Lecture 14
Tue. 10.10.2017

*treb-yew-shay

# *Introducing GCC Lagrangian `a la Trebuchet* Dynamics*

*Ch. 1-3 of Unit 2 and Unit 3 (Mostly Unit 2.)*

*The trebuchet (or ingenium) and its cultural relevancy (3000 BCE to 21st See Sci. Am. **273**, 66 (July 1995))*
- *The medieval ingenium (9th to 14th century) and modern re-enactments*
- *Human kinesthetics and sports kinesiology*

*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*
- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* $\gamma_{mn}$
- *Structure of dynamic metric tensor* $\gamma_{mn}$
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* $\gamma_{mn}$ *tensor*

*Summary of Lagrange equations and force analysis (Mostly Unit 2.)*
- *Forces: total, genuine, potential, and/or fictitious*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)*
- *Multivalued functionality and connections*
- *Covariant and contravariant relations*
    - *Tangent space vs. Normal space*
- *Metric* $g_{mn}$ *tensor geometric relations to length, area, and volume*

## Chapter 1. The Trebuchet: A dream problem for Galileo?

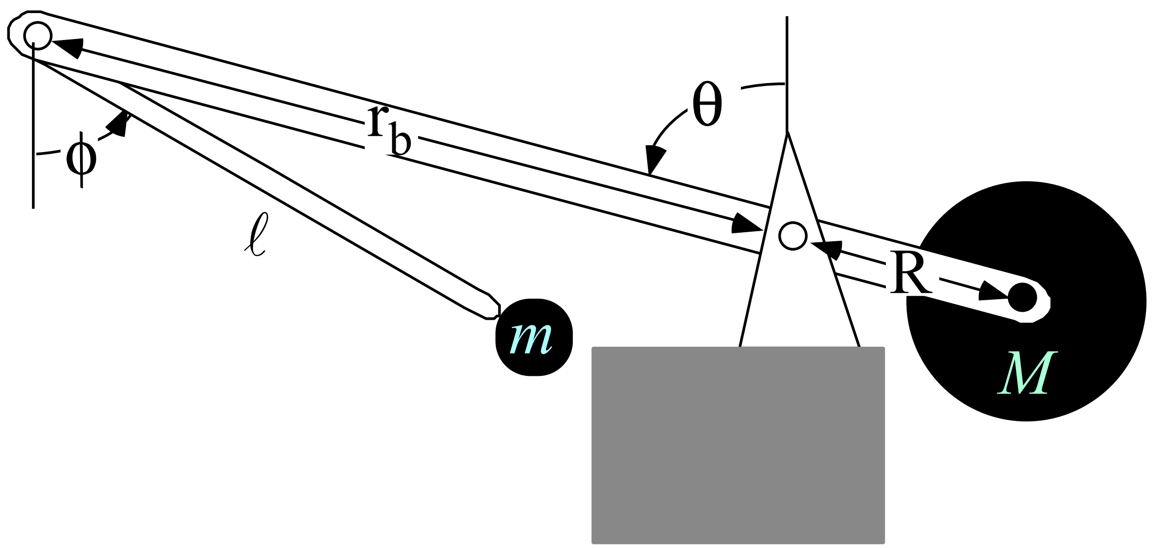
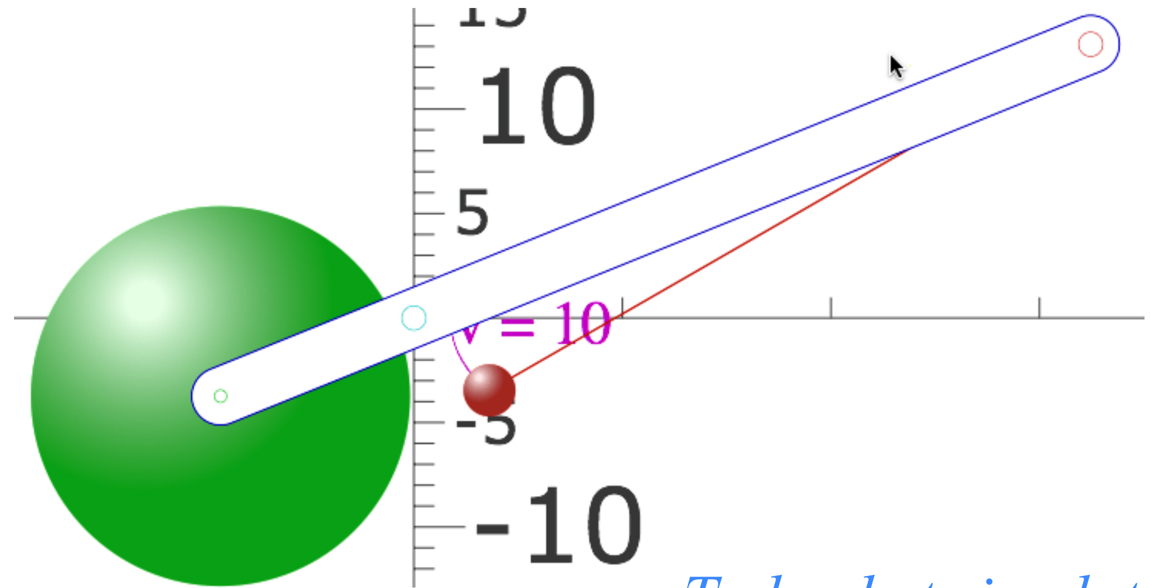



*Fig. 2.1.1 An elementary ground-fixed trebuchet*




*Trebuchet simulator*

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html

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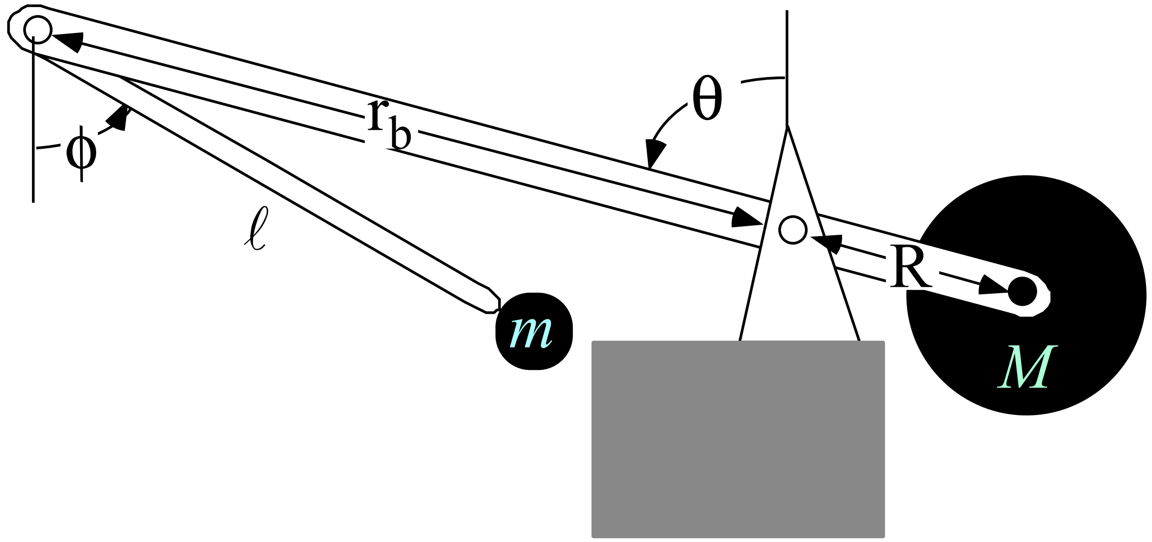


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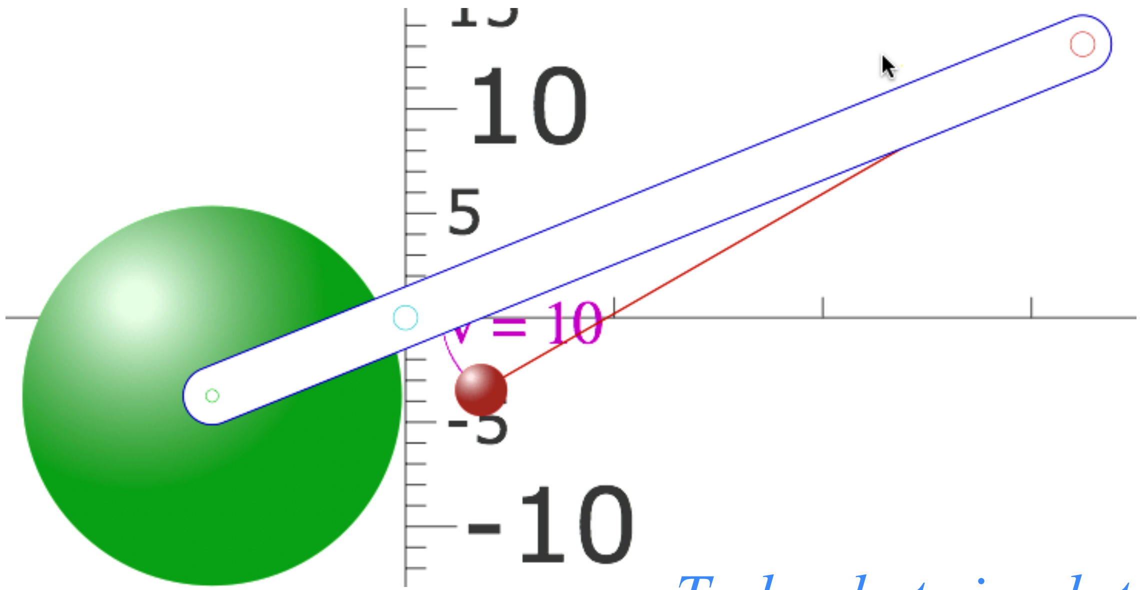


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http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=MontezumasRevenge

Trebuchet Web App, use a URL with the string after the equal sign replaced with the desired scenario.

PlotPosSpaceCourse
PlotPosSpaceFine
AnimateFlinger
AnimateTrebuchet
MontezumasRevenge
SeigeOfKenilworth

http://www.uark.edu/ua/modphys/markup/TrebuchetWeb.html?scenario=XXX =>

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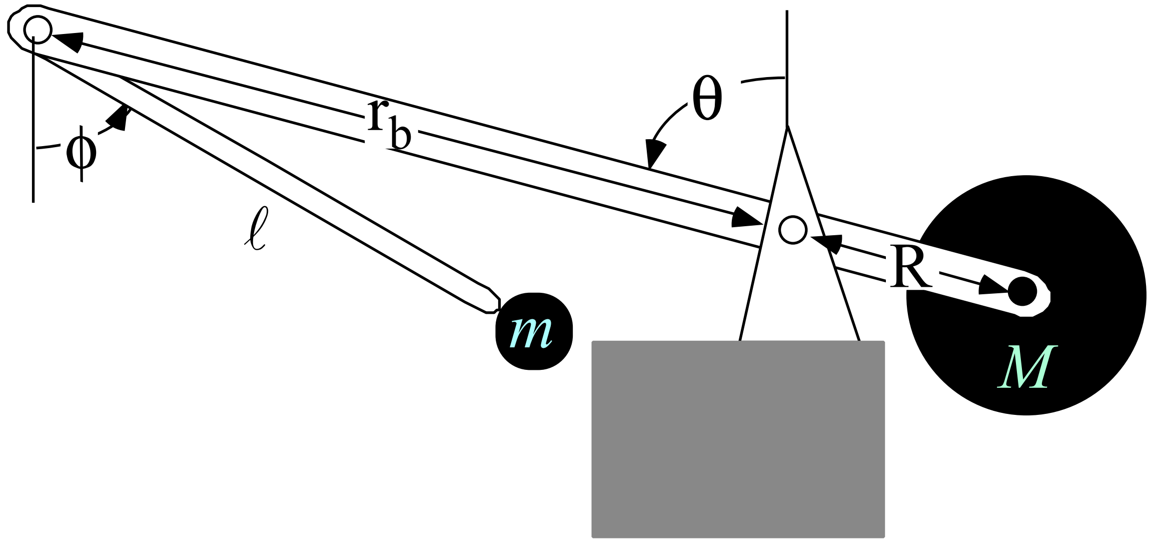


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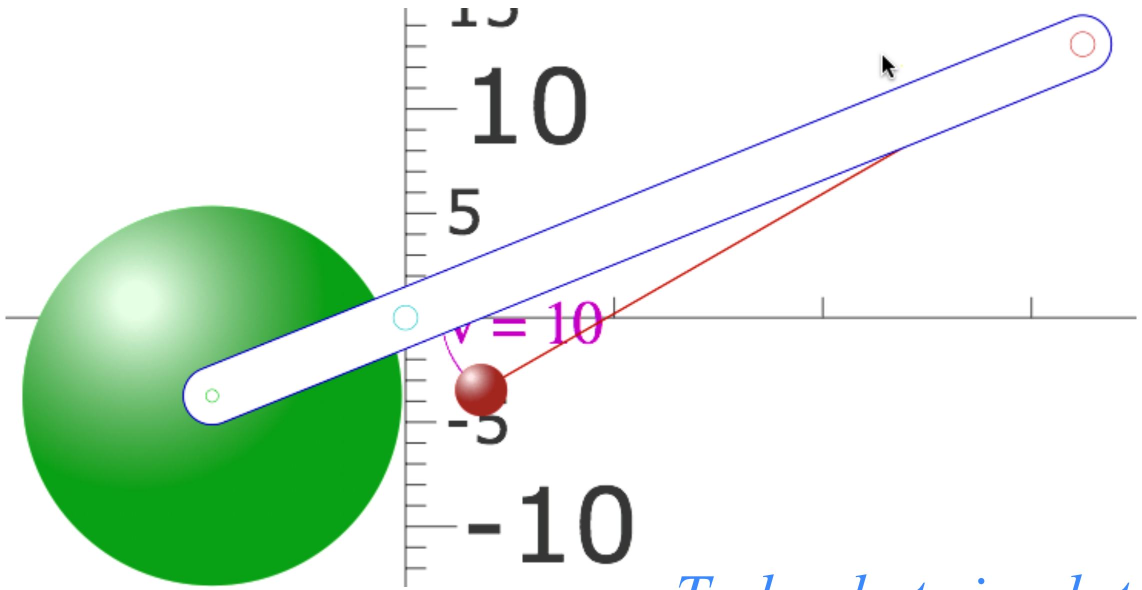


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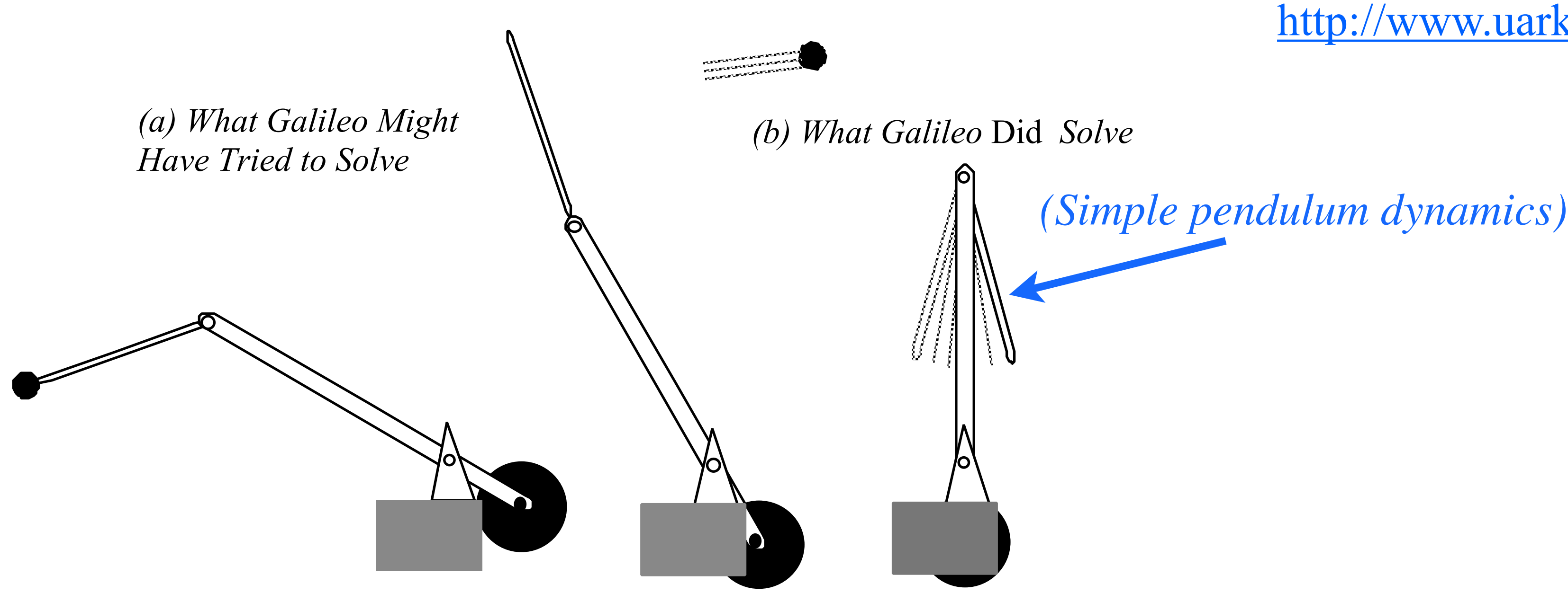


*Fig. 2.1.2 Galileo's (supposed fictitious) problem*

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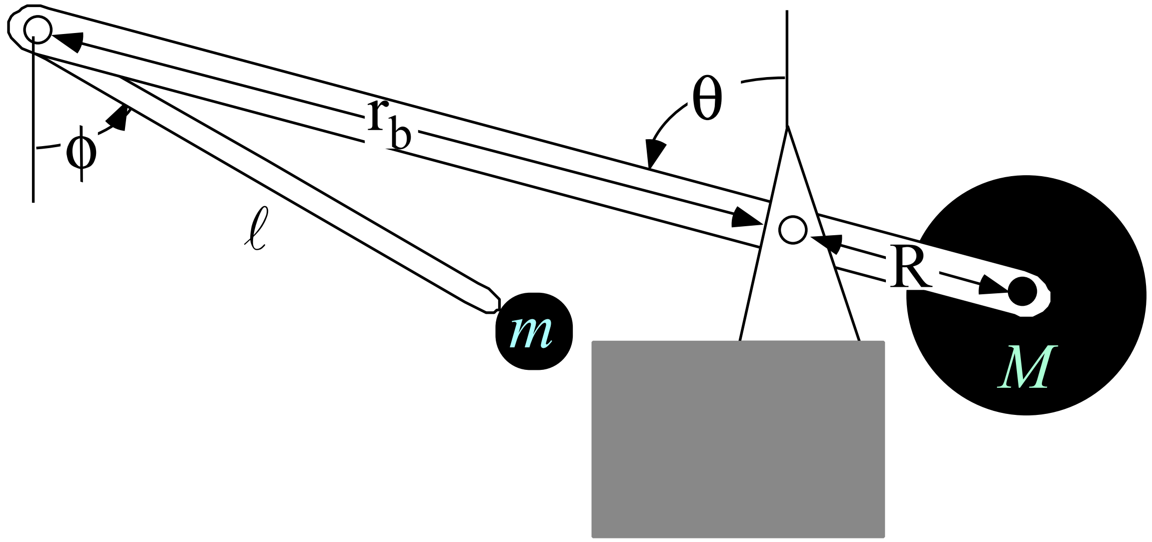


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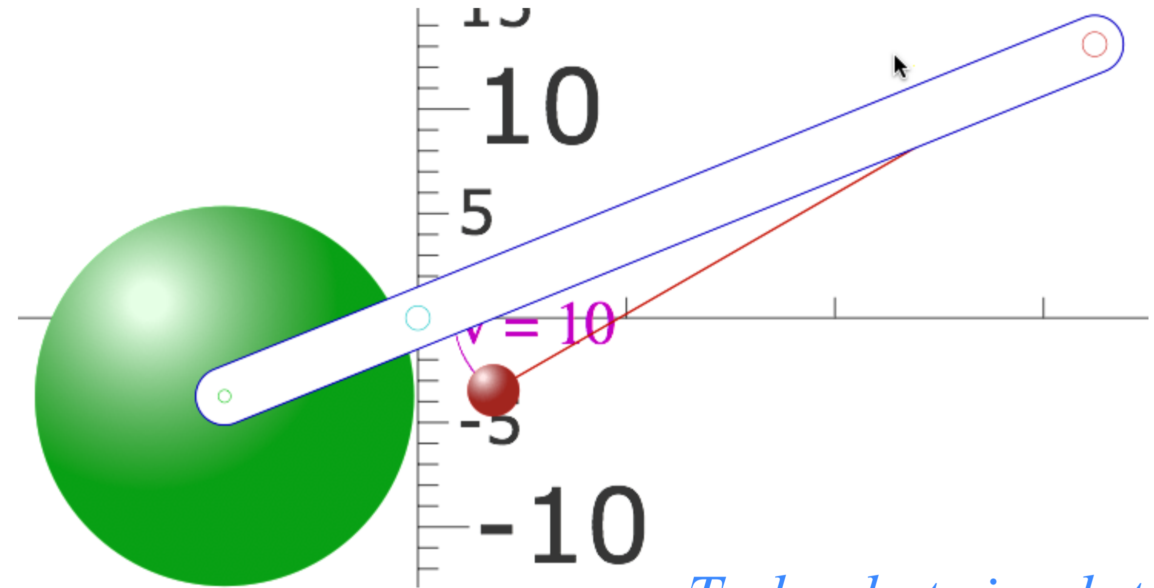


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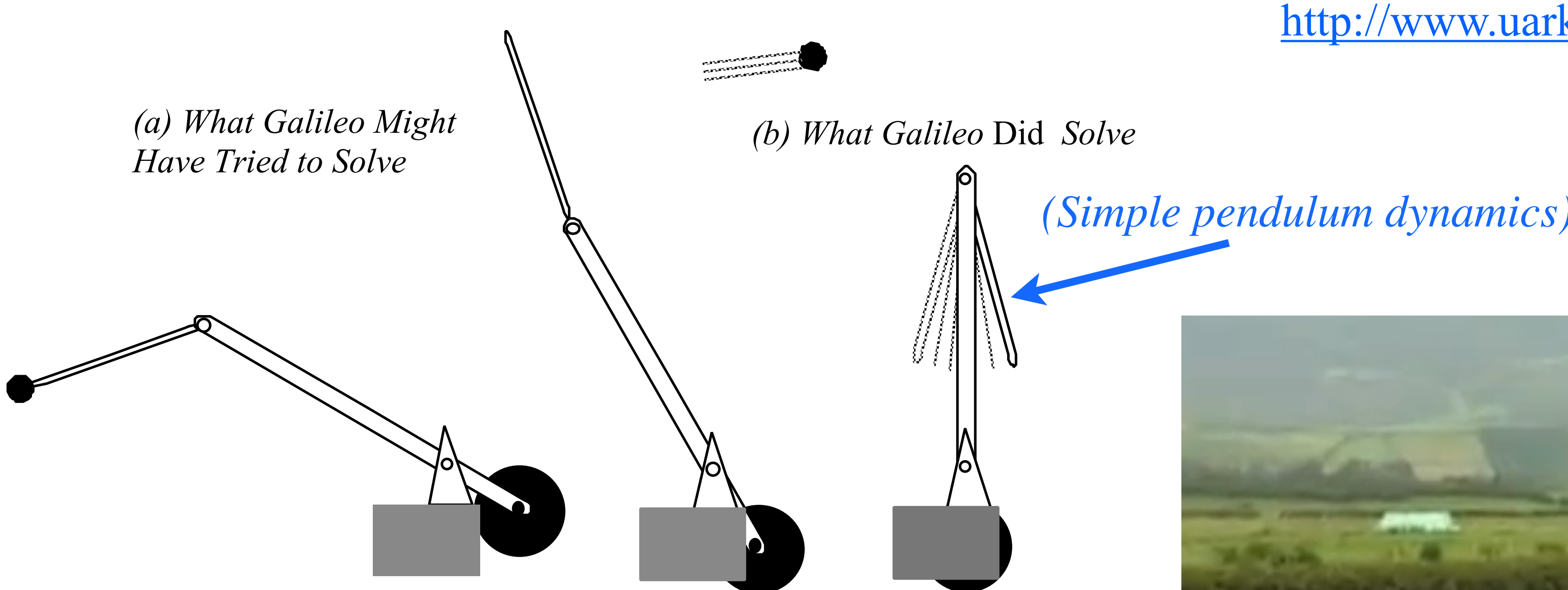


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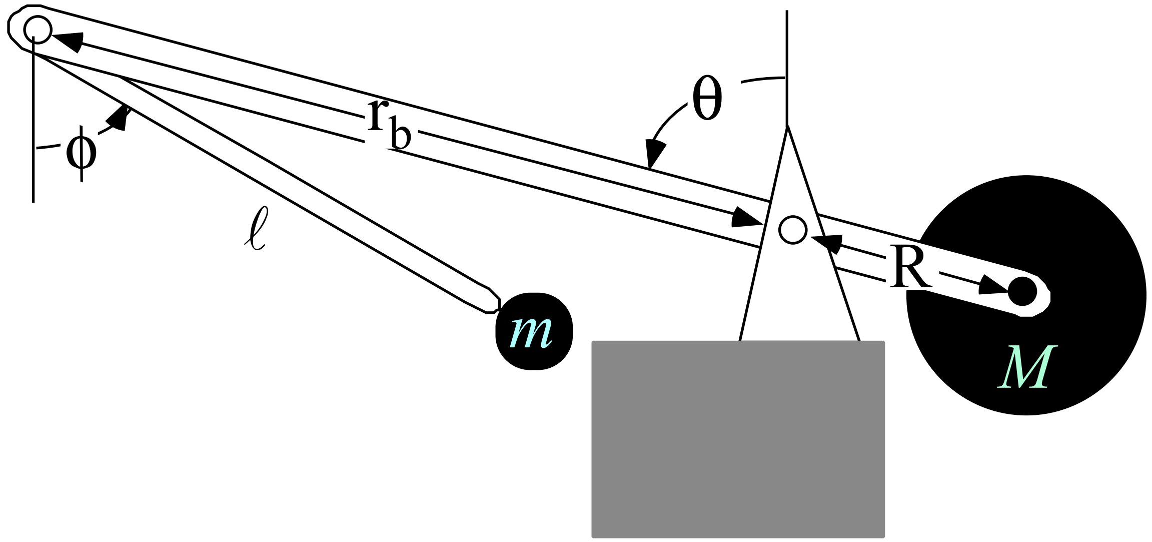


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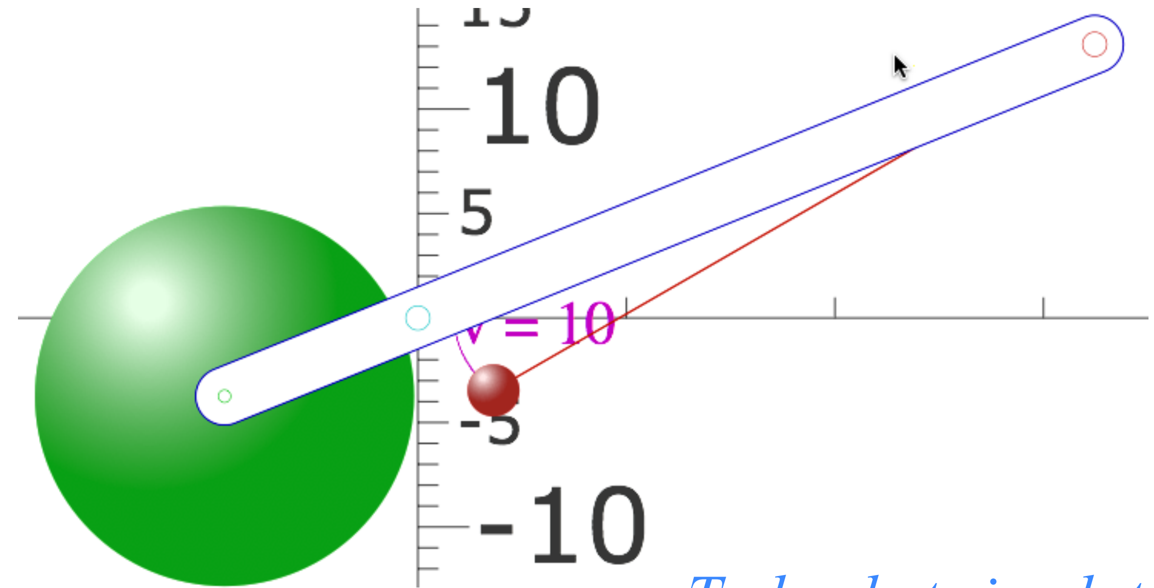


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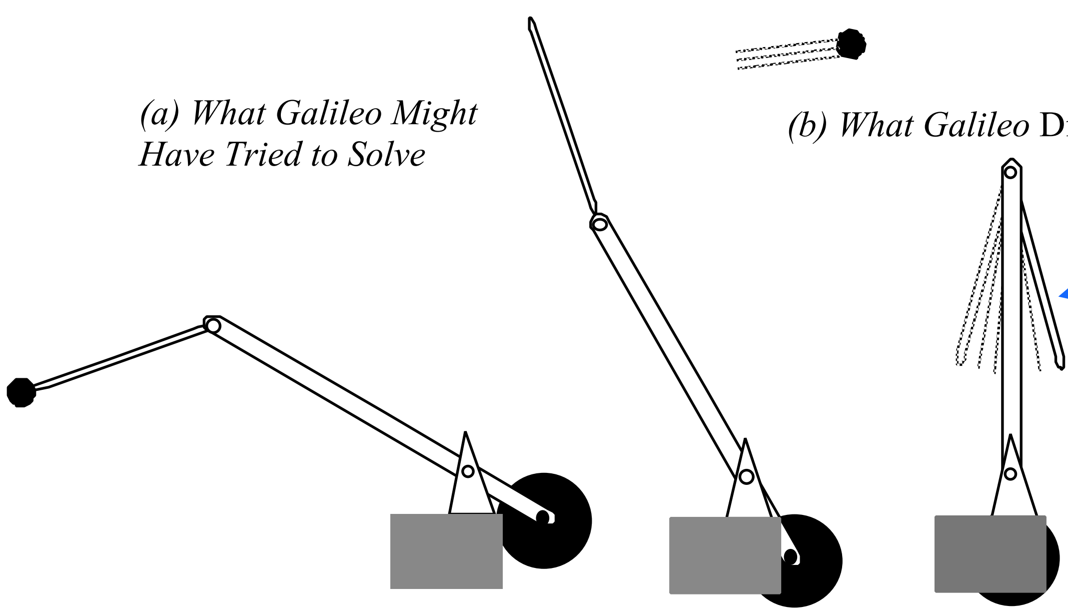



*Fig. 2.1.2 Galileo's (supposed) problem*

*(Simple pendulum dynamics)*

## *It's Halloween!...and time for Punkin' Chunkin' Trebuchets*



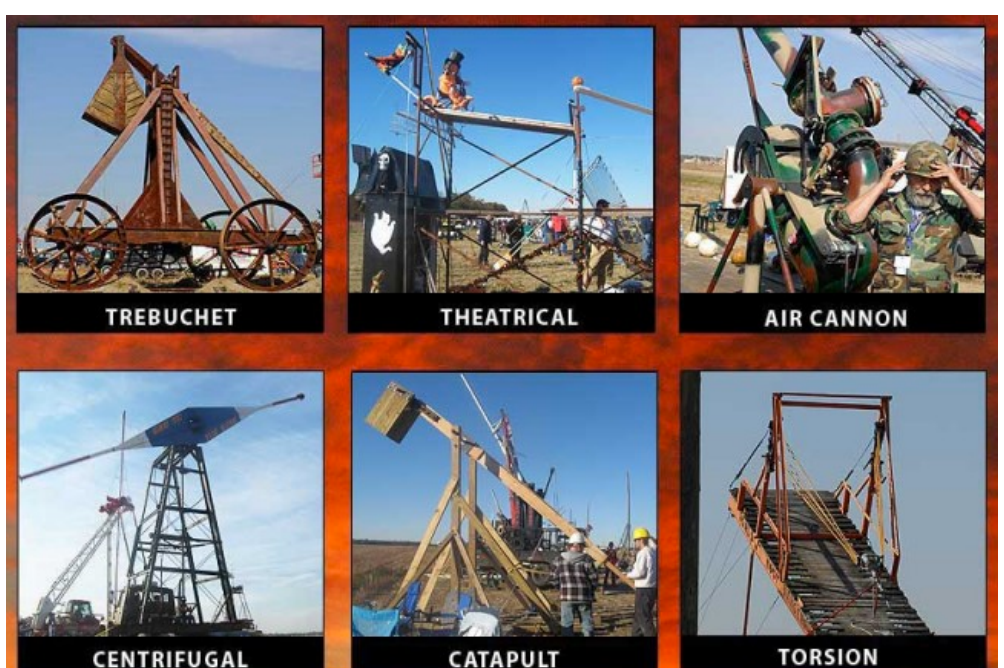
http://thearmchaircritic.blogspot.com/2011/11/punkin-chunkin.html

## *As happened in history...Trebuchet is replaced by higher-tech (or lower tech)*

## *Giant cannons can chunk-a-punkin over 4,000 ft.* *Trebuchet range max ~1,200ft.*

http://www.twcenter.net/forums/showthread.php?358315-Shooting-range-for-medieval-siege-weapons-Anybody-knows


http://www.sussexcountyonline.com/news/photos/punkinchunkin.html

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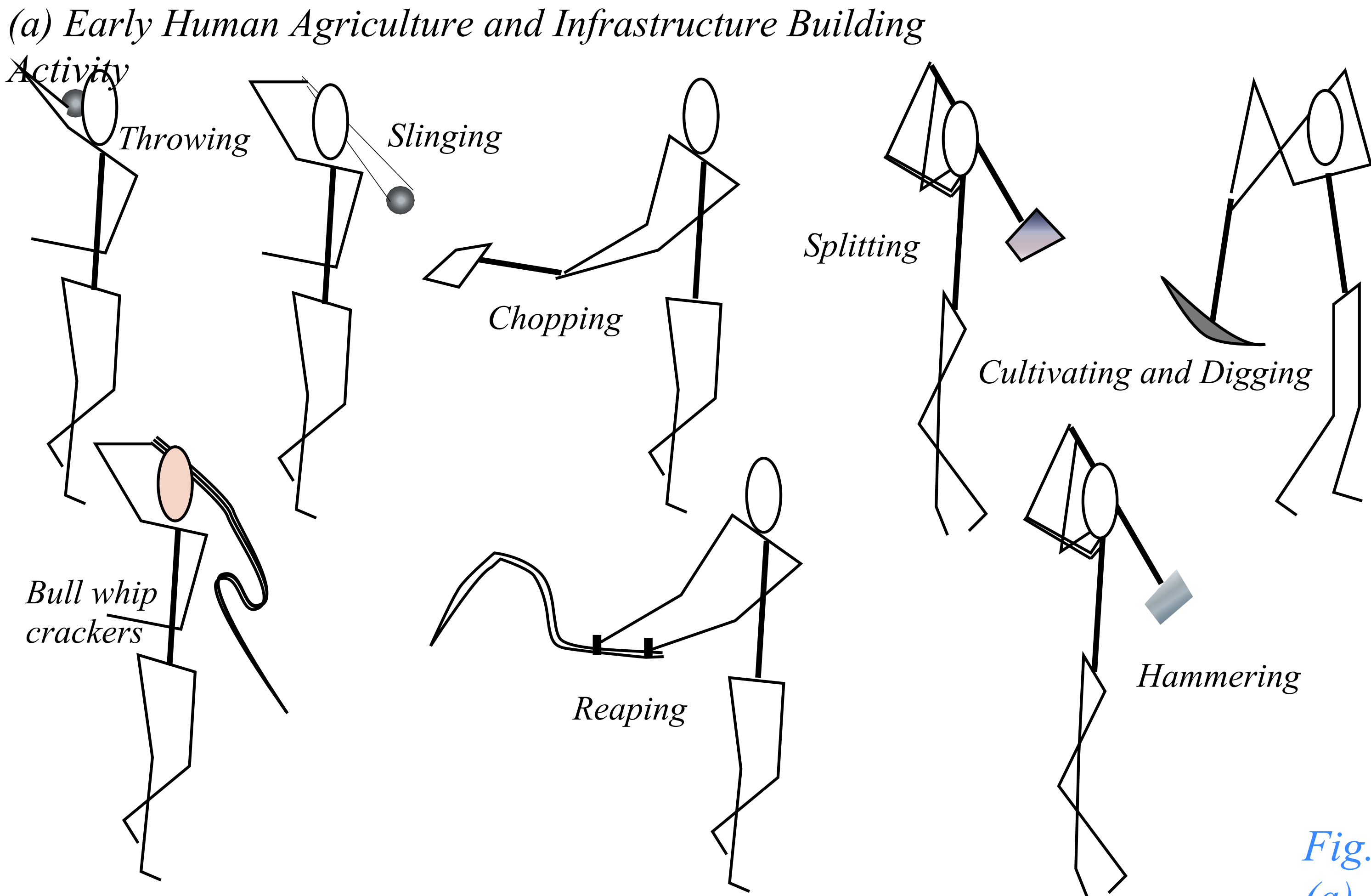


Some technique required!
KE achieved by non-linear whip action
Must avoid injury

Fig. 2.1.3 Trebuchet-like motion of humans.
(a) Early work.

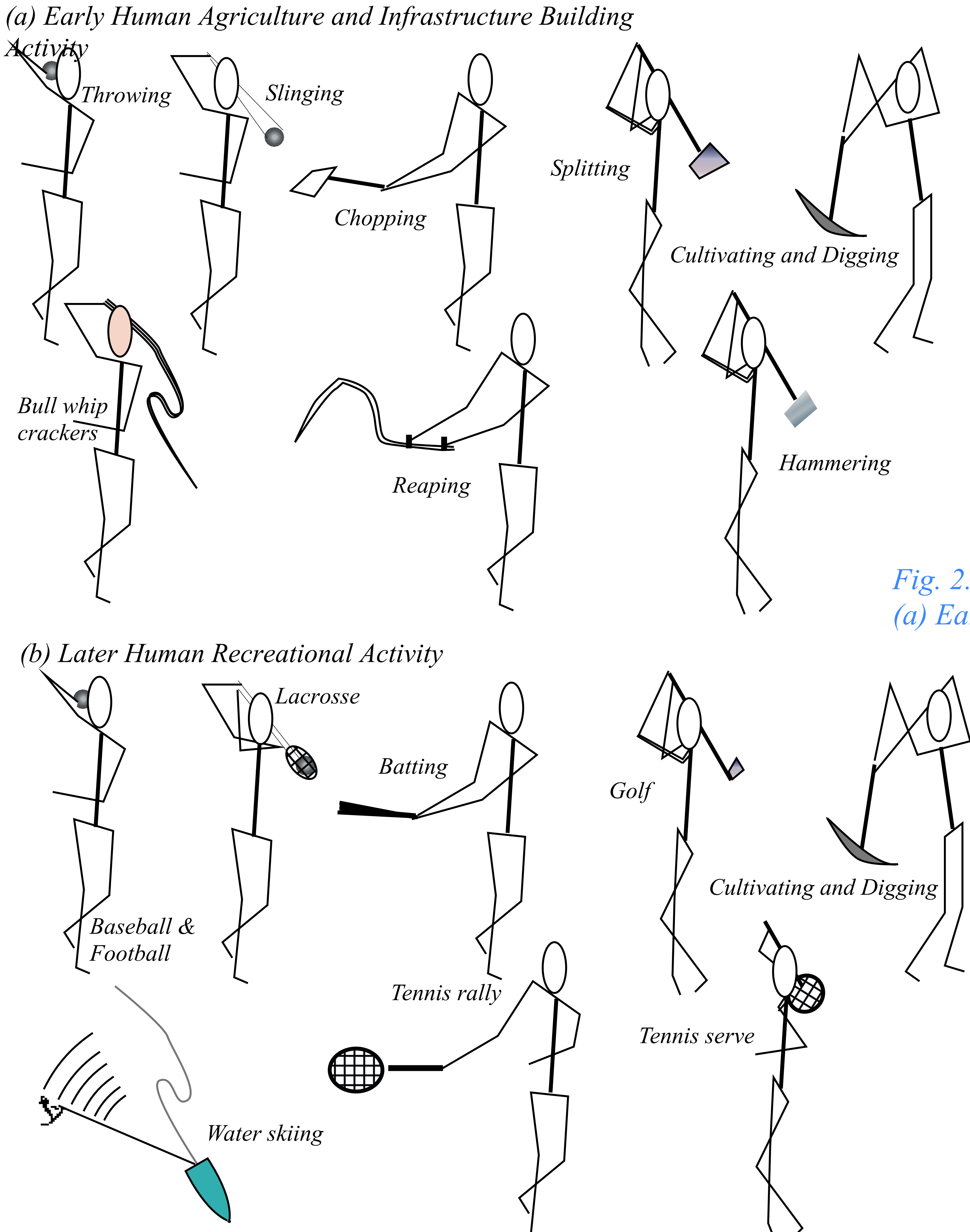


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*Fig. 2.1.3 Trebuchet-like motion of humans.*
*(a) Early work. (b) Later recreational kinesthetics.*

*Some technique required!*
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*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*

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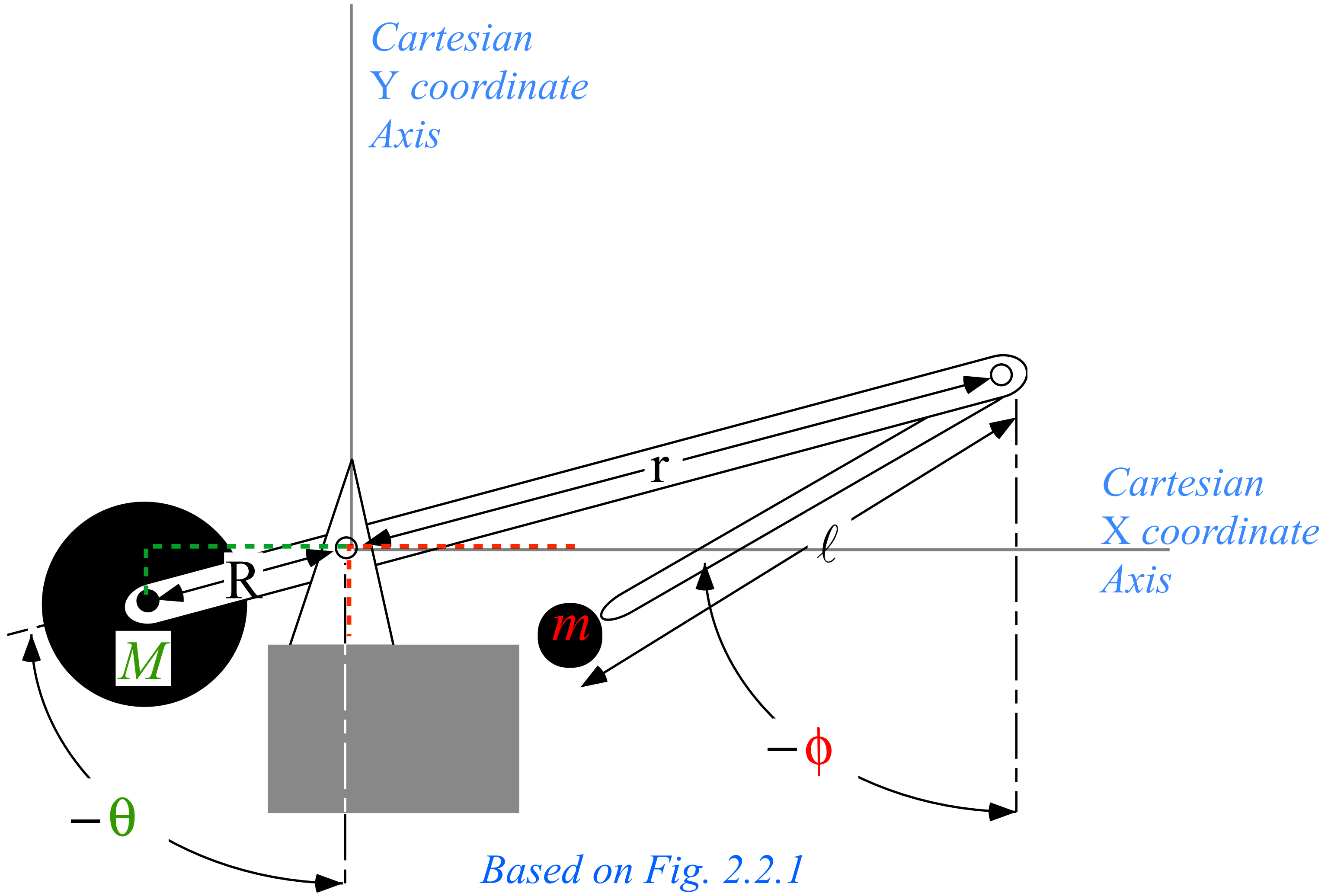
# *Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$*

*geometry of trebuchet*

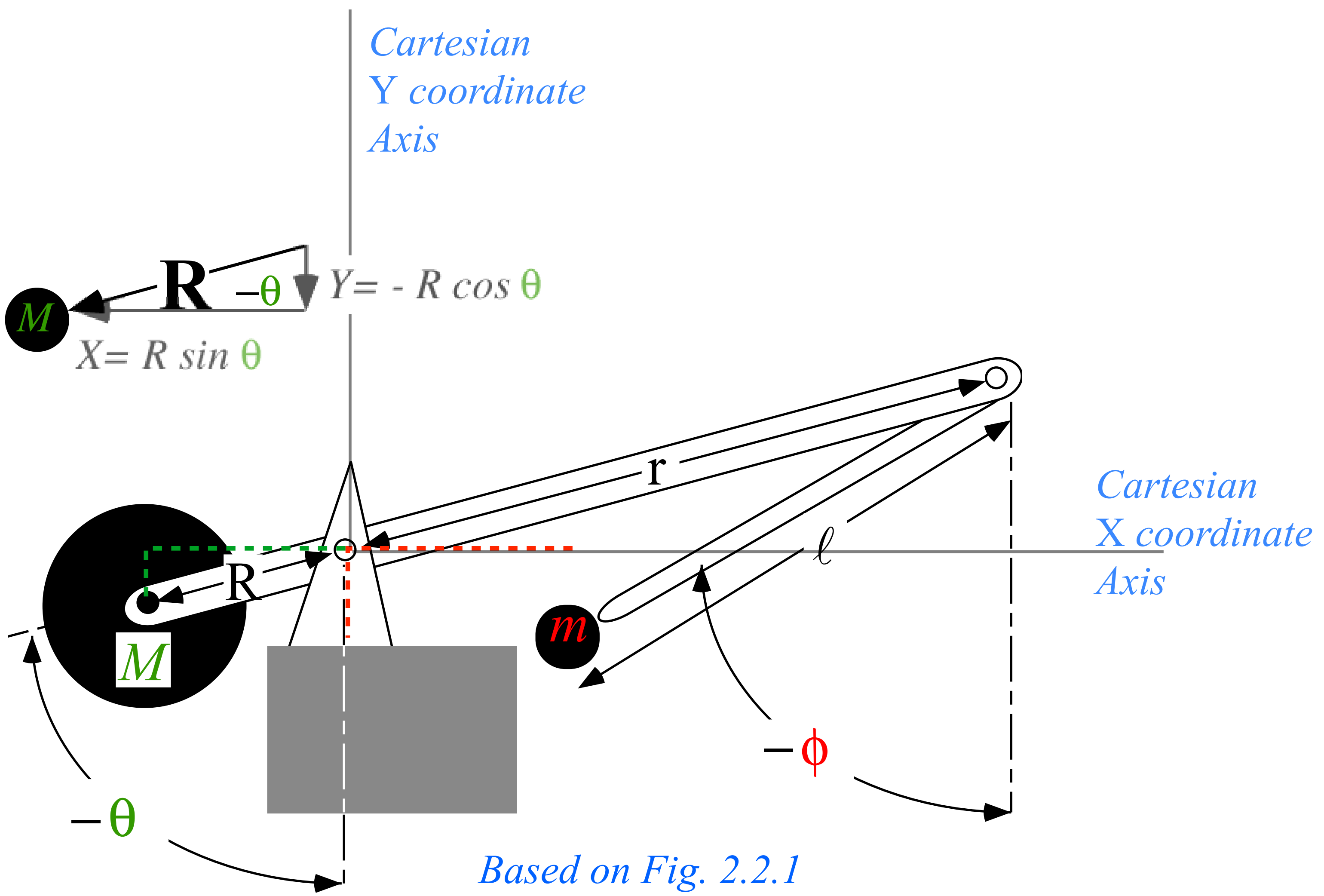



*Based on Fig. 2.2.1*

# Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$

Coordinates of *M*
*(Driving weight Mg)*:
$X = R \sin \theta$
$Y = -R \cos \theta$

*geometry of trebuchet*

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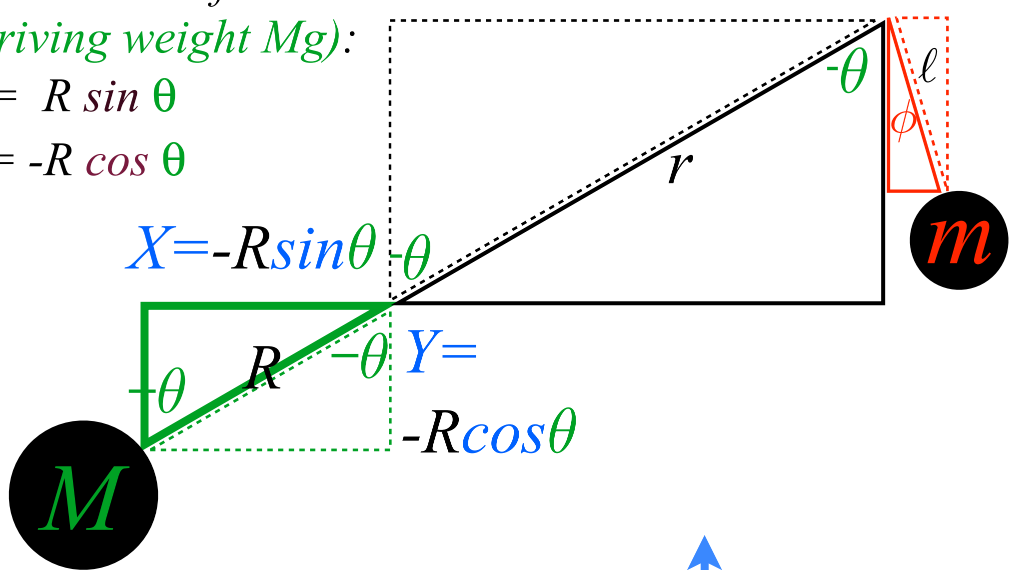
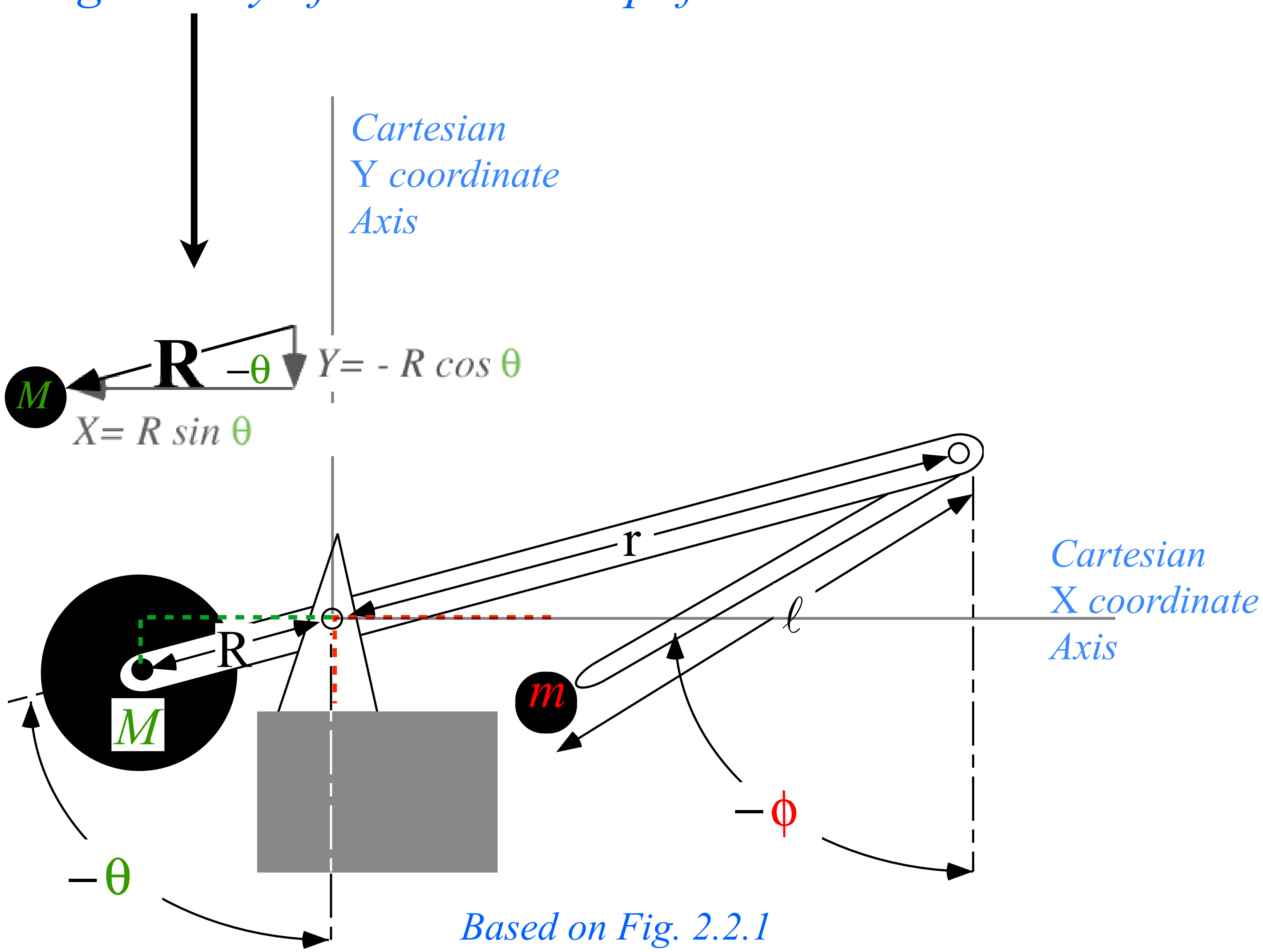



*geometry of trebuchet simplified somewhat...*

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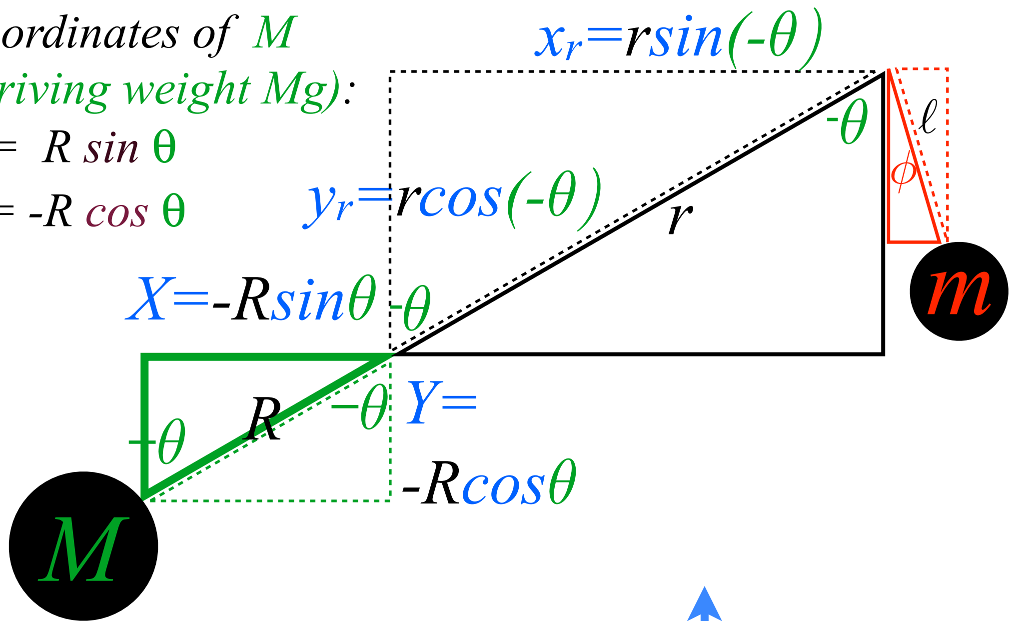


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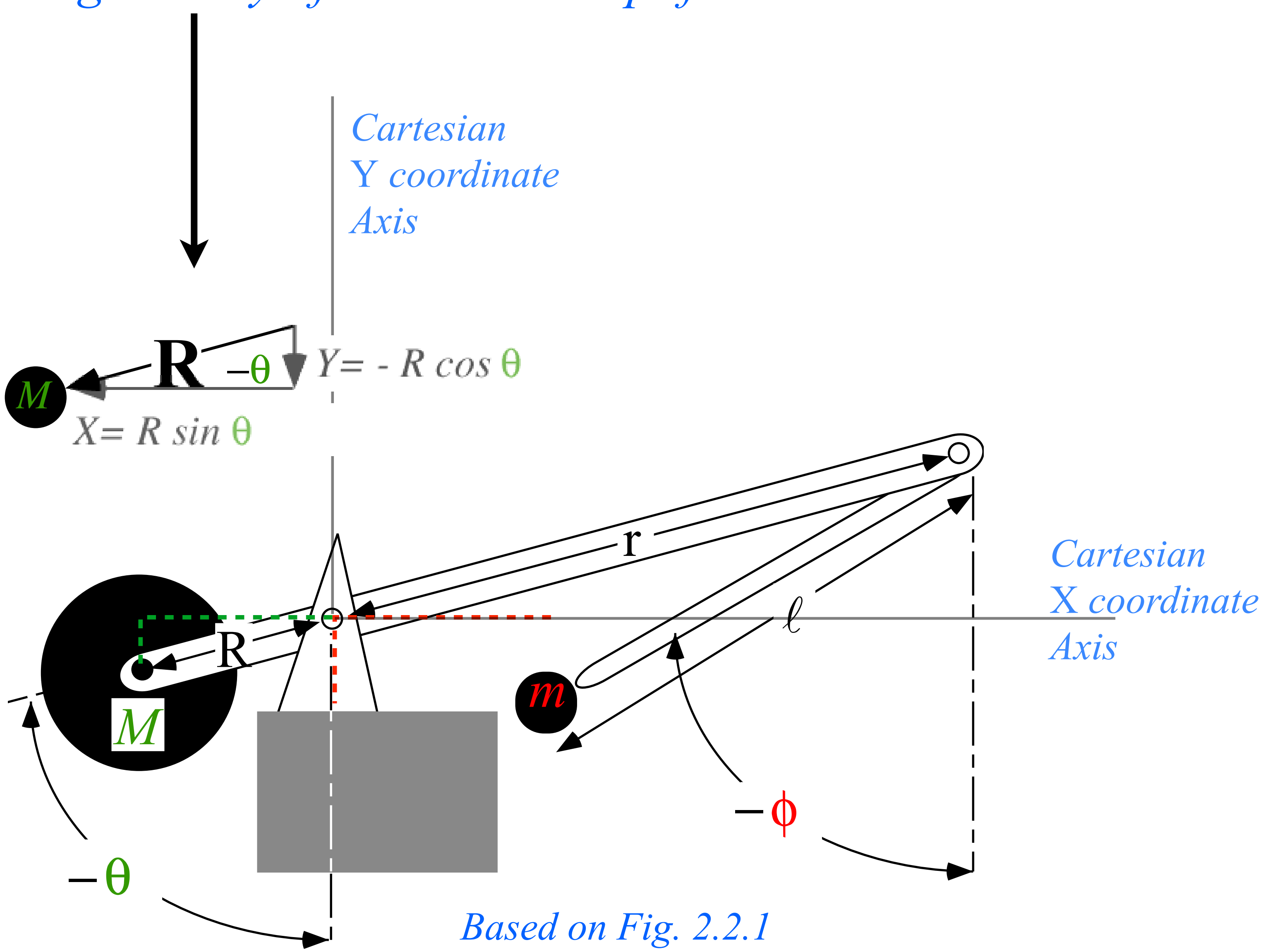


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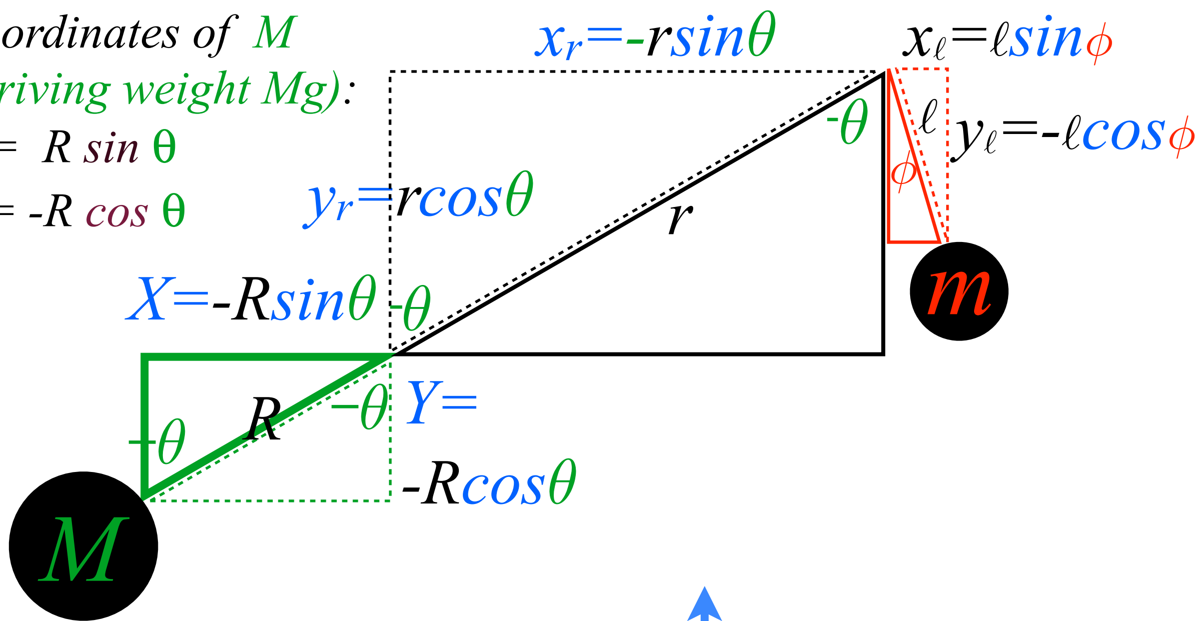


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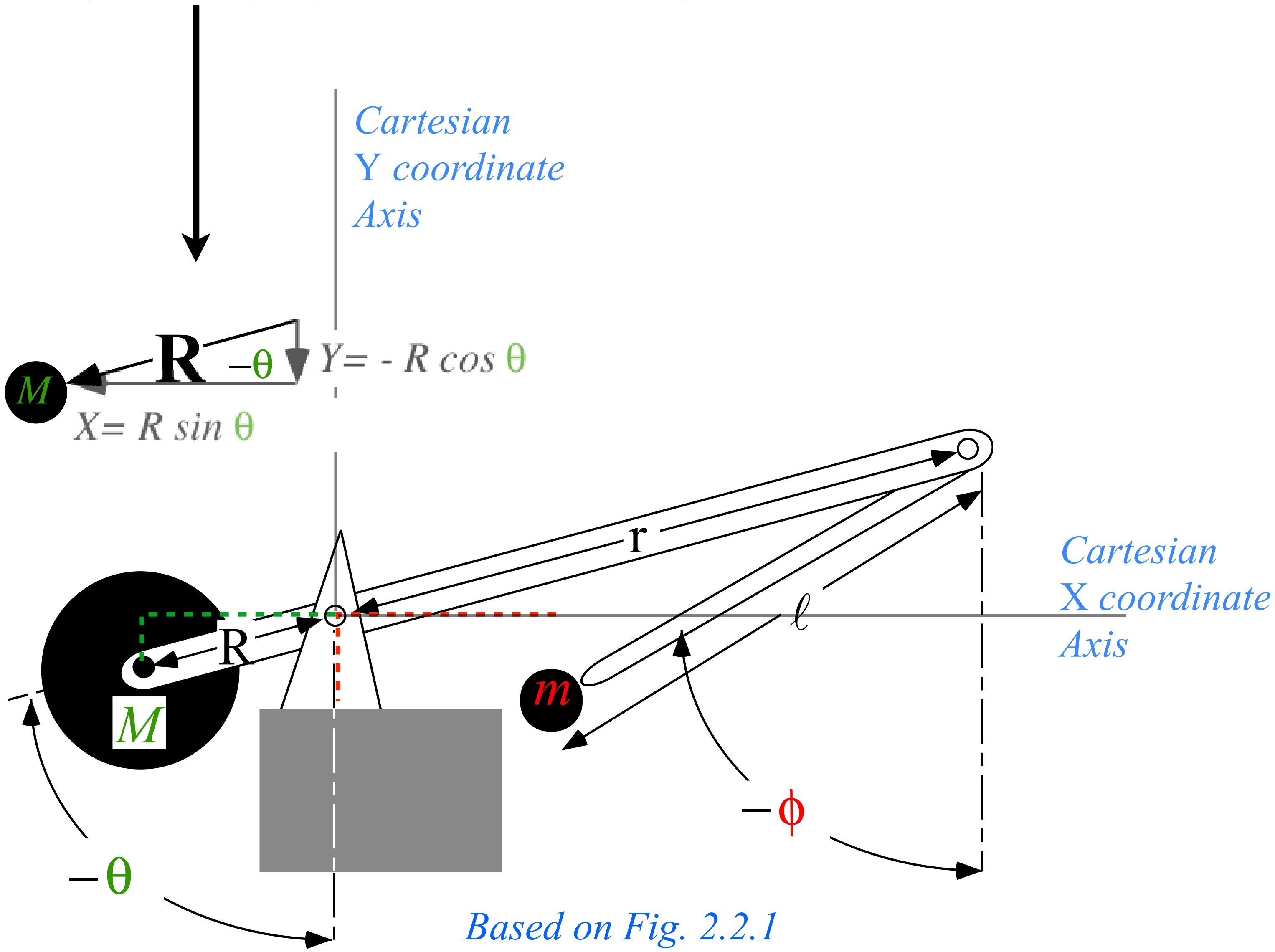


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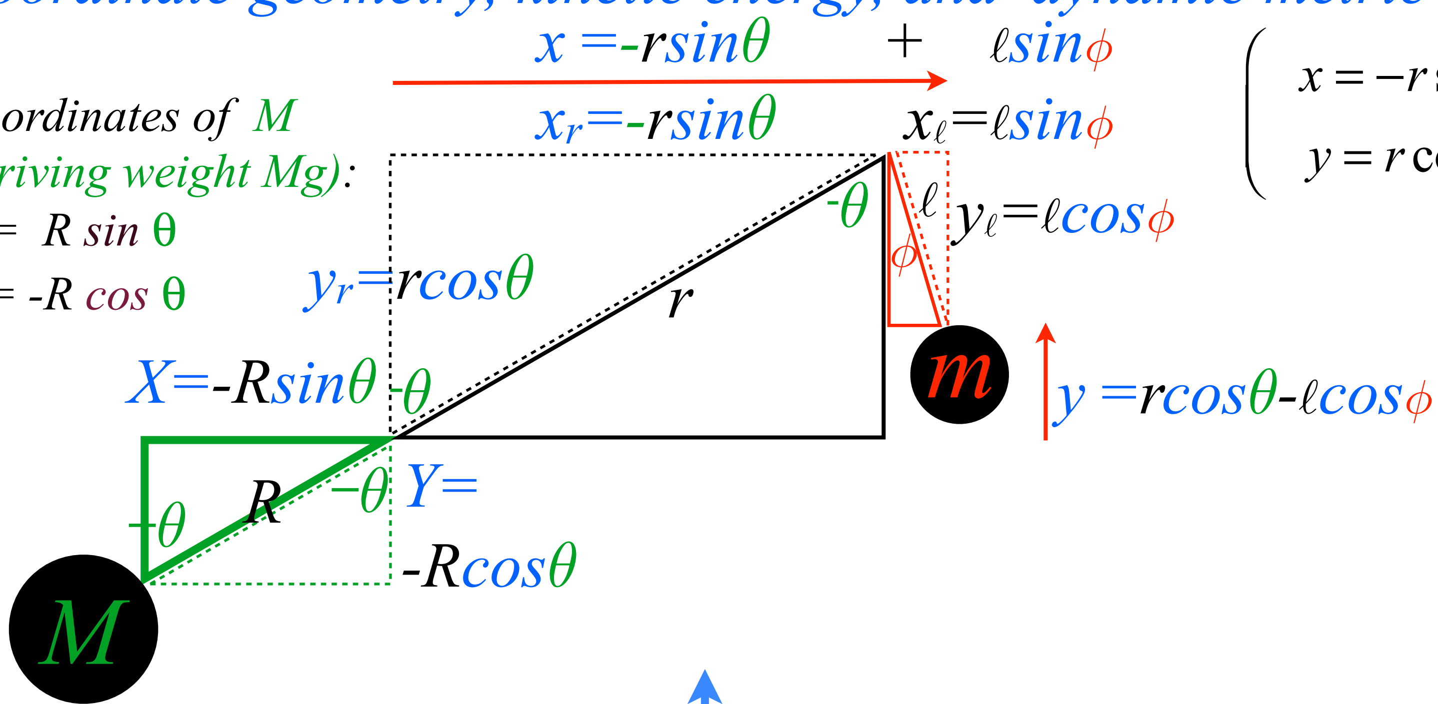


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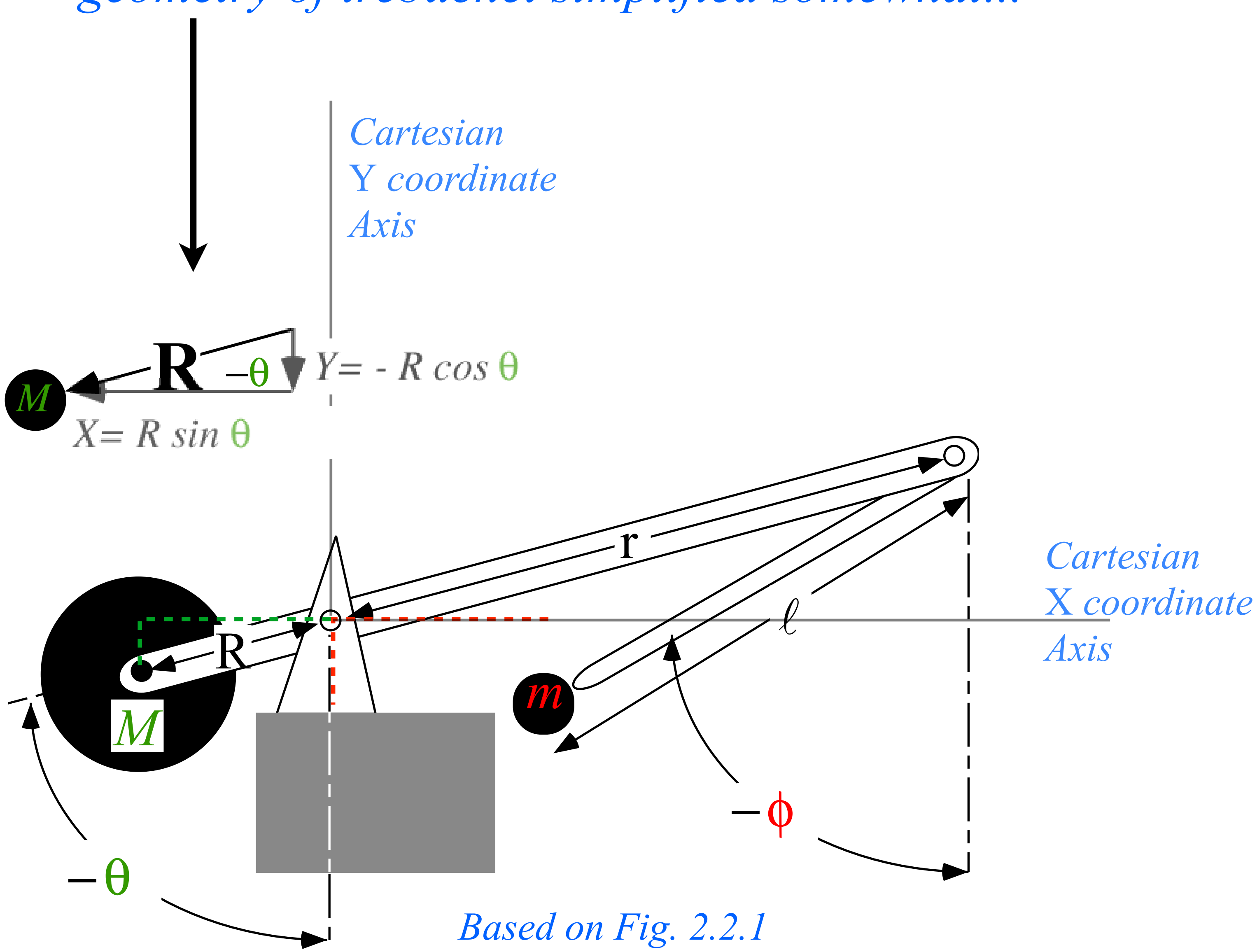


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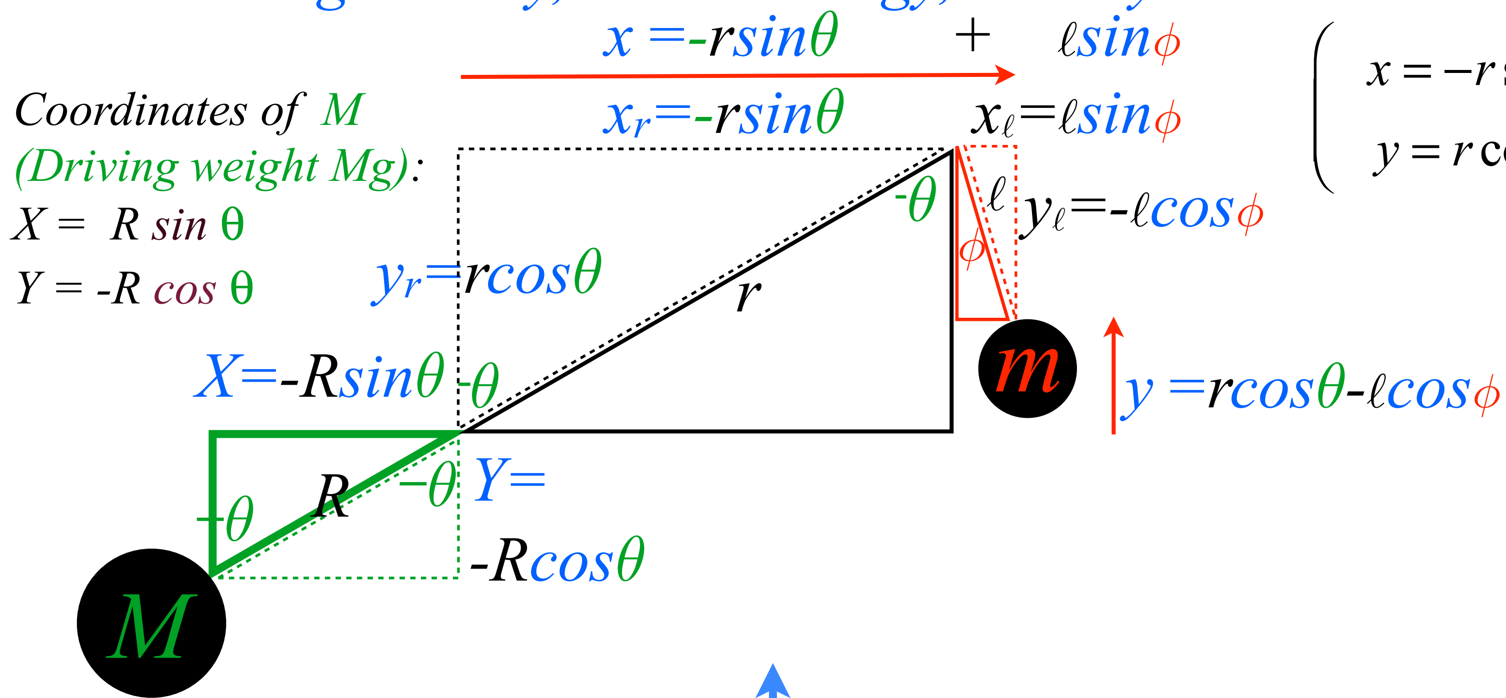


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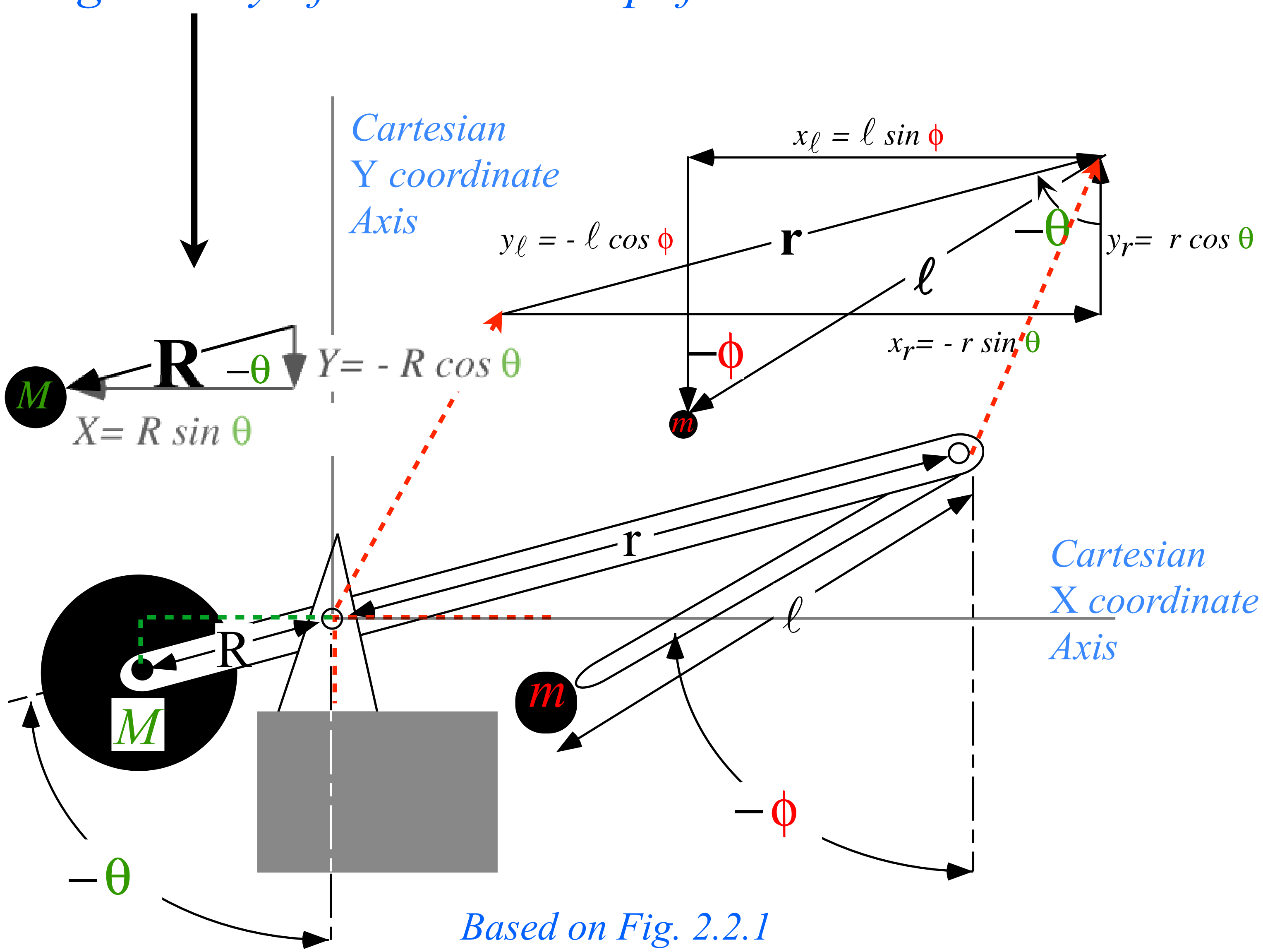


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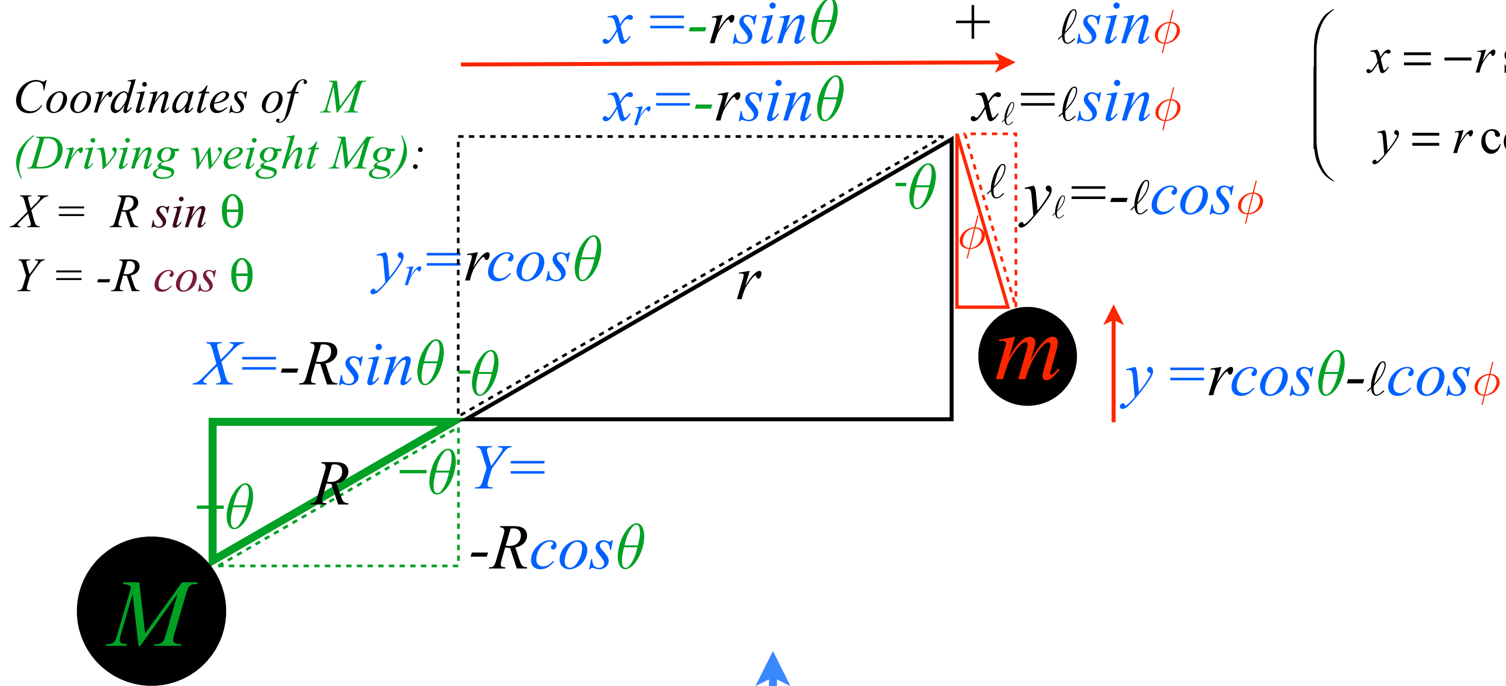


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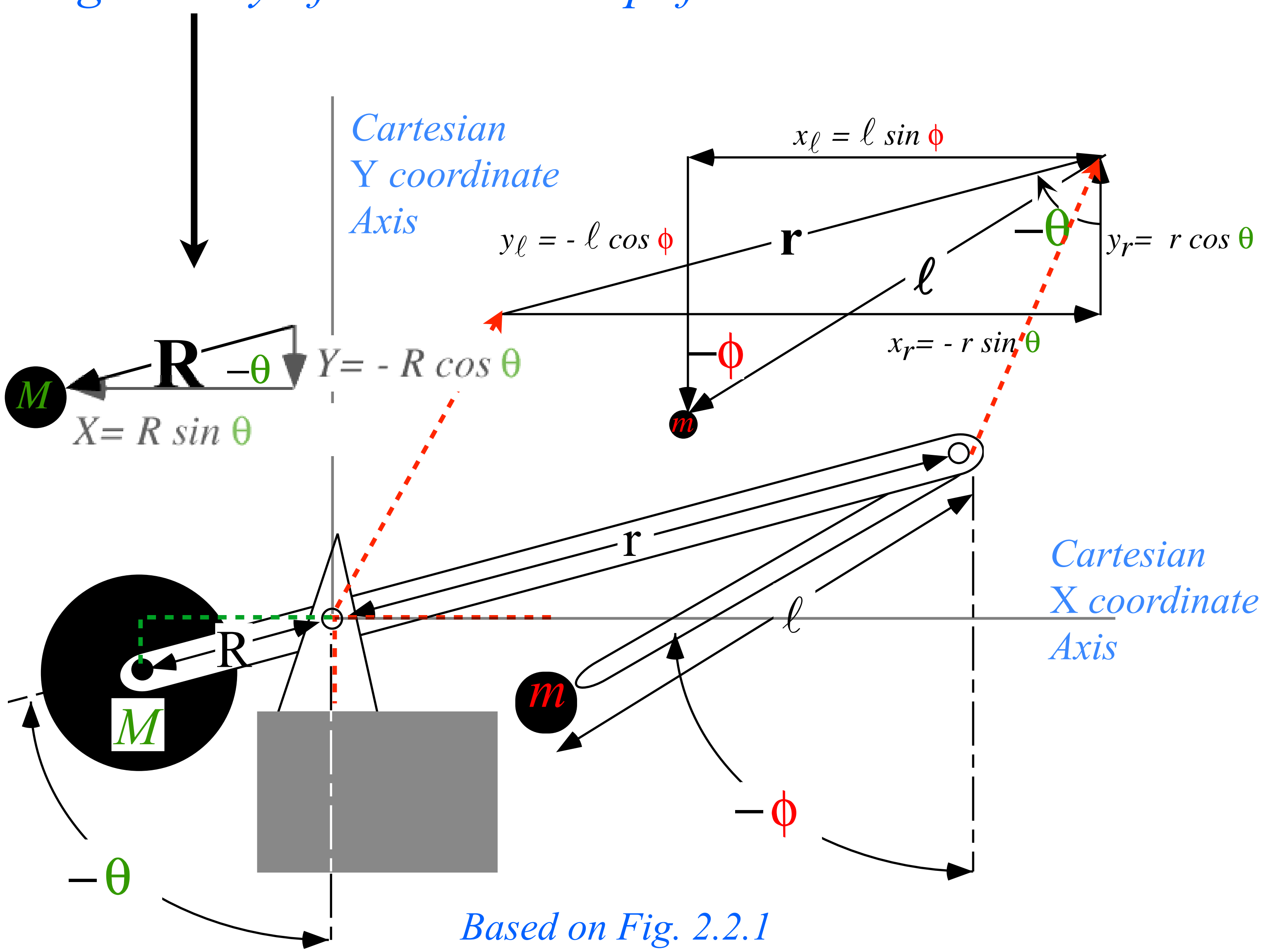


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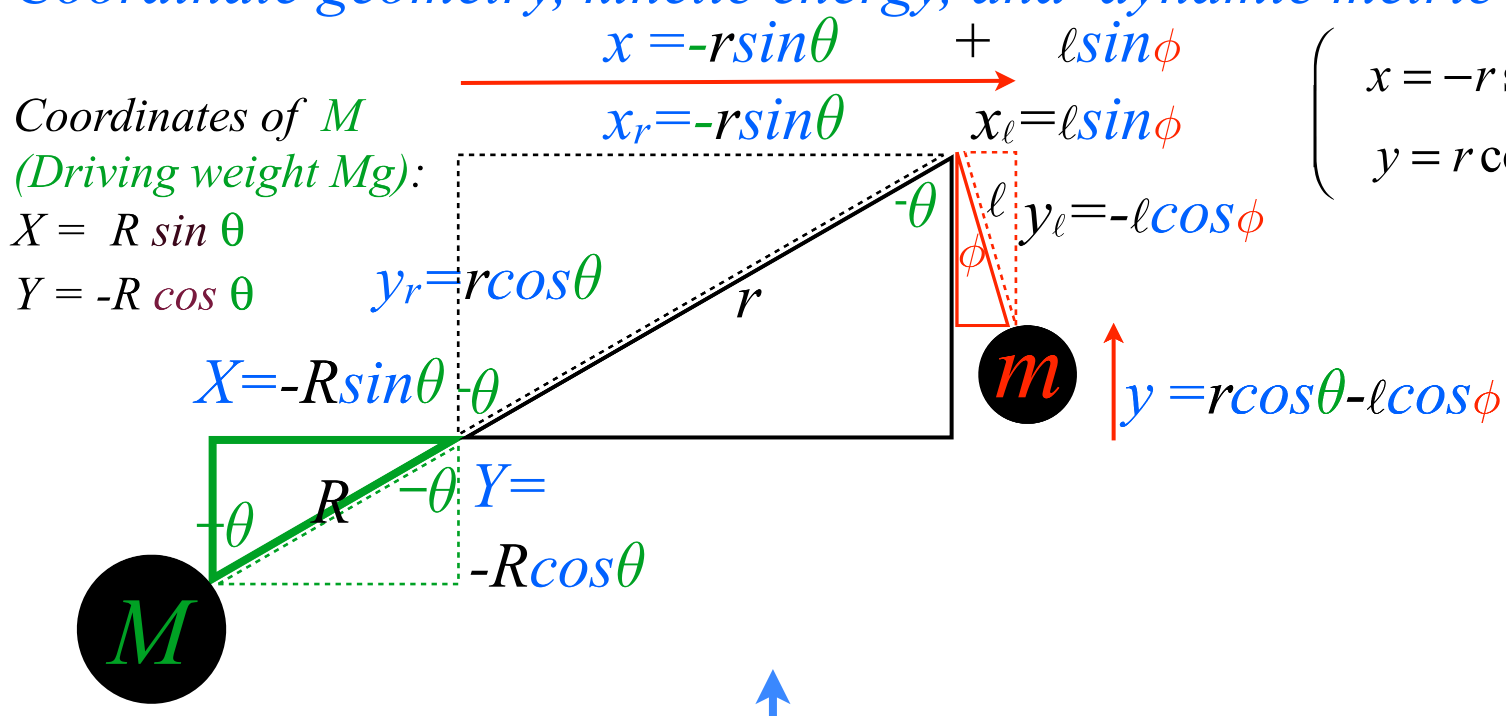


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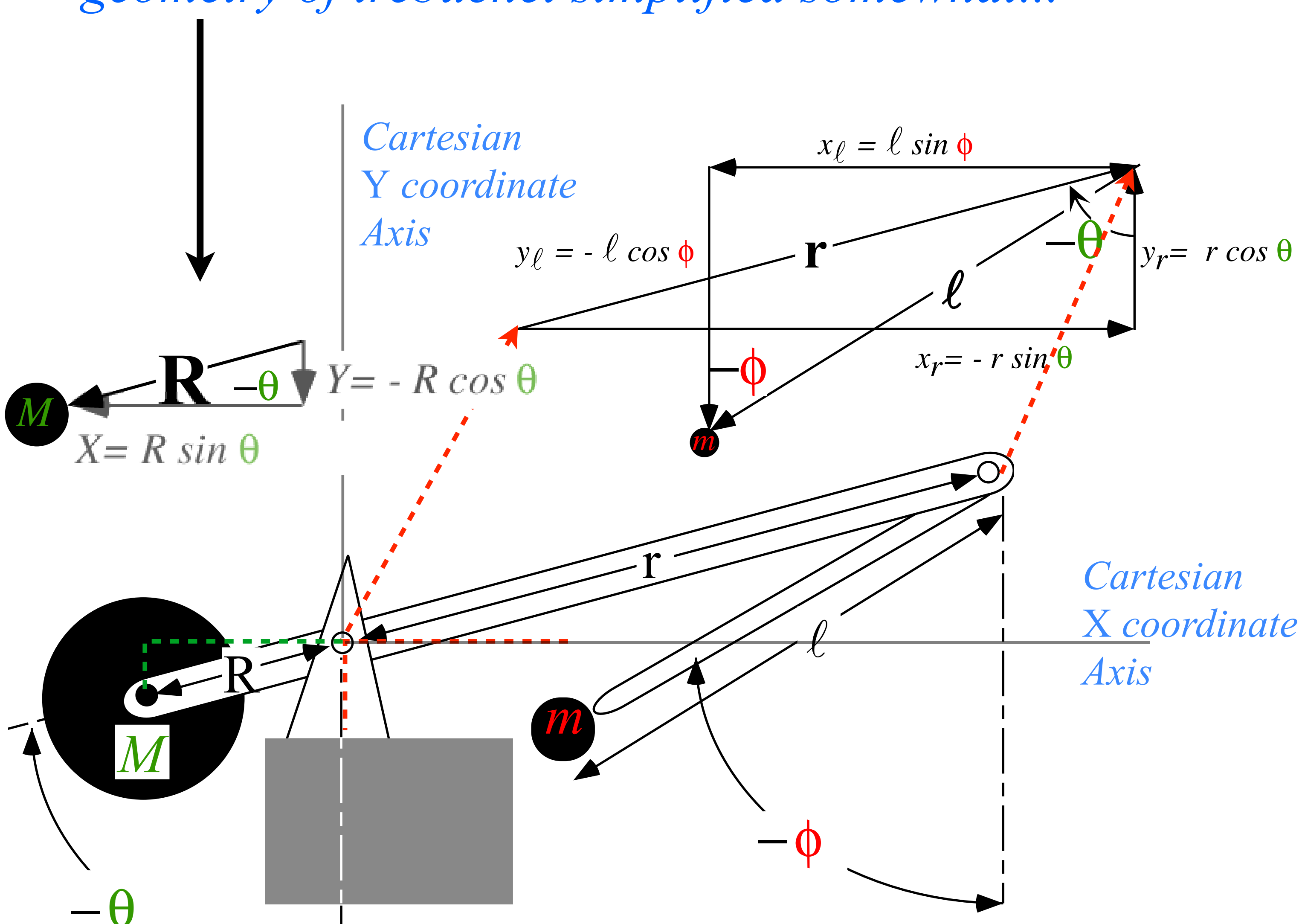


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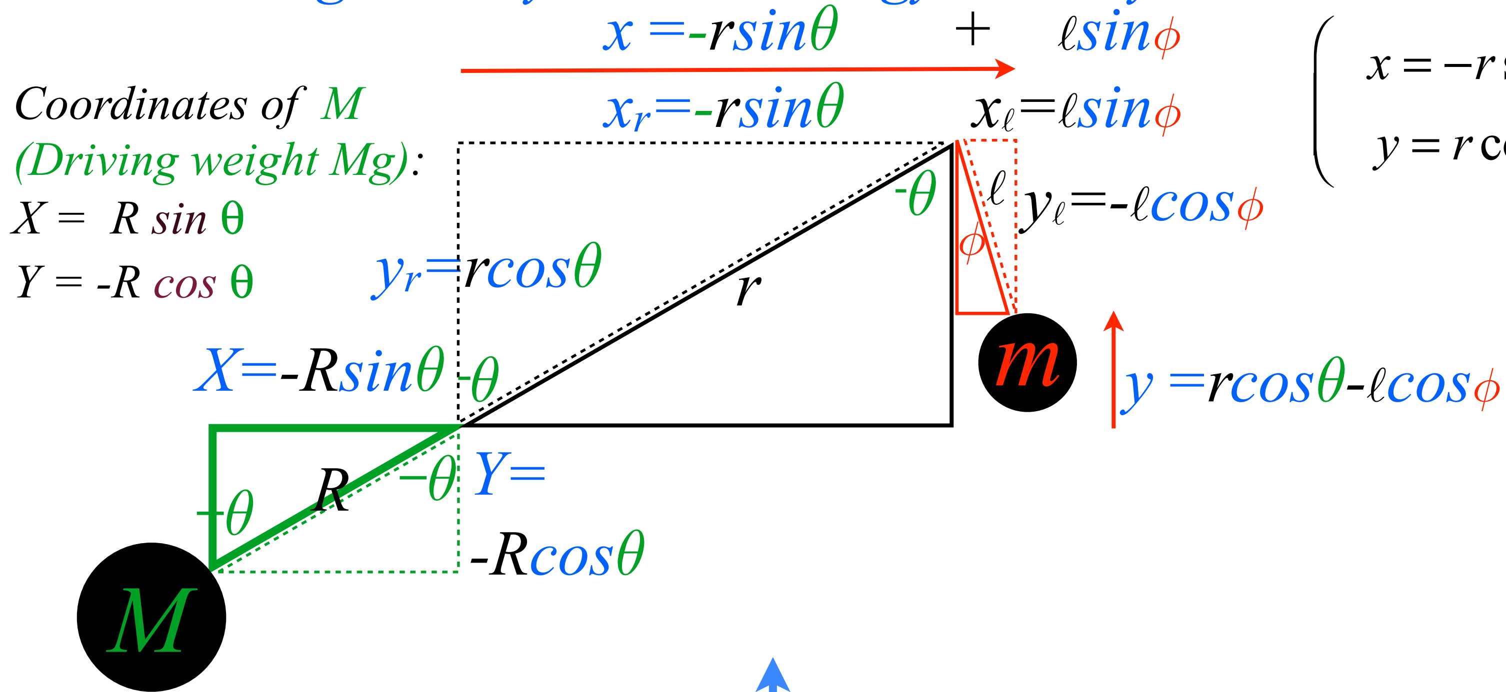


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geometry of trebuchet simplified somewhat...

Constraint relations:

$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$

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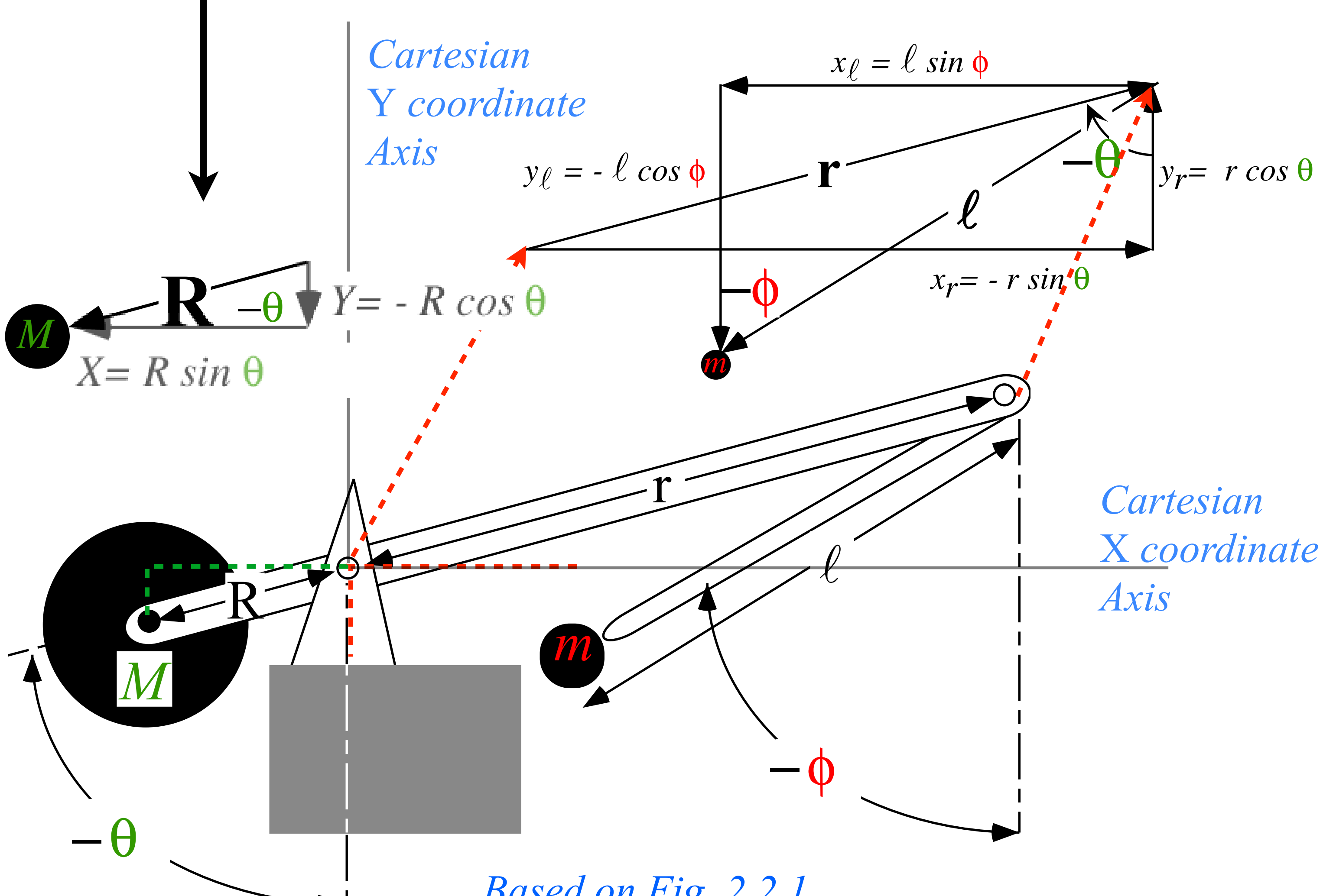
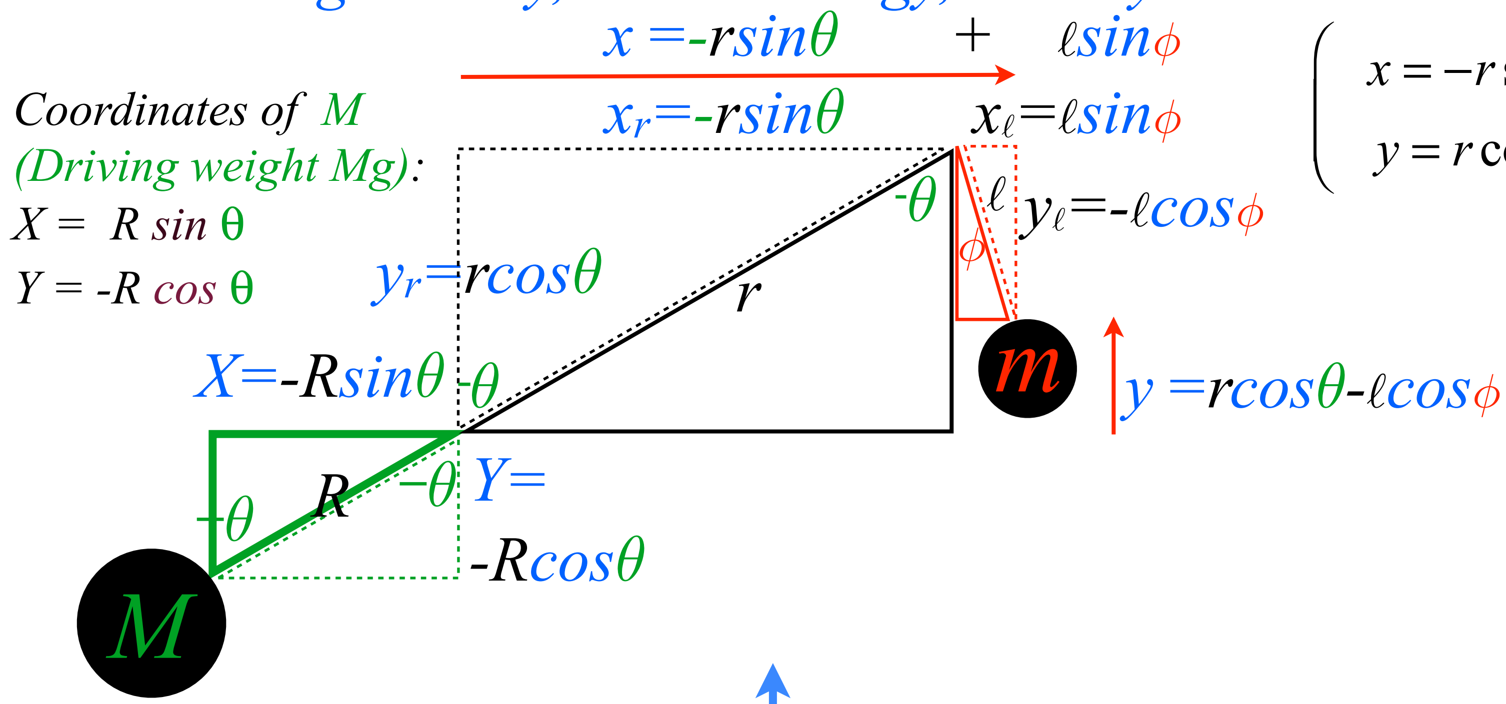



Based on Fig. 2.2.1

## 'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

# Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$

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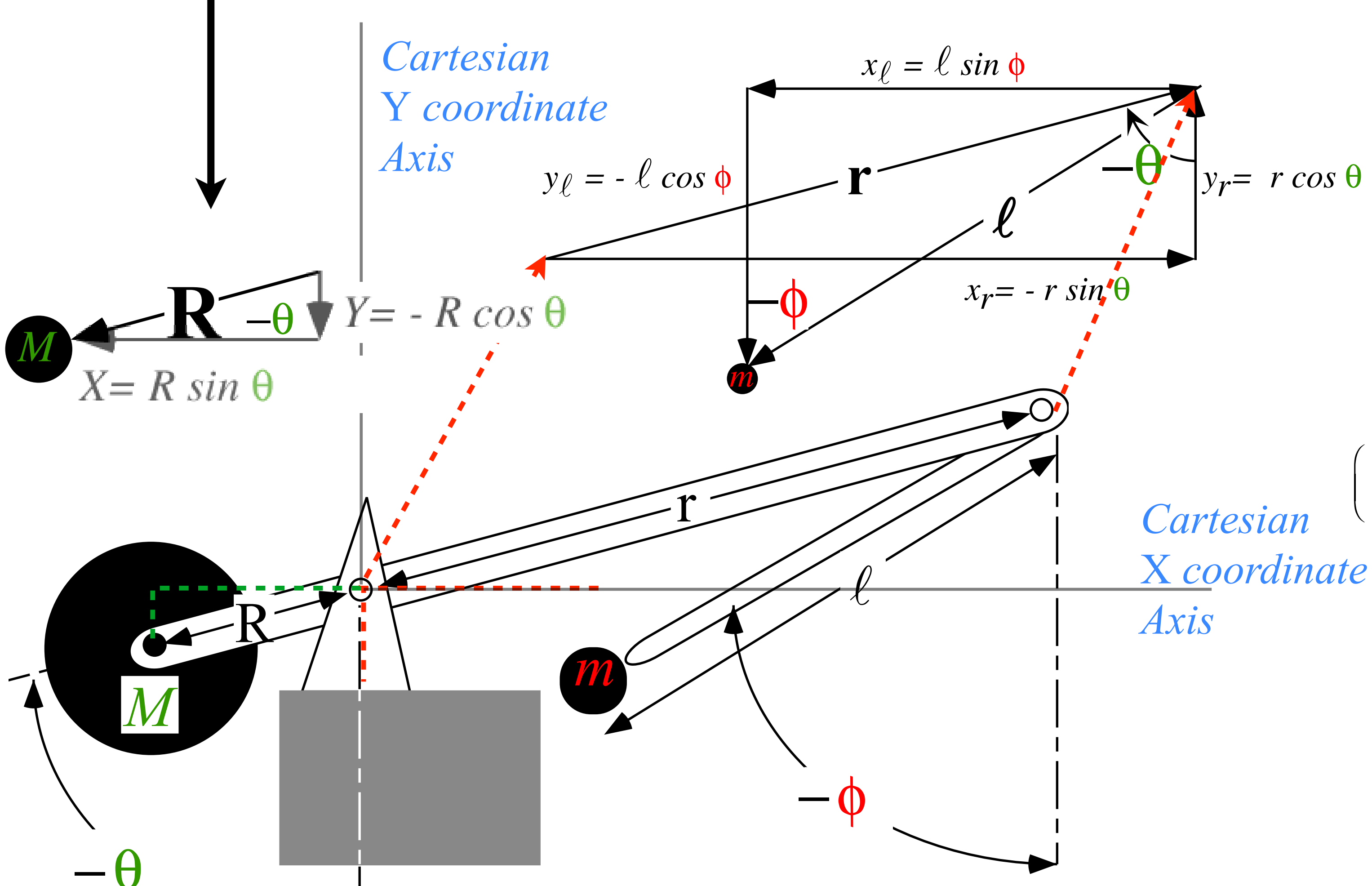



*Based on Fig. 2.2.1*

## 'Raw' Jacobian form

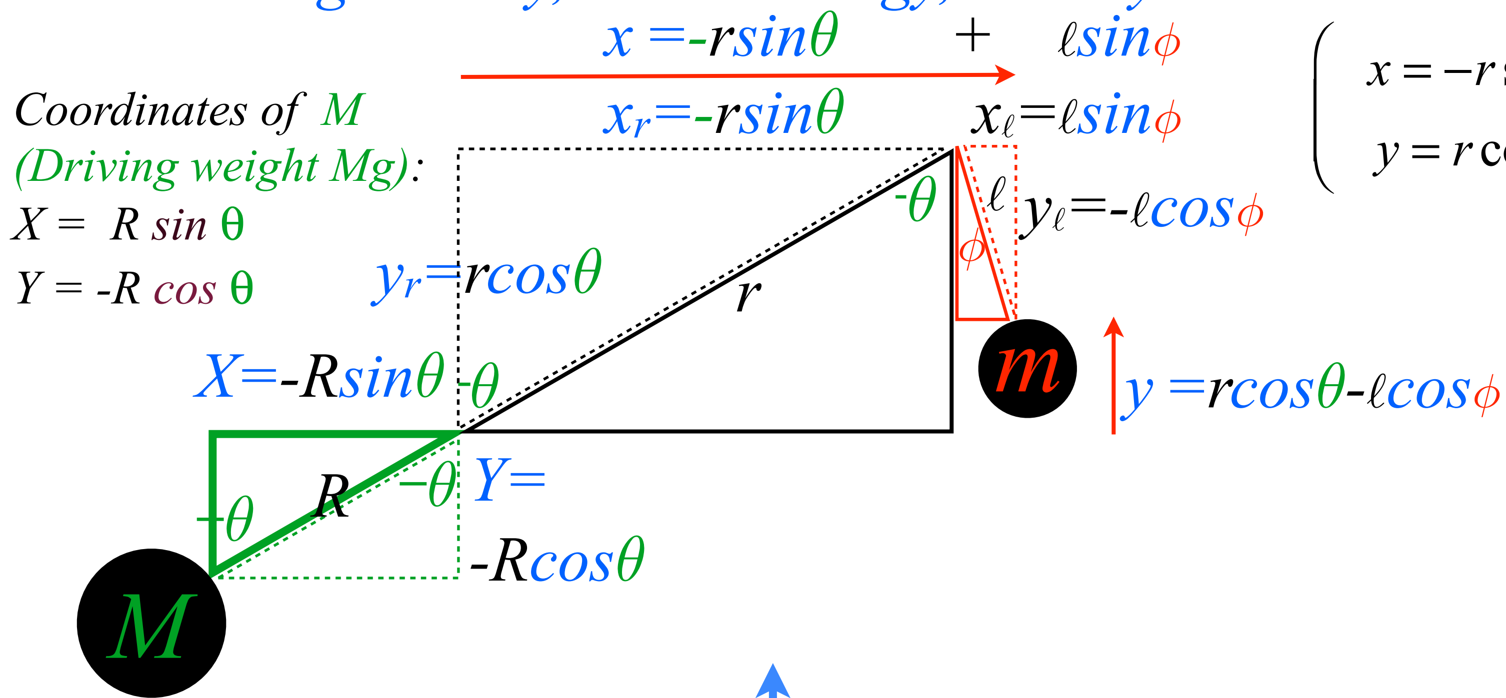
$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

## Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$ FAILS since: $\det \begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$

*FAILS! (Always singular)*

# Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$



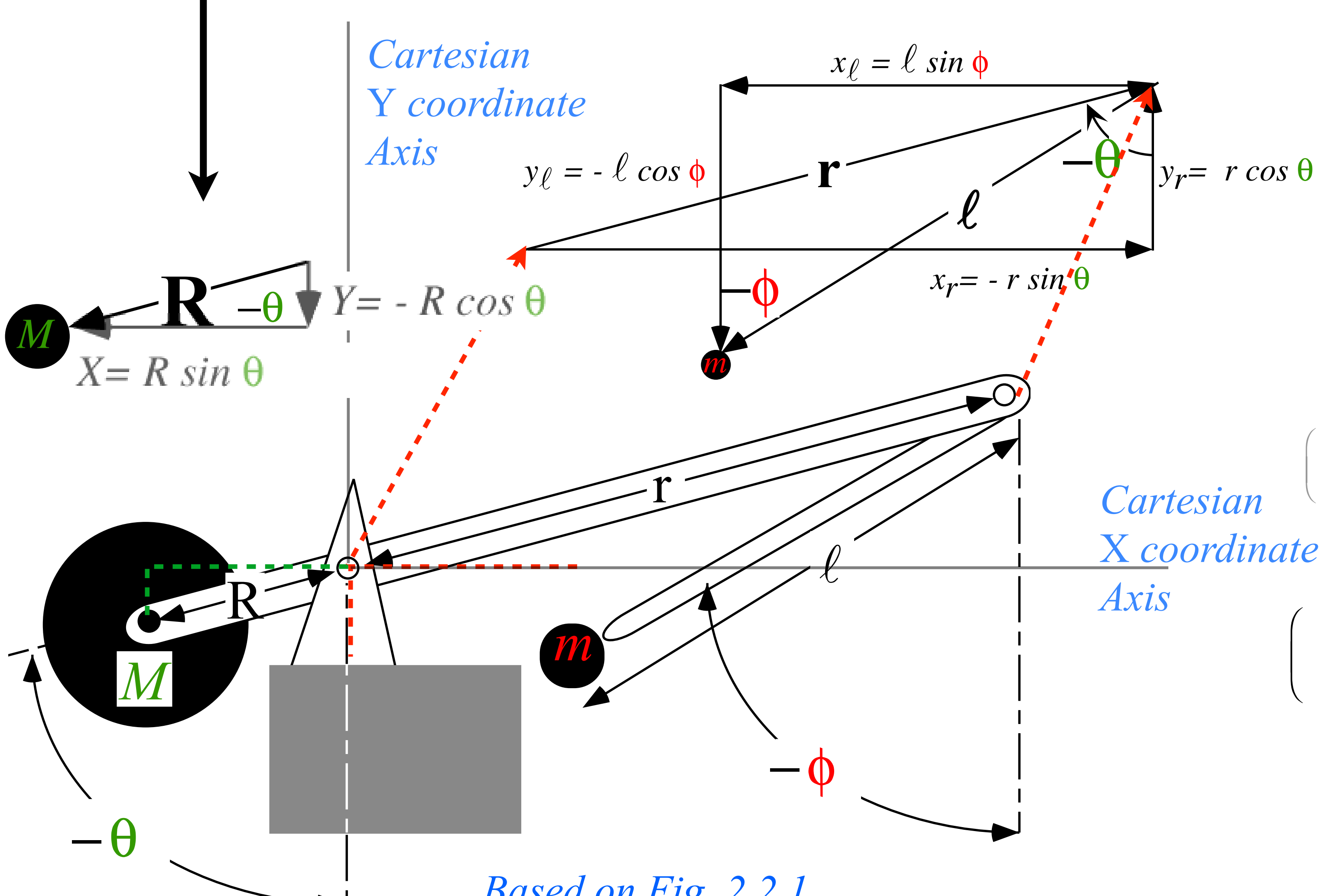

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass $m$ (Payload or projectile):

$x = x_r + x_\ell = -\, r \sin\theta + \ell \sin\phi$

$y = y_r + y_\ell = \;\; r \cos\theta - \ell \cos\phi$

geometry of trebuchet simplified somewhat...




Based on Fig. 2.2.1

*1st differential and Jacobian relations:*

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

*Constraint relations:*

$$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$$

$$c_\ell(x_\ell,y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$$

$$c_r(x_r,y_r) = x_r^2 + y_r^2 = r^2 = const.$$

## 'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

## Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$ FAILS since: $\det\begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$

FAILS! (Always singular)

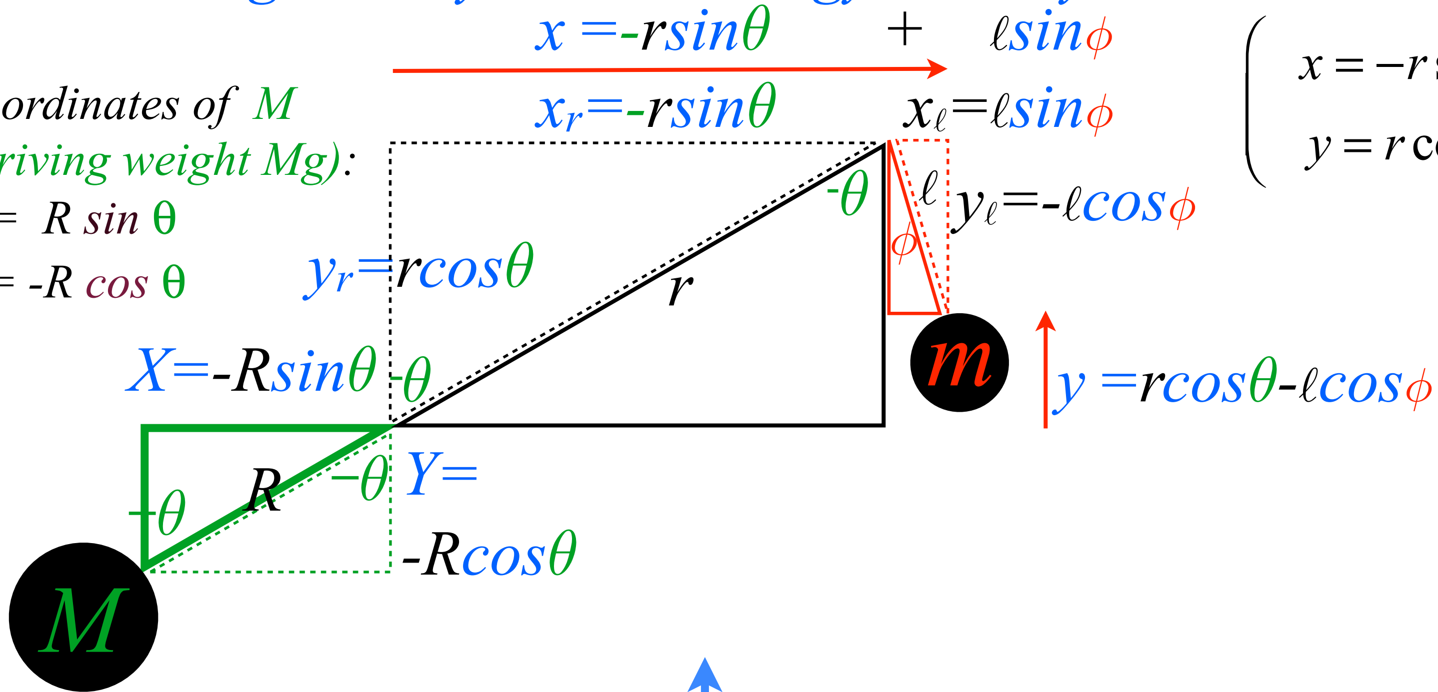
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

# Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$

Coordinates of $M$ (Driving weight $Mg$):

$X = R\sin\theta$

$Y = -R\cos\theta$




$$x = -r\sin\theta + \ell\sin\phi$$
$$y = r\cos\theta - \ell\cos\phi$$

Coordinates of mass $m$ (Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

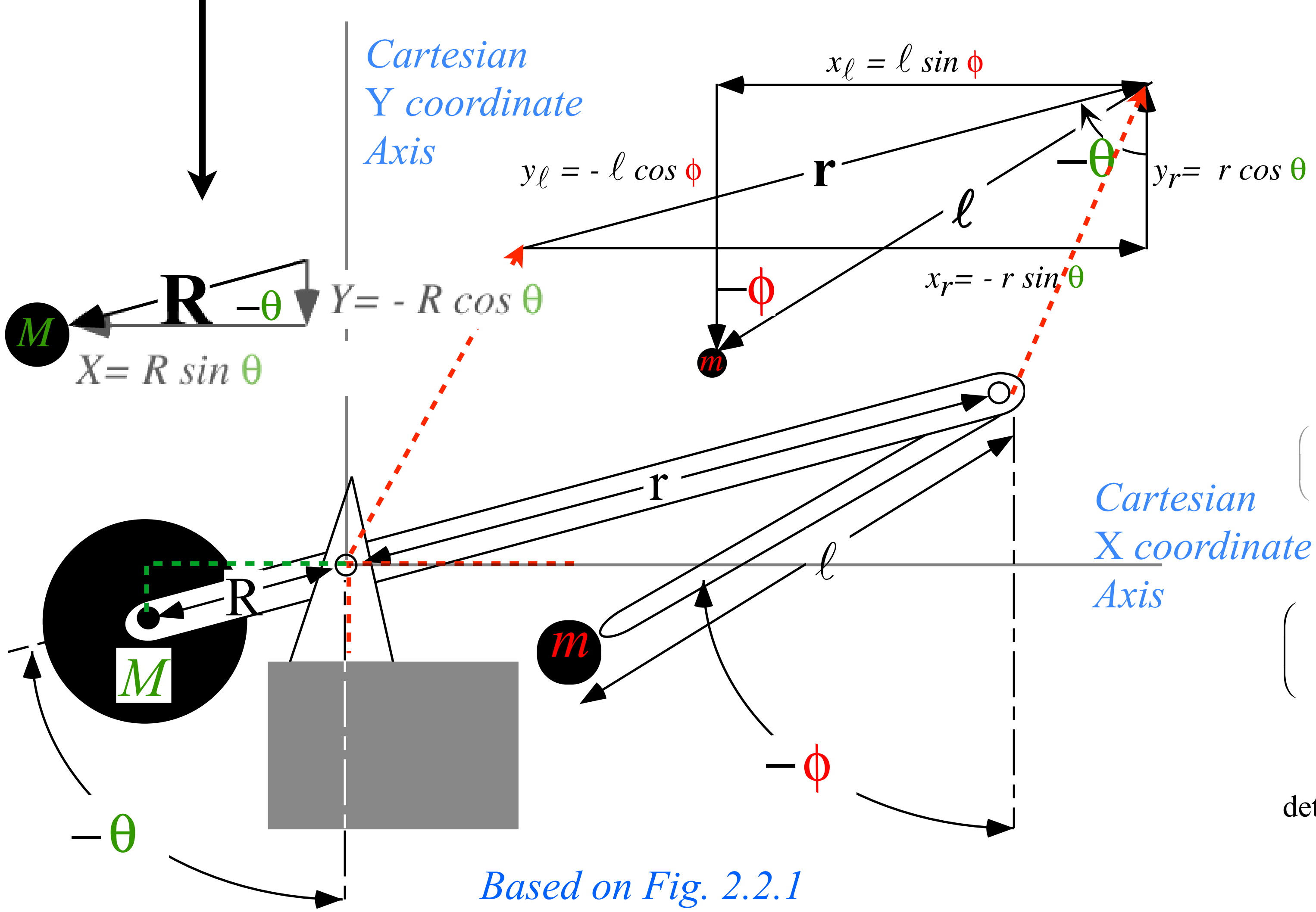
$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$
$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$
$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

geometry of trebuchet simplified somewhat...

Constraint relations:

$$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$$
$$c_\ell(x_\ell,y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$$
$$c_r(x_r,y_r) = x_r^2 + y_r^2 = r^2 = const.$$




Based on Fig. 2.2.1

## 'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

## Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$ FAILS since: $\det\begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$

FAILS! (Always singular)

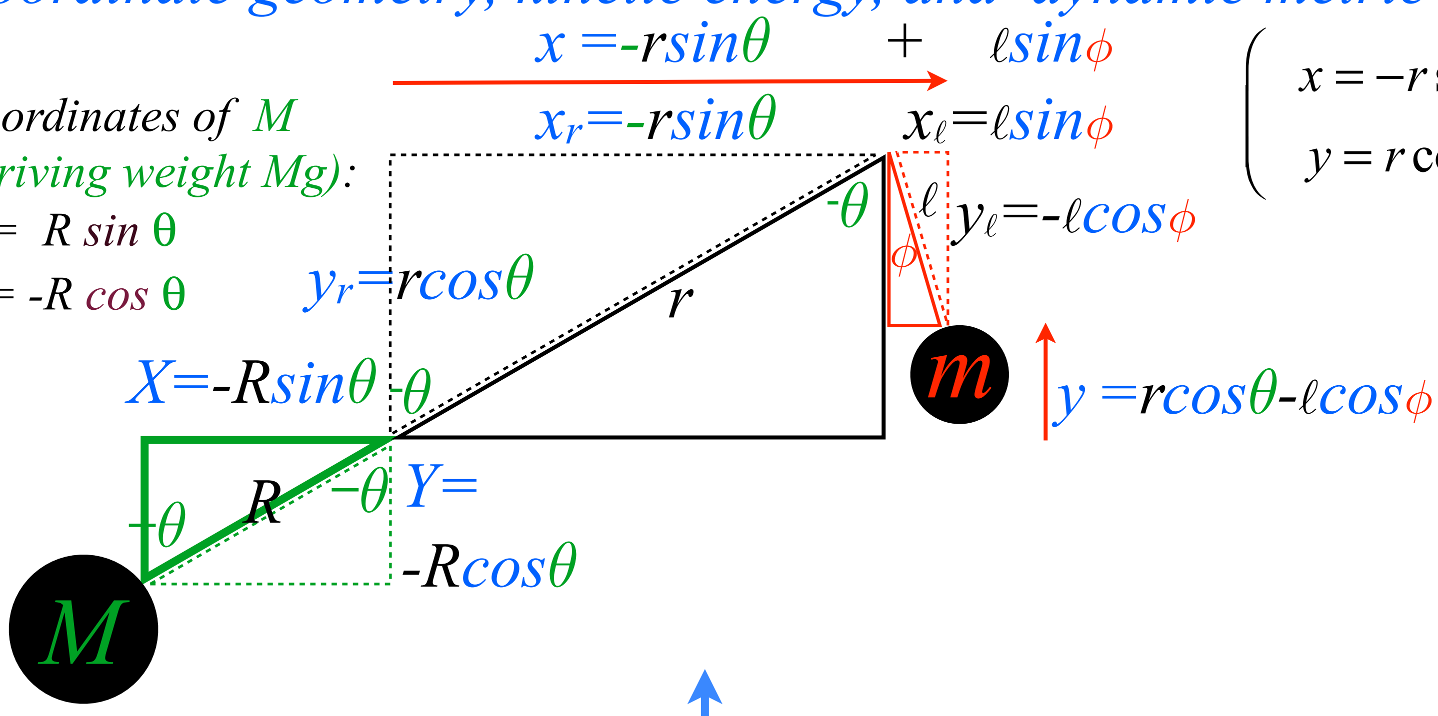
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

SUCCESS! (Usually non-singular)

# Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$

Coordinates of $M$
(Driving weight Mg):
$X = R \sin\theta$
$Y = -R \cos\theta$




geometry of trebuchet simplified somewhat...

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass $m$
(Payload or projectile):
$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$
$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

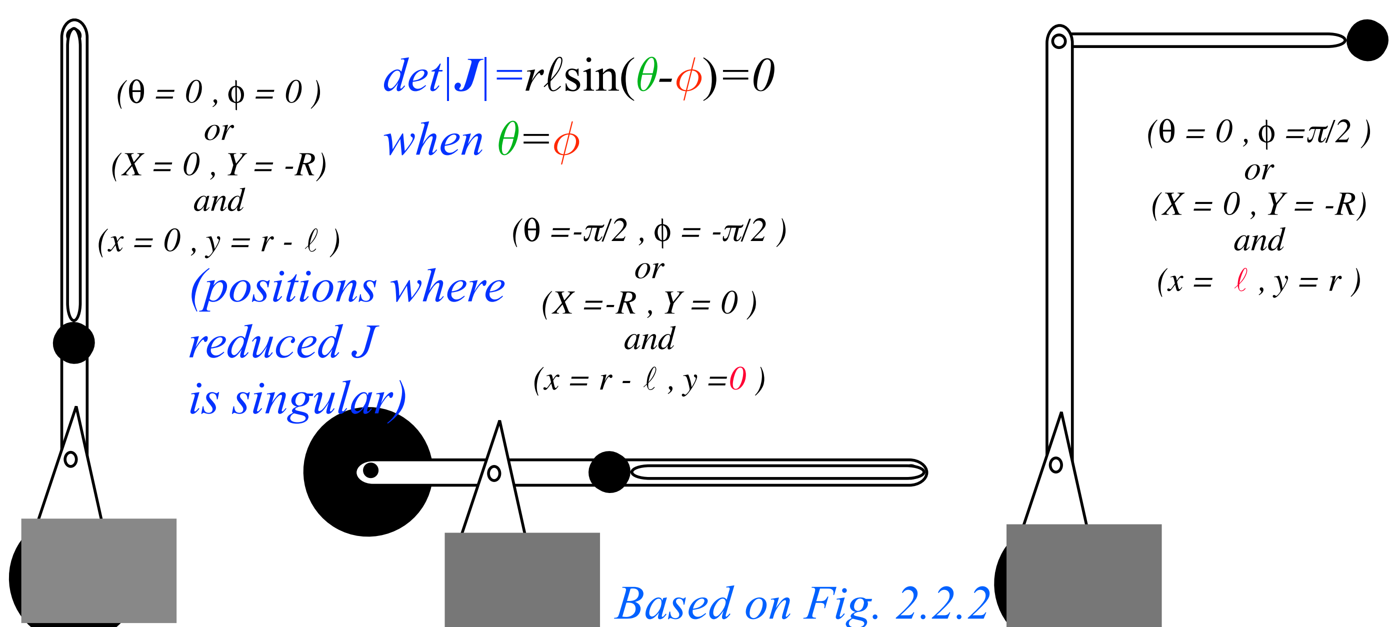
$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

Constraint relations:
$c_R(X,Y) = X^2 + Y^2 = R^2 = const.$
$c_\ell(x_\ell,y_\ell) = x_\ell^2 + y_\ell^2 = \ell^2 = const.$
$c_r(x_r,y_r) = x_r^2 + y_r^2 = r^2 = const.$

## 'Raw' Jacobian form

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

Fig. 2.2.2 Singular positions of the trebuchet




Based on Fig. 2.2.2

## Finding a reduced Jacobian form

$$\begin{pmatrix} dX \\ dY \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$ FAILS since: $\det\begin{vmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \end{vmatrix} = 0$

FAILS! (Always singular)

Jacobian
**J**-matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

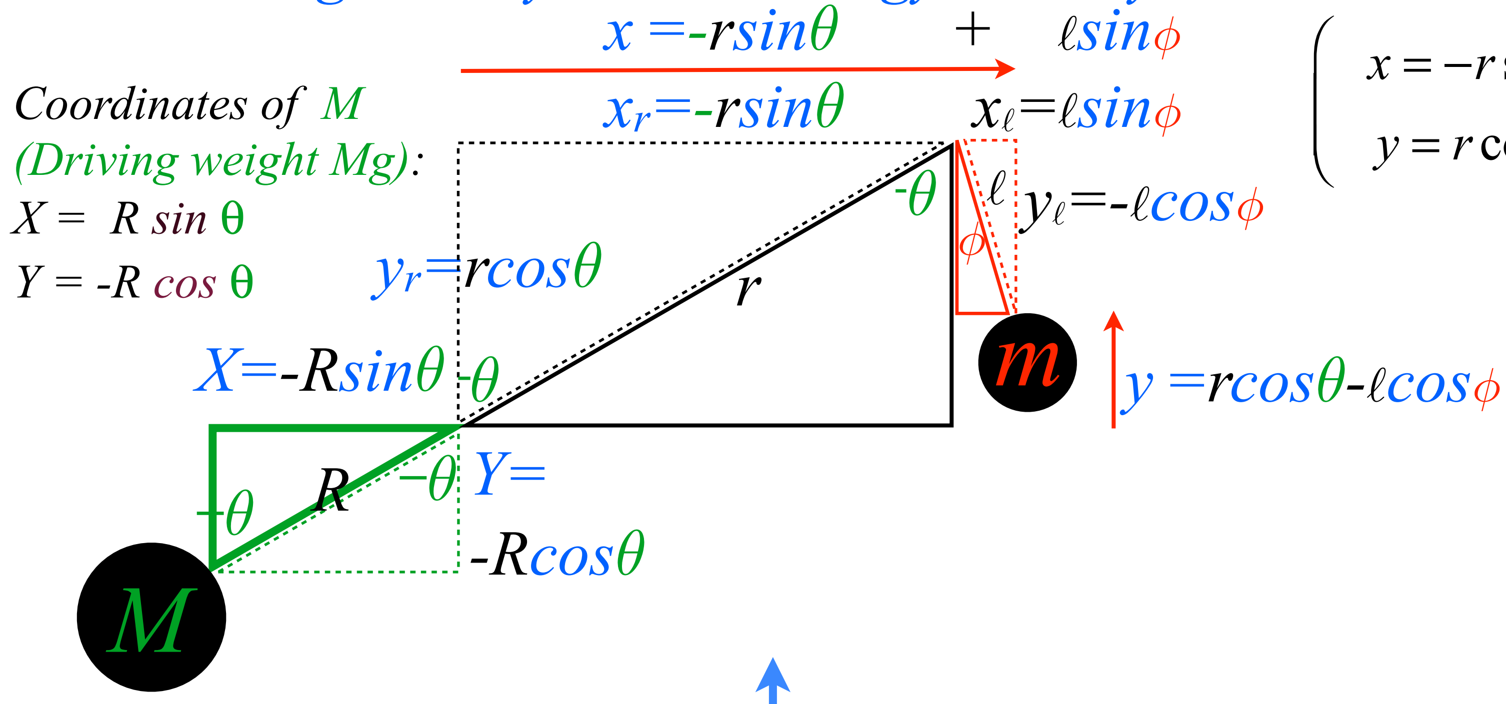
$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

SUCCESS! (Usually non-singular)

*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor $\gamma_{mn}$*
- *Structure of dynamic metric tensor $\gamma_{mn}$*
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and $\gamma_{mn}$ tensor*

# Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$




geometry of trebuchet simplified somewhat...

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass $m$
(Payload or projectile):

$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$

$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

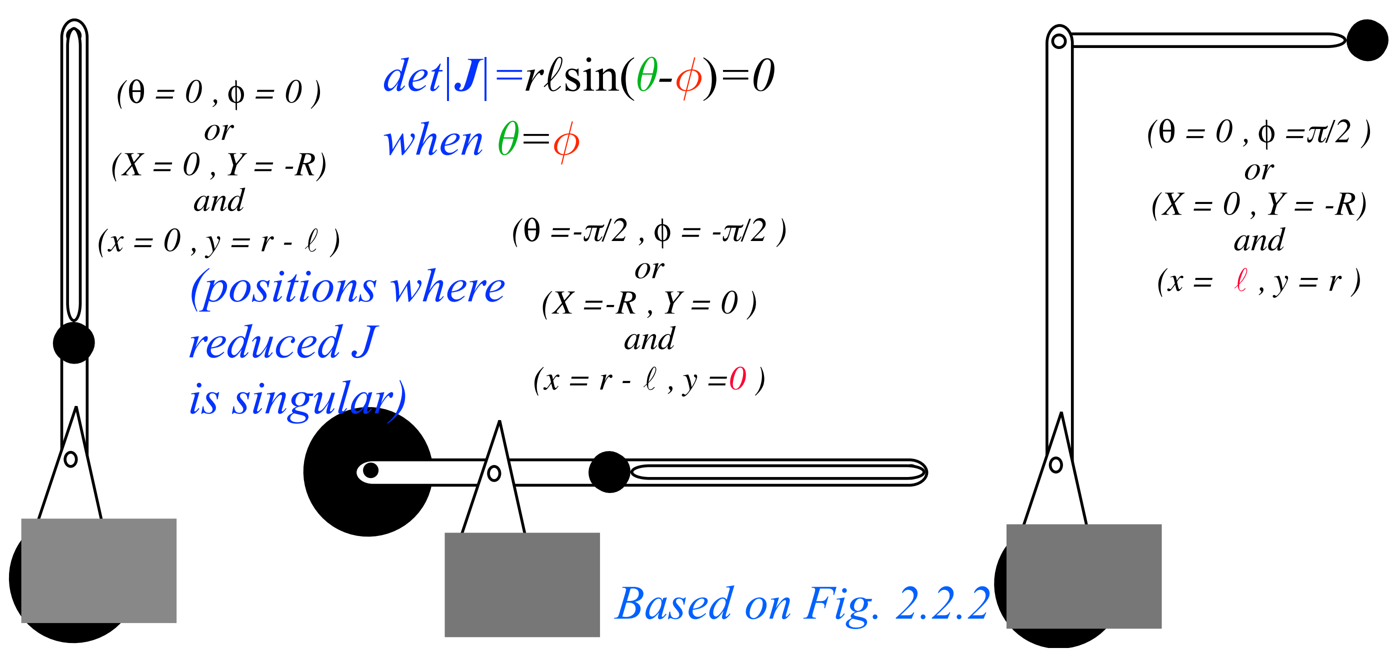
$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

GCC Velocity relations:

$$\dot{X} = R\cos\theta\ \dot\theta + 0, \qquad \dot{x} = -r\cos\theta\ \dot\theta + \ell\cos\phi\ \dot\phi$$

$$\dot{Y} = R\sin\theta\ \dot\theta + 0, \qquad \dot{y} = -r\sin\theta\ \dot\theta + \ell\sin\phi\ \dot\phi$$

Fig. 2.2.2 Singular positions of the trebuchet




Jacobian
$J$-matrix

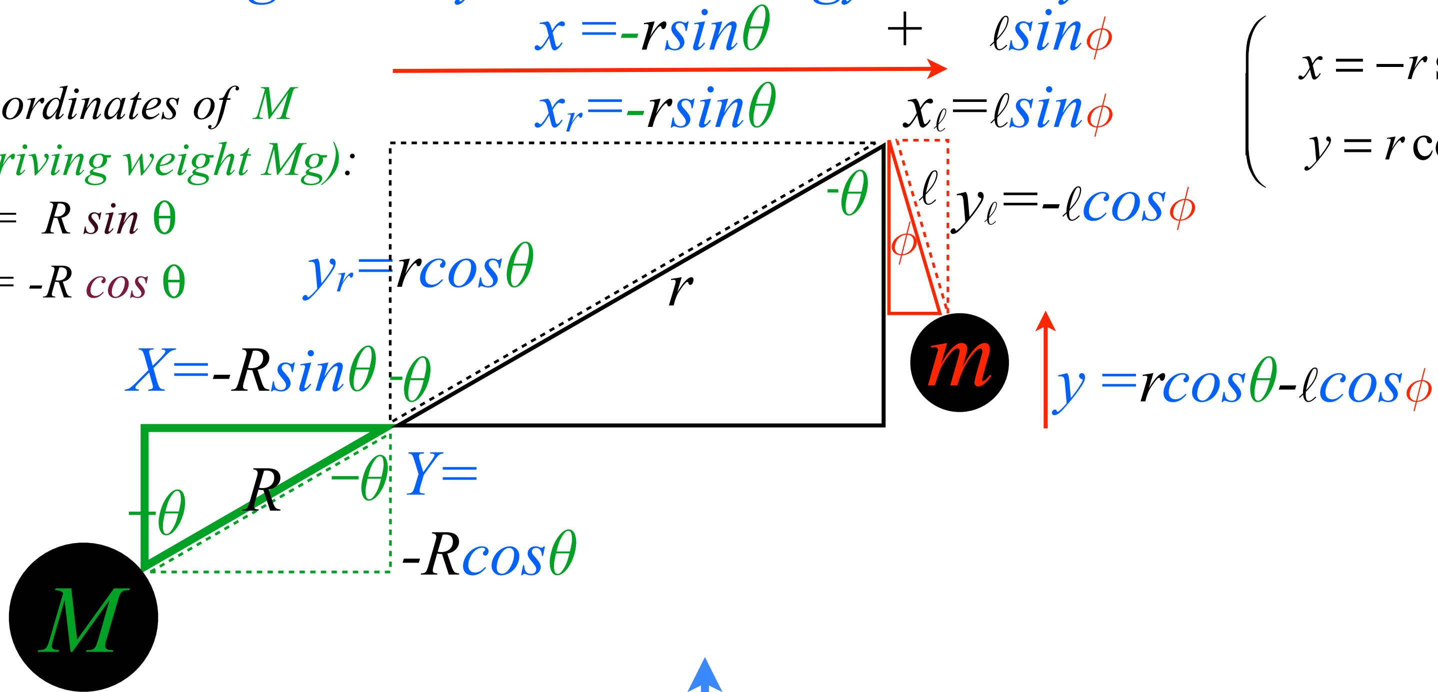
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

SUCCESS! (Usually non-singular)

# Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$

Coordinates of $M$
(Driving weight $Mg$):
$X = R \sin\theta$
$Y = -R \cos\theta$




$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass $m$
(Payload or projectile):
$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$
$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

GCC Velocity relations:

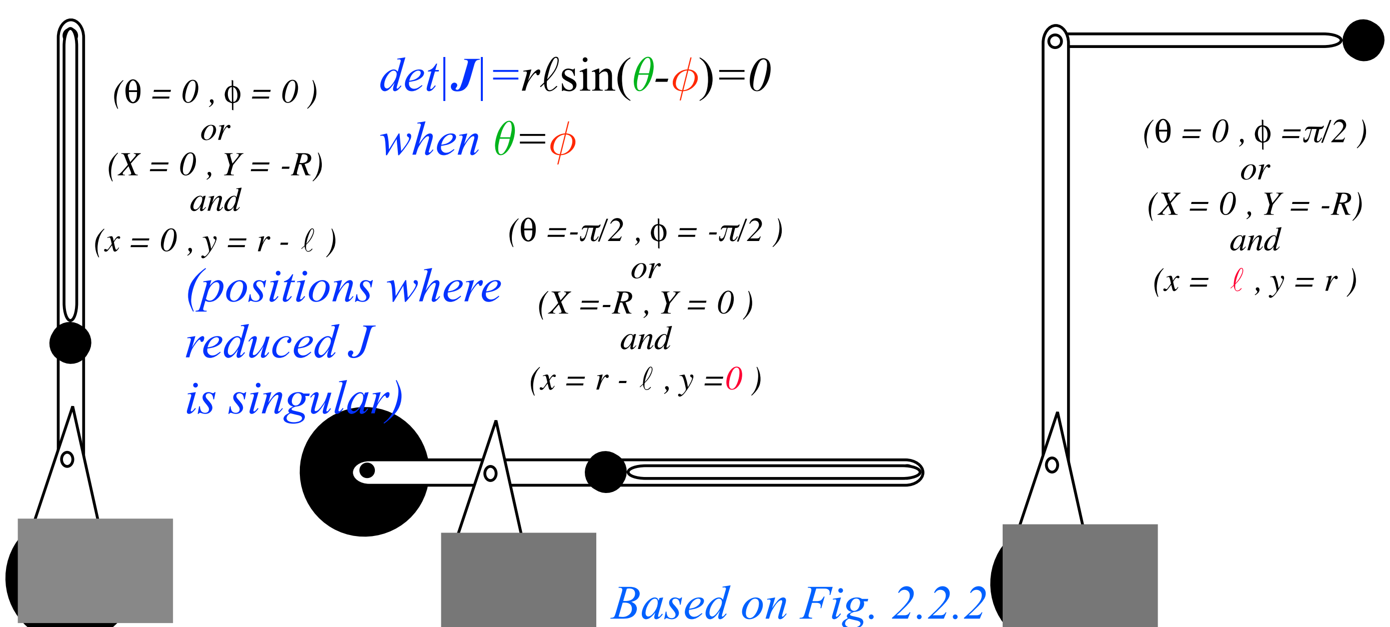
$$\dot{X} = R\cos\theta\,\dot\theta + 0, \qquad \dot{x} = -r\cos\theta\,\dot\theta + \ell\cos\phi\,\dot\phi$$

$$\dot{Y} = R\sin\theta\,\dot\theta + 0, \qquad \dot{y} = -r\sin\theta\,\dot\theta + \ell\sin\phi\,\dot\phi$$

Jacobian ***J***-matrix velocity relations:

$$\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \underbrace{\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}}_{\textbf{\textit{J}}\text{-matrix}} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

Fig. 2.2.2 Singular positions of the trebuchet




Jacobian ***J***-matrix

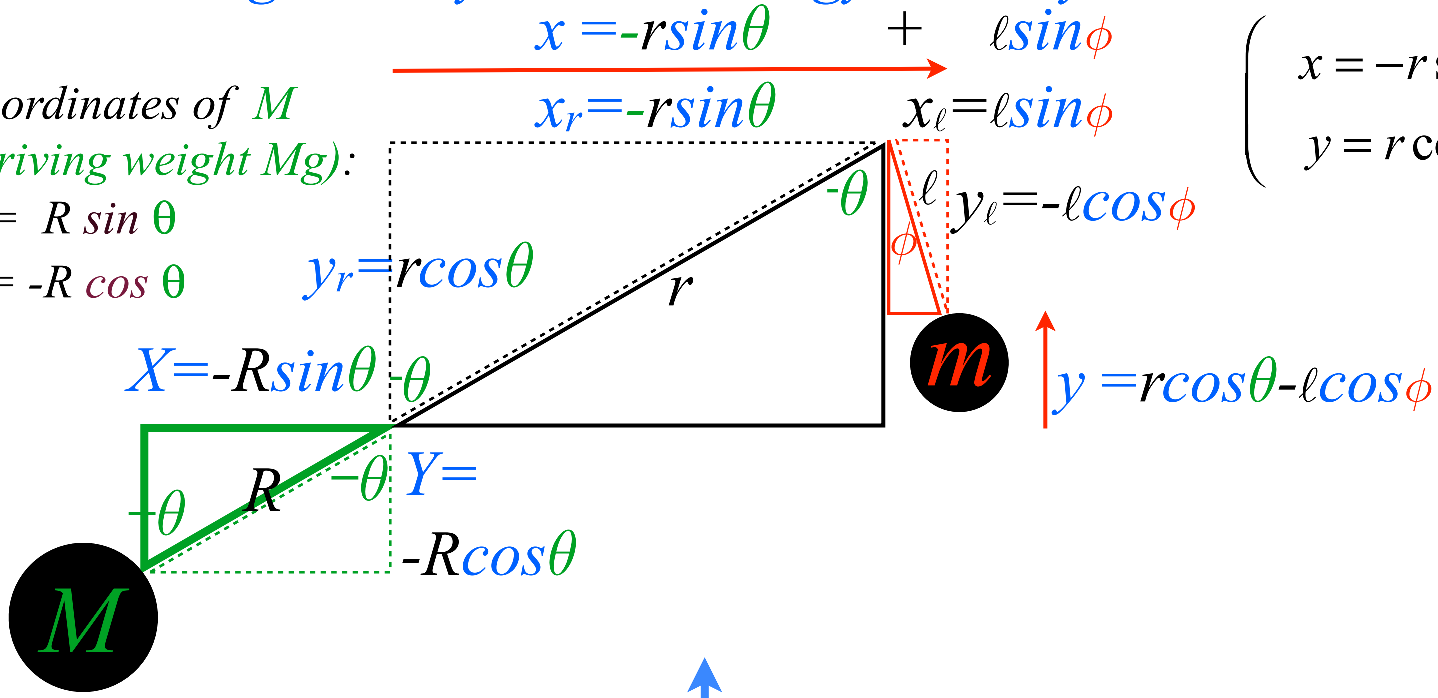
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

SUCCESS! (Usually non-singular)

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$

Coordinates of $M$
(Driving weight $Mg$):
$X = R \sin\theta$
$Y = -R\cos\theta$




geometry of trebuchet simplified somewhat...

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

Coordinates of mass $m$
(Payload or projectile):
$x = x_r + x_\ell = -r\sin\theta + \ell\sin\phi$
$y = y_r + y_\ell = r\cos\theta - \ell\cos\phi$

### 1st differential and Jacobian relations:

$$dX = \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial\phi}d\phi, \qquad dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi,$$

$$dY = \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial\phi}d\phi, \qquad dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi.$$

$$dX = R\cos\theta\, d\theta + 0, \qquad dx = -r\cos\theta\, d\theta + \ell\cos\phi\, d\phi$$

$$dY = R\sin\theta\, d\theta + 0, \qquad dy = -r\sin\theta\, d\theta + \ell\sin\phi\, d\phi$$

### GCC Velocity relations:

$$\dot X = R\cos\theta\ \dot\theta + 0, \qquad \dot x = -r\cos\theta\ \dot\theta + \ell\cos\phi\ \dot\phi$$

$$\dot Y = R\sin\theta\ \dot\theta + 0, \qquad \dot y = -r\sin\theta\ \dot\theta + \ell\sin\phi\ \dot\phi$$

Jacobian **J**-matrix velocity relations:

$$\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

**J**-matrix

Transpose Jacobian $\mathbf{J}^T$-matrix velocity relations:

$\mathbf{J}^T$-matrix

$$\begin{pmatrix} \dot x & \dot y \end{pmatrix} = \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}$$

Jacobian **J**-matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

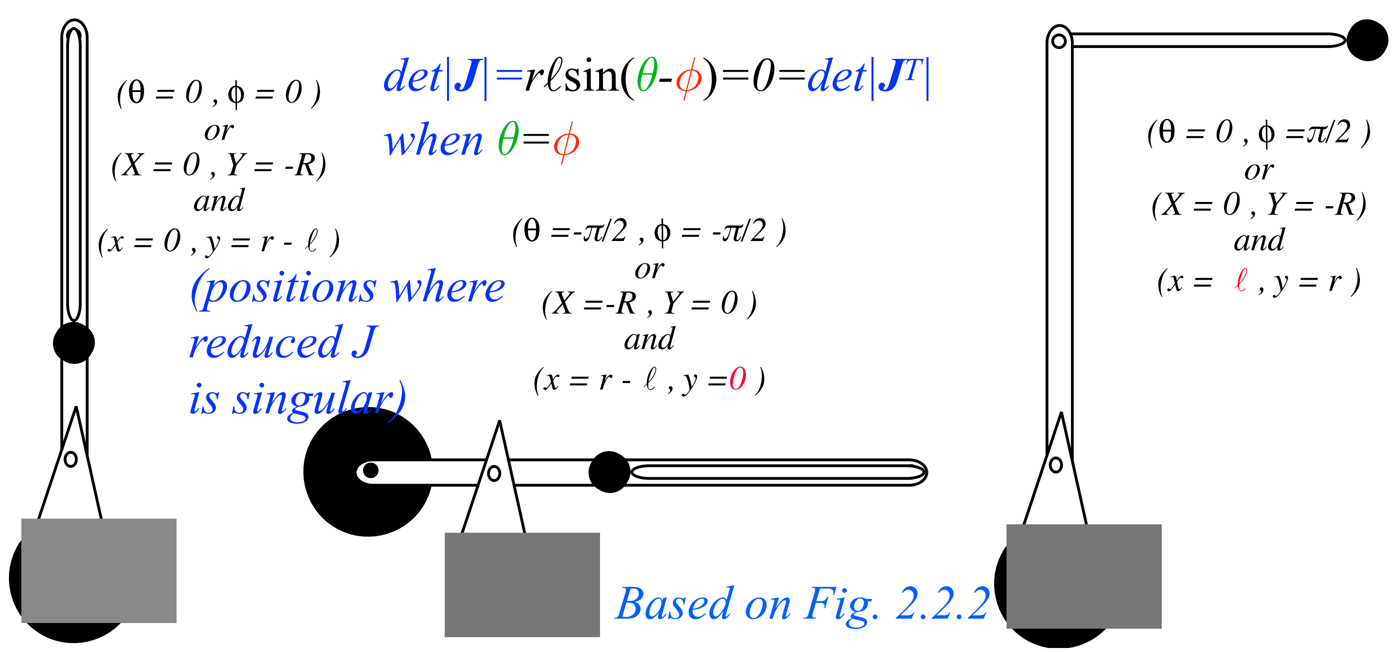
$$\det\begin{vmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{vmatrix} = -r\ell\cos\theta\sin\phi + r\ell\sin\theta\cos\phi = r\ell\sin(\theta-\phi)$$

SUCCESS! (Usually non-singular)

Fig. 2.2.2 Singular positions of the trebuchet

$\det|\mathbf{J}| = r\ell\sin(\theta-\phi) = 0 = \det|\mathbf{J}^T|$ when $\theta = \phi$

(positions where reduced $J$ is singular)

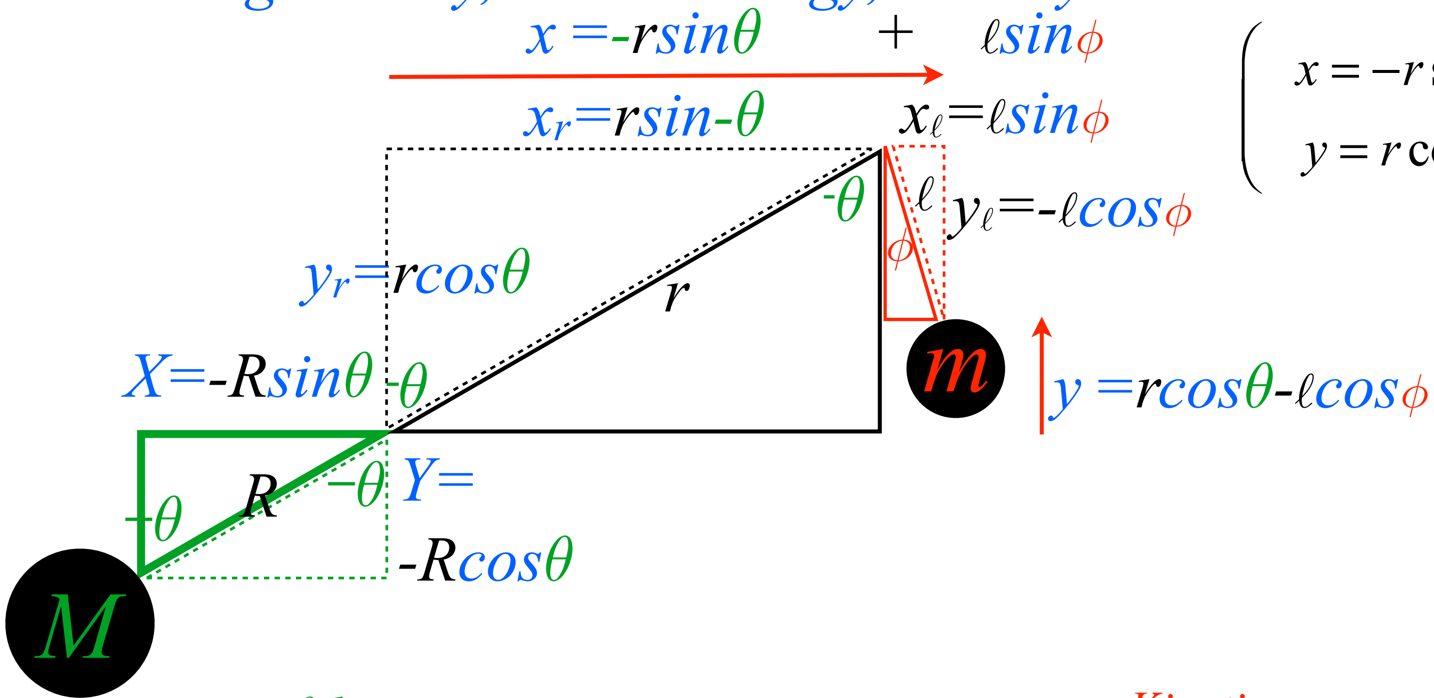



Based on Fig. 2.2.2

*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*

- → *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* $\gamma_{mn}$
- *Structure of dynamic metric tensor* $\gamma_{mn}$
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* $\gamma_{mn}$ *tensor*

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$ Velocity, Jacobian, and KE




$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

**J**-matrix

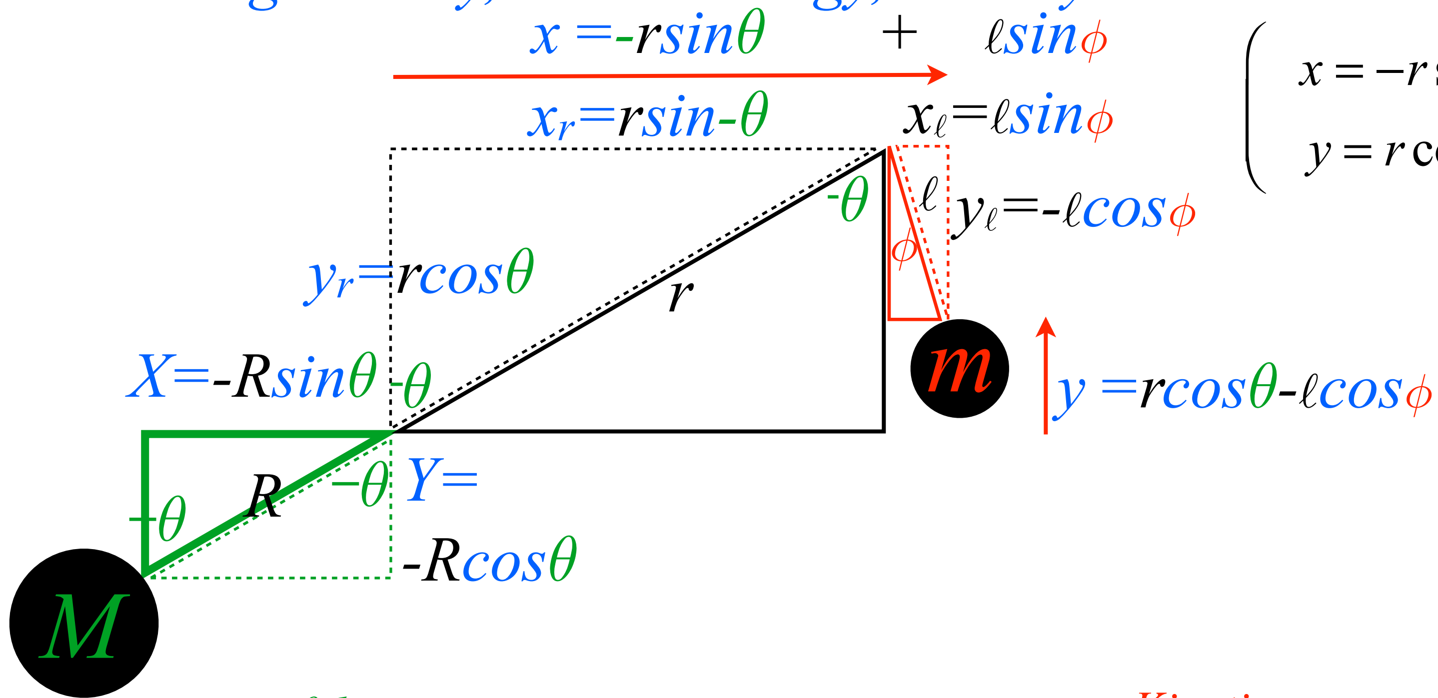
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

*Kinetic energy of driver M*

$$T(M) = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} M\dot{Y}^2$$

*Kinetic energy of projectile m* $T(m) = \frac{1}{2}\left[ m\dot{x}^2 + m\dot{y}^2 \right]$

## *Coordinate geometry, kinetic energy, and dynamic metric tensor* $\gamma_{mn}$




*Kinetic energy of driver M*

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

## *Velocity, Jacobian, and KE*

$$\begin{pmatrix} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{pmatrix}$$

**J**-*matrix*

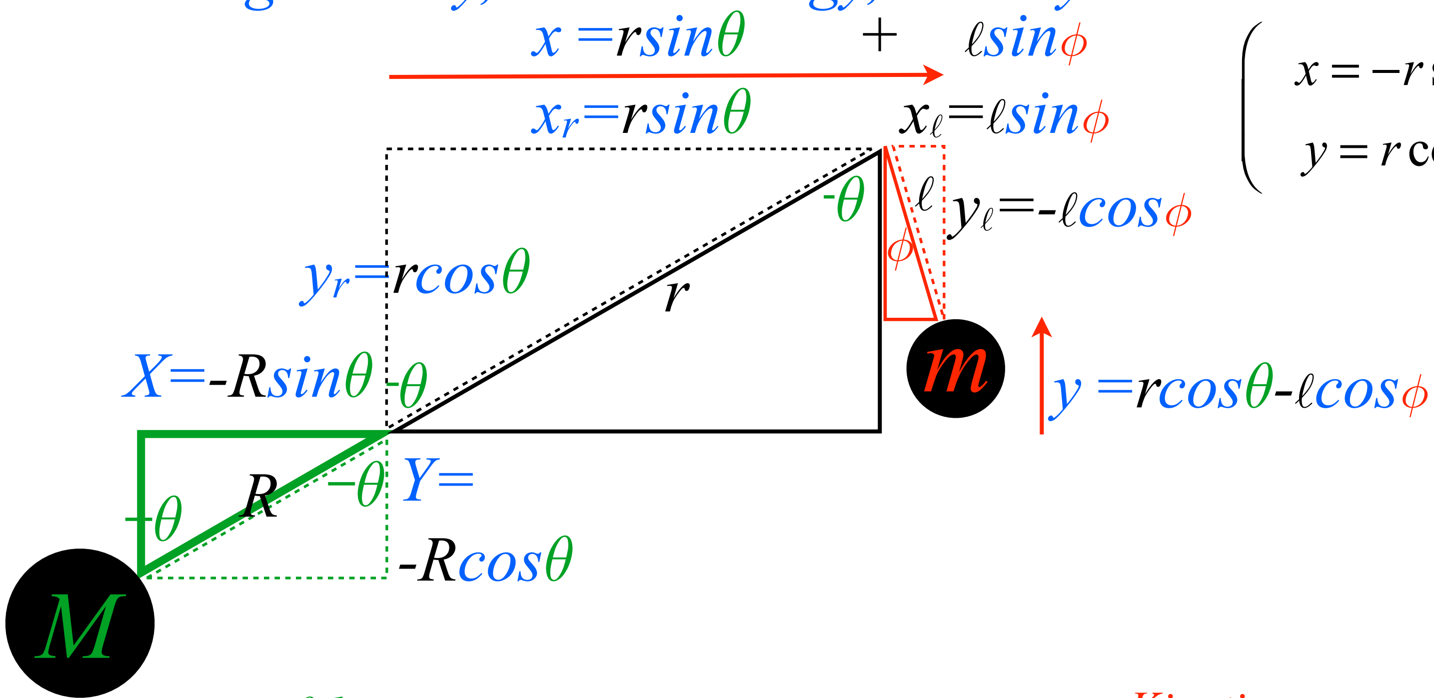
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}-r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

**J**$^T$-*matrix*

$$\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}=\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}$$

*Kinetic energy of projectile m* $T(m)=\frac{1}{2}\left[m\dot{x}^2+m\dot{y}^2\right]=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}$

# *Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$*




$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

## *Velocity, Jacobian, and KE*

***J**-matrix*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

***J**$^T$-matrix*

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}$$

*Kinetic energy of driver M*

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

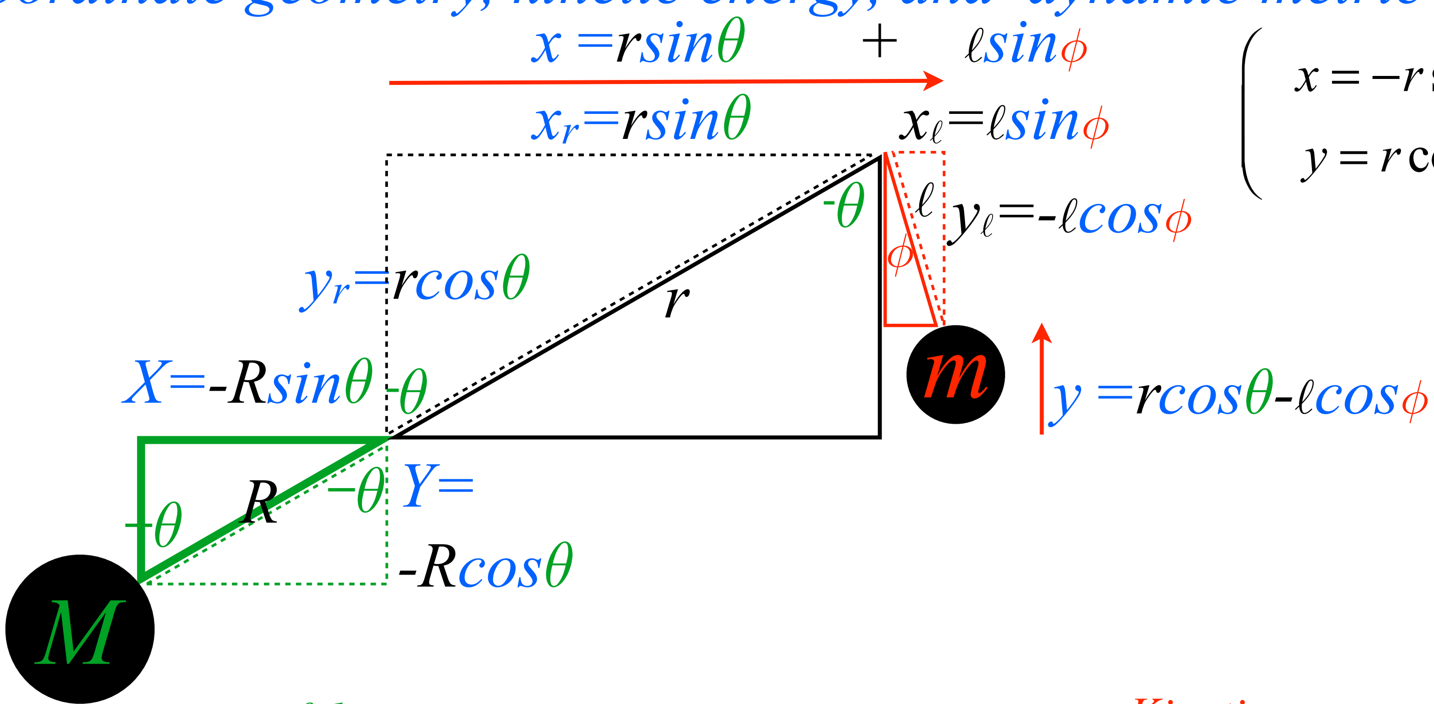
*Kinetic energy of projectile m*

$$T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

***J**$^T$-matrix* ***J**-matrix*

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$ Velocity, Jacobian, and KE




$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

**J**-matrix

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

**J**$^T$-matrix

$$\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix} = \begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}$$

*Kinetic energy of driver M*

*Kinetic energy of projectile m*

$$T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2 = \frac{1}{2}MR^2\dot{\theta}^2$$

**J**$^T$-matrix, **J**-matrix

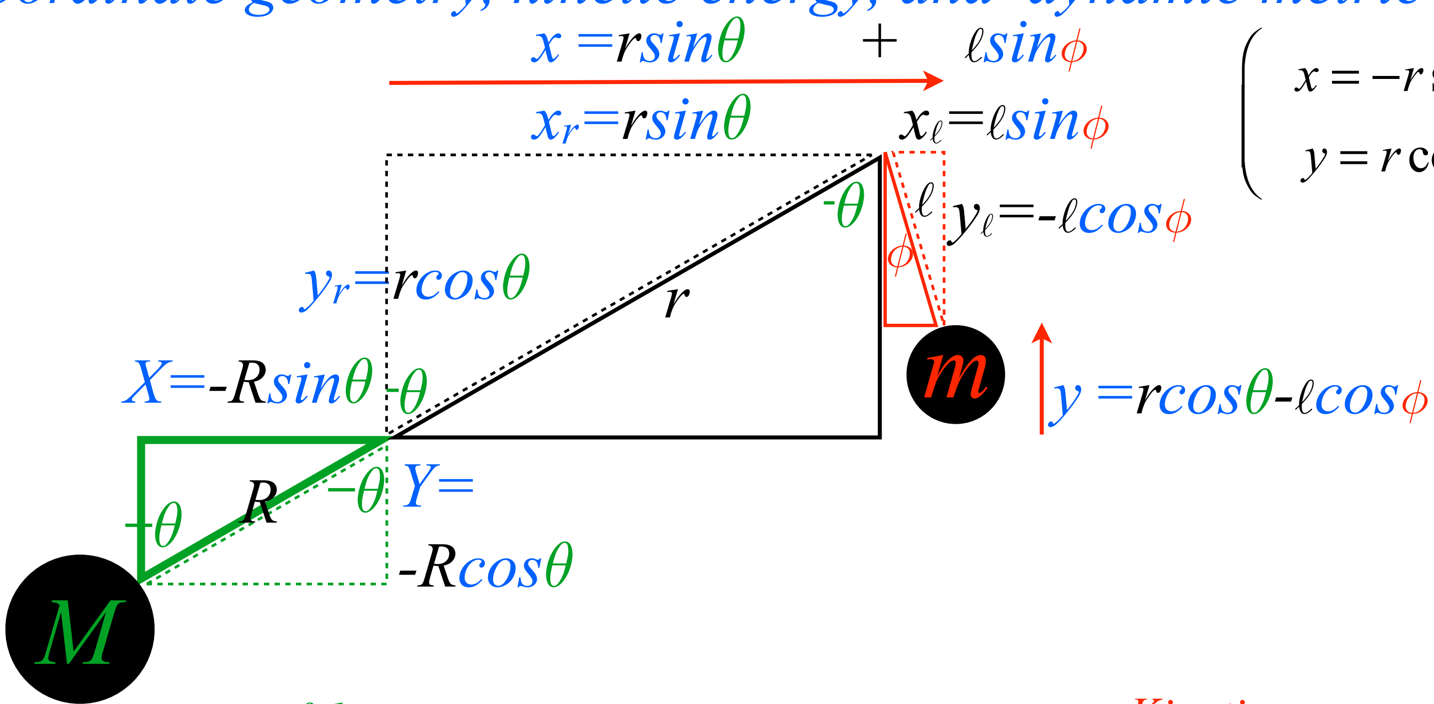
$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\textit{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\textit{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$




## Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

### 'Raw' Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix}\begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

*Kinetic energy of driver M*

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

*Kinetic energy of projectile m* $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right] = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$

*$J^T$-matrix* *J-matrix*

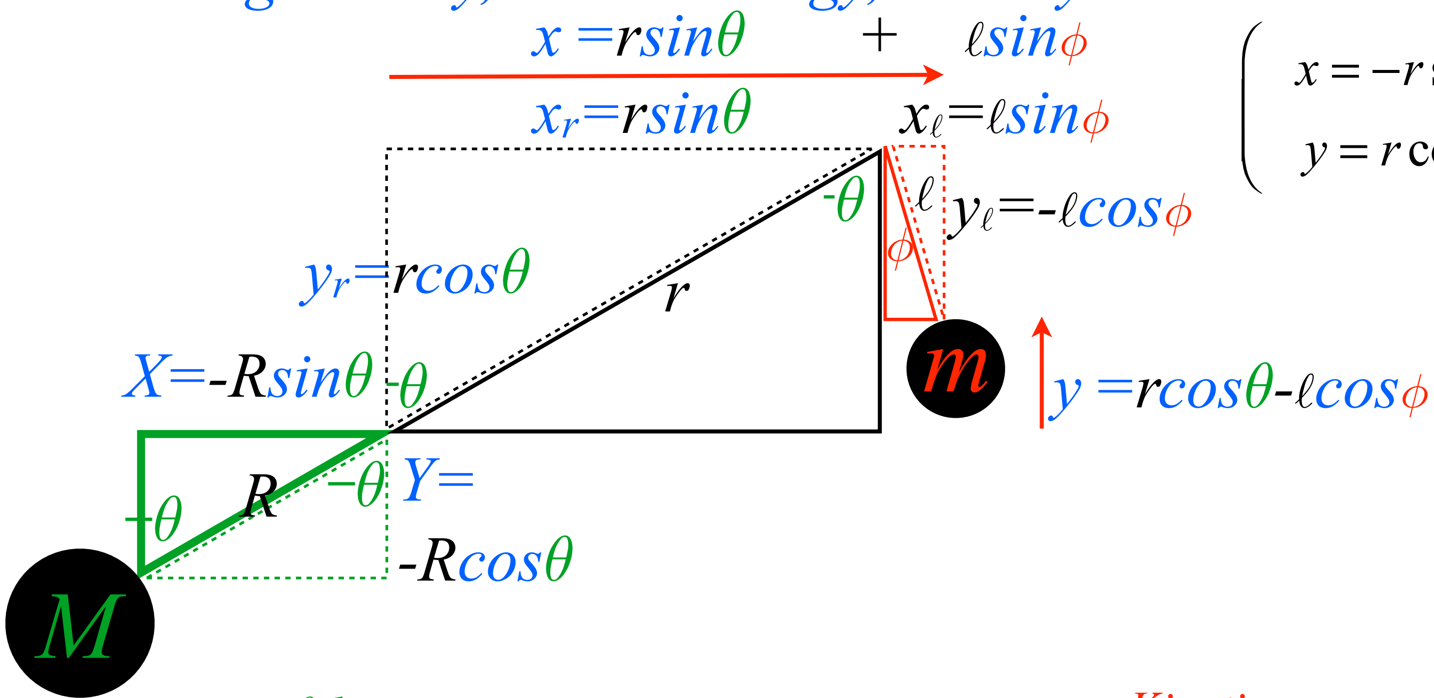
$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$




$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

## Velocity, Jacobian, and KE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

(x,y) to (θ,φ) Jacobian

Kinetic energy of driver M

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

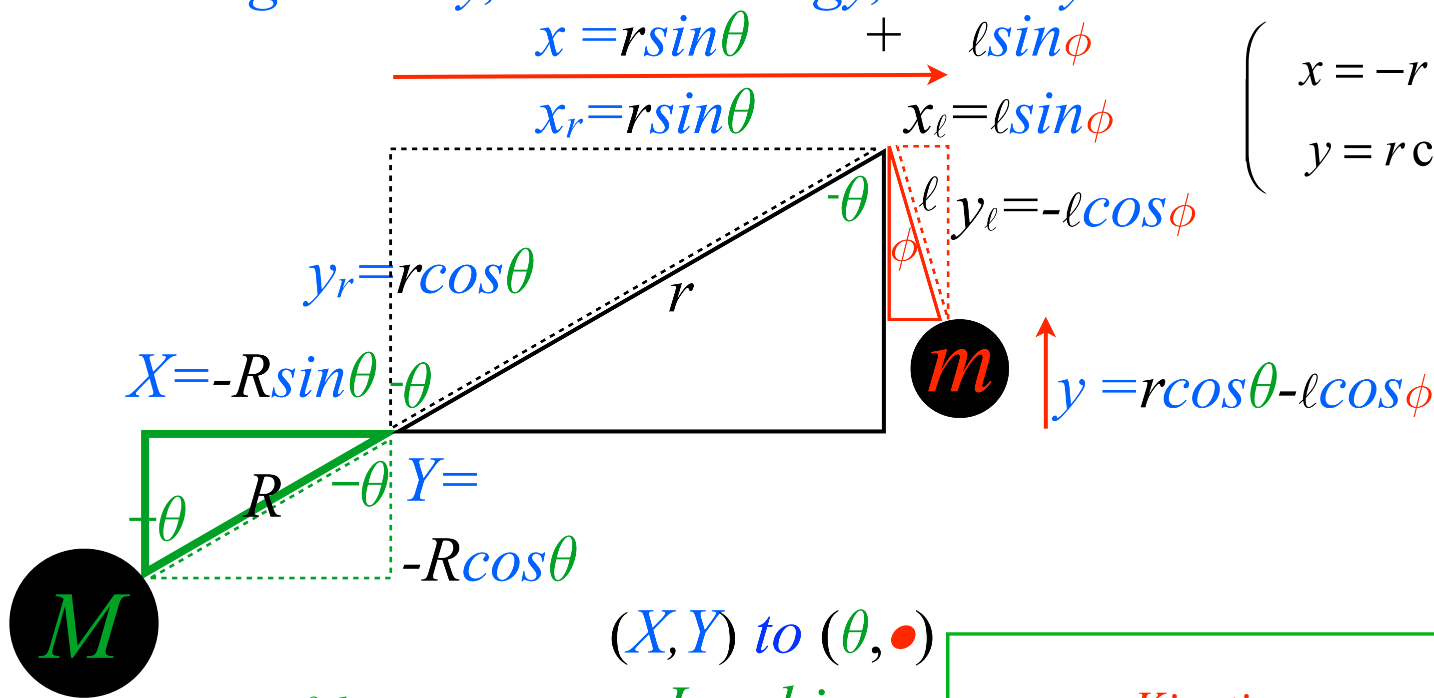
$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$




## Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

### Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$(x,y)$ to $(\theta,\phi)$ Jacobian

$(X,Y)$ to $(\theta,\bullet)$ Jacobian

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m $\quad T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

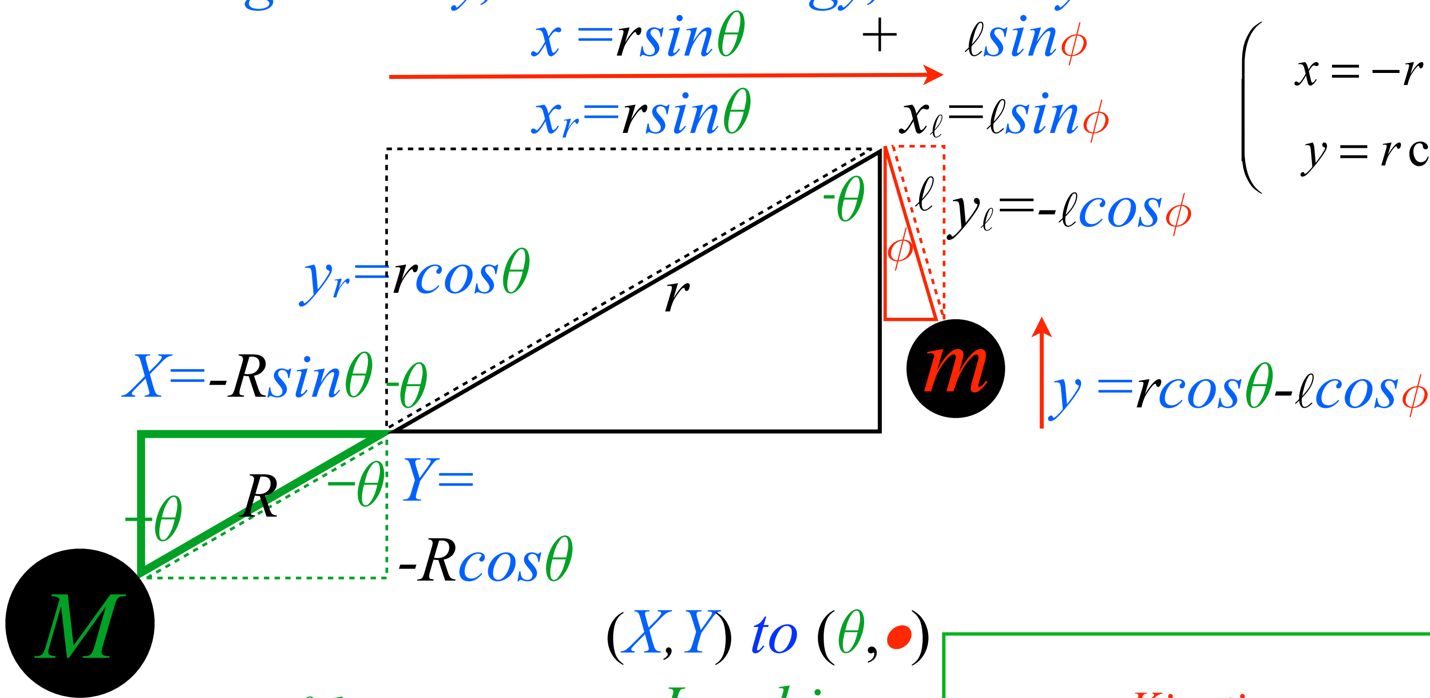
$$\text{Total KE} = T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* $\gamma_{mn}$
- *Structure of dynamic metric tensor* $\gamma_{mn}$
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* $\gamma_{mn}$ *tensor*

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$




$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix}$$

## Velocity, Jacobian, and KE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$(X,Y)$ to $(\theta,\bullet)$ Jacobian

$(x,y)$ to $(\theta,\phi)$ Jacobian

Kinetic energy of driver M

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

$J^T$-matrix, $J$-matrix

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

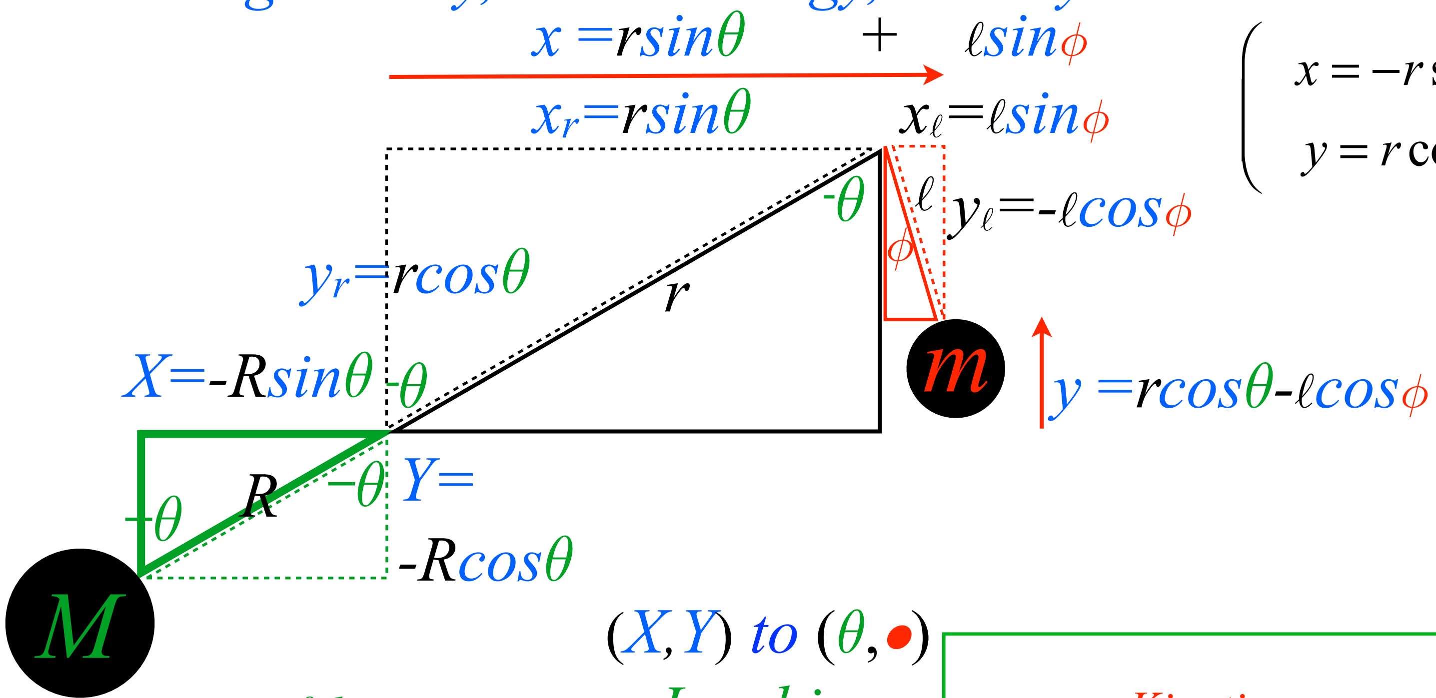
$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor $\gamma_{mn}$
in raw Cartesian $X,Y$ and $x,y$

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$




$$\begin{pmatrix} x=-r\sin\theta+\ell\sin\phi \\ y=r\cos\theta-\ell\cos\phi \end{pmatrix}$$

## Velocity, Jacobian, and KE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

### Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$(X,Y)$ to $(\theta,\bullet)$ Jacobian

$(x,y)$ to $(\theta,\phi)$ Jacobian

Kinetic energy of driver M

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m $T(m)=\frac{1}{2}\left[m\dot{x}^2+m\dot{y}^2\right]$

$J^T$-matrix, $J$-matrix

$$T(m)=\frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}=\frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2+mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

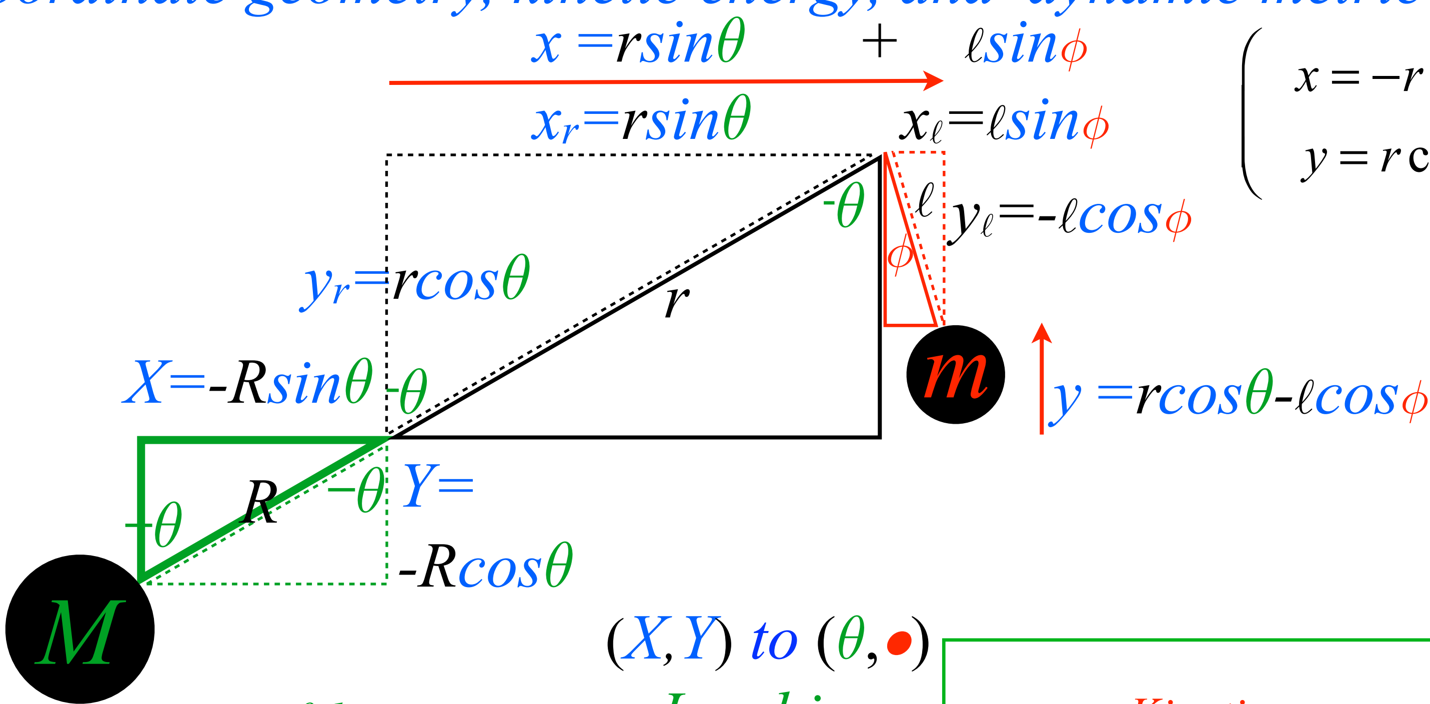
$$\text{Total KE}=T=\frac{1}{2}\left[M\dot{X}^2+M\dot{Y}^2+m\dot{x}^2+m\dot{y}^2\right]$$

$$T=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor $\gamma_{mn}$
in raw Cartesian $X,Y$ and $x,y$

## Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$




$$x = -r\sin\theta + \ell\sin\phi \qquad y = r\cos\theta - \ell\cos\phi$$

## Velocity, Jacobian, and KE

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

## Raw Jacobian

$$\begin{pmatrix}dX\\ dY\\ dx\\ dy\end{pmatrix} = \begin{pmatrix}\frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi}\\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi}\\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}d\theta\\ d\phi\end{pmatrix} = \begin{pmatrix}R\cos\theta & 0\\ R\sin\theta & 0\\ -r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}d\theta\\ d\phi\end{pmatrix}$$

$(X,Y)$ to $(\theta,\bullet)$ Jacobian

$(x,y)$ to $(\theta,\phi)$ Jacobian

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2 = \frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

$J^T$-matrix, $J$-matrix

$$T(m) = \frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2 + mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

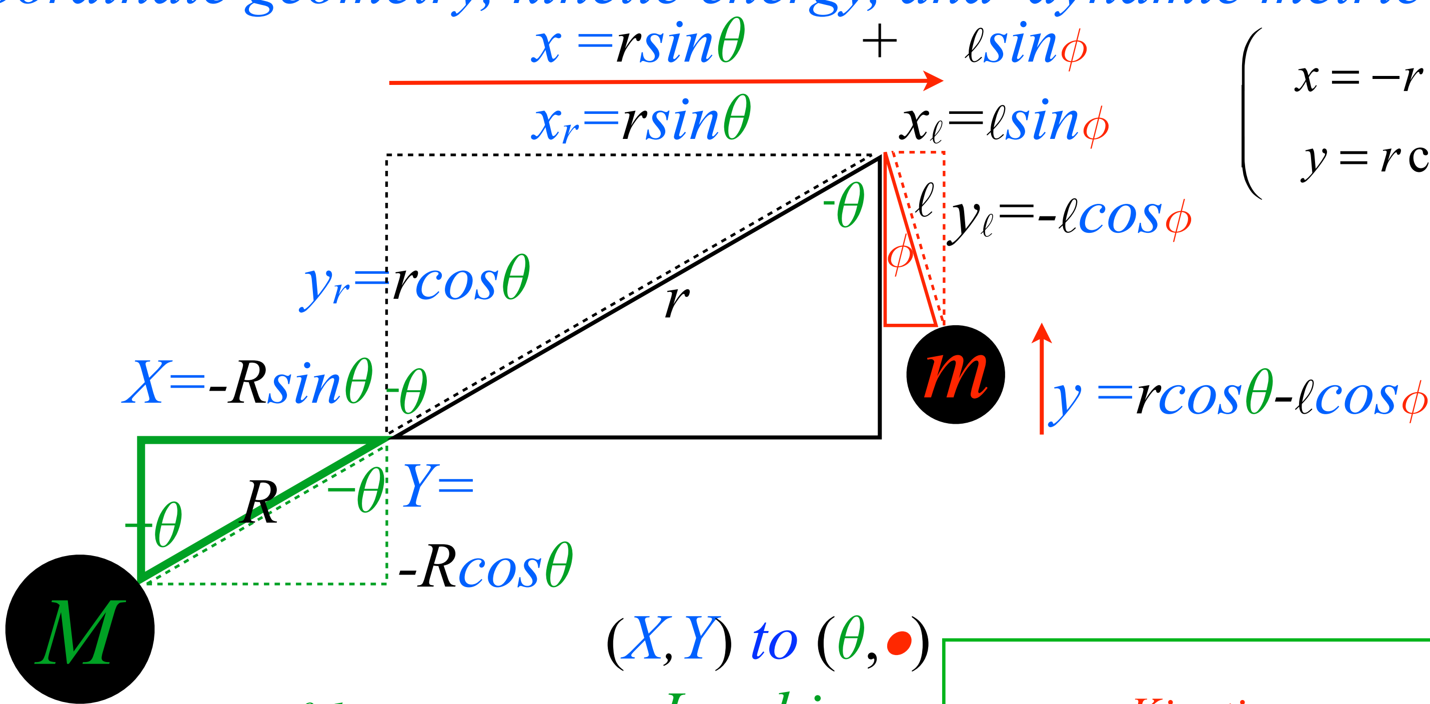
$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

$$T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right] = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix}\gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y}\end{pmatrix} = \begin{pmatrix}M & & & \\ & M & & \\ & & m & \\ & & & m\end{pmatrix}$$

Dynamic metric tensor $\gamma_{mn}$
in raw Cartesian $X,Y$ and $x,y$

# Coordinate geometry, kinetic energy, and dynamic metric tensor $\gamma_{mn}$




## Velocity, Jacobian, and KE

$$\begin{pmatrix} x = -r\sin\theta + \ell\sin\phi \\ y = r\cos\theta - \ell\cos\phi \end{pmatrix} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

$(X,Y)$ to $(\theta,\bullet)$ Jacobian

$(x,y)$ to $(\theta,\phi)$ Jacobian

Kinetic energy of driver M

$$T(M) = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}M\dot{Y}^2$$

$$= \frac{1}{2}MR^2\dot{\theta}^2$$

Kinetic energy of projectile m $T(m) = \frac{1}{2}\left[m\dot{x}^2 + m\dot{y}^2\right]$

$J^T$-matrix $\qquad$ $J$-matrix

$$T(m) = \frac{1}{2}m\begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{2}m\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

$$\text{Total KE} = T = T(M) + T(m) = \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\text{Total KE} = T = \frac{1}{2}\left[M\dot{X}^2 + M\dot{Y}^2 + m\dot{x}^2 + m\dot{y}^2\right]$$

$$T = \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right] = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

$$\begin{pmatrix} \gamma_{X,X} & & & \\ & \gamma_{Y,Y} & & \\ & & \gamma_{x,x} & \\ & & & \gamma_{y,y} \end{pmatrix} = \begin{pmatrix} M & & & \\ & M & & \\ & & m & \\ & & & m \end{pmatrix}$$

Dynamic metric tensor $\gamma_{mn}$
in raw Cartesian $X,Y$ and $x,y$

$J^T$-matrix $\qquad$ $J$-matrix

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta\text{-}\phi) \\ -mr\ell\cos(\theta\text{-}\phi) & m\ell^2 \end{pmatrix} = \begin{pmatrix} MR^2 & 0 \\ 0 & 0 \end{pmatrix} + m\begin{pmatrix} -r\cos\theta & -r\sin\theta \\ \ell\cos\phi & \ell\sin\phi \end{pmatrix}\begin{pmatrix} -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix}$$

Dynamic metric tensor $\gamma_{mn}$ in GCC $\theta$ and $\phi$

*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* $\gamma_{mn}$
- *Structure of dynamic metric tensor* $\gamma_{mn}$
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* $\gamma_{mn}$ *tensor*

*Kinetic energy of driver M*

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

*Kinetic energy of projectile m*

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

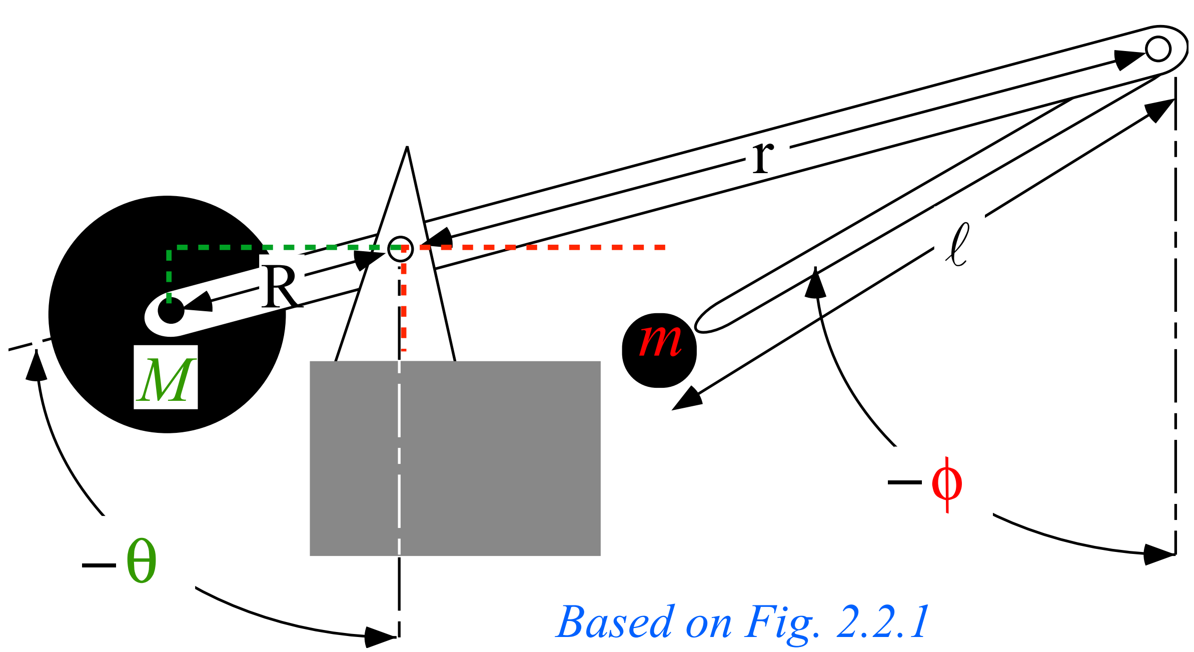
$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

*Total kinetic energy of M and m*

$$Total\ KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

*Dynamic metric tensor* $\gamma_{mn}$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$




*Based on Fig. 2.2.1*

*Kinetic energy of driver M*

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

*Kinetic energy of projectile m*

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

*Total kinetic energy of M and m*

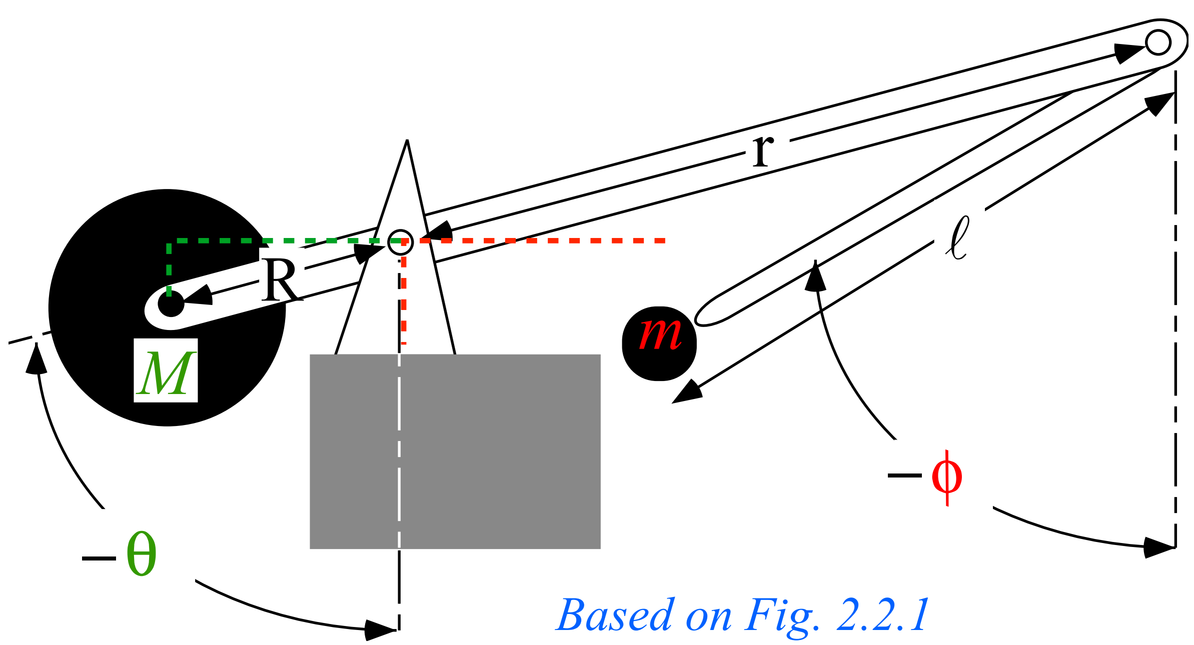
$$\text{Total } KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

*Dynamic metric tensor* $\gamma_{mn}$ $\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$

$$\gamma_{mn}=\sum_{mass\ \mu}m(\mu)\frac{\partial x^j(\mu)}{\partial q^m}\frac{\partial x^j(\mu)}{\partial q^n}$$

$$=\sum_{mass\ \mu}m(\mu)\frac{\partial\mathbf{r}(\mu)}{\partial q^m}\bullet\frac{\partial\mathbf{r}(\mu)}{\partial q^n}$$

$$=\sum_{mass\ \mu}m(\mu)\mathbf{E_m}(\mu)\bullet\mathbf{E_n}(\mu)$$




*Based on Fig. 2.2.1*

$$KE=\sum_{mass\ \mu}\frac{1}{2}m(\mu)\dot{x}^j(\mu)\dot{x}^j(\mu)=\sum_{mass\ \mu}\frac{1}{2}m(\mu)\frac{\partial x^j(\mu)}{\partial q^m}\frac{\partial x^j(\mu)}{\partial q^n}\dot{q}^m\dot{q}^n$$

$$=\frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

*Kinetic energy of driver M*

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

*Kinetic energy of projectile m*

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

*Total kinetic energy of M and m*

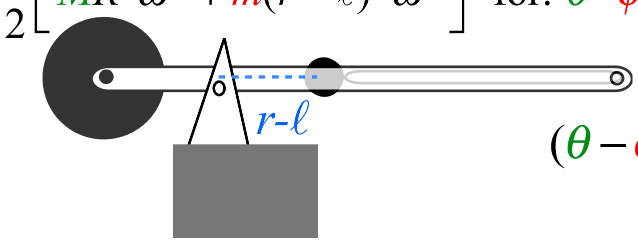
$$\textit{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

## *Dynamic metric tensor $\gamma_{mn}$*

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

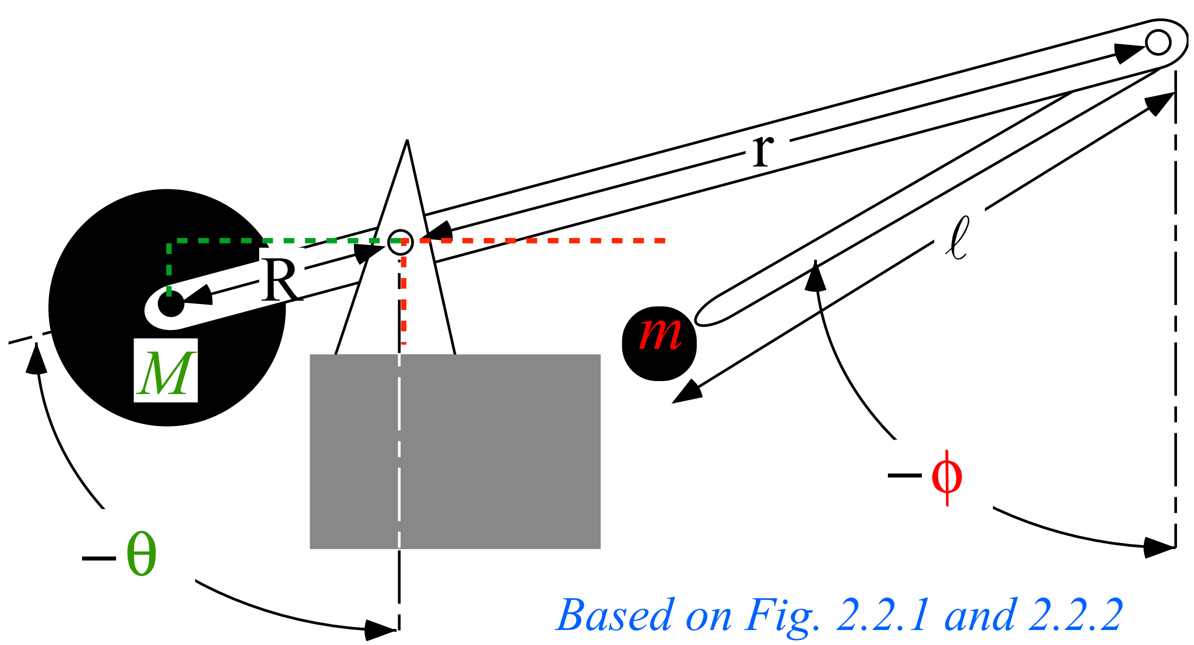
*Special cases (rigid rotation)*

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=0$$




*(J is Singular)*

$(\theta-\phi)=0$




*Based on Fig. 2.2.1 and 2.2.2*

*Kinetic energy of driver M*

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

*Kinetic energy of projectile m*

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

*Total kinetic energy of M and m*

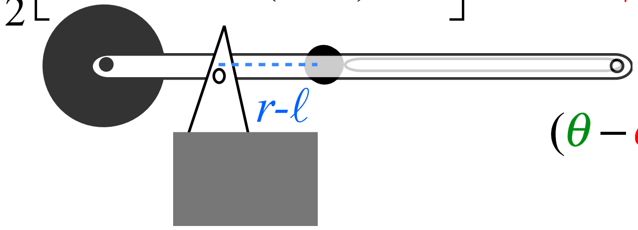
$$\text{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

## Dynamic metric tensor $\gamma_{mn}$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

*Special cases (rigid rotation)*

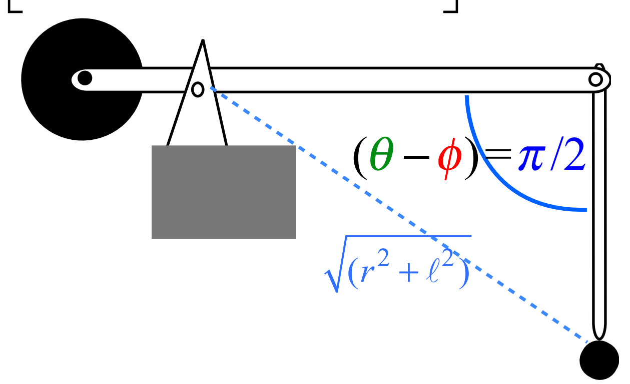
$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=0$$



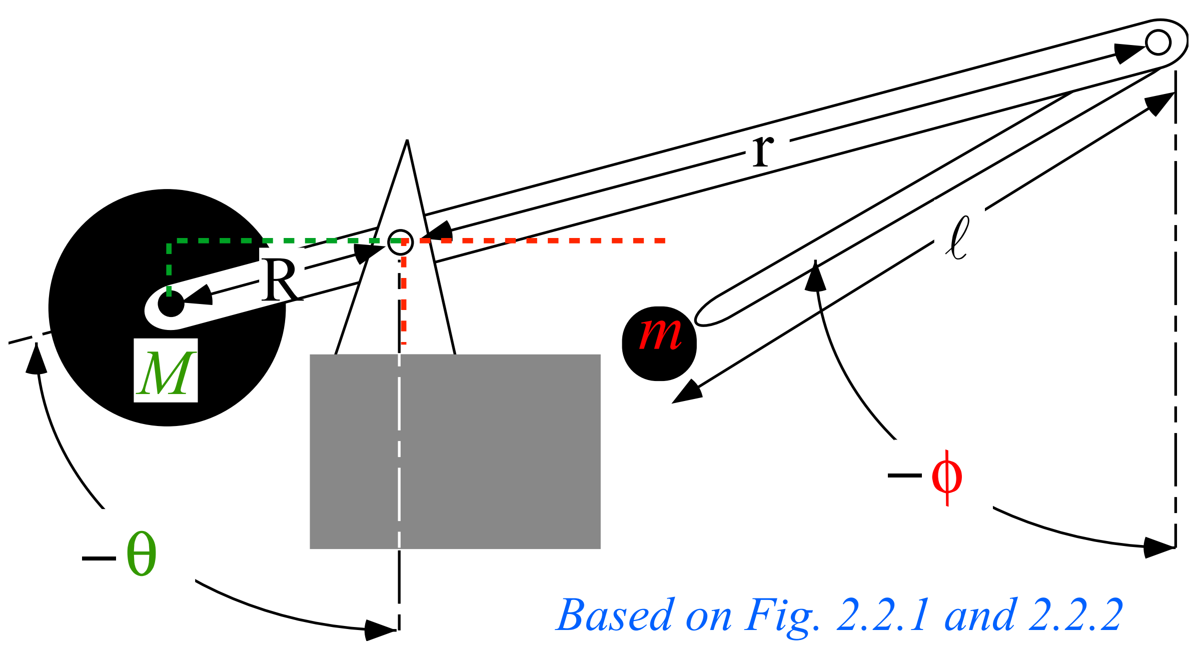

*(J is Singular)*

$(\theta-\phi)=0$

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad \text{for: } \dot{\theta}=\dot{\phi}=\omega \text{ and } (\theta-\phi)=\pi/2$$




*(J is Orthogonal)*




*Based on Fig. 2.2.1 and 2.2.2*

*Kinetic energy of driver M*

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

*Kinetic energy of projectile m*

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

*Total kinetic energy of M and m*

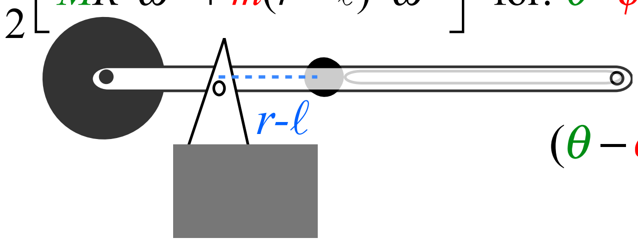
$$\text{Total }KE=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

## *Dynamic metric tensor $\gamma_{mn}$*

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

*Special cases (rigid rotation)*

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=0$$

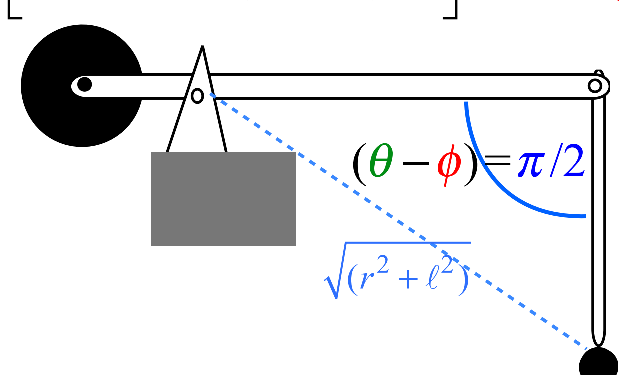



*(J is Singular)*

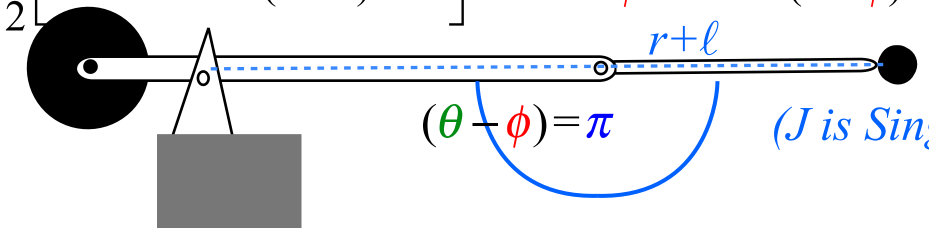
$(\theta-\phi)=0$

$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi/2$$

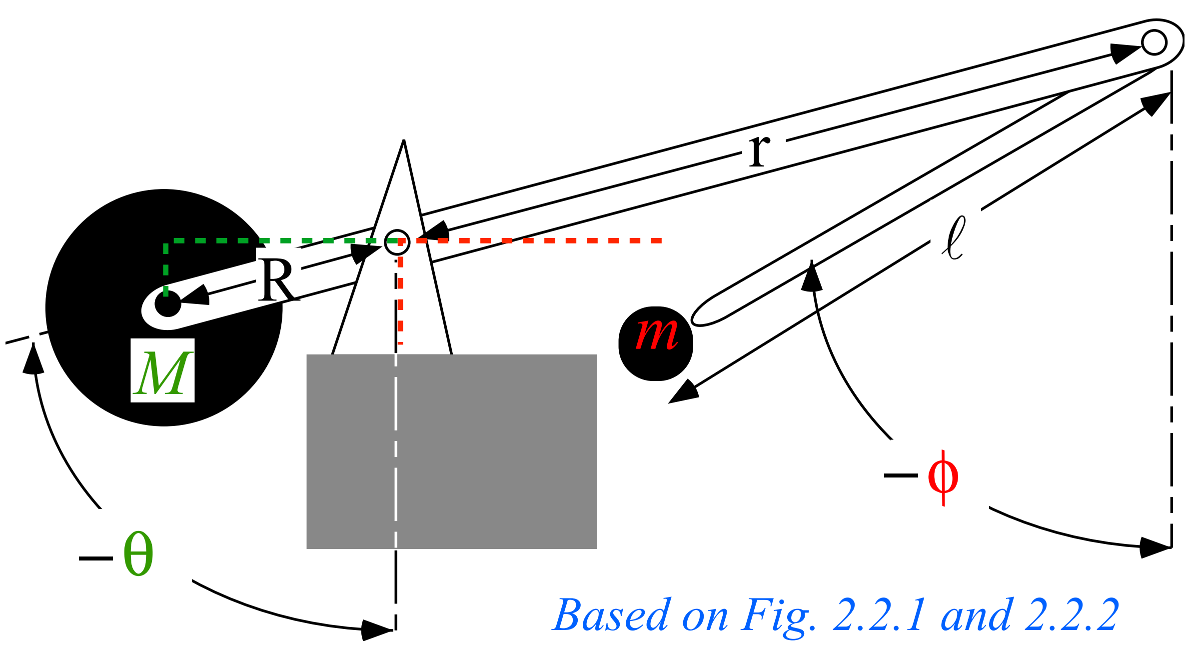
*(J is Orthogonal)*




$$T=\frac{1}{2}\left[MR^2\omega^2+m(r+\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi$$




*(J is Singular)*




*Based on Fig. 2.2.1 and 2.2.2*

*Kinetic energy of driver M*

$$T(M)=\frac{1}{2}M\dot{X}^2+\frac{1}{2}M\dot{Y}^2$$

$$=\frac{1}{2}MR^2\dot{\theta}^2$$

*Kinetic energy of projectile m*

$$T(m)=\frac{1}{2}m\begin{pmatrix}\dot{x} & \dot{y}\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}^T\begin{pmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}-r\cos\theta & -r\sin\theta\\ \ell\cos\phi & \ell\sin\phi\end{pmatrix}\begin{pmatrix}-r\cos\theta & \ell\cos\phi\\ -r\sin\theta & \ell\sin\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}r^2\cos^2\theta+r^2\sin^2\theta & -r\ell\cos\theta\cos\phi-r\ell\sin\theta\sin\phi\\ -\ell r\cos\phi\cos\theta-r\ell\sin\theta\sin\phi & \ell^2\cos^2\phi+\ell^2\sin^2\phi\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}$$

*Total kinetic energy of M and m*

$$\textit{Total KE}=T=T(M)+T(m)=\frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}MR^2+mr^2 & -mr\ell\cos(\theta-\phi)\\ -mr\ell\cos(\theta-\phi) & m\ell^2\end{pmatrix}\begin{pmatrix}\dot{\theta}\\ \dot{\phi}\end{pmatrix}=\frac{1}{2}\left[(MR^2+mr^2)\dot{\theta}^2-2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi}+m\ell^2\dot{\phi}^2\right]$$

## Dynamic metric tensor $\gamma_{mn}$

$$\begin{pmatrix}\gamma_{\theta,\theta} & \gamma_{\theta,\phi}\\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi}\end{pmatrix}$$

SPECIAL CASE

(a) When (θ,ϕ) coordinates are orthogonal

USUAL CASE

(b) When (θ,ϕ) coordinates are not orthogonal

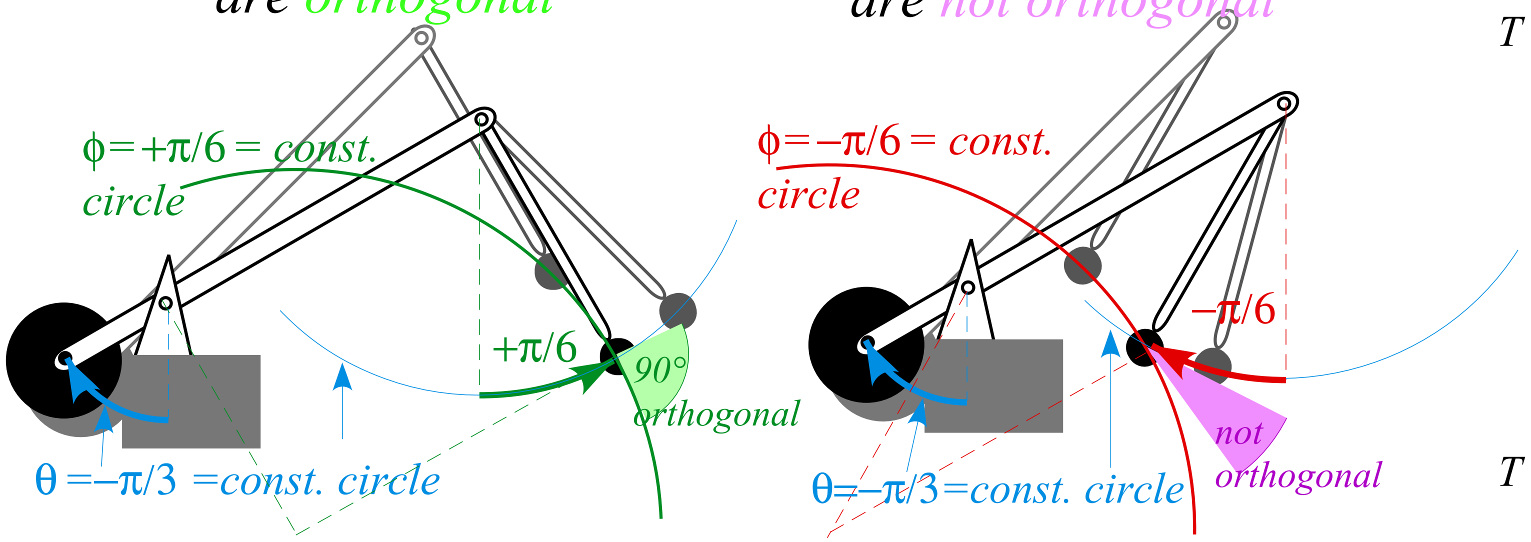


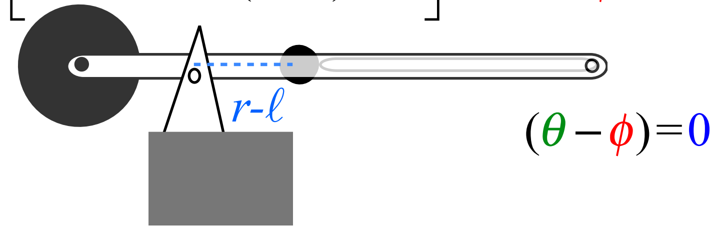

*Fig. 2.3.1 Examples of (θ,ϕ) intersections (a) othogonal (special case), (b) non-orthogonal (typical).*

Based on Fig. 2.3.1 and 2.2.2

*Special cases (rigid rotation)*

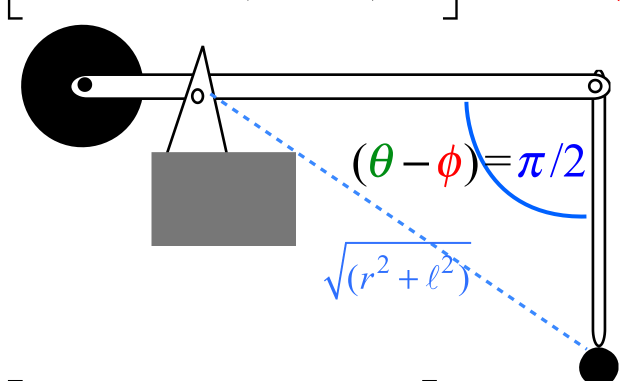
$$T=\frac{1}{2}\left[MR^2\omega^2+m(r-\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=0$$

(J is Singular)

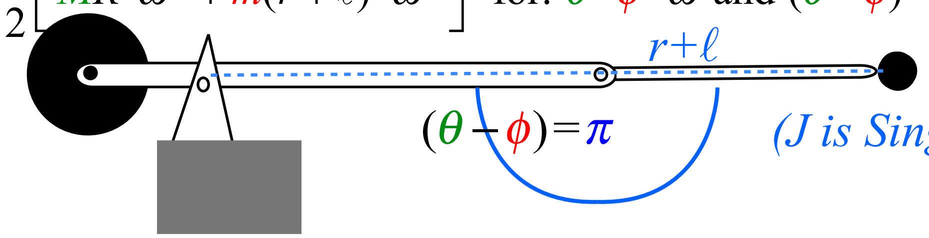



$$T=\frac{1}{2}\left[MR^2\omega^2+m(r^2+\ell^2)\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi/2$$

(J is Orthogonal)




$$T=\frac{1}{2}\left[MR^2\omega^2+m(r+\ell)^2\omega^2\right]\quad\text{for: }\dot{\theta}=\dot{\phi}=\omega\text{ and }(\theta-\phi)=\pi$$




(J is Singular)

*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* $\gamma_{mn}$
- *Structure of dynamic metric tensor* $\gamma_{mn}$
- → *Basic force, work, and acceleration*
- *Lagrangian force equation*
- *Canonical momentum and* $\gamma_{mn}$ *tensor*

# Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

## Write work-sums in columns:

$$dW = F_X\, dX \quad = \quad M\ddot{X}\, dX$$

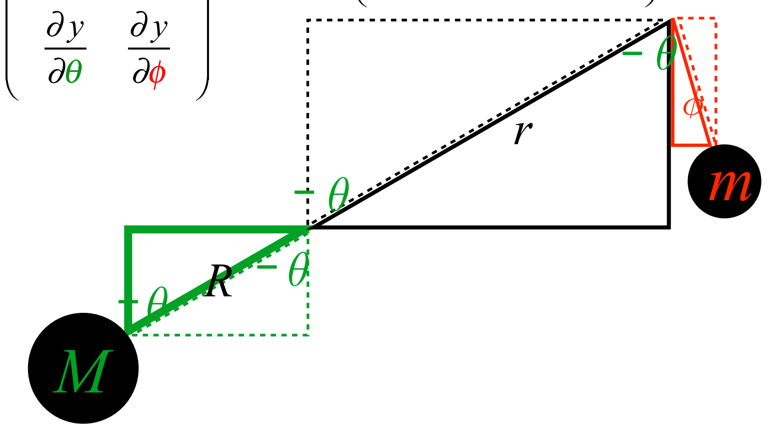
$$+ F_Y\, dY \qquad + M\ddot{Y}\, dY$$

$$+ F_x\, dx \qquad + m\ddot{x}\, dx$$

$$+ F_y\, dy \qquad + m\ddot{y}\, dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$



# Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

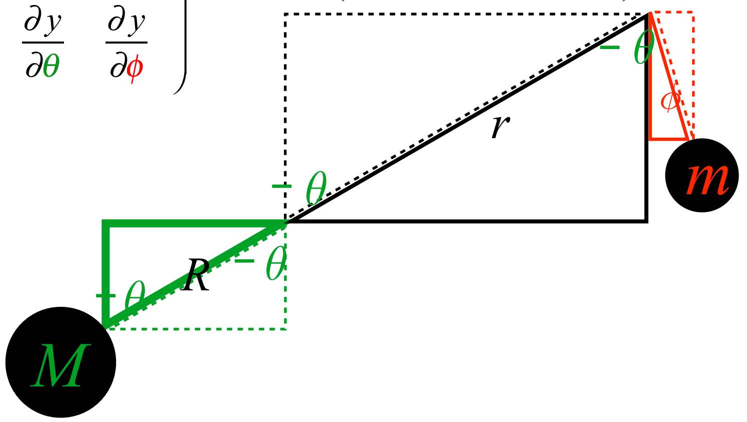
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$

## Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\ dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\ dY &\quad +M\ddot{Y}\,dY &\quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\ dx &\quad +m\ddot{x}\,dx &\quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &\quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\ dy &\quad +m\ddot{y}\,dy &\quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$



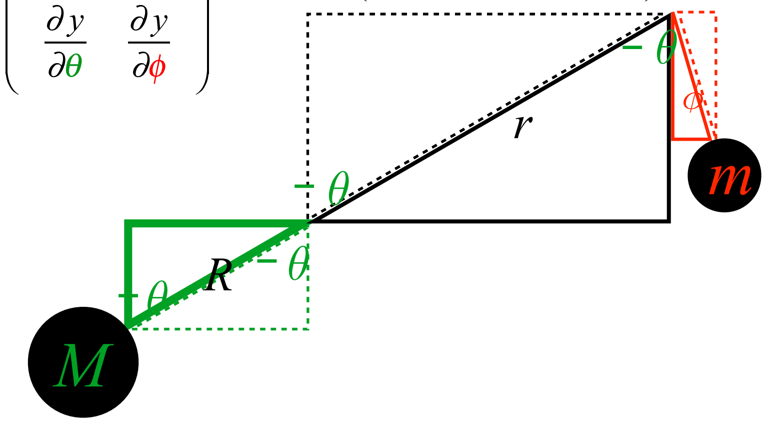
# Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\ dX = M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\ dY \quad + M\ddot{Y}\,dY \quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\ dx \quad + m\ddot{x}\,dx \quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\ dy \quad + m\ddot{y}\,dy \quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

*Assuming variables* $\theta$ *and* $\phi$ *are independent...*

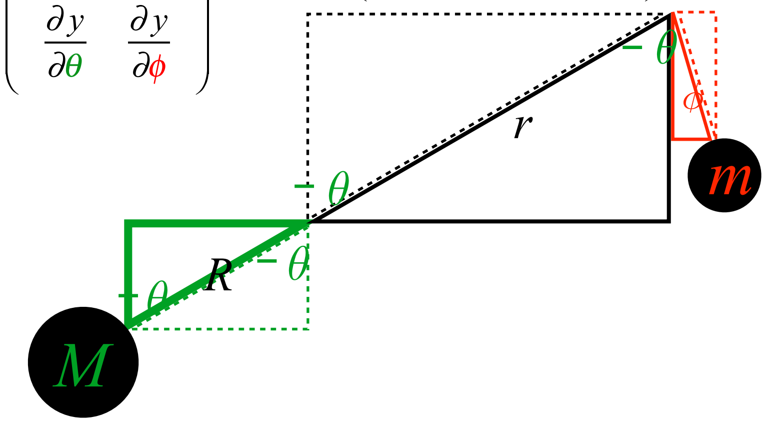
# Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns:(Using GCC dθ and dφ in Jacobian)

$$\begin{aligned}
dW = \; & F_X\, dX &=\;& M\ddot{X}\,dX &=\;& F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &=\;& M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
& + F_Y\, dY && + M\ddot{Y}\,dY && + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi && + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
& + F_x\, dx && + m\ddot{x}\,dx && + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi && + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
& + F_y\, dy && + m\ddot{y}\,dy && + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi && + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

*Assuming variables θ and φ are independent...*

*Set:* $d\theta=1 \quad d\phi=0$

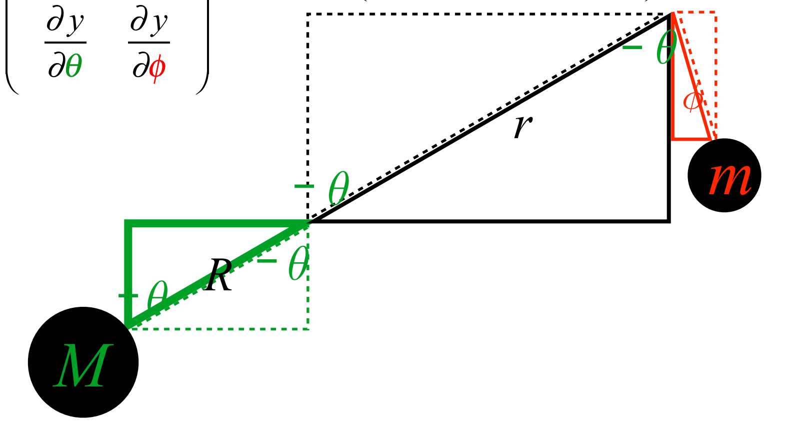
$$\begin{aligned}
& F_X\frac{\partial X}{\partial\theta} &=\;& M\ddot{X}\frac{\partial X}{\partial\theta} \\
& + F_Y\frac{\partial Y}{\partial\theta} && + M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
& + F_x\frac{\partial x}{\partial\theta} && + m\ddot{x}\frac{\partial x}{\partial\theta} \\
& + F_y\frac{\partial y}{\partial\theta} && + m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

## Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$
$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




### Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\,dX &= M\ddot{X}\,dX &= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\,dY &\quad +M\ddot{Y}\,dY &\quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\,dx &\quad +m\ddot{x}\,dx &\quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &\quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\,dy &\quad +m\ddot{y}\,dy &\quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

*Assuming variables $\theta$ and $\phi$ are independent...*

### Set: $d\theta=1$ $d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad +m\ddot{x}\frac{\partial x}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad +m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

### Set: $d\theta=0$ $d\phi=1$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad +M\ddot{Y}\frac{\partial Y}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad +m\ddot{x}\frac{\partial x}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad +m\ddot{y}\frac{\partial y}{\partial\phi}
\end{aligned}$$

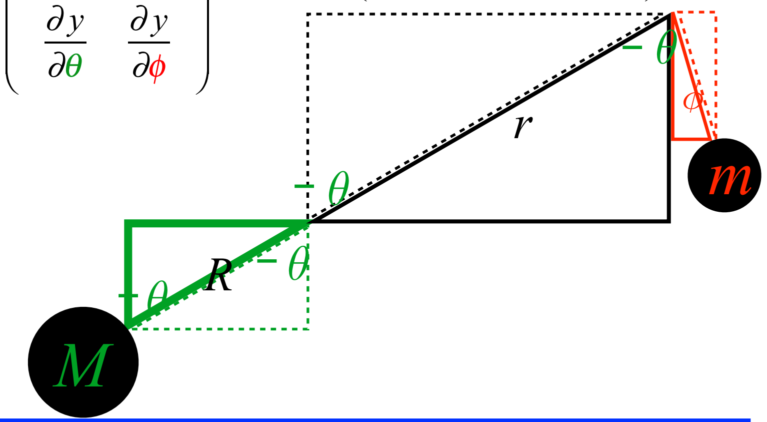
# Force, Work, and Acceleration

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns:(Using GCC dθ and dϕ in Jacobian)

$$\begin{aligned}
dW = F_X\ dX &= M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\ dY &\quad + M\ddot{Y}\,dY \quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\ dx &\quad + m\ddot{x}\,dx \quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\ dy &\quad + m\ddot{y}\,dy \quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

## Lagrange trickery:

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta}$$

$$\text{(using } \frac{d}{dt}\left(\dot{X}U\right) = \ddot{X}U + \dot{X}\dot{U}\text{)}$$

STEP
**A**

### Set: dθ=1 dϕ=0

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad + m\ddot{x}\frac{\partial x}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad + m\ddot{y}\frac{\partial y}{\partial\theta}
\end{aligned}$$

### Set: dθ=0 dϕ=1

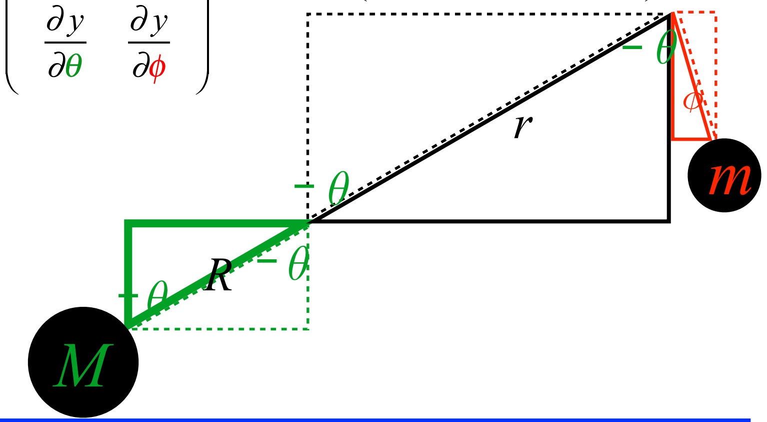
$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad + m\ddot{x}\frac{\partial x}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad + m\ddot{y}\frac{\partial y}{\partial\phi}
\end{aligned}$$

# Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$
$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns:(Using GCC dθ and dϕ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\, dY \qquad +M\ddot{Y}\, dY \qquad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \qquad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\, dx \qquad +m\ddot{x}\, dx \qquad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \qquad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\, dy \qquad +m\ddot{y}\, dy \qquad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \qquad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

## Lagrange trickery:

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial\dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial\dot{X}}{\partial\theta}$$

(using $\frac{d}{dt}(\dot{X}U) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **A**

STEP **B**

by *lemma 1*: $\frac{\partial\dot{X}}{\partial\dot{q}} = \frac{\partial X}{\partial q}$

and *lemma 2*: $\frac{\partial\dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

*Lemmas from Lect.9*
*lemma 1: p.9.13*
*lemma 2: p.9.24*

### Set: $d\theta=1 \quad d\phi=0$

$$F_X\frac{\partial X}{\partial\theta} = M\ddot{X}\frac{\partial X}{\partial\theta}$$
$$+F_Y\frac{\partial Y}{\partial\theta} \qquad +M\ddot{Y}\frac{\partial Y}{\partial\theta}$$
$$+F_x\frac{\partial x}{\partial\theta} \qquad +m\ddot{x}\frac{\partial x}{\partial\theta}$$
$$+F_y\frac{\partial y}{\partial\theta} \qquad +m\ddot{y}\frac{\partial y}{\partial\theta}$$

### Set: $d\theta=0 \quad d\phi=1$

$$F_X\frac{\partial X}{\partial\phi} = M\ddot{X}\frac{\partial X}{\partial\phi}$$
$$+F_Y\frac{\partial Y}{\partial\phi} \qquad +M\ddot{Y}\frac{\partial Y}{\partial\phi}$$
$$+F_x\frac{\partial x}{\partial\phi} \qquad +m\ddot{x}\frac{\partial x}{\partial\phi}$$
$$+F_y\frac{\partial y}{\partial\phi} \qquad +m\ddot{y}\frac{\partial y}{\partial\phi}$$

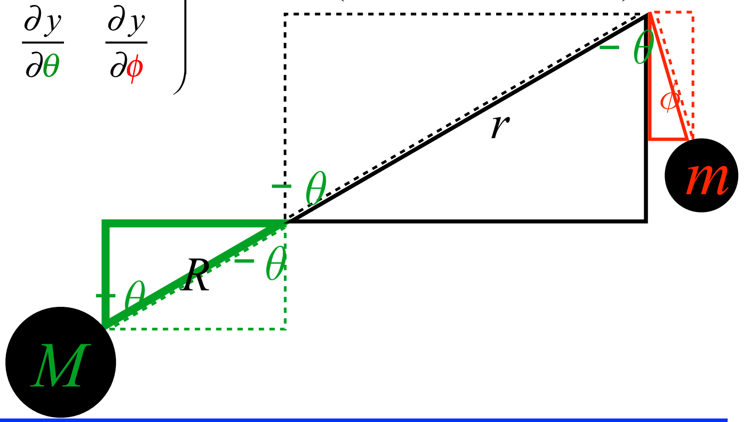
# Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\,dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X}\frac{\partial X}{\partial \theta} d\theta + M\ddot{X}\frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\,dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y}\frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\,dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x}\frac{\partial x}{\partial \theta} d\theta + m\ddot{x}\frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\,dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y}\frac{\partial y}{\partial \theta} d\theta + m\ddot{y}\frac{\partial y}{\partial \phi} d\phi$$

## Lagrange trickery:

STEP **A**

$$\ddot{X}\frac{\partial X}{\partial \theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial \theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial \theta}$$

(using $\frac{d}{dt}\left(\dot{X}U\right) = \ddot{X}U + \dot{X}\dot{U}$)

STEP **B**

$$= \frac{d}{dt}\left(\dot{X}\frac{\partial \dot{X}}{\partial \dot{\theta}}\right) - \dot{X}\frac{\partial \dot{X}}{\partial \theta}$$

*by lemma 1:* $\frac{\partial X}{\partial q} = \frac{\partial \dot{X}}{\partial \dot{q}}$

*and lemma 2:* $\frac{\partial \dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

STEP **C**

$$= \frac{d}{dt}\left(\frac{\partial\left(\dot{X}^2/2\right)}{\partial \dot{\theta}}\right) - \frac{\partial\left(\dot{X}^2/2\right)}{\partial \theta}$$

(using $\frac{\partial\left(U^2/2\right)}{\partial q} = U\frac{\partial U}{\partial q}$)

### Set: $d\theta=1$ $d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X}\frac{\partial X}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} \quad + m\ddot{x}\frac{\partial x}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} \quad + m\ddot{y}\frac{\partial y}{\partial \theta}$$

### Set: $d\theta=0$ $d\phi=1$

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X}\frac{\partial X}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} \quad + M\ddot{Y}\frac{\partial Y}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} \quad + m\ddot{x}\frac{\partial x}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} \quad + m\ddot{y}\frac{\partial y}{\partial \phi}$$

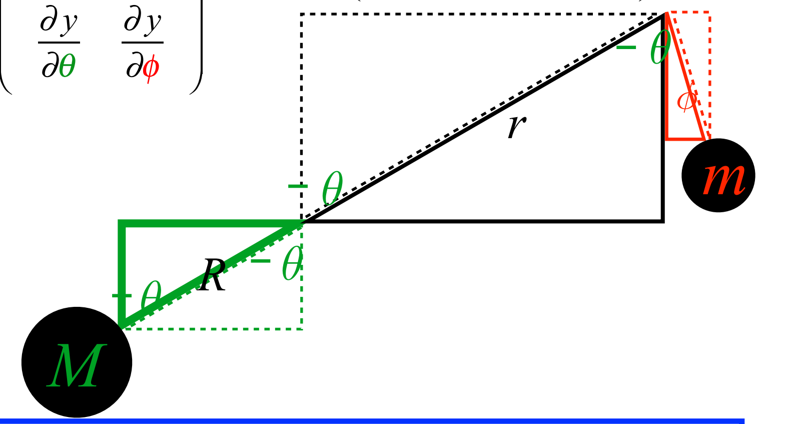
# *Force, Work, and Acceleration*

$$dW = F_X\ dX + F_Y\ dY + F_x\ dx + F_x\ dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## *Raw Jacobian*

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## *Write work-sums in columns:(Using GCC dθ and dφ in Jacobian)*

$$dW = F_X\ dX = M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$

$$+F_Y\ dY \quad + M\ddot{Y}\,dY \quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$

$$+F_x\ dx \quad + m\ddot{x}\,dx \quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$

$$+F_y\ dy \quad + m\ddot{y}\,dy \quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

## *Lagrange trickery:*

$$\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\left(\dot{X}\frac{\partial X}{\partial\theta}\right) - \dot{X}\frac{d}{dt}\frac{\partial X}{\partial\theta} \qquad = \frac{d}{dt}\left(\dot{X}\frac{\partial \dot{X}}{\partial\dot{\theta}}\right) - \dot{X}\frac{\partial \dot{X}}{\partial\theta} \qquad = \frac{d}{dt}\left(\frac{\partial\left(\dot{X}^2/2\right)}{\partial\dot{\theta}}\right) - \frac{\partial\left(\dot{X}^2/2\right)}{\partial\theta}$$

(using $\frac{d}{dt}\left(\dot{X}U\right) = \ddot{X}U + \dot{X}\dot{U}$)

STEP ***A***

*by lemma 1*: $\frac{\partial X}{\partial q} = \frac{\partial \dot{X}}{\partial \dot{q}}$

STEP ***B*** *and lemma 2*: $\frac{\partial \dot{X}}{\partial q} = \frac{d}{dt}\frac{\partial X}{\partial q}$

STEP ***C*** (using $\frac{\partial\left(U^2/2\right)}{\partial q} = U\frac{\partial U}{\partial q}$)

### *Set: dθ=1  dφ=0*

$$F_X\frac{\partial X}{\partial\theta} = M\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\theta}$$

$$+F_Y\frac{\partial Y}{\partial\theta} \quad + M\ddot{Y}\frac{\partial Y}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\theta}$$

$$+F_x\frac{\partial x}{\partial\theta} \quad + m\ddot{x}\frac{\partial x}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\theta}$$

$$+F_y\frac{\partial y}{\partial\theta} \quad + m\ddot{y}\frac{\partial y}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\theta}$$

### *Set: dθ=0  dφ=1*

$$F_X\frac{\partial X}{\partial\phi} = M\ddot{X}\frac{\partial X}{\partial\phi} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\phi}$$

$$+F_Y\frac{\partial Y}{\partial\phi} \quad + M\ddot{Y}\frac{\partial Y}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\phi}$$

$$+F_x\frac{\partial x}{\partial\phi} \quad + m\ddot{x}\frac{\partial x}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\phi}$$

$$+F_y\frac{\partial y}{\partial\phi} \quad + m\ddot{y}\frac{\partial y}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\phi}$$

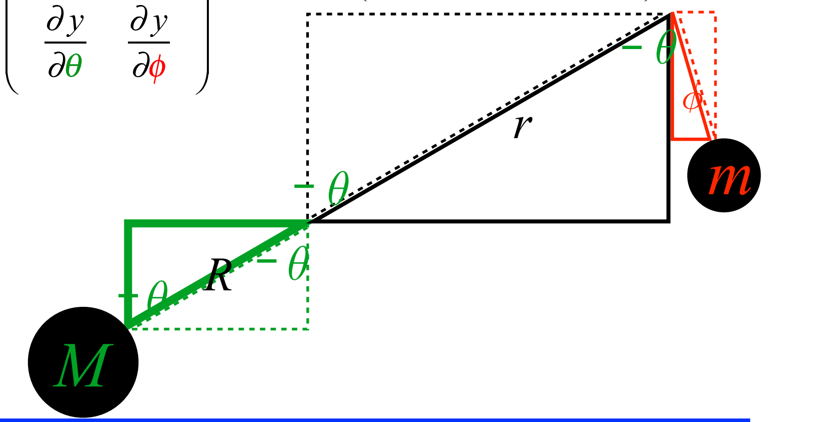
# Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$

## Lagrange trickery:

STEP **D** Add up first and last columns for each variable $\theta$ and $\phi$

### Set: $d\theta=1 \quad d\phi=0$

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$

$$+ F_Y \frac{\partial Y}{\partial \theta} \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} \quad + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$

$$+ F_x \frac{\partial x}{\partial \theta} \quad + m\ddot{x} \frac{\partial x}{\partial \theta} \quad + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$

$$+ F_y \frac{\partial y}{\partial \theta} \quad + m\ddot{y} \frac{\partial y}{\partial \theta} \quad + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

### Set: $d\theta=0 \quad d\phi=1$

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt} \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$

$$+ F_Y \frac{\partial Y}{\partial \phi} \quad + M\ddot{Y} \frac{\partial Y}{\partial \phi} \quad + \frac{d}{dt} \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$

$$+ F_x \frac{\partial x}{\partial \phi} \quad + m\ddot{x} \frac{\partial x}{\partial \phi} \quad + \frac{d}{dt} \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \phi}$$

$$+ F_y \frac{\partial y}{\partial \phi} \quad + m\ddot{y} \frac{\partial y}{\partial \phi} \quad + \frac{d}{dt} \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \phi}$$

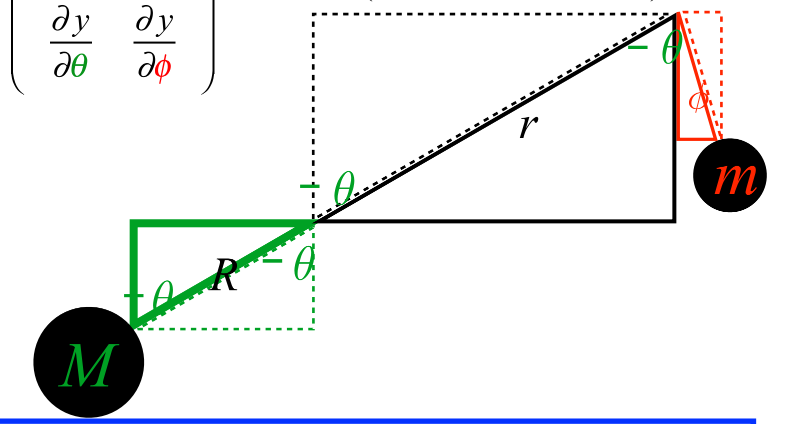
# Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\,dX &= M\ddot{X}\,dX &&= F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &&= M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\
+F_Y\,dY &\quad + M\ddot{Y}\,dY &&\quad + F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi &&\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\
+F_x\,dx &\quad + m\ddot{x}\,dx &&\quad + F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi &&\quad + m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\
+F_y\,dy &\quad + m\ddot{y}\,dy &&\quad + F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi &&\quad + m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi
\end{aligned}$$

STEP **D**

## Add up first and last columns for each variable $\theta$ and $\phi$ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

*Lagrange trickery:*

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta$$

Defines $F_\theta$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

### Set: $d\theta=1$ $d\phi=0$

$$\begin{aligned}
F_X\frac{\partial X}{\partial\theta} &= M\ddot{X}\frac{\partial X}{\partial\theta} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\theta} \\
+F_Y\frac{\partial Y}{\partial\theta} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\theta} \\
+F_x\frac{\partial x}{\partial\theta} &\quad + m\ddot{x}\frac{\partial x}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\theta} \\
+F_y\frac{\partial y}{\partial\theta} &\quad + m\ddot{y}\frac{\partial y}{\partial\theta} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\theta}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\theta}
\end{aligned}$$

### Set: $d\theta=0$ $d\phi=1$

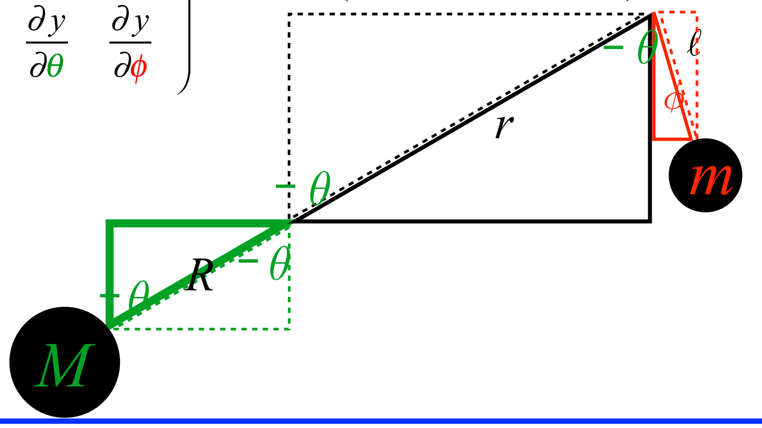
$$\begin{aligned}
F_X\frac{\partial X}{\partial\phi} &= M\ddot{X}\frac{\partial X}{\partial\phi} = \frac{d}{dt}\frac{\partial\frac{M\dot{X}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{X}^2}{2}}{\partial\phi} \\
+F_Y\frac{\partial Y}{\partial\phi} &\quad + M\ddot{Y}\frac{\partial Y}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{Y}^2}{2}}{\partial\phi} \\
+F_x\frac{\partial x}{\partial\phi} &\quad + m\ddot{x}\frac{\partial x}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{x}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{x}^2}{2}}{\partial\phi} \\
+F_y\frac{\partial y}{\partial\phi} &\quad + m\ddot{y}\frac{\partial y}{\partial\phi} \quad + \frac{d}{dt}\frac{\partial\frac{M\dot{y}^2}{2}}{\partial\dot{\phi}} - \frac{\partial\frac{M\dot{y}^2}{2}}{\partial\phi}
\end{aligned}$$

## Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$
$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns: (Using GCC dθ and dϕ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$
$$+ F_Y\, dY \quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$
$$+ F_x\, dx \quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$
$$+ F_y\, dy \quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$

### Lagrange trickery:

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$\text{Let}: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta \quad \text{Defines } F_\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$\text{Let}: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi \quad \text{Defines } F_\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

### Set: dθ=1 dϕ=0

$$F_X \frac{\partial X}{\partial \theta} = M\ddot{X} \frac{\partial X}{\partial \theta} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \theta}$$
$$+ F_Y \frac{\partial Y}{\partial \theta} \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \theta}$$
$$+ F_x \frac{\partial x}{\partial \theta} \quad + m\ddot{x} \frac{\partial x}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{x}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{x}^2}{2}}{\partial \theta}$$
$$+ F_y \frac{\partial y}{\partial \theta} \quad + m\ddot{y} \frac{\partial y}{\partial \theta} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{y}^2}{2}}{\partial \dot{\theta}} - \frac{\partial \frac{M\dot{y}^2}{2}}{\partial \theta}$$

### Set: dθ=0 dϕ=1

$$F_X \frac{\partial X}{\partial \phi} = M\ddot{X} \frac{\partial X}{\partial \phi} = \frac{d}{dt}\frac{\partial \frac{M\dot{X}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{X}^2}{2}}{\partial \phi}$$
$$+ F_Y \frac{\partial Y}{\partial \phi} \quad + M\ddot{Y} \frac{\partial Y}{\partial \phi} \quad + \frac{d}{dt}\frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \dot{\phi}} - \frac{\partial \frac{M\dot{Y}^2}{2}}{\partial \phi}$$
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*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor* $\gamma_{mn}$
- *Structure of dynamic metric tensor* $\gamma_{mn}$
- *Basic force, work, and acceleration*
- → *Lagrangian force equation*
- *Canonical momentum and* $\gamma_{mn}$ *tensor*

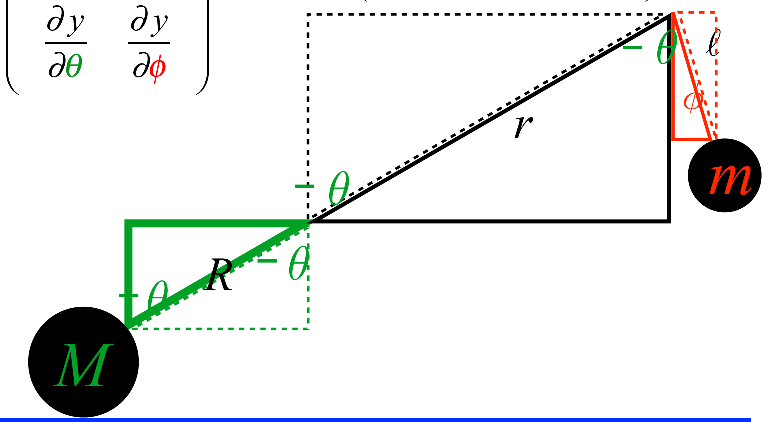
# Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_x\, dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\,dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X} \frac{\partial X}{\partial \theta} d\theta + M\ddot{X} \frac{\partial X}{\partial \phi} d\phi$$

$$+ F_Y\, dY \quad + M\ddot{Y}\,dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y} \frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y} \frac{\partial Y}{\partial \phi} d\phi$$

$$+ F_x\, dx \quad + m\ddot{x}\,dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x} \frac{\partial x}{\partial \theta} d\theta + m\ddot{x} \frac{\partial x}{\partial \phi} d\phi$$

$$+ F_y\, dy \quad + m\ddot{y}\,dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y} \frac{\partial y}{\partial \theta} d\theta + m\ddot{y} \frac{\partial y}{\partial \phi} d\phi$$

## Lagrange trickery:

STEP **D** Add up first and last columns for each variable $\theta$ and $\phi$ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$Let: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable $\theta$ and $\phi$.

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta \equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi \equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

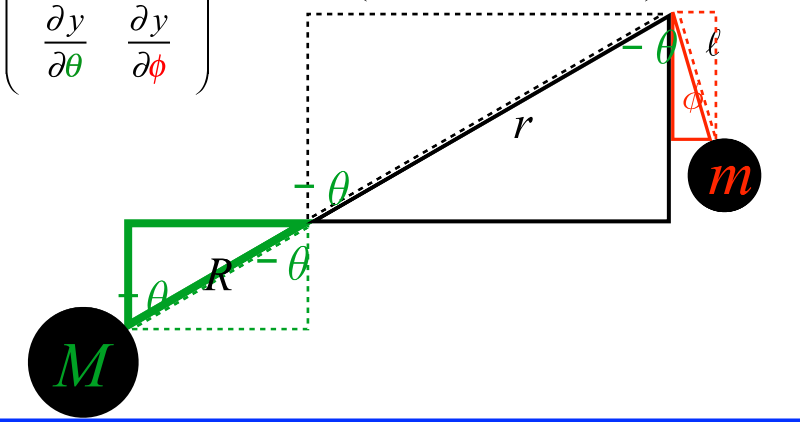
## Force, Work, and Acceleration

$$dW = F_X\, dX + F_Y\, dY + F_x\, dx + F_y\, dy$$

$$= M\ddot{X}\, dX + M\ddot{Y}\, dY + m\ddot{x}\, dx + m\ddot{y}\, dy$$

## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




### Write work-sums in columns:(Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$\begin{aligned}
dW = F_X\, dX &= M\ddot{X}\, dX = F_X \frac{\partial X}{\partial \theta} d\theta + F_X \frac{\partial X}{\partial \phi} d\phi = M\ddot{X}\frac{\partial X}{\partial \theta} d\theta + M\ddot{X}\frac{\partial X}{\partial \phi} d\phi \\
+ F_Y\, dY &\quad + M\ddot{Y}\, dY \quad + F_Y \frac{\partial Y}{\partial \theta} d\theta + F_Y \frac{\partial Y}{\partial \phi} d\phi \quad + M\ddot{Y}\frac{\partial Y}{\partial \theta} d\theta + M\ddot{Y}\frac{\partial Y}{\partial \phi} d\phi \\
+ F_x\, dx &\quad + m\ddot{x}\, dx \quad + F_x \frac{\partial x}{\partial \theta} d\theta + F_x \frac{\partial x}{\partial \phi} d\phi \quad + m\ddot{x}\frac{\partial x}{\partial \theta} d\theta + m\ddot{x}\frac{\partial x}{\partial \phi} d\phi \\
+ F_y\, dy &\quad + m\ddot{y}\, dy \quad + F_y \frac{\partial y}{\partial \theta} d\theta + F_y \frac{\partial y}{\partial \phi} d\phi \quad + m\ddot{y}\frac{\partial y}{\partial \theta} d\theta + m\ddot{y}\frac{\partial y}{\partial \phi} d\phi
\end{aligned}$$

Lagrange trickery:

STEP **D** Add up first and last columns for each variable $\theta$ and $\phi$ for: $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X \frac{\partial X}{\partial \theta} + F_Y \frac{\partial Y}{\partial \theta} + F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \equiv F_\theta$$

Defines $F_\theta$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$Let: F_X \frac{\partial X}{\partial \phi} + F_Y \frac{\partial Y}{\partial \phi} + F_x \frac{\partial x}{\partial \phi} + F_y \frac{\partial y}{\partial \phi} \equiv F_\phi$$

Defines $F_\phi$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

Completes derivation of Lagrange covariant-force equation for each GCC variable $\theta$ and $\phi$ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}$$

$$F_X \cdot 0 + F_Y \cdot 0 + F_x \ell\cos\phi + F_y \ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi}$$

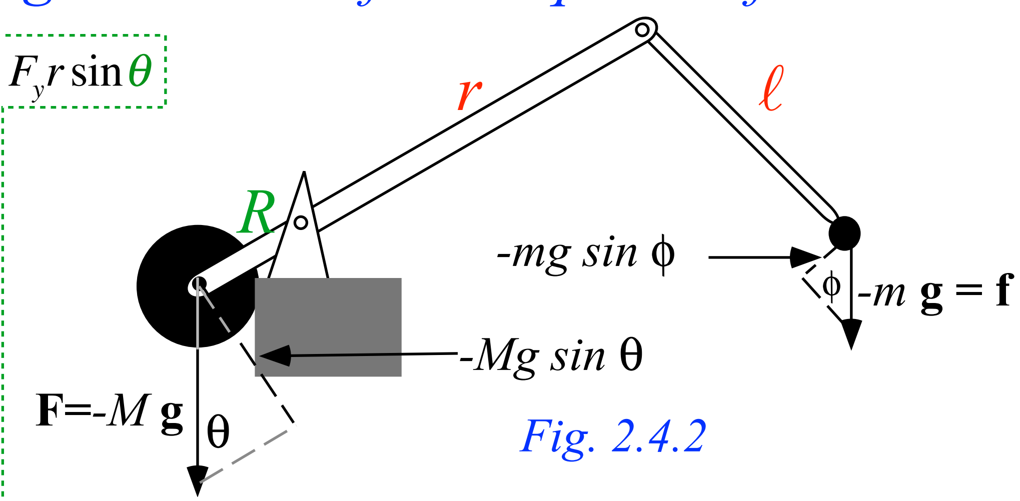



Fig. 2.4.2

Add $F_\theta$ gravity given

($F_X = 0$ , $F_Y = -Mg$)

($F_x = 0$ , $F_y = -mg$)

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = -MgR\sin\theta + mgr\sin\theta$$

These are competing torques on main beam R

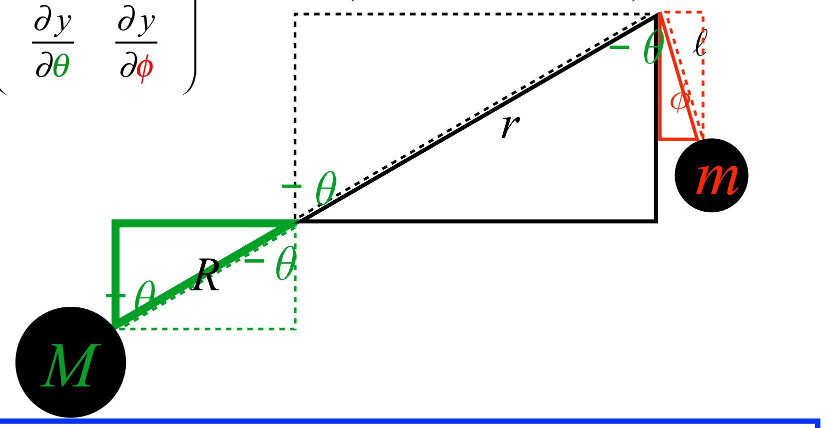
## *Force, Work, and Acceleration*

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_y\,dy$$

$$= M\ddot{X}\,dX + M\ddot{Y}\,dY + m\ddot{x}\,dx + m\ddot{y}\,dy$$

### *Raw Jacobian*

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




### *Write work-sums in columns:(Using GCC dθ and dϕ in Jacobian)*

$$\begin{aligned} dW = F_X\,dX &= M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\ +F_Y\,dY &\quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\ +F_x\,dx &\quad +m\ddot{x}\,dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\ +F_y\,dy &\quad +m\ddot{y}\,dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi \end{aligned}$$

*Lagrange trickery:*

STEP **D** *Add up first and last columns for each variable θ and ϕ for:*

$$T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$$

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Defines $F_\theta$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

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Defines $F_\phi$

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*Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .*

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

*Add $F_\theta$ gravity given*

*($F_X$ =0 , $F_Y$ =-Mg)*

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$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

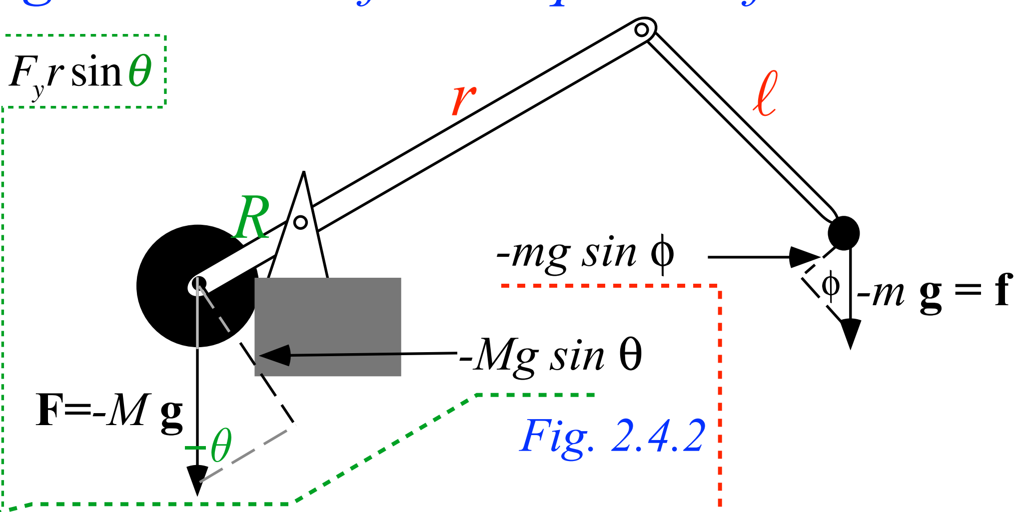


*Fig. 2.4.2*

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

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*These are competing torques on main beam R...*

*... and a torque on throwing lever ℓ*

# Force, Work, and Acceleration

$$dW = F_X\,dX + F_Y\,dY + F_x\,dx + F_x\,dy$$

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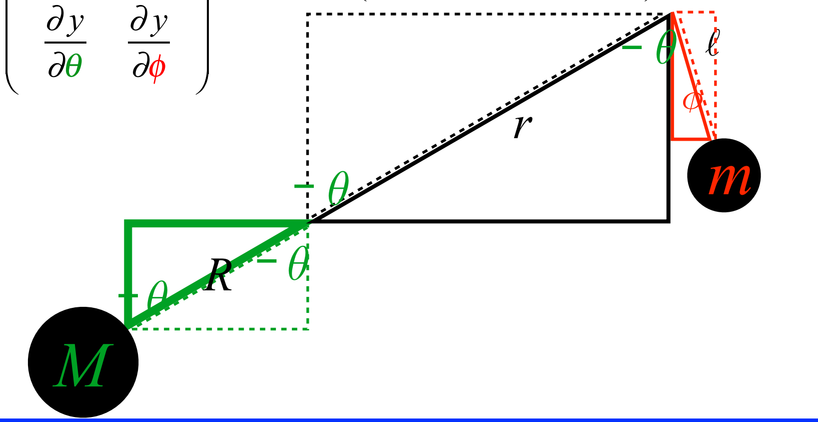
## Raw Jacobian

$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




## Write work-sums in columns: (Using GCC dθ and dϕ in Jacobian)

$$\begin{aligned} dW = \; & F_X\,dX &=\; & M\ddot{X}\,dX &=\; & F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi &=\; & M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi \\ & +F_Y\,dY && +M\ddot{Y}\,dY && +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi && +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi \\ & +F_x\,dx && +m\ddot{x}\,dx && +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi && +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi \\ & +F_y\,dy && +m\ddot{y}\,dy && +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi && +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi \end{aligned}$$

## Lagrange trickery:

STEP **D**

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## Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$

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*Add $F_\theta$ gravity given*

$(F_X = 0\,,\ F_Y = -Mg)$

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$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

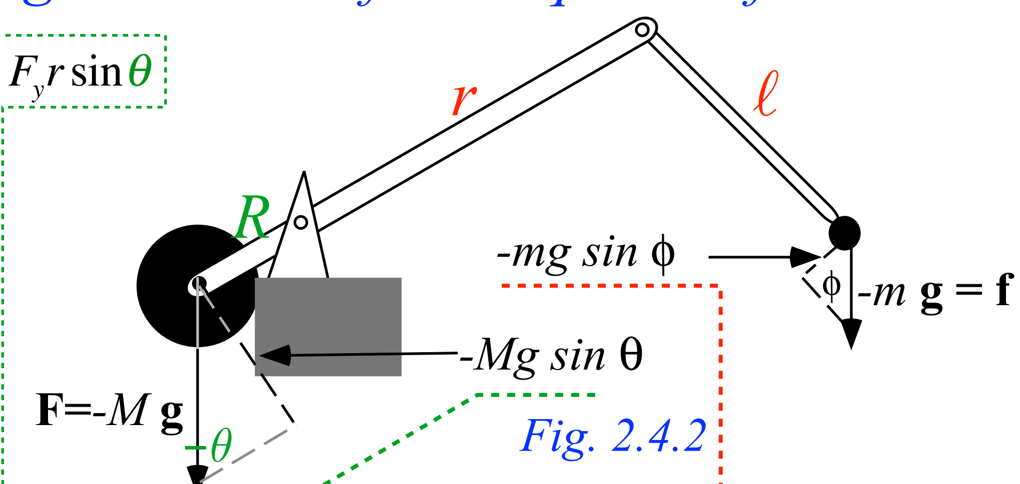


Fig. 2.4.2

Q: Are there ± sign errors here?

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$

$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

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*These are competing torques on main beam R...* *... and a torque on throwing lever ℓ*

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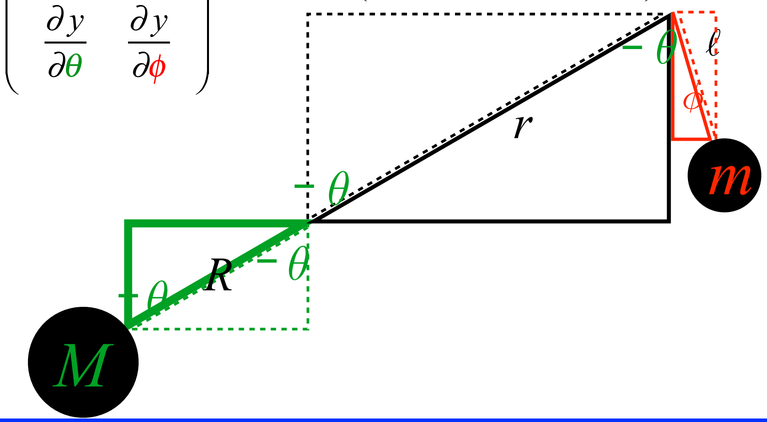
$$\begin{pmatrix} dX \\ dY \\ dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial X}{\partial\theta} & \frac{\partial X}{\partial\phi} \\ \frac{\partial Y}{\partial\theta} & \frac{\partial Y}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} R\cos\theta & 0 \\ R\sin\theta & 0 \\ -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{pmatrix} \begin{pmatrix} d\theta \\ d\phi \end{pmatrix}$$




### Write work-sums in columns: (Using GCC $d\theta$ and $d\phi$ in Jacobian)

$$dW = F_X\, dX = M\ddot{X}\,dX = F_X\frac{\partial X}{\partial\theta}d\theta + F_X\frac{\partial X}{\partial\phi}d\phi = M\ddot{X}\frac{\partial X}{\partial\theta}d\theta + M\ddot{X}\frac{\partial X}{\partial\phi}d\phi$$
$$+F_Y\, dY \quad +M\ddot{Y}\,dY \quad +F_Y\frac{\partial Y}{\partial\theta}d\theta + F_Y\frac{\partial Y}{\partial\phi}d\phi \quad +M\ddot{Y}\frac{\partial Y}{\partial\theta}d\theta + M\ddot{Y}\frac{\partial Y}{\partial\phi}d\phi$$
$$+F_x\, dx \quad +m\ddot{x}\,dx \quad +F_x\frac{\partial x}{\partial\theta}d\theta + F_x\frac{\partial x}{\partial\phi}d\phi \quad +m\ddot{x}\frac{\partial x}{\partial\theta}d\theta + m\ddot{x}\frac{\partial x}{\partial\phi}d\phi$$
$$+F_y\, dy \quad +m\ddot{y}\,dy \quad +F_y\frac{\partial y}{\partial\theta}d\theta + F_y\frac{\partial y}{\partial\phi}d\phi \quad +m\ddot{y}\frac{\partial y}{\partial\theta}d\theta + m\ddot{y}\frac{\partial y}{\partial\phi}d\phi$$

*Lagrange trickery:*

STEP **D** *Add up first and last columns for each variable θ and ϕ for:* $T = \frac{M\dot{X}^2}{2} + \frac{M\dot{Y}^2}{2} + \frac{M\dot{x}^2}{2} + \frac{M\dot{y}^2}{2}$

$$Let: F_X\frac{\partial X}{\partial\theta} + F_Y\frac{\partial Y}{\partial\theta} + F_x\frac{\partial x}{\partial\theta} + F_y\frac{\partial y}{\partial\theta} \equiv F_\theta \quad \text{Defines } F_\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

$$Let: F_X\frac{\partial X}{\partial\phi} + F_Y\frac{\partial Y}{\partial\phi} + F_x\frac{\partial x}{\partial\phi} + F_y\frac{\partial y}{\partial\phi} \equiv F_\phi \quad \text{Defines } F_\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

### Completes derivation of Lagrange covariant-force equation for each GCC variable θ and ϕ .

$$F_X R\cos\theta + F_Y R\sin\theta - F_x r\cos\theta - F_y r\sin\theta$$
$$\equiv F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta}$$

*Add $F_\theta$ gravity given*
*($F_X = 0$ , $F_Y = -Mg$)*
*($F_x = 0$ , $F_y = -mg$)*

$$F_\theta = \frac{d}{dt}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = -MgR\sin\theta + mgr\sin\theta$$

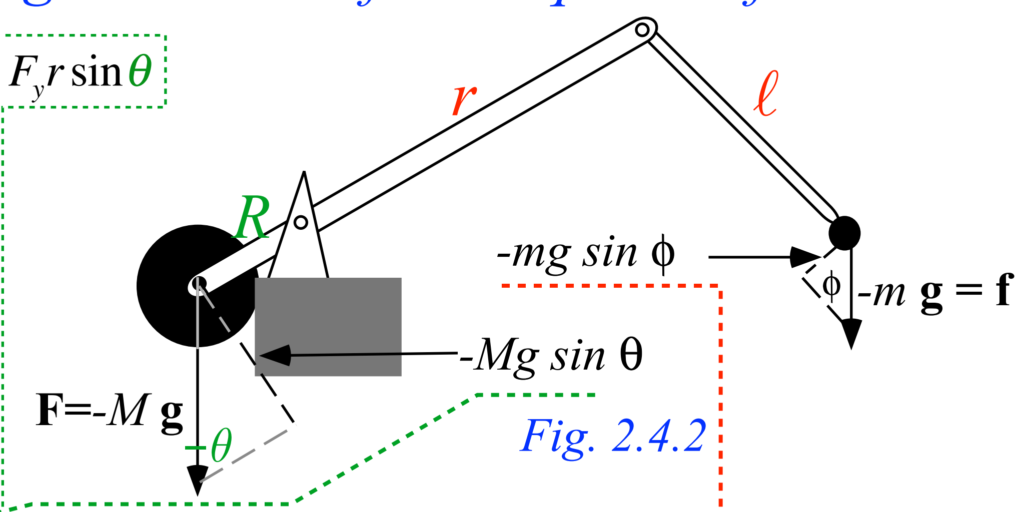



*Fig. 2.4.2*

Q: Are there ± sign errors here?
A: No. Beam in $-\theta$ position.

$$F_X\cdot 0 + F_Y\cdot 0 + F_x\ell\cos\phi + F_y\ell\sin\phi$$
$$\equiv F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi}$$

*Add $F_\phi$ gravity given*
*($F_X = 0$ , $F_Y = -Mg$)*
*($F_x = 0$ , $F_y = -mg$)*

$$F_\phi = \frac{d}{dt}\frac{\partial T}{\partial\dot{\phi}} - \frac{\partial T}{\partial\phi} = -mg\ell\sin\phi$$

*These are competing torques on main beam R...* *... and a torque on throwing lever ℓ*

*Review of Lagrangian equation derivation from Lecture 10 (Now with trebuchet model)*

- *Coordinate geometry, Jacobian, velocity, kinetic energy, and dynamic metric tensor $\gamma_{mn}$*
- *Structure of dynamic metric tensor $\gamma_{mn}$*
- *Basic force, work, and acceleration*
- *Lagrangian force equation*
- → *Canonical momentum and $\gamma_{mn}$ tensor*

# *Canonical momentum and $\gamma_{mn}$ tensor*

## *Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$*

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[ (MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

## *The $\gamma_{mn}$ tensor/matrix formulation*

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

*where:*
*$\gamma_{mn}$ tensor is*

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

# Canonical momentum and $\gamma_{mn}$ tensor

## Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[ (MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

## The $\gamma_{mn}$ tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: $\gamma_{mn}$ tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

# Canonical momentum and $\gamma_{mn}$ tensor

## Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$Total\ KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[ (MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

## The $\gamma_{mn}$ tensor/matrix formulation

$$Total\ KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: $\gamma_{mn}$ tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

### Momentum $\gamma_{mn}$-matrix theorem: *(matrix-proof on page 78)*

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

# Canonical momentum and $\gamma_{mn}$ tensor

## Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[ (MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

## The $\gamma_{mn}$ tensor/matrix formulation

$$\text{Total } KE = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: $\gamma_{mn}$ tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

### Momentum $\gamma_{mn}$-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

### Momentum $\gamma_{mn}$-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

*proof:*

$$\text{Given: } p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where: } T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then: } p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

# Canonical momentum and $\gamma_{mn}$ tensor

## Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[(MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2\right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left(\frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2\right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

## The $\gamma_{mn}$ tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: $\gamma_{mn}$ tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

### Momentum $\gamma_{mn}$-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

### Momentum $\gamma_{mn}$-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

$$\text{Given}: p_m = \frac{\partial T}{\partial \dot{q}^m} \quad \text{where}: \quad T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$$

$$\text{Then}: \quad p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m$$

# Canonical momentum and $\gamma_{mn}$ tensor

## Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

Total KE $= T = T(M) + T(m)$

$$= \frac{1}{2}\left[ (MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

## The $\gamma_{mn}$ tensor/matrix formulation

Total KE $= T = T(M) + T(m)$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: $\gamma_{mn}$ tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

### Momentum $\gamma_{mn}$-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

### Momentum $\gamma_{mn}$-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof: Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$

Then: $$p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

# Canonical momentum and $\gamma_{mn}$ tensor

## Standard formulation of $p_m = \frac{\partial T}{\partial \dot{q}^m}$

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\left[ (MR^2 + mr^2)\dot{\theta}^2 - 2mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + m\ell^2\dot{\phi}^2 \right]$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= (MR^2 + mr^2)\dot{\theta} - mr\ell\dot{\phi}\cos(\theta-\phi)$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}}\left( \frac{1}{2}(MR^2 + mr^2)\dot{\theta}^2 - mr\ell\cos(\theta-\phi)\,\dot{\theta}\dot{\phi} + \frac{1}{2}m\ell^2\dot{\phi}^2 \right)$$

$$= m\ell^2\dot{\phi} - mr\ell\dot{\theta}\cos(\theta-\phi)$$

## The $\gamma_{mn}$ tensor/matrix formulation

$$\text{Total KE} = T = T(M) + T(m)$$

$$= \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{\phi} \end{pmatrix} \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \frac{1}{2}\gamma_{mn}\dot{q}^m\dot{q}^n$$

where: $\gamma_{mn}$ tensor is

$$\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} = \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}$$

### Momentum $\gamma_{mn}$-matrix theorem: (matrix-proof on page 43)

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot{\theta}} \\ \frac{\partial T}{\partial \dot{\phi}} \end{pmatrix} = \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix}$$

### Momentum $\gamma_{mn}$-tensor theorem: (proof here)

$$p_m = \gamma_{mn}\,\dot{q}^n$$

proof:

Given: $p_m = \frac{\partial T}{\partial \dot{q}^m}$ where: $T = \frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k$

Then: $$p_m = \frac{\partial}{\partial \dot{q}^m}\frac{1}{2}\gamma_{jk}\dot{q}^j\dot{q}^k = \frac{1}{2}\gamma_{jk}\frac{\partial \dot{q}^j}{\partial \dot{q}^m}\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\frac{\partial \dot{q}^k}{\partial \dot{q}^m}$$

$$= \frac{1}{2}\gamma_{jk}\delta^j_m\dot{q}^k + \frac{1}{2}\gamma_{jk}\dot{q}^j\delta^k_m = \frac{1}{2}\gamma_{mk}\dot{q}^k + \frac{1}{2}\gamma_{jm}\dot{q}^j$$

$$= \gamma_{mn}\dot{q}^n \quad \text{if: } \gamma_{mn} = \gamma_{nm}$$

QED

## *Momentum $\gamma_{mn}$-matrix theorem:* *(matrix-proof here on page 43)*

$$\begin{pmatrix} p_\theta \\ p_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial \dot\theta} \\ \frac{\partial T}{\partial \dot\phi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial \dot\theta}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\frac{\partial}{\partial \dot\theta}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \\ \frac{\partial}{\partial \dot\phi}\begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\frac{\partial}{\partial \dot\phi}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \begin{pmatrix} \dot\theta & \dot\phi \end{pmatrix}\bullet\gamma\bullet\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\phi,\theta} \\ \gamma_{\theta,\phi} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix}$$

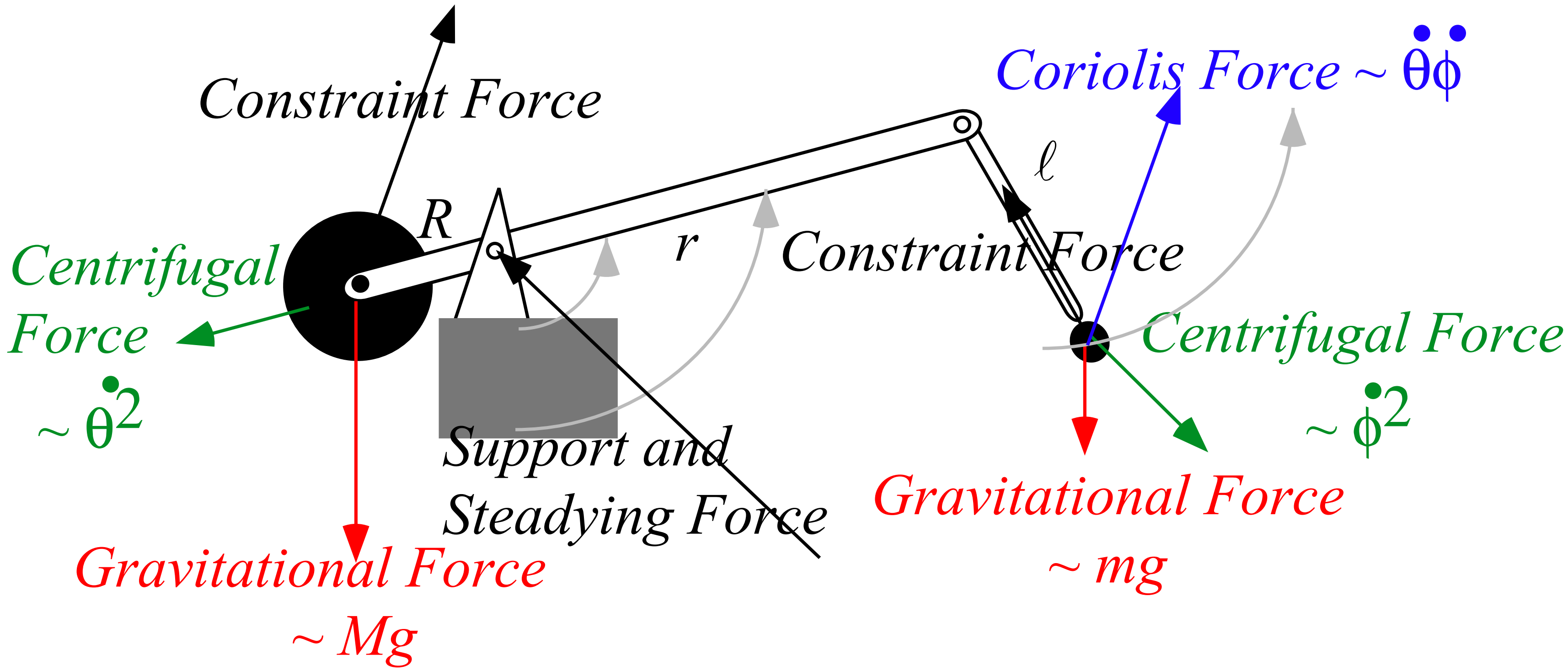
$$= \begin{pmatrix} \gamma_{\theta,\theta} & \gamma_{\theta,\phi} \\ \gamma_{\phi,\theta} & \gamma_{\phi,\phi} \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \quad \text{if: } \gamma_{\phi,\theta} = \gamma_{\theta,\phi} \text{ (symmetry)}$$

$$= \begin{pmatrix} MR^2 + mr^2 & -mr\ell\cos(\theta-\phi) \\ -mr\ell\cos(\theta-\phi) & m\ell^2 \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\phi \end{pmatrix} \qquad QED$$

*Summary of Lagrange equations and force analysis (Mostly Unit 2.)*

→ *Forces: total, genuine, potential, and/or fictitious*

## *Forces: total, genuine, potential, and/or fictitious*




*Acceleration and 'Fictitious' Forces:*
*Coriolis*
*Centrifugal*

*Applied 'Real' Forces:*
*Gravity*
*Stimuli*
*Friction...*

*Constraint 'Internal' Forces:*
*Stresses*
*Support...*
*(Do not contribute. Do no work.)*

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

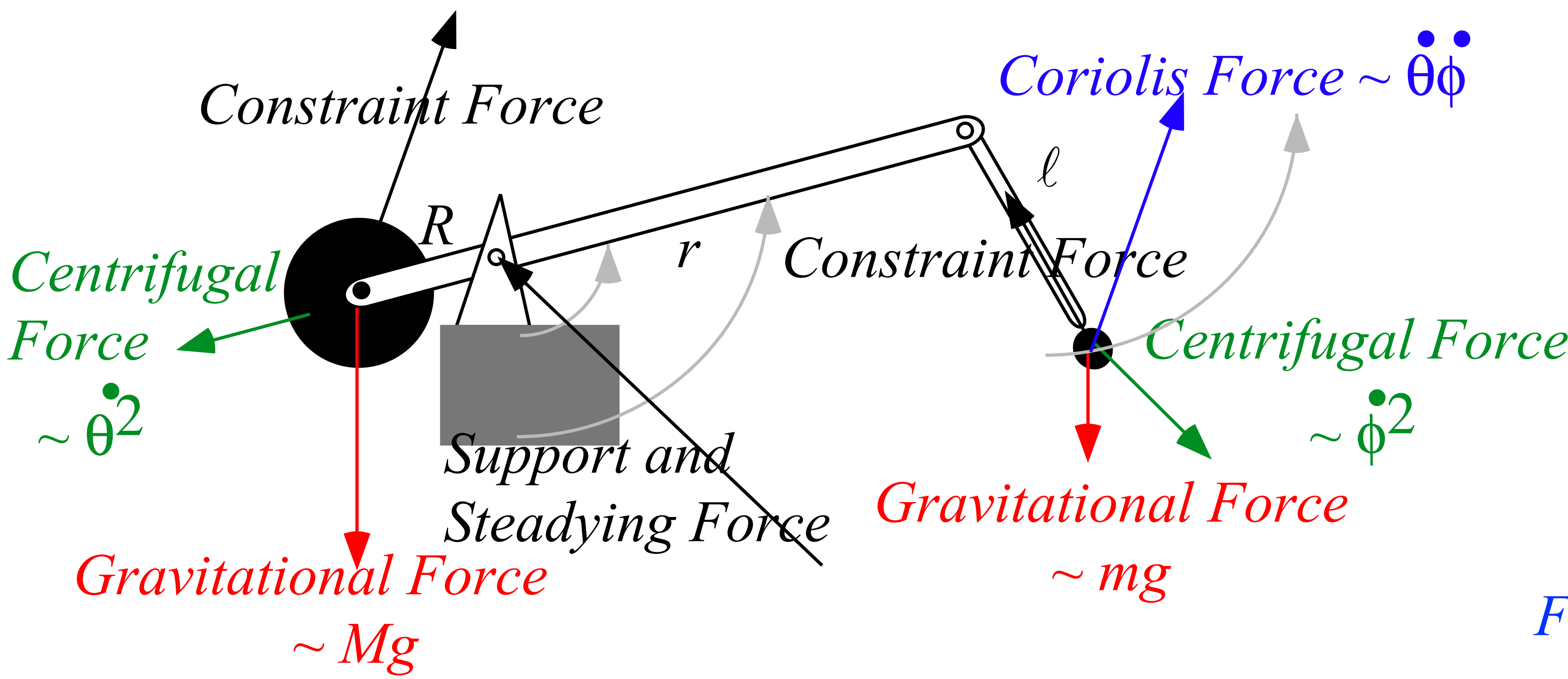
*Lagrange Force equations*

*(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)*

*Fig. 2.5.2 (modified)*

*Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.*

# Forces: total, genuine, potential, and/or fictitious




*For conservative forces*

*where:* $F_\theta = -\dfrac{\partial V}{\partial \theta}$ and: $\dfrac{\partial V}{\partial \dot{\theta}} = 0$

$F_\phi = -\dfrac{\partial V}{\partial \phi}$ and: $\dfrac{\partial V}{\partial \dot{\phi}} = 0$

**Acceleration and 'Fictitious' Forces:** Coriolis, Centrifugal

**Applied 'Real' Forces:** Gravity, Stimuli, Friction...

**Constraint 'Internal' Forces:** Stresses, Support... (Do not contribute. Do no work.)

$$\dot{p}_\theta = \frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} + F_\theta + 0$$

$$\dot{p}_\phi = \frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial \phi} + F_\phi + 0$$

*Lagrange Force equations*

*(See also derivation Eq. (2.4.7) on p. 23 , Unit 2)*

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_\theta = \frac{\partial L}{\partial \theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \qquad \dot{p}_\phi = \frac{\partial L}{\partial \phi}$$

*Lagrange Potential equations*

$L = T - V$

*Fig. 2.5.2 (modified)*

*Compare to derivation Eq (12.25a) in Ch. 12 of Unit 1 and Eq. (3.5.10) in Unit 3.*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)*

→ *Multivalued functionality and connections*

*Covariant and contravariant relations*

*Tangent space vs. Normal space*

*Metric $g_{mn}$ tensor geometric relations to length, area, and volume*

## *Trebuchet Cartesian projectile coordinates are double-valued*

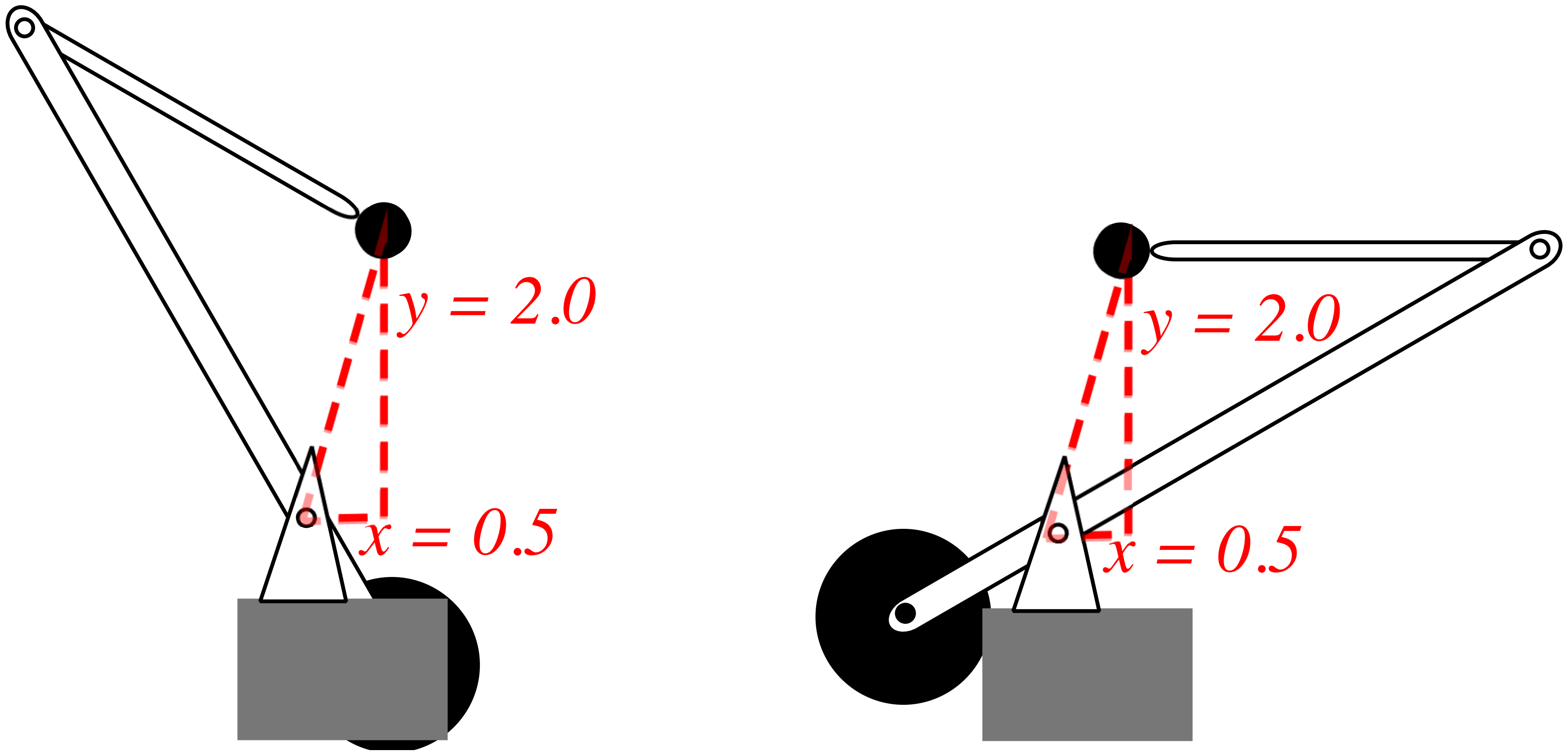



*Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y* of projectile *m*.

*Trebuchet Cartesian projectile coordinates are double-valued…(Belong to 2 distinct manifolds)*

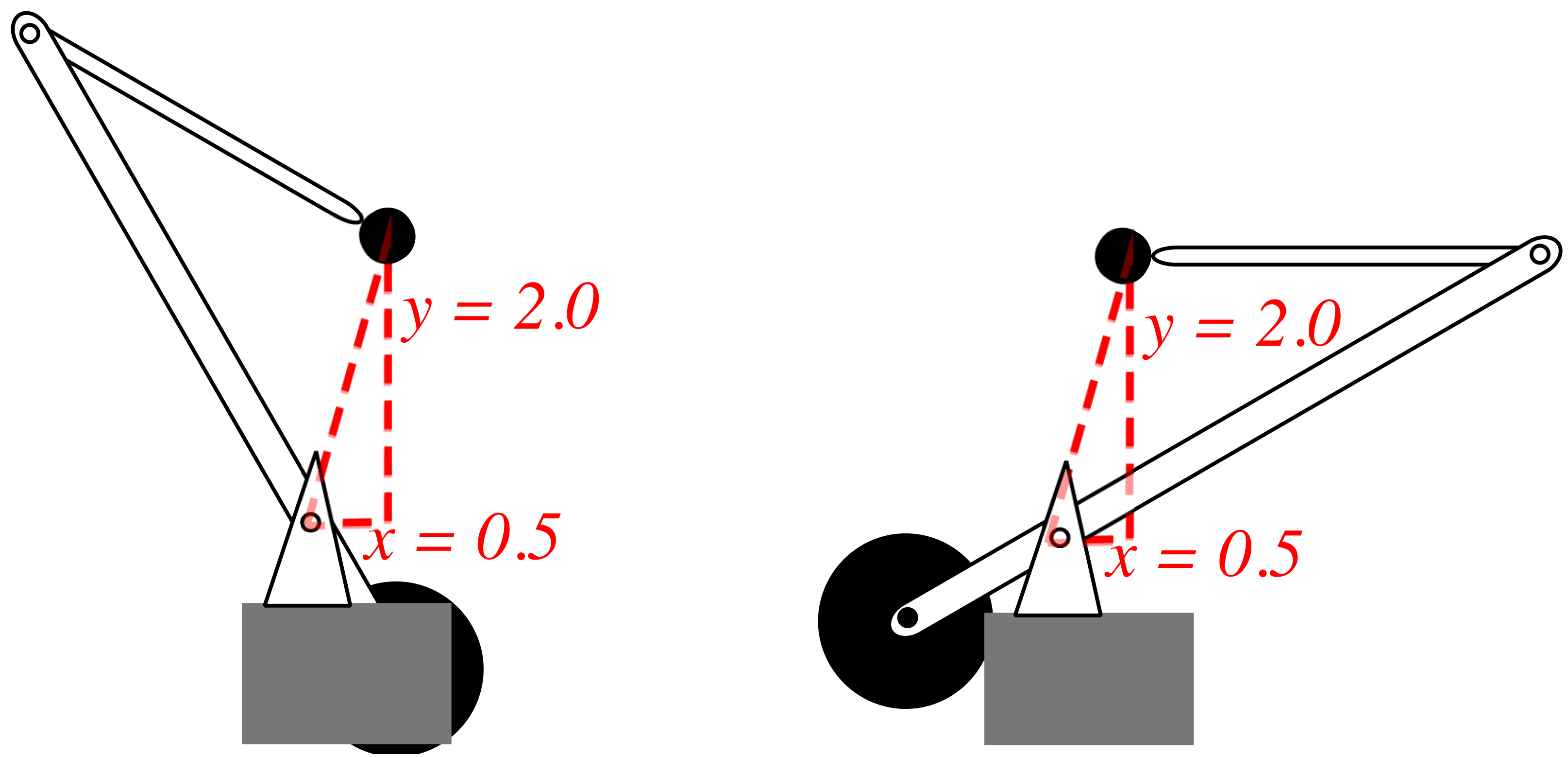



*Fig. 2.2.3 Trebuchet configurations with the same coordinates x and y* of projectile *m*.

*So, for example, are polar coordinates … (for each angle there are two r-values)*

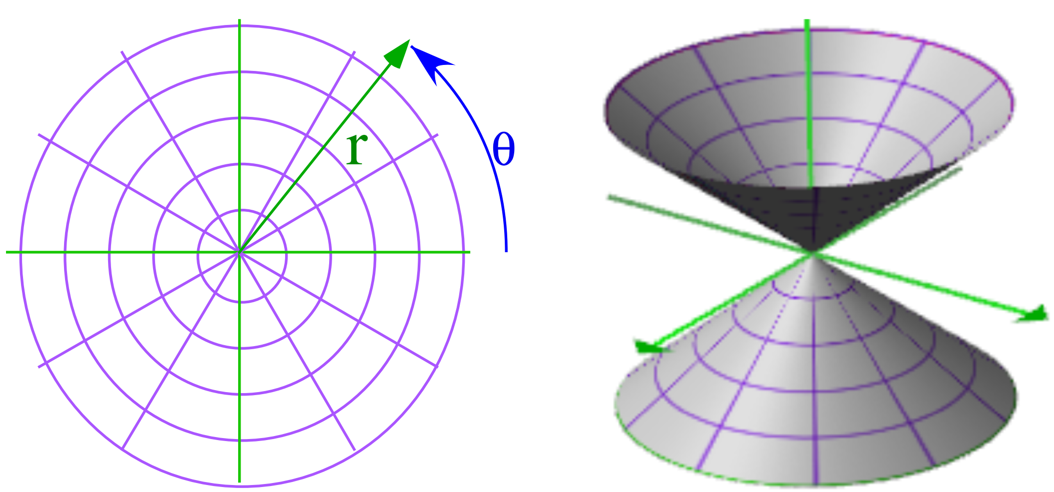



*Fig. 3.1.4 Polar coordinates and possible embedding space on conical surface.*

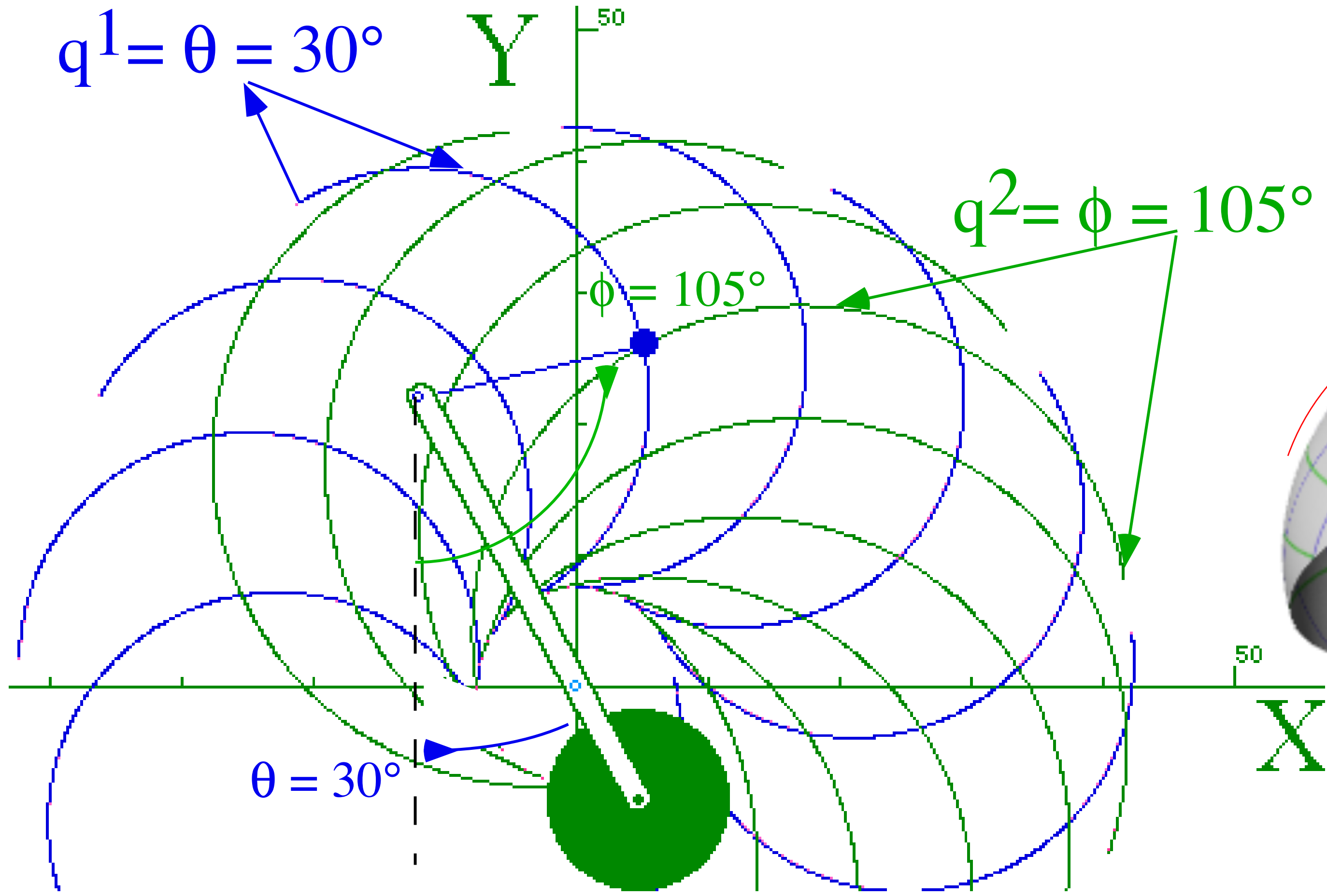



*Fig. 3.1.1a ($q^1$=θ, $q^2$=φ )Coordinate manifold for trebuchet (Left handed sheet.)*

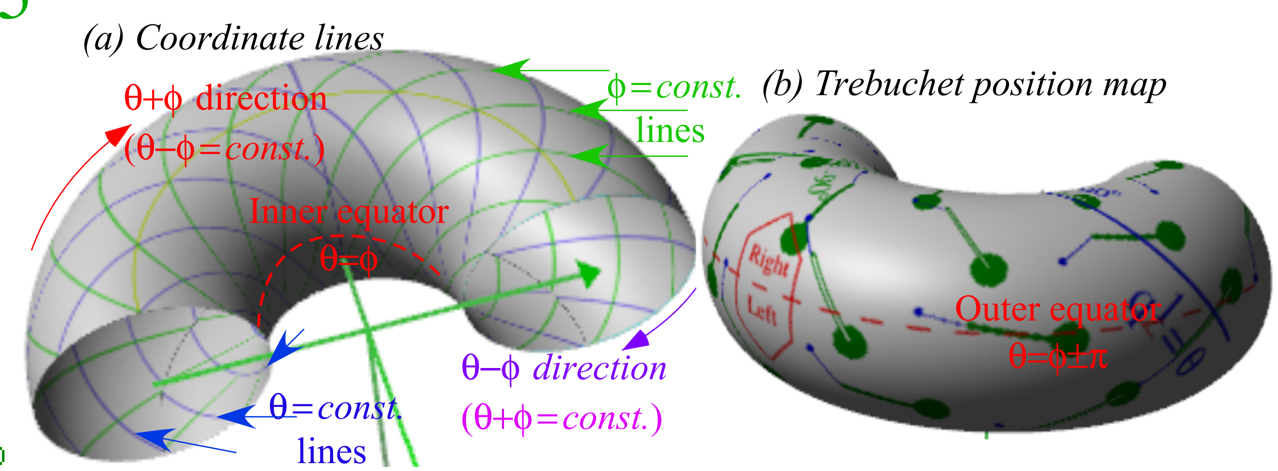



*Fig. 3.1.2 Trebuchet torus.*
*(a) ($q^1$=θ, $q^2$=φ ) coordinate lines. (b)Trebuchet position map and equators.*

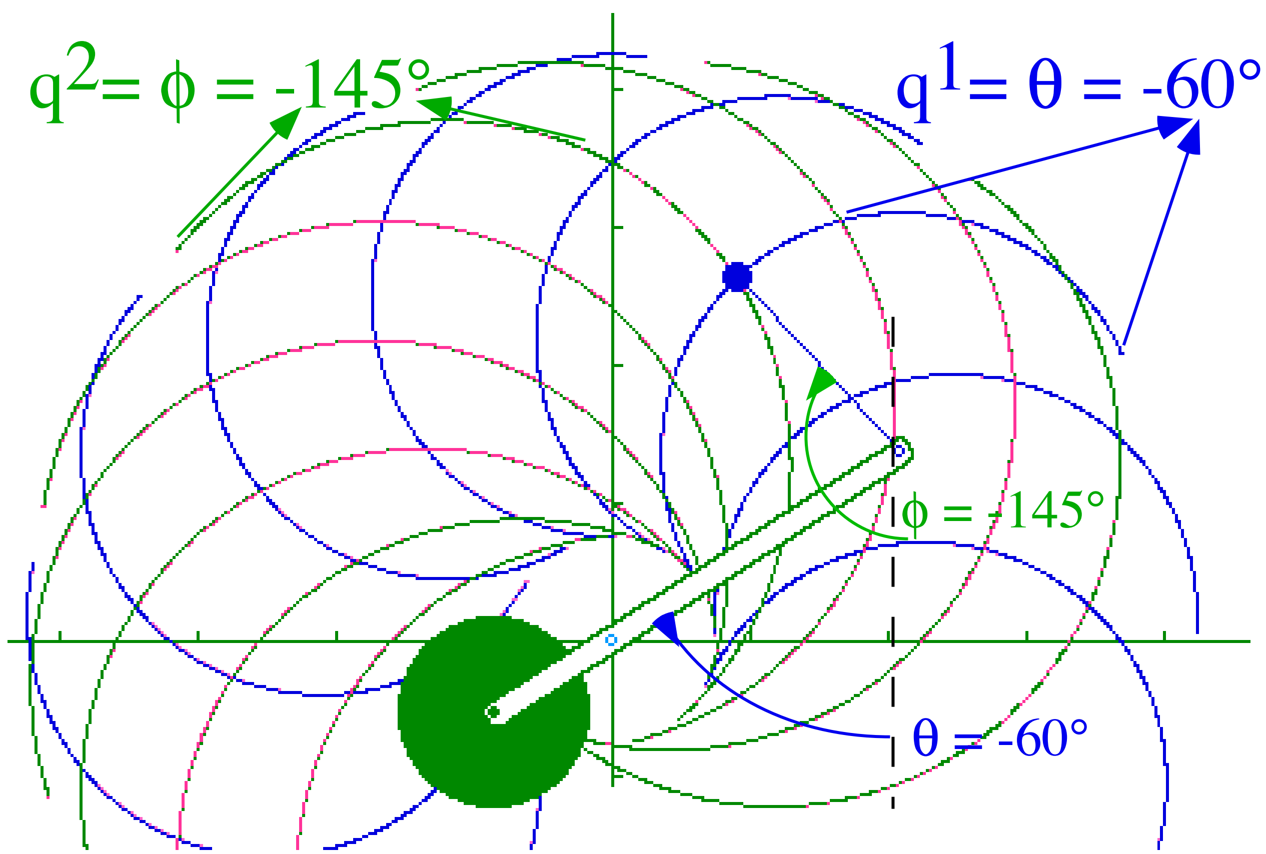



*Fig. 3.1.1b ($q^1$=θ, $q^2$=φ )Coordinate manifold for trebuchet (Right handed sheet.)*

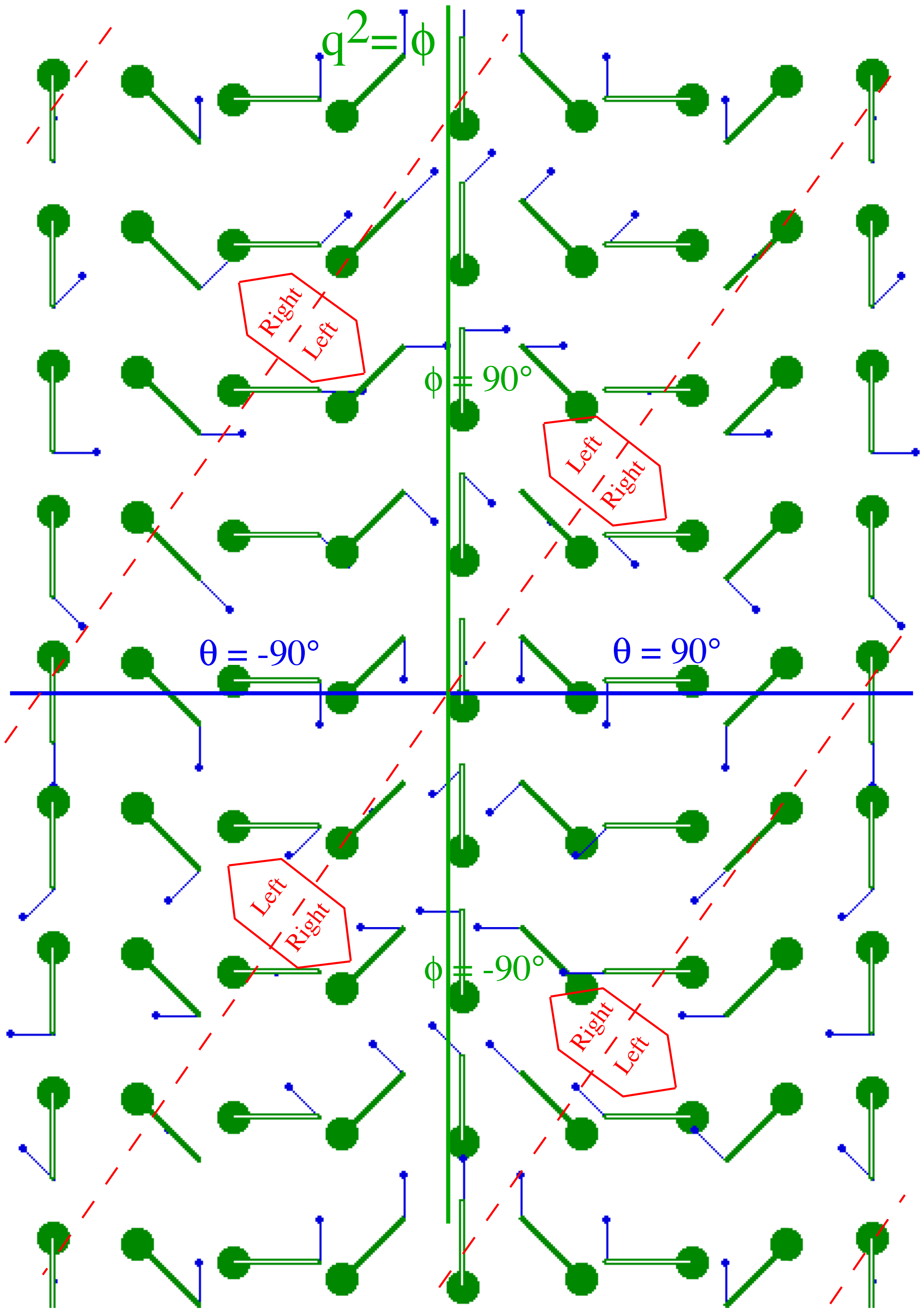



*Fig. 3.1.3 "Flattened" ($q^1=\theta$, $q^2=\phi$ ) coordinate manifold for trebuchet*

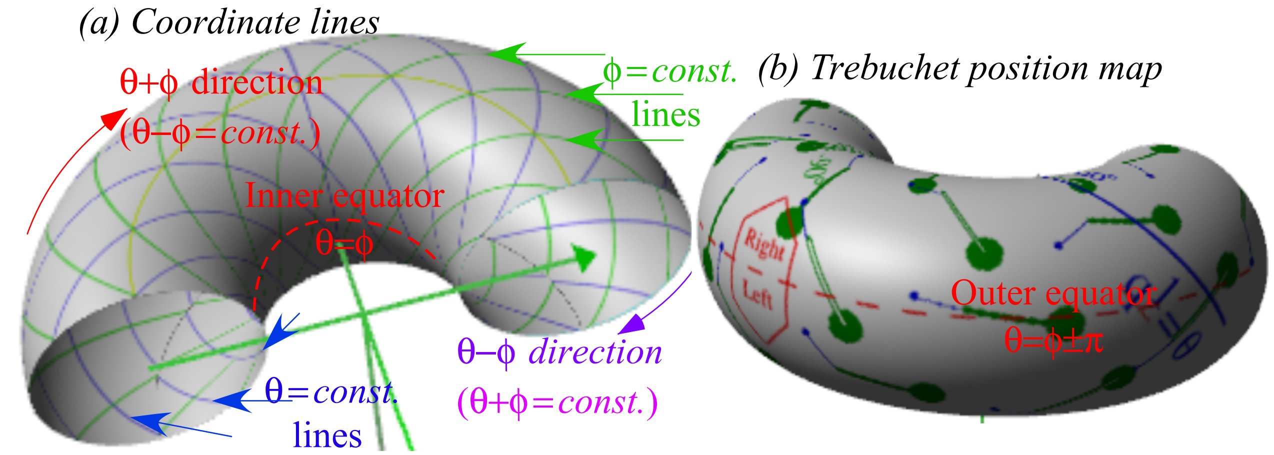



*Fig. 3.1.2 Trebuchet torus.*
*(a) ($q^1=\theta$, $q^2=\phi$ ) coordinate lines.(b)Trebuchet position map and equators.*

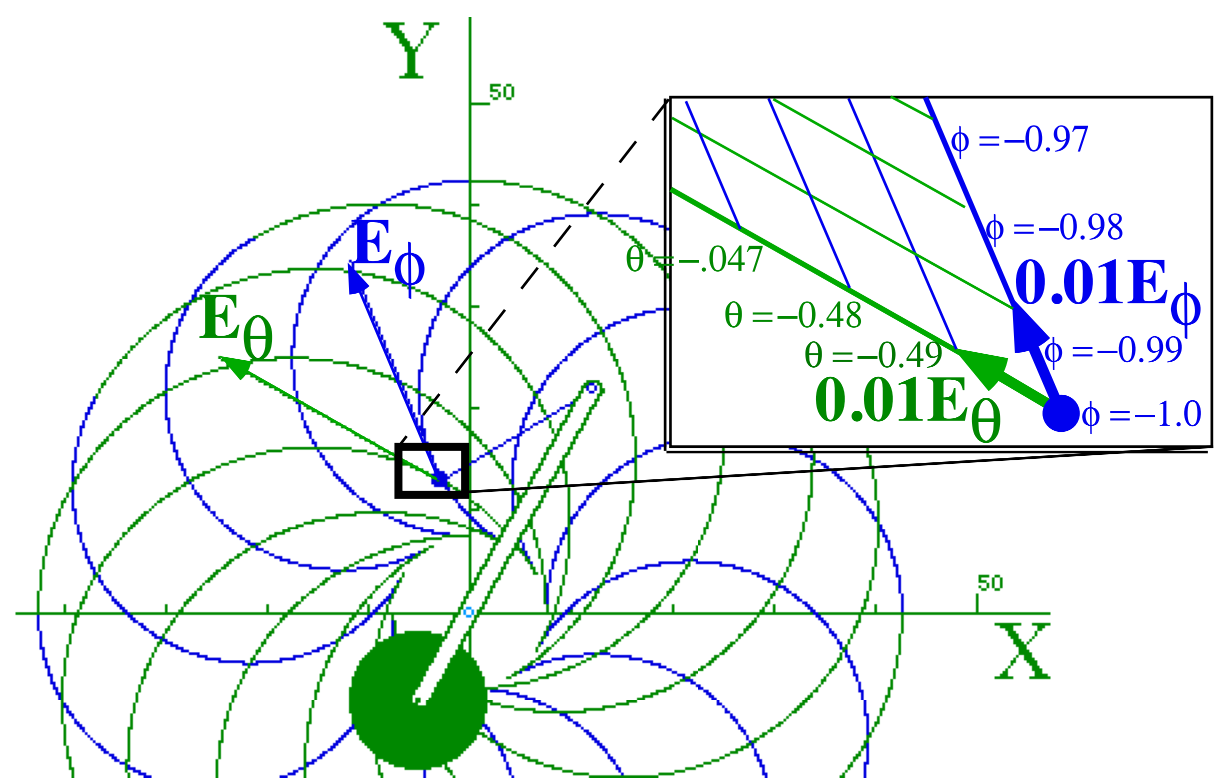



*Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)*

*Multivalued functionality and connections*

→ *Covariant and contravariant relations*

*Tangent space vs. Normal space*

*Metric $g_{mn}$ tensor geometric relations to length, area, and volume*

*Kajobian transformation matrix* *versus* *Jacobian transformation matrix*

$$\left\langle \frac{\partial q^m}{\partial x^j} \right\rangle = \begin{array}{|ccc|c} \frac{\partial q^1}{\partial x^1} & \frac{\partial q^1}{\partial x^2} & \cdots & \mathbf{E}^1 \\ \frac{\partial q^2}{\partial x^1} & \frac{\partial q^2}{\partial x^2} & \cdots & \mathbf{E}^2 \\ \vdots & \vdots & \ddots & \vdots \end{array} = \begin{pmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{pmatrix} = \frac{\begin{array}{|cc|c} \ell \sin\phi & -\ell\cos\phi & \mathbf{E}^\theta \\ r\sin\theta & -r\cos\theta & \mathbf{E}^\phi \end{array}}{r\ell\sin(\theta-\phi)}$$

$$\left\langle \frac{\partial x^j}{\partial q^m} \right\rangle = \begin{array}{|ccc|} \mathbf{E}_1 & \mathbf{E}_2 & \cdots \\ \hline \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \cdots \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \cdots \\ \vdots & \vdots & \ddots \end{array} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \begin{array}{|cc|} \mathbf{E}_\theta & \mathbf{E}_\phi \\ \hline -r\cos\theta & \ell\cos\phi \\ -r\sin\theta & \ell\sin\phi \end{array}$$

*Contravariant vectors* $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

$$\mathbf{E}^\theta = \begin{pmatrix} \ell\sin\phi & -\ell\cos\phi \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}^\phi = \begin{pmatrix} r\sin\theta & -r\cos\theta \end{pmatrix} / r\ell\sin(\theta-\phi)$$

$$\mathbf{E}_\theta = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}, \quad \mathbf{E}_\phi = \begin{pmatrix} \ell\cos\phi \\ \ell\sin\phi \end{pmatrix}$$

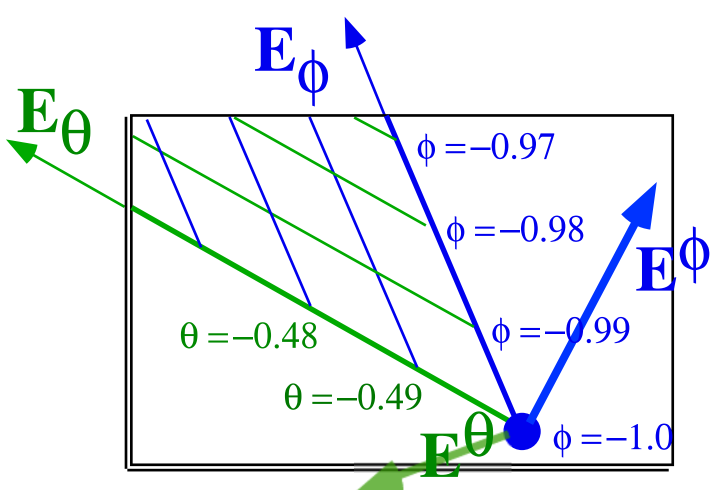



*Fig. 3.2.3 Example of contravariant unitary vectors and their normal space.*

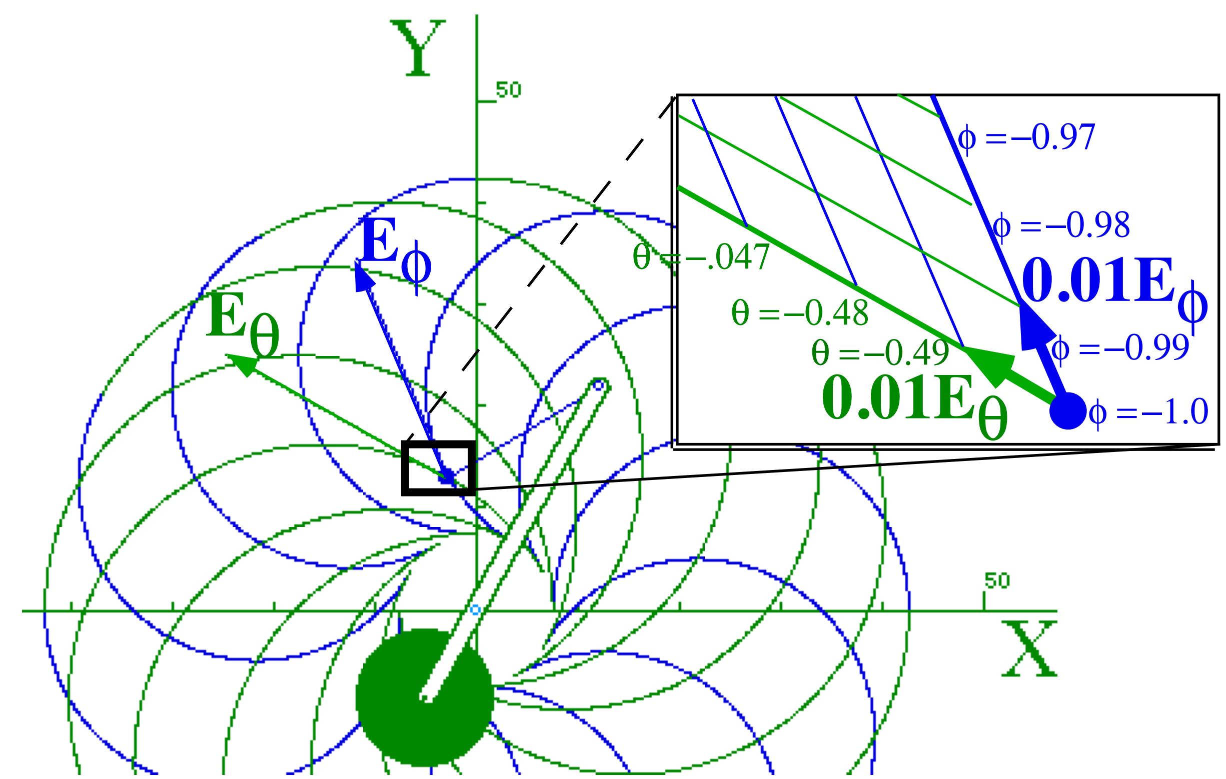



*Fig. 3.2.2 Example of covariant unitary vectors and their tangent space.*

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)*

*Multivalued functionality and connections*

*Covariant and contravariant relations*

→ *Tangent space vs. Normal space*

*Metric $g_{mn}$ tensor geometric relations to length, area, and volume*

## *Contravariant vectors* $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

*Any* vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *$U^m$, $V^m$,...are contravariant components* and the *$U_n$ , $V_n$ ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, , $\quad U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

*Normal space (Contravariant)*

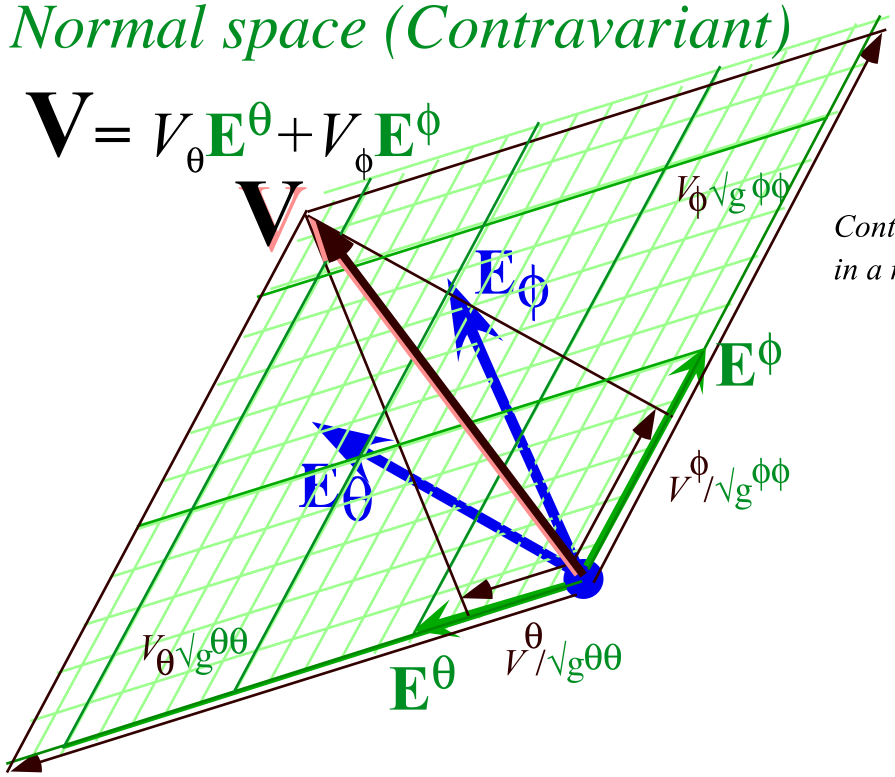



*Fig. 3.3.2*
*Contravariant vector geometry in a normal space* $(\mathbf{E}^\theta,\mathbf{E}^\phi)$.

*Tangent space (Covariant)*

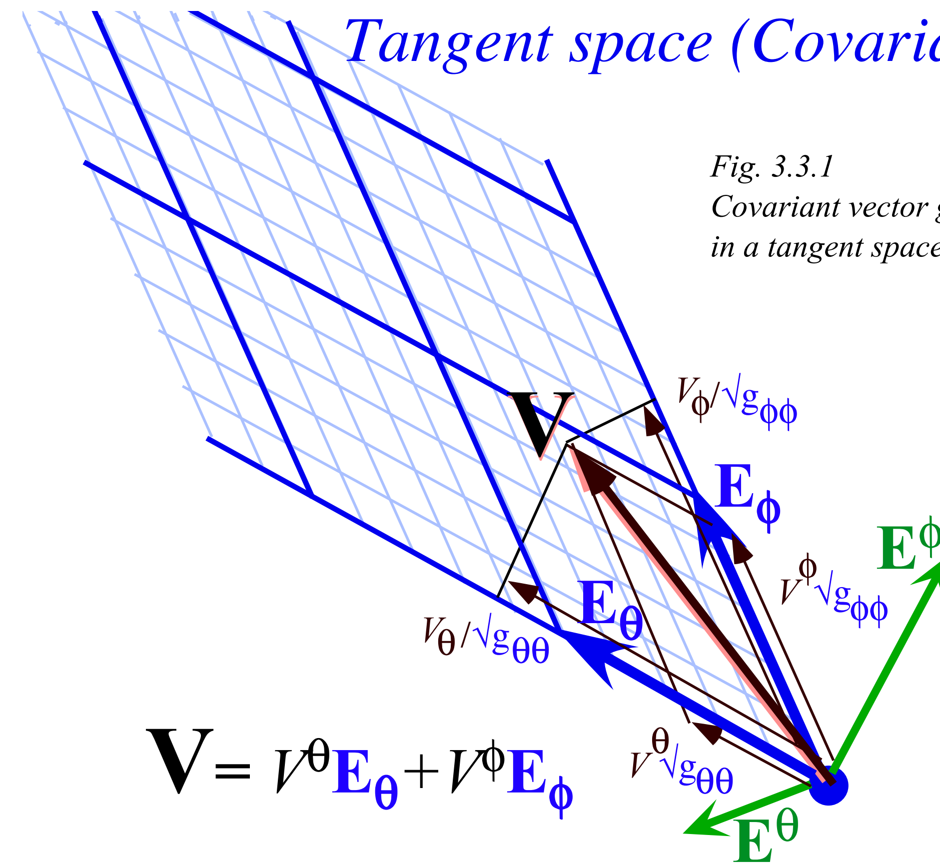



*Fig. 3.3.1*
*Covariant vector geometry in a tangent space* $(\mathbf{E}_\theta,\mathbf{E}_\phi)$.

## *Contravariant vectors* $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

*Any* vector **U**,**V**,... is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *$U^m$, $V^m$,...are contravariant components* and the *$U_n$ , $V_n$ ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\,\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

*Normal space (Contravariant)*

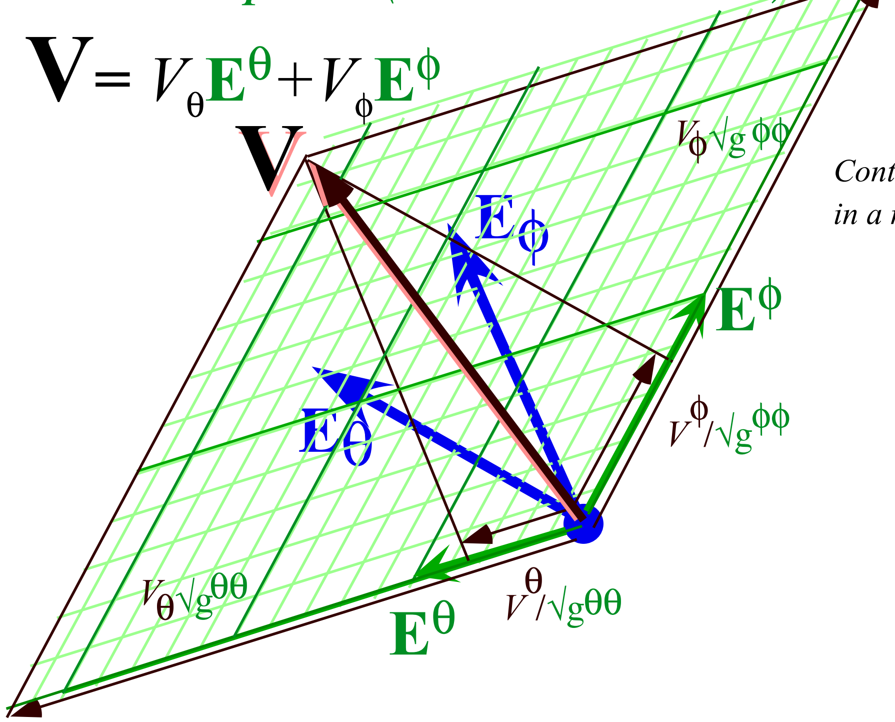



*Fig. 3.3.2*
*Contravariant vector geometry in a normal space ( $\mathbf{E}^\theta$,$\mathbf{E}^\phi$ ).*

*Tangent space (Covariant)*

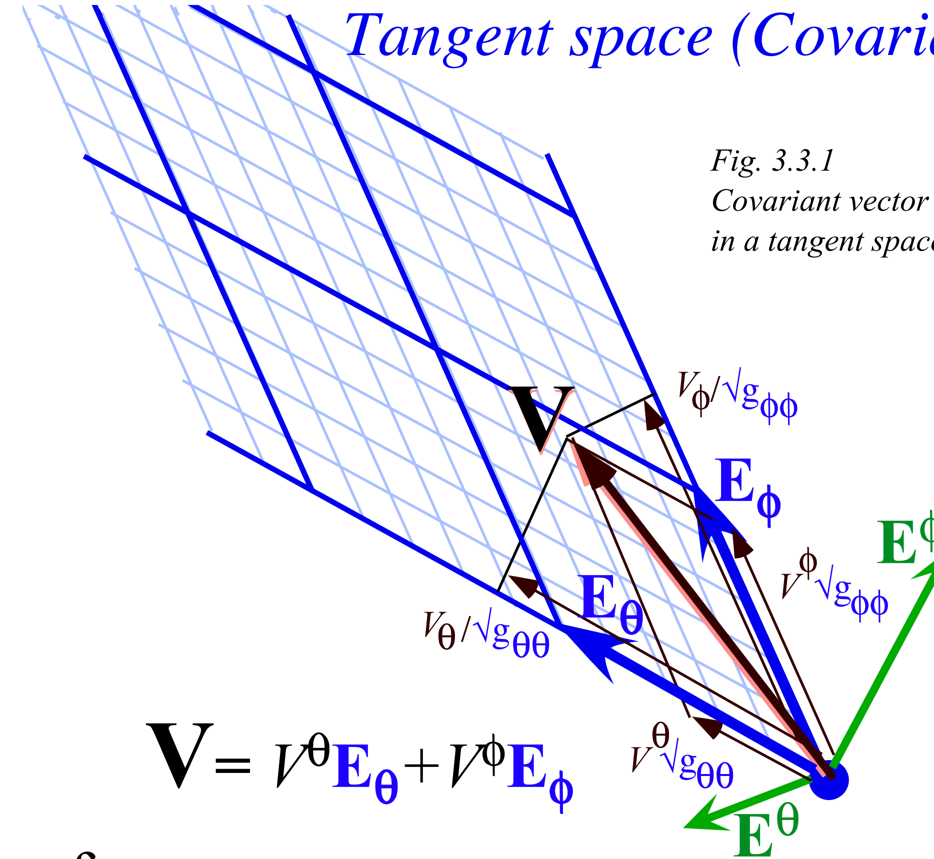



*Fig. 3.3.1*
*Covariant vector geometry in a tangent space ( $\mathbf{E}_\theta$,$\mathbf{E}_\phi$ ).*

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*....

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}\ , \quad \text{or:}\ \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

## Contravariant vectors $\mathbf{E}^m$ versus Covariant vectors $\mathbf{E}_n$

*Any* vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

$$\mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *$U^m$, $V^m$,...are contravariant components* and the *$U_n$, $V_n$ ...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}}$, ,

$U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\mathbf{\bar{E}_{\bar{n}}}$, *etc.*

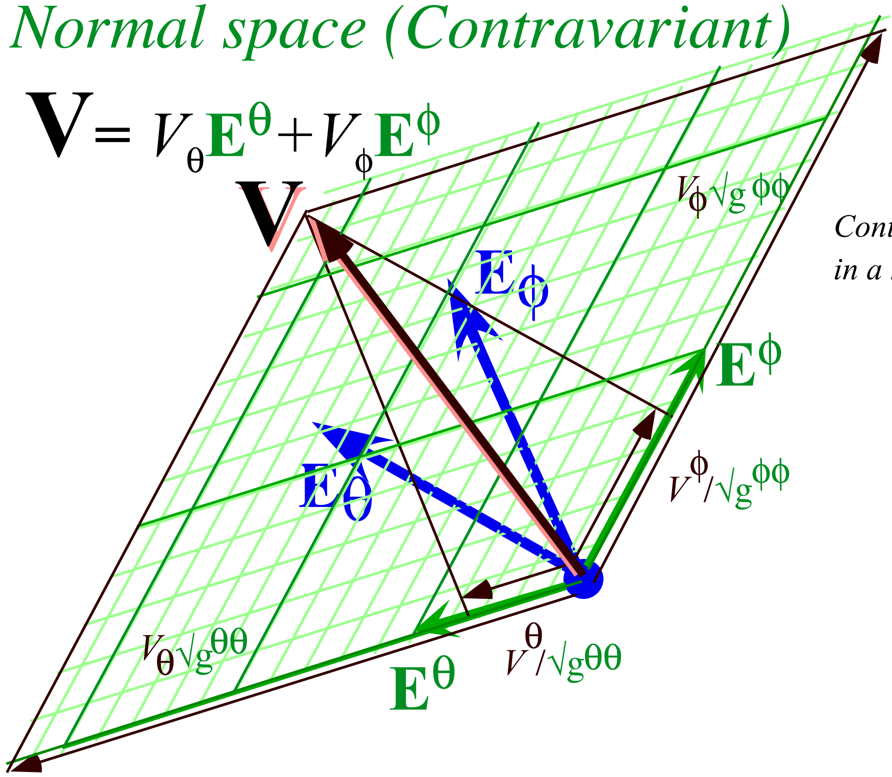



*Fig. 3.3.2*
*Contravariant vector geometry in a normal space ( $\mathbf{E}^\theta$,$\mathbf{E}^\phi$ ).*

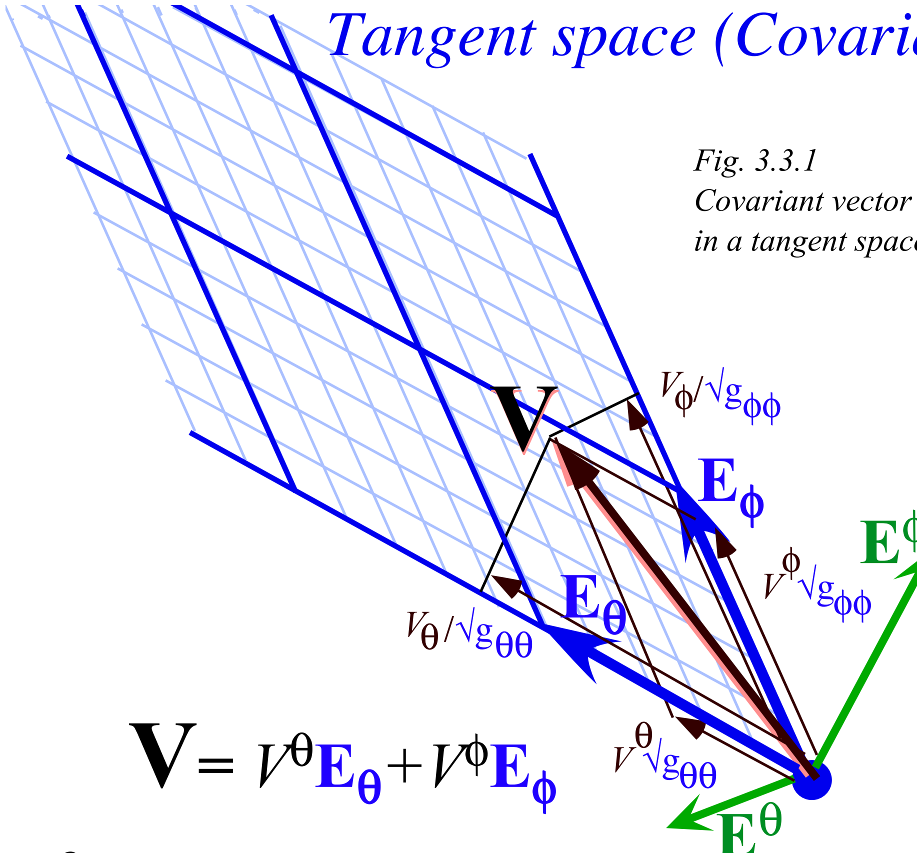



*Fig. 3.3.1*
*Covariant vector geometry in a tangent space ( $\mathbf{E}_\theta$,$\mathbf{E}_\phi$ ).*

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*….

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

...and the same for covariant vectors $\mathbf{E_m}$ and $\mathbf{\bar{E}_{\bar{m}}}$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}}}$$

## *Contravariant vectors* $\mathbf{E}^m$ *versus* *Covariant vectors* $\mathbf{E}_n$

*Any* vector **U,V,...** is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{U}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\bar{\mathbf{E}}_{\bar{\mathbf{m}}} = \bar{V}_{\bar{n}}\,\bar{\mathbf{E}}^{\bar{\mathbf{n}}}$$

where the *Um, Vm,...are contravariant components* and the *Un , Vn ,...are covariant components*

$U^m = \mathbf{U}\bullet\mathbf{E^m}$, $V^m = \mathbf{V}\bullet\mathbf{E^m}$, and $\bar{U}^{\bar{m}} = \mathbf{U}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, $\bar{V}^{\bar{m}} = \mathbf{V}\bullet\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$, , $U_n = \mathbf{U}\bullet\mathbf{E_n}$, $V_n = \mathbf{V}\bullet\mathbf{E_n}$, and $\bar{U}_{\bar{n}} = \mathbf{U}\bullet\bar{\mathbf{E}}_{\bar{\mathbf{n}}}$, *etc.*

*Normal space (Contravariant)*

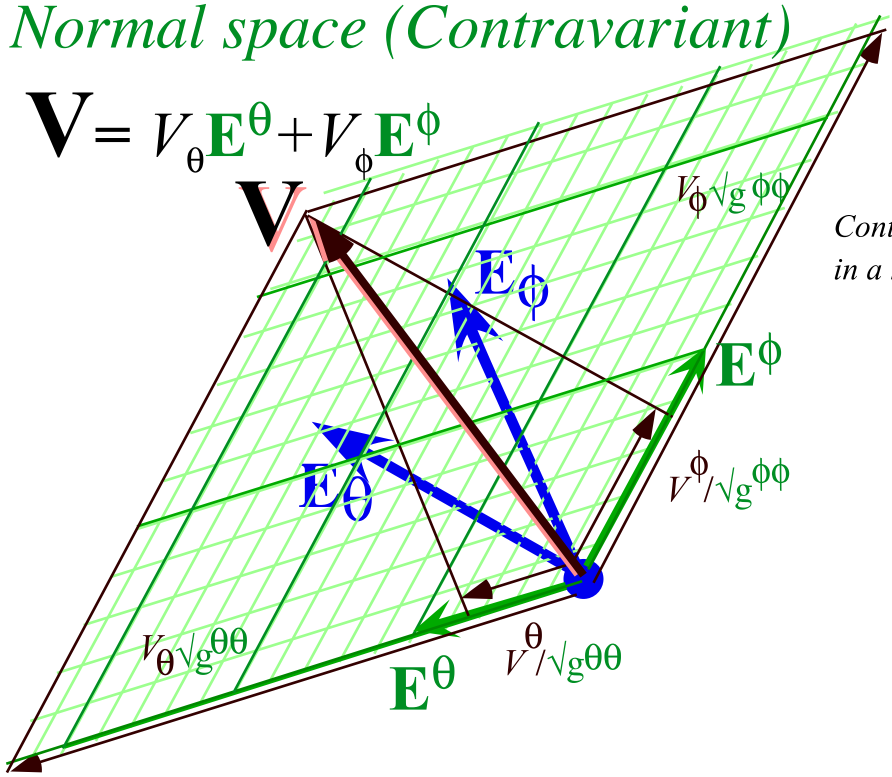
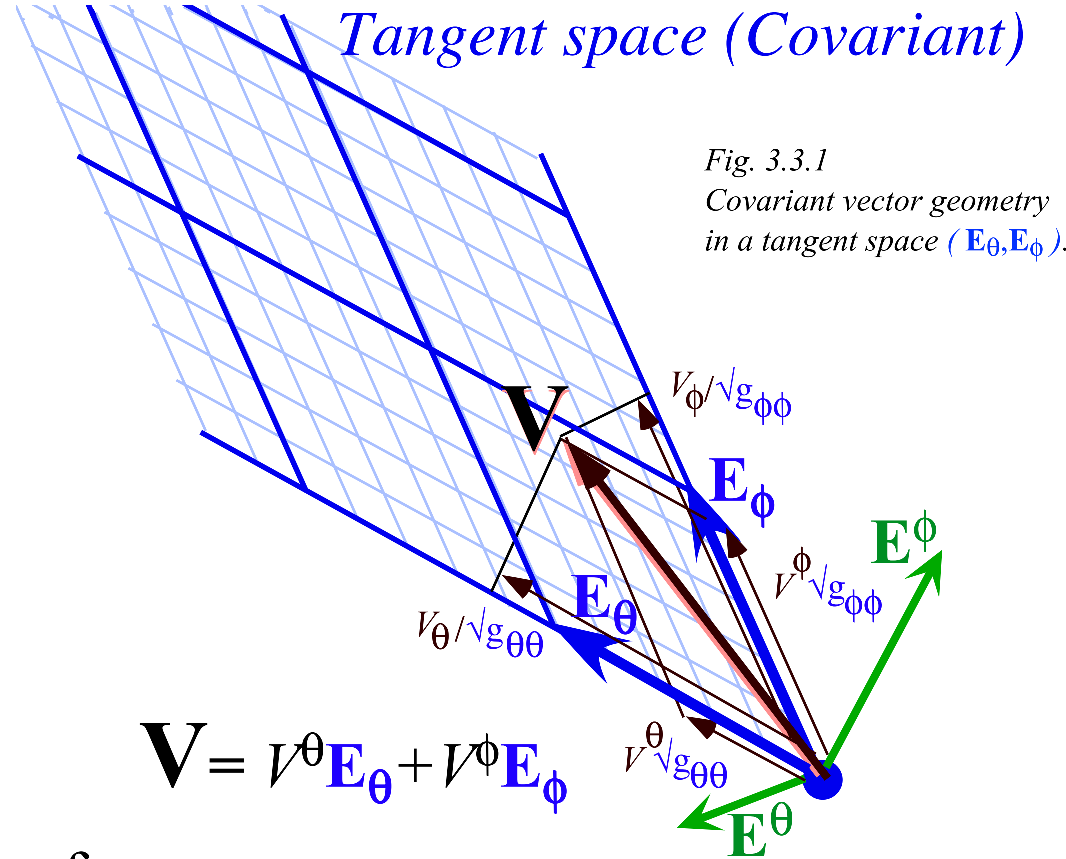



*Fig. 3.3.2*
*Contravariant vector geometry in a normal space* ( $\mathbf{E}^\theta,\mathbf{E}^\phi$ ).

*Tangent space (Covariant)*




Contravariant vector $\mathbf{E^m}$ for frame $\{q^1,q^2,\cdots\}$ is written in terms of new vectors $\bar{\mathbf{E}}^{\bar{\mathbf{m}}}$ for a new "barred" frame $\{\bar{q}^1,\bar{q}^2,\cdots\}$ using a *"chain-saw-sum rule"*....

...and the same for covariant vectors $\mathbf{E_m}$ and $\bar{\mathbf{E}}_{\bar{\mathbf{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}, \quad \text{or: } \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{\mathbf{E}}^{\bar{\mathbf{m}}}}$$

implies: $V^m = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}, \quad \text{or: } \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{\mathbf{E}}_{\bar{\mathbf{m}}}}$$

implies: $V_m = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\bar{V}_{\bar{m}}$

## Contravariant vectors $\mathbf{E}^m$ versus Covariant vectors $\mathbf{E}_n$

*Any* vector **U**,**V**,... is expressed using *either* set from any viewpoint, coordinate system, or *frame*,

$$\mathbf{U} = U^m\,\mathbf{E_m} = U_n\,\mathbf{E^n} = \bar{U}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{U}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}} \qquad \mathbf{V} = V^m\,\mathbf{E_m} = V_n\,\mathbf{E^n} = \bar{V}^{\bar{m}}\,\mathbf{\bar{E}_{\bar{m}}} = \bar{V}_{\bar{n}}\,\mathbf{\bar{E}^{\bar{n}}}$$

where the *$U^m$, $V^m$,...are contravariant components* and the *$U_n$ , $V_n$ ...are covariant components*

$$U^m = \mathbf{U}\bullet\mathbf{E^m},\ V^m = \mathbf{V}\bullet\mathbf{E^m},\ \text{and}\ \bar{U}^{\bar{m}} = \mathbf{U}\bullet\mathbf{\bar{E}^{\bar{m}}},\ \bar{V}^{\bar{m}} = \mathbf{V}\bullet\mathbf{\bar{E}^{\bar{m}}},, \qquad U_n = \mathbf{U}\bullet\mathbf{E_n},\ V_n = \mathbf{V}\bullet\mathbf{E_n},\ \text{and}\ \bar{U}_{\bar{n}} = \mathbf{U}\bullet\mathbf{\bar{E}_{\bar{n}}},\ etc.$$

*Normal space (Contravariant)*

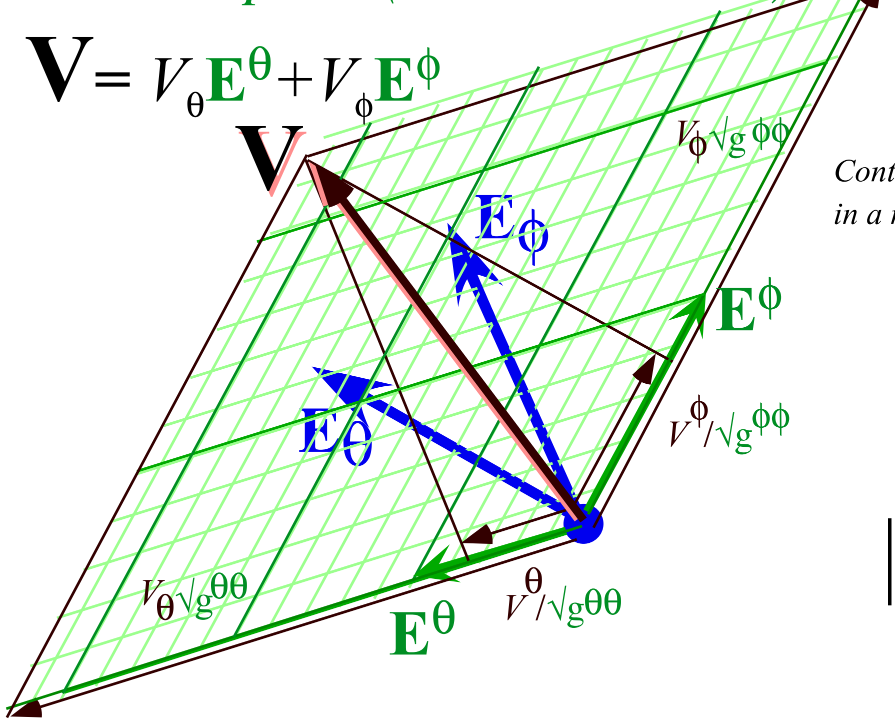



*Fig. 3.3.2*
*Contravariant vector geometry in a normal space ( $\mathbf{E}^\theta$,$\mathbf{E}^\phi$ ).*

*Tangent space (Covariant)*

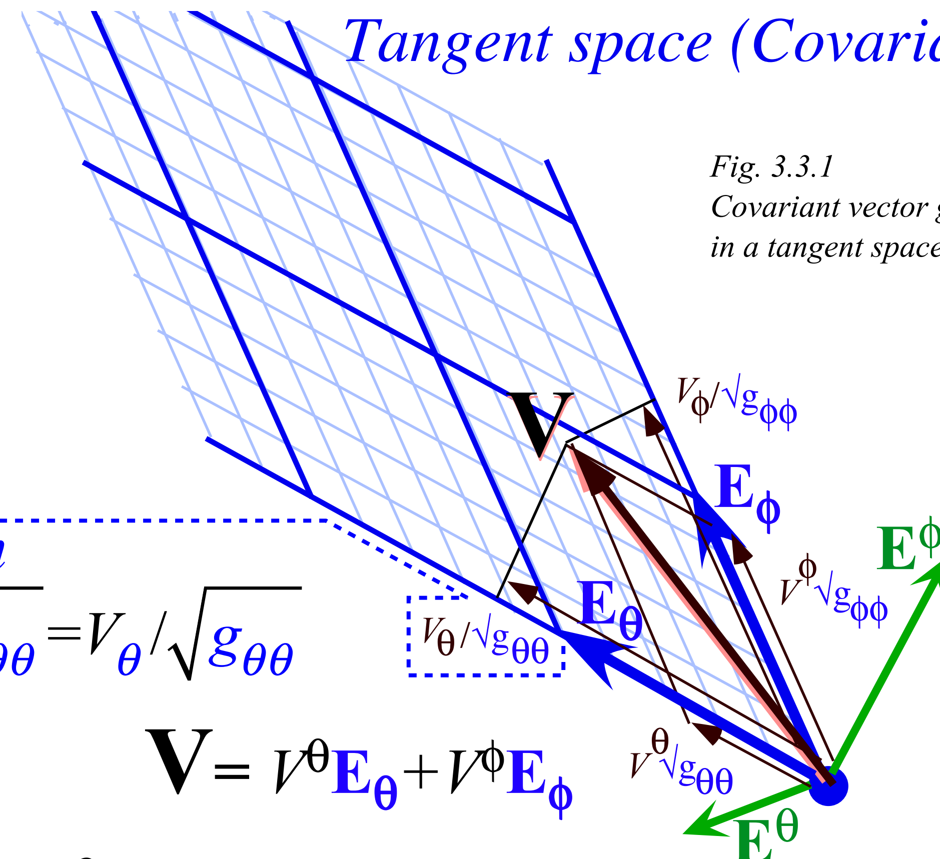



*Fig. 3.3.1*
*Covariant vector geometry in a tangent space ( $\mathbf{E}_\theta$,$\mathbf{E}_\phi$ ).*

covariant projection

$$\left|\mathbf{V}\bullet\mathbf{E}_\theta\right| = \mathbf{V}\bullet\mathbf{\hat{E}}_\theta = \mathbf{V}\bullet\mathbf{E}_\theta/\sqrt{g_{\theta\theta}} = V_\theta/\sqrt{g_{\theta\theta}}$$

Contravariant vector $\mathbf{E^m}$ for frame $\{q^1, q^2, \cdots\}$ is written in terms of new vectors $\mathbf{\bar{E}^{\bar{m}}}$ for a new "barred" frame $\{\bar{q}^1, \bar{q}^2, \cdots\}$ using a *"chain-saw-sum rule"*....

...and the same for covariant vectors $\mathbf{E_m}$ and $\mathbf{\bar{E}_{\bar{m}}}$

$$\mathbf{E^m} = \frac{\partial q^m}{\partial \mathbf{r}} = \frac{\partial q^m}{\quad}\frac{\quad}{\partial \mathbf{r}} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\frac{\partial \bar{q}^{\bar{m}}}{\partial \mathbf{r}}\ ,\quad \text{or:}\ \boxed{\mathbf{E^m} = \frac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\mathbf{\bar{E}^{\bar{m}}}}$$

implies: $V^m = \dfrac{\partial q^m}{\partial \bar{q}^{\bar{m}}}\bar{V}^{\bar{m}}$

$$\mathbf{E_m} = \frac{\partial \mathbf{r}}{\partial q^m} = \frac{\quad}{\partial q^m}\frac{\partial \mathbf{r}}{\quad} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\frac{\partial \mathbf{r}}{\partial \bar{q}^{\bar{m}}}\ ,\quad \text{or:}\ \boxed{\mathbf{E_m} = \frac{\partial \bar{q}^{\bar{m}}}{\partial q^m}\mathbf{\bar{E}_{\bar{m}}}}$$

Dirac notation equivalents:

$$\langle m| = \langle m|\cdot\mathbf{1} = \langle m|\cdot\sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}| = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}| \quad \text{implies:}\ \langle m|\Psi\rangle = \sum_{\bar{m}}\langle m|\bar{m}\rangle\langle\bar{m}|\Psi\rangle$$

Dirac notation equivalents:

$$|m\rangle = \mathbf{1}\cdot|m\rangle = \sum_{\bar{m}}|\bar{m}\rangle\langle\bar{m}||m\rangle = \sum_{\bar{m}}\langle\bar{m}|m\rangle|\bar{m}\rangle$$

*Geometric and topological properties of GCC transformations (Mostly from Unit 3.)*

*Multivalued functionality and connections*

*Covariant and contravariant relations*

*Tangent space vs. Normal space*

→ *Metric $g_{mn}$ tensor geometric relations to length, area, and volume*

*Metric tensor* **g** *covariant (and contravariant) metric components* $g_{mn}$ *(and* $g^{mn}$ *)*

$$g_{mn} = \mathbf{E_m} \bullet \mathbf{E_n} = g_{nm} \ , \quad g^{mn} = \mathbf{E^m} \bullet \mathbf{E^n} = g^{nm} \ .$$

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"Mixed" covariant-contravariant metric components

$$g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = g_m^n = \mathbf{E_m} \bullet \mathbf{E^n} = \delta_m^n = \begin{cases} 0 & \text{if: } m \neq n \\ 1 & \text{if: } m = n \end{cases}$$

Caution: $\delta_{mn}$ is $g_{mn}$ and not $\delta_n^m$ in GCC.

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Metric coefficients express covariant unitary vectors in terms of contras and vice-versa

$$\mathbf{E_m} = g_{mn}\mathbf{E^n} \;, \qquad \mathbf{E^m} = g^{mn}\mathbf{E_n} \;.$$

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$$\mathbf{E_m} = g_{mn}\mathbf{E^n} \;, \qquad \mathbf{E^m} = g^{mn}\mathbf{E_n} \;.$$

Co-and-Contra vector and tensor components are related by $g$-transformation. (So are $g$'s themselves.)

$$V_m = g_{mn}V^n, \qquad V^m = g^{mn}V_n \;, \quad T^{mm'} = g^{mn}g^{m'n'}T_{nn'} \;, \; etc.$$

*Metric tensor* **g** *covariant (and contravariant) metric components* $g_{mn}$ *(and* $g^{mn}$ *)*

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Metric coefficients express covariant unitary vectors in terms of contras and vice-versa

$$\mathbf{E_m} = g_{mn}\mathbf{E^n} \ , \qquad \mathbf{E^m} = g^{mn}\mathbf{E_n} \ .$$

Co-and-Contra vector and tensor components are related by $g$-transformation. (So are $g$'s themselves.)

$$V_m = g_{mn}V^n, \qquad V^m = g^{mn}V_n \ , \quad T^{mm'} = g^{mn}g^{m'n'}V_{nn'} \ , \ etc.$$

Diagonal square roots $\sqrt{g_{mm}}$ are the lengths of the covariant unitary vectors.

$$\left|\mathbf{E_m}\right| = \sqrt{\mathbf{E_m} \bullet \mathbf{E_m}} = \sqrt{g_{mm}}$$

$$\left|\mathbf{E^m}\right| = \sqrt{\mathbf{E^m} \bullet \mathbf{E^m}} = \sqrt{g^{mm}}$$

tangent space area spanned by $V^1\mathbf{E}_1$ and $V^2\mathbf{E}_2$

$$Area\left(V^1 E_1, V^2 E_2\right) = V^1 V^2 \left|\mathbf{E_1} \times \mathbf{E_2}\right| = V^1 V^2 \sqrt{\left(\mathbf{E_1} \times \mathbf{E_2}\right) \bullet \left(\mathbf{E_1} \times \mathbf{E_2}\right)}$$

$$Area\left(V^1 E_1, V^2 E_2\right) = V^1 V^2 \sqrt{\left(\mathbf{E_1} \bullet \mathbf{E_1}\right)\left(\mathbf{E_2} \bullet \mathbf{E_2}\right) - \left(\mathbf{E_1} \bullet \mathbf{E_2}\right)\left(\mathbf{E_1} \bullet \mathbf{E_2}\right)}$$

$$= V^1 V^2 \sqrt{g_{11} g_{22} - g_{12} g_{12}} = V^1 V^2 \sqrt{\det \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}}$$

3D Jacobian determinant $J$-columns are $\mathbf{E}1$, $\mathbf{E}2$ and $\mathbf{E}3$.

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1 V^2 V^3 \left|\mathbf{E_1} \times \mathbf{E_2} \bullet \mathbf{E_3}\right| = V^1 V^2 V^3 \det \begin{vmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{vmatrix}$$

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^2}{\partial q^1} & \frac{\partial x^3}{\partial q^1} \\ \frac{\partial x^1}{\partial q^2} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^3}{\partial q^2} \\ \frac{\partial x^1}{\partial q^3} & \frac{\partial x^2}{\partial q^3} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial x^1}{\partial q^1} & \frac{\partial x^1}{\partial q^2} & \frac{\partial x^1}{\partial q^3} \\ \frac{\partial x^2}{\partial q^1} & \frac{\partial x^2}{\partial q^2} & \frac{\partial x^2}{\partial q^3} \\ \frac{\partial x^3}{\partial q^1} & \frac{\partial x^3}{\partial q^2} & \frac{\partial x^3}{\partial q^3} \end{pmatrix} = J^T \bullet J$$

Determinant product ($det|A|\ det|B| = det|A \bullet B|$) and symmetry ($det|A^T| = det|A|$) gives

$$Volume\left(V^1\mathbf{E}_1, V^2\mathbf{E}_2, V^3\mathbf{E}_3\right) = V^1 V^2 V^3 \det|J| = V^1 V^2 V^3 \sqrt{\det|g|}$$