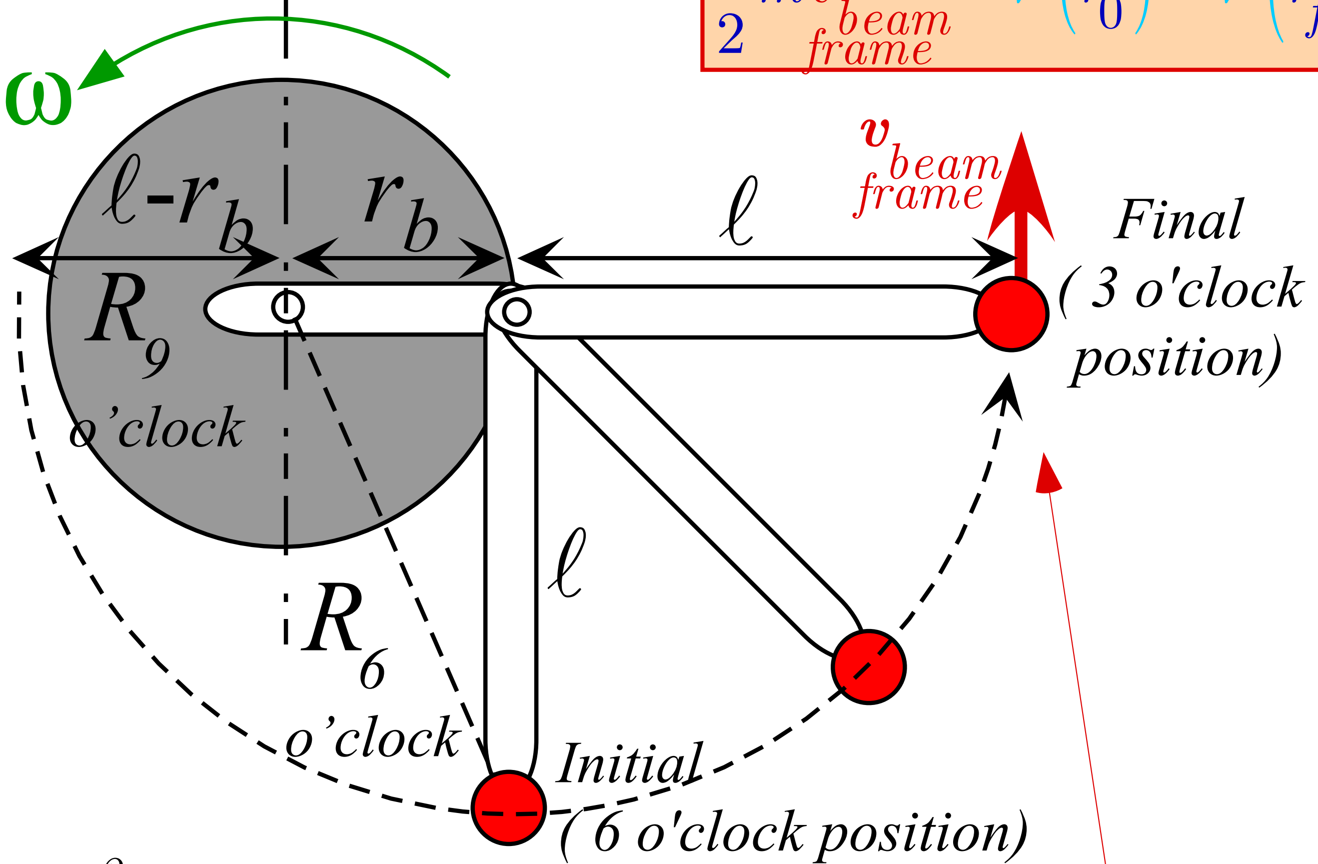
$$\underset{Final}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{Initial}{\frac{1}{2}m\omega^2 r_b^2} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) - V(r_f) = \frac{1}{2}m\omega^2 r_f^2 - \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω

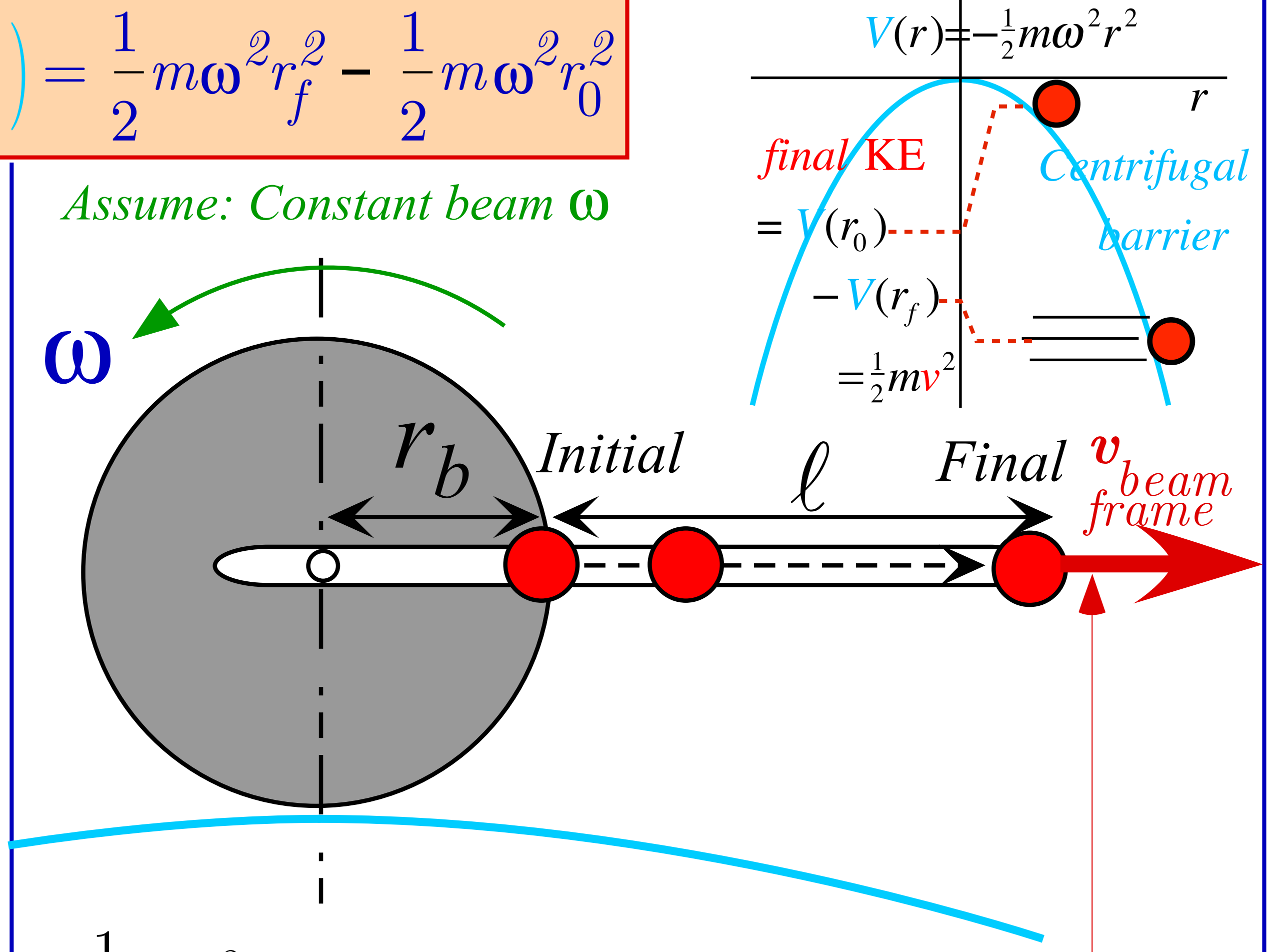


$$\frac{1}{2}mv^2_{\substack{beam\\frame}}\text{(trebuchet)} = \text{Final Trebuchet KE}_{\substack{beam\\frame}}$$

$$\underset{\substack{\text{Final}\\ \text{3 o'clock}}}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{\substack{\text{Initial}\\ \text{6 o'clock}}}{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)} = \frac{1}{2}m\omega^2\left(2r_b\,\ell\right)$$

$$R^2_{\substack{6\\o'clock}} = r_b^2 + \ell^2$$

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}\text{(flinger)} = \text{Final Flinger KE}_{\substack{beam\\frame}}$$

$$\underset{\substack{\text{Final}\\ \text{3 o'clock}}}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{\substack{\text{Initial}\\ \text{3 o'clock}}}{\frac{1}{2}m\omega^2 r_b^2} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

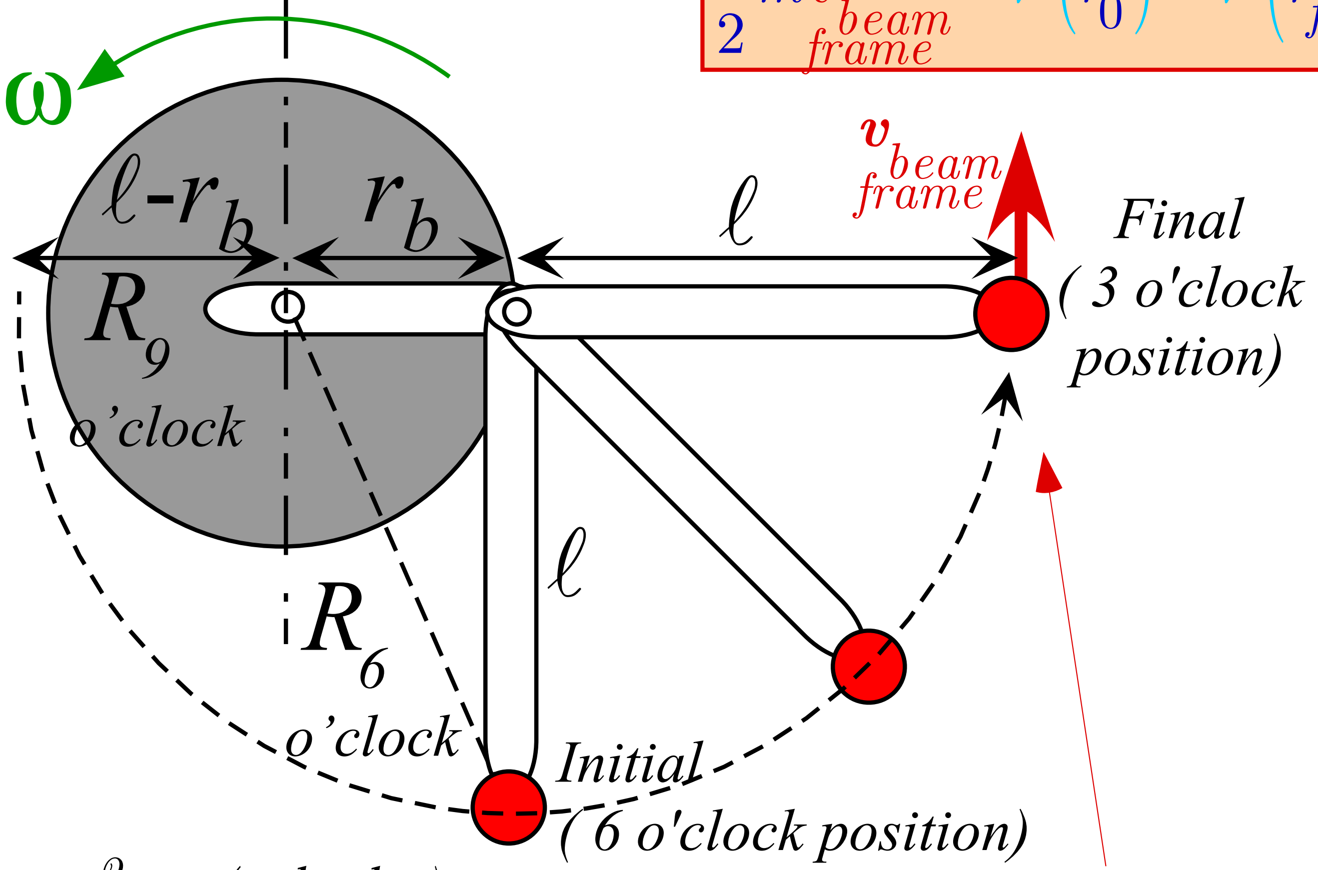
Flinger KE is $\frac{m\omega^2}{2}\ell^2$ more than 6 o'clock trebuchet
but misdirected

Trebuchet model in rotating beam frame

Flinger model in rotating beam frame

$$\frac{1}{2}mv^2_{\substack{beam\\frame}} = V(r_0) \qquad V(r_f) = \frac{1}{2}m\omega^2 r_f^2 \qquad \frac{1}{2}m\omega^2 r_0^2$$

Assume: Constant beam ω



$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{trebuchet}) = \text{Final Trebuchet } KE_{\substack{beam\\frame}}$$

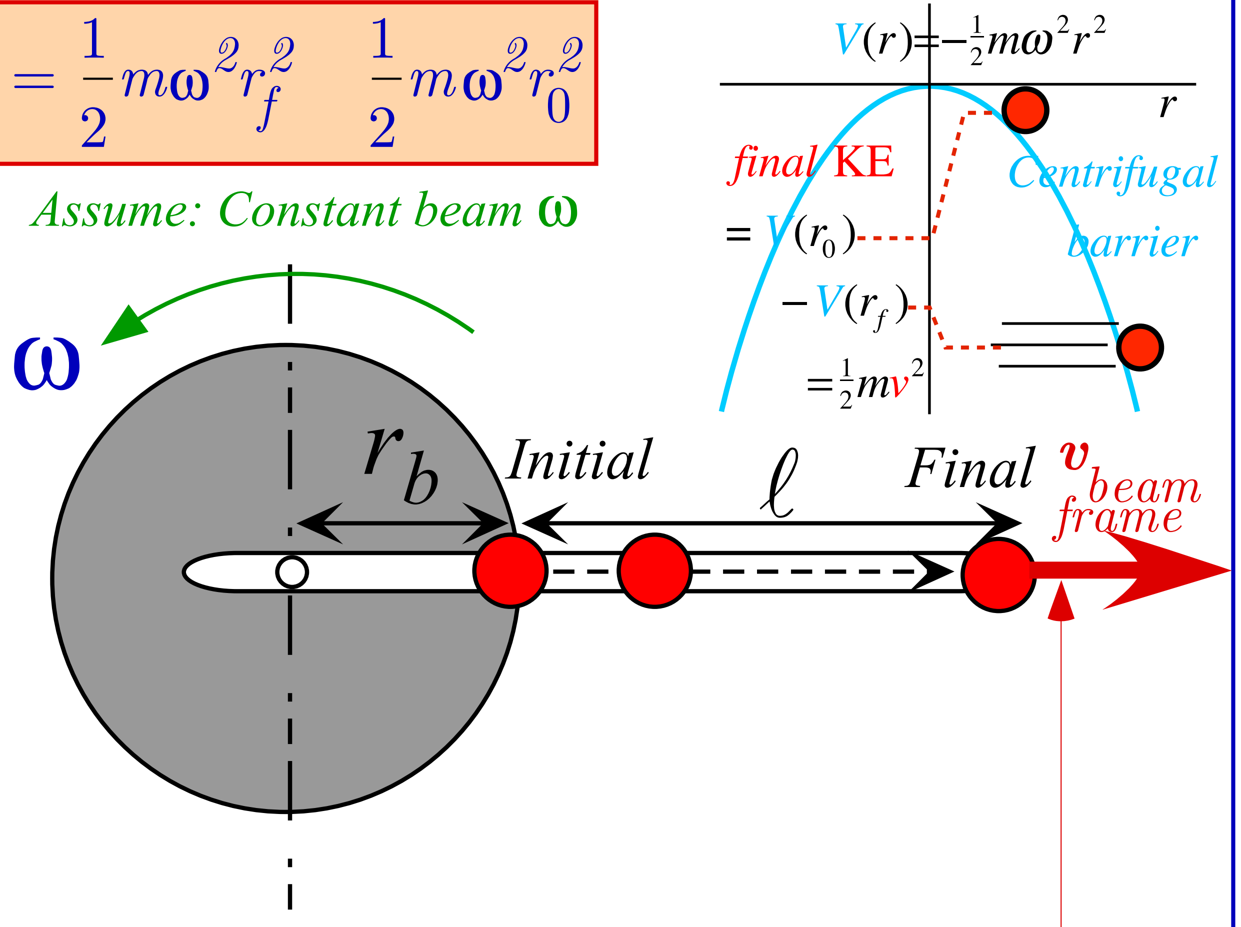
$$\underset{\substack{Final\\3\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{\substack{Initial\\6\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b^2+\ell^2\right)} = \frac{1}{2}m\omega^2\left(2r_b\ell\right)$$

$$R^2_{\substack{6\\o'clock}} = r_b^2+\ell^2$$

Initial 9 o'clock $= \frac{1}{2}m\omega^2\left(4r_b\ell\right)$

$$R^2_{\substack{9\\o'clock}} = r_b^2+\ell^2-2r_b\ell$$

Assume: Constant beam ω



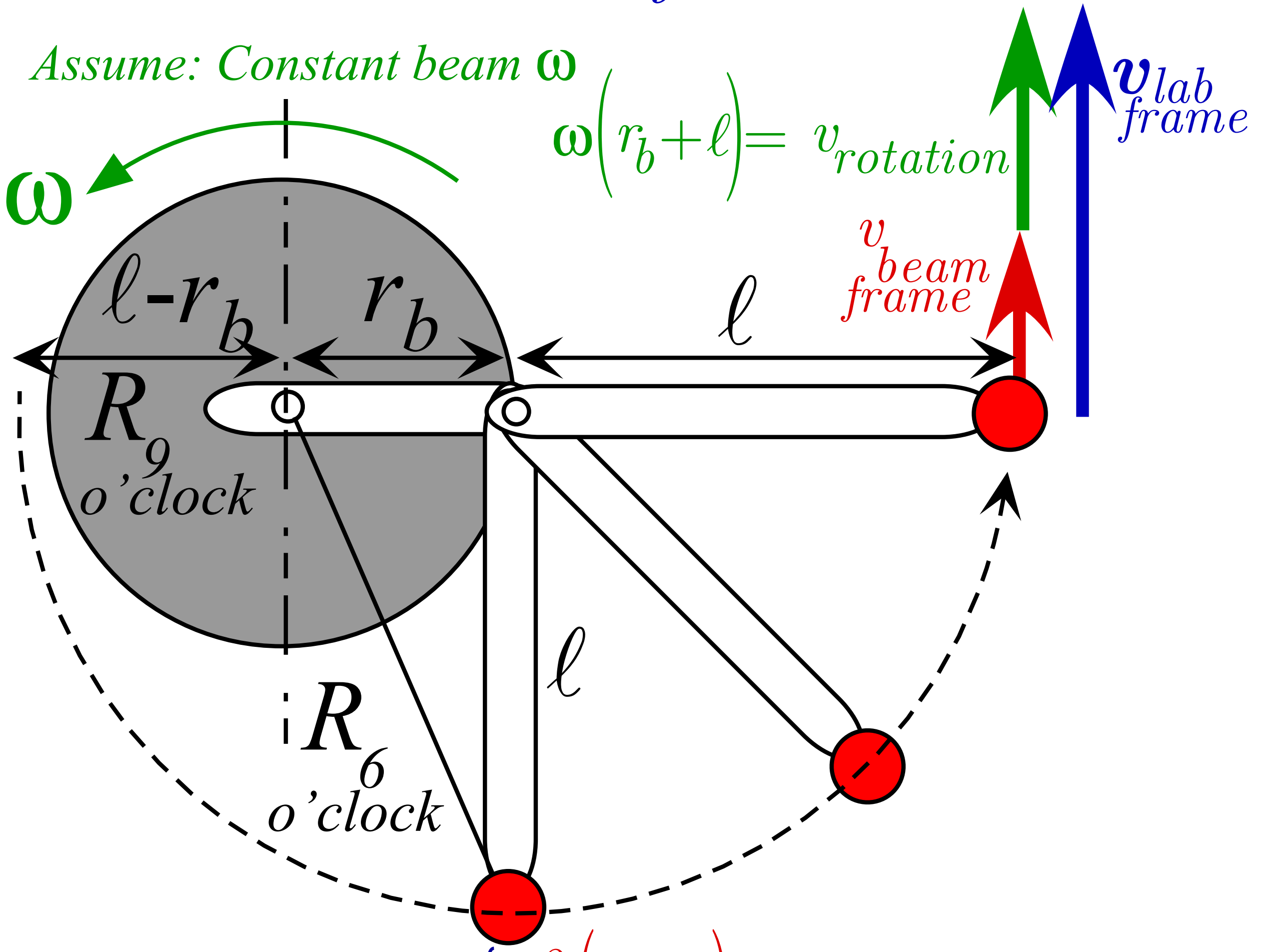
$$\frac{1}{2}mv^2_{\substack{beam\\frame}}(\text{flinger}) = \text{Final Flinger } KE_{\substack{beam\\frame}}$$

$$\underset{\substack{Final\\3\ o'clock}}{\frac{1}{2}m\omega^2\left(r_b+\ell\right)^2} - \underset{\substack{Initial\\3\ o'clock}}{\frac{1}{2}m\omega^2 r_b^2} = \frac{1}{2}m\omega^2\ell\left(2r_b+\ell\right)$$

Flinger KE is $\frac{m\omega^2}{2}\ell^2$ more than 6 o'clock trebuchet but misdirected

Flinger KE is $\frac{m\omega^2}{2}\left(2r_b\ell-\ell^2\right)$ less than 9 o'clock trebuchet and misdirected

Trebuchet model in lab frame



$$v^2_{\substack{beam\\frame}}(\text{trebuchet}) = \begin{cases} \omega^2(2r_b\ \ell) & \textit{half-cocked 6 o'clock}\\ \omega^2(4r_b\ \ell) & \textit{fully-cocked 9 o'clock}\end{cases}$$

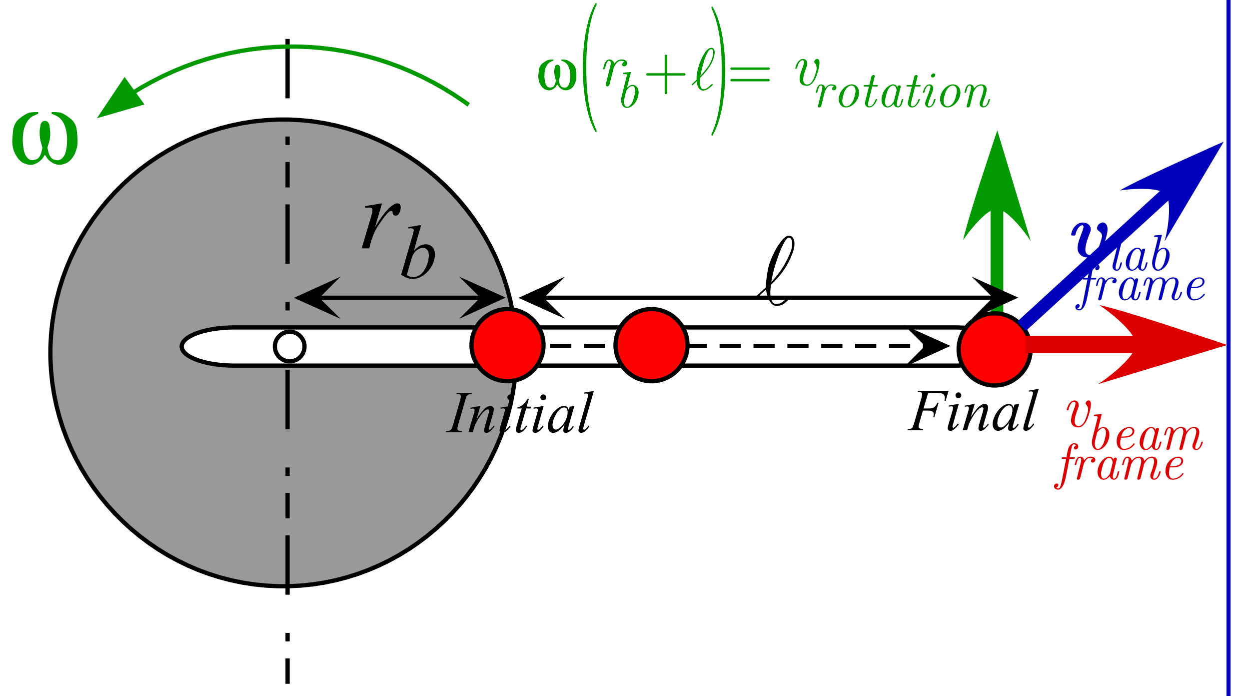
$$v_{lab\ \ frame}(trebuchet) =$$

$$\begin{cases} \omega\left(r_b+\ell+\sqrt{2\ell r_b}\right) & \textit{half-cocked 6 o'clock}\\ \omega\left(r_b+\ell+2\sqrt{\ell r_b}\right) & \textit{fully-cocked 9 o'clock}\end{cases}$$

$$=\begin{cases}5.00\omega\\5.82\omega\end{cases} \qquad =\begin{cases}5.16\omega\\6.00\omega\end{cases} \qquad =\begin{cases}5.00\omega\\5.82\omega\end{cases}$$

$$(r_b=2,\ell=1),\ (r_b=1.5,\ell=1.5),(r_b=1,\ell=2)$$

Flinger model in lab frame



$$v^2_{\substack{beam\\frame}}(\text{flinger}) = \omega^2\ell(2r_b+\ell)$$

$$v_{lab\ \ frame}(flinger) =$$

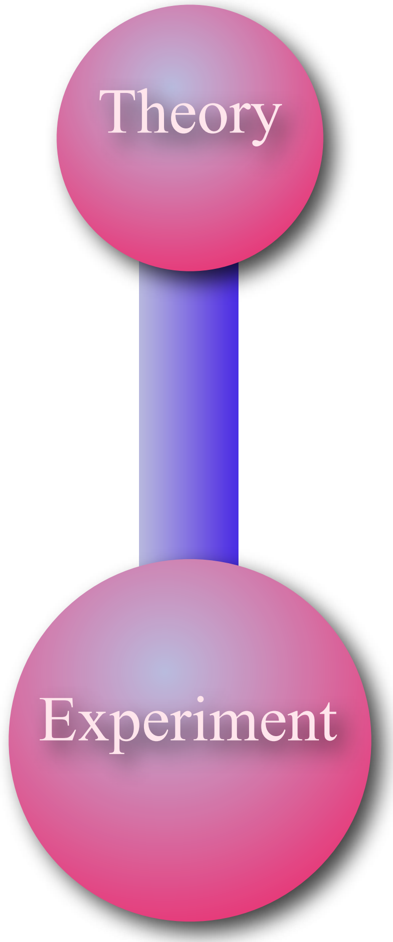
$$=\omega\sqrt{(r_b+\ell)^2+\ell(2r_b+\ell)}=\omega\sqrt{2(r_b+\ell)^2-r_b{}^2}$$

(compare)

$$=3.74\omega \qquad =3.96\omega \qquad =4.12\omega$$

$$(r_b=2,\ell=1),\ (r_b=1.5,\ell=1.5),(r_b=1,\ell=2)$$

Physics used to be pretty much bi-polar...



Theory

Experiment

Now that situation is changing...

Many Approaches to Mechanics (Trebuchet Equations)

Each has advantages and disadvantages

- U.S. Approach

  Quick'n dirty

  Newton F=Ma Equations

  Cartesian coordinates

- French Approach

  Tres elegant

  Lagrange Equations

  in Generalized Coordinates

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

- German Approach

  Pride and Precision

  Riemann Christoffel Equations

  in Differential Manifolds

$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$$

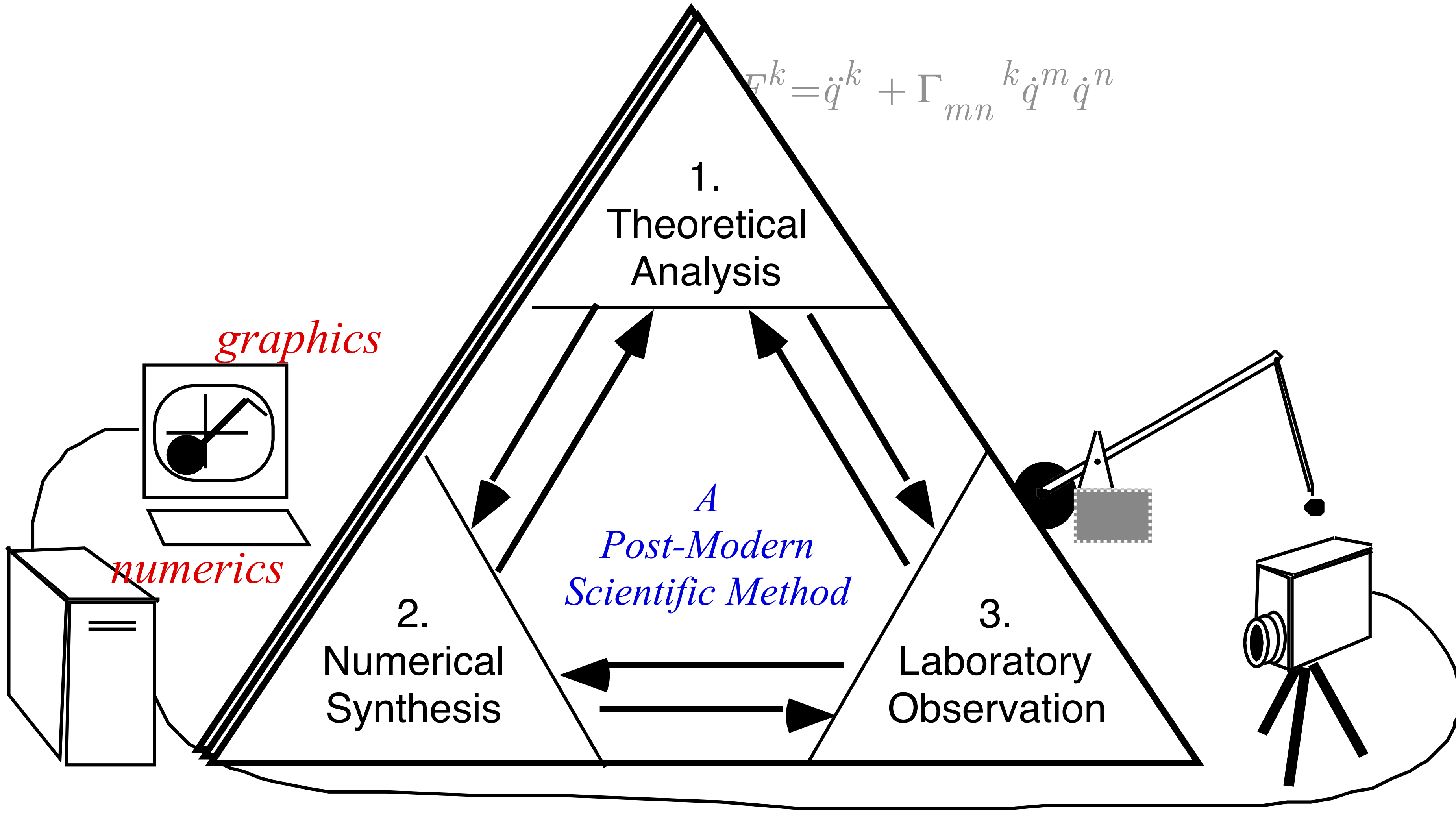
- Anglo-Irish Appproach

  Powerfully Creative

  Hamilton's Equations

  Phase Space $\dot{p}_j = -\frac{\partial H}{\partial q^j}, \qquad \dot{q}^k = \frac{\partial H}{\partial p^k}.$

- Unified Approach



$$F^k = \ddot{q}^k + \Gamma_{mn}{}^k \dot{q}^m \dot{q}^n$$

All approaches have one thing in common:

The Art of Approximation

Physics lives and dies by the art of approximate models and analogs.

$dS = Ldt = p_\mu dq^\mu - Hdt$

Hamilton-Jacobi-Poincare: $p_\mu = \frac{\partial S}{\partial q^\mu}$, $-H = \frac{\partial S}{\partial t}$

Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

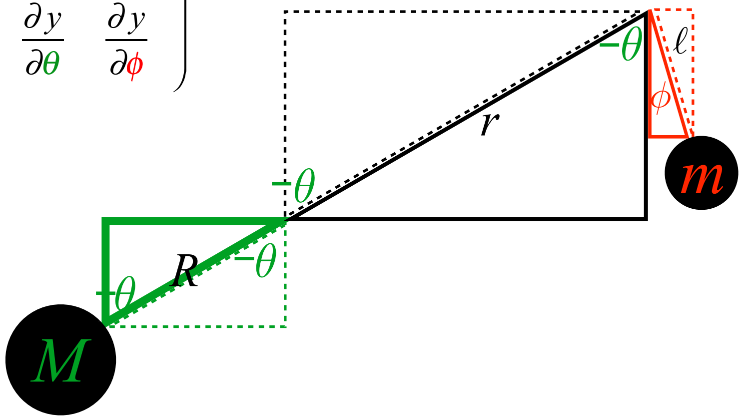
Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\,dX \;=\; M\ddot{X}\,dX \;=\; F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi \;=\; M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\,dY \quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\,dx \quad +m\ddot{x}\,dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\,dy \quad +m\ddot{y}\,dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$



Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$\text{Let}: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$\text{Let}: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

Add F_θ gravity given
(F_X=0 , F_Y=-Mg)
(F_x=0 , F_y=-mg)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

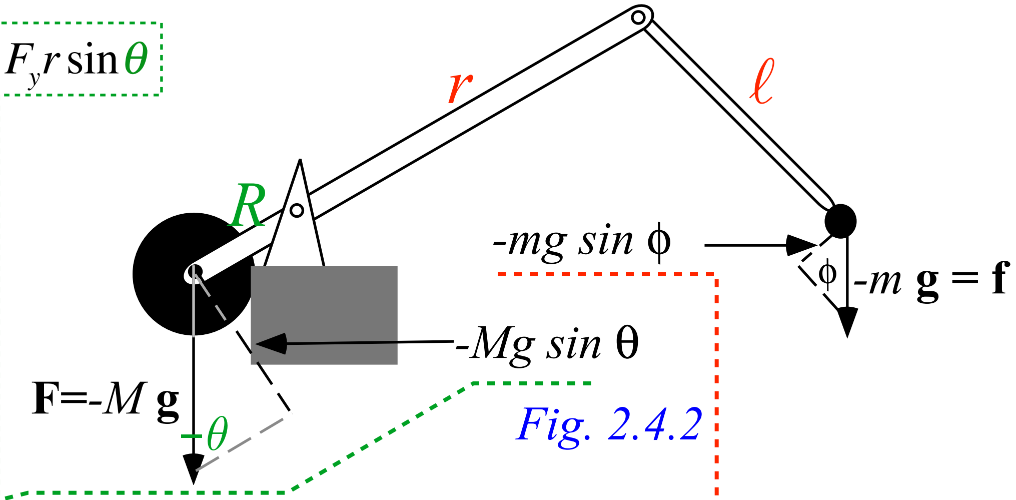


Fig. 2.4.2

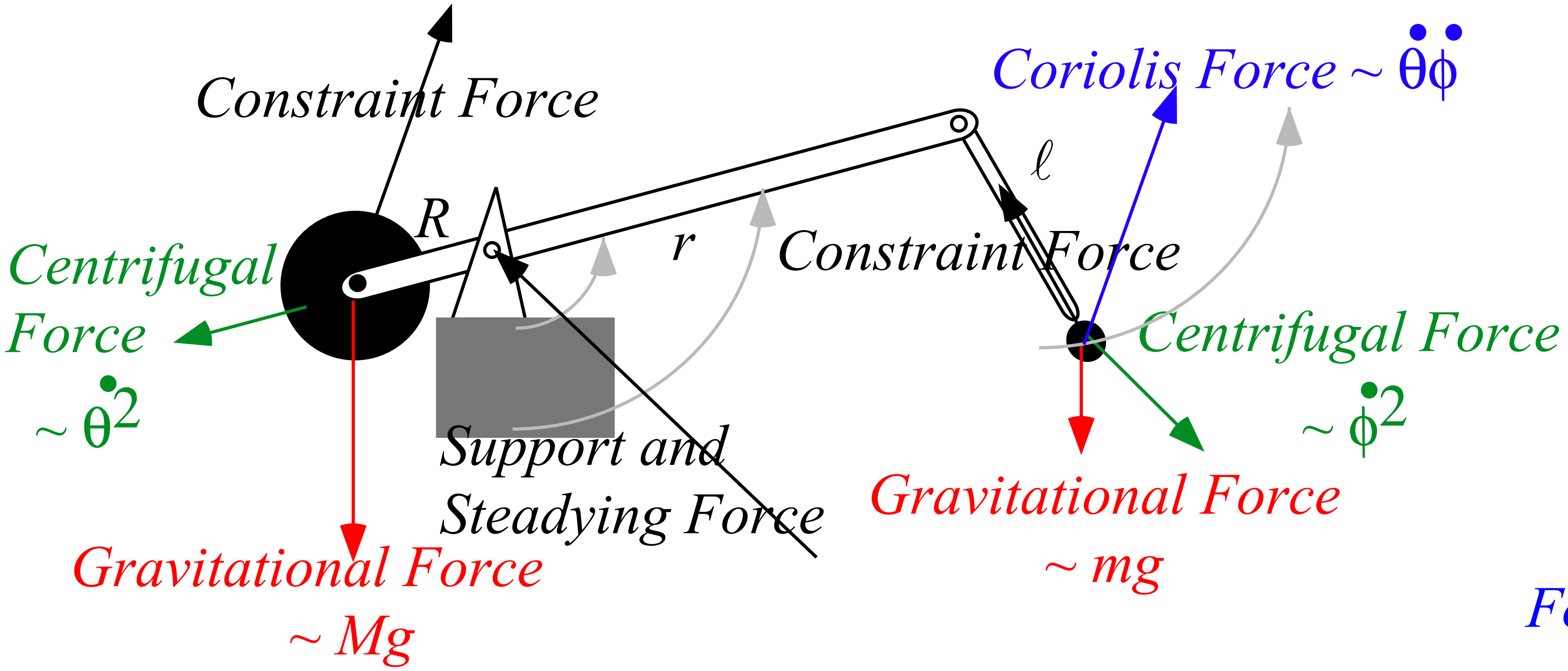
$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

Add F_ϕ gravity given
(F_X=0 , F_Y=-Mg)
(F_x=0 , F_y=-mg)

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

These are competing torques on main beam R... *... and a torque on throwing lever ℓ*

Forces: total, genuine, potential, and/or fictitious



Acceleration and 'Fictitious' Forces:
Coriolis
Centrifugal

Applied 'Real' Forces:
Gravity
Stimuli
Friction...

Constraint 'Internal' Forces:
Stresses
Support...
(Do not contribute. Do no work.)

$$\dot{p}_{\theta} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_{\theta} + 0$$

$$\dot{p}_{\phi} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_{\phi} + 0$$

Lagrange Force equations

(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)

For conservative forces

where: $F_{\theta} = -\frac{\partial V}{\partial \theta}$ and: $\frac{\partial V}{\partial \dot{\theta}} = 0$

$F_{\phi} = -\frac{\partial V}{\partial \phi}$ and: $\frac{\partial V}{\partial \dot{\phi}} = 0$

$$p_{\theta} = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_{\theta} = \frac{\partial L}{\partial \theta}$$

$$p_{\phi} = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_{\phi} = \frac{\partial L}{\partial \phi}$$

Lagrange Potential equations

$L=T-V$

Fig. 2.5.2 (modified)

Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.