Lecture 17
Thur 10.20.2016

Reimann-Christoffel equations and covariant derivative (Ch. 4-7 of Unit 3)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

Christoffel g-derivative formula
What's a tensor? What's not?

General Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)
Christoffel relation to Coriolis coefficients
Mechanics of ideal fluid vortex

Separation of GCC Equations: Effective Potentials

Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations
2D Spherical pendulum "Bowl-Bowling" and the "I-Ball"
(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

Cycloidal ruler&compass geometry

(To be applied to mechanics in electromagnetic fields and collisional rotation in following lectures.)

→ *Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}{}^{k}$*

Christoffel g-derivative formula
What's a tensor? What's not?

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

GCC q^m derivatives of vectors $\mathbf{U}$ are due to:

(1) changing U^m components *(2)* curving GCC vectors $\mathbf{E_n}$.

$$\frac{\partial \mathbf{U}}{\partial q^i} = \frac{\partial}{\partial q^i}\left(U^j \mathbf{E}_j\right) = \frac{\partial U^m}{\partial q^i}\left(\mathbf{E}_m\right) + U^n \frac{\partial \mathbf{E}_n}{\partial q^i}$$

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(Note funny semi-colon ; notation)

Derivative of $\mathbf{E}_n$ is expressed using $\mathbf{E}^\ell$ or else $\mathbf{E}_m$

$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \Gamma_{in;\ell}\,\mathbf{E}^\ell$$

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Christoffel coefficients $\Gamma_{ij;k}$ of the first kind

defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

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i,n to *n,i*
symmetry
guaranteed here

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$$\frac{\partial \mathbf{E_n}}{\partial q^i} = \frac{\partial^2 \mathbf{r}}{\partial q^i \partial q^n} = \frac{\partial^2 \mathbf{r}}{\partial q^n \partial q^i} = \frac{\partial \mathbf{E_i}}{\partial q^n}$$

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Q: Do we need a third kind of Γ-coefficient or a Λ-coefficient?

$$\frac{\partial \mathbf{E}^n}{\partial q^i} = \Lambda^n_{im}\mathbf{E}^m \text{ , where: } \Lambda^n_{im} = \frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m$$

(to differentiate *contra*variant-$\mathbf{E}^n$ or *co*variant U_n)

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}^{k}$

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(to differentiate <u>*contra*</u>variant-$\mathbf{E}^n$ or <u>*co*</u>variant U_n)

A: NO! That Λ-coefficient is just a Γ-coefficient with a (-).

$$0 = \frac{\partial\left(\delta_m^n\right)}{\partial q^i} = \frac{\partial\left(\mathbf{E}^n \bullet \mathbf{E}_m\right)}{\partial q^i} = \frac{\partial \mathbf{E}^n}{\partial q^i}\bullet \mathbf{E}_m + \mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda_{im}^{n} = -\Gamma_{im}^{n}$

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$$\boxed{\Gamma_{in;}{}^{m}} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^{m}$$

Any vector derivative can be expressed using $\Gamma_{ij;}{}^{k}$ in terms of $\mathbf{E}_m$

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m}\right)\mathbf{E}_m$$

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$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}^{\ \ \ m}\right)\mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}^{\ \ \ n}\right)\mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m = -\mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda_{im}^{n} = -\Gamma_{im}^{n}$

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$$= \quad U^m_{\ ;i}\ \mathbf{E}_m \quad = \quad U_{m;i}\ \mathbf{E}^m$$

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So: $\Lambda^n_{im} = -\Gamma^n_{im}$

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$$= \quad U^m{}_{;i}\ \mathbf{E}_m \quad = \quad U_{m;i}\ \mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m = -\mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

Defining *covariant derivative $U^m{}_{;i}$*
of a contravariant component U^m

(Note more funny semi-colon ; notation)

$$U^m{}_{;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m}$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ *and* $\Gamma_{ij;}{}^{k}$

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Christoffel coefficients $\Gamma_{ij;k}$ *of the first kind* defined by:

$$\Gamma_{in;\ell} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}_\ell = \Gamma_{ni;\ell}$$

Christoffel coefficients $\Gamma_{ij;}{}^{k}$ *the second kind* defined by:

$$\Gamma_{in;}{}^{m} = \frac{\partial \mathbf{E}_n}{\partial q^i} \bullet \mathbf{E}^m = \Gamma_{ni;}{}^{m}$$

Any vector derivative can be expressed using $\Gamma_{ij;}{}^{k}$ in terms of $\mathbf{E}_m$ or $\mathbf{E}^m$

$$\frac{\partial \mathbf{U}}{\partial q^i} = \left(\frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m}\right)\mathbf{E}_m = \left(\frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^{n}\right)\mathbf{E}^m$$

$$= \quad U^m_{\ ;i}\ \mathbf{E}_m \quad = \quad U_{m;i}\ \mathbf{E}^m$$

$$\frac{\partial \mathbf{E}^n}{\partial q^i} \bullet \mathbf{E}_m = -\mathbf{E}^n \bullet \frac{\partial \mathbf{E}_m}{\partial q^i}$$

So: $\Lambda^n_{im} = -\Gamma^n_{im}$

Defining *covariant derivative* $U^m{}_{;i}$ *of a contravariant component* U^m

$$U^m_{\ ;i} = \frac{\partial U^m}{\partial q^i} + U^n \Gamma_{in;}{}^{m}$$

...and *covariant derivative* $U_{m;i}$ *of a covariant component* U_m

$$U_{m;i} = \frac{\partial U_m}{\partial q^i} - U_n \Gamma_{im;}{}^{n}$$

Intrinsic derivatives:

(Mathematicians being cute)

Defining *intrinsic derivative of contravariant vector components*.

$$\frac{\delta V^k}{\delta t}=\frac{dV^k}{dt}+\Gamma^k_{mn}V^m\dot{q}^n=\frac{\partial V^k}{\partial q^n}\dot{q}^n+\Gamma^k_{mn}V^m\dot{q}^n=V^k_{;n}\dot{q}^n$$

$$F_k=\frac{\delta p_k}{\delta t}$$

Defining *intrinsic derivative of covariant vector components*.

$$\frac{\delta V_k}{\delta t}=\frac{dV_k}{dt}-\Gamma^m_{kn}V_m\dot{q}^n=\frac{\partial V_k}{\partial q^n}\dot{q}^n-\Gamma^m_{kn}V_m\dot{q}^n=V_{k\,;\,n}\dot{q}^n$$

$$F^k=\frac{\delta p^k}{\delta t}$$

Tensor chain rules.

$\frac{\delta V^k}{\delta t}=V^k_{;n}\dot{q}^n$, replaces: $\frac{dV^k}{dt}=\frac{\partial V^k}{\partial q^n}\dot{q}^n$ where: $V^k_{;n}=\frac{\partial V^k}{\partial q^n}+\Gamma^k_{mn}V^m$

$\frac{\delta V_k}{\delta t}=V_{k\,;\,n}\dot{q}^n$, replaces: $\frac{dV_k}{dt}=\frac{\partial V_k}{\partial q^n}\dot{q}^n$ where: $V_{k\,;\,n}=\frac{\partial V_k}{\partial q^n}-\Gamma^m_{kn}V_m$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}{}^{k}$

→ *Christoffel g-derivative formula*

What's a tensor? What's not?

Christoffel g-derivative formula

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\quad\Gamma_{im;n}\quad+\quad\Gamma_{in;m}$$

Christoffel g-derivative formula

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$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\Gamma_{in;m}$$

$$\frac{\partial g_{mi}}{\partial q^n}=\Gamma_{nm;i}+\Gamma_{in;m}\qquad\text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\Gamma_{mn;i}\qquad\text{(switched i}\leftrightarrow\text{m)}$$

Christoffel g-derivative formula

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

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$$-\frac{\partial g_{mi}}{\partial q^n}=-\Gamma_{nm;i}-\Gamma_{in;m}\qquad\text{(switched i} \leftrightarrow \text{n)}$$

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Christoffel g-derivative formula

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$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\cancel{\Gamma_{in;m}}$$

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$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\cancel{\Gamma_{mn;i}}\qquad\text{(switched i}\leftrightarrow\text{m)}$$

Gives the Christoffel formula

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

Covariant derivative and Christoffel Coefficients $\Gamma_{ij;k}$ and $\Gamma_{ij;}{}^{k}$

Christoffel g-derivative formula

→ *What's a tensor? What's not?*

What's a tensor? What's not?

$$\frac{\partial(\mathbf{E_m} \bullet \mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i} \bullet \mathbf{E_n} + \mathbf{E_m} \bullet \frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} \cancel{+ \Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n} = \cancel{-\Gamma_{nm;i}} \cancel{- \Gamma_{in;m}} \qquad \text{(switched i} \leftrightarrow \text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} \cancel{+ \Gamma_{mn;i}} \qquad \text{(switched i} \leftrightarrow \text{m)}$$

Gives the Christoffel formula

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n}\right)}$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar m}{}_{;\bar n} = \frac{\partial \bar{\mathbf{U}}}{\partial \bar q^{\bar n}} \bullet \bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ of covariant derivative $U^m{}_{;n} = \frac{\partial \mathbf{U}}{\partial q^n} \bullet \mathbf{E_m}$

$$\bar{U}^{\bar m}{}_{;\bar n} = \frac{\partial \bar{\mathbf{U}}}{\partial \bar q^{\bar n}} \bullet \bar{\mathbf{E}}^{\bar m} = \frac{}{\partial \bar q^{\bar n}}\frac{\partial \bar{\mathbf{U}}}{} \bullet \bar{\mathbf{E}}^{\bar m} = \frac{\partial q^n}{\partial \bar q^{\bar n}}\frac{\partial \bar{\mathbf{U}}}{\partial q^n} \bullet \bar{\mathbf{E}}_{\bar m} = \frac{\partial q^n}{\partial \bar q^{\bar n}}\frac{\partial \bar{\mathbf{U}}}{\partial q^n} \bullet \frac{\partial q^m}{\partial \bar q^{\bar m}}\mathbf{E}_m$$

What's a tensor? What's not?

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

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But, still need to write $\frac{\partial\bar{U}^{\bar{m}}}{\partial q^n}$ in terms of $\frac{\partial U^m}{\partial q^n}$.

What's a tensor? What's not?

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n} + \mathbf{E_m}\bullet\frac{\partial \mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i} = \Gamma_{im;n} + \cancel{\Gamma_{in;m}}$$

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$$\frac{\partial g_{in}}{\partial q^m} = \Gamma_{im;n} + \cancel{\Gamma_{mn;i}} \quad \text{(switched i} \leftrightarrow \text{m)}$$

Gives the Christoffel formula

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{im}}{\partial q^n}\right)}$$

Chain-saw-sums transform a "bar-frame" view $\bar U^{\bar m}{}_{;\bar n} = \frac{\partial \bar{\mathbf{U}}}{\partial \bar q^{\bar n}}\bullet \bar{\mathbf{E}}^{\bar m}$ of covariant derivative $U^m{}_{;n} = \frac{\partial \mathbf{U}}{\partial q^n}\bullet \mathbf{E_m}$

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What's a tensor? What's not?

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$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

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1st term is OK, but 2nd term is zero
only if Jacobian is constant matrix!

What's a tensor? What's not?

$$\frac{\partial\left(\mathbf{E_m}\bullet\mathbf{E_n}\right)}{\partial q^i} = \frac{\partial \mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n} + \mathbf{E_m}\bullet\frac{\partial \mathbf{E_n}}{\partial q^i}$$

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1[st] term is OK, but 2[nd] term is zero only if Jacobian is constant matrix!

$\frac{\partial \bar U^{\bar m}}{\partial \bar q^{\bar n}} = \frac{\partial \bar q^{\bar m}}{\partial q^m}\frac{\partial q^n}{\partial \bar q^{\bar n}}\frac{\partial U^m}{\partial q^n}$ holds if and only if $\frac{\partial}{\partial q^n}\left(\frac{\partial \bar q^{\bar m}}{\partial q^m}\right) \equiv 0$

What's a tensor? What's not?

$$\frac{\partial(\mathbf{E_m}\bullet\mathbf{E_n})}{\partial q^i}=\frac{\partial\mathbf{E_m}}{\partial q^i}\bullet\mathbf{E_n}+\mathbf{E_m}\bullet\frac{\partial\mathbf{E_n}}{\partial q^i}$$

$$\frac{\partial g_{mn}}{\partial q^i}=\Gamma_{im;n}+\cancel{\Gamma_{in;m}}$$

$$-\frac{\partial g_{mi}}{\partial q^n}=-\cancel{\Gamma_{nm;i}}-\cancel{\Gamma_{in;m}}\qquad \text{(switched i}\leftrightarrow\text{n)}$$

$$\frac{\partial g_{in}}{\partial q^m}=\Gamma_{im;n}+\cancel{\Gamma_{mn;i}}\qquad \text{(switched i}\leftrightarrow\text{m)}$$

Gives the Christoffel formula

$$\boxed{\Gamma_{im;n}=\frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i}+\frac{\partial g_{in}}{\partial q^m}-\frac{\partial g_{im}}{\partial q^n}\right)}$$

Chain-saw-sums transform a "bar-frame" view $\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\bar{\mathbf{U}}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ of covariant derivative $U^m{}_{;n}=\frac{\partial\mathbf{U}}{\partial q^n}\bullet\mathbf{E_m}$

$$\bar{U}^{\bar{m}}{}_{;\bar{n}}=\frac{\partial\bar{\mathbf{U}}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}}=\frac{\partial\bar{\mathbf{U}}}{\partial\bar{q}^{\bar{n}}}\bullet\bar{\mathbf{E}}^{\bar{m}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\bar{\mathbf{U}}}{\partial q^n}\bullet\bar{\mathbf{E}}_{\bar{m}}=\frac{\partial q^n}{\partial\bar{q}^{\bar{n}}}\frac{\partial\bar{\mathbf{U}}}{\partial q^n}\bullet\frac{\partial q^m}{\partial\bar{q}^{\bar{m}}}\mathbf{E}_m$$

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Otherwise, $U^m{}_{,n}$ needs "correction" $U^\ell\Gamma^m_{n\ell;}$. And, that $U^\ell\Gamma^m_{n\ell;}$ <u>*cannot*</u> be a $T^m{}_n$-tensor either!

General Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Christoffel relation to Coriolis coefficients

Mechanics of ideal fluid vortex

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $$T = \frac{1}{2}M_{jk}\dot{x}^j\dot{x}^k$$ to GCC $$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are <u>*zero*</u>

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Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2} M_{jk} \left(\frac{\partial x^j}{\partial q^m} \dot{q}^m + \cancel{\left\{ \frac{\partial x^j}{\partial t} \right\}} \right) \left(\frac{\partial x^k}{\partial q^n} \dot{q}^n + \cancel{\left\{ \frac{\partial x^k}{\partial t} \right\}} \right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

$$F_\ell = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2} \frac{d}{dt} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial \dot{q}^\ell} - \frac{1}{2} \frac{\partial \left(\gamma_{mn} \dot{q}^m \dot{q}^n \right)}{\partial q^\ell}$$

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1st term involves *covariant momentum* p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

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All explicit-t-dependent terms are *zero*
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Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

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Lagrange equations for fixed GCC convert to tensor form

$$T=\frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m+\cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n+\cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

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$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

1st term involves *covariant momentum* p_ℓ.

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Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell \gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

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Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

$$F_\ell = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^\ell} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$

Converts Cartesian kinetic energy $T=\frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T=\frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m+\left\{\cancel{\frac{\partial x^j}{\partial t}}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n+\left\{\cancel{\frac{\partial x^k}{\partial t}}\right\}\right)$$

All explicit-t-dependent terms are zero
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$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{d\gamma_{\ell n}}{dt} - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial\gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are <u>zero</u>
(Time must be included as a <u>dimension</u>)

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

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Canonical Lagrange equations valid for *<u>all</u>* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for <u>fixed</u> GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

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$$\boxed{\frac{d\gamma_{\ell n}}{dt}} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

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Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\cancel{\frac{\partial x^j}{\partial t}}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\cancel{\frac{\partial x^k}{\partial t}}\right\}\right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

1st term involves ***covariant momentum*** p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell\gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\boxed{\frac{d\gamma_{\ell n}}{dt}} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{\partial \gamma_{\ell n}}{\partial q^m}}\dot{q}^m - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Rearrange to expose
Christoffel coefficients (from p 22):

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \left\{\cancel{\frac{\partial x^j}{\partial t}}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \left\{\cancel{\frac{\partial x^k}{\partial t}}\right\}\right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

1st term involves ***covariant momentum*** p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

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Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

Lagrange Force Equation
The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial\gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{\partial\gamma_{\ell n}}{\partial q^m}\dot{q}^m - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Rearrange to expose
Christoffel coefficients (from p 22):

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial\gamma_{n\ell}}{\partial q^m} + \frac{\partial\gamma_{\ell n}}{\partial q^m} - \frac{\partial\gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial\gamma_{n\ell}}{\partial q^m} + \frac{\partial\gamma_{\ell m}}{\partial q^n} - \frac{\partial\gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T=\frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$T=\frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m+\left\{\cancel{\frac{\partial x^j}{\partial t}}\right\}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n+\left\{\cancel{\frac{\partial x^k}{\partial t}}\right\}\right)$$

All explicit-t-dependent terms are <u>zero</u>
(Time must be included as a <u>dimension</u>)

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

1st term involves ***covariant momentum*** p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse ***contravariant kinetic metric*** γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell\gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *<u>all</u>* GCC, fixed or explicit in time *t:*

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

Following is for <u>fixed</u> GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\frac{d\gamma_{\ell n}}{dt} = \frac{\partial\gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\frac{\partial\gamma_{\ell n}}{\partial q^m}\dot{q}^m - \frac{1}{2}\frac{\partial\gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Rearrange to expose
Christoffel coefficients (from p 22):

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial\gamma_{n\ell}}{\partial q^m} + \frac{\partial\gamma_{\ell n}}{\partial q^m} - \frac{\partial\gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial\gamma_{n\ell}}{\partial q^m} + \frac{\partial\gamma_{\ell m}}{\partial q^n} - \frac{\partial\gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

This gives ***covariant Riemann equations***

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\,\dot{q}^m\dot{q}^n$$

Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Kinetic metric γ_{mn} is a covariant tensor transform of an original Cartesian inertia tensor M_{ij}

$$\gamma_{mn} = M_{jk}\frac{\partial x^j}{\partial q^m}\frac{\partial x^k}{\partial q^n}$$ Converts Cartesian kinetic energy $T = \frac{1}{2}M_{jk}\dot{x}^j\dot{q}^k$ to GCC $T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$

Lagrange equations for fixed GCC convert to tensor form

$$T = \frac{1}{2}M_{jk}\left(\frac{\partial x^j}{\partial q^m}\dot{q}^m + \cancel{\left\{\frac{\partial x^j}{\partial t}\right\}}\right)\left(\frac{\partial x^k}{\partial q^n}\dot{q}^n + \cancel{\left\{\frac{\partial x^k}{\partial t}\right\}}\right)$$

All explicit-t-dependent terms are zero
(Time must be included as a dimension)

$$F_\ell = \frac{d}{dt}\boxed{\frac{\partial T}{\partial \dot{q}^\ell}} - \frac{\partial T}{\partial q^\ell} = \frac{1}{2}\frac{d}{dt}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} - \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial q^\ell}$$

1st term involves *covariant momentum* p_ℓ.

$$p_\ell \equiv \frac{\partial T}{\partial \dot{q}^\ell} = \frac{1}{2}\frac{\partial\left(\gamma_{mn}\dot{q}^m\dot{q}^n\right)}{\partial \dot{q}^\ell} = \gamma_{\ell n}\dot{q}^n$$

Inverse *contravariant kinetic metric* γ^{mn} gives velocity $\dot{q}^n$

$$\dot{q}^n = p_\ell\gamma^{\ell n} \equiv p^n$$

Canonical Lagrange equations valid for *all* GCC, fixed or explicit in time *t*:

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}}$$

The "4-wheel-drive garbage truck"

Following is for fixed GCC only:

$$F_\ell = \frac{d}{dt}\left(\gamma_{\ell n}\dot{q}^n\right) - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{d\gamma_{\ell n}}{dt}} - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Time derivative of kinetic metric is expanded by chain rule.

$$\boxed{\frac{d\gamma_{\ell n}}{dt}} = \frac{\partial \gamma_{\ell n}}{\partial q^m}\dot{q}^m$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \dot{q}^n\boxed{\frac{\partial \gamma_{\ell n}}{\partial q^m}}\dot{q}^m - \frac{1}{2}\frac{\partial \gamma_{mn}}{\partial q^\ell}\dot{q}^m\dot{q}^n$$

Rearrange to expose Christoffel coefficients:

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\boxed{\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell n}}{\partial q^m}} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \frac{1}{2}\left[\frac{\partial \gamma_{n\ell}}{\partial q^m} + \frac{\partial \gamma_{\ell m}}{\partial q^n} - \frac{\partial \gamma_{mn}}{\partial q^\ell}\right]\dot{q}^m\dot{q}^n$$

This gives *covariant Riemann equations* and *contravariant Riemann equations*.

$$F_\ell = \gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n$$

$$F^k = \ddot{q}^k + \Gamma^k_{mn}\dot{q}^m\dot{q}^n$$

General Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

→ *Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)*

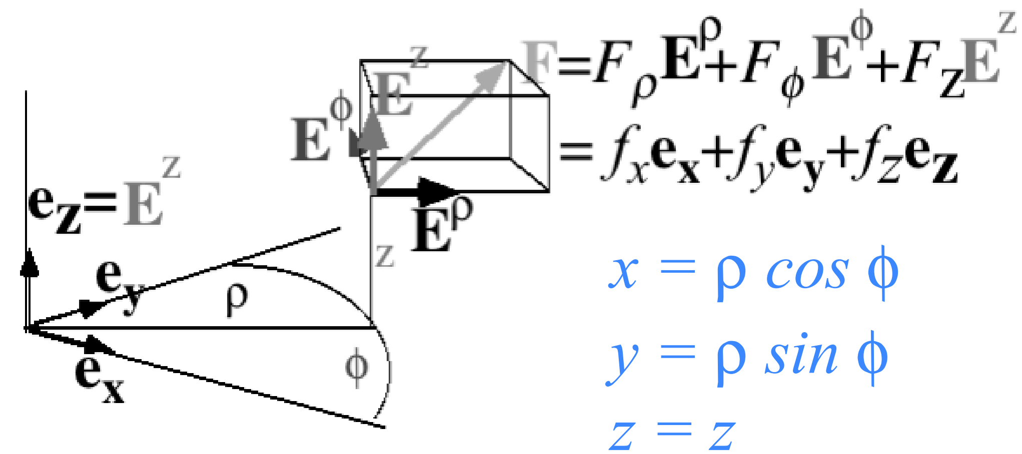
Christoffel relation to Coriolis coefficients

Mechanics of ideal fluid vortex

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$
\langle J \rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad
\begin{array}{c} \langle K \rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array}
$$

$$
\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array}
\qquad = \langle J^{-1} \rangle
$$



$x = \rho \cos \phi$

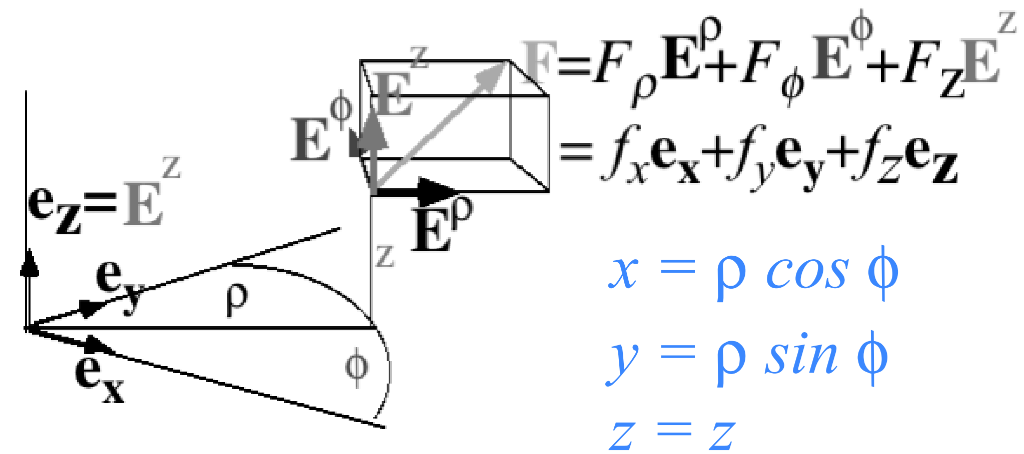
$y = \rho \sin \phi$

$z = z$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix}, \quad \langle K\rangle = \begin{array}{cl} \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} & \begin{array}{l} \leftarrow \mathbf{E}^\rho \\[2em] \leftarrow \mathbf{E}^\phi \\[2em] \leftarrow \mathbf{E}^z \end{array} \end{array}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \qquad\qquad = \langle J^{-1}\rangle$$



$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

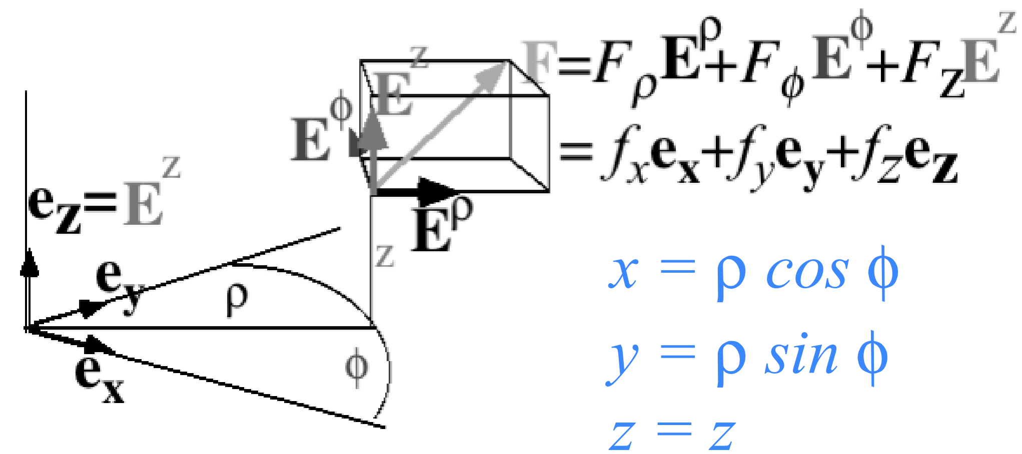
$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho \cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K \rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} & \begin{array}{c} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array}$$

$$= \langle J^{-1} \rangle$$



$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho \cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

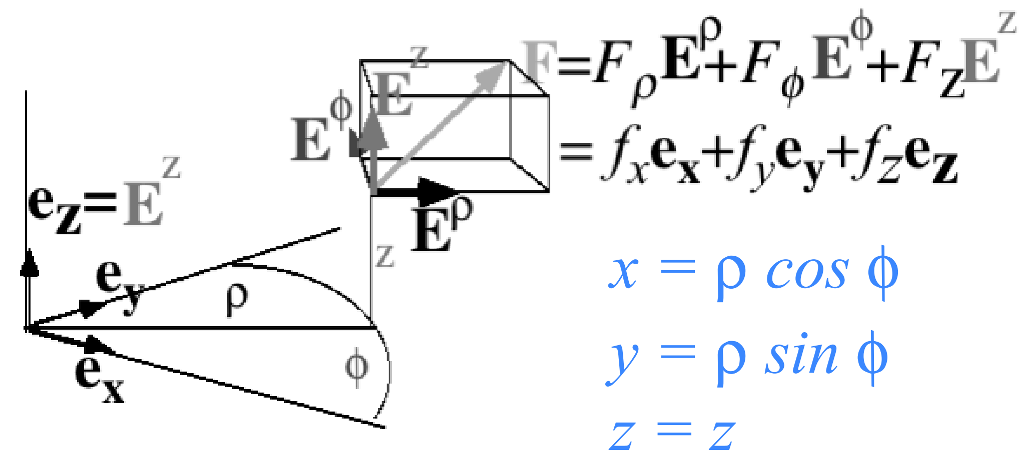
$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \bullet \mathbf{E}_z = m$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} & \begin{array}{c} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array}$$

$$= \langle J^{-1}\rangle$$



$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial \rho} + f_y\frac{\partial y}{\partial \rho} + f_z\frac{\partial z}{\partial \rho} = \quad f_x\cos\phi + f_y\sin\phi \quad + 0$$

$$F_\phi = f_x\frac{\partial x}{\partial \phi} + f_y\frac{\partial y}{\partial \phi} + f_z\frac{\partial z}{\partial \phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z} + f_y\frac{\partial y}{\partial z} + f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

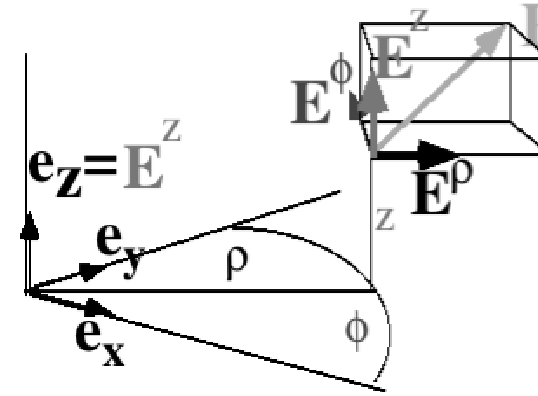
$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$
\langle J \rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho \sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array} , \quad \langle K \rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} & \begin{array}{c} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array} \end{array}
$$

$$= \langle J^{-1} \rangle$$



$$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_z \mathbf{E}^z$$
$$= f_x \mathbf{e_x} + f_y \mathbf{e_y} + f_z \mathbf{e_z}$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m \mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m \mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m \mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

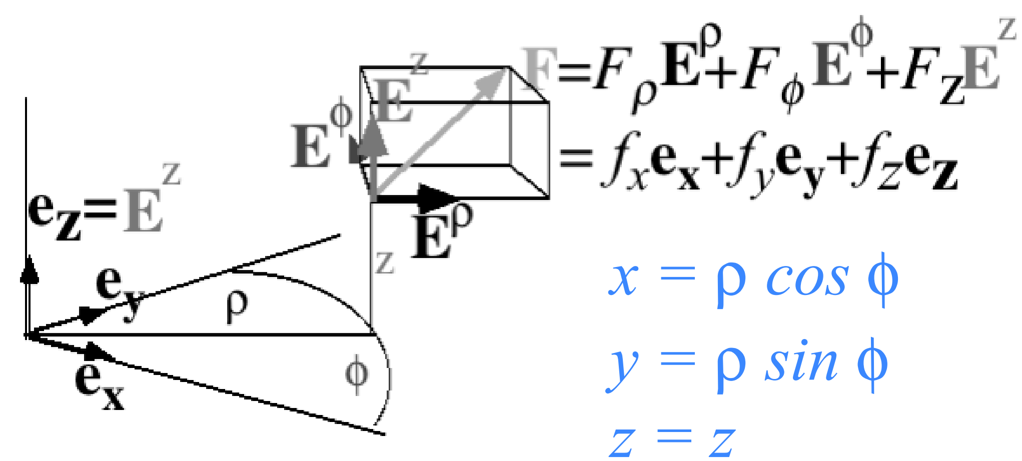
Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m \dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J \rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho} = \cos\phi & \frac{\partial x}{\partial \phi} = -\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho} = \sin\phi & \frac{\partial y}{\partial \phi} = \rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \end{array} , \quad \langle K \rangle = \begin{array}{cc} \begin{pmatrix} \frac{\partial \rho}{\partial x} = \cos\phi & \frac{\partial \rho}{\partial y} = \sin\phi & 0 \\ \frac{\partial \phi}{\partial x} = \frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z} = 1 \end{pmatrix} & \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix} \end{array}$$

$$= \langle J^{-1} \rangle$$



$x = \rho \cos \phi$

$y = \rho \sin \phi$

$z = z$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad + 0$$

$$F_\phi = f_x \frac{\partial x}{\partial \phi} + f_y \frac{\partial y}{\partial \phi} + f_z \frac{\partial z}{\partial \phi} = -f_x \rho \sin\phi + f_y \rho\cos\phi + 0$$

$$F_z = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m \frac{\partial x_j}{\partial \rho} \frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m \frac{\partial x_j}{\partial \phi} \frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m \frac{\partial x_j}{\partial z} \frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z}$$

$$= m\dot{\rho} \qquad\qquad = m\rho^2\dot{\phi} \qquad\qquad = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

General Riemann equations of motion *(No explicit t-dependence and fixed GCC)*

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

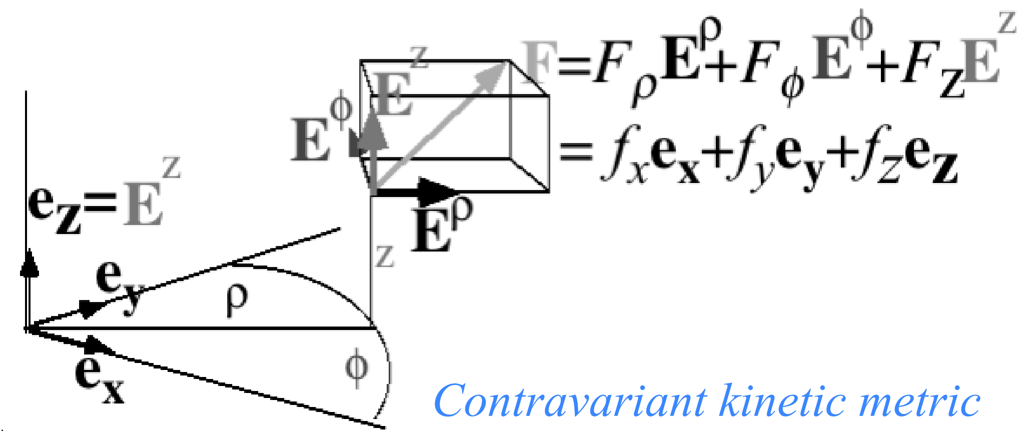
→ *Christoffel relation to Coriolis coefficients*

Mechanics of ideal fluid vortex

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \end{array}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{matrix} = \langle J^{-1}\rangle$$

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial \rho}+f_y\frac{\partial y}{\partial \rho}+f_z\frac{\partial z}{\partial \rho} = \quad f_x\cos\phi + f_y\sin\phi \quad +0$$

$$F_\phi = f_x\frac{\partial x}{\partial \phi}+f_y\frac{\partial y}{\partial \phi}+f_z\frac{\partial z}{\partial \phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial \rho}\frac{\partial x^j}{\partial \rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial \phi}\frac{\partial x^j}{\partial \phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial \dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial \dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann *co*variant *force* equations

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}} = \boxed{\gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

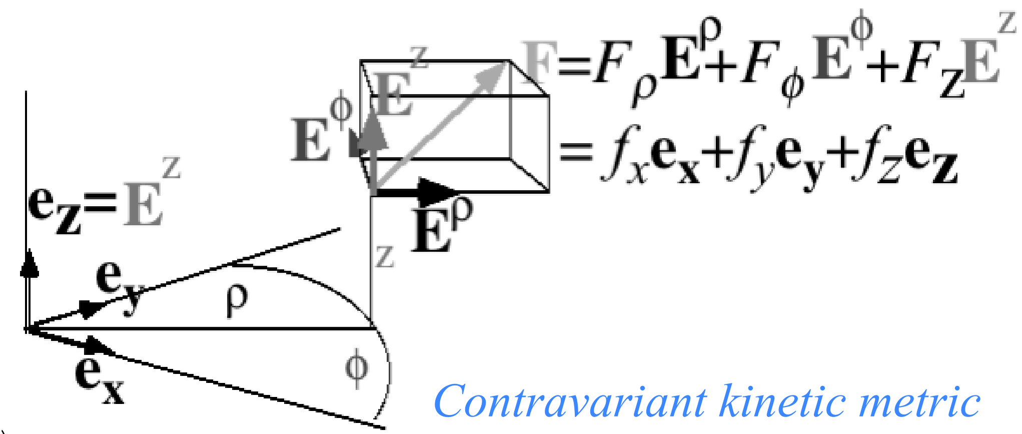
$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{array}{c} \begin{pmatrix} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \qquad\qquad =\langle J^{-1}\rangle$$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$



Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho} + f_y\frac{\partial y}{\partial\rho} + f_z\frac{\partial z}{\partial\rho} = f_x\cos\phi + f_y\sin\phi + 0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi} + f_y\frac{\partial y}{\partial\phi} + f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z} + f_y\frac{\partial y}{\partial z} + f_z\frac{\partial z}{\partial z} = 0 + 0 + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho \bullet \mathbf{E}_\rho = m\left(\cos^2\phi + \sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi \bullet \mathbf{E}_\phi = m\left(\rho^2\cos^2\phi + \rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z \bullet \mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann *co*variant *force* equations

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}} = \boxed{\gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n}$$

Note: This a much more efficient way to derive Γ-coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial\rho} = \gamma_{\rho\rho}\ddot{\rho} + \Gamma_{mn;\rho}\dot{q}^m\dot{q}^n$$

$$F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial\phi} = \gamma_{\phi\phi}\ddot{\phi} + \Gamma_{mn;\phi}\dot{q}^m\dot{q}^n$$

Christoffel g-formula (from p. 22 and pp. 46-49):

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{array}{l} \leftarrow \mathbf{E}^\rho \\ \leftarrow \mathbf{E}^\phi \\ \leftarrow \mathbf{E}^z \end{array}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \qquad = \langle J^{-1}\rangle$$

$x = \rho \cos\phi$

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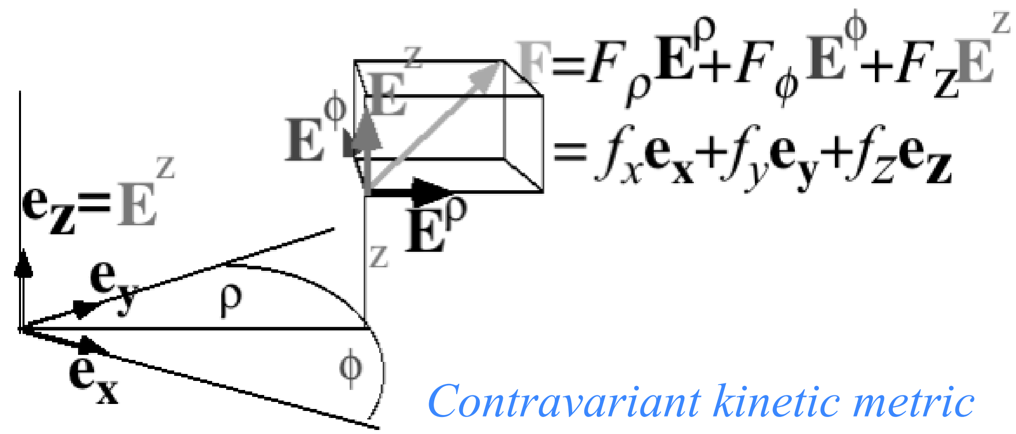

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho} + f_y\frac{\partial y}{\partial\rho} + f_z\frac{\partial z}{\partial\rho} = f_x\cos\phi + f_y\sin\phi + 0$$

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Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing **Lagrange** and the Riemann *covariant force* equations

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}} = \boxed{\gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n}$$

Note: This a much more efficient way to derive Γ-coefficients than the g-formula.

Only three non-zero Christoffel coefficients appear, and only two are independent.

$$F_\rho = \frac{dp_\rho}{dt} - \frac{\partial T}{\partial\rho} = \gamma_{\rho\rho}\ddot{\rho} + \Gamma_{mn;\rho}\dot{q}^m\dot{q}^n$$

$$= \frac{d\left(m\dot{\rho}\right)}{dt} - \frac{\partial}{\partial\rho}\left(\frac{1}{2}m\rho^2\dot{\phi}^2\right) = m\ddot{\rho} - m\rho\dot{\phi}^2 \quad \text{so:}\ \Gamma_{\phi\phi;\rho} = -m\rho$$

$$F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial\phi} = \gamma_{\phi\phi}\ddot{\phi} + \Gamma_{mn;\phi}\dot{q}^m\dot{q}^n$$

Christoffel g-formula
(from p. 22 and pp. 46-49):

$$\boxed{\Gamma_{im;n} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial q^i} + \frac{\partial g_{in}}{\partial q^m} - \frac{\partial g_{mi}}{\partial q^n}\right)}$$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{array}{c} \left(\begin{array}{ccc} \frac{\partial x}{\partial\rho}=\cos\phi & \frac{\partial x}{\partial\phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial\rho}=\sin\phi & \frac{\partial y}{\partial\phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{array} \end{array}, \quad \langle K\rangle = \begin{array}{cc} \left(\begin{array}{ccc} \frac{\partial\rho}{\partial x}=\cos\phi & \frac{\partial\rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial\phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{array}\right) & \begin{array}{c} \leftarrow\mathbf{E}^\rho \\ \leftarrow\mathbf{E}^\phi \\ \leftarrow\mathbf{E}^z \end{array} \end{array} = \langle J^{-1}\rangle$$

$x = \rho\cos\phi$

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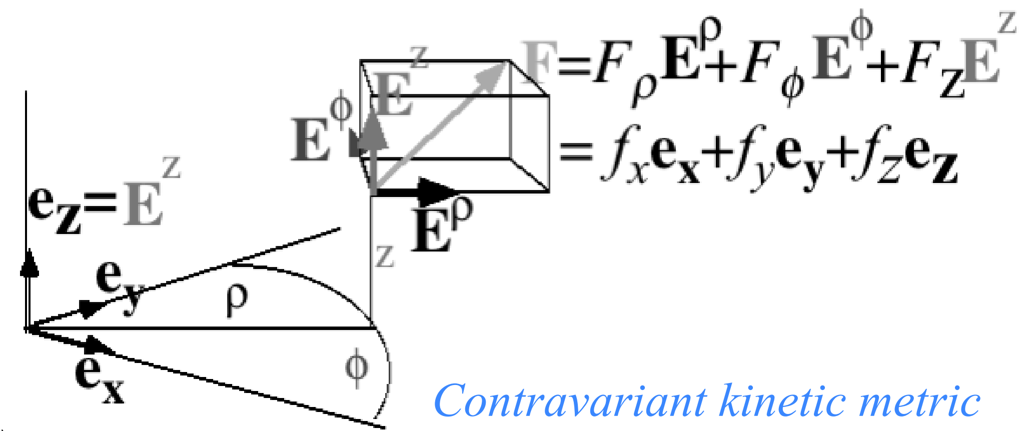

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho}+f_y\frac{\partial y}{\partial\rho}+f_z\frac{\partial z}{\partial\rho} = f_x\cos\phi + f_y\sin\phi \;+0$$

$$F_\phi = f_x\frac{\partial x}{\partial\phi}+f_y\frac{\partial y}{\partial\phi}+f_z\frac{\partial z}{\partial\phi} = -f_x\rho\sin\phi + f_y\rho\cos\phi + 0$$

$$F_z = f_x\frac{\partial x}{\partial z}+f_y\frac{\partial y}{\partial z}+f_z\frac{\partial z}{\partial z} = \quad 0 \quad + \quad 0 \quad + f_z$$

Covariant kinetic metric

$$\gamma_{\rho\rho} = m\frac{\partial x_j}{\partial\rho}\frac{\partial x^j}{\partial\rho} = m\mathbf{E}_\rho\bullet\mathbf{E}_\rho = m\left(\cos^2\phi+\sin^2\phi\right) = m$$

$$\gamma_{\phi\phi} = m\frac{\partial x_j}{\partial\phi}\frac{\partial x^j}{\partial\phi} = m\mathbf{E}_\phi\bullet\mathbf{E}_\phi = m\left(\rho^2\cos^2\phi+\rho^2\sin^2\phi\right) = m\rho^2$$

$$\gamma_{zz} = m\frac{\partial x_j}{\partial z}\frac{\partial x^j}{\partial z} = m\mathbf{E}_z\bullet\mathbf{E}_z = m$$

Contravariant kinetic metric

$$\gamma^{\rho\rho} = 1/m$$

$$\gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\gamma^{zz} = 1/m$$

Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

Covariant momenta

$$p_\rho = \frac{\partial T}{\partial\dot{\rho}} = \gamma_{\rho\rho}\dot{\rho} = m\dot{\rho} \qquad p_\phi = \frac{\partial T}{\partial\dot{\phi}} = \gamma_{\phi\phi}\dot{\phi} = m\rho^2\dot{\phi} \qquad p_z = \frac{\partial T}{\partial\dot{z}} = \gamma_{zz}\dot{z} = m\dot{z}$$

Contravariant momenta

$$p^\rho = \dot{\rho}$$

$$p^\phi = \dot{\phi}$$

$$p^z = \dot{z}$$

Comparing Lagrange and the Riemann covariant force equations

$$\boxed{F_\ell = \frac{dp_\ell}{dt} - \frac{\partial T}{\partial q^\ell}} = \boxed{\gamma_{\ell n}\ddot{q}^n + \Gamma_{mn;\ell}\dot{q}^m\dot{q}^n}$$

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$$= \frac{d\left(m\dot{\rho}\right)}{dt} - \frac{\partial}{\partial\rho}\left(\frac{1}{2}m\rho^2\dot{\phi}^2\right) = m\ddot{\rho} - m\rho\dot{\phi}^2 \quad \text{so: } \Gamma_{\phi\phi;\rho} = -m\rho$$

$$F_\phi = \frac{dp_\phi}{dt} - \frac{\partial T}{\partial\phi} = \gamma_{\phi\phi}\ddot{\phi} + \Gamma_{mn;\phi}\dot{q}^m\dot{q}^n$$

$$= \frac{d\left(m\rho^2\dot{\phi}\right)}{dt} - 0 = m\rho^2\ddot{\phi} + 2m\rho\dot{\rho}\dot{\phi} \quad \text{so: } \Gamma_{\rho\phi;\phi} = m\rho = \Gamma_{\phi\rho;\phi}$$

Note: $\Gamma_{pq;r} = \Gamma_{qp;r}$ symmetry gives 2 factor for $q \neq p$

Example of Riemann-Christoffel forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

$$\langle J\rangle = \begin{pmatrix} \frac{\partial x}{\partial \rho}=\cos\phi & \frac{\partial x}{\partial \phi}=-\rho\sin\phi & 0 \\ \frac{\partial y}{\partial \rho}=\sin\phi & \frac{\partial y}{\partial \phi}=\rho\cos\phi & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix}, \quad \langle K\rangle = \begin{pmatrix} \frac{\partial \rho}{\partial x}=\cos\phi & \frac{\partial \rho}{\partial y}=\sin\phi & 0 \\ \frac{\partial \phi}{\partial x}=\frac{-\sin\phi}{\rho} & \frac{\partial \phi}{\partial y}=\frac{\cos\phi}{\rho} & 0 \\ 0 & 0 & \frac{\partial z}{\partial z}=1 \end{pmatrix} \begin{matrix} \leftarrow \mathbf{E}^{\rho} \\ \leftarrow \mathbf{E}^{\phi} \\ \leftarrow \mathbf{E}^{z} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{E}_\rho & \mathbf{E}_\phi & \mathbf{E}_z \end{matrix} \qquad = \langle J^{-1}\rangle$$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

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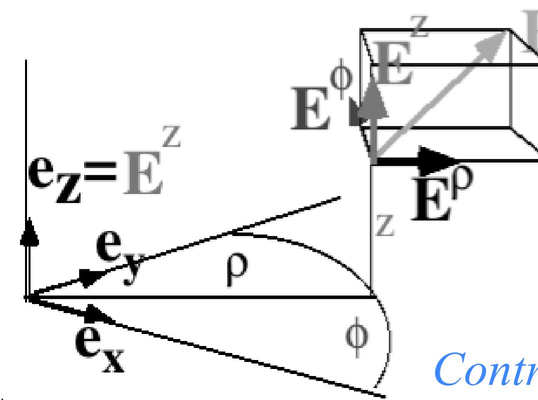

$$\mathbf{F} = F_\rho \mathbf{E}^\rho + F_\phi \mathbf{E}^\phi + F_Z \mathbf{E}^Z = f_x \mathbf{e_x} + f_y \mathbf{e_y} + f_Z \mathbf{e_Z}$$

Covariant forces

$$F_\rho = f_x \frac{\partial x}{\partial \rho} + f_y \frac{\partial y}{\partial \rho} + f_z \frac{\partial z}{\partial \rho} = \quad f_x \cos\phi + f_y \sin\phi \quad +0$$

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Contravariant kinetic metric

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Lagrangian

$$T = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2$$

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Comparing Lagrange and the Riemann *co*variant *force* equations

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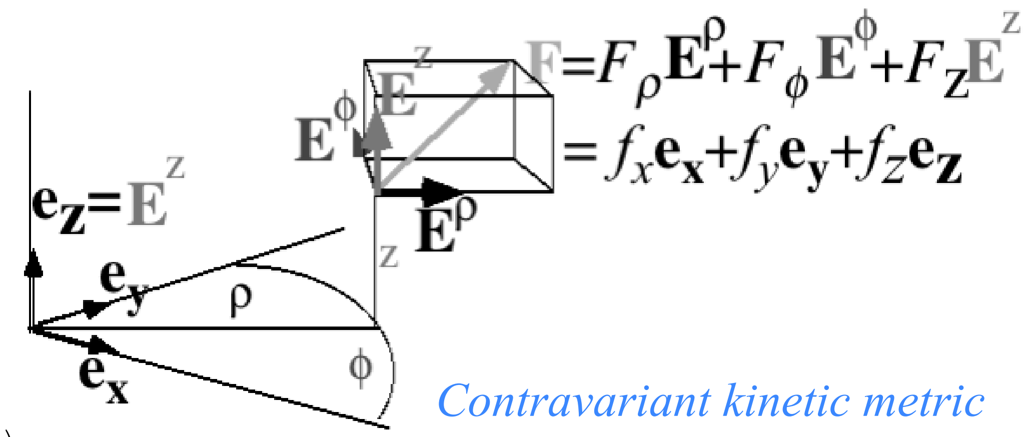

Covariant forces

$$F_\rho = f_x\frac{\partial x}{\partial\rho} + f_y\frac{\partial y}{\partial\rho} + f_z\frac{\partial z}{\partial\rho} = \quad f_x\cos\phi + f_y\sin\phi \ +0$$

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Lagrangian

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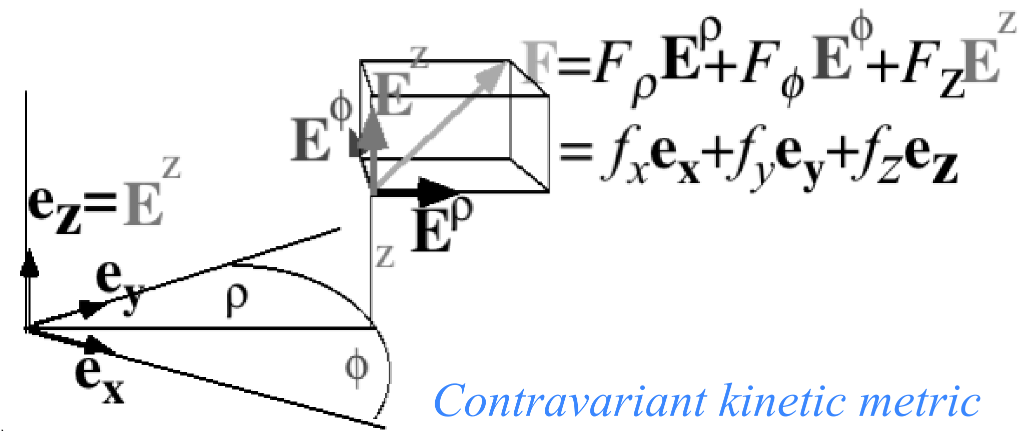

Covariant forces

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Contravariant momenta

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Comparing Lagrange and the Riemann *covariant force* equations

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$$\ddot\rho = F^\rho + \rho\dot\phi^2 \qquad \textit{(Centrifugal acceleration)}$$

$$F^\phi = \gamma^{\phi\phi}F_\phi = \ddot{q}^\rho + \Gamma^\rho_{mn}\dot{q}^m\dot{q}^n$$

$$= \ddot\phi + 2\dot\rho\dot\phi/\rho \quad \text{so: } \Gamma^\phi_{\rho\phi} = 1/\rho = \Gamma^\phi_{\phi\rho} \qquad \gamma^{\phi\phi} = 1/\left(m\rho^2\right)$$

$$\ddot\phi = F^\phi - 2\dot\rho\dot\phi/\rho \qquad \textit{(Coriolis acceleration)}$$

Rewriting GCC Lagrange equations :

(Review of Lecture 11)

$$\dot{p}_r \equiv \frac{dp_r}{dt} = M\ddot{r}$$
$$= M\,r\,\dot{\phi}^2 - \frac{\partial U}{\partial r}$$

Centrifugal (center-fleeing) force equals total Centripetal (center-pulling) force

$$\dot{p}_\phi \equiv \frac{dp_\phi}{dt} = 2Mr\dot{r}\dot{\phi} + Mr^2\ddot{\phi}$$
$$= 0 - \frac{\partial U}{\partial \phi}$$

Torque relates to two distinct parts: Coriolis and angular acceleration

Angular momentum p_ϕ is conserved if potential U has no explicit ϕ-dependence

Conventional forms

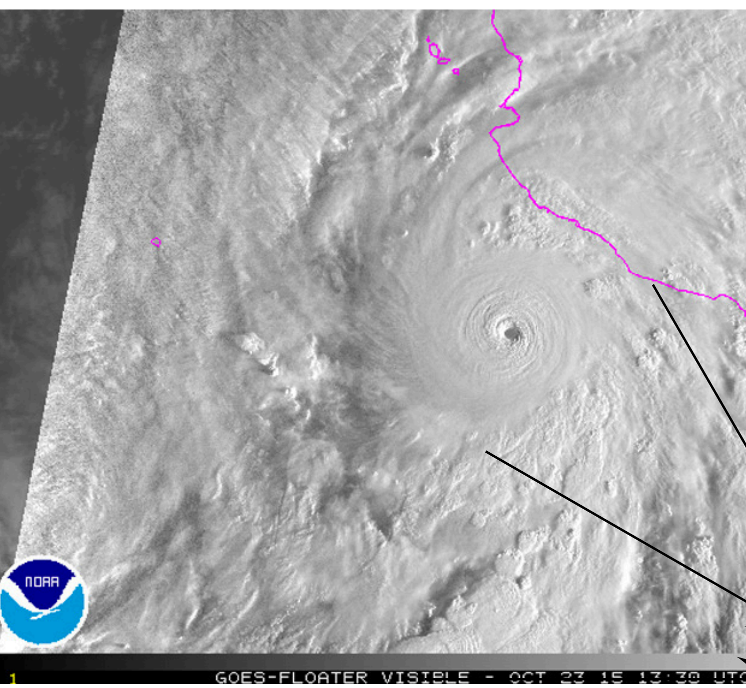
radial force: $M\ddot{r} = M\,r\,\dot{\phi}^2 - \dfrac{\partial U}{\partial r}$

angular force or torque: $Mr^2\ddot{\phi} = -2Mr\dot{r}\dot{\phi} - \dfrac{\partial U}{\partial \phi}$

Field-free (U=0)

radial acceleration: $\ddot{r} = r\,\dot{\phi}^2$

angular acceleration: $\ddot{\phi} = -2\dfrac{\dot{r}\dot{\phi}}{r}$

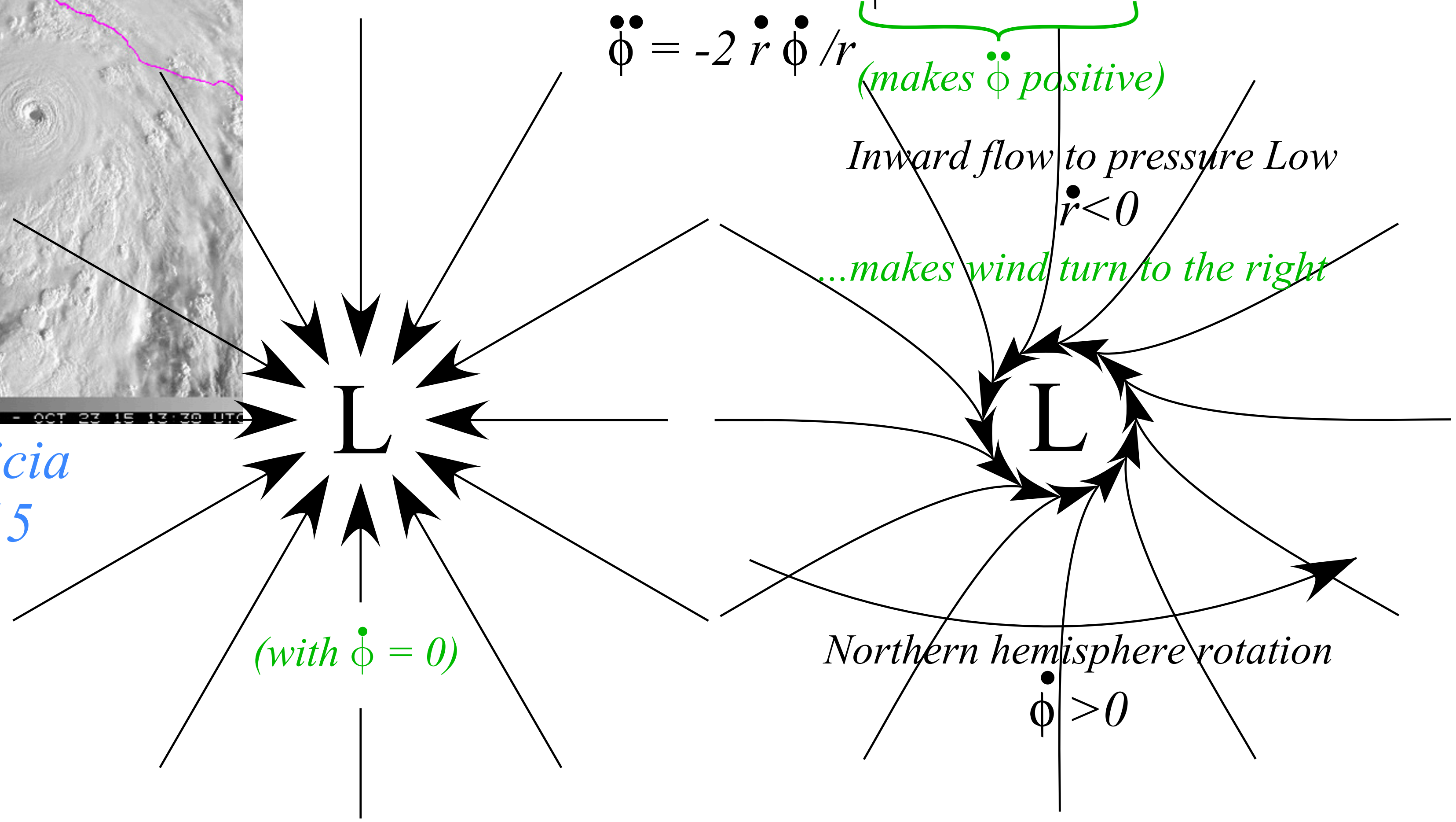


Hurricane Patricia October 23, 2015

Because Earth rotation is <u>counter</u>-clockwise (positive) in North

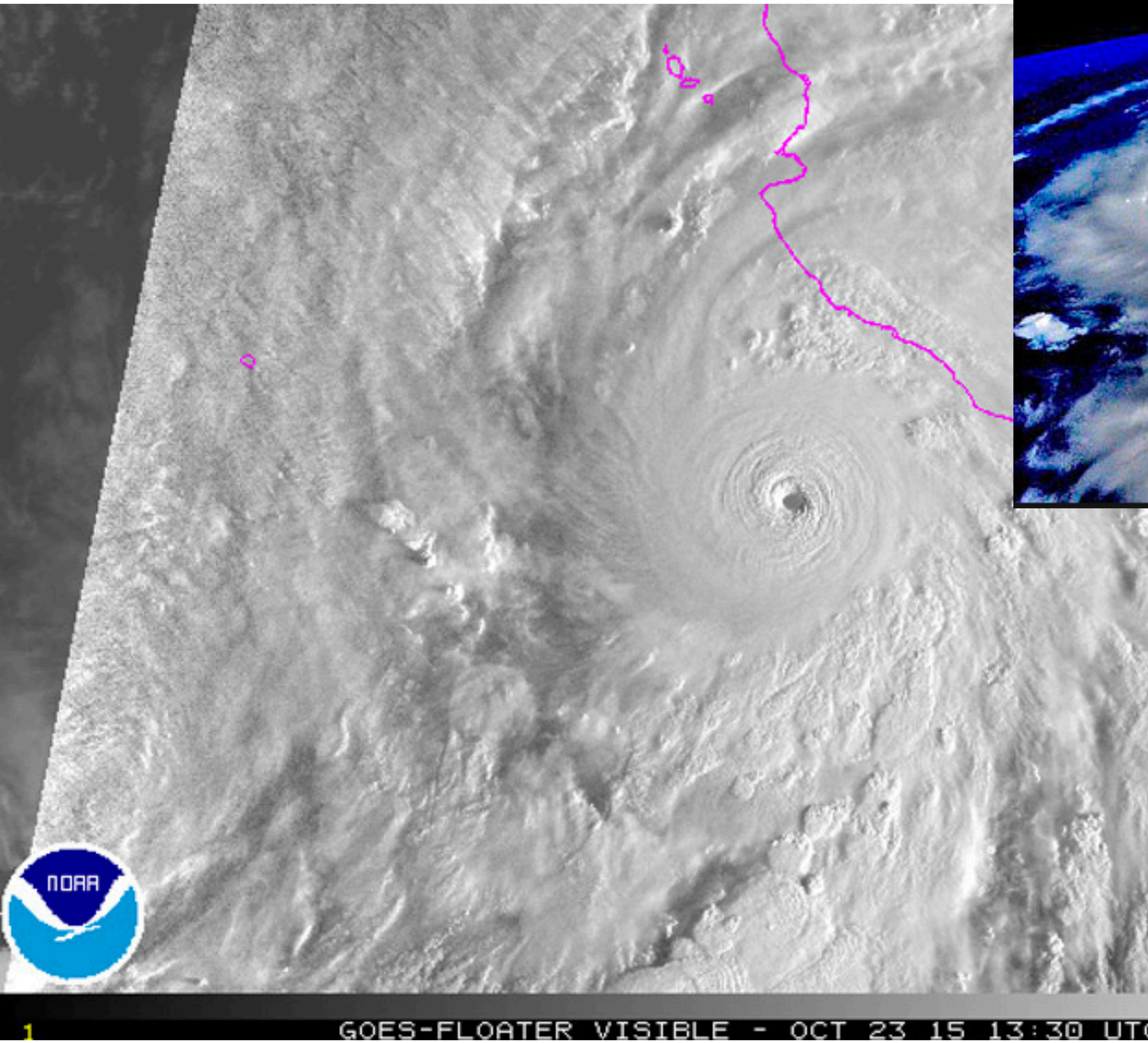
Coriolis acceleration with $\dot{\phi} > 0$ and $\dot{r} < 0$

$\ddot{\phi} = -2\,\dot{r}\,\dot{\phi}/r$ *(makes $\ddot{\phi}$ positive)*



Effect on <u>Northern</u> Hemisphere local weather

Cyclonic flow around lows



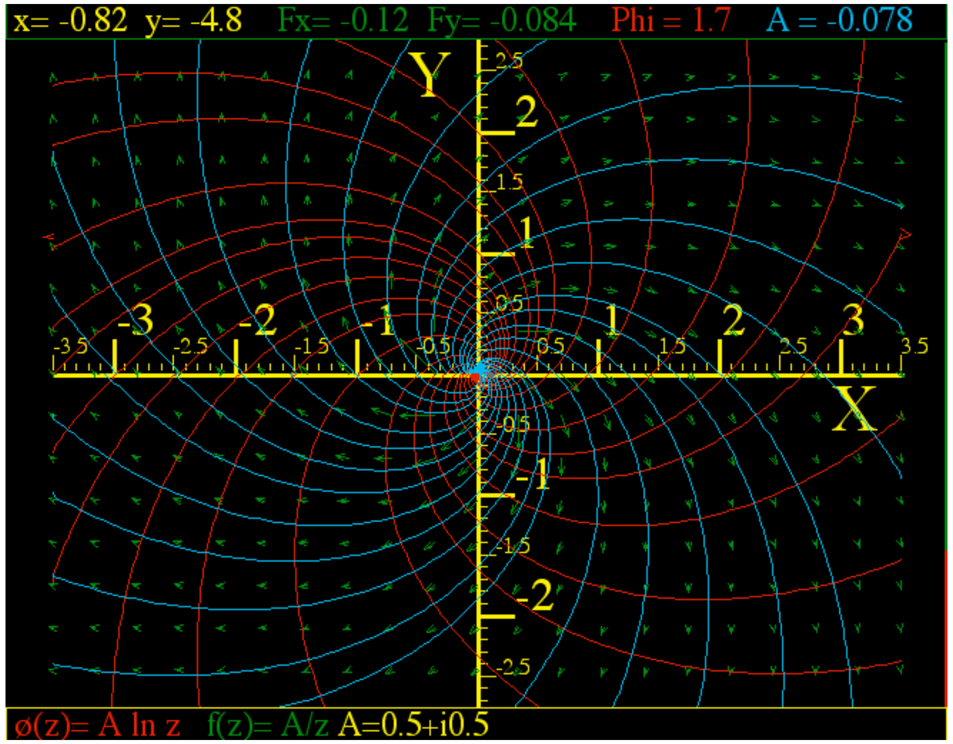
Hurricane Patricia
October 23, 2015

https://www.google.com/search?q=Satellite+view+of+Patricia&biw=1811&bih=1247&tbm=isch&tbo=u&source=univ&sa=X&ved=0CD0QsARqFQoTCLb17N728sgCFdA0iAodl4kMsg

Riemann-forms in cylindrical polar OCC ($q^1 = \rho$, $q^2 = \phi$, $q^3 = z$)

Christoffel relation to Coriolis coefficients

Mechanics of ideal fluid vortex

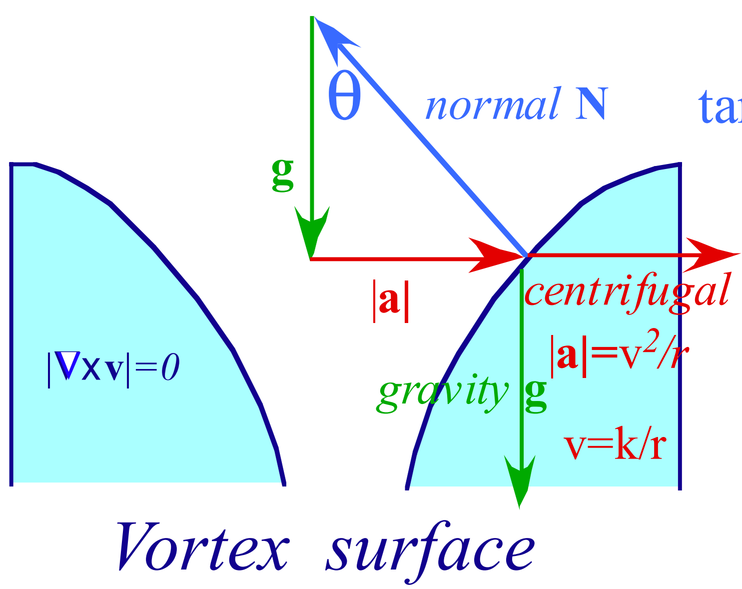


Mechanics of ideal fluid vortex

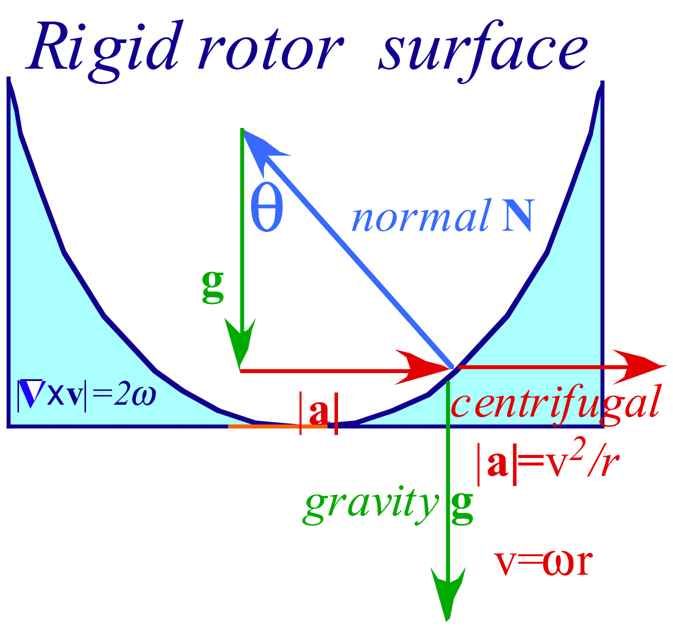
Rotating fluid surface has normal acceleration **N** *be sum of gravity* **g** *and centrifugal* $\mathbf{a}=\mathbf{e}_a\, v^2/r$

Case 1: Vortex with velocity field $v=k/r$

Case 2: Rigidly rotating fluid with velocity field $v=\omega\, r$



$\tan\theta = |\mathbf{a}|/\mathbf{g}$



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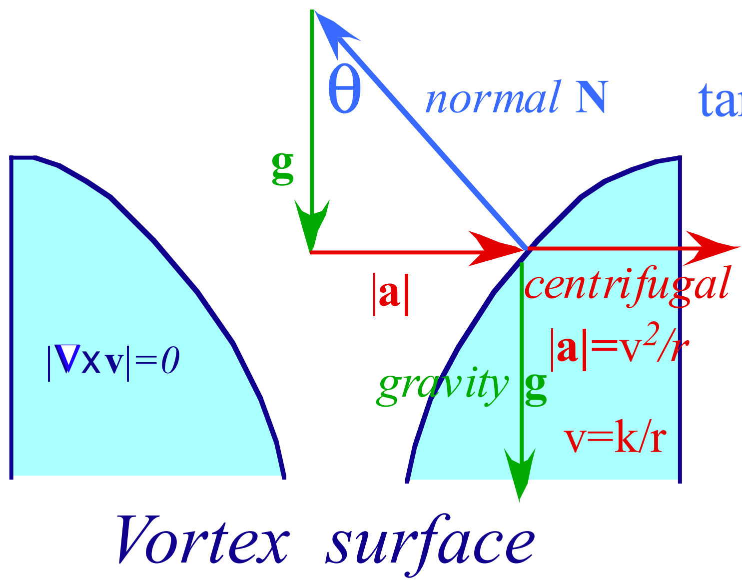
In either case slope θ *of normal* **N** *is:*

$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{v^2/r}{g}$$



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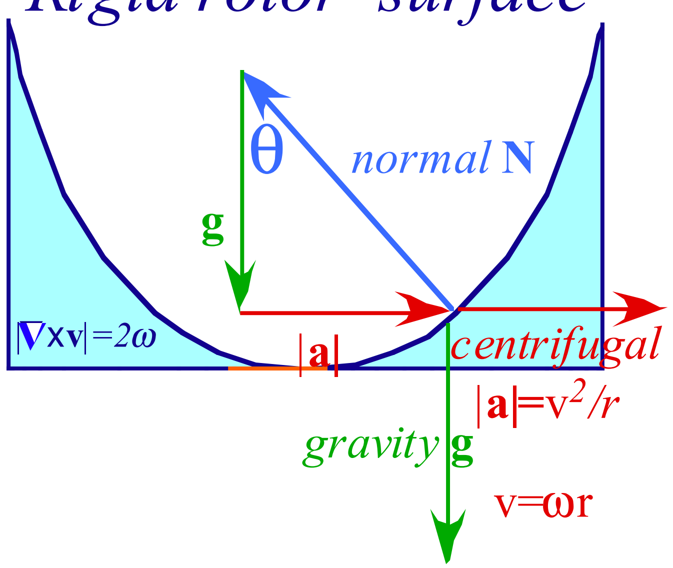

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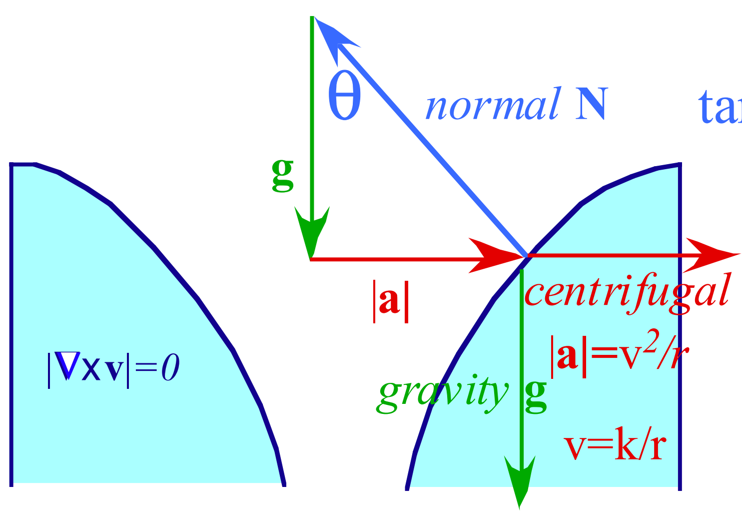
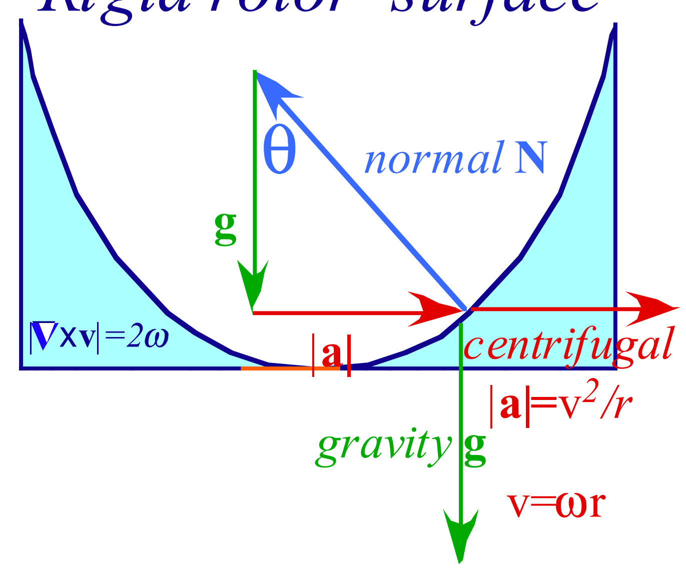


Vortex surface

$\tan\theta = |\mathbf{a}|/\mathbf{g}$

Rigid rotor surface



$$\tan\theta=\frac{dz}{dr}=\frac{|a|}{g}=\frac{v^2/r}{g}=\frac{k^2}{gr^3}$$

$$\tan\theta=\frac{dz}{dr}=\frac{|a|}{g}=\frac{v^2/r}{g}=\frac{\omega^2}{g}r^1$$

Mechanics of ideal fluid vortex

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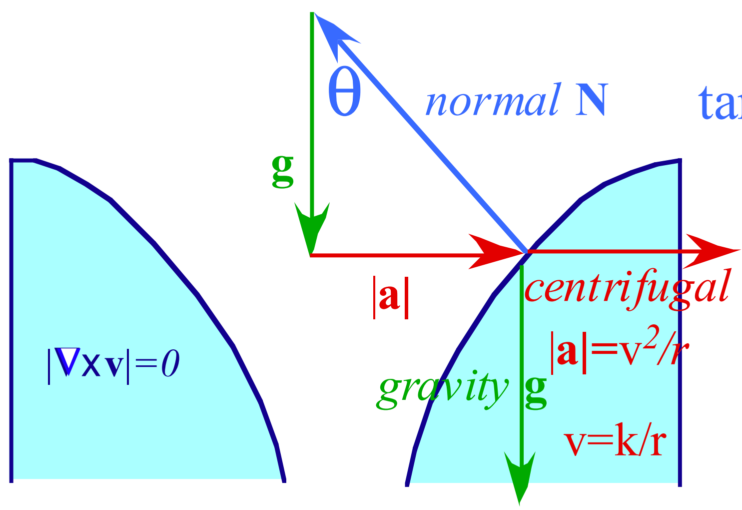
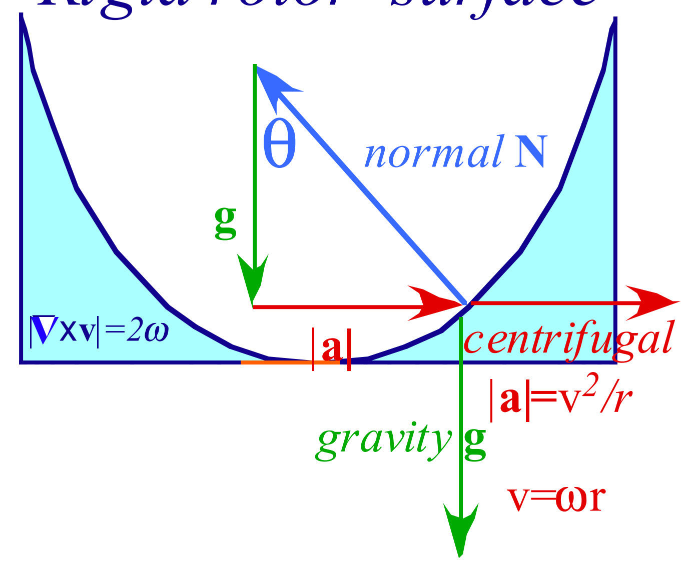


Vortex surface

tanθ= |**a**|/**g**

Rigid rotor surface



$$\tan\theta=\frac{dz}{dr}=\frac{|a|}{g}=\frac{v^2/r}{g}=\frac{k^2}{gr^3}$$

Integrating:

$$z(r)=\int\frac{dz}{dr}dr=\int\frac{k^2}{gr^3}dr=-\frac{k^2}{2gr^2}$$

$$\tan\theta=\frac{dz}{dr}=\frac{|a|}{g}=\frac{v^2/r}{g}=\frac{\omega^2}{g}r^1$$

Integrating:

$$z(r)=\int\frac{dz}{dr}dr=\int\frac{\omega^2}{g}r^1\,dr=\frac{\omega^2}{2g}r^2$$

Mechanics of ideal fluid vortex

Rotating fluid surface has normal acceleration **N** *be sum of gravity* **g** *and centrifugal* $\mathbf{a}=\mathbf{e}_a\, v^2/r$

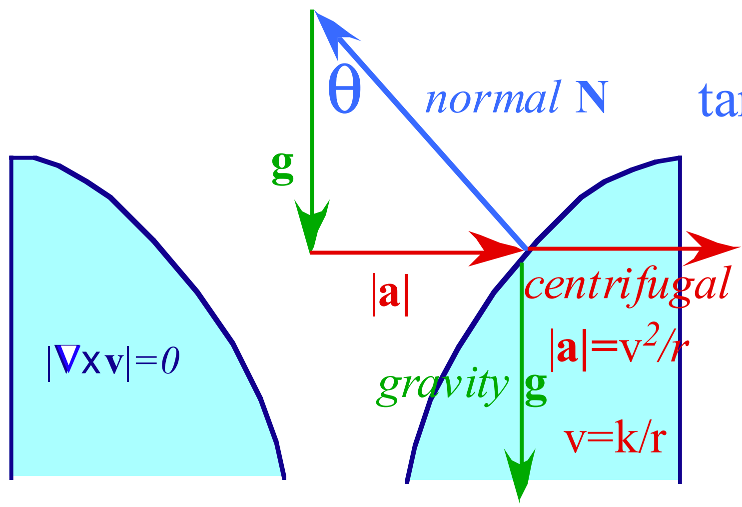
Case 1: Vortex with velocity field $v=k/r$

Case 2: Rigidly rotating fluid with velocity field $v=\omega\, r$

In either case slope θ *of normal* **N** *is:*

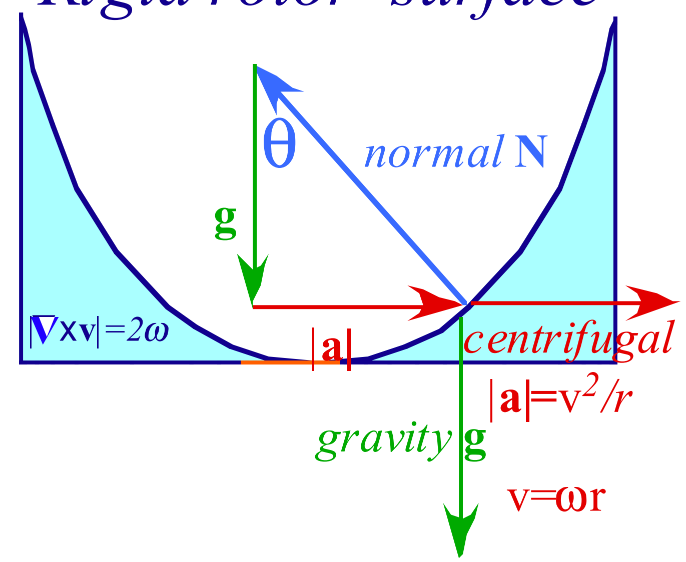
$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{v^2/r}{g}$$

$\tan\theta = |\mathbf{a}|/\mathbf{g}$

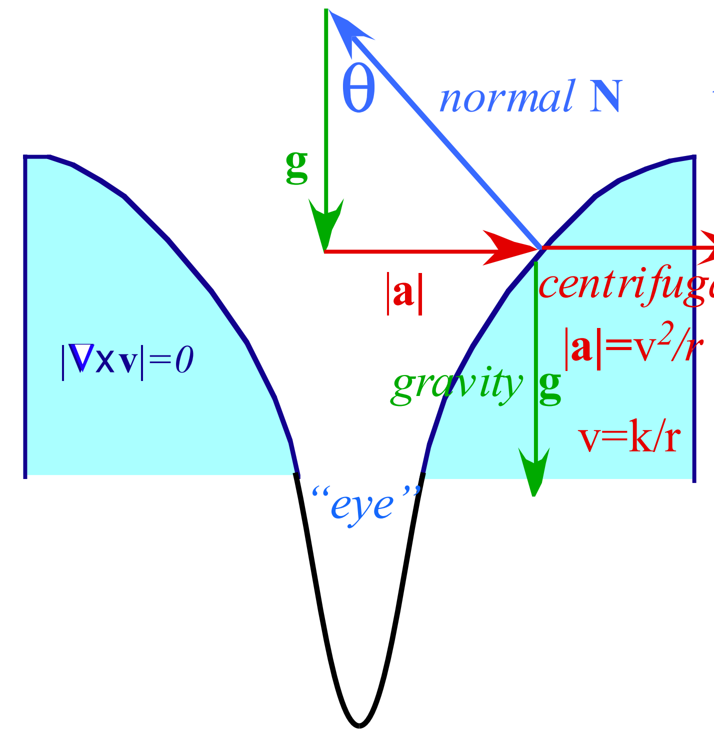


Vortex surface

Rigid rotor surface



Ideal vortex without drain has a parabolic "eye"



$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{v^2/r}{g} = \frac{k^2}{gr^3}$$

Integrating:

$$z(r) = \int \frac{dz}{dr}dr = \int \frac{k^2}{gr^3}dr = -\frac{k^2}{2gr^2}$$

$$\tan\theta = \frac{dz}{dr} = \frac{|a|}{g} = \frac{v^2/r}{g} = \frac{\omega^2}{g}r^1$$

Integrating:

$$z(r) = \int \frac{dz}{dr}dr = \int \frac{\omega^2}{g}r^1\,dr = \frac{\omega^2}{2g}r^2$$

Somewhat analogous to the "Sophomore-Physics Earth"

→ *Separation of GCC Equations: Effective Potentials*

Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations

2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"

(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

(Numerically correct ONLY!)

$$= \frac{1}{2}\gamma^{mn}p_m p_n + V = \frac{1}{2m}p_\rho^2 + \frac{1}{2m\rho^2}p_\phi^2 + \frac{1}{2m}p_z^2 + V$$

(Formally **and** Numerically correct)

Separation of GCC Equations: Effective Potentials (For isotropic $H(r,p_r,\phi,p_\phi)$)

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

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(Formally **and** Numerically correct)

If potential V is *isotropic* (cylindrical) function of radius ρ. ($V = V(\rho)$)

H has no explicit ϕ–dependence and the ϕ–momenta is constant.

$$m\rho^2\dot{\phi} = p_\phi = const. = \mu$$

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(Let $k = 0$)

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 \quad + \quad \frac{1}{2}m\rho^2\dot{\phi}^2 \quad + \quad \frac{1}{2}m\dot{z}^2 + V$$

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Symmetry reduces problem to a one-dimensional form.

$$H = \frac{1}{2m}p_\rho^2 + V^{eff}(\rho) = E = const.$$

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Separation of GCC Equations: Effective Potentials

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$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+V(\rho)$$

Velocity relations:

$$\dot{\phi}=\mu/\left(m\rho^2\right) \qquad \dot{\rho}=\frac{d\rho}{dt}=\frac{\partial H}{\partial p_\rho}=\frac{p_\rho}{m}=\pm\sqrt{\frac{2}{m}\left(E-V^{eff}(\rho)\right)}$$

Separation of GCC Equations: Effective Potentials

$$H = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n + V = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + \frac{1}{2}m\dot{z}^2 + V$$ (Numerically correct ONLY!)

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$$\dot{\rho} = \frac{d\rho}{dt} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} = \pm\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}$$

Equations solved by a *quadrature integral* for time versus radius.

$$\int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\frac{2}{m}\left(E - V^{eff}(\rho)\right)}} = \left(\text{Travel time } \rho_0 \text{ to } \rho_1\right) = t_1 - t_0$$

Separation of GCC Equations: Effective Potentials

→ *Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations*

2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"

(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

Small radial oscillations

Stable minimal-energy radius will satisfy a zero-slope equation.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_0} = 0\ , \quad \text{with: } \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0\ .$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_0) + 0 + \frac{1}{2}(\rho-\rho_0)^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0}$$

Stable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} > 0$ Unstable flat $\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_0} < 0$

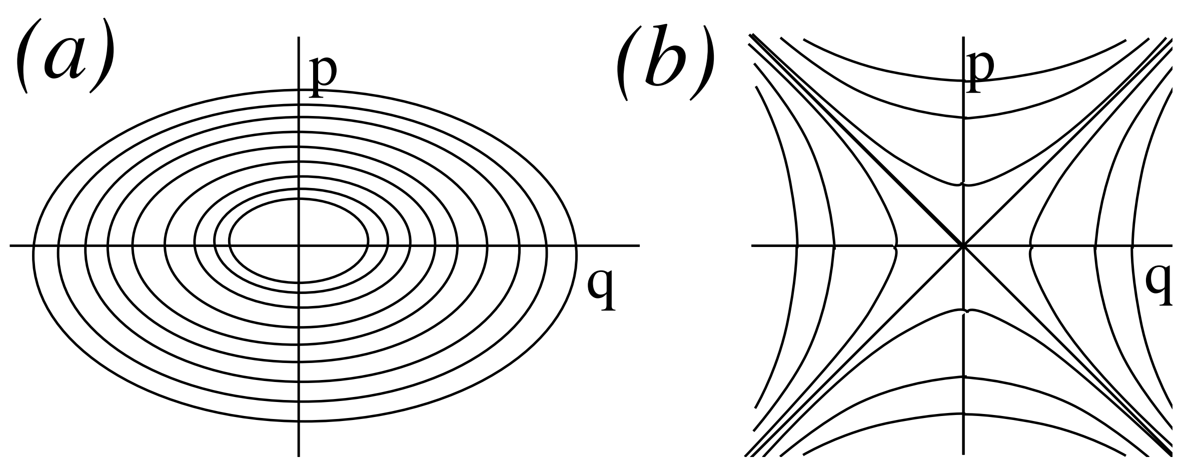

Fig. 2.7.4 Phase paths around fixed points (a) Stable point (b) Unstable saddle point

Small radial oscillations

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$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0 \,, \quad \text{with:} \quad \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} > 0 \,.$$

A Taylor series around this minimum can be used to estimate orbit properties for small oscillations.

$$V^{eff}(\rho) = V^{eff}(\rho_{stable}) + 0 + \frac{1}{2}(\rho - \rho_{stable})^2 \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}$$

An effective "spring constant" at the stable point giving approximate frequency of oscillation.

$$k^{eff} = \left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}} \qquad \omega_{\rho_{stable}} = \sqrt{\frac{k^{eff}}{m}} = \sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}$$

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Small oscillation orbits are closed if and only if the ratio of the two is a rational (fractional) number.

$$\frac{\omega_{\rho_{stable}}}{\omega_{\phi}} = \frac{\omega_{\rho_{stable}}}{\dot{\phi}(\rho_{stable})} = \frac{n_{\rho}}{n_{\phi}} \Leftrightarrow \text{Orbit is closed-periodic}$$

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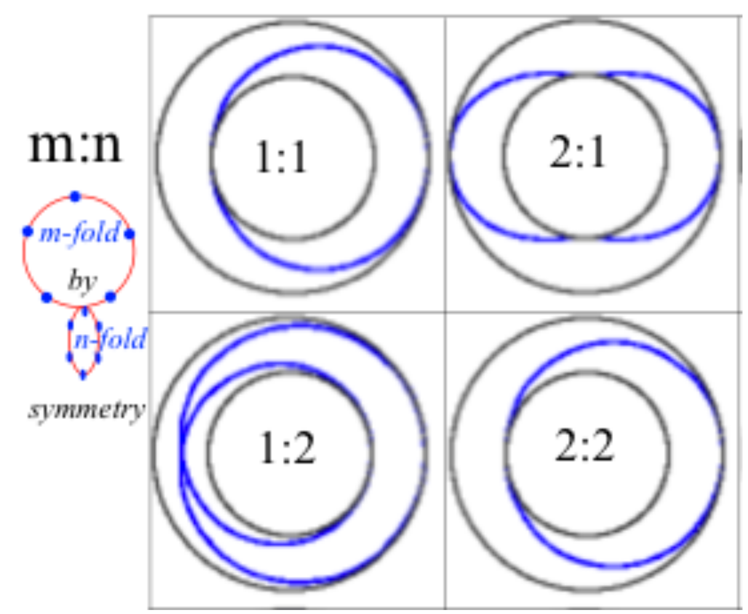
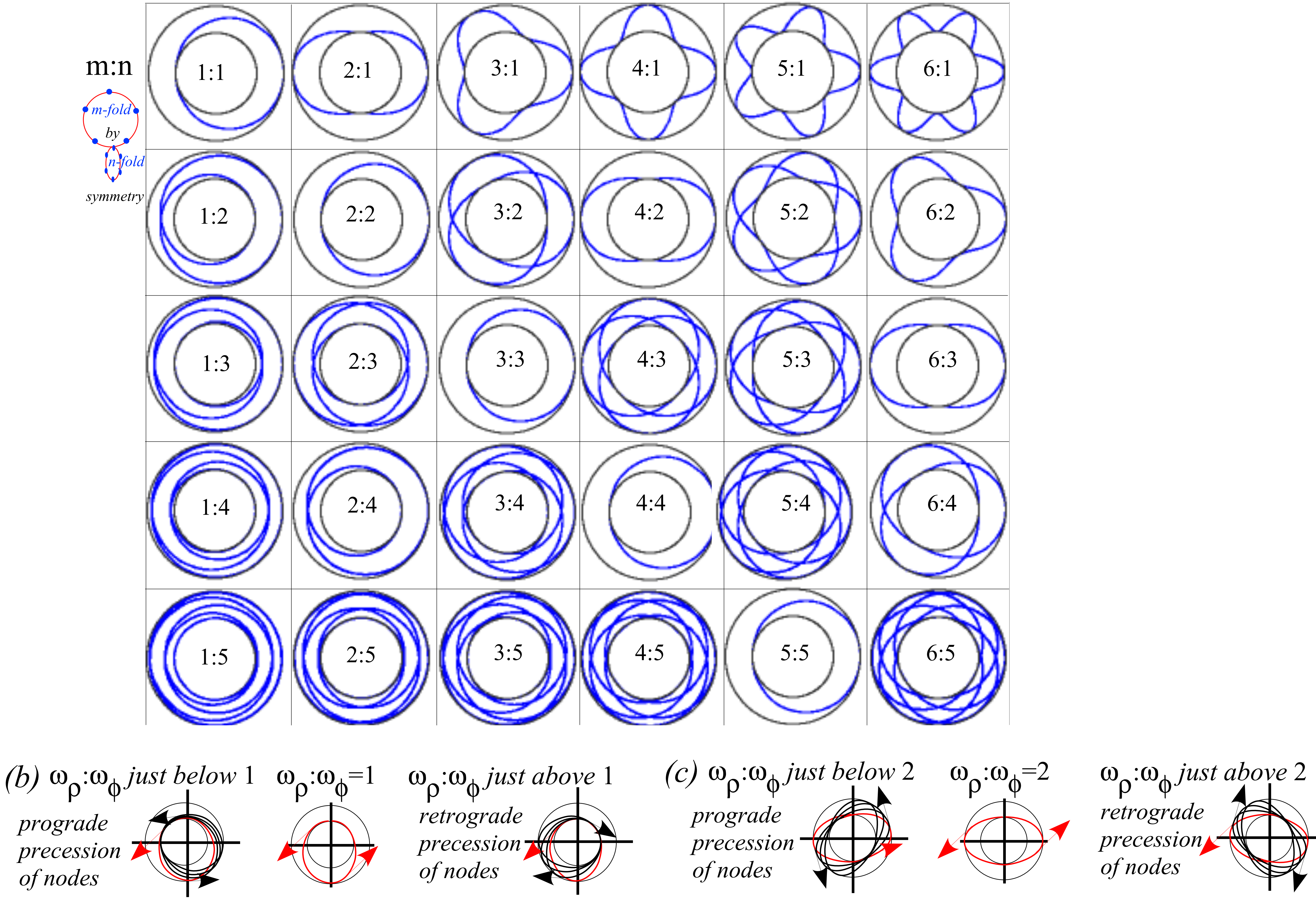
Some generic shapes resulting from various ratios $n\rho : n\phi$





m:n
m-fold
by
n-fold
symmetry
1:1
2:1
3:1
4:1
5:1
6:1
1:2
2:2
3:2
4:2
5:2
6:2
1:3
2:3
3:3
4:3
5:3
6:3
1:4
2:4
3:4
4:4
5:4
6:4
1:5
2:5
3:5
4:5
5:5
6:5
(b) $\omega_\rho:\omega_\phi$ just below 1
prograde precession of nodes
$\omega_\rho:\omega_\phi=1$
$\omega_\rho:\omega_\phi$ just above 1
retrograde precession of nodes
(c) $\omega_\rho:\omega_\phi$ just below 2
prograde precession of nodes
$\omega_\rho:\omega_\phi=2$
$\omega_\rho:\omega_\phi$ just above 2
retrograde precession of nodes

Separation of GCC Equations: Effective Potentials

Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations

2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"

(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

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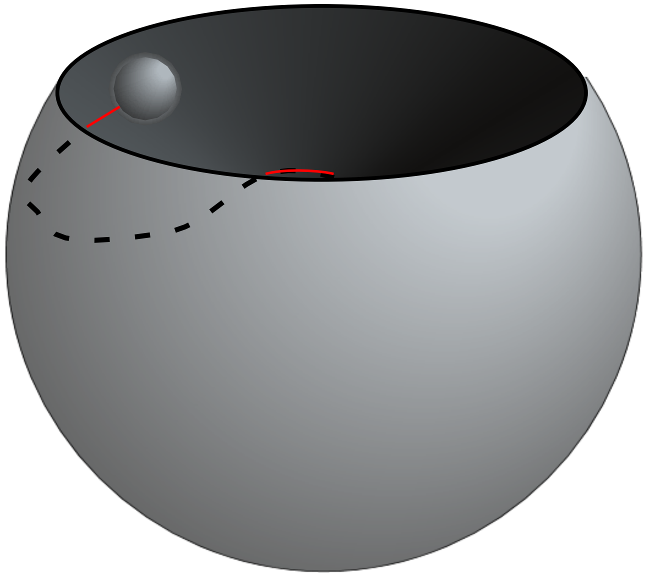
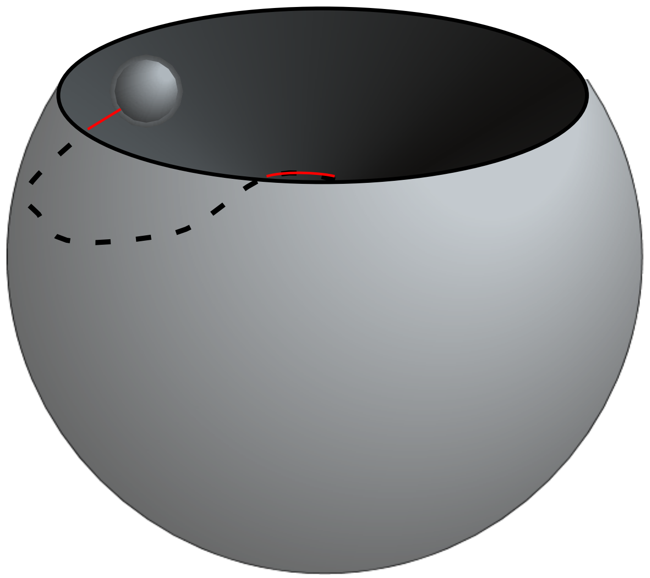


Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

$x=x^1=r\sin\theta\cos\phi, \quad y=x^2=r\sin\theta\sin\phi, \quad z=x^3=r\cos\theta,$

2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"



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$x=x^1=r\sin\theta\cos\phi,\quad y=x^2=r\sin\theta\sin\phi,\quad z=x^3=r\cos\theta,$

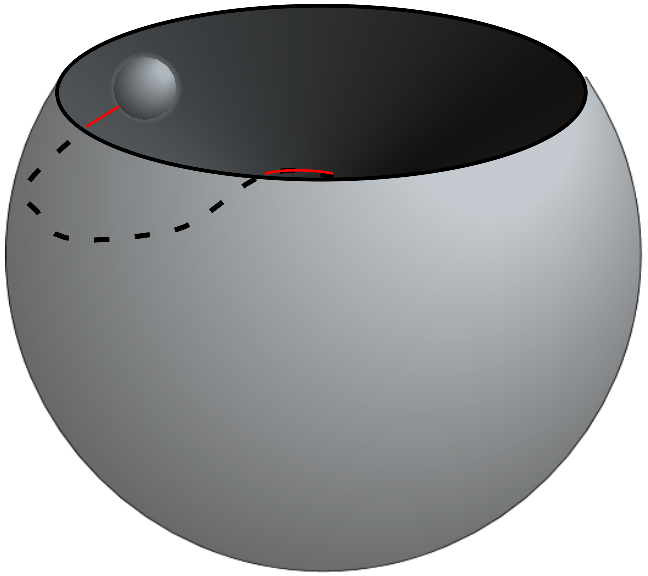
Jacobian matrices and determinants:

Reduced to cylindrical coordinates:

$$J=\begin{array}{c} \quad\mathbf{E}_r \quad \mathbf{E}_\theta \quad \mathbf{E}_\phi \\ \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}\end{array} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \xrightarrow[r=\rho]{\theta=\pi/2} \begin{pmatrix} \cos\phi & 0 & -\rho\sin\phi \\ \sin\phi & 0 & \rho\cos\phi \\ 0 & -\rho & 0 \end{pmatrix}$$

$$\det J = \det J^T = \frac{\partial\{xyz\}}{\partial\{r\theta\phi\}} = r^2\sin\theta \xrightarrow[r=\rho]{\theta=\pi/2} \rho^2$$

2D Spherical pendulum or "Bowl-Bowling"



Spherical coordinates: $\{q^1=r, q^2=\theta, q^3=\phi\}$ obvious choice:

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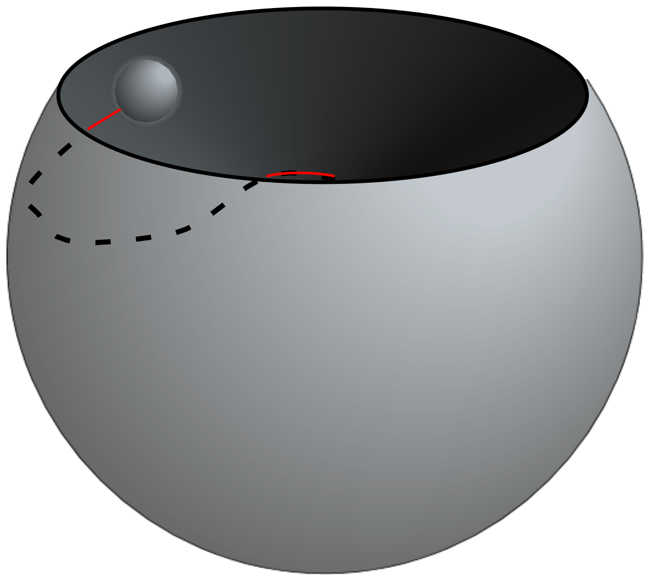
$$\det J=\det J^{\mathrm{T}}=\frac{\partial\{xyz\}}{\partial\{r\theta\phi\}}=r^2\sin\theta\xrightarrow[r=\rho]{\theta=\pi/2}\rho^2$$

Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T\cdot J$ of Jacobian and its transpose. OCC g's are diagonal.

Covariant: $g_{rr}=\mathbf{E}_r\bullet\mathbf{E}_r=1,\ \ g_{\theta\theta}=\mathbf{E}_\theta\bullet\mathbf{E}_\theta=r^2,\ \ g_{\phi\phi}=\mathbf{E}_\phi\bullet\mathbf{E}_\phi=r^2\sin^2\theta,$

Contravariant: $g^{rr}=1,\qquad g^{\theta\theta}=1/r^2,\qquad g^{\phi\phi}=1/r^2\sin^2\theta.$

2D Spherical pendulum or "Bowl-Bowling"



Spherical coordinates: $\{q^1=r,\ q^2=\theta,\ q^3=\phi\}$ obvious choice:

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$$J=\begin{array}{c}\begin{array}{ccc}\mathbf{E}_r & \mathbf{E}_\theta & \mathbf{E}_\phi\end{array}\\ \begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{pmatrix}\end{array}=\begin{pmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{pmatrix}\xrightarrow[r=\rho]{\theta=\pi/2}\begin{pmatrix}\cos\phi & 0 & -\rho\sin\phi\\ \sin\phi & 0 & \rho\cos\phi\\ 0 & -\rho & 0\end{pmatrix}$$

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Covariant metric $g_{\mu\nu}$ is matrix product $g=J^T\cdot J$ of Jacobian and its transpose. OCC g's are diagonal.

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(Lagrangian form) *(Hamiltonian form)*

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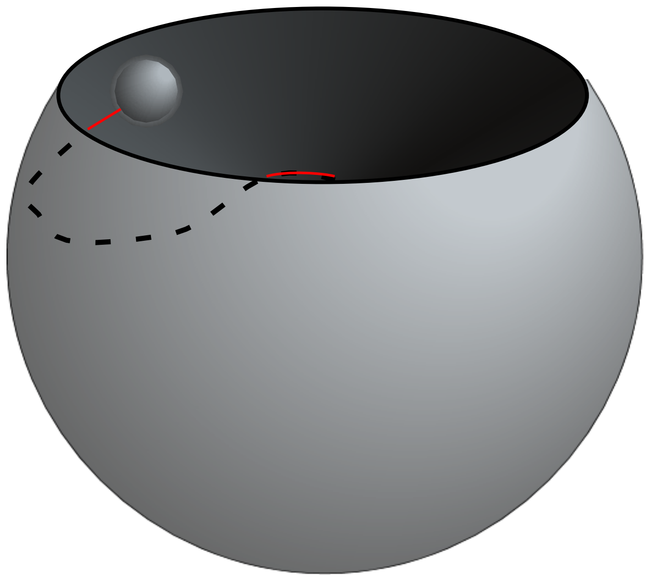

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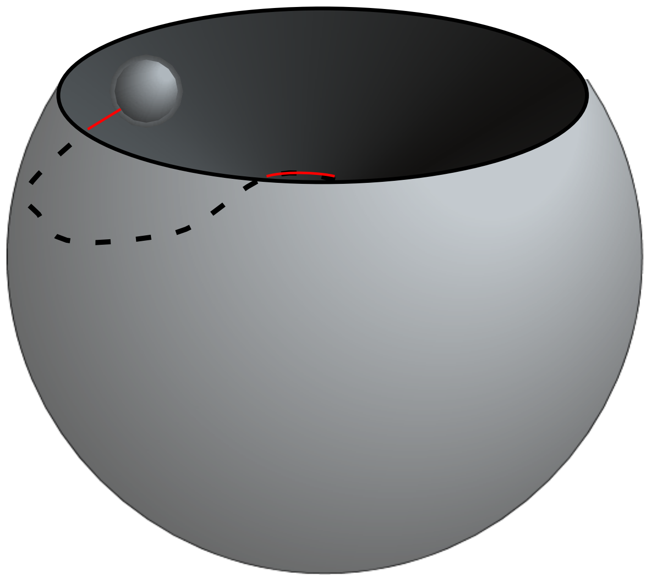

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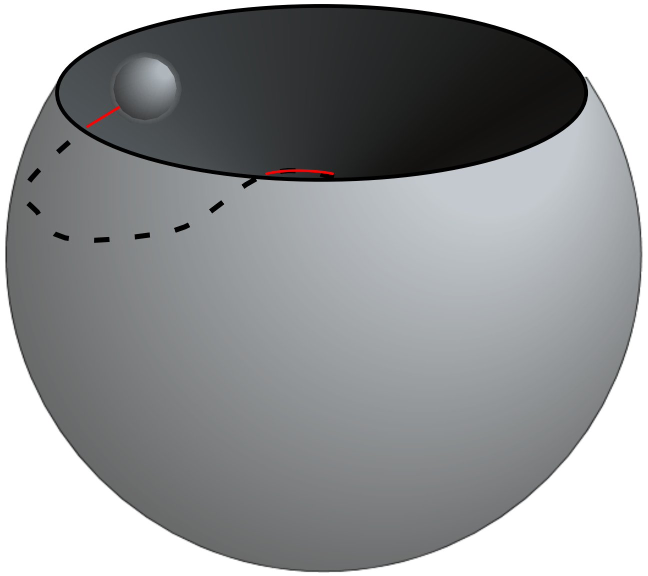
Total Energy from Hamiltonian $E=T+V(gravity)=const.$:

$$E=\frac{mR^2}{2}\dot\theta^2+V^{effective}(\theta)=\frac{mR^2}{2}\dot\theta^2+\frac{p_\phi^2}{2mR^2\sin^2\theta}+mgR\cos\theta\quad=\quad\alpha\dot\theta^2+\frac{\delta}{\sin^2\theta}+\gamma\cos\theta$$

Let : $\alpha=\frac{mR^2}{2}$, $\delta=\frac{p_\phi^2}{2mR^2}$, $\gamma=mgR$ where: $p_\phi=mR^2\sin^2\theta(\dot\phi\ )$

2D Spherical pendulum or "Bowl-Bowling"



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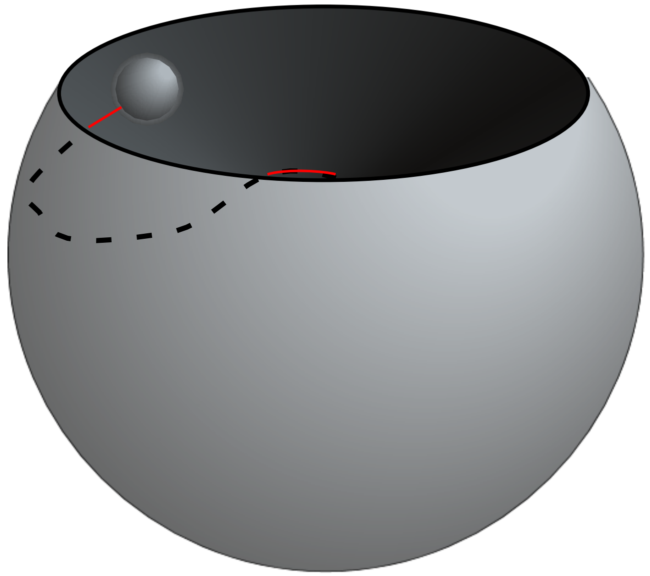

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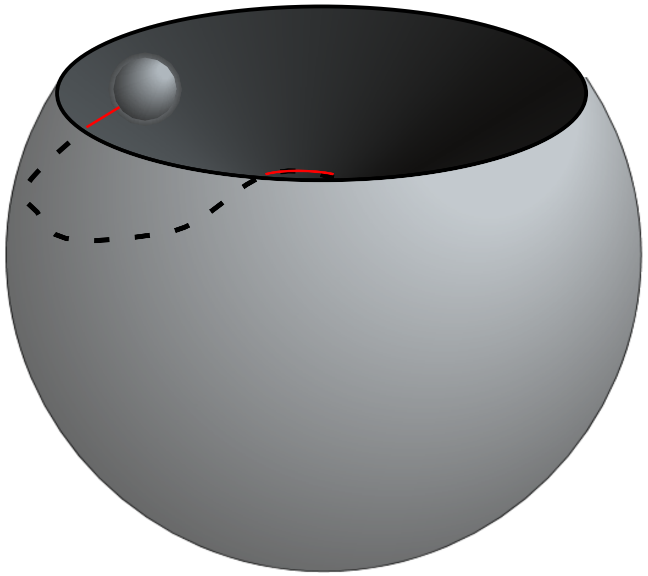

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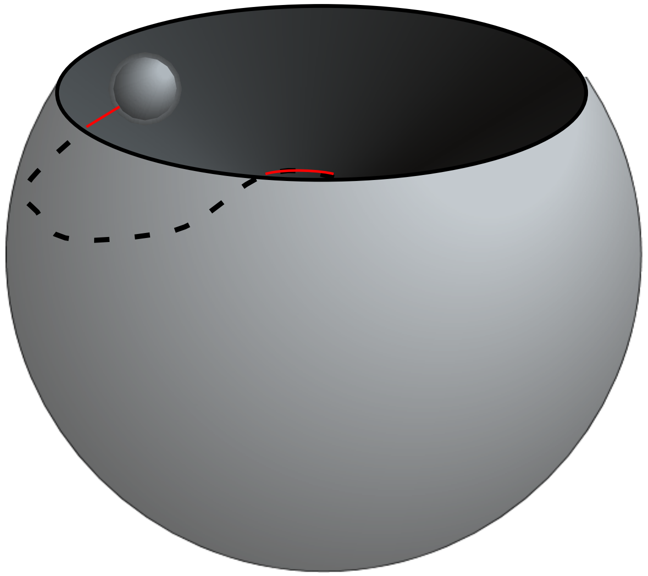

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$$0=(mR^2\sin\theta)\dot{\phi}^2\cos\theta-mgR\sin\theta$$ or: $$\dot{\phi}_{equil}^2=-\frac{g}{R\cos\theta_{equil}}$$

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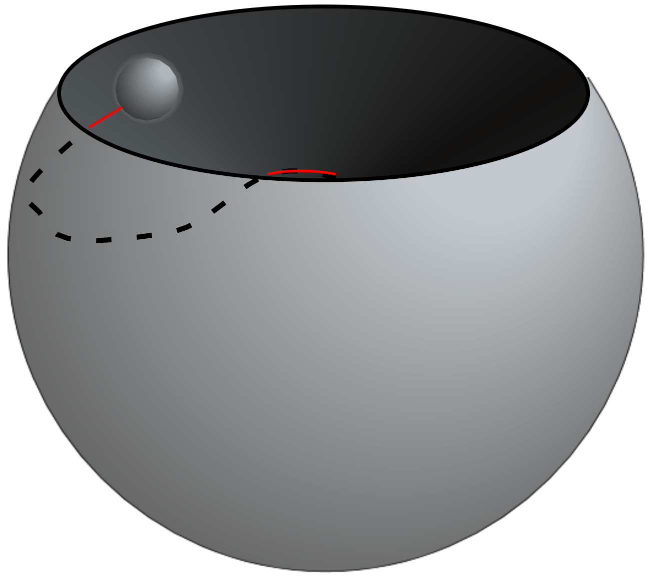

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V-Derivative for small oscillation frequency:

$$\frac{d^2V^{effective}(\theta)}{d\theta^2}=-\gamma\cos\theta+\frac{2\delta\sin\theta}{\sin^3\theta}+\frac{3\cdot 2\delta\cos^2\theta}{\sin^4\theta}=-\gamma\cos\theta+2\delta\frac{\sin^2\theta+3\cos^2\theta}{\sin^4\theta}$$

$$=-mgR\cos\theta+\frac{2\left(mR^2\sin^2\theta\quad\dot{\phi}\right)^2}{2mR^2}\frac{1+2\cos^2\theta}{\sin^4\theta}$$

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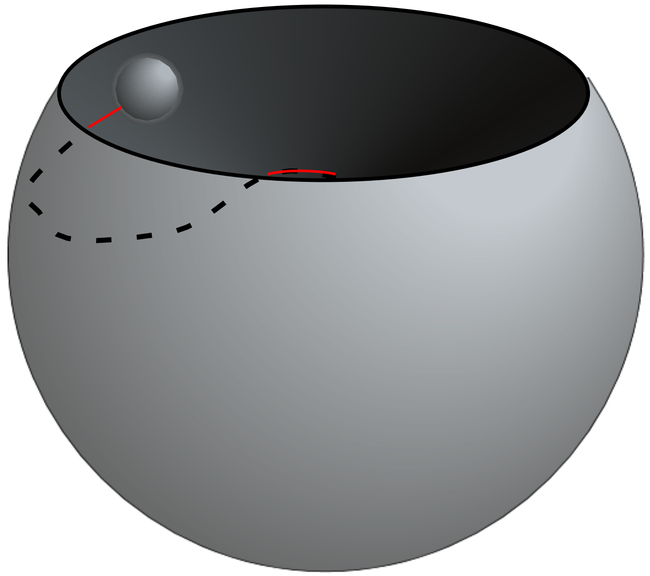

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At equilibrium:

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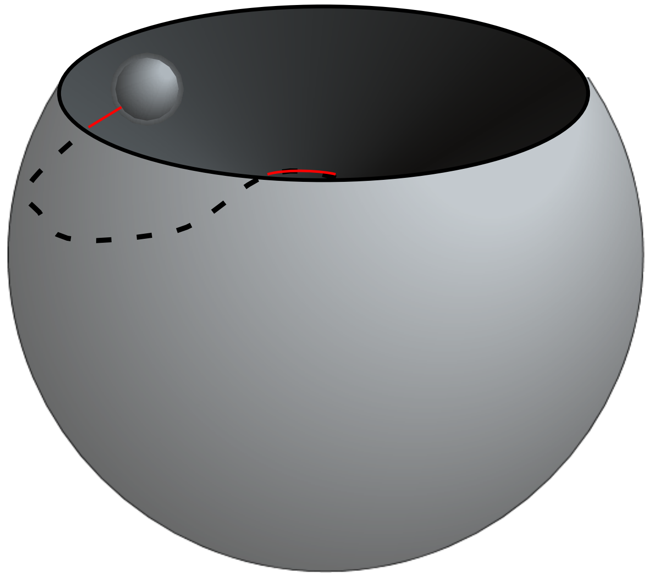

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Separation of GCC Equations: Effective Potentials

Small (n_ρ:m_ϕ)-periodic and quasi-periodic oscillations

→ *2D Spherical pendulum or "Bowl-Bowling" and the "I-Ball"*

(n_ρ:m_ϕ)=(2:1) vs (1:1) periodic and quasi-periodic orbits

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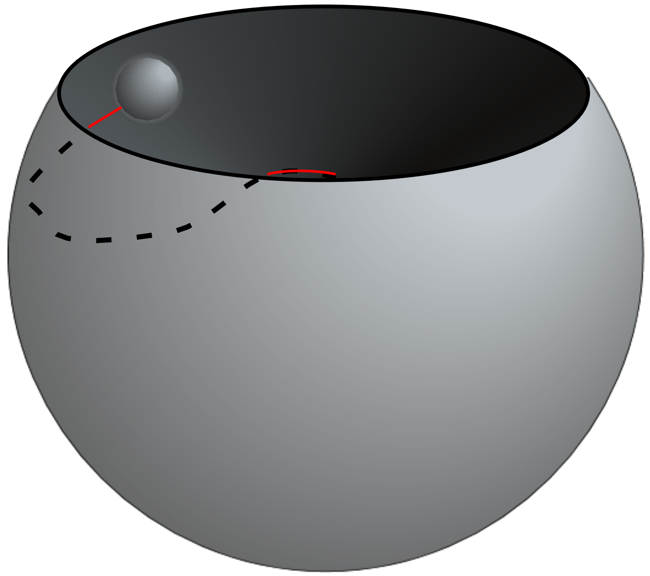

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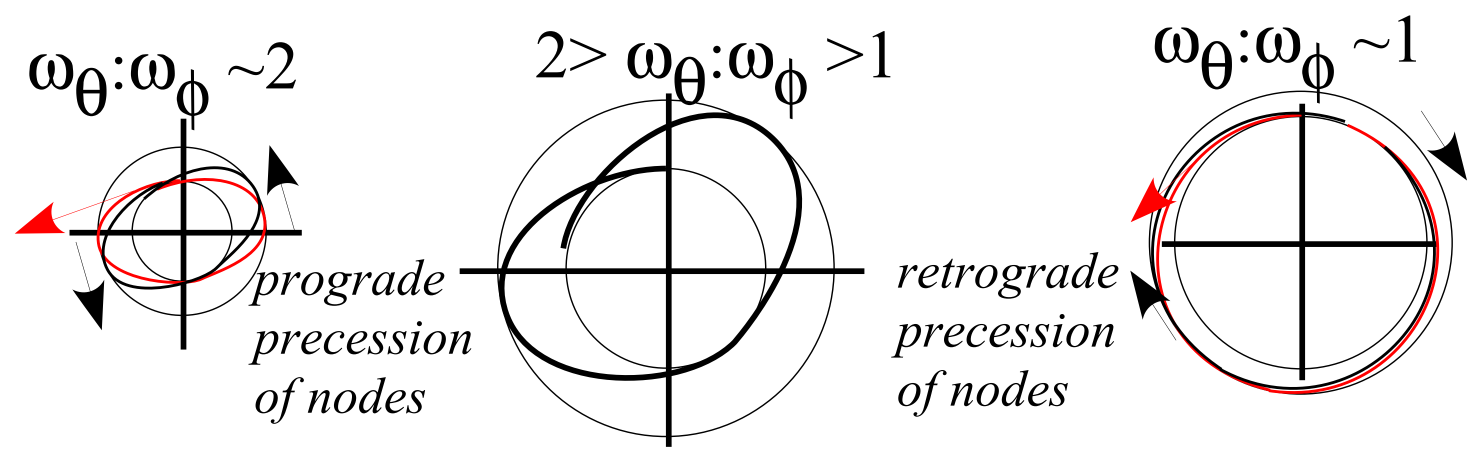
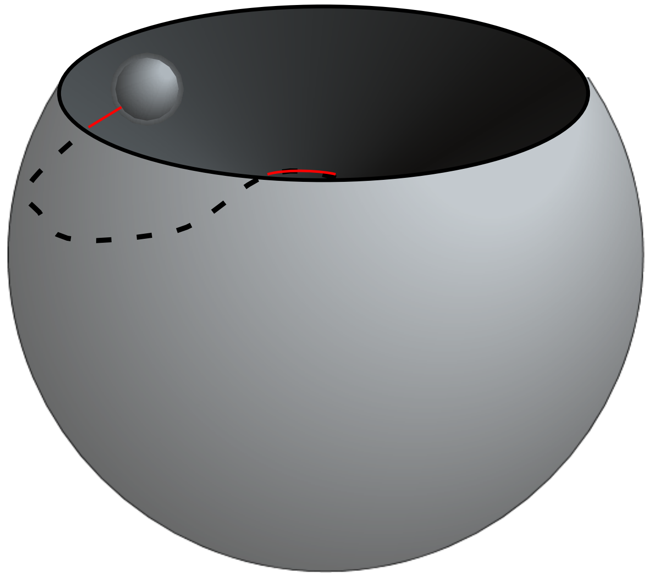
At bottom θ→π the ratio of in-out ω_θ to circle ω_ϕ approaches 2:1

At equator θ→π/2 the ratio approaches 1:1.



2D Spherical pendulum or "Bowl-Bowling"



Total Energy from Hamiltonian $E=T+V(gravity)=const.$:

$$E=\frac{mR^2}{2}\dot{\theta}^2+V^{effective}(\theta)=\ \alpha\dot{\theta}^2+\frac{\delta}{\sin^2\theta}+\gamma\cos\theta$$

Let : $\alpha=\frac{mR^2}{2}$, $\delta=\frac{p_\phi^2}{2mR^2}$, $\gamma=mgR$ where: $p_\phi=mR^2\sin^2\theta(\dot{\phi}\ )$

Equilibrium point of stable orbit and small oscillation frequency near equilibrium:

$$\frac{dV^{effective}(\theta)}{d\theta}=\frac{-2\delta\cos\theta}{\sin^3\theta}-\gamma\sin\theta=0=\frac{-2p_\phi^2\cos\theta}{2mR^2\sin^3\theta}-mgR\sin\theta$$

$$\left(\omega_\theta^{equil}\right)^2=\frac{1}{mR^2}\left.\frac{d^2V^{effective}(\theta)}{d\theta^2}\right|_{equil}$$

$$0=(mR^2\sin\theta)\dot{\phi}^2\cos\theta-mgR\sin\theta$$ or: $$\dot{\phi}_{equil}^2=-\frac{g}{R\cos\theta_{equil}}$$

(Polar angle librational frequency ω_θ^{equil} is related to azimuthal frequency $\dot{\phi}_{equil}^2$.)

V-Derivative for small oscillation frequency:

$$\frac{d^2V^{effective}(\theta)}{d\theta^2}=-\gamma\cos\theta+\frac{2\delta\sin\theta}{\sin^3\theta}+\frac{3\cdot2\delta\cos^2\theta}{\sin^4\theta}=-\gamma\cos\theta+2\delta\frac{\sin^2\theta+3\cos^2\theta}{\sin^4\theta}$$

$$=-mgR\cos\theta+\frac{2\left(mR^2\sin^2\theta\ \ \dot{\phi}\right)^2}{2mR^2}\frac{1+2\cos^2\theta}{\sin^4\theta}$$

$$=-mgR\cos\theta+mR^2\dot{\phi}^2\left(1+2\cos^2\theta\right)$$

At equilibrium:

$$\left.\frac{d^2V^{effective}(\theta)}{d\theta^2}\right|_{equil}=-mgR\cos\theta_{equil}+mR^2\left(-\frac{g}{R\cos\theta_{equil}}\right)\left(1+2\cos^2\theta_{equil}\right)$$

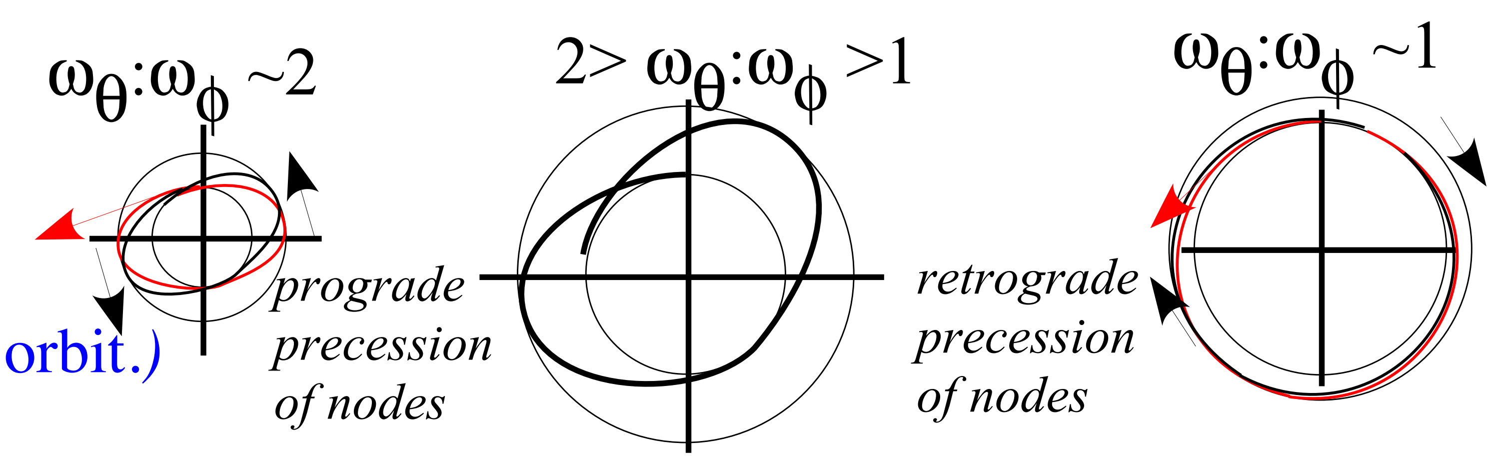
$$=-\frac{mgR}{\cos\theta_{equil}}\left(1+3\cos^2\theta_{equil}\right)$$

$$\boxed{\left(\omega_\theta^{equil}\right)^2/(\ \dot{\phi}_{equil}^2)=\left(1+3\cos^2\theta_{equil}\right)}$$

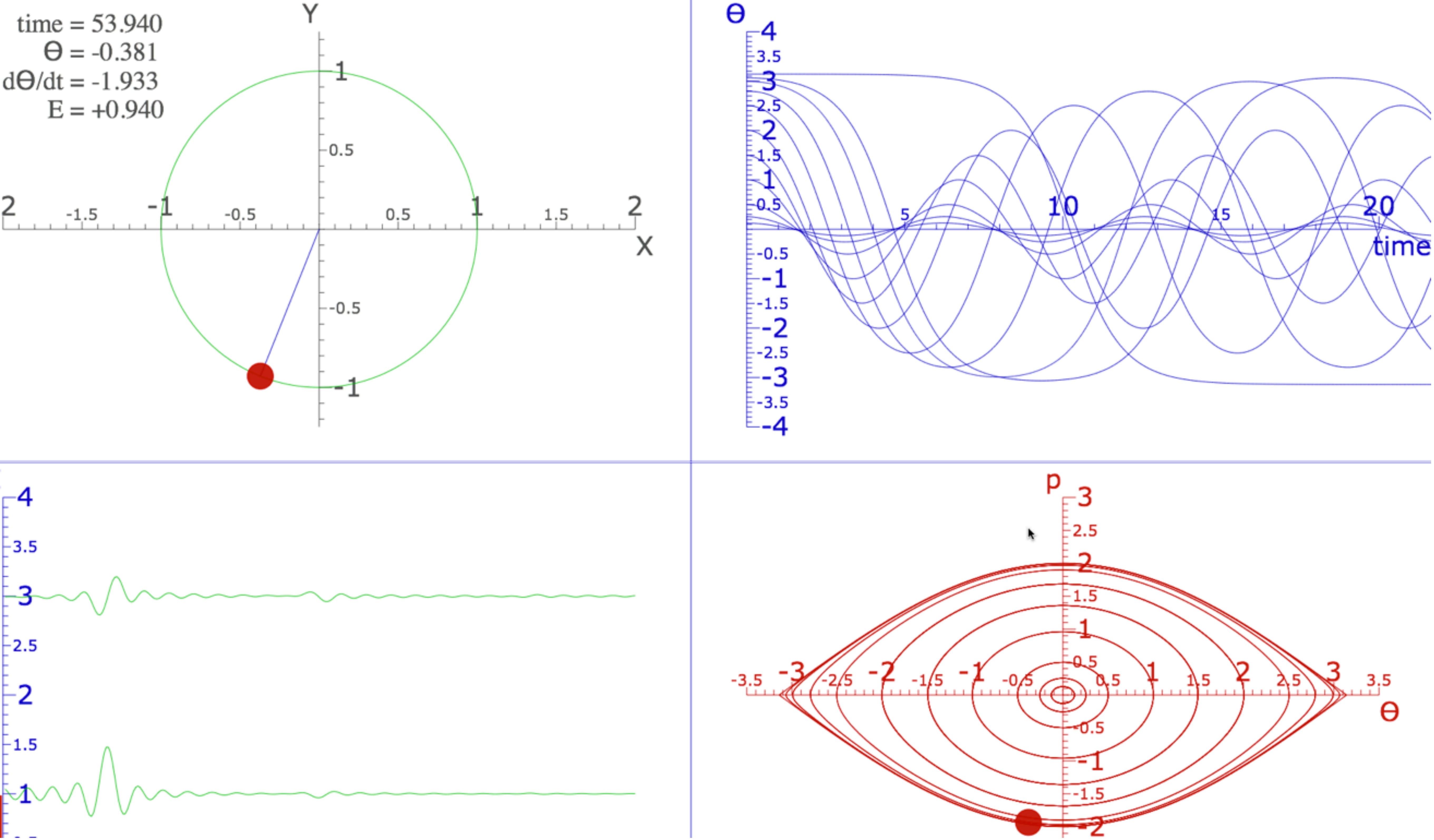
At bottom $\theta\to\pi$ the ratio of in-out ω_θ to circle ω_ϕ approaches 2:1

At equator $\theta\to\pi/2$ the ratio approaches 1:1.

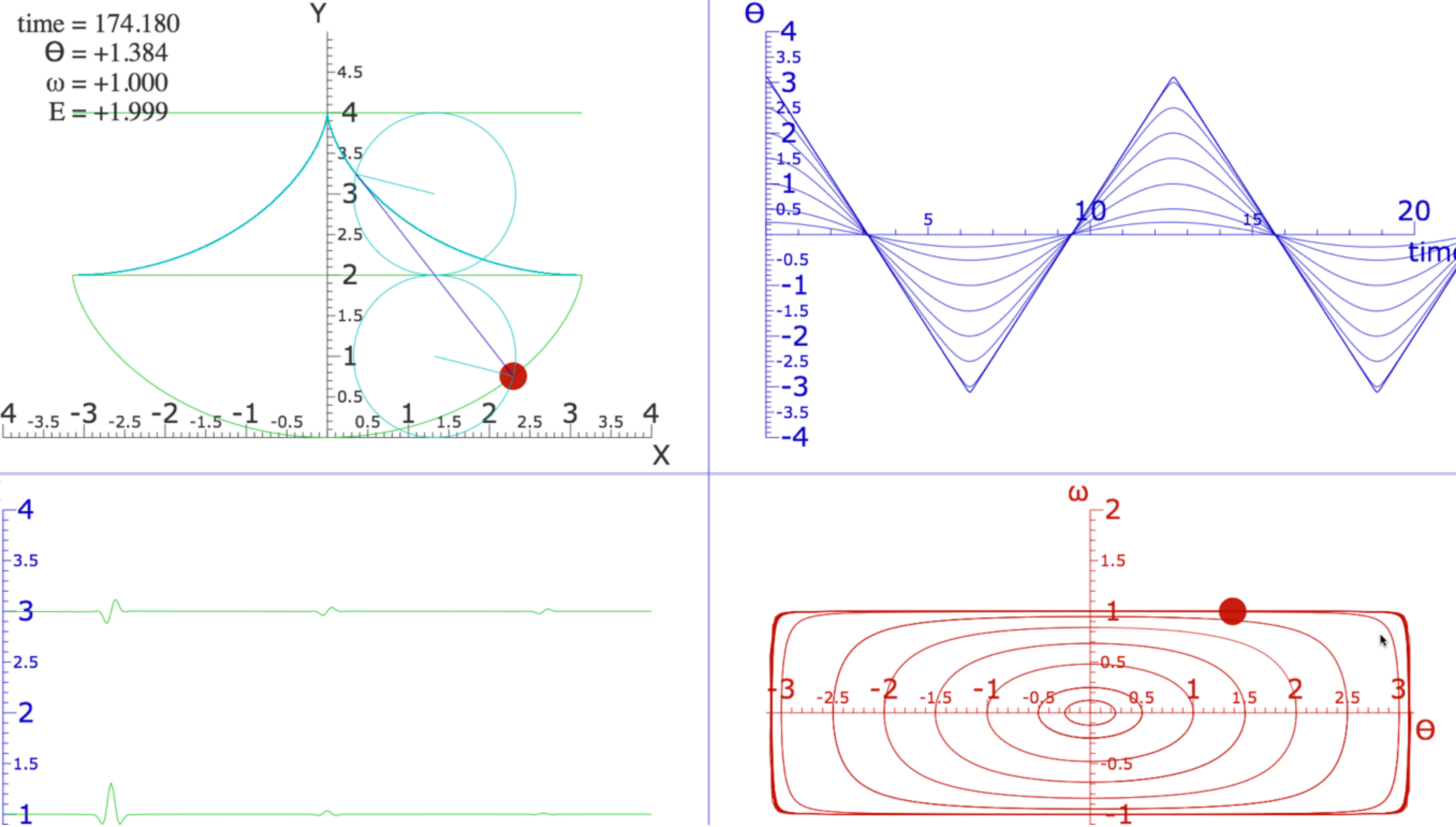
Ratio is between 2 and 1
(Usually irrational non-closed orbit).
(2:1 is like 2D IHO, but 1:1 is like coulomb orbit.)



→ *Cycloidal ruler&compass geometry*

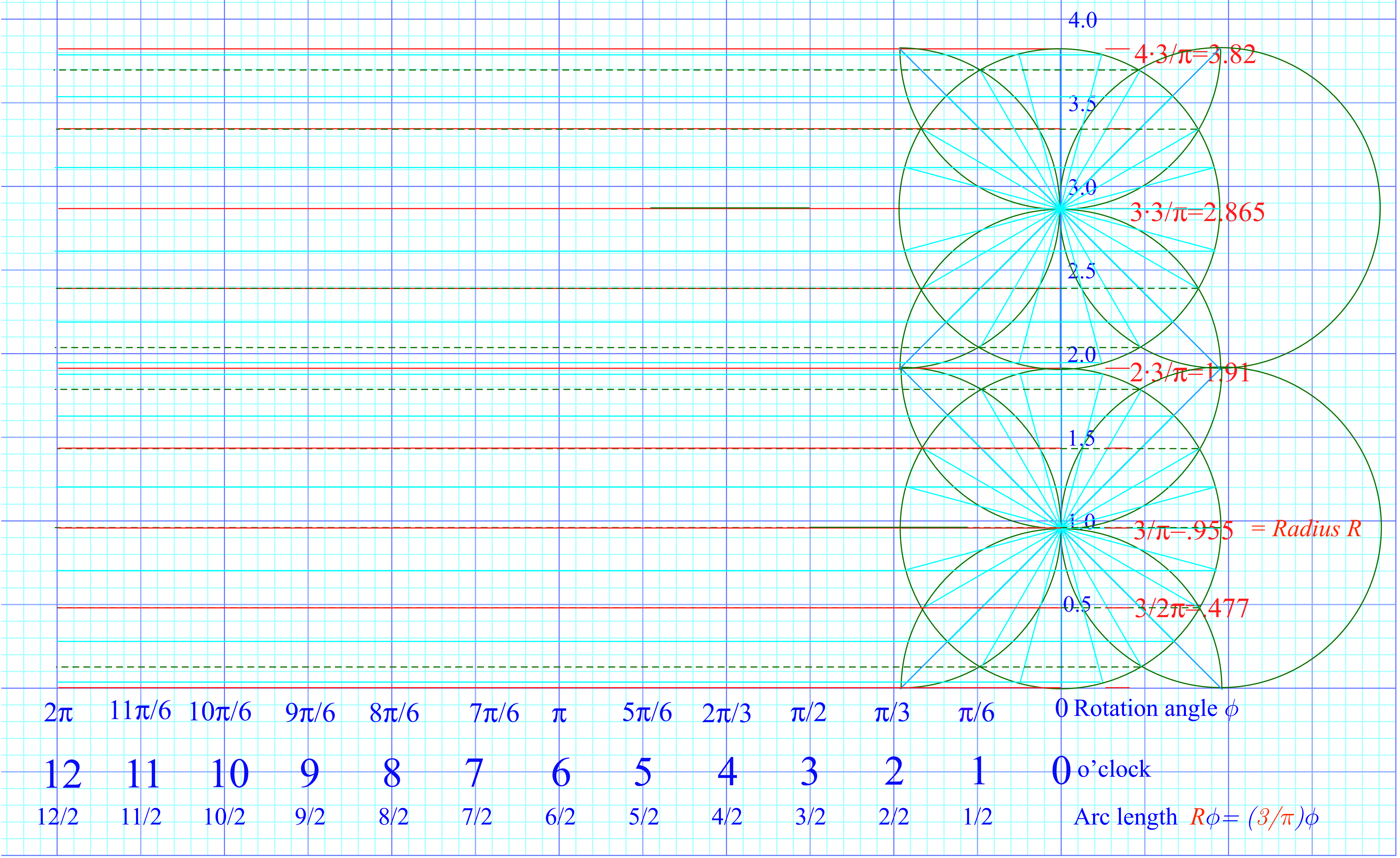


http://www.uark.edu/ua/modphys/markup/PendulumWeb.html



http://www.uark.edu/ua/modphys/markup/CycloidulumWeb.html

Here the radius is plotted as an irrational $R=3/\pi=0.955$ length so rolling by rational angle $\phi = m\pi/n$ is a rational length of rolled -out circumference $R\phi = (3/\pi)m\pi/n = 3m/n$. Diameter is $2R=6/\pi=1.91$



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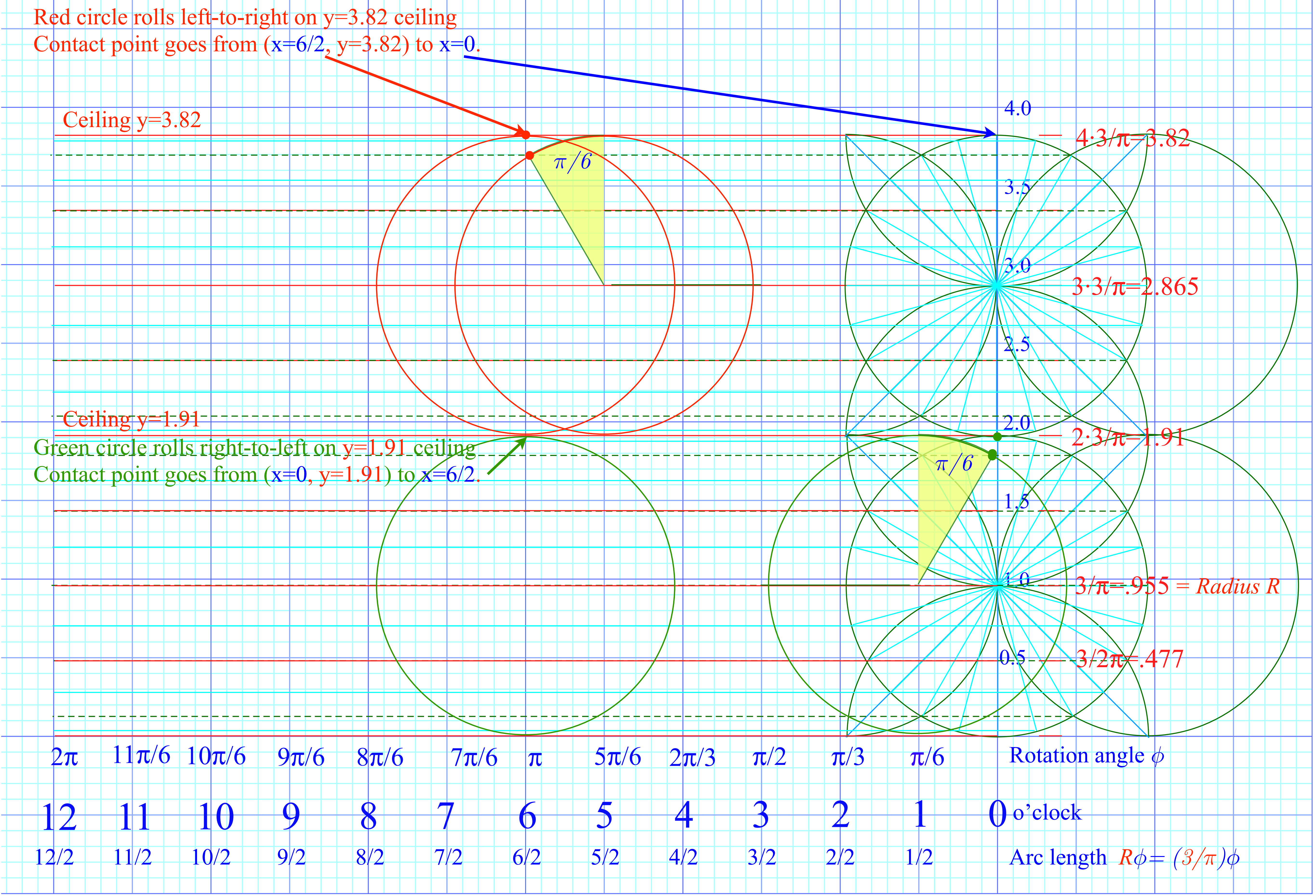

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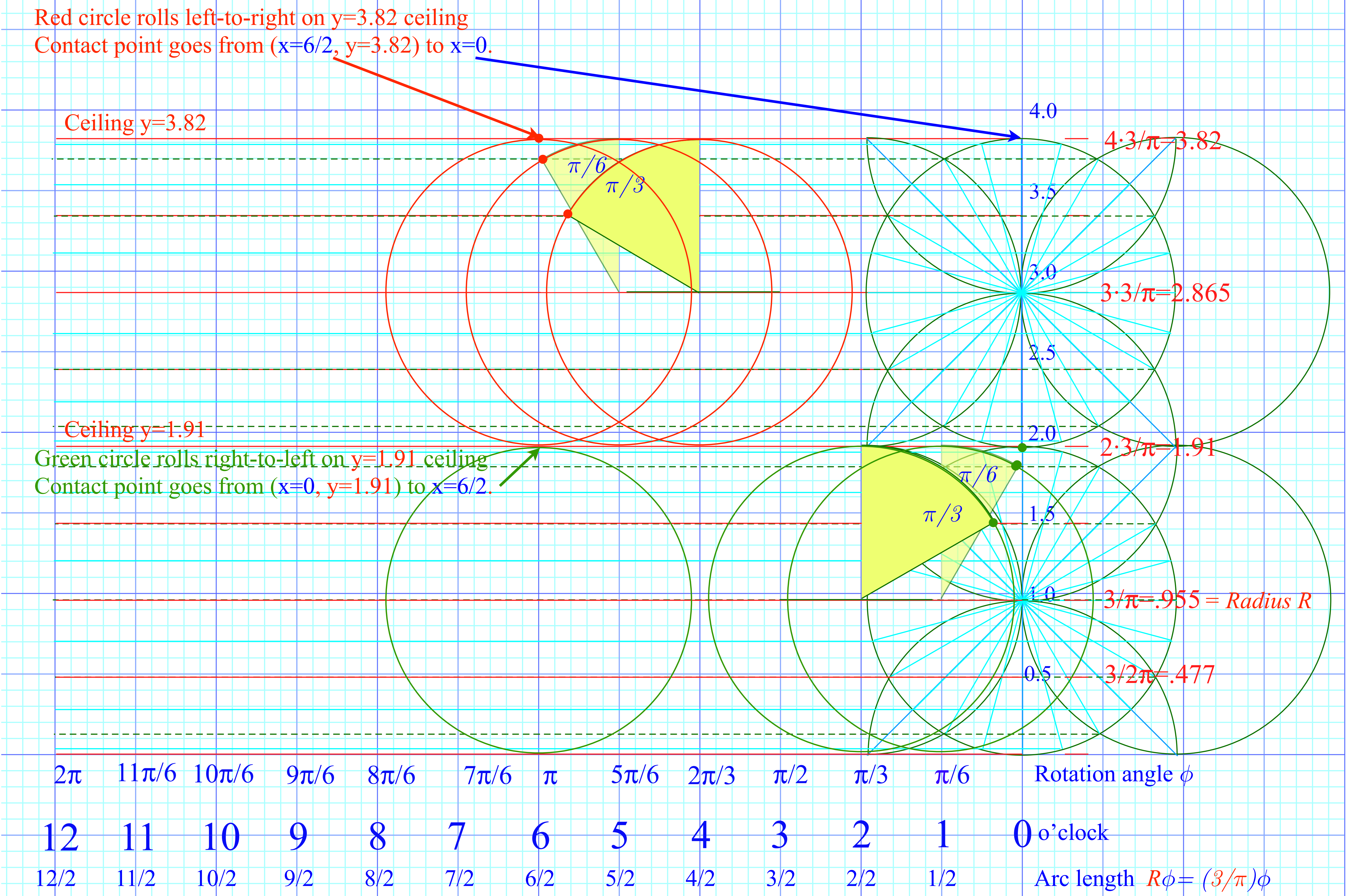

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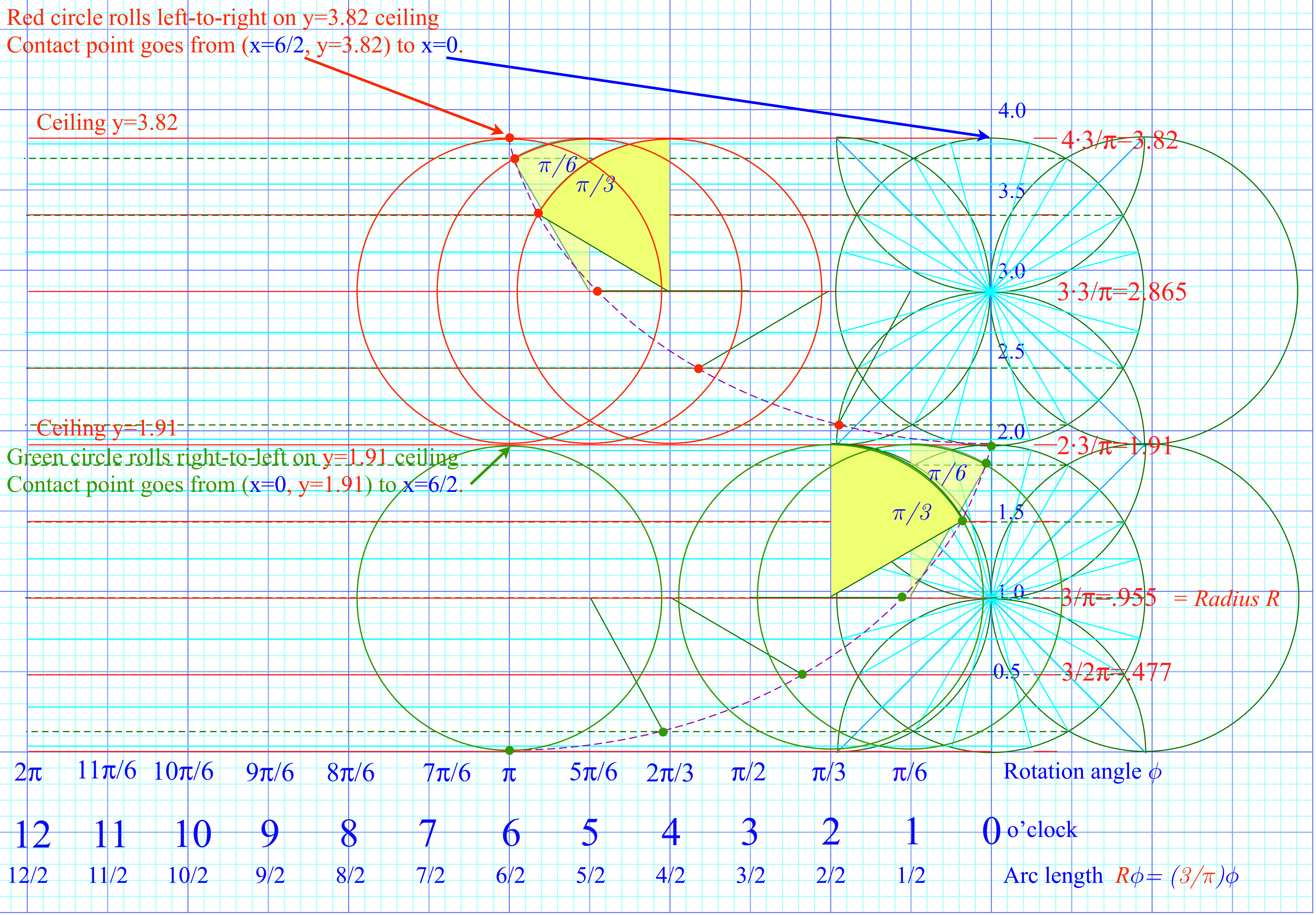

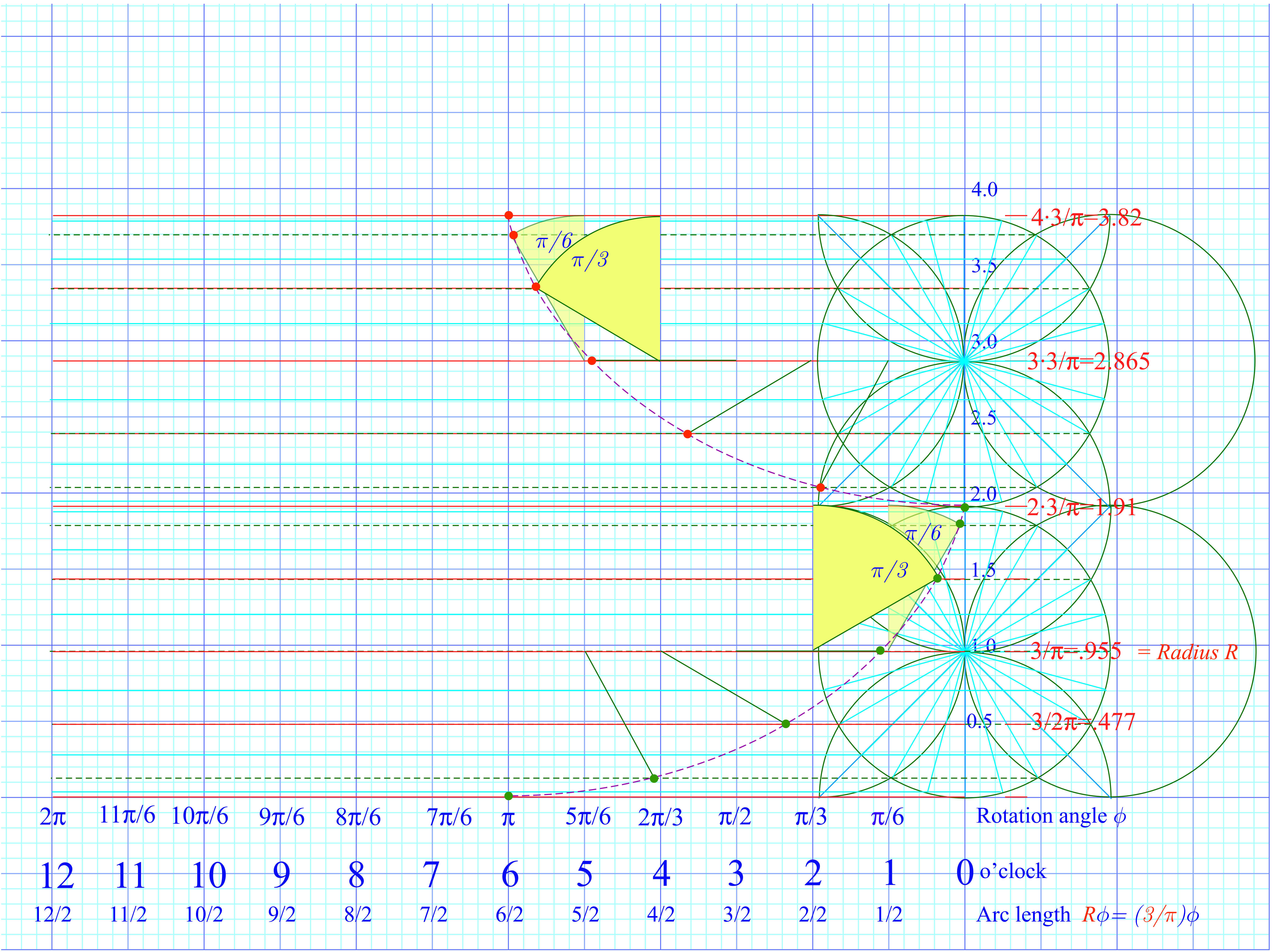

4.0
4·3/π=3.82
3.5
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π/3
3.0
3·3/π=2.865
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2.0
2·3/π=1.91
π/6
π/3
1.5
1.0
3/π=.955 = Radius R
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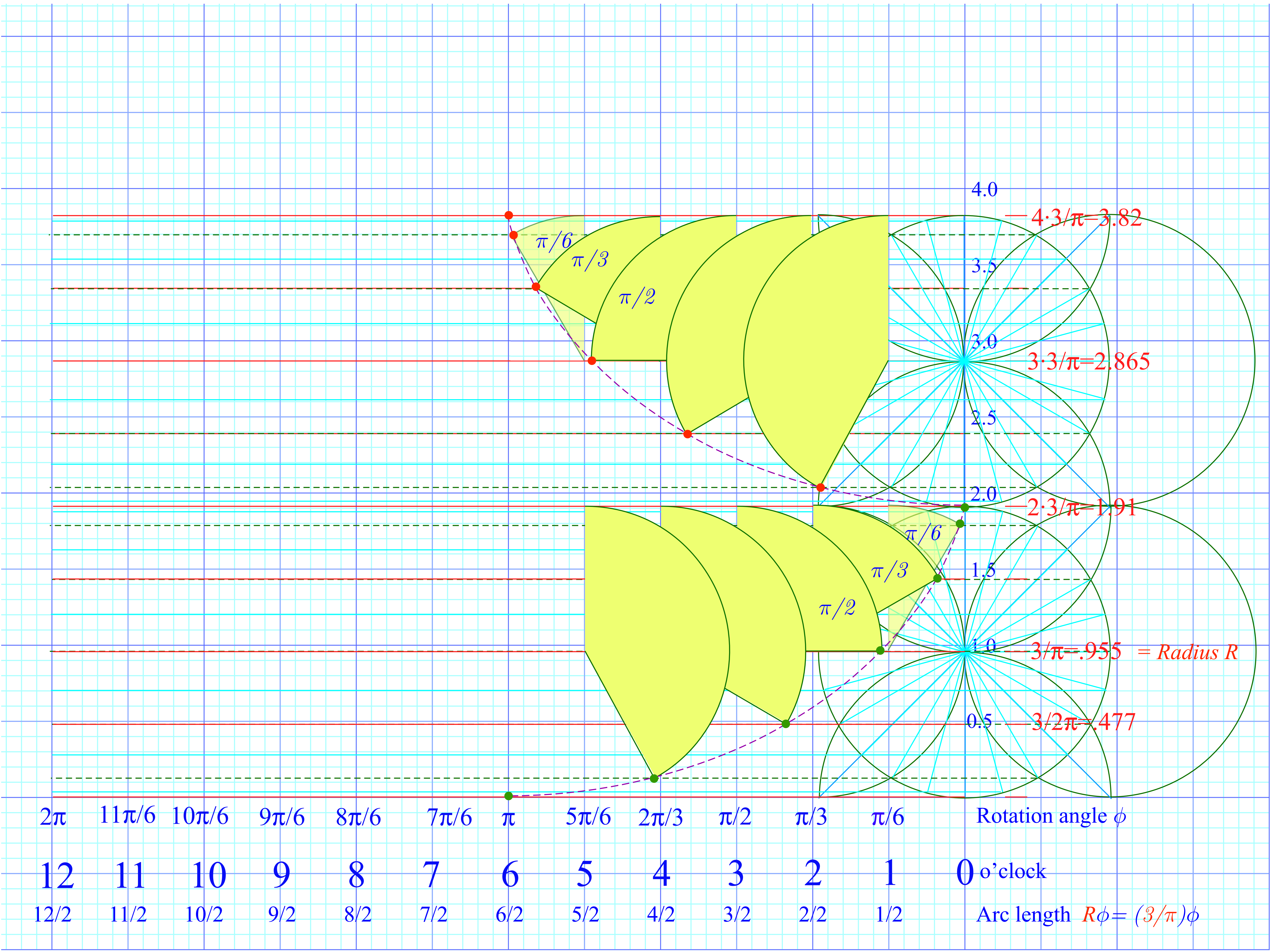
3/2π=.477
2π 11π/6 10π/6 9π/6 8π/6 7π/6 π 5π/6 2π/3 π/2 π/3 π/6
Rotation angle φ
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2
Arc length Rφ= (3/π)φ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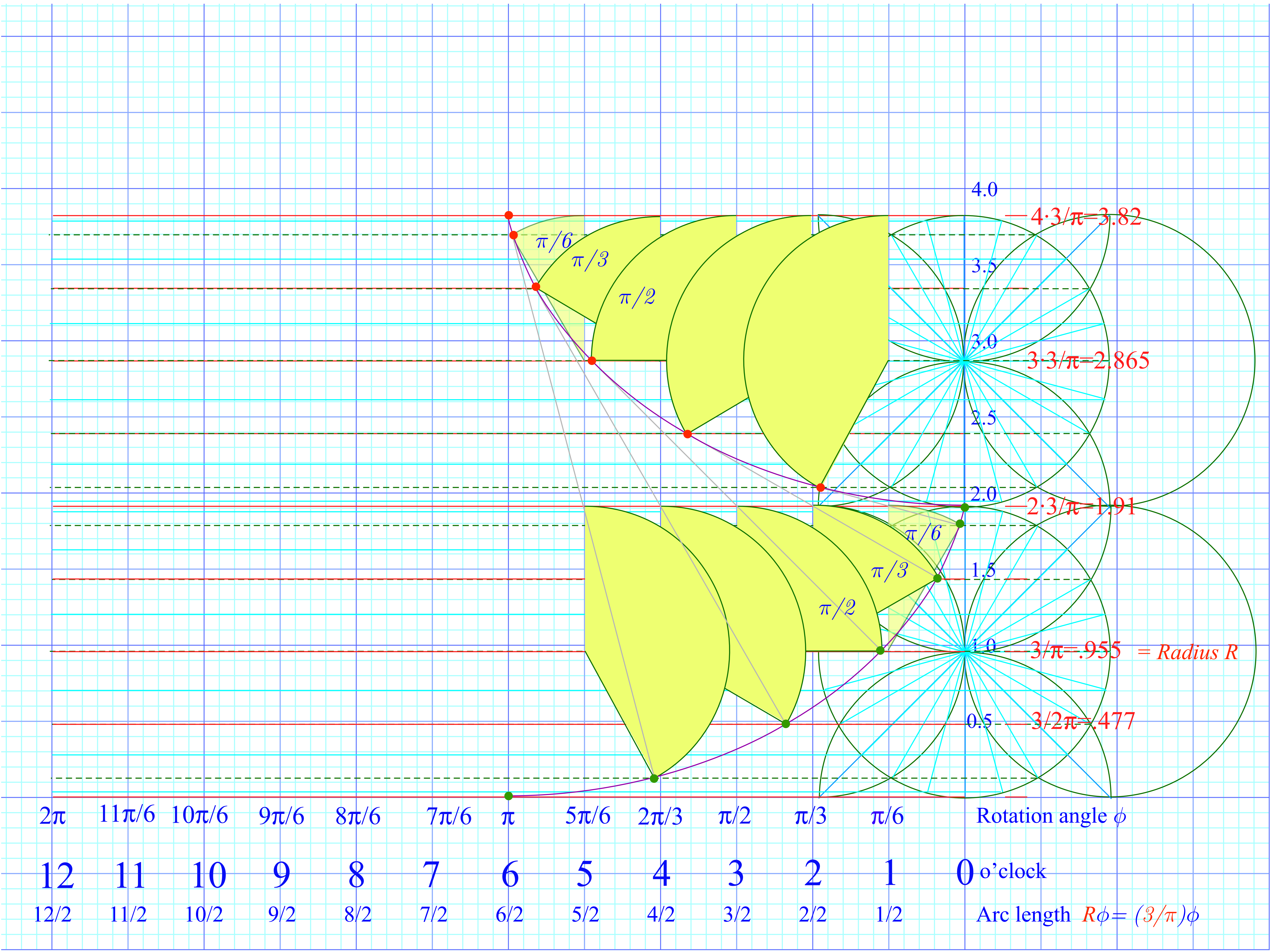
12 11 10 9 8 7 6 5 4 3 2 1 0 o'clock
12/2 11/2 10/2 9/2 8/2 7/2 6/2 5/2 4/2 3/2 2/2 1/2 Arc length Rϕ= (3/π)ϕ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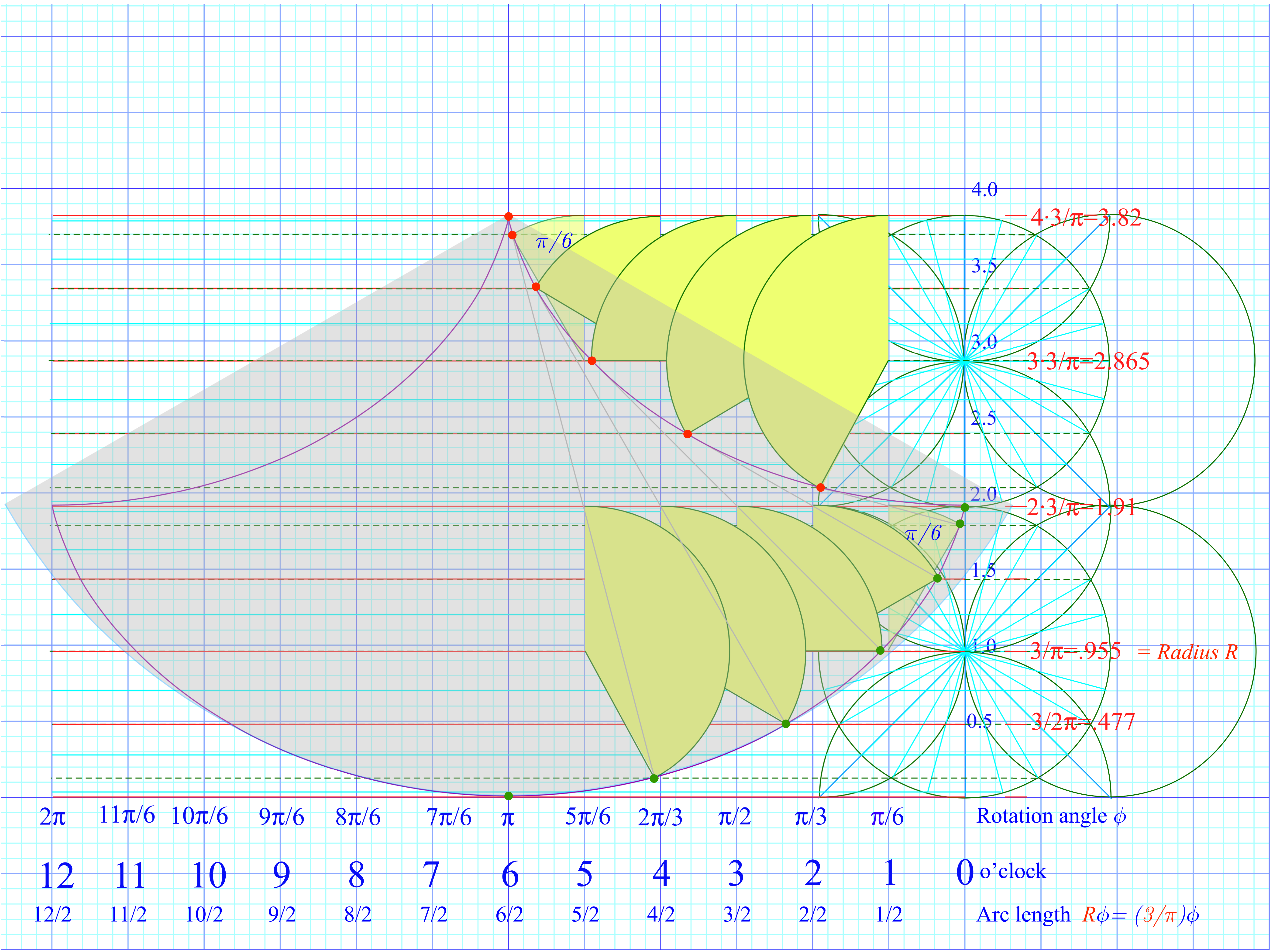

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