Lecture 26

Introduction to Orbital Dynamics

(Ch. 2-4 of Unit 5 11.29.12)

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

➔ *Effective potentials for IHO and Coulomb orbits*

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

(A mystery similarity appears)

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

Geometry and Symmetry of Coulomb orbits

Rutherford scattering and differential scattering crossections

Ruler & compass construction

Eccentricity vector ε *and orbital phase geometry*

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu \qquad \boxed{\dot{\phi} = \frac{\mu}{m\rho^2}}$$

For ALL central forces

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

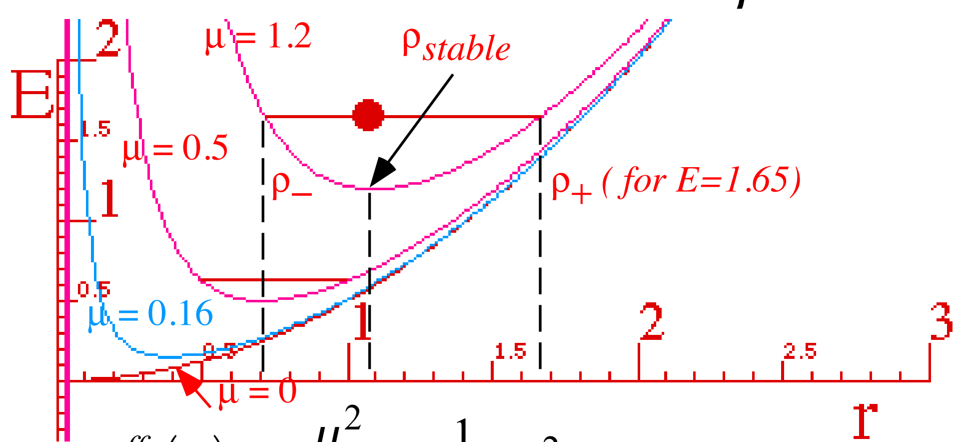
$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu \quad \boxed{\dot{\phi} = \frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$



$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

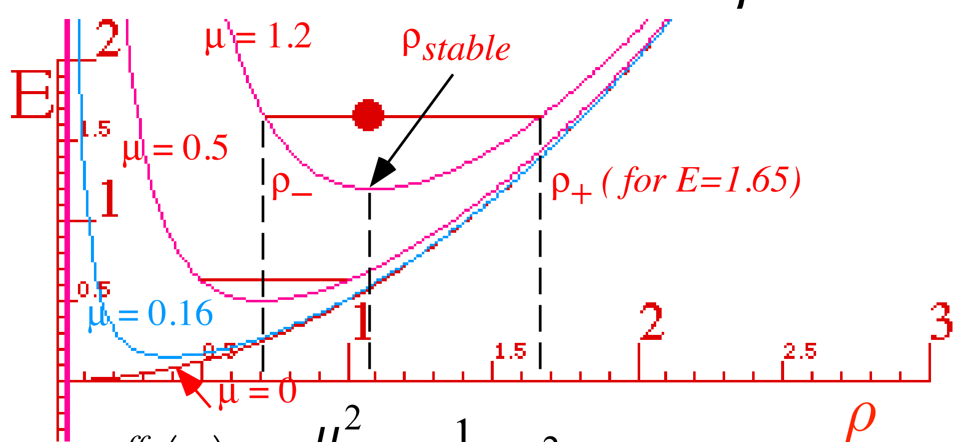
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

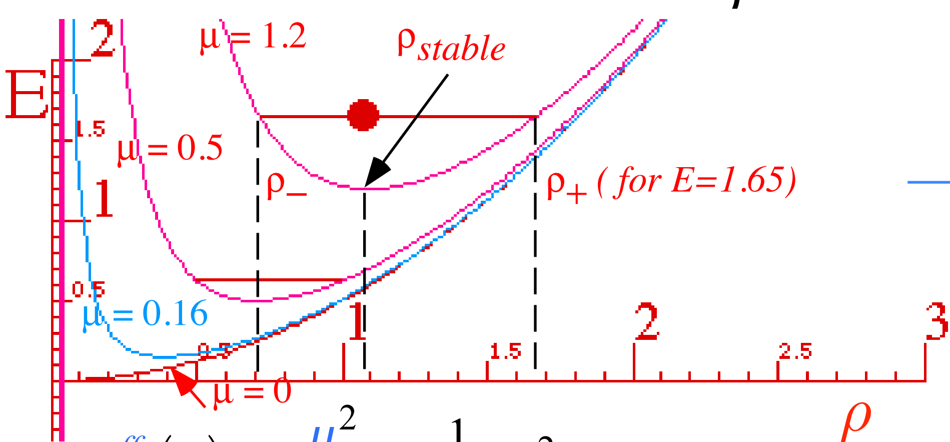
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

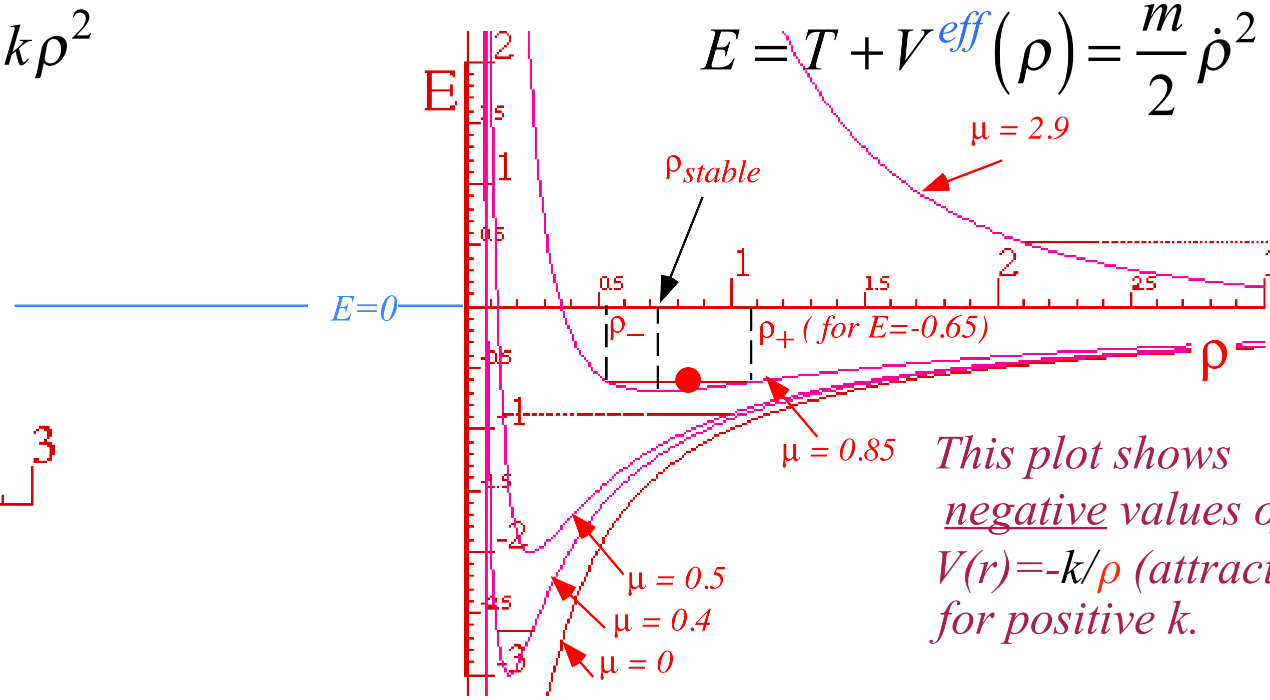
Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ | **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

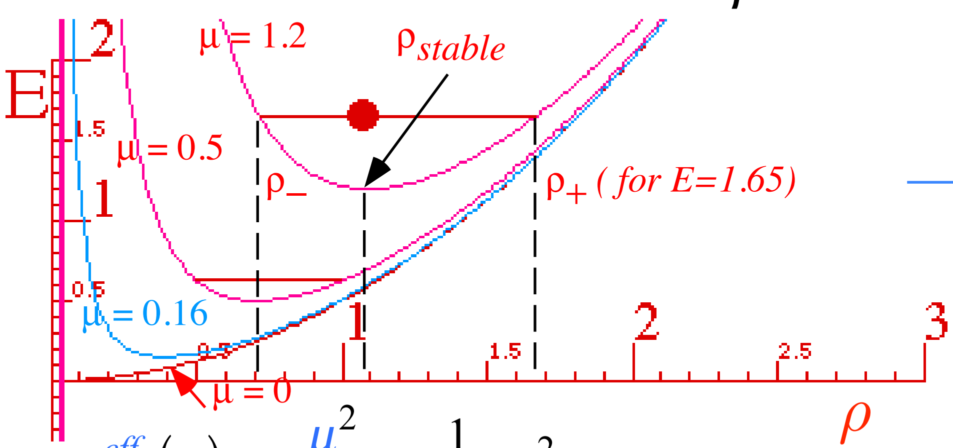
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

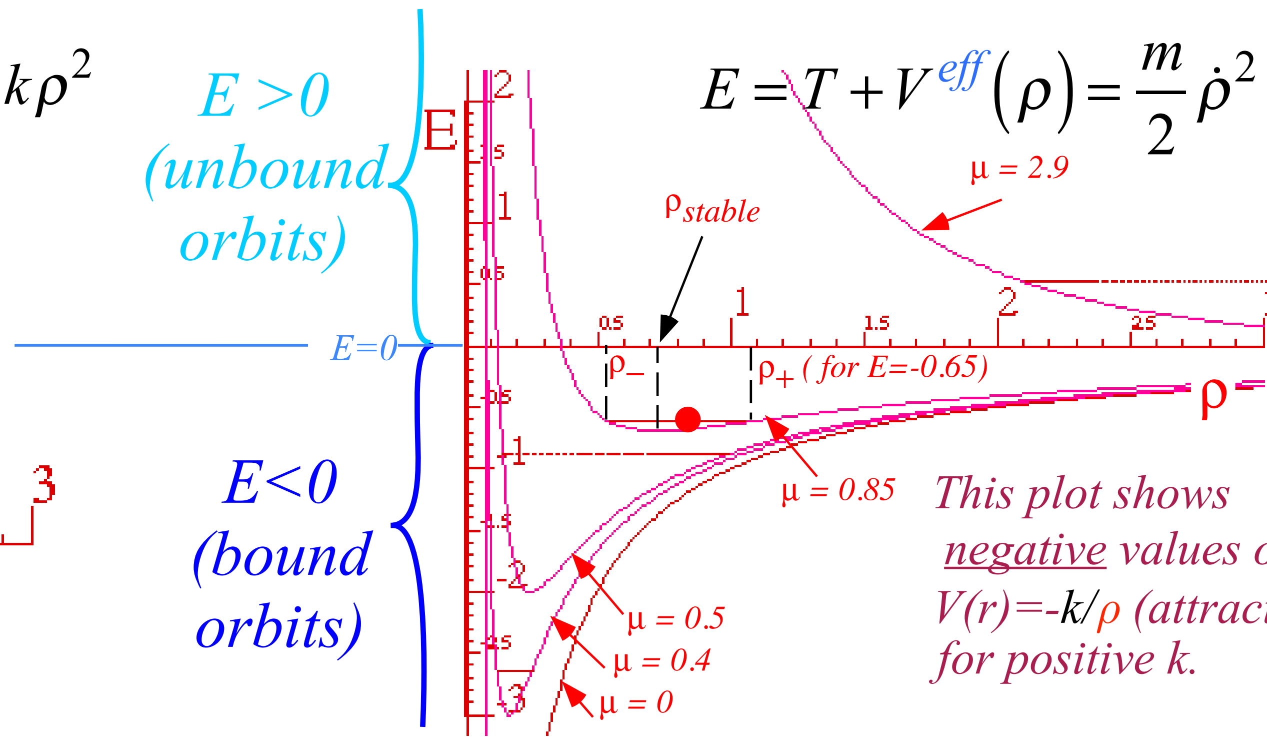
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

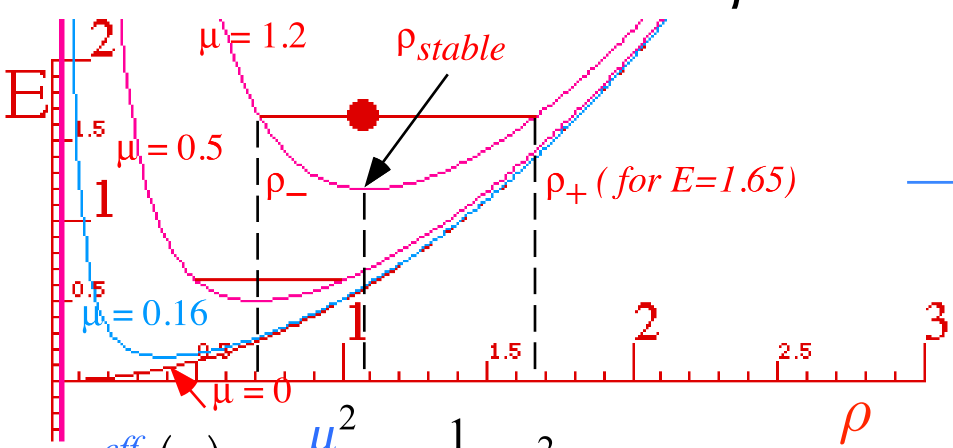
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

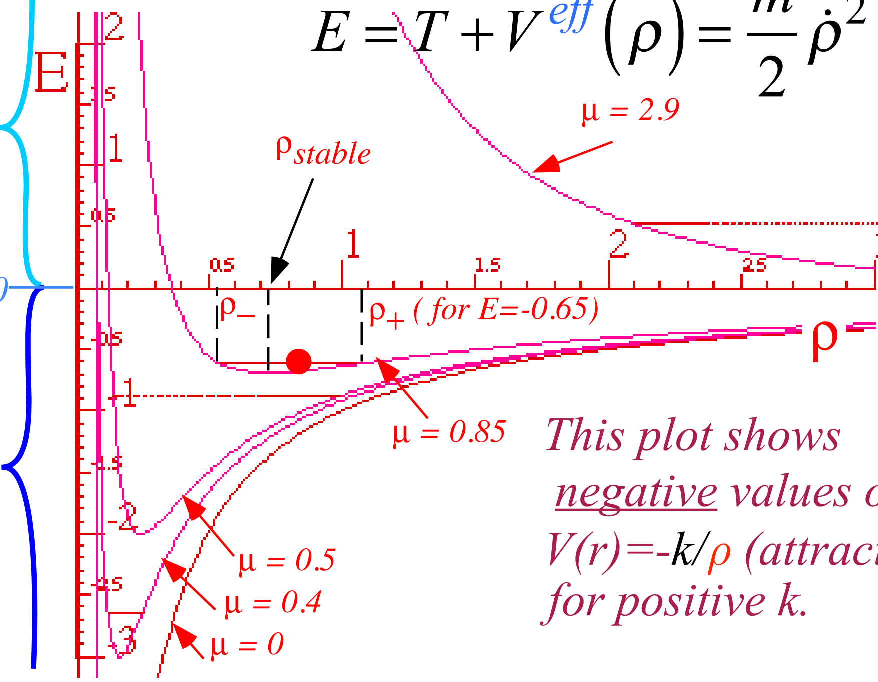
E >0 (unbound orbits)

E=0

E<0 (bound orbits)

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive) for positive k.

In either case: *IHO or Coulomb orbit blows up if* k *is negative.*

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

➔ *Stable equilibrium radii and radial/angular frequency ratios*

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

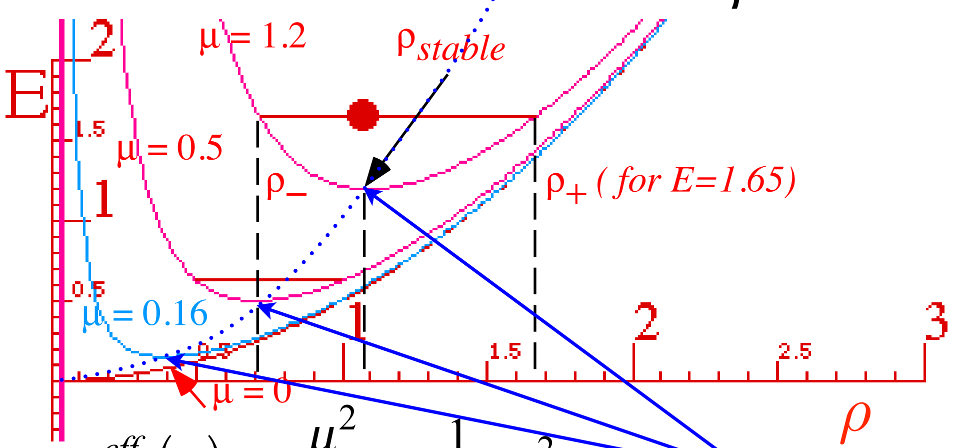
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

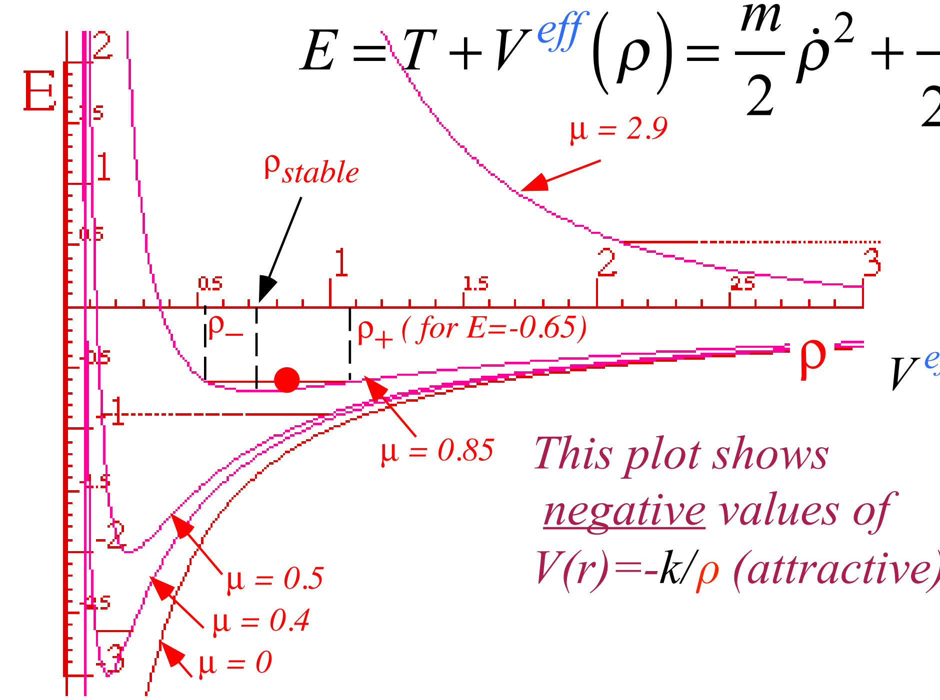
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\frac{\mu^2}{m}=+k\rho^4$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

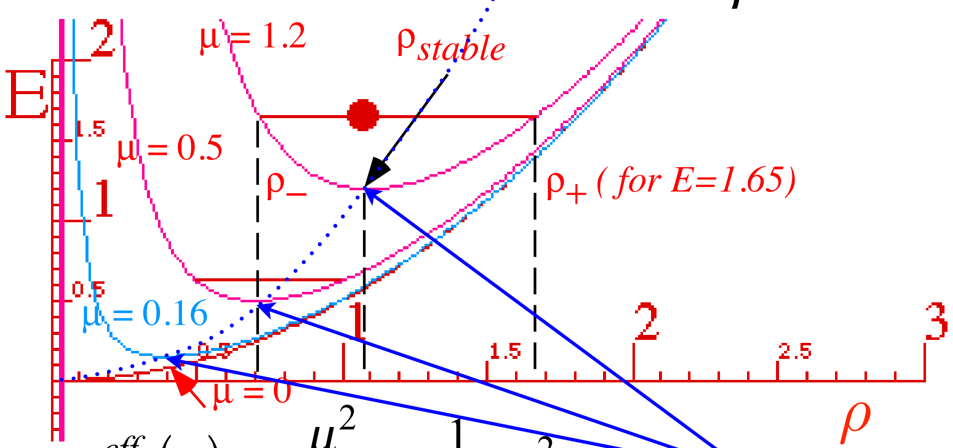
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

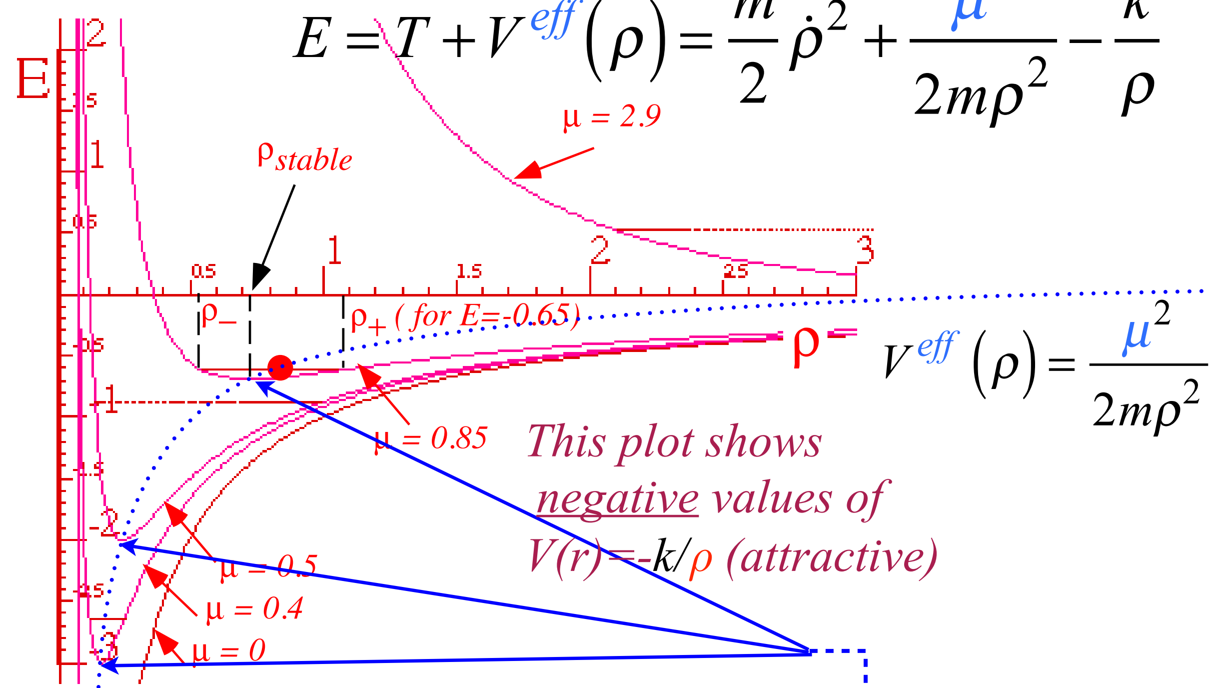
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \qquad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\frac{\mu^2}{m}=+k\rho^4$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \qquad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

$$\frac{\mu^2}{m}=+k\rho$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

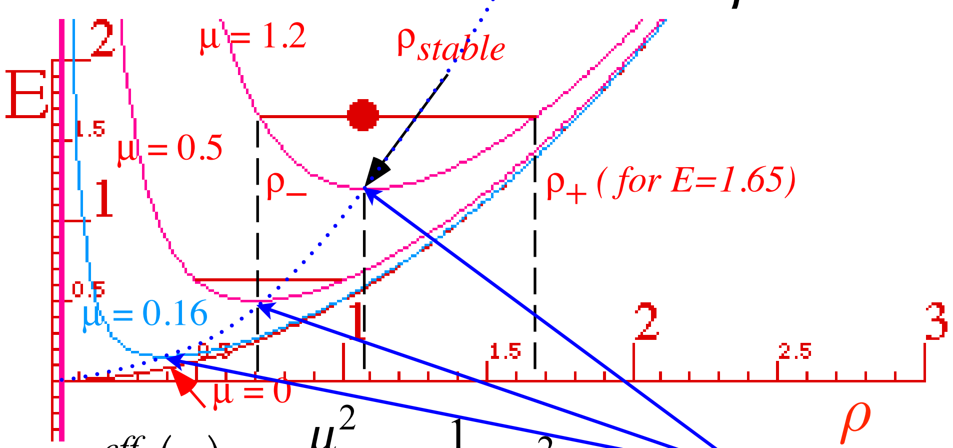
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

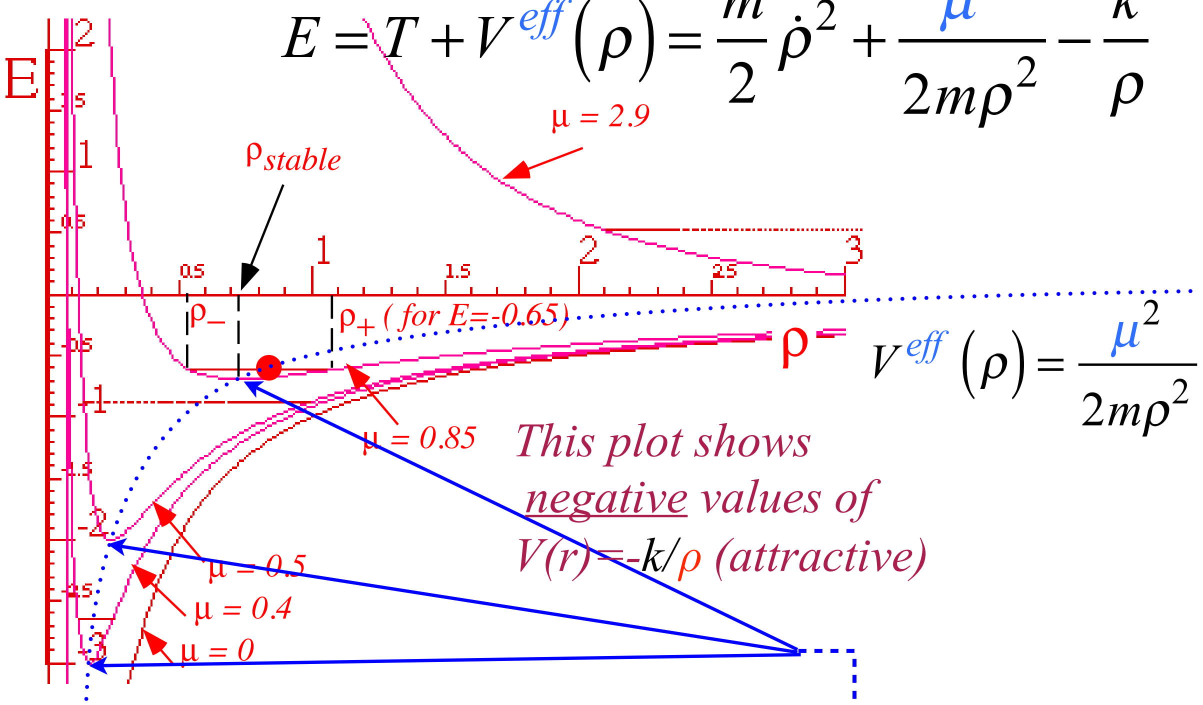
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows <u>negative</u> values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \quad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

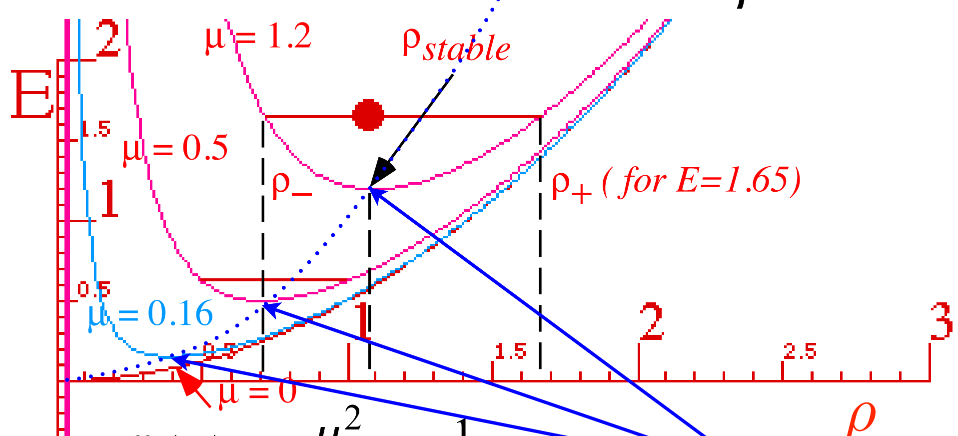
$$T = \frac{m}{2} g_{\rho\rho}\dot{\rho}^2 + \frac{m}{2} g_{\phi\phi}\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{m}{2}\rho^2\dot{\phi}^2 = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} = const = \mu \qquad \boxed{\dot{\phi} = \frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho) = k\rho^2/2$

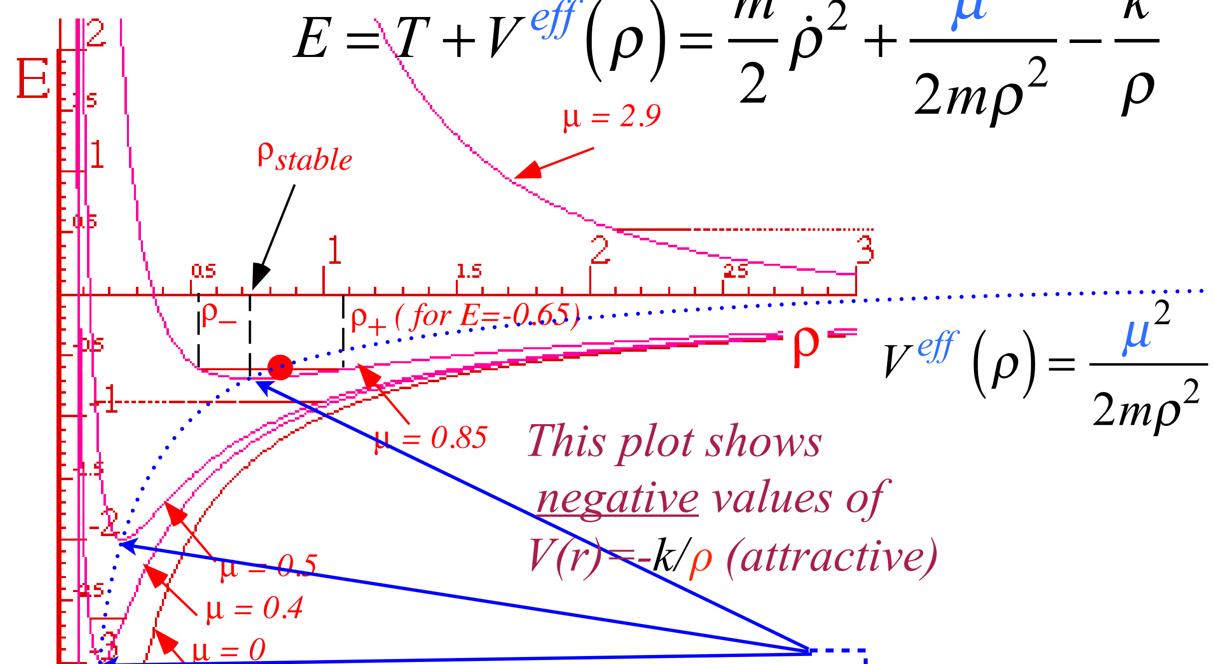
$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$



$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} + \frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho) = -k/\rho$

$$E = T + V^{eff}(\rho) = \frac{m}{2}\dot{\rho}^2 + \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$



$$V^{eff}(\rho) = \frac{\mu^2}{2m\rho^2} - \frac{k}{\rho}$$

This plot shows negative values of $V(r)=-k/\rho$ (attractive)

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0 = \frac{-\mu^2}{m\rho^3} + k\rho \quad \text{or: } \rho_{stable} = \sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}} = 0 = \frac{-\mu^2}{m\rho^3} + \frac{k}{\rho^2} \quad \text{or: } \rho_{stable} = \frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}} = \sqrt{\left.\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}} = \sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4} + k\right)} = \sqrt{\frac{1}{m}(3k+k)} = 2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}} = \sqrt{\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}} = \sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^4} - \frac{k}{\rho_{stable}^3}\right)} = \sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6} - \frac{2m^3k^4}{\mu^6}\right)} = \frac{mk^2}{\mu^3}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

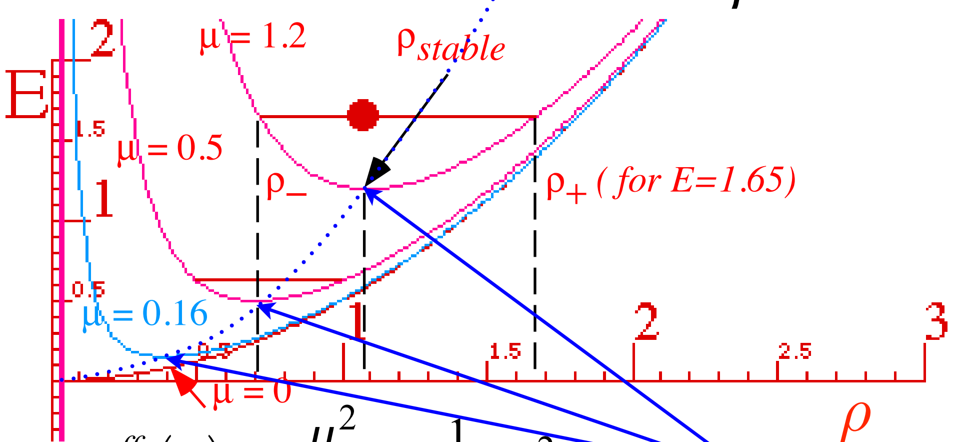
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

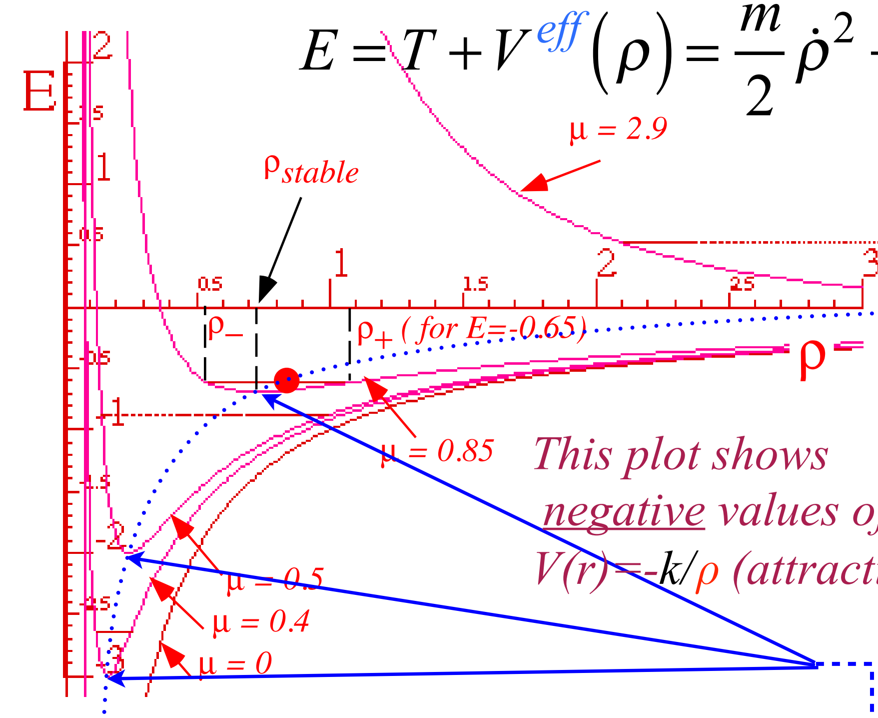
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

This plot shows *negative values of* $V(r)=-k/\rho$ *(attractive)*

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \quad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^{\ 4}}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^{\ 4}}-\frac{k}{\rho_{stable}^{\ 3}}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}^{\ 2}}=\sqrt{\dfrac{k}{m}}$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

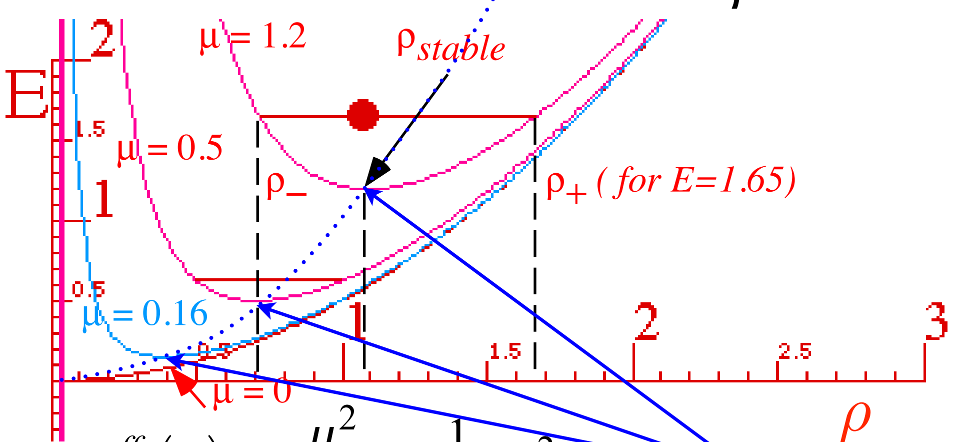
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

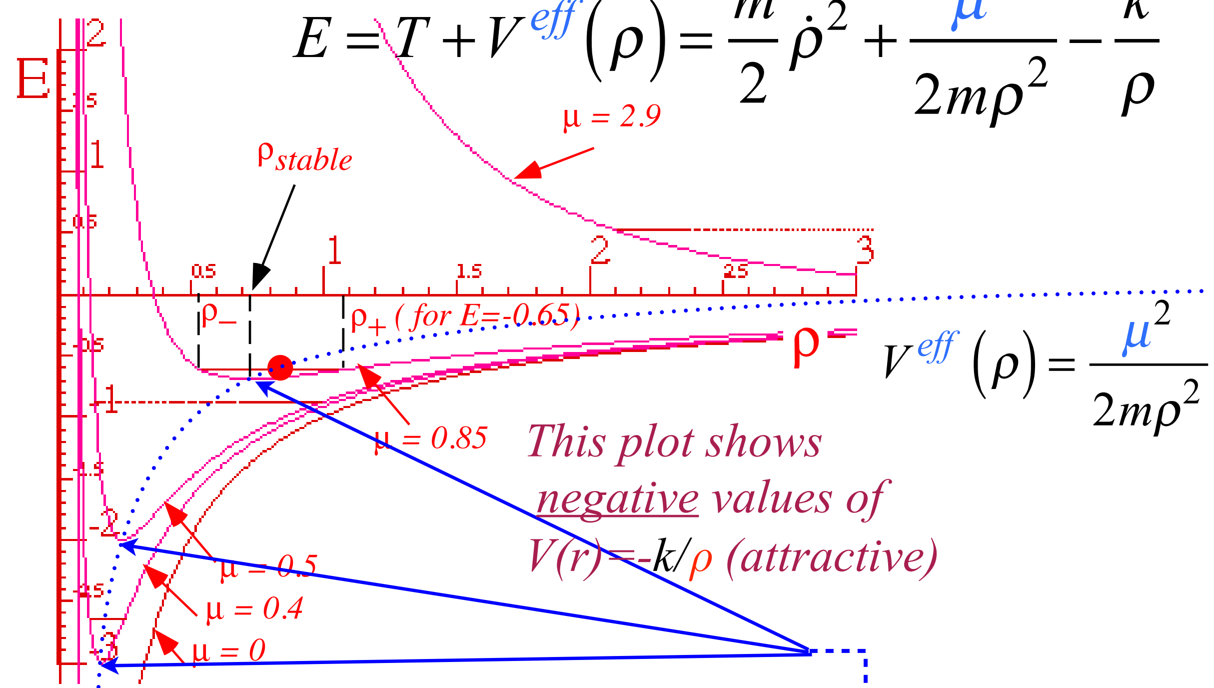
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \quad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\left.\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\frac{d^2V^{eff}}{d\rho^2}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}{}^4}-\frac{k}{\rho_{stable}{}^3}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}{}^2}=\sqrt{\dfrac{k}{m}}$

...angular orbit frequency: $\omega_\phi=\dot{\phi}=\dfrac{\mu}{m\rho_{stable}{}^2}=\dfrac{\mu}{m}\dfrac{m^2k^2}{\mu^4}=\dfrac{mk^2}{\mu^3}$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

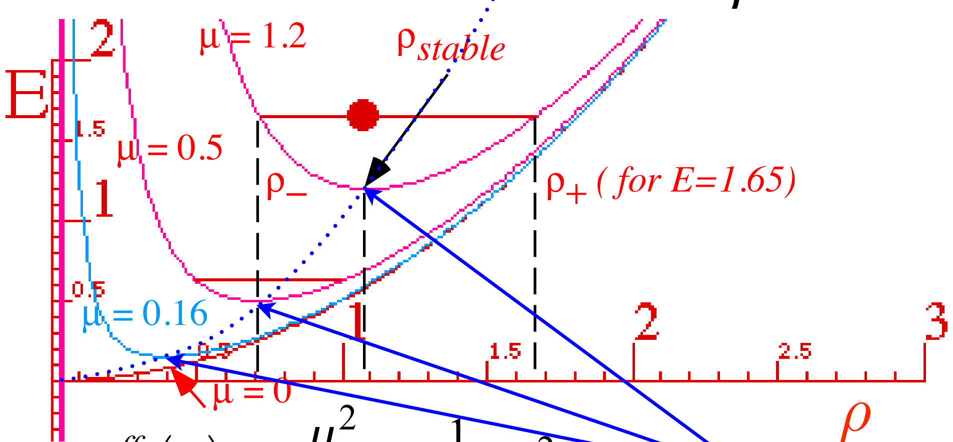
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

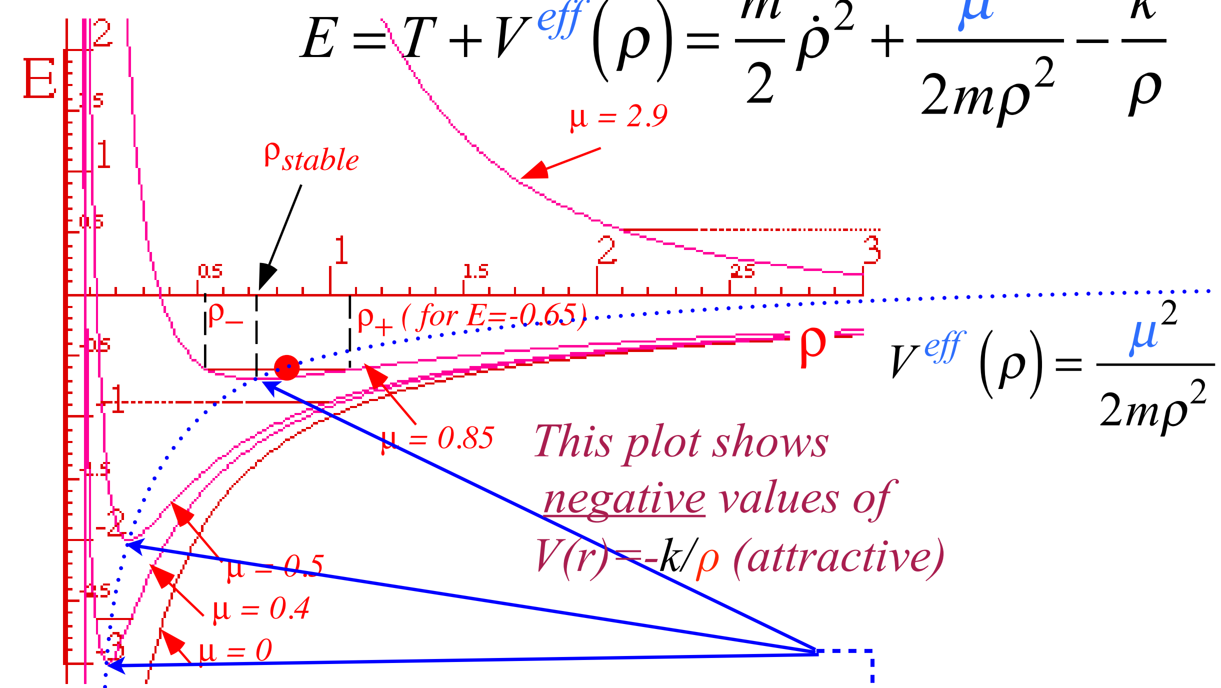
$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



$$V^{eff}(\rho)=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Stability radius ρ_{stable} for circular orbits: force or V^{eff} derivative is zero.

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+k\rho \quad \text{or: } \rho_{stable}=\sqrt{\frac{\mu}{\sqrt{mk}}}$$

$$\left.\frac{dV^{eff}(\rho)}{d\rho}\right|_{\rho_{stable}}=0=\frac{-\mu^2}{m\rho^3}+\frac{k}{\rho^2} \quad \text{or: } \rho_{stable}=\frac{\mu^2}{mk}$$

Radial oscillation frequency for orbit circle is square root of 2nd V^{eff}-derivative divided by mass m.

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|_{\rho_{stable}}}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^{\ 4}}+k\right)}=\sqrt{\frac{1}{m}(3k+k)}=2\sqrt{\frac{k}{m}}$$

$$\omega_{\rho_{stable}}=\sqrt{\frac{1}{m}\left.\frac{d^2V^{eff}}{d\rho^2}\right|}=\sqrt{\frac{1}{m}\left(\frac{3\mu^2}{m\rho_{stable}^{\ 4}}-\frac{k}{\rho_{stable}^{\ 3}}\right)}=\sqrt{\frac{1}{m}\left(\frac{3m^3k^4}{\mu^6}-\frac{2m^3k^4}{\mu^6}\right)}=\frac{mk^2}{\mu^3}$$

Compare angular orbit frequency: $\omega_\phi=\dot{\phi}=\frac{\mu}{m\rho_{stable}^{\ 2}}=\sqrt{\frac{k}{m}}$

$\omega_{\rho_{stable}}:\omega_\phi=2:1$

...angular orbit frequency: $\omega_\phi=\dot{\phi}=\frac{\mu}{m\rho_{stable}^{\ 2}}=\frac{\mu}{m}\frac{m^2k^2}{\mu^4}=\frac{mk^2}{\mu^3}$

$\omega_{\rho_{stable}}:\omega_\phi=1:1$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

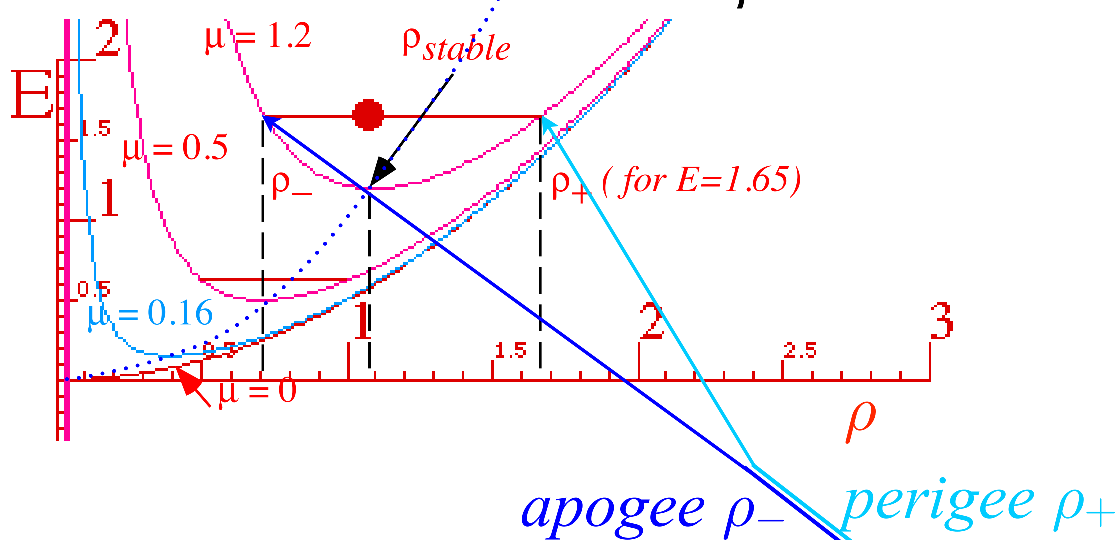
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

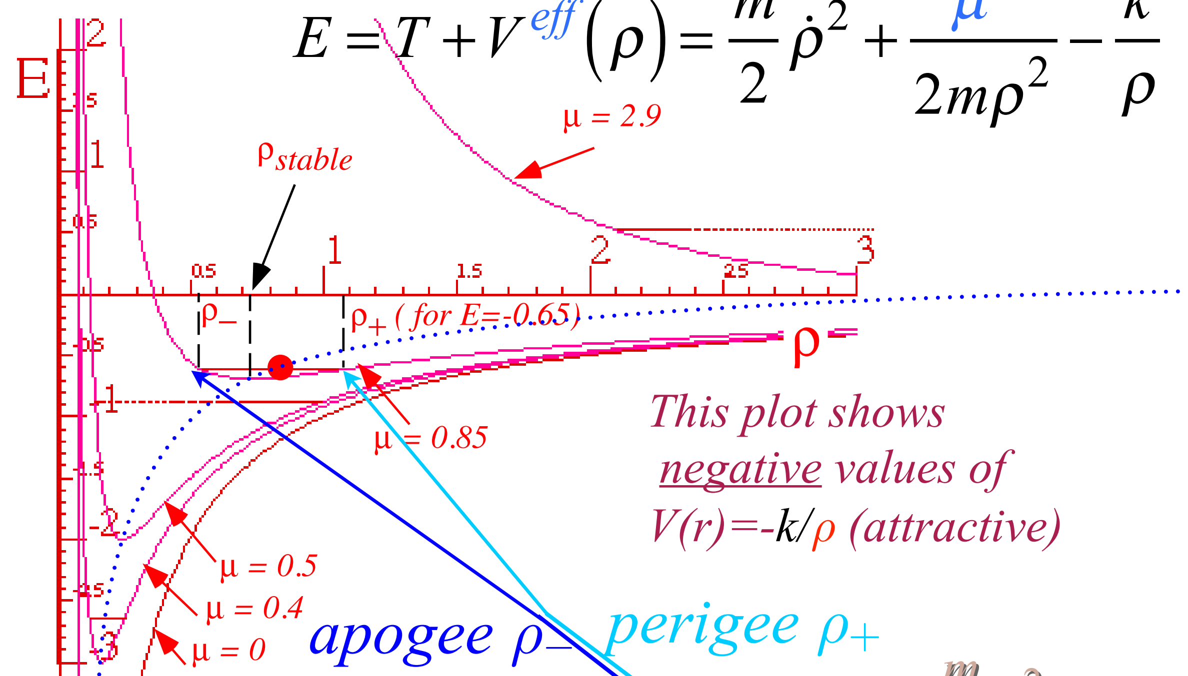
Effective potential for IHOscillator $V(\rho)=k\rho^2/2$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



Effective potential for Coulomb $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

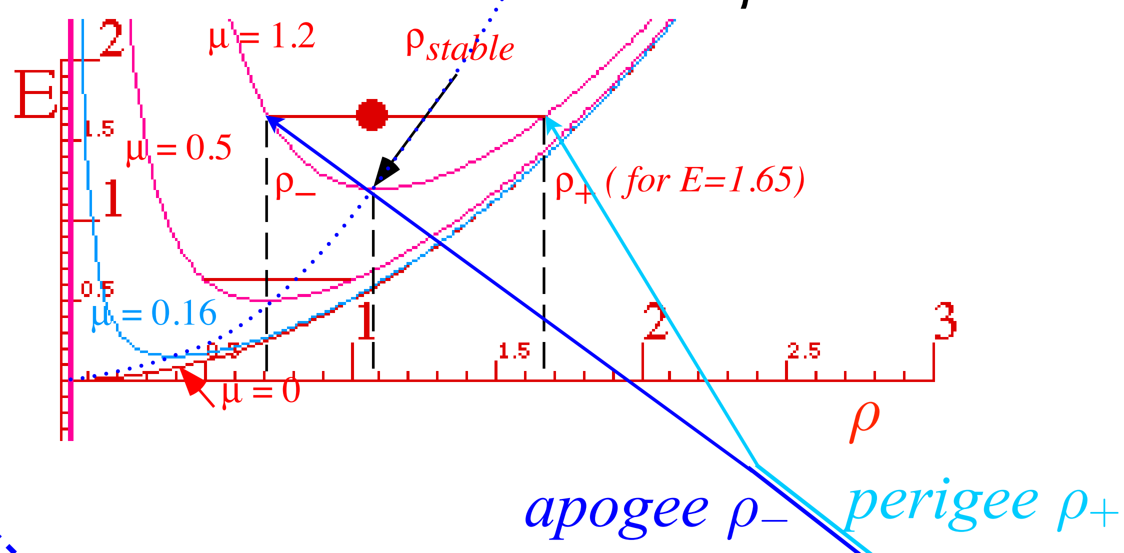
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

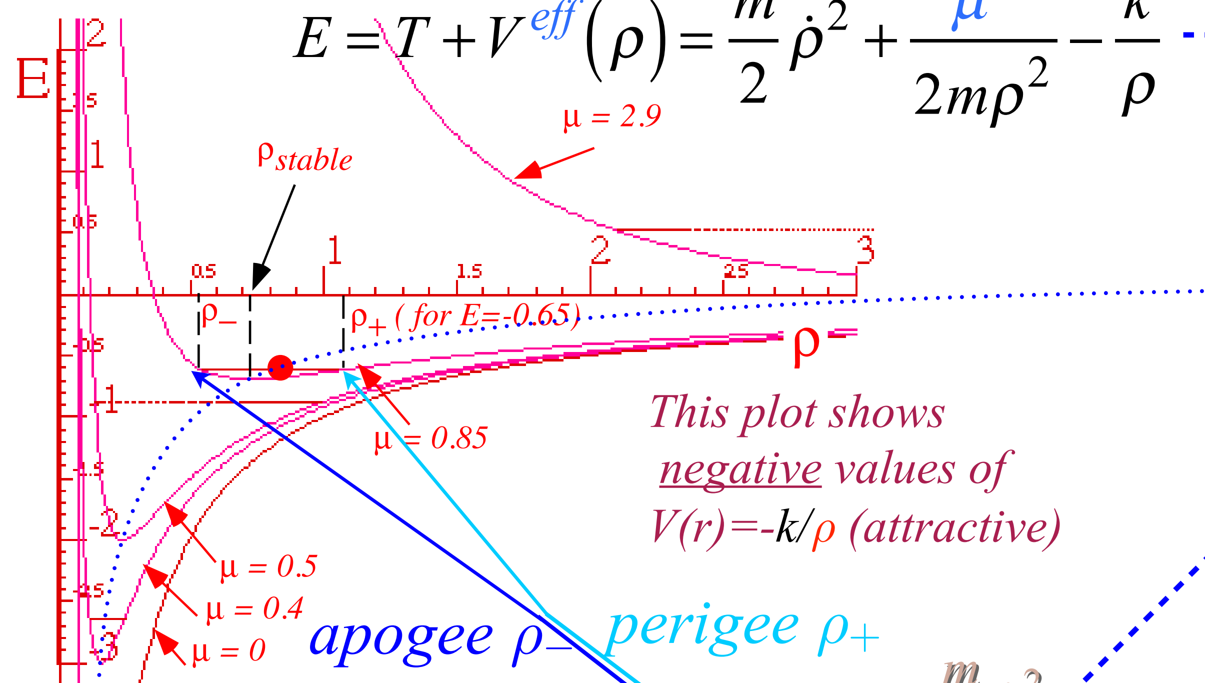
Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad\qquad 0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

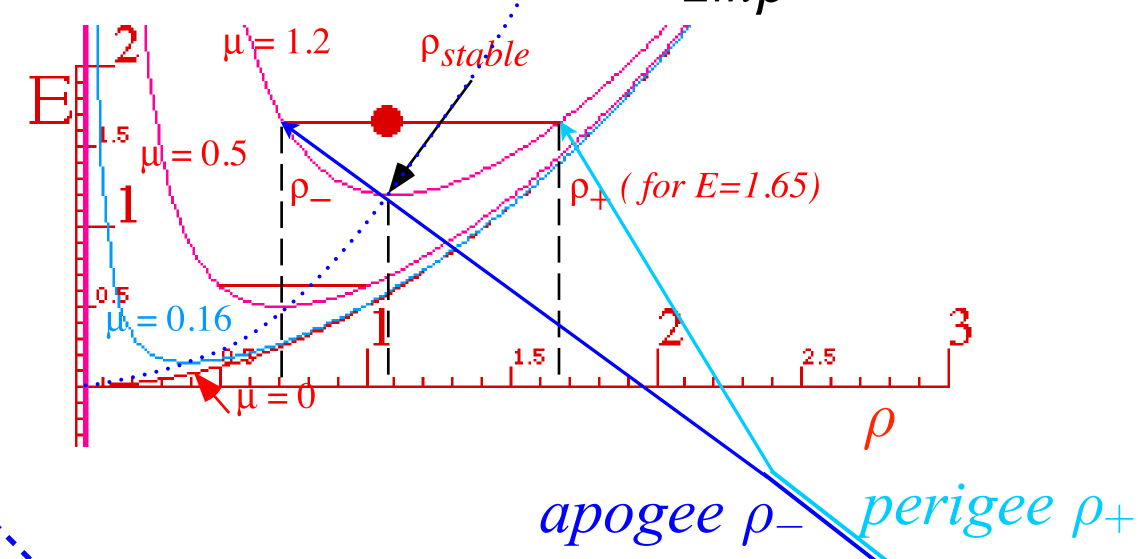
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

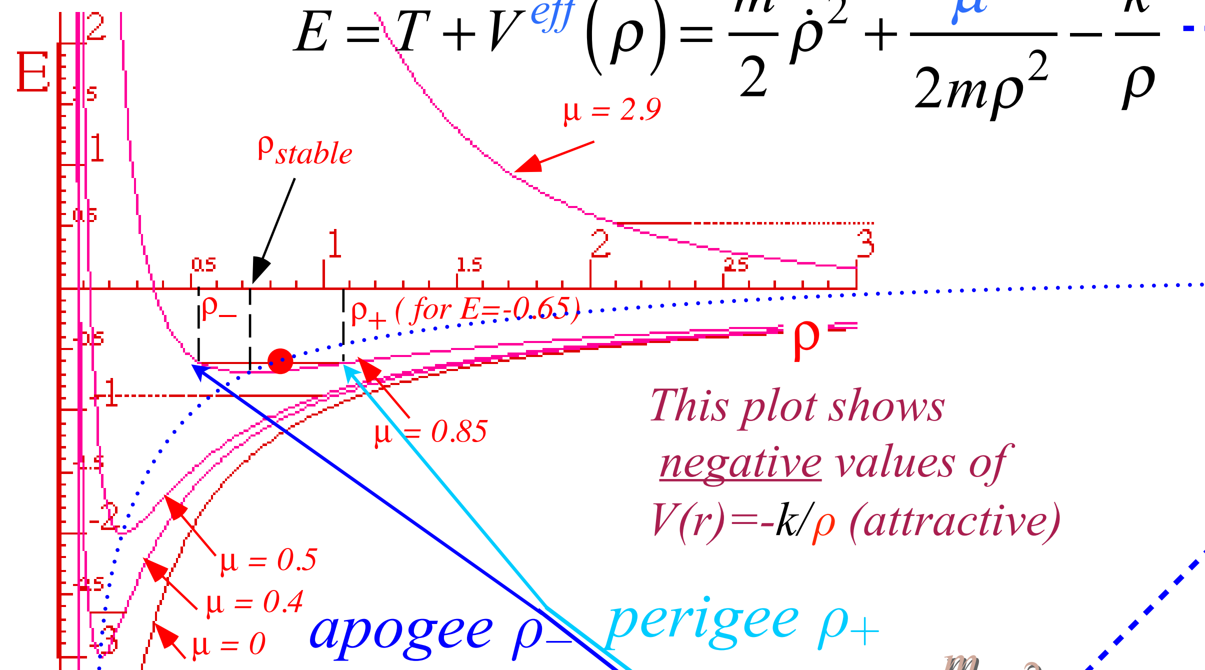
Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad 0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \quad \text{or else: } 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

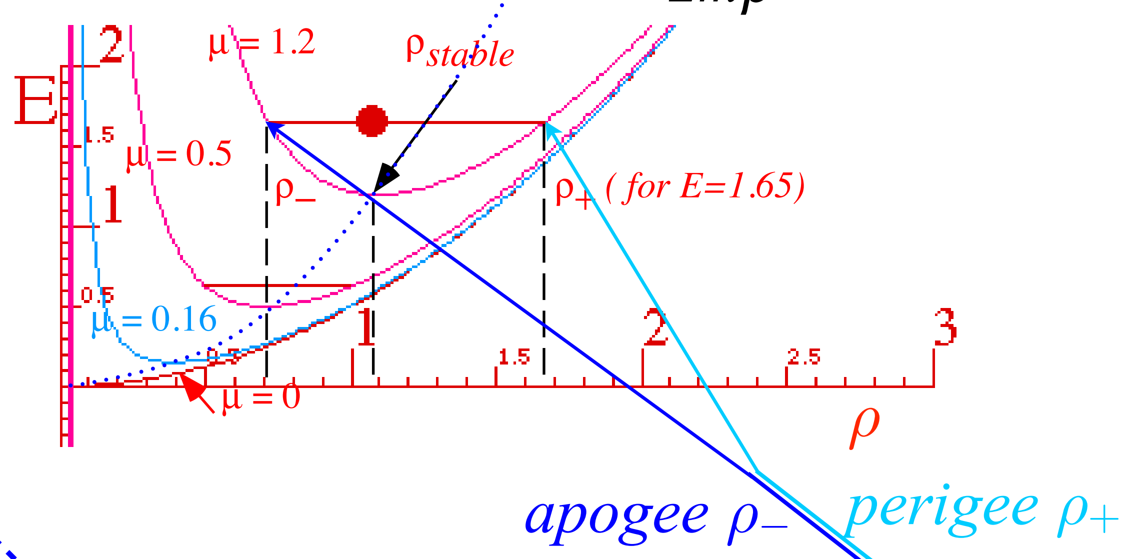
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

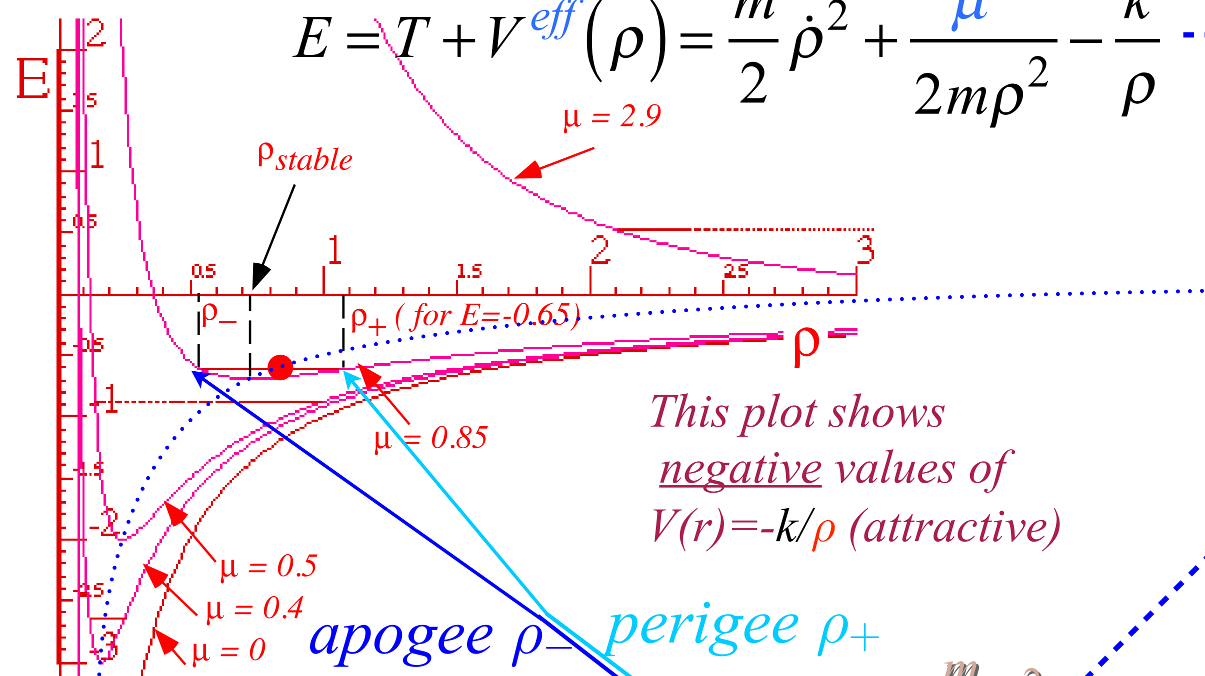
Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ | **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad\qquad 0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \quad \text{or else: } 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2}$$

$$0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0 \quad \text{or else: } 0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot\rho^2+\frac{m}{2}g_{\phi\phi}\dot\phi^2=\frac{m}{2}\dot\rho^2+\frac{m}{2}\rho^2\dot\phi^2=\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot\phi}=m\rho^2\dot\phi=const=\mu \qquad \boxed{\dot\phi=\frac{\mu}{m\rho^2}}$$

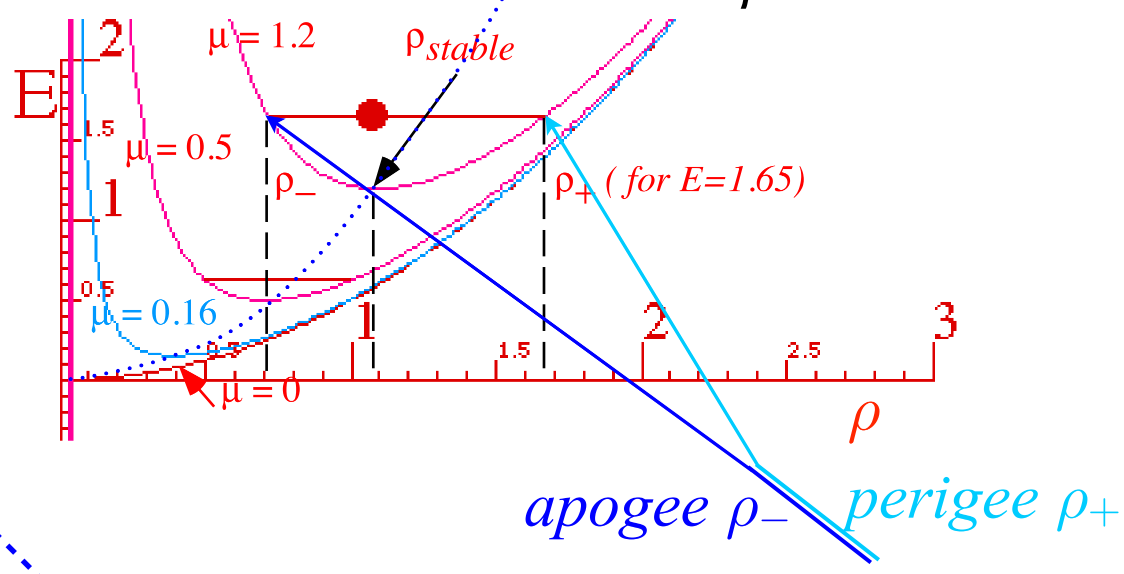
For ALL central forces

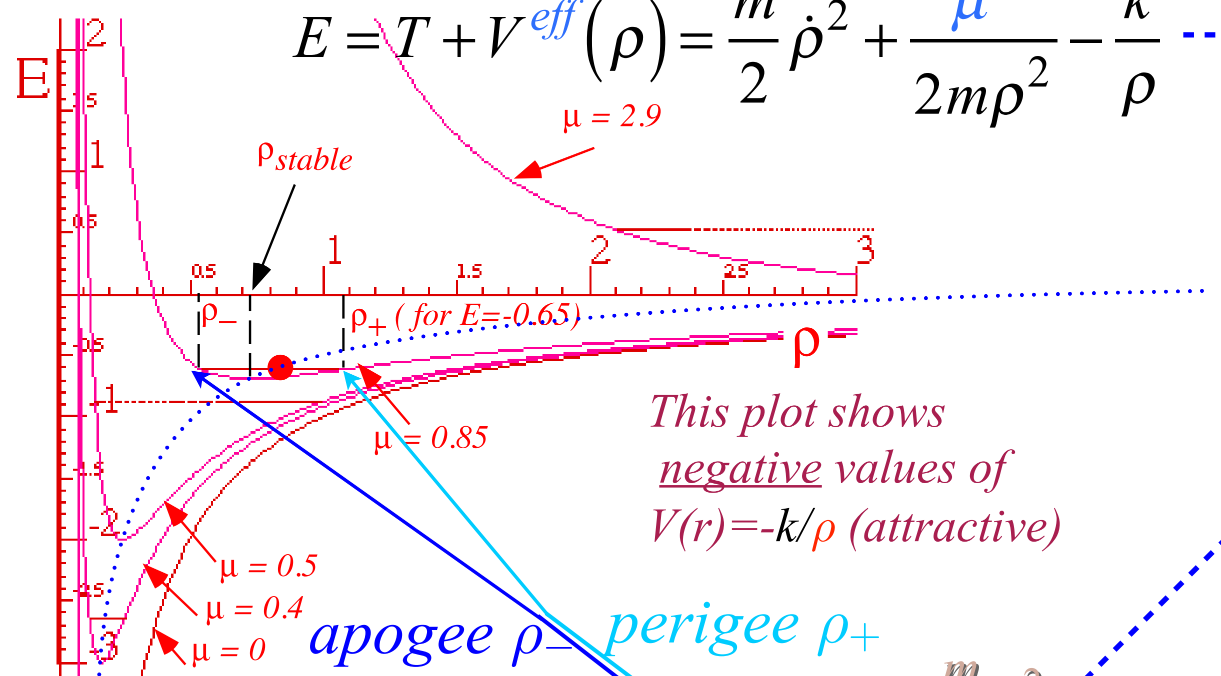
Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$





Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot\rho^2$ is zero.

$$0=-E+\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4\quad\text{or else: } 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2}$$

$${\rho_\pm}^2=\frac{E\pm\sqrt{E^2-k\mu^2/m}}{k}\quad\text{or else: }\frac{1}{{\rho_\pm}^2}=\frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$0=-E+\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0\quad\text{or else: } 0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

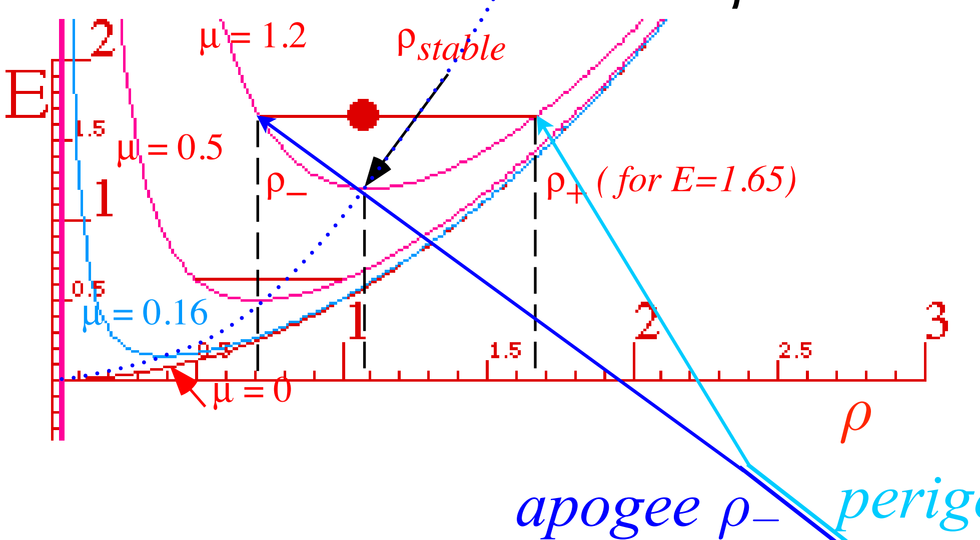
$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

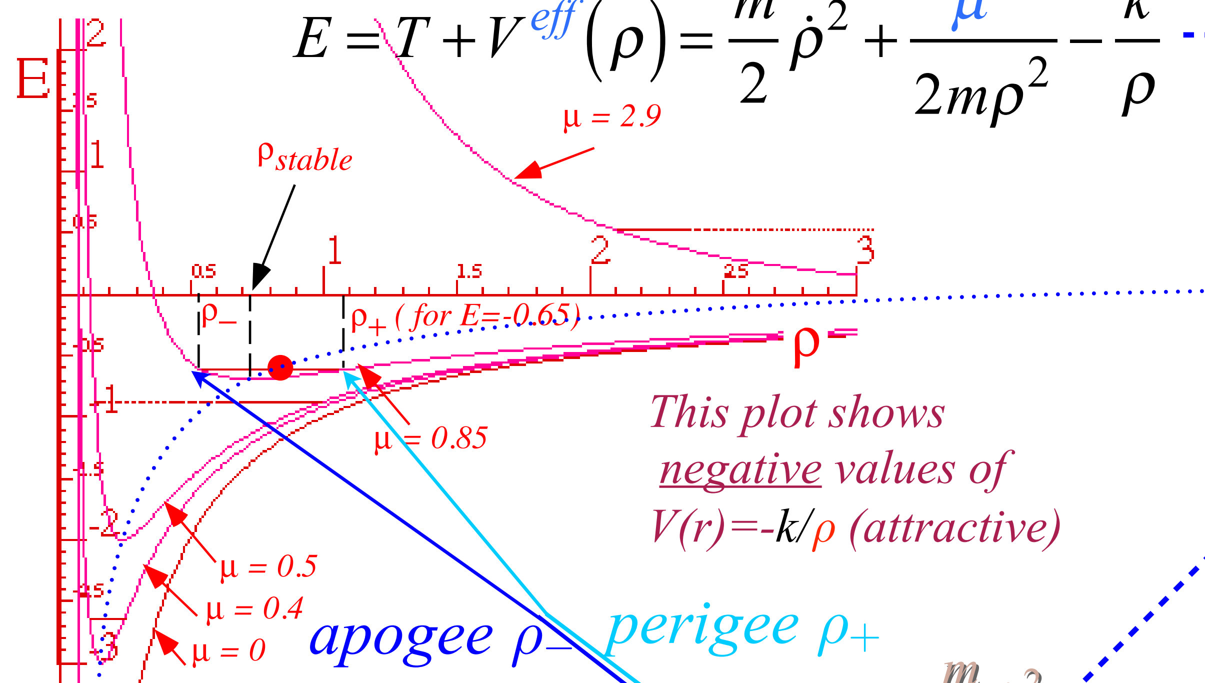
Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

Effective potential for IHOscillator $V(\rho)=k\rho^2/2$ **Effective potential for Coulomb** $V(\rho)=-k/\rho$

$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2$$



$$E=T+V^{eff}(\rho)=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$



Classical turning radii $\rho_\pm$ for bound orbits are where radial kinetic energy $\frac{m}{2}\dot{\rho}^2$ is zero.

$$0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}+\frac{1}{2}k\rho^2 \qquad\qquad 0=-E+\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}$$

$$0=\frac{\mu^2}{2m}-E\rho^2+\frac{k}{2}\rho^4 \quad\text{or else: }\quad 0=\frac{\mu^2}{2m}\frac{1}{\rho^4}-E\frac{1}{\rho^2}+\frac{k}{2}$$

$$0=\frac{-\mu^2}{2m}+k\rho+E\rho^2 0 \quad\text{or else: }\quad 0=\frac{\mu^2}{2m\rho^2}-\frac{k}{\rho}-E$$

$$\rho_\pm{}^2=\frac{E\pm\sqrt{E^2-k\mu^2/m}}{k}\quad\text{or else: }\quad\frac{1}{\rho_\pm{}^2}=\frac{E\mp\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$\rho_\pm=\frac{-k\pm\sqrt{k^2+2E\mu^2/m}}{2E}\quad\text{or else: }\quad\frac{1}{\rho_\pm}=\frac{k\pm\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

Notice mysterious similarity: E→k and k→2E

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2 \qquad \dot{\phi}=\frac{\mu}{m\rho^2} \qquad \frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$ (ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2 \qquad \dot{\phi}=\frac{\mu}{m\rho^2} \qquad \frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}} \qquad \frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}} \qquad \dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

$$d\phi=\frac{\mu\, d\rho}{m\rho^2\dot{\rho}}$$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **equations for IHOscillator** $V(\rho)=k\rho^2/2$

(ρ,ϕ) **equations for Coulomb** $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2 \qquad \dot{\phi}=\frac{\mu}{m\rho^2} \qquad \frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}} \qquad \frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}} \qquad \dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$\text{Let: } \frac{1}{\rho}=u \quad \text{so: } \begin{cases} -\dfrac{d\rho}{\rho^2}=du \\ d\rho=-\dfrac{du}{u^2} \end{cases}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu\qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2\qquad \dot{\phi}=\frac{\mu}{m\rho^2}\qquad \frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}\qquad \frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}\qquad \dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}\qquad d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$\text{Let: }\frac{1}{\rho}=u\quad\text{so: }\begin{cases}-\dfrac{d\rho}{\rho^2}=du\\[2ex] d\rho=-\dfrac{du}{u^2}\end{cases}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}\qquad d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}}$$

$$\dot{\phi}=\frac{\mu}{m\rho^2}$$

$$\frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}}$$

Let: $\dfrac{1}{\rho}=u$ so: $\begin{cases}-\dfrac{d\rho}{\rho^2}=du\\[2ex] d\rho=-\dfrac{du}{u^2}\end{cases}$

Let: $x=u^2=\dfrac{1}{\rho^2}$ so: $\begin{cases}dx=2u\,du\\[2ex] du=\dfrac{dx}{2\sqrt{x}}\end{cases}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$ | **(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$**

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2 \qquad \dot{\phi}=\frac{\mu}{m\rho^2} \qquad \frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}} \qquad \frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}} \qquad \dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}} \qquad d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

Let: $\frac{1}{\rho}=u$ so: $\begin{cases}-\frac{d\rho}{\rho^2}=du\\ d\rho=-\frac{du}{u^2}\end{cases}$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}} \qquad d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

Let: $x=u^2=\frac{1}{\rho^2}$ so: $\begin{cases}dx=2u\,du\\ du=\frac{dx}{2\sqrt{x}}\end{cases}$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{k}{mx}}} \qquad d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{2k\sqrt{x}}{m}}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$ | (ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$\frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-\frac{1}{2}k\rho^2 \qquad \dot{\phi}=\frac{\mu}{m\rho^2} \qquad \frac{m}{2}\dot{\rho}^2=E-\frac{\mu^2}{2m\rho^2}-k/\rho$$

$$\dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}} \qquad \frac{d\phi}{d\rho}=\frac{\dot{\phi}}{\dot{\rho}}=\frac{\mu}{m\rho^2\dot{\rho}} \qquad \dot{\rho}=\frac{d\rho}{dt}=\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}$$

$$d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{k\rho^2}{m}}} \qquad d\phi=\frac{\mu}{m}\frac{d\rho}{\rho^2\dot{\rho}}=\frac{\mu}{m}\frac{d\rho}{\rho^2\sqrt{\frac{2E}{m}-\frac{\mu^2}{m^2\rho^2}-\frac{2k}{m\rho}}}$$

$$\text{Let: } \frac{1}{\rho}=u \quad \text{so: } \begin{cases} -\dfrac{d\rho}{\rho^2}=du \\ d\rho=-\dfrac{du}{u^2}\end{cases}$$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{k}{mu^2}}} \qquad d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{\frac{2E}{m}-\frac{\mu^2u^2}{m^2}-\frac{2ku}{m}}}$$

$$\text{Let: } x=u^2=\frac{1}{\rho^2} \quad \text{so: } \begin{cases} dx=2u\,du \\ du=\dfrac{dx}{2\sqrt{x}}\end{cases}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{k}{mx}}} \qquad d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{x}\sqrt{\frac{2E}{m}-\frac{\mu^2x}{m^2}-\frac{2k\sqrt{x}}{m}}}$$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}} \qquad d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

(A mystery similarity appears)

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-\left(z-z_+\right)\left(x-x_-\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\left(z_+-z\right)\left(z-z_-\right)}}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

$$z_\pm=\alpha\pm\beta \text{ , where: } \alpha=\frac{z_++z_-}{2}=\frac{-B}{2A} \text{ , and: } \beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

Solution based on quadratic roots of $Az^2+Bz+C=0$.

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot\rho^2+\frac{m}{2}g_{\phi\phi}\dot\phi^2=\frac{m}{2}\dot\rho^2+\frac{m}{2}\rho^2\dot\phi^2=\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot\phi}=m\rho^2\dot\phi=const=\mu \qquad \boxed{\dot\phi=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

$$z_\pm=\alpha\pm\beta \text{ , where: } \alpha=\frac{z_++z_-}{2}=\frac{-B}{2A} \text{ , and: } \beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

Solution based on quadratic roots of $Az^2+Bz+C=0$.

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

$$z_\pm=\alpha\pm\beta\text{ , where: }\alpha=\frac{z_++z_-}{2}=\frac{-B}{2A}\text{ , and: }\beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

Solution based on quadratic roots of $Az^2+Bz+C=0$. Variable z may be ρ or $u=1/\rho$ or ρ^2 or $x=1/\rho^2$...

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}\qquad z(\phi)=\frac{\sqrt{B^2-4AC}}{2A}\sin\frac{\sqrt{A}}{D}\phi-\frac{B}{2A}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ $\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho)=k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

$$z_\pm=\alpha\pm\beta\text{ , where: }\alpha=\frac{z_++z_-}{2}=\frac{-B}{2A}\text{ , and: }\beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

Solution based on quadratic roots of $Az^2+Bz+C=0$. Variable z may be ρ or $u=1/\rho$ or ρ^2 or $x=1/\rho^2$...

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

$$z(\phi)=\frac{\sqrt{B^2-4AC}}{2A}\sin\frac{\sqrt{A}}{D}\phi-\frac{B}{2A}$$

$$z(\phi)=\beta\cdot\sin\frac{\sqrt{A}}{D}\phi-\alpha$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V{=}V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E{=}T{+}V^{eff}(\rho){=}T{+}\dfrac{\mu^2}{2m\rho^2}{+}V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) equations for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) equations for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Some radial $V(\rho){=}k\rho^n$ repeatedly enjoy the integral $\phi(z)$ below. *(Introduced briefly in Unit 3)*

$$\phi(z)=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(x-x_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-{=}\alpha{-}\beta$, *apogee* $z_+{=}\alpha{+}\beta$). Solve integral $\phi(z)$ for $z(\phi)$.

$$z_\pm=\alpha\pm\beta\text{ , where: }\alpha=\frac{z_++z_-}{2}=\frac{-B}{2A}\text{ , and: }\beta=\frac{z_+-z_-}{2}=\frac{\sqrt{B^2-4AC}}{2A}$$

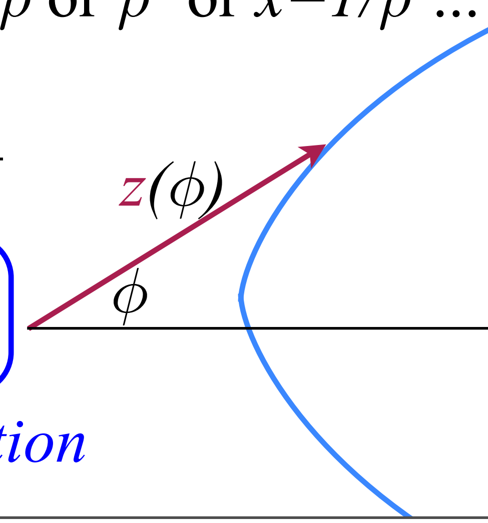
Solution based on quadratic roots of $Az^2{+}Bz{+}C{=}0$. Variable z may be ρ or $u{=}1/\rho$ or ρ^2 or $x{=}1/\rho^2$...

$$\frac{\sqrt{A}}{D}\phi(z)=\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\sin^{-1}\frac{z-\alpha}{\beta}$$

$$z(\phi)=\frac{\sqrt{B^2-4AC}}{2A}\sin\frac{\sqrt{A}}{D}\phi-\frac{B}{2A}$$

$$\boxed{z(\phi)=\beta\cdot\sin\frac{\sqrt{A}}{D}\phi-\alpha}$$

radial-polar-coordinate orbit function



Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

➔ *Detailed orbital functions*

Relating orbital energy-momentum to conic-sectional orbital geometry

Kepler equation of time and phase geometry

(A mystery similarity appears)

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}\qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}\qquad \text{Let: } u=\frac{1}{\rho}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}\qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$\text{Let: } u=\frac{1}{\rho}\qquad d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}$$

where: $p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu$ $\quad\boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) **orbits for IHOscillator** $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\frac{1}{\rho^2}$

(ρ,ϕ) **orbits for Coulomb** $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\frac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\quad \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad\text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}$$

Let: $x=u^2=\dfrac{1}{\rho^2}$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

Let: $u=\dfrac{1}{\rho}$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\ \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \qquad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}\qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}\qquad \text{Let: } u=\frac{1}{\rho}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A},\quad \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

IHOscillator:

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=\boxed{-\frac{\mu}{2m}}$$

$$\alpha=\frac{E}{\mu^2/m}$$

Coulomb:

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=\boxed{-\frac{\mu}{m}}$$

$$\alpha=\frac{-k}{\mu^2/m},$$

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot{\rho}^2+\frac{m}{2}g_{\phi\phi}\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{m}{2}\rho^2\dot{\phi}^2=\frac{m}{2}\dot{\rho}^2+\frac{\mu^2}{2m\rho^2} \quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot{\phi}}=m\rho^2\dot{\phi}=const=\mu \quad \boxed{\dot{\phi}=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\frac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}} \qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}} \qquad \text{Let: } u=\frac{1}{\rho}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\quad \beta=\frac{\sqrt{B^2-4AC}}{2A} \qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

IHOscillator:

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

Coulomb:

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

$$\alpha=\frac{-k}{\mu^2/m}\ ,$$

$$\beta=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

Algebra details on following page 48

Orbits in Isotropic Oscillator and Coulomb Potentials

Kinetic energy T in polar coordinates: Orbital momentum p_ϕ conserved for isotropic potential $V=V(\rho)$

$$T=\frac{m}{2}g_{\rho\rho}\dot\rho^2+\frac{m}{2}g_{\phi\phi}\dot\phi^2=\frac{m}{2}\dot\rho^2+\frac{m}{2}\rho^2\dot\phi^2=\frac{m}{2}\dot\rho^2+\frac{\mu^2}{2m\rho^2}\quad \text{where: } p_\phi=\frac{\partial T}{\partial\dot\phi}=m\rho^2\dot\phi=const=\mu \quad \boxed{\dot\phi=\frac{\mu}{m\rho^2}}$$

For ALL central forces

Total energy $E=T+V^{eff}(\rho)=T+\dfrac{\mu^2}{2m\rho^2}+V(\rho)$ conserved for constant parameters m and k of T and $V(\rho)$.

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

$$d\phi=\frac{\mu}{m}\frac{-dx}{2\sqrt{-\left(\frac{\mu^2}{m^2}x^2-\frac{2E}{m}x+\frac{k}{m}\right)}}\qquad \text{Let: } x=u^2=\frac{1}{\rho^2}$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$

$$\text{Let: } u=\frac{1}{\rho}\qquad d\phi=\frac{\mu}{m}\frac{-du}{\sqrt{-\left(\frac{\mu^2}{m^2}u^2+\frac{2k}{m}u-\frac{2E}{m}\right)}}$$

$$\phi=D\int\frac{dz}{\sqrt{-\left(Az^2+Bz+C\right)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{-(z-z_+)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{(z_+-z)(z-z_-)}}=\frac{D}{\sqrt{A}}\int\frac{dz}{\sqrt{\beta^2-(z-\alpha)^2}}=\frac{D}{\sqrt{A}}\sin^{-1}\frac{z-\alpha}{\beta}$$

Roots $z_\pm$ are *classical turning points* (*perigee* $z_-=\alpha-\beta$, *apogee* $z_+=\alpha+\beta$).

$$\alpha=\frac{-B}{2A}\ ,\quad \beta=\frac{\sqrt{B^2-4AC}}{2A}\qquad z=\alpha+\beta\sin\frac{\phi\sqrt{A}}{D}$$

IHOscillator:

$$A=\frac{\mu^2}{m^2}\quad B=-\frac{2E}{m}\quad C=\frac{k}{m}\quad D=-\frac{\mu}{2m}$$

$$\alpha=\frac{E}{\mu^2/m}$$

$$\beta=\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}$$

$$\boxed{x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)}$$

Coulomb:

$$A=\frac{\mu^2}{m^2}\quad B=\frac{2k}{m}\quad C=-\frac{2E}{m}\quad D=-\frac{\mu}{m}$$

$$\alpha=\frac{-k}{\mu^2/m}\ ,$$

$$\beta=\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}$$

$$\boxed{u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)}$$

Algebra details on following page 53

Algebra details and checks

$$\alpha = \frac{-B}{2A}\ ,\ \beta = \frac{\sqrt{B^2 - 4AC}}{2A}$$

(ρ,ϕ) orbits for IHOscillator $V(\rho) = k\rho^2/2$

$$A = \frac{\mu^2}{m^2} \quad B = -\frac{2E}{m} \quad C = \frac{k}{m} \quad D = -\frac{\mu}{2m}$$

$$\alpha = \frac{\frac{2E}{m}}{2\frac{\mu^2}{m^2}} = \frac{E}{\mu^2/m} = \frac{z_+ + z_-}{2}$$

$$\beta = \frac{\sqrt{\left(\frac{2E}{m}\right)^2 - 4\frac{\mu^2}{m^2}\frac{k}{m}}}{2\frac{\mu^2}{m^2}}$$

$$\beta = \frac{\sqrt{\left(\frac{2Em}{m\cdot m}\right)^2 - 4\frac{\mu^2}{m^2}\frac{km}{m^2}}}{2\frac{\mu^2}{m^2}} = \frac{\sqrt{E^2 - k\mu^2/m}}{\mu^2/m} = \frac{z_+ - z_-}{2}$$

(ρ,ϕ) equations for Coulomb $V(\rho) = -k/\rho$

$$A = \frac{\mu^2}{m^2} \quad B = \frac{2k}{m} \quad C = -\frac{2E}{m} \quad D = -\frac{\mu}{m}$$

$$\alpha = \frac{-\frac{2k}{m}}{2\frac{\mu^2}{m^2}} = \frac{-k}{\mu^2/m} = \frac{z_+ + z_-}{2}$$

$$\beta = \frac{\sqrt{\left(\frac{2k}{m}\right)^2 + 4\frac{\mu^2}{m^2}\frac{2E}{m}}}{2\frac{\mu^2}{m^2}} = \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m} = \frac{z_+ - z_-}{2}$$

Checking that roots $z_\pm$ are *classical turning points* (*perigee* $z_- = \alpha - \beta$, *apogee* $z_+ = \alpha + \beta$) derived before.

$$\rho_\pm^{\ 2} = \frac{E \pm \sqrt{E^2 - k\mu^2/m}}{k} \quad \text{or else:} \quad \frac{1}{\rho_\pm^{\ 2}} = \frac{E \mp \sqrt{E^2 - k\mu^2/m}}{\mu^2/m}$$

$$\rho_\pm = \frac{-k \pm \sqrt{k^2 + 2E\mu^2/m}}{2E} \quad \text{or else:} \quad \frac{1}{\rho_\pm} = \frac{k \pm \sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}$$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

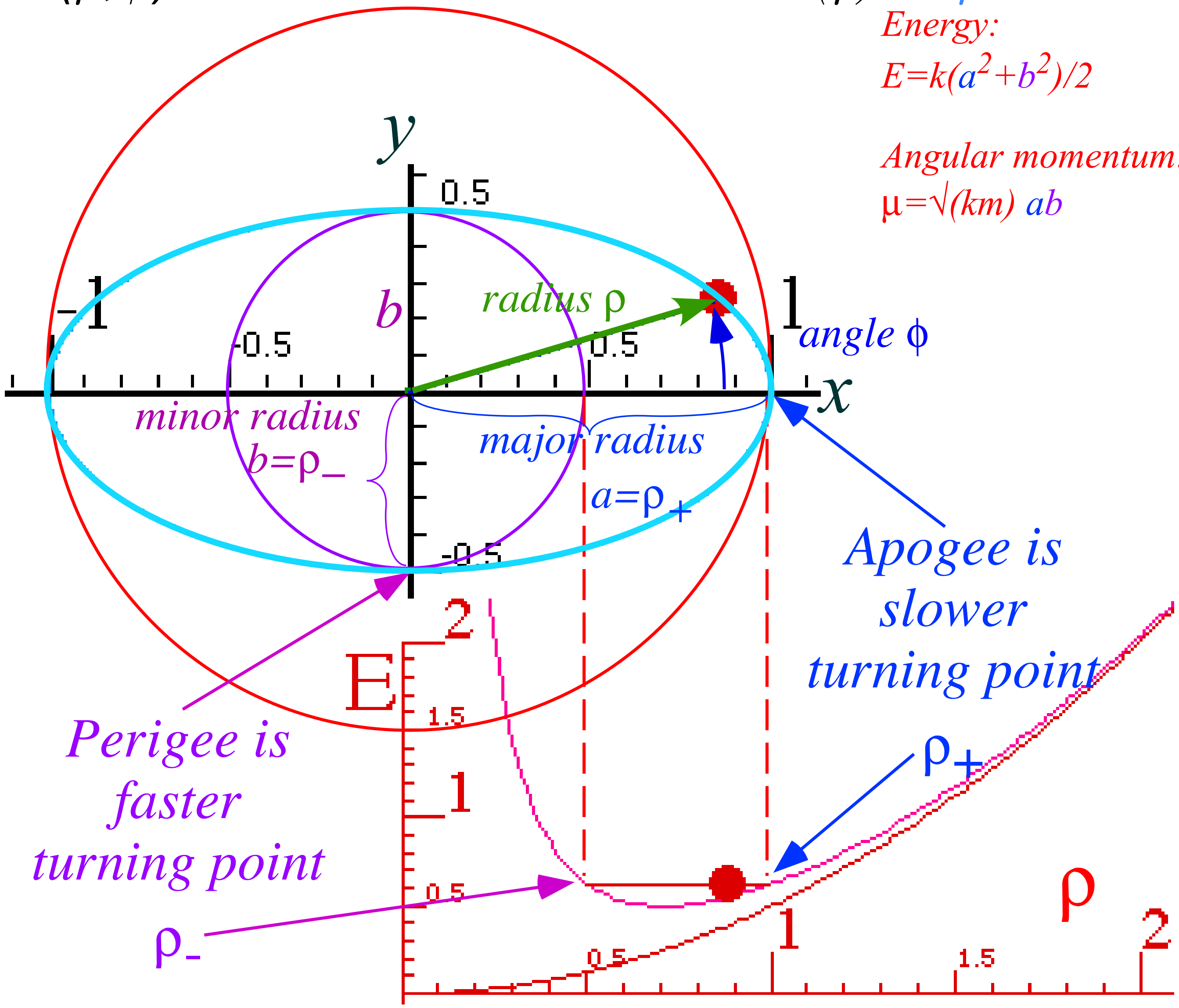
Detailed orbital functions

➤ *Relating orbital energy-momentum to conic-sectional orbital geometry*

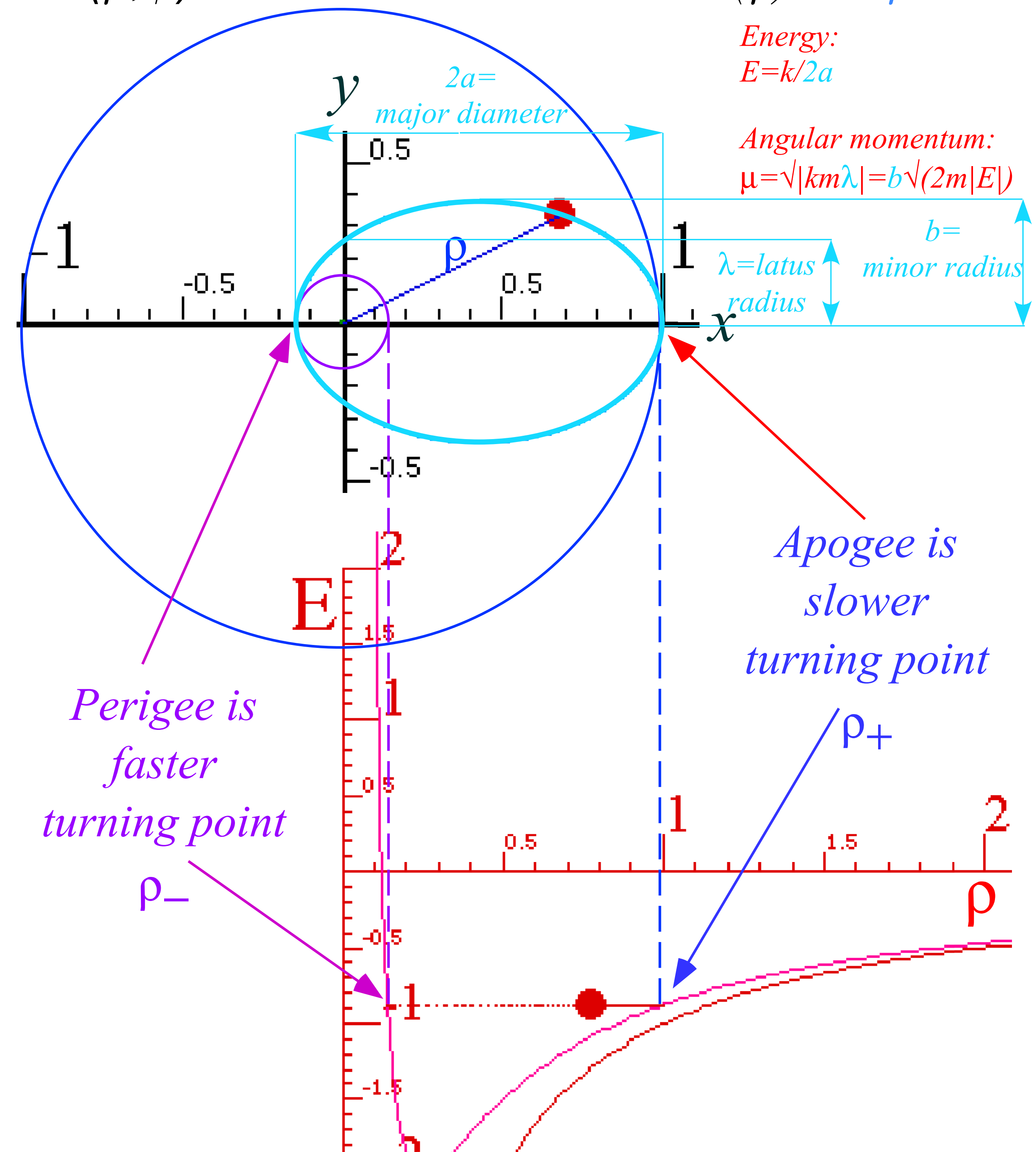
Kepler equation of time and phase geometry

Orbits in Isotropic Oscillator and Coulomb Potentials

(ρ,ϕ) orbits for IHOscillator $V(\rho) = k\rho^2/2$



(ρ,ϕ) orbits for Coulomb $V(\rho) = -k/\rho$



Just derived equation of IHO orbit ellipse

$$x = \frac{1}{\rho^2} = \frac{E}{\mu^2/m} + \frac{\sqrt{E^2 - k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

$$\frac{1}{\rho^2} = +\frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+\frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\cos(2\phi)$$

One of many equations of center-centered ellipse

Just derived equation of Coulomb orbit ellipse

$$u = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

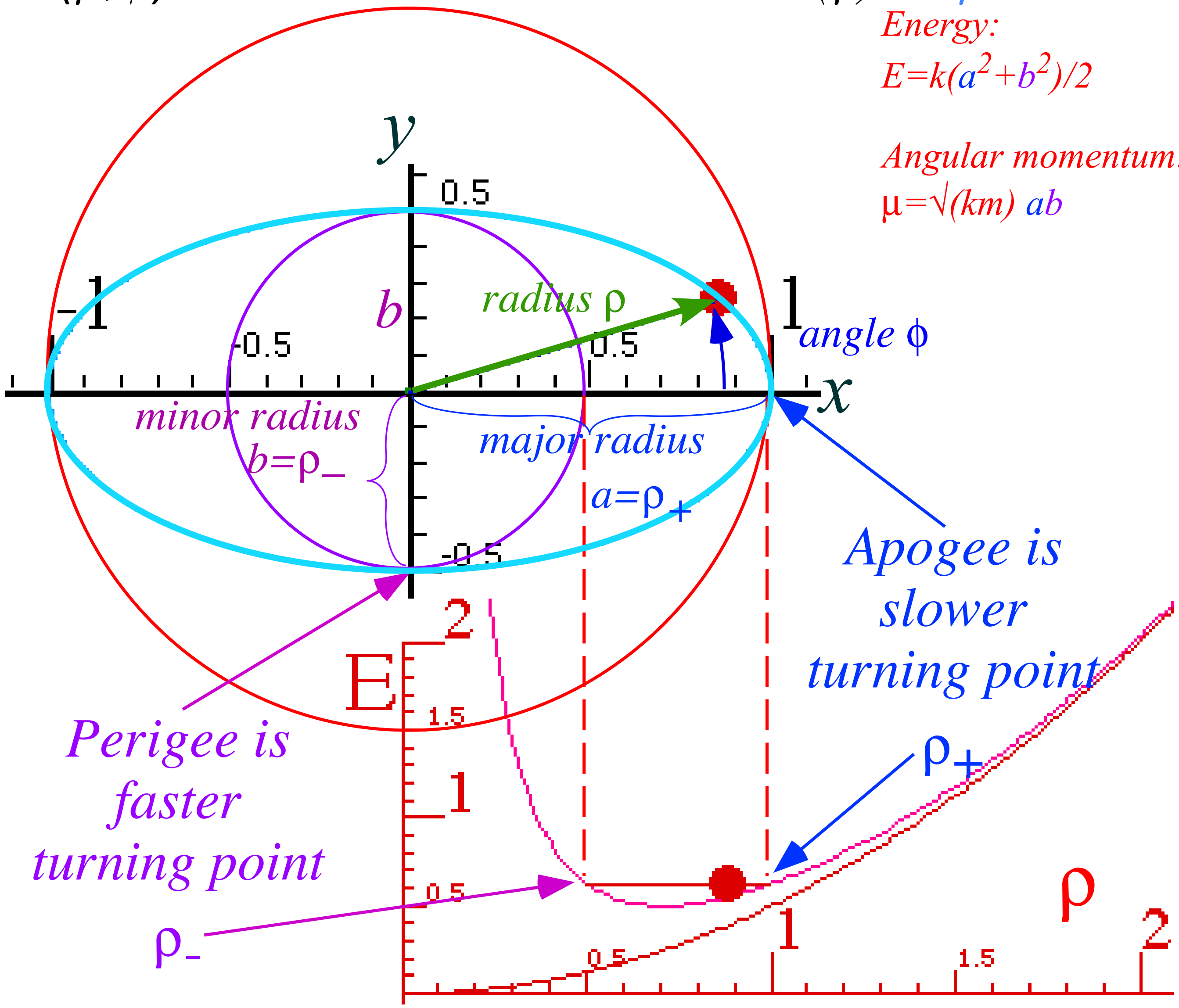
(to be discussed shortly)

$$\frac{1}{\rho} = \frac{a}{b^2} - \frac{\sqrt{a^2-b^2}}{b^2}\cos\phi$$

One of many equations of focus-centered ellipse

Orbits in Isotropic Oscillator and Coulomb Potentials

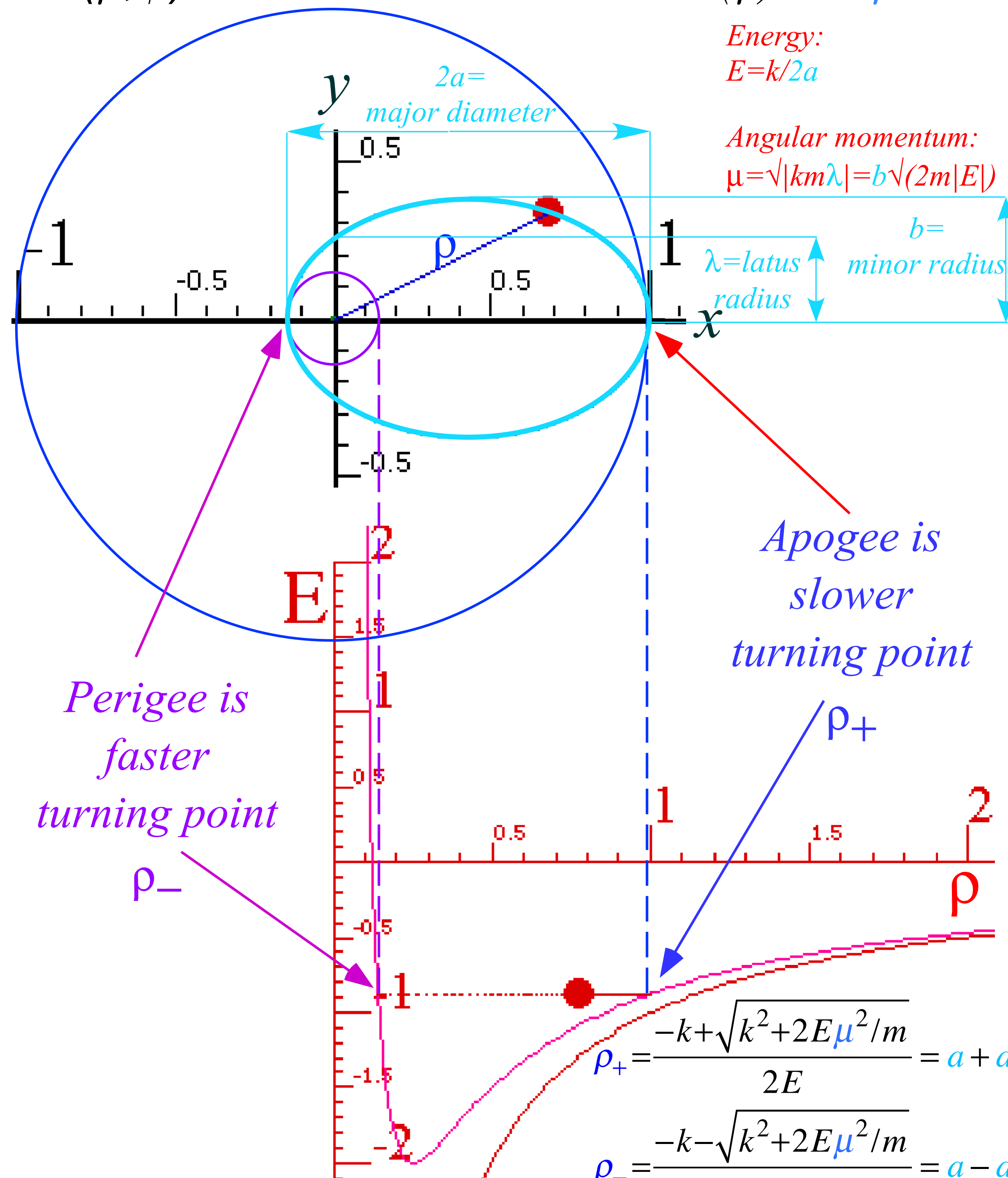
(ρ,ϕ) orbits for IHOscillator $V(\rho) = k\rho^2/2$



$$\rho_+{}^2 = \frac{E+\sqrt{E^2 - k\mu^2/m}}{k} = a^2$$

$$\rho_-{}^2 = \frac{E-\sqrt{E^2 - k\mu^2/m}}{k} = b^2$$

(ρ,ϕ) orbits for Coulomb $V(\rho) = -k/\rho$



$$\rho_+ = \frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E} = a + a\varepsilon$$

$$\rho_- = \frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E} = a - a\varepsilon$$

(to be discussed first: turning point relations)

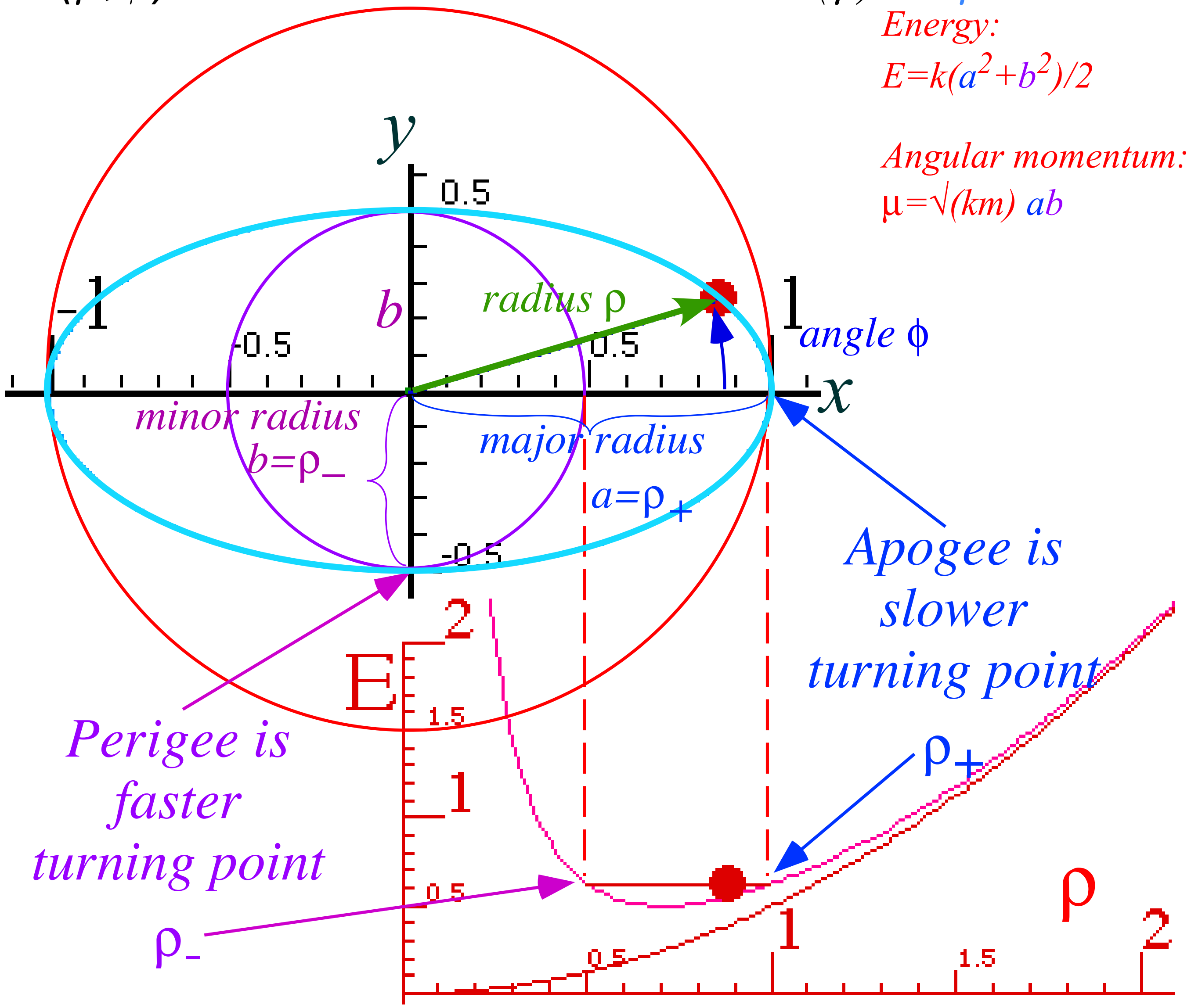
Just derived equation of IHO orbit ellipse

$$x = \frac{1}{\rho^2} = \frac{E}{\mu^2/m} + \frac{\sqrt{E^2 - k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

$$u = \frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

(ρ,ϕ) orbits for IHOscillator $V(\rho)=k\rho^2/2$

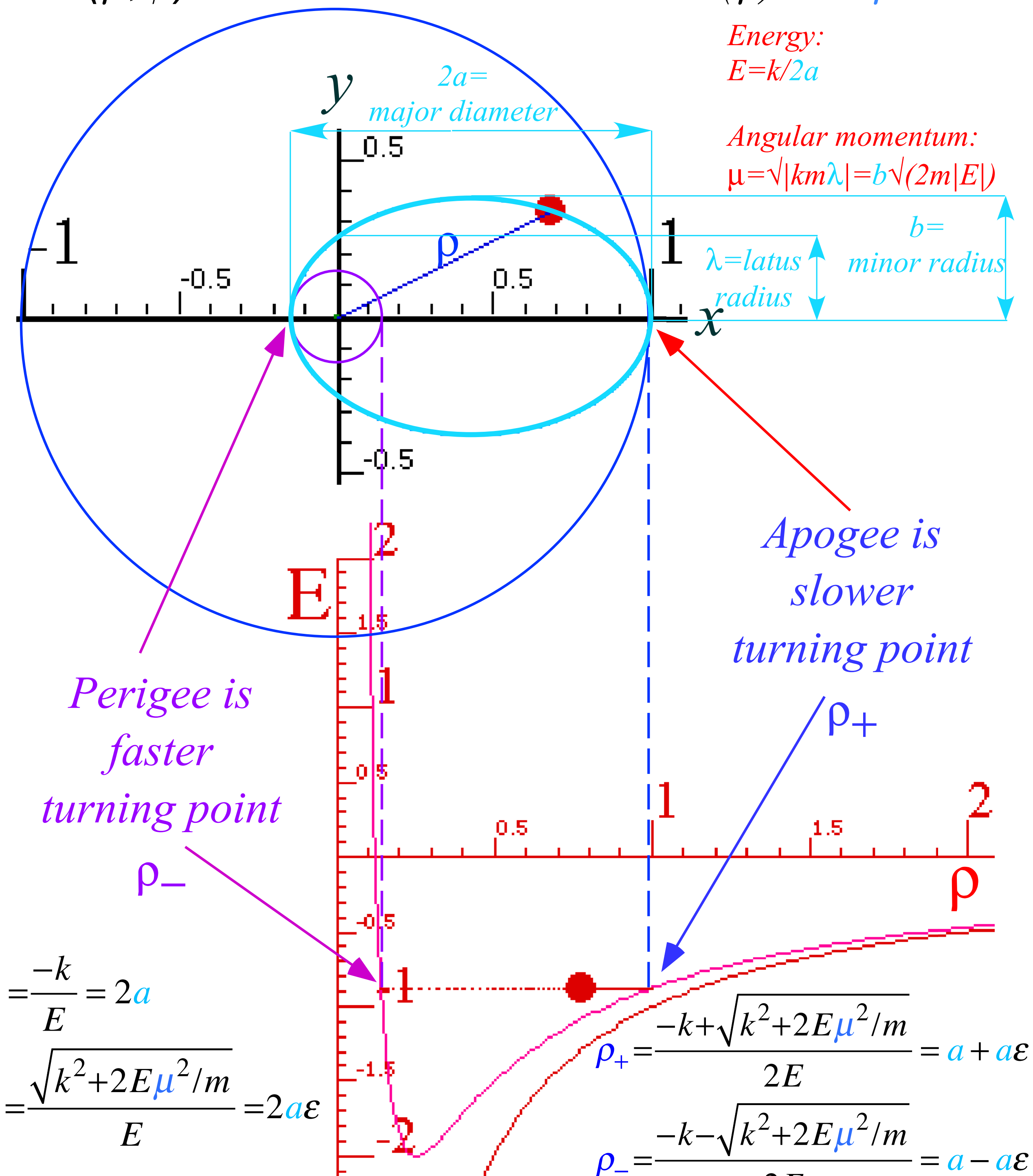


$$\rho_+^2=\frac{E+\sqrt{E^2-k\mu^2/m}}{k}=a^2 \qquad \rho_+^2+\rho_-^2=\frac{2E}{k}=a^2+b^2$$

$$\rho_-^2=\frac{E-\sqrt{E^2-k\mu^2/m}}{k}=b^2 \qquad \rho_+^2-\rho_-^2=\frac{2\sqrt{E^2-k\mu^2/m}}{k}=a^2-b^2=a\varepsilon$$

$$\rho_+^2\rho_-^2=\frac{E^2-E^2-k\mu^2/m}{k^2}=a^2b^2=\frac{-\mu^2}{km}$$

(ρ,ϕ) orbits for Coulomb $V(\rho)=-k/\rho$



$$\rho_++\rho_-=\frac{-k}{E}=2a$$

$$\rho_+-\rho_-=\frac{\sqrt{k^2+2E\mu^2/m}}{E}=2a\varepsilon$$

$$\rho_+=\frac{-k+\sqrt{k^2+2E\mu^2/m}}{2E}=a+a\varepsilon$$

$$\rho_-=\frac{-k-\sqrt{k^2+2E\mu^2/m}}{2E}=a-a\varepsilon$$

$$\rho_+\rho_-=\frac{k^2-k^2-2E\mu^2/m}{(2E)^2}=a^2-a^2\varepsilon^2=\frac{-\mu^2}{2Em}=b^2$$

(to be discussed first: turning point relations)

Just derived equation of IHO orbit ellipse

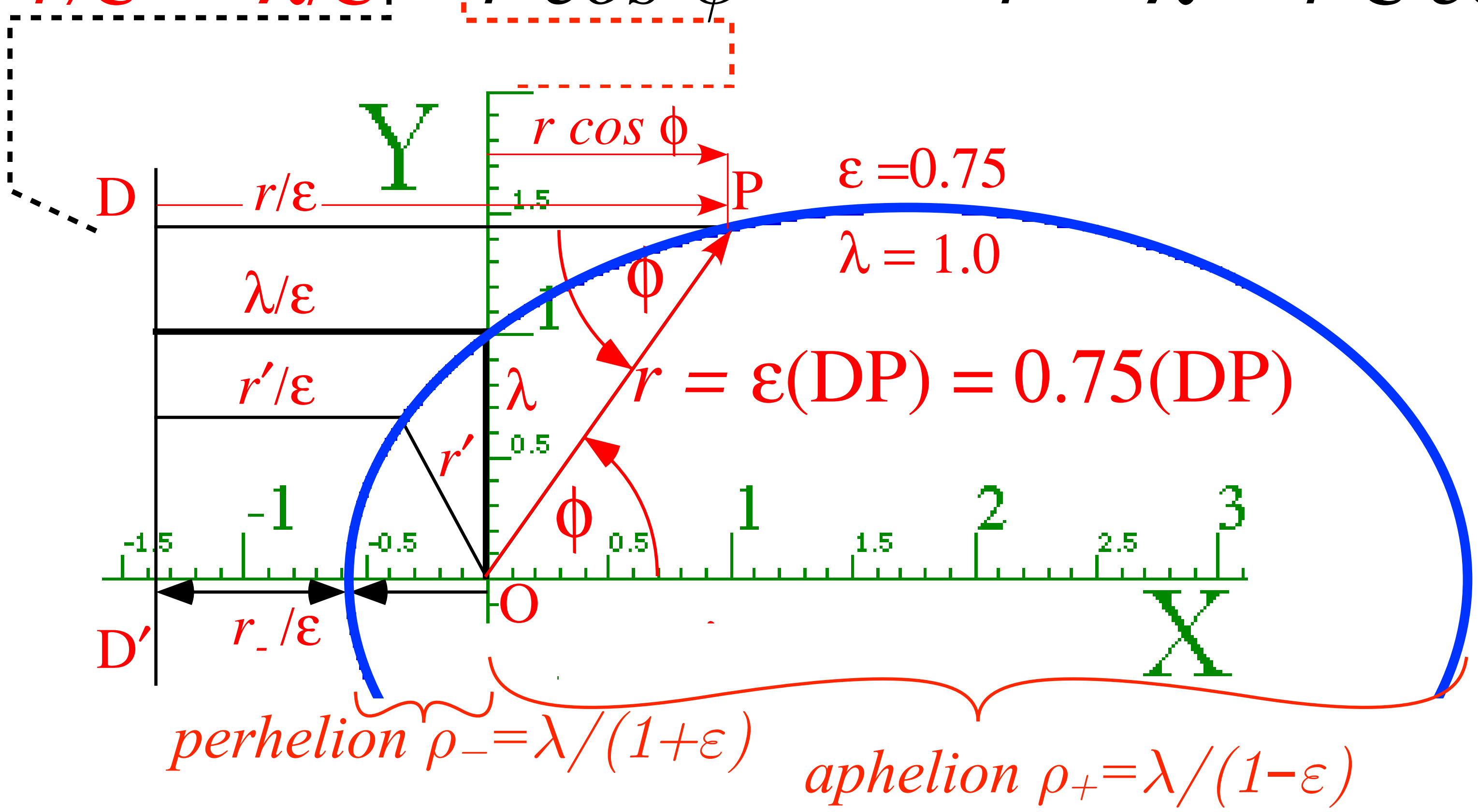
$$x=\frac{1}{\rho^2}=\frac{E}{\mu^2/m}+\frac{\sqrt{E^2-k\mu^2/m}}{\mu^2/m}\sin(-2\phi)$$

Just derived equation of Coulomb orbit ellipse

$$u=\frac{1}{\rho}=\frac{-k}{\mu^2/m}+\frac{\sqrt{k^2+2E\mu^2/m}}{\mu^2/m}\sin(-\phi)$$

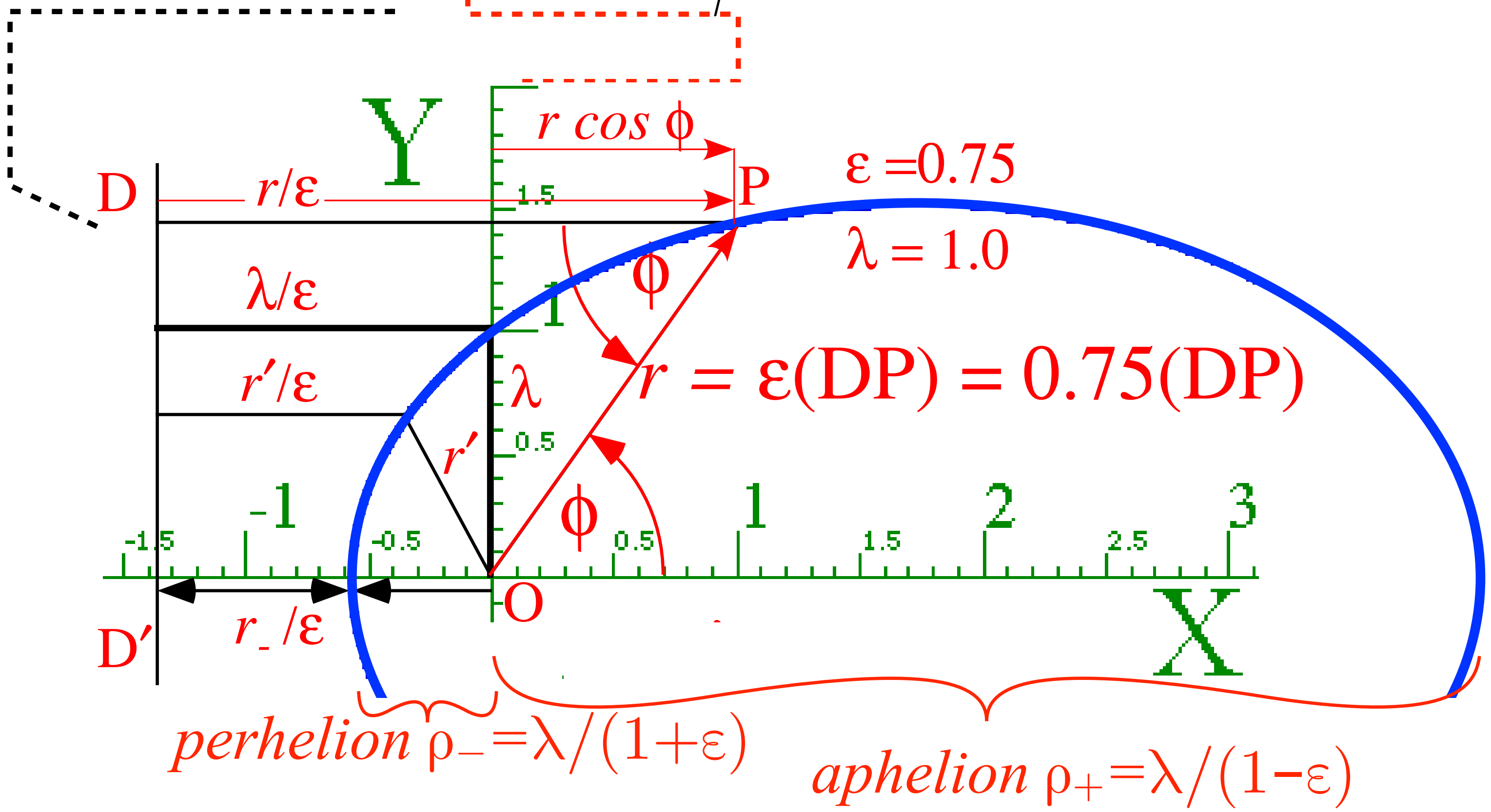
Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi$$



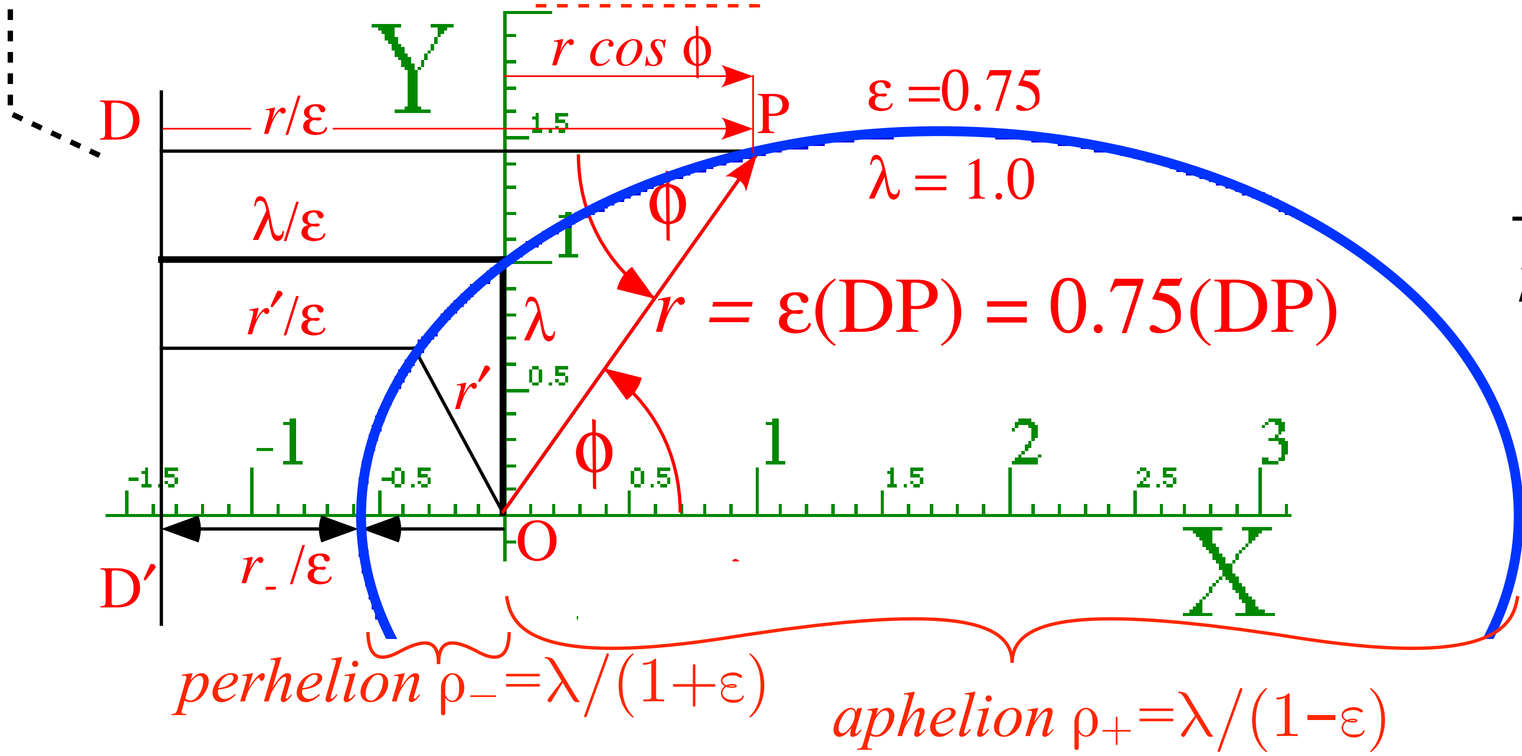
Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

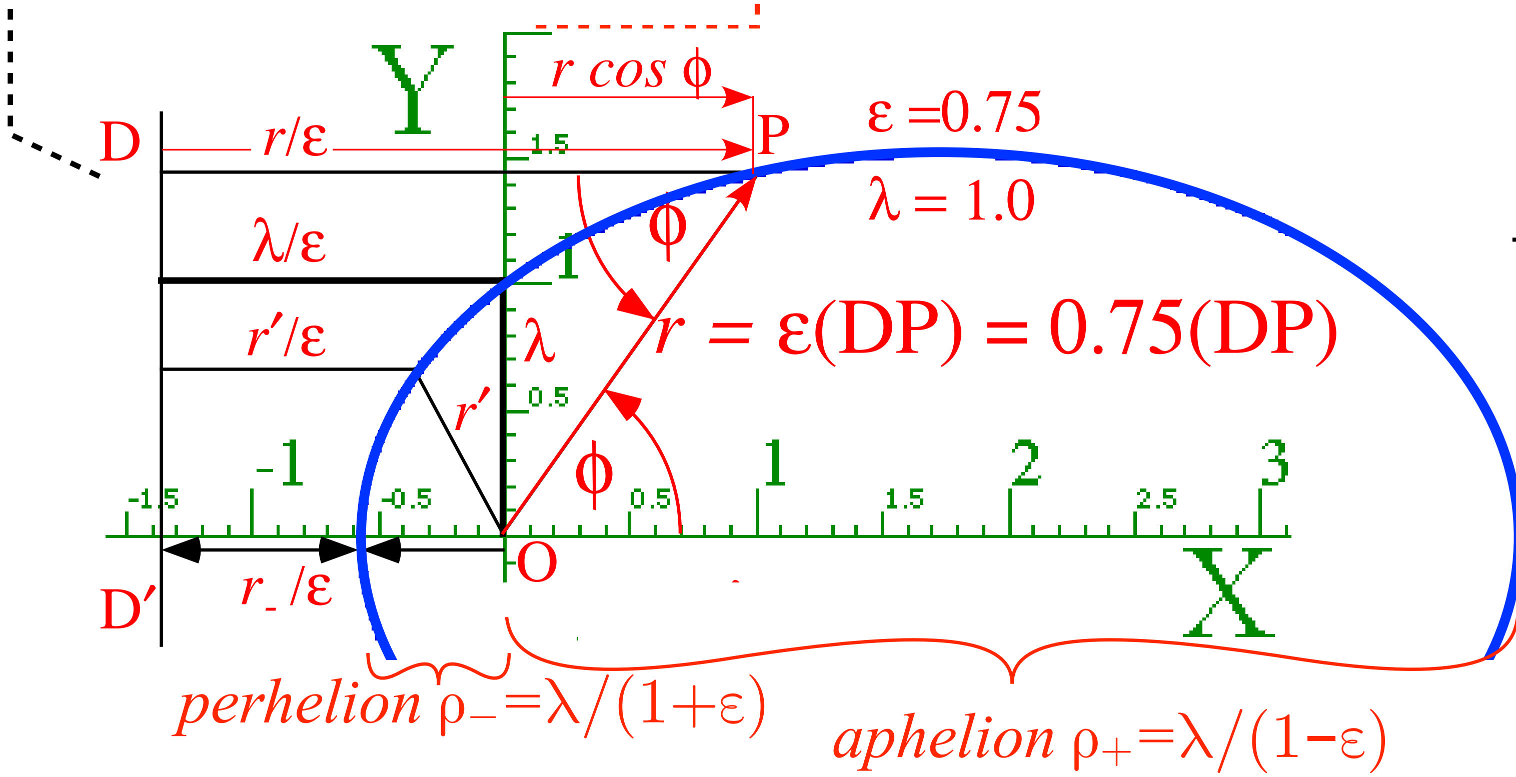
$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$

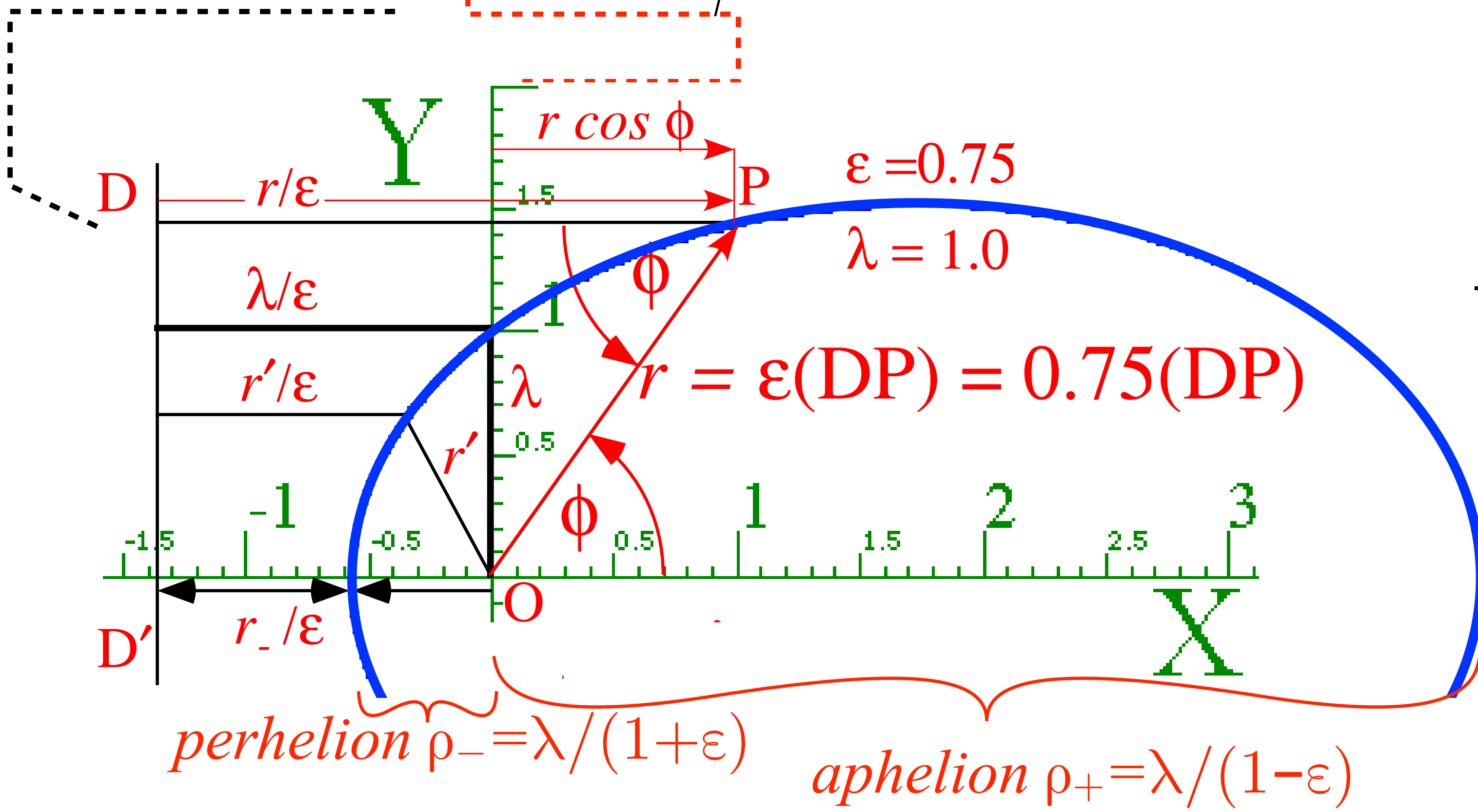


$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

Geometry of Coulomb conic section orbits (Let: $r = \rho$ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$



$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

Cartesian Parameters	Physics	Polar Parameters
Semi-major axis $a=k/\|2E\|$	Energy E	Eccentricity $\varepsilon=\sqrt{(1+2\mu^2E/(k^2m))}$
Semi-minor axis $b=\mu/\sqrt{\|2mE\|}$	Angular momentum $\mu=\ell$	Latus radius $\lambda=\mu^2/(km)$

Orbits in Isotropic Harmonic Oscillator and Coulomb Potentials

Effective potentials for IHO and Coulomb orbits

Stable equilibrium radii and radial/angular frequency ratios

Classical turning radii and apogee/perigee parameters

Polar coordinate differential equations

Quadrature integration techniques

Detailed orbital functions

Relating orbital energy-momentum to conic-sectional orbital geometry

➔ *Kepler equation of time and phase geometry*

Kepler equation of time for Coulomb orbits

Throughout the history of astronomy a most important consideration was the timing of orbits.

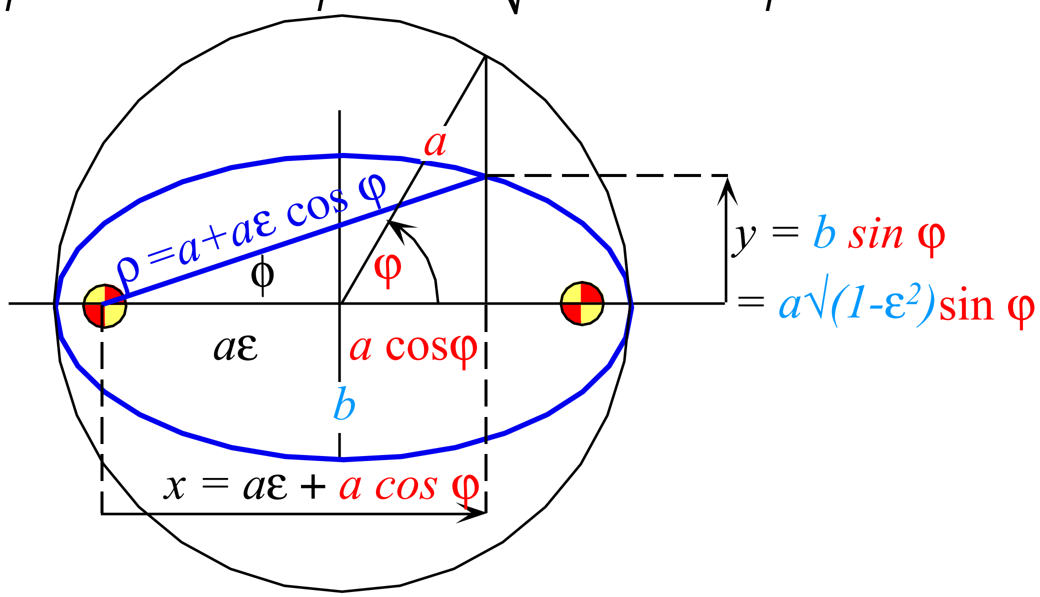
$$t_1 - t_0 = \int_{t_0}^{t_1} dt = \int_{\rho_0}^{\rho_1} \frac{d\rho}{\sqrt{\left(\frac{2E}{m} - \frac{\mu^2}{m^2\rho^2} + \frac{2k}{m\rho}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_0}^{\rho_1} \frac{\rho d\rho}{\sqrt{\left(\frac{E}{k}\rho^2 + \rho - \frac{\mu^2}{2km}\right)}} = \sqrt{\frac{m}{2k}} \int_{\rho_{apogee}}^{\rho_{perigee}} \frac{-\rho d\rho}{\sqrt{\left(\frac{-1}{2a}\rho^2 + \rho - \frac{b^2}{2a}\right)}}$$

$$x = a\varepsilon + a\cos\varphi\ , \quad y = a\sqrt{1-\varepsilon^2}\,\sin\varphi\ ,$$

$$\rho = \sqrt{x^2 + y^2} = \sqrt{a^2\varepsilon^2 + 2a^2\varepsilon a\cos\varphi + a^2\cos^2\varphi + a^2\sin^2\varphi - a^2\varepsilon^2\sin^2\varphi}$$

$$= \sqrt{a^2\varepsilon^2 - a^2\varepsilon^2\sin^2\varphi + 2a^2\varepsilon a\cos\varphi + a^2} = \sqrt{a^2\varepsilon^2\cos^2\varphi + 2a^2\varepsilon a\cos\varphi + a^2}$$

$$\rho = a\left(1 + \varepsilon\cos\varphi\right)$$



$$t = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi d\varphi}{\sqrt{\left(\frac{-\left(a + a\varepsilon\cos\varphi\right)^2}{2a} + a + a\varepsilon\cos\varphi - \frac{a^2\left(1-\varepsilon^2\right)}{2a}\right)}} = \sqrt{\frac{m}{2k}} \int \frac{(a + a\varepsilon\cos\varphi)a\varepsilon\sin\varphi d\varphi}{\sqrt{\left(\frac{-a^2 - 2a^2\varepsilon\cos\varphi - a^2\varepsilon^2\cos^2\varphi + 2a^2 + 2a^2\varepsilon\cos\varphi - a^2 + a^2\varepsilon^2}{2a}\right)}}$$

$$t = \sqrt{\frac{ma^3}{k}} \int (1 + \varepsilon\cos\varphi)d\varphi = \sqrt{\frac{ma^3}{k}}(\varphi + \varepsilon\sin\varphi)$$

$$Orbit\ Period = T = \frac{2\pi}{\omega_\varphi} = 2\pi\sqrt{\frac{ma^3}{k}}$$

Geometry and Symmetry of Coulomb orbits

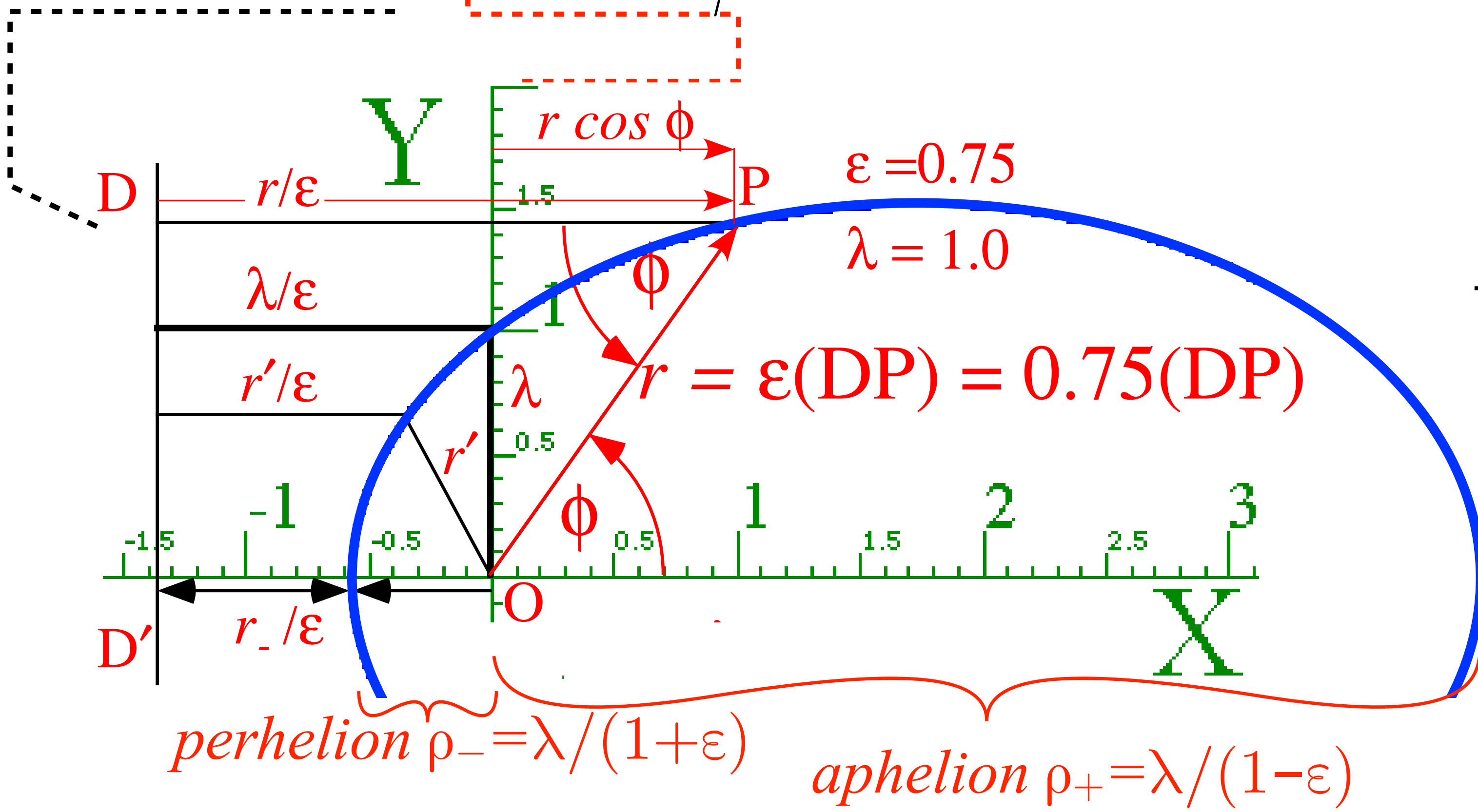
Rutherford scattering and differential scattering crossections

Ruler & compass construction

Eccentricity vector ε *and orbital phase geometry*

Geometry of Coulomb conic section orbits (Let: r =ρ here)

$$r/\varepsilon = \lambda/\varepsilon + r\cos\phi \qquad r = \lambda + r\,\varepsilon\cos\phi \qquad r = \frac{\lambda}{1-\varepsilon\cos\phi}$$

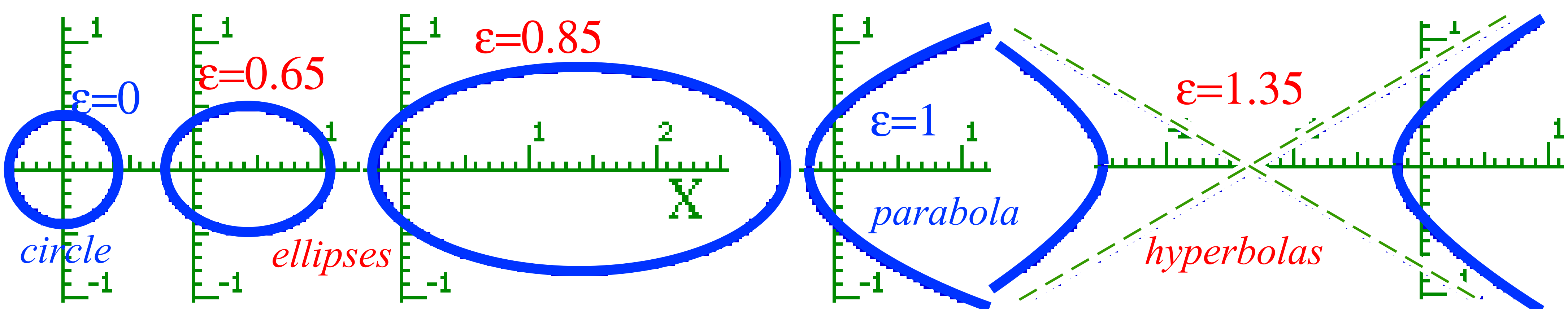


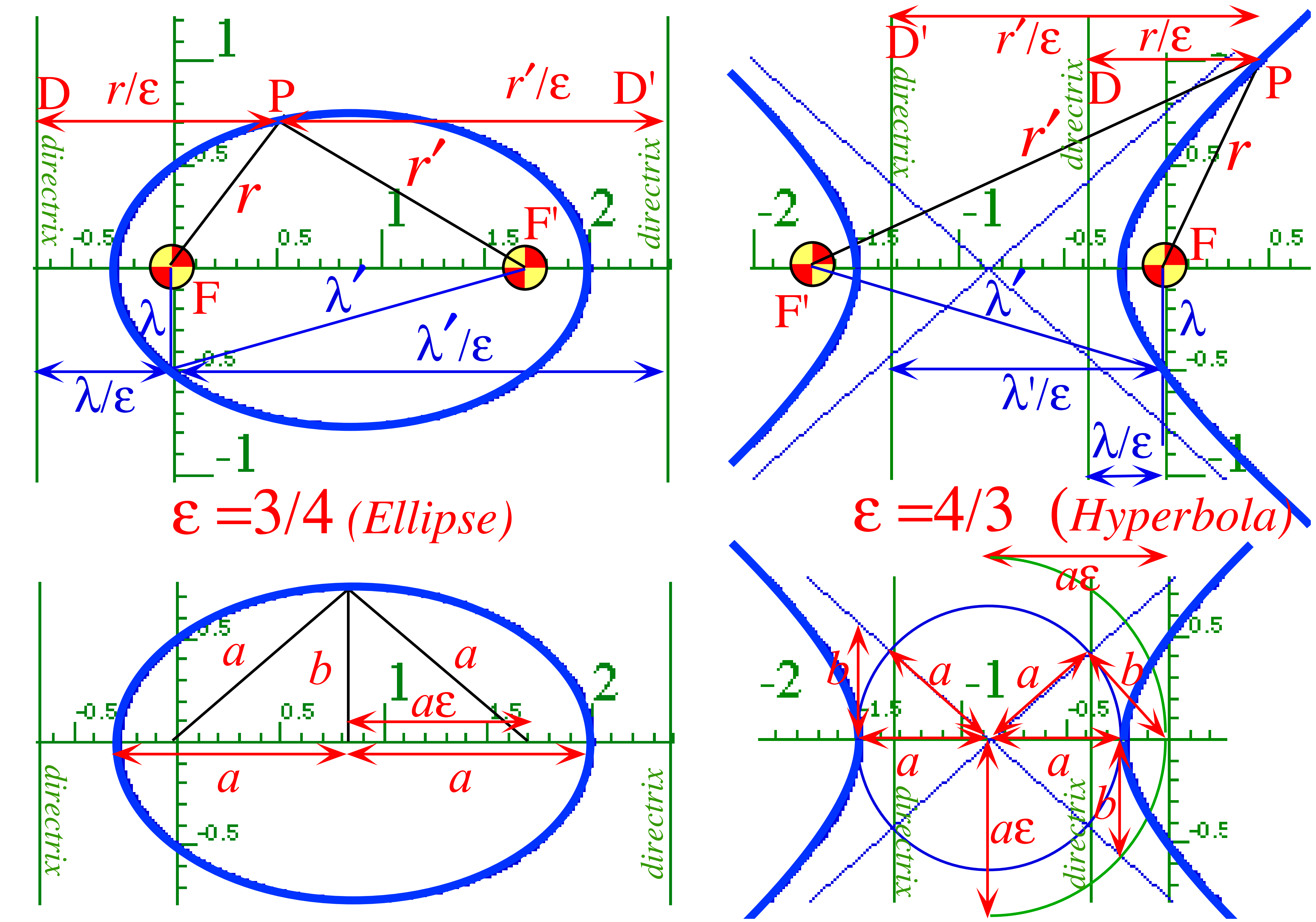
$$\frac{1}{r} = \frac{1-\varepsilon\cos\phi}{\lambda} = \frac{1}{\lambda} - \frac{\varepsilon}{\lambda}\cos\phi$$

$$\frac{1}{\rho} = \frac{-k}{\mu^2/m} + \frac{\sqrt{k^2 + 2E\mu^2/m}}{\mu^2/m}\cos\phi$$

Becoming more and more eccentric...

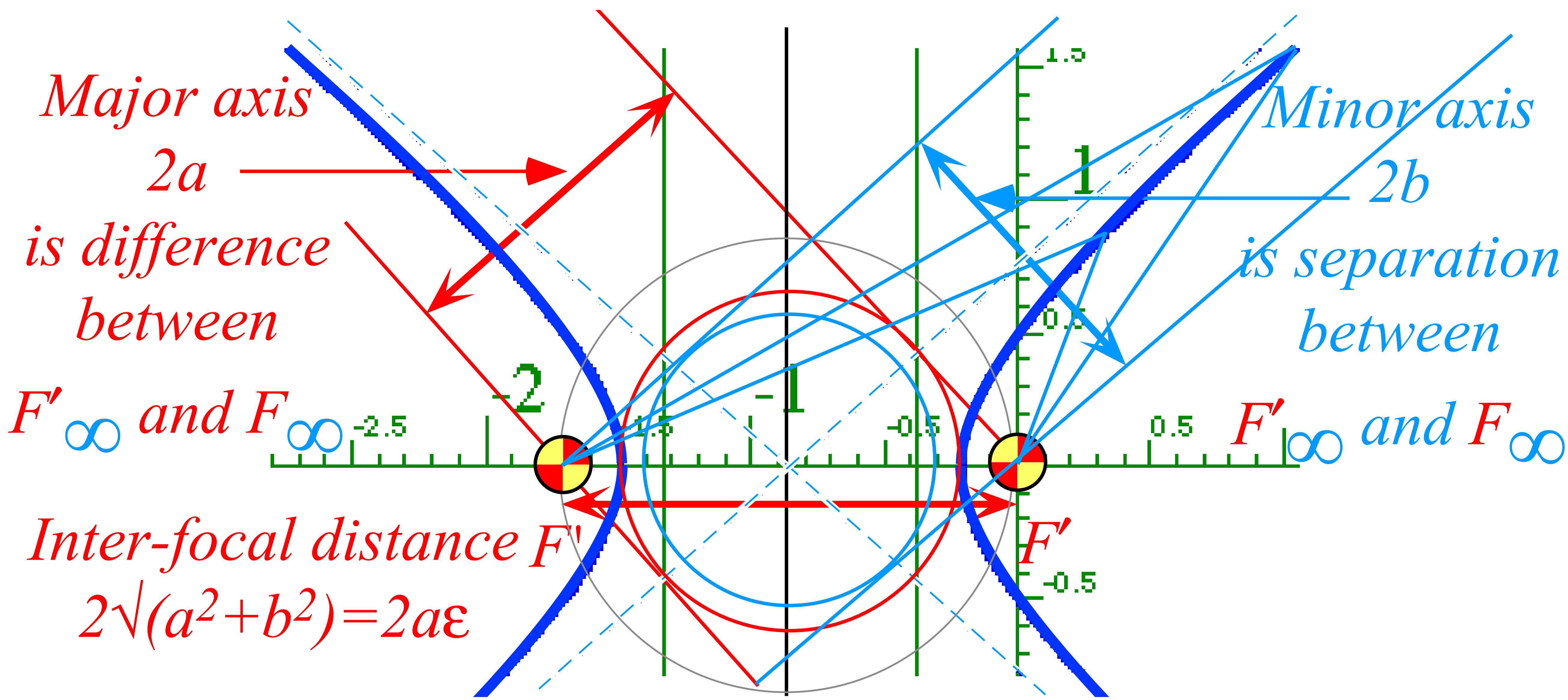
Eccentricity ε=0 (circle) to 0<ε<1 (ellipse) to ε=1 (parabola) to ε >1 (hyperbola)

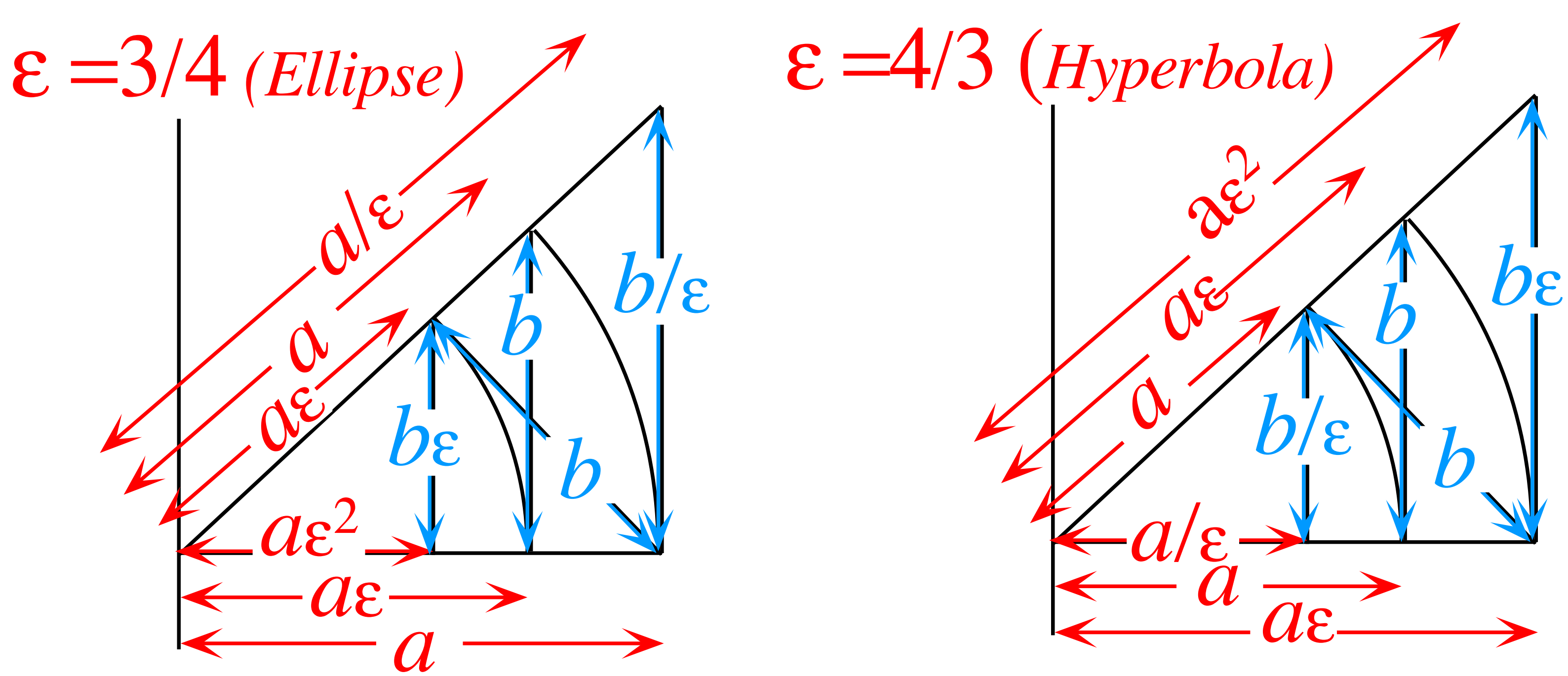


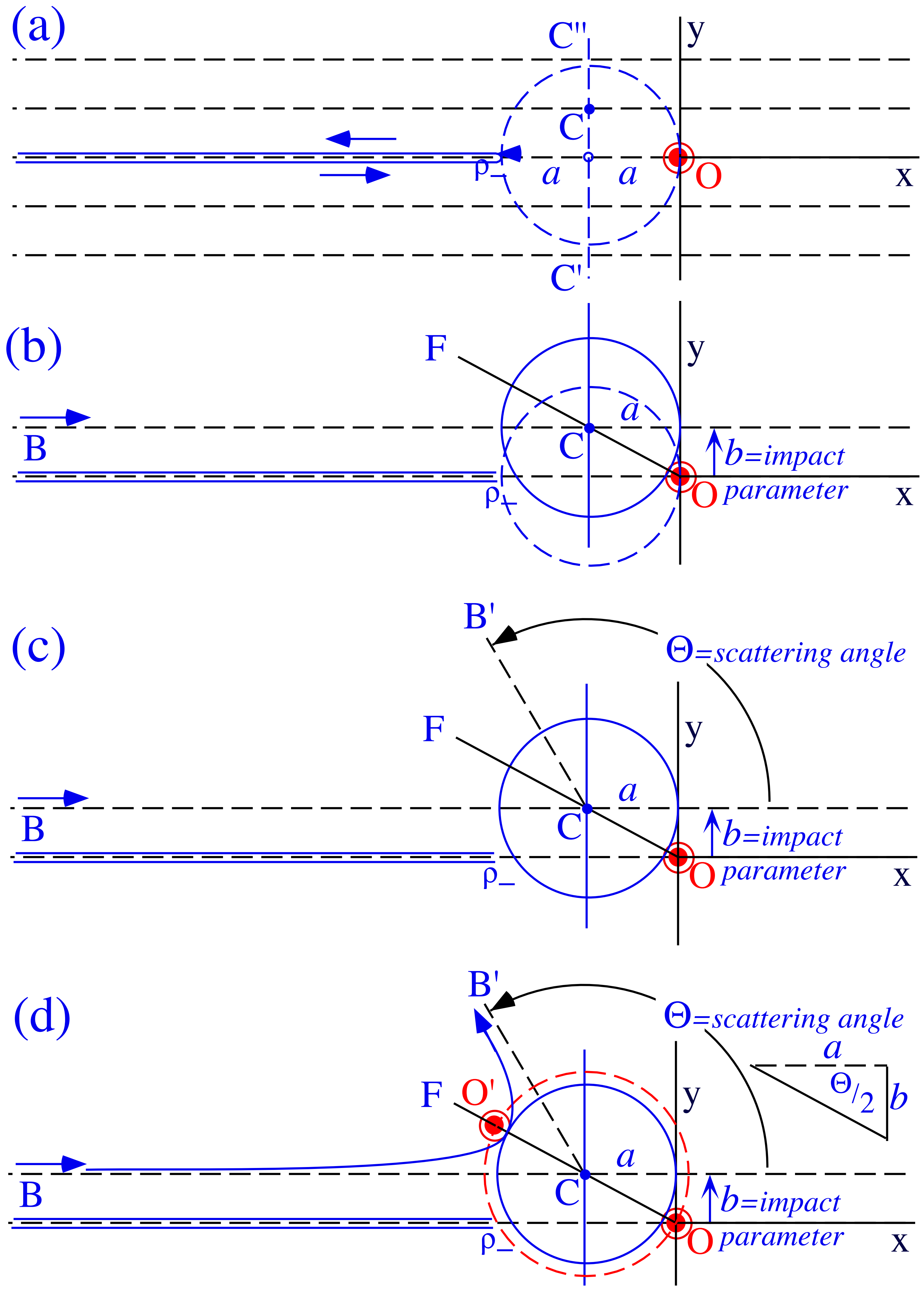


$$2a = |r_+ + r_-| = |\lambda/(1-\varepsilon) + \lambda/(1+\varepsilon)| = |2\lambda/(1-\varepsilon^2)|$$

$$FF' = |r_+ - r_-| = |\lambda/(1-\varepsilon) - \lambda/(1+\varepsilon)| = |2\lambda\varepsilon/(1-\varepsilon^2)| = 2a\varepsilon$$


Major axis
2a
is difference
between
F′∞ and F∞
Minor axis
2b
is separation
between
F′∞ and F∞
Inter-focal distance F′F′
2√(a²+b²)=2aε
-2
-1
1
-2.5
-1.5
-0.5
0.5
1.5
0.5
-0.5


ε =3/4 (Ellipse)
a/ε
a
aε
b/ε
b
bε
b
aε²
aε
a
ε =4/3 (Hyperbola)
aε²
aε
a
bε
b
b/ε
b
a/ε
a
aε


(a)
C''
C
y
a
a
O
x
ρ_
C'
(b)
F
y
a
B
C
b=impact
O parameter
x
ρ+
(c)
B'
Θ=scattering angle
F
y
a
B
C
b=impact
O parameter
x
ρ_
(d)
B'
Θ=scattering angle
a
Θ/2
b
F
O'
y
a
B
C
b=impact
O parameter
x
ρ_