

Lecture 17
Tue 10.24.2017

Riemann-Christoffel equations and covariant derivative (Ch. 4-7 of Unit 3)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

Christoffel g-derivative formula

What's a tensor? What's not?

General Riemann equations of motion (No explicit t-dependence and fixed GCC)

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Christoffel relation to Coriolis coefficients

Mechanics of ideal fluid vortex

Separation of GCC Equations: Effective Potentials

Small $(n_\rho:m_\phi)$ -periodic and quasi-periodic oscillations

2D Spherical pendulum "Bowl-Bowling" and the "I-Ball"

$(n_\rho:m_\phi)=(2:1)$ vs $(1:1)$ periodic and quasi-periodic orbits

Cycloidal ruler&compass geometry

(To be applied to mechanics in electromagnetic fields and collisional rotation in following lectures.)

→ *Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$*

Christoffel g-derivative formula

What's a tensor? What's not?

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;k}$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m $\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}{}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij}{}^k$ the second kind

defined by:

$$\Gamma_{in;}{}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^m$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}{}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij}{}^k$ the second kind

defined by:

$$\Gamma_{in;}{}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^m$$

$$\Gamma_{in;\ell} = g_{m\ell} \Gamma_{in;}{}^m \text{ since: } \mathbf{E}_\ell = g_{m\ell} \mathbf{E}^m$$

$$\Gamma_{in;}{}^m = g^{m\ell} \Gamma_{in;\ell} \text{ since: } \mathbf{E}^m = g^{m\ell} \mathbf{E}_\ell$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}{}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij}{}^k$ the second kind

defined by:

$$\Gamma_{in;}{}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^m$$

$$\Gamma_{in;\ell} = g_{m\ell} \Gamma_{in;}{}^m \text{ since: } \mathbf{E}_\ell = g_{m\ell} \mathbf{E}^m$$

i,n to n,i
symmetry
guaranteed here

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E}_i}{\partial q^n}$$

i,n to n,i
symmetry
guaranteed here

$$\Gamma_{in;}{}^m = g^{m\ell} \Gamma_{in;\ell} \text{ since: } \mathbf{E}^m = g^{m\ell} \mathbf{E}_\ell$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and Γ_{ij}^k

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients Γ_{ij}^k the second kind

defined by:

$$\Gamma_{in;}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}^m$$

$$\Gamma_{in;\ell} = g_{m\ell} \Gamma_{in;}^m \text{ since: } \mathbf{E}_\ell = g_{m\ell} \mathbf{E}^m$$

i,n to n,i
symmetry
guaranteed here

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E}_i}{\partial q^n}$$

i,n to n,i
symmetry
guaranteed here

$$\Gamma_{in;}^m = g^{m\ell} \Gamma_{in;\ell} \text{ since: } \mathbf{E}^m = g^{m\ell} \mathbf{E}_\ell$$

Q: Do we need a third kind of Γ -coefficient or a Λ -coefficient?

(to differentiate contravariant- \mathbf{E}^n or covariant U_n)

$$\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda_{im}^n \mathbf{E}^m, \text{ where: } \Lambda_{im}^n = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and Γ_{ij}^k

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients Γ_{ij}^k the second kind

defined by:

$$\Gamma_{in;}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}^m = \Gamma_{ni;}^m$$

i,n to n,i
symmetry
guaranteed here

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E}_i}{\partial q^n}$$

i,n to n,i
symmetry
guaranteed here

Q: Do we need a third kind of Γ -coefficient or a Λ -coefficient? $\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda_{im}^n \mathbf{E}^m$, where: $\Lambda_{im}^n = \frac{\partial \mathbf{E}^n}{\partial q^i} \cdot \mathbf{E}_m$
(to differentiate contravariant- \mathbf{E}^n or covariant U_n)

A: NO! That Λ -coefficient is just a Γ -coefficient with a (-). $0 = \frac{\partial(\delta_m^n)}{\partial q^i} = \frac{\partial(\mathbf{E}^n \cdot \mathbf{E}_m)}{\partial q^i} = \frac{\partial \mathbf{E}^n}{\partial q^i} \cdot \mathbf{E}_m + \mathbf{E}^n \cdot \frac{\partial \mathbf{E}_m}{\partial q^i}$
So: $\Lambda_{im}^n = -\Gamma_{im}^n$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}{}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij}{}^k$ the second kind

defined by:

$$\Gamma_{in;}{}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^m$$

Any vector derivative can be expressed using $\Gamma_{ij}{}^k$ in terms of \mathbf{E}_m

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^m \right) \mathbf{E}_m$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}{}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij}{}^k$ the second kind

defined by:

$$\Gamma_{in;}{}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^m$$

Any vector derivative can be expressed using $\Gamma_{ij}{}^k$ in terms of \mathbf{E}_m or \mathbf{E}^m

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^m \right) \mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^n \right) \mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m = -\mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda_{im}{}^n = -\Gamma_{im;}{}^n$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and Γ_{ij}^k

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients Γ_{ij}^k the second kind

defined by:

$$\Gamma_{in;}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}^m = \Gamma_{ni;}^m$$

Any vector derivative can be expressed using Γ_{ij}^k in terms of \mathbf{E}_m or \mathbf{E}^m

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial q^i} &= \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m \right) \mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}^n \right) \mathbf{E}^m \\ &= \underbrace{\left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^m \right)}_{U^m_{;i}} \mathbf{E}_m = \underbrace{\left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}^n \right)}_{U_{m;i}} \mathbf{E}^m \end{aligned}$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \cdot \mathbf{E}_m = -\mathbf{E}^n \cdot \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda_{im}^n = -\Gamma_{im}^n$

(Note more funny semi-colon ; notation)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}{}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij}{}^k$ the second kind

defined by:

$$\Gamma_{in;}{}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}^m = \Gamma_{ni;}{}^m$$

Any vector derivative can be expressed using $\Gamma_{ij}{}^k$ in terms of \mathbf{E}_m or \mathbf{E}^m

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial q^i} &= \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^m \right) \mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^n \right) \mathbf{E}^m \\ &= U^m{}_{;i} \mathbf{E}_m = U_{m;i} \mathbf{E}^m \end{aligned}$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \cdot \mathbf{E}_m = -\mathbf{E}^n \cdot \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda_{im}{}^n = -\Gamma_{im}{}^n$

Defining *covariant derivative* $U^m{}_{;i}$
of a *contravariant component* U^m

(Note more funny semi-colon ; notation)

$$U^m{}_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^m \equiv \frac{\partial \mathbf{U}}{\partial q^i} \cdot \mathbf{E}^m$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

GCC q^m derivatives of vectors \mathbf{U} are due to:

(1) changing U^m components

(2) curving GCC vectors \mathbf{E}_n .

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i} (U^j \mathbf{E}_j) = \frac{\partial U^m}{\partial q^i} (\mathbf{E}_m) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

(Note funny semi-colon ; notation)

Derivative of \mathbf{E}_n is expressed using \mathbf{E}^ℓ or else \mathbf{E}_m

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \Gamma_{in;\ell} \mathbf{E}^\ell = \Gamma_{in;}{}^m \mathbf{E}_m$$

Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij}{}^k$ the second kind

defined by:

$$\Gamma_{in;}{}^m = \frac{\partial \mathbf{E}_n}{\partial q^i} \cdot \mathbf{E}^m = \Gamma_{ni;}{}^m$$

Any vector derivative can be expressed using $\Gamma_{ij}{}^k$ in terms of \mathbf{E}_m or \mathbf{E}^m

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial q^i} &= \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^m \right) \mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^n \right) \mathbf{E}^m \\ &= U^m{}_{;i} \mathbf{E}_m = U_{m;i} \mathbf{E}^m \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{E}^n}{\partial q^i} \cdot \mathbf{E}_m &= -\mathbf{E}^n \cdot \frac{\partial \mathbf{E}_m}{\partial q^i} \\ \text{So: } \Lambda_{im}{}^n &= -\Gamma_{im}{}^n \end{aligned}$$

Defining *covariant derivative* $U^m{}_{;i}$ of a *contravariant* component U^m

$$U^m{}_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^m \equiv \frac{\partial \mathbf{U}}{\partial q^i} \cdot \mathbf{E}^m$$

...and *covariant derivative* $U_{m;i}$ of a *covariant* component U_m

$$U_{m;i} = \frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^n \equiv \frac{\partial \mathbf{U}}{\partial q^i} \cdot \mathbf{E}_m$$

Intrinsic derivatives:
(Mathematicians being cute)

Intrinsic derivatives: (Mathematicians being cute)

Defining *intrinsic derivative of contravariant vector components*.

$$\begin{aligned}\frac{\delta V^k}{\delta t} &= \frac{dV^k}{dt} + \Gamma_{mn}{}^k V^m \dot{q}^n = \frac{\partial V^k}{\partial q^n} \dot{q}^n + \Gamma_{mn}{}^k V^m \dot{q}^n \\ &= V^k{}_{;n} \dot{q}^n\end{aligned}$$

$$F^k = \frac{\delta p^k}{\delta t} \quad \text{“Cute” contravariant Newton II}$$

Defining *intrinsic derivative of covariant vector components*.

$$\begin{aligned}\frac{\delta V_k}{\delta t} &= \frac{dV_k}{dt} - \Gamma_{kn}{}^m V_m \dot{q}^n = \frac{\partial V_k}{\partial q^n} \dot{q}^n - \Gamma_{kn}{}^m V_m \dot{q}^n \\ &= V_{k;n} \dot{q}^n\end{aligned}$$

$$F_k = \frac{\delta p_k}{\delta t} \quad \text{“Cute” covariant Newton II}$$

Tensor chain rules replace Cartesian chain rules.

$$\frac{\delta V^k}{\delta t} = V^k{}_{;n} \dot{q}^n \quad \text{replaces:} \quad \frac{dV^k}{dt} = \frac{\partial V^k}{\partial q^n} \dot{q}^n$$

$$\text{where: } V^k{}_{;n} = \frac{\partial V^k}{\partial q^n} + \Gamma_{mn}{}^k V^m$$

contravariant

versus

$$\frac{\delta V_k}{\delta t} = V_{k;n} \dot{q}^n \quad \text{replaces:} \quad \frac{dV_k}{dt} = \frac{\partial V_k}{\partial q^n} \dot{q}^n$$

$$\text{where: } V_{k;n} = \frac{\partial V_k}{\partial q^n} - \Gamma_{kn}{}^m V_m$$

covariant

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

 *Christoffel g-derivative formula*
What's a tensor? What's not?

Christoffel ($g_{mn}=\mathbf{E}_m \cdot \mathbf{E}_n$)-derivative formula

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

Christoffel ($g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$)-derivative formula

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\begin{aligned} \frac{\partial g_{mn}}{\partial q^i} &= \Gamma_{im;n} + \Gamma_{in;m} \\ \frac{\partial g_{mi}}{\partial q^n} &= \Gamma_{nm;i} + \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n) \\ \frac{\partial g_{in}}{\partial q^m} &= \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m) \end{aligned}$$

*Christoffel symmetry $\Gamma_{in;\alpha} = \Gamma_{ni;\alpha}$
due to:*

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E}_i}{\partial q^n}$$

Christoffel ($g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$)-derivative formula

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\begin{aligned} \frac{\partial g_{mn}}{\partial q^i} &= \Gamma_{im;n} + \Gamma_{in;m} \\ \frac{\partial g_{mi}}{\partial q^n} &= -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n) \\ \frac{\partial g_{in}}{\partial q^m} &= \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m) \end{aligned}$$

*Christoffel symmetry $\Gamma_{in;\alpha} = \Gamma_{ni;\alpha}$
due to:*

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E}_i}{\partial q^n}$$

Christoffel ($g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$)-derivative formula

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$- \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

*Christoffel symmetry $\Gamma_{in;\alpha} = \Gamma_{ni;\alpha}$
due to:*

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E}_i}{\partial q^n}$$

Christoffel ($g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$)-derivative formula

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

Christoffel symmetry $\Gamma_{in;\alpha} = \Gamma_{ni;\alpha}$
due to:

$$\frac{\partial \mathbf{E}_n}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E}_i}{\partial q^n}$$

Gives the Christoffel formula

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij}{}^k$

Christoffel g-derivative formula

 *What's a tensor? What's not?*

What's a tensor? What's not? $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$ is example of 2nd rank covariant tensor

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$- \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

... Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

What's a tensor? What's not? $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}} = \bar{\mathbf{E}}_{\bar{m}} \cdot \bar{\mathbf{E}}_{\bar{n}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$- \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

... Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

What's a tensor? What's not? $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}} = \bar{\mathbf{E}}_{\bar{m}} \cdot \bar{\mathbf{E}}_{\bar{n}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$- \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

... Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

What's a tensor? What's not? $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\begin{aligned} \bar{g}_{\bar{m}\bar{n}} &= \bar{\mathbf{E}}_{\bar{m}} \cdot \bar{\mathbf{E}}_{\bar{n}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \\ &= \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} g_{mn} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \end{aligned}$$

$$\begin{aligned} \frac{\partial g_{mn}}{\partial q^i} &= \Gamma_{im;n} + \Gamma_{in;m} \\ \frac{\partial g_{mi}}{\partial q^n} &= -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n) \\ \frac{\partial g_{in}}{\partial q^m} &= \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m) \end{aligned}$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

$$\bar{g}_{\bar{m}\bar{n}} = g_{mn} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

...are NOT tensorial

What's a tensor? What's not? $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}} = \bar{\mathbf{E}}_{\bar{m}} \cdot \bar{\mathbf{E}}_{\bar{n}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} g_{mn} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

...are NOT tensorial

$$\bar{g}_{\bar{m}\bar{n}} = g_{mn} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

C-saw-sum gets "bar-frame" view $\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}}$ from covar-derivative $U^m_{;n} = \frac{\partial \mathbf{U}}{\partial q^n} \cdot \mathbf{E}^m$ of **contra**var- U^m :

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m_{;n}$$

How to derive this?

From p.13-14:

$$\frac{\partial \mathbf{U}}{\partial q^n} = \left(\frac{\partial U^m}{\partial q^n} + U^k \Gamma_{kn}^m \right) \mathbf{E}_m = U^m_{;n} \mathbf{E}_m$$

$U^m_{;n} \mathbf{E}_m$

What's a tensor? What's not? $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}} = \bar{\mathbf{E}}_{\bar{m}} \cdot \bar{\mathbf{E}}_{\bar{n}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} g_{mn} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

...are NOT tensorial

$$\bar{g}_{\bar{m}\bar{n}} = g_{mn} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

C-saw-sum gets "bar-frame" view $\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}}$ from covar-derivative $U^m_{;n} = \frac{\partial \mathbf{U}}{\partial q^n} \cdot \mathbf{E}^m$ of **contra**var- U^m :

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial \mathbf{U}}{\partial \bar{q}^{\bar{n}}} \cdot \frac{\partial \mathbf{U}}{\partial q^n} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \mathbf{U}}{\partial q^n} \cdot \bar{\mathbf{E}}^{\bar{m}}$$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m_{;n}$$

How to derive this?

From p.13-14:

$$\frac{\partial \mathbf{U}}{\partial q^n} = \left(\frac{\partial U^m}{\partial q^n} + U^k \Gamma_{kn}^m \right) \mathbf{E}_m = U^m_{;n} \mathbf{E}_m$$

What's a tensor? What's not? $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}} = \bar{\mathbf{E}}_{\bar{m}} \cdot \bar{\mathbf{E}}_{\bar{n}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} g_{mn} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

...are NOT tensorial

$$\bar{g}_{\bar{m}\bar{n}} = g_{mn} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

C-saw-sum gets "bar-frame" view $\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial U}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}}$ from covar-derivative $U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot \mathbf{E}^m$ of **contra**var- U^m :

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial U}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial U}{\partial \bar{q}^{\bar{n}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U}{\partial q^n} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U}{\partial q^n} \cdot \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \mathbf{E}^m = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial U}{\partial q^n} \cdot \mathbf{E}^m$$

contravariant \mathbf{E} transform

$$\bar{\mathbf{E}}^{\bar{m}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \mathbf{E}^m$$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m_{;n}$$

How to derive this?
...by C-saw sums...

From p.13-14:

$$\frac{\partial U}{\partial q^n} = \left(\frac{\partial U^m}{\partial q^n} + U^k \Gamma_{kn}^m \right) \mathbf{E}_m = U^m_{;n} \mathbf{E}_m$$

What's a tensor? What's not? $g_{mn} = \mathbf{E}_m \cdot \mathbf{E}_n$ is example of 2nd rank covariant tensor whose transformation to "bar-frame" is *cov-cov* product found by chain-saw-sum

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\bar{g}_{\bar{m}\bar{n}} = \bar{\mathbf{E}}_{\bar{m}} \cdot \bar{\mathbf{E}}_{\bar{n}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial \mathbf{r}}{\partial q^m} \cdot \frac{\partial \mathbf{r}}{\partial q^n} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} g_{mn} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m} \quad (\text{switched } i \leftrightarrow n)$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i} \quad (\text{switched } i \leftrightarrow m)$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right)$$

...are NOT tensorial

$$\bar{g}_{\bar{m}\bar{n}} = g_{mn} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}}$$

C-saw-sum gets "bar-frame" view $\bar{U}_{\bar{m};\bar{n}} = \frac{\partial U}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}_{\bar{m}}$ from covar-derivative $U_{m;n} = \frac{\partial U}{\partial q^n} \cdot \mathbf{E}_m$ of covar- U_m :

$$\bar{U}_{\bar{m};\bar{n}} = \frac{\partial U}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}_{\bar{m}} = \frac{\partial U}{\partial \bar{q}^{\bar{n}}} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U}{\partial q^n} \cdot \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U}{\partial q^n} \cdot \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \mathbf{E}_m = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial U}{\partial q^n} \cdot \mathbf{E}_m$$

covariant \mathbf{E} transform

$$\bar{\mathbf{E}}_{\bar{m}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} = \frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \mathbf{E}_m$$

$$\bar{U}_{\bar{m};\bar{n}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U_{m;n}$$

How to derive this?
...by C-saw sums...

From p.13-14:

$$\frac{\partial U}{\partial q^n} = \left(\frac{\partial U^m}{\partial q^n} + U^k \Gamma_{kn}^m \right) \mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^n} - U_k \Gamma_{mn}^k \right) \mathbf{E}^m = U^m_{;n} \mathbf{E}_m = U_{m;n} \mathbf{E}^m$$

What's a tensor? What's not?

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right) \dots \text{are NOT tensorial}$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}}$ of covariant derivative $U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot \mathbf{E}^m$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial U}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial U}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U}{\partial q^n} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U}{\partial q^n} \cdot \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \mathbf{E}^m = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial U}{\partial q^n} \cdot \mathbf{E}^m$$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m_{;n}$$

The transformation of $U^m_{;n} = \frac{\partial U^m}{\partial q^n} + U^l \Gamma_{nl}^m$ is that of mixed 2nd-rank tensor T^m_n

$$T^{\bar{m}}_{\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} T^m_n$$

What's a tensor? What's not?

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right) \dots \text{are NOT tensorial}$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}}$ of covariant derivative $U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot \mathbf{E}^m$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}}{\partial q^n} \cdot \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \mathbf{E}_m$$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m_{;n}$$

The transformation of $U^m_{;n} = \frac{\partial U^m}{\partial q^n} + U^l \Gamma_{nl}{}^m$ is that of mixed 2nd-rank tensor T^m_n

$$T^{\bar{m}}_{\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} T^m_n$$

The transformation of $U^m_{,n} = \frac{\partial U^m}{\partial q^n}$ is NOT so simple. At first it looks possible. $\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}^{\bar{m}}}{\partial q^n}$

But, still need to write $\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}}$ in terms of $\frac{\partial U^m}{\partial q^n}$.
 standard contra-tran: $\bar{U}^{\bar{m}}$

$$\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial}{\partial q^n} \left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} U^m \right) = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U^m}{\partial q^n} + U^m \frac{\partial}{\partial q^n} \left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \right)$$

What's a tensor? What's not?

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right) \dots \text{are NOT tensorial}$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}}$ of covariant derivative $U^m_{;n} = \frac{\partial U}{\partial q^n} \cdot \mathbf{E}^m$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}}{\partial q^n} \cdot \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \mathbf{E}_m$$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m_{;n}$$

The transformation of $U^m_{;n} = \frac{\partial U^m}{\partial q^n} + U^l \Gamma_{nl}{}^m$ is that of mixed 2nd-rank tensor T^m_n

$$T^{\bar{m}}_{\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} T^m_n$$

The transformation of $U^m_{,n} = \frac{\partial U^m}{\partial q^n}$ is NOT so simple. At first it looks possible. $\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}^{\bar{m}}}{\partial q^n}$
standard contra-tran: $\bar{U}^{\bar{m}}$

But, still need to write $\frac{\partial \bar{U}^{\bar{m}}}{\partial q^n}$ in terms of $\frac{\partial U^m}{\partial q^n}$. $\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial}{\partial q^n} \left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} U^m \right) = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U^m}{\partial q^n} + U^m \frac{\partial}{\partial q^n} \left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \right)$

$$\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U^m}{\partial q^n} \text{ holds if and only if } \frac{\partial}{\partial q^n} \left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \right) \equiv 0$$

1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

What's a tensor? What's not?

From p.13-14:

$$\frac{\partial \mathbf{U}}{\partial q^n} = \left(\underbrace{\frac{\partial U^m}{\partial q^n} + U^k \Gamma_{kn; m}}_{U^m_{;n}} \right) \mathbf{E}_m$$

$$\frac{\partial(\mathbf{E}_m \cdot \mathbf{E}_n)}{\partial q^i} = \frac{\partial \mathbf{E}_m}{\partial q^i} \cdot \mathbf{E}_n + \mathbf{E}_m \cdot \frac{\partial \mathbf{E}_n}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \Gamma_{in;m}$$

$$- \frac{\partial g_{mi}}{\partial q^n} = -\Gamma_{nm;i} - \Gamma_{in;m}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \Gamma_{mn;i}$$

...but Christoffel coefficients...

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n} \right) \dots \text{are NOT tensorial}$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}}$ of covariant derivative $U^m_{;n} = \frac{\partial \mathbf{U}}{\partial q^n} \cdot \mathbf{E}^m$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial \bar{U}}{\partial \bar{q}^{\bar{n}}} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}}{\partial q^n} \cdot \bar{\mathbf{E}}^{\bar{m}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}}{\partial q^n} \cdot \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}} \mathbf{E}_m$$

$$\bar{U}^{\bar{m}}_{;\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} U^m_{;n}$$

The transformation of $U^m_{;n} = \frac{\partial U^m}{\partial q^n} + U^l \Gamma_{nl; m}$ is that of mixed 2nd-rank tensor T^m_n

$$T^{\bar{m}}_{\bar{n}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} T^m_n$$

The transformation of $U^m_{,n} = \frac{\partial U^m}{\partial q^n}$ is NOT so simple. At first it looks possible. $\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial \bar{U}^{\bar{m}}}{\partial q^n}$

But, still need to write $\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}}$ in terms of $\frac{\partial U^m}{\partial q^n}$. $\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial}{\partial q^n} \left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} U^m \right) = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U^m}{\partial q^n} + U^m \frac{\partial}{\partial q^n} \left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \right)$

$\frac{\partial \bar{U}^{\bar{m}}}{\partial \bar{q}^{\bar{n}}} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \frac{\partial q^n}{\partial \bar{q}^{\bar{n}}} \frac{\partial U^m}{\partial q^n}$ holds if and only if $\frac{\partial}{\partial q^n} \left(\frac{\partial \bar{q}^{\bar{m}}}{\partial q^m} \right) = 0$ 1st term is OK, but 2nd term is zero only if Jacobian is constant matrix!

Otherwise, $U^m_{,n}$ needs "Christoffel correction" $U^l \Gamma_{nl; m}$. That $U^l \Gamma_{nl; m}$ cannot be a T_n^m -tensor either!

→ *General Riemann equations of motion (No explicit t -dependence and fixed GCC)*

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Christoffel relation to Coriolis coefficients

Mechanics of ideal fluid vortex

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

γ_{mn} replaces g_{mn}
when mass matrix
 M_{ij} is involved

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit- t -dependent terms are zero

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

γ_{mn} replaces g_{mn}
when mass matrix
 M_{ij} is involved

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

*All explicit- t -dependent terms are zero
(Time must be included as a dimension)*

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

*All explicit- t -dependent terms are zero
(Time must be included as a dimension)*

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

1st term involves *covariant momentum* p_ℓ .

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit- t -dependent terms are zero
(Time must be included as a dimension)

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit- t -dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

Inverse *contravariant kinetic metric* γ^{mn} gives velocity \dot{q}^n

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit- t -dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

Inverse *contravariant kinetic metric* γ^{mn} gives velocity \dot{q}^n

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time t :

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

Lagrange Force Equation

The “4-wheel-drive garbage truck”

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit- t -dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

Inverse *contravariant kinetic metric* γ^{mn} gives velocity \dot{q}^n

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time t :

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} (\gamma_{\ell n} \dot{q}^n) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Lagrange Force Equation

The "4-wheel-drive garbage truck"

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit- t -dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

Inverse *contravariant kinetic metric* γ^{mn} gives velocity \dot{q}^n

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time t :

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} (\gamma_{\ell n} \dot{q}^n) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Lagrange Force Equation

The "4-wheel-drive garbage truck"

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit- t -dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

Inverse *contravariant* kinetic metric γ^{mn} gives velocity \dot{q}^n

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time t :

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} (\gamma_{\ell n} \dot{q}^n) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Time derivative of kinetic metric is expanded by chain rule.

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n$$

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Riemann equations of motion (No explicit t -dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit- t -dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

Inverse *contravariant kinetic metric* γ^{mn} gives velocity \dot{q}^n

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time t :

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} (\gamma_{\ell n} \dot{q}^n) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

Rearrange to expose

Christoffel coefficients (from p 23):

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

Inverse *contravariant kinetic metric* γ^{mn} gives velocity \dot{q}^n

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time t:

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} (\gamma_{\ell n} \dot{q}^n) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Rearrange to expose

Christoffel coefficients (from p 23):

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

Inverse *contravariant kinetic metric* γ^{mn} gives velocity \dot{q}^n

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time t:

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} (\gamma_{\ell n} \dot{q}^n) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Rearrange to expose

Christoffel coefficients (from p 23):

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

This gives *covariant Riemann equations*

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

Riemann equations of motion (No explicit t-dependence and fixed GCC)

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk} \frac{\partial x^j}{\partial q^m} \frac{\partial x^k}{\partial q^n} \quad \text{Converts Cartesian kinetic energy } T = \frac{1}{2} M_{jk} \dot{x}^j \dot{x}^k \quad \text{to GCC } T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n$$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt} \left[\frac{\partial T}{\partial \dot{q}^\ell} \right] - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial q^\ell}$$

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \left\{ \frac{\partial x^j}{\partial t} \right\} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \left\{ \frac{\partial x^k}{\partial t} \right\} \right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

1st term involves *covariant momentum* p_ℓ .

Inverse *contravariant* kinetic metric γ^{mn} gives velocity \dot{q}^n

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2} \frac{\partial (\gamma_{mn} \dot{q}^m \dot{q}^n)}{\partial \dot{q}^\ell} = \gamma_{\ell n} \dot{q}^n$$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for all GCC, fixed or explicit in time t:

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell}$$

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt} (\gamma_{\ell n} \dot{q}^n) - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m$$

Rearrange to expose
Christoffel coefficients:

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \dot{q}^n \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} \dot{q}^m - \frac{1}{2} \frac{\partial \gamma_{mn}}{\partial q^\ell} \dot{q}^m \dot{q}^n \right]$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \frac{1}{2} \left[\frac{\partial \gamma_{\ell n}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell} \right] \dot{q}^m \dot{q}^n$$

This gives *covariant Riemann equations*

$$F_\ell = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

and *contravariant Riemann equations*.

$$F^k = \ddot{q}^k + \Gamma_{mn}^k \dot{q}^m \dot{q}^n$$

General Riemann equations of motion (No explicit t -dependence and fixed GCC)

 *Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

Christoffel relation to Coriolis coefficients

Mechanics of ideal fluid vortex

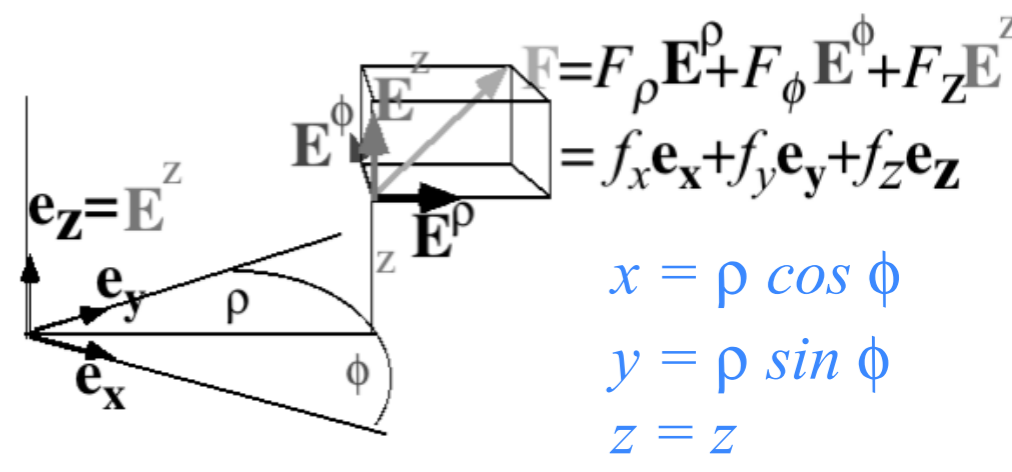
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}$$

$\leftarrow \mathbf{E}^\rho$
 $\leftarrow \mathbf{E}^\phi$
 $\leftarrow \mathbf{E}^z$

\uparrow
 \mathbf{E}_ρ \uparrow
 \mathbf{E}_ϕ \uparrow
 \mathbf{E}_z

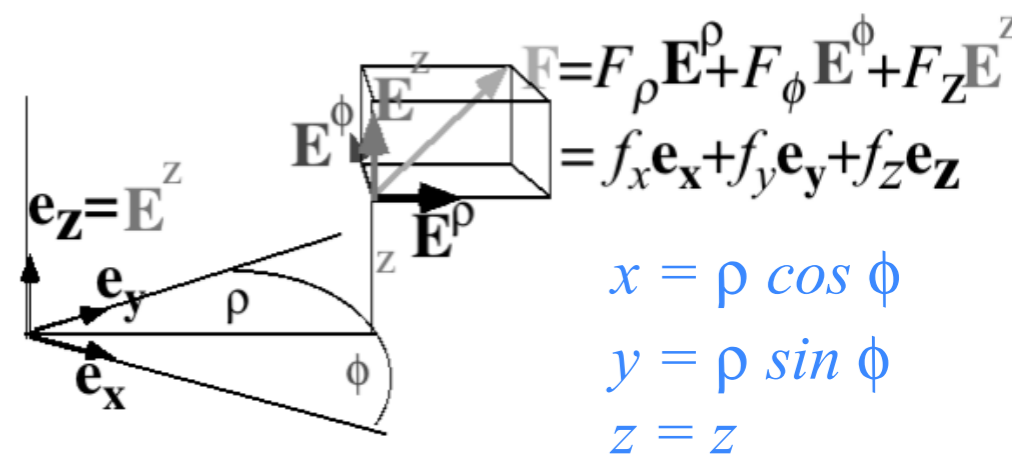
$= \langle J^{-1} \rangle$



Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \quad = \langle J^{-1} \rangle$



Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$= \langle J^{-1} \rangle$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix}$

$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z$

$x = \rho \cos \phi$
 $y = \rho \sin \phi$
 $z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$= \langle J^{-1} \rangle$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix}$

$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z$

$x = \rho \cos \phi$
 $y = \rho \sin \phi$
 $z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/(m\rho^2)$$

$$\gamma^{zz} = 1/m$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$= \langle J^{-1} \rangle$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix}$

$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z$

$x = \rho \cos \phi$
 $y = \rho \sin \phi$
 $z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/(m\rho^2)$$

$$\gamma^{zz} = 1/m$$

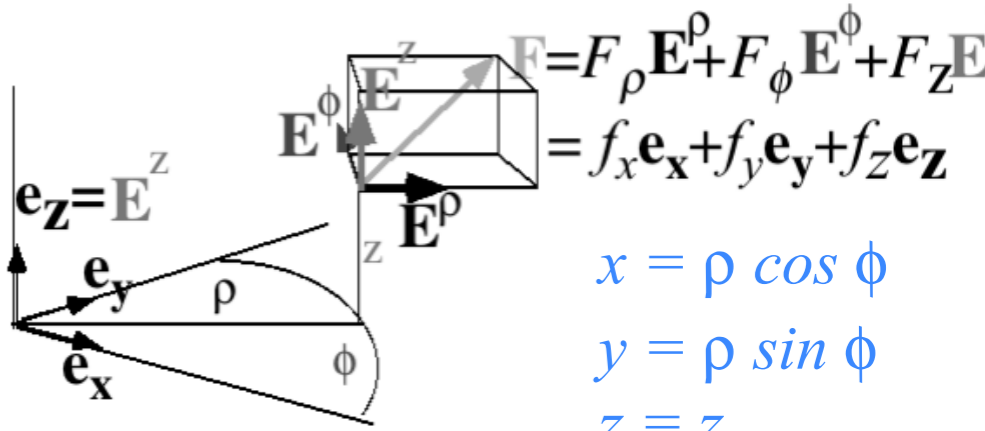
Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$\mathbf{E}_\rho \quad \mathbf{E}_\phi \quad \mathbf{E}_z \quad = \langle J^{-1} \rangle$



$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z = f_x \mathbf{e}_x + f_y \mathbf{e}_y + f_z \mathbf{e}_z$

$x = \rho \cos \phi$
 $y = \rho \sin \phi$
 $z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/(m\rho^2)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} = m \dot{\rho}$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} = m \rho^2 \dot{\phi}$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} = m \dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

General Riemann equations of motion (No explicit t -dependence and fixed GCC)

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

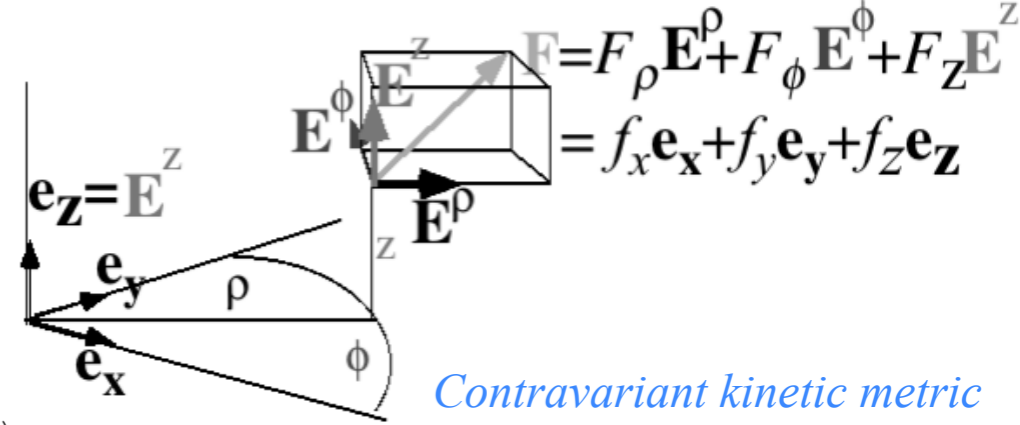
 *Christoffel relation to Coriolis coefficients*
Mechanics of ideal fluid vortex

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} = \langle J^{-1} \rangle$

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned}$$



Covariant forces

$$\begin{aligned} F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\ F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\ F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z \end{aligned}$$

Covariant kinetic metric

$$\begin{aligned} \gamma_{\rho\rho} &= m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m \\ \gamma_{\phi\phi} &= m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2 \\ \gamma_{zz} &= m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m \end{aligned}$$

Contravariant kinetic metric

$$\begin{aligned} \gamma^{\rho\rho} &= 1/m \\ \gamma^{\phi\phi} &= 1/(m\rho^2) \\ \gamma^{zz} &= 1/m \end{aligned}$$

Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Covariant momenta

$$\begin{aligned} p_\rho &= \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} = m \dot{\rho} \\ p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} = m \rho^2 \dot{\phi} \\ p_z &= \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} = m \dot{z} \end{aligned}$$

Contravariant momenta

$$\begin{aligned} p^\rho &= \dot{\rho} \\ p^\phi &= \dot{\phi} \\ p^z &= \dot{z} \end{aligned}$$

Comparing Lagrange and the Riemann covariant force equations

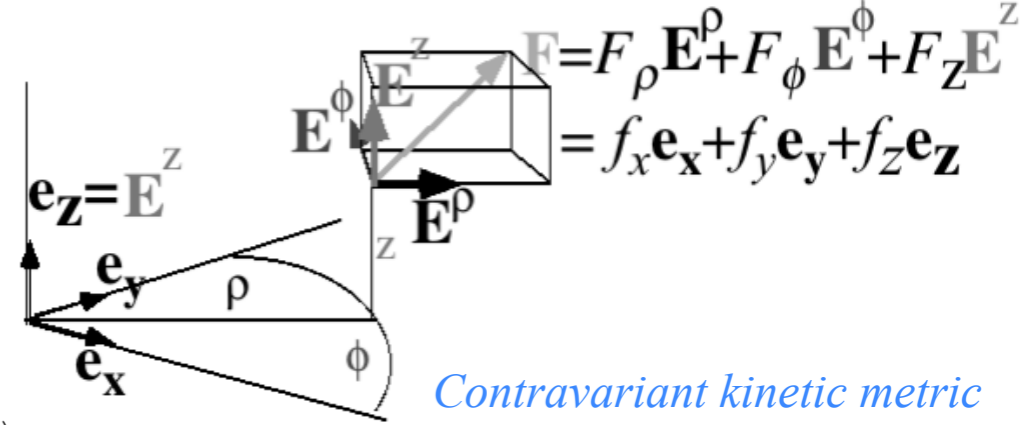
$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} = \langle J^{-1} \rangle$

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned}$$



Covariant forces

$$\begin{aligned} F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\ F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\ F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z \end{aligned}$$

Covariant kinetic metric

$$\begin{aligned} \gamma_{\rho\rho} &= m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m \\ \gamma_{\phi\phi} &= m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2 \\ \gamma_{zz} &= m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m \end{aligned}$$

Contravariant kinetic metric

$$\begin{aligned} \gamma^{\rho\rho} &= 1/m \\ \gamma^{\phi\phi} &= 1/(m\rho^2) \\ \gamma^{zz} &= 1/m \end{aligned}$$

Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Covariant momenta

$$\begin{aligned} p_\rho &= \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} = m \dot{\rho} \\ p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} = m \rho^2 \dot{\phi} \\ p_z &= \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} = m \dot{z} \end{aligned}$$

Contravariant momenta

$$\begin{aligned} p^\rho &= \dot{\rho} \\ p^\phi &= \dot{\phi} \\ p^z &= \dot{z} \end{aligned}$$

Comparing Lagrange and the Riemann covariant force equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

Note: This is a much more efficient way to derive Γ -coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent. Christoffel g-formula (from p. 23 to p. 36):

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho} \ddot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n$$

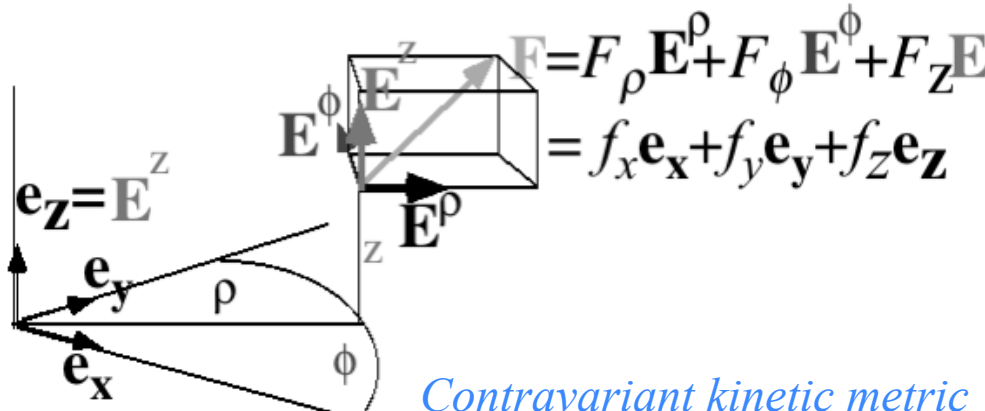
$$F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi} \ddot{\phi} + \Gamma_{mn;\phi} \dot{q}^m \dot{q}^n$$

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$\begin{matrix} x = \rho \cos \phi \\ y = \rho \sin \phi \\ z = z \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} = \langle J^{-1} \rangle$$


Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/(m\rho^2)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} = m \dot{\rho}$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} = m \rho^2 \dot{\phi}$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} = m \dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann covariant force equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

Note: This is a much more efficient way to derive Γ -coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent.

Christoffel g-formula (from p. 23 to p. 36):

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho} \ddot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n$$

$$F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi} \ddot{\phi} + \Gamma_{mn;\phi} \dot{q}^m \dot{q}^n$$

$$= \frac{d(m\dot{\rho})}{dt} - \frac{\partial}{\partial \rho} \left(\frac{1}{2} m \rho^2 \dot{\phi}^2 \right) = m \ddot{\rho} - m \rho \dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho$$

$$\Gamma_{im;n} = \frac{1}{2} \left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n} \right)$$

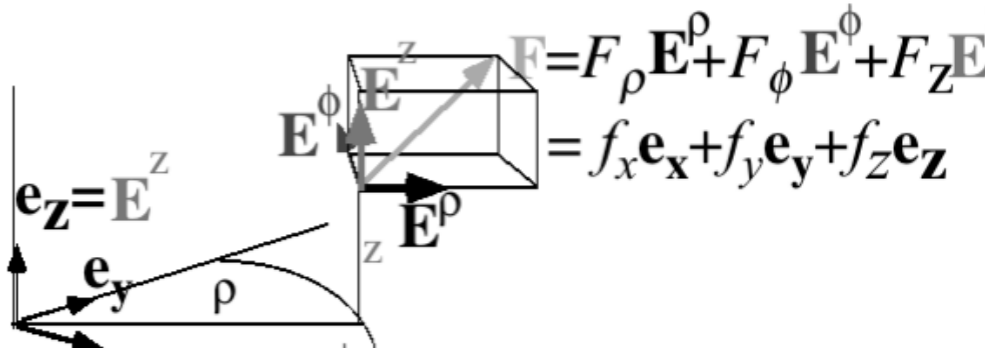
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho, q^2 = \phi, q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} = \langle J^{-1} \rangle$$


Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/(m\rho^2)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} = m \dot{\rho}$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} = m \rho^2 \dot{\phi}$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} = m \dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann covariant force equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

Note: This is a much more efficient way to derive Γ -coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent.

Note: $\Gamma_{pq;r} = \Gamma_{qp;r}$ symmetry gives 2 factor for $q \neq p$

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho} \ddot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n$$

$$= \frac{d(m\dot{\rho})}{dt} - \frac{\partial}{\partial \rho} \left(\frac{1}{2} m \rho^2 \dot{\phi}^2 \right) = m \ddot{\rho} - m \rho \dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho$$

$$F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi} \ddot{\phi} + \Gamma_{mn;\phi} \dot{q}^m \dot{q}^n$$

$$= \frac{d(m\rho^2 \dot{\phi})}{dt} - 0 = m\rho^2 \ddot{\phi} + 2m\rho \dot{\rho} \dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi}$$

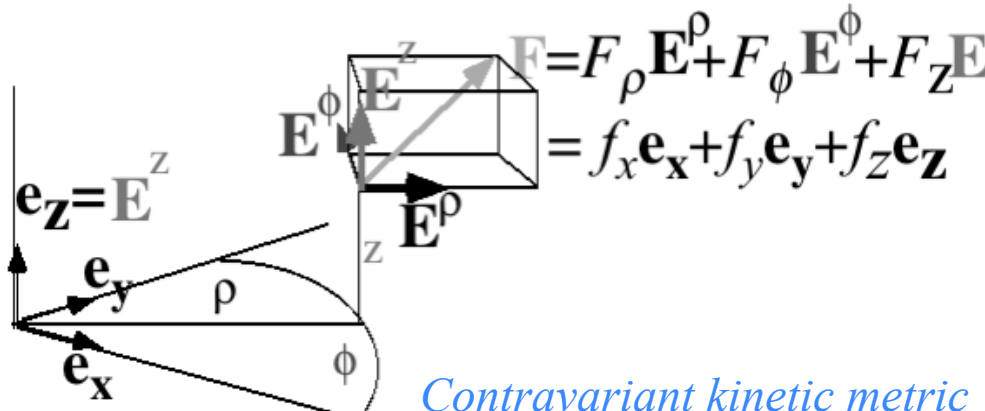
Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho, q^2 = \phi, q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix}$$

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} = \langle J^{-1} \rangle$$


Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/(m\rho^2)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} = m \dot{\rho}$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} = m \rho^2 \dot{\phi}$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} = m \dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann covariant force equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

Note: This is a much more efficient way to derive Γ -coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent.

Note: $\Gamma_{pq;r} = \Gamma_{qp;r}$ symmetry gives 2 factor for $q \neq p$

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho} \ddot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n$$

$$= \frac{d(m\dot{\rho})}{dt} - \frac{\partial}{\partial \rho} \left(\frac{1}{2} m \rho^2 \dot{\phi}^2 \right) = m \ddot{\rho} - m \rho \dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho$$

$$F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi} \ddot{\phi} + \Gamma_{mn;\phi} \dot{q}^m \dot{q}^n$$

$$= \frac{d(m\rho^2 \dot{\phi})}{dt} - 0 = m\rho^2 \ddot{\phi} + 2m\rho \dot{\rho} \dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi}$$

Contravariant equations are acceleration equations. $F^k = \gamma^{jk} F_j = \ddot{q}^k + \Gamma_{mn}^k \dot{q}^m \dot{q}^n$

$$F^\rho = \gamma^{\rho\rho} F_\rho = \ddot{q}^\rho + \Gamma_{mn}^\rho \dot{q}^m \dot{q}^n$$

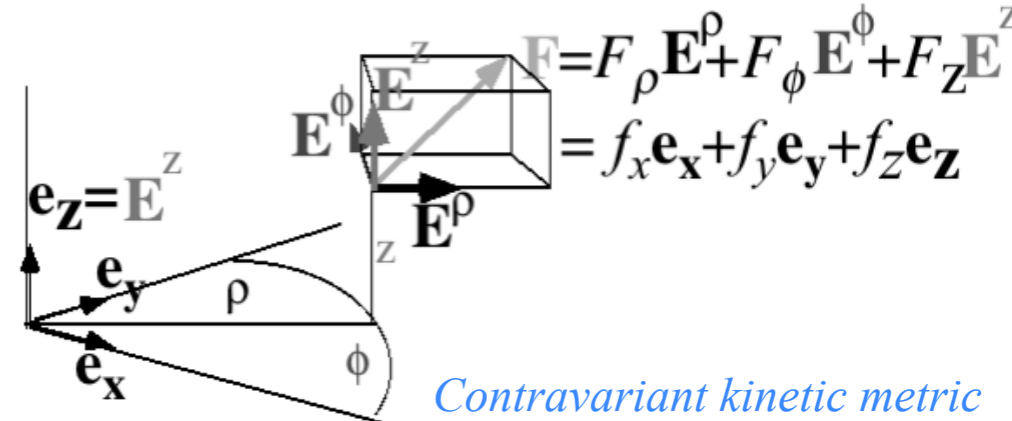
$$F^\phi = \gamma^{\phi\phi} F_\phi = \ddot{q}^\phi + \Gamma_{mn}^\phi \dot{q}^m \dot{q}^n$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho, q^2 = \phi, q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} = \langle J^{-1} \rangle$

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned}$$



Covariant forces

$$\begin{aligned} F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\ F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\ F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z \end{aligned}$$

Covariant kinetic metric

$$\begin{aligned} \gamma_{\rho\rho} &= m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m \\ \gamma_{\phi\phi} &= m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2 \\ \gamma_{zz} &= m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m \end{aligned}$$

Contravariant kinetic metric

$$\begin{aligned} \gamma^{\rho\rho} &= 1/m \\ \gamma^{\phi\phi} &= 1/(m\rho^2) \\ \gamma^{zz} &= 1/m \end{aligned}$$

Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Covariant momenta

$$\begin{aligned} p_\rho &= \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} = m \dot{\rho} \\ p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} = m \rho^2 \dot{\phi} \\ p_z &= \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} = m \dot{z} \end{aligned}$$

Contravariant momenta

$$\begin{aligned} p^\rho &= \dot{\rho} \\ p^\phi &= \dot{\phi} \\ p^z &= \dot{z} \end{aligned}$$

Comparing Lagrange and the Riemann covariant force equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

Note: This is a much more efficient way to derive Γ -coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent.

Note: $\Gamma_{pq;r} = \Gamma_{qp;r}$ symmetry gives 2 factor for $q \neq p$

$$\begin{aligned} F_\rho &= \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho} \ddot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n \\ &= \frac{d(m\dot{\rho})}{dt} - \frac{\partial}{\partial \rho} \left(\frac{1}{2} m \rho^2 \dot{\phi}^2 \right) = m \ddot{\rho} - m \rho \dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m \rho \end{aligned}$$

$$\begin{aligned} F_\phi &= \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi} \ddot{\phi} + \Gamma_{mn;\phi} \dot{q}^m \dot{q}^n \\ &= \frac{d(m\rho^2 \dot{\phi})}{dt} - 0 = m \rho^2 \ddot{\phi} + 2m\rho \dot{\rho} \dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi} \end{aligned}$$

Contravariant equations are acceleration equations. $F^k = \gamma^{jk} F_j = \ddot{q}^k + \Gamma_{mn}^k \dot{q}^m \dot{q}^n$

$$\begin{aligned} F^\rho &= \gamma^{\rho\rho} F_\rho = \ddot{\rho} + \Gamma_{mn}^\rho \dot{q}^m \dot{q}^n \\ &= \ddot{\rho} - \rho \dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi}^\rho = -\rho \quad \gamma^{\rho\rho} = 1/m \end{aligned}$$

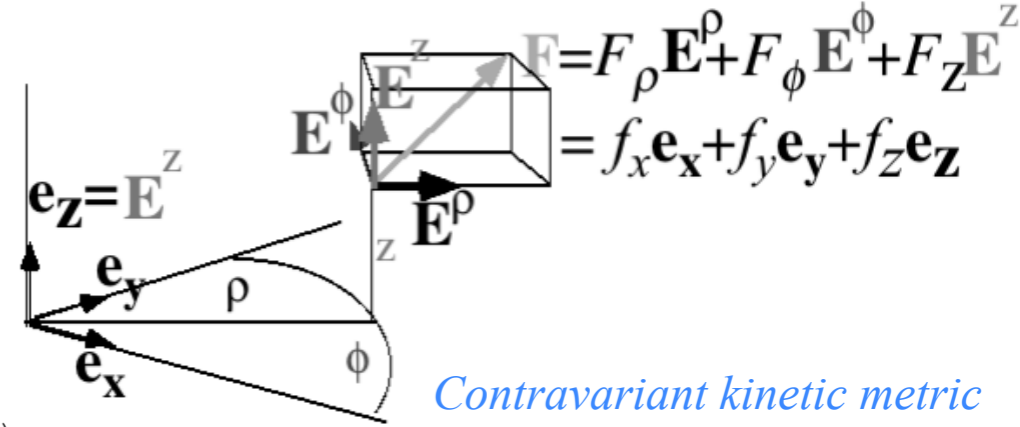
$$\begin{aligned} F^\phi &= \gamma^{\phi\phi} F_\phi = \ddot{\phi} + \Gamma_{mn}^\phi \dot{q}^m \dot{q}^n \\ &= \ddot{\phi} + 2\dot{\rho} \dot{\phi} / \rho \quad \text{so: } \Gamma_{\rho\phi}^\phi = 1/\rho = \Gamma_{\phi\rho}^\phi \quad \gamma^{\phi\phi} = 1/(m\rho^2) \end{aligned}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho, q^2 = \phi, q^3 = z$)

$$\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos \phi & \frac{\partial x}{\partial \phi} = -\rho \sin \phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin \phi & \frac{\partial y}{\partial \phi} = \rho \cos \phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos \phi & \frac{\partial \rho}{\partial y} = \sin \phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin \phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos \phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} = \langle J^{-1} \rangle$

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned}$$



Covariant forces

$$\begin{aligned} F_\rho &= f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = f_x \cos \phi + f_y \sin \phi + 0 \\ F_\phi &= f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin \phi + f_y \rho \cos \phi + 0 \\ F_z &= f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = 0 + 0 + f_z \end{aligned}$$

Covariant kinetic metric

$$\begin{aligned} \gamma_{\rho\rho} &= m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \cdot \mathbf{E}_\rho = m (\cos^2 \phi + \sin^2 \phi) = m \\ \gamma_{\phi\phi} &= m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \cdot \mathbf{E}_\phi = m (\rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi) = m \rho^2 \\ \gamma_{zz} &= m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \cdot \mathbf{E}_z = m \end{aligned}$$

Contravariant kinetic metric

$$\begin{aligned} \gamma^{\rho\rho} &= 1/m \\ \gamma^{\phi\phi} &= 1/(m\rho^2) \\ \gamma^{zz} &= 1/m \end{aligned}$$

Lagrangian

$$T = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$$

Covariant momenta

$$\begin{aligned} p_\rho &= \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho} \dot{\rho} = m \dot{\rho} \\ p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi} \dot{\phi} = m \rho^2 \dot{\phi} \\ p_z &= \frac{\partial T}{\partial \dot{z}} = \gamma_{zz} \dot{z} = m \dot{z} \end{aligned}$$

Contravariant momenta

$$\begin{aligned} p^\rho &= \dot{\rho} \\ p^\phi &= \dot{\phi} \\ p^z &= \dot{z} \end{aligned}$$

Comparing Lagrange and the Riemann covariant force equations

$$F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell} = \gamma_{\ell n} \ddot{q}^n + \Gamma_{mn;\ell} \dot{q}^m \dot{q}^n$$

Note: This is a much more efficient way to derive Γ -coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent.

Note: $\Gamma_{pq;r} = \Gamma_{qp;r}$ symmetry gives 2 factor for $q \neq p$

$$\begin{aligned} F_\rho &= \frac{dp_\rho}{dt} - \frac{\partial T}{\partial \rho} = \gamma_{\rho\rho} \ddot{\rho} + \Gamma_{mn;\rho} \dot{q}^m \dot{q}^n \\ &= \frac{d(m\dot{\rho})}{dt} - \frac{\partial}{\partial \rho} \left(\frac{1}{2} m \rho^2 \dot{\phi}^2 \right) = m\ddot{\rho} - m\rho\dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho \end{aligned}$$

$$\begin{aligned} F_\phi &= \frac{dp_\phi}{dt} - \frac{\partial T}{\partial \phi} = \gamma_{\phi\phi} \ddot{\phi} + \Gamma_{mn;\phi} \dot{q}^m \dot{q}^n \\ &= \frac{d(m\rho^2\dot{\phi})}{dt} - 0 = m\rho^2\ddot{\phi} + 2m\rho\dot{\rho}\dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi} \end{aligned}$$

Contravariant equations are acceleration equations. $F^k = \gamma^{jk} F_j = \ddot{q}^k + \Gamma_{mn}^k \dot{q}^m \dot{q}^n$

$$\begin{aligned} F^\rho &= \gamma^{\rho\rho} F_\rho = \ddot{\rho} + \Gamma_{mn}^\rho \dot{q}^m \dot{q}^n \\ &= \ddot{\rho} - \rho\dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi}^\rho = -\rho \quad \gamma^{\rho\rho} = 1/m \end{aligned}$$

$$\begin{aligned} F^\phi &= \gamma^{\phi\phi} F_\phi = \ddot{\phi} + \Gamma_{mn}^\phi \dot{q}^m \dot{q}^n \\ &= \ddot{\phi} + 2\dot{\rho}\dot{\phi}/\rho \quad \text{so: } \Gamma_{\rho\phi}^\phi = 1/\rho = \Gamma_{\phi\rho}^\phi \quad \gamma^{\phi\phi} = 1/(m\rho^2) \end{aligned}$$

$$\ddot{\rho} = F^\rho + \rho\dot{\phi}^2 \quad (\text{Centrifugal acceleration})$$

$$\ddot{\phi} = F^\phi - 2\dot{\rho}\dot{\phi}/\rho \quad (\text{Coriolis acceleration})$$

Rewriting GCC Lagrange equations :

(Review of Lecture 11)

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M \ddot{r}$$

Centrifugal (center-fleeing) force equals total

$$= M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{\phi} + Mr^2\ddot{\phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

$$= 0 - \frac{\partial U}{\partial \phi}$$

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ -dependence

Conventional forms

radial force: $M \ddot{r} = M r \dot{\phi}^2 - \frac{\partial U}{\partial r}$

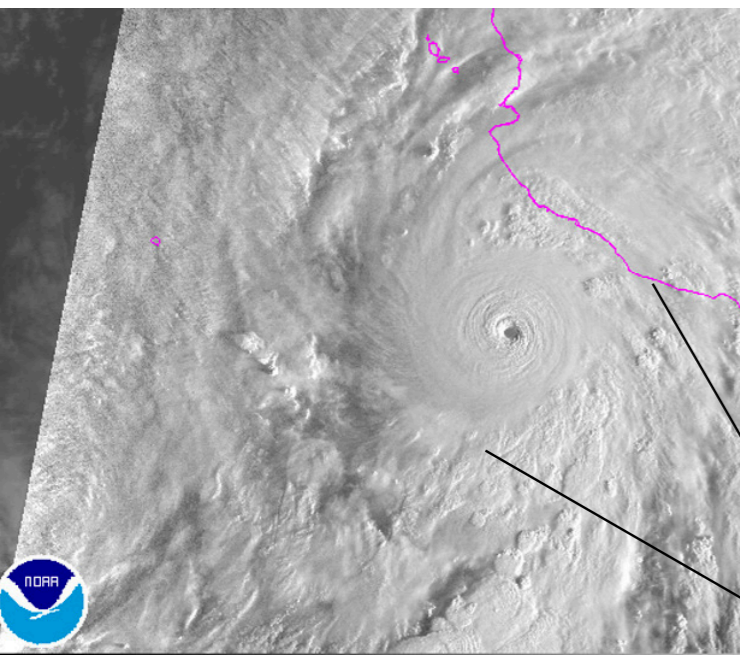
angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{\phi} - \frac{\partial U}{\partial \phi}$

Field-free ($U=0$)

radial acceleration: $\ddot{r} = r \dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2 \frac{\dot{r}\dot{\phi}}{r}$

Because Earth rotation is counter-clockwise (positive) in North



Hurricane Patricia
October 23, 2015

Coriolis acceleration with $\dot{\phi} > 0$ and $\dot{r} < 0$

$$\ddot{\phi} = -2 \dot{r} \dot{\phi} / r$$

(makes $\ddot{\phi}$ positive)

Inward flow to pressure Low
 $\dot{r} < 0$

...makes wind turn to the right

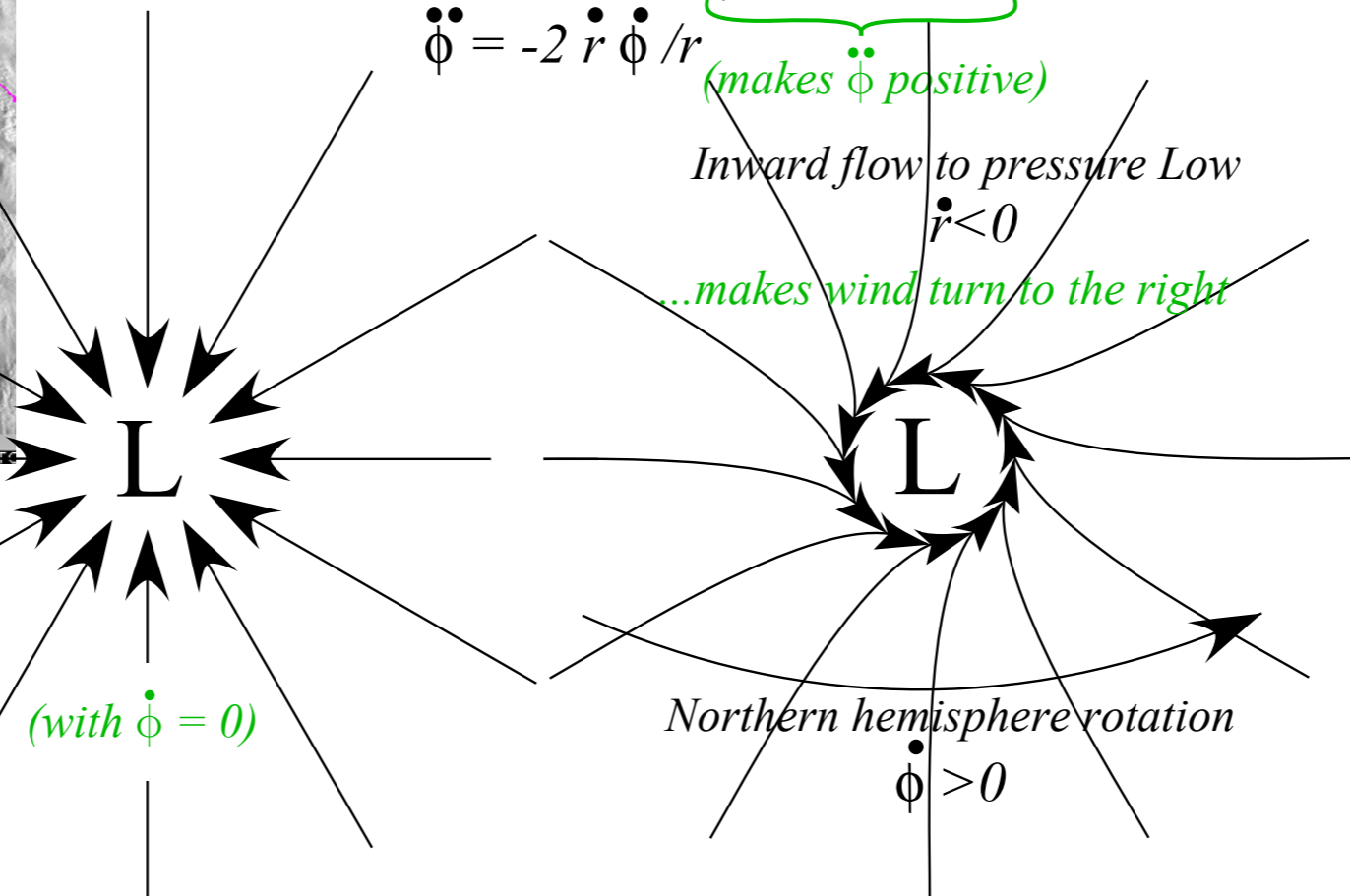
Effect on Northern Hemisphere local weather

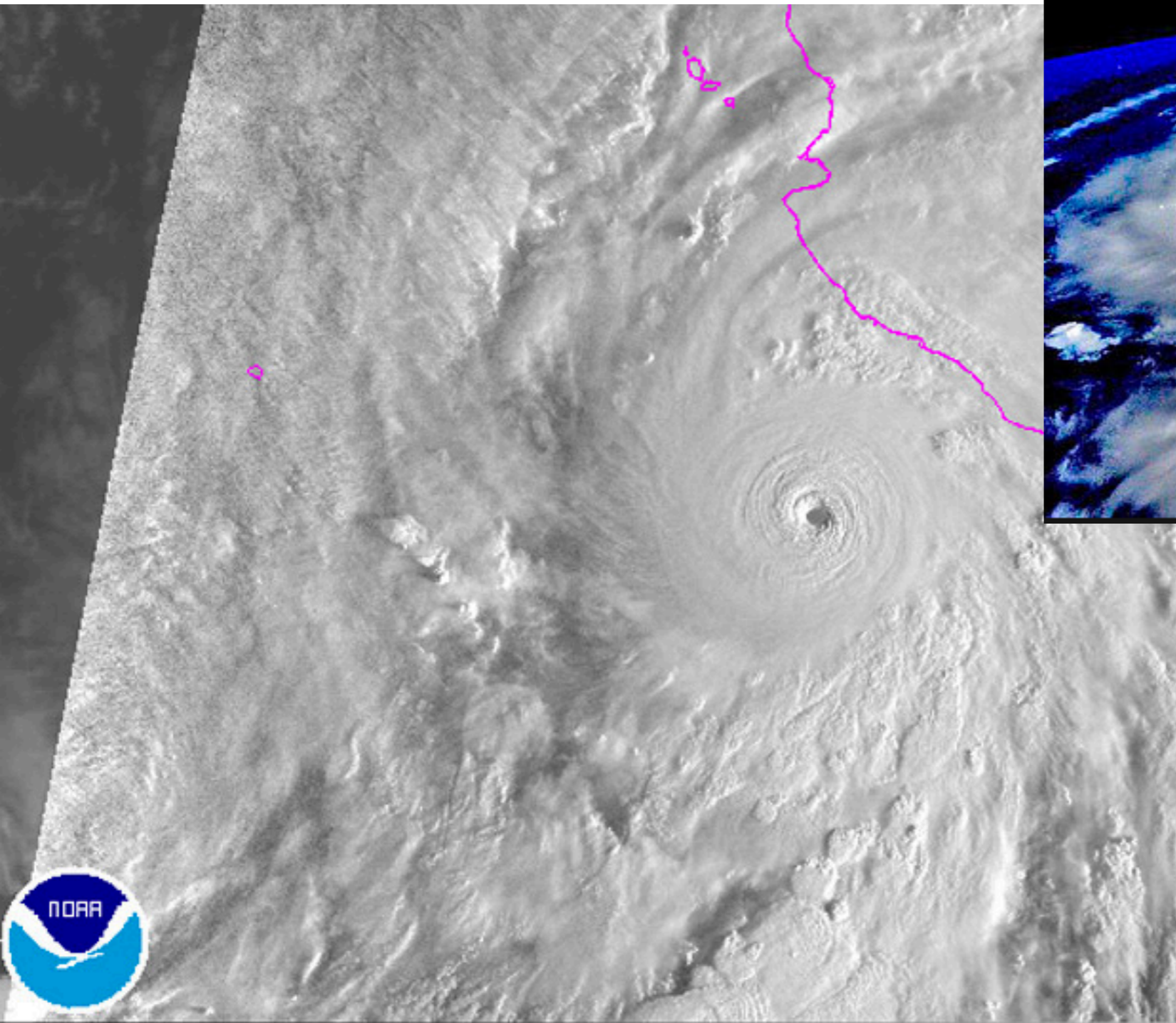
Cyclonic flow around lows

(with $\dot{\phi} = 0$)

Northern hemisphere rotation

$\dot{\phi} > 0$





1 GOES-FLOATER VISIBLE - OCT 23 15 13:30 UTC

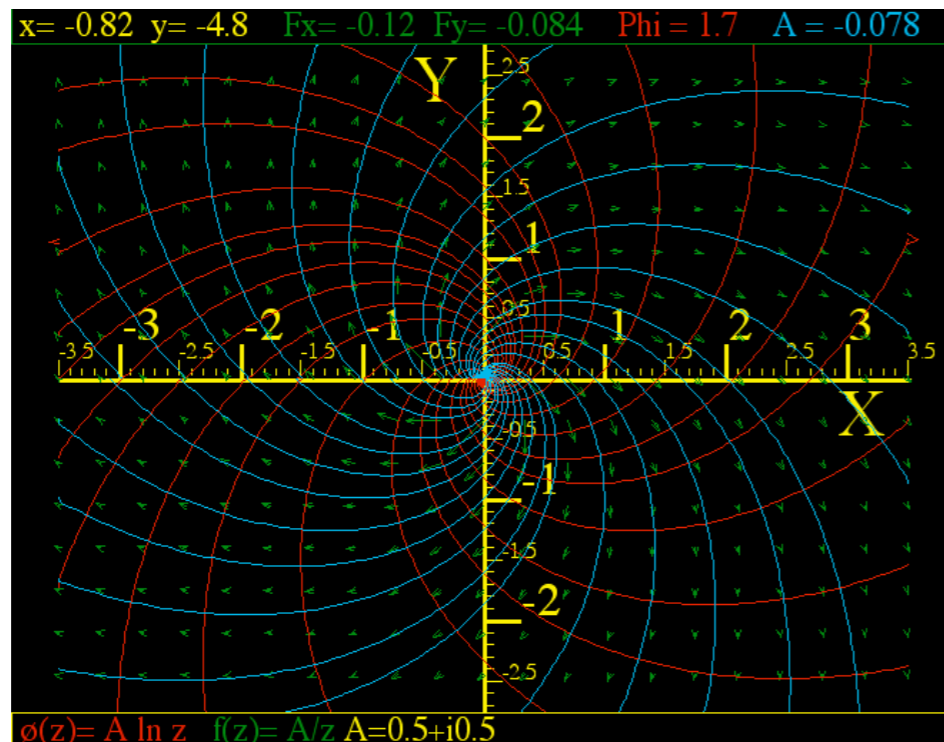
Hurricane Patricia
October 23, 2015

<https://www.google.com/search?q=Satellite+view+of+Patricia&biw=1811&bih=1247&tbm=isch&tbo=u&source=univ&sa=X&ved=0CD0QsARqFQoTCLbI7N728sgCFdA0iAodl4kMg>

Riemann-forms in cylindrical polar OCC ($q1 = \rho$, $q2 = \phi$, $q3 = z$)

Christoffel relation to Coriolis coefficients

→ Mechanics of ideal fluid vortex



Mechanics of ideal fluid vortex

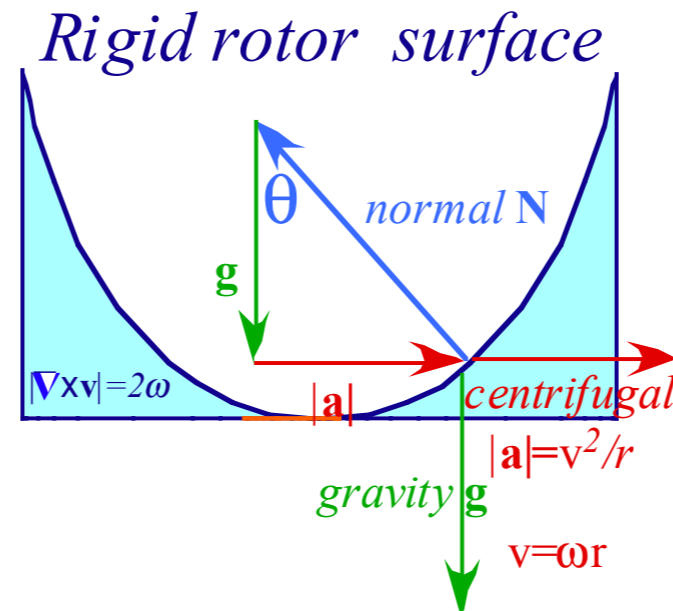
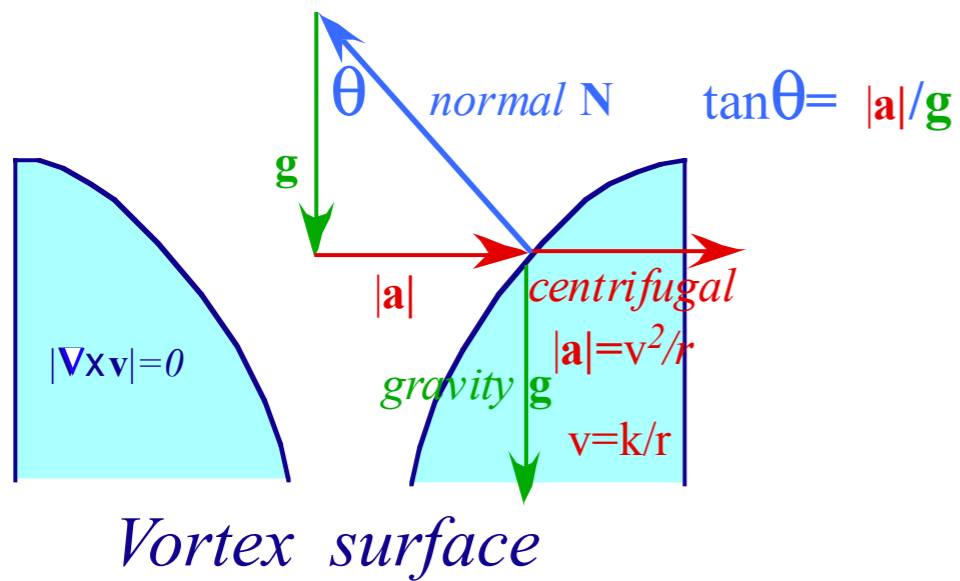
Rotating fluid surface has normal acceleration \mathbf{N} be sum of gravity \mathbf{g} and centrifugal $\mathbf{a} = \mathbf{e}_a v^2/r$

Case 1: Vortex with velocity field

$$v = k/r$$

Case 2: Rigidly rotating fluid with velocity field

$$v = \omega r$$



Mechanics of ideal fluid vortex

Rotating fluid surface has normal acceleration \mathbf{N} be sum of gravity \mathbf{g} and centrifugal $\mathbf{a} = \mathbf{e}_a v^2/r$

Case 1: Vortex with velocity field

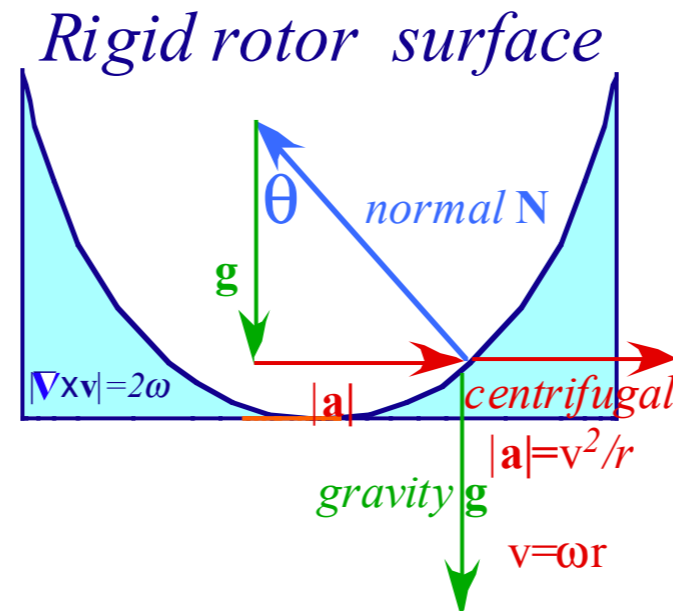
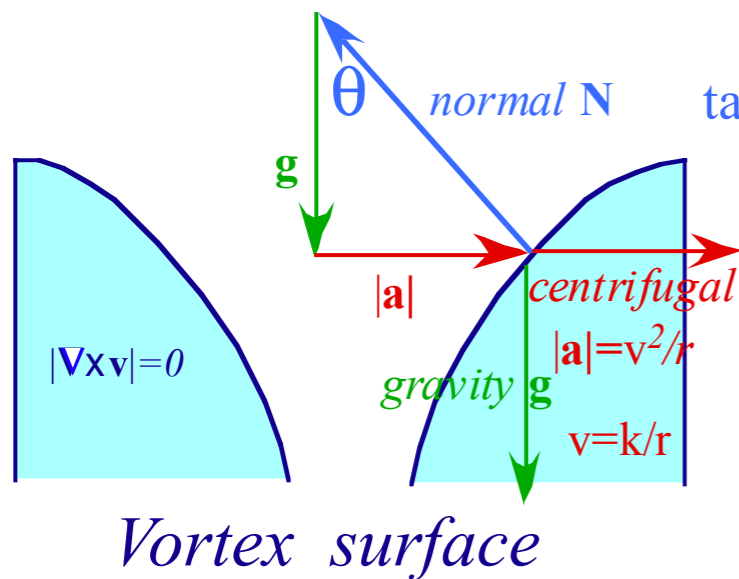
$$v = k/r$$

Case 2: Rigidly rotating fluid with velocity field

$$v = \omega r$$

In either case slope θ of normal \mathbf{N} is:

$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2 / r}{g}$$



Mechanics of ideal fluid vortex

Rotating fluid surface has normal acceleration \mathbf{N} be sum of gravity \mathbf{g} and centrifugal $\mathbf{a} = \mathbf{e}_a v^2/r$

Case 1: Vortex with velocity field

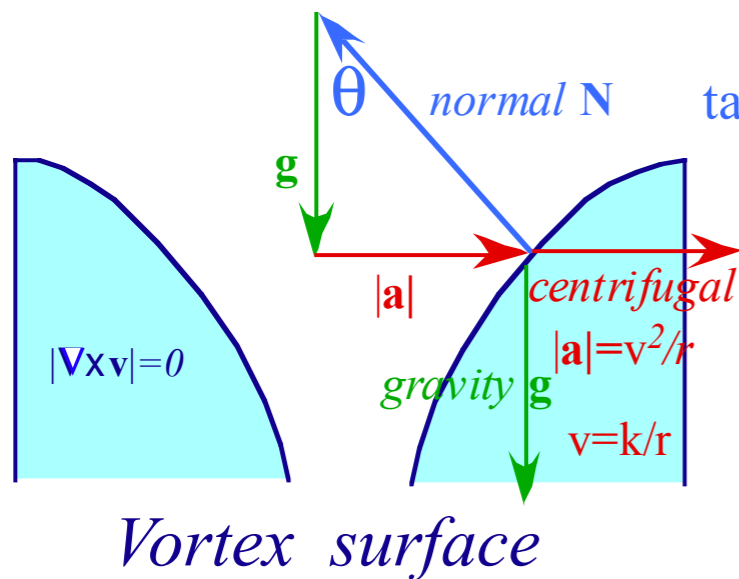
$$v = k/r$$

Case 2: Rigidly rotating fluid with velocity field

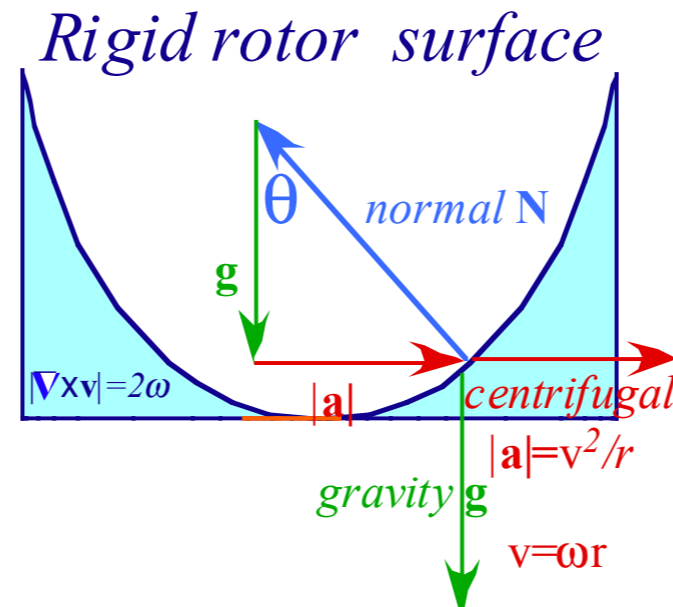
$$v = \omega r$$

In either case slope θ of normal \mathbf{N} is:

$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2 / r}{g}$$



$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2 / r}{g} = \frac{k^2}{gr^3}$$



$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2 / r}{g} = \frac{\omega^2}{g} r^1$$

Mechanics of ideal fluid vortex

Rotating fluid surface has normal acceleration \mathbf{N} be sum of gravity \mathbf{g} and centrifugal $\mathbf{a} = \mathbf{e}_a v^2/r$

Case 1: Vortex with velocity field

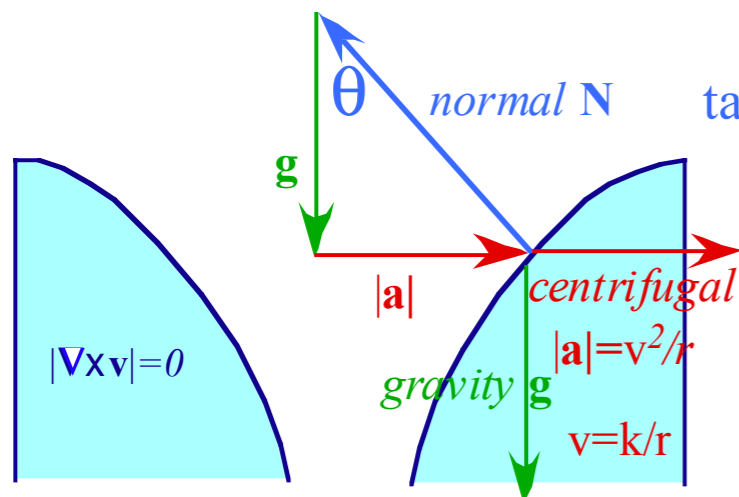
$$v = k/r$$

Case 2: Rigidly rotating fluid with velocity field

$$v = \omega r$$

In either case slope θ of normal \mathbf{N} is:

$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2 / r}{g}$$



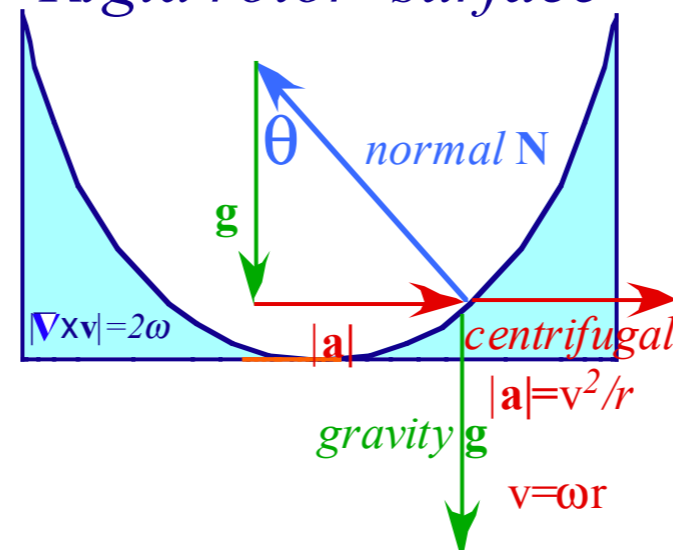
Vortex surface

$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2 / r}{g} = \frac{k^2}{gr^3}$$

Integrating:

$$z(r) = \int \frac{dz}{dr} dr = \int \frac{k^2}{gr^3} dr = -\frac{k^2}{2gr^2}$$

Rigid rotor surface



$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2 / r}{g} = \frac{\omega^2}{g} r^1$$

Integrating:

$$z(r) = \int \frac{dz}{dr} dr = \int \frac{\omega^2}{g} r^1 dr = \frac{\omega^2}{2g} r^2$$

Mechanics of ideal fluid vortex

Rotating fluid surface has normal acceleration \mathbf{N} be sum of gravity \mathbf{g} and centrifugal $\mathbf{a} = \mathbf{e}_a v^2/r$

Case 1: Vortex with velocity field

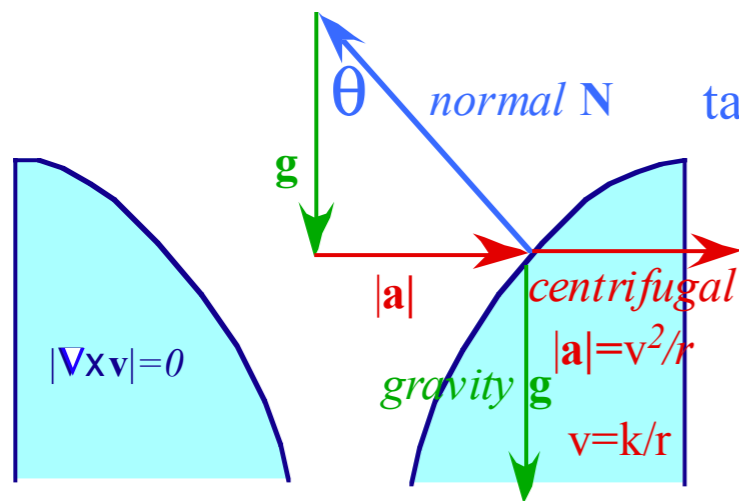
$$\mathbf{v} = k/r$$

Case 2: Rigidly rotating fluid with velocity field

$$\mathbf{v} = \omega r$$

In either case slope θ of normal \mathbf{N} is:

$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2/r}{g}$$



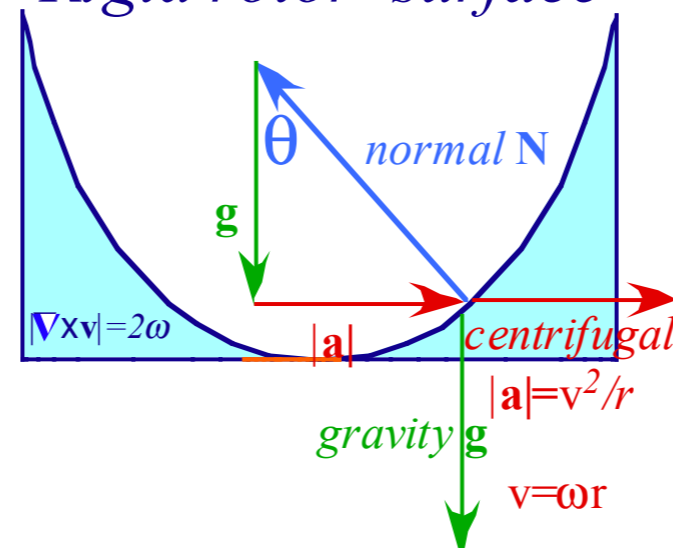
Vortex surface

$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2/r}{g} = \frac{k^2}{gr^3}$$

Integrating:

$$z(r) = \int \frac{dz}{dr} dr = \int \frac{k^2}{gr^3} dr = -\frac{k^2}{2gr^2}$$

Rigid rotor surface

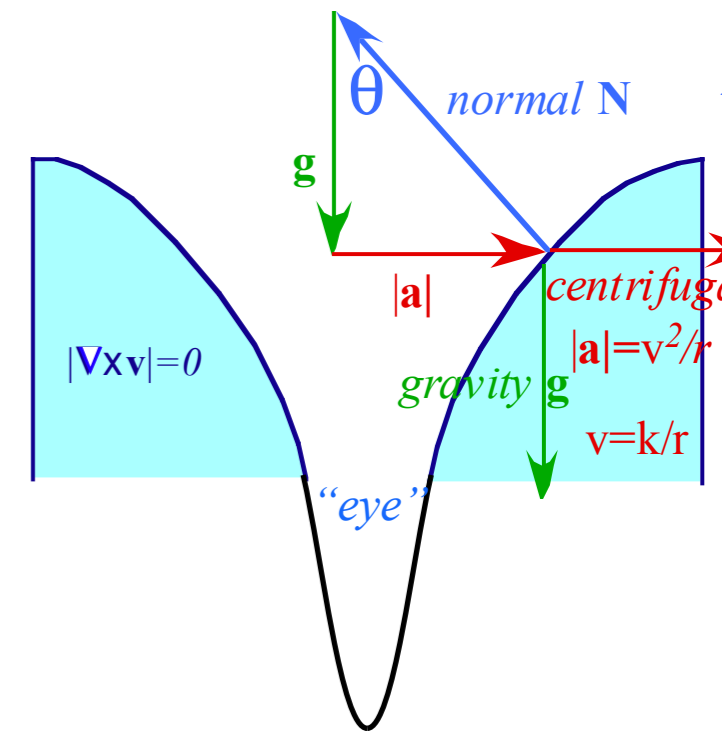


$$\tan \theta = \frac{dz}{dr} = \frac{|\mathbf{a}|}{g} = \frac{v^2/r}{g} = \frac{\omega^2}{g} r^1$$

Integrating:

$$z(r) = \int \frac{dz}{dr} dr = \int \frac{\omega^2}{g} r^1 dr = \frac{\omega^2}{2g} r^2$$

Ideal vortex without drain has a parabolic "eye"



Somewhat analogous to the "Sophomore-Physics Earth"

Pool vortex movie

→ *Separation of GCC Equations: Effective Potentials*

Small $(n_\rho:m_\phi)$ -periodic and quasi-periodic oscillations

2D Spherical pendulum or “Bowl-Bowling” and the “I-Ball”

$(n_\rho:m_\phi)=(2:1)$ vs $(1:1)$ periodic and quasi-periodic orbits

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{Numerically correct ONLY!})$$

$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{Formally and Numerically correct})$$

Separation of GCC Equations: Effective Potentials (For isotropic $H(r, p_r, \phi, p_\phi)$)

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{Numerically correct ONLY!})$$
$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{Formally and Numerically correct})$$

If potential V is *isotropic* (cylindrical) function of radius ρ . ($V = V(\rho)$)

H has no explicit ϕ -dependence and the ϕ -momenta is constant.

$$m\rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

Separation of GCC Equations: Effective Potentials (For isotropic $H(r, p_r, \phi, p_\phi)$)

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{ Numerically correct ONLY! })$$
$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{ Formally and Numerically correct })$$

Potential V is *isotropic* (cylindrical) function of radius ρ . ($V = V(\rho)$)

H has no explicit ϕ -dependence and the ϕ -momenta is constant.

If H has no explicit z -dependence then the z -momenta is constant, too.

$$m\rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

$$m\dot{z} = p_z = \text{const.} = k$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{ Numerically correct ONLY! })$$
$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{ Formally and Numerically correct })$$

Potential V is *isotropic* (cylindrical) function of radius ρ . ($V = V(\rho)$)
 H has no explicit ϕ -dependence and the ϕ -momenta is constant.

If H has no explicit z -dependence then the z -momenta is constant, too.

$$m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

$$m \dot{z} = p_z = \text{const.} = k$$

$$H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{ Numerically correct ONLY! })$$
$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{ Formally and Numerically correct })$$

Potential V is *isotropic* (cylindrical) function of radius ρ . ($V = V(\rho)$)
 H has no explicit ϕ -dependence and the ϕ -momenta is constant.

If H has no explicit z -dependence then the z -momenta is constant, too.

$$m \rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

$$H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}$$

$$m \dot{z} = p_z = \text{const.} = k$$

(Let $k=0$)

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{ Numerically correct ONLY! })$$
$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{ Formally and Numerically correct })$$

Potential V is *isotropic* (cylindrical) function of radius ρ . ($V = V(\rho)$)
 H has no explicit ϕ -dependence and the ϕ -momenta is constant.

If H has no explicit z -dependence then the z -momenta is constant, too.

$$m\rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

$$H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}$$

$$m\dot{z} = p_z = \text{const.} = k$$

(Let $k=0$)

Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.}$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{Numerically correct ONLY!})$$
$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{Formally and Numerically correct})$$

Potential V is *isotropic* (cylindrical) function of radius ρ . ($V = V(\rho)$)
 H has no explicit ϕ -dependence and the ϕ -momenta is constant.

If H has no explicit z -dependence then the z -momenta is constant, too.

$$m\rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

$$m\dot{z} = p_z = \text{const.} = k$$

$$H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}$$

(Let $k=0$)

Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.}$$

An *effective potential* $V^{\text{eff}}(\rho)$ has a *centrifugal barrier*.

$$V^{\text{eff}}(\rho) = \frac{\mu^2}{2m\rho^2} + V(\rho)$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{Numerically correct ONLY!})$$
$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{Formally and Numerically correct})$$

Potential V is *isotropic* (cylindrical) function of radius ρ . ($V = V(\rho)$)
 H has no explicit ϕ -dependence and the ϕ -momenta is constant.

If H has no explicit z -dependence then the z -momenta is constant, too.

$$m\rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

$$m\dot{z} = p_z = \text{const.} = k$$

$$H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}$$

(Let $k=0$)

Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.}$$

An effective potential $V^{\text{eff}}(\rho)$ has a centrifugal barrier.

$$V^{\text{eff}}(\rho) = \frac{\mu^2}{2m\rho^2} + V(\rho)$$

Velocity relations:

$$\dot{\phi} = \mu / (m\rho^2) \quad \dot{\rho} = \frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm \sqrt{\frac{2}{m} (E - V^{\text{eff}}(\rho))}$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2} \gamma_{mn} \dot{q}^m \dot{q}^n + V = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + V \quad (\text{Numerically correct ONLY!})$$

$$= \frac{1}{2} \gamma^{mn} p_m p_n + V = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} p_\phi^2 + \frac{1}{2m} p_z^2 + V \quad (\text{Formally and Numerically correct})$$

Potential V is *isotropic* (cylindrical) function of radius ρ . ($V = V(\rho)$)
 H has no explicit ϕ -dependence and the ϕ -momenta is constant.

If H has no explicit z -dependence then the z -momenta is constant, too.

$$m\rho^2 \dot{\phi} = p_\phi = \text{const.} = \mu$$

$$m\dot{z} = p_z = \text{const.} = k$$

$$H = \frac{1}{2m} p_\rho^2 + \frac{\mu^2}{2m\rho^2} + \frac{k^2}{2m} + V(\rho) = E = \text{const.}$$

(Let $k=0$)

Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m} p_\rho^2 + V^{\text{eff}}(\rho) = E = \text{const.}$$

An *effective potential* $V^{\text{eff}}(\rho)$ has a *centrifugal barrier*.

$$V^{\text{eff}}(\rho) = \frac{\mu^2}{2m\rho^2} + V(\rho)$$

Velocity relations:

$$\dot{\phi} = \mu / (m\rho^2) \quad \dot{\rho} = \frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm \sqrt{\frac{2}{m} (E - V^{\text{eff}}(\rho))}$$

Equations solved by a *quadrature integral* for time versus radius.

$$\int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m} (E - V^{\text{eff}}(\rho))}} = (\text{Travel time } \rho_0 \text{ to } \rho_1) = t_1 - t_0$$

Separation of GCC Equations: Effective Potentials

- *Small $(n_\rho:m_\phi)$ -periodic and quasi-periodic oscillations*
 - 2D Spherical pendulum or “Bowl-Bowling” and the “I-Ball”*
 - $(n_\rho:m_\phi)=(2:1)$ vs $(1:1)$ periodic and quasi-periodic orbits*

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left. \frac{dV^{eff}(\rho)}{d\rho} \right|_{\rho_0} = 0, \quad \text{with: } \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_0} > 0.$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_0) + 0 + \frac{1}{2}(\rho - \rho_0)^2 \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_0}$$

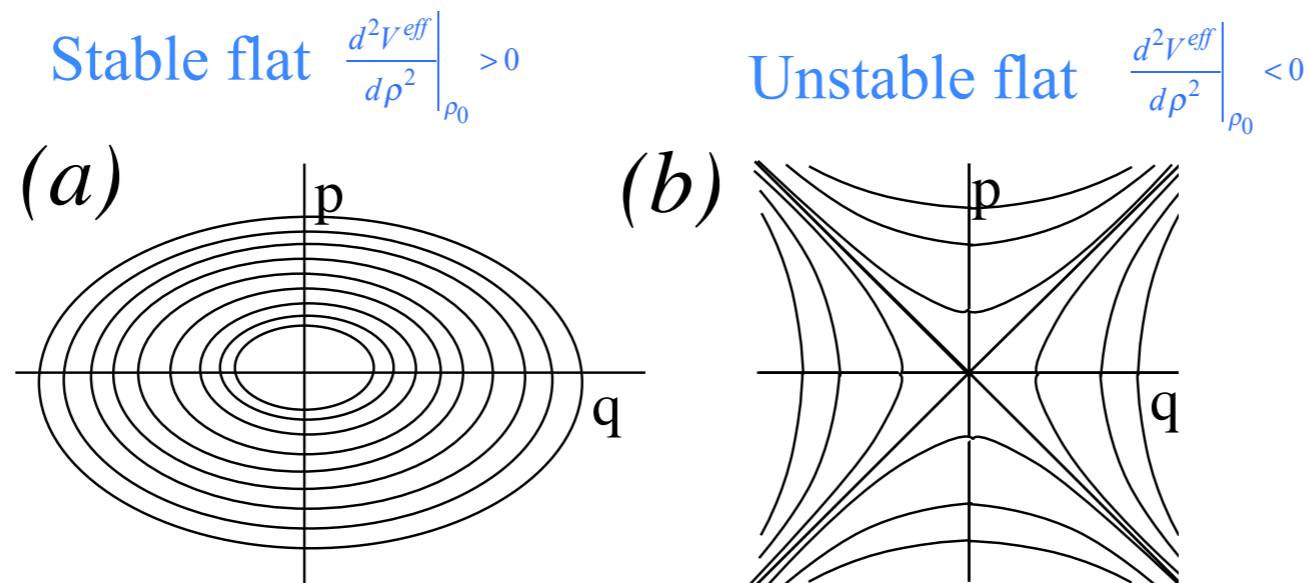


Fig. 2.7.4 Phase paths around fixed points (a) Stable point (b) Unstable saddle point

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left. \frac{dV^{eff}(\rho)}{d\rho} \right|_{\rho_{stable}} = 0, \quad \text{with:} \quad \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}} > 0.$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}}$$

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

$$k^{eff} = \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}} \quad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m} \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}}}$$

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left. \frac{dV^{eff}(\rho)}{d\rho} \right|_{\rho_{stable}} = 0, \quad \text{with:} \quad \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}} > 0.$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}}$$

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

$$k^{eff} = \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}} \quad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m} \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}}}$$

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

$$\frac{\omega_{\rho_{stable}}}{\omega_{\phi}} = \frac{\omega_{\rho_{stable}}}{\dot{\phi}(\rho_{stable})} = \frac{n_{\rho}}{n_{\phi}} \Leftrightarrow \text{Orbit is closed-periodic}$$

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left. \frac{dV^{eff}(\rho)}{d\rho} \right|_{\rho_{stable}} = 0, \quad \text{with:} \quad \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}} > 0.$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}}$$

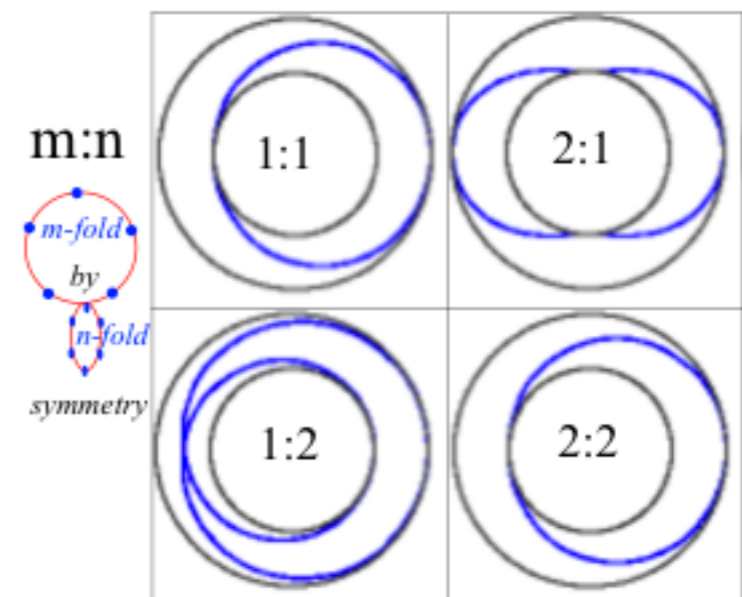
An effective "spring constant" at the stable point giving approximate frequency of oscillation.

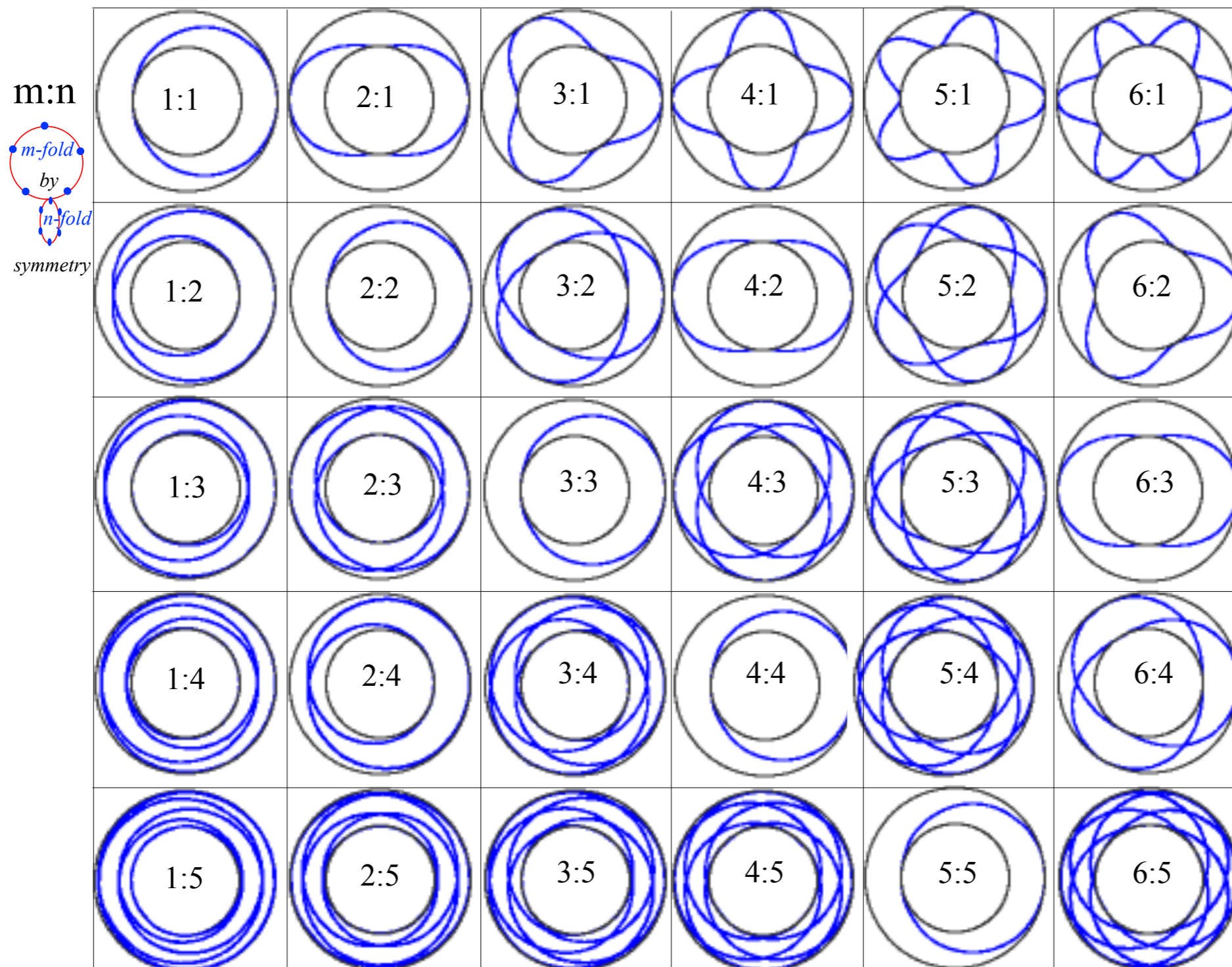
$$k^{eff} = \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}} \quad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m} \left. \frac{d^2V^{eff}}{d\rho^2} \right|_{\rho_{stable}}}$$

Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

$$\frac{\omega_{\rho_{stable}}}{\omega_{\phi}} = \frac{\omega_{\rho_{stable}}}{\dot{\phi}(\rho_{stable})} = \frac{n_{\rho}}{n_{\phi}} \Leftrightarrow \text{Orbit is closed-periodic}$$

Some generic shapes resulting from various ratios $n_{\rho} : n_{\phi}$





(b) $\omega_\rho:\omega_\phi$ just below 1

$\omega_\rho:\omega_\phi=1$

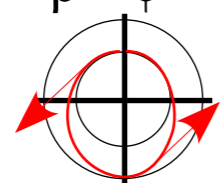
$\omega_\rho:\omega_\phi$ just above 1

(c) $\omega_\rho:\omega_\phi$ just below 2

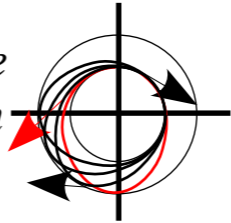
$\omega_\rho:\omega_\phi=2$

$\omega_\rho:\omega_\phi$ just above 2

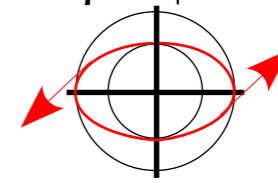
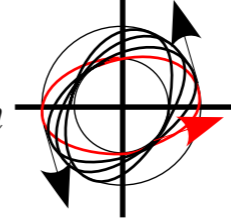
prograde
precession
of nodes



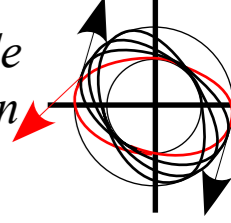
retrograde
precession
of nodes



prograde
precession
of nodes




retrograde
precession
of nodes



Separation of GCC Equations: Effective Potentials

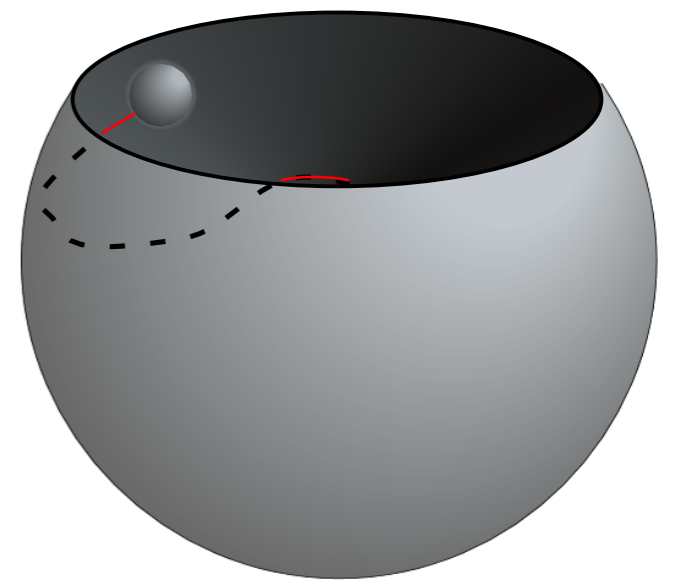
Small $(n_\rho:m_\phi)$ -periodic and quasi-periodic oscillations

 *2D Spherical pendulum or “Bowl-Bowling” and the “I-Ball”*
 $(n_\rho:m_\phi)=(2:1)$ vs $(1:1)$ periodic and quasi-periodic orbits

2D Spherical pendulum or “Bowl-Bowling” and the “I-Ball”

Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

$$x=x^1=r\sin\theta\cos\phi, \quad y=x^2=r\sin\theta\sin\phi, \quad z=x^3=r\cos\theta,$$



2D Spherical pendulum or “Bowl-Bowling” and the “I-Ball”

Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

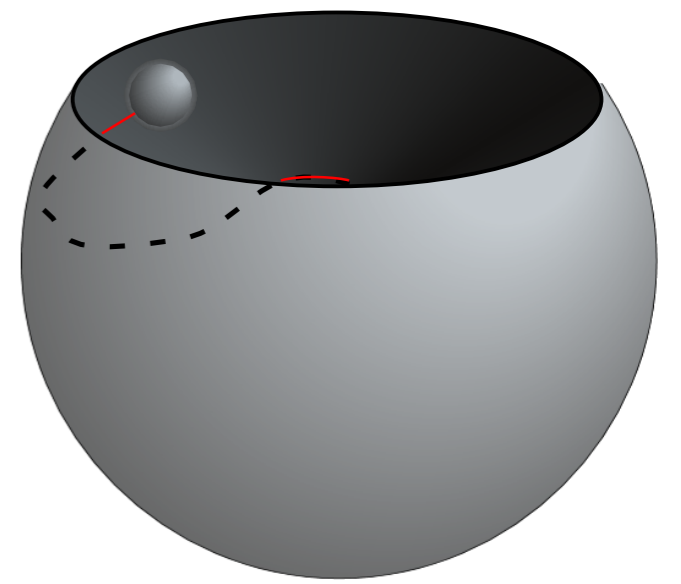
$$x=x^1=r\sin\theta\cos\phi, \quad y=x^2=r\sin\theta\sin\phi, \quad z=x^3=r\cos\theta,$$

Jacobian matrices and determinants:

$$J = \begin{pmatrix} \mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \xrightarrow[r=\rho]{\theta=\pi/2} \begin{pmatrix} \cos\phi & 0 & -\rho\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \\ 0 & -\rho & 0 \end{pmatrix}$$

Reduced to cylindrical coordinates:

$$\det J = \det J^T = \frac{\partial\{xyz\}}{\partial\{r\theta\phi\}} = r^2 \sin\theta \xrightarrow[r=\rho]{\theta=\pi/2} \rho^2$$



2D Spherical pendulum or “Bowl-Bowling”

Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

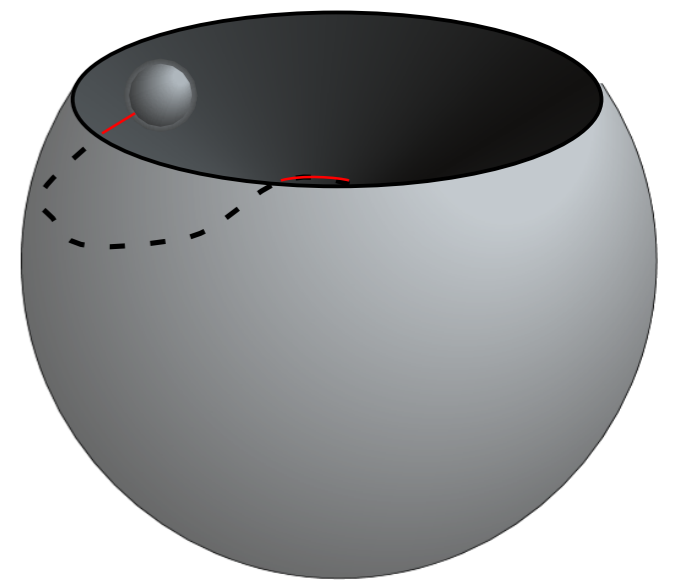
$$x=x^1=r\sin\theta\cos\phi, \quad y=x^2=r\sin\theta\sin\phi, \quad z=x^3=r\cos\theta,$$

Jacobian matrices and determinants:

$$J = \begin{pmatrix} \mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \xrightarrow[r=\rho]{\theta=\pi/2} \begin{pmatrix} \cos\phi & 0 & -\rho\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \\ 0 & -\rho & 0 \end{pmatrix}$$

Reduced to cylindrical coordinates:

$$\det J = \det J^T = \frac{\partial\{xyz\}}{\partial\{r\theta\phi\}} = r^2 \sin\theta \xrightarrow[r=\rho]{\theta=\pi/2} \rho^2$$



Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T \cdot J$ of Jacobian and its transpose. OCC g 's are diagonal.

$$\text{Covariant: } g_{rr} = \mathbf{E}_r \cdot \mathbf{E}_r = 1, \quad g_{\theta\theta} = \mathbf{E}_\theta \cdot \mathbf{E}_\theta = r^2, \quad g_{\phi\phi} = \mathbf{E}_\phi \cdot \mathbf{E}_\phi = r^2 \sin^2 \theta,$$

$$\text{Contravariant: } g^{rr}=1, \quad g^{\theta\theta}=1/r^2, \quad g^{\phi\phi}=1/r^2 \sin^2 \theta.$$

2D Spherical pendulum or "Bowl-Bowling"

Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

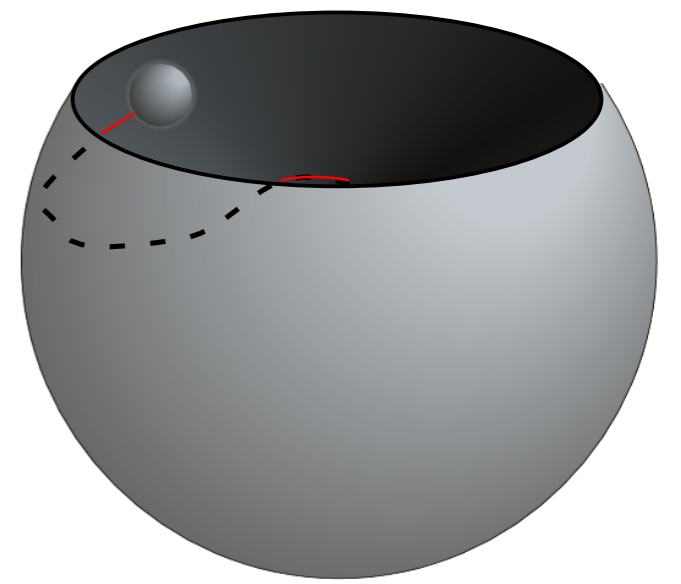
$$x=x^1=r\sin\theta\cos\phi, \quad y=x^2=r\sin\theta\sin\phi, \quad z=x^3=r\cos\theta,$$

Jacobian matrices and determinants:

$$J = \begin{pmatrix} \mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \xrightarrow[r=\rho]{\theta=\pi/2} \begin{pmatrix} \cos\phi & 0 & -\rho\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \\ 0 & -\rho & 0 \end{pmatrix}$$

Reduced to cylindrical coordinates:

$$\det J = \det J^T = \frac{\partial\{xyz\}}{\partial\{r\theta\phi\}} = r^2 \sin\theta \xrightarrow[r=\rho]{\theta=\pi/2} \rho^2$$



Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T \cdot J$ of Jacobian and its transpose. OCC g's are diagonal.

Covariant: $g_{rr} = \mathbf{E}_r \cdot \mathbf{E}_r = 1, \quad g_{\theta\theta} = \mathbf{E}_\theta \cdot \mathbf{E}_\theta = r^2, \quad g_{\phi\phi} = \mathbf{E}_\phi \cdot \mathbf{E}_\phi = r^2 \sin^2 \theta,$

Contravariant: $g^{rr}=1, \quad g^{\theta\theta}=1/r^2, \quad g^{\phi\phi}=1/r^2 \sin^2 \theta.$

(Lagrangian form)

(Hamiltonian form)

$$\begin{aligned} T &= \frac{m}{2}(g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2) = \frac{1}{2m}(g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + g^{\phi\phi}p_\phi^2) \\ &= \frac{1}{2}(\gamma_{rr}\dot{r}^2 + \gamma_{\theta\theta}\dot{\theta}^2 + \gamma_{\phi\phi}\dot{\phi}^2) = \frac{1}{2}(\gamma^{rr}p_r^2 + \gamma^{\theta\theta}p_\theta^2 + \gamma^{\phi\phi}p_\phi^2) \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) \end{aligned}$$

2D Spherical pendulum or "Bowl-Bowling"

Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

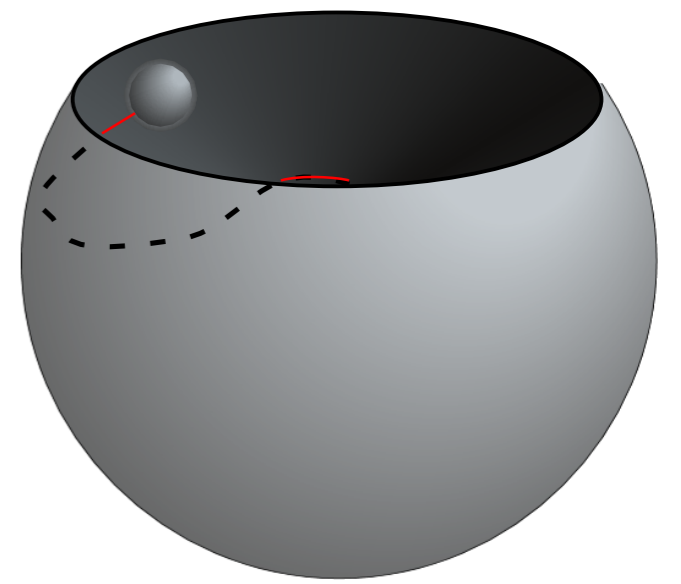
$$x=x^1=r\sin\theta\cos\phi, \quad y=x^2=r\sin\theta\sin\phi, \quad z=x^3=r\cos\theta,$$

Jacobian matrices and determinants:

$$J = \begin{pmatrix} \mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \xrightarrow[r=\rho]{\theta=\pi/2} \begin{pmatrix} \cos\phi & 0 & -\rho\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \\ 0 & -\rho & 0 \end{pmatrix}$$

Reduced to cylindrical coordinates:

$$\det J = \det J^T = \frac{\partial\{xyz\}}{\partial\{r\theta\phi\}} = r^2 \sin\theta \xrightarrow[r=\rho]{\theta=\pi/2} \rho^2$$



Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T \cdot J$ of Jacobian and its transpose. OCC g 's are diagonal.

Covariant: $g_{rr} = \mathbf{E}_r \cdot \mathbf{E}_r = 1, \quad g_{\theta\theta} = \mathbf{E}_\theta \cdot \mathbf{E}_\theta = r^2, \quad g_{\phi\phi} = \mathbf{E}_\phi \cdot \mathbf{E}_\phi = r^2 \sin^2 \theta,$

Contravariant: $g^{rr}=1, \quad g^{\theta\theta}=1/r^2, \quad g^{\phi\phi}=1/r^2 \sin^2 \theta.$

(Lagrangian form)

(Hamiltonian form)

$$\begin{aligned} T &= \frac{m}{2}(g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2) = \frac{1}{2m}(g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + g^{\phi\phi}p_\phi^2) \\ &= \frac{1}{2}(\gamma_{rr}\dot{r}^2 + \gamma_{\theta\theta}\dot{\theta}^2 + \gamma_{\phi\phi}\dot{\phi}^2) = \frac{1}{2}(\gamma^{rr}p_r^2 + \gamma^{\theta\theta}p_\theta^2 + \gamma^{\phi\phi}p_\phi^2) \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}) \end{aligned}$$

Spherical coordinates with constant radius r implies conserved azimuthal momentum:

$$p_\phi \equiv \frac{\partial T}{\partial \dot{\phi}} = m(R^2 \sin^2 \theta)\dot{\phi} = \text{const.}$$

2D Spherical pendulum or "Bowl-Bowling"

Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

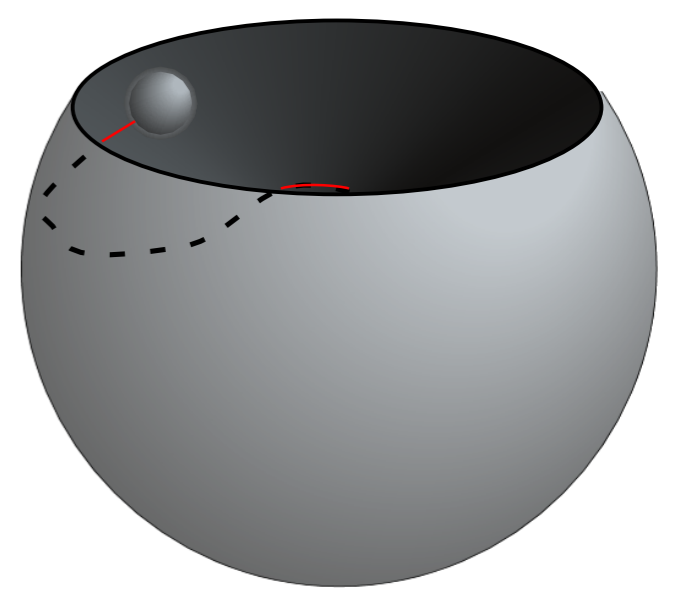
$$x=x^1=r\sin\theta\cos\phi, \quad y=x^2=r\sin\theta\sin\phi, \quad z=x^3=r\cos\theta,$$

Jacobian matrices and determinants:

$$J = \begin{pmatrix} \mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \xrightarrow[r=\rho]{\theta=\pi/2} \begin{pmatrix} \cos\phi & 0 & -\rho\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \\ 0 & -\rho & 0 \end{pmatrix}$$

Reduced to cylindrical coordinates:

$$\det J = \det J^T = \frac{\partial\{xyz\}}{\partial\{r\theta\phi\}} = r^2 \sin\theta \xrightarrow[r=\rho]{\theta=\pi/2} \rho^2$$



Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T \cdot J$ of Jacobian and its transpose. OCC g 's are diagonal.

Covariant: $g_{rr} = \mathbf{E}_r \cdot \mathbf{E}_r = 1, \quad g_{\theta\theta} = \mathbf{E}_\theta \cdot \mathbf{E}_\theta = r^2, \quad g_{\phi\phi} = \mathbf{E}_\phi \cdot \mathbf{E}_\phi = r^2 \sin^2 \theta,$

Contravariant: $g^{rr}=1, \quad g^{\theta\theta}=1/r^2, \quad g^{\phi\phi}=1/r^2 \sin^2 \theta.$

(Lagrangian form)

(Hamiltonian form)

$$\begin{aligned} T &= \frac{m}{2}(g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2) = \frac{1}{2m}(g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + g^{\phi\phi}p_\phi^2) \\ &= \frac{1}{2}(\gamma_{rr}\dot{r}^2 + \gamma_{\theta\theta}\dot{\theta}^2 + \gamma_{\phi\phi}\dot{\phi}^2) = \frac{1}{2}(\gamma^{rr}p_r^2 + \gamma^{\theta\theta}p_\theta^2 + \gamma^{\phi\phi}p_\phi^2) \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}) \end{aligned}$$

Spherical coordinates with constant radius r implies conserved azimuthal momentum:

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}} = m(R^2 \sin^2 \theta)\dot{\phi} = \text{const.}$$

Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.} :$

$$E = \frac{mR^2}{2}\dot{\theta}^2 + V^{\text{effective}}(\theta) = \frac{mR^2}{2}\dot{\theta}^2 + \frac{p_\phi^2}{2mR^2\sin^2\theta} + mgR\cos\theta = \alpha\dot{\theta}^2 + \frac{\delta}{\sin^2\theta} + \gamma\cos\theta$$

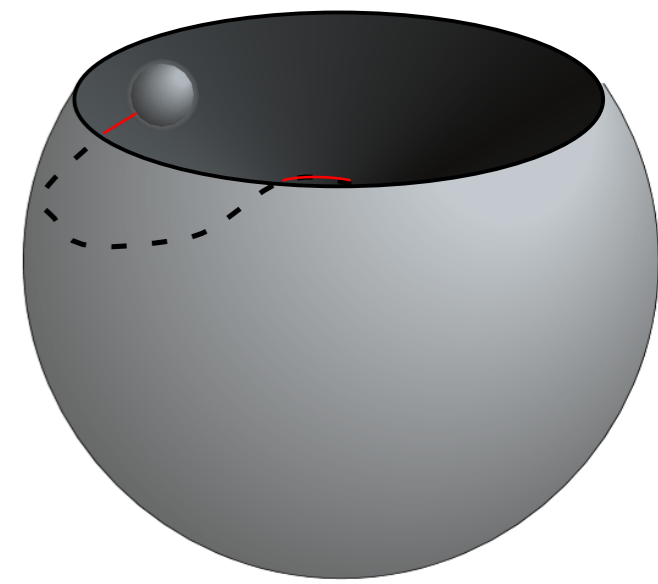
Let: $\alpha = \frac{mR^2}{2}, \quad \delta = \frac{p_\phi^2}{2mR^2}, \quad \gamma = mgR$ where: $p_\phi = mR^2\sin^2\theta(\dot{\phi})$

2D Spherical pendulum or “Bowl-Bowling”

Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$



2D Spherical pendulum or “Bowl-Bowling”

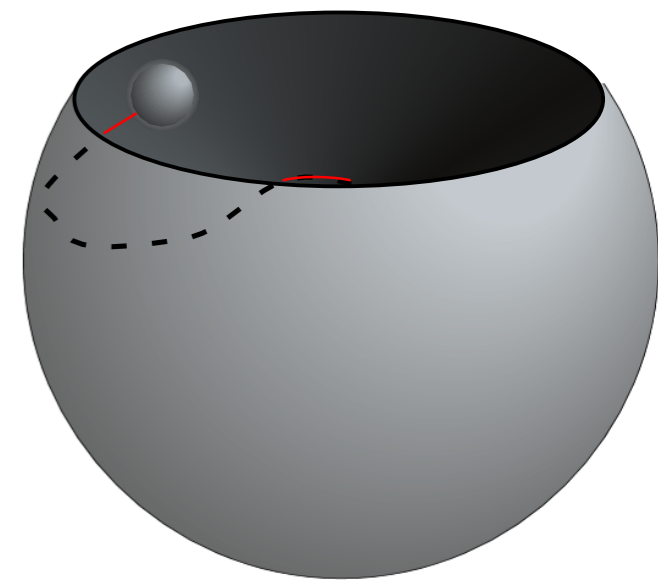
Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

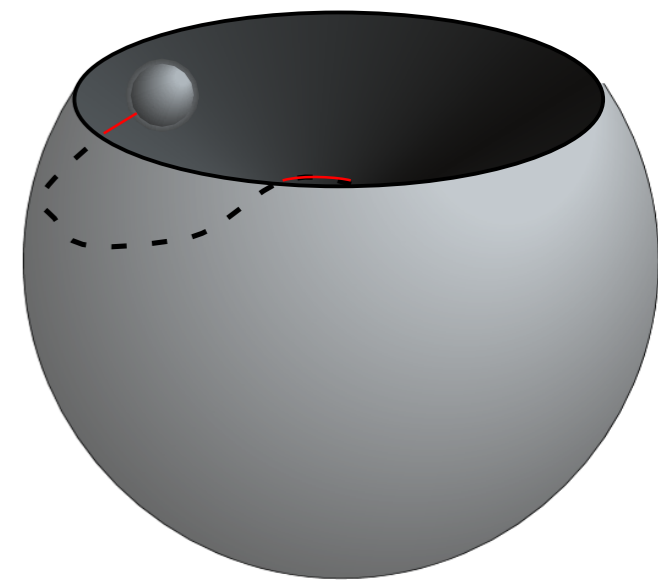
Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$



2D Spherical pendulum or "Bowl-Bowling"



Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

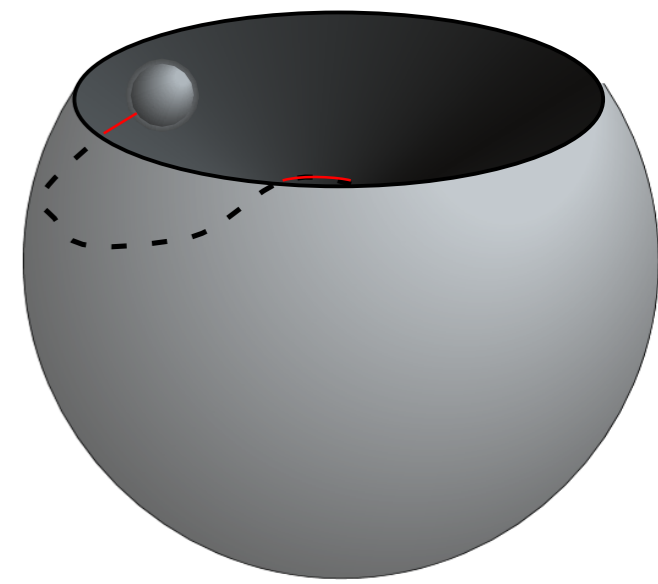
Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$\left(\omega_\theta^{\text{equil}}\right)^2 = \frac{1}{mR^2} \left. \frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$

2D Spherical pendulum or “Bowl-Bowling”



Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

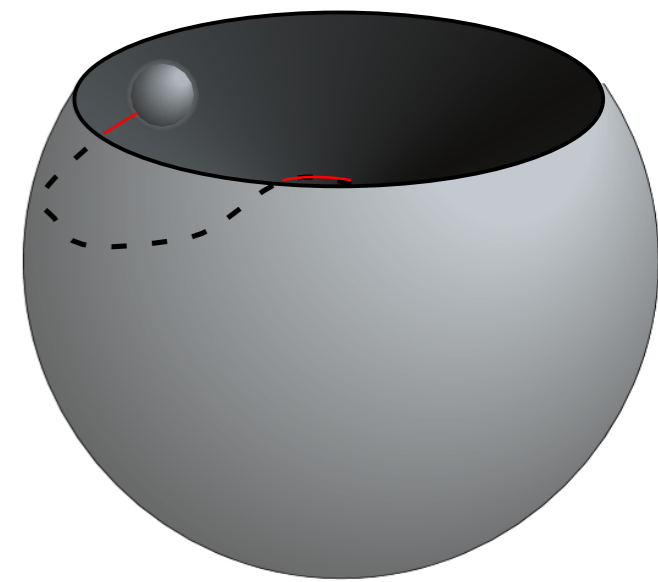
$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or:} \quad \dot{\phi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$$

$$\left(\omega_\theta^{\text{equil}}\right)^2 = \frac{1}{mR^2} \left. \frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$

(Polar angle librational frequency $\omega_\theta^{\text{equil}}$ is related to azimuthal frequency $\dot{\phi}_{\text{equil}}^2$.)

2D Spherical pendulum or "Bowl-Bowling"



Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or:} \quad \dot{\phi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$$

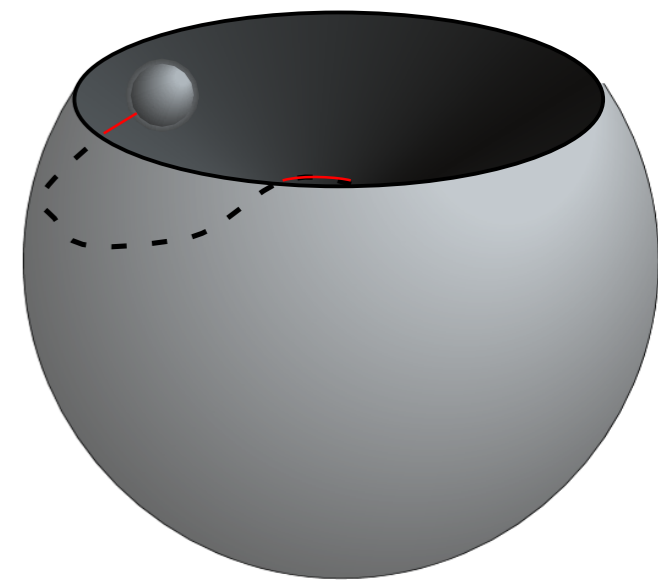
$$\left(\omega_\theta^{\text{equil}}\right)^2 = \frac{1}{mR^2} \left. \frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$

(Polar angle librational frequency $\omega_\theta^{\text{equil}}$ is related to azimuthal frequency $\dot{\phi}_{\text{equil}}^2$.)

V-Derivative for small oscillation frequency:

$$\begin{aligned} \frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} &= -\gamma \cos \theta + \frac{2\delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2\delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2\delta \frac{\sin^2 \theta + 3\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + \frac{2(mR^2 \sin^2 \theta \dot{\phi})^2}{2mR^2} \frac{1+2\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + mR^2 \dot{\phi}^2 (1+2\cos^2 \theta) \end{aligned}$$

2D Spherical pendulum or "Bowl-Bowling"



Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$\left(\omega_\theta^{\text{equil}}\right)^2 = \frac{1}{mR^2} \left. \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or:} \quad \dot{\phi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$$

(Polar angle librational frequency $\omega_\theta^{\text{equil}}$ is related to azimuthal frequency $\dot{\phi}_{\text{equil}}^2$.)

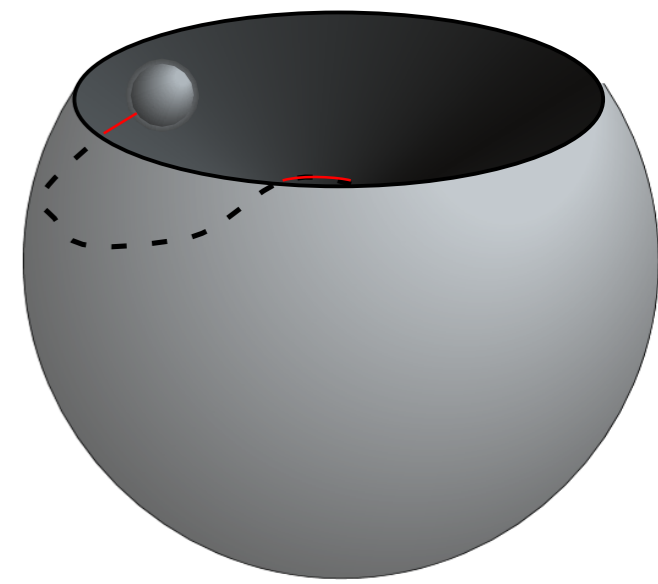
V-Derivative for small oscillation frequency:

$$\begin{aligned} \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} &= -\gamma \cos \theta + \frac{2\delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2\delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2\delta \frac{\sin^2 \theta + 3\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + \frac{2(mR^2 \sin^2 \theta \dot{\phi})^2}{2mR^2} \frac{1+2\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + mR^2 \dot{\phi}^2 (1+2\cos^2 \theta) \end{aligned}$$

At equilibrium:

$$\begin{aligned} \left. \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}} &= -mgR \cos \theta_{\text{equil}} + mR^2 \left(-\frac{g}{R \cos \theta_{\text{equil}}} \right) (1+2\cos^2 \theta_{\text{equil}}) \\ &= -\frac{mgR}{\cos \theta_{\text{equil}}} (1+3\cos^2 \theta_{\text{equil}}) \end{aligned}$$

2D Spherical pendulum or "Bowl-Bowling"



Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$\left(\omega_\theta^{\text{equil}} \right)^2 = \frac{1}{mR^2} \left. \frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or:} \quad \dot{\phi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$$

(Polar angle librational frequency $\omega_\theta^{\text{equil}}$ is related to azimuthal frequency $\dot{\phi}_{\text{equil}}$.)

V-Derivative for small oscillation frequency:

$$\begin{aligned} \frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} &= -\gamma \cos \theta + \frac{2\delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2\delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2\delta \frac{\sin^2 \theta + 3\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + \frac{2(mR^2 \sin^2 \theta \dot{\phi})^2}{2mR^2} \frac{1+2\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + mR^2 \dot{\phi}^2 (1+2\cos^2 \theta) \end{aligned}$$


At equilibrium:

$$\begin{aligned} \left. \frac{d^2 V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}} &= -mgR \cos \theta_{\text{equil}} + mR^2 \left(-\frac{g}{R \cos \theta_{\text{equil}}} \right) (1+2\cos^2 \theta_{\text{equil}}) \\ &= -\frac{mgR}{\cos \theta_{\text{equil}}} (1+3\cos^2 \theta_{\text{equil}}) \end{aligned}$$

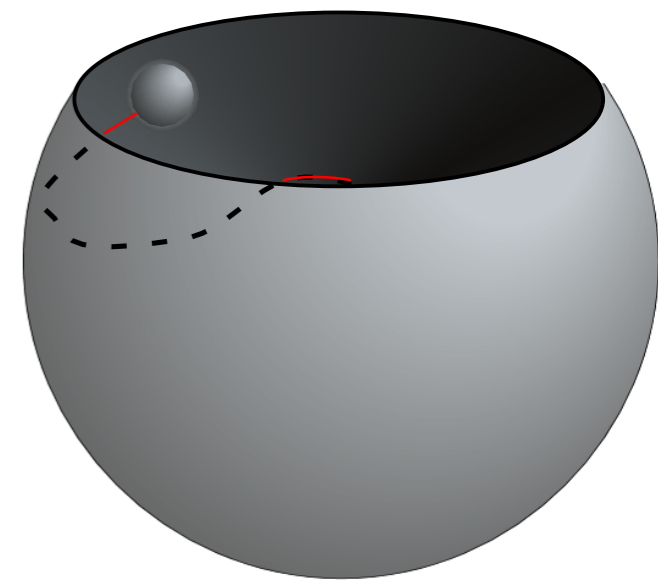
$$\left(\omega_\theta^{\text{equil}} \right)^2 / \left(\dot{\phi}_{\text{equil}}^2 \right) = \left(1+3\cos^2 \theta_{\text{equil}} \right)$$

Separation of GCC Equations: Effective Potentials

Small $(n_\rho:m_\phi)$ -periodic and quasi-periodic oscillations

 *2D Spherical pendulum or “Bowl-Bowling” and the “I-Ball”*
 $(n_\rho:m_\phi)=(2:1)$ vs $(1:1)$ periodic and quasi-periodic orbits

2D Spherical pendulum or "Bowl-Bowling"



Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$\left(\omega_\theta^{\text{equil}}\right)^2 = \frac{1}{mR^2} \left. \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or:} \quad \dot{\phi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$$

(Polar angle librational frequency $\omega_\theta^{\text{equil}}$ is related to azimuthal frequency $\dot{\phi}_{\text{equil}}$.)

V-Derivative for small oscillation frequency:

$$\begin{aligned} \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} &= -\gamma \cos \theta + \frac{2\delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2\delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2\delta \frac{\sin^2 \theta + 3\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + \frac{2(mR^2 \sin^2 \theta \dot{\phi})^2}{2mR^2} \frac{1+2\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + mR^2 \dot{\phi}^2 (1+2\cos^2 \theta) \end{aligned}$$

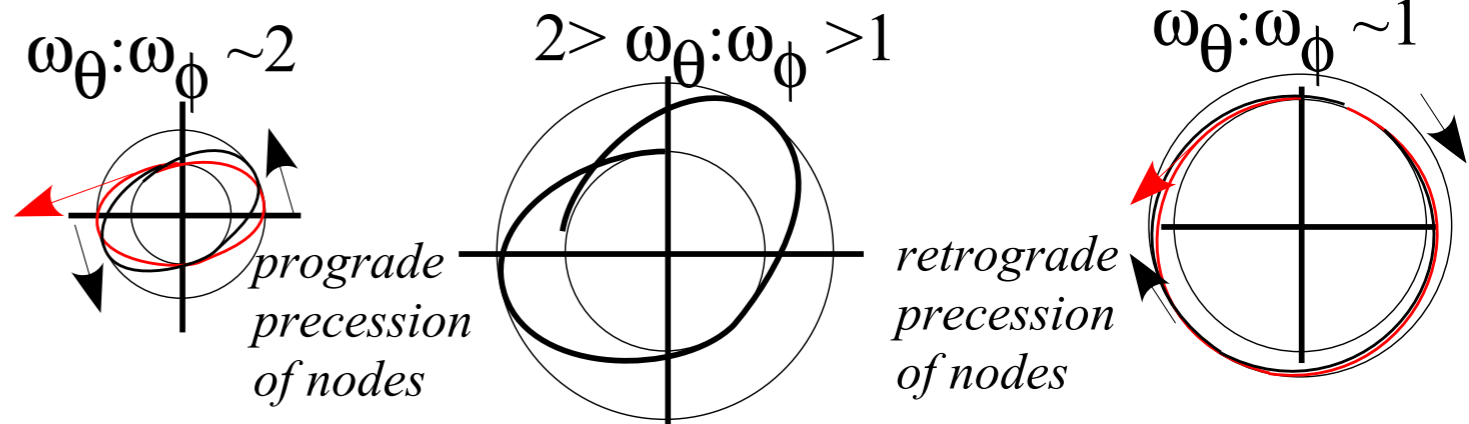
At equilibrium:

$$\begin{aligned} \left. \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}} &= -mgR \cos \theta_{\text{equil}} + mR^2 \left(-\frac{g}{R \cos \theta_{\text{equil}}} \right) (1+2\cos^2 \theta_{\text{equil}}) \\ &= -\frac{mgR}{\cos \theta_{\text{equil}}} (1+3\cos^2 \theta_{\text{equil}}) \end{aligned}$$

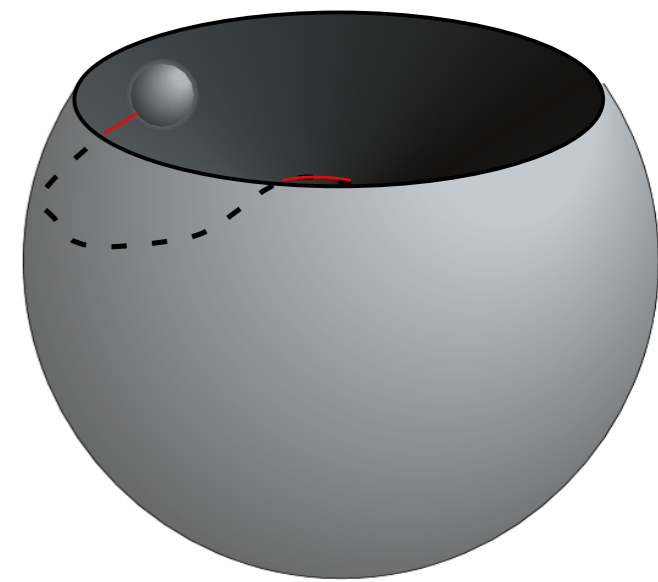
$$\left(\omega_\theta^{\text{equil}}\right)^2 / (\dot{\phi}_{\text{equil}}^2) = (1+3\cos^2 \theta_{\text{equil}})$$

At bottom $\theta \rightarrow \pi$ the ratio of in-out ω_θ to circle ω_ϕ approaches 2:1

At equator $\theta \rightarrow \pi/2$ the ratio approaches 1:1.



2D Spherical pendulum or "Bowl-Bowling"



Total Energy from Hamiltonian $E=T+V(\text{gravity})=\text{const.}$:

$$E = \frac{mR^2}{2} \dot{\theta}^2 + V^{\text{effective}}(\theta) = \alpha \dot{\theta}^2 + \frac{\delta}{\sin^2 \theta} + \gamma \cos \theta$$

Let: $\alpha = \frac{mR^2}{2}$, $\delta = \frac{p_\phi^2}{2mR^2}$, $\gamma = mgR$ where: $p_\phi = mR^2 \sin^2 \theta (\dot{\phi})$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{\text{effective}}(\theta)}{d\theta} = \frac{-2\delta \cos \theta}{\sin^3 \theta} - \gamma \sin \theta = 0 = \frac{-2p_\phi^2 \cos \theta}{2mR^2 \sin^3 \theta} - mgR \sin \theta$$

$$\left(\omega_\theta^{\text{equil}}\right)^2 = \frac{1}{mR^2} \left. \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}}$$

$$0 = (mR^2 \sin \theta) \dot{\phi}^2 \cos \theta - mgR \sin \theta \quad \text{or:} \quad \dot{\phi}_{\text{equil}}^2 = -\frac{g}{R \cos \theta_{\text{equil}}}$$

(Polar angle librational frequency $\omega_\theta^{\text{equil}}$ is related to azimuthal frequency $\dot{\phi}_{\text{equil}}$.)

V-Derivative for small oscillation frequency:

$$\begin{aligned} \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} &= -\gamma \cos \theta + \frac{2\delta \sin \theta}{\sin^3 \theta} + \frac{3 \cdot 2\delta \cos^2 \theta}{\sin^4 \theta} = -\gamma \cos \theta + 2\delta \frac{\sin^2 \theta + 3\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + \frac{2(mR^2 \sin^2 \theta \dot{\phi})^2}{2mR^2} \frac{1+2\cos^2 \theta}{\sin^4 \theta} \\ &= -mgR \cos \theta + mR^2 \dot{\phi}^2 (1+2\cos^2 \theta) \end{aligned}$$

At equilibrium:

$$\begin{aligned} \left. \frac{d^2V^{\text{effective}}(\theta)}{d\theta^2} \right|_{\text{equil}} &= -mgR \cos \theta_{\text{equil}} + mR^2 \left(-\frac{g}{R \cos \theta_{\text{equil}}} \right) (1+2\cos^2 \theta_{\text{equil}}) \\ &= -\frac{mgR}{\cos \theta_{\text{equil}}} (1+3\cos^2 \theta_{\text{equil}}) \end{aligned}$$

$$\left(\omega_\theta^{\text{equil}}\right)^2 / (\dot{\phi}_{\text{equil}}^2) = (1+3\cos^2 \theta_{\text{equil}})$$

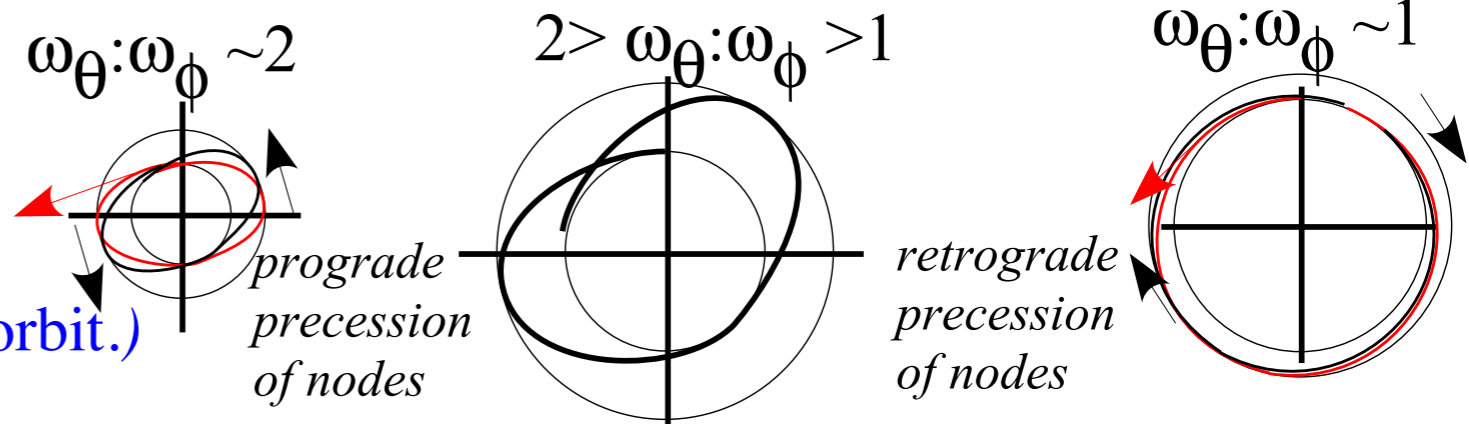
At bottom $\theta \rightarrow \pi$ the ratio of in-out ω_θ to circle ω_ϕ approaches 2:1

At equator $\theta \rightarrow \pi/2$ the ratio approaches 1:1.

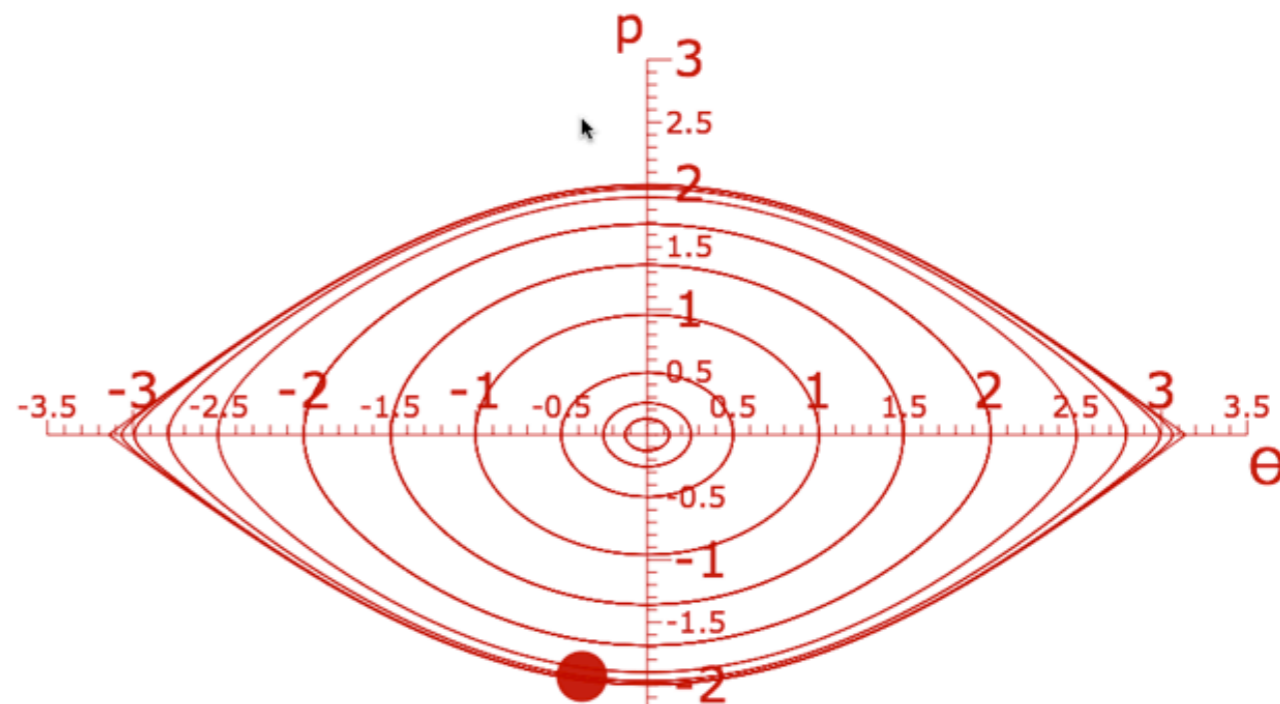
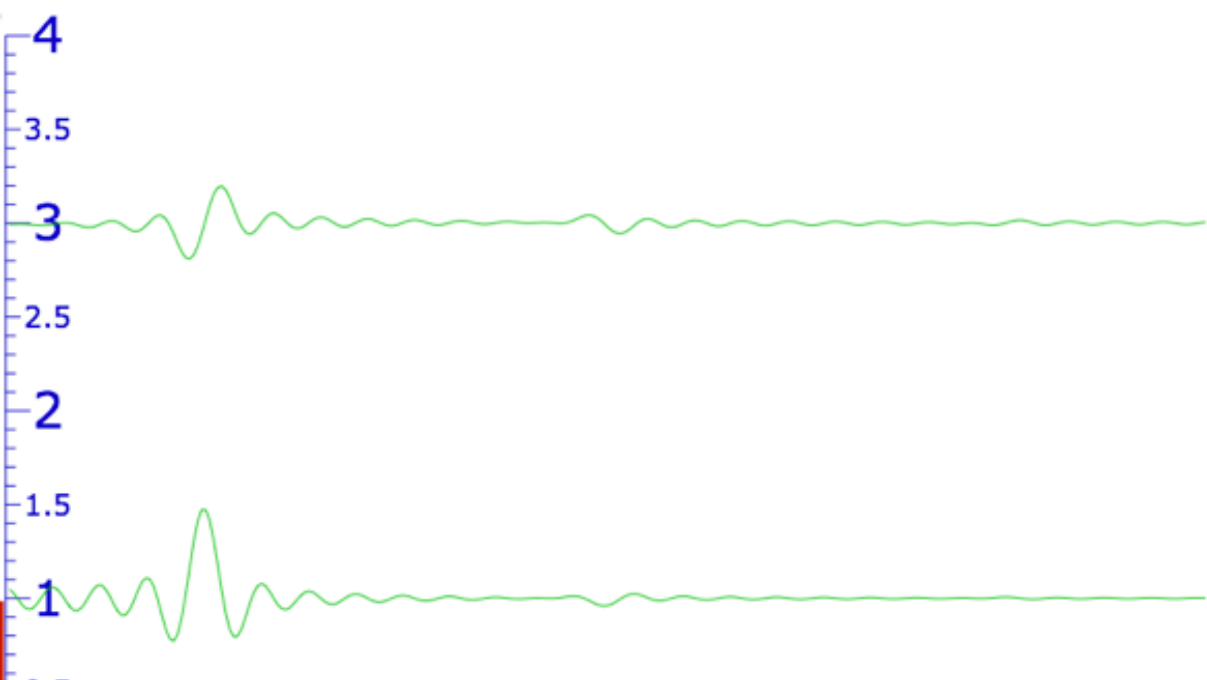
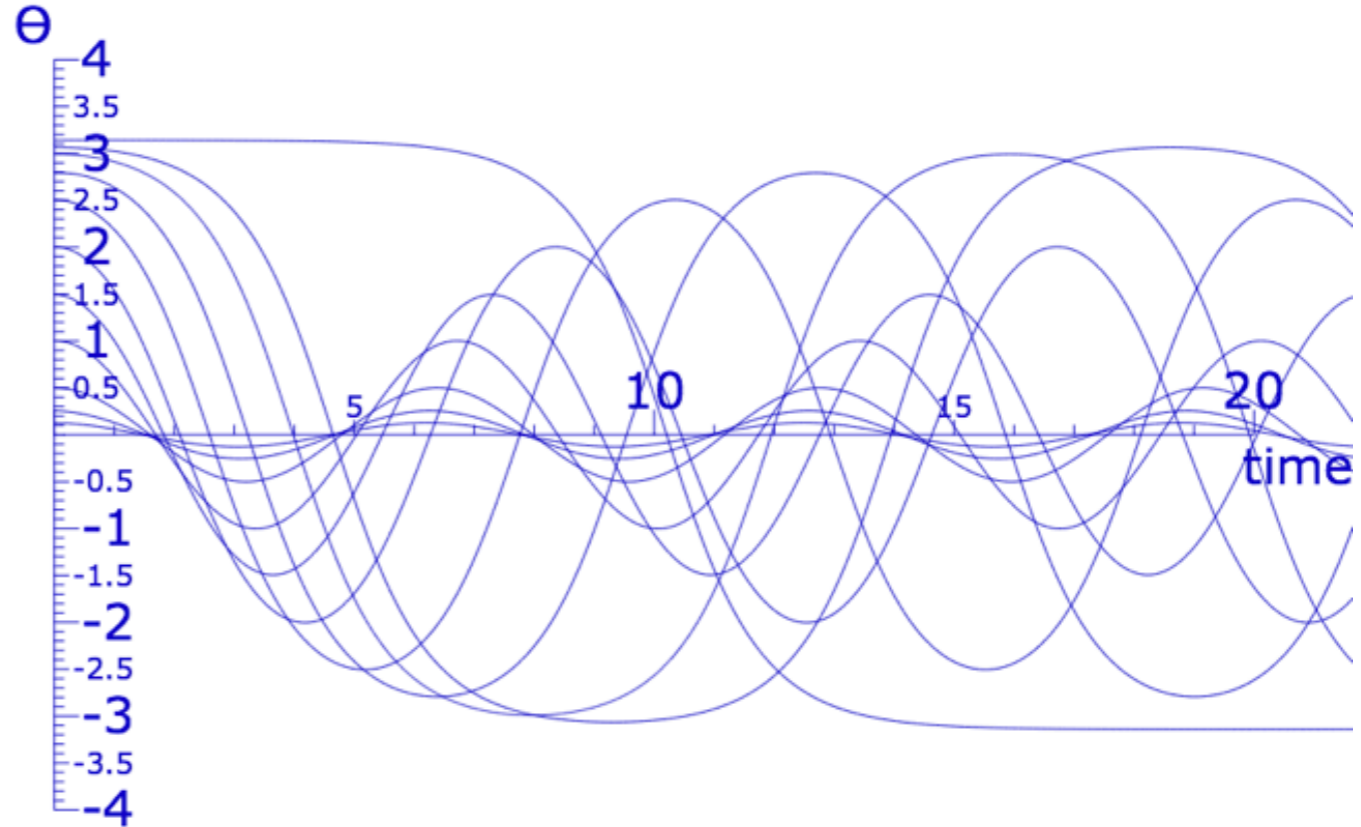
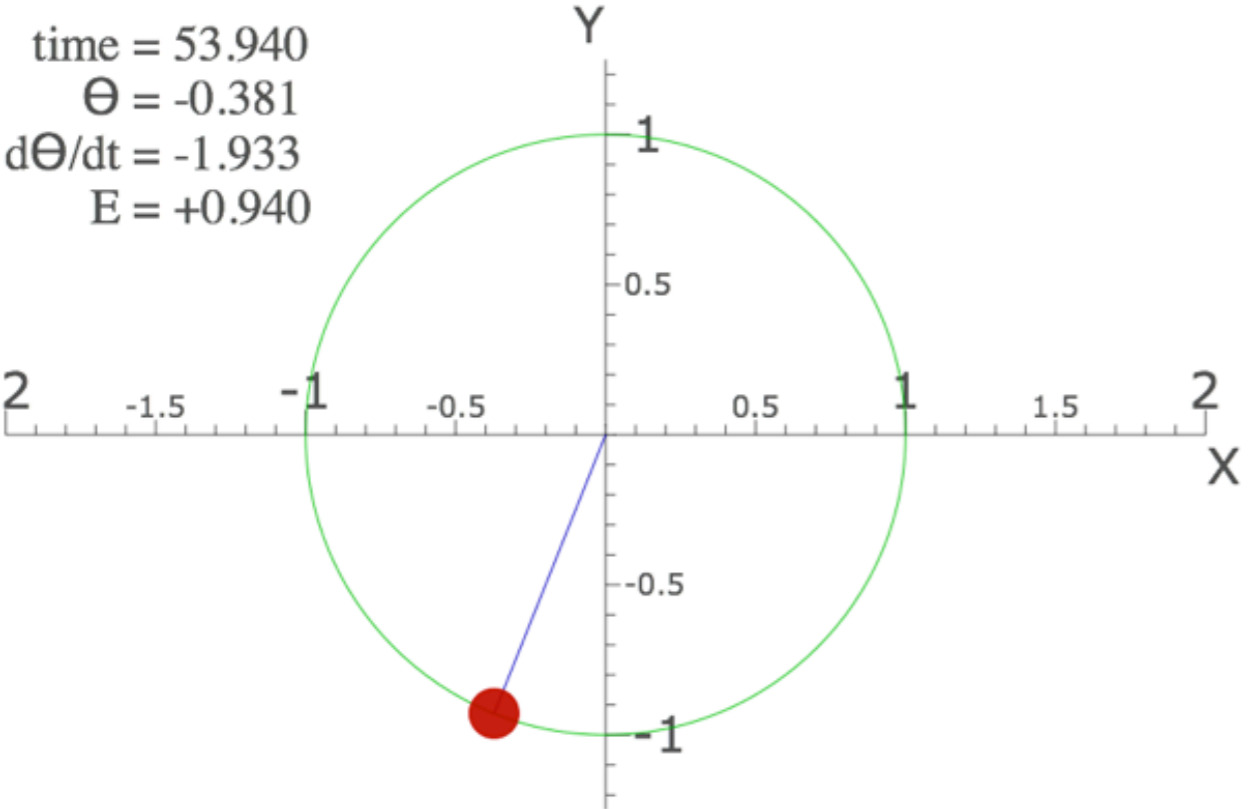
Ratio is between 2 and 1

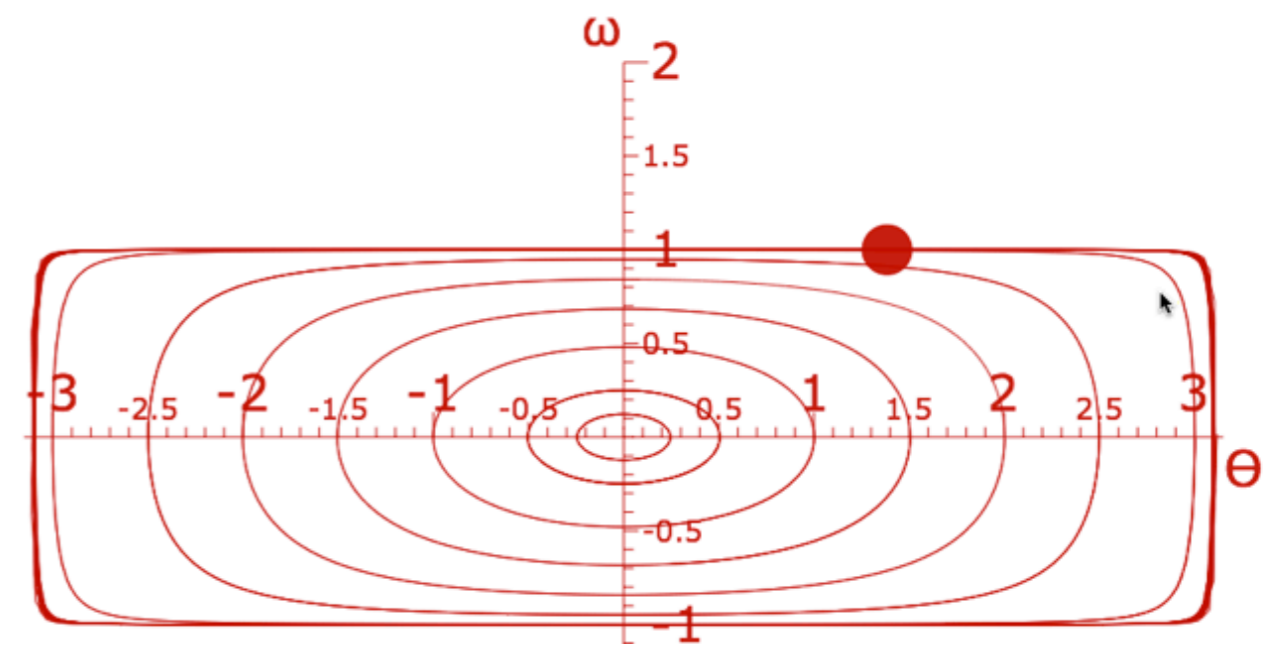
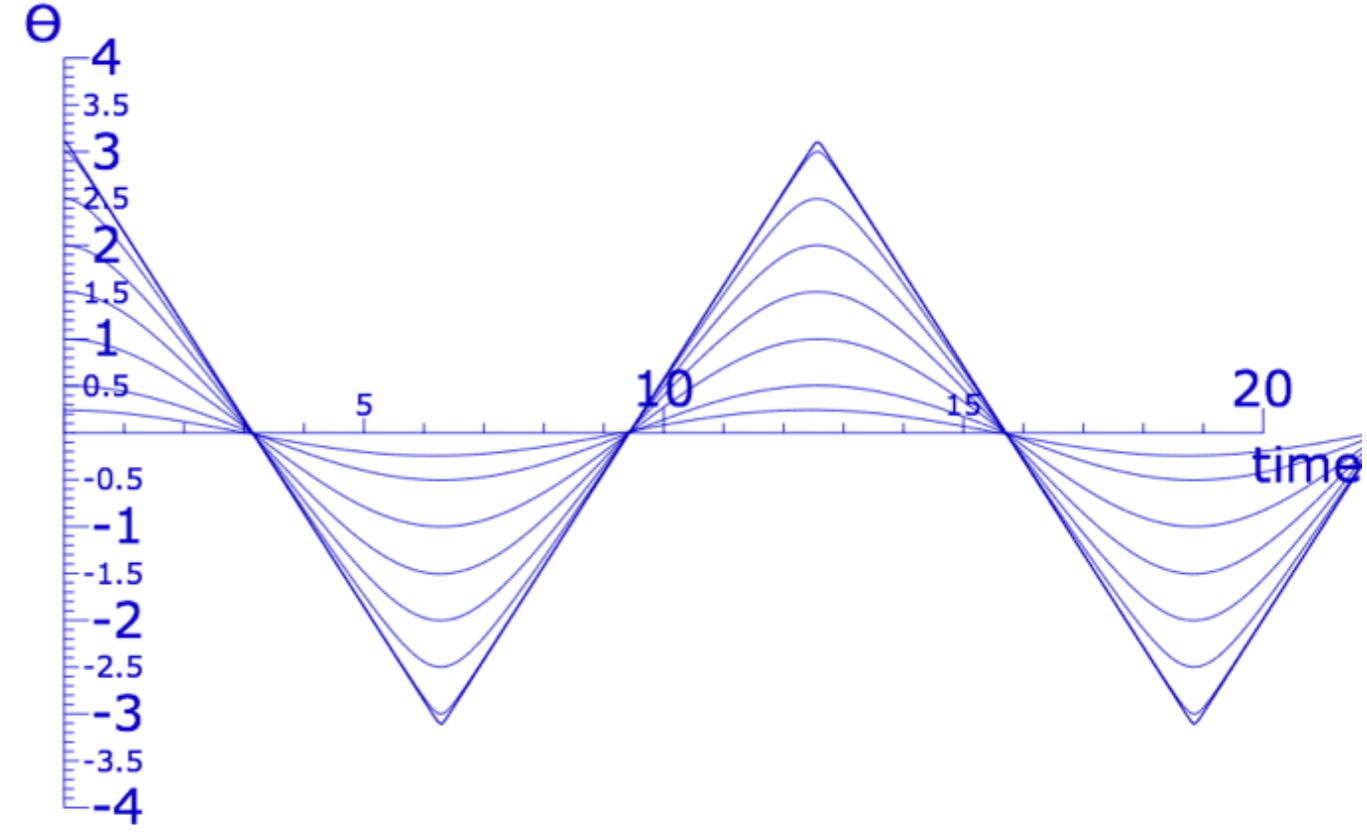
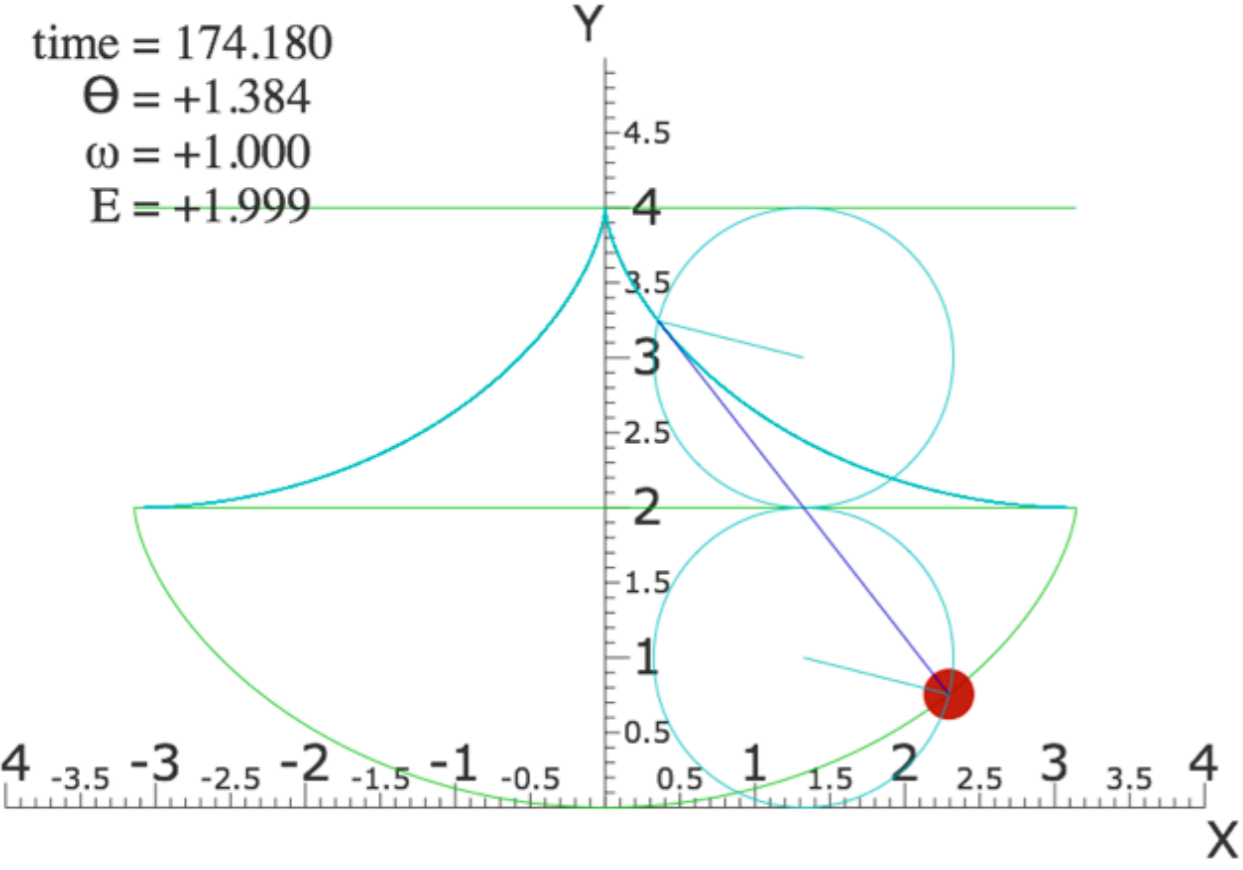
(Usually irrational non-closed orbit).

(2:1 is like 2D IHO, but 1:1 is like coulomb orbit.)



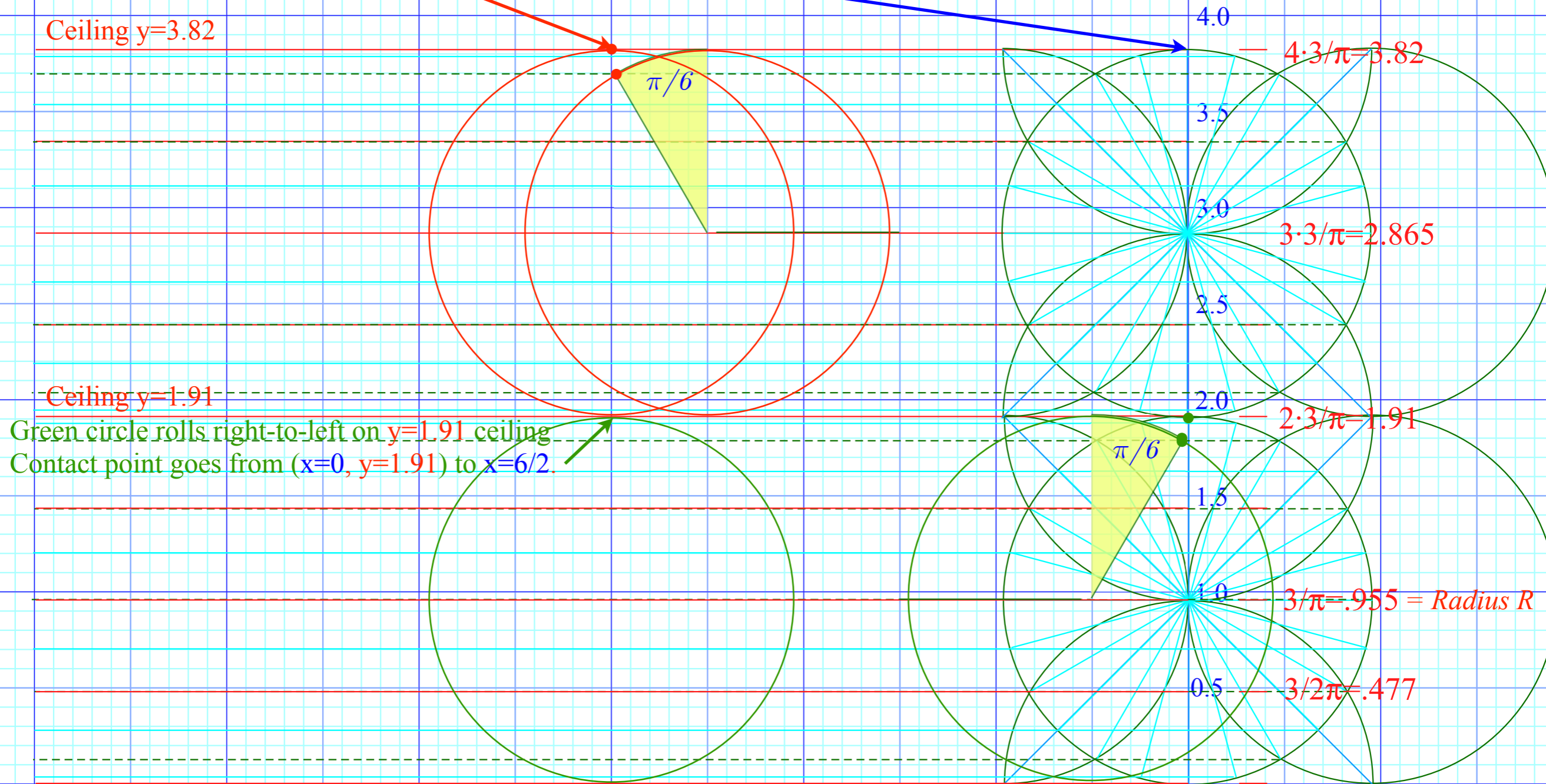
→ *Cycloidal ruler & compass geometry*





Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled-out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. Diameter is $2R=6/\pi=1.91$

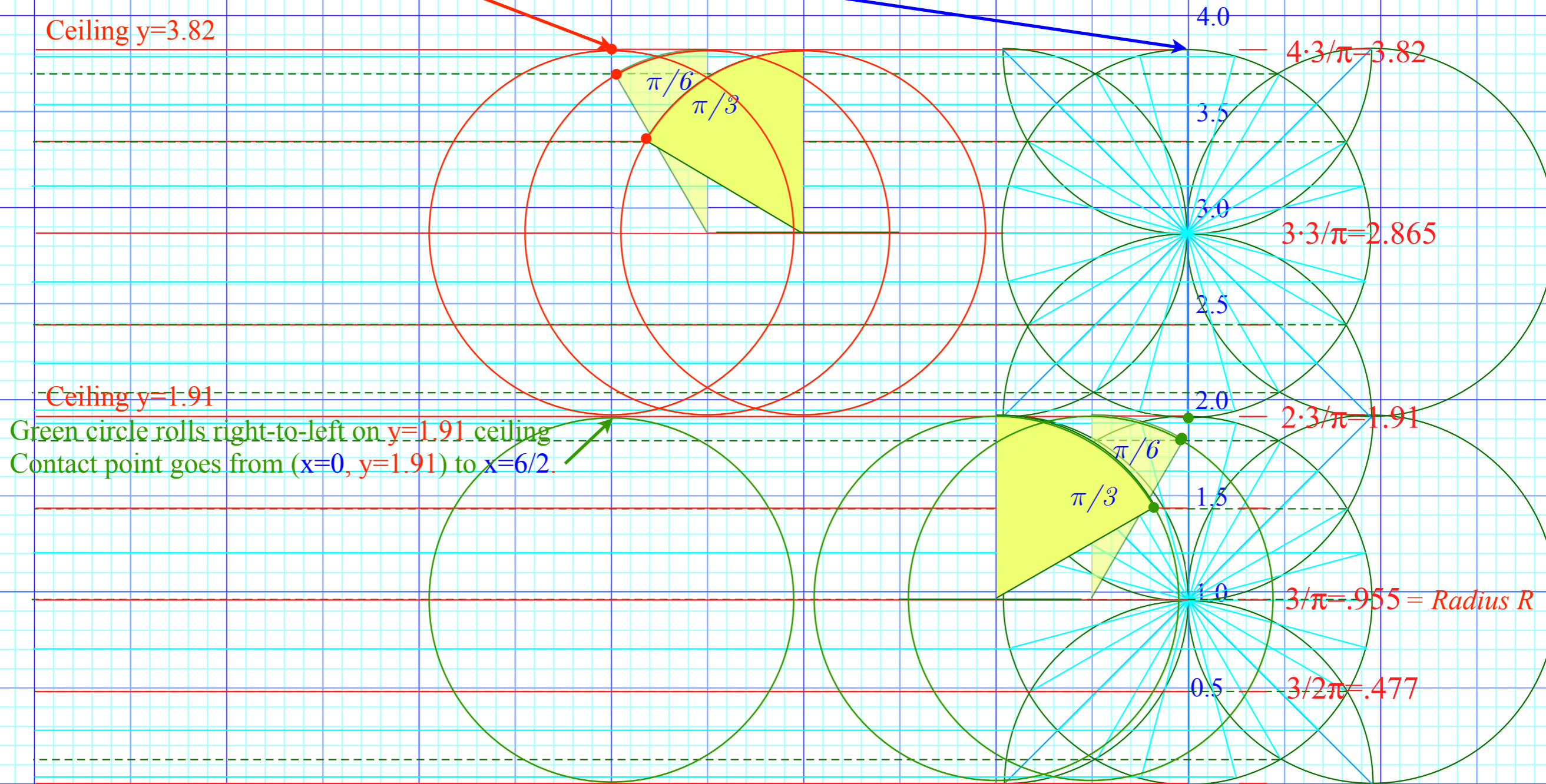
Red circle rolls left-to-right on $y=3.82$ ceiling
 Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.



2π	$11\pi/6$	$10\pi/6$	$9\pi/6$	$8\pi/6$	$7\pi/6$	π	$5\pi/6$	$2\pi/3$	$\pi/2$	$\pi/3$	$\pi/6$	Rotation angle ϕ
12	11	10	9	8	7	6	5	4	3	2	1	0 o'clock
$12/2$	$11/2$	$10/2$	$9/2$	$8/2$	$7/2$	$6/2$	$5/2$	$4/2$	$3/2$	$2/2$	$1/2$	Arc length $R\phi = (3/\pi)\phi$

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled-out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. Diameter is $2R=6/\pi=1.91$

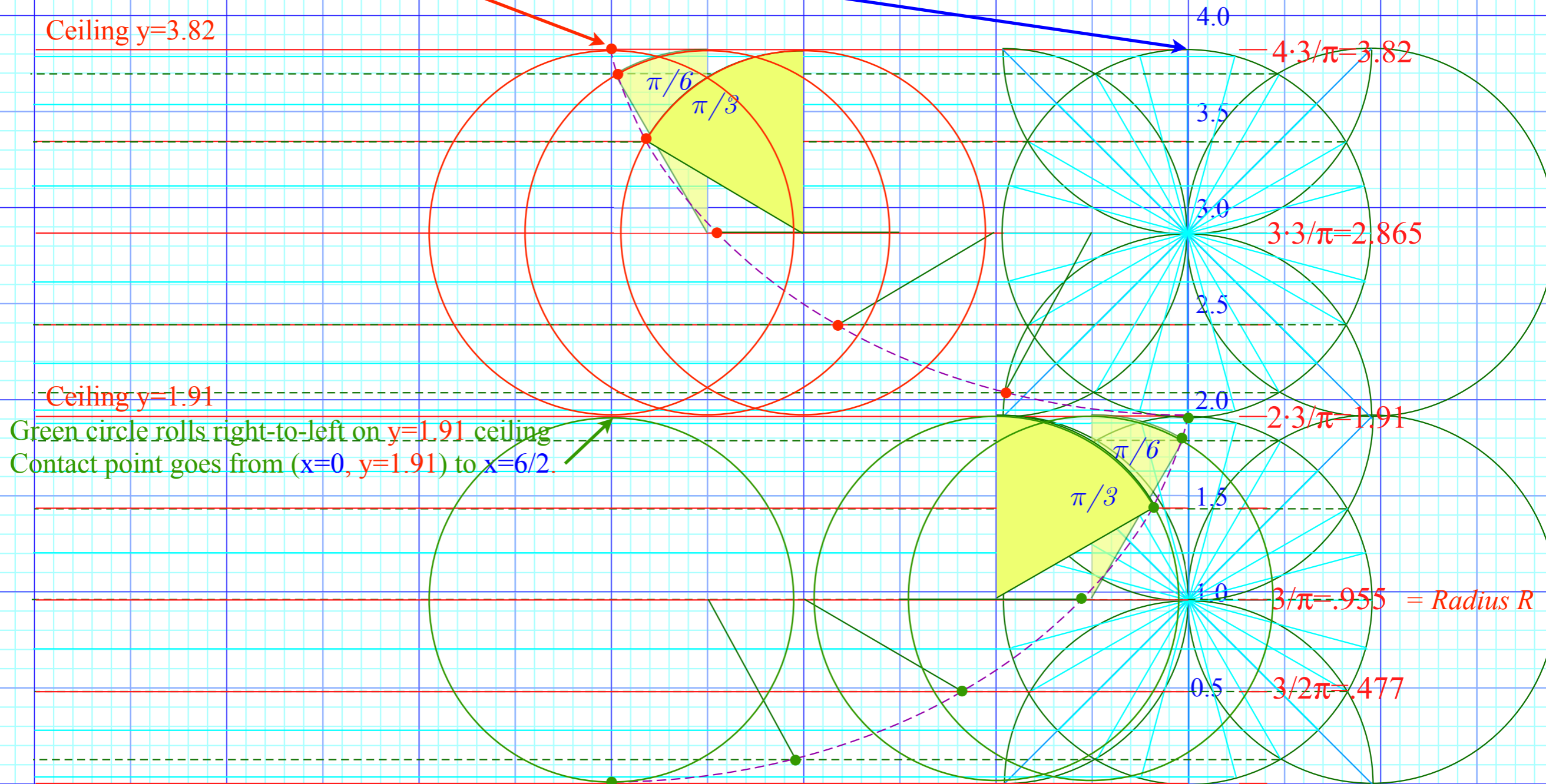
Red circle rolls left-to-right on $y=3.82$ ceiling
 Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.



2π	$11\pi/6$	$10\pi/6$	$9\pi/6$	$8\pi/6$	$7\pi/6$	π	$5\pi/6$	$2\pi/3$	$\pi/2$	$\pi/3$	$\pi/6$	Rotation angle ϕ
12	11	10	9	8	7	6	5	4	3	2	1	0 o'clock
$12/2$	$11/2$	$10/2$	$9/2$	$8/2$	$7/2$	$6/2$	$5/2$	$4/2$	$3/2$	$2/2$	$1/2$	Arc length $R\phi = (3/\pi)\phi$

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled-out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. Diameter is $2R=6/\pi=1.91$

Red circle rolls left-to-right on $y=3.82$ ceiling
 Contact point goes from $(x=6/2, y=3.82)$ to $x=0$.



2π	$11\pi/6$	$10\pi/6$	$9\pi/6$	$8\pi/6$	$7\pi/6$	π	$5\pi/6$	$2\pi/3$	$\pi/2$	$\pi/3$	$\pi/6$	Rotation angle ϕ
12	11	10	9	8	7	6	5	4	3	2	1	0 o'clock
$12/2$	$11/2$	$10/2$	$9/2$	$8/2$	$7/2$	$6/2$	$5/2$	$4/2$	$3/2$	$2/2$	$1/2$	Arc length $R\phi = (3/\pi)\phi$

