Poincare, Lagrange, Hamiltonian, and Jacobi mechanics

(Unit 1 Ch. 12, Unit 2 Ch. 2-7, Unit 3 Ch. 1-3)

Review of Lecture 12 relations:

Examples of Hamiltonian mechanics in phase plots
1D Pendulum and phase plot (Simulation)
1D-HO phase-space control (Simulation of “Catcher in the Eye”)

Exploring phase space and Lagrangian mechanics more deeply
A weird “derivation” of Lagrange’s equations
Poincare identity and Action, Jacobi-Hamilton equations
How Classicists might have “derived” quantum equations
Huygen’s contact transformations enforce minimum action
How to do quantum mechanics if you only know classical mechanics
Examples of Hamiltonian mechanics in phase plots

1D Pendulum and phase plot *(Simulation)*
1D-HO phase-space control *(Simulation of “Catcher in the Eye”)***
1D Pendulum and phase plot

(a) Force geometry

\[ x = R \sin \theta \sim \theta R \]

\[ -MgR \sin \theta = F_\theta = -Mg \dot{x} \]

(b) Energy geometry

\[ \text{PE:} \]
\[ V = MgY = -MgR \cos \theta \]

\[ \frac{1}{2} (Mg/R) x^2 \quad \sim Mgh \]

\[ x^2 = h(2R-h) \sim 2hR \quad \text{(Euclid mean)} \]

(c) Time geometry

\[ \varepsilon = \frac{\theta}{2} \]

\[ R \]

Lagrangian function

\[ L = KE - PE = T - U \] where potential energy is

\[ U(\theta) = -MgR \cos \theta \]

\[ L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta \]

NOTE: Very common loci of ± sign blunders
Lagrangian function $L = KE - PE = T - U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$L(\dot{\theta},\theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR\cos\theta$$

Hamiltonian function $H = KE + PE = T + U$ where potential energy is $U(\theta) = -MgR\cos\theta$

$$H(p_\theta,\theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR\cos\theta = E = \text{const.}$$
1D Pendulum and phase plot

(a) Force geometry

\[ x = R \sin \theta \approx R \theta \]
\[ -MgR \sin \theta = F_\theta \]
\[ \theta \]
\[ Mg \]

(b) Energy geometry

\[ \frac{1}{2} (Mg/R) x^2 \approx Mgh \]
\[ x^2 = h(2R-h) \sim 2hR \]
\[ (Euclid \ mean) \]

(c) Time geometry

\[ \varepsilon = \theta/2 \]
\[ \theta \]

\[ PE: \]
\[ V = MgY \]
\[ = -MgR \cos \theta \]

Lagrangian function \( L = KE - PE = T - U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[ L(\dot{\theta}, \theta) = \frac{1}{2} I \dot{\theta}^2 - U(\theta) = \frac{1}{2} I \dot{\theta}^2 + MgR \cos \theta \]

Hamiltonian function \( H = KE + PE = T + U \) where potential energy is \( U(\theta) = -MgR \cos \theta \)

\[ H(p_\theta, \theta) = \frac{1}{2I} p_\theta^2 + U(\theta) = \frac{1}{2I} p_\theta^2 - MgR \cos \theta = E = \text{const.} \]

implies: \[ p_\theta = \sqrt{2I(E + MgR \cos \theta)} \]
Example of plot of Hamilton for 1D-solid pendulum in its Phase Space \((\theta,p_\theta)\)

\[
H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta , \quad \text{or} \quad p_\theta = \sqrt{2I(E + MgR \cos \theta)}
\]
Example of plot of Hamilton for 1D-solid pendulum in its Phase Space \((\theta, p_\theta)\)

\[
H(p_\theta, \theta) = E = \frac{1}{2I} p_\theta^2 - MgR \cos \theta, \quad \text{or:} \quad p_\theta = \sqrt{2I \left( E + MgR \cos \theta \right)}
\]

Funny way to look at Hamilton's equations:

\[
\begin{pmatrix}
\dot{q} \\
\dot{p}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial H}{\partial p} \\
-\frac{\partial H}{\partial q}
\end{pmatrix} = e_H \times (-\nabla H) = (\text{H-axis}) \times (\text{fall line}), \text{ where: }
\begin{cases}
(H-\text{axis}) = e_H = e_q \times e_p \\
(\text{fall line}) = -\nabla H
\end{cases}
\]
Examples of Hamiltonian dynamics and phase plots

1D Pendulum and phase plot (Simulation)

Phase control (Simulation of “Catcher in the Eye”)

Thursday, October 4, 2012
Simulation of atomic classical (or semi-classical) dynamics using varying phase control

Unit 1
Fig. 7.4

$F(Y) = -kY - Mg$

$U(Y) = (1/2)kY^2 + MgY$

$u(y) = (1/4)y^2 + y$

$f(y) = -(1/2)y - 1$

$Y_{shift} = -Mg/k$

$U_{shift} = -(Mg)^2 / 2k$

$F(x)$ or $V(x)$ or velocity

Reset atom $K$

Reset Field $qE$
Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations
Poincare identity and Action, Jacobi-Hamilton equations
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Huygen’s contact transformations enforce minimum action
How to do quantum mechanics if you only know classical mechanics
Variational calculus finds extreme (minimum or maximum) values to entire integrals.

Minimize (or maximize): \( S(q) = \int_{t_0}^{t_1} dt \, L(q(t), \dot{q}(t), t) \).

A strange “derivation” of Lagrange’s equations by Calculus of Variation

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. (1)

\[ \delta q(t_0) = 0 = \delta q(t_1) \quad (1) \]

1st order \( L(q+\delta q) \) approximate:

\[ S(q+\delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \]

where: \( \delta \dot{q} = \frac{d}{dt} \delta q \)
An arbitrary but small variation function $\delta q(t)$ is allowed at every point $t$ in the figure along the curve except at the end points $t_0$ and $t_1$. There we demand it not vary at all.

**1st order $L(q+\delta q)$ approximate:**

$$S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$$

where: $\delta \dot{q} = \frac{d}{dt} \delta q$

Replace $\frac{\partial L}{\partial \dot{q}} \delta q$ with $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) \delta q$

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**A weird “derivation” of Lagrange’s equations**

Variational calculus finds extreme (minimum or maximum) values to entire integrals

$$S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t).$$

An arbitrary but small variation function $\delta q(t)$ is allowed at every point $t$ in the figure along the curve except at the end points $t_0$ and $t_1$. There we demand it not vary at all.

**1st order $L(q+\delta q)$ approximate:**

$$\delta q(t_0) = 0 = \delta q(t_1) \quad (1)$$

Replace $\frac{\partial L}{\partial \dot{q}} \delta q$ with $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) \delta q$
A weird “derivation” of Lagrange’s equations
Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[ S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t) . \]

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. \( (1) \)

1st order \( L(q+\delta q) \) approximate:

\[ \delta q(t_0) = 0 = \delta q(t_1) \quad (1) \]

\[ S(q+\delta q) = \int_{t_0}^{t_1} dt \ \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \]

where: \( \delta \dot{q} = \frac{d}{dt} \delta q \)

Replace \( \frac{\partial L}{\partial \dot{q}} \delta q \) with \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \)
A weird “derivation” of Lagrange’s equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[
S(q) = \int_{t_0}^{t_1} dt \, L(q(t), \dot{q}(t), t).
\]

An arbitrary but small variation function \(\delta q(t)\) is allowed at every point \(t\) in the figure along the curve except at the end points \(t_0\) and \(t_1\). There we demand it not vary at all. (1)

1st order \(L(q+\delta q)\) approximate:

\[
S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]
\]

where: \(\delta \dot{q} = \frac{d}{dt} \delta q\)

Replace \(\frac{\partial L}{\partial \dot{q}} \delta q\) with \(\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) \delta q\)

\[
S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) \left[ \frac{\partial L}{\partial \dot{q}} \delta q \right] \right] + \int_{t_0}^{t_1} dt \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right)
\]

\[
= \int_{t_0}^{t_1} dt \, L(q, \dot{q}, t) + \int_{t_0}^{t_1} dt \left[ \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) \bigg|_{t_0}^{t_1}
\]
A weird “derivation” of Lagrange’s equations

Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[ S(q) = \int_{t_0}^{t_1} dt \, L(q(t), \dot{q}(t), t). \]

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all. \((1)\)

1st order \( L(q+\delta q) \) approximate:

\[ S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \]

where: \( \delta \dot{q} = \frac{d}{dt} \delta q \)

Replace \( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \) with \( \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt}\left( \frac{\partial L}{\partial q} \right) \delta q \)

Third term vanishes by \((1)\). This leaves first order variation:

\[ \delta S = S(q + \delta q) - S(q) = \int_{t_0}^{t_1} dt \left[ \frac{\partial L}{\partial q} \delta q - \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \right) \delta q \right] \]

Extreme value (actually minimum value) of \( S(q) \) occurs if and only if Lagrange equation is satisfied!

\[ \delta S = 0 \Rightarrow \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)} \]
A weird “derivation” of Lagrange’s equations
Variational calculus finds extreme (minimum or maximum) values to entire integrals

\[ S(q) = \int_{t_0}^{t_1} dt \ L(q(t), \dot{q}(t), t). \]

An arbitrary but small variation function \( \delta q(t) \) is allowed at every point \( t \) in the figure along the curve except at the end points \( t_0 \) and \( t_1 \). There we demand it not vary at all.

\[ S(q + \delta q) = \int_{t_0}^{t_1} dt \ L(q + \delta q(t), \dot{q} + \delta \dot{q}, t) \]

1st order \( L(q + \delta q) \) approximate:

\[ \delta q(t_0) = 0 = \delta q(t_1) \]  \( (1) \)

\[ S(q + \delta q) = \int_{t_0}^{t_1} dt \left[ L(q, \dot{q}, t) + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right] \]

where: \( \delta \dot{q} = \frac{d}{dt} \delta q \)

Replace \( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \) with \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \delta q \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial q} \right) \delta q \)

Third term vanishes by (1). This leaves first order variation: \( \delta S = S(q + \delta q) - S(q) = \int_{t_0}^{t_1} dt \left[ \frac{\partial L}{\partial q} \delta q \right] \)

Extreme value (actually minimum value) of \( S(q) \) occurs if and only if Lagrange equation is satisfied!

\[ \delta S = 0 \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{Euler-Lagrange equation(s)} \]

But, WHY is nature so inclined to fly JUST SO as to minimize the Lagrangian \( L = T - U \)??
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How Classicists might have “derived” quantum equations

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How to do quantum mechanics if you only know classical mechanics
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes Poincare's invariant differential if $dt$ is cleared.

$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt$

$v = \frac{dr}{dt}$ implies: $v \cdot dt = dr$
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes Poincare’s invariant differential if $dt$ is cleared.

$$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt$$

$v = \frac{dr}{dt}$ implies: $v \cdot dt = dr$

This is the time differential $dS$ of action $S = \int L \cdot dt$ whose time derivative is rate $L$ of quantum phase.

$$dS = L \cdot dt = p \cdot dr - H \cdot dt$$

where: $L = \frac{dS}{dt}$
Legendre-Poincare identity and Action

Legendre transform \( L(v) = p \cdot v - H(p) \) becomes Poincare’s invariant differential if \( dt \) is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt
\]

\[ v = \frac{dr}{dt} \]

This is the time differential \( dS \) of action \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of quantum phase.

\[ dS = L \cdot dt = p \cdot dr - H \cdot dt \]

where:

\[ L = \frac{dS}{dt} \]

Unit 8 shows [DeBroglie law \( p = \hbar k \)] and [Planck law \( H = \hbar \omega \)] make quantum plane wave phase \( \Phi \):

\[ \Phi = \frac{S}{\hbar} = \int \frac{L \cdot dt}{\hbar} \]
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes Poincare’s invariant differential if $dt$ is cleared.

$$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt$$

$v = \frac{dr}{dt}$

This is the time differential $dS$ of action $S = \int L \cdot dt$ whose time derivative is rate $L$ of quantum phase.

$$dS = L \cdot dt = p \cdot dr - H \cdot dt$$

where: $L = \frac{dS}{dt}$

Unit 2 shows DeBroglie law $p = \hbar k$ and Planck law $H = \hbar \omega$ make quantum plane wave phase $\Phi$:

$$\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}$$
Legendre-Poincare identity and Action

Legendre transform $L(v) = p \cdot v - H(p)$ becomes Poincare’s invariant differential if $dt$ is cleared.

$$L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt$$

This is the time differential $dS$ of action $S = \int L \cdot dt$ whose time derivative is rate $L$ of quantum phase.

$$dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}$$

Unit 2 shows DeBroglie law $p = \hbar k$ and Planck law $H = \hbar \omega$ make quantum plane wave phase $\Phi$:

$$\psi(r, t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}$$

Q: When is the Action-differential $dS$ integrable?
A: A differential $dW=f_x(x,y)dx+f_y(x,y)dy$ is integrable to a $W(x,y)$ if $f_x = \frac{\partial W}{\partial x}$ and $f_y = \frac{\partial W}{\partial y}$.
**Legendre-Poincare identity and Action**

Legendre transform \( L(v) = p \cdot v - H(p) \) becomes **Poincare’s invariant differential** if \( dt \) is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt
\]

This is the time differential \( dS \) of **action** \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of **quantum phase**.

\[
dS = L \cdot dt = p \cdot dr - H \cdot dt
\]

where:

\[
L = \frac{dS}{dt}
\]

**Unit 2 shows**

- **DeBroglie law** \( p = \hbar k \)
- **Planck law** \( H = \hbar \omega \)

make **quantum plane wave phase** \( \Phi \):

\[
\psi(r, t) = e^{i S/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}
\]

**Q:** When is the **Action**-differential \( dS \) integrable?

**A:** Differential \( dW = f_x(x, y) dx + f_y(x, y) dy \) is **integrable** to a \( W(x, y) \) if:

\[
\frac{\partial W}{\partial x} \quad \text{and} \quad \frac{\partial W}{\partial y}
\]

That condition is **no curl allowed** \( \nabla \times f = 0 \) or \( \partial \)-symmetry of \( W \):

\[
\frac{\partial f_x}{\partial y} = \frac{\partial^2 W}{\partial y \partial x} = \frac{\partial^2 W}{\partial x \partial y} = \frac{\partial f_y}{\partial x}
\]
Legendre-Poincare identity and Action

Legendre transform \( L(v) = p \cdot v - H(p) \) becomes *Poincare’s invariant differential* if \( dt \) is cleared.

\[
L \cdot dt = p \cdot v \cdot dt - H \cdot dt = p \cdot dr - H \cdot dt
\]

This is the time differential \( dS \) of *action* \( S = \int L \cdot dt \) whose time derivative is rate \( L \) of *quantum phase*.

\[
dS = L \cdot dt = p \cdot dr - H \cdot dt \quad \text{where:} \quad L = \frac{dS}{dt}
\]

Unit 2 shows **DeBroglie law** \( p = \hbar k \) and **Planck law** \( H = \hbar \omega \) make *quantum plane wave phase* \( \Phi \):

\[
\psi(r,t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)}
\]

Q: When is the *Action*-differential \( dS \) integrable?
A: Differential \( dW = f_x(x,y)dx + f_y(x,y)dy \) is *integrable* to a \( W(x,y) \) if: \( f_x = \frac{\partial W}{\partial x} \) and: \( f_y = \frac{\partial W}{\partial y} \)

\[
dS \text{ is integrable if: } \frac{\partial S}{\partial r} = p \quad \text{and:} \quad \frac{\partial S}{\partial t} = -H
\]

*These conditions are known as Jacobi-Hamilton equations*
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How Jacobi-Hamilton could have “derived” Schrodinger equations

Given “quantum wave”

$$\psi(r,t) = e^{iS/\hbar} = e^{i(p\cdot r - H\cdot t)/\hbar} = e^{i(k\cdot r - \omega\cdot t)}$$

dS is integrable if: $$\frac{\partial S}{\partial r} = p$$ and: $$\frac{\partial S}{\partial t} = -H$$

These conditions are known as Jacobi-Hamilton equations
How Jacobi-Hamilton could have “derived” Schrodinger equations

(Given “quantum wave”)

\[ \psi(r, t) = e^{iS/\hbar} = e^{i(p \cdot r - H \cdot t)/\hbar} = e^{i(k \cdot r - \omega \cdot t)} \]

\( dS \) is integrable if:

\[ \frac{\partial S}{\partial r} = p \quad \text{and} \quad \frac{\partial S}{\partial t} = -H \]

These conditions are known as Jacobi-Hamilton equations

Try 1st \( r \)-derivative of wave \( \psi \)

\[ \frac{\partial}{\partial r} \psi(r, t) = \frac{\partial}{\partial r} e^{iS/\hbar} = \frac{\partial}{\partial r} \left( \frac{iS}{\hbar} \right) e^{iS/\hbar} = \left( \frac{i}{\hbar} \right) \frac{\partial S}{\partial r} \psi(r, t) \]

\[ \frac{\partial}{\partial r} \psi(r, t) = \left( \frac{i}{\hbar} \right) p \psi(r, t) \quad \text{or} \quad \frac{\partial}{\partial r} \psi(r, t) = \frac{\hbar}{i} \frac{\partial}{\partial r} \psi(r, t) = p \psi(r, t) \]
How Jacobi-Hamilton could have “derived” Schrodinger equations

(Given “quantum wave”)

$$\psi(\mathbf{r}, t) = e^{i S / \hbar} = e^{i (\mathbf{p} \cdot \mathbf{r} - H \cdot t) / \hbar} = e^{i (\mathbf{k} \cdot \mathbf{r} - \omega \cdot t)}$$

$dS$ is integrable if: $$\frac{\partial S}{\partial \mathbf{r}} = \mathbf{p}$$ and: $$\frac{\partial S}{\partial t} = -H$$

These conditions are known as Jacobi-Hamilton equations

Try 1st $\mathbf{r}$-derivative of wave $\psi$

$$\frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r}, t) = \frac{\partial}{\partial \mathbf{r}} e^{i S / \hbar} = \frac{\partial (i S / \hbar)}{\partial \mathbf{r}} e^{i S / \hbar} = \left( i / \hbar \right) \frac{\partial S}{\partial \mathbf{r}} \psi(\mathbf{r}, t)$$

$$\frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r}, t) = \left( i / \hbar \right) \mathbf{p} \psi(\mathbf{r}, t)$$ or: $$\frac{\hbar}{i} \frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r}, t) = \mathbf{p} \psi(\mathbf{r}, t)$$

Try 1st $t$-derivative of wave $\psi$

$$\frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \frac{\partial}{\partial t} e^{i S / \hbar} = \frac{\partial (i S / \hbar)}{\partial t} e^{i S / \hbar} = \left( i / \hbar \right) \frac{\partial S}{\partial t} \psi(\mathbf{r}, t)$$

$$\frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left( i / \hbar \right) (-H) \psi(\mathbf{r}, t)$$ or: $$i \hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = H \psi(\mathbf{r}, t)$$
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**Huygen’s contact transformations enforce minimum action**
How to do quantum mechanics if you only know classical mechanics
Huygen’s contact transformations enforce minimum action

**Each point** \( \mathbf{r}_k \) **on a wavefront** “broadcasts” **in all directions.**

**Only minimum action** path interferes constructively.

**Time-independent action** (Hamilton’s *reduced action*)

\[
S_H = \int_{r_0}^{r_1} p \cdot d\mathbf{r}
\]

\( S_H(\mathbf{r}_0; \mathbf{r}) = 30 \)

\( S_H(\mathbf{r}_0; \mathbf{r}) = 20 \)

\( S_H(\mathbf{r}_0; \mathbf{r}) = 10 \)

**Time-dependent action** (Hamilton’s *principle action*)

\[
S_p = \int_{t_0}^{t_1} (p \cdot d\mathbf{r} - H \cdot dt)
\]

\( S_p = \int_{r_0}^{r_1} (p \cdot d\mathbf{r} - H \cdot dt) \)

\( S_p = \int_{r_0}^{r_1} (p \cdot d\mathbf{r} - H \cdot dt) \)

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**Optimal path** \( \mathbf{r}_0 \) to \( \mathbf{r}_{20} \)

**Non-optimal path** \( \mathbf{r}_0 \) to \( \mathbf{r}_{20} \)

(Least action possible)

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Fig. 12.12
Huygen’s contact transformations enforce minimum action

Each point \( \mathbf{r}_k \) on a wavefront “broadcasts” in all directions. Only **minimum action** path interferes constructively

**Time-independent action** (Hamilton’s *reduced action*) is a purely spatial integral.

\[
S_H = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{p} \cdot d\mathbf{r}
\]

\( S_H(\mathbf{r}_0: \mathbf{r}) = 30 \)

\( S_H(\mathbf{r}_0: \mathbf{r}) = 20 \)

\( S_H(\mathbf{r}_0: \mathbf{r}) = 10 \)

...because action is quantum wave phase

**Time-dependent action** (Hamilton’s *principle action*) is space-time integral.

\[
S_p = \int_{\mathbf{r}_0}^{\mathbf{r}_1} (\mathbf{p} \cdot d\mathbf{r} - H \cdot dt)
\]

\[ \left\langle \mathbf{r}_1, t_1 | \mathbf{r}_0, t_0 \right\rangle = e^{i S(\mathbf{r}_0, t_0: \mathbf{r}_1, t_1)/\hbar} \]

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\[
\int \mathbf{r}_0 \rightarrow \mathbf{r}_1 \quad e^{i S_H(\mathbf{r}_0: \mathbf{r}_1)/\hbar} \quad \left\langle \mathbf{r}_1 \right\rangle = e^{i S_H(\mathbf{r}_0: \mathbf{r}_1)/\hbar}
\]

\[ \sum_{\mathbf{r}'} \left\langle \mathbf{r}_1 | \mathbf{r}' \right\rangle \left\langle \mathbf{r}' | \mathbf{r}_0 \right\rangle \equiv \sum_{\mathbf{r}'} e^{i \left( S_H(\mathbf{r}_0: \mathbf{r}') + S_H(\mathbf{r}' : \mathbf{r}_1) \right)/\hbar} = e^{i S_H(\mathbf{r}_0: \mathbf{r}_1)/\hbar} = \left\langle \mathbf{r}_1 | \mathbf{r}_0 \right\rangle
\]

Feynman’s path-sum closure relation

Optimal path \( \mathbf{r}_0 \) to \( \mathbf{r}_{20} \)

(Least action possible)

Non-optimal path \( \mathbf{r}_0 \) to \( \mathbf{r}_{20} \)

accumulates 30
Exploring phase space and Lagrangian mechanics more deeply

A weird “derivation” of Lagrange’s equations
Poincare identity and Action, Jacobi-Hamilton equations
How Classicists might have “derived” quantum equations
Huygen’s contact transformations enforce minimum action

How to do quantum mechanics if you only know classical mechanics
Bohr quantization requires quantum phase $s_H/\hbar$ in amplitude to be an integral multiple $n$ of $2\pi$ after a closed loop integral $S_H(r_0 : r_0) = \int_{r_0}^{0} p \cdot dr$. The integer $n (n = 0, 1, 2, \ldots)$ is a quantum number.

$$I = \langle r_0 | r_0 \rangle = e^{iS_H(r_0 : r_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \quad \text{for:} \quad \Sigma_H = 2\pi \hbar n = \hbar n$$

Numerically integrate Hamilton's equations and Lagrangian $L$. Color the trajectory according to the current accumulated value of action $S_H(0 : r)/\hbar$. Adjust energy to quantized pattern (if closed system*)

$$S_H(0 : r) = S_p(0, 0 : r, t) + Ht = \int_0^t L\, dt + Ht.$$
**Bohr quantization** requires quantum phase \( s_H/\hbar \) in amplitude to be an integral multiple \( n \) of \( 2\pi \) after a closed loop integral \( S_H(r_0: r_0) = \int_{r_0}^{r_0} p \cdot dr \). The integer \( n (n = 0, 1, 2,...) \) is a *quantum number*.

\[
I = \langle r_0 | r_0 \rangle = e^{iS_H(r_0: r_0)/\hbar} = e^{i \Sigma_H/\hbar} = 1 \quad \text{for: } \Sigma_H = 2\pi \hbar n = \hbar n
\]

Numerically integrate Hamilton's equations and Lagrangian \( L \). Color the trajectory according to the current accumulated value of action \( S_H(0 : r)/\hbar \). Adjust energy to quantized pattern (if closed system*)

\[
S_H(0 : r) = S_p(0, 0 : r, t) + Ht = \int_0^t L \, dt + Ht.
\]

The hue should represent the phase angle \( S_H(0 : r)/\hbar \) modulo \( 2\pi \) as, for example, 

\( 0 = \text{red}, \pi/4 = \text{orange}, \pi/2 = \text{yellow}, 3\pi/4 = \text{green}, \pi = \text{cyan} \) (opposite of red), \( 5\pi/4 = \text{indigo}, 3\pi/2 = \text{blue}, 7\pi/4 = \text{purple}, \) and \( 2\pi = \text{red} \) (full color circle).

Interpolating action on a palette of 32 colors is enough precision for low quanta.
Bohr quantization requires quantum phase $S_H/\hbar$ in amplitude to be an integral multiple $n$ of $2\pi$ after a closed loop integral $S_H(r_0:r_0) = \int_{r_0}^{r_0} p \cdot dr$. The integer $n (n = 0, 1, 2,...)$ is a quantum number.

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Interpolating action on a palette of 32 colors is enough precision for low quanta.

*open system has continuous energy
A moving wave has a *quantum phase velocity* found by setting $S=\text{const.}$ or $dS(0,0;r,t)=0=\mathbf{p} \cdot d\mathbf{r} - H dt$.

$$\mathbf{V}_{\text{phase}} = \frac{d\mathbf{r}}{dt} = \frac{H}{p} = \frac{\omega}{k}$$

**Quantum “phase wavefronts”**

(a) $S_H=0.3$

(b) $S_H=0.35$

(c) $S_H=0.4$

(d) $S_H=0.9$

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quantum *phase velocity*

wavefront “cat ears” scoot outward..
A moving wave has a *quantum phase velocity* found by setting $S=\text{const.}$ or $dS(0,0;r,t)=0=p\cdot dr-Hdt$.

$$V_{\text{phase}} = \frac{dr}{dt} = \frac{H}{p} = \frac{\omega}{k}$$

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

$$V_{\text{group}} = \frac{dr}{dt} = \dot{r} = \frac{\partial H}{\partial p} = \frac{\partial \omega}{\partial k}$$

Note: This is Hamilton’s 1st Equation
A moving wave has a *quantum phase velocity* found by setting \( S = \text{const.} \) or \( dS(0,0;r,t)=0=\mathbf{p} \cdot d\mathbf{r} - H \, dt \).

\[
\nabla S_H = \mathbf{p}
\]

This is quite the opposite of classical particle velocity which is *quantum group velocity*.

\[
V_{\text{phase}} = \frac{d\mathbf{r}}{dt} = \frac{H}{p} = \frac{\omega}{k}
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Note: This is Hamilton’s 1st Equation

**Quantum “phase wavefronts”**

(a) \( S_H = 0.3 \)

(b) \( S_H = 0.35 \)

(c) \( S_H = 0.4 \)

(d) \( S_H = 0.9 \)

After a while … nothing left but a smile!

From *Alice's Adventures in Wonderland* by Lewis Carrol (1865)
A moving wave has a \textit{quantum phase velocity} found by setting \( S = \text{const.} \) or \( dS(0,0:r,t) = 0 = \mathbf{p} \cdot d\mathbf{r} - H dt \).

This is quite the opposite of classical particle velocity which is \textit{quantum group velocity}.

\[
V_{\text{phase}} = \frac{d\mathbf{r}}{dt} = \frac{H}{p} = \frac{\omega}{k}
\]

\[
V_{\text{group}} = \frac{d\mathbf{r}}{dt} = \mathbf{r} = \frac{\partial H}{\partial p} = \frac{\partial \omega}{\partial k}
\]

Note: This is Hamilton’s 1\textsuperscript{st} Equation

\textit{Quantum “phase wavefronts”}

(a) \( S_H = 0.3 \)  
\[ \nabla S_H = \mathbf{p} \]

(b) \( S_H = 0.35 \)

(c) \( S_H = 0.4 \)

(d) \( S_H = 0.9 \)

\textit{Classical “blast wavefronts”}

(a) \( T = 0.4 \)

(b) \( T = 1.0 \)

(c) \( T = 2.3 \)

\( \nabla S_H = \mathbf{p} \)

higher \( V_{\text{phase}} \) up here

quantum \textit{phase velocity}

...not to be confused with...

...quantum \textit{group velocity}...

that is classical particle velocity

lower \( V_{\text{group}} \) up here

higher \( V_{\text{group}} \) down here

lower \( V_{\text{phase}} \) down here

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Fig. 12.15

Thursday, October 4, 2012