Exercise Set 7 Due Wednesday Oct. 16: Based on Unit 1 Chapter 10 and 12 and Lectures 12-13 (2018).

"Professional" Parabolic and Hyperbolic Coordinates (Relates to Fig. 1.10.7)

1. Consider GCC definition: $q^1=\Phi= x^2 - y^2$, $q^2 =A= 2xy$. Both $(x^1=x,x^2=y)$ and $(q^1=u=\Phi,q^2=v=A)$ are Orthogonal Curvilinear Coordinates (OCC) related by an analytic function $w=z^2$ or $(u+iv)=(x+iy)^2$. You can treat *either one* as Cartesian. (This is based on the analytic function $f(z)=2z$ whose complex potential is ϕ=__________)

(a) Plot $(q^1=u,q^2=v)$ coordinate curves in a Cartesian $(x^1=x,x^2=y)$ graph. Derive the Jacobian, Kajobian, unitary vectors $\mathbf{E}_k$ and $\mathbf{E}^k$ and metric tensors g_{mn} and g^{mn} for this GCC.
(b) Plot $(x^1=x,x^2=y)$ coordinate curves in a Cartesian $(q^1=u,q^2=v)$ graph. Derive the Jacobian, Kajobian, unitary vectors and metric tensors for this GCC.

Galaxy Grids

2. Consider the monopole field function $f(z)=e^{i\alpha}/z$ with complex source $e^{i\alpha}$ discussed in Lectures 12-13.

(a) Derive its $(q^1=\Phi,q^2=A)$ scalar and vector potential coordinate functions.
(b) Plot examples for angle $\alpha=30°$ and $\alpha=45°$.

Fun with Exponentials & more from The Story of e

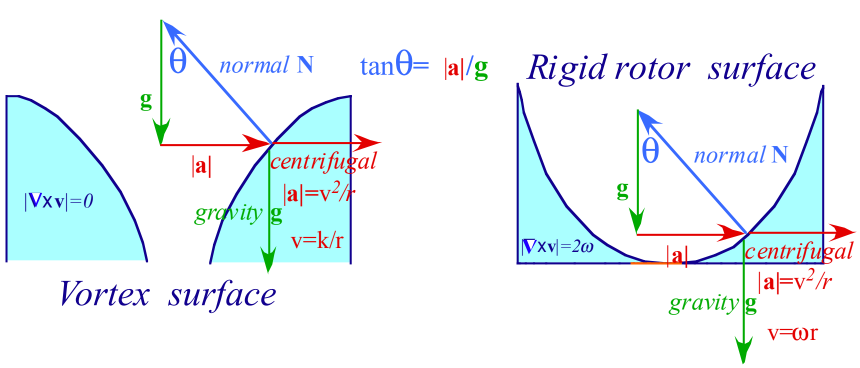
3. Consider a sequence of functions , $f_1(z)=z^z, f_2(z)=z^{f_1(z)}=z^{z^z}, f_3(z)=z^{f_2(z)}=z^{z^{z^z}}$,.... The function $f_N(z)$ has a finite limit $f_\infty(z)$ for N approaching infinity if argument z is small enough . ($z=1$ works! But, so does $z=\sqrt{2}$.)

(a) Find $f_\infty(\sqrt{2})$=____?
(b) Find analytic expression for limiting real z_{max} to give finite $f_\infty(z)$ .It involves Euler constant. $e=2.718281828...$

Fun in the bathtub (This has a peculiar connection to "Sophomore-Physics-Earth" potential.)

4. Derive surface shape of rotating fluid subject to constraints on curl function $\nabla\times\mathbf{v}$ for velocity field. From this you should be able to derive surface altitude $S=S(r)$ as a function of radius r by relating balanced forces to differential slope. (Objects floating on these surfaces would not move up or down their $S(r)$ surface.)

(a) $\nabla\times\mathbf{v}=0$ (Whirlpool or Vortex) Complex vortex field $f(z^*)=v_x(x,y)+i\,v_y(x,y)=i/z^*$ has zero z-derivative and zero divergence (flux derivative $\nabla\bullet\mathbf{v}=0$) and zero curl (circulation derivative $\nabla\times\mathbf{v}=\mathbf{0}$).
(b) $\nabla\times\mathbf{v}=const$. (Rigid rotation) Complex vortex field $f(z)=v_x(x,y)+i\,v_y(x,y)=i\omega\, z$ has constant imaginary z-derivative and therefore zero divergence (flux derivative $\nabla\bullet\mathbf{v}=0$) and constant curl (circulation derivative $\nabla\times\mathbf{v}=\boldsymbol{\omega}$).



(c) How might the *"Sophomore-Physics-Earth"* potential be related to a surface whirlpool in deep water

Solutions: Assignment 6 "Professional" Parabolic and Hyperbolic Coordinates

1 Consider the GCC(Cartesian) definition: $q^1 = (x^1)^2 - (x^2)^2$, $q^2 = 2(x^1)(x^2)$. Both $(x^1=x, x^2=y)$ and $(q^1=u, q^2=v)$ are Orthogonal Curvilinear Coordinates (OCC) related by an analytic function $w=z^2$ or $(u+iv)=(x+iy)^2$.

(a) Plot $(q^1=u, q^2=v)$ coordinate curves in a Cartesian $(x^1=x, x^2=y)$ graph. Derive the Jacobian, Kajobian, unitary vectors and metric tensors for this GCC. See below.

(b) Plot $(x^1=x, x^2=y)$ coordinate curves in a Cartesian $(q^1=u, q^2=v)$ graph. Derive the Jacobian, Kajobian, unitary vectors and metric tensors for this GCC.

$w=(u+iv)=z^2=(x+iy)^2$ is analytic function of z and w

Expansion: $u=x^2-y^2$ and $v=2xy$ may be solved using $|w|=|z^2|=|z|^2$

Expansion: $|w|=\sqrt{u^2+v^2}=x^2+y^2=|z|^2$

Solution: $x^2=\dfrac{u+\sqrt{u^2+v^2}}{2}$ $\quad y^2=\dfrac{-u+\sqrt{u^2+v^2}}{2}$

Easier to get Kajobian first:

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}^u \\ \bar{\mathbf{E}}^v \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ +2y & 2x \end{pmatrix}$$

Jacobian follows by inversion:

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{E}}_u & \bar{\mathbf{E}}_v \end{pmatrix} = \frac{\begin{pmatrix} 2x & +2y \\ -2y & 2x \end{pmatrix}}{4\left(x^2+y^2\right)} = \frac{\sqrt{2}\begin{pmatrix} \sqrt{u+\sqrt{u^2+v^2}} & \sqrt{\sqrt{u^2+v^2}-u} \\ -\sqrt{\sqrt{u^2+v^2}-u} & \sqrt{u+\sqrt{u^2+v^2}} \end{pmatrix}}{4\sqrt{u^2+v^2}}$$

$\det J = 4\left(x^2+y^2\right)=0$ when: $x=y=0$

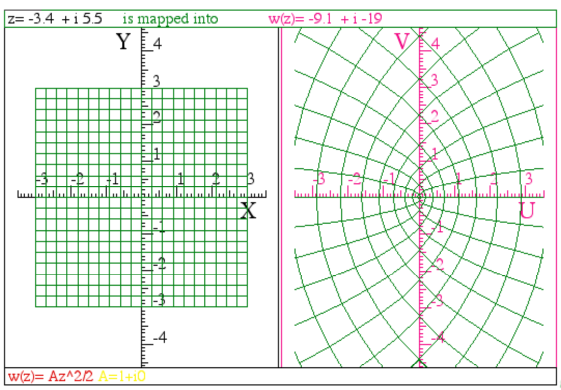
$g^{11}=\bar{\mathbf{E}}^u\bullet\bar{\mathbf{E}}^u=\left(2x \quad -2y\right)\bullet\left(2x \quad -2y\right)=4x^2+4y^2$

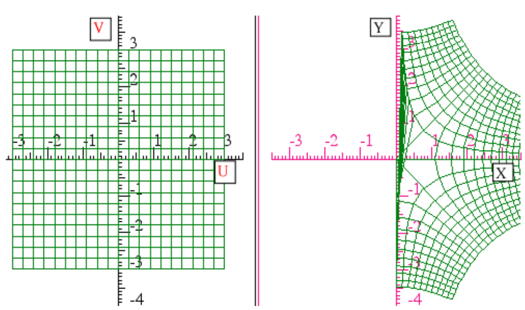
$g^{12}=\bar{\mathbf{E}}^u\bullet\bar{\mathbf{E}}^v=\left(2x \quad -2y\right)\bullet\left(2y \quad 2x\right)=0$

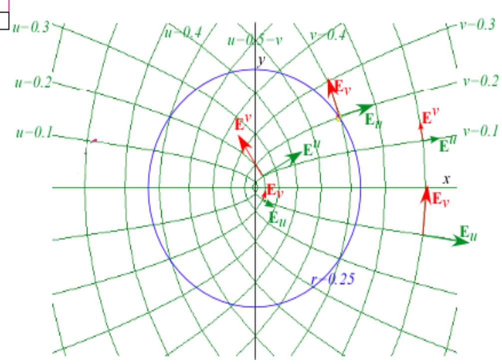
$g^{22}=\bar{\mathbf{E}}^v\bullet\bar{\mathbf{E}}^v=\left(2y \quad 2x\right)\bullet\left(2y \quad 2x\right)=4x^2+4y^2$

$\det g = g_{11}g_{22}-g_{12}g_{12}=(\det J)^2=(4x^2+4y^2)^2$

OCC everywhere: $g^{12}=0$







Note how parabolic covariant vectors grow with distance from origin while contravariant vectors shrink.

Galaxy Grids-Solutions:

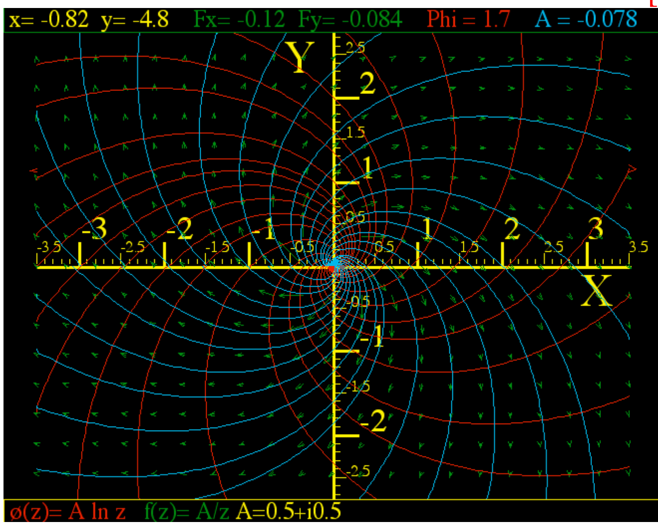
2. Consider the monopole field function $f(z)=e^{i\alpha}/z$ with complex source $e^{i\alpha}$ discussed in Lect. 12-13.

(a) Derive its $(q^1=\Phi, q^2=A)$ scalar and vector potential coordinate functions.

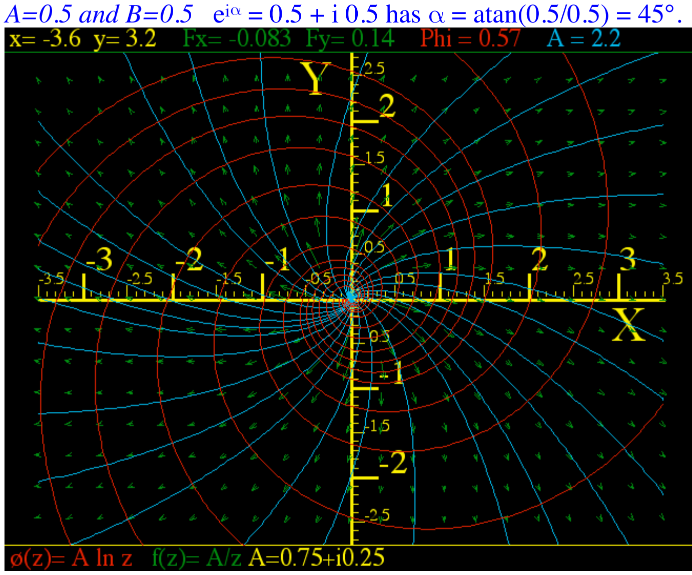
(b) Plot examples for angle α=30° and α=45°.

$$\phi=\int f(z)dz=\int (A+iB)/z\,dz=(A+iB)\ln(z)=(A+iB)(\ln(r)+i\theta)=\left[A\ln(r)-B\theta\right]+i\left[A\theta+B\ln(r)\right]$$

$A=\cos\alpha$ and: $B=\sin\alpha$ $\Phi=\left[A\ln(r)-B\theta\right]=const.$ $|\mathbf{A}|=\left[A\theta+B\ln(r)\right]=const.$



A=0.5 and B=0.5 $e^{i\alpha}$ = 0.5 + i 0.5 has α = atan(0.5/0.5) = 45°.



A=0.75 and B=0.25 A=$e^{i\alpha}$ = 0.75 + i 0.25 has α = atan(0.25/0.75) = 18°.

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3. Consider a sequence of functions , $f_1(z)=z^z, f_2(z)=z^{f_1(z)}=z^{z^z}, f_3(z)=z^{f_2(z)}=z^{z^{z^z}}$,.... The function $f_N(z)$ has a limit for N approaching infinity if argument z is small enough . ($z=1$ works! But, so does $z=\sqrt{2}$.) Find an analytic expression for the limiting real z that involves the Euler constant. $e=2.718281828...$

Solution: Assume limiting case can be reached. Then: $f(z)=z^{f(z)}$ or: $z=f^{\frac{1}{f}}$. Let's plot this function: $y(x)=x^{\frac{1}{x}}$

$1^{\frac{1}{1}}=1.00,\quad 1.5^{\frac{1}{1.5}}=1.30,\quad 2^{\frac{1}{2}}=1.414,\quad 3^{\frac{1}{3}}=1.442,\quad 4^{\frac{1}{4}}=1.414$,...Has max when: $\frac{dy}{dx}=0\ with: \ln y=\ln x^{\frac{1}{x}}=\frac{\ln x}{x}$

Take implicit derivative of $x\ln y=\ln x$: then set $\frac{dy}{dx}=0$:

$$\frac{d}{dx}x\ln y=\frac{d}{dx}\ln x \quad or: \quad \ln y+\frac{x}{y}\frac{dy}{dx}=\ln y=\frac{1}{x},\ so: x^{\frac{1}{x}}\equiv y=e^{\frac{1}{x}}=e^{\frac{1}{e}}=1.44466786$$

Fun in the bathtub (This has a peculiar connection to "Sophomore-Physics-Earth" potential.)
Solution

4. Derive surface shape of rotating fluid subject to constraints on curl function $\nabla\times\mathbf{v}$ on velocity field.

(a) $\nabla\times\mathbf{v}=0$ (Whirlpool or Vortex) Complex vortex field $f(z^*)=v_x(x,y)+i\,v_y(x,y)=i/z^*$ has zero z-derivative and therefore zero divergence (flux derivative $\nabla\bullet\mathbf{v}=0$) and zero curl (circulation derivative $\nabla\times\mathbf{v}=\mathbf{0}$).

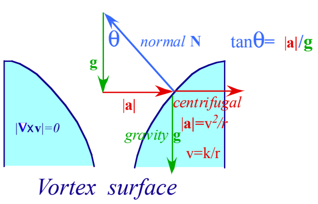
$$0=\frac{dv(z^*)}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(v_x+iv_y)=\frac{1}{2}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)+\frac{i}{2}\left(\frac{\partial v_x}{\partial y}-\frac{\partial v_y}{\partial x}\right)=\frac{1}{2}\left(\nabla\bullet\mathbf{v}\right)+\frac{i}{2}\left|\nabla\times\mathbf{v}\right|_{\perp x,y}$$

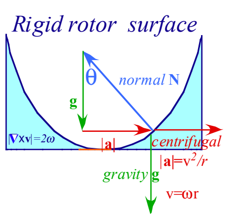
$$for: v(z^*)=\frac{i}{z^*}=\frac{i}{x-iy}=\frac{i}{x-iy}\frac{x+iy}{x+iy}=i\left(\frac{x}{x^2+y^2}+i\frac{y}{x^2+y^2}\right)=v_x+iv_y \text{ where: } \begin{array}{l} v_x=-\dfrac{y}{r^2}=-\dfrac{\sin\theta}{r} \\ v_y=+\dfrac{x}{r^2}=+\dfrac{\cos\theta}{r}\end{array}$$

(b) $\nabla\times\mathbf{v}=const.$ (Rigid rotation) Complex vortex field $f(z)=v_x(x,y)+i\,v_y(x,y)=i\omega\,z$ has constant imaginary z-derivative and therefore zero divergence (flux derivative $\nabla\bullet\mathbf{v}=0$) and <u>constant</u> curl (circulation derivative $\nabla\times\mathbf{v}=\boldsymbol{\omega}$).

$$i\omega=\frac{d(i\omega z)}{dz}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)(v_x+iv_y)=\frac{1}{2}\left(\nabla\bullet\mathbf{v}\right)+\frac{i}{2}\left|\nabla\times\mathbf{v}\right|_{\perp x,y} \quad \text{So: } \nabla\bullet\mathbf{v}=0 \text{ and: } \left|\nabla\times\mathbf{v}\right|_{\perp x,y}=2\omega$$

$$for: v(z)=i\omega z=i\omega(x+iy)=v_x+iv_y \text{ where: } \begin{array}{l} v_x=-\omega y=-\omega r\sin\theta \\ v_y=+\omega x=+\omega r\cos\theta\end{array}$$





$$\tan\theta=\frac{dz}{dr}=\frac{|a|}{g}=\frac{v^2/r}{g}=\begin{cases}(k^2/g)r^{-3} & \text{for vortex } (v=k/r)\\ (\omega^2/g)r^{+1} & \text{for rotor } (v=\omega r)\end{cases} \qquad \text{surface: } z(r)=\begin{cases}-(k^2/2g)r^{-2} & \text{for vortex } (v=k/r)\\ (\omega^2/2g)r^{+2} & \text{for rotor } (v=\omega r)\end{cases}$$

(c) Ideal whirlpools have parabolic core-bottoms starting wherever viscosity wins over curl-free flow.